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Limiting Cases of the Black-Scholes Type Asymptotics of Call Option Pricing in the Generalised CRR Model

Abstract: The article concerns the generalised Cox-Ross-Rubinstein (CRR) option pricing model with new formulas for changes in upper and lower stock prices. The formula for option pricing in this model, which is the Black-Scholes type formula, and its asymptotics are presented. The aim of the paper is to analyse limiting cases of the obtained asymptotics using probability theory and later data from the Warsaw Stock Exchange. Empirical analyses of option pricing in the generalised CRR model confirm the calculated limits.

Keywords: Cox-Ross-Rubinstein model (CRR model), binomial model, Black-Scholes formula, option pricing

JEL: C01, G13

1. Introduction

The problem of option pricing is one of central issues in financial mathematics. As a *European call option*, we understand the right, without the obligation, to buy a stock on a particular date T (the expiry date) for a guaranteed price K (the strike price). The strike price is determined at the moment $t = 0$. The holder of a call option has to pay a premium for getting this right (the option price).

The following definition presents a European call option.

Definition. A European call option is a pair (T, C_T) where $T > 0$ and $C_T(\cdot): \mathfrak{R}_+ \to \mathfrak{R}$ is the function:

$$C_T(s) = (s-K)^+ = \begin{cases} s-K & \text{if } s > K \\ 0 & \text{if } s \le K \end{cases} \text{ for some } K \in \mathfrak{R}_+.$$

In 1979, Cox, Ross and Rubinstein (Cox, Ross, Rubinstein, 1979) presented a discrete time option pricing formula (CRR model). They assumed that changes in the upper and lower stock prices are the same at each point in time, and they got the following possible changes of the stock price:

$$S_0 = s_0,$$
$$S_t = u \cdot S_{t-1} \text{ or } S_t = d \cdot S_{t-1},\ 1 \le t \le T,\ t \in \mathrm{N},$$

where:

T is a fixed natural number of short period (the expiry date),

t is the number of the present point in time,

s_0 is a positive constant (the stock price at the moment 0),

u is the upper stock price change during one short period,

d is the lower stock price change during one short period.

Cox, Ross and Rubinstein (1979) proved the following theorem describing option pricing:

Theorem 1.1. CRR option pricing. In the CRR model, the fair price $C_0(s_0)$ at the moment 0 of a European call option with the expiry date T and the strike price $K = s_0 u^{k_0} d^{T-k_0}$ for a certain $k_0 = 0,1,\dots,T$ is given by:

$$C_0(s_0) = s_0 \overline{D} - \frac{K}{\hat{r}^T} D^*, \tag{1}$$

where:

$$\overline{D} = \sum\nolimits_{k=k_0}^{T} \binom{T}{k} \cdot \overline{p}^{k} \cdot \overline{q}^{T-k},\ D^* = \sum\nolimits_{k=k_0}^{T} \binom{T}{k} \cdot p^{*k} \cdot q^{*T-k},$$

$$k_0 = \frac{\ln\frac{K}{S_0} - T \cdot \ln d}{\ln\left(\frac{u}{d}\right)}, \; p^* = \frac{\hat{r}-d}{u-d}, \; q^* = \frac{u-\hat{r}}{u-d}, \; \bar{p} = p^* \cdot \frac{u}{\hat{r}}, \; \bar{q} = q^* \cdot \frac{d}{\hat{r}},$$

$\hat{r}$ is the value of one unit of money in a bank account at the time $t = 1$.

Moreover, it is assumed that: $0 < d < \hat{r} < u$ (Jakubowski, 2006).

Remark 1.1. The formula (1) is not dependent on the correlation between the upper and lower stock price change.

Cox, Ross and Rubinstein (1979) have proven that their formula converges to the following Black-Scholes formula (Black, Scholes, 1973) when the number of moments of portfolio change between the moment 0 and the expiry date T is large:

Theorem 1.2. Black-Scholes option pricing. The time 0 fair price $C_0^{BS}(s_0)$ of a European call option with the strike price K and the expiry date T in the Black-Scholes model is given by:

$$C_0^{BS}(s_0) = s_0 \phi\left(\frac{\ln\frac{s_0}{K} + T\left(r + \frac{\sigma^2}{2}\right)}{\sigma\sqrt{T}}\right) - \frac{K}{e^{rT}} \phi\left(\frac{\ln\frac{s_0}{K} + T\left(r - \frac{\sigma^2}{2}\right)}{\sigma\sqrt{T}}\right),$$

where $\phi(\cdot)$ is the cumulative normal distribution function.

Black and Scholes (1973) considered a continuous-time market in which the stock price $S(t)$ at the time t is defined as:

$$S(t) = s_0 \cdot \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right), \; t \geq 0,$$

where W_t is a Wiener process, $s_0 > 0$, $\sigma > 0$ and μ are constants, s_0 is the stock price observed at the time 0 and σ is the volatility. Moreover, the instantaneous interest rate $r > 0$ of a bank account is also assumed to be constant (Jakubowski, 2006).

The literature on option pricing is abundant and presents the CRR model (Hull, 1998; Shreve, 2004; Elliot, Kopp, 2005; Jakubowski et al., 2006; Musiela, Rutkowski, 2008; Capiński, Kopp, 2012), the CRR model with transaction costs (Cox, Rubinstein, 1985; Stettner, 1997), the estimate of the rate of convergence of the CRR formula to the Black-Scholes formula (Walsh, 2003; Ratibenyakool, Neammanee, 2019), modified CRR models (Rendleman, Bartter, 1979; Rachev, Ruschendorff, 1994; Karandikar, Rachev, 1995; Rubinstein, 2000; Jabbour, Kramin, Young, 2001), and its asymptotic formulas (Heston, Zhou, 2000; Diener, Diener, 2004; Leisen, Reimer, 2006; Chang, Palmer, 2007; Joshi, 2010; Xiao, 2010).

In chapter 2.6 of the paper entitled *Financial Markets in Continuous Time* (Dana, Jeanblanc, 2007), there are similar assumptions for the possible upper and lower stock price change to the one presented by the author, however, there are different transformations of random variables used.

The aim of the article is to show the proven formula for option pricing in the generalised Cox-Ross-Rubinstein (CRR) model (Fraszka-Sobczyk, 2014; 2020) and to calculate limiting cases of this formula's asymptotics, which is the Black-Scholes type formula, using probability theory. The next objective is to compute these limiting cases based on real data from the Warsaw Stock Exchange and check if the same limits will be obtained.

The proposed generalised CRR model assumes that the expected value of the random variable $\ln(S(n)/s_0)$ is ρ, $\rho \in \Re$ ($S(n)$ is the stock price after *n* moments of portfolio change during one time unit, s_0 is the beginning stock price). This assumption $E[\ln S(n)/s_0] = \rho$ gives the following formula for stock price changes u_n and d_n: $u_n = e^{\sigma\frac{1}{\sqrt{n}}} \cdot e^{\rho\frac{1}{n}}$, $d_n = \frac{e^{2\rho\frac{1}{n}}}{u_n} = e^{-\sigma\frac{1}{\sqrt{n}}} \cdot e^{\rho\frac{1}{n}}$, where σ is the standard deviation of $\ln(S(n)/s_0)$, *n* is the number of moments of portfolio change in each time unit. In the known CRR model, the possible stock price changes u_n and d_n are given by formulas: $u_n = e^{\sigma\frac{1}{\sqrt{n}}}$ and $d_n = \frac{1}{u_n} = e^{-\sigma\frac{1}{\sqrt{n}}}$. These formulas are the consequences of the following assumptions: $E\left[\ln\frac{S(n)}{s_0}\right] = 0$. The famous CRR formula for option pricing is given according to the risk-neutral measure. By contrast, in the proposed generalised CRR model, the expected value of the random variable $\ln(S(n)/s_0)$ ρ is calculated according to the measure of probability based on historical data. This measure makes the difference between the presented generalised CRR model and the CRR model. The starting point for both models is the assumption that probabilities of the upper and lower stock price changes are the same and they equal 0.5. The presented generalisation of CRR model involves the coefficient ρ which describes the tendency for stock price changes in the financial market. In comparison to the CRR model, this assumption reflects reality. An investor (a holder of a call option) observes stock price changes and defines the coefficient ρ. Then the investor decides to buy or not to buy a call option.

The paper consists of five parts. The second section describes the generalised CRR model: its assumptions, option pricing and the limit of option pricing (the Black-Scholes type formula). In the third section, there is an analysis of limiting cases of the Black-Scholes type formula. The author also demonstrates limiting cases when the coefficient ρ is equal to zero (limiting cases for the Black-Scholes formula). The next section includes empirical research which focuses on determining the presented limits. This study is based on data obtained from the Warsaw Stock Exchange and concerns two periods of time and two options with different strike prices. The last part of the article contains conclusions.

2. Generalised CRR model

Let τ denote the number of time units to the expiry time (for example, the number of months) and let n be the number of moments of portfolio change in each time unit. Then $T = n \cdot \tau$ is the expiry date. The stock price $S(n)$ after n moments totals:

$$S(n) = s_0 \cdot u_n^{S_n} \cdot d_n^{n-S_n},$$

where S_n is the random variable describing the number of stock price rises during one-time unit (for example, during one month), u_n is the upper stock price change between next moments of portfolio change and d_n is the lower stock price change between next moments of portfolio change. There is the assumption that empirical probabilities of the upper and lower stock price change are the same and they are equal to 0.5.

First, the generalised CRR model will be considered. It assumes that the expected value of the random variable $\ln(S(n)/s_0)$ is ρ, $\rho \in \Re$ and the variance $\sigma^2 := D^2\left[\ln \frac{S(n)}{s_0}\right]$.

This assumption results in the new following formulas for the upper u_n and lower d_n stock price changes:

$$u_n = e^{\sigma \frac{1}{\sqrt{n}}} \cdot e^{\rho \frac{1}{n}} \text{ and } d_n = \frac{e^{2\rho \frac{1}{n}}}{u_n} = e^{-\sigma \frac{1}{\sqrt{n}}} \cdot e^{\rho \frac{1}{n}}.$$

In the CRR model, it is assumed that $E\left[\ln S(n)/s_0\right] = 0$ and the upper and lower stock price changes are as follows:

$$u_n = e^{\sigma \frac{1}{\sqrt{n}}} \text{ and } d_n = \frac{1}{u_n} = e^{-\sigma \frac{1}{\sqrt{n}}}.$$

From this and **Remark 1.1**, the following corollary is obtained:

Corollary 2.1. The pricing for a European call option in the generalised CRR model is described by the following formula:

$$C_{0,n}\left(s_0\right)=s_0\overline{D}_n-\frac{K}{e^{r\tau}}D_n^*, \tag{2}$$

where:

$$\overline{D}_n=\sum_{k=k_{0,n}}^{n\tau}\binom{n\tau}{k}\cdot\overline{p}_n^{\,k}\cdot\overline{q}_n^{\,n\tau-k},$$

$$D_n^*=\sum_{k=k_{0,n}}^{n\tau}\binom{n\tau}{k}\cdot p_n^{*k}\cdot q_n^{*n\tau-k},$$

$$k_{0,n}=\frac{\ln\frac{K}{s_0}-\tau\cdot n\cdot\ln d_n}{\ln\left(\frac{u_n}{d_n}\right)}=\frac{\sqrt{n}\ln\frac{K}{s_0}+\sigma\cdot\tau\cdot n-\sqrt{n}\cdot\tau\cdot\rho}{2\sigma},$$

$$p_n^*=\frac{\hat{r}_n-d_n}{u_n-d_n},\ q_n^*=\frac{u_n-\hat{r}_n}{u_n-d_n},\ \overline{p}_n=p_n^*\cdot\frac{u_n}{\hat{r}_n},\ \overline{q}_n=q_n^*\cdot\frac{d_n}{\hat{r}_n},$$

$$u_n=e^{\sigma\frac{1}{\sqrt{n}}}\cdot e^{\rho\frac{1}{n}}\ \text{and}\ d_n=\frac{e^{2\rho\frac{1}{n}}}{u_n}=e^{-\sigma\frac{1}{\sqrt{n}}}\cdot e^{\rho\frac{1}{n}},$$

$$\hat{r}_n=e^{r\cdot\frac{1}{n}},$$

r is the interest rate of the bank account (or credit) for one-time unit.

The option pricing formula of the generalised CRR model in **Corollary 2.1** converges next to $n\to\infty$ to the Black-Scholes type formula (Fraszka-Sobczyk, 2014; 2020):

Theorem 2.1. For a European call option with the strike price K, the number τ of time units to the expiry time, the beginning stock price s_0, the interest rate of the bank account (or credit) for one-unit r, the coefficient ρ and the volatility σ, defined in (2), the following limit is given:

$$\lim_{n\to\infty}C_{o,n}\left(s_0\right)=s_0\phi\left(\frac{\ln\frac{s_0}{K}+\tau\left(r-\rho+\frac{\sigma^2}{2}\right)}{\sigma\sqrt{\tau}}\right)-\frac{K}{e^{r\tau}}\phi\left(\frac{\ln\frac{s_0}{K}+\tau\left(r-\rho-\frac{\sigma^2}{2}\right)}{\sigma\sqrt{\tau}}\right),$$

where:

$$\frac{\ln\frac{s_0}{K}+\tau\left(r-\rho+\frac{\sigma^2}{2}\right)}{\sigma\sqrt{\tau}}=:\ d_1,$$

$$\frac{\ln\frac{s_0}{K}+\tau\left(r-\rho-\frac{\sigma^2}{2}\right)}{\sigma\sqrt{\tau}}=:d_2.$$

3. Limiting cases

Let us note that $C_o(s_0):=\lim_{n\to\infty} C_{o,n}(s_0)$. This section will present an analysis of some limits of $C_o(s_0)$, taking into account the coefficient ρ, which can be interpreted as the average change of logarithm of stock prices after one time unit (for example, after one month).

Lemma 3.1. There are the following limiting cases of the Black-Scholes type formula that is the asymptotics (next to $n\to\infty$) of the option pricing formula in the generalised CRR model:

a) $\lim_{\tau\to 0^+} C_o(s_0)=(s_0-K)^+$;

b) $$\lim_{\tau\to\infty} C_o(s_0)=\begin{cases} s_0 & \text{if} \quad \sigma^2>2(\rho-r) \\ 0 & \text{if} \quad \sigma^2<2(\rho-r); \\ \frac{1}{2}s_0 & \text{if} \quad \sigma^2=2(\rho-r) \end{cases}$$

c) $$\lim_{\sigma\to 0} C_o(s_0)=\begin{cases} 0 & \text{if} \quad s_0<\frac{K}{e^{(r-\rho)\tau}} \\ s_0-\frac{K}{e^{r\tau}} & \text{if} \quad s_0>\frac{K}{e^{(r-\rho)\tau}}; \\ \frac{1}{2}\left(s_0-\frac{K}{e^{r\tau}}\right) & \text{if} \quad s_0=\frac{K}{e^{(r-\rho)\tau}} \end{cases}$$

d) $$\lim_{s_0\to\infty}\left[C_o(s_0)-\left(s_0-\frac{K}{e^{r\tau}}\right)\right]=0.$$

The following property of normal distribution that will be used in the proof of d) will be shown before the proof of **Lemma 3.1**.

Property 3.1. Let $\phi(\cdot)$ be the cumulative normal $N(0, 1)$ distribution function and $s>1$. For fixed $a\in\Re$, $b>0$, we have the following limit:

$$\lim_{s\to\infty}\left[s\cdot\phi(a-b\cdot\ln s)\right]=0.$$

Proof of Property 3.1. Let $f(\cdot)$ be the density of the standard normal distribution and $\gamma < 0$. Then:

$$\int_{-\infty}^{\gamma} f(x)dx = \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\gamma+u)^2} du = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\gamma^2} \int_{-\infty}^{0} e^{-\gamma u} \cdot e^{-\frac{1}{2}u^2} du \leq \\ \leq \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\gamma^2} \int_{-\infty}^{0} e^{-\gamma u} du = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\gamma^2} \cdot \frac{1}{-\gamma}. \tag{3}$$

Let us note that there exists $s_0 > 0$ such that $a - b\ln s < 0$ for all $s \geq s_0$ (because $b > 0$). From this and (3) for $s \geq s_0$:

$$0 \leq \phi(a - b \cdot \ln s) \cdot s \leq \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(a - b\ln s)^2} \cdot \frac{1}{-a + b\ln s} \cdot s = \\ = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}a^2} \cdot e^{-\ln s\left(\frac{1}{2}b^2 \ln s - ab - 1\right)} \cdot \frac{1}{-a + b\ln s} \underset{s\to\infty}{\to} 0.$$

Finally:

$$\lim_{s\to\infty}\left[\phi(a - b \cdot \ln s) \cdot s\right] = 0.$$

Remark 3.1. Let us note that if $f(\cdot)$ is the density of the standard normal distribution and $\gamma < 0$, then the following estimate is given:

$$\int_{-\infty}^{\gamma} f(x)dx = \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\gamma+u)^2} du = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\gamma^2} \int_{-\infty}^{0} e^{-\gamma u} \cdot e^{-\frac{1}{2}u^2} du \leq \\ \leq e^{-\frac{1}{2}\gamma^2} \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du \leq \frac{1}{2} e^{-\frac{1}{2}\gamma^2}.$$

Proof of Lemma 3.1. Denote: $\phi(\infty) := \lim_{t\to\infty}\phi(t)$, $\phi(-\infty) := \lim_{t\to-\infty}\phi(t)$.

a) The following limit will be calculated

$$\lim_{\tau\to 0^+} C_o(s_0) = (s_0 - K)^+.$$

Let us observe that:

$$\lim_{\tau\to 0^+} C_o(s_0) = \lim_{\tau\to 0^+} s_0\left(1-\phi\left(\frac{\ln\frac{K}{s_0}}{\sigma\sqrt{\tau}}-\sqrt{\tau}\left(\frac{r}{\sigma}-\frac{\rho}{\sigma}+\frac{1}{2}\sigma\right)\right)\right)-$$

$$-\frac{K}{e^{r\tau}}\left(1-\varphi\left(\frac{\ln\frac{K}{s_0}}{\sigma\sqrt{\tau}}-\sqrt{\tau}\left(\frac{r}{\sigma}-\frac{\rho}{\sigma}-\frac{1}{2}\sigma\right)\right)\right)=$$

$$=\begin{cases} s_0\left(1-\phi(\infty)\right)-\frac{K}{e^0}\left(1-\phi(\infty)\right) & \text{if } s_0<K \\ s_0\left(1-\phi(-\infty)\right)-\frac{K}{e^0}\left(1-\phi(-\infty)\right) & \text{if } s_0>K \end{cases} = \begin{cases} 0 & \text{if } s_0<K \\ s_0-K & \text{if } s_0>K \end{cases} = (s_0-K)^+ .$$

b) This limit will be proven

$$\lim_{\tau\to\infty} C_o(s_0) = \begin{cases} s_0 & \text{if } \sigma^2>2(\rho-r) \\ 0 & \text{if } \sigma^2<2(\rho-r) \\ \frac{1}{2}s_0 & \text{if } \sigma^2=2(\rho-r) \end{cases}.$$

Next equalities are given:

$$\lim_{\tau\to\infty} C_o(s_0) = \lim_{\tau\to\infty} s_0\left(1-\phi\left(\frac{\ln\frac{K}{s_0}}{\sigma\sqrt{\tau}}-\sqrt{\tau}\left(\frac{r}{\sigma}-\frac{\rho}{\sigma}+\frac{1}{2}\sigma\right)\right)\right)-$$

$$-\frac{K}{e^{r\tau}}\left(1-\phi\left(\frac{\ln\frac{K}{s_0}}{\sigma\sqrt{\tau}}-\sqrt{\tau}\left(\frac{r}{\sigma}-\frac{\rho}{\sigma}-\frac{1}{2}\sigma\right)\right)\right)=$$

$$=\begin{cases} s_0\left(1-\phi(-\infty)\right)-0 & \text{if } \sigma^2>2(\rho-r) \\ s_0\left(1-\phi(\infty)\right)-0 & \text{if } \sigma^2<2(\rho-r) \\ s_0\left(1-\phi(0)\right)-0 & \text{if } \sigma^2=2(\rho-r) \end{cases} = \begin{cases} s_0 & \text{if } \sigma^2>2(\rho-r) \\ 0 & \text{if } \sigma^2<2(\rho-r) \\ \frac{1}{2}s_0 & \text{if } \sigma^2=2(\rho-r) \end{cases}.$$

c) The following limit will be calculated

$$\lim_{\sigma\to 0} C_o(s_0) = \begin{cases} 0 & \text{if} \quad s_0 < \dfrac{K}{e^{(r-\rho)\tau}} \\ s_0 - \dfrac{K}{e^{r\tau}} & \text{if} \quad s_0 > \dfrac{K}{e^{(r-\rho)\tau}} \\ \dfrac{1}{2}\left(s_0 - \dfrac{K}{e^{r\tau}}\right) & \text{if} \quad s_0 = \dfrac{K}{e^{(r-\rho)\tau}} \end{cases}.$$

The following two cases will be considered.

I. $\dfrac{K}{s_0} \neq e^{(r-\rho)\tau}$.

Then:

$$\lim_{\sigma\to 0} C_o(s_0) = \lim_{\sigma\to 0} s_0\left(1-\phi\left(\frac{\ln\frac{K}{s_0}}{\sigma\sqrt{\tau}} - \sqrt{\tau}\left(\frac{r}{\sigma}-\frac{\rho}{\sigma}+\frac{1}{2}\sigma\right)\right)\right) -$$

$$-\frac{K}{e^{r\tau}}\left(1-\phi\left(\frac{\ln\frac{K}{s_0}}{\sigma\sqrt{\tau}} - \sqrt{\tau}\left(\frac{r}{\sigma}-\frac{\rho}{\sigma}-\frac{1}{2}\sigma\right)\right)\right) =$$

$$= \lim_{\sigma\to 0} s_0\left(1-\phi\left(\frac{1}{\sigma}\left(\frac{\ln\frac{K}{s_0}}{\sqrt{\tau}} - r\sqrt{\tau}+\rho\sqrt{\tau}\right) - \frac{1}{2}\sqrt{\tau}\sigma\right)\right) -$$

$$-\frac{K}{e^{r\tau}}\left(1-\phi\left(\frac{1}{\sigma}\left(\frac{\ln\frac{K}{s_0}}{\sqrt{\tau}} - r\sqrt{\tau}+\rho\sqrt{\tau}\right) + \frac{1}{2}\sqrt{\tau}\sigma\right)\right) =$$

$$= \begin{cases} s_0\left(1-\phi(\infty)\right) - \dfrac{K}{e^{r\tau}}\left(1-\phi(\infty)\right) = 0 & \text{if} \quad \dfrac{1}{\sqrt{\tau}}\ln\dfrac{K}{s_0} - r\sqrt{\tau}+\rho\sqrt{\tau} > 0 \\ s_0\left(1-\phi(-\infty)\right) - \dfrac{K}{e^{r\tau}}\left(1-\phi(-\infty)\right) = s_0 - \dfrac{K}{e^{r\tau}} & \text{if} \quad \dfrac{1}{\sqrt{\tau}}\ln\dfrac{K}{s_0} - r\sqrt{\tau}+\rho\sqrt{\tau} < 0 \end{cases} =$$

$$= \begin{cases} 0 & \text{if} \quad \ln\dfrac{K}{s_0} > (r-\rho)\tau \\ s_0 - \dfrac{K}{e^{r\tau}} & \text{if} \quad \ln\dfrac{K}{s_0} < (r-\rho)\tau \end{cases} = \begin{cases} 0 & \text{if} \quad s_0 < \dfrac{K}{e^{(r-\rho)\tau}} \\ s_0 - \dfrac{K}{e^{r\tau}} & \text{if} \quad s_0 > \dfrac{K}{e^{(r-\rho)\tau}} \end{cases}.$$

II. $\frac{K}{s_0} = e^{(r-\rho)\tau}$.

Then:

$$\lim_{\sigma\to 0} C_o(s_0) = \lim_{\sigma\to 0} s_0\left(1-\phi\left(\frac{\ln\frac{K}{s_0}}{\sigma\sqrt{\tau}} - \sqrt{\tau}\left(\frac{r}{\sigma}-\frac{\rho}{\sigma}+\frac{1}{2}\sigma\right)\right)\right)-$$

$$-\frac{K}{e^{r\tau}}\left(1-\phi\left(\frac{\ln\frac{K}{s_0}}{\sigma\sqrt{\tau}} - \sqrt{\tau}\left(\frac{r}{\sigma}-\frac{\rho}{\sigma}-\frac{1}{2}\sigma\right)\right)\right)=$$

$$=\lim_{\sigma\to 0} s_0\left(1-\phi\left(-\frac{1}{2}\sigma\sqrt{\tau}\right)\right)-\frac{K}{e^{r\tau}}\left(1-\phi\left(\frac{1}{2}\sigma\sqrt{\tau}\right)\right)=s_0(1-0.5)-\frac{K}{e^{r\tau}}(1-0.5)=$$

$$=0.5\cdot\left(s_0-\frac{K}{e^{r\tau}}\right).$$

a) The following limit should be proven

$$\lim_{s_0\to\infty}\left[C_o(s_0)-\left(s_0-\frac{K}{e^{r\tau}}\right)\right]=0.$$

From **Property 3.1**, where $b=\frac{1}{\sigma\sqrt{\tau}}>0$, $a=\frac{\ln K}{\sigma\sqrt{\tau}}-\sqrt{\tau}\left(\frac{r-\rho}{\sigma}+\frac{\sigma}{2}\right)$, it is given:

$$\lim_{s_0\to\infty}\left[C_o(s_0)-\left(s_0-\frac{K}{e^{r\tau}}\right)\right]=\lim_{s_0\to\infty}[s_0\left(1-\phi\left(-\frac{1}{\sigma\sqrt{\tau}}\ln s_0+\frac{\ln K}{\sigma\sqrt{\tau}}-\sqrt{\tau}\left(\frac{r-\rho}{\sigma}+\frac{\sigma}{2}\right)\right)\right)-$$

$$-\frac{K}{e^{r\tau}}\left(1-\phi\left(-\frac{1}{\sigma\sqrt{\tau}}\ln s_0+\frac{\ln K}{\sigma\sqrt{\tau}}-\sqrt{\tau}\left(\frac{r-\rho}{\sigma}-\frac{\sigma}{2}\right)\right)\right)-\left(s_0-\frac{K}{e^{r\tau}}\right)]=$$

$$=s_0\cdot 1-0-\frac{K}{e^{r\tau}}-\left(s_0-\frac{K}{e^{r\tau}}\right)=0.$$

Corollary 3.1. For $\rho=0$, the following limiting cases for the Black-Scholes formula are given:

a) $\lim_{\tau\to 0^+} C_o(s_0)=(s_0-K)^+$;

b) $\lim_{\tau\to\infty} C_o(s_0)=\begin{cases} s_0 & \text{if} \quad \sigma^2>-2r \\ 0 & \text{if} \quad \sigma^2<-2r \\ \frac{1}{2}s_0 & \text{if} \quad \sigma^2=-2r \end{cases}$;

Taking into account $r > 0$ and $\sigma > 0$, it is given: $\lim_{\tau \to \infty} C_o(s_0) = s_0$.

c) $$\lim_{\sigma \to 0} C_o(s_0) = \begin{cases} 0 & \text{if} \quad s_0 < \frac{K}{e^{r\tau}} \\ s_0 - \frac{K}{e^{r\tau}} & \text{if} \quad s_0 > \frac{K}{e^{r\tau}} \\ 0 & \text{if} \quad s_0 = \frac{K}{e^{r\tau}} \end{cases};$$

d) $$\lim_{s_0 \to \infty} \left[C_o(s_0) - \left(s_0 - \frac{K}{e^{r\tau}} \right) \right] = 0.$$

4. Empirical analysis

The research presented focuses on determining the limits in Lemma 3.1 using data obtained from the Warsaw Stock Exchange. The study of option pricing covers the period from 22nd December 2021 to 21st January 2022. For this period of time, the option price has been calculated according to the generalised CRR model. The expiry time of the option OW20A222200 with the strike price $K = 2200$ is 21st January 2022. The number of days from 22nd December 2021 to the expiry time totals 30 ($\tau = 30/360$). On 22nd December 2021, the WIG20 index was equal to 2212.32 points (the beginning stock price $s_0 = 2212.32$) and the WIBOR reference rate (Warsaw Interbank Offered Rate) for PLN deposits in the Polish interbank market for the one-month (WIBOR 1M) tenor was equal to 1.95% ($r = 0.0195$). The daily coefficient ρ and the daily volatility σ are calculated from 22nd September 2021 to 21st December 2021. Next, they are multiplied by 250 (the number of trading sessions in one year) to get the annual ρ and σ which equal –0.16474 and 0.03214 respectively.

Empirical analysis approaches for calculating the option price according to the generalised CRR model and its limits are presented next. Results are given in the tables and figures presented below.

In the first case, the limit of the option price is calculated when the period of time to the expiry time is very short.

Table 1. The limit of the option price OW20A222200 next to $\tau \to 0^+$

s_0	K	σ	The number of days to the expiry time	r	ρ	d_1	$\phi(d_1)$	d_2	$\phi(d_2)$	The limit option price $C_0(s_0)$	$(s_0-K)^+$
2212.32	2200	0.03	30.0000	0.02	– 0.16	0.43	0.67	0.38	0.65	52.37	12.32
2212.32	2200	0.03	14.0000	0.02	– 0.16	0.38	0.65	0.34	0.63	38.11	12.32
2212.32	2200	0.03	1.0000	0.02	– 0.16	0.65	0.74	0.64	0.74	15.97	12.32
2212.32	2200	0.03	0.2500	0.02	– 0.16	1.21	0.89	1.21	0.89	12.95	12.32
2212.32	2200	0.03	0.0156	0.02	– 0.16	4.74	1.00	4.73	1.00	12.32	12.32
2212.32	2200	0.03	0.0020	0.02	– 0.16	13.38	1.00	13.38	1.00	12.32	12.32
2212.32	2200	0.03	0.0001	0.02	– 0.16	53.50	1.00	53.50	1.00	12.32	12.32

Source: the author's own elaboration

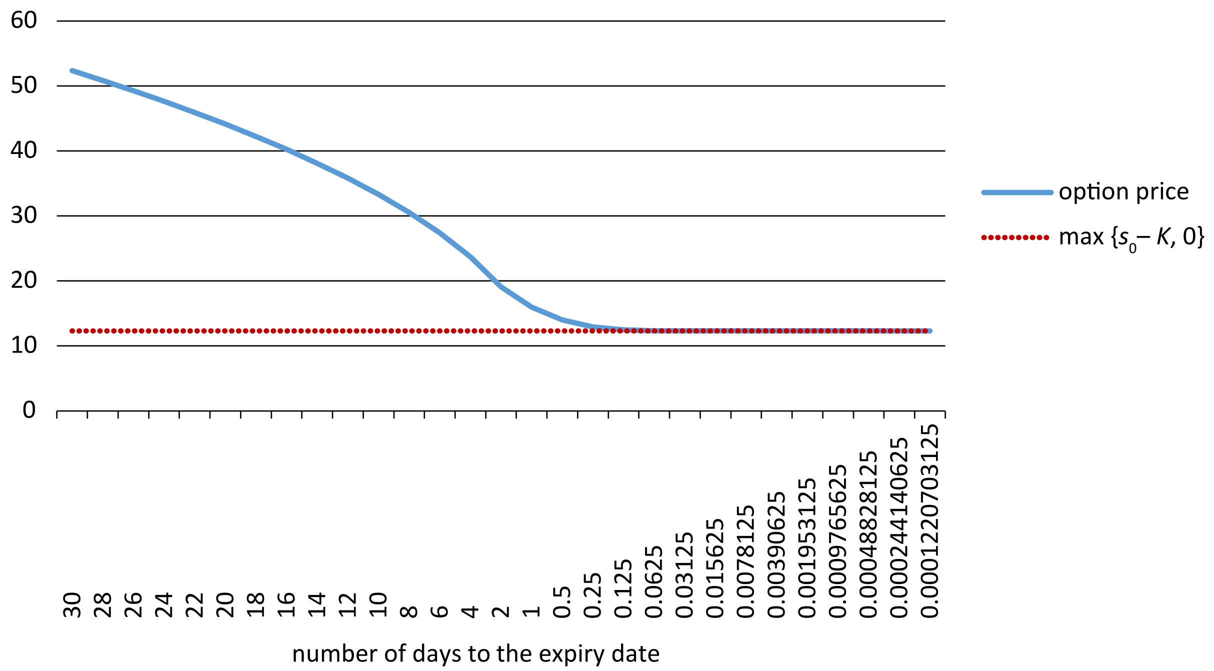

Figure 1. The limit of the option price OW20A222200 next to $\tau \to 0^+$
Source: the author's own elaboration

Considering a lot of numbers of time units to the expiry time (a very long period of time to the expiry time) and data obtained from the Warsaw Stock Exchange, the following calculations of the limit of the option price next to $\tau \to \infty$ relate the case when $\sigma^2 > 2(\rho - r)$.

Table 2. The limit of the option price OW20A222200 next to $\tau \to \infty$

s_0	K	σ	The number of days to the expiry time	r	ρ	d_1	$\phi(d_1)$	d_2	$\phi(d_2)$	The limit option price $C_0(s_0)$
2 212.32	2 200	0.03	30	0.02	– 0.16	0.43	0.67	0.38	0.65	52.37
2 212.32	2 200	0.03	120	0.02	– 0.16	0.70	0.76	0.60	0.72	93.31
2 212.32	2 200	0.03	960	0.02	– 0.16	1.84	0.97	1.55	0.94	177.91
2 212.32	2 200	0.03	7 680	0.02	– 0.16	5.17	1.00	4.34	1.00	761.03
2 212.32	2 200	0.03	61 440	0.02	– 0.16	14.60	1.00	12.26	1.00	2 133.42
2 212.32	2 200	0.03	491 520	0.02	– 0.16	41.29	1.00	34.66	1.00	2 212.32
2 212.32	2 200	0.03	7 864 320	0.02	– 0.16	165.15	1.00	138.65	1.00	2 212.32

Source: the author's own elaboration

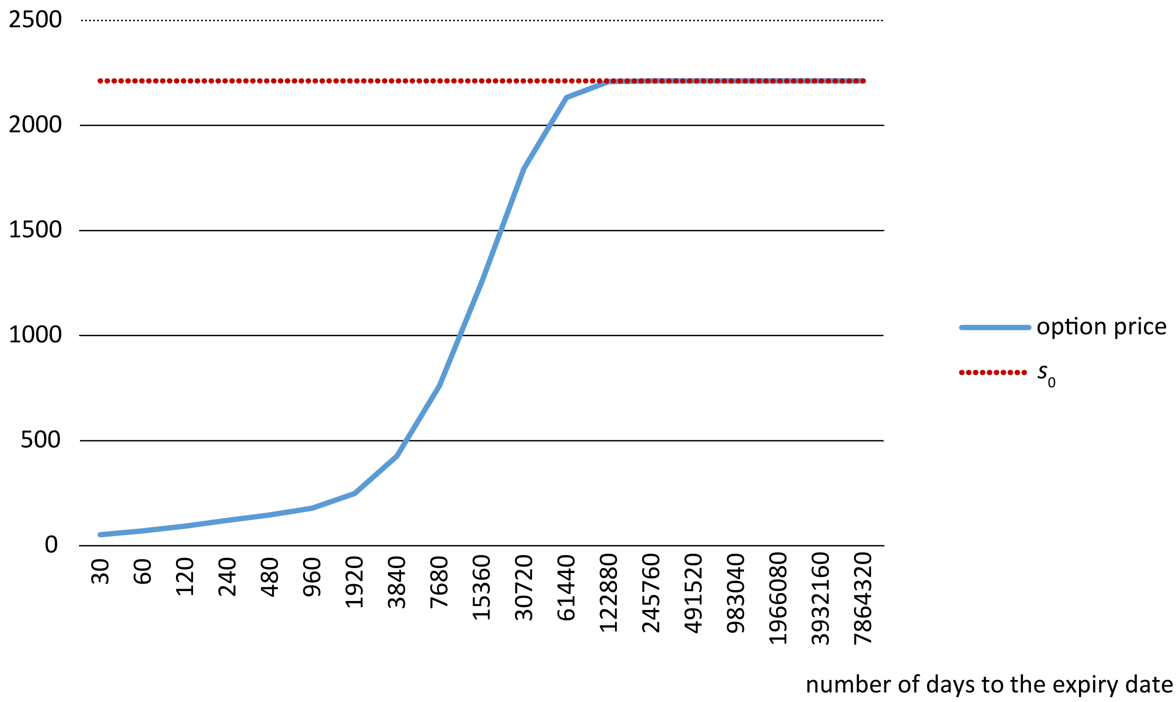

Figure 2. The limit of the option price OW20A222200 next to $\tau \to \infty$
Source: the author's own elaboration

Taking into account data from the Warsaw Stock Exchange and considering very small volatility $\sigma \to 0$, the inequality $s_0 > K / e^{(r-\rho)\tau}$ is given, so the limit of the option price next to $\sigma \to 0$ is equal to $s_0 - K / e^{r\tau}$ (**Lemma 3.1**).

Table 3. The limit of the option price OW20A222200 next to $\sigma \to 0$

s_0	K	σ	The number of days to the expiry time	r	ρ	d_1	$\phi(d_1)$	d_2	$\phi(d_2)$	The limit option price $C_0(s_0)$
2212.32	2200	0.0321	30.00	0.02	– 0.16	0.43	0.67	0.38	0.65	52.37
2212.32	2200	0.0206	30.00	0.02	– 0.16	0.53	0.70	0.49	0.69	43.06
2212.32	2200	0.0105	30.00	0.02	– 0.16	0.72	0.76	0.69	0.76	32.37
2212.32	2200	0.0054	30.00	0.02	– 0.16	1.00	0.84	0.98	0.84	24.77
2212.32	2200	0.0028	30.00	0.02	– 0.16	1.39	0.92	1.37	0.92	19.71
2212.32	2200	0.0014	30.00	0.02	– 0.16	1.93	0.97	1.92	0.97	16.95
2212.32	2200	0.0005	30.00	0.02	– 0.16	3.37	1.00	3.37	1.00	15.90

Source: the author's own elaboration

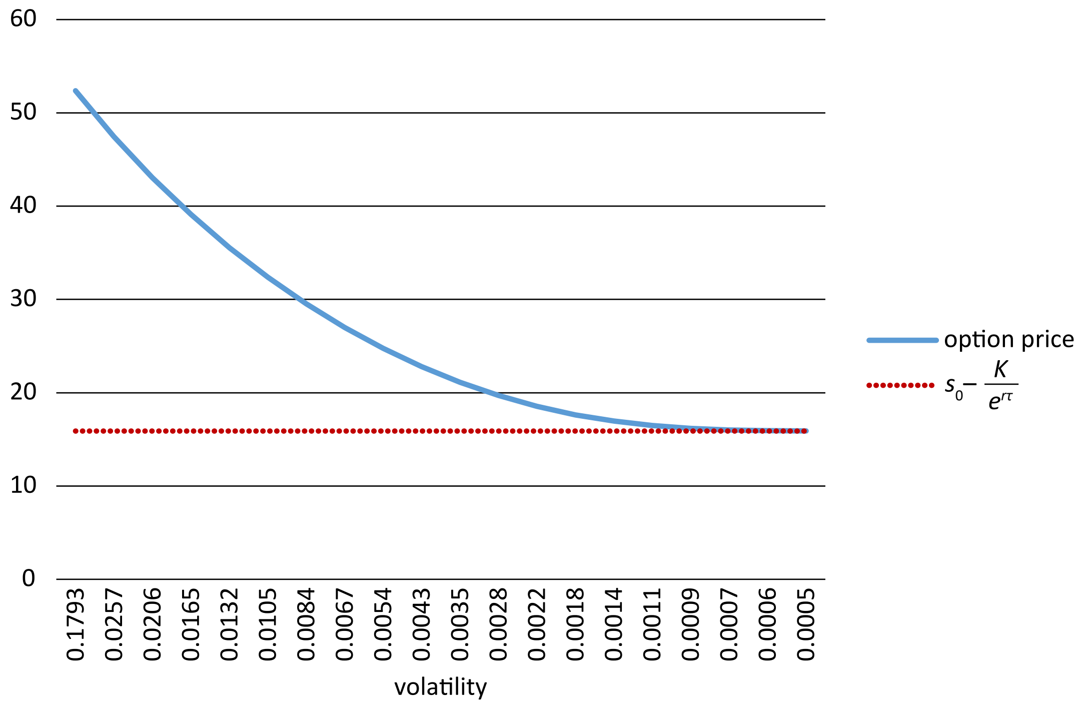

Figure 3. The limit of the option price OW20A222200 next to $\sigma \to 0$
Source: the author's own elaboration

Next, a simulation where the initial stock price is very large is presented.

Table 4. The limit of $C_o(s_0)-\left(s_0-\frac{K}{e^{r\tau}}\right)$ next to $s_0\to\infty$ for the option price OW20A222200

s_0	K	σ	The number of days to the expiry time	r	ρ	d_1	$\phi(d_1)$	d_2	$\phi(d_2)$	The limit option price $C_0(s_0)$
2212.32	2200	0.03	30.00	0.02	– 0.16	0.43	0.67	0.38	0.65	52.37
2256.79	2200	0.03	30.00	0.02	– 0.16	0.82	0.79	0.76	0.78	81.05
2325.17	2200	0.03	30.00	0.02	– 0.16	1.39	0.92	1.34	0.91	136.03
2395.63	2200	0.03	30.00	0.02	– 0.16	1.97	0.98	1.92	0.97	201.19
2468.22	2200	0.03	30.00	0.02	– 0.16	2.55	0.99	2.49	0.99	272.20
2543.00	2200	0.03	30.00	0.02	– 0.16	3.12	1.00	3.07	1.00	346.64
2646.26	2200	0.03	30.00	0.02	– 0.16	3.89	1.00	3.84	1.00	449.84

Source: the author's own elaboration

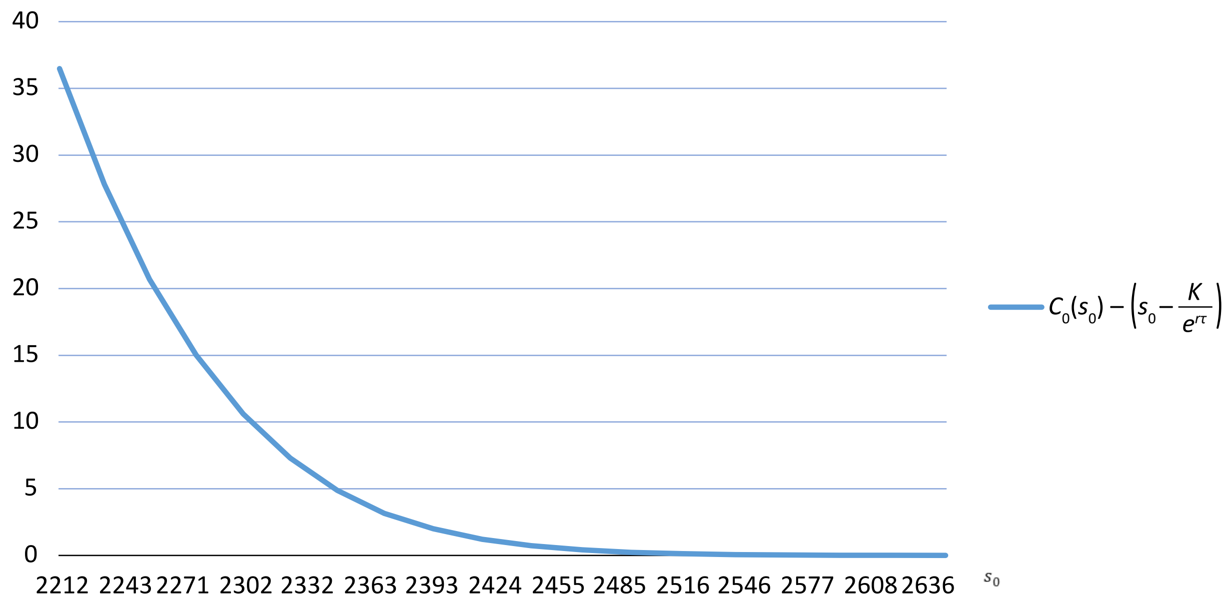

Figure 4. The limit of $C_o(s_0)-\left(s_0-\frac{K}{e^{r\tau}}\right)$ next to $s_0\to\infty$ for the option price OW20A222200

Source: the author's own elaboration

The subsequent empirical analysis concerns a longer period of time from 17th December 2021 to 16th December 2022. A new option OW20X222100 is considered with the strike price $K = 2100$ and the expiry time 16th December 2022. On 17th December 2021, the WIG20 index equalled 2194.56 points and the WIBOR reference rate for PLN deposits in the Polish interbank market for the one-year (WIBOR 1Y) tenor equalled 2.82% ($r = 0.0282$). The daily coefficient ρ and the daily volatility σ are calculated from 17th September 2021 to 16th December 2022. The annual ρ and σ equal –0.24904 and 0.03485 respectively.

Results of calculating the option price according to the generalised CRR model and its limits for the option OW20X222100 with the expiry time 16th December 2022 are presented in the following tables and figures.

Table 5. The limit of the option price OW20X222100 next to $\tau \to 0^+$

s_0	K	σ	The number of days to the expiry time	r	ρ	d_1	$\phi(d_1)$	d_2	$\phi(d_2)$	The limit option price $C_0(s_0)$	$(s_0-K)^+$
2194.56	2100	0.03	364.0000	0.03	– 0.25	1.82	0.97	1.63	0.95	182.77	94.56
2194.56	2100	0.03	84.6548	0.03	– 0.25	1.25	0.89	1.16	0.88	133.40	94.56
2194.56	2100	0.03	23.6256	0.03	– 0.25	1.33	0.91	1.28	0.90	106.48	94.56
2194.56	2100	0.03	16.4067	0.03	– 0.25	1.44	0.93	1.40	0.92	102.13	94.56
2194.56	2100	0.03	7.9122	0.03	– 0.25	1.83	0.97	1.80	0.96	97.10	94.56
2194.56	2100	0.03	4.5788	0.03	– 0.25	2.27	0.99	2.25	0.99	95.58	94.56
2194.56	2100	0.03	2.2081	0.03	– 0.25	3.14	1.00	3.12	1.00	94.93	94.56

Source: the author's own elaboration

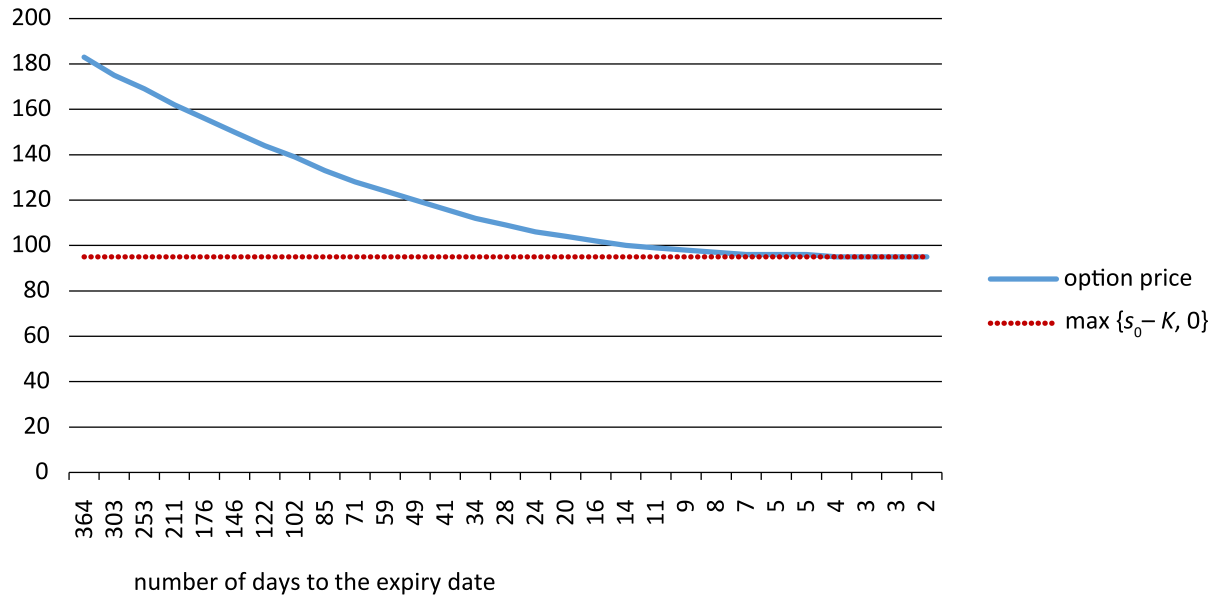

Figure 5. The limit of the option price OW20X222100 next to $\tau \to 0^+$
Source: the author's own elaboration

Table 6. The limit of the option price OW20X222100 next to $\tau \to \infty$

s_0	K	σ	The number of days to the expiry time	r	ρ	d_1	$\phi(d_1)$	d_2	$\phi(d_2)$	The limit option price $C_0(s_0)$
2 194.56	2 100	0.03	364	0.03	– 0.25	1.82	0.97	1.63	0.95	182.77
2 194.56	2 100	0.03	1 456	0.03	– 0.25	3.29	1.00	2.92	1.00	323.14
2 194.56	2 100	0.03	11 648	0.03	– 0.25	9.02	1.00	7.96	1.00	1 351.31
2 194.56	2 100	0.03	93 184	0.03	– 0.25	25.41	1.00	22.41	1.00	2 193.14
2 194.56	2 100	0.03	745 472	0.03	– 0.25	71.84	1.00	63.34	1.00	2 194.56
2 194.56	2 100	0.03	5 963 776	0.03	– 0.25	203.17	1.00	179.14	1.00	2 194.56
2 194.56	2 100	0.03	95 420 416	0.03	– 0.25	812.66	1.00	716.56	1.00	2 194.56

Source: the author's own elaboration

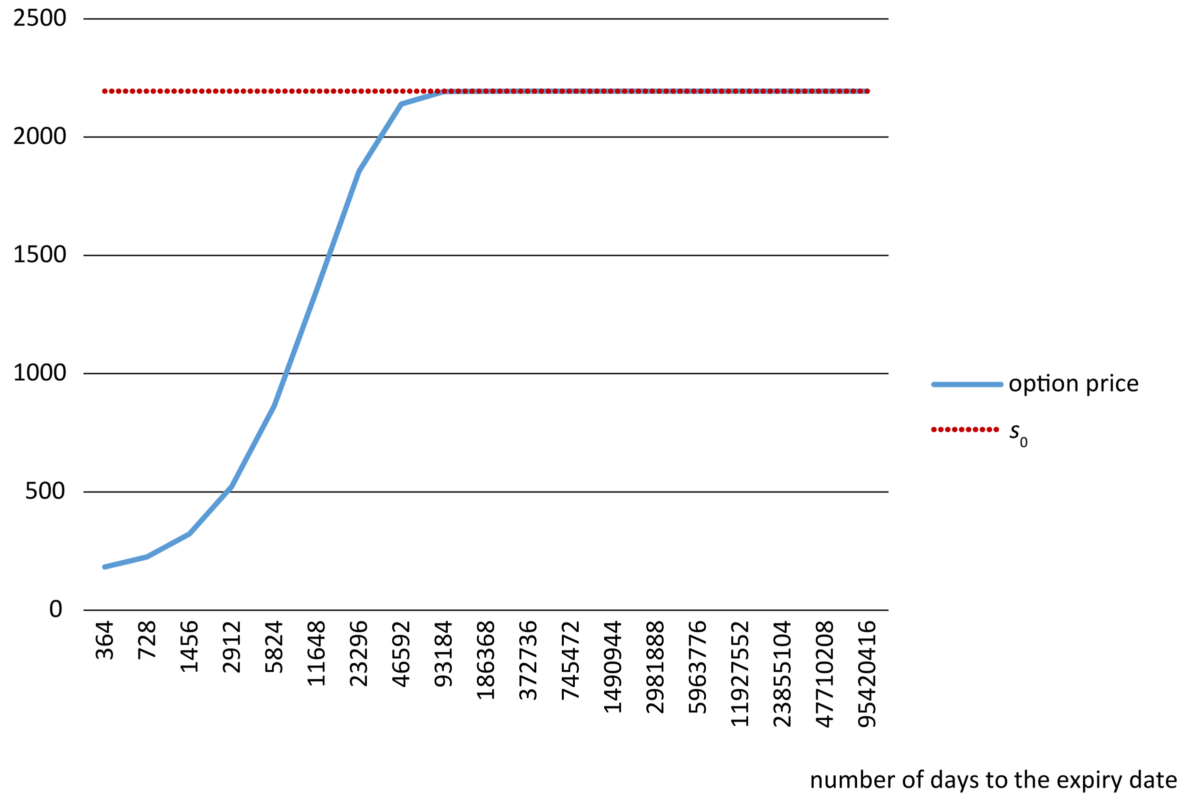

Figure 6. The limit of the option price OW20X222100 next to $\tau \to \infty$
Source: the author's own elaboration

Table 7. The limit of the option price OW20X222100 next to $\sigma \to 0$

s_0	K	σ	The number of days to the expiry time	r	ρ	d_1	$\phi(d_1)$	d_2	$\phi(d_2)$	The limit option price $C_0(s_0)$
2194.56	2100	0.0348	364.00	0.03	– 0.25	1.82	0.97	1.63	0.95	182.77
2194.56	2100	0.0223	364.00	0.03	– 0.25	2.24	0.99	2.08	0.98	163.54
2194.56	2100	0.0114	364.00	0.03	– 0.25	3.07	1.00	2.97	1.00	154.35
2194.56	2100	0.0058	364.00	0.03	– 0.25	4.26	1.00	4.18	1.00	153.60
2194.56	2100	0.0030	364.00	0.03	– 0.25	5.92	1.00	5.87	1.00	153.59
2194.56	2100	0.0015	364.00	0.03	– 0.25	8.26	1.00	8.22	1.00	153.59
2194.56	2100	0.0005	364.00	0.03	– 0.25	14.41	1.00	14.38	1.00	153.59

Source: the author's own elaboration

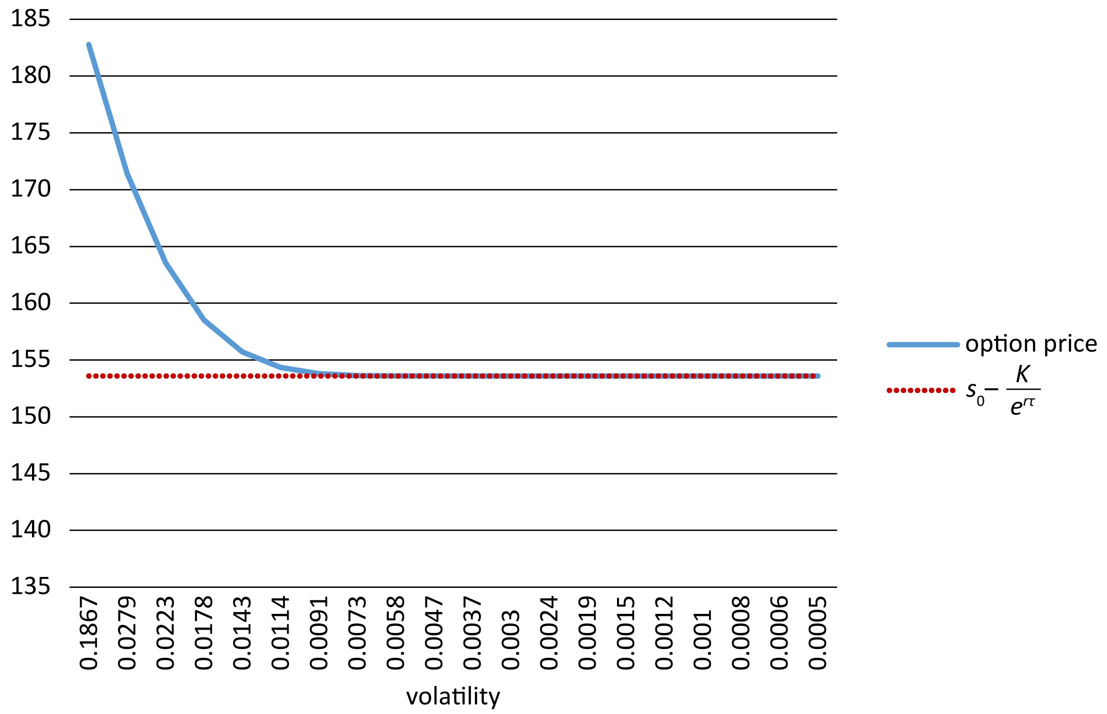

Figure 7. The limit of the option price OW20X222100 next to $\sigma \to 0$

Source: the author's own elaboration

Table 8. The limit of $C_o(s_0) - \left(s_0 - \frac{K}{e^{r\tau}}\right)$ next to $s_0 \to \infty$ for the option price OW20X222100

s_0	K	σ	The number of days to the expiry time	r	ρ	d_1	$\phi(d_1)$	d_2	$\phi(d_2)$	The limit option price $C_0(s_0)$
2194.56	2100	0.03	364.00	0.03	– 0.25	1.82	0.97	1.63	0.95	182.77
2283.22	2100	0.03	364.00	0.03	– 0.25	2.03	0.98	1.85	0.97	260.58
2422.97	2100	0.03	364.00	0.03	– 0.25	2.35	0.99	2.16	0.98	390.49
2571.28	2100	0.03	364.00	0.03	– 0.25	2.67	1.00	2.48	0.99	533.91
2728.66	2100	0.03	364.00	0.03	– 0.25	2.98	1.00	2.79	1.00	689.09
2895.68	2100	0.03	364.00	0.03	– 0.25	3.30	1.00	3.11	1.00	855.20
3134.37	2100	0.03	364.00	0.03	– 0.25	3.72	1.00	3.53	1.00	1093.51

Source: the author's own elaboration

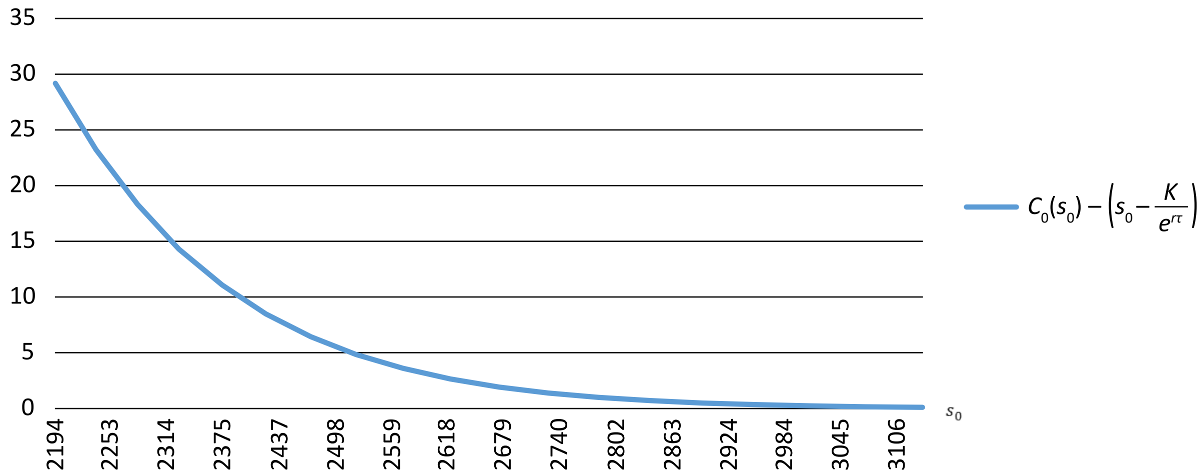

Figure 8. The limit of $C_o(s_0)-\left(s_0-\frac{K}{e^{r\tau}}\right)$ next to $s_0\to\infty$ for the option price OW20X222100

Source: the author's own elaboration

5. Conclusions

After comparing the limiting cases of the Black-Scholes type formula (**Lemma 3.1**) with the limiting cases of the Black-Scholes formula (**Corollary 3.1**), the following conclusions can be drawn:

1. Limiting cases of both formulas are the same when the number of time units to the expiry time is close to 0 (for short call option contracts).
2. Limiting cases of both formulas are different when the number of time units to the expiry time is very large (for long call option contracts). In the generalised case, three formulas (not one) are obtained according to the comparison between the volatility, the interest rate of a bank account and the coefficient ρ (the average change of logarithm of stock prices after one-time unit).
3. Limiting cases of both formulas are different when stock price volatility is close to 0 (the stock price after one-time unit is close to the beginning stock price at the moment 0). This approximate formula is simple and can be applied to financial markets with small volatility.
4. The limiting case (when the beginning stock price is very large) of the difference between the Black-Scholes option price $C_0^{BS}(s_0)$ and $s_0-Ke^{r\tau}$ is the same as the limiting case of the difference between the Black-Scholes type option price $C_o(s_0)$ and $s_0-Ke^{r\tau}$.

Empirical analyses of option pricing and its limits in the generalised CRR model which are based on data obtained from the Warsaw Stock Exchange confirm the limits in **Lemma 3.1**. The limit next to $s_0 \to \infty$ and the limit next to $\sigma \to 0$ relate untypical situations in the financial market when the initial stock price is very high or stock prices almost never change. However, it is worth considering these cases because trading sessions are unpredictable and they involve extreme price fluctuations.

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Przypadki graniczne przejścia granicznego typu Blacka-Scholesa wyceny opcji kupna w uogólnionym modelu CRR

Streszczenie: Artykuł przedstawia uogólniony model Coxa-Rossa-Rubinsteina (CRR) wyceny opcji, uwzględniający nowe formuły na górne i dolne zmiany cen akcji. Zaprezentowano formułę na wycenę opcji w rozważanym modelu oraz jej przejście graniczne typu Blacka-Scholesa. Głównym celem artykułu jest wyznaczenie przypadków granicznych uzyskanego przejścia granicznego z wykorzystaniem teorii prawdopodobieństwa, a następnie danych z Giełdy Papierów Wartościowych w Warszawie. Empiryczne badania wyceny opcji w uogólnionym modelu CRR potwierdzają uzyskane granice.

Słowa kluczowe: model Coxa-Rossa-Rubinsteina (model CRR), model dwumianowy, formuła Blacka-Scholesa, wycena opcji

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