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Notes on D-optimal Spring Balance Weighing Designs

Abstract: Spring balance weighing design is a model of an experiment in which the result can be presented as a linear combination of unknown measurements of objects with factors of this combination equalling zero or one. In this paper, we assume that the variances of measurement errors are not equal and errors are not correlated. We consider D-optimal designs, i.e. designs in which the determinant of the information matrix for the design attains the maximal value. The upper bound of its value is obtained and the conditions for the upper bound to be attained are proved. The value of the upper bound depends on whether the number of objects in the experiment is odd or even. Some methods of construction of regular D-optimal spring balance weighing designs are demonstrated.

Keywords: D-optimal design, spring balance weighing design

JEL: C02, C18, C90

1. Introduction

Let us introduce a model of spring balance weighing design. Assume that $\mathbf{\Phi}_{n \times p}(0, 1)$ denotes a set of all $n \times p$ matrices in which each element equals 0 or 1. We estimate true unknown weights $w_1, w_2, \ldots, w_p$ using n measuring operations from a spring balance. Let $y_1, y_2, \ldots, y_n$ be the observations in these n operations, respectively. Suppose that the observations follow the linear model $\mathbf{y} = \mathbf{Xw} + \mathbf{e}$. The vector $\mathbf{y} = (y_1, y_2, \ldots, y_n)'$ is an $n \times 1$ vector of observations. The matrix $\mathbf{X} = (x_{ij}) \in \mathbf{\Phi}_{n \times p}(0, 1)$ is the design matrix. If the jth object is placed on the pan during the ith operation then $x_{ij} = 1$, when a particular object is absent, then $x_{ij} = 0$. $\mathbf{w} = (w_1, w_2, \ldots, w_p)'$

is the vector of true unknown weights of objects. The vector $\mathbf{e} = (e_1, e_2, ..., e_n)'$ is the vector of errors components such that $E(\mathbf{e}) = \mathbf{0}_n$ and $\mathrm{Cov}(\mathbf{e}) = \sigma^2 \mathbf{G}$, $\mathbf{0}_n$ is $n \times 1$ null vector, σ^2 is the constant, $\mathbf{G}$ is the $n \times n$ positive definite diagonal matrix of known elements. The problem is to estimate the vector $\mathbf{w}$. If the design matrix is of full column rank then the generalized least squares estimator of $\mathbf{w}$ is $\hat{\mathbf{w}} = (\mathbf{X'G^{-1}X})^{-1}\mathbf{X'G^{-1}y}$. The matrix $\mathbf{M} = \mathbf{X'G^{-1}X}$ is named the information matrix of the design $\mathbf{X}$. Any weighing design is singular or non-singular if $\mathbf{M}$ is singular or non-singular, respectively.

Weighing designs with other forms of $\mathrm{Cov}(\mathbf{e})$ and their applications are considered in the literature, see Masaro and Wong (2008), Cheng (2014) and Ceranka and Graczyk (2013; 2015; 2016).

Many different optimality criteria are considered in the literature. One of the most popular is D-optimality. The design is called D-optimal if the determinant of the inverse of information matrix $\mathbf{M}^{-1}$ is minimal in the set of all matrices $\mathbf{X} \in \mathbf{\Phi}_{n \times p} (0, 1)$. It is known that the determinant of $\mathbf{M}^{-1}$ is minimal if and only if determinant of $\mathbf{M}$ is maximal. Moreover, if the determinant of information matrix attains the upper bound then $\mathbf{X}$ is called a regular D-optimal design. It is worth noting that each regular D-optimal design is D-optimal. The concept of D-optimality was presented in books of Raghavarao (1971), Banerjee (1975) and in the paper of Jacroux and Notz (1983). In the case $\mathbf{G} = \mathbf{I}_n$, Neubauer, Watkins and Zeitlin (1998) gave some construction methods of the D-optimal spring balance weighing designs. In the D-optimal design, the product of variances of estimators is minimal. Therefore, there is the possibility of applying these designs in economic investigation. For example, when we study the value of shares $\mathbf{y} = (y_1, y_2, ..., y_n)'$ of listed companies $\mathbf{w} = (w_1, w_2, ..., w_p)'$. The elements of the design matrix $\mathbf{X} \in \mathbf{\Phi}_{n \times p} (0, 1)$ indicate if a company is chosen or not. The investor preferences are given in the matrix $\mathbf{G}$.

In the next Section, we will consider the experiment in which we have five kinds of spring balances with different precisions. We give the upper bound for $\det(\mathbf{M})$ and the necessary and sufficient conditions to this bound to be attained when

$$\mathbf{G} = \begin{bmatrix} \mathbf{I}_{n-4} & \mathbf{0}_{n-4} & \mathbf{0}_{n-4} & \mathbf{0}_{n-4} & \mathbf{0}_{n-4} \\ \mathbf{0}'_{n-4} & g_1^{-1} & 0 & 0 & 0 \\ \mathbf{0}'_{n-4} & 0 & g_2^{-1} & 0 & 0 \\ \mathbf{0}'_{n-4} & 0 & 0 & g_3^{-1} & 0 \\ \mathbf{0}'_{n-4} & 0 & 0 & 0 & g_4^{-1} \end{bmatrix}, \; g_1, g_2, g_3, g_4 > 0. \qquad (1)$$

Moreover, we give the construction method of the regular D-optimal spring balance weighing design.

2. The upper bound of det(M)

The problems of determining the regular D-optimal spring balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n \times p}(0, 1)$ based on the matrix from class $\mathbf{\Lambda}_{(n-1) \times p}(0, 1)$ or $\mathbf{\Xi}_{(n-2) \times p}(0, 1)$ of the regular D-optimal spring balance weighing design were presented in Katulska and Przybył (2007), while based on the matrix from class $\mathbf{\Theta}_{(n-3) \times p}(0, 1)$ in Ceranka and Graczyk (2014).

For any number of measurements n and any number of objects p, we are not able to determine regular D-optimal design. That is why, in order to determine the regular D-optimal spring balance weighing design in the class $\mathbf{\Phi}_{n \times p}(0, 1)$, we consider the design matrix $\mathbf{X}_1 \in \mathbf{\Psi}_{(n-4) \times p}(0, 1)$ which is regular D-optimal for $n - 4$ measurements and p objects, and we add four measurements. So, let $\mathbf{X} \in \mathbf{\Phi}_{n \times p}(0, 1)$ be given in the following form

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{x}_1' \\ \mathbf{x}_2' \\ \mathbf{x}_3' \\ \mathbf{x}_4' \end{bmatrix}. \tag{2}$$

The problem is to formulate the relations between $\mathbf{X}_1$ and the $p \times 1$ vectors $\mathbf{x}_h$ of elements 0 or 1, $h = 1, 2, 3, 4$, and to determine the forms of additionally weighing operations. In order to give the D-optimal design $\mathbf{X}$ in the class $\mathbf{\Phi}_{n \times p}(0, 1)$, we have to give the upper bound of the determinant of the information matrix $\mathbf{M}$ for this design. Thus, for $\mathbf{X} \in \mathbf{\Phi}_{n \times p}(0, 1)$ in the form (2) with the covariance matrix of errors $\sigma^2 \mathbf{G}$, where $\mathbf{G}$ is of the form (1), we calculate

$$\det(\mathbf{M}) = \det\left(\mathbf{X}_1'\mathbf{X}_1 + \begin{bmatrix} g_1\mathbf{x}_1 & g_2\mathbf{x}_2 & g_3\mathbf{x}_3 & g_4\mathbf{x}_4 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 & \mathbf{x}_4 \end{bmatrix}'\right) =$$

$$\det\left(\mathbf{X}_1'\mathbf{X}_1\right)\det\left(\mathbf{I}_4 + \begin{bmatrix} g_1\mathbf{x}_1 & g_2\mathbf{x}_2 & g_3\mathbf{x}_3 & g_4\mathbf{x}_4 \end{bmatrix}\left(\mathbf{X}_1'\mathbf{X}_1\right)^{-1}\begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 & \mathbf{x}_4 \end{bmatrix}'\right)$$

according to the Theorem 18.1.1 in Harville (1997). We assume that the vectors $\mathbf{x}_h$ are of the form

$$\begin{cases} \mathbf{x}_h'\mathbf{1}_p = t_h, & 0 < t_h \le p \\ \mathbf{x}_h'\mathbf{x}_{h'} = \mathbf{x}_{h'}'\mathbf{x}_h = u_{hh'}, & 0 \le u_{hh'} \le \min(t_h, t_{h'}) \\ h, h' = 1, 2, 3, 4, h \ne h' \end{cases} \tag{3}$$

The value of det($\mathbf{M}$) depends on the number of objects p. Therefore, the proof falls naturally into two parts.

2.1. The case: p is an odd number

Theorem 1. Let p be odd. In any non-singular spring balance weighing design $\mathbf{X}_1 \in \boldsymbol{\Psi}_{(n-4) \times p}(0,1)$, $\det(\mathbf{X}_1'\mathbf{X}_1) \le (p+1)m^p$, where $m = (p+1)(n-4)(4p)^{-1}$ and the equality holds if and only if m is an integer and $\mathbf{X}_1'\mathbf{X}_1 = m(\mathbf{I}_p + \mathbf{1}_p \mathbf{1}_p')$.

This Theorem was given by Hudelson, Klee and Larman (1996).

From Theorem 1 we obtain $(\mathbf{X}_1'\mathbf{X}_1)^{-1} = \dfrac{1}{m}\left(\mathbf{I}_p - \dfrac{1}{p+1}\mathbf{1}_p \mathbf{1}_p'\right)$ and

$$\det(\mathbf{M}) \le (p+1)m^p \det(\boldsymbol{\Omega}), (4), \qquad (4)$$

where $\boldsymbol{\Omega} = \mathbf{I}_4 + \dfrac{1}{m}\begin{bmatrix} g_1\mathbf{x}_1 & g_2\mathbf{x}_2 & g_3\mathbf{x}_3 & g_4\mathbf{x}_4 \end{bmatrix}'\left(\mathbf{I}_p - \dfrac{1}{p+1}\mathbf{1}_p \mathbf{1}_p'\right)\begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 & \mathbf{x}_4 \end{bmatrix}$.
From (3) it follows that

$$\det(\boldsymbol{\Omega}) = \prod_{h=1}^{4}\left(1 + \frac{g_h t_h(p+1-t_h)}{m(p+1)}\right) + \frac{2\upsilon - \gamma + g_1 g_2 g_3 g_4 (\rho - 2\delta)}{m^4(p+1)^4},$$

where

$$\upsilon = \sum_h \left(m(p+1) + g_h t_h(p+1-t_h)\right)\prod_s\prod_z g_s\left(u_{sz}(p+1) - t_s t_z\right), \quad h,s,z = 1,2,3,4, \quad h \ne s \ne z$$

$$\rho = \sum_h\sum_s\sum_z\left(u_{1h}(p+1) - t_1 t_h\right)^2\left(u_{sz}(p+1) - t_s t_z\right)^2, \quad h,s,z = 2,3,4, h \ne s \ne z,$$

$$\gamma = \sum_h\sum_{h'}\sum_s\sum_z g_h g_{h'}\left(m + g_s t_s(p+1-t_s)\right)\left(m + g_z t_z(p+1-t_z)\right)\left(u_{hh'}(p+1) - t_h t_{h'}\right)^2,$$
$h = 1,2,3$, $h' = h+1$, $s,z = 1,2,3,4$, $h \ne h' \ne s \ne z$,

$$\begin{aligned}\delta = &\left(u_{12}(p+1) - t_1 t_2\right)\left(u_{13}(p+1) - t_1 t_3\right)\left(u_{24}(p+1) - t_2 t_4\right)\left(u_{34}(p+1) - t_3 t_4\right) + \\ &\left(u_{12}(p+1) - t_1 t_2\right)\left(u_{14}(p+1) - t_1 t_4\right)\left(u_{23}(p+1) - t_2 t_3\right)\left(u_{34}(p+1) - t_3 t_4\right) + \\ &\left(u_{13}(p+1) - t_1 t_3\right)\left(u_{14}(p+1) - t_1 t_4\right)\left(u_{23}(p+1) - t_2 t_3\right)\left(u_{24}(p+1) - t_2 t_4\right)\end{aligned}$$

For given number of objects p, we have to maximize $\det(\boldsymbol{\Omega})$. In an attempt to achieve this aim, we should determine in regard to t_k, the maximal value of $t_h(p+1-t_h)$ and the minimal value of $\left(u_{hh'}(p+1) - t_h t_{h'}\right)^2$, simultaneously for any $h, h' = 1,2,3,4$, $h \ne h'$. Then $t_h(p+1-t_h)(p+1)^{-1} \le 0.25(p+1)$ and the equality holds if and only if $t_h = 0.5(p+1)$. Therefore, we have

$$\det(\mathbf{\Omega}) \le \prod_{h=1}^{4}\left(1+\frac{g_h p}{n-4}\right) + \frac{8(n-4)m\chi + (n-4)^2 g_1 g_2 g_3 g_4 (\varphi - 2\vartheta) - 16m^2\kappa}{256\,m^4(n-4)^2}, \quad (5)$$

where

$$\chi = \sum_h (g_h p + n - 4)\prod_s\prod_z (4u_{sz} - p - 1),\ h,s,z = 1,2,3,4,\ s \ne z\,,$$

$$\varphi = \sum_s\sum_h\sum_z (4u_{1h} - p - 1)^2 (4u_{sz} - p - 1)^2,\ s,h,z = 2,3,4,\ h \ne s \ne z\,,$$

$$\kappa = \sum_h\sum_{h'}\sum_s\sum_z g_h g_{h'}(g_s p + n - 4)(g_z p + n - 4)(4u_{hh'} - p - 1)^2,\ h = 1,2,3,$$

$$h' = h+1,\ s,z = 1,2,3,4,\ h \ne h' \ne s \ne z\,,$$

$$\vartheta = (4u_{12} - p - 1)(4u_{13} - p - 1)(4u_{24} - p - 1)(4u_{34} - p - 1) +$$
$$(4u_{12} - p - 1)(4u_{14} - p - 1)(4u_{23} - p - 1)(4u_{34} - p - 1) +$$
$$(4u_{13} - p - 1)(4u_{14} - p - 1)(4u_{23} - p - 1)(4u_{24} - p - 1).$$

The equality in (5) is attained if and only if $t_h = 0.5(p+1)$, $h = 1, 2, 3, 4$. The value of **Błąd! Nieprawidłowy obiekt osadzony.** depends on the value of $\det(\mathbf{\Omega})$. Let us consider two cases $p+1 \equiv 0 \bmod(4)$ and $p+3 \equiv 0 \bmod(4)$.

The case: $p+1 \equiv 0 \bmod(4)$.

If $p+1 \equiv 0 \bmod(4)$, then the minimal value of $4u_{hh'} - p - 1$ is zero for $u_{hh'} = 0.25(p+1)$, $h, h' = 1,2,3,4$, $h \ne h'$. We conclude that

$$\det(\mathbf{\Omega}) \le \prod_{h=1}^{4}\left(1+\frac{g_h p}{n-4}\right) = \tau\,.$$

Thus (4) takes the form $\det(\mathbf{M}) \le (p+1)m^p\tau$.

The case: $p+3 \equiv 0 \bmod(4)$.

If $p+3 \equiv 0 \bmod(4)$, then the minimal value of $(4u_{hh'} - p - 1)^2$ is 0.25 for $u_{hh'} = 0.25(p-1)$ or $u_{hh'} = 0.25(p+3)$ $h, h' = 1,2,3,4$, $h \ne h'$. Therefore, we obtain $\det(\mathbf{\Omega}) \le \tau - \psi - \beta$, where

$$\psi = \frac{1}{4(n-4)m^3}\sum_h (g_h p + n - 4)\prod_s g_s,\ h,s = 1,2,3,4,\ s \ne h\,,$$

$$\beta = \frac{1}{4(n-4)^2 m^2}\sum_h\sum_{h'}\sum_s\sum_z g_h g_{h'}(g_s p + n - 4)(g_z p + n - 4) + \frac{3}{8m^4}\prod_h g_h\,,$$

$$h = 1,2,3,\quad h' = h+1,\quad s,z = 1,2,3,4,\ h \ne h' \ne s \ne z\,.$$

As the consequence we obtain that $\det(\mathbf{M}) \le (p+1)m^p(\tau - \psi - \beta)$. If $u_{hh'} = 0.25(p+3)$ for any $h, h' = 1,2,3,4$, $h \ne h'$, then we obtain $\det(\mathbf{\Omega}) \le \tau + \psi - \beta$ and

following on from this $\det(\mathbf{M}) \le (p+1)m^p(\tau + \psi - \beta)$. Since $\tau - \psi - \beta < \tau + \psi - \beta$, so for the case $p+3 \equiv 0 \bmod(4)$, the upper bound is given as $\det(\mathbf{M}) \le (p+1)m^p (\tau + \psi - \beta)$. From above considerations, we have the following Theorem.

Theorem 2. Let p be odd. In any non-singular spring balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n \times p}(0, 1)$ in (2) with the covariance matrix of errors $\sigma^2 \mathbf{G}$, where $\mathbf{G}$ is of the form (1), we have

1) if $p+1 \equiv 0 \bmod(4)$ then $\det(\mathbf{M}) \le (p+1)\, m^p \tau$ and the equality is attained if and only if $u_{hh'} = 0.25(p+1)$,
2) if $p+3 \equiv 0 \bmod(4)$ then $\det(\mathbf{M}) \le (p+1)m^p(\tau + \psi - \beta)$ and the equality is attained if and only if $u_{hh'} = 0.25(p+3)$,

where $t_h = 0.5(p+1)$, $h, h' = 1,2,3,4$, $h \ne h'$.

In the special case when $\mathbf{G} = \mathbf{I}_n$ and if

1) $p+1 \equiv 0 \bmod(4)$, then $\det(\mathbf{M}) = (p+1)m^p \iota^4$,
2) $p+3 \equiv 0 \bmod(4)$, then $\det(\mathbf{M}) = (p+1)m^p(\iota^4 + \iota^2 m^{-3} - 0,375\,m^{-4} - 1.5\iota^2 m^{-2})$, where $\iota = (p+n-4)(n-4)^{-1}$.

2.2. The case: p is an even number

Neubauer, Watkins and Zeitlin (1997) gave the following Theorem.

Theorem 3. Let p be even. In any non-singular spring balance weighing design $\mathbf{X}_1 \in \mathbf{\Psi}_{(n-4) \times p}(0,1)$ $\det(\mathbf{X}_1' \mathbf{X}_1) \le (p+1)q^p$, where $q = (p+2)(n-4)(4(p+1))^{-1}$ and the equality holds if and only if q is an integer and $\mathbf{X}_1' \mathbf{X}_1 = q(\mathbf{I}_p + \mathbf{1}_p \mathbf{1}_p')$.

From Theorem 3 we obtain $(\mathbf{X}_1' \mathbf{X}_1)^{-1} = \dfrac{1}{q}\left(\mathbf{I}_p - \dfrac{1}{p+1}\mathbf{1}_p \mathbf{1}_p'\right)$ and $\det(\mathbf{M}) \le$

$(p+1)q^p \det(\mathbf{\Gamma})$,

where

$$\mathbf{\Gamma} = \prod_{h=1}^{4}\left(1 + \frac{g_h t_h(p+1-t_h)}{q(p+1)}\right) + \frac{2\phi - \nu}{q^4(p+1)^4} + \frac{g_1 g_2 g_3 g_4(\rho - 2\delta)}{q^4(p+1)^4},$$

$$\phi = \sum_h \left(q(p+1) + g_h t_h(p+1-t_h)\right)\prod_s\prod_h g_s\left(u_{sz}(p+1) - t_s t_z\right),$$

$$h, s, z = 1,2,3,4, \ h \ne s \ne z,$$

$$\nu = \sum_h \sum_{h'} \sum_s \sum_z g_h g_{h'}\left(q + g_s t_s(p+1-t_s)\right)\left(q + g_z t_z(p+1-t_z)\right)\left(u_{hh'}(p+1) - t_h t_{h'}\right)^2,$$

$$h = 1,2,3, \ h' = h+1, \ s,z = 1,2,3,4, \ h \ne h' \ne s \ne z.$$

Owing to the fact that we have to maximize $\det(\boldsymbol{\Gamma})$, we should determine the maximal value of $t_h(p+1-t_h)$ and the minimal value of $\left(u_{hh'}(p+1)-t_ht_{h'}\right)^2$ $h, h' = 1,2,3,4$, $h \neq h'$, simultaneously. Then $t_h(p+1-t_h)(p+1)^{-1} \leq 0.25p(p+2)(p+1)^{-1}$ and the equality holds if and only if $t_h = 0.5p$ or $t_h = 0.5(p+2)$. Therefore, we have

$$\det(\boldsymbol{\Gamma}) \leq \prod_{h=1}^{4}\left(1+\frac{g_hp}{n-4}\right) +$$
$$\frac{2(n-4)(p+1)q\chi + (n-4)^2 g_1g_2g_3g_4\left(\rho-2\delta\right) - q^2(p+1)^2\kappa}{q^4(n-4)^2(p+1)^4}. \tag{6}$$

The equality in (6) holds if and only if $t_h = 0.5p$ or $t_h = 0.5(p+2)$. Next, we consider two cases $p \equiv 0\,\mathrm{mod}(4)$ and $p+2 \equiv 0\,\mathrm{mod}(4)$.

The case: $p \equiv 0\,\mathrm{mod}(4)$

If $p \equiv 0\,\mathrm{mod}(4)$, then the minimal value of $\left(u_{hh'}(p+1)-t_ht_{h'}\right)^2$ equals

$$\left(0.25p(p+1)^{-1}\right)^2 \text{ for } \begin{cases} u_{hh'} = 0.25p, & h, h' = 1,2,3,4,\ h \neq h' \\ t_h = 0.5p, & h = 1,2,3 \\ t_h = 0.5(p+2), & h = 4 \end{cases} \cdot \text{ For } h, h' = 1,2,3,4,\ h \neq h',$$

the product of three expressions of the form $\left(u_{hh'}(p+1)-t_ht_{h'}\right)(p+1)^{-1}$ equals $\left(0.25p(p+1)^{-1}\right)^3$ and the product of four expressions in this form equals $\left(0.25p(p+1)^{-1}\right)^4$. Hence $\det(\boldsymbol{\Gamma}) \leq \tau + 2\mu\varepsilon^{-3} - 3\varepsilon^{-4}\prod_{h=1}^{4}g_h - \varepsilon^{-2}\eta$, where

$$\varepsilon = (p+2)(n-4)p^{-1},$$

$$\mu = (n-4)^{-1}\sum_h(pg_h+n-4)\prod_s g_s,\ h, s = 1,2,3,4,\ s \neq h$$

$$\eta = (n-4)^{-2}\sum_h\sum_{h'}\sum_s\sum_z g_hg_{h'}(g_sp+n-4)(g_zp+n-4),$$
$$h = 1,2,3,\quad h' = h+1,\quad s, z = 1,2,3,4,\quad h \neq h' \neq s \neq z,$$

Finally, $\det(\mathbf{M}) \leq (p+1)q^p\left(\tau + 2\mu\varepsilon^{-3} - 3\varepsilon^{-4}\prod_{h=1}^{4}g_h - \varepsilon^{-2}\eta\right)$.

The case: $p+2 \equiv 0\,\mathrm{mod}(4)$

If $p+2 \equiv 0\,\mathrm{mod}(4)$, then the minimal value of $\left(u_{hh'}(p+1)-t_ht_{h'}\right)^2$ equals

$$\left(0.25(p+2)(p+1)^{-1}\right)^2 \ \text{for} \ \begin{cases} u_{hh'} = 0.25(p+2), & h,h'=1,2,3,4, \ h \neq h' \\ t_h = 0.5(p+2), & h=1,2,3 \\ t_h = 0.5p, & h=4 \end{cases}.$$

For $h, h' = 1,2,3,4, \ h \neq h'$, the product of three expressions in the form $\left(u_{hh'}(p+1) - t_h t_{h'}\right)(p+1)^{-1}$ equals $-\left(0.25(p+2)(p+1)^{-1}\right)^3$ and the product of four expressions in this form equals $\left(0.25(p+2)(p+1)^{-1}\right)^4$.

Hence $\det(\mathbf{\Gamma}) \leq \tau - 2\mu(n-4)^{-3} - 3\prod_{h=1}^{4} g_h (n-4)^{-4} - (n-4)^{-2}\eta$

and finally $\det(\mathbf{M}) \leq (p+1)q^p \left(\tau - 2(n-4)^{-3}\mu - 3(n-4)^{-4}\prod_{h=1}^{4} g_h - \eta(n-4)^{-2} \right).$

From above considerations we have

Theorem 4. Let p be even. In any non-singular spring balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n \times p}(0, 1)$ in (2) with the covariance matrix of errors $\sigma^2 \mathbf{G}$, where $\mathbf{G}$ is of the form (1) we have

1) if $p \equiv 0 \bmod(4)$ then $\det(\mathbf{M}) \leq (p+1)q^p \left(\tau + 2\varepsilon^{-3}\mu - 3\varepsilon^{-4}\prod_{h=1}^{4} g_h - \eta\varepsilon^{-2} \right)$ and the

equality is attained if and only if, $\begin{cases} u_{hh'} = 0.25p, & h,h'=1,2,3,4, \ h \neq h' \\ t_h = 0.5p, & h=1,2,3 \\ t_h = 0.5(p+2), & h=4 \end{cases}.$

2) if $p+2 \equiv 0 \bmod(4)$ then

$$\det(\mathbf{M}) \leq (p+1)q^p \left(\tau - 2(n-4)^{-3}\mu - 3(n-4)^{-4}\prod_{h=1}^{4} g_h - \eta(n-4)^{-2} \right)$$

and the equality is attained

if and only if $\begin{cases} u_{hh'} = 0.5(p+2), & h,h'=1,2,3,4, \ h \neq h' \\ t_h = 0.5p, & h=1,2,3 \\ t_h = 0.5(p+2), & h=4 \end{cases}$

In the special case when $\mathbf{G} = \mathbf{I}_n$ and if

1) $p \equiv 0 \bmod(4)$, then $\det(\mathbf{M}) = (p+1)q^p \left(\iota^4 + 8\iota\varepsilon^{-3} - 3\varepsilon^{-4} - 6\varepsilon^{-2}\iota^2 \right)$,

2) $p+2 \equiv 0 \bmod(4)$,
    then $\det(\mathbf{M}) = (p+1)q^p \left(\iota^4 - 8\iota(n-4)^{-3} - 3(n-4)^{-4} - 6\iota^2(n-4)^{-2} \right)$.

Definition 1. Any non-singular spring balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n \times p} (0, 1)$ in (2) with the covariance matrix of errors $\sigma^2 \mathbf{G}$, where $\mathbf{G}$ is of the form (1), is regular D-optimal if $\det(\mathbf{M})$ attains the upper bound given in Theorem 2 for odd p or in Theorem 4 for even p.

3. Construction and examples

In order to construct regular D-optimal spring balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n \times p}$ (0, 1) in (2) with the covariance matrix of errors $\sigma^2 \mathbf{G}$, where $\mathbf{G}$ is of the form (1), we have to form the matrix $\mathbf{X}_1 \in \mathbf{\Psi}_{(n-4) \times p}(0, 1)$ and, as the next step, add four measurements of the form given in Theorem 2 or in Theorem 4 for odd and even number of objects, respectively. Some methods of construction $\mathbf{X}_1 \in \mathbf{\Psi}_{(n-4) \times p}(0, 1)$ are given in Ceranka, Graczyk and Katulska (2009) and Ceranka and Graczyk (2017).

Let us consider the experiment in which we determine unknown measurements of $p = 7$ objects in $n = 11$ measurement operations. Thus, we have to determine the regular D-optimal spring balance weighing design in $\mathbf{X} \in \mathbf{\Phi}_{11 \times 7}$ (0, 1). The variance matrix of errors $\sigma^2 \mathbf{G}$, is given by the matrix $\mathbf{G}$ of the form (1) in which $g_1 = \frac{1}{5}$, $g_2 = \frac{1}{3}$, $g_3 = \frac{1}{7}$, $g_4 = \frac{1}{2}$. It is easy to see that $m = n(p+1)(4p)^{-1} = \frac{22}{7}$ is not integer. Hence, the matrix $\mathbf{X} \in \mathbf{\Phi}_{11 \times 7}$ (0, 1) for which $\mathbf{X}'\mathbf{X} = m(\mathbf{I}_7 + \mathbf{1}_7 \mathbf{1}_7')$ does not exist. Now, let us consider $\mathbf{X}_1 \in \mathbf{\Psi}_{7 \times 7}(0, 1)$ and in this case $m = 1$. For the matrix $\mathbf{X}_1 \in \mathbf{\Psi}_{7 \times 7}(0, 1)$ in the form

$$\mathbf{X}_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \text{ the condition } \mathbf{X}'\mathbf{X} = 2(\mathbf{I}_7 + \mathbf{1}_7 \mathbf{1}_7')$$

is fulfilled. From Theorem 2 we have $u_{hh'} = 2$ and $t_h = 4$, $h = 1, 2, 3, 4$. Therefore, the matrix

$$\mathbf{X} = \begin{bmatrix} & & & \mathbf{X}_1 & & & \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

is a regular D-optimal spring balance weighing design for which $\det(\mathbf{M}) = 589824$.

Let us consider the problem of estimating of $p = 6$ objects using $n = 11$ measurement operations. The variance matrix of errors $\sigma^2\mathbf{G}$, is given by the matrix $\mathbf{G}$ of the form as above. Thus, we have to determine the regular D-optimal spring balance weighing design in $\mathbf{X} \in \mathbf{\Phi}_{11 \times 6}(0, 1)$. We have $q = n(p+2)(4(p+1))^{-1} = 22\frac{2}{7}$ is not integer and the matrix $\mathbf{X} \in \mathbf{\Phi}_{11 \times 6}(0, 1)$ for which $\mathbf{X}'\mathbf{X} = q(\mathbf{I}_6 + \mathbf{1}_6\mathbf{1}_6')$ does not exist. Thus, let us take the matrix $\mathbf{X}_1 \in \mathbf{\Psi}_{7 \times 6}(0, 1)$. For the matrix $\mathbf{X}_1 \in \mathbf{\Psi}_{7 \times 6}(0, 1)$ in the form

$$\mathbf{X}_1 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}, \quad q = 2 \quad \text{and the condition} \quad \mathbf{X}'\mathbf{X} = 2(\mathbf{I}_6 + \mathbf{1}_6\mathbf{1}_6')$$

holds. From Theorem 4 we have $u_{hh'} = 2$ and $t_1 = t_2 = t_3 = 4$ and $t_4 = 3$ the matrix

$$\mathbf{X} = \begin{bmatrix} & & \mathbf{X}_1 & & & \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \text{ is regular D-optimal and } \det(\mathbf{M}) = 142848.$$

4. Conclusions

For any number of objects p and measurements n, it is not possible to construct regular D-optimal spring balance weighing design. Some methods of construction of regular D-optimal designs in the class $\mathbf{\Phi}_{n \times p}(0, 1)$ based on the design matrix $\mathbf{X}_1$ which is regular D-optimal in the class $\mathbf{\Lambda}_{(n-1) \times p}(0, 1)$, $\mathbf{\Xi}_{(n-2) \times p}(0, 1)$ were presented in Katulska and Przybył (2007), Ceranka, Graczyk, Katulska (2009), while for $\mathbf{\Theta}_{(n-3) \times p}(0, 1)$ were given in Ceranka and Graczyk (2014). In the present paper we give new construction method of the design matrix $\mathbf{X} \in \mathbf{\Phi}_{n \times p}(0, 1)$, based on the matrix $\mathbf{X}_1$ of the regular D-optimal design in the class $\mathbf{\Psi}_{(n-4) \times p}(0, 1)$. It means, we have enlarged the set of classes in that we are able to determine regular D-optimal spring balance weighing design.

Particularly, in order to determine the regular D-optimal spring balance weighing design in the class $\Phi_{11\times7}(0, 1)$ it is impossible to use regular D-optimal matrix X_1 from the class $\Lambda_{10\times7}(0, 1)$, $\Xi_{9\times7}(0, 1)$ or $\Theta_{8\times7}(0, 1)$. For the determining regular D-optimal designs in the class $\Phi_{11\times7}(0, 1)$, we have to use the design matrix X_1 which is regular D-optimal in the class $\Psi_{7\times7}(0, 1)$ and add four measurements. Similarly, to determine the optimal design in the class $\Phi_{11\times6}(0, 1)$ it is impossible to use regular D-optimal matrix X_1 from the class $\Lambda_{10\times6}(0, 1)$, $\Xi_{9\times6}(0, 1)$ or $\Theta_{8\times6}(0, 1)$. We have to form this matrix based on the optimal design form the class $\Psi_{7\times6}(0, 1)$ and add four measurements.

References

Banerjee K.S. (1975), *Weighing Designs for Chemistry, Medicine, Economics, Operations Research, Statistics*, Marcel Dekker Inc., New York.

Ceranka B., Graczyk M. (2013), *Construction of E-optimal spring balance weighing designs for even number of objects*, "Acta Universitatis Lodziensis. Folia Oeconomica", vol. 285, pp. 141–148.

Ceranka B., Graczyk M. (2014), *Regular D-optimal spring balance weighing designs: construction*, "Acta Universitatis Lodziensis. Folia Oeconomica", vol. 302, pp. 111–125.

Ceranka B., Graczyk M. (2015), *On D-optimal chemical balance weighing designs*, "Acta Universitatis Lodziensis. Folia Oeconomica", vol. 311, pp. 71–84.

Ceranka B., Graczyk M. (2016), *About some properties and constructions of experimental designs*, "Acta Universitatis Lodziensis. Folia Oeconomica", vol. 333, pp. 73–85.

Ceranka B., Graczyk M. (2017), *Recent developments in D–optimal spring balance weighing designs*, "Communication in Statistics-Theory and Methods", accepted to publication.

Ceranka B., Graczyk M., Katulska K. (2009), *On some constructions of regular D–optimal spring balance weighing designs*, "Biometrical Letters", vol. 46, pp. 103–112.

Cheng C.S. (2014), *Optimal biased weighing designs and two-level main effect plans*, "Journal of Statistical Theory and Practice", vol. 8, pp. 83–99.

Harville D.A. (1997), *Matrix Algebra from a Statistician's Perspective*, Springer Verlag, New York.

Hudelson M., Klee V., Larman D. (1996), *Largest j-simplices in d-cubes: Some relatives to the Hadamard determinant problem*, "Linear Algebra and its Applications", vol. 24, pp. 519–598.

Jacroux M., Notz W. (1983), *On the optimality of spring balance weighing designs*, "The Annals of Statistics", vol. 11, pp. 970–978.

Katulska K., Przybył K. (2007), *On certain D-optimal spring balance weighing designs*, "Journal of Statistical Theory and Practice", vol. 1, pp. 393–404.

Masaro J., Wong Ch.S. (2008), *D-optimal designs for correlated random vectors*, "Journal of Statistical Planning and Inference", vol. 138, pp. 4093–4106.

Neubauer M.G., Watkins W., Zeitlin J. (1997), *Maximal j-simplices in the real dimensional unit cube*, "Journal of Combinatorial Theory", Ser. A, vol. 80, pp. 1–12.

Neubauer G.N., Watkins W., Zeitlin J. (1998), *Notes on D-optimal designs*, "Linear Algebra and its Applications", vol. 280, pp. 109–127.

Raghavarao D. (1971), *Constructions and combinatorial problems in design of experiment*, John Wiley and Sons, New York.

Uwagi o D-optymalnych sprężynowych układach wagowych

Streszczenie: Sprężynowy układ wagowy to model doświadczenia, którego wynik można opisać jako liniową kombinację nieznanych miar obiektów o współczynnikach równych zero lub jeden. W artykule rozważamy układy, dla których błędy pomiarów są nieskorelowane i mają różne wariancje. Rozważamy D-optymalne sprężynowe układy wagowe, tzn. takie układy, w których wyznacznik macierzy informacji układu jest maksymalny. Podano górne ograniczenie jego wartości oraz warunki konieczne i dostateczne, przy spełnieniu których to ograniczenie jest osiągnięte. Ponadto zaprezentowane zostały metody konstrukcji macierzy D-optymalnych układów.

Słowa kluczowe: sprężynowy układ wagowy, układ D-optymalny

JEL: C02, C18, C90