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ON DIFFERENT NOTIONS OF QUANTIFIERS

1. Introduction

Among the most central notions of logic seem to be quantifier and quantification. Although these terms were introduced in the 19th century, the very idea of quantification is older and it can be traced to antiquity. The present paper will attempt an explication of that notion. Aristotle's ideas of quantification have close correspondence with modern notion of generalized quantifier formulated by Lindström in opposition to classical quantifiers of predicate calculus. Modern notion of a generalized quantifier enabled the development of practical applications in linguistics and logic of induction. It is obvious that the two classical quantifiers are useless in these subjects.

2. The notion of quantification
in Aristotle's syllogistic

The earliest attempt to create a system of formal logic was Aristotle's syllogistic. Aristotle recognizes four categorical statements and specifies the rules of inference for them. Categorical statements represent relations between the interpretations of subjects and predicative words.

$$\begin{array}{l} S a P \quad \|S\| - \|P\| = \emptyset \\ S e P \quad \|S\| \cap \|P\| = \emptyset \\ S i P \quad \|S\| \cap \|P\| \neq \emptyset \\ S o P \quad \|S\| - \|P\| \neq \emptyset \end{array}$$

In set-theoretical terms copulas "a", "e", "i", "o" can be interpreted as binary relations between the subsets of a domain of

interpretation E , (i.e. the interpretation of "a" in E is the relation of inclusion: for any $X, Y \subseteq E$ holds $(X, Y) \in \|\text{all}\|$ iff $X \subseteq Y$, where $\|\text{all}\|$ denotes the symbol "a"). Making use of Lindström's generalized quantifier the above symbols can be treated as generalized quantifier symbols characteristic for syllogistic language. These symbols have set-theoretical denotations in the modified model-theory.

On account of the development of mathematics, modern predicate logic with the classical notion of quantifier has been formulated. However, in spite of its usefulness in mathematics and metamathematics, Aristotle's intuition of quantification has been lost.

3. Classical predicate calculus and its extensions

First-order predicate calculus is not a mere extension of syllogistic. It has different and richer dictionary and syntax. There is a significant difference of syntactical roles between copulas in syllogistic and classical quantifiers in predicate calculus. Symbols \forall and \exists are operators that convert propositional functions to propositions; in the classical theory of models they have no explicit interpretation, they are usually interpreted in the context of a definition of satisfying a formula beginning with quantifier symbol by a valutional function.

From the point of view of linguistics, predicate calculus seems artificial and insufficient, because of its poor possibility to express "quantitative words" of natural language.

The first step towards the extension of "expressive power" of predicate calculus consisted in the reformulating of syllogistic in terms of predicate logic. It was possible by introduction of the symbols of quantifiers of limited range¹.

$$S a P \text{ iff } (\forall s_{\langle x \rangle}) P(x) \equiv (\forall x)(S(x) \rightarrow P(x))$$

$$S i P \text{ iff } (\exists s_{\langle x \rangle}) P(x) \equiv (\forall x)(S(x) \wedge \sim P(x))$$

$$S e P \text{ iff } (\forall s_{\langle x \rangle}) \sim P(x) \equiv (\forall x)(S(x) \rightarrow P(x))$$

$$S o P \text{ iff } (\exists s_{\langle x \rangle}) \sim P(x) \equiv (\forall x)(S(x) \wedge \sim P(x))$$

¹ See: Z. Siupecki, L. Borkowski, Elementy logiki matematycznej i teorii mnogości, Warszawa 1984.

Thus the formulas of predicate logic become shorter and their syntax more adequate for natural language description.

The next extension was the introduction of numerical quantifiers².

- For exactly m holds... $-\Sigma\langle m \rangle$, ($m > 0$)

$$(\Sigma\langle 0 \rangle)x A(x) \equiv \sim \exists x A(x) \text{ for } m = 0$$

$$(\Sigma\langle m \rangle)x A(x) \equiv \exists x_1 \dots x_m (A(x_1)) \wedge \dots \wedge A(x_m) \wedge x_1 \neq x_2$$

$$\wedge \dots \wedge x_1 \neq x_m \wedge \dots \wedge x_{m-1} \neq x_m \wedge \forall x_{m+1} (A(x_{m+1}) \rightarrow (x_{m+1} = x_1 \vee \dots \vee x_{m+1} = x_m)) \text{ for } m > 0$$

- For exactly n doesn't hold... $-\Pi\langle n \rangle$, ($n > 0$)

$$\Pi\langle n \rangle x A(x) \equiv \Sigma\langle n \rangle x (\sim A(x)) \text{ for } n > 0$$

- For at most m holds... $-\Sigma[m]$, $m > 0$

$$\Sigma[m] x A(x) = \Sigma\langle 0 \rangle x A(x) \vee \dots \vee \Sigma\langle m \rangle x A(x)$$

- For at most n doesn't hold... $-\Pi[n]$, $n > 0$

$$\Pi[n] x A(x) = \Pi\langle 0 \rangle x A(x) \vee \dots \vee \Pi\langle m \rangle x A(x)$$

It is clear that both extensions are abbreviations of complex formulas of predicate calculus.

The symbols belong to the class of first-order quantifier symbols and can be defined in terms of classical quantifiers \forall and \exists , within first-order predicate logic. The first real extension of predicate calculus is due to Mostowski³, who introduced quantifiers of other classes.

4. Generalized quantifiers of Mostowski

One of the most important steps towards modern notion of quantifier was made by Mostowski⁴. Mostowski presents an explicit model-theoretic definition of quantifier as an interpretation of a quantifier symbol.

² See e.g. A. Mostowski, On a Generalization of Quantifiers, "Fundamenta Mathematica" 1957, vol. 44, pp. 12-36.

³ Ibidem.

⁴ Ibidem.

Definition 1⁵:

A quantifier limited to the domain of interpretation E is a function Q_E which assigns one of truth values 1, 0: to each interpretation of one-argument propositional function F on E , which satisfies the invariance condition: $Q(F) = Q(F\phi)$, where ϕ is a permutation of E , such that for $a \in E$ holds $F\phi(\phi(a)) = F(a)$.

By this definition, for every quantifier symbol there is a family of subsets of E that "satisfies" its interpretation (a quantifier limited to E).

The invariance condition enables to formulate a very interesting number - theoretic representation of a quantifier. Let (m, n) be the sequence of pairs of cardinal numbers satisfying the equation $m + n = \text{card}(E)$. For each function T which assigns 1 or 0 to each pair we put:

$$Q_T(F) = T(\text{card}\{a \in E: F(a) = 1\}, \text{card}\{a \in E: F(a) = 0\})$$

Theorem 1⁶:

a) Q_T is a quantifier limited to E .

b) For each quantifier Q_E limited to E there is a function T such that $Q_T = Q_E$.

If $Q_T = Q_E$, then T is associated function of Q_E . This notion is very useful in defining the specific quantifiers by means of the conditions under which the associated function of a quantifier possesses value 1.

Examples:

1. Classical quantifiers

$$\exists E: m \neq 0$$

$$\forall E: n = 0$$

2. Numerical quantifiers

$$\sum \langle m \rangle E: m = m$$

$$\sum [m] E: m < m$$

$$\prod \langle n \rangle E: n = n$$

$$\prod [n] E: n < n$$

⁵ Ibidem.

⁶ Ibidem.

3. Non-first-order quantifiers

$\exists \epsilon: m < X_0$ - for denumerably many holds

$\exists \sim \epsilon: m < X_0 \vee n < X_0$ - for finitely many holds or for finitely many doesn't hold

4. Trivial quantifiers (defined in set-theoretical terms)

P_E : for every F holds $P_E(F) = 1$

\emptyset_E : for every F holds $\emptyset_E(E) = 0$

It is also possible to define duals and boolean combinations of quantifiers.

Definition 2⁷:

Let $T(m, n)$ be associated function of Q_E . A quantifier Q^*_E with associated function $T^*(m, n) = \sim T(n, m)$ is a dual of Q_E (e.g. $\exists \epsilon = \forall^* \epsilon$).

Definition 3⁸:

$Q_E = \sim Q^*_E$ iff $Q_E(F) = \sim Q^*_E(F)$

$Q_E = Q^*_E \vee Q^{**}_E$ iff $Q_E(F) = Q^*_E(F) \vee Q^{**}_E(F)$

$Q_E = Q^*_E \wedge Q^{**}_E$ iff $Q_E(F) = Q^*_E(F) \wedge Q^{**}_E(F)$

for every F.

The present theoretical background enables to examine the boolean algebra \mathcal{U} of quantifiers limited to denumerable set E. \mathcal{U} is isomorphic to the product of three boolean algebras:

$$\mathcal{U} = \mathfrak{K} \times \mathfrak{K} \times (\{1, 0\}, \sim, \vee, \wedge)$$

where $\mathfrak{K} = (P(N), ', \cup, \cap)$ and $P(N)$ is a powerset of all real numbers (including 0). It occurs that quantifiers of \mathcal{U} belong to one of the four classes: $\{\emptyset_E\}$, $\{S_E\}$, $\{S^*_E\}$ and $\{S^{\sim}_E\}$ represented by the triples of \mathcal{U} respectively:

$$\begin{array}{l} P(N)_{fin} \times P(N)_{fin} \times \{0\} \\ P(N)_{infin} \times P(N)_{fin} \times \{0\} \\ P(N)_{infin} \times P(N)_{fin} \times \{1\} \end{array}$$

⁷ Ibidem.

⁸ Ibidem.

$$\begin{array}{l} P(N)_{\text{infin}} \times P(N)_{\text{fin}} \times \{1\} \\ P(N)_{\text{infin}} \times P(N)_{\text{infin}} \times \{0\} \end{array}$$

The class $\|\emptyset\epsilon\|$ is the class of first-order quantifiers. Other quantifiers can't be defined in terms of $\forall\epsilon$ and $\exists\epsilon$ only.

Clearly, associated functions of Mostowski's generalized quantifiers of \mathcal{U} are of the form mRn , nRn and their boolean combinations, where R is a relation on numbers: "=", "<", ">", "<=", ">=" and $n, m < \text{card}(E)$. It is possible to express other interesting quantifiers not considered here, e. g. for more than half holds...: $m > n$

Mostowski managed to give an explicit definition of a quantifier as an interpretation of a quantifier symbol and pointed out how to define specific quantifiers by means of set theory or associated functions. Mostowski didn't use terms of theory of models but his ideas were continued by Lindström who found their model-theoretical form.

5. A formal model-theoretical interpretation of quantifiers

A final generalization of a notion of a quantifier was introduced by Lindström. Lindström⁹ has given a formal model-theoretical definition of a generalized quantifier as an interpretation of a quantifier symbol.

Definition 4¹⁰:

A quantifier Q is a class of relational structures of type $t \in N^n$ such that Q is closed under isomorphism.

Choosing a special domain of interpretation we get a definition of a generalized quantifier limited to E as a relational structure of type $t \in N^n$.

$$Q_E = \langle E, R_1 \subseteq P(E^{t_1}), \dots, R_n \subseteq P(E^{t_n}) \rangle$$

⁹ P. Lindström, First Order Predicate Logic with Generalized Quantifiers, "Theoria" 1966, pp. 186-195.

¹⁰ Ibidem.

closed under permutations of E . The type $t = \langle t_1, \dots, t_n \rangle (t_i \geq 0)$ is a sequence of arities of relations R_1, \dots, R_n .

It is obvious that Mostowski's generalized quantifiers are a special case of Lindström's, as they are relational structures of type $\langle 1 \rangle$.

$$Q_E = \langle E, R \subseteq P(E) \rangle$$

The copulas of Aristotle's syllogistics are quantifier symbols of type $\langle 1, 1 \rangle$, e.g.

$$\forall a \parallel_E = \langle E, R_1 \subseteq P(E), R_2 \subseteq P(E): R_1 - R_2 = \emptyset \rangle$$

Lindström's idea of quantification seems to be a generalization of ancient intuitions of Aristotle, it will be obvious when examining logic of induction based on generalized quantifiers.

The notion of a type of a quantifier is bound to syntactical role of a respective quantifier symbol Q of a generalized first-order predicate logic. If $t = \langle t_1, \dots, t_n \rangle$, then n expresses number of formulas and t_i number of variables in each formula, bound by Q , e.g.

$$\phi = Qx_{11} \dots x_{1t_1} \dots x_{n1} \dots x_{nt_n} (\phi_1, \dots, \phi_n)$$

Let M_{iE} be a valuational function (under interpretation I in a domain E). Every function M_{iE} and formula ϕ fix a relational structure of a type t .

$$[M_{iE}, \phi] = \langle E, \{ (a_{11}, \dots, a_{1t_1}) \in E^{t_1}: \text{stsf}[M_{iE}(a_{11}/x_{11}, \dots, a_{1t_1}/x_{1t_1}), \phi_1] \}, \dots, \{ (a_{n1}, \dots, a_{nt_n}) \in E^{t_n}: \text{stsf}[M_{iE}(a_{n1}/x_{n1}, \dots, a_{nt_n}/x_{nt_n}), \phi_n] \} \rangle$$

where a_{ij} is an object of E , such that $M_{iE}(x_{ij}) = a_{ij}$ and $\text{stsf}[M_{iE}(\dots), \phi_i]$ means that M_{iE} satisfies the sub-formula ϕ_i .

Let P_{jE} and Q_E be interpretations I in E of a predicate symbol P_i of arity m and a quantifier Q . Generalized first-order predicate logic (with identity), by adding generalized quantifiers of various types.

The definition below is an inductive definition of satisfying a formula by a valutional function M_{IE} .

Definition 5¹¹:

1. $\text{stsf}[M_{IE}, P_j(x_{j1}, \dots, x_{jm})]$ iff $\langle M_{IE}(x_{j1}), \dots, M_{IE}(x_{jm}) \rangle \in P_{jE}$.
2. $\text{stsf}[M_{IE}, x_k = x_1]$ iff $M_{IE}(x_k) = M_{IE}(x_1)$.
3. $\text{stsf}[M_{IE}, \phi]$ where ϕ begins with quantifier symbol Q iff $[M_{IE}, \phi] \in Q_E$.

Examples:

1. Well known sentential connectives: $\sim, \vee, \rightarrow, \wedge, \leftrightarrow$, are in terms of Lindström quantifier symbols of type $\langle 0 \rangle$ and $\langle 0, 0 \rangle$, as their interpretations are relational structures of arities 0. This resembles Łukasiewicz's notation of sentential connectives.

2. Let W_E be a quantifier of type $\langle 1, 1 \rangle$, $\langle E, R_1, R_2 \rangle \in W_E$ iff $\text{card}(R_1) \geq \text{card}(R_2)$. The quantifier denotes: There are no less... then...

The examples show the utility of Lindström's concept of a quantifier. The next chapter deals with the practical application of this idea.

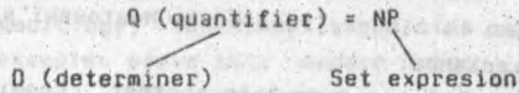
6. Practical applications of a modern notion of a generalized quantifier

Accordingly to Aristotle's intuition, the most important quantifiers in the natural language investigations are of the type $\langle 1, 1 \rangle$ since they fit for the specific syntax of a natural language. They were thoroughly examined during last ten years.

The studies were started by Barwise and Cooper¹², who treat a quantifier as a noun phrase -NP:

¹¹ Ibidem.

¹² J. Barwise, R. Cooper, Generalized Quantifiers and Natural Language, "Linguistics and Philosophy" 1981, No. 4, pp. 159-219.

Examples:

|| Every man || = $\{x: \text{|| man ||} \subseteq x\}$

|| Two boys || = $\{x: \text{card}(\text{|| boy ||} \cap x) = 2\}$

|| most women || = $\{x: \text{card}(\text{|| woman ||} \cap x) > \text{card}(\text{|| woman ||} - x)\}$.

|| Every (thing) || = $\{x: x = E\}$.

where every, two, most etc. are d e t e r m i n e r s. In terms of Barwise and Cooper interpretation every quantifier is of type <1> and is closely related to the quantifier of limited range.

The main goal of Barwise and Cooper's paper was to formulate a formal language LGQ capable of formalizing a fragment of English. The LGQ syntax is closely related to the English syntax, e.g. the special function in LGQ is assigned to a term "thing" which will manifold compound expressions like "everything", "nothing", "something".

Other papers concerning quantifiers in natural language¹³ deal with a quantifier identified with a determiner (it is of type <1, 1>). As quantifiers are binary relations on subsets of E it is possible to impose some conditions on them. First condition is a u n i v e r s a l one, it is satisfied by all quantifiers.

CONSERV. (conservativity)

For every E and every A, B $\subseteq E$ holds: $Q_E AB$ iff $Q_E A(ANB)$.

This condition implies for example that e.g. some men are running iff some men are running men. Second condition is not a universal one. It is not satisfied by some special quantifiers e.g. many, some, few, a few.

QUANT. (quantity)

For every E, E', every bijection $\phi: E \rightarrow E'$ and every A, B $\subseteq E$ holds: $Q_E AB$ iff $Q_{E'} \phi(A)\phi(B)$.

¹³ See: J. van B e n t h e m, Questions about Quantifiers, "The Journal of Symbolic Logic" 1984, vol. 49, pp. 443-446; i d e m, Essays in Logical Semantics, Amsterdam 1985; D. W e s t e r s t å h l, Some Remarks on Quantifiers, Göteborg 1982.

This condition is closely related to Mostowski's and Lindström's condition of invariance.

It is also possible to formulate another condition satisfied only by special groups of quantifiers. The monotonicity conditions of quantifiers are fundamental in defining the class of first-order quantifiers of type $\langle 1, 1 \rangle$. For quantifiers that satisfy CONSERV. and QUANT. there exists possibility to introduce number-theoretical representation¹⁴, that enables to examine their relational properties.

Logic of induction is another domain of application of generalized quantifiers. A quantifier as a "relation on relations" is very useful to express various correlations searched by empirical sciences. Let us look at Aristotle's scheme of induction:

S_1 is P

S_2 is P

.....

S_n is P

S a P (every S is P)

All other schemes of induction have the same form but as other quantifiers are used it is possible to express various correlations of interest.

The practical utility of generalized induction was possible thanks to computers. The group of Czech scientists have invented special computer methods of obtaining induction hypothesis, called GUHA methods - General Unary Hypothesis Automaton¹⁵. The GUHA methods are based on number-theoretic representation of generalized quantifiers. With support of associated functions of quantifiers it is possible for a computer to search interesting correlations in empirical data. Very important sort of generalized quantifiers used in GUHA methods are statistical quantifiers. Their associated functions ex-

¹⁴ See: A. Mostowski, op. cit.

¹⁵ See: P. Hájek, J. Havráněk, Mechanizing Hypothesis Formation, Springer Verlag, Heidelberg 1977.

press well known statistical tests. Statistical quantifiers were applicated in sociology, medicine, linguistics and even industry.

The given examples prove that modern notion of a quantifier due to Lindström is a very important issue of some branches of contemporary logic.

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O RÓŻNYCH POJĘCIACH KWANTYFIKATORA

Jednym z podstawowych pojęć logiki jest pojęcie kwantyfikacji i związane z nim pojęcie kwantifikatora. Pierwsze problemy związane z kwantyfikacją pojawiły się wraz z powstaniem sylogistyki Arystotelesa, chociaż nie istniało wówczas pojęcie kwantifikatora. Powstanie rachunku predykatów I rzędu oraz wprowadzenie klasycznych kwantyfikatorów \forall i \exists zmieniło zupełnie sens pojęcia kwantyfikacji.

Prace Mostowskiego i Lindströma poświęcone kwantyfikatorom uogólnionym wprowadziły zupełnie nowe rozumienie kwantyfikacji, które zaowocowało wieloma praktycznymi zastosowaniami w lingwistyce logicznej i logice indukcji.

Głównym celem pracy jest wykazanie, że koncepcja kwantyfikatorów uogólnionych Lindströma nawiązuje bezpośrednio do intuicji Arystotelesa, zaś spójki "a", "e", "i", "o" występujące w języku sylogistyki mogą być uznane za symbole kwantyfikatorów uogólnionych typu $\langle 1, 1 \rangle$.