A C T A   U N I V E R S I T A T I S   L O D Z I E N S I S
FOLIA OECONOMICA 3(302), 2014

*Dorota Pekasiewicz**

PROPERTIES OF SELECTED SIGNIFICANCE TESTS FOR EXTREME VALUE INDEX

Abstract. The paper presents two tests verifying the hypothesis about the shape parameter of the generalized distribution of maximum statistic. It is called the extreme value index. The inverse of the positive index is called the tail index and determines the degree of fatness of the tail. The asymptotic properties of the Pickands and the Hill estimator of the shape parameter are used to construct the test statistics. Simulation studies of the properties of these significance tests allow us to formulate some conclusions regarding their applications.
Keywords: extreme index, size of test, power of test, Pickands estimator, Hill estimator.

I. INTRODUCTION

The extreme statistic distributions and the tail distributions are characterized by the shape parameter denoted by ξ. The value of ξ determines the class of maximum statistic distribution. When $\xi = 0$ the generalized maximum statistic distribution is Gumbel distribution, when $\xi > 0$ the generalized maximum statistic distribution is Frechet distribution and when $\xi < 0$ – Weibull distribution. This parameter is called the extreme value index. In case of a heavy tailed distribution of a random variable the parameter ξ is positive and its inverse is called the tail index. Significance tests for the extreme index can be useful in financial studies, in which we are frequently dealing with heavy tailed distributions, trying to approximate their tails and calculate risk measures based on the theory of extreme events.

We consider two tests based on asymptotic properties of Pickands and Hill estimators of ξ parameter. The size and the power of each test depend on the number of the order statistics used for extreme value index estimation. Through the simulation analysis we study these properties for various numbers of used order statistics. The obtained results can be treated as a clue about the test selection and choice of the number of order statistics in practical application.

* Ph.D., Department of Statistical Methods, University of Łódź.

II. SELECTED ESTIMATORS OF EXTREME VALUE INDEX AND THEIR PROPERTIES

We consider two estimators of the extreme index: the Pickands and the Hill estimator.

Pickands estimator has the following form:

$$\hat{\xi}_{k,n}^{P} = (\ln 2)^{-1} \ln \left(\frac{X_{(n-k+1)}^{(n)} - X_{(n-2k+1)}^{(n)}}{X_{(n-2k+1)}^{(n)} - X_{(n-4k+1)}^{(n)}} \right), \tag{1}$$

where $X_{(n-k+1)}^{(n)}, X_{(n-2k+1)}^{(n)}$ and $X_{(n-4k+1)}^{(n)}$ are the order statistics obtained from the random independent sample $X_1, X_2, ..., X_n$, $1 \le k \le \frac{n}{4}$ and $k = k(n) \to \infty$, $k/n \to 0$.

P. Embrechts, C. Klüppelberg, T. Mikosch [1997] present properties of the Pickands estimator. It is a consistent estimator and it is asymptotically normally distributed.

Theorem 1. Let $\hat{\xi}_{k,n}^{P}$ be the Pickands estimator of the extreme value index ξ obtained from the random sample $X_1, X_2, ..., X_n$, where $n \to \infty$, if $k \to \infty$ and $k/n \to 0$. Then

$$\sqrt{k}\left(\hat{\xi}_{k,n}^{P} - \xi\right) \sim as.N(0, v), \tag{2}$$

where $v = \frac{\xi\sqrt{2^{2\xi+1}+1}}{2\left(2^{\xi}-1\right)\ln 2}$.

The Hill estimator is the most popular, but it can only be applied to the positive extreme value index estimation (see B. M. Hill [1975]). It uses *k* largest order statistics and has the following form:

$$\hat{\xi}_{k,n}^{H} = k^{-1} \sum_{i=1}^{k} \ln X_{(n-i+1)}^{(n)} - \ln X_{(n-k)}^{(n)}, \tag{3}$$

where $X_{(n-i+1)}^{(n)}$ are order statistics from the random sample $X_1, X_2, ..., X_n$, $i = 1, ..., k+1$ and $1 \le k < n$.

This estimator is consistent but biased. The bias increases with increasing *k*. Asymptotic convergence of the standardized Hill estimator is formulated in the following theorem (cf. R. Davis, S. Resnick [1984]).

Theorem 2. Let $X_1, X_2, ..., X_n$ be the random sample and $\hat{\xi}_{k,n}^{H}$ the Hill estimator of the extreme value index ξ calculated from this sample. If $\frac{k}{\ln \ln n} \to \infty$ and $k/n \to 0$, then

$$\sqrt{k}\left(\hat{\xi}_{k,n}^{H} - \xi\right) \sim as.N(0,\xi). \tag{4}$$

III. SELECTED SIGNIFICANCE TESTS FOR EXTREME VALUE INDEX

We consider the following parametric statistical hypotheses about extreme value index ξ:

$$H_0 : \xi = \xi_0, \tag{5}$$

$$H_1 : \xi \neq \xi_0, \tag{6}$$

where ξ_0 is the fixed positive real value.

To verify the above hypotheses we apply tests based on asymptotic properties of the Pickands and the Hill estimators of the parameter ξ.

The first test has the statistic:

$$U^P = \frac{2\ln 2\sqrt{k}\left(\hat{\xi}_{k,n}^{P} - \xi_0\right)\left(2^{\xi_0} - 1\right)}{\xi_0\sqrt{2^{2\xi_0+1} + 1}} \tag{7}$$

and the second:

$$U^H = \frac{\sqrt{k}\left(\hat{\xi}_{k,n}^{H} - \xi_0\right)}{\xi_0}. \tag{8}$$

Under the null hypothesis these statistics have the asymptotic standard normal distribution $N(0,1)$.

The value of the Pickands and the Hill estimators depend on the fixed value *k* and the properties of tests, specifically the size and the power depend on *k*.

IV. SIMULATION STUDY OF SIZE AND POWER FOR EXTREME INDEX

Analysis of the size and the power of tests is carried out through the Monte Carlo methods. In the first step the dependence of the size of test on the value of *k* was studied and next the power of test was estimated for chosen *k*.

At first, the test constructed on the basis of the properties of the Pickands estimator of ξ is considered. The procedure of hypothesis (5) and (6) verification was repeated 10 000 times for significance level $\alpha = 0.05$. Figure 1 presents dependence of the size of the test on the value of *k* for the following distributions:

– *t*-Student distribution with degree of freedom 0.5;
– Pareto distribution with parameters $\theta = 2$, $\alpha = 0.5$;
– *t*-Student distribution with degree of freedom 0.25;
– Pareto distribution with parameters $\theta = 2$, $\alpha = 0.25$.

The sample size was equal to *n* = 200.

For the first two distributions the hypothesis $H_0 : \xi = 2$ and $H_1 : \xi \neq 2$ was verified and for the next distrbutions $H_0 : \xi = 4$ and $H_1 : \xi \neq 4$.

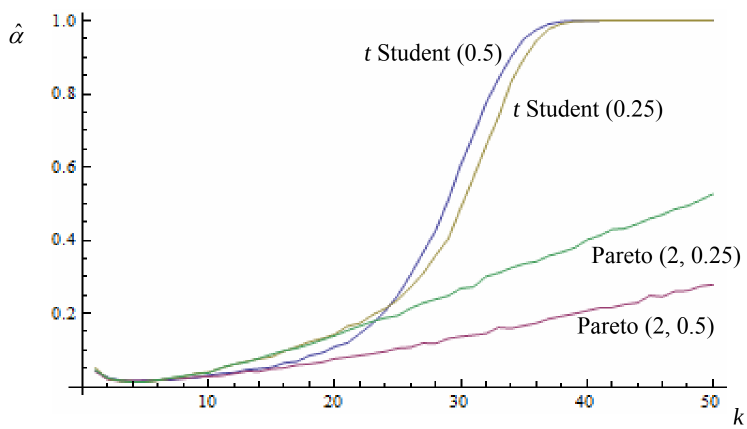

Figure 1. Estimated size of test with statistic U^P for different *k* and *n* = 200

Source: Own elaboration.

For the value of *k* not exceeding 10 the size of the test was the smallest. Large *k* values may result in major departures from the chosen significance level.

In the next step the power of the test was studied for given *k* = 10, for which the test size was approximately 0.05. Figure 2 presents estimated power of the test $M(\xi)$ for *n* = 200.

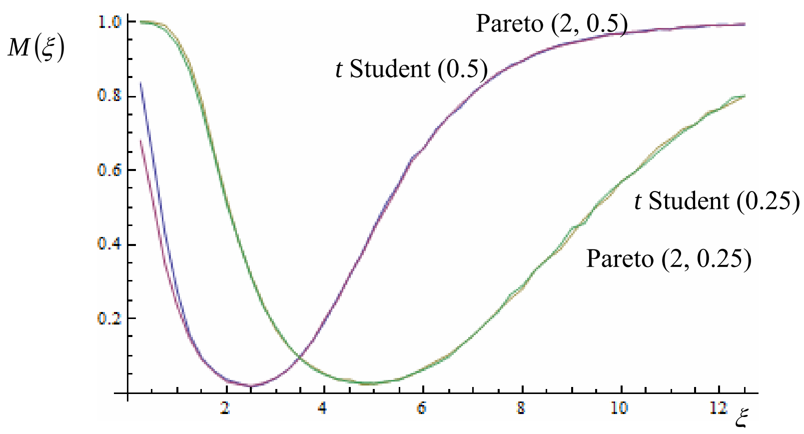

Figure 2. Estimated power of test with statistic U^P for n =200

Source: Own elaboration.

The study shows that the test statistic is biased and this test does not offer the highest power. In the next step, the analysis of its properties was carried out for larger sample sizes. For n = 1000 the estimated power of the test is shown in Figure 3. Increasing the sample size did not improve substantially the power of the test.

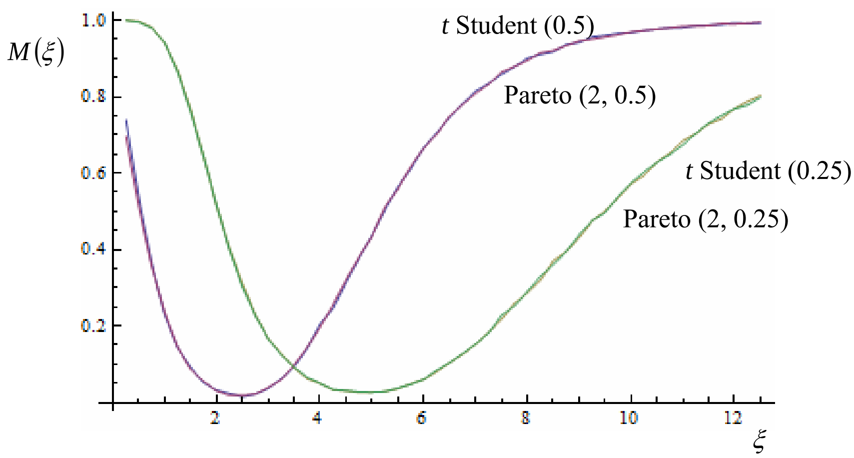

Figure 3. Estimated power of test with statistic U^P for n =1000

Source: Own elaboration.

An analogous simulation study was carried out for the second test The test size depends on *k*, which is shown in Figure 4.

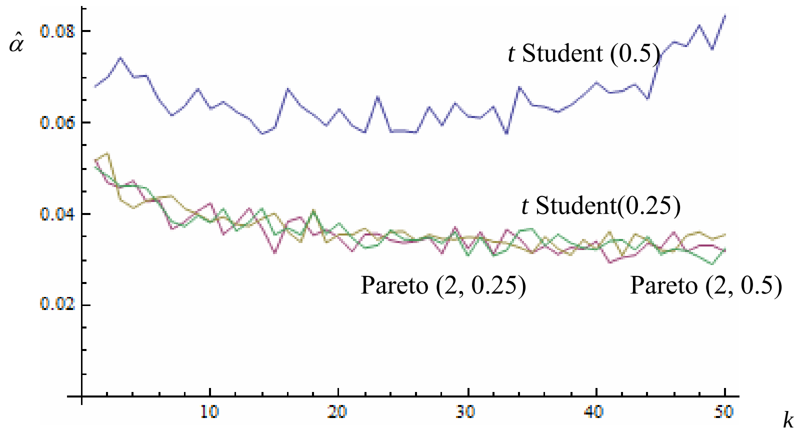

Figure 4. Estimated size of test with statistic U^H for different k and n =200
Source: Own elaboration.

From the obtained results we can conclude that the size of the test based on the Hill estimator is not as sensitive to the change of *k* as the previous test. From the three considered distributions, both Pareto and Student with 0.25 degrees of freedom gave the test size smaller than 0.05, regardless of the choice of *k*.

In case of the Student distribution with 0.5 degrees of freedom the test size was a little larger and, except for *k* close 50, it did not exceed 0.08. For k = 20 the power of the test was estimated, which is shown in Figure 5.

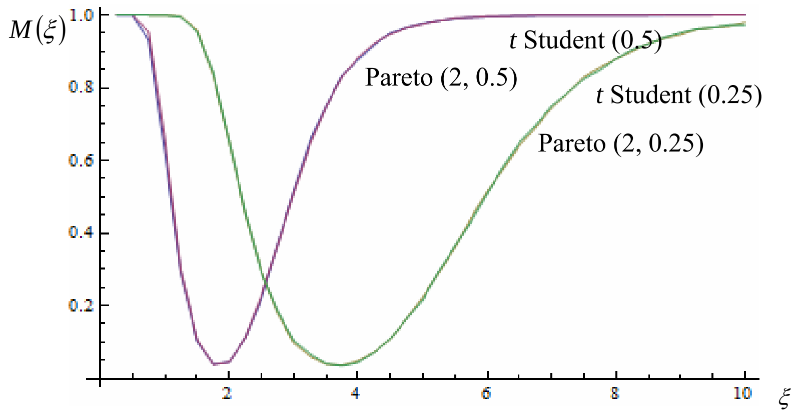

Figure 5. Estimated power of test with statistic U^H for n =200
Source: Own elaboration.

V. CONCLUSIONS

In the paper two significance tests for the extreme value index were considered. The asymptotic properties of the Pickands and the Hill estimators were used to construct test statistics. The results of simulation studies of the properties of these tests allowed us to formulate some conclusions. The results of the Monte Carlo methods showed that the test statistic based on asymptotic properties of Pickands estimator of the extreme value index is biased and this test did not produce the highest power in considered cases. The test with statistic based on the Hill estimator has better properties than the test with statistic based on the Pickands estimator. It can be applied to verify the hypotheses about the value of the shape parameter of Frechet distribution of maximum statistic and the parameter of the tail distribution for heavy tailed distributions. The study presented results for selected distribution parameters, but for other parameters of considered distributions the conclusions were similar. Further generalization requires subsequent simulation studies on other distributions with fat tails.

ACKNOWLEDGEMENTS
This work was supported by the project number DEC-2011/01/B/HS4/02746 from the National Science Centre.

REFERENCES

Davis R. A., Resnick S. T. (1984), *Tail estimates motivated by extreme value theory,* The Annals of Statistics 17, 1467–1487.

Embrechts P., Klüppelberg C., Mikosch T. (1997), *Modelling extremal events for insurance and finance*, Springer-Verlag.

Hill B.M. (1975), *A simple general approach to inference about the tail of a distribution*, The Annals of Statistics, Vol. 3, No. 5, 1163–1174.

Pickands J. (1975), *Statistical inference using extreme order statistics,* The Annals of Statistics, Vol. 3, No. 1, 119–131.

Dorota Pekasiewicz

WŁASNOŚCI WYBRANYCH TESTÓW ISTOTNOŚCI DLA INDEKSU EKSTEMALNEGO

W pracy przedstawione są testy weryfikujące hipotezy o parametrze kształtu rozkładu statystyki maksimum zwanego indeksem ekstremalnym. Odwrotnością dodatniego indeksu ekstremalnego jest indeks ogona rozkładu związany z grubością ogona rozkładu. Rozważane testy wykorzystują asymptotyczne własności estymatorów Pickandsa i Hilla. Badania symulacyjne własności tych testów istotności pozwoliły sformułować wnioski dotyczące ich zastosowań.