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## On the Power of Some Nonparametric Isotropy Tests

**Abstract:** In this paper, properties of nonparametric significance tests verifying the random field isotropy hypothesis are discussed. In particular, the subject of the conducted analysis is the probability of rejecting the null hypothesis when it is true. A potential significant difference of empirical rejection probability from the assumed significance level could distort the results of statistical inference. The tests proposed by Guan, Sherman, Calvin (2004) and Lu, Zimmerman (2005) are considered. A simulation study has been carried out through generating samples from a given theoretical distribution and repeatedly testing the null hypothesis. Isotropic distributions are considered, among others, those based on a multidimensional normal distribution. The main aim of the paper is to compare both considered nonparametric significance tests verifying the random field isotropy hypothesis. For this purpose, the empirical rejection probabilities for both tests have been calculated and compared with the assumed significance level.

**Keywords:** isotropy, anisotropy, significance tests

**JEL:** C15

## 1. Introduction

In spatial statistics, observations of the study variables are treated as realisations of the spatial stochastic process, understood as a collection of random variables  $X = (X_t)_{t \in T}$  indexed by a coordinate vector  $T \subset \mathbb{R}^n$ . In this paper, weakly stationary spatial processes are considered. These are processes for which the expected value is constant and the covariance function depends only on the shift vector, i.e.

$Cov(X_s, X_t) = C(s - t) = E((X_s - E(X_s))(X_t - E(X_t)))$ . One of the basic tools used to study the variability structure of studied phenomena is the variogram. It is

a measure defined in locations shifted by the vector  $h$  as  $2\gamma(h) = Var(X_s - X_t)$ , where  $h = s - t$ . An important assumption used in the estimation of variograms is random field isotropy (Sherman, 2010).

The aim of the article is to investigate the properties of nonparametric significance tests verifying the random field isotropy hypothesis. The subject of the analysis will be the empirical probability of rejecting the null hypothesis estimated by percentage of null hypothesis rejections recorded in a simulation. A potential significant difference of empirical rejection probability from the assumed significance level could distort the results of statistical inference.

## 2. Isotropy

Isotropy is a property of the stochastic process that occurs when covariance depends only on the distance between locations. In other words, the covariance between realisations of the spatial stochastic process in locations shifted by the vector  $(x_1, x_2, \dots, x_n)$  depends only on the length of the vector  $(x_1, x_2, \dots, x_n)$ , (in this work, understood as the Euclidean norm) i.e.  $C(x_1, x_2, \dots, x_n) = C^*(\|(x_1, x_2, \dots, x_n)\|)$ , where  $C^*$  is the covariance function of a one-dimensional random process. However, in reality, anisotropy is common. It is a situation in which the covariance between realisations of the stochastic process in any fixed location and realisations in at least two other locations away by the same distance is different. Anisotropy is a subject of research in many areas, such as computer graphics, chemistry, geology, or physics. In physics, anisotropy in cosmic blackbody radiation is a good example (Smoot, Gorenstein, Muller, 1977). Figures 1 and 2 respectively show examples of the isotropic and anisotropic function of covariance, where  $C(x, y)$  represents the covariance between realisations in locations shifted by the vector  $(x, y)$ .

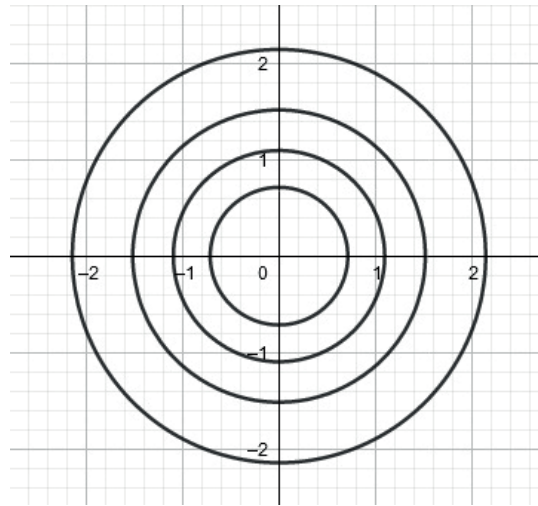


Figure 1.  $C(x, y) = \exp(-x^2 - y^2)$

Source: own calculations

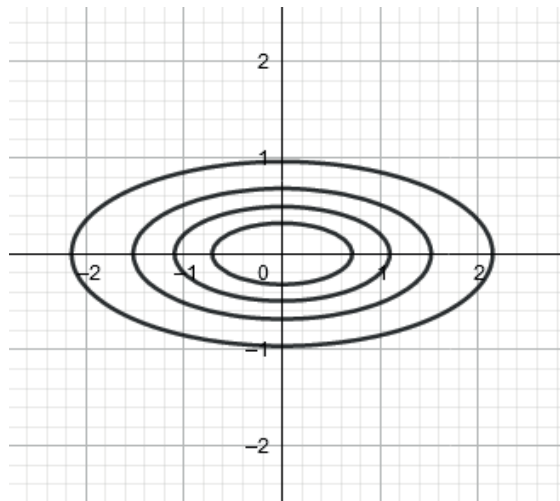


Figure 2.  $C(x, y) = \exp(-x^2 - 5y^2)$

Source: own calculations

### 3. Simulation

The purpose of the simulation was to estimate the empirical rejection probability. The simulation was carried out in the following steps:

- generation of the realisation of a random field using a given isotropic theoretical distribution on a  $16 \times 16$  square grid;
- verification of the null hypothesis about random field isotropy;
- repetition of the first two steps 10,000 times;
- calculation of the empirical rejection probability.

Three isotropic theoretical distributions were used to generate the realisations of the random field:

- 256 – dimensional normal distribution with mean 0 and covariance between locations shifted by the vector  $(x, y)$  given by the formula

$$C(x, y) = \exp\left(-\frac{1}{4}\sqrt{x^2 + y^2}\right);$$

- 256 – dimensional normal distribution with mean 0 and a unit variance-covariance matrix;
- for each location, a realisation was independently generated from the uniform distribution over the range  $(-1, 1)$ .

The *spTest* package (Weller, 2015) in the R environment was used to verify the hypothesis  $H_0: \exists C^* \forall (x, y) C(x, y) = C^*(\|(x, y)\|)$ . The two tests used were

proposed by: Guan, Sherman and Calvin (2004) and Lu and Zimmerman (2005). Four  $p$ -values were calculated. In the first test, those were  $p$ -values calculated on the basis of the asymptotic distribution of test statistics, taking into account and excluding the correction for the finite sample size. Let  $\Gamma$  be a set of lags which

is used in the estimate of the variogram. Define  $G = \{2\gamma(h): h \in \Gamma\}$ . Consider

a sequence of increasing index sets  $T_n$ , with  $\{X(s): s \in T_n\}$ . Let  $2\hat{\gamma}(h)$  and let

$\hat{G}_n = \{2\hat{\gamma}(h): h \in \Gamma\}$  be the estimators of  $2\hat{\gamma}(h)$  and  $G$  be obtained over  $T_n$ .

Moreover, if  $H_0$  is true, then there exists a full rank matrix  $A$  such that  $AG = 0$  (Lu, Zimmerman, 2001), where  $0$  is the zero matrix. Then the test statistic is given

by the formula:  $TS_n = |T_n| x(A\hat{G}_n)'(A\hat{\Sigma}_n A')^{-1}(A\hat{G}_n)$ , where  $\hat{\Sigma}_n$  is an estimator

of the variance-covariance matrix and  $|T_n|$  is the cardinality of the index set  $T_n$ .

According to multivariate Slutsky's theorem (Ferguson, 1996), the test statistic has an asymptotic chi-square distribution with  $d$  degrees of freedom, where  $d$  is the row rank of  $A$ . In the Lu and Zimmerman test, symmetry tests were used to study isotropy. The  $p$ -values were calculated by verifying the hypothesis about reflective symmetry and complete symmetry (Hoeting, Weller, 2016). They are defined as follows:

**Definition 1**

A weakly stationary spatial process on a grid is reflection symmetric if

$$\forall (x, y) C(x, y) = C(-x, y).$$

**Definition 2**

A weakly stationary spatial process on a grid is completely symmetric if

$$\forall (x, y) C(x, y) = C(-x, y) = C(y, x) = C(-y, x).$$

Both symmetries are field properties weaker than isotropy. Therefore, by rejecting the null hypothesis of symmetry, we have reasons to reject the hypothesis of isotropy. Lu and Zimmerman (2005) used the periodogram as an estimator of the spectral density. They took advantage of the fact that under certain conditions and at certain frequencies and when the null hypothesis of reflection or complete symmetry is true, ratios of periodogram values at different frequencies follow an  $F(2, 2)$  distribution, where  $F(2, 2)$  means Snedecor's  $F$ -distribution with parameters 2 and 2. To calculate the  $p$ -value, it is preferable to use a Cramér–von Mises goodness-of-fit test (Csörgo, Faraway, 1996) using the appropriate set of periodogram ratios.

## 4. Results of experiments

The first of the distributions used is a multidimensional normal distribution with exponential covariance.

### 4.1. Multidimensional normal distribution – exponential covariance

Figure 3 shows an example of realisation. This realisation was generated using the *mvtnorm* package.

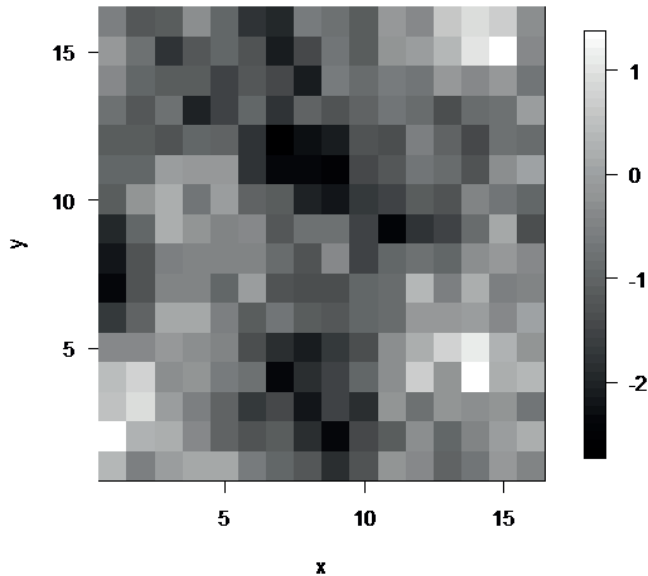


Figure 3. Multidimensional normal distribution – exponential covariance  
Source: own calculations

Empirical rejection probabilities depending on the significance level for Guan, Sherman, Calvin and Lu, Zimmerman tests were calculated (10,000 realisations were used). The results are shown in Tables 1 and 2 respectively.

Table 1. Empirical rejection probabilities – Guan, Sherman and Calvin test

Significance level	1%	2%	5%	10%
Empirical rejection probability (correction)	0.0037	0.0126	0.0556	0.1404
Empirical rejection probability	0.0344	0.0541	0.1056	0.1794

Source: own calculations

Table 2. Empirical rejection probabilities – Lu and Zimmerman test

Significance level	1%	2%	5%	10%
Empirical rejection probability (reflective symmetry)	0.0075	0.0178	0.0464	0.0978
Empirical rejection probability (complete symmetry)	0.0264	0.0444	0.0901	0.1558

Source: own calculations

Empirical rejection probabilities significantly differ from the assumed significance level. Empirical rejection probability values greater than the significance level mean that the test rejects the null hypothesis more often than the user is willing to accept.

### 4.2. Multidimensional normal distribution – lack of correlation

Figure 4 shows an example of a multidimensional normal distribution with mean 0 and a unit variance-covariance matrix.

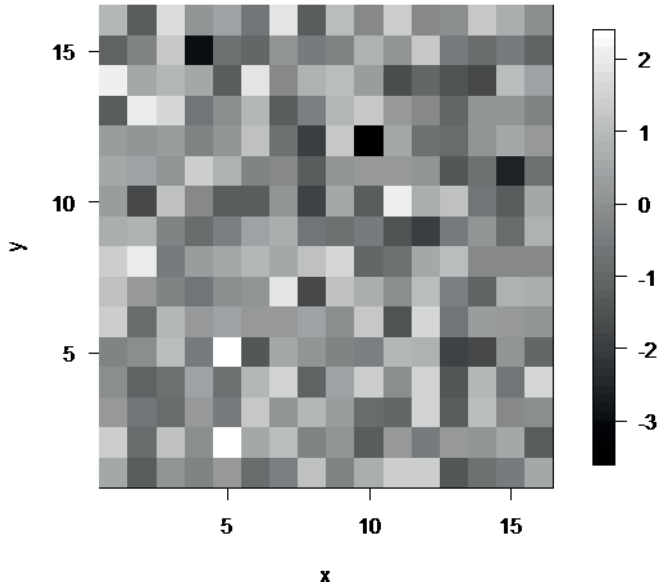


Figure 4. Multidimensional normal distribution – lack of correlation

Source: own calculations

Empirical rejection probabilities depending on the significance level for Guan, Sherman, Calvin and Lu, Zimmerman tests were calculated. The results are shown in Tables 3 and 4 respectively.

Table 3. Empirical rejection probabilities – Guan, Sherman and Calvin test

Significance level	1%	2%	5%	10%
Empirical rejection probability (correction)	0.0003	0.0017	0.0181	0.0629
Empirical rejection probability	0.0081	0.0151	0.0396	0.0837

Source: own calculations

Table 4. Empirical rejection probabilities – Lu and Zimmerman test

Significance level	1%	2%	5%	10%
Empirical rejection probability (reflective symmetry)	0.0109	0.0204	0.0474	0.0995
Empirical rejection probability (complete symmetry)	0.0079	0.0166	0.0463	0.0918

Source: own calculations

It is worth noting that in the case of the Guan, Sherman and Calvin test, the empirical rejection probabilities are lower than the significance level in each simulated case. This may suggest that it is possible to shift the critical value increasing the test power.

### 4.3. Uniform distribution

Figure 5 shows an example of a realisation created by independently generating (for each location from a  $16 \times 16$  grid) realisations from a uniform distribution over the interval  $(-1.1)$ .

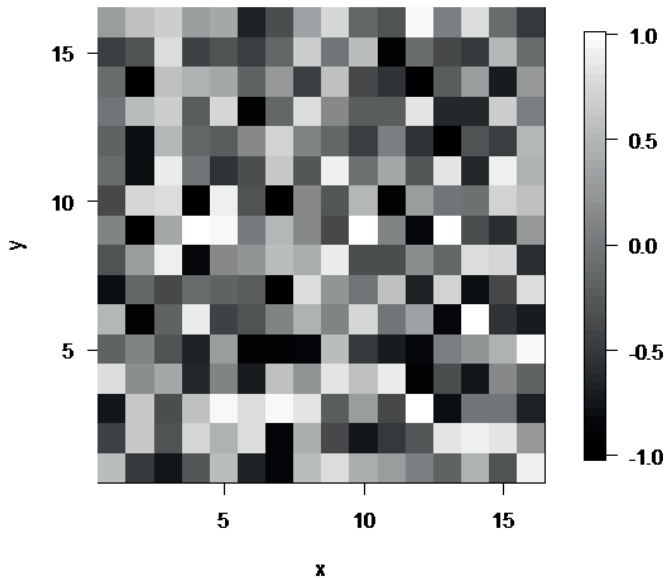


Figure 5. Uniform distribution

Source: own calculations

Empirical rejection probabilities depending on the significance level for the Guan, Sherman and Lu, Zimmerman tests were calculated. The results are shown in Tables 5 and 6 respectively.

Table 5. Empirical rejection probabilities – Guan, Sherman and Calvin test

Significance level	1%	2%	5%	10%
Empirical rejection probability (correction)	0.0060	0.0123	0.0411	0.0982
Empirical rejection probability	0.0153	0.0295	0.0620	0.1164

Source: own calculations



Table 6. Empirical rejection probabilities – Lu and Zimmerman test

Significance level	1%	2%	5%	10%
Empirical rejection probability (reflective symmetry)	0.0095	0.0186	0.0499	0.0967
Empirical rejection probability (complete symmetry)	0.0110	0.0215	0.0534	0.1033

Source: own calculations

The difference between empirical rejection probabilities and significance levels is smaller than in the normal distributions case.

## 5. Conclusions

In the case of the Guan, Sherman and Calvin test, including correction for the finite sample and the Lu and Zimmerman test (complete symmetry), relative percentage errors were calculated as

$$\frac{\text{empirical rejection probability} - \text{significance level}}{\text{significance level}} 100\%.$$

The results are shown in Tables 7 and 8 respectively.

Table 7. Empirical rejection probabilities – Guan, Sherman and Calvin test – correction

Significance level	1%	2%	5%	10%
Multidimensional normal distribution – exponential covariance	–63%	–37%	11%	40%
Multidimensional normal distribution – lack of correlation	–97%	–92%	–64%	37%
Uniform distribution	–40%	–39%	–18%	–2%

Source: own calculations

Table 8. Empirical rejection probabilities – Lu and Zimmerman test – complete symmetry

Significance level	1%	2%	5%	10%
Multidimensional normal distribution – exponential covariance	164%	122%	80%	56%
Multidimensional normal distribution – lack of correlation	–21%	–17%	–7%	–8%
Uniform distribution	10%	7%	7%	3%

Source: own calculations

Empirical rejection probabilities significantly differ from the assumed significance level. For the Lu and Zimmerman test and the multivariate normal distribution

with exponential covariance, the percentage relative error is up to 164%. A downward trend is visible – the higher the level of significance, the smaller the percentage relative error is. It is advisable to be extremely cautious when using the tests presented in this work.

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### Moc wybranych nieparametrycznych testów izotropii

**Streszczenie:** W artykule zbadano własności wybranych nieparametrycznych testów istotności, weryfikujących prawdziwość hipotezy o izotropii pola losowego. Przedmiotem analiz było w szczególności prawdopodobieństwo odrzucenia hipotezy zerowej w przypadku, gdy jest ona prawdziwa. Ewentualna znaczna różnica empirycznego prawdopodobieństwa odrzucenia od zakładanego poziomu istotności testu mogłaby świadczyć o zniekształceniu wyników wnioskowania statystycznego.

Testy, które rozważono w badaniu, to testy zaproponowane przez Guana, Shermana i Calvina (2004) oraz Lu i Zimmermana (2005). W artykule przeprowadzono symulację polegającą na wygenerowaniu ciągów realizacji pola losowego o zadanym rozkładzie teoretycznym, dla których testowano hipotezę zerową stanowiącą o izotropii. Rozważano procesy izotropowe – między innymi oparte na wielowymiarowym rozkładzie normalnym. Głównym celem artykułu było porównanie obu rozważanych nieparametrycznych testów istotności, weryfikujących hipotezę izotropii pola losowego. W tym celu wyznaczono empiryczne prawdopodobieństwa odrzuceń dla obu testów i porównano je z zakładanym z góry poziomem istotności.

**Słowa kluczowe:** izotropia, anizotropia, testy istotności

**JEL:** C15

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