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THE PROBLEM OF DEGREES OF MAXIMALITY
(A survey)

The paper is a complete survey of the methods for proving theorems on degrees of maximality and of the results obtained up to 1979 by the authors working in the area. In the first two sections the reader will find the whole conceptual and notational apparatus necessary for the further discussion of the problem of strengthenings and, in particular, of degrees of maximality of sentential calculi.

§1. Preliminaries. Let L be a set of formulas formed by means of sentential variables p, q, r, ..., and a finite number of connectives $f_1, f_2, \ldots, f_n$. Then the algebra

(1) $$\underline{L} = (L, f_1, f_2, \ldots f_n).$$

is called a sentential language. Endomorphisms e of $\underline{L}$ are called substitutions ($e \in End(\underline{L})$). For every $X \subseteq L$ we set $Sb(X) = \{e\alpha : \alpha \in X$ and $e \in End(\underline{L})\}$. By a consequence operation on $\underline{L}$, or simply a consequence on $\underline{L}$, we understand an operation C defined on 2^L (the power set) of L such that for any X, Y ⊆ L

$$X \subseteq C(X) = C(C(X)) \quad \text{and} \quad X \subseteq Y \rightarrow C(X) \subseteq C(Y).$$

A consequence C is said to be structural provided that for any substitution $e \in End(L)$, and for any $X \subseteq L$, $eC(X) \subseteq C(eX)$.

Every relation

$$R \subseteq 2^L \times L$$

is referred to as a rule of inference of L, or simply a rule. When $X \subseteq L$ and $\alpha \in L$, instead of $R(X,\alpha)$ we shall sometimes write $^X/\alpha \in R$, and call $^X/\alpha$ a sequent of R. Any rule of the form

(2) $$Sb(^X/\alpha) = \{{}^{eX}/e\alpha : e \text{ is a substitution of } L\}$$

will be called sequential. All rules of the form (2) where X is the empty set, $X = \emptyset$, i.e. rules of the form $Sb(^\emptyset/\alpha)$ will be called axiomatic. In that case the elements of the set $\{e\alpha : e \in End(L)\}$ are called axioms.

Given a set of rules of inference θ of a given sentential language $\underline{L}$, let us define an operation Cn_θ on 2^L as follows: for every $X \subseteq L$, $Cn_\theta(X)$ is the least superset of X closed under the rules in θ. Cn_θ is a consequence operation on $\underline{L}$, and, moreover, for every consequence operation on $\underline{L}$, there exists a set rules θ, such that $C = Cn_\theta$, cf. [4]. Any such θ will be called a basis for C. We also have

LEMMA 1. (cf. [28]). A consequence C on $\underline{L}$ is structural if and only if it has a sequential basis.

Given a sentential language $\underline{L} = (L, f_1, f_2, \dots, f_n)$, any couple

(3) $$M = (\underline{A}_M, I_M),$$

where $\underline{A}_M = (A_M, f_1, f_2, \dots, f_n)$ is an algebra similar to $\underline{L}$ and $I \subseteq A_M$, is called an (elementary) matrix corresponding to $\underline{L}$, and the elements of I_M are sometimes called the distinguished values of M, cf. [5]. In turn, let us use the symbol $HOM(\underline{L}, \underline{A}_M)$ to denote the class of all homorphisms of $\underline{L}$ into $\underline{A}_M$ – the elements of $Hom(\underline{L}, \underline{A}_M)$ are called valuations of $\underline{L}$ in M. For every $X \subseteq L$ let us put

(4) $$Cn_M(X) = \{\alpha \in L : \text{for every } h \in Hom(\underline{L}, A_M),\ h\alpha \in I_M \text{ wherever } h(X) \subseteq I_M\}.$$

Cn_M is a structural operation on $\underline{L}$. If K is a class of matrices corresponding to a given language $\underline{L}$, then by a consequence operation determined by K on $\underline{L}$, we shall understand the operation Cn_K defined as follows (cf. [28]): For every $X \subseteq L$,

(5) $$Cn_K(X) = \bigcap_{M \in K} Cn_M(X).$$

It turns out, cf. [29], that for every structural consequence C there is a set of matrices K such that $C = Cn_K$.

Given a consequence C defined on $\underline{L}$, denote by Matr (C) the set of all matrices corresponding to $\underline{L}$ such that $C \leq Cn_M$. We then have

LEMMA 2 (cf. [28]). Each structural consequence C is uniquely determined by Matr(C), i.e. for any two structural consequence operations C, C' defined on $\underline{L}$, C = C' if and only if Matr(C) = = Matr(C').

Two matrices M, N corresponding to the same language are <u>equivalent</u>, $M \sim N$, provided that the consequence operations which they define are identical, i.e. if $Cn_M = Cn_N$. If $\approx$ is a congruence of the algebra $\underline{A}_M$ such that $|a| \subseteq I_M$ for any $a \in I_M$, then it is called a <u>congruence of the matrix</u> $M = (\underline{A}_M, I_M)$. It turns out that for any matrix congruence $\approx$ of M, the quotient matrix $M/_{\approx} = (\underline{A}_{M/\approx}, I_{M/\approx})$ is equivalent to M; $M/_{\approx} \sim M$.

Any couple

(6) $$S = (\underline{L}, C),$$

where $\underline{L}$ is a sentential language and C is a structural consequence operation on $\underline{L}$, is called a <u>sentential calculus</u>. In the sequel the elements of the class Matr(C) for a given calculus $S = (\underline{L}, C)$ will be called S-matrices cf. [28].

Dealing with the calculi implicative in the sense of R a s i o w a [17] (cf. also [28]), one can improve Lemma 2 replacing the class of S-matrices, Matr(C), by a class of quotient matrices. Assume that $S = (\underline{L}, C)$ is an implicative calculus. Let $\rightarrow$ denote the implication connective of $\underline{L}$. Then for every $M \in$ Matr(C) the relation defined on A_M as follows:

(7) $$a \equiv_M b \text{ if and only if } a \rightarrow b,\ b \rightarrow a \in I_M$$

is a congruence of M, and therefore $M/_{\equiv_M} \sim M$. Let us put

(8) $$\mathrm{Alg}^R(C) = \{M/_{\equiv_M} : M \in \mathrm{Matr}(C)\}.$$

The elements of $\mathrm{Alg}^R(C)$ will be called <u>S-algebras</u> (cf. [17], [28]).

LEMMA 3 (cf. [17]). A consequence C of implicative sentential calculus $S = (\underline{L}, C)$ is uniquely determined by the class $\mathrm{Alg}^R(C)$.

§2. Two kinds of strengthenings of a sentential calculus. The notion of degree of maximality versus the notion of degree of completeness. Given a sentential language $\underline{L}$, the class of all structural consequence operations on $\underline{L}$, to be denoted here as $\underline{C}(\underline{L})$, forms a complete lattice with the order $\leqslant$ defined as follows:

(9) $C_1 \leqslant C_2$ if and only if $C_1(X) \subseteq C_2(X)$ for every $X \subseteq L$,

cf. [29]. In the paper the symbols sup (C_1, C_2) and inf (C_1, C_2) will be used to denote the supremum and infimum of C_1 and C_2, respectively. In the case when $C_1 \leqslant C_2$ $(C_1, C_2 \in \underline{C}\ \underline{L})$, C_2 is called a strengthening of C_1. If, moreover, for some $X \subseteq L$, $C_1(X) \subsetneq C_2(X)$, then we say that C_2 is a proper strengthening of C_1, and write $C_1 < C_2$. In the sequel the symbol L will also be used to denote the so-called inconsistent consequence defined as follows:

(10) $$L(X) = L \text{ for every } X \subseteq L.$$

Obviously, L is the greatest element of $\underline{C}(\underline{L})$. Every consequence operation on $\underline{L}$ which is not inconsistent is called consistent. Finally, a consequence $C \in \underline{C}(\underline{L})$ will be said to be maximal provided that it does not have proper consistent strengthenings.

Very often the notions introduced in the last paragraph refer also to sentential calculi - the correspondence between structural consequence operations of a given language $\underline{L}$ and sentential calculi formalized in that language is, under the definition, quite obvious. Thus, given a calculus $S = (\underline{L}, C)$, we shall say that a calculus $S' = (\underline{L}, C')$ is a strengthening of S provided that $C \leqslant C'$, and so on... In this sense all notions which we are still going to introduce will also be used ambiguosly.

According to Lemma 1, every strengthening of a sentential calculus $S = (\underline{L}, C)$ can be obtained from S by adding to the set of rules of C some set of sequential rules Θ - this strengthening will be denoted as $S^{\Theta} = (\underline{L}, C^{\Theta})$ and if $\Theta = \{R\}$ also as $S^R = (\underline{L}, C^R)$. When all rules in Θ are axiomatic, S^{Θ} (C^{Θ}) will be called an axiomatic strengthening of S (of C). In that case the set $C(A)$, where

$$A = \{Sb(\alpha) : Sb(\emptyset/\alpha) \in \Theta\}$$

is an invariant system of C, i.e. the following holds:

$$C(A) = Sb(C(A)) \text{ and } C(C(A)) = C(A)$$

and C^{θ} can be defined as follows: for every $X \subseteq L$, $C^{\theta}(X) = C(X \cup A)$.

Given a sentential calculus $S = (\underline{L}, C)$, the cardinal number of all axiomatic strengthenings of S is called the degree of completeness of S, dc(S), cf. T a r s k i [18]. On the other hand, by the degree of maximality of S, dm(S), we shall understand, following W ó j c i c k i [27], the cardinal number of all strengthenings of S, i.e. both axiomatic and non-axiomatic. Obviously, dm(S) is at least as great as the degree of completeness of S and it turns out that in many cases dm(S) > dc(S).

Given a sentential calculus $S = (\underline{L}, C)$, from Lemma 2 it follows that any structural consequence operation $C' \geqslant C$ is determined by some subclass of Matr(C). Consequently,

(11) $$dm(S) \leqslant card\left\{K : K \subseteq {}^{Matr(C)}/_{\sim}\right\}$$

or, more precisely,

(12) $$dm(S) = card\left\{Cn_K : K \subseteq {}^{Matr(C)}/_{\sim}\right\}.$$

And, according to Lemma 3, for the case of implicative sentential calculi the last formulas can be improved to

(11^1) $$dm(S) \leqslant card\left\{K : K \subseteq {}^{Alg^R(C)}/_{\sim}\right\}$$

and

(12^1) $$dm(S) = card\left\{Cn_K : K \subseteq {}^{Alg^R(C)}/_{\sim}\right\},$$

respectively.

Finally, the counterparts of (12) and (12^1) for the notion of degree of completeness are the following:

(13) $$dc(S) = card\left\{Cn_K(\emptyset) : K \subseteq {}^{Matr(C)}/_{\sim}\right\},$$

(14) $$dc(S) = card\left\{Cn_K(\emptyset) : K \subseteq {}^{Alg^R(C)}/_{\sim}\right\}.$$

§3. Historical account of particular studies of the problem of degrees of maximality. In the present section all explict contributions to the topic are listed in the chronological order and the main methods for proving theorems on degrees of maximality are briefly reported. All undefined notation concerning n-valued Łukasiewicz sentential calculi comes from [8].

The paper by W ó j c i c k i, [27], in which the notion of degree of maximality was introduced was, at the same time, the first contribution to the studies on the problem. The main theorem of [27] says that the degree of maximality of the three-valued Łukasiewicz sentential calculus $Ł_3 = (\underline{L}, C_3)$ equals 4, i.e.

(I) $$dm(Ł_3) = 4.$$

The crucial point of the original method of proof applied by R. Wójcicki is the reduction of the whole problem to the problem of strengthenings of $Ł_3$ which can be obtained by the use of rules of inference determined by sequents of the sublanguage of $\underline{L}$ generated by a single sentential variable p, $\underline{L}_p = (L_p, \to, \vee, \wedge, \neg)$. Accordingly, the first step was to prove the following assertion:

(p) For every $\alpha \in L$, $X \subseteq L$, $\alpha \in C_3(X)$ if and only if for every substitution $e: \underline{L} \to \underline{L}_p$ $e\alpha \in C_3(eX)$.

Next, using some properties of C_3, it is possible to define an equivalence relation $\approx_3$ having the two properties: (i) $\alpha \approx_3 \beta$ if and only if $h(\alpha) = h(\beta)$ for every $h: \underline{L} \to A_3$ (ii) If $\alpha \approx_3 \beta$ then $C_3(\alpha) = C_3(\beta)$.

It turned out that the quotient set $L_p/\approx_3$ had 12 elements - in [27] their representatives were denoted as $\varphi_1, \varphi_2, \ldots, \varphi_{12}$. Subsequently, from the properties of $\approx_3$ it follows that every strengthening $Ł_3^R$ of $Ł_3$ by a sequential rule $R = Sb(^X/\alpha)$ determined by a sequent $^X/\alpha$ of $\underline{L}_p$ is equal to some strengthening of $Ł_3$ by a rule of the form $Sb(^{\varphi_i}/\varphi_j)$. Thus, the full investigation of the set of all sequential rules of $\underline{L}$ determined by sequents of $\underline{L}_p$ can be replaced by a study of the set of 144 rules of inference of the form $Sb(^{\varphi_i}/\varphi_j)$. To do this, the author of [27] used some matrix methods and finally reached the conclusion that each of the rules R of the form $Sb(^{\varphi_i}/\varphi_j)$ falls into one of the categories defined by the following conditions:

(a) R is a rule of $Ł_3$ thus $Ł_3^R = Ł_3$

(b) $Ł_3^R = Ł_2$

(c) $Ł_3^R = Ł_1$

(d) $Ł_3^R = Ł_3^*$,

where $Ł_3^* = (\underline{L}, C_3^*)$ with C_3^* defined as follows:

$$C_3^*(X) = \begin{cases} C_3(X) & \text{if } C_2(X) \neq L \\ L & \text{otherwise.} \end{cases}$$

Thus $Ł_3$ has at least four strengthenings: $Ł_3, Ł_3^*, Ł_2, Ł_1 = L$. The fact that these are the only possible strengthenings of $Ł_3$, and thus that $dm(Ł_3) = 4$, follows easily from the following results:

- $C_3 < C_3^* < C_2 < L$ and therefore, by the use of (p) one can prove that every proper strengthening of C_3 is not weaker than C_3^*;
- $Ł_3^\Theta$ is consistent if and only if Θ is the set of rules of C_2;
- $Ł_3^*$ is $\{\emptyset\}$ - complete (Theorem 2 in [27]), and therefore for every $C > C_3^*$, $C(\emptyset) \supsetneq C_3(\emptyset)$;
- $Ł_2$ is the only proper consistent strengthening of $Ł_3$ (Wajsberg's theorem on degrees of completeness of $Ł_3$).

Next, using the fact that Łukasiewicz sentential calculi $Ł_n$ are implicative in the sense of R a s i o w a [17], the author of the present review gave in [10] an algebraic proof of (I) and, moreover, showed that the degree of maximality of the four-valued Łukasiewicz calculus $Ł_4$ also equals 4,

(II) $$dm(Ł_4) = 4.$$

Both the method and the results of [10] were subsequently generalized in [11] for a wider class of Łukasiewicz logics, namely, for those calculi $Ł_n$ for which n-1 is prime. The main result of [11] says that

(III) $dm(Ł_n) = 4$ for any $n \geq 2$ such that n-1 is prime.

The origin of the proof of (III) given in [11] is the use of some algebraic structures corresponding to n-valued Łukasiewicz matrices, namely so-called MV_n-algebras introduced by R. S. Grigolia. Given finite $n \geq 2$, MV_n-algebra is a structure $\pi = (A, +, \cdot, -, 0, 1)$ of the type $(2,2,1,0,0)$ fulfiling a number of equations, cf. [3]. The primary correspondence between Łukasiewicz matrices and MV_n algebras runs as follows: For every Łukasiewicz matrix $M_n = (A_n, \to, \vee, \wedge, \neg, \{1\})$ one can define MV_n algebra

$$\underline{A}_n = (A_n, +, \cdot, -, 0, 1)$$

putting $x + y = \neg x \to y$, $x \cdot y = \neg(x \to \neg y)$ and $\overline{x} = \neg x$. Moreover, this correspondence is one-to-one, since conversely: $x \to y = \overline{x} + y$, $x \vee y = x \cdot \overline{y} + y$, $x \wedge y = (x + \overline{y}) \cdot y$ and, obviously, $\neg x = \overline{x}$. Secondly, we have (cf. [3]):

(Rt) Every MV_n algebra with more than one element is isomorphic with a subdirect product of a number of copies of algebras $\underline{A}_m$, where $m \leq n$ and $m-1$ is a divisor of $n-1$.

Therefore, it turns out that each MV_n algebra π can be considered as the algebra of the form $\pi^{*} = (A, \to, \cup, \cap, -, 1)$, where $\to, \cup, \cap, -$ are the natural counterparts of $\to, \vee, \wedge, \neg$, respectively, defined in the appropriate algebras $\underline{A}_m$. In the sequel we shall use the symbol C_π to denote the consequence C_{π^*} determined on the language $\underline{L}$ of Łukasiewicz sentential calculi by the matrix $(\pi^{*}, \{1\})$.

Given $n \geq 2$, let $\underset{n}{\approx}$ be the relation on $\underline{L} = (L, \to, \vee, \wedge, \neg)$ defined as follows: For every $\alpha, \beta \in L$,

$$\alpha \underset{n}{\approx} \beta \quad \text{if and only if } \alpha \to \beta, \beta \to \alpha \in C_n(\emptyset).$$

Using the fact that $\underline{A}_n$ correlated with the Łukasiewicz matrix M_n is an MV_n algebra, one can easily verify that the Lindenbaum algebra $\underline{L}/\underset{n}{\approx}$ is an MV_n algebra. Now, recall that the Łukasiewicz calculus $Ł_n = (\underline{L}, C_n)$ is implicative. Thus, as a particular case of a general result of R a s i o w a cf. [17], p. 184 we obtain that $\underline{L}/\underset{n}{\approx}$ is a free algebra in the class $Alg^R(C_n)$ of all $Ł_n$-algebras. Using this fact and the representation theorem (Rt) one can prove that for a given $n \geq 2$, the class of all $Ł_n$-algebras coincides with the class MV_n of all MV_n algebras, i.e. that

(15) $$\mathrm{Alg}^R(C_n) = MV_n.$$

Now, let us assume that n is a natural number such that n-1 is prime. If so, $\underline{A}_n$ has only one non-trivial subalgebra - $\underline{A}_2$. Using this fact and (Rt) one can prove cf. [11] that for any MV_n algebra $\mathfrak{n}$, the consequence operation C_n coincides with one of the following consequence operations:

$$Cn_{\underline{A}_1},\ Cn_{\underline{A}_2},\ Cn_{\underline{A}_2\times\underline{A}_n},\ Cn_{\underline{A}_n},$$

where $\underline{A}_2\times\underline{A}_n$ denotes the direct product of $\underline{A}_2$ and $\underline{A}_n$. In turn, we have

$$L = Cn_{\underline{A}_1} > Cn_{\underline{A}_2} > Cn_{\underline{A}_2\times\underline{A}_n} > Cn_{\underline{A}_n}.$$

And therefore

$$dm(Ł_n) = \left\{ Cn_K : K \subseteq {}^{MV}n/_{\sim} \right\} = \left\{ Cn_{\underline{A}_1},\ Cn_{\underline{A}_2},\ Cn_{\underline{A}_2\times\underline{A}_n},\ CN_{\underline{A}_n} \right\}.$$

This ends a sketch of the proof of (III).

REMARK. To give a similar proof of (I) and (II), in [10] the author uses the so-called three-and four-valued Łukasiewicz algebras introduced by M o i s i l cf. e.g. [13]. Incidentaly, it can be proved that in the cases n=3 and n=4, both the notions - that of n-valued Łukasiewicz Moisil algebra and that of MV_n algebra - coincide.

The examination of the problem of degrees of maximality in the whole class of n-valued Łukasiewicz calculi was carried out by W ó j c i c k i in [31], where the following result was proved:

$$dm(Ł_n) \text{ is finite for every finite } n \geq 2.$$

To get this result, Wójcicki modified the algebraic technique of [11] and introduced a very handy notion of the characteristic element of an $Ł_n$-algebra. Below, a sketch of the method used in [31] is given.

First, using properties of the consequence C_n, it is possible to give a purely logical proof of the following equality:

(16) $$\mathrm{Alg}^R(C_n) = HSP(M_n),$$

where HSP(M_n) denotes the variety generated by the n-valued Łukasiewicz matrix (the least class of matrices containing M_n and closed under the operations of forming direct products, subalgebras and homomorphic images). Notice, that (16) is another version of (15). In turn, using the one-to-one correspondence between simple MV_n algebras $\underline{A}_n$ and matrices M_n, and the equational definability of MV_n algebras, it is possible to translate Grigolia's representation result (Rt) into

$$HSP(M_n) = SP(M_n).$$

Thus, cf. (12^1), for every $n \geqslant 2$

$$\text{(17)} \qquad dm(Ł_n) = card\left\{K : K \subseteq SP(M_n)/_{\sim}\right\}.$$

Where $a \in A$ is an element of some algebra $\underline{A} = (A, \to, \vee, \wedge, \neg, \{1\})$ in HSP(M_n), let us denote by $[a]$ the subalgebra of $\underline{A}$ generated by a, for every $Ł_n$-algebra $\underline{A} \in Sp(M_n)$ it is possible to find an element a^* in A a "characteristic element" of $\underline{A}$, cf. [31], with the following two properties:

1°. $[a^*] \sim \underline{A}$

2°. There are pairwise different submatrices M_{m_1}, M_{m_2}, ..., M_{m_k} of M_n such that $[a^*] \sim M_{m_1} \times M_{m_2} \times \ldots \times M_{m_k}$.

Therefore, denoting by V_n the set of all direct products of the form $M_{m_1} \times M_{m_2} \times \ldots \times M_{m_k}$ of pairwise different submatrices of M_n, we may pass from (17) to

$$\text{(17')} \qquad dm(Ł_n) \leqslant card\left\{K : K \subseteq {}^{V_n}/_{\sim}\right\},$$

and since V_n is finite, dm($Ł_n$) is finite. This ends the proof of (IV).

Now, we are going to discuss [9], in which the solution to the dm-problem for some non-implicative sentential calculi was given. As the title of [9] makes plain, the paper concerns the so-called dual counterparts of n-valued Łukasiewicz calculi, cf. [26], and [12]. Where $\underline{L} = (\to, \vee, \wedge, \neg)$ is the language of Łukasiewicz calculi and $n \geqslant 2$, the calculus dual to $Ł_n$, $dŁ_n$, is a pair

$$d\text{Ł}_n = (\underline{L}, \overline{C}_n),$$

where $\overline{C}n$ is the consequence operation determined by the matrix $\overline{M}_n = (\underline{A}_n, A_n - \{1\})$, cf. [12]. The fact that the calculi $d\text{Ł}_n$ are not implicative easily follows from the definition of the class S of R a s i o w a cf. [17], p. 179. In spite of this, the main theorem of [9] saying that the degrees of maximality of all calculi $d\text{Ł}_n$ are finite, i.e.

(V) $\qquad dm(d\text{Ł}_n)$ is finite for every finite $n \geq 2$.

was obtained by the use of matrices analogous to S-algebras (Ł_n-algebras) of Rasiowa. In the sequel the class of all such matrices will be denoted as $Matr^R(dC_n)$.

Given $n \geq 2$, for every $x \in A_n$, let us put

$$\neg x = \begin{cases} 0 & \text{if } x = 1 \\ 1 & \text{otherwise.} \end{cases}$$

Using the criterion in [14], it is easy to verify that $\neg$ is definable in $\underline{A}_n$. By the same symbol, $\neg$, we shall denote a sentential connective in $\underline{L}$ corresponding to $\neg$. Where $M = (\underline{A}_M, I_M) \in Matr\ (dC_n)$, let us put:

$$a \doteq_M b \text{ if and only if } \neg(a \to b), \neg(b \to a) \in I_M.$$

$\doteq_M$ is a congruence of the matrix M and therefore $M \sim {}^M/\doteq_M$. Consequently, the class

$$Matr^R(dC_n) = \left\{ {}^M/\doteq_M : M \in Matr(dC_n) \right\}$$

can be used to replace $Matr(dC_n)$ in (11), and thus we get:

(18) $$dm(d\text{Ł}_n) = card\left\{ K : K \subseteq {}^{Matr^R(dC_n)}/_\sim \right\}.$$

It turns out that all matrices in $Matr^R(dC_n)$ have, among others, a very special property - in every $M \in Matr^R(dC_n)$ there is a subset V_M such that the quotient set ${}^{V_M}/\doteq_M$ is the one-element set, namely, ${}^{V_M}/\doteq_M = 1_M$ and that $(\underline{A}_M, 1_M) \in HSP(M_n)$ cf. p. 46. Using this fact and the representation theorem for $HSP(M_n)$, one can prove that

(19) $$Matr^R(dC_n) = HSP(\overline{M}_n),$$

compare (16). Moreover, there is a one-to-one correspondence between matrices from $HSP(\overline{M}_n)$ and those from $HSP(M_n)$: For every $M = (\underline{A}_M, I_M) \in HSP(\overline{M}_n)$ and $M_+ = (\underline{A}_M, 1_M)$ and for any $\alpha \in L$, $X \subseteq L$,

(+) $$\alpha \in Cn_M(X) \text{ if and only if } \neg\alpha \in Cn_{M_+}(\neg X)$$
$$\neg\alpha \in Cn_M(\neg X) \text{ if and only if } \alpha \in Cn_{M_+}(X),$$

where $\neg X$ denotes the set of formulas resulting from X by preceding each of its formulas by $\neg$. In turn, let $\overline{V}_n$ be the "natural" counterpart of the set of product matrices V_n used on p. 46. Then, using (19) and (+) one can prove that for every $M \in Matr^R(dC_n)$ there is a matrix $M_d \in \overline{V}_n$ such that $M_d \sim M$. Consequently, (18) can be improved to

(18') $$dm(dŁ_n) = card\{K : K \subseteq {}^{\overline{V}_n}/_{\sim}\}$$

and since $\overline{V}_n$ is finite, $dm(Ł_n)$ is finite.

The method of [9] described in the last sequence of paragraphs was also used to some extent in [8] to give a characterization of strenghenings of the so-called Łukasiewicz-like sentential calculi, $Ł_n^I$, which are determined by n-valued Łukasiewicz matrices M_n^I with superdesignated logical values: $1 \in I$, $0 \notin I$. The main result of [8] says that the degree of maximality of any n-valued Łukasiewicz-like sentential calculus if finite and equal to the degree of maximality of the corresponding n-valued Łukasiewicz calculus.

(VI) $dm(Ł_n^I) = dm(Ł_n)$ every finite $n \geq 2$, every $I \subseteq A_n$, $1 \in I$ and $0 \notin I$.

In his abstract [6], M a d u c h reproted briefly some results of studies on pure implicational sentential calculi of Łukasiewicz. Given a finite $n \geq 2$, the n-valued Łukasiewicz implicational calculus $Ł_n^*$ is the pair $(\underline{L}, C_n^*)$ consisting of the pure implicational sentential language $\underline{L}^* = (L^*, \to)$ and $C_n^* = Cn_{M_n^*}$, where M_n^* is the implicational reduct of the Łukasiewicz matrix M_n. Among others, one can find in [6] the following theorem:

(VII) $dm(Ł_n^*) = dc(Ł_n^*) = n$ (for every finite $n \geq 2$).

The original method of proof is based on some results concerning Lindenbaum's algebras determined by the so-called irreducible theories of $Ł_n^*$. Unfortunately, the proof contains some gaps not easy to remove. The result, however, is certainly valied - a very simple proof of (VII) will be given in Section 4.

M. Tokarz in [20] examined the problem of structural strengthenings of the consequence operations C_3^s and C_4^s determined by the so-called Sugihara matrices M_3^s and M_4^s, respectively ($M_3^s = (\{-1,0,1\}, \to, \vee, \wedge, \neg, \{0,1\})$ and $M_4^s = (\{-2,-1,1,2\}, \to, \vee, \wedge, \neg, \{1,2\})$, where $\neg x = -x$, $x \vee y = \max(x,y)$, $x \wedge y = \min(x,y)$, $x \to y = -x \vee y$ if $x \leq y$ and $x \to y = -x \wedge y$ otherwise). The main result of [20] says that the degrees of maximality of C_3^s and C_4^s are both equal to 4,

(VIII) $dm(C_3^s) = dm(C_4^s) = 4.$

The details of proof of (VIII) are strongly based on particular properties of Sugihara matrices. The method, however, seems to be more universal - it has some points of contact with Wójcicki's method of proof of (I). Accordingly, the proof for the case of C_3^s can be sketched so as to consist of the following four steps:

Step 1. If $C_3^s \leq C$, where C is a consistent structural consequence operation, then $C \leq C_2$ (C_2 being the classical consequence operation - Lemma 2 in [20]).

Step 2. If C_3^{s*} is used to denote the struturally complete consequence cf. [15] for which $C_3^{s*}(\emptyset) = C_3^s(\emptyset)$, then for every C, $C_3 < C < C_2$ we have $C \leq C_3^{s*}$ (a particular case of the general rèsult concerning structural completeness, cf. Section 4, p. 52).

Step 3. Let us now assume again that $C_3^s < C < C_2$. Then, if for some $\alpha \in L$, $X \subseteq L$, $\alpha \in C(X)$ and $\alpha \notin C_3^s(X)$, then by Lemma 4 in [20] we get $C_2(X) \neq L$ and this together with Lemma 8 in [20] implies that for some sentential variables p_0 and p_1, $p_0 \in C(p_1 \wedge \neg p_1)$. Finally, the strengthening $C_3^{s\{\varrho\}}$ of C_3^s by the rule $\varrho = \{p_1 \wedge \neg p_1 / p_0\}$ is structurally complete, and thus $C_3^{s\{\varrho\}} = C_3^{3*}$ (Lemma 7 of [20]). So, $C_3^{s*} \leq C$.

Step 4. Steps 2 and 3 together imply that C_3^{s*} is the only structural consequence operation between C_3^s and C_2. On the other hand, C_2 is maximal and therefore C_3^s, C_3^s and C_2 are all consistent strengthenings of C_3^s. Thus, $dm(C_3^s) = 4$.

As early as in [27] W ó j c i c k i posed the following conjecture:

(H) The degree of maximality of any strongly finite sentential calculus is finite.

Recall, cf. [30], that a calculus $S = (\underline{L}, C)$ is strongly finite if there is a finite set of finite matrices K strongly adequate to C, i.e. such that $C = Cn_K$. Incidentally, it was proved in [25] that the degree of completeness of a strongly finite sentential calculus is always finite.

The conjecture (H) appeared to be not true - T o k a r z succeded [23] in constructing a strongly finite logic, whose degree of maximality is infinite, more exactly, he showed that the consequence operation determined by the four-valued implicational-negational Sugihara matrix (INSA) has infinitely many structural strengthenings. The basic Tokarz's idea was subsequently modified by W r o ń s k i, who in [32] have a similar counterexample to (H) by the use of a three-element matrix. From the two examples, Wroński's is much easier to describe. It runs as follows:

Let $\underline{A} = (\{0,1,2\}, \cdot)$ be an algebra of type (2), whose binary operation $\cdot$ is defined by the conditions: $0\cdot 0 = 2\cdot 2 = 2$, $1\cdot 1 = 1$ and $x\cdot y = 0$ otherwise. It is easy to see that $\underline{B} = (\{0,2\}, \cdot)$ is a subalgebra of $\underline{A}$. In the sequel we shall consider the two matrices

$$A = (\underline{A}, \{0\}) \quad \text{and} \quad B = (\underline{B}, \{0\}).$$

For every n = 1, 2, ... define a set of sequential rules

$R_n = \{(X, \alpha)$: (i) X is a finite set of formulas built up from at most n sentential variables $p_1, \ldots, p_n$, (ii) $Cn_B(X) = L\}$.

Subsequently, let us put $\underline{C}_n =^{df} Cn_A^{R_n}$ ($\underline{C}_n$ is the strengthening of Cn_A by the set of rules R_n). One can prove that for any $n > 1$,

$$Cn_A \leq \underline{C}_n < \underline{C}_{n+1},$$

cf. [32] - Lemma 1.1. This immediately implies that Cn_A has infinitely many structural strengthenings.

§4. Some related topics. In the logical literature there is a number of results which provide us with very convenient methods of establishing degrees of maximality of some special sentential calculi. Especially important are the results concerning such notions as maximality, almost maximality and structural completeness for the extended discussion of this sort of things see e. g. [21].

Maximality. If $S = (\underline{L}, C)$ is a maximal sentential calculus, i.e. if C does not have proper structural strengthenings except L, then obviously $dm(S) = 2$. In [22] one can find a very useful matrix criterion of maximality of consequence operation.

M1 (cf. [19]). If every constant function of $\underline{A}_M$ is definable in the matrix $M = (\underline{A}_M, I_M)$, then Cn_M is maximal.

A very illustrative example of the use of the criterion (M1) is a pretty short proof of maximality of the classical sentential calculus $Ł_2 = (\underline{L}, C_2)$. Accordingly, we have that the matrix $M_2 = (\{0,1\}, \rightarrow, \vee, \wedge, \neg, \{1\})$ is strongly adequate for C_2, $Cn_{M_2} = C_2$ and both the constants: 0 and 1 are definable in M_2, e. g. as $0 = \neg(x \rightarrow x)$ and $1 = x \rightarrow x$.

Almost-maximality. Given a sentential language $\underline{L}$ let us put

$$(19) \qquad L_\emptyset(X) = \begin{cases} \emptyset & \text{if } X = \emptyset \\ L & \text{otherwise.} \end{cases}$$

$L_\emptyset$ is a (structural) consequence operation on $\underline{L}$ and it is called, cf. [22], almost-inconsistent. In turn, a consequence C on $\underline{L}$ will be called almost-maximal whenever for every structural consequence C', $C < C'$ implies that $C' = L_\emptyset$ or $C' = L$, cf. [22]. From the definition it immediately follows that the degree of maximality of any almost-maximal sentential calculus $S = (\underline{L}, C)$ equals 3 or 2. The following matrix criterion of almost-maximality can be found in [22]:

AM_1. If $M = (\underline{A}_M, \{a\})$, where $a \in A_M$ is a matrix such that for every $b \in A_M$ there is a function f_b definable in $\underline{A}_M$ such that $f_b(a) = b$, then Cn_M is almost-maximal.

Another criterion, which, originally, was stated earlier by Wójcicki and Wroński, can be treated as a corollary to AM_1:

AM_2 (Wójcicki-Wroński, unpublished). Let $\underline{A}_M$ have no proper subalgebra and let $a \in A_M$. Then $Cn_{(\underline{A}_M, \{a\})}$ is almost-maximal.

Notice that if a consequence operation C is almost-maximal and $C(\emptyset) \neq \emptyset$, then C is maximal. Thus, any maximal sentential calculus can serve as an example of almost-maximal calculus with the degree of maximality 2. Finally, the following theorem seems to be of some interest:

THEOREM 1. (Wroński, unpublished). If M is a two-element matrix, then Cn_M is almost-maximal.

A proof of Theorem 1 can be found in [21].

Structural completeness. Given a sentential language $\underline{L}$, a rule of inference R of $\underline{L}$ is called structural if and only if for every sequent X/α, $X/\alpha \in R$ implies that $eX/e\alpha \in R$ for every substitution $e \in End(\underline{L})$. Where C is a consequence operation on $\underline{L}$, R is called permissible in $C(\emptyset)$ if and only if for every $X/\alpha \in R$, $\alpha \in C(\emptyset)$ whenever $X \subseteq C(\emptyset)$. A sentential calculus $S = (\underline{L}, C)$ is structurally complete, cf. [18], if and only if every structural rule which is permissible in $C(\emptyset)$ is a rule of C.

A very useful criterion of structural completeness was given by D. Makinson:

SC (cf. [7]). $S = (\underline{L}, C)$ is structurally complete if and only if for every structural consequence operation C' on $\underline{L}$, $C'(\emptyset) = C(\emptyset)$ implies that $C' \leqslant C$.

The property mentioned in SC can sometimes be used for establishig the degree of maximality - in the sequel this problem will be discussed in two examples, namely for n-valued Gödel's calculi and for n-valued pure implicational Łukasiewicz calculi.

Given a finite $n \geqslant 2$, the n-valued matrix of Gödel is defined as follows cf. e. g. [2]:

$$g_n = (\{1,2,\dots,n\}, \dot{\rightarrow}, \dot{\vee}, \dot{\wedge}, \dot{\neg}, \{1\}),$$

where for every $x,y \in \{1,\dots,n\}$

$$\dot{\neg}x = \begin{cases} n & \text{if } x<n \\ 1 & \text{if } x=n \end{cases}, \quad x \dot{\rightarrow} y = \begin{cases} 1 & \text{if } x \geqslant y \\ y & \text{if } x<y \end{cases}$$

and $x \dot{\vee} y = \max(x,y)$, $x \dot{\wedge} y = \min(x,y)$. Where $\underline{L}$ is the appropriate sentential language, the n-valued Gödel's calculus, $\underline{G}_n$, can be defined as the pair

$$\underline{G}_n = (\underline{L}, G_n)$$

with $G_n = Cn_{g_n}$, cf. [30] p. 65. It is immediately seen that for every $n \geqslant 2$,

(.) $$G_n < G_{n-1} < \dots < G_2 < G_1 = L$$

and, in particular, also

(..) $$G_n(\emptyset) \subsetneq \dots \subsetneq G_2(\emptyset) \subsetneq G_1(\emptyset) = L.$$

Secondly, A n d e r s o n proved in [1] that the degree of completeness of $\underline{G}_n$ equals n, $dc(\underline{G}_n) = n$, what together with (..) implies that the calculi $\underline{G}_k$ with $k \leqslant n$ are the only axiomatic strengthenings of $\underline{G}_n$. On the other hand, all Gödel's calculi $\underline{G}_n$ are structurally complete, cf. [2], and thus, according to (SC) the number of structural strengthenings of G_n is not greater than n. Finally, using (.) we obtain

(IX) $$dm(G_n) = dc(G_n) = n \text{ for every finite } n \geqslant 2 .$$

An extended version of the method applied for the proof of (IX) can also be used for proving Maduch's result (VII), p. 49. Now, we shall also use some result by P. Wojtylak, namely, the following:

THEOREM 2 (cf. [24]). Let M be a matrix corresponding to a given sentential language $\underline{L}$. Then, for every $X \subseteq L$,

$$Cn_M(Sb(X)) = \bigcap\{Cn_N(\emptyset): N \text{ is a submatrix of } M \text{ and } X \subseteq Cn_N(\emptyset)\}.$$

Given a finite $n \geq 2$, one can easily verify that the purely implicational Łukasiewicz matrix M_n^* has the following submatrices: $M_n^*, M_{n-1}^*, \ldots, M_1^*$. In turn, one can also verify that

$$C_n^* < C_{n-1}^* < \ldots < C_2^* < C_1^* = L$$

$$C_n(\emptyset) \subsetneq C_{n-1}(\emptyset) \subsetneq \ldots \subsetneq C_2(\emptyset) \subsetneq C_1(\emptyset) = L.$$

Making use of Theorem 2, from the latter inclusion we obtain that any invariant system of C_n must be equal to one of the following sets: $C_n^*(\emptyset)$, $C_{n-1}^*(\emptyset)$, ..., $C_2^*(\emptyset)$, L. Thus, $dc(Ł_n^*) = n$. Finally, all $Ł_n^*$'s are structurally complete, cf. [16], so, by the same type of argument as for Gödels calculi we obtain $dm(Ł_n^*) = n$.

Note. The notion of structural completeness of a sentential calculus can be understand in two ways, according to how the notions of a rule of inference is defined. First, if we restrict the notion of a rule to its finitary sense assuming the rule to be a subset of $L^n \times L$ for some finite n, then we obtain the notion of structural completeness in the finitary sense. And, if the rule is assumed as in our paper, as a subset of 2^L, we obtain the notion of structural completeness in the infinitary sense. An extended discussion of this distinction can be found e. g. in [7]. I would like to stress that the notion of structural completeness was used here in the latter sense. Incidentally, one can show by an easy argument that the two notions coincide in the case of strongly finite sentential calculi - just for this reason we could use the results of Dzik-Wroński and Prucnal concerning finitary structural completeness of Gödel's and Łukasiewicz calculi.

Bibliography

[1] Anderson J. G., The degree of completeness of Dummet LC and Thomas LC_n, "The Journal of the London Mathematical Society", 3(1971), pp. 558-569.

[2] Dzik W., Wroński A., Structural completeness of Gödel's and Dummet's propositional calculi, "Studia Logica", 32(1973), pp. 69-73.

[3] G r i g o l i a R. S., Algebraic analysis of Łukasiewicz-Tarski's n-valued logical calculi (in Russian), "Proceedings of Tbilisi University", A 6-7, 149-150. Tbilisi 1973, pp. 121-132.

[4] Ł o ś J., S u s z k o R., Remarks on sentential logics, "Indagationes Mathematicae", 20 (1958), pp. 177-183.

[5] Ł u k a s i e w i c z J., T a r s k i A., Untersuchungen über den Aussagenkalkul, "Comptes-rendus des séances de la Societé des Sciences et de Lettres de Varsovie", Cl. III, 23 (1930), pp. 30-50.

[6] M a d u c h M., On Lindenbaum's algebras of finite implicational Łukasiewicz logics, "Bulletin of the Section of Logic", 5(1976), No 1, pp. 29-32.

[7] M a k i n s o n D., A characterization of structural completeness of a structural consequence operations, "Reports on Mathematical Logic", 6(1976), No 4, pp. 99-102.

[8] M a l i n o w s k i G., Degrees of maximality of Łukasiewicz-like sentential calculi, "Studia Logica", 36(1977), No 3, pp. 213-228.

[9] M a l i n o w s k i G., Matrix representations for the dual counterparts of Łukasiewicz n-valued sentential calculi and the problem of their degrees of maximality, "Proceedings of the V-th International Symposium on Multiple-Valued Logics", Indiana University, Bloomington, May 1975, pp. 252-261.

[10] M a l i n o w s k i G., S-algebras and the degrees of maximality of three and four valued logics of Łukasiewicz, "Studia Logica", 33(1974), No 4, pp. 201-213.

[11] M a l i n o w s k i G., S-algebras for n-valued sentential calculi of Łukasiewicz. The degrees of maximality of some Łukasiewicz's logics. [in:] Selected papers on Łukasiewicz sentential calculi, Ed, R. W ó j c i c k i, G. M a l i n o w s k i, Ossolineum 1977, pp. 149-159.

[12] M a l i n o w s k i G., S p a s o w s k i M., Dual counterparts of Łukasiewicz's sentential calculi, "Studia Logica", 33(1974), No 2, pp. 153-162.

[13] M o i s i l Gr. C., Essais sur les logiques non chrysippiennes, Editions de l'Académie de la Republique Socialiste de Roumania, Bucarest 1972.

[14] M c N a u g h t o n, A theorem about infinite-valued sentential logic, "Journal of Symbolic Logic", 16(1951), pp. 1-13.

[15] P o g o r z e l s k i W. A., Structural completeness of the propositional calculus, "Bulletin de l'Académie Polonaise des Sciences", Série de mathématiques, astronomiques et physiques, 19(1971), No 5, pp. 349-351.

[16] P r u c n a l T., Proof of structural completeness of a certain class of implicative propositional calculi, "Studia Logica", 32(1973), pp. 93-98.

[17] R a s i o w a H., An algebraic approach to non-classical logics, North Holland Publishing Company, Amsterdam, Polish Scientific Publishers, Warszawa 1974.

[18] T a r s k i A., Fundamentale Begriffe der Methodologie der deduktiven Wissenschaften, "Monatshefte für Mathematik und Physik", 37(1930), pp. 361-404.

[19] T o k a r z M., Connections between some notions of completeness of structural propositional calculi, "Studia Logica", 32(1973), pp. 77-89.

[20] T o k a r z M., Degrees of maximality of three- and four-valued- RM-extensions, "Reports on Mathematical Logic", 9(1977), pp. 63-69.

[21] T o k a r z M., Maximality, post-completeness and structural completeness, "Technical Report", No 2/sf, Institute of Philosophy and Sociology, Polish Academy of Sciences, Wrocław, February 1978.

[22] T o k a r z M., A remark on maximal matrix consequences, "Bulletin of the Section of Logic", 6(1977), No 4, pp. 150-192.

[23] T o k a r z M., A strongly finite logic with infinite degree of maximality, "Studia Logica", 35(1976), No 4, pp. 447-451.

[24] W o j t y l a k P., On structural completeness of many-valued logics, "Studia Logica", 37(1978), No 2, pp. 139-147.

[25] W ó j c i c k i R., The degree of completeness of the finite-valued propositional calculi, Zeszyty Naukowe U. J., Prace z Logiki, z. 7, Kraków 1972, pp. 77-85.

[26] W ó j c i c k i R., Dual counterparts of consequence operations, "Bulletin of the Section of Logic", 2(1973), No 1, pp. 55-57.

[27] W ó j c i c k i R., The logics stronger than three-valued sentential calculus. The notion of degree of maximality versus the notion of degree of completeness, "Studia Logica", 33(1974), No 2, pp. 201-214.

[28] W ó j c i c k i R., Matrix approach in methodology of sentential calculi, "Studia Logica", 32 (1973), pp. 7-37.

[29] W ó j c i c k i R., Some remarks on the consequence operations in sentential logics, "Fundamenta Mathematicae", 58(1970), pp. 269-279.

[30] W ó j c i c k i R., Strongly finite sentential calculi, [in:] Selected papers of Łukasiewicz sentential calculi, Ed. R. W ó j c i c k i, G. M a l i n o w s k i, Ossolineum 1977, pp. 53-77.

[31] W ó j c i c k i R., A theorem on the finiteness of the degree of maximality of the n-valued Łukasiewicz logics, "Proceedings of the V-thenternational Symposium on Multiple-Valued Logics", Indiana University, Bloomington, May 1975, pp. 240-251.

[32] W r o ń s k i A., On finitely based consequence operations, "Studia Logica", 35(1976), No 4, pp. 453-458.

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ZAGADNIENIE STOPNI MAKSYMALNOŚCI
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