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TOWARDS THE CONCEPT OF LOGICAL MANY-VALUEDNESS

Introduction

Abolition of the Fregean Axiom by R. Suszko opens a possibility for making distinction between reference assignements and logical valuations. Further to this one may conclude that each logic, i.e. an inference relation conforming Tarski's conditions, is logically two-valued. Constructing a three-valued inference relation we show that introduction of the concept of logical many-valuedness is inevitably connected with revision of the fundamentals of theory of consequence operation.

1. Matrix and consequence[1]

B y a p r o p o s i t i o n a l l a n g u a g e we shall mean an algebra of formulas

$$\mathbb{L} = (For, f_1, f_2, \ldots, f_n)$$

freely generated by a denumerable set of propositional variables $Var \subseteq For$. Obviously, $\mathbb{L}$ is absolutely free in its similarity class.

While interpreting a language to each formula α a semantic correlate $h(\alpha)$ from a set M is associated in such a way that each interpretation structure

[1] J. C z e l a k o w s k i, G. M a l i n o w s k i, Key Notions of Tarski's Methodology of Deductive Systems, "Studia Logica" 1985, vol. 44, No. 4, pp. 321-351.

$$\underline{M} = (M, F_1, F_2, \ldots, F_n)$$

is an algebra similar to $\underline{L}$ and that h: For $\longrightarrow M$ is a homomorphism from $\underline{L}$ to $\underline{M}$, $h \in Hom(\underline{L}, \underline{M})$. Any couple of the form

$$M = (\underline{M}, B)$$

where $\underline{M}$ is an algebra similar to $\underline{L}$ and $B \subseteq M$ is called a m a t r i x for $\underline{L}$. Elements of B are referred to as d i s t i n g u i s h e d v a l u e s of M. Each such M determines in $\underline{L}$ a c o n s e q u e n c e r e l a t i o n $\vdash_M \subseteq 2^{For} \times For$ defined for any $X \subseteq For$ and $\alpha \in For$ by

$X \vdash_M \alpha$ iff for every $h \in Hom(\underline{L}, \underline{M})(h(X) \subseteq B$ implies $h\alpha \in B)$.

Subsequently, the operation $Cn_M: 2^{For} \longrightarrow 2^{For}$ defined for every $X \subseteq For$ by

$$Cn_M(X) = \{\alpha : X \vdash \alpha\}$$

is a s t r u c t u r a l c o n s e q u e n c e o p e r a t i o n, i.e. satisfies the Tarski's conditions

(T0) $X \subseteq C(X)$,

(T1) $C(X) \subseteq C(Y)$ whenever $X \subseteq Y$,

(T2) $C(C(X)) = C(X)$,

and for every endomorphism i. e. s u b s t i t u t i o n e of $\underline{L}$, $e \in End(\underline{L})$,

(S) $eC(X) \subseteq C(eX)$.

2. Logical and algebraic valuations

It was stated above that homomorphisms i.e. elements of $Hom(\underline{L}, \underline{M})$ represent reference assignements. In what follows, elements of algebra $\underline{M}$ will be referred to as s i t u a t i o n s and elements of B as s i t u a t i o n s, w h i c h o b t a i n[2]. Obviously, l o g i c a l v a l u a t i o n s i.e. zero-one valued functions defined on For are of quite different conceptual nature: Given a

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[2] Cf. R. S u s z k o, Abolition of the Fregean Axiom, [in:] Logic Colloquium. Symposium on Logic held in Boston, 1972-1973, ed. R. Parikh, "Lecture Notes in Mathematics", vol. 453, pp. 169-239.

matrix M, the corresponding set of logical valuations TV_M: For $\longrightarrow \{0, 1\}$ may be defined as

$$TV_M = \{t_h : h \in Hom(\underline{L}, \underline{M})\}$$

where

$$t_h(a) = \begin{cases} 1 & \text{if } h(a) \in B \\ 0 & \text{otherwise.} \end{cases}$$

Consequently

[T] $X \vdash_M \alpha$ iff for each $t \in TV_M$ ($t(X) \subseteq \{1\}$ implies $t\alpha = 1$).

The construction of TV may in a straightforward way be repeated for any structural consequence operation C (or, equivalently, for $\vdash_C$) since for each such C there exists a class K of matrices such that

$$C = \bigcap \{Cn_M : M \in K\}^3.$$

This justifies the thesis stating that e a c h (s t r u c t u r a l) l o g i c ($\underline{L}$, C) i s l o g i c a l l y t w o - - v a l u e d[4]. Notice that that quality being entirely independent from properties of a particular ontological interpretation is rather a result of the division of the universe of interpretation into two subsets: the set od distinguished elements and its complement.

One may naturally ask whether and how is it possible to find recursive conditions describing the set of logical valuations LV for a given propositional logic. One may also ask for small subsets S of For such that any mapping f: S $\longrightarrow \{0, 1\}$ can be extended to a valuation. The general answer to these questions is difficult and, hopefully, unimportant for the aims of the present paper - the thorough discussion on this problem provided is confi-

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[3] R. W ó j c i c k i, Some Remarks on the Consequence Operation in Sentential Logics, "Fundamenta Mathematicae" 1970, No. 68, pp. 269-279.

[4] R. S u s z k o, The Fregean Axiom and Polish Mathematical Logic in the 1920s, "Studia Logica" 1977, vol. 36, No. 4, pp. 377--380.

ned to finite Łukasiewicz logics alone[5]. On the other hand, for several propositional calculi logical valuations are definable directly form algebraic valuations and special formulas. This means that in some cases two-valuedness is an internal property of logical construction.

Example[6].

Consider propositional language $\underline{L} = (For, \rightarrow, \sim)$ with implication ($\rightarrow$) and negation ($\sim$) and the n-valued ($n \geqslant 2$, n finite) Łukasiewicz matrix:

$$N = (\{0, 1/n-1, \ldots, n-2/n-1, 1\}, \rightarrow, \sim, \{1\}),$$

$x \rightarrow y = \min(1, 1 - x + y)$ and $\sim x = 1 - x$. For every $h \in Hom(\underline{L}, \underline{N})$ define $t_h : For \longrightarrow \{0, 1\}$ putting

$$t_h(\alpha) = h(\sim[\alpha \rightarrow_{n-2} \sim((\sim\alpha \rightarrow \alpha) \rightarrow \alpha))]) \quad \text{any } \alpha \in For.$$

Then, $LV_N = \{t_h : h \in Hom(\underline{L}, \underline{N})\}$ is the set of logical valuations for $\vdash_N$ i.e.

$$X \vdash_N \alpha \text{ iff for every } t_h \in LV_N \ (t_h(X) \subseteq \{1\} \text{ implies } t_h(\alpha) = 1).$$

3. Three-valued inference

Where $\underline{M} = (M, F_1, F_2, \ldots, F_n)$ is an algebra similar to a given propositional language $\underline{L}$ and A, B are disjoint subsets of M ($A \cap B = \emptyset$), the triple

$$M = (\underline{M}, A, B)$$

will be called a q-matrix for $\underline{L}$. A and B may be then interpreted as sets of r e j e c t e d and d i s t i n g u i s h e d values of M, respectively.

For any such M we define the q - c o n s e q u e n c e r e l a t i o n $\vdash\!-_M \subseteq 2^{For} \times For$ as follows:

[5] G. M a l i n o w s k i, Classical Characterization of n-Valued Łukasiewicz Calculi, "Reports on Mathematical Logic" 1977, vol. 9, pp. 44-45.

[6] Ibidem.

$X \vdash\!\!-_{M} \alpha$ iff for any $h \in Hom(\underline{L}, \underline{M})$: $hX \cap A = \emptyset$ implies $h\alpha \in B$.

Obiously, with each relation $\vdash\!\!-_{M}$ one may associate the operation $Wn_{M}: 2^{For} \longrightarrow 2^{For}$ putting

$$Wn_{M}(X) = \{\alpha : X \vdash\!\!-_{M} \alpha\}.$$

Notice that when $A \cup B = M$ Wn_{M} coincides with the consequence operation Cn_{M}, determined by the matrix $M = (\underline{M}, B)$. In other cases, however, the two operations differ from each other - to see this consider e.g. any q-matrix of the form $(\{e_1, e_2, e_3\}, F_1, F_2, \ldots, F_n, \{e_1\}, \{e_3\})$.

It is easy to see that for any q-matrix M for which $A \cup B \neq M$ no class TV of functions t: For $\longrightarrow \{0, 1\}$ exists such that for all X and α, $X \vdash\!\!-_{M} \alpha$ iff for each $t \in TV$ ($t(X) \subseteq 1$ implies $t\alpha = 1$). Therefore, some propositional logics $(\underline{L}, Wn_{M})$ are not logically two-valued (cf. Section 2).

Now, for every $h \in Hom(\underline{L}, \underline{M})$ let us define the three-valued function k_h: For $\longrightarrow \{0, 1/2, 1\}$ putting

$$k_h(\alpha) = \begin{cases} 0 & \text{if } h(\alpha) \in A \\ 1/2 & \text{if } h(\alpha) \in M-(A \cup B) \\ 1 & \text{if } h(\alpha) \in B. \end{cases}$$

Given a q-matrix M for $\underline{L}$ let $KV_{M} = \{k_h : h \in Hom(\underline{L}, \underline{M})\}$. We obtain

[K] $X \vdash\!\!-_{M} \alpha$ iff for every $k_h \in KV_{M}$: $k_h(X) \cap \{0\} = \emptyset$ implies $k_h(\alpha) = 1$).

This is a kind of three-valued description of the q-consequence relation $\vdash\!\!-_{M}$. Notice that KV_{M} reduces to TV_{M} as well as [K] to [T] when $A \cup B = M$.

Wn_{M} introduced above is a prototype of the concept of q-consequence operation[7]. An operation $W: 2^{For} \longrightarrow 2^{For}$ is a q-consequence provided that for every $X, Y \subseteq For$

(W1) $W(X) \subseteq W(Y)$ whenever $X \subseteq Y$,

(W2) $W(X \cup W(X)) = W(X)$.

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[7] G. M a l i n o w s k i, Q-Consequence Operation (to appear).

If for every substitution $e \in End\ \underline{L})$

(S) $eW(X) \subseteq W(eX)$

W is called s t r u c t u r a l.

Since for every structural q-consequence W there exists a class of q-matrices K such that

$$W = \bigcap \{Wn_{M} : M \in K\}^{8},$$

taking [K] into account we conclude that e a c h (s t r u c t u r a l) l o g i c ($\underline{L}$, W) i s l o g i c a l l y t w o o r t h r e e - v a l u e d.

Clearly, logically three-valued logics exist - see the example of three-element q-matrix given in this Section.

4. Prospectus

The concept of logical three-valuedness elaborated in the paper is obviously related to the division of the universe of interpretation into three subsets of elements: d i s t i n g u i s h e d, r e j e c t e d and i n d i f f e r e n t. Then, if we referred to elements of algebra $\underline{M}$ as situations, cf. Section 2, we should say that three kinds of situations are considered: those which obtain, those which do not obtain and those which are p o s s i b l e. It is worth to notice that the two terms "indifferent" and "possible" were used by Łukasiewicz[9] for supporting ontological base of construction of three-valued logic. Consequently, our proposal mirrors, in a sense, Łukasiewicz views upon non-classical logic.

The generalization of Tarski's concept of consequence operation received as a result of introducing of q-matrices is consistent with common understanding of l o g i c a l s y s t e m as a set of formulas closed under substitutions, usually defined as c o n t e n t of a logical matrix. Namely, if we take a q-matrix $M = (\underline{M}, A, B)$ and the corresponding matrix $M' = (\underline{M}, B)$ then

$$Wn_{M}(\phi) = Cn_{M'}(\phi).$$

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[8] Ibidem.

[9] J. Ł u k a s i e w i c z, O logice trójwartościowej, "Ruch Filozoficzny" 1920, t. 5, pp. 170-171.

This means that any logical system may equally well be extended to a two-valued logic $(\underline{L}, Cn_{M})$ as to a three-valued logic $(\underline{L}, Wn_{M})$. And, depending on the quality and cardinality of M three-valued extensions may take different shapes. Moreover, for some extensions defined through three-element algebras referential assignements $h \in Hom(\underline{L}, \underline{M})$ and logical valuations $k_h \in KV_M$ coincide. Therefore, we may claim that such extensions satisfy the Fregean Axiom related to a new logical paradigm.

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W KIERUNKU POJĘCIA WIELOWARTOŚCIOWOŚCI LOGICZNEJ

Obalenie aksjomatu Fregego przez R. Suszko otwiera możliwość odróżnienia wartościowań logicznych od odwzorowań referencyjnych. Podążając tą drogą można wywnioskować, że każda logika, tzn. relacja inferencji spełniająca warunki Tarskiego, jest logicznie dwuwartościowa. Konstruując trójwartościową relację inferencji pokazujemy, że wprowadzenie pojęcia wielowartościowości logicznej jest nierozerwalnie związane z rewizją podstaw teorii konsekwencji logicznej.