A C T A U N I V E R S I T A T I S L O D Z I E N S I S
FOLIA PHILOSOPHICA 7, 1990
https://doi.org/10.18778/0208-6107.07.09

Marek Nowak, Piotr Rydzewski

ON A GENERALIZED INFERENCE OPERATION

1. The aim of the paper is to present some formal operation useful for a formalization of the large class of reasonings. The analysis of various kinds of logically valuable reasonings employed in science and everyday situations[1] results in their essential characterization:

(1) The accepted premise must not be rejected in the result of inference. Therefore, it gets the status of conclusion.

(2) The enlargement of the inconsistent set X of premises must not remove inconsistency.

(3) If a proposition α is inferred from the set X of premises, it can be also concluded from every larger set of premises Y if only $Y \cup \{\alpha\}$ is consistent.

(4) A proposition α cannot be inferred from a consistent set X of premises, when $X \cup \{\alpha\}$ is inconsistent.

It is straightforward that any deductive reasoning possesses the above properties. For induction, let us consider the well-known swan's example. The set of premises:

$X = \{$swan no. i is white and its neck is long: $i = 1, 2, \ldots, 100\}$

From X there can be inductively drawn that:

(α) Every swan is white.

(β) Every swan's neck is long.

Next premise: (γ) Swan no. 101 is grey and its neck is long, is added to the set X. The new set of premises $X \cup \{\gamma\}$ still al-

[1] Cf. e.g. K. A j d u k i e w i c z, Klasyfikacja rozumowań, [in:] Język i poznanie, t. 2, Warszawa 1965, pp. 206-225; "Artifical Intelligence" 1980, No. 13; H. M o r t i m e r, Logika indukcji, Wybrane problemy, Warszawa 1982.

lows to infer (β) since $X \cup \{\gamma\} \cup \{\beta\}$ is consistent, but (α) is no more a conclusion, since $(X \cup \{\gamma\}) \cup \{\alpha\}$ is inconsistent. That proves that (3) and (4) refer to induction.

Now we shall provide another example to ilustrate non-monotonic reasoning. The set of premises:

$X = \{$a patient has a sharp ache on the right side of his belly$\}$

makes the doctor to draw the conclusion: (α) the patient has appendicitis. If X is enlarged by (β) the patient cannot rise his leg, α is still valid for $(X \cup \{\beta\}) \cup \{\alpha\}$ is consistent. Hereby (3). But when the next premise is that: (γ) the patient had the appendix cut off, then no doctor can infer (α). Notice that $(X \cup \{\gamma\}) \cup \{\alpha\}$ is inconsistent, thus (4) is obviously fulfilled.

We conclude that (1) - (4) conditions are necessary for reasoning to be logically valuable. Albeit all the practically used reasonings posses many other properties, however the formalization of the class of all the reasonings for which (1) - (4) are valid seems to be justified.

2. Where $\underline{S} = (S, F_1, \ldots, F_n)$ is a propositional language and $P(S)$ is the power set of S we shall say that a function $C: P(S) \rightarrow P(S)$ is a g e n e r a l i z e d i n f e r e n c e o p e r a t i o n (g.i.-operation for short) on $\underline{S}$ iff for any $X, Y \subseteq S$, $\alpha \in S$ the following conditions are satisfied:

(i) $X \subseteq C(X)$,

(ii) $C(Y) = S$ whenever $X \subseteq Y$ and $C(X) = S$,

(iii) $\alpha \in C(X)$, $X \subseteq Y$, $C(Y, \alpha) \neq S$ imply that $\alpha \in C(Y)$,

(iv) $\alpha \notin C(X)$ whenever $C(X) \neq S$ and $C(X,\alpha) = S$,

set of formulas $X \subseteq S$ is called i n c o n s i s t e n t w i t h r e s p e c t t o C (C-inconsistent) whenever $C(X) = S$.

Lemma 1. For any function $C: P(S) \rightarrow P(S)$ such that (ii) holds true, (iii) and (iv) are also satisfied if and only if for any $X \subseteq S$ the following conditions are equivalent:

(.) $C(X) \neq S$,

(..) $C(X) = \bigcup\{Y \subseteq S: C(X \cup Y) \neq S$ and $\exists Z \subseteq X: Y \subseteq C(Z)\}$.

<u>Proof:</u>

Denote for any $X \subseteq S$, $\bigcup\{Y \subseteq S: C(X \cup Y) \neq S$ and $\exists Z \subseteq X: Y \subseteq C(Z)\} = K(X)$.

($\Rightarrow$) Assume that the conditions (ii), (iii), (iv) hold for C, suppose that $C(X) \neq S$ and let $\alpha \in C(X)$. So $C(X, \alpha) \neq S$ due to (iv), thus $\alpha \in K(X)$.

On the other hand assume that $\alpha \in K(X)$. Then for some $Y \subseteq S$ we have: $\alpha \in Y$, $C(X \cup Y) \neq S$, $\exists Z \subseteq X$: $Y \subseteq C(Z)$. So $\alpha \in C(Z)$, $Z \subseteq X$ and according to (ii): $C(X, \alpha) \neq S$. Thus $\alpha \in C(X)$ due to (iii).

Now assume that: (..) $C(X) = K(X)$ and $C(X) = S$ for some $X \subseteq S$. So for any $Y \subseteq S$, $C(X \cup Y) = S$ due to (ii), thus $K(X) = \emptyset$. A contradiction.

($\Leftarrow$) Assume that (ii) holds true for C and the conditions (.), (..) are equivalent.

Ad (iii): suppose that $\alpha \in C(X)$, $X \subseteq Y$, $C(Y, \alpha) \neq S$. Then, according to (ii): $C(Y) \neq S$, hence also $C(X) \neq S$. So $C(Y) = K(Y)$ and $C(X) = K(X)$. Hence we obtain that $\alpha \in C(Z)$ for some $Z \subseteq X$. Thus $\alpha \in C(Y)$ since $X \subseteq Y$ and $C(Y, \alpha) \neq S$.

Ad (iv): let $\alpha \in C(X)$ and $C(X) \neq S$. Then we have: $\alpha \in K(X)$, so due to (ii): $C(X, \alpha) \neq S$. □

Let R, T be binary relations on P(S). Consider the following conditions:

$(A)_R$ R is reflexive;

$(B)_R$ $\langle X, Y\rangle \in R$ iff for each $\beta \in Y$: $\langle X, \{\beta\}\rangle \in R$;

$(A)_T$ $\langle X, Y\rangle \in T$ and $X \cup Y \subseteq X' \cup Y'$ imply that $\langle X', Y'\rangle \in T$;

(A) $\langle X, X\rangle \in T$ iff $\langle X, S\rangle \in R$;

(B) $\langle X, X\rangle \notin T$ and $\langle X, \{\alpha\}\rangle \in R$ imply that $\langle X, \{\alpha\}\rangle \notin T$;

(C) $\langle X, \{\alpha\}\rangle \in R$, $X \subseteq Y$, $\langle Y, \{\alpha\}\rangle \notin T$ imply that $\langle Y, \{\alpha\}\rangle \in R$;

for any $X, X', Y, Y' \subseteq S$, $\alpha \in S$.

Let $C_{R,T}: P(S) \longrightarrow P(S)$ be a function defined as follows: for any $X \subseteq S$

$$C_{R,T}(X) = \begin{cases} S & \text{if } \langle X, X\rangle \in T \\ \bigcup\{Y \subseteq S: \langle X, Y\rangle \notin T \text{ and } \exists Z \subseteq X: \langle Z, Y\rangle \in R\} & \text{oth.} \end{cases}$$

<u>Lemma 2</u>. For any binary relations R, T on P(S) fulfilling $(B)_R$, $(A)_T$, (A), (B), (C) and any $X, Y \subseteq S$ the following conditions are satisfied:

(1) $\langle X, Y\rangle \in T$ iff $C_{R,T}(X \cup Y) = S$;

(2) $\langle X, Y\rangle \in R$ iff $Y \subseteq C_{R,T}(X)$.

<u>Proof:</u>

We first show that for any $X \subseteq S$,

(3) $\bigcup\{Y \subseteq S: \langle X, Y\rangle \notin T$ and $\exists Z \subseteq X: \langle Z, Y\rangle \in R\} \neq S$ which implies $(*)$ $C_{R,T}(X) = S$ iff $\langle X, X\rangle \in T$.

Suppose that (3) does not hold. Then according to $(A)_T$ and $(B)_R$ for some $X \subseteq S$, for any $\alpha \in S$, $\langle X, \{\alpha\}\rangle \notin T$ and $\langle Z, \{\alpha\}\rangle \in R$ for some $Z \subseteq X$. So for any $\alpha \in S$, $\langle X, \{\alpha\}\rangle \in R$ due to (C). Hence according to $(B)_R$ and (A) we have: $\langle X, X\rangle \in T$. Thus a contradiction by $(A)_T$.

Now, we immediately have (1) by $(A)_T$ and $(*)$.

To prove (2) assume that $\langle X, Y\rangle \in R$ and $C_{R,T}(X) \neq S$. Then from $(*)$: $\langle X, X\rangle \notin T$ and from $(B)_R$: $\langle X, \{\alpha\}\rangle \in R$ for any $\alpha \in Y$. Thus $Y \subseteq C_{R,T}(X)$ due to the definition of $C_{R,T}$. On the other side assume that $Y \subseteq C_{R,T}(X)$. If $C_{R,T}(X) = S$, then from $(*)$ and (A): $\langle X, S\rangle \in R$, hence according to $(B)_R$: $\langle X, Y\rangle \in R$. So suppose that $C_{R,T}(X) \neq S$. Then $\langle X, X\rangle \notin T$ due to $(*)$ and for any $\alpha \in Y$ there exists $U \subseteq S$ such that $\alpha \in U$, $\langle X, U\rangle \notin T$ and $\langle Z, U\rangle \in R$ for some $Z \subseteq X$. Hence for any $\alpha \in Y$, $\langle X, \{\alpha\}\rangle \notin T$ due to $(A)_T$ and $\langle Z, \{\alpha\}\rangle \in R$ from $(B)_R$. Therefore, according to (C), for any $\alpha \in Y$, $\langle X, \{\alpha\}\rangle \in R$, thus $\langle X, Y\rangle \in R$ by $(B)_R$. □

Using lemma 1 and the condition $(*)$ from the proof of lemma 2 one may prove the following

<u>Theorem 1.</u> For two binary relations R, T on P(S) fulfilling $(A)_R$, $(B)_R$, $(A)_T$, (A), (B), (C), $C_{R,T}$ is a generalized inference operation on <u>S</u>. □

Now, denote by $\mathfrak{R}$ the family of all pairs $\langle R, T\rangle$ of relations for which the conditions $(A)_R$, $(B)_R$, $(A)_T$, (A), (B), (C) are satisfied and by $\mathfrak{I}$ the class of all g.i.-operations on <u>S</u>.

<u>Theorem 2.</u> For any $C \in \mathfrak{I}$ there exists a pair $\langle R, T\rangle \in \mathfrak{R}$ such that $C = C_{R,T}$. Moreover, the correspondence $\langle R, T\rangle \longrightarrow C_{R,T}$ is unique.

<u>Proof:</u>

Using lemma 1 one may choose for given g.i.-operation C the pair $\langle R, T\rangle$ such that $C = C_{R,T}$ in the following way: for any $X, Y \subseteq S$ –

$\langle X, Y\rangle \in T$ iff $C(X \cup Y) = S$,

$\langle X, Y\rangle \in R$ iff $Y \subseteq C(X)$.

It is easy varification that $\langle R, T\rangle \in \mathcal{R}$. Finally, it is obvious due to lemma 2, that for any $\langle R1, T1\rangle$, $\langle R2, T2\rangle \in \mathcal{R}$, $C_{R1, T1} = C_{R2, T2}$ implies that $R1 = R2$ and $T1 = T2$. □

3. Each consequence on $\underline{S}$ i.e. a function $C: P(S) \longrightarrow P(S)$ such that for any $X, Y \subseteq S$, $X \subseteq C(X)$, $C(X) \subseteq C(Y)$ whenever $X \subseteq Y$ and $C(C(X)) \subseteq C(X)$ proves to be a g.i.-operation. Therefore the relational description of consequence operation is possible. To get the additional conditions for $\langle R, T\rangle$'s, the following lemma is useful.

Lemma 3. An operation $C: P(S) \longrightarrow P(S)$ defined for every $X \subseteq S$ by the condition:

$$C(X) = \begin{cases} S & \text{if } X \notin \mathcal{D} \\ K(X) & \text{if } X \in \mathcal{D} \end{cases}$$

with $\mathcal{D} \subseteq P(S)$ and $K: P(S) \longrightarrow P(S)$ being any function, is a consequence on $\underline{S}$ if and only if

(1) for any $X, Y \subseteq S$, $X \subseteq Y$, $Y \in \mathcal{D}$, $K(X) \neq S$ imply that $X \in \mathcal{D}$,

(2) for any $X \in \mathcal{D}$, $K(X) \in \mathcal{D}$ whenever $K(X) \neq S$,

(3) the restriction $K \upharpoonright \mathcal{D}$ satisfies the conditions for a closure operation.

Proof by easy verification. □

Theorem 3. A g.i.-operation $C_{R, T}$ on $\underline{S}$ for $\langle R, T\rangle \in \mathcal{R}$ is a consequence operation if and only if the following conditions are satisfied for any $X, Y, Z \subseteq S$:

$(C)_R$ $\langle Z, Y\rangle \in R$ whenever $\langle X, Y\rangle \in R$ and $X \subseteq Z$,

$(D)_R$ R is transitive,

$(B)'$ $\langle X, X\rangle \notin T$ and $\langle X, Y\rangle \in R$ imply that $\langle X, Y\rangle \notin T$.

Proof:

($\Rightarrow$) According to lemma 2, the conditions $(C)_R$, $(D)_R$, $(B)'$ follow immediately from the assumtion that $C_{R,T}$ is a consequence operation.

($\Leftarrow$) Assume that $\langle R, T\rangle \in \mathcal{R}$ fulfils the conditions $(C)_R$, $(D)_R$, $(B)'$. Due to lemma 3 it is sufficient to show that the conditions (1), (2), (3) from it hold true for

$\mathcal{D} = \{X \subseteq S: \langle X, X\rangle \notin T\}$ and (using (C_R))

$K(X) = \bigcup\{Y \subseteq S: \langle X, Y\rangle \in R - T\}$, $X \subseteq S$.

Notice that for any $X \subseteq S$, $K(X) \neq S$ (cf. the proof of lemma 2).

So the condition (1) follows immediately from $(A)_T$. To prove (2) assume that $X \in \mathcal{D}$, that is $\langle X, X\rangle \notin T$. Notice that for any $\alpha \in K(X)$, $\langle X, \{\alpha\}\rangle \in R$ by $(B)_R$, so also from $(B)_R$: $\langle X, K(X)\rangle \in R$, thus $\langle X, K(X)\rangle \notin T$ by $(B)'$, so from $(A)_T$: $\langle K(X), K(X)\rangle \notin T$ i.e. $K(X) \in \mathcal{D}$.

Naturally, for any $X \in \mathcal{D}$, $X \subseteq K(X)$ due to $(A)_R$.

To prove the monotonicality of $k\restriction\mathcal{D}$ assume that $\langle Y, Y\rangle \notin T$, $X \subseteq Y$ and $\alpha \in K(X)$. Hence $\langle X, \{\alpha\}\rangle \in R$ by $(B)_R$ and therefore, according to $(C)_R$ we have: $\langle Y, \{\alpha\}\rangle \in R$. Moreover, due to $(B)'$, $\langle Y, \{\alpha\}\rangle \notin T$, thus $\alpha \in K(Y)$.

To the end assume that $\langle X, X\rangle \notin T$ and $\alpha \in K(K(X))$. Then $\langle K(X)\{\alpha\}\rangle \notin T$ and $\langle K(X), \{\alpha\}\rangle \in R$ due to $(A)_T$ and $(B)_R$. Since $X \subseteq K(X)$, so according to $(A)_T$: $\langle X, \{\alpha\}\rangle \notin T$. Moreover, from $(B)_R$ we have: $\langle X, K(X)\rangle \in R$ which together with $\langle K(X), \{\alpha\}\rangle \in R$, leads by $(D)_R$ to the conclusion that $\langle X, \{\alpha\}\rangle \in R$. Thus $\alpha \in K(X)$. □

University of Łódź
Poland

Marek Nowak, Piotr Rydzewski

O UOGÓLNIONEJ OPERACJI INFERENCJI

Celem pracy jest analiza formalna pewnych ogólnych własności charakteryzujących wnioskowania. Autorzy twierdzą, że posiadanie tych własności jest warunkiem koniecznym, aby wnioskowanie było logicznie wartościowe. Wprowadzają aksjomatycznie pojęcie "uogólnionej operacji inferencji", które formalnie ujmuje wnioskowanie mające owe cechy. Następnie reprezentują uogólnioną operację inferencji przy użyciu relacji binarnych określonych na podzbiorach języka. Podają również taką reprezentację dla logicznej operacji konsekwencji (każda logiczna operacja konsekwencji jest uogólnioną operacją inferencji).