ACTA UNIVERSITATIS LODZIENSIS
FOLIA PHILOSOPHICA 9, 1993

https://doi.org/10.18778/0208-6107.09.13

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ANALYTICITY AND METAMATHEMATICS

The distinction of analytic/synthetic was explicitly stated for the first time by Kant who referred it to judgements. Other authors applied the distinction also to sentences propositions and statements; in what follows, I shall use the form 'sentence' even if reviewed authors employed another names. Before Kant, related ideas concerned with the distinctions a priori/a posteriori and necessary/contingent had been developed mainly by Hume and Leibniz. Although pre-Kantians did not used the terms 'analytic' and 'synthetic', it is common to regard Leibnitian definition of necessary truth (as a sentence true in all possible worlds) or Humean treatment of relations between ideas (as recorded by tautologies) as important proposals concerning the concept of analyticity.

For Kant, the linguistic structure „A is B" is the general form of sentence. Now a sentence S is an analytic if and only if its predicate A is 'contained' in its subject B; otherwise S is a synthetic sentence. It follows from Kant's definition that negations of analytic sentences are self-contradictory. Moreover, analytic truths are uninformative (tautologous) because they merely analyse the relevant subject concept. Formal logic for Kant consists of analytic sentences. On the other hand, synthetic sentences consist in a synthesis of concepts and provide an information. All analytic sentences are for Kant a priori by definition but synthetic ones can be either a priori or a posteriori. The celebrated problem of Kant's philosophy concerned the possibility of sentences which would be are both synthetic and a priori. Kant himself was entirely convinced that such sentences exist.

The post-Kantian philosophers proposed many definitions of analyticity. Several of them are included in the following list[1] (analytic = analytically true):

[1] See: B. Mates, *Analytic sentences*, „Philosophical Review" 1951, No. 60, p. 525.

(1a) S is analytic iff S is true in all possible worlds;

(1b) S is analytic iff S could not be false;

(1c) S is analytic iff not-S is self-contradictory;

(1d) S is analytic iff S is true by virtue of meanings and independently of facts;

(1e) S is analytic iff either S is logically true or S can be turned into a logical truth by putting synonyms for synonyms;

(1f) S is analytic iff S comes out true under every state-description;

(1g) S is analytic iff S can be reduced to logical truth by definition;

(1h) S is analytic in a language L iff S is true according to the semantical rules of L.

The definition (1a) goes back to Leibniz, (1b) and (1e) are mentioned as possible explications by Quine in his very famous criticism of analyticity[2] (1c) is proposed by Strawson[3], (1d) records a typical positivistic treatment of analyticity[4], (1f) and (1h) are taken from Carnap[5], and (1g) expresses Frege's definition of analyticity.

Various general logical terms occur in definitions (1a) – (1h). Truth, logical truth, definition and contradiction appear explicitly in them but other, for instance model, provability or consistency – implicitly via possible worlds (state descriptions), logical truth and contradiction respectively. We can rewrite for instance (1a) and (1f) as

(2) S is analytic iff S is true in all models

and (1g) as (note that (3) is closer to Frege's original formulation than (1g))

(3) S analytic iff S is provable exclusively by logic and definitions.

Important aspects of metalogical concepts like truth, consistency or provability are formally regulated by metamathematical theorems; for simplicity, I assume that metamathematics comprises metalogic and formal semantics. So we can ask what follows from metamathematics for the „philosophy of analyticity". My aim in this paper is to put together (with some comments) various observations on analyticity which have been made by several contemporary logicians from the metamathematical point of view. I shall center on so called limitative theorems, in particular

(4) if X contains formalized Peano arithmetic, then X is incomplete if consistent (the first Gödel incompleteness theorem);

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[2] See: V. van Quine, *Two dogmas of empiricism*, „Philosophical Review" 1951, No. 60, p. 20–43.

[3] See: P. Strawson, *Introduction to Logical Theory*, Methuen, London 1952, p. 21.

[4] See: A. Ayer, *Language, Truth and Logic*, Penguin Books, Harmodwordth 1971, p. 104–106.

[5] See: R. Carnap, *Logical Foundations of Probability*, Routledge and Kegan Paul, London 1962, p. 83 and R. Carnap, *Meaning and Necessity*, The University of Chicago Press, Chicago 1956, p. 8, 10.

(5) if S contains formalized Peano arithmetic, then consistency of S is unprovable in S (the second Gödel incompleteness theorem);

(6) Peano arithmetic and first order logic are not decidable (the Church undecidability theorem).

The first analysis of analyticity with the help of metamathematics was given by Carnap[6]. He distuinguished Language I consisting of elementary logic together with the portion of arithmetic sufficient for arithmetization (in the sens of Gödel) and Language II which contains all means which are needed for expressing classical mathematics in it. Now analyticity in Language I is defined by

(7) S is analytic in LI iff S is a consequence of the null class of sentences (or every sentence);

However, (7) is too narrow for LII because arithmetic is incomplete what causes that „in every sufficiently rich system for which the method of derivation is prescribed, sentences can be constructed which, though they consist of symbols of the system, are yet not resoluble in accordance with the method of the system – that is to say, are neither demonstrable nor refutable in it. And, in particular, for every system in which mathematics can be formulated, sentences can be constructed which are valid in the sense of classical mathematics but not demonstrable within the system"[7]. So we have sentences which are not consequences of every sentence. To solve this difficulty, Carnap (he wanted to have all mathematical truths amont analytic sentences) proposes to admit infinite sets of premises and supplement rules of proof by non-effective ones, for instance ω-rule. Carnap's definition of analyticity for Language II is too complicated in order to present it here in a detailed way but the general idea is captured by

(8) S is analytic in LII iff S is derivable from analytic sentences by rules of proof which are admissible in LII.

As far as I know, Gödel adressed to the problem of analyticity only once in his published works, namely in his paper on Russell's mathematical logic[8]. According to Gödel

(9) S is analytic iff A is a special case of the law of identity in virtue of explicite definitions of terms or rules of their elimination.

However, Gödel observes that (6) implies non-analyticity of arithmetic. Admitting sentences of infinite length does not save the situation because to prove that some important mathematical theorems (for instance, the axiom of choice) are analytic, one would have to assume analyticity of the whole

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[6] See: R. Carnap, *Logical Syntax of Language*, Routledge and Kegan Paul, London.

[7] *Ibid.*, p. 100.

[8] See: K. Gödel, *Russell's Mathematical Logic*, [in:] *The Philosophy of Bertrand Russelll*, ed. P. Schilpp, Open Court, La Salle 1944, p. 123–153.

mathematics in advance. Gödel also considers another definition of analyticity, namely a version of (1d) but he does not link it with any metamathematical fact.

The next step in the history in question was made by Copi and Turquette[9]. Copi examines the following definition of analyticity

(10) S is analytic iff its truth or validity follows from the syntactical or grammatical rules governing a language rules in which it is expresed.

from the point of view (4) formulated by him as

(11) given any reasonably rich language, there is non-empirical, non-inductive proposition expressible within it which is not decidable on the basis of the syntactical rules of that langauge.

Then Copi says that (11) leads to

(12) there are a priori non-analytic truths,

which destroys the analytic theory of a priori (all a priori sentences are analytic).

Turquette makes several objection against Copi. Let me mention two. The first is general: „In fact, the claim that there are Gödel synthetic *a priori* truths then amounts to nothing more than a restatement in misleading philosophical language of some well-established logical results, notably of what is usually called Gödel's second incompleteness theorem"[10]. Secondly, Turquette observes that undecidable statements could be interpreted as empirical or well-formed but devoid of meaning.

Copi in his answer[11] says that his theses are not derived from undecidable sentences but from the fact that „there are such statements as Gödel's which are a priori true but not analytic"[12]. Moreover, he rejects the empirical theory of mathematics and observes that regarding undecidable sentences as devoid of meaning is untenable because we understand them.

Turquette positive solutions require either accepting that mathematics is empirical or a revision of logic; both proposal must meet several well-known objections. Turquette's general objection against Copi raises a serious methodological problem. Gödel's theorems (like other limitative results) says nothing on analyticity or apriority. So Copi's formulation of (4) is in fact its certain philosophical interpretation which should be separatately justified. Moreover, (12) is derived by Copi from (10) but it may not hold under other definitions of analyticity.

[9] See: I. C o p i, *Modern Logic and the Synthetic a priori*, „Journal of Philosophy" 1949, No. 46, p. 243–245; I. C o p i, *Gödel and the Synthetic a priori: a Rejoinder*. „Journal of Philosophy" 1950, No. 47, p. 633–636; A. T u r q u e t t e, *Gödel and the Synthetic a priori*, „Journal of Philosophy" 1950, No. 47, p. 125–128.

[10] T u r q u e t t e, *Gödel...*, p. 126.

[11] See: C o p i, *Gödel...*

[12] *Ibid.*, p. 634.

Kemeny[13] argues that the concept of intended model (interpretation) forms an adequate conceptual base for formal semantics. Assume that we define analytic propositions as those which are universally valid, i.e. hold in all models. This definition is too narrow in virtue of incompleteness of arithmetic. Kemeny argues that a more satisfactory account of analyticity is to be obtained with the help of the concept of intended model.

Let T be a theory, i.e. a set of sentences closed under the consequence operation. Now all intended models of T have exactly the same universes. Moreover, if M and M' are intended models, then both can mutually differ only with respect to valuation of extralogical constants; Kemeny considers arithmetical constants as logical. Then

(13) S is analytic in L iff A is true in all L-interpretations, i.e. L intended models of L.

Assume that S is analytic in L. A theory T is complete (Kemeny says that it is the most natural concept of completeness) if and only if

(14) S belongs to Cn(T) if A is analytic in T.

If T is complete, then its analytic truths can be defined as valid in all models. But if T is incomplete, this definition must be replaced by (13) because for instance we have arithmetical truths which are not valid in all models of arithmetic.

Kemeny's approach raises some doubts. Let S be an undecidable formula in its intended meaning. Consider its negation not-S. We can easy define a set of models in which S holds. One can even claim that models of not-S (not those of S) are intended. This means that not-S is analytic on this claim. So we obtain that two mutually contradictory sentences are analytic. This reasoning shows that the concept of analyticity via L-interpetations is rather pragmatic and relativised than semantic and absolute.

Borkowski[14] considers two definitions of analyticity, namely

(15) S is analytic in the syntactic sense iff S is provable exclusively by logic;

(16) S is analytic in the semantic sense iff S is true in all models.

According to Borkowski, the first Gödel theorem implies that not every sentence semantically analytic is also syntactically analytic. However, this thesis is dubious. If sentence S is true in all models, it is (by completeness theorem) provable exclusively by logic. This means that both classes of analytic sentences mutually coincide.

[13] See: J. K e m e n y, *A New Approach to Semantics*, „Journal of Symbolic Logic" 1956, No. 21, Part 1, p. 1-27; Part 2, p. 149-161.

[14] L. B o r k o w s k i, *Deductive Foundation and Analytic Propositions*, „Studia Logica" 1966, No. 19, p. 59-72; L. B o r k o w s k i, *Logika formalna*, PWN, Warszawa 1970; L. B o r k o w s k i, *Wprowadzenie do logiki i teorii mnogości*, Towarzystwo Naukowe Katolickiego Uniwersytetu Lubelskiego, Lublin 1990.

DeLong[15] argues that (the formula Con(Ar) means „arithmetic is consistent").

(16) The sentence Con(Ar) under its intended interpretation is synthetic *a priori*.

The formula expressing consistency of arithemic is synthetic because it is not provable exclusively by general logic and definitions and a priori because if arithmetic is consistent, it is necessarily so.

Now assume that Con(Ar) is necessary true. Let M is the standard model of Ar. So Nec(Con(Ar)) is true iff and only if Con(Ar) is true in all models accessible from M. However, these models are not determined a priori but with respect to pragmatic criteria of standardeness. To obtain (16) one has to show that Con(Ar) holds in all models in which Peano axioms hold but it would be inconsistent with undecidability of Con(Ar).

Castonguay[16] claims that Church's theorem (together with Church's thesis) implies that mathematical knowledge is synthetic a priori. However, this is too strong claim because (6) implies only that mathematical knowledge is nor reducible to purely algorithmic procedures. Castonguay seems to assume

(17) if X is a set of analytic sentences, than X is decidable.

But this supposition is by no means obvious.

There is not systematic treatment of analyticity from the point of view metamathematics. On the other hand, metamathematical seem to be of a fundamental importance for any analysis of analyticity. Let me finish this survery with some very general observations[17]. Metamathematics suggests two divisions of analytic sentences: (I) into syntactic, semantic and pragmatic (note however that my proposals in this respect considerably differ from those of Borkowski[18]), and (II) into absolute and relative. The proposed definitions are as follows:

(17) S is an absolute semantic analytic sentence iff S is universally valid;

(18) S is an absolute syntactic analytic sentences iff S is an absolute semantic analytic sentence and S belongs to a decidable set of logical truths;

(19) S is a relative semantic analytic sentence in a theory T iff S is true in all models of T;

(20) S is a relative syntactic analytic sentence in a theory T iff S is a relative syntactic analytic sentence in a theory T and S belongs to a decidable set of truths of T;

[15] See: H. DeLong, *A Profile of Mathematical Logic*, Addison-Wesley, Reading, Mass. 1970.

[16] See: Ch. Castonguay, *Church's Theorem and the Analytic-synthetic Distinction in Mathematics*, „Philosophica" 1976, No. 18, p. 77–89.

[17] See: J. Woleński, *Metamatematyka i epistemologia*, PWN, Warszawa (forthcoming).

[18] See: Borkowski, *Deductive Foundation...*

(21) S is a pragmatic analytic sentence in a theory T iff S is true in all standard models of T.

Obviously we have,

(22) absolute syntactic analytic sentences $\subseteq$ absolute semantic analytic sentences $\subseteq$ pragmatic analytic sentences (the same holds if 'absolute' will be replaced by 'relative').

So syntactic analytic sentences are those which can be resolved by algorithmic methods. Moreover only logic consists of absolute analytic sentences. These consequences are consistent with many traditional accounts concerning analytic sentences.

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ANALITYCZNOŚĆ I METAMATEMATYKA

Chociaż rozróżnienie sądów analitycznych i syntetycznych pojawiło się po raz pierwszy u Kanta, to pokrewne pojęcia można odnaleźć już u Hume'a i Leibniza. Autor zestawia i analizuje różne definicje i charakterystyki pojęcia analityczności, jakie proponowali m. in.: Kant, pozytywiści, Frege, Carnap, Strawson i Quine. Wskazuje się, że w badaniach nad zagadnieniem analityczności często odwoływano się do takich pojęć metalogicznych, jak: prawdziwość, niesprzeczność, czy dowiedlność, a te z kolei zostały scharakteryzowane na gruncie metamatematyki przez tzw. twierdzenia limitacyjne, w szczególności przez twierdzenia Gödla o niezupełności i twierdzenie Churcha o nierozstrzygalności. W związku z tym referowano dyskusję nad związkiem ww. twierdzeń z zagadnieniem rozstrzygalności prowadzoną przez samego Gödla, a także przez Turquette'a, Copiego, Kemeny'ego, Borkowskiego i in.