A C T A U N I V E R S I T A T I S L O D Z I E N S I S
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REGULAR D-OPTIMAL SPRING BALANCE WEIGHING DESIGNS: CONSTRUCTION

Abstract. Spring balance weighing design is the model of experiment in which the result of experiment can be presented as linear combination of unknown measurements of objects with factors of this combination equal to zero or one. In the paper we consider such designs under the basic assumption that errors of measurement are uncorrelated and they have different variances, which means that measurements are taken in different conditions or with the use of different measurement equipment. We consider D-optimal designs i.e. the designs in which the determinant of the information matrix for the design attains the maximal value. We give the bounds of its value depending on whether the number of objects in experiment, is odd or even. The theoretical considerations are illustrated with examples of construction of respective designs.

Keywords: D-optimal design, spring balance weighing design.

I. INTRODUCTION

Spring balance weighing design is a model $\mathbf{y} = \mathbf{X}\mathbf{w} + \mathbf{e}$ of the experiment in that

(i) $\mathbf{y}$ is an $n \times 1$ vector of observations,

(ii) $\mathbf{X} = (x_{ij})$, $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(0,1)$,

(iii) $\mathbf{\Phi}_{n\times p}(0,1)$ is the class of $n \times p$ matrices of elements $x_{ij} = 0$ or 1,

(iv) $\mathbf{w}$ is a $p \times 1$ vector of unknown parameters,

(v) $\mathbf{e}$ is an $n \times 1$ vector of random errors,

(vi) $\mathrm{E}(\mathbf{e}) = \mathbf{0}_n$ and $\mathrm{Cov}(\mathbf{e}) = \sigma^2\mathbf{G}$, σ^2 is the constant.

In addition, assume $\mathbf{G}$ is the $n \times n$ symmetric positive definite diagonal matrix of known elements.

The problem is to estimate the vector $\mathbf{w}$. We use the normal equations $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\hat{\mathbf{w}} = \mathbf{X}'\mathbf{G}^{-1}\mathbf{y}$. Any weighing design is nonsingular if the matrix $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$

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is nonsingular. It is obvious that if $\mathbf{G}$ is the symmetric positive definite matrix of known elements then any weighing design is nonsingular if and only if $\mathbf{X}$ is of full column rank and in that case $\hat{\mathbf{w}} = \left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{G}^{-1}\mathbf{y}$, $\mathrm{Var}(\hat{\mathbf{w}}) = \sigma^2\left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)^{-1}$.

Some problems dealing with the estimation in weighing designs have been considered in the literature, see Raghavarao (1971), Banerjee (1975). Among many optimality criteria there is D-optimality. The problems related to the D-optimality of the design have been considered in Pukelsheim (1993). Some aspects of the D-optimal spring balance weighing designs for the case $\mathbf{G} = \mathbf{I}_n$, were discussed in Neubauer et al. (1998, 2000), Neubauer and Pace (2010), whereas for the case $\mathbf{G}$ is known positive definite diagonal matrix in Katulska and Przybył (2007). Under the assumption that the errors are correlated, some theoretical results on D-optimal designs are given in Masaro and Wong (2008).

Definition 1. Any nonsingular spring balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(0,1)$ with the covariance matrix of errors $\sigma^2\mathbf{G}$ is said to be D-optimal if $\det\left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)^{-1}$ is minimal. Moreover, if $\det\left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)^{-1}$ attains the lower bound then $\mathbf{X}$ is called regular D-optimal.

It worth underlining that each regular D-optimal design is D-optimal. Moreover, the minimizing of $\det\left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)^{-1}$ is equivalent to the maximizing $\det\left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)$. Thus in order to determine regular D-optimal spring balance weighing design we have to give the upper bound for $\det\left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)$.

II. THE UPPER BOUND OF $\det\left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)$

The problems of determining the regular D-optimal spring balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(0,1)$ based on the design matrix from the class $\mathbf{\Lambda}_{(n-1)\times p}(0,1)$ or from the class $\mathbf{\Xi}_{(n-2)\times p}(0,1)$ of the regular D-optimal spring balance weighing design were presented in Katulska and Przybył (2007).

For any number of objects p and any number of measurements n, we are not able to determine regular D-optimal design. That is why, in order to determine the regular D-optimal design in the class $\mathbf{\Phi}_{n\times p}(0,1)$, we consider the design matrix $\mathbf{X}_1 \in \mathbf{\Psi}_{(n-3)\times p}(0,1)$ which is regular D-optimal spring balance weighing design for p objects and $n-3$ measurements we add three measurements. So, let $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(0,1)$ be given in the following form

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{x}' \\ \mathbf{y}' \\ \mathbf{z}' \end{bmatrix}. \tag{1}$$

Suppose that the covariance matrix of errors $\sigma^2\mathbf{G}$ is similar partitioned, i.e.

$$\mathbf{G} = \begin{bmatrix} \mathbf{I}_{n-3} & \mathbf{0}_{n-3} & \mathbf{0}_{n-3} & \mathbf{0}_{n-3} \\ \mathbf{0}_{n-3}' & g_1^{-1} & 0 & 0 \\ \mathbf{0}_{n-3}' & 0 & g_2^{-1} & 0 \\ \mathbf{0}_{n-3}' & 0 & 0 & g_3^{-1} \end{bmatrix}, \quad g_1, g_2, g_3 > 0. \tag{2}$$

The problem is to formulate the relations between $\mathbf{X}_1$ and the vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$ and determine the forms of additionally weighing operations. In order to give the D-optimal design $\mathbf{X}$ in the class $\mathbf{\Phi}_{n\times p}(0,1)$ we have to give the upper bound of the determinant of the information matrix for the design, i.e. the matrix $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$. Thus for $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(0,1)$ in (1) with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is of the form (2), we calculate

$$\det\left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right) = \det\left(\mathbf{X}_1'\mathbf{X}_1 + [g_1\mathbf{x} \quad g_2\mathbf{y} \quad g_3\mathbf{z}][\mathbf{x} \quad \mathbf{y} \quad \mathbf{z}]'\right) =$$
$$\det\left(\mathbf{X}_1'\mathbf{X}_1\right)\det\left(\mathbf{I}_3 + [\mathbf{x} \quad \mathbf{y} \quad \mathbf{z}]'\left(\mathbf{X}_1'\mathbf{X}_1\right)^{-1}[g_1\mathbf{x} \quad g_2\mathbf{y} \quad g_3\mathbf{z}]\right)$$

according to the Theorem 18.1.1 in Harville (1997). We assume that $\mathbf{x}, \mathbf{y}$ and $\mathbf{z}$ are $p\times 1$ vectors of 0's and 1's and let

$$\begin{cases} \mathbf{x}'\mathbf{x} = k, & 1 \le k \le p, & \mathbf{y}'\mathbf{y} = l, & 1 \le l \le p \\ \mathbf{z}'\mathbf{z} = q, & 1 \le q \le p, & \mathbf{x}'\mathbf{y} = m, & 0 \le m \le \min\{k,l\}. \\ \mathbf{x}'\mathbf{z} = h, & 0 \le h \le \min\{k,q\}, & \mathbf{y}'\mathbf{z} = s, & 0 \le s \le \min\{l,q\} \end{cases} \tag{3}$$

The value of $\det\left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)$ depends on the number of objects p. So, the proof falls naturally into two parts.

II.1 The case: p is an odd number.

Theorem 1. (Hudelson et al. (1996)) Let p be odd. In any nonsingular spring balance weighing design $\mathbf{X}_1 \in \mathbf{\Psi}_{(n-3)\times p}(0,1)$

$$\det\left(\mathbf{X}_1'\mathbf{X}_1\right) \leq (p+1)\left(\frac{(p+1)(n-3)}{4p}\right)^p \tag{4}$$

and the equality in (4) is fulfilled if and only if

$$\mathbf{X}_1'\mathbf{X}_1 = t\left(\mathbf{I}_p + \mathbf{1}_p\mathbf{1}_p'\right) \text{ and } t = \frac{(p+1)(n-3)}{4p} \text{ is integer.} \tag{5}$$

From Theorem 1 we obtain

$$\det\left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right) \leq (p+1)\left(\frac{(p+1)(n-3)}{4p}\right)^p \det\left(\mathbf{I}_3 + [\mathbf{x}\ \mathbf{y}\ \mathbf{z}]'\left(\mathbf{X}_1'\mathbf{X}_1\right)^{-1}[g_1\mathbf{x}\ g_2\mathbf{y}\ g_3\mathbf{z}]\right)$$

and the equality is attained if and only if the conditions in (5) hold. Because $\left(\mathbf{X}_1'\mathbf{X}_1\right)^{-1} = \frac{1}{t}\left(\mathbf{I}_p - \frac{1}{p+1}\mathbf{1}_p\mathbf{1}_p'\right)$ then we obtain

$\det\left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right) \leq (p+1)\left(\frac{(p+1)(n-3)}{4p}\right)^p \det(\mathbf{\Omega})$, where

$\mathbf{\Omega} = \mathbf{I}_3 + \frac{1}{t}[\mathbf{x} \quad \mathbf{y} \quad \mathbf{z}]'\left(\mathbf{I}_p - \frac{1}{p+1}\mathbf{1}_p\mathbf{1}_p'\right)[g_1\mathbf{x} \quad g_2\mathbf{y} \quad g_3\mathbf{z}]$. From (3) it follows that

$$
\begin{aligned}
\det(\boldsymbol{\Omega}) = {} & \\
& \left(1+\frac{g_1}{t}\left(k-\frac{1}{p+1}k^2\right)\right)\left(1+\frac{g_2}{t}\left(l-\frac{1}{p+1}l^2\right)\right)\left(1+\frac{g_3}{t}\left(q-\frac{1}{p+1}q^2\right)\right)+ \\
& \frac{2g_1g_2g_3}{t^3}\left(m-\frac{1}{p+1}kl\right)\left(h-\frac{1}{p+1}kq\right)\left(s-\frac{1}{p+1}lq\right)- \\
& \frac{g_1g_2}{t^2}\left(1+\frac{g_3}{t}\left(q-\frac{1}{p+1}q^2\right)\right)\left(m-\frac{1}{p+1}kl\right)^2- \\
& \frac{g_1g_3}{t^2}\left(1+\frac{g_2}{t}\left(l-\frac{1}{p+1}l^2\right)\right)\left(h-\frac{1}{p+1}kq\right)^2- \\
& \frac{g_2g_3}{t^2}\left(1+\frac{g_1}{t}\left(k-\frac{1}{p+1}k^2\right)\right)\left(s-\frac{1}{p+1}lq\right)^2.
\end{aligned} \tag{6}
$$

As we want to maximize $\det\left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)$, we should determine the maximum value of $k-(p+1)^{-1}k^2$, $l-(p+1)^{-1}l^2$ and $q-(p+1)^{-1}q^2$ and the minimum value of $\left(m-(p+1)^{-1}kl\right)^2$, $\left(h-(p+1)^{-1}kq\right)^2$ and $\left(s-(p+1)^{-1}lq\right)^2$ simultaneously. Therefore we have

$$
\begin{aligned}
\det(\boldsymbol{\Omega}) \le {} & \left(1+\frac{g_1(p+1)}{4t}\right)\left(1+\frac{g_2(p+1)}{4t}\right)\left(1+\frac{g_3(p+1)}{4t}\right)+ \\
& \frac{2g_1g_2g_3}{t^3}\left(m-\frac{p+1}{4}\right)\left(h-\frac{p+1}{4}\right)\left(s-\frac{p+1}{4}\right)- \\
& \frac{g_1g_2}{t^2}\left(1+\frac{g_3(p+1)}{4t}\right)\left(m-\frac{p+1}{4}\right)^2- \\
& \frac{g_1g_3}{t^2}\left(1+\frac{g_2(p+1)}{4t}\right)\left(h-\frac{p+1}{4}\right)^2-\frac{g_2g_3}{t^2}\left(1+\frac{g_1(p+1)}{4t}\right)\left(s-\frac{p+1}{4}\right)^2.
\end{aligned} \tag{7}
$$

The equality in (7) is attained if and only if $k = l = q = 0.5(p+1)$. The value of $\det(\boldsymbol{\Omega})$ depends on the value of p. Let us consider two cases $p+1 \equiv 0 \bmod(4)$ and $p+3 \equiv 0 \bmod(4)$.

II.1.1.If $p+1 \equiv 0 \bmod(4)$, then minimal value of $(m-0.25(p+1))^2$, $(h-0.25(p+1))^2$ and $(s-0.25(p+1))^2$ is zero for $m = h = s = 0.25(p+1)$. We conclude that

$\det(\boldsymbol{\Omega}) \le \left(1+\frac{g_1(p+1)}{4t}\right)\left(1+\frac{g_2(p+1)}{4t}\right)\left(1+\frac{g_3(p+1)}{4t}\right) = \kappa$, where

$\kappa = \left(1+\frac{g_1 p}{n-3}\right)\left(1+\frac{g_2 p}{n-3}\right)\left(1+\frac{g_3 p}{n-3}\right)$.

Thus $\det(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}) \le (p+1)\left(\frac{(p+1)(n-3)}{4p}\right)^p \kappa$.

II.1.2.If $p+3 \equiv 0 \bmod(4)$, then the minimal value of $(m-0.25(p+1))^2$, $(h-0.25(p+1))^2$ and $(s-0.25(p+1))^2$ is 0.25 for $m = h = s = 0.25(p-1)$ or $m = h = s = 0.25(p+3)$. In that case $\det(\boldsymbol{\Omega}) \le$ $\kappa - \eta + \frac{2g_1g_2g_3}{t^3}\left(m-\frac{p+1}{4}\right)\left(h-\frac{p+1}{4}\right)\left(s-\frac{p+1}{4}\right)$, where

$\eta = \frac{g_1g_2}{4t^2}\left(1+\frac{g_3 p}{n-3}\right)+\frac{g_1g_3}{4t^2}\left(1+\frac{g_2 p}{n-3}\right)+\frac{g_2g_3}{4t^2}\left(1+\frac{g_1 p}{n-3}\right)$. In any case

$$\begin{cases} m = h = s = 0.25(p-1), \\ m = h = 0.25(p+3) \quad \text{and} \quad s = 0.25(p-1), \\ m = s = 0.25(p+3) \quad \text{and} \quad h = 0.25(p-1) \quad \text{or} \\ s = h = 0.25(p+3) \quad \text{and} \quad m = 0.25(p-1) \end{cases}$$

we obtain $(m-0.25(p+1))(h-0.25(p+1))(s-0.25(p+1)) = -\frac{1}{8}$ and under above assumptions $\det(\boldsymbol{\Omega}) \le \kappa - \eta - \frac{g_1g_2g_3}{4t^3}$. There is a consequence that

$\det(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}) \le (p+1)\left(\frac{(p+1)(n-3)}{4p}\right)^p \left(\kappa - \eta - \frac{g_1g_2g_3}{4t^3}\right)$. In any case

$$\begin{cases} m = h = s = 0.25(p+3), \\ m = h = 0.25(p-1) \quad \text{and} \quad s = 0.25(p+3), \\ m = s = 0.25(p-1) \quad \text{and} \quad h = 0.25(p+3) \quad \text{or} \\ s = h = 0.25(p-1) \quad \text{and} \quad m = 0.25(p+3) \end{cases}$$

we conclude that $(m - 0.25(p+1))(h - 0.25(p+1))(s - 0.25(p+1)) = \frac{1}{8}$ and hence under above assumptions that $\det(\mathbf{\Omega}) \le \kappa - \eta + \frac{g_1 g_2 g_3}{4t^3}$ and finally that $\det(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}) \le (p+1)\left(\frac{(p+1)(n-3)}{4p}\right)^p \left(\kappa - \eta + \frac{g_1 g_2 g_3}{4t^3}\right)$.

Under above assumptions $\kappa - \eta - \frac{g_1 g_2 g_3}{4t^3} < \kappa - \eta + \frac{g_1 g_2 g_3}{4t^3}$, so for the case $p + 3 \equiv 0 \bmod(4)$, the upper bound equals $(p+1)\left(\frac{(p+1)(n-3)}{4p}\right)^p \left(\kappa - \eta + \frac{g_1 g_2 g_3}{4t^3}\right)$.

II.2. The case: p is an even number.

Theorem 2. (Neubauer et al., 1997) If p is even and $\mathbf{X}_1 \in \mathbf{\Psi}_{(n-3)\times p}(0,1)$ then

$$\det(\mathbf{X}_1'\mathbf{X}_1) \le (p+1)\left(\frac{(p+2)(n-3)}{4(p+1)}\right)^p \tag{8}$$

and the equality in (8) holds if, and only if

$$\mathbf{X}_1'\mathbf{X}_1 = c(\mathbf{I}_p + \mathbf{1}_p\mathbf{1}_p') \text{ and } c = \frac{(p+2)(n-3)}{4(p+1)} \text{ is integer.} \tag{9}$$

Let us apply the formulas given in Theorem 2 to compute the determinant of the information matrix. So, we have $\det(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}) \le (p+1)\left(\frac{(p+2)(n-3)}{4(p+1)}\right)^p \det\left(\mathbf{I}_3 + [\mathbf{x}\ \mathbf{y}\ \mathbf{z}]'(\mathbf{X}_1'\mathbf{X}_1)^{-1}[g_1\mathbf{x}\ g_2\mathbf{y}\ g_3\mathbf{z}]\right)$ and the equality is attained if and only if (9) holds. Because $(\mathbf{X}_1'\mathbf{X}_1)^{-1} = \frac{1}{c}\left(\mathbf{I}_p - \frac{1}{p+1}\mathbf{1}_p\mathbf{1}_p'\right)$ then we obtain

$\det\left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right) \leq (p+1)\left(\frac{(p+2)(n-3)}{4(p+1)}\right)^{p}\det(\boldsymbol{\Gamma})$, where $\boldsymbol{\Gamma} = \mathbf{I}_3 + \frac{1}{c}[\mathbf{x} \quad \mathbf{y} \quad \mathbf{z}]'\left(\mathbf{I}_p - \frac{1}{p+1}\mathbf{1}_p\mathbf{1}_p'\right)[g_1\mathbf{x} \quad g_2\mathbf{y} \quad g_3\mathbf{z}]$. Applying the formulas given in (3) we are now in a position to show

$$\begin{aligned}
&\det(\boldsymbol{\Gamma}) = \\
&\left(1+\frac{g_1}{c}\left(k-\frac{1}{p+1}k^2\right)\right)\left(1+\frac{g_2}{c}\left(l-\frac{1}{p+1}l^2\right)\right)\left(1+\frac{g_3}{c}\left(q-\frac{1}{p+1}q^2\right)\right)+ \\
&\qquad \frac{2g_1g_2g_3}{c^3}\left(m-\frac{1}{p+1}kl\right)\left(h-\frac{1}{p+1}kq\right)\left(s-\frac{1}{p+1}lq\right)- \\
&\qquad \frac{g_1g_2}{c^2}\left(1+\frac{g_3}{c}\left(q-\frac{1}{p+1}q^2\right)\right)\left(m-\frac{1}{p+1}kl\right)^2- \\
&\qquad \frac{g_1g_3}{c^2}\left(1+\frac{g_2}{c}\left(l-\frac{1}{p+1}l^2\right)\right)\left(h-\frac{1}{p+1}kq\right)^2- \\
&\qquad \frac{g_2g_3}{c^2}\left(1+\frac{g_1}{c}\left(k-\frac{1}{p+1}k^2\right)\right)\left(s-\frac{1}{p+1}lq\right)^2.
\end{aligned} \tag{10}$$

As we want to maximize $\det\left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)$, we should determine the maximum value of $k-(p+1)^{-1}k^2$, $l-(p+1)^{-1}l^2$ and $q-(p+1)^{-1}q^2$ and the minimum value of $\left(m-(p+1)^{-1}kl\right)^2$, $\left(h-(p+1)^{-1}kq\right)^2$ and $\left(s-(p+1)^{-1}lq\right)^2$ simultaneously. Then we obtain

$$\begin{aligned}
\det(\boldsymbol{\Gamma}) \leq &\left(1+\frac{g_1p(p+2)}{4c(p+1)}\right)\left(1+\frac{g_2p(p+2)}{4c(p+1)}\right)\left(1+\frac{g_3p(p+2)}{4c(p+1)}\right)+ \\
&\frac{2g_1g_2g_3}{c^3}\left(m-\frac{1}{p+1}kl\right)\left(h-\frac{1}{p+1}qk\right)\left(s-\frac{1}{p+1}lq\right)- \\
&\quad \frac{g_1g_2}{c^2}\left(1+\frac{g_3p(p+2)}{4c(p+1)}\right)\left(m-\frac{1}{p+1}kl\right)^2-
\end{aligned} \tag{11}$$

$$\frac{g_1g_3}{c^2}\left(1+\frac{g_2p(p+2)}{4c(p+1)}\right)\left(h-\frac{1}{p+1}qk\right)^2 -$$

$$\frac{g_2g_3}{c^2}\left(1+\frac{g_1p(p+2)}{4c(p+1)}\right)\left(s-\frac{1}{p+1}lq\right)^2.$$

The equality in (11) holds if and only if $k, l, q \in \left\{\frac{p}{2}, \frac{p+2}{2}\right\}$. Let us consider two cases $p \equiv 0 \bmod(4)$ and $p+2 \equiv 0 \bmod(4)$.

II.2.1.If $p \equiv 0 \bmod(4)$, then the minimal value of $\left(m-(p+1)^{-1}kl\right)^2$ equals $\frac{p^2}{16(p+1)^2}$ in any case $\begin{cases} k = l = 0.5p, \\ k = 0.5p \quad \text{and} \quad l = 0.5(p+2) \\ k = 0.5(p+2) \quad \text{and} \quad l = 0.5p \end{cases}$. Analogous considerations lead to the result that minimal value of $\left(h-(p+1)^{-1}qk\right)^2$ is $\frac{p^2}{16(p+1)^2}$ for any case $\begin{cases} k = q = 0.5p, \\ k = 0.5p \quad \text{and} \quad q = 0.5(p+2) \\ k = 0.5(p+2) \quad \text{and} \quad q = 0.5p \end{cases}$. Moreover, minimal value of $\left(s-(p+1)^{-1}lq\right)^2$ is $\frac{p^2}{16(p+1)^2}$ for any case $\begin{cases} l = q = 0.5p, \\ l = 0.5p \quad \text{and} \quad q = 0.5(p+2) \\ l = 0.5(p+2) \quad \text{and} \quad q = 0.5p \end{cases}$. It is easily seen that

$$\det(\mathbf{\Gamma}) \le \left(1+\frac{g_1p(p+2)}{4c(p+1)}\right)\left(1+\frac{g_2p(p+2)}{4c(p+1)}\right)\left(1+\frac{g_3p(p+2)}{4c(p+1)}\right)+$$

$$\frac{2g_1g_2g_3}{c^3}\left(m-\frac{1}{p+1}kl\right)\left(h-\frac{1}{p+1}qk\right)\left(s-\frac{1}{p+1}lq\right)-$$

$$\frac{p^2}{16c^2(p+1)^2}\left(g_1g_2+g_1g_3+g_2g_3+\frac{3g_1g_2g_3p(p+2)}{4c(p+1)}\right).$$

For $m=h=s=0.25p$ we consider cases

$$\begin{cases} k=l=q=0.5p, \\ k=l=0.5p \quad \text{and} \quad q=0.5(p+2) \\ k=q=0.5p \quad \text{and} \quad l=0.5(p+2) \\ l=q=0.5p \quad \text{and} \quad k=0.5(p+2) \end{cases}.$$

For any combination of k, l, q we obtain

$\left(m-\frac{1}{p+1}kl\right)\left(h-\frac{1}{p+1}qk\right)\left(s-\frac{1}{p+1}lq\right)=\frac{p^3}{64(p+1)^3}$. Therefore

$\det(\mathbf{\Gamma}) \le \kappa - \frac{p^2\chi}{(p+2)^2(n-3)^2} + \frac{2g_1g_2g_3p^3}{(p+2)^3(n-3)^3}$, where

$\chi = g_1g_2 + g_1g_3 + g_2g_3 + \frac{3g_1g_2g_3p}{n-3}$. So, we have

$$\det\left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right) \le (p+1)\left(\frac{(p+2)(n-3)}{4(p+1)}\right)^p \left(\kappa + \frac{2g_1g_2g_3p^3}{(p+2)^3(n-3)^3} - \frac{p^2\chi}{(p+2)^2(n-3)^2}\right).$$

II.2.2.If $p+2 \equiv 0 \bmod(4)$, then the minimal value of $\left(m-(p+1)^{-1}kl\right)^2$ equals $\frac{(p+2)^2}{16(p+1)^2}$ for any case $\begin{cases} k=l=0.5(p+2), \\ k=0.5p \quad \text{and} \quad l=0.5(p+2) \\ k=0.5(p+2) \quad \text{and} \quad l=0.5p \end{cases}$. Analogous considerations lead to the result that the minimal value of $\left(h-(p+1)^{-1}qk\right)^2$ is $\frac{(p+2)^2}{16(p+1)^2}$ for $\begin{cases} k=q=0.5(p+2), \\ k=0.5p \quad \text{and} \quad q=0.5(p+2) \\ k=0.5(p+2) \quad \text{and} \quad q=0.5p \end{cases}$. For $\begin{cases} l=q=0.5(p+2), \\ l=0.5p \quad \text{and} \quad q=0.5(p+2) \\ l=0.5(p+2) \quad \text{and} \quad q=0.5p \end{cases}$ the minimal value of $\left(s-(p+1)^{-1}lq\right)^2$ is $\frac{(p+2)^2}{16(p+1)^2}$.

Thus $\det(\mathbf{\Gamma}) \le \kappa + \frac{2g_1g_2g_3}{c^3}\left(m - \frac{1}{p+1}kl\right)\left(h - \frac{1}{p+1}qk\right)\left(s - \frac{1}{p+1}lq\right) -$

$$\frac{(p+2)^2}{16c^2(p+1)^2}\left(g_1g_2 + g_1g_3 + g_2g_3 + \frac{3g_1g_2g_3p(p+2)}{4c(p+1)}\right).$$

For $m = h = s = 0.25(p+2)$ we consider cases

$\begin{cases} k = l = q = 0.5(p+2), \\ k = l = 0.5(p+2) \quad \text{and} \quad q = 0.5p \\ k = q = 0.5(p+2) \quad \text{and} \quad l = 0.5p \\ l = q = 0.5(p+2) \quad \text{and} \quad k = 0.5p \end{cases}$. For each combination of k, l, q we obtain

$\left(m - \frac{1}{p+1}kl\right)\left(h - \frac{1}{p+1}qk\right)\left(s - \frac{1}{p+1}lq\right) = \frac{(p+2)^3}{64(p+1)^3}$. Therefore

$\det(\mathbf{\Gamma}) \le \kappa - \frac{\chi}{(n-3)^2} - \frac{2g_1g_2g_3}{(n-3)^3}$. We have

$\det\left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right) \le (p+1)\left(\frac{(p+2)(n-3)}{4(p+1)}\right)^p \left(\kappa - \frac{\chi}{(n-3)^2} - \frac{2g_1g_2g_3}{(n-3)^3}\right)$. On the basis of the results given above we can formulate Theorem

Theorem 3. In any nonsingular spring balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(0,1)$ in (1) with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is of the form (2), we obtain

1) if p is an odd number, $k = l = q = 0.5(p+1)$ and

1.1) if $p+1 \equiv 0 \bmod(4)$ then $\det\left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right) \le \alpha\kappa$ and the equality is attained if and only if $m = h = s = 0.25(p+1)$,

1.2) if $p+3 \equiv 0 \bmod(4)$ then $\det\left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right) \le \alpha\left(\kappa - \eta + \frac{g_1g_2g_3}{4t^3}\right)$ and the equality is attained if and only if $\begin{cases} m = h = s = 0.25(p+3) \\ m = h = 0.25(p-1) \text{ and } s = 0.25(p+3) \\ m = s = 0.25(p-1) \text{ and } h = 0.25(p+3) \\ s = h = 0.25(p-1) \text{ and } m = 0.25(p+3) \end{cases}$.

2) If p is an even number and

2.1) if $p \equiv 0 \bmod(4)$ then

$\det\left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right) \le \beta\left(\kappa - \frac{p^2\chi}{(p+2)^2(n-3)^2} + \frac{2g_1g_2g_3p^3}{(p+2)^3(n-3)^3}\right)$ and the equality is attained if and only if $m = h = s = \frac{p}{4}$ and $\begin{cases} k = l = q = 0.5p \\ k = l = 0.5p \quad \text{and} \quad q = 0.5(p+2) \\ k = q = 0.5p \quad \text{and} \quad l = 0.5(p+2) \\ l = q = 0.5p \quad \text{and} \quad k = 0.5(p+2) \end{cases}$,

2.2) if $p + 2 \equiv 0 \bmod(4)$ then

$$\det\left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right) \le \beta\left(\kappa - \frac{\chi}{(n-3)^2} - \frac{2g_1g_2g_3}{(n-3)^3}\right) \text{ and}$$

the equality is attained if and only if $m = h = s = \frac{p+2}{4}$ and

$$\begin{cases} k = l = q = 0.5(p+2) \\ k = l = 0.5(p+2) \quad \text{and} \quad q = 0.5p \\ k = q = 0.5(p+2) \quad \text{and} \quad l = 0.5p \\ l = q = 0.5(p+2) \quad \text{and} \quad k = 0.5p \end{cases},$$

where $\alpha = (p+1)\left(\frac{(p+1)(n-3)}{4p}\right)^p$, $\beta = (p+1)\left(\frac{(p+2)(n-3)}{4(p+1)}\right)^p$,

$$\eta = \frac{g_1g_2}{4t^2}\left(1 + \frac{g_3p}{n-3}\right) + \frac{g_1g_3}{4t^2}\left(1 + \frac{g_2p}{n-3}\right) + \frac{g_2g_3}{4t^2}\left(1 + \frac{g_1p}{n-3}\right),$$

$$\kappa = \left(1 + \frac{g_1p}{n-3}\right)\left(1 + \frac{g_2p}{n-3}\right)\left(1 + \frac{g_3p}{n-3}\right),\ \chi = g_1g_2 + g_1g_3 + g_2g_3 + \frac{3g_1g_2g_3p}{n-3},$$

$$\begin{cases} \mathbf{x}'\mathbf{x} = k, \ 1 \le k \le p, & \mathbf{y}'\mathbf{y} = l, \ 1 \le l \le p, & \mathbf{z}'\mathbf{z} = q, \ 1 \le q \le p, \\ \mathbf{x}'\mathbf{y} = m, \ 0 \le m \le \min\{k,l\}, & \mathbf{x}'\mathbf{z} = h, \ 0 \le h \le \min\{k,q\}, & \mathbf{y}'\mathbf{z} = s, \ 0 \le s \le \min\{l,q\} \end{cases}$$

Definition 2. Any nonsingular spring balance weighing is regular D-optimal design with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is of the form (2), if $\det\left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)$ attains the upper bound given in Theorem 3.

III. CONSTRUCTION

The method of construction of regular D-optimal spring balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(0,1)$ in (1) with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is of the form (2), is given in the next Theorem.

Theorem 4. Any nonsingular spring balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(0,1)$ in (1) with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is of (2), is regular D-optimal

1) if p is an odd number and $\mathbf{X}_1'\mathbf{X}_1 = t\left(\mathbf{I}_p + \mathbf{1}_p\mathbf{1}_p'\right)$ and moreover

1.1) $p+1 \equiv 0\,\mathrm{mod}(4)$ and $k = l = q = 0.25(p+1)$ if and only if $m = h = s = 0.25(p+1)$,

1.2) $p+3 \equiv 0\,\mathrm{mod}(4)$ if and only if $\begin{cases} m = h = s = 0.25(p+3) \\ m = h = 0.25(p-1) \text{ and } s = 0.25(p+3) \\ m = s = 0.25(p-1) \text{ and } h = 0.25(p+3) \\ s = h = 0.25(p-1) \text{ and } m = 0.25(p+3) \end{cases}$,

2) if p is an even number and $\mathbf{X}_1'\mathbf{X}_1 = c\left(\mathbf{I}_p + \mathbf{1}_p\mathbf{1}_p'\right)$ and

2.1) $p \equiv 0\,\mathrm{mod}(4)$ if and only if $m = h = s = \frac{p}{4}$ and

$$\begin{cases} k = l = q = 0.5p \\ k = l = 0.5p \quad \text{and} \quad q = 0.5(p+2) \\ k = q = 0.5p \quad \text{and} \quad l = 0.5(p+2) \\ l = q = 0.5p \quad \text{and} \quad k = 0.5(p+2) \end{cases},$$

2.2) $p+2 \equiv 0\,\mathrm{mod}(4)$ if and only if $m = h = s = \frac{p+2}{4}$ and

$$\begin{cases} k = l = q = 0.5(p+2) \\ k = l = 0.5(p+2) \quad \text{and} \quad q = 0.5p \\ k = q = 0.5(p+2) \quad \text{and} \quad l = 0.5p \\ l = q = 0.5(p+2) \quad \text{and} \quad k = 0.5p \end{cases}.$$

Proof. The Theorem is a direct consequence of above considerations.

In order to construct regular D-optimal spring balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(0,1)$ in (1) with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is of (2), we have to form the matrix $\mathbf{X}_1 \in \mathbf{\Psi}_{(n-3)\times p}(0,1)$ and, as the next step add three measurements of the form given in Theorem 3. Some methods of construction of $\mathbf{X}_1 \in \mathbf{\Psi}_{(n-3)\times p}(0,1)$ are given in Ceranka et al. (2009)**.** Let us consider the experiment in which we determine unknown measurements of $p = 6$ objects in $n = 10$ measurements. Thus we have to determine the regular D-optimal spring balance weighing design in $\mathbf{X} \in \mathbf{\Phi}_{10\times 6}(0,1)$. We consider the matrix $\mathbf{X}_1 \in \mathbf{\Psi}_{7\times 6}(0,1)$ of the regular D-optimal spring balance weighing design in the form $\mathbf{X}_1 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}$. Next, we add tree measurements $\mathbf{x} = [1 \;\; 1 \;\; 1 \;\; 1 \;\; 0 \;\; 0]'$, $\mathbf{y} = [1 \;\; 1 \;\; 0 \;\; 0 \;\; 1 \;\; 1]'$, $\mathbf{z} = [0 \;\; 0 \;\; 1 \;\; 1 \;\; 1 \;\; 1]'$ and finally we obtain $\mathbf{X}' = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}$, $\det\left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right) = 2816$.

IV. DISCUSSION

For any number of objects p and measurements n it is not possible to construct the regular D-optimal spring balance weighing design. Some methods of construction of D-optimal design in the class $\mathbf{\Phi}_{n\times p}(0,1)$ based on the design matrix $\mathbf{X}_1$ which is regular D-optimal in the class $\mathbf{\Xi}_{(n-2)\times p}(0,1)$) or $\mathbf{\Lambda}_{(n-1)\times p}(0,1)$ are presented in Ceranka et al. (2009). In the present paper we give new construction method of the design matrix $\mathbf{X} \in \mathbf{\Phi}_{n \times p}(0,1)$ which is based on the matrix $\mathbf{X}_1$, regular D-optimal in the class $\mathbf{\Psi}_{(n-3) \times p}(0,1)$. For that reason the main result of this paper is widening the class of matrices in which we determine the regular D-optimal spring balance weighing design.

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REGULARNE D-OPTYMALNE UKŁADY WAGOWE: KONSTRUKCJA

Sprężynowe układy wagowe są to takie układy doświadczalne, w których wynik eksperymentu jest liniową kombinacją nieznanych miar obiektów ze współczynnikami tej kombinacji równymi zero lub jeden. W pracy przedstawiamy te układy przy założeniu, że błędy pomiarów są nieskorelowane i maja różne wariancje, co może oznaczać, że pomiary zostały wykonane w różnych miejscach lub przy użyciu różnej aparatury. Rozważamy D-optymalne układy wagowe, tzn. układy w których wyznacznik macierzy informacji dla układu jest maksymalny. Podajemy oszacowanie jego wartości w zależności od tego, czy liczba obiektów biorących udział w doświadczeniu jest parzysta czy nieparzysta. Rozważania teoretyczne zostały zilustrowane przykładami konstrukcji odpowiednich układów.