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CONDITIONALS, BASED ON STRICT ENTAILMENT

Conditionals can be obtaind by several ways:
- as a result of empirical investigations;
- from other conditional or nonconditional propositions by logical rules;
- from definitions or other terminological statements;
- postulated;
- from sentences about logical entailment.

The last way of obtaining conditional propositions is the main topic of the paper which deals only with first degree conditionals (conditionals, containing only one occurence of the conditional operator). Such conditionals are important: they are logically true and are used to draw conclusions from facts to get facts.

Every true entailment A ⊢ B corresponds to a true conditional A → B. What kind of conditionals we get depends obviously on the system of logical entailment, which rules the entailments, and on the logical rules, governing conditionals. The basic system is in this case the system F^s of strict entailment constructed by Wessel. The proposed introduction-rule for conditionals allowes to use two implicative structures: entailments and conditionals, with different properties. Using some conditional principles the class of conditionals can change while the class of entailments remains unchanged.

The alphabet of F^s consists of

1) countable many propositional variables p, q, r, p_1,...;

2) truth-functional connectives ∧ (conjunction), ∨ (disjunction), ~ (negation);

3) the predicate of entailment ⊢;

4) parentheses.

D1. A formula is a truth-functional formula, if it is constructed by the usual rules with truth-functional connectives only.

D2. A formula is a formula of logical entailment, if it has the structure A $\vdash$ B, and A and B are truth-functional formulas.

The postulates for F^s are all formulas of logical entailment having the form of one of the following schemata and meeting the conditions E1 and E2:

E1. If A $\vdash$ B, then B contains only such propositional variables, which are also in A.

E2. If A $\vdash$ B, then A is not a contradiction and B is not a tautology.

A1. $A \vdash \sim \sim A$

A2. $\sim \sim A \vdash A$

A3. $A \wedge B \vdash A$

A4. $A \wedge B \vdash B \wedge A$

A5. $\sim (A \wedge B) \vdash \sim A \vee \sim B$

A6. $\sim A \vee \sim B \vdash \sim (A \wedge B)$

A7. $(A \vee B) \wedge C \vdash (A \wedge C) \vee B$

A8. $(A \wedge C) \vee (B \wedge C) \vdash (A \wedge B) \wedge C$

A9. $A \vdash A \wedge (B \vee \sim B)$

The rules of F^s are:

R1. If A $\vdash$ B and B $\vdash$ C, then A $\vdash$ C.

R2. If A $\vdash$ B and A $\vdash$ C, then A $\vdash$ B $\wedge$ C.

R3. If A $\supset$ B and B $\supset$ A are tautologies, then C $\vdash$ C[A/B], where C[A/B] means that in C all or some (including no one) occurences of A are to be replaced by B, and C is not a contradiction and C[A/B] is not a tautology.

Wessel proved: A formula of logical entailment A $\vdash$ B is a theorem in F^s if and only if: A $\supset$ B is a tautology, B contains only such variables, which are also in A, A is not a contradiction and B is not a tautology[1].

To get a conditional system we introduce a non-truth-functional connective $\rightarrow$ (conditional operator: if – then) into the language:

D3. A formula is a sentence, if the following conditions are satisfied: 1. Truth-functional formulas are sentences. 2. If A and B are sentences, $\sim$ A, (A $\wedge$ B), (A $\vee$ B), (A $\rightarrow$ B) are sentences.

The construction is completed by the conditional axiom and the conditional rule:

A10. $\vdash A \rightarrow A$

R4. If A $\vdash$ B and $\vdash$ B $\rightarrow$ C, then $\vdash$ A $\rightarrow$ C.

For the resulting system F^{SK} it is easy to show:

S1. If $\vdash$ A in F^{SK}, then A is a conditional.

S2. A $\vdash$ B is theorem in F^S if and only if A $\vdash$ B is theorem in F^{SK}.

S3. A $\vdash$ B is theorem in F^S if and only if $\vdash$ A $\rightarrow$ B is theorem in F^{SK}.

S4. If $\vdash$ A $\rightarrow$ C and $\vdash$ B $\rightarrow$ C are theorems in F^{SK}, then $\vdash$ (A $\vee$ B) $\rightarrow$ C is theorem in F^{SK}.

[1] Cf. H. Wessel, *Logik*, Berlin 1984, p. 170–173.

S5. If $\vdash (A \wedge B) \rightarrow C$ is theorem in F^{SK}, and B contains no variables, occuring in A or in C, then $\vdash A \rightarrow C$ is theorem in F^{SK}.

■If $A \wedge B \vdash C$ is theorem, A ist not a contradiction and C is not a tautology. Under this condition and because of the restriction on B in C occur only such variables, which are also in A. Let *W* be a valuation, which prescribes A the value *T* and C the value *F*. In any case *W* can be extended to a valuation *W* including the variables of B, that prescribes the value *T* to A B too. Because of S3 and Wessel's result mentioned above the sentence is proved. ■

S6. If $\vdash A \rightarrow B$ is theorem in F^{SK}, then $\vdash (A \wedge C) \rightarrow B$ is theorem in F^{SK}, where $A \wedge C$ is not a contradiction.

■Use A3 and R1. ■

Obviously $A \rightarrow B$ is not a theorem in F^{SK}, if A is a contradiction or if B is a tautology. In a direct sense this system is a paraconsistent logic: the appearance of contradictory data does not force the system to be explosive, to derive any formula. The unusual restriction not to conclude from contradictions is a difference between relevant and paraconsistent logics and F^{SK} and has to be explained. In relevant logic from $p \wedge \sim p$ does not follow q, but it follows p and also $\sim p$. Even if one stipulates that there are true contradictions probably not all contradictions are true, therefore in some cases from a contradiction does not follow all nonsense you want (as in classical logic), but a little nonsense anyway. In order to avoid any nonsense the restriction on the antecedents is made. On the other hand the restriction on the consequents is understandable at once: why we should conclude tautologies, if we already know that they are tautologies? Such implications are often funny, so there is an old german rule: If the cock crows on the dunghill, the weather is changing or it remains unchanged.

There are good reasons for the restrictions, but sometimes they seem to be very hard. Systematically violating them we construct weaker systems.

We start to build up several systems of conditional logic by adding conditional rules. In all systems the set of entailments remains unchanged, it is the set of theorems of F^s. The concrete choice of rules, which we want to use, depends of course on practical purposes. So it may, for example, be useful to have the non-monotonic relation of entailment together with a monotonic conditional operator. Such things can be done, as we want to show.

A disadvantage of F^{SK} is the absence of the substitution rule. So it is necessary to distinguish logically between $(p \wedge q) \rightarrow p$ (what is valid on the base of A3) and $(p \wedge \sim p) \rightarrow p$ (what is invalid because of E2), though the letter is derivable from the former by substitution. Logicians working in relevant logic would argue, that substitution is a logical rule and therefore the set of conditionals, obtained from sentences about entailments, should consist of not only the corresponding conditionals, but also of all substitutions in such conditionals.

We get F^{SK5} by adding the following rule to F^{SK}:

R5. If $\vdash A \to B$, then $\vdash C \to D$, where $C \to D$ is the result of substituting propositional variables of $A \to B$ by truth-functional formulas.

In this system we can prove conditionals which do not meet the condition E2: it is possible to derive conditionals with contradictory antecedents and tautological consequents.

One of Wessel's systems allows to prove entailments, fulfilling the condition E1 but failing to meet restriction E2. His system of logical entailment S^S can be obtained from F^S simply by rejecting E2, a system S^{SK} can be constructed adding A10 and R4. Obviously F^{SK5} is a system between F^{SK} and S^{SK}: all theorems of F^{SK5} are provable in S^{SK}, but $p \to p \vee \sim p$ is theorem in the latter and not in the former system.

In F^{SK5} theorems are all conditionals, corresponding to F^S-entailments, and all conditionals being substitutions in such „innocent" formulas. Such a construction is useful, if we want to introduce counterfactuals with logically false antecedents into the system.

Adding rule R6 to F^{SK} we get F^{SK6}:

R6. If $\vdash A \to B$, then $\vdash \sim B \to \sim A$.

It is easy to see that some formulas being provable with R6 are violating E1. So formulas like $\sim A \to \sim (A \wedge B)$ are theorems, but not $A \to (A \vee B)$ (because there is no ordinary transitivity-rule). In some connections it makes sense to distinguish between these formulas. One may argue, that $A \to (A \vee B)$ means: on the base of A it is possible to introduce into the discourse what you want (If roses are red, then roses are red or the moon is a green piece of cheese); but $\sim A \to \sim (A \wedge B)$ means only something like the „monotonicity of negative information" (If something is not the case, then it is not the case whatever happens).

Systems like the mentioned one may be used in deontic logic. The well known principle:

From $A \vdash B$ follows $O(A) \vdash O(B)$

produces paradoxical situations in classical, relevant and most of modal logics. The reason is not only the Ross-paradox:

If the secretary has to mail the letter, she has to mail or to burn the letter;

but also the possible occurence of contradictory A. Of course, there are contradictory false normative contexts, but then it is necessary to decide, which norms one has to meet. In no case it is in a rational sense possible to oblige someone to generate a contradictory situation. This is, by the way, the sense of a important philosophical principle in political and social philosophy: All, what is orderd, is possible. Remembering political practice it should be added: but be careful in ordering.

Concerning the Ross- paradox confer the mentioned sentence with

If the secretary has to mail the letter, she has to mail the letter or to go to dinner.

Because the secretary may first go to have a dinner and then mail the letter or vice versa, there is nothing paradoxical at all. The paradox in the famous secretary-example raises up from the fact, that burn the letter means not to mail it, and mail it means not to burn the letter. Therefore it is a tertium-non-datur-construction in the conclusion of A $\vdash$ B, what makes the mentioned deontic principle leading to paradox. Such constructions are explicitely excluded by E2.

With similar result it is possible to add R6 to F^{SK5} and S^{SK}. The following rule

R7. If $\vdash A \rightarrow B$, then $\vdash (A \wedge C) \rightarrow B$
added to F^{SK} allows to prove in the resulting system F^{SK7} conditionals with contradictory antecedents. It is a system between F^{SK} and S^{SK}, different from F^{SK5}. In F^{SK7} one may use the monotonic conditional or the non-monotonic entailment and also both together. This may be interesting in data systems, where the data are arriving from different sources: conclusions within the different pools should be drawn with the help of the monotonic conditional, conclusions with data from different sources whould be drawn on the base of the entailments.

The conditional in F^{SK67}, constructed by adding R7 to F^{SK6}, is also a monotonic one. In this system conditionals with tautological consequents are provable, it is another system between F^{SK} and S^{SK}, different from F^{SK5}.

Together with F^{SK} the following rule constitutes F^{SK8}:

R8. If $\vdash A \rightarrow B$, then $\vdash A \rightarrow (B \vee C)$; where $B \vee C$ contains only such variables, which occur in A.

By R8 conditionals with tautological consequents are derivable, the system is not equivalent to one of the former mentioned.

Let F^{SR} be the system, constructed by adding R5 – R8 to F^{SK}. The conditional operator, occuring in probable conditional sentences of this system, is not the material implication. This is shown by an easy sentence:

S7. If $\vdash A \rightarrow B$, then there is a propositional variable, occuring in A and in B.

■Use induction: the postulates have the property, the rules hand it down. ■

S8. By adding the transitivity-rule for conditionals (If $\vdash A \rightarrow B$ and $\vdash B \rightarrow C$, then $\vdash A \rightarrow C$) the conditional operator becomes material implication.

■1. $(B \wedge \sim B) \vee \sim A \vdash \sim A$ (F^{S})

2. $\vdash A \rightarrow (A \wedge (B \vee \sim B))$ $(F^{SK}$, R6, Trans.)

3. $\vdash (A \wedge (B \vee \sim B)) \rightarrow (B \vee \sim B)$ (F^{SR})

4. $\vdash A \rightarrow (B \vee \sim B)$ (Trans.) ■

In order to get the last system of the paper we have to accept two additional rules:

R9. If $A \vdash B$ and $B \vdash A$ and $\vdash C \rightarrow D$, then $\vdash C \rightarrow D[A/B]$.

R10. If $\vdash (A \vee B) \rightarrow C$, then $\vdash A \rightarrow C$, if A and C are sharing a common propositional variable.

These rules together with F^{SR} constitute the system $F^{SR'}$, containing all means to construct normal forms.

The rule R10 without restriction is one of the often discussed rules in conditional logic. There are some counterexamples against this rule, for instance:

From „If the secretary has to write a letter or to go home, she would go home" follows by unrestricted R10 „If the secretary has to write a letter, she would go home".

The restriction on R10 prevents the appearance of such examples, formally it prevents the validity of $(p \wedge \sim p) \rightarrow q$. Therefore S7 holds also for $F^{SR'}$.

S9. If $\vdash A \rightarrow B$, there is a formula C such, that $\vdash A \rightarrow C$ and $\vdash C \rightarrow B$, and in C are only these variables, which occur also in A and in B.

■In $F^{SR'}$ are all means to construct for any formula the corresponding formula in extended disjunctive normal form (a disjunctive normal form such, that for all occuring variables holds: they occur – with or without negation – in all elementary conjunctions). Because of R9 it is sufficient to show S9 for formules in extended normal form.

Let A and B be formulas in extanded disjunctive normal form and $\vdash A \rightarrow B$. Let C be the result of erasing in A all propositional variables, which do not occur in B. C exists because of S7.

For all elementary conjunctions A_i of A there is an elementary conjunction C_j of C such, that for the sets of occuring atomic formulas $\{A_i\}$ and $\{C_j\}$ holds $\{C_j\} \subseteq \{A_i\}$. For these A_i and C_j the conditional $A_i \rightarrow C_j$ is provable because of A3, and so is $\vdash A_i \rightarrow C$ for all A_i. By S4 follows $\vdash A \rightarrow C$.

Since $\vdash A \rightarrow B$, for all A_i because of R10 is valid $\vdash A_i \rightarrow B$. Any A_i is a conjunction $C_i \wedge D_i$, and D_i does not share variables with C_i and B; therefore (by S5) follows $\vdash C_i \rightarrow B$. Since this holds for all C_i, $\vdash C \rightarrow B$ is valid because of S4. ■

The Interpolation-theorem S9 can be proved also in the formulation:

If $\vdash A \to B$ and A is not a contradiction, then there is a formula C such, that $A \vdash C$ and $\vdash C \to B$, and C contains only variables, occuring in B.

$F^{SR'}$ is not equivalent to the relevant system FDE of first degree entailment. In all mentioned systems $\vdash ((\sim p \quad q) \quad p) \to q$ (the γ-principle) is valid, in FDE not. In FDE we have unrestricted transitivity, S8 shows, that $F^{SR'}$ together with transitivity collapses to a system of material implication.

$F^{SR'}$ is not equivalent to the first degree fragment of the system SI of strict implication. The so called paradoxes of strict implication are not provable in $F^{SR'}$, but it is easy to see, that $F^{SR'}$ is a subsystem of SI.

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OKRESY WARUNKOWE OPARTE NA ŚCISŁYM „ENTAILMENT"

W artykule rozważa się okresy warunkowe oparte na systemie ścisłego entailment F^{S} skonstruowanym przez Wessela. Poprzez uzupełnienie aksjomatyki i reguł inferencji F^{S} otrzymuje się system F^{SK}, posiadający dwie struktury implikacyjne, typu: entailment i okresu warunkowego. Konsekwencją dalszej modyfikacji systemu F^{SK} poprzez wprowadzenie dodatkowych reguł inferencji, systemy F^{SK5}, F^{SK6} i F^{SK7}, jest zmiana odpowiednich klas okresów warunkowych bez zmiany entailment.