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A Regular D-optimal Weighing Design with Negative Correlations of Errors

Abstract: The issues concerning optimal estimation of unknown parameters in the model of chemical balance weighing designs with negative correlated errors are considered. The necessary and sufficient conditions determining the regular D-optimal design and some new construction methods are presented. They are based on the incidence matrices of balanced incomplete block designs and balanced bipartite weighing designs.

Keywords: balanced bipartite weighing design, balanced incomplete block design, chemical balance weighing design, optimal design

JEL: C02, C18, C90

1. Introduction

This paper discusses some studies of optimal chemical balance weighing designs. The possibility of using the proposed methodology for measuring economic phenomena is presented in the literature (see Banerjee, 1975: 33–48; Ceranka, Graczyk, 2014a: 317–320).

Any chemical balance weighing design is defined as a design in which we determine unknown measurements of p objects in n measurement operations according to the model $\mathbf{y} = \mathbf{Xw} + \mathbf{e}$, where $\mathbf{y}$ is a $n \times 1$ random vector of the recorded results of measurements, $\mathbf{X} = (x_{ij}) \in \mathbf{\Psi}_{n\times p}(-1, 0, 1)$ denotes the class of matrices with elements $x_{ij} = 1, -1$ or 0, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, p$. Next, $\mathbf{w}$ is a $p \times 1$ vector of unknown measurements of objects, $\mathbf{e}$ is an $n \times 1$ random vector of errors, $E(\mathbf{e}) = \mathbf{0}_n$ and $E(\mathbf{ee}') = \sigma^2\mathbf{G}$, $\mathbf{G}$ is a known positive definite matrix.

The problem is to determine all unknown measurements of p objects using exactly n measurements. Due to this fact, we use normal equations $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\hat{\mathbf{w}} = \mathbf{X}'\mathbf{G}^{-1}\mathbf{y}$, where $\hat{\mathbf{w}}$ is the vector estimated by the least squares method. If $\mathbf{X}$ is of full column rank, then the least squares estimator of $\mathbf{w}$ is given by $\hat{\mathbf{w}} = \left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{G}^{-1}\mathbf{y}$ and the covariance matrix of $\hat{\mathbf{w}}$ is equal to $V(\hat{\mathbf{w}}) = \sigma^2\left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)^{-1}$.

If we assume that experimental errors are equally negatively correlated, then we are working with the matrix $E(\mathbf{ee}') = \sigma^2\mathbf{G}$, where $\mathbf{G}$ is of the form:

$$\mathbf{G} = g\left((1-\rho)\mathbf{I}_n + \rho\mathbf{1}_n\mathbf{1}_n'\right), g > 0, \frac{-1}{n-1} < \rho < 0, \tag{1}$$

where $\mathbf{I}_n$ is the identity matrix of rank n, $\mathbf{1}_n$ is $n \times 1$ vector of ones.

2. D-optimal design

The issues concerning the determination of D-optimal designs were presented in the literature. The classical works include Raghavarao (1971: 315–321), Jacroux, Wong and Masaro (1983: 213–230), Shah and Sinha (1989: 1–15).

If we consider the class of the chemical balance weighing designs $\mathbf{\Psi}_{n\times p}(-1, 0, 1)$, then the optimality criteria are the functions of the matrix $(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-1}$ and the elements of the determined matrix $\mathbf{X}$ of the optimal design have to belong to the set $\{-1, 0, 1\}$. For a complete theory, we refer the reader to the papers Masaro, Wong (2008a: 1392–1400, 2008b: 4093–4101), Ceranka, Graczyk (2016: 73–82).

The design $\mathbf{X}_d \in \mathbf{\Psi}_{n\times p}(-1, 0, 1)$ is D-optimal if $\det(\mathbf{X}_d'\mathbf{G}^{-1}\mathbf{X}_d) = \max\{\det(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}): \mathbf{X} \in \mathbf{\Psi}_{n\times p}(-1, 0, 1)\}$.

From the study of Ceranka, Graczyk (2014b: 11–13), we have:

Definition 1. Any non-singular chemical balance weighing design $\mathbf{X} \in \mathbf{\Psi}_{n\times p}(-1, 0, 1)$ is regular D-optimal if

$$\det\left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)^{-1} = \left(\frac{g(1-\rho)}{m - \dfrac{\rho(m-2u)^2}{1+\rho(n-1)}}\right)^p,$$

where $m = \max\{m_1, m_2, \ldots, m_p\}$, m_j denotes the number of elements equal to 1 or –1 in j-th column of $\mathbf{X}$, $u = \min\{u_1, u_2, \ldots, u_p\}$, u_j denotes the number of elements equal to –1 in j-th column of $\mathbf{X}$.

Theorem 1. Any non-singular chemical balance weighing design $\mathbf{X} \in \mathbf{\Psi}_{n\times p}(-1, 0, 1)$ is regular D-optimal if:

a) $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X} = m\mathbf{I}_p - \dfrac{\rho(m-2u)^2}{1+\rho(n-1)}\left(\mathbf{I}_p - \mathbf{1}_p\mathbf{1}_p\right),$

b) $\mathbf{X}'\mathbf{1}_n = \pm(m-2u)\mathbf{1}_p.$

The problem is to provide a regular D-optimal design in the class $\mathbf{\Psi}_{n\times p}(-1, 0, 1)$. For any pair of the number of objects and number of measurements, it is not possible to determine a regular D-optimal design. Therefore, the aim of the study presented here is to give a new construction method of such a matrix and thus expand the class $\mathbf{\Psi}_{n\times p}(-1, 0, 1)$ in which optimal designs exist. It is worth noting that some problems related to the regular D-optimal design are given in Ceranka, Graczyk (2015: 36–39; 2016: 74–77; 2018: 5–16). Now, we suggest forming the matrix of the optimal design based on the set of incidence matrices of balanced incomplete block designs and balanced bipartite weighing designs.

3. Construction of regular D-optimal designs

3.1. Block designs

In this section, we present the definitions of the balanced incomplete block design given by Raghavarao, Padgett (2005: 54–79) and the balanced bipartite weighing design given in Huang (1976: 20–30).

A balanced incomplete block design given by the incidence matrix $\mathbf{N}$ is an arrangement of v treatments in b blocks, each of size k, arranged in such a way that each treatment occurs at most once in each block, occurs in exactly r blocks, and each pair of treatments occurs together in exactly λ blocks. The integers v, b, r, k, λ

are called the parameters of the balanced incomplete block design. The parameters satisfy the following relations $vr = bk$, $\lambda(v - 1) = r(k - 1)$, $\mathbf{NN}' = (r-\lambda)\mathbf{I}_v + \lambda\mathbf{1}_v\mathbf{1}_v'$.

A balanced bipartite weighing design given by the incidence matrix $\mathbf{N}^*$ is an arrangement of v treatments into b blocks, such that each block containing k distinct treatments is divided into 2 subblocks containing k_1 and k_2 treatments, respectively, $k = k_1 + k_2$. Each treatment appears in r blocks. Each pair of treatments from different subblocks appears together in λ_1 blocks and each pair of treatments from the same subblock appears together in λ_2 blocks. The integers v, b, r, k_1, k_2, λ_1, λ_2 are called the parameters of the balanced bipartite weighing design. Let $\mathbf{N}^*$ be the incidence matrix of such a design. If $k_1 \neq k_2$, then each object occurs $r_1 = \lambda_1(v - 1)/2k_2$ times in the first subblock and $r_2 = \lambda_1(v - 1)/2k_1$ times in the second subblock. The parameters satisfy

$$vr = bk,\ b = 0.5\lambda_1 v(v-1)k_1^{-1}k_2^{-1},\ \lambda_2 = 0.5\lambda_1\left(k_1(k_1-1)+k_2(k_2-1)\right)k_1^{-1}k_2^{-1},$$

$$r = r_1 + r_2 = 0.5\lambda_1 k(v-1)k_1^{-1}k_2^{-1},\ \mathbf{N}^*\mathbf{N}^{*'} = (r-\lambda_1-\lambda_2)\mathbf{I}_v + (\lambda_1+\lambda_2)\mathbf{1}_v\mathbf{1}_v'.$$

3.2. The design construction

Now, we will construct the design matrix $\mathbf{X} \in \mathbf{\Psi}_{n\times p}(-1, 0, 1)$ of the regular D-optimal design more precisely. We take into our account the incidence matrix $\mathbf{N}_1$ of the balanced incomplete block design with the parameters v, b_1, r_1, k_1, λ_1 and the incidence matrix $\mathbf{N}_2^*$ of the balanced bipartite weighing design with the parameters v, b_2, r_2, k_{12}, k_{22}, λ_{12}, λ_{22}. From the incidence matrix $\mathbf{N}_2^*$ we form the matrix $\mathbf{N}_2$ by multiplying each element belonging to the first subblock by –1. Thus, let us consider any chemical balance weighing design $\mathbf{X}$ in the form:

$$\mathbf{X} = \begin{bmatrix} 2\mathbf{N}'_1 - \mathbf{1}_v\mathbf{1}'_v \\ \mathbf{N}'_2 \end{bmatrix}. \tag{2}$$

Each column of the design matrix $\mathbf{X}$ in (2) contains $r_1 + r_{22}$ elements equal to +1, $b_1 - r_1 + r_{12}$ elements equal to –1 and $b_2 - r_2$ elements equal to 0. Moreover, for $\mathbf{X}$ in the form (2), we have $p = v$, $n = b_1 + b_2$, $m = b_1 + r_2$.

The chemical balance weighing design in the form (2) is non-singular if and only if the matrix $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$ is non-singular. According to the form (1) of the matrix $\mathbf{G}$, this condition is fulfilled if and only if $\mathbf{X}'\mathbf{X}$ is non-singular. From Ceranka, Graczyk (2018: 5), we have the following theorem.

Theorem 2. Any chemical balance weighing design $\mathbf{X} \in \mathbf{\Psi}_{n\times p}(-1, 0, 1)$ in the form (2) is non-singular if and only if $v \neq 2k_1$ or $k_{12} \neq k_{22}$.

Theorem 3. Any non-singular chemical balance weighing design $\mathbf{X} \in \mathbf{\Psi}_{n\times p}(-1, 0, 1)$ in the form (2) is regular D-optimal if and only if:

a) $$\mathbf{X}'\mathbf{G}^{-1}\mathbf{X} = (b_1 + r_2)\mathbf{I}_p - \frac{\rho(2r_1 - b_1 + r_2 - 2r_{12})^2}{1+\rho(n-1)}(\mathbf{I}_p - \mathbf{1}_p\mathbf{1}_p'),$$

b) $$\mathbf{X}'\mathbf{1}_n = \pm(2r_1 - b_1 + r_2 - 2r_{12})\mathbf{1}_p$$

Proof. Let us first note that for the design matrix $\mathbf{X} \in \mathbf{\Psi}_{n\times p}(-1, 0, 1)$ in the form (2) ρ is expressed as

$$\rho = \frac{b_1 - 4(r_1 - \lambda_1) + \lambda_{22} - \lambda_{12}}{(2r_1 - b_1 + r_2 - 2r_{12})^2 - (b_1 + b_2 - 1)(b_1 - 4(r_1 - \lambda_1) + \lambda_{22} - \lambda_{12})}.$$

Taking into consideration the relations between the parameters of the balanced incomplete block design and the balanced bipartite weighing design, proof is completed when we observe that the optimality conditions given in Theorem 1 determine the forms of $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$ and $\mathbf{X}'\mathbf{1}_n$ as presented above. So, if these conditions are simultaneously fulfilled, then $\mathbf{X} \in \mathbf{\Psi}_{n\times p}(-1, 0, 1)$ is a regular D-optimal chemical balance weighing design.

Theorem 4. If for a given ρ, the parameters of the balanced incomplete block designs are equal to $v = 4s + 1$, $b_1 = 2(4s + 1)$, $k_1 = 2s$, $r_1 = 4s$, $\lambda_1 = 2s - 1$, $4s + 1$ is a prime or a prime power, and the parameters of the balanced bipartite weighing designs are equal to:

a) $\rho = -(13s^2 - 3s + 5)^{-1}$, $v = 4s + 1$, $b_2 = s(4s + 1)$, $r_2 = 5s$, $k_{12} = 1$, $k_{22} = 4$, $\lambda_{12} = 2$, $\lambda_{22} = 3$, $s = 1, 2, \ldots$,

b) $\rho = -3(13s^2 + 23s + 7)^{-1}$, $v = 4s + 1$, $b_2 = s(4s + 1)$, $r_2 = 5s$, $k_{12} = 2$, $k_{22} = 3$, $\lambda_{12} = 3$, $\lambda_{22} = 2$, $s = 2, 3, \ldots$,

c) $\rho = -3(28s^2 + 22s + 7)^{-1}$, $v = 4s + 1$, $b_2 = 2s(4s + 1)$, $r_2 = 6s$, $k_{12} = 1$, $k_{22} = 2$, $\lambda_{12} = 2$, $\lambda_{22} = 1$, $s = 1, 2, \ldots$,

d) $\rho = -3(40s^2 + 14s + 7)^{-1}$, $v = 4s + 1$, $b_2 = 2s(4s + 1)$, $r_2 = 12s$, $k_{12} = 2$, $k_{22} = 4$, $\lambda_{12} = 8$, $\lambda_{22} = 7$, $s = 2, 3, \ldots$,

e) $\rho = -(44s^2 - 14s + 5)^{-1}$, $v = 4s + 1$, $b_2 = 2s(4s + 1)$, $r_2 = 14s$, $k_{12} = 2$, $k_{22} = 5$, $\lambda_{12} = 10$, $\lambda_{22} = 11$, $s = 1, 2, \ldots$,

f) $\rho = -0.5(6s^2 + 3s + 1)^{-1}$, $v = 4s + 1$, $b_2 = 2s(4s + 1)$, $r_2 = 16s$, $k_{12} = 3$, $k_{22} = 5$, $\lambda_{12} = 15$, $\lambda_{22} = 13$, $s = 2, 3, \ldots$,

g) $\rho = -(2(2st - 1)^2 + 8s^2t + 8s + 2st + 1)^{-1}$, $v = 4s + 1$, $b_2 = 2st(4s + 1)$, $r_2 = 8st$, $k_{12} = 1$, $k_{22} = 3$, $\lambda_{12} = 3t$, $\lambda_{22} = 3t$, $s, t = 1, 2, \ldots$,

then $\mathbf{X} \in \mathbf{\Psi}_{n\times p}(-1, 0, 1)$ in the form (2) with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is of the form (1), is a regular D-optimal design.

Proof. It is obvious that the parameters given above satisfy conditions (a)–(b) of Theorem 3.

Theorem 5. If for a given ρ, the parameters of the balanced incomplete block and the balanced bipartite weighing designs are equal to:

1) $\rho = -3(10s^2 + 5s - 8)^{-1}$, $v = 2s$, $b_1 = 2(2s - 1)$, $r_1 = 2s - 1$, $k_1 = s$, $\lambda_1 = s - 1$ and $v = 2s$, $b_2 = s(2s - 1)$, $r_2 = 3(2s - 1)$, $k_{12} = 2$, $k_{22} = 4$, $\lambda_{12} = 8$, $\lambda_{22} = 7$, $s = 3, 4, \ldots$,

b) $\rho = -4(180s^2 + 12s - 11)^{-1}$, $v = 6s$, $b_1 = 2(6s - 1)$, $r_1 = 6s - 1$, $k_1 = 3s$, $\lambda_1 = 3s - 1$ and $v = 6s$, $b_2 = 6s(6s - 1)$, $r_2 = 3(6s - 1)$, $k_{12} = 1$, $k_{22} = 2$, $\lambda_{12} = 4$, $\lambda_{22} = 2$, $s = 1, 2, \ldots$,

then $\mathbf{X} \in \mathbf{\Psi}_{n\times p}(-1, 0, 1)$ in the form (2) with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is of the form (1), is a regular D-optimal design.

Proof. Clearly, the parameters given above satisfy conditions (a)–(b) of Theorem 3.

Theorem 6. If for a given ρ, the parameters of the balanced incomplete block designs are equal to $v = 4s^2 - 1$, $b_1 = 4s^2 - 1$, $k_1 = 2s^2 - 1$, $r_1 = 2s^2 - 1$, $\lambda_1 = 2s^2 - 1$ and the parameters of the balanced bipartite weighing designs are equal to:

a) $\rho = -(10s^4 - 6s^2 + 1)^{-1}$, $v = 4s^2 - 1$, $b_2 = (2s^2 - 1)(4s^2 - 1)$, $r_2 = 3(2s^2 - 1)$, $k_{12} = 1$, $k_{22} = 2$, $\lambda_{12} = 2$, $\lambda_{22} = 1$,

b) $\rho = -2(32s^4 - 28s^2 + 7)^{-1}$, $v = 4s^2 - 1$, $b_2 = (2s^2 - 1)(4s^2 - 1)$, $r_2 = 6(2s^2 - 1)$, $k_{12} = 2$, $k_{22} = 4$, $\lambda_{12} = 8$, $\lambda_{22} = 7$,

c) $\rho = -3(40s^4 - 30s^2 + 6)^{-1}$, $v = 4s^2 - 1$, $b_2 = (2s^2 - 1)(4s^2 - 1)$, $r_2 = 8(2s^2 - 1)$, $k_{12} = 3$, $k_{22} = 5$, $\lambda_{12} = 15$, $\lambda_{22} = 13$,

d) $\rho = -(4t^2(2s^2 - 1)^2 + t(2s^2 - 1)(4s^2 - 5))^{-1}$, $v = 4s^2 - 1$, $b_2 = t(2s^2 - 1)(4s^2 - 1)$, $r_2 = 4t(2s^2 - 1)$, $k_{12} = 1$, $k_{22} = 3$, $\lambda_{12} = 3t$, $\lambda_{22} = 3t$, $t = 1, 2, \ldots$,

$s = 1, 2, \ldots$ then $\mathbf{X} \in \mathbf{\Psi}_{n\times p}(-1, 0, 1)$ in the form (2) with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is of the form (1), is a regular D-optimal design.

Proof. It is a simple matter to deduce that the parameters given above satisfy conditions (a)–(b) of Theorem 3.

Theorem 7. If for a given ρ, the parameters of the balanced incomplete block designs are equal to $v = 4s + 3$, $b_1 = 4s + 3$, $r_1 = 2s + 1$, $k_1 = 2s + 1$, $\lambda_1 = s$, $4s+3$ is a prime or a prime power, and the parameters of the balanced bipartite weighing designs are equal to:

a) $\rho = -(10s^2 + 14s + 5)^{-1}$, $v = 4s+3$, $b_2 = (2s + 1)(4s + 3)$, $r_2 = 3(2s + 1)$, $k_{12} = 1$, $k_{22} = 2$, $\lambda_{12} = 2$, $\lambda_{22} = 1$,

b) $\rho = -2(32s^2 + 36s + 11)^{-1}$, $v = 4s + 3$, $b_2 = (2s + 1)(4s + 3)$, $r_2 = 6(2s + 1)$, $k_{12} = 2$, $k_{22} = 4$, $\lambda_{12} = 8$, $\lambda_{22} = 7$,

c) $\rho = -3(40s^2 + 50s + 7)^{-1}$, $v = 4s + 3$, $b_2 = (2s + 1)(4s + 3)$, $r_2 = 8(2s + 1)$, $k_{12} = 3$, $k_{22} = 5$, $\lambda_{12} = 15$, $\lambda_{22} = 13$,

d) $\rho = -(4t^2(2s + 1)^2 + t(2s + 1)(4s - 1) + 4s + 3)^{-1}$, $v = 4s + 3$, $b_2 = t(2s + 1)(4s + 3)$, $r_2 = 4t(2s + 1)$, $k_{12} = 1$, $k_{22} = 3$, $\lambda_{12} = 3t$, $\lambda_{22} = 3t$,

$s, t = 1, 2, \ldots$ then $\mathbf{X} \in \mathbf{\Psi}_{n\times p}(-1, 0, 1)$ in the form (2) with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is of the form (1), is a regular D-optimal design.

Proof. One can easily check that the parameters (a)–(d) satisfy conditions (a)–(b) of Theorem 3.

Theorem 8. If for a given ρ, the parameters of the balanced incomplete block designs are equal to $v = 8s + 7$, $b_1 = 8s + 7$, $r_1 = 4s + 3$, $k_1 = 4s + 3$, $\lambda_1 = 2s + 1$, and the parameters of the balanced bipartite weighing designs are equal to:

a) $\rho = -(2(2s + 1)^2 + 4(s + 1)(8s + 7)-1)^{-1}$, $v = 8s + 7$, $b_2 = (4s + 3)(8s + 7)$, $r_2 = 3(4s + 3)$, $k_{12} = 1$, $k_{22} = 2$, $\lambda_{12} = 2$, $\lambda_{22} = 1$,

b) $\rho = -2(128s^2 + 200s + 79)^{-1}$, $v = 8s + 7$, $b_2 = (4s + 3)(8s + 7)$, $r_2 = 6(4s + 3)$, $k_{12} = 2$, $k_{22} = 4$, $\lambda_{12} = 8$, $\lambda_{22} = 7$,

c) $\rho = -3(8(4s + 3)(2s + 1) + 12(s + 1)(8s + 7)-2)^{-1}$, $v = 8s + 7$, $b_2 = (4s + 3)(8s + 7)$, $r_2 = 8(4s + 3)$, $k_{12} = 3$, $k_{22} = 5$, $\lambda_{12} = 15$, $\lambda_{22} = 13$,

d) $\rho = -(4t^2(4s + 3)^2 + t(4s + 3)(8s + 3) + 8s + 7)^{-1}$, $v = 8s + 7$, $b_2 = t(4s + 3)(8s + 7)$, $r_2 = 4t(4s + 3)$, $k_{12} = 1$, $k_{22} = 3$, $\lambda_{12} = 3t$, $\lambda_{22} = 3t$,

$s, t = 1, 2, \ldots$ then $\mathbf{X} \in \mathbf{\Psi}_{n\times p}(-1, 0, 1)$ in the form (2) with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is of the form (1), is a regular D-optimal design.

Proof. It is evident that the parameters given above satisfy conditions (a)–(b) of Theorem 3.

4. Example

Here, we consider the experiment in which we determine unknown measurements of $p = 5$ objects using $n = 20$ measurements. We are interested in determining the design having the best statistical properties in the class $\mathbf{X} \in \mathbf{\Psi}_{20\times 5}(-1, 0, 1)$ for $\rho = -1/19$. In order to construct the design matrix of a regular D-optimal chemical balance weighing design, we consider the balanced incomplete block design and the balanced bipartite weighing design given in Theorem 4(c). Let $\mathbf{N}_1$ and $\mathbf{N}_2^*$ be the incidence matrices of appropriate designs with the parameters $v = 5$, $b_1 = 1$ $r_1 = 4$, $k_1 = 2$, $\lambda_1 + 1$ and $v = 5$, $b_2 = 10$, $r_2 = 6$, $k_{12} = 1$, $k_{22} = 2$, $\lambda_{12} = 2$, $\lambda_{22} = 1$:

$$\mathbf{N}_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix},$$

$$\mathbf{N}_2^* = \begin{bmatrix} 1_2 & 1_2 & 1_2 & 1_2 & 0 & 1_1 & 0 & 0 & 1_1 & 0 \\ 1_2 & 0 & 0 & 1_1 & 1_2 & 1_2 & 1_2 & 1_1 & 0 & 0 \\ 1_1 & 1_2 & 0 & 0 & 1_2 & 0 & 0 & 1_2 & 1_2 & 1_1 \\ 0 & 1_1 & 1_2 & 0 & 0 & 1_2 & 1_1 & 1_2 & 0 & 1_2 \\ 0 & 0 & 1_1 & 1_2 & 1_1 & 0 & 1_2 & 0 & 1_2 & 1_2 \end{bmatrix}.$$

Here, 1_q denotes the element belonging to the q-th subblock, $q = 1, 2$. Thus design matrix of the regular D-optimal chemical balance weighing design $\mathbf{X} \in \mathbf{\Psi}_{20\times5}(-1, 0, 1)$ is given in the form:

$$\mathbf{X} = \begin{pmatrix} + & + & - & - & - \\ + & - & + & - & - \\ + & - & - & + & - \\ + & - & - & - & + \\ - & + & + & - & - \\ - & + & - & - & - \\ - & + & - & - & + \\ - & - & + & + & - \\ - & - & + & + & + \\ - & - & - & - & + \\ + & + & - & - & 0 \\ + & 0 & + & + & 0 \\ + & 0 & 0 & 0 & - \\ + & - & 0 & 0 & + \\ 0 & + & + & + & - \\ - & + & 0 & 0 & 0 \\ 0 & + & 0 & 0 & + \\ 0 & - & + & + & 0 \\ - & 0 & + & + & + \\ 0 & 0 & - & - & + \end{pmatrix},$$

where "+" denotes the element equal to 1 and "–" denotes element the equal to –1.

5. Discussion

The principal significance of Theorems 3–8 is that they allow for widening the possible classes $\mathbf{\Psi}_{n\times p}(-1, 0, 1)$ for any n and p in which a regular D-optimal chemical balance weighing design exists. However, the conditions given in Theorem 3 imply that for any class $\mathbf{\Psi}_{n\times p}(-1, 0, 1)$ and for any given ρ we cannot construct a regular D-optimal chemical balance weighing design. For example, in the class $\mathbf{\Psi}_{15\times5}(-1, 0, 1)$, we determine a regular D-optimal chemical balance weighing design for $\rho = -1/10$ (Ceranka, Graczyk, 2016: 78–82; Theorem 6 (iii)) and here for $\rho = -1/51$. Based on the same paper, we determine the optimal design for $\rho = -1/120$ in the class $\mathbf{\Psi}_{120\times15}(-1, 0, 1)$, see Theorem 7 (ii). Besides, based on the presented theoretical results, we set the optimal design for $\rho = -1/992$. Next, determining unknown measurements of $p = 7$ objects, we are able to use $n = 35$ measurements for $\rho = -1/37$, see Ceranka, Graczyk (2016: 82–83), Theorem 8 (ii). In addition, based on the results obtained above, we are able to use $n = 28$ measurements for $\rho = -3/97$. Summarising, we add new classes but the problem of determining an optimal design in any class is still open and requires further study.

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Regularny D-optymalny układ wagowy z ujemnie skorelowanymi błędami

Streszczenie: W artykule rozważa się problematykę dotyczącą istnienia regularnego D-optymalnego chemicznego układu wagowego przy założeniu, że błędy pomiarów są ujemnie skorelowane i mają takie same wariancje. Przedstawiono warunki konieczne i dostateczne, wyznaczające układ regularnie D-optymalny oraz podano nowe metody konstrukcji. Są one oparte na macierzach incydencji układów zrównoważonych o blokach niekompletnych oraz dwudzielnych układów bloków.

Słowa kluczowe: dwudzielny układ bloków, chemiczny układ wagowy, układ optymalny, układ zrównoważony o blokach niekompletnych

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