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On Estimation of Bi-liner Regression Parameters

Abstract: Two non-parallel lines will be named as bi-lines. The relationship between the definition of the bi-lines function and linear regression functions of the distribution mixture is considered. The bi-lines function parameters are estimated using the least squares method for an implicit interdependence. In general, values of parameter estimators are evaluated by means of an approximation numerical method. In a particular case, the exact expressions for the parameter estimators were derived. In this particular case, the properties of the estimators are examined in details. The bi-lines are also used to estimate the regression functions of the distribution mixture. The accuracy of the parameter estimation is analyzed.

Keywords: bi-lines function, linear regression, least squares method for an implicit interdependence, mixture of probability distribution

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1. Introduction

The classic simple regression does not represent a two-dimensional population if it is a bimodal population. Antoniewicz (1988; 2001) proposed to approximate probability distribution of a one-dimensional random variable by means of two points. Generalizing this result he approximates a two-dimensional distribution by means of two lines, a technique which he called "bi-linear regression". In statistical literature, this term is rather used to define a specific linear model, see e.g. Gabriel (1998). In this paper, we will call Antoniewicz's model simply "bi-lines function", because in general it leads to a two-dimensional data spread approximation by means of two lines. In the next part of this paper, the bi-lines are compared with regressions of two-dimensional probability mixture distributions, or more simply regressions of mixture distributions, see e.g. Quandt (1972). The mixture of Gaussian distributions has been extensively studied e.g. by McLachlan and Peel (2000) and Lindsay and Basak (1993).

The main purpose of the paper is finding differences between the bi-lines function and regressions of mixture distributions. Moreover, we show that the estimators of bi-lines function parameters are biased estimators of appropriate parameters of regressions of mixture distribution. In a particular case, the exact expressions for the estimators of the bi-lines function parameters, as well as for their variances will be derived.

Let a bivariate distribution of (X, Y) be a mixture of two bivariate distributions of (X_1, Y_1) and (X_2, Y_2). Hence:

$$f(x,y) = pf_1(x,y) + (1-p)f_2(x,y)\,,\ 0 < p < 1, \tag{1}$$

where p is the mixing parameter. Let:

$$\begin{aligned} h(x) = \int f(x,y)dy = p\int f_1(x,y)dy + (1-p)\int f_2(x,y)dy = \\ = ph_1(x) + (1-p)h_2(x), \end{aligned} \tag{2}$$

where: $h_i(x) = \int f_i(x,y)dy$, $i = 1, 2$,

$$f(y\,|\,x) = \frac{pf_1(x,y) + (1-p)f_2(x,y)}{ph_1(x) + (1-p)h_2(x)} = f_1(y\,|\,x)w_1(x) + f_2(y\,|\,x)w_2(x). \tag{3}$$

$$w_1(x) = \frac{ph_1(x)}{h(x)}, \qquad w_2(x) = \frac{(1-p)h_2(x)}{h(x)}.$$

It is obvious that if $h_1(x) = h_2(x)$ then $h(x) = h_1(x) = h_2(x)$ and $w_1(x) = p$, $w_2(x) = 1 - p$. Moreover, in this case:

$$f(y \mid x) = pf_1(y \mid x) + (1 - p) f_2(y \mid x), \tag{4}$$

$$E(Y \mid x) = pE(Y_1 \mid x) + (1 - p)E(Y_2 \mid x). \tag{5}$$

Particularly, if $h_1(x) = h_2(x)$ and $E(Y_i \mid x) = a_i x + b_i$, $i = 1, 2$, then expression (5) reduces to the following:

$$E(Y \mid x) = p(a_1 x + b_1) + (1 - p)(a_2 x + b_2). \tag{6}$$

The above function we will treat as a linear regression function of the mixture distribution while $E(Y_i \mid x) = a_i x + b_i$, $i = 1, 2$, are linear regression functions of the mixture distribution.

2. The least squares method for an implicit interdependence

Let (X, Y) be a two-dimensional random variable. Antoniewicz (1988) proposed an original method of approximation distribution by means of two lines, neither of which is parallel to the axis of the system. Parameters a, b, c and d of the lines minimize the following function:

$$\Phi(a, \beta, \gamma, \delta) = E(Y - aX - c)^2 (Y - bX - d)^2. \tag{7}$$

Based on the available data, the parameters a, b, c and d will be estimated. This is equivalent to finding the straight lines that give the best fit (representation) of the points in the scatter plot of the response versus the predictor variable. We estimate the parameters using the popular least squares method, which gives the lines that minimize the sum of squares of the vertical distances from each point to the lines. The vertical distances represent the errors in the response variable. The sum of squares of these distances can then be written as follows:

$$S(a, b, c, d) = \sum_{i=1}^{n} [(y_i - a \cdot x_i - c)(y_i - b \cdot x_i - d)]^2. \tag{8}$$

The values of estimators $\hat{a}, \hat{b}, \hat{c}, \hat{d}$ which minimize $S(a, b, c, d)$ are derived by Antoniewicz (1988). In general, values of $\hat{a}, \hat{b}, \hat{c}, \hat{d}$ can be obtained only numerically. This problem is considered in details by Sitek (2016).

If $c = d = 0$, expression (8) reduces to the following:

$$S(a,b)=\sum_{i=1}^{n}[(y_i-ax_i)(y_i-bx_i)]^2. \tag{9}$$

Values of $\hat{a}, \hat{b}$, that minimize $S(a, b)$ are given by the solution (roots) of the following nonlinear system of two equations (10).

$$\begin{cases} m_{13}-2bm_{22}+b^2m_{31}-am_{22}+2abm_{31}-ab^2m_{40}=0 \\ m_{13}-2am_{22}+a^2m_{31}-bm_{22}+2abm_{31}-a^2bm_{40}=0 \end{cases} \tag{10}$$

where $m_{uv}=\sum_{i=1}^{n}x_i^u y_i^v$, $u = 0, 1, 2, \ldots, v = 0, 1, 2, \ldots$

Under the assumption that $a \neq b$, after appropriate transformations we have:

$$-(b-a)m_{22}+(b^2-a^2)m_{31}-ab(b-a)m_{40}=0/(b-a)$$

$$a=\frac{m_{22}-bm_{31}}{m_{31}-bm_{40}}. \tag{11}$$

After putting the right site of equation (11) into the first equation of system (10) we obtain the following quadratic equation,

$$b^2(m_{40}m_{22}-m_{31}^2)+b(m_{31}m_{22}-m_{13}m_{40})+m_{31}m_{13}-m_{22}^2=0. \tag{12}$$

Next, we have:

$$\Delta=(m_{31}m_{22}-m_{13}m_{40})^2-4(m_{40}m_{22}-m_{31}^2)(m_{31}m_{13}-m_{22}^2).$$

If $\Delta > 0$ then there are the following two distinct roots:

$$\hat{b}_1=\frac{m_{22}m_{31}-m_{13}m_{40}-\sqrt{\Delta}}{2(m_{31}^2-m_{22}m_{40})}, \tag{13}$$

$$\hat{b}_2=\frac{m_{22}m_{31}-m_{13}m_{40}+\sqrt{\Delta}}{2(m_{31}^2-m_{22}m_{40})}. \tag{14}$$

Substituting the results into the equation (12), we get:

$$\hat{a}_1 = \frac{m_{22} - \hat{b}_1 m_{31}}{m_{31} - \hat{b}_1 m_{40}}.$$

Using the well-known Vieta's formulas, it can be shown that $\hat{a}_1 = \hat{b}_2$:

$$\frac{m_{22} - \hat{b}_1 m_{31}}{m_{31} - \hat{b}_1 m_{40}} = \hat{b}_2 \,/\cdot (m_{31} - \hat{b}_1 m_{40}),$$

$$m_{22} - \hat{b}_1 m_{31} = \hat{b}_2 \cdot (m_{31} - \hat{b}_1 m_{40}),$$

$$m_{22} - \hat{b}_1 m_{31} = \hat{b}_2 \cdot m_{31} - \hat{b}_2 \hat{b}_1 m_{40},$$

$$m_{22} + \hat{b}_2 \hat{b}_1 m_{40} = (\hat{b}_1 + \hat{b}_2) \cdot m_{31}.$$

Let us introduce notation: $A = m_{40} m_{22} - m_{31}^2$, $B = m_{31} m_{22} - m_{13} m_{40}$, $C = m_{31} m_{13} - m_{22}^2$.

Hence, from Viete's formula we have:

$$\hat{b}_1 + \hat{b}_2 = \frac{-B}{A},$$

$$\hat{b}_1 \cdot \hat{b}_2 = \frac{C}{A},$$

$$m_{22} - \frac{C}{A} m_{40} = \frac{-B}{A} \cdot m_{31} \,/\cdot (A),$$

$$m_{22} A - C m_{40} = -B \cdot m_{31},$$

$$m_{40} m_{22}^2 - m_{22} m_{31}^2 + m_{40} \cdot m_{31} m_{13} - m_{40} m_{22}^2 = m_{40} \cdot m_{31} m_{13} - m_{22} m_{31}^2,$$

$$m_{40} \cdot m_{31} m_{13} - m_{22} m_{31}^2 = m_{40} \cdot m_{31} m_{13} - m_{22} m_{31}^2.$$

Hence, we have proved that $\hat{a}_1 = \hat{b}_2$.

3. The expected value and variance of estimators

Let $\mu_{uv} = E(X^u Y^v)$, $u = 1, 2, \ldots$ and $v = 1, 2, \ldots$ be a mixed moment of random variable X and Y distribution. The estimators $\hat{b}_i$, $i = 1, 2$, are a function F of sample moments m_{uv}, $u = 0, 1, 2, \ldots$, $v = 0, 1, 2, \ldots$ (Cramér, 1946) proved that when the following conditions are satisfied:

1) in some neighborhood of the points $m_{uv} = \mu_{uv}$, the function $F(m_{uv}, \ldots)$ is continuous and has the continuous first and second order derivatives with respect to the arguments m_{uv},
2) the sample size $n \to \infty$,

then $F(m_{uv}, \ldots)$ has a normal distribution with the expected value $F(\mu_{uv}, \ldots)$ and variance:

$$Var(\hat{b}_i) = \sum_{\{u,v,\chi,\delta\}} \frac{\partial F}{\partial m_{uv}} \frac{\partial F}{\partial m_{\chi\delta}} Cov(m_{uv}, m_{\chi\delta}), \tag{15}$$

where $\delta = 0, 1, 2, \ldots$, $\chi = 0, 1, 2, \ldots$

In our case, this let us derive the following expected value and variance of the estimators $\hat{b}_i$:

$$E(\hat{b}_i) = b_i + O(n^{-1}), \tag{16}$$

$$Var(\hat{b}_i) = \sum_{\{m_{uv}\}} \left(\frac{\partial \hat{b}_i}{\partial m_{uv}} \right)_{m_{uv}=\mu_{uv}} \left(\frac{\partial \hat{b}_i}{\partial m_{\chi\delta}} \right)_{m_{\chi\delta}=\mu_{\chi\delta}} Cov(m_{uv}, m_{\chi\delta}) = \mathrm{O}(n^{-1}), \tag{17}$$

$$Cov(\hat{b}_1, \hat{b}_2) = \sum_{\{m_{uv}\}} \left(\frac{\partial \hat{b}_1}{\partial m_{uv}} \right)_{m_{uv}=\mu_{uv}} \left(\frac{\partial \hat{b}_2}{\partial m_{\chi\delta}} \right)_{m_{\chi\delta}=\mu_{\chi\delta}} Cov(m_{uv}, m_{\chi\delta}) = \mathrm{O}(n^{-1}), \tag{18}$$

where $\mathrm{O}(n^{-1}) = en^{-1}$, $e \neq 0$ is a constant.

$$Var(\hat{b}_1 - \hat{b}_2) = Var(\hat{b}_1) + Var(\hat{b}_2) - 2Cov(\hat{b}_1, \hat{b}_2), \tag{19}$$

$$Cov(m_{uv}, m_{\chi\delta}) = \frac{1}{n} \left(\mu_{u+\chi, u+\delta} - \mu_{uv}\mu_{\chi\delta} \right). \tag{20}$$

The derivatives are as follows:

$$\hat{b}_i = \frac{m_{22}m_{31} - m_{13}m_{40} + (-1)^i \sqrt{\Delta}}{2(m_{31}^2 - m_{22}m_{40})},$$

$$\left(\frac{\partial \hat{b}_i}{\partial m_{40}}\right)_{m_{uv}=\mu_{uv}} = \frac{\mu_{13}}{2A(\mu_{uv})} + (-1)^i \frac{2\mu_{22}C(\mu_{uv}) + \mu_{13}B(\mu_{uv})}{2\sqrt{\Delta(\mu_{uv})}A(\mu_{uv})} - \frac{\mu_{22}\hat{b}_i}{A(\mu_{uv})},$$

$$\left(\frac{\partial \hat{b}_i}{\partial m_{13}}\right)_{m_{uv}=\mu_{uv}} = \frac{\mu_{40}}{2A(\mu_{uv})} + (-1)^i \frac{\mu_{40}B(\mu_{uv}) + 2\mu_{31}A(\mu_{uv})}{2\sqrt{\Delta(\mu_{uv})}A(\mu_{uv})},$$

$$\left(\frac{\partial \hat{b}_i}{\partial m_{31}}\right)_{m_{uv}=\mu_{uv}} = \frac{-\mu_{22}}{2A(\mu_{uv})} - (-1)^i \frac{4\mu_{31}C(\mu_{uv}) - 2\mu_{13}A(\mu_{uv}) + \mu_{22}B(\mu_{uv})}{2\sqrt{\Delta(\mu_{uv})}A(\mu_{uv})} + \frac{2\mu_{31}\hat{b}_i}{A(\mu_{uv})},$$

$$\left(\frac{\partial \hat{b}_i}{\partial m_{22}}\right)_{m_{uv}=\mu_{uv}} = -\frac{\mu_{31}}{2A(\mu_{uv})} - (-1)^i \frac{-2\mu_{40}C(\mu_{uv}) + \mu_{31}B(\mu_{uv}) + 4\mu_{22}A(\mu_{uv})}{2\sqrt{\Delta(\mu_{uv})}A(\mu_{uv})} - \frac{\mu_{40}\hat{b}_i}{A(\mu_{uv})}.$$

The above result let us assess the accuracy of estimation of the parameter b_i by means of $\hat{b}_i$, $i = 1, 2$. This let us test the following hypotheses:

$$H_0 : b_1 - b_2 = \delta_0 \neq 0, \qquad H_1 : b_1 - b_2 \neq \delta_0$$

by means of the following test statistic:

$$Z = \frac{\hat{b}_1 - \hat{b}_2 - \delta_0}{\sqrt{Var(\hat{b}_1 - \hat{b}_2)}}.$$

For a large sample size, statistic Z has a normal distribution with the unit variance.

4. Bi-lines of two-dimensional normal distribution

The above results will be considered in the particular case of two-dimensional random variables. Moreover, distributions are defined as mixtures of two-dimensional normal distributions with assumed parameters. Let us consider the following particular cases:

Case 1. We consider the bi-variable normal distribution: $N(0, 0, 1, 1, \rho)$. It is well-known:

$$\mu_{40} = 3, \mu_{31} = \mu_{13} = 3\rho,\ \mu_{22} = 1 + 2\rho^2.$$

On the basis of previously obtained results we have:

$$\Delta = 12(1-\rho^2)^3, \quad A = 3(1-\rho^2), \quad B = -6\rho(1-\rho^2).$$

The parameters of bi-lines are as follows:

$$b_1 = \frac{-B-\sqrt{\Delta}}{2A} = \rho - \frac{\sqrt{12(1-\rho^2)^3}}{6(1-\rho^2)} = \rho - \frac{\sqrt{3(1-\rho^2)}}{3},$$

$$b_2 = \frac{-B+\sqrt{\Delta}}{2A} = \rho + \frac{\sqrt{3(1-\rho^2)}}{3}.$$

Case 2. We consider the bivariate normal distribution $N(0, 0, \sigma_1, \sigma_2, \rho)$. In this case we have:

$$\mu_{40} = 3\sigma_1^4, \mu_{31} = 3\rho\sigma_1^3\sigma_2, \mu_{13} = 3\rho\sigma_2^3\sigma_1,\ \mu_{22} = \sigma_1^2\sigma_2^2(1+2\rho^2),$$

$$\Delta = 12(1-\rho^2)^3\sigma_1^{10}\sigma_2^6,\quad A = 3(1-\rho^2)\sigma_1^6\sigma_2^2,\quad B = -6\rho(1-\rho^2)\sigma_1^5\sigma_2^3.$$

where $\mu_{20} = \sigma_1^2,\ \ \mu_{02} = \sigma_2^2$.

The results evaluated in Chapter 3 of this paper lead to the following:

$$b_1 = \frac{-B-\sqrt{\Delta}}{2A} = \frac{\sigma_2}{\sigma_1}\rho - \frac{\sigma_1^5\sigma_2^3\sqrt{12(1-\rho^2)^3}}{6(1-\rho^2)\sigma_1^6\sigma_2^2} = \frac{\sigma_2}{\sigma_1}\rho - \frac{\sigma_2\sqrt{3(1-\rho^2)}}{3\sigma_1},$$

$$b_2 = \frac{-B+\sqrt{\Delta}}{2A} = \frac{\sigma_2}{\sigma_1}\rho + \frac{\sigma_2\sqrt{3(1-\rho^2)}}{3\sigma_1}.$$

Finally, we get the following slope coefficients of the lines:

$$b_i = \frac{\sigma_2}{\sigma_1}\rho \mp \sigma_2\sqrt{1-\rho^2}\sqrt{\frac{\sigma_1^2}{3\sigma_1^3}} = \frac{\sigma_2}{\sigma_1}\rho \mp \sigma_{Y|X}\sqrt{\frac{\mu_{20}}{\mu_{40}}},\ i = 1, 2.$$

Case 3. Let the distribution be defined by the following mixture of bivariate normal distribution: $pN(0, 0, 1, 1, r_1) + (1-p)N(0, 0, 1, 1, r_2)$. In this case the moments are as follows:

$$\mu_{40} = 3, \mu_{31} = \mu_{13} = 3p\rho_1 + 3(1-p)\rho_2,\ \mu_{22} = p(1+2\rho_1^2) + (1-p)(1+2\rho_2^2),$$

$$\Delta = 36(p\rho_1 + (1-p)\rho_2)^2(1-p\rho_1^2 + (p-1)\rho_2^2)^2 - 4(3 + 6p\rho_1^2 + (1-p)\rho_2^2 -$$
$$-(9(p\rho_1 + (1-p)\rho_2)^2 - (1 + 2p\rho_1^2 + 2(1-p)\rho_2^2)^2),$$

$$A = 3 + 6p\rho_1^2 + (1-p)\rho_2^2 - 9(p\rho_1 + (1-p)\rho_2)^2,$$
$$B = 6(p\rho_1 + (1-p)\rho_2)(p\rho_1^2 + (1-p)\rho_2^2 - 1).$$

This leads to the following slope coefficients:

$$b_1 = \frac{-B-\sqrt{\Delta}}{2A}, \qquad b_2 = \frac{-B+\sqrt{\Delta}}{2A}.$$

5. Simulation analysis of accuracy estimation

Let us consider the relationship between the definition of the bi-lines function and the regression mixture of two normal distributions: $pN(0, 0, 1, 1, r_1) + (1-p)$ $N(0, 0, 1, 1, r_2)$, where $p = 0.5$ and where $pN(0, 0, 1, 1, r_1)$ is a two-dimensional normal distribution with expected values equal to zero, variances equal to one and the correlation coefficient r_i, $i = 1, 2$. The regression of mixture distributions are: $y = r_i x$ where $i = 1, 2$. We examine the bias of estimators of the slope coefficients r_1 and r_2 of the regression using coefficients by means of estimators b_1 and b_2. For the accuracy analysis we evaluate bias according to the formula $d = (r_1 - b_1)^2 + (r_2 - b_2)^2$ and the sum of variances $g = D^2(\hat{b}_1^2) + D^2(\hat{b}_2^2)$. For the assumed and r_i, $i = 1, 2$, parameters g and d are calculated on the basis of an n-element sample generated according to the above-defined mixture distribution. This operation is replicated independently $N = 10000$ times. Finally, means of the calculated values of d and g are evaluated and denoted by δ and λ.

Table 1. The relationship between the bi-lines and the bi-regression

	$r_1 = -0.1, r_2 = 0.1$		$r_1 = -0.5, r_2 = 0.5$		$r_1 = -0.8, r_2 = 0.8$		$r_1 = -0.95, r_2 = 0.95$	
	δ	**λ**	**δ**	**λ**	**δ**	**λ**	**δ**	**λ**
$n = 5$	1.87	0.497	0.89	0.419	0.34	0.160	0.098	0.046
$n = 10$	1.05	0.263	0.4	0.215	0.15	0.096	0.036	0.023
$n = 20$	0.75	0.143	0.24	0.129	0.07	0.055	0.016	0.013
$n = 50$	0.58	0.064	0.15	0.058	0.035	0.029	0.007	0.007
$n = 100$	0.52	0.033	0.12	0.032	0.022	0.015	0.004	0.004
$n = 200$	0.49	0.015	0.1	0.015	0.016	0.008	0.002	0.002
$n = 500$	0.47	0.010	0.092	0.006	0.012	0.003	0.001	0.001
$n = 1000$	0.47	0.005	0.089	0.003	0.011	0.002	0.0009	0.000

Source: own calculation

The analysis of Table 1 and 2 leads to the conclusion that accuracy of estimators $\hat{b}_1$ and $\hat{b}_2$ of the parameters b_1 and b_2 increases because the sum λ of their variances decreases. Moreover, on the basis of the previously cited Cramér theorem, we infer that $\hat{b}_i$, $i = 1, 2$, are asymptotically unbiased estimators of the appropriate bi-lines function parameters. Hence, they are also consistent because their variances tend to zero.

Table 2. The relationship between the bi-lines and the bi-regression

	$r_1 = 0.1, r_2 = 0.2$		$r_1 = 0.1, r_2 = 0.4$		$r_1 = 0.1, r_2 = 0.6$		$r_1 = 0.1, r_2 = 0.8$	
	δ	λ	δ	λ	δ	λ	δ	λ
$n = 5$	2	0.469	1.64	0.490	1.33	0.492	1.09	0.467
$n = 10$	1.15	0.252	0.88	0.248	0.66	0.244	0.56	0.252
$n = 20$	0.83	0.135	0.61	0.134	0.43	0.136	0.35	0.143
$n = 50$	0.66	0.057	0.45	0.056	0.31	0.059	0.25	0.063
$n = 100$	0.6	0.032	0.41	0.031	0.27	0.033	0.21	0.031
$n = 200$	0.57	0.018	0.38	0.016	0.25	0.016	0.2	0.015
$n = 500$	0.55	0.011	0.37	0.008	0.24	0.007	0.193	0.006
$n = 1000$	0.55	0.006	0.36	0.004	0.23	0.002	0.19	0.004

Source: own calculation

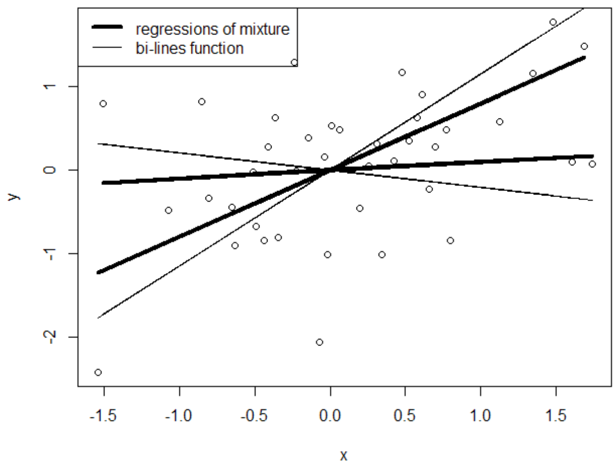

Figure 1. Bi-lines function and regressions of normal distribution mixture, $r_1 = 0.1, r_2 = 0.8, n = 20$

Source: own calculations

In general, on the basis of these tables we can say that estimators $\hat{b}_1$ and $\hat{b}_2$ of the bi-lines function parameters b_1 and b_2 are biased estimators of parameters r_1 and r_2 of the regressions of the considered mixture distribution. The bias (measured by means of factor d) decreases, when the module of the difference between the regression slopes (correlation) coefficients increases. Similarly, when sample size increases, the bias decreases but it does not disappear. Hence, the estimators of bi-lines parameters can be useful for the estimation of bi-linear regression when the slope coefficients of a bi-linear regression differ from each other significantly.

6. Conclusion

In this article, we considered the problem of the estimation of bi-lines function parameters by means of an implicit least square method. Usually, this method needs some numerical methods because it is not usually possible to get exact results. But in the particular case described in the paper, the exact expressions for the estimators were derived as well as expressions for the estimators of their variances. These estimators are consistent for the slope coefficients of the bi-lines function, but they are biased for slope coefficients of the mixture distribution regressions. This last conclusion could be expected since the bilinear function is defined as the specific method of approximation of spread of observations of a two-dimensional variable, which is not usually compatible with the conditional expected value definition which is one of the bases of the ordinary linear regression function construction. This and the result of the above analysis do not let us recommend the methodology used to estimate the bi-lines function parameters for the estimation of the regression parameters of mixture distributions.

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O estymacji parametrów bi-prostych

Streszczenie: W artykule rozważano problem estymacji parametrów bi-prostych, które z geometrycznego punktu widzenia są dwiema przecinającymi się prostymi. Za pomocą tzw. metody najmniejszych kwadratów dla zależności niejawnych wyznaczono przybliżone wartości estymatorów parametrów bi-prostych. W szczególnym przypadku wyznaczono dokładne wzory na estymatory parametrów, dowodząc ich łącznej asymptotycznej normalności rozkładu prawdopodobieństwa. Ponadto analizowano związki między bi-prostymi oraz liniowymi regresjami rozkładów prawdopodobieństwa, które są składowymi mieszanki dwuwymiarowych rozkładów prawdopodobieństw. W wyniku symulacyjnych eksperymentów wykazano, że brane pod uwagę zgodne estymatory parametrów bi-prostych dają jednak obciążone estymatory parametrów regresji rozkładów będących składowymi mieszanki dwuwymiarowych rozkładów prawdopodobieństwa.

Słowa kluczowe: bi-proste, regresja, metoda najmniejszych kwadratów dla zależności niejawnych, mieszanki rozkładów prawdopodobieństwa

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