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ON THE CATEGORY OF EQ-ALGEBRAS

Abstract

In this paper, we studied the category of EQ-algebras and showed that it is complete, but it is not cocomplete, in general. We proved that multiplicatively relative EQ-algebras have coequlizers and we calculated coproduct and pushout in a special case. Also, we constructed a free EQ-algebra on a singleton.

Keywords: EQ-algebras, free EQ-algebras, category theory, universal algebra, variety.

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1. Introduction

Fuzzy type theory was developed as a counterpart of the classical higher-order logic. Since the algebra of truth values is no longer a residuated lattice, a specific algebra called an EQ-algebra was proposed by Novák [16, 17, 18]. The main primitive operations of EQ-algebras are meet, multiplication, and fuzzy equality. Implication is derived from fuzzy equality and it is not a residuation with respect to multiplication. Consequently, EQ-algebras overlap with residuated lattices but are not identical to them. Novák and De Baets in [18] introduced various kinds of EQ-algebras.

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Novák and El-Zekey in [14], proved that the class of EQ-algebras is a variety. In [19], J. Yang and X. Zhang introduced a new class of EQ-algebras, i.e., multiplicatively relative EQ -algebras. Also, they defined the notion of a filter generated by a nonempty subset.

Category theory is a powerful language, or conceptual framework, allowing us to see the universal components of a family of structures of a given kind, and how structures of different kinds are interrelated. Category theory is both an interesting object of philosophical study, and a potentially powerful formal tool for philosophical investigations of concepts such as space, system, and even truth. In [1], it has shown that the variety algebras, together with its homomorphisms, form a category and also, every non-trivial variety of algebras contains a free object on a given set. The category of some algebraic structures are studied. It is well known that the category of groups, rings, modules, and vector spaces are complete and cocomplete. The category of some logical algebraic structures have been studied well, too. For example, it has been proved that the category of Boolean algebras is isomorphic to the subcategory of rings named as Boolean rings [8]. Also, it is known that the category of MV-algebras is equivalent to that of unital lattice ordered groups (ℓ-groups). This equivalence, which depends in large part on the natural algebraic addition of MV-algebras [9] has been an essential tool in the study of MV-algebras. The categories of some other algebraic structures such as BCK(BCI)-algebras, BL-algebras, soft and rough sets have been studied, too (see [5, 6, 7, 10, 11, 12]).

With these inspirations, we studied the category of EQ-algebras and showed that it is complete, but it is not cocomplete, in general. We proved that multiplicatively relative EQ-algebras have coequlizers and we calculated coproduct and pushout in a special case. Also, we constructed a free EQ-algebra on a singleton.

2. Preliminaries

In this section, we recollect some definitions and results which will be used in this paper (See [13, 14, 19]).

An *EQ-algebra* is an algebraic structure $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ of type $(2, 2, 2, 0)$, where for any $a, b, c, d \in E$, the following statements hold:

($E1$) $(E, \wedge, 1)$ is a $\wedge$-semilattice with top element 1. For any $a, b \in E$, we set $a \leqslant b$ if and only if $a \wedge b = a$.

$(E2)$ $(E, \otimes, 1)$ is a (commutative) monoid and $\otimes$ is isotone with respect to $\leqslant$.

$(E3)$ $a \sim a = 1$.

$(E4)$ $((a \wedge b) \sim c) \otimes (d \sim a) \leqslant (c \sim (d \wedge b))$.

$(E5)$ $(a \sim b) \otimes (c \sim d) \leqslant (a \sim c) \sim (b \sim d)$.

$(E6)$ $(a \wedge b \wedge c) \sim a \leqslant (a \wedge b) \sim a$.

$(E7)$ $a \otimes b \leqslant a \sim b$.

The operations " $\wedge$ ", " $\otimes$ ", and " $\sim$ " are called *meet*, *multiplication*, and *fuzzy equality*, respectively. For any $a, b \in E$, we defined the binary operation *implication* on E by, $a \to b = (a \wedge b) \sim a$. Also, in particular $1 \to a = 1 \sim a = \tilde{a}$.

Let $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ be an EQ-algebra and $a, b, c, d \in E$ are arbitrary elements. Then $\mathcal{E}$ is called *separated*, if $a \sim b = 1$, then $a = b$, *good*, if $\tilde{a} = a$, *residuated*, where $(a \otimes b) \wedge c = a \otimes b$ if and only if $a \wedge ((b \wedge c) \sim b) = a$, *lattice-ordered EQ-algebra*, if it has a lattice reduct,[1], *lattice EQ-algebra (or ℓEQ-algebra).* if it is a lattice-ordered EQ-algebra and

$$((a \vee b) \sim c) \otimes (d \sim a) \leqslant ((d \vee b) \sim c),$$

multiplicatively relative EQ-algebra, if $a \sim b \leqslant (a \otimes c) \sim (b \otimes c)$.

Proposition 2.1 ([14]). Let $\mathcal{E}$ be an EQ-algebra. Then for any $a, b, c \in E$, $\mathcal{E}$ is residuated if and only if $\mathcal{E}$ is good and $(a \otimes b) \to c \leqslant a \to (b \to c)$.

Proposition 2.2 ([19]). Each linear and residuated EQ-algebra is multiplicatively relative.

Proposition 2.3 ([14]). Let $\mathcal{E}$ be an EQ-algebra. Then, for all $a, b, c \in E$ (i) $a \sim b = b \sim a$ and (ii) $a \otimes (a \sim b) = \tilde{b}$.

Let $\mathcal{E} = (E, \otimes_{\mathcal{E}}, \wedge_{\mathcal{E}}, \sim_{\mathcal{E}}, 1_{\mathcal{E}})$ and $\mathcal{G} = (G, \otimes_{\mathcal{G}}, \wedge_{\mathcal{G}}, \sim_{\mathcal{G}}, 1_{\mathcal{G}})$ be two EQ-algebras. A map $f : E \to G$ is an EQ-homomorphism, if for any $a, b \in E$, $f(a \otimes_{\mathcal{E}} b) = f(a) \otimes_{\mathcal{G}} f(b)$, $f(a \wedge_{\mathcal{E}} b) = f(a) \wedge_{\mathcal{G}} f(b)$, $f(a \sim_{\mathcal{E}} b) = f(a) \sim_{\mathcal{G}} f(b)$,

[1] Given an algebra $\langle E, F\rangle$, where F is a set of operations on E and $F' \subseteq F$, then the algebra $\langle E, F'\rangle$ is called the F'-reduct of $\langle E, F\rangle$.

and $f(1_{\mathcal{E}}) = 1_{\mathcal{G}}$. A nonempty subset F of $\mathcal{E}$ is called a filter of $\mathcal{E}$ if for any $a, b, c \in E$: (F1) if $a \in F$ and $a \leqslant b$, then $b \in F$, (F2) if $a, b \in F$, then $a \otimes b \in F$, (F3) if $a \sim b \in F$, then $(a \otimes c) \sim (b \otimes c) \in F$.

Proposition 2.4 ([19]). Let $f : \mathcal{E} \to \mathcal{G}$ be an EQ-homomorphism. If $\mathcal{G}$ is separated, then $ker(f) = \{a \in E | f(a) = 1\}$ is a filter of $\mathcal{E}$.

Proposition 2.5 ([19]).

(i) Let $\{F_i | i \in I\}$ be a family of filters of an EQ-algebra $\mathcal{E}$. Then $\bigcap\limits_{i \in I} F_i$ is a filter of $\mathcal{E}$.

(ii) Let $\{\mathcal{E}_i | i \in I\}$ be a family of EQ-algebras and F_i be a filter of E_i for any $i \in I$. Then $F = \prod F_i$ is a filter of $\mathcal{E} = \prod \mathcal{E}_i$.

Theorem 2.6 ([14]). *Let F be a filter of EQ-algebra $\mathcal{E}$. A binary relation $\approx_F$ on E which is defined by $a \approx_F b$ if and only if $a \sim b \in F$, is a congruence relation on $\mathcal{E}$ and $\mathcal{E}/F = (E/F, \wedge_F, \otimes_F, \sim_F, F)$ is a separated EQ-algebra, where, for any $a, b \in E$, we have,*

$$[a] \wedge_F [b] = [a \wedge b], \quad [a] \otimes_F [b] = [a \otimes b], \quad [a] \sim_F [b] = [a \sim b].$$

A binary relation $\leqslant_F$ on E/F which is defined by $[a] \leqslant_F [b]$ if and only if $[a] \wedge_F [b] = [a]$ is a partial order on E/F and for any $[a], [b] \in \mathcal{E}/F$, $[a] \leqslant_F [b]$ if and only if $a \to b \in F$.

Theorem 2.7. [19] *Let X be a nonempty subset of a multiplicatively relative EQ-algebra $\mathcal{E}$. Then*

$$\langle X \rangle = \{a \in E \mid \exists n \in \mathbb{Z}^+, x_i \in X \ \ s.t. \ \ x_1 \otimes x_2 \otimes \cdots \otimes x_n \leqslant a\}$$

is a generated filter by X.

Now, we present some definitions and results in category theory which will be used in this paper (see [1, 2, 3, 8, 15]). A *category* consists of *objects*: $A, B, X, \ldots$ and *morphisms* (arrows): $f, g, h, \ldots$. For each morphism f, there are given objects $dom(f), cod(f)$ called the *domain* and *codomain* of f. We write $f : A \to B$ to indicate that $A = dom(f)$ and $B = cod(f)$. Given morphisms $f : A \to B$ and $g : B \to C$, where $cod(f) = dom(g)$ there is given a morphism $g \circ f : A \to C$ called the composite of f and g. For each object A, there is given a morphism $id_A : A \to A$ called the *identity* morphism of A. These data are required to satisfy the laws, for all $f : A \to B$, $g : B \to C$, $h : C \to D$, $h \circ (g \circ f) = (h \circ g) \circ f$ and $f \circ id_A = f = id_B \circ f$.

A category $\mathcal{C}$ is called *locally small* if for all objects X, Y in $\mathcal{C}$, the collection $Hom_{\mathcal{C}}(X, Y) = \{f \in \mathcal{C} \mid f : X \to Y\}$ is a set (called a hom-set). A category is said to be *small* if its objects form a set. A category $\mathcal{C}$ is called *connected*, if for all objects X, Y in $\mathcal{C}$, $Hom_{\mathcal{C}}(X, Y) \neq \emptyset$. A *functor* F from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a prescription that assigns to every object A of $\mathcal{C}$ an object $F(A)$ of $\mathcal{D}$, and to every morphism $\alpha : A \to B$ of $\mathcal{C}$ a morphism $F(\alpha) : F(A) \to F(B)$ of $\mathcal{D}$, such that (i): $F(id_A) = id_{F(A)}$, for every object A of $\mathcal{C}$ and (ii): if $\beta \circ \alpha$ is defined in $\mathcal{C}$, then $F(\beta) \circ F(\alpha)$ is defined in $\mathcal{D}$ and $F(\beta) \circ F(\alpha) = F(\beta \circ \alpha)$. A *monomorphism* (also called a *monic* morphism) is a left-cancellative morphism. That is, a morphism $f : X \to Y$ such that for all objects Z and all morphisms $g_1, g_2 : Z \to X$, if $f \circ g_1 = f \circ g_2$, then $g_1 = g_2$. An *epimorphism* (also called an *epic* morphism) is a morphism $f : X \to Y$ that is right-cancellative in the sense that, for all objects Z and all morphisms $g_1, g_2 : Y \to Z$, if $g_1 \circ f = g_2 \circ f$, then $g_1 = g_2$. An *initial object* is an object I such that for every object X, there exists precisely one morphism $I \to X$. A *terminal object* is an object T such that for every object X, there exists precisely one morphism $X \to T$. If an object is both initial and terminal, it is called a *zero object*. An object Q in a category $\mathcal{C}$ is said to be *injective* if for every monomorphism $f : X \to Y$ and every morphism $g : X \to Q$ there exists a morphism $h : Y \to Q$ such that $h \circ f = g$. If B is an object of a category $\mathcal{C}$, then by a *subobject* of B we mean a pair of (A, f) consisting of an object A of $\mathcal{C}$ and a morphism $f : A \to B$ that is monic. Let X be a set (called a basis), A be an object, and $i : X \to A$ be an injective map between sets (called the canonical insertion). We say that A is the *free object* on X (with respect to i) if and only if it satisfies the following universal property: for any object B and any map between sets $f : X \to B$, there exists a unique morphism $g : A \to B$ such that $f = g \circ i$. Let A_1 and A_2 be two algebras of the same type F. The *product* $A_1 \times A_2$ is an algebraic structure whose universe is the set $A_1 \times A_2$, such that for $f \in F_n$ and $a_i \in A_1$, $a'_i \in A_2$, $1 \leqslant i \leqslant n$,

$$f^{A_1 \times A_2}(\langle a_1, a'_1 \rangle, \cdots, \langle a_n, a'_n \rangle) = \langle f^{A_1}(a_1, ..., a_n), f^{A_2}(a'_1, ..., a'_n) \rangle.$$

Let A and B be two objects and $f, g : A \to B$ be two morphisms. The *equalizer* of f and g is an object E and a morphism $eq : E \to A$ satisfying $f \circ eq = g \circ eq$ such that, for any object O and morphism $m : O \to A$, if $f \circ m = g \circ m$, then there exists a unique morphism $u : O \to E$ such that $eq \circ u = m$. Let A be an object and $f : Y \to A$ and $g : Z \to A$ be two morphisms. The *pullback* of f, g is an object P with to morphisms $p_1 : P \to Y$ and $p_2 : P \to Z$ such that $f \circ p_1 = g \circ p_2$ and for any $q_1 : Q \to Y$ and $q_2 : Q \to Z$ with $f \circ q_1 = g \circ q_2$, there exists a unique $u : Q \to P$ such that $q_1 = p_1 \circ u$ and $q_2 = p_2 \circ u$. Let $\mathcal{J}$ and $\mathcal{C}$ be categories. A *diagram* of type $\mathcal{J}$ in $\mathcal{C}$ is a functor $D : \mathcal{J} \to \mathcal{C}$. We will write the objects in the index category $\mathcal{J}$ lower case, $i, j, \cdots$ and the values of the functor $D : \mathcal{J} \to \mathcal{C}$ in the form D_i, D_j, etc. If $\mathcal{J}$ is a small category, then $D : \mathcal{J} \to \mathcal{C}$ is a *small diagram*. A *cone* to a diagram D consists of an object C in $\mathcal{C}$ and a family of morphisms in $\mathcal{C}$, $c_j : C \to D_j$ one for each object $j \in \mathcal{J}$, such that for each morphism $\alpha : i \to j$ in $\mathcal{J}$, such that $D_\alpha \circ c_i = c_j$. A *morphism of cones* $\vartheta : (C, c_j) \to (C', c'_j)$ is a morphism ϑ in $\mathcal{C}$ such that for any $j \in \mathcal{J}$, $c_j = c'_j \circ \vartheta$. Thus, we have an apparent category $\mathbf{Cone}(D)$ of cones to D. A *limit* for a diagram $D : J \to C$ is a terminal object in $\mathbf{Cone}(D)$. A category $\mathcal{C}$ is called *small-complete* if all small diagrams in $\mathcal{C}$ have limits in $\mathcal{C}$.

Suppose that $(A_i)_{i \in I}$ is a family of subobjects of a given object B. Constructing a category $\mathcal{K}$ as follows: for the objects of $\mathcal{K}$ take those subobjects (D, d) of B for which there exists a commutative triangle as follows.

$$\begin{tikzcd} D \arrow[r, "d_i"] \arrow[dr, "d"'] & A_i \arrow[d, "f_i"] \\ & B \end{tikzcd}$$

For the morphisms from (D, d) to (E, e) take those morphisms $\alpha : D \to E$ in $\mathcal{C}$ such that the following diagram is commutative.

$$\begin{tikzcd} D \arrow[dr, "d"] \arrow[dd, "\alpha"'] & \\ & B \\ E \arrow[ur, "e"'] & \end{tikzcd}$$

The terminal object in $\mathcal{K}$ is called an *intersection* of the family $(A_i, f_i)_i \in I$ of subobjects of B.

Proposition 2.8 ([3]). A category $\mathcal{C}$ has finite products if and only if it has a terminal object and every pair of objects in $\mathcal{C}$ has a product.

THEOREM 2.9 ([2, 3]). *Let $\mathcal{C}$ be a category. The following statements are equivalent:*

(i) *$\mathcal{C}$ has finite products and equalizers.*

(ii) *$\mathcal{C}$ has finite products and finite intersections.*

(iii) *$\mathcal{C}$ has pullbacks and a terminal object.*

(iv) *$\mathcal{C}$ has all finite limits.*

Let A and B be two objects and $f, g : A \to B$ be two morphisms. The *coequalizer* of f and g is an object Q and a morphism $q : B \to Q$ such that $q \circ f = q \circ g$. Moreover, for any object Q' and morphism $q' : B \to Q'$ there exists a unique morphism $u' : Q \to Q'$ such that $u' \circ q = q'$. Let X be an object and $f : X \to Y$ and $g : X \to Z$ be two morphisms. The *pushout* of f, g is an object P with to morphisms $i_1 : Y \to P$ and $i_2 : Z \to P$ such that $i_1 \circ f = i_2 \circ g$ and for any $q_1 : Y \to Q$ and $q_2 : Z \to Q$ with $q_1 \circ f = q_2 \circ g$, there exists a unique $u : P \to Q$ such that $q_1 = u \circ i_1$ and $q_2 = u \circ i_2$.

THEOREM 2.10 ([8]). *A nonempty class K of algebraic structures of type F is called a variety if it is closed under subalgebras, homomorphic images, and direct products.*

THEOREM 2.11 ([8]). *Every variety has free objects.*

Notation. From now on, in this paper, $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ or simply $\mathcal{E}$ is an EQ-algebra, unless otherwise state.

3. Category of EQ-algebras

If we consider EQ-algebras and EQ-homomorphisms between them as objects and morphisms, then class of all EQ-algebras and the EQ-homomorphisms with the usual composition of maps forms a locally small category which is denoted by $\mathcal{EQ}$. In the rest of this article, we study the morphims, objects, limits and colimits of $\mathcal{EQ}$.

3.1. Morphisms

In this subsection, we give the conditions that when an injective EQ-homomorphism is monic. Also, we show that an onto EQ-homomorphism is epic but the connverse is not true, in general.

PROPOSITION 3.1. $\mathcal{EQ}$ is connected.

PROOF: Let $\mathcal{E}$ and $\mathcal{G}$ be two EQ-algebras. Then the map $e : \mathcal{E} \to \mathcal{G}$, where for any $a \in E$, $e(a) = 1_{\mathcal{G}}$ is a homomorphism. Thus, $Hom(\mathcal{E}, \mathcal{G}) \neq \emptyset$. □

PROPOSITION 3.2. Let $f : \mathcal{E} \to \mathcal{G}$ and $ker f = \{a \in E | f(a) = 1_{\mathcal{G}}\} = \{1_{\mathcal{E}}\}$. If $\mathcal{E}$ is separated, then f is injective.

PROOF: Suppose that for some $a, b \in E$, $f(a) = f(b)$. Since f is a homomorphism, by $(E3)$ we have, $1_{\mathcal{G}} = f(a) \sim f(b) = f(a \sim b)$. Then $a \sim b \in ker f$ and so $a \sim b = 1$. Since $\mathcal{E}$ is separated, we have $a = b$ and f is injective. □

In the following example, we show that the separated condition in Proposition 3.2 is necessary.

Example 3.3. Let $H = \{0, a, b, 1\}$ be a chain where $0 \leqslant a \leqslant b \leqslant 1$. For any $x, y \in H$, we define the operations $\otimes$ and $\sim$ on H as Table 1 and Table 2:

Table 1

$\otimes$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	0	b
1	0	a	b	1

Table 2

$\sim$	0	a	b	1
0	1	0	0	0
a	0	1	a	a
b	0	a	1	1
1	0	a	1	1

Then $\mathcal{H} = (H, \wedge, \otimes, \sim, 1)$ is a non-separated EQ-algebra. Let $G = \{0, d, 1\}$ be a chain where $0 \leqslant d \leqslant 1$. For any $x, y \in G$, we define the operations $\otimes$ and $\sim$ on G as Table 3 and Table 4:

Table 3

$\otimes$	0	d	1
0	0	0	0
d	0	0	d
1	0	d	1

Table 4

$\sim$	0	d	1
0	1	0	0
d	0	1	1
1	0	1	1

Then $\mathcal{G} = (G, \wedge, \otimes, \sim, 1)$ is an EQ-algebra and $f : H \to G$ is an EQ-homomorphism where $f(0) = f(a) = 0$, $f(b) = d$ and $f(1) = 1$. Thus, it is clear that $kerf = \{1\}$ but f is not injective.

PROPOSITION 3.4. Let $f : \mathcal{E} \to \mathcal{G}$. If f is injective, then $kerf = \{1_{\mathcal{E}}\}$.

THEOREM 3.5. *Let $f \in Hom(\mathcal{E}, \mathcal{G})$ in $\mathcal{EQ}$.*

(i) *If f is injective, then f is monic.*

(ii) *If $\mathcal{E}$ is a separated EQ-algebra and f is monic, then f is an injective map.*

(iii) *If f is onto, then f is epic.*

PROOF: (i) The proof is clear.

(ii) Let $H = \{a \in E | f(a) = 1_{\mathcal{G}}\}$. It is easy to see that $\mathcal{H} = (H, \otimes_{\mathcal{H}}, \wedge, \sim_{\mathcal{H}}, 1_{\mathcal{E}})$ is a sub-algebra of $\mathcal{E}$. Suppose $i : \mathcal{H} \to \mathcal{E}$ is an inclusion morphism and $g : \mathcal{H} \to \mathcal{E}$ is an EQ-homomorphism such that for any $a \in H$, $g(a) = 1_{\mathcal{E}}$. Since f is monic if $f \circ i = f \circ g$, then $H = \{1\}$. Since $\mathcal{E}$ is separated, by Proposition 3.2, f is injective.

(iii) Suppose that $g, h : \mathcal{G} \to \mathcal{H}$ are two morphisms such that $g \circ f = h \circ f$. Since f is onto, for any $b \in G$, there is $a \in E$ where $f(a) = b$. Thus, for any $b \in G$,

$$g(b) = g(f(a)) = g \circ f(a) = h \circ f(a) = h(f(a)) = h(b).$$

Hence f is epic. $\square$

In the following example we show that the converse of Theorem 3.5 (iii) is not true, in general.

Example 3.6. Let $E = \{0, a, b, c, d, 1\}$ be a lattice with a Hesse diagram as Figure 1. For any $x, y \in E$, we define the operations $\otimes$ and $\sim$ on E as Table 5 and Table 6:

Then $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ is a good EQ-algebra. Let $\mathcal{H}$ be the EQ-algebra as in Example 3.3. Then the map $f : H \to E$ where $f(0) = 0$, $f(a) = b$, $f(b) = a$ and $f(1) = 1$ is a non-onto homomorphism. Let $id : E \to E$ be the identity map and $t : E \to E$ be the trivial EQ-homomorphism. It is clear that $h \neq g$. Since $id \circ f(a) = b$ and $t \circ f(a) = 1$, we get $h \circ f \neq g \circ f$.

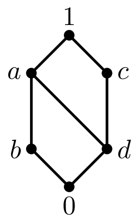

Figure 1

Table 5

$\otimes$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	b	b	d	0	a
b	0	b	b	0	0	b
c	0	d	0	c	d	c
d	0	0	0	d	0	d
1	0	a	b	c	d	1

Table 6

$\sim$	0	a	b	c	d	1
0	1	d	c	b	a	0
a	d	1	a	d	c	a
b	c	a	1	0	d	b
c	b	d	0	1	a	c
d	a	c	d	a	1	d
1	0	a	b	c	d	1

3.2. Objects

In this subsection, we show that $\mathcal{EQ}$ has zero objects and introduce the free EQ-algebra on the singleton.

THEOREM 3.7. *In $\mathcal{EQ}$, $\{1\}$ is the zero object.*

PROOF: Let $\mathcal{E}$ be an arbitrary EQ-algebra. Then $f : \{1\} \to \mathcal{E}$ where $f(1) = 1_{\mathcal{E}}$ is an EQ-homomorphism. On the other hand, let $g : \mathcal{E} \to \{1\}$ be a map where for any $a \in E$, $g(a) = 1$. Since g preserves the operations $\otimes$, $\wedge$ and $\sim$, we obtain that g is an EQ-homomorphism. □

COROLLARY 3.8. The zero objects are the only injective object in $\mathcal{EQ}$.

In [14], Novák and El-zekey proved that the class of EQ-algebras is a variety. Thus, by Theorem 2.11, there exists a free EQ-algebra on a given set.

In [3], Blyth showed that the free monoid on a singleton is isomorphic to the additive monoid $\mathbb{Z}^+$. So we can consider free monoid on element $\{x\}$, by $(\mathcal{E}_x, \otimes)$ where $\mathcal{E}_x = \{e = x^0, x^1, x^2, \cdots, x^i, \cdots\}$. For any $i \in \mathbb{N}$,

$x^i \otimes x^j = x^{i+j}$ and e is the identity element. Now, we define a relation on $\mathcal{E}_x$ as follows: for any $x^i, x^j \in \mathcal{E}_x$, we say that $x^i \leqslant_f x^j$ if and only if $j \leqslant i$.

LEMMA 3.9. *Let $(\mathcal{E}_x, \otimes)$ be a free monoid on $\{x\}$.*

(i) *The relation $\leqslant_f$ on $\mathcal{E}_x$ is an order.*

(ii) *For any $i, j, k \in \mathbb{Z}^+$, if $x^i \leqslant_f x^j$, then $x^i \otimes x^k \leqslant_f x^j \otimes x^k$.*

PROOF: (i) We show that $\leqslant_f$ is reflexive, antisymmetric, and transitive. Since $\mathbb{Z}$ is an ordered set, for any $i \in \mathbb{Z}^+$, we have $x^i \leqslant_f x^i$ and $\leqslant_f$ is reflexive.

Suppose that for $i, j \in \mathbb{N}$, $x^i \leqslant_f x^j$ and $x^j \leqslant_f x^i$. By definition of $\leqslant_f$, we obtain $j \leqslant i$ and $i \leqslant j$. Thus $i = j$, then $x^i = x^j$ and so $\leqslant_f$ is antisymmetric.

Suppose for some $i, j, k \in \mathbb{N}$, $x^i \leqslant_f x^j$ and $x^j \leqslant_f x^k$. By definition of $\leqslant_f$, we have $j \leqslant i$ and $k \leqslant j$. Hence $k \leqslant i$, $x^i \leqslant_f x^k$ and so $\leqslant_f$ is transitive.

Since $\mathbb{N}$ is a chain, we can see that $\mathcal{E} = (E, \leqslant_f)$ is a chain with maximum element. Therefore, it is meet semilattice with upper bound.

(ii) Now suppose that $x^i \leqslant_f x^j$, then $j \leqslant i$. For any $k \in \mathbb{Z}^+$, $j + k \leqslant i + k$ and so $x^{i+k} \leqslant x^{j+k}$. Thus, $x^i \otimes x^k \leqslant_f x^j \otimes x^k$. Therefore, $\leqslant_f$ is an order relation on $\mathcal{E}$. □

THEOREM 3.10. *Let $X = \{x\}$ be a set and $\mathcal{E} = (E, \otimes, \wedge)$ be a free monoid on X with an order we define in Lemma 3.9. If for any $i, j \in \mathbb{Z}^+$, we define a fuzzy equality on $\mathcal{E}$ as $x^i \sim x^j = x^{|i-j|}$, then $\mathcal{E}_x = (E, \otimes, \wedge, \sim, e)$ is an EQ-algebra.*

PROOF: By Lemma 3.9, $(E1)$ and $(E2)$ are satisfied. For any $i \in \mathbb{Z}^+$, $x^i \sim x^i = x^{|i-i|} = x^0 = e$, and so $(E3)$ is satisfied. Without loss of generality, in the rest of proof we suppose that $i \leqslant j \leqslant k \leqslant w$. Then

$$((x^i \wedge x^j) \sim x^k) \otimes (x^w \sim x^i) = (x^j \sim x^k) \otimes (x^w \sim x^i) = x^{w+k-(i+j)}.$$

On the other hand, $x^k \sim (x^j \wedge x^w) = x^{w-k}$. Moreover, since $i, j \leqslant k$, we have $i + j \leqslant 2k$ and so $w - k \leqslant w + k - (i + j)$. Thus, $x^{w+k-(i+j)} \leqslant x^{w-k}$. Hence $(E4)$ holds.

We can see that $(x^i \sim x^j) \otimes (x^k \sim x^w) = x^{w+j-(i+k)}$ and $(x^i \sim x^k) \sim (x^j \sim x^w) = x^{|(w-j)-(k-i)|}$. To show that $(E5)$ is satisfied, we consider two following cases.

Case 1. If $k+j \leqslant w+i$, then $k+j-(w+i) \leqslant w+j-(i+k)$ and so

$$(x^i \sim x^j) \otimes (x^k \sim x^w) \leqslant (x^i \sim x^k) \sim (x^j \sim x^w).$$

Case 2. If $w+i \leqslant k+j$, then $x^{w+j-(i+k)} \leqslant x^{(w-j)-(k-i)}$ and so $(E5)$ holds.

Moreover, since $j-i \leqslant k-i$, we get that $x^{k-i} \leqslant x^{j-i}$ and so $(x^i \wedge x^j \wedge x^k) \sim x^i \leqslant (x^i \wedge x^j) \sim x^i$. Thus, $(E6)$ is satisfied.
Also, from $i, j \leqslant j+i$, we have $x^i \otimes x^j \leqslant x^i \wedge x^j$, and $(E7)$ is satisfied. Therefore, $\mathcal{E}_x = (E, \otimes, \wedge, \sim, e)$ is an EQ-algebra. □

Remark 3.11. For any $i, j \in \mathbb{Z}^+$, the implication operation on $\mathcal{E}_x$ is

$$x^i \to x^j = \begin{cases} e & j \leqslant i \\ x^{j-i} & i < j. \end{cases}$$

Proposition 3.12. Let $X = \{x\}$ be a set. Then the following statements hold:

(i) $\mathcal{E}_x$ is a good EQ-algebra.

(ii) $\mathcal{E}_x$ is a residuated EQ-algebra.

(iii) $\mathcal{E}_x$ is an ℓEQ-algebra.

(iv) $\mathcal{E}_x$ is a multiplicatively relative EQ-algebra.

Proof: (i) By the definition of $\sim$ on $\mathcal{E}_x$, the proof is clear.

(ii) Let $i, j, w \in \mathbb{Z}^+$. If $w \leqslant i+j$, then $(x^i \otimes x^j) \to x^w = e$. Thus, we consider two following cases.

Case 1. If $w \leqslant j$, then $x^j \to x^w = e$ and so for any $i \in \mathbb{Z}^+$, we have

$$(x^i \otimes x^j) \to x^w = x^i \to (x^j \to x^w) = e.$$

Case 2. If $j < w$, then $x^j \to x^w = x^{w-j}$ and since $w-j \leqslant i$ we have $x^i \to (x^j \to x^w) = e$.

Now, if $i+j < w$, then $(x^i \otimes x^j) \to x^w = x^{w-(i+j)}$. Moreover, since $i < w$ and $j < w$, we get that $x^i \to (x^j \to x^w) = x^{w-j-i}$. Hence, by Proposition 2.1 (ii), $\mathcal{E}_x$ is residuated.

(iii) Without loss of generality, we consider that $i \leqslant j \leqslant k \leqslant w$. Since $2i \leqslant w + j$, we have

$$((x^i \vee x^j) \sim x^k) \otimes (x^w \sim x^i) \leqslant ((x^j \vee x^w) \sim x^k).$$

(iv) By Lemma 3.9, we know that $\mathcal{E}_x$ is a chain. Then by Proposition 2.2, it is multiplicatively relative. □

Remark 3.13. Since $\mathcal{E}_x$ dose not have least element, it is not a residuated lattice.

In the following example, we show that $\mathcal{E}_x$ is not free in general.

Example 3.14. Let $\mathcal{E}$ be an EQ-algebra as in Example 3.6. Let $X = \{x\}$ be an arbitrary set and $h : X \to E$ be a map where $h(x) = b$. If $\mathcal{E}_x$ is a free EQ-algebra, then $f : \mathcal{E}_x \to \mathcal{E}$ is an EQ-homomorphism which $f(x) = h(x)$ and so for any $i \in \mathbb{N}$, we should have $f(x^i) = h(x) \otimes h(x) \otimes \cdots \otimes h(x) = b$. We claim that f is not an EQ-algebra homomorphism. Because, for $j > i$, $f(x^i \sim x^j) = f(x^{j-i}) = b$ but, $f(x^i) \sim f(x^j) = b \sim b = 1 \neq b$. Thus, f is not preserves the both operations $\otimes$ and $\sim$ at the same time.

Remark 3.15. Let $X = \{x\}$ be a set and $\mathcal{E}_x$ be the EQ-algebra as in Theorem 3.10. For any $i, j, k \in \mathbb{Z}^+$, $x^i \sim x^j = x^{i+k} \sim x^{j+k} = (x^i \otimes x^j) \sim (x^j \otimes x^k)$.

Definition 3.16. An EQ-algebra $\mathcal{E}$ is *multiplicatively equal* if for any $a, b, c \in E$,

$$a \sim b = (a \otimes c) \sim (b \otimes c).$$

Example 3.17. By Remark 3.15, $\mathcal{E}_x$ is multiplcatively equal.

Theorem 3.18. *The EQ-algebra $\mathcal{E}_x$ is a free object on the class of good multiplicatively equal EQ-algebras.*

Proof: Let $\mathcal{H} = (H, \otimes_H, \wedge_H, \sim_H, e_H)$ be a multiplicatively equal EQ-algebra, $X = \{x\}$ be a set and $g : X \to H$ be a map. We define a map $f : \mathcal{E}_x \to H$ such that for any $i \in \mathbb{N}$, $f(x^i) = g(x)^i$ and $f(e_{\mathcal{E}_x}) = e_H$. Now, we show that f is an EQ-homomorphism. Let $i, j \in \mathbb{N}$,

$$f(x^i \otimes x^j) = f(x^{i+j}) = g(x)^{i+j} = g(x)^i \otimes_H g(x)^j.$$

Without loss of generality, we can consider that $i \leqslant j$ and so $g(x)^j = g(x)^i \otimes g(x)^{j-i}$. By $(E7)$, we have $g(x)^i \wedge_H g(x)^j = g(x)^j$. Thus, $f(x^i \wedge x^j) = f(x^j) = g(x)^j = g(x)^i \wedge g(x)^j$.

Since $\mathcal{H}$ is good multiplicatively equal, for any $j > i \in \mathbb{N}$, we get that,

$$g(x)^{j-i} = e_H \sim g(x)^{j-i} = g(x)^i \sim g(x)^j = f(x^i) \sim f(x^j).$$

On the other hand, $f(x^i \sim x^j) = f(x^{j-i}) = g(x)^{j-i}$ and so f preserves the operation "$\sim$". Therefore, f is an EQ-homomorphism. □

3.3. Limits

In this subsection, we show $\mathcal{EQ}$ has products and also all finite limits.

THEOREM 3.19. *$\mathcal{EQ}$ has*

(i) *product,*

(ii) *equilizers,*

(iii) *pullbacks,*

(iv) *all finite limits,*

(v) *finite intersections.*

PROOF: (i) Since the class of EQ-algebras is a variety, by Theorem 2.10, it has products. Then for any EQ-algebras $\mathcal{E}$ and $\mathcal{G}$, $\mathcal{E} \times \mathcal{G}$ with pointwise operations is an EQ-algebra. Thus, $\mathcal{E} \times \mathcal{G}$ with projection maps (p_1, p_2) is the product of $\mathcal{E}$ and $\mathcal{G}$.

(ii) Let $f, g : \mathcal{E} \to \mathcal{G}$ be two EQ-homomorphisms and let $H = \{a \in| f(a) = g(a)\}$. Since f, g are homomorphisms, $\mathcal{H} = (H, \otimes_{\mathcal{E}}, \wedge_{\mathcal{E}}, \sim_{\mathcal{E}}, 1_{\mathcal{E}})$ is an EQ-algebra. Let $i : \mathcal{H} \to \mathcal{E}$ be the inclusion map. Then for any $a \in H$, $f \circ i(a) = f(a) = g(a) = g \circ i(a)$.

$$\begin{tikzcd}
\mathcal{H} \arrow[r, "i"] & \mathcal{E} \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \mathcal{G} \\
\mathcal{K} \arrow[u, dotted, "l"] \arrow[ur, "j"] & &
\end{tikzcd}$$

Now, suppose that $\mathcal{K}$ is an EQ-algebra and $j : \mathcal{K} \to \mathcal{E}$ is a morphism such that $f \circ j = g \circ j$. Then for any $x \in K$, $j(x) \in H$ and $Im(j) \subseteq H$. Thus, we can define a morphism $l : \mathcal{K} \to \mathcal{H}$ where for any $x \in K$, $l(x) = j(x)$. It is clear that l is an unique EQ-homomorphism and $i \circ l = j$. Hence $(\mathcal{H}, i)$ is the equilizer of f, g.

(iii) Let $f \in Hom(\mathcal{E}, \mathcal{H})$ and $g \in Hom(\mathcal{G}, \mathcal{H})$. Since $\mathcal{EQ}$ has a binary product, $\mathcal{P} = \{(a, b) \in E \times G | f(a) = g(b)\}$ is an EQ-algebra and so (P, p_1, p_2) is the pull back of (f, g).

$$\begin{tikzcd}
\mathcal{Q} \arrow[drr, bend left, "q_1"] \arrow[dr, dotted, "h"] \arrow[ddr, bend right, "q_2"'] & & \\
& \mathcal{P} \arrow[r, "p_1"] \arrow[d, "p_2"] & \mathcal{E} \arrow[d, "f"] \\
& \mathcal{G} \arrow[r, "g"] & \mathcal{H}
\end{tikzcd}$$

Now suppose that $(\mathcal{Q}, q_1, q_2)$ is an EQ-algebra with two morphisms where $f \circ q_1 = g \circ q_2$. Let $h : \mathcal{Q} \to \mathcal{P}$ be a map where for any $x \in Q$, $h(x) = (q_1(x), q_2(x))$. Since q_1 and q_2 are homomorphisms, h is homomorphism. By considering the definition of h, we can see that $p_1 \circ h = q_1$ and $p_2 \circ h = q_2$. Moreover, since p_1 and p_2 are onto, they are epic and so h is unique.

(iv), (v) Since $\mathcal{EQ}$ has all finite products and equlizers, by Theorem 2.9, it has all finite limits. $\square$

3.4. Co-limits

In this subsection, we show that $\mathcal{EQ}$ does not have co-limits such as co-equlizers, coproduct and pushout, in general. In the rest of this article, we introduce a method to extend any good EQ-algebra and by using this method we calculate coprodcuts and push out of EQ-algebras in special cases.

THEOREM 3.20. *Let $f, g : \mathcal{E} \to \mathcal{G}$ be two EQ-homomorphisms. If $\mathcal{G}$ is a multiplicatively relative EQ-algebra, then f, g have co-equilizer.*

PROOF: Let

$$\mathcal{F} = \{F \mid F \text{ is a filter of } \mathcal{G} \text{ such that for any } x \in E, \quad f(x) \sim g(x) \in F\}.$$

Since $\mathcal{G}$ is a multiplicatively relative EQ-algebra, $\langle Imf \cup Img \rangle$ is a filter of $\mathcal{G}$ and so $\mathcal{F}$ is not empty. By Proposition 2.5, $\cap\mathcal{F}$ is a filter of $\mathcal{G}$. Then by Theorem 2.6, $\frac{\mathcal{G}}{\cap\mathcal{F}}$ is an EQ-algebra. Let $\pi : \mathcal{G} \to \mathcal{G}/\cap\mathcal{F}$ be a map such that for any $a \in G$, $\pi(a) = [a]$. For any $x \in E$, $\pi \circ f(x) = [f(x)]$ and $\pi \circ g(x) = [g(x)]$. Since $f(x) \sim g(x) \in \cap\mathcal{F}$, we have $[f(x)] = [g(x)]$.

Suppose that there exists a separated EQ-algebra and a homomorphism such as $(\mathcal{J}, j)$ where $j \circ f = j \circ g$. Since, for any $x \in E$, $j(f(x) \sim g(x)) = (j \circ f(x)) \sim (j \circ g(x)) = 1$, we get that $f(x) \sim g(x) \in kerj$.
Now, let $k : \mathcal{G}/\cap\mathcal{F} \to \mathcal{J}$ be a map such that $k([a]) = j(a)$ for any $[a] \in \mathcal{G}/\cap\mathcal{F}$. We show that k is a homomorphism.

$$\begin{tikzcd} \mathcal{E} \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \mathcal{G} \arrow[r, "\pi"] \arrow[dr, "j"'] & \frac{\mathcal{G}}{\cap\mathcal{F}} \arrow[d, dotted, "k"] \\ & & \mathcal{J} \end{tikzcd}$$

Suppose that $[a] = [b]$. By Proposition 2.6 we have, $a \sim b \in \cap\mathcal{F}$. By Proposition 2.4, $kerj$ is a filter of $\mathcal{G}$ and so $j(a) \sim j(b) = 1$. Since $\mathcal{J}$ is separated, $j(a) = j(b)$ and so $k([a]) = k([b])$.
By considering the definition of k, it is clear that $k \circ \pi = j$ and k is an unique EQ-homomorphism. Therefore, $(\mathcal{G}/\cap\mathcal{F}, \pi)$ is a co-equilizer. $\square$

THEOREM 3.21. *Let $\mathcal{E}$ be a good. If $e \notin E$, then $\mathcal{E}' = (E \cup \{e\}, \otimes_{\mathcal{E}'}, \wedge_{\mathcal{E}'}, \sim_{\mathcal{E}'}, e)$ is a good EQ-algebra where $\otimes_{\mathcal{E}'}$, $\wedge_{\mathcal{E}'}$, and $\sim_{\mathcal{E}'}$ define as follows:*

$$a \otimes_{\mathcal{E}'} b = \begin{cases} a \otimes b & a, b \in E \\ a & a \in E, b = e \\ b & b \in E, a = e \\ e & a = b = e. \end{cases} \qquad a \wedge_{\mathcal{E}'} b = \begin{cases} a \wedge b & a, b \in E \\ a & a \in E, b = e \\ b & b \in E, a = e \\ e & a = b = e. \end{cases}$$

$$a \sim_{\mathcal{E}'} b = \begin{cases} a \sim b & a, b \in E, a \neq b \\ e & a, b \in E, a = b \\ a & a \in E, b = e \\ b & b \in E, a = e \\ e & a = b = e. \end{cases}$$

PROOF: By considering the definition of $\otimes_{\mathcal{E}'}$, we can see that $(\mathcal{E}', \otimes_{\mathcal{E}'}, e)$ is a commutative monoid [4]. Also, $(\mathcal{E}', \wedge_{\mathcal{E}'}, e)$ is a meet semilattice with upper bound e. Now, we show $\otimes_{\mathcal{E}'}$ is isotone with respect to $\leqslant_{\mathcal{E}'}$. Let $a, b \in E \cup \{e\}$ such that $a \leqslant_{\mathcal{E}'} b$. We can consider two cases.

Case 1. We suppose that $a \leqslant b < e$. If $c \in E$, then $a \otimes_{\mathcal{E}'} c = a \otimes c \leqslant b \otimes c = b \otimes_{\mathcal{E}'} c$. If $c = e$, then $a \otimes_{\mathcal{E}'} e = a \leqslant b = b \otimes_{\mathcal{E}'} e$.

Case 2. Since for any $a \in E$, $a \leqslant_{\mathcal{E}'} e$, we have $a \otimes_{\mathcal{E}'} c = a \otimes c \leqslant c = e \otimes_{\mathcal{E}'} c$, for any $c \in E$. If $c = e$, then $a = a \otimes_{\mathcal{E}'} e \leqslant_{\mathcal{E}'} e = e \otimes_{\mathcal{E}'} e$. Thus, $(E2)$ holds.

By considering the definition of $\sim_{\mathcal{E}'}$, we can see that $(E3)$ is satisfied. To show that $(E4)$ is satisfied on $\mathcal{E}'$ we can consider four following cases.

Case 1. Suppose that $a, b, c \in E$. Since $\mathcal{E}$ is good, we have,

$$\begin{aligned}((a \wedge b) \sim_{\mathcal{E}'} e) \otimes_{\mathcal{E}'} (c \sim a) &= (a \wedge b) \otimes (c \sim a)\\ &= ((a \wedge b) \sim 1) \otimes (c \sim a)\\ &\leqslant (1 \sim (c \wedge b))\\ &= c \wedge b\\ &= (e \sim_{\mathcal{E}'} (c \wedge_{\mathcal{E}'} b)).\end{aligned}$$

Case 2. Since $\mathcal{E}$ is an EQ-algebra, by $(E5)$ we have,

$$\begin{aligned}((a \wedge_{\mathcal{E}'} e) \sim_{\mathcal{E}'} c) \otimes_{\mathcal{E}'} (b \sim_{\mathcal{E}'} a) &= (a \sim c) \otimes (b \sim a)\\ &\leqslant (a \sim a) \sim (c \sim b)\\ &= 1 \sim (c \sim b)\\ &= (c \sim_{\mathcal{E}'} (b \wedge_{\mathcal{E}'} e)).\end{aligned}$$

Case 3. Since $\mathcal{E}$ is good, we have

$$\begin{aligned}((a \wedge_{\mathcal{E}'} b) \sim_{\mathcal{E}'} c) \otimes_{\mathcal{E}'} (e \sim_{\mathcal{E}'} a) &= ((a \wedge b) \sim c) \otimes (1 \sim a)\\ &\leqslant (c \sim (1 \wedge b))\\ &= (c \sim_{\mathcal{E}'} (e \wedge_{\mathcal{E}'} b)).\end{aligned}$$

Case 4. Since $\mathcal{E}$ is good by Proposition 2.3 (ii), we have,

$$\begin{aligned}((a \wedge_{\mathcal{E}'} e) \sim_{\mathcal{E}'} e) \otimes_{\mathcal{E}'} (d \sim_{\mathcal{E}'} a) &= a \otimes (d \sim a) \leqslant d\\ &= e \sim_{\mathcal{E}'} (d \wedge_{\mathcal{E}'} e).\end{aligned}$$

For any $a, b, c, d \in E$, $(E5)$ is satisfied. Now, we show that $(E6)$ holds.

$$\begin{aligned}(a \sim_{\mathcal{E}'} b) \otimes_{\mathcal{E}'} (c \sim_{\mathcal{E}} e) &= (a \sim b) \otimes (c \sim 1)\\ &\leqslant (a \sim c) \sim (b \sim 1)\\ &= (a \sim_{\mathcal{E}'} c) \sim_{\mathcal{E}'} (b \sim_{\mathcal{E}'} e).\end{aligned}$$

Since for any $a, b, c \in E$, $(a \wedge_{\mathcal{E}'} b \wedge_{\mathcal{E}'} e) \sim_{\mathcal{E}'} a = (a \wedge_{\mathcal{E}'} b) \sim_{\mathcal{E}'} a$ and $(e \wedge_{\mathcal{E}'} b \wedge_{\mathcal{E}'} c) \sim_{\mathcal{E}'} e = (b \wedge_{\mathcal{E}'} c) \sim_{\mathcal{E}'} e$, $(E6)$ is satisfied. For any $a \in E$, $a \otimes_{\mathcal{E}'} e = a = a \sim_{\mathcal{E}'} e$, and so $(E7)$ is satisfied. □

COROLLARY 3.22. Let X be a countable chain with maximum element x. If $\mathcal{E}$ is a good EQ-algebra, then $\mathcal{E}' = (E \cup X, \otimes_{\mathcal{E}'}, \wedge_{\mathcal{E}'}, \sim_{\mathcal{E}'}, x)$ is a good EQ-algebra.

PROOF: By induction on the cardinality of X and Theorem 3.21, the proof is clear. □

THEOREM 3.23. *Let $\mathcal{E}$ be a good EQ-algebra and $\mathcal{G} = \{e\}$. Then $\mathcal{E}$ and $\mathcal{G}$ have co-product.*

PROOF: By Theorem 3.21, $\mathcal{E} \cup \mathcal{G}$ is an EQ-algebra. Let $i_2 : \mathcal{G} \to \mathcal{E} \cup \mathcal{G}$ be the inclusion map and $i_1 : \mathcal{E} \to \mathcal{E} \cup \mathcal{G}$ be a map such that $i_1(1) = e$ and for any $a \in E - \{1\}$, $i_1(a) = a$. We claim that $(\mathcal{E} \cup \mathcal{G}, i_1, i_2)$ is the co-product of $\mathcal{E}$ and $\mathcal{G}$.

$$\begin{tikzcd} & \mathcal{S} & \\ \mathcal{E} \arrow[ur, "f_1"] \arrow[r, "i_1"'] & \mathcal{E} \cup \mathcal{G} \arrow[u, dashed, "h"] & \mathcal{G} \arrow[ul, "f_2"'] \arrow[l, "i_2"] \end{tikzcd}$$

Suppose that $(\mathcal{S}, f_1, f_2)$ is an EQ-algebra with two homomorphisms such that $f_1 : \mathcal{E} \to \mathcal{S}$ and $f_2 : \mathcal{G} \to \mathcal{S}$. We define a map $h : \mathcal{E} \cup \mathcal{G} \to \mathcal{S}$ as follows:

$$h(a) = \begin{cases} f_1(a) & a \in E\\ f_2(a) & a \in G. \end{cases}$$

Since f_1 and f_2 are EQ-homomorphisms, h is an EQ-homomorphism and by definition of h we can see that $h \circ i_1 = f_1$ and $h \circ i_2 = f_2$ and so, h is unique. □

In the following example, we show that $\mathcal{EQ}$ does not have co-product, in general.

Example 3.24. Let $H = \{d, e_{\mathcal{H}}\}$ be an EQ-algebra and $E = \{0, a, b, 1\}$ be a chain where $0 \leqslant a \leqslant b \leqslant 1$. For any $\alpha, \beta \in E$, we define the operations $\otimes$ and $\sim$ on E as Table 7 and Table 8:

Table 7

$\otimes$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	0	b
1	0	a	b	1

Table 8

$\sim$	0	a	b	1
0	1	0	0	0
a	0	1	a	a
b	0	a	1	b
1	0	a	b	1

By routine calculations, we can see that $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ is a good EQ-algebra. Let $G = \{0, x, 1\}$ be a chain where $0 \leqslant x \leqslant 1$. For any $\alpha, \beta \in E$, we define the operations $\otimes$ and $\sim$ on E as Table 9 and Table 10.

Table 9

$\otimes$	0	x	1
0	0	0	0
x	0	0	x
1	0	x	1

Table 10

$\sim$	0	x	1
0	1	0	0
x	0	1	x
1	0	x	1

We can see that $\mathcal{G} = (G, \wedge, \otimes, \sim, 1)$ is a good EQ-algebra. By routine calculations we can see that $f_1 : E \to G$ such that $f_1(0) = 0$, $f_1(a) = x$, and $f_1(b) = f_1(1) = 1$ is an EQ-homomorphism. By Corollary 3.22, $\mathcal{E} \cup \mathcal{H}$ is an EQ-algebra. Suppose that $f_2 : \mathcal{H} \to \mathcal{G}$ be a map where $f_2(d) = 0$ and $f_2(e_{\mathcal{H}}) = 1_{\mathcal{G}}$. If $(\mathcal{E} \cup \mathcal{H}, i_1, i_2)$ is the co-product of $\mathcal{E}$ and $\mathcal{H}$, there exists a homomorphism $h : \mathcal{E} \cup \mathcal{H} \to \mathcal{G}$ such that $i_2 \circ h = f_2$ and $i_1 \circ h = f_1$. Since i_2 is a homomorphism we get that $i_2(e_{\mathcal{H}}) = 1_{\mathcal{G}}$. Now, we consider three cases for $i_2(d)$.

Case 1. Suppose that $i_2(d) = 0$. Then we should have $h \circ i_2(d) = h(0) = f_2(d) = 0$. By Theorem 3.21, $a \otimes d = a$ and so we should have $h(a \otimes d) = h(a) \otimes h(d) = h(a) \otimes 0 = h(a)$. According to Table 12, $h(a) = 0$. On the other hand $h(0) \sim h(a) = 1_{\mathcal{G}}$ and $h(0 \sim a) = h(0) = 0$, which means that h is not a homomorphism.

Case 2. If $i_2(d) \in \{a, b, d, 1\}$, then $h(0) = h(a) = 0$ with similar way in Case 1, we can see that h is not homomorphism.

Case 3. Suppose that $i_2(d) = e_{\mathcal{H}}$. Then $h \circ i_2(d) = h(e_{\mathcal{H}}) = f_2(d) = 0$. But if h is a homomorphism, then we should have $h(e_{\mathcal{H}}) = 1_{\mathcal{G}}$.

THEOREM 3.25. *Let $\mathcal{G}$ be a good EQ-algebra and $\mathcal{H} = \{e\}$. If $f : \mathcal{E} \to \mathcal{G}$ and $g : \mathcal{E} \to \mathcal{H}$ are an arbitrary and trivial EQ-homomorphisms, respectively, then $(\mathcal{E}, f, g)$ has pushout.*

PROOF: By Theorem 3.23, $(\mathcal{G} \cup \mathcal{H})$ is the co-product of $\mathcal{G}$ and $\mathcal{H}$. Let $t : \mathcal{G} \to \mathcal{G} \cup \mathcal{H}$ be the trivial homomorphism. For any $a \in E$, we have $t \circ f(a) = t(f(a)) = e_{\mathcal{H}} = id \circ g(a)$ and the following diagram is commutative.

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "f"] \arrow[d, "g"'] & \mathcal{G} \arrow[d, "t"'] \arrow[ddr, bend left, "q_1"] & \\
\mathcal{H} \arrow[r, "id"] \arrow[drr, bend right, "q_2"'] & \mathcal{G} \cup \mathcal{H} \arrow[dr, dashed, "k"] & \\
& & \mathcal{Q}
\end{tikzcd}$$

Suppose that $(\mathcal{Q}, q_1, q_2)$ is an EQ-algebra with two homomorphisms where $q_1 \circ f = q_2 \circ g$. We can see that q_1 is trivial EQ-homomorphism, too. If $k : \mathcal{G} \cup \mathcal{H} \to \mathcal{Q}$ is the trivial map, then the above diagram is commutative and also k is unique. □

4. Conclusions and future works

In this paper, the category of EQ-algebras is studied and showed that it is complete, but it is not cocomplete, in general. It is proved that the multiplicatively relative EQ-algebras have coequlizers and coprodut and pushout in a special case. Also, the free EQ-algebra on a singleton is constructed. Since every good EQ-algebra is an equality algebra [20], most results of this paper hold for equality algebras, too. For the future work, we can find free EQ-algebra on any set. Maybe, there exist some special kind of EQ-algebras which have co-product and pushout in general.

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THE (GREATEST) FRAGMENT OF CLASSICAL LOGIC THAT RESPECTS THE VARIABLE-SHARING PRINCIPLE (IN THE FMLA-FMLA FRAMEWORK)

Abstract

We examine the set of formula-to-formula valid inferences of Classical Logic, where the premise and the conclusion share at least a propositional variable in common. We review the fact, already proved in the literature, that such a system is identical to the first-degree entailment fragment of R. Epstein's Relatedness Logic, and that it is a non-transitive logic of the sort investigated by S. Frankowski and others. Furthermore, we provide a semantics and a calculus for this logic. The semantics is defined in terms of a p-matrix built on top of a 5-valued extension of the 3-element weak Kleene algebra, whereas the calculus is defined in terms of a Gentzen-style sequent system where the left and right negation rules are subject to linguistic constraints.

Keywords: Relevant logics, non-transitive logics, p-matrix, weak Kleene algebra, infectious logics.

1. Background and aim

In the wake of the so-called paradoxes of strict implication, characteristic of the systems presented by C. I. Lewis in the early decades of the last century, many logics were proposed whose featured notions of implication did not suffer such inconveniences. In contemporary terminology, systems

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of this sort are referred to as *relevant* or *relevance logics*—see, e.g., [28]. Work around these logics was usually done in a rather idiosyncratic way, having in mind a particular understanding of the characteristic *relevance* of an implication free of the paradoxes. For example, in [29] E. J. Nelson proposed a relevant implication, defined as the impossibility of the truth of the antecedent and the falsity of the consequent, the relevance of which lied in the requirement that both components be accessory for said impossibility to obtain—contrary to Lewis' implication, where the impossibility of either of these conditions above was sufficient for the impossibility of their conjunction. Alternatively, in [33] W. T. Parry proposed a relevant implication, called analytic implication, the relevance of which lied in the requirement that the content of the consequent is included or contained in the content of the antecedent—according to the exegesis of some scholars, which was nevertheless disputed by W. T. Parry himself.

Despite the debates that took place in the decades following Lewis' work, it is nowadays widely assumed that when working with propositional languages an implication is *relevant* only if its *antecedent and consequent share some propositional variable in common.* This seems to reflect the fact that these terms should not totally diverge with regard to their subject-matters, meaning by this that there should be some common subject-matter connecting the former and the latter—with systems satisfying this condition only sometimes being called "weakly" relevant. Granting a few idealized but relatively standard assumptions about the formalization of subject-matters in the context of propositional languages, this constraint is usually formalized by the so-called Variable-Sharing Principle (VSP, for short).[1] This principle requires the following of a theorem that is an implication of the form $\varphi \to \psi$, where $Var(\chi)$ refers to the set of propositional variables appearing in a formula χ:

$$Var(\varphi) \cap Var(\psi) \neq \emptyset$$

[1]In particular, granting that the subject-matter of a complex proposition is to be identified with the sum or collection of the subject matter of all the propositional letters appearing in it—an idealized but relatively standard assumption, as discussed, e.g., in [5, p. 563]. Furthermore, in this respect it should be said that we are not considering languages with propositional constants—like *Verum* or the Ackermann constant—for different considerations need to be taken into account in such cases. We would like to thank Shawn Standefer for urging us to clarify this issue. As suggested by an anonymous reviewer, for references on this issue see [22] and [2].

As it is known, there are many logics that respect the Variable-Sharing Principle—a paradigmatic example being A. Anderson and N. Belnap's logic R, for which see [1, pp. 252–254]. In this vein, although it has been pointed out that the satisfaction of this criterion is only *necessary but not sufficient* to establish the relevance of a target notion of implication, it could be interesting to consider its satisfaction as an appropriate filter on a previously given independent notion of implication—thus rendering an (at least weakly) relevant subsystem thereof.[2] In this vein we could conceive, for example, filtering Classical Logic (CL, hereafter). Then, although truth-preservation in CL is an unacceptable guide to implication (due to its permeability to irrelevancies in the form of the paradoxes of material and strict implication), it might well be the case that the simultaneous satisfaction of truth-preservation and the Variable-Sharing Principle is an acceptable criterion. In fact, this is exactly the path followed by R. Epstein in [16] where his propositional Relatedness Logic (REL, henceforth) is introduced.

Now, when working with relevant logics, it is standard to denote as "first-degree entailments" those implications of the form $\varphi \to \psi$ where φ and ψ contain no occurrence of the implication connective. As noted in [16], [30], and [31], it can be observed that whenever a first-degree entailment is valid in REL, the consequent preserves the truth of the antecedent and, moreover, the implication respects the Variable-Sharing Principle. More formally, when $\varphi \to \psi$ is a first-degree entailment:

$$\vdash_{\mathsf{REL}} \varphi \to \psi \quad \Longleftrightarrow \quad \begin{cases} \varphi \vdash_{\mathsf{CL}} \psi, \text{ and} \\ Var(\varphi) \cap Var(\psi) \neq \emptyset \end{cases}$$

It is also sometimes customary to think about such a set of valid first-degree entailments as a logical system on its own right. This can be easily

[2] That the Variable-Sharing Principle can be seen as necessary but not sufficient is salient by noticing that there have been many relevant logicians (Anderson and Belnap among them) who rejected implications that are valid in Classical Logic and comply with the Variable-Sharing Principle—e.g. those going from $\neg\varphi \wedge (\varphi \vee \psi)$ to ψ, or from $\varphi \wedge \neg\varphi$ to $\varphi \wedge \psi$. We would like to thank Francesco Paoli for urging us to highlight this fact. Also, as pointed out by an anoynmous reviewer, it should be noticed that the VSP should be mainly predicated of systems and not of formulae. It may only be used metaphorically in the latter cases—and even then, not without some risk, since for example the schema $\varphi \to (\psi \to \varphi)$ "satisfies" the VSP but proves $\psi \to (\varphi \to \varphi)$ in any system with $\varphi \to \varphi$ as an axiom and and Modus Ponens as a rule.

done by considering the "first-degree fragment" $\mathsf{L}_{\mathsf{fde}}$ of a logic L formulated in a language with an implication connective, where $\varphi \to \psi$ is a first-degree entailment:

$$\vdash_{\mathsf{L}} \varphi \to \psi \quad \Longleftrightarrow \quad \varphi \vdash_{\mathsf{L}_{\mathsf{fde}}} \psi$$

In this respect, it is instructive to notice that valid first-degree entailments in REL encode certain validities in CL. Indeed, it can be easily seen that $\mathsf{REL}_{\mathtt{fde}}$ is the FMLA-FMLA fragment of $\mathsf{CL}_{\mathsf{VSP}}$—that is to say, the fragment of CL that respects the Variable-Sharing Principle.[3] That we choose to denote this fragment by $\mathsf{CL}_{\mathsf{VSP}}$ can be explained by noting that, in general, we may denote with the FMLA-FMLA fragment of $\mathsf{L}_{\mathsf{VSP}}$ the subsystem of a given logic L whose valid inferences are only those valid inferences of L that satisfy the Variable-Sharing Principle.[4] That is to say:

$$\varphi \vdash_{\mathsf{L}_{\mathsf{VSP}}} \psi \quad \Longleftrightarrow \quad \begin{cases} \varphi \vdash_{\mathsf{L}} \psi, \text{ and} \\ Var(\varphi) \cap Var(\psi) \neq \emptyset \end{cases}$$

With these clarifications in mind, let us state what our goals are with regard to $\mathsf{REL}_{\mathtt{fde}}$—i.e., the FMLA-FMLA fragment of $\mathsf{CL}_{\mathsf{VSP}}$. We aim at providing, first, an extensional semantics and, second, a simple Gentzen-style sequent calculus for it. Before detailing how the paper is structured in order to achieve our goals, let us discuss two aspects of the title of this article which are connected to said objectives. As an anonymous reviewer pointed out, the term "fragment" is often used to signal language restriction, instead of delimitation of a certain concrete and precisely delineated subsystem. However, in absence of a better and widespread term for this purpose, we prefer to stick to it and hope that the reader does not fall prey of any ambiguity—thus, in what follows fragments will not be linguistic but deductive restrictions of logical systems. Then, as another anonymous reviewer points out, as there are many such fragments of Classical Logic

[3]The FMLA-FMLA fragment of a logic L is the restriction of L to what is called, e.g., in [25, p. 108] the FMLA-FMLA framework. That is to say, set of inferences that are valid in such a logic which have exactly one formula as a premise and exactly one formula as a conclusion.

[4]As an anonymous reviewer points out, this constitutes a slightly different variant of the VSP—a deductive version of the VSP, one may claim. Here we are not concerned with logics and their theorems involving an implication connective, but in logics and their valid inferences, regardless of whether the system in question has a certified implication connective or not.

that respect the VSP, one may doubt the definite description element of the title—i.e., calling $\mathsf{CL}_{\mathsf{VSP}}$ *the* fragment of said logic that respects the VSP. However, we think it is clear enough that singling out the system that has *all* the deductive validities of Classical Logic that also comply with the VSP makes it an unequivocal qualification for this denomination. Furthermore, taking into account that this is the greatest collection of such valid inferences of Classical Logic that respect the VSP, also explains why our target subsystem of Classical Logic is denoted by this definite description.

Thus, for the purpose of achieving our goals, our work is structured as follows. In Section 2, we analyze with a certain degree of generality the fragment of any Tarskian logic that respects the Variable-Sharing Principle, establishing that in some important cases the resulting systems belong to a peculiar family—that of the non-transitive p-logics. In Section 3, we provide appropriate semantics for $\mathsf{REL}_{\mathtt{fde}}$ with the help of certain structures called p-matrices that generalize the so-called regular logical matrices. In Section 4, we present a sound and complete Gentzen-style sequent calculus for $\mathsf{REL}_{\mathtt{fde}}$ whose rules are bound to certain linguistic restrictions, guaranteeing the satisfaction of the Variable-Sharing Principle. Finally, in Section 5 we wrap up some concluding remarks and point towards directions of future work.

This being said, before delving into the proper contents of the article, let us briefly make explicit that we will be working with a propositional language $\mathcal{L}$ counting with a denumerable set Var of propositional variables $p, q, r, \ldots$ and with logical connectives $\neg, \wedge, \vee$—intended to represent negation, conjunction, and disjunction, respectively. Thus, $\mathbf{FOR}(\mathcal{L})$ will be the algebra of well-formed formulae, standardly defined, whose carrier set is the set of well-formed formulae $FOR(\mathcal{L})$. In this respect, lower case Roman letters $\varphi, \psi, \chi, \ldots$ will be considered as schematic formulae, whereas upper case Greek letters $\Gamma, \Delta, \Theta, \ldots$ will be considered as schematic sets of formulae.

2. The fragment of a Tarskian logic that respects the Variable-Sharing Principle

In this section we analyze the fragment of any Tarskian logic that respects the Variable-Sharing Principle, paying special attention to the kind of systems that results from applying such a sieve, and to the semantic structures usually associated with said fragments. Thus, we notice that sometimes constraining Tarskian logics in this way results in a peculiar kind of systems called non-transitive p-logics. In this vein, we discuss logical matrices and related structures generalizing them, called p-matrices, furthermore focusing on some sufficient conditions that guarantee the satisfaction of the Variable-Sharing Principle in the systems induced by such matrices.

To begin with, let us recall what the literature usually understands by a Tarskian logic. By this it is usually meant a logical system formulated in the SET-FMLA framework, whose underlying consequence relation $\vdash$ has the following properties, where $\Gamma, \Delta \subseteq FOR(\mathcal{L})$ and $\varphi, \psi \in FOR(\mathcal{L})$:[5]

- $\Gamma \vdash \varphi$ if $\varphi \in \Gamma$ (Reflexivity)
- If $\Gamma \vdash \varphi$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash \varphi$ (Monotonicity)
- If $\Delta \vdash \varphi$ and $\Gamma \vdash \psi$ for every $\psi \in \Delta$, then $\Gamma \vdash \varphi$ (Transitivity)

In our case though—since we are interested in discussing semantics and calculi for $\mathsf{REL}_{\mathtt{fde}}$ which is the FMLA-FMLA fragment of $\mathsf{CL}_{\mathsf{VSP}}$—we are interested in the definition of Tarskian logics in the FMLA-FMLA framework. Whence, if a logic counts with connectives $\wedge$ and $\vee$ (to be interpreted, respectively, as conjunction and disjunction) we may say that a Tarskian logic is a logical system whose consequence relation $\vdash$ enjoys the following features, where $\varphi, \psi, \gamma, \delta \in FOR(\mathcal{L})$:

- $\varphi \vdash \varphi$ (Reflexivity)
- If $\varphi \vdash \psi$, then $\varphi \wedge \gamma \vdash \psi$ and $\varphi \vdash \psi \vee \delta$ (Monotonicity)
- If $\varphi \vdash \psi$ and $\psi \vdash \gamma$, then $\varphi \vdash \gamma$ (Transitivity)

Now, regarding the semantic interpretation of Tarskian logics, it is interesting to notice that all such systems can be semantically characterized by *logical matrices*. For a given propositional language $\mathcal{L}$ a logical matrix $\mathcal{M}$ is a pair $\langle \mathbf{A}, D \rangle$, where $\mathbf{A}$ is an algebra of the same similarity type than $\mathcal{L}$, and D is a subset of A, the universe or carrier set of $\mathbf{A}$. Letting an

[5] As pointed out by an anonymous reviewer, if we take into account sets of formulae—as opposed to sequences, lists, or multisets thereof—Reflexivity and Transitivity below imply Monotonicity. These properties are expressed here as standardly defined, e.g., in [19, p. 12].

$\mathcal{M}$-valuation v be an homomorphism from $\mathbf{FOR}(\mathcal{L})$ to $\mathbf{A}$, a logical matrix $\mathcal{M}$ induces a Tarskian consequence relation $\vDash_{\mathcal{M}}$ in the following standard manner, where $\Gamma \cup \{\varphi\} \subseteq FOR(\mathcal{L})$:

$$\Gamma \vDash_{\mathcal{M}} \varphi \iff \text{for every } \mathcal{M}\text{-valuation } v\text{: if } v(\Gamma) \subseteq D, \text{ then } v(\varphi) \in D$$

In this vein, it is a well-known result in Abstract Algebraic Logic—proved by R. Wójcicki in [44]—that for any Tarskian logic whose underlying consequence relation is $\vdash_{\mathsf{L}}$, there is a class $\mathbb{M}$ of logical matrices such that $\vdash_{\mathsf{L}} = \cap\{\vDash_{\mathcal{M}} \mid \mathcal{M} \in \mathbb{M}\}$. Whenever such a class is a singleton $\{\mathcal{M}\}$, we may say that $\vdash_{\mathsf{L}} = \vDash_{\mathcal{M}}$. In such a case, we will take the liberty of referring to $\vDash_{\mathcal{M}}$ as $\vDash_{\mathsf{L}}$. Thus, logical matrices allow understanding logical consequence in the context of Tarskian logics as preservation of designated values. Whence, *if all the premises are assigned a designated value, so must the conclusion.* This generalizes the idea, dear to Classical Logic, that valid arguments are such that if the premises are all true, so must be the conclusion. Of course, all the previous remarks apply equally to a Tarskian logic formulated in the Fmla-Fmla framework—just that, instead of talking of a plurality of premises, we just need to consider a single premise.

Having clarified what Tarskian logics are, we may now move on to consider the main question of this section, namely, what kind of system results from focusing on the fragment of a Tarskian logic that respects the Variable-Sharing Principle. We hope that answering this question will provide us some clarity with regard to the semantic and proof-theoretic characterization of our target logic, $\mathsf{REL}_{\mathtt{fde}}$. But, to answer this question we must consider two scenarios. In the first, the Tarskian logic in question already satisfies the Variable-Sharing Principle. In the second, it does not. It is obvious then, that applying such a constraint to a logic in the first scenario does not change anything. Thus, we obtain the same system we started with.[6] It is the second scenario that is more interesting, because if the Tarskian logic we start with does not respect the Variable-Sharing Principle, then the system resulting from filtering out all its irrelevant impurities can be quite non-standard.

To observe why this may be the case, consider the following. For a logic whose underlying consequence relation is $\vdash$ let us a say that a *theorem* is a

[6] One case of this sort is the logic $\mathsf{E}_{\mathtt{fde}}$, induced by a logical matrix built on top of the 4-element Belnap-Dunn algebra—discussed, e.g., in [15] by J. M. Dunn and in [4] by N. Belnap.

formula φ such that $\psi \vdash \varphi$, for all $\psi \in FOR(\mathcal{L})$, whereas an *anti-theorem* is a formula φ such that $\varphi \vdash \psi$, for all $\psi \in FOR(\mathcal{L})$. It should be clearly noticeable that a logic L cannot satisfy the Variable-Sharing Principle if it has either theorems or anti-theorems. Furthermore, as we will show below, if L has either theorems or anti-theorems, its fragment satisfying the Variable-Sharing Principle results in a logic that is *not Tarskian*—for it is non-transitive.

Interestingly enough, although non-transitive systems are not Tarskian logics, some of them belong to a special kind that generalizes Tarskian logics. These are the so-called *p-logics*, developed firstly by S. Frankowksi in [20]. When formulated either in the Set-Fmla or the Fmla-Fmla framework, *p*-logics should be considered as systems whose underlying consequence relation *respects both Reflexivity and Monotonicity, although it does not necessarily respect Transitivity.* By this, we mean that *p*-logics that are transitive are Tarskian logics, whereas *p*-logics that are non-transitive are not—and can be, thus, regarded as "proper" *p*-logics in some sense. In this spirit, consequence relations underlying proper *p*-logics can be rightfully referred to as proper *p*-consequence relations.[7]

Along these lines, it can be easily shown that whenever we start with a Tarskian logic L and later focus on its fragment satisfying the Variable-Sharing Principle—that is on $\mathsf{L}_{\mathsf{VSP}}$—there are some conditions that L may have which guarantee that $\mathsf{L}_{\mathsf{VSP}}$ be a non-transitive *p*-logic. These can be summarized as follows.

Observation 2.1. If L is a Tarskian logic and has either theorems or anti-theorems, then the system $\mathsf{L}_{\mathsf{VSP}}$ is a non-transitive *p*-logic.

Proof: We first establish the Reflexivity and Monotonicity of $\mathsf{L}_{\mathsf{VSP}}$, for which it is important to remember the meaning that these properties have in the context of Tarskian logics formulated in the Fmla-Fmla framework. To prove the former, suppose $\varphi \vdash_{\mathsf{L}} \varphi$. Trivially, $Var(\varphi) \cap Var(\varphi) \neq \emptyset$. Whence, $\varphi \vdash_{\mathsf{L}_{\mathsf{VSP}}} \varphi$. To prove the latter, suppose $\varphi \vdash_{\mathsf{L}} \psi$ and $Var(\varphi) \cap Var(\psi) \neq \emptyset$, whence $\varphi \vdash_{\mathsf{L}_{\mathsf{VSP}}} \psi$. Since L is assumed to be Tarskian, in the Fmla-Fmla framework its Monotonicity amounts to the following inference being valid, for all $\gamma \in FOR(\mathcal{L})$: $\varphi \wedge \gamma \vdash_{\mathsf{L}} \psi$. Simple set-theoretic

[7] The remark that some non-transitive systems are *p*-logics has substance to it as the latter comprises, e.g., reflexive systems. Thus, non-transitive systems that are also non-reflexive cannot be regarded as *p*-logics and therefore not all non-transitive systems are of this kind.

reasoning allows to establish that $Var(\varphi \wedge \gamma) \cap Var(\psi) \neq \emptyset$. Whence, $\varphi \wedge \gamma \vdash_{\mathsf{L_{VSP}}} \psi$. Similar reasoning establishes that $\varphi \vdash_{\mathsf{L_{VSP}}} \psi \vee \delta$.

We now prove that if L has either theorems or anti-theorems, then $\mathsf{L_{VSP}}$ is non-transitive—and, thus, a "proper" p-logic. For this purpose, consider first that L has theorems, letting ψ be a theorem, and φ and γ be arbitrary formulae, such that $Var(\varphi) \cap Var(\psi) = \emptyset$, but $Var(\varphi) \cap Var(\gamma) \neq \emptyset$ and $Var(\gamma) \cap Var(\psi) \neq \emptyset$. Since L is assumed to be Tarskian, in the FMLA-FMLA framework its Monotonicity implies the validity of $\varphi \vdash_{\mathsf{L}} \varphi \vee \gamma$, for all $\gamma \in FOR(\mathcal{L})$. Because of ψ being a theorem, we know that $\varphi \vee \gamma \vdash_{\mathsf{L}} \psi$. Given $Var(\varphi) \cap Var(\varphi \vee \gamma) \neq \emptyset$, and $Var(\varphi \vee \gamma) \cap Var(\psi) \neq \emptyset$—the latter by hypothesis—the previous remarks guarantee that $\varphi \vdash_{\mathsf{L_{VSP}}} \varphi \vee \gamma$ and $\varphi \vee \gamma \vdash_{\mathsf{L_{VSP}}} \psi$, although $\varphi \nvdash_{\mathsf{L_{VSP}}} \psi$. Thus, if L is a Tarskian logic that has theorems, $\mathsf{L_{VSP}}$ is a non-transitive p-logic.

The case for anti-theorems is analogous, and thus we leave it to the reader as an exercise. □

Now, let us recall that the aim of this article is to provide a simple semantics and calculus for $\mathsf{REL_{fde}}$—that is to say, the FMLA-FMLA fragment of $\mathsf{CL_{VSP}}$. With the information of the previous result in hand, we may safely claim that $\mathsf{REL_{fde}}$ is a non-transitive p-logic.[8] But, if this is the case, it would be interesting to know whether p-logics in general (and non-transitive p-logics as a special case) can be associated with certain semantic structures, just like Tarskian logics can be identified with logical matrices.

Happily, the answer is affirmative in this respect. Indeed, there is a correspondence between p-logics and a family of structures that generalizes logical matrices—opportunely called *logical p-matrices*. Thus, for a given propositional language $\mathcal{L}$ a logical p-matrix $\mathcal{M}$ is a triple $\langle \mathbf{A}, D_p, D_c \rangle$, where $\mathbf{A}$ is an algebra of the same similarity type than $\mathcal{L}$, and D_p, D_c are subsets of A, the universe or carrier set of $\mathbf{A}$, such that $D_p \subseteq D_c$. These sets should be understood as a set of *designated* values for formulae conceived *as premises*, and a set of *designated* values for formulae conceived *as conclusions*. Hence, by the restrictions imposed above, if a formula is designated as a premise it must be designated as a conclusion—although if it is not designated as

[8] It should be duly noted that the non-transitive nature of $\mathsf{REL_{fde}}$ *as a deductive system* stems from the non-transitivity of the implication involved in the first-degree entailments that are valid in Epstein's logic REL, which was already discussed in [16] and [30]. We would like to thank an anonymous reviewer for urging us to clarify this.

a premise, it may well be designated as a conclusion.[9] Letting an $\mathcal{M}$-valuation v be an homomorphism from $\mathbf{FOR}(\mathcal{L})$ to $\mathbf{A}$, a logical p-matrix $\mathcal{M}$ induces a p-consequence relation $\vDash_{\mathcal{M}}$ in the following, standard manner, where $\Gamma \cup \{\varphi\} \subseteq FOR(\mathcal{L})$:

$$\Gamma \vDash_{\mathcal{M}} \varphi \Longleftrightarrow \text{for every } \mathcal{M}\text{-valuation } v\text{: if } v(\Gamma) \subseteq D_p, \text{ then } v(\varphi) \in D_c$$

In this vein, S. Frankowski's shows in [20, p. 47] that for any p-logic whose underlying consequence relation is $\vdash_{\mathsf{L}}$, there is a class $\mathbb{M}$ of logical p-matrices such that $\vdash_{\mathsf{L}} = \bigcap\{\vDash_{\mathcal{M}} \mid \mathcal{M} \in \mathbb{M}\}$. Whenever such a class is a singleton $\{\mathcal{M}\}$, we may say that $\vdash_{\mathsf{L}} = \vDash_{\mathcal{M}}$. In such a case, we will take the liberty of referring to $\vDash_{\mathcal{M}}$ as $\vDash_{\mathsf{L}}$. It should be noticed that whenever $D_p = D_c$, the corresponding p-matrix is actually a regular logical matrix—justifying the claim that the former kind of structures generalizes the latter. In a nutshell, if being designated as a premise is the same as being designated as a conclusion, then we are in the presence of a regular logical matrix. When this is not the case and the p-matrix in question is not a regular logical matrix, it is interesting to observe that logical consequence cannot be understood as preservation of designated values, in the traditional sense. It is perhaps better to say that it can be understood in terms of preservation in a more liberal or generalized reading. Whence, *if all the premises are assigned a designated value for premises, then the conclusion must be assigned a designated value for conclusions.* Once again, the previous remarks apply equally to a p-logic formulated in the Fmla-Fmla framework.

Information of this sort is useful, as it suggests to us that $\mathsf{REL}_{\mathtt{fde}}$ being a proper p-logic, its semantics should be given in terms of a proper p-matrix. This, of course, does not suggest in itself the features of the semantics in question.[10] For this purpose, let us review a number of remarks that will make our approximation below more intelligible. These observations concern some sufficient—although not necessary—features that a

[9] In S. Frankowski's words, this formalizes the idea that p-consequence relations represent the transition from premises which may be held to a stricter standard (of acceptance, or belief, or truth) to conclusions which may be held to a more tolerant standard—constituting plausible (whence the "p") conclusions rather than strictly certain conclusions thereof.

[10] Notice that a proper p-logic cannot receive other than proper p-matrix semantics. Were someone to offer regular matrix semantics for it, then the resulting system will be transitive, and thus not a proper p-logic. Therefore, it will not be a semantics *for it*. We would like to thank an anonymous reviewer for urging us to clarify this.

logical matrix, and a p-matrix, in turn, may have that will make the system thereby induced to comply with the Variable-Sharing Principle. In this regard, adapting some of the terminology used in their article, we may paraphrase G. Robles and J. Mendez in [37] (see also [36] and [38]) by stating the following result.

LEMMA 2.2 ([37]). *Let* L *be a Tarskian logic induced by the logical matrix* $\langle \mathbf{A}, D \rangle$*, formulated in the propositional language counting with connectives* $\neg, \wedge, \vee$*. If there are* $a_1, a_2 \in A$ *such that:*

- $a_1 \in D$ *and* $\neg^{\mathbf{A}}(a_1) = \wedge^{\mathbf{A}}(a_1, a_1,) = \vee^{\mathbf{A}}(a_1, a_1,) = a_1$
- $a_2 \notin D$ *and* $\neg^{\mathbf{A}}(a_2) = \wedge^{\mathbf{A}}(a_2, a_2,) = \vee^{\mathbf{A}}(a_2, a_2,) = a_2$*.*

Then, L *satisfies the Variable-Sharing Principle.*

We can easily see that these remarks can be straightforwardly generalized so as to provide an analogous result concerning p-matrices, instead of regular logical matrices. To discuss such a generalization we now turn.

LEMMA 2.3. *Let* L *be a* p*-logic induced by the* p*-logical matrix* $\langle \mathbf{A}, D_p, D_c \rangle$*, formulated in the propositional language counting with connectives* $\neg, \wedge, \vee$*. If there are* $a_1, a_2 \in A$ *such that:*

- $a_1 \in D_p$ *and* $\neg^{\mathbf{A}}(a_1) = \wedge^{\mathbf{A}}(a_1, a_1,) = \vee^{\mathbf{A}}(a_1, a_1,) = a_1$
- $a_2 \notin D_c$ *and* $\neg^{\mathbf{A}}(a_2) = \wedge^{\mathbf{A}}(a_2, a_2,) = \vee^{\mathbf{A}}(a_2, a_2,) = a_2$*.*

Then, L *satisfies the Variable-Sharing Principle.*

PROOF: Assume L is a p-logic induced by the p-logical matrix $\langle \mathbf{A}, D_p, D_c \rangle$, where all the operations and the truth-values involved have the conditions outlined above. Suppose, then that there is a valid inference $\varphi \vDash_{\mathsf{L}} \psi$ such that $Var(\varphi) \cap Var(\psi) = \emptyset$. Then, consider an L-valuation v such that:

$$v(p) = \begin{cases} a_1 & \text{if } p \in \varphi \\ a_2 & \text{otherwise} \end{cases}$$

By the conditions assumed above, we know that $v(\varphi) = a_1$, whereas $v(\psi) = a_2$. Thus, $v(\varphi) \in D_p$ while $v(\psi) \notin D_c$, whence v is a valuation witnessing $\varphi \nvDash_{\mathsf{L}} \psi$. This contradicts our initial assumption, which then implies that if the aforementioned conditions are met, then every valid inference satisfies the Variable-Sharing Principle. □

In what follows, we will use these remarks in the investigation of semantic structures that will induce the fragment of Classical Logic that respects the Variable-Sharing Principle—i.e., in introducing semantics for $\mathsf{REL}_{\mathtt{fde}}$. By this, we mean that we will build a p-matrix that will induce the logic in question, where such a p-matrix will have two truth-values behaving in the way described by Lemma 2.3.

3. Semantics

The aim of this section is to present a simple extensional semantics for $\mathsf{REL}_{\mathtt{fde}}$. In this regard, it should be noted that algebraic semantics—particularly, semantics where logical consequence is defined in terms of certain order-theoretic relations holding between the elements of the carrier set of a given algebra as, e.g., in L. Humberstone's [25, p. 246]—have been introduced both for the full system REL by R. Epstein in [16] and for the restricted fragment $\mathsf{REL}_{\mathtt{fde}}$ that concerns us, by F. Paoli in [31]. Additionally, F. Paoli presents a more traditional algebraic semantics for it in [30], in the form of a class of products of Boolean algebras and τ-semilattices.

However, no extensional semantics where logical consequence is understood in terms of the assignment of designated values of some kind to premises and conclusions has been discussed so far, whence the material below constitutes a novel development in this respect.[11] Of course, since $\mathsf{REL}_{\mathtt{fde}}$ is a non-transitive p-logic and therefore a non-Tarskian logic, if it happens to be possible for it to be induced by a logical matrix of sorts, such a structure will not be a regular logical matrix, but rather a proper p-matrix. Thus, in what follows we present a route to arrive at such a p-matrix, highlighting that there might be other equally interesting manners of landing the same results.

In particular, we will go through a two-step process in order to define our target p-matrix. This process will consist, on the one hand, in finding a proper p-matrix that induces CL and, on the other hand, in extending said

[11] As an anonymous reviewer points out, the semantics by Paoli in [30] are extensional, although logical consequence there is not understood in terms of the usual notion of preservation of designated values for logical matrices, but instead in terms of the satisfaction of a binary relation *Imp*.

p-matrix with additional truth-values so as to guarantee the satisfaction of the Variable-Sharing Principle—without causing any other logical side-effects, as invalidating classically valid inferences that satisfy this principle.

Our *first step in the way to arriving at a p-matrix semantics for* $\mathsf{REL}_{\mathtt{fde}}$ is the presentation of a proper p-matrix that will induce CL. This already suggests a few discussions in itself. To wit, if the matrix in question is a proper p-matrix but not a regular matrix, one may wonder whether the resulting logic will be identical to CL, or if it will differ with this system in some respect. Lengthy debates have been had in the past few years in this regard, mostly revolving around the logic ST defended by Cobreros, Égré, Ripley and van Rooij in many works—some of which include [9], [8], [10], [11], [34] and [35]. For future reference, the logic ST is induced by the p-matrix $\langle \mathbf{SK}, \{\mathbf{t}\}, \{\mathbf{t}, \mathbf{n}\}\rangle$ built on top of the 3-element strong Kleene algebra $\mathbf{SK}$ from S. Kleene's [26], whose carrier set is $\{\mathbf{t}, \mathbf{n}, \mathbf{f}\}$ and whose characteristic operations can be presented in the form of the "truth-tables" appearing in Figure 1. These authors championed the view that Classical Logic can be legitimately seen as induced by a structure of this sort, whereas other scholars contested that although the resulting system called by them ST coincided with CL with regard to its set of valid inferences it did not coincide in what regards to its valid *metainferences*—which, roughly speaking, refers to inferences between inferences themselves. The jury is still out in this trial, as it is in a related meta-discussion, that of trying to determine whether the question itself is substantial or terminological.[12]

	$\neg$
t	**f**
n	**n**
f	**t**

$\wedge$	**t**	**n**	**f**
t	**t**	**n**	**f**
n	**n**	**n**	**f**
f	**f**	**f**	**f**

$\vee$	**t**	**n**	**f**
t	**t**	**t**	**t**
n	**t**	**n**	**n**
f	**t**	**n**	**f**

Figure 1. The strong Kleene truth-tables

For the purpose of this article, however, we will admit that certain p-matrices can characterize CL, at least in what concerns to its set of valid inferences. This is instrumental for us, given the task we set for ourselves of trying to find a simple semantics for those inferences that not only are valid

[12]Some of the crucial works on this debate are B. Dicher and F. Paoli's [14], E. Barrio, F. Pailos and D. Szmuc's [3], and C. Scambler's [39].

in Classical Logic, but that also respect the Variable-Sharing Principle. As a consequence of adopting this point of view, we will entertain these p-matrices as inducing CL, although we will sometimes refer to the systems thereby induced with other names—because this will be useful for matters of clarity below, when we extend these structures to arrive at semantics for $\mathsf{REL}_{\mathtt{fde}}$.

Interestingly enough, recently p-matrix semantics for CL, different from those discussed by Cobreros, Égré, Ripley and van Rooij, have been presented. This alternative option is built on top of the 3-element weak Kleene algebra **WK**—instead of the aforementioned strong Kleene algebra. Thus, either implicitly or explicitly it is possible to find semantics along these lines in F. Correia's [13], F. Paoli and M. Pra Baldi's [32] and in our own [42]. In order for things to be clear in what follows, let us state here that the **WK** algebra is the structure whose carrier set is $\{\mathbf{t}, \mathbf{e}, \mathbf{f}\}$ and whose characteristic operations can be presented in the form of the "truth-tables" appearing in Figure 2. In this respect, it was either shown or mentioned in the previously referred works that an interesting logic that we call wST can be shown to have the same valid inferences that CL, thereby offering a proper p-matrix semantics for it.

	$\neg$
t	**f**
e	**e**
f	**t**

$\wedge$	**t**	**e**	**f**
t	**t**	**e**	**f**
e	**e**	**e**	**e**
f	**f**	**e**	**f**

$\vee$	**t**	**e**	**f**
t	**t**	**e**	**t**
e	**e**	**e**	**e**
f	**t**	**e**	**f**

Figure 2. The weak Kleene truth-tables

Definition 3.1. wST is the p-logic induced by the following p-matrix:

$$\langle \mathbf{WK}, \{\mathbf{t}\}, \{\mathbf{t}, \mathbf{e}\} \rangle$$

Lemma 3.2 ([32], [42]). *For all $\varphi, \psi \in FOR(\mathcal{L})$:*

$$\varphi \vDash_{\mathsf{CL}} \psi \quad \Longleftrightarrow \quad \varphi \vDash_{\mathsf{wST}} \psi$$

Before moving on to the further extension of this p-matrix in order to arrive at a structure inducing $\mathsf{REL}_{\mathtt{fde}}$, let us take a moment to understand why the introduction of the third value **e** is not disruptive, i.e., why the resulting logic has the same valid inferences than CL. The explanation appearing next is a straightforward adaptation of the one used to explain why ST has the same valid inferences than CL, in many places of the literature.

Let us first observe the *exclusion of* $\mathbf{e}$ *from the set of designated values for premises* guarantees that no inferences will be rendered invalid because the premises were assigned this new value. In other words, that only classically-satisfiable premises can be the premises of an inference having a counterexample. Secondly, the *inclusion of* $\mathbf{e}$ *in the set of designated values for conclusions* guarantees that no inference will be invalid because the conclusion was assigned this new value. Again, this means that only classically-falsifiable conclusions can be the conclusion of an inference having a counterexample. In a nutshell, with the help of the linguistic resources available, the introduction of the non-classical value $\mathbf{e}$ is ineffective for the generation of new counterexamples to classically valid inferences. Furthermore, whenever a wST-valuation constitutes a counterexample to some inference, the fact that the operations in **WK** are monotonic with regard to the partial order $\mathbf{i} \leq \mathbf{t}, \mathbf{i} \leq \mathbf{f}, \mathbf{i} \leq \mathbf{i}, \mathbf{t} \leq \mathbf{t}, \mathbf{f} \leq \mathbf{f}$ guarantees that these valuations can be transformed into Boolean valuations without altering the values of complex formulae assigned $\mathbf{t}$ and $\mathbf{f}$.[13]

Our *second step in the way to arriving at a proper p-matrix for* $\mathsf{REL}_{\mathtt{fde}}$ will be, then, to appropriately extend the previously discussed p-matrix in the spirit of the remarks made in Lemma 2.3. That is to say, we will have a p-matrix whose underlying algebra has two additional values with respect to the **WK** algebra—one of such values will be designated for premises and conclusions, while the other will be undesignated for premises and conclusions. In addition, these two elements will behave in the way described by Lemma 2.3, that is to say, whenever they are negated, conjoined with themselves, or disjoined with themselves, they will respectively return the same value. For reasons that will be clear below, let us refer to these truth-values as $\mathbf{o}_1^{\mathbf{e}}$ and $\mathbf{o}_2^{\mathbf{e}}$, respectively.

[13]From these remarks one may take away the fact that for any 3-element algebra $\mathbf{A}$ with carrier set $\{\mathbf{t}, \mathbf{i}, \mathbf{f}\}$ having the 2-element Boolean algebra as a subalgebra, the p-matrix $\langle \mathbf{A}, \{\mathbf{t}\}, \{\mathbf{t}, \mathbf{i}\}\rangle$ induces a logic that has the same valid inferences that CL, as long as all the operations in $\mathbf{A}$ are monotonic with regard to the aforementioned partial order—an observation already present *in nuce* in K. Schütte's [40], as reviewed by J.-Y. Girard in [23, p. 162]. As mentioned in several places, among them in S. Kripke's [27], both the operations in the **SK** algebra and the operations in the **WK** algebra are monotonic in this way.

However, on top of securing this behavior, we need to make sure that the inclusion of such values is as effective and as innocuous as desired. In other words, that their inclusion renders invalid all inferences that are valid in CL which do not comply with the Variable-Sharing Principle, without invalidating some inferences that do comply with said principle. For this purpose, one way to extend the **WK** algebra to satisfy this demands is to allow for two additional elements working exactly like the non-classical value $\mathbf{e}$ whenever premises and conclusions share a propositional variable. Thus, it should be understood that, whenever premises and conclusions share a propositional variable, it should be impossible to generate counterexamples to the validity of the inference in question by assigning the formulae involved the newly introduced truth-values in a convenient way.

This can be done by letting the result of every operation in which the elements $\mathbf{o}^{\mathbf{e}}_1$ and $\mathbf{o}^{\mathbf{e}}_2$ are some, but not all of the inputs, be calculated as if these truth-values were replaced by $\mathbf{e}$—additionally, letting the result be $\mathbf{o}^{\mathbf{e}}_1$ if all inputs were $\mathbf{o}^{\mathbf{e}}_1$, and $\mathbf{o}^{\mathbf{e}}_2$ if all inputs were $\mathbf{o}^{\mathbf{e}}_2$, respectively. This guarantees that *new counterexamples to classically valid inferences* will only emerge when premises can be assigned the truth-value $\mathbf{o}^{\mathbf{e}}_1$ and conclusions can be assigned the truth-value $\mathbf{o}^{\mathbf{e}}_2$. A situation only possible if premises and conclusions do not share any propositional variable.

Finally, before moving on to defining the ingredients of the p-matrix inducing $\mathsf{REL}_{\mathtt{fde}}$, let us observe that the requirements above can be translated into general algebraic terminology, as follows.

Definition 3.3. An algebra $\mathbf{A}$ has distinct elements $k, \mathbf{o}^k \in A$ such that $\mathbf{o}^k$ "mimics" k if and only if for all n-ary operations ¶ and all $\{a_1, \ldots, a_n\} \subseteq A$:

$$\text{if } \{\mathbf{o}^k\} \subsetneq \{a_1, \ldots, a_n\}, \text{ then } ¶^{\mathbf{A}}(a_1, \ldots, a_n) = ¶^{\mathbf{A}}((a_1, \ldots, a_n)[\mathbf{o}^k/k])$$

where $(a_1, \ldots, a_n)[\mathbf{o}^k/k]$ is the result of replacing each occurrence of $\mathbf{o}^k$ for an occurrence of k in $a_1, \ldots, a_n$.

Naturally, this can be generalized to algebras with a set $\{\mathbf{o}^k_1, \ldots, \mathbf{o}^k_n\}$ of elements that "mimic" an element k.

Definition 3.4. An algebra $\mathbf{A}$ has a universally idempotent element k if and only if for all n-ary operations ¶ and all $\{a_1, \ldots, a_n\} \subseteq A$:

$$\text{if } \{k\} = \{a_1, \ldots, a_n\}, \text{ then } ¶^{\mathbf{A}}(a_1, \ldots, a_n) = k$$

Definition 3.5. Given an algebra $\mathbf{A}$, the algebra $\mathbf{A}[\mathbf{o}^k]$ is its extension with a universally idempotent element $\mathbf{o}^k \notin A$ that "mimics" an element $k \in A$, such that for all n-ary operations ¶ and all $\{a_1, \ldots, a_n\} \subseteq A \cup \{\mathbf{o}^k\}$:

$$\P^{\mathbf{A}[\mathbf{o}^k]}(a_1, \ldots, a_n) = \begin{cases} \mathbf{o}^k & \text{if } \{\mathbf{o}^k\} = \{a_1, \ldots, a_n\} \\ \P^{\mathbf{A}}((a_1, \ldots, a_n)[\mathbf{o}^k/k]) & \text{if } \{\mathbf{o}^k\} \subsetneq \{a_1, \ldots, a_n\} \\ \P^{\mathbf{A}}(a_1, \ldots, a_n) & \text{otherwise} \end{cases}$$

where $(a_1, \ldots, a_n)[\mathbf{o}^k/k]$ is the result of replacing each occurrence of $\mathbf{o}^k$ for an occurrence of k in $a_1, \ldots, a_n$.

Again, this can be generalized to extended algebras $\mathbf{A}[\mathbf{o}^k_1, \ldots, \mathbf{o}^k_n]$ with a set of universally idempotent elements $\{\mathbf{o}^k_1, \ldots, \mathbf{o}^k_n\}$ that "mimic" an element k previously available in the starting algebra $\mathbf{A}$—which can be redescribed as $\mathbf{A}[\mathbf{o}^k_1, \ldots, \mathbf{o}^k_{n-1}][\mathbf{o}^k_n] = \ldots = \mathbf{A}[\mathbf{o}^k_1] \ldots [\mathbf{o}^k_n]$.

Having clarified this, our requirements above concerning a semantic structure for $\mathsf{REL}_{\mathtt{fde}}$ can otherwise be phrased as saying that we need to extend the **WK** algebra with two universally idempotent elements that mimic $\mathbf{e}$, one of which should be designated for premises and conclusions in the context of the extended p-matrix, whereas the other should be undesignated for premises and conclusions in the context of the extended p-matrix. This algebra we call, correspondingly, the 5-element algebra $\mathbf{WK}[\mathbf{o}^{\mathbf{e}}_1 \mathbf{o}^{\mathbf{e}}_2]$, whose carrier set can be conspicuously described as $\{\mathbf{t}, \mathbf{o}^{\mathbf{e}}_1, \mathbf{e}, \mathbf{o}^{\mathbf{e}}_2, \mathbf{f}\}$ and whose operations can be described by the "truth-tables" in Figure 3.[14]

With these tools in hand, we turn to defining our target non-transitive p-logic and to proving that its Fmla-Fmla fragment is equal to $\mathsf{REL}_{\mathtt{fde}}$, that is to say, to the Fmla-Fmla fragment of $\mathsf{CL}_{\mathsf{VSP}}$.

Definition 3.6. $\mathsf{wST}[\mathbf{o}^{\mathbf{e}}_1 \mathbf{o}^{\mathbf{e}}_2]$ is the logic induced by the following p-matrix:

$$\langle \mathbf{WK}[\mathbf{o}^{\mathbf{e}}_1 \mathbf{o}^{\mathbf{e}}_2], \{\mathbf{t}, \mathbf{o}^{\mathbf{e}}_1\}, \{\mathbf{t}, \mathbf{o}^{\mathbf{e}}_1, \mathbf{e}\} \rangle$$

[14]Notice that this algebra can be seen as the extension with a universally idempotent element that mimics the infectious element of a 4-element algebra appearing in the article [31] by F. Paoli, referred to as the **FP** algebra in [41] by us. Whence, in turn, this last structure can be equally described as the extension of the **WK** algebra with a universally idempotent element that mimics the infectious element $\mathbf{e}$—that is to say, as $\mathbf{WK}[\mathbf{o}^{\mathbf{e}}]$.

	$\neg$
$\mathbf{t}$	$\mathbf{f}$
$\mathbf{o}_1^{\mathbf{e}}$	$\mathbf{o}_1^{\mathbf{e}}$
$\mathbf{e}$	$\mathbf{e}$
$\mathbf{o}_2^{\mathbf{e}}$	$\mathbf{o}_2^{\mathbf{e}}$
$\mathbf{f}$	$\mathbf{t}$

$\wedge$	$\mathbf{t}$	$\mathbf{o}_1^{\mathbf{e}}$	$\mathbf{e}$	$\mathbf{o}_2^{\mathbf{e}}$	$\mathbf{f}$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{f}$
$\mathbf{o}_1^{\mathbf{e}}$	$\mathbf{e}$	$\mathbf{o}_1^{\mathbf{e}}$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{e}$
$\mathbf{e}$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{e}$
$\mathbf{o}_2^{\mathbf{e}}$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{o}_2^{\mathbf{e}}$	$\mathbf{e}$
$\mathbf{f}$	$\mathbf{f}$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{f}$

$\vee$	$\mathbf{t}$	$\mathbf{o}_1^{\mathbf{e}}$	$\mathbf{e}$	$\mathbf{o}_2^{\mathbf{e}}$	$\mathbf{f}$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{t}$
$\mathbf{o}_1^{\mathbf{e}}$	$\mathbf{e}$	$\mathbf{o}_1^{\mathbf{e}}$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{e}$
$\mathbf{e}$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{e}$
$\mathbf{o}_2^{\mathbf{e}}$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{o}_2^{\mathbf{e}}$	$\mathbf{e}$
$\mathbf{f}$	$\mathbf{t}$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{f}$

Figure 3. The five-valued $\mathsf{wST}[\mathbf{o}_1^{\mathbf{e}}\mathbf{o}_2^{\mathbf{e}}]$ truth-tables

LEMMA 3.7. *For every $\varphi, \psi \in FOR(\mathcal{L})$ if there is a $\mathsf{wST}[\mathbf{o}_1^{\mathbf{e}}\mathbf{o}_2^{\mathbf{e}}]$-valuation v such that either $v(\varphi) = \mathbf{t}$ and $v(\psi) = \mathbf{o}_2^{\mathbf{e}}$, or $v(\varphi) = \mathbf{o}_1^{\mathbf{e}}$ and $v(\psi) = \mathbf{f}$, then $Var(\varphi) \cap Var(\psi) = \emptyset$.*

PROOF: Firstly, that $v(\varphi) = \mathbf{t}$ implies that for all $p \in Var(\varphi)$, $v(p) \in \{\mathbf{t}, \mathbf{f}\}$. Simultaneously, that $v(\psi) = \mathbf{o}_2^{\mathbf{e}}$ implies that for all $q \in Var(\psi)$, $v(q) = \mathbf{o}_2^{\mathbf{e}}$. Whence, $Var(\varphi) \cap Var(\psi) = \emptyset$. Secondly, that $v(\varphi) = \mathbf{o}_1^{\mathbf{e}}$ implies that for all $p \in Var(\varphi)$, $v(p) = \mathbf{o}_1^{\mathbf{e}}$. Simultaneously, that $v(\psi) = \mathbf{f}$ implies that for all $q \in Var(\psi)$, $v(q) \in \{\mathbf{t}, \mathbf{f}\}$. Whence, $Var(\varphi) \cap Var(\psi) = \emptyset$. $\square$

THEOREM 3.8. *The* FMLA-FMLA *fragment of* $\mathsf{wST}[\mathbf{o}_1^{\mathbf{e}}\mathbf{o}_2^{\mathbf{e}}] = \mathsf{REL}_{\mathtt{fde}}$

PROOF: On the one hand, assume $\varphi \nvDash_{\mathsf{wST}[\mathbf{o}_1^{\mathbf{e}}\mathbf{o}_2^{\mathbf{e}}]} \psi$. There are four ways in which this can happen. Either there is a $\mathsf{wST}[\mathbf{o}_1^{\mathbf{e}}\mathbf{o}_2^{\mathbf{e}}]$-valuation v such that (i) $v(\varphi) = \mathbf{t}$ and $v(\psi) = \mathbf{f}$, or (ii) $v(\varphi) = \mathbf{t}$ and $v(\psi) = \mathbf{o}_2^{\mathbf{e}}$, or (iii) $v(\varphi) = \mathbf{o}_1^{\mathbf{e}}$ and $v(\psi) = \mathbf{f}$, or (iv) $v(\varphi) = \mathbf{o}_1^{\mathbf{e}}$ and $v(\psi) = \mathbf{o}_2^{\mathbf{e}}$. In case (i), we are guaranteed that v is a Boolean valuation, whence we know that $\varphi \nvDash_{\mathsf{CL}} \psi$. In cases (ii) and (iii), we know by Lemma 3.7 that $Var(\varphi) \cap Var(\psi) = \emptyset$. In case (iv), we know by Lemma 2.3 that $Var(\varphi) \cap Var(\psi) = \emptyset$. From all these considerations, it follows that $\varphi \nvDash_{\mathsf{REL}_{\mathtt{fde}}} \psi$.

On the other hand, assume that $\varphi \nvDash_{\mathsf{REL}_{\mathtt{fde}}} \psi$. That is to say, that either $\varphi \nvDash_{\mathsf{CL}} \psi$, or $Var(\psi) \cap Var(\varphi) = \emptyset$. If the former is the case, then there is a CL-valuation v such that $v(\varphi) = \mathbf{t}$ and $v(\psi) = \mathbf{f}$. However, given CL-valuations are a subset of $\mathsf{wST}[\mathbf{o}_1^{\mathbf{e}}\mathbf{o}_2^{\mathbf{e}}]$-valuations, this establishes that there is a $\mathsf{wST}[\mathbf{o}_1^{\mathbf{e}}\mathbf{o}_2^{\mathbf{e}}]$-valuation v' such that $v'(\varphi) = \mathbf{t}$ and $v'(\psi) = \mathbf{f}$. From this it follows that $\varphi \nvDash_{\mathsf{wST}[\mathbf{o}_1^{\mathbf{e}}\mathbf{o}_2^{\mathbf{e}}]} \psi$. If the latter is the case, it is possible to construct a $\mathsf{wST}[\mathbf{o}_1^{\mathbf{e}}\mathbf{o}_2^{\mathbf{e}}]$-valuation v such that:

$$v(p) = \begin{cases} \mathbf{o}_1^{\mathbf{e}} & \text{if } p \in Var(\varphi) \\ \mathbf{o}_2^{\mathbf{e}} & \text{otherwise} \end{cases}$$

For such a valuation it is possible to show, as in Lemma 2.3, that $v(\varphi) = \mathbf{o}_1^{\mathbf{e}}$ while $v(\psi) = \mathbf{o}_2^{\mathbf{e}}$. From this it follows that $\varphi \nvDash_{\mathsf{wST}[\mathbf{o}_1^{\mathbf{e}}\mathbf{o}_2^{\mathbf{e}}]} \psi$. □

Matters of interpretation of the truth-values involved are rather difficult. To wit, whereas usually the characteristic value of the weak Kleene algebra is understood as representing meaninglessness or nonsense of some sort, as in D. Bochvar's [6] and S. Halldén's [24], it is saliently complicated to explain how this reading spills into the interpretation of the mimicking values $\mathbf{o}_1^{\mathbf{e}}$ and $\mathbf{o}_2^{\mathbf{e}}$. Our intention here is not, however, to provide a cogent philosophical reading of the truth-values involved in a semantic presentation of $\mathsf{REL}_{\mathtt{fde}}$—the FMLA-FMLA fragment of $\mathsf{CL}_{\mathsf{VSP}}$—but simply to offer a semantic structure that will induce this target non-transitive p-logic. In this respect, an in-depth discussion of these matters, hoping to determine if there is a p-matrix with a cogent and perspicuous philosophical reading for $\mathsf{REL}_{\mathtt{fde}}$, will have to wait for another time.

A further question regarding this semantic rendering of $\mathsf{REL}_{\mathtt{fde}}$ lies in its being an extension of a proper p-matrix inducing a system with the same valid inferences that CL. Our semantics for this fragment of $\mathsf{CL}_{\mathsf{VSP}}$ consisted of extending a p-matrix built on top of the **WK** algebra with mimicking values appropriately taken to be designated or undesignated for premises and conclusions. One may, then, ask whether it is possible to build another different p-matrix for $\mathsf{REL}_{\mathtt{fde}}$ by means of extending a proper p-matrix for CL built on top of another structure. We leave this question for future research, although we provide some preliminary conjectures in Section 5.

Having provided a semantics for our target logic, in the following section, we devote ourselves to defining an appropriate calculus for this system.

4. Sequent calculus

In this section we provide a sound and complete sequent calculus for $\mathsf{REL}_{\mathtt{fde}}$, that is to say, for the set of first-degree entailments valid in Epstein's Relatedness Logic, which incidentally coincides with the FMLA-FMLA fragment of $\mathsf{CL}_{\mathsf{VSP}}$.[15] Proof-systems for Epstein's logic as a whole have been given by R. Epstein himself in [16] in the form of an axiom system, by W. Carnielli in [7] in the form of a tableaux system, and by L. Fari˜nas del Cerro and V. Lugardon in [17] in the form of a Gentzen-style sequent calculus.

As regards $\mathsf{REL}_{\mathtt{fde}}$, a Hilbert-style axiomatization has been presented by F. Paoli in [30] , whereas a tableaux system is presented by him in [31]. Here, with the purpose of endowing our target logic with a Gentzen-style sequent calculus we will follow the ideas and techniques discussed by M. I. Corbalán and M. Coniglio in [12], and by R. French in [21], where calculi with linguistic restrictions are presented for 3-valued systems based on the **WK** algebra, as well as for subsystems thereof like the first-degree entailments of R. Angell's logic of Analytic Containment.

For this task, we will work with sequents of the form $\Gamma \succ \Delta$ defined as pairs $\langle \Gamma, \Delta \rangle$ where Γ and Δ are finite sets of formulae. In this context, sequents will receive a concrete interpretation, as we will establish that $\varphi_1, \ldots, \varphi_n \succ \psi_1, \ldots, \psi_m$ is provable in the target calculus if and only if the first-degree entailment $\varphi_1 \wedge \cdots \wedge \varphi_n \rightarrow \psi_1 \vee \cdots \vee \psi_m$ is valid in REL—in other words, if and only if $\varphi_1 \wedge \cdots \wedge \varphi_n \vDash_{\mathsf{REL}_{\mathtt{fde}}} \psi_1 \vee \cdots \vee \psi_m$. It will be important to bear this in mind when conducting the soundness and completeness proofs.

The main idea behind the calculus that we introduce below is to have sequent rules (be it initial sequents, operational or structural rules), that are bound to linguistic restrictions. That is to say, rules that can be applied only if certain constraints regarding the parametric or active formulae are met. These restrictions guarantee that the rules preserve the satisfaction of the Variable-Sharing Principle or, put differently, that the rules guarantee that there is subject-matter overlap between premises and conclusions.

We will now proceed to present the set of rules that define our calculus $\mathcal{G}_{\mathsf{REL}_{\mathtt{fde}}}$, later showing the adequacy of the formalism. Let us note, in passing, that for $\Theta \subseteq FOR(\mathcal{L})$, $Var(\Theta) = \bigcup\limits_{\theta \in \Theta} Var(\theta)$.

[15] I would like to thank Bruno Da Ré, Francesco Paoli and Shawn Standefer for very illuminating and insightful discussions on the content this section.

DEFINITION 4.1. The calculus $\mathcal{G}_{\mathsf{REL}_{\mathtt{fde}}}$ is constituted by the following rules:

Initial Sequents:

$$[Initial] \;\; \Gamma, p \succ p, \Delta$$

Structural Rules:

$$\frac{\Gamma, \varphi \succ \Delta \quad \Gamma \succ \varphi, \Delta}{\Gamma \succ \Delta}\ [Cut]^{\ddagger}$$

$$\ddagger: \text{ where } Var(\Gamma) \cap Var(\Delta) \neq \emptyset$$

Operational Rules:

$$\frac{\Gamma \succ \varphi, \Delta}{\Gamma, \neg\varphi \succ \Delta}\ [\neg L]^{\dagger} \qquad\qquad \frac{\Gamma, \varphi \succ \Delta}{\Gamma \succ \neg\varphi, \Delta}\ [\neg R]^{\ddagger}$$

$\dagger:$ where $Var(\Gamma, \varphi) \cap Var(\Delta) \neq \emptyset$ $\quad$ $\ddagger:$ where $Var(\Delta, \varphi) \cap Var(\Gamma) \neq \emptyset$

$$\frac{\Gamma, \varphi \succ \Delta \quad \Gamma, \psi \succ \Delta}{\Gamma, \varphi \vee \psi \succ \Delta}\ [\vee L] \qquad\qquad \frac{\Gamma \succ \varphi, \psi, \Delta}{\Gamma \succ \varphi \vee \psi, \Delta}\ [\vee R]$$

$$\frac{\Gamma, \varphi, \psi \succ \Delta}{\Gamma, \varphi \wedge \psi \succ \Delta}\ [\wedge L] \qquad\qquad \frac{\Gamma \succ \varphi, \Delta \quad \Gamma \succ \psi, \Delta}{\Gamma \succ \varphi \wedge \psi, \Delta}\ [\wedge R]$$

Regarding the structural rules, it shall be noted that $[Initial]$ is a form of the structural rule of Identity or Reflexivity, with Left and Right Weakening "absorbed"—to some extent. Indeed, as we remark below, adopting these initial sequents allows for the left and right Weakening rules to be admissible in their unrestricted forms.

LEMMA 4.2. *The following form of the Weakening rules are admissible in* $\mathcal{G}_{\mathsf{REL}_{\mathtt{fde}}}$*:*

$$\frac{\Gamma \succ \Delta}{\Gamma, \varphi \succ \Delta}\ [KL] \qquad\qquad \frac{\Gamma \succ \Delta}{\Gamma \succ \varphi, \Delta}\ [KR]$$

Proof: Regarding $[KL]$, suppose we have a derivation of $\Gamma \succ \Delta$. We can turn this into a derivation of $\Gamma, \varphi \succ \Delta$ by adding φ to the left-hand side of each of the nodes of the derivation, as the uppermost node will still constitute a rightful instance of $[Initial]$. Similarly, regarding $[KR]$, suppose we have a derivation of $\Gamma \succ \Delta$. We can turn this into a derivation of $\Gamma \succ \varphi, \Delta$ by adding φ to the right-hand side of each of the nodes of the derivation, as the uppermost node will still constitute a rightful instance of $[Initial]$. □

The next result we discuss shows that every provable sequent of $\mathcal{G}_{\mathsf{REL_{fde}}}$ encodes a corresponding first-degree entailment that is valid in REL—or, what is the same, a valid inference of $\mathsf{REL_{fde}}$. For the purpose of proving this, we will appeal to the characterization of said set of valid entailments in the paragraphs above.

Lemma 4.3. *All the rules of* $\mathcal{G}_{\mathsf{REL_{fde}}}$ *preserve* $\mathsf{REL_{fde}}$*-validity. In other words, for each of the rules of the calculus, if the premise sequents are valid in* $\mathsf{REL_{fde}}$*, so is the conclusion sequent of that rule.*

Proof: We show this by cases—focusing on the restricted rules and leaving the rest as exercises to the reader—assuming the premise sequents of a rule are valid in $\mathsf{REL_{fde}}$, and later proving that its conclusion sequent is also valid in said logic. In all cases below, we will assume that Γ can be redescribed as $\gamma_1, \dots, \gamma_n$, and that Δ can be redescribed as $\delta_1, \dots, \delta_m$.

> $[\neg L]^{\dagger}$: Assume $\gamma_1 \wedge \cdots \wedge \gamma_n \vDash_{\mathsf{REL_{fde}}} \varphi \vee \delta_1 \vee \cdots \vee \delta_m$, and $Var(\gamma_1 \wedge \cdots \wedge \gamma_n, \varphi) \cap Var(\delta_1 \vee \cdots \vee \delta_m) \neq \emptyset$. By simple reasoning this allows to establish that $\gamma_1 \wedge \cdots \wedge \gamma_n \vDash_{\mathsf{CL}} \varphi \vee \delta_1 \vee \cdots \vee \delta_m$ and, concomitantly, that $\gamma_1 \wedge \cdots \wedge \gamma_n \wedge \neg\varphi \vDash_{\mathsf{CL}} \delta_1 \vee \cdots \vee \delta_m$. Furthermore, that $Var(\gamma_1 \wedge \cdots \wedge \gamma_n, \varphi) \cap Var(\delta_1 \vee \cdots \vee \delta_m) \neq \emptyset$ guarantees that $Var(\gamma_1 \wedge \cdots \wedge \gamma_n \wedge \neg\varphi) \cap Var(\delta_1 \vee \cdots \vee \delta_m) \neq \emptyset$. Whence, $\gamma_1 \wedge \cdots \wedge \gamma_n \wedge \neg\varphi \vDash_{\mathsf{REL_{fde}}} \delta_1 \vee \cdots \vee \delta_m$.
>
> $[\neg R]^{\ddagger}$: Assume $\gamma_1 \wedge \cdots \wedge \gamma_n \wedge \varphi \vDash_{\mathsf{REL_{fde}}} \delta_1 \vee \cdots \vee \delta_m$, and $Var(\delta_1 \vee \cdots \vee \delta_m, \varphi) \cap Var(\gamma_1 \wedge \cdots \wedge \gamma_n) \neq \emptyset$. By simple reasoning this allows to establish that $\gamma_1 \wedge \cdots \wedge \gamma_n \wedge \varphi \vDash_{\mathsf{CL}} \delta_1 \vee \cdots \vee \delta_m$ and, concomitantly, that $\gamma_1 \wedge \cdots \wedge \gamma_n \vDash_{\mathsf{CL}} \neg\varphi \vee \delta_1 \vee \cdots \vee \delta_m$. Furthermore, that $Var(\delta_1 \vee \cdots \vee \delta_m, \varphi) \cap Var(\gamma_1 \wedge \cdots \wedge \gamma_n) \neq \emptyset$ guarantees that $Var(\gamma_1 \wedge \cdots \wedge \gamma_n) \cap Var(\neg\varphi \vee \delta_1 \vee \cdots \vee \delta_m) \neq \emptyset$. Whence, $\gamma_1 \wedge \cdots \wedge \gamma_n \vDash_{\mathsf{REL_{fde}}} \neg\varphi \vee \delta_1 \vee \cdots \vee \delta_m$.

$[Cut]^{\ddagger}_{\ddagger}$: Assume that $\gamma_1 \wedge \cdots \wedge \gamma_n \wedge \varphi \vDash_{\mathsf{REL}_{\mathtt{fde}}} \delta_1 \vee \cdots \vee \delta_m$, that $\gamma_1 \wedge \cdots \wedge \gamma_n \vDash_{\mathsf{REL}_{\mathtt{fde}}} \varphi \vee \delta_1 \vee \cdots \vee \delta_m$, and that $Var(\gamma_1 \wedge \cdots \wedge \gamma_n) \cap Var(\delta_1 \vee \cdots \vee \delta_m) \neq \emptyset$. By simple reasoning this allows to establish that $\gamma_1 \wedge \cdots \wedge \gamma_n \wedge \varphi \vDash_{\mathsf{CL}} \delta_1 \vee \cdots \vee \delta_m$ and $\gamma_1 \wedge \cdots \wedge \gamma_n \vDash_{\mathsf{CL}} \varphi \vee \delta_1 \vee \cdots \vee \delta_m$. In CL these two facts imply that $\gamma_1 \wedge \cdots \wedge \gamma_n \vDash_{\mathsf{CL}} \delta_1 \vee \cdots \vee \delta_m$, which together with the assumption that $Var(\gamma_1 \wedge \cdots \wedge \gamma_n) \cap Var(\delta_1 \vee \cdots \vee \delta_m) \neq \emptyset$ implies that $\gamma_1 \wedge \cdots \wedge \gamma_n \vDash_{\mathsf{REL}_{\mathtt{fde}}} \delta_1 \vee \cdots \vee \delta_m$.

The case of $[Initial]$, the $[\wedge]$ and $[\vee]$ rules are straightforward and thus are left to the reader as an exercise. $\square$

THEOREM 4.4 (Soundness). *If the sequent $\varphi_1, \ldots, \varphi_n \succ \psi_1, \ldots, \psi_m$ is provable in $\mathcal{G}_{\mathsf{REL}_{\mathtt{fde}}}$, then $\varphi_1 \wedge \cdots \wedge \varphi_n \vDash_{\mathsf{REL}_{\mathtt{fde}}} \psi_1 \vee \cdots \vee \psi_m$.*

PROOF: We know that the initial sequents are valid in $\mathsf{REL}_{\mathtt{fde}}$ and that all rules preserve $\mathsf{REL}_{\mathtt{fde}}$ validity. A straightforward induction on the height of the derivation shows (using Lemma 4.3 in the inductive step) that all provable sequents encode inferences that are valid in $\mathsf{REL}_{\mathtt{fde}}$. Thus, if $\varphi_1, \ldots, \varphi_n \succ \psi_1, \ldots, \psi_m$ is provable in $\mathcal{G}_{\mathsf{REL}_{\mathtt{fde}}}$, then the corresponding inference is valid in $\mathsf{REL}_{\mathtt{fde}}$—in other words, $\varphi_1 \wedge \cdots \wedge \varphi_n \vDash_{\mathsf{REL}_{\mathtt{fde}}} \psi_1 \vee \cdots \vee \psi_m$. $\square$

Now, having discussed the soundness of our calculus, we will now turn to the more tiresome task of providing a completeness proof for $\mathcal{G}_{\mathsf{REL}_{\mathtt{fde}}}$. For this purpose, we will show that whenever $\varphi_1 \wedge \cdots \wedge \varphi_n \vDash_{\mathsf{REL}_{\mathtt{fde}}} \psi_1 \vee \cdots \vee \psi_m$, there is a respective sequent that is provable in our Gentzen-style sequent calculus $\mathcal{G}_{\mathsf{REL}_{\mathtt{fde}}}$.

THEOREM 4.5 (Completeness). *If $\varphi_1 \wedge \cdots \wedge \varphi_n \vDash_{\mathsf{REL}_{\mathtt{fde}}} \psi_1 \vee \cdots \vee \psi_m$, then the sequent $\varphi_1, \ldots, \varphi_n \succ \psi_1, \ldots, \psi_m$ is provable in $\mathcal{G}_{\mathsf{REL}_{\mathtt{fde}}}$.*

PROOF: In the Appendix. $\square$

COROLLARY 4.6 (Cut-elimination). The restricted version of the Cut rule is eliminable from $\mathcal{G}_{\mathsf{REL}_{\mathtt{fde}}}$

PROOF: In the Appendix. $\square$

5. Conclusions

In this article, we discussed $\mathsf{REL}_{\mathtt{fde}}$, the first-degree fragment of R. Epstein's Relatedness Logic—which is identical to the FMLA-FMLA fragment of $\mathsf{CL_{VSP}}$. In this respect, we presented a p-matrix semantics and a Gentzen-style sequent calculus for this logic.

A couple of venues for further research are left open in this regard. First, our p-matrix semantics are based on the extension of the **WK** algebra with two universally idempotent elements that "mimic" the characteristic infectious element $\mathbf{e}$. It would be important to know whether it is possible to offer different semantics for $\mathsf{REL_{fde}}$, which are not built on top of the **WK** algebra, but on top of a different algebraic structure. For example, extending a p-matrix for CL built on top of the **SK** algebra. This may as well be possible, but we should notice that an extension thereof like the one discussed above, with two mimicking values will not work. In fact, it is easy to check that the logic $\mathsf{ST}[\mathbf{o}_1^{\mathbf{n}}\mathbf{o}_2^{\mathbf{n}}]$ induced by the p-matrix $\langle \mathbf{SK}[\mathbf{o}_1^{\mathbf{n}}\mathbf{o}_2^{\mathbf{n}}], \{\mathbf{t}, \mathbf{o}_1^{\mathbf{n}}\}, \{\mathbf{t}, \mathbf{o}_1^{\mathbf{n}}, \mathbf{n}\}\rangle$ will invalidate the inference schema $\varphi \vee \psi \vDash_{\mathsf{ST}[\mathbf{o}_1^{\mathbf{n}}\mathbf{o}_2^{\mathbf{n}}]} \psi \vee \neg\psi$, which nevertheless satisfies the Variable-Sharing Principle.[16] Other routes may be available that make no appeal to mimicking values, starting from the **SK** algebra and obtaining a structure on top of which a proper p-matrix for $\mathsf{REL_{fde}}$ can be built—these will definitely be interesting to explore.

Furthermore, it would be illuminating to learn, where L is a subclassical logic characterizable by a single finite matrix (like, e.g., S. Kleene's $\mathsf{K_3}$ or G. Priest's LP) whether p-matrix semantics for the FMLA-FMLA fragment of $\mathsf{L_{VSP}}$ can be obtained, in the spirit of the semantics for $\mathsf{REL_{fde}}$. In other words, by expanding their characteristic regular matrix semantics to proper p-matrix semantics inducing systems having the same valid inferences, and later extending said conforming p-matrix semantics with two mimicking values of the appropriate kind. In this vein, a systematic and general way of obtaining proper p-matrix semantics for subclassical systems may be useful—and can be found in some recent developments by M. Fitting's works, like [18]. We hope to investigate these and other questions in the near future.

[16] As a means of an example, let us assume that φ is p and ψ is q, and consider a $\mathsf{ST}[\mathbf{o}_1^{\mathbf{n}}\mathbf{o}_2^{\mathbf{n}}]$-valuation v such that $v(p) = \mathbf{t}$ and $v(q) = \mathbf{o}_2^{\mathbf{n}}$. This valuation is such that $v(p \vee q) = \mathbf{t}$, whereas $v(q \vee \neg q) = \mathbf{o}_2^{\mathbf{n}}$, witnessing $p \vee q \nvDash_{\mathsf{ST}[\mathbf{o}_1^{\mathbf{n}}\mathbf{o}_2^{\mathbf{n}}]} q \vee \neg q$, whence the invalidity of the aforementioned schema.

Appendix: Completeness and cut-elimination

Completeness proof

We start by assuming that the sequent $\varphi_1, \ldots, \varphi_n \succ \psi_1, \ldots, \psi_m$ is unprovable in $\mathcal{G}_{\mathsf{REL_{fde}}}$. We, then, consider two cases:

(i) $Var(\psi_1, \ldots, \psi_m) \cap Var(\varphi_1, \ldots, \varphi_n) = \emptyset$

(ii) $Var(\psi_1, \ldots, \psi_m) \cap Var(\varphi_1, \ldots, \varphi_n) \neq \emptyset$

showing that in both cases we can design valuations that witness $\varphi_1 \wedge \cdots \wedge \varphi_n \nvDash_{\mathsf{REL_{fde}}} \psi_1 \vee \cdots \vee \psi_m$.

Case (i): if $Var(\psi_1, \ldots, \psi_m) \cap Var(\varphi_1, \ldots, \varphi_n) = \emptyset$ is the case, consider a $\mathsf{REL_{fde}}$-valuation v such that:

$$v(p) = \begin{cases} \mathbf{o}_1^{\mathbf{e}} & \text{if } p \in Var(\varphi_1, \ldots, \varphi_n) \\ \mathbf{o}_2^{\mathbf{e}} & \text{if } p \in Var(\psi_1, \ldots, \psi_m) \\ \mathbf{e} & \text{otherwise} \end{cases}$$

It is then straightforward to notice, as in Lemma 2.3, that all $\varphi_j \in \{\varphi_1, \ldots, \varphi_n\}$ will be such that $v(\varphi_j) = \mathbf{o}_1^{\mathbf{e}}$, whereas all $\psi_i \in \{\psi_1, \ldots, \psi_m\}$ will be such that $v(\psi_i) = \mathbf{o}_2^{\mathbf{e}}$. A quick inspections of the $\mathbf{WK}[\mathbf{o}_1^{\mathbf{e}}\mathbf{o}_2^{\mathbf{e}}]$ algebra allows to notice that this renders $v(\varphi_1 \wedge \cdots \wedge \varphi_n) = \mathbf{o}_1^{\mathbf{e}}$, while at the same time giving $v(\psi_1 \vee \cdots \vee \psi_m) = \mathbf{o}_2^{\mathbf{e}}$. Whence, $\varphi_1 \wedge \cdots \wedge \varphi_n \nvDash_{\mathsf{REL_{fde}}} \psi_1 \vee \cdots \vee \psi_m$.

Case (ii): if $Var(\psi_1, \ldots, \psi_m) \cap Var(\varphi_1, \ldots, \varphi_n) \neq \emptyset$ is the case, in order to show that $\varphi_1 \wedge \cdots \wedge \varphi_n \nvDash_{\mathsf{REL_{fde}}} \psi_1 \vee \cdots \vee \psi_m$ we will apply a slight modification of the method of reduction trees as explored, e.g., in [43] by G. Takeuti and in [35] by D. Ripley.

The idea is to start with a sequent that we assume to be unprovable later extending it in a finite series of steps with the help of *reduction rules* that will finally render a *reduction tree*. Thus, we start with an unprovable sequent and build a tree above it, with each node consisting of a sequent that results from an application of the reduction rules to the sequent below it. As we extend the tree, we will sometimes find that the tip of a branch is an instance of one of $[Initial]$—in such a case we will consider this branch *closed* and will stop performing reductions on it. Contrary to that, if a branch is not closed after applying all the possible reduction rules, we will consider this branch *open*.

Below, we detail the rules that we apply to the sequents at the top of each branch of the tree, at each stage of the reduction process. Let us note, in passing, that this technique requires an enumeration of the formulae of our language, and that when the same sequent appears at the tip of some branch of more than one tree, they are simultaneously reduced.

- To reduce a sequent of the form $\Gamma, \varphi \wedge \psi \succ \Delta$, extend the branch with the sequent $\Gamma, \varphi, \psi \succ \Delta$.
- To reduce a sequent of the form $\Gamma \succ \varphi \wedge \psi, \Delta$, extend the branch by splitting in two. To one new branch, add the sequent $\Gamma \succ \varphi, \Delta$; to the other, add the sequent $\Gamma \succ \psi, \Delta$.
- To reduce a sequent of the form $\Gamma \succ \varphi \vee \psi, \Delta$, extend the branch with the sequent $\Gamma \succ \varphi, \psi, \Delta$.
- To reduce a sequent of the form $\Gamma, \varphi \vee \psi \succ \Delta$, extend the branch by splitting in two. To one new branch, add the sequent $\Gamma, \varphi \succ \Delta$; to the other, add the sequent $\Gamma, \psi \succ \Delta$.
- To reduce a sequent of the form $\Gamma, \neg\varphi \succ \Delta$, consider whether it is the case that $Var(\Gamma, \varphi) \cap Var(\Delta) \neq \emptyset$. If this is the case, extend the branch with the sequent $\Gamma, \neg\varphi \succ \varphi, \Delta$; otherwise, do nothing and proceed to reduce the next sequent, if there is one.
- To reduce a sequent of the form $\Gamma \succ \neg\varphi, \Delta$, consider whether it is the case that $Var(\Delta, \varphi) \cap Var(\Gamma) \neq \emptyset$. If this is the case, extend the branch with the sequent $\Gamma, \varphi \succ \neg\varphi, \Delta$; otherwise, do nothing and proceed to reduce the next sequent, if there is one.

Suppose we start with a sequent of the form $\varphi_1, \dots, \varphi_n \succ \psi_1, \dots, \psi_m$ where $Var(\psi_1, \dots, \psi_m) \cap Var(\varphi_1, \dots, \varphi_n) \neq \emptyset$ and follow this process as many times as necessary for there to be no more legal applications of the reduction rules. Then, either all branches of the tree will be closed (whence, we have a proof of the sequent that was assumed to be unprovable, contradicting our initial hypothesis), or some branch will be open. Suppose the latter is the case.

The next step in our proof is to show that it is possible to find a $\mathsf{REL}_{\mathtt{fde}}$-valuation that witnesses $\varphi_1 \wedge \dots \wedge \varphi_n \nvDash_{\mathsf{REL}_{\mathtt{fde}}} \psi_1 \vee \dots \vee \psi_m$. For this purpose, let us temporarily relabel the sequents in the open branch as $\Gamma_1 \succ \Delta_1, \dots, \Gamma_k \succ \Delta_k$, letting $\Gamma_1 \succ \Delta_1$ be $\varphi_1, \dots, \varphi_n \succ \psi_1, \dots, \psi_m$ and letting

$\Gamma_k \succ \Delta_k$ be the sequent at the tip of the open branch. Furthermore, let the sequent $\Gamma \succ \Delta$—where $\Gamma = \cup\{\Gamma_i \mid 1 \leq i \leq k\}$ and $\Delta = \cup\{\Delta_i \mid 1 \leq i \leq k\}$—be the sequent that "collects" all the sequents appearing in the nodes of the open branch.

Before going into the final stage of this proof, lets us highlight a number of facts regarding our newly defined Γ and Δ. These are: (i) for all propositional variables p, $p \notin \Gamma \cap \Delta$; (ii) there are $\Gamma', \Delta' \subseteq Var$ such that $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$; (iii) for all formulae φ, if $\neg\varphi \in \Gamma$ then $\varphi \in \Delta$; (iv) for all formulae φ, if $\neg\varphi \in \Delta$ then $\varphi \in \Gamma$. All these can be derived from the definition of Γ and Δ, the fact that none of the $\Gamma_i \succ \Delta_i$ ($1 \leq i \leq k$) is an instance of [*Initial*], and the fact that the reduction rules preserve the satisfaction of the Variable-Sharing Principle.

We prove here remark (iii), noting that the proof for remark (iv) is perfectly analogous.[17] Thus, suppose $\neg\varphi \in \Gamma$. By construction of Γ, either $\neg\varphi \in \Gamma_1$ or $\neg\varphi \in \Gamma_j$, for $j > 1$. We now reason focusing on when an appearance of $\neg\varphi$ is being reduced.

- Suppose it is being reduced in the sequent $\Gamma_1 \succ \Delta_1$, and that $\neg\varphi \in \Gamma_1$. Then, in this case, the restriction to reduce $\neg\varphi$ amounts to $Var(\Gamma_1 \setminus \{\neg\varphi\}, \varphi) \cap Var(\Delta_1) \neq \emptyset$. However, $Var(\Gamma_1 \setminus \{\neg\varphi\}, \varphi)$ is just $Var(\Gamma_1)$. Whence, given we know by hypothesis that $Var(\Gamma_1) \cap Var(\Delta_1) \neq \emptyset$, this restriction is satisfied and it is guaranteed by the reduction rules that $\varphi \in \Delta_{1+1}$. Therefore, by the construction process above $\varphi \in \Delta$. (Notice that this would not be guaranteed if it were not the case that, by hypothesis, $Var(\Gamma_1) \cap Var(\Delta_1) \neq \emptyset$)

- Suppose, alternatively, that it is being reduced in the sequent $\Gamma_j \succ \Delta_j$, for $j > 1$, and that $\neg\varphi \in \Gamma_j$. Then, in this case, the restriction to reduce $\neg\varphi$ amounts to $Var(\Gamma_j \setminus \{\neg\varphi\}, \varphi) \cap Var(\Delta_j) \neq \emptyset$. Recall that, by construction, $\Gamma_1 \subseteq \Gamma_j$ and $\Delta_1 \subseteq \Delta_j$. There are, now, two cases: either $\varphi \notin \Gamma_1$, or $\varphi \in \Gamma_1$. If the former, then by the above $Var(\Gamma_j \setminus \{\neg\varphi\}, \varphi) \cap Var(\Delta_j) \neq \emptyset$. If the latter, then once again $Var(\Gamma_j \setminus \{\neg\varphi\}, \varphi)$ is just $Var(\Gamma_j)$, and then by the above $Var(\Gamma_j \setminus \{\neg\varphi\}, \varphi) \cap Var(\Delta_j) \neq \emptyset$. Therefore, the restriction is satisfied and it is guaranteed by the reduction rules that $\varphi \in \Delta_{j+1}$. Finally, by

[17] In turn, we stress here that remark (i) is true because otherwise $\Gamma \succ \Delta$ would be an instance of [*Initial*], and remark (ii) is true because otherwise $Var(\Gamma) \cap Var(\Delta) = \emptyset$, which would contradict the assumptions holding at this point of the proof.

the construction process above $\varphi \in \Delta$. (Notice that this would not be guaranteed if it were not the case that, by hypothesis, $Var(\Gamma_1) \cap Var(\Delta_1) \neq \emptyset$)

Now, for the final stage of the proof, take the aforementioned sequent $\Gamma \succ \Delta$ and consider the $\mathsf{REL}_{\mathtt{fde}}$-valuation v such that:

$$v(p) = \begin{cases} \mathbf{t} & \text{if } p \in \Gamma \text{ or } \neg p \in \Delta \\ \mathbf{f} & \text{otherwise} \end{cases}$$

We now prove by induction on the complexity of φ that v is a $\mathsf{REL}_{\mathtt{fde}}$-valuation such that $v(\varphi) = \mathbf{t}$ if and only if $\varphi \in \Gamma$ and $v(\varphi) = \mathbf{f}$ if and only if $\varphi \in \Delta$.

Base case:

- $\varphi = p$. If $p \in \Gamma$, $v(p) = \mathbf{t}$ by definition of v. Otherwise, if $p \in \Delta$, for example, $v(p) = \mathbf{f}$ by definition. Notice that, by the remarks above, we know that either $p \notin \Gamma$, or $p \notin \Delta$—granting the well-definedness of v.

Inductive step: we assume that for all formulae of lesser complexity than φ, the hypothesis holds and show that it also holds for φ.

- $\varphi = \neg\psi$. If $\neg\psi \in \Gamma$, we know that $\psi \in \Delta$ by the remarks above. By the IH we know that $v(\psi) = \mathbf{f}$, whence $v(\neg\psi) = \mathbf{t}$. Otherwise, if $\neg\psi \in \Delta$, we know that $\psi \in \Gamma$ by the remarks above. By the IH we know that $v(\psi) = \mathbf{t}$, whence $v(\neg\psi) = \mathbf{f}$.

- $\varphi = \psi \wedge \chi$. $\psi \wedge \chi \in \Gamma$ we know that $\psi, \chi \in \Gamma$. By the IH we know that $v(\psi) = v(\chi) = \mathbf{t}$. Thus, $v(\psi \wedge \chi) = \mathbf{t}$. Otherwise, if $\psi \wedge \chi \in \Delta$, then either $\psi \in \Delta$ or $\chi \in \Delta$. By the IH we know that either $v(\psi) = \mathbf{f}$ or $v(\chi) = \mathbf{f}$. Whence, $v(\psi \wedge \chi) = \mathbf{f}$.

- $\varphi = \psi \vee \chi$. If $\psi \vee \chi \in \Gamma$ we know that either $\psi \in \Gamma$ or $\chi \in \Gamma$. By the IH we know that either $v(\psi) = \mathbf{t}$ or $v(\chi) = \mathbf{t}$. Whence, $v(\psi \vee \chi) = \mathbf{t}$. Otherwise, if $\psi \vee \chi \in \Delta$, we know that $\psi, \chi \in \Delta$. By the IH this implies that $v(\psi) = v(\chi) = \mathbf{f}$. Whence, $v(\psi \vee \chi) = \mathbf{f}$.

Given this, and since $\{\varphi_1, \ldots, \varphi_n\} \subseteq \Gamma$ and $\{\psi_1, \ldots, \psi_m\} \subseteq \Delta$, we know that for all i such that $1 \leq i \leq n$, $v(\varphi_i) = \mathbf{t}$, and for all j such that $1 \leq j \leq m$, $v(\psi_j) = \mathbf{f}$. Whence, by looking at the Boolean reduct of the $\mathbf{WK}[\mathbf{o}_1^{\mathbf{e}}\mathbf{o}_2^{\mathbf{e}}]$ algebra it is easy to notice that $v(\varphi_1 \wedge \cdots \wedge \varphi_n) = \mathbf{t}$ and $v(\psi_1 \vee \cdots \vee \psi_m) = \mathbf{f}$. Therefore, v is a $\mathsf{REL}_{\mathtt{fde}}$-valuation witnessing the fact that $\varphi_1 \wedge \cdots \wedge \varphi_n \nvDash_{\mathsf{REL}_{\mathtt{fde}}} \psi_1 \vee \cdots \vee \psi_m$.

This establishes that if $\varphi_1 \wedge \cdots \wedge \varphi_n \vDash_{\mathsf{REL}_{\mathtt{fde}}} \psi_1 \vee \cdots \vee \psi_m$, then a sequent of the form $\varphi_1, \ldots, \varphi_n \succ \psi_1, \ldots, \psi_m$ is provable in $\mathcal{G}_{\mathsf{REL}_{\mathtt{fde}}}$.[18]

Cut-elimination proof

By Theorem 4.4, if there is a proof of the sequent $\varphi_1, \ldots, \varphi_n \succ \psi_1, \ldots, \psi_m$ in $\mathcal{G}_{\mathsf{REL}_{\mathtt{fde}}}$, then $\varphi_1 \wedge \cdots \wedge \varphi_n \vDash_{\mathsf{REL}_{\mathtt{fde}}} \psi_1 \vee \cdots \vee \psi_m$. Furthermore, by Theorem 4.5, if $\varphi_1 \wedge \cdots \wedge \varphi_n \vDash_{\mathsf{REL}_{\mathtt{fde}}} \psi_1 \vee \cdots \vee \psi_m$, then applying the method of reduction trees gives a proof of the sequent $\varphi_1, \ldots, \varphi_n \succ \psi_1, \ldots, \psi_m$ in $\mathcal{G}_{\mathsf{REL}_{\mathtt{fde}}}$. However, notice that this proof does not feature any instance of the restricted version of the Cut rule and is, thus, a Cut-free proof. Whence, for any sequent provable in $\mathcal{G}_{\mathsf{PAI}_{\mathtt{fde}}}$, there is a proof of it that does not use the restricted version of the Cut rule.

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[18]Notice that this proof does not give full CL, because when considering whether we are in Case (i) or Case (ii), some classically valid inferences are discarded right away—namely, those inferences which do not satisfy the Variable-Sharing Principle.

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A SOUND INTERPRETATION OF LEŚNIEWSKI'S EPSILON IN MODAL LOGIC KTB

Abstract

In this paper, we shall show that the following translation I^M from the propositional fragment $\mathbf{L_1}$ of Leśniewski's ontology to modal logic **KTB** is sound: for any formula ϕ and ψ of $\mathbf{L_1}$, it is defined as

(M1) $I^M(\phi \vee \psi) = I^M(\phi) \vee I^M(\psi)$,
(M2) $I^M(\neg\phi) = \neg I^M(\phi)$,
(M3) $I^M(\epsilon ab) = \Diamond p_a \supset p_a. \wedge .\Box p_a \supset \Box p_b. \wedge .\Diamond p_b \supset p_a$,

where p_a and p_b are propositional variables corresponding to the name variables a and b, respectively. In the last section, we shall give some comments including some open problems and my conjectures.

Keywords: Leśniewski's ontology, propositional ontology, translation, interpretation, modal logic, KTB, soundness, Grzegorczyk's modal logic.

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1. Introduction and I^M

Inoué [9] initiated a study of interpretations of Leśniewski's epsion ϵ in the modal logic **K** and its certain extensions. That is, Ishimoto's propositional fragment $\mathbf{L_1}$ (Ishimoto [12]) of Leśniewski's ontology **L** (refer to Urbaniak [19]) is partially embedded in **K** and in the extensions, respectively, by the following translation I from $\mathbf{L_1}$ to them: for any formula ϕ and ψ of $\mathbf{L_1}$, it is defined as

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(I1) $I(\phi \vee \psi) = I(\phi) \vee I(\psi)$,

(I2) $I(\neg\phi) = \neg I(\phi)$,

(I3) $I(\epsilon ab) = p_a \wedge \Box(p_a \equiv p_b)$,

where p_a and p_b are propositional variables corresponding to the name variables a and b, respectively. Here, "$\mathbf{L_1}$ is partially embedded in $\mathbf{K}$ by I" means that for any formula ϕ of a certain decidable nonempty set of formulas of $\mathbf{L_1}$ (i.e. decent formulas (see § 3 of Inoué [10])), ϕ is a theorem of $\mathbf{L_1}$ if and only if $I(\phi)$ is a theorem of $\mathbf{K}$. Note that I is sound. The paper [10] also proposed similar partial interpretations of Leśniewski's epsilon in certain von Wright-type deontic logics, that is, ten Smiley-Hanson systems of monadic deontic logic and in provability logic **GL**, respectively. (See Åqvist [1] and Boolos [3] for those logics.)

The interpretation I is however not faithful. A counterexample for the faithfulness is, for example, $\epsilon ac \wedge \epsilon bc. \supset .\epsilon ab \vee \epsilon cc$ (for the details, see [10]). Blass [2] gave a modification of the interpretation and showed that his interpretation T is faithful, using Kripke models. Inoué [11] called the translation *Blass translation* (for short, *B-translation*) or *Blass interpretation* (for short, *B-interpretation*). The translation B from $\mathbf{L_1}$ to $\mathbf{K}$ is defined as follows: for any formula ϕ and ψ of $\mathbf{L_1}$,

(B1) $B(\phi \vee \psi) = B(\phi) \vee B(\psi)$,

(B2) $B(\neg\phi) = \neg B(\phi)$,

(B3) $B(\epsilon ab) = p_a \wedge \Box(p_a \supset p_b) \wedge .p_b \supset \Box(p_b \supset p_a)$,

where p_a and p_b are propositional variables corresponding to the name variables a and b, respectively. Inoué [11] extended Blass's faithfulness result for many normal modal logics, provability logic and von Wright-type deontic logics including **K4**, **KD**, **KB**, **KD4**, etc, **GL** and ten Smiley-Hanson systems of monadic deontic logic, using model constructions based on Hintikka formula (cf. Kobayashi and Ishimoto [13]).

In this paper, we first propose a translation I^M from $\mathbf{L_1}$ in modal logic **KTB**, which will be specified in § 2.

Definition 1.1. A translation I^M of Leśniewski's propositional ontology $\mathbf{L_1}$ in modal logic **KTB** is defined as follows: for any formula ϕ and ψ of $\mathbf{L_1}$,

(M1) $I^M(\phi \vee \psi) = I^M(\phi) \vee I^M(\psi)$,
(M2) $I^M(\neg\phi) = \neg I^M(\phi)$,
(M3) $I^M(\epsilon ab) = \Diamond p_a \supset p_a. \wedge .\Box p_a \supset \Box p_b. \wedge .\Diamond p_b \supset p_a$,

where p_a and p_b are propositional variables corresponding to the name variables a and b, respectively.

We call I^M to be *M-translation* or *M-interpretation*.

In the following § 2, we shall collect the basic preliminaries for this paper. In § 3, using proof theory, we shall show that I^M is sound, as the main theorem of this paper. In § 4, we shall give some comments including some open problems and my conjectures.

2. Propositional ontology $\mathbf{L_1}$ and modal logic KTB

Let us recall a formulation of $\mathbf{L_1}$, which was introduced in [12]. The Hilbert-style system of it, denoted again by $\mathbf{L_1}$, consists of the following axiom-schemata with a formulation of classical propositional logic **CP** as its axiomatic basis:

(Ax1) $\epsilon ab \supset \epsilon aa$,

(Ax2) $\epsilon ab \wedge \epsilon bc. \supset \epsilon ac$,

(Ax3) $\epsilon ab \wedge \epsilon bc. \supset \epsilon ba$,

where we note that every atomic formula of $\mathbf{L_1}$ is of the form ϵab for some name variables a and b and a possible intuitive interpretation of ϵab is 'the a is b'. We note that (Ax1), (Ax2) and (Ax3) are theorems of Leśniewski's ontology (see Słupecki [17]).

The modal logic **K** is the smallest logic which contains all instances of classical tautology and all formulas of the forms $\Box(\phi \supset \psi) \supset .\Box\phi \supset \Box\psi$ being closed under modus ponens and the rule of necessitation (for **K** and basics for modal logic, see Bull and Segerberg [4], Chagrov and Zakharyaschev [5], Fitting [6], Hughes and Cresswell [8] and so on).

We recall the naming of modal logics as follows (refer to e.g. Poggiolesi [15] and Ono [14], also see Bull and Segerberg [4]):

KT: **K** + $\Box\phi \supset \phi$ (**T**, reflexive relation)
KB: **K** + $\phi \supset \Box\Diamond\phi$ (**B**, symmetric relation)
KTB: **KT** + **B** (reflexive and symmetric relation).

3. The soundness of I^M

THEOREM 3.1. (Soundness) *For any formula* ϕ *of* $\mathbf{L_1}$*, we have*

$$\vdash_{\mathbf{L_1}} \phi \ \Rightarrow \vdash_{\mathbf{KTB}} I^M(\phi).$$

PROOF: Let ϕ be a formula of $\mathbf{L_1}$. We shall prove the meta-implication by induction on derivation.

BASIS.

(Case 1) We shall first treat the case for (Ax1). Let a and b be name variables. Then we have the following inferences in **KTB**:

(∗) $I^M(\epsilon ab)$ (Assumption)

(1.1) $\Diamond p_a \supset p_a$ from (∗) and Definition 1.1) †

(1.2) $\Box p_a \supset \Box p_a$ (true in **K**) †

(1.3) $\Diamond p_a \supset p_a. \wedge .\Box p_a \supset \Box p_a. \wedge .\Diamond p_a \supset p_a$ (from (1.1) and (1.2))

(1.4) $I^M(\epsilon aa)$ (from (1.3) and Definition 1.1)

(1.5) $I^M(\epsilon ab \supset \epsilon aa)$ (from (∗), (1.4) and Definition 1.1).

(Case 2) Next we shall deal with the case of (Ax2). Let a, b and c be name variables. Then we have the following inferences in **KTB**:

(∗∗) $I^M(\epsilon ab \wedge \epsilon bc)$ (Assumption)

(2.1) $I^M(\epsilon ab)$ (from (∗∗) and Definition 1.1)

(2.2) $I^M(\epsilon bc)$ (from (∗∗) and Definition 1.1)

(2.3) $\Diamond p_a \supset p_a. \wedge .\Box p_a \supset \Box p_b. \wedge .\Diamond p_b \supset p_a$ (from (2.1) and Def 1.1)

(2.4) $\Diamond p_b \supset p_b. \wedge .\Box p_b \supset \Box p_c. \wedge .\Diamond p_c \supset p_b$ (from (2.2) and Def 1.1)

(2.5) $\Diamond p_a \supset p_a$ (from (2.3)) †

(2.6) $\Box p_a \supset \Box p_b$ (from (2.3))

(2.7) $\Box p_b \supset \Box p_c$ (from (2.4))

(2.8) $\Box p_a \supset \Box p_c$ (from (2.6) and (2.7)) †

(2.9) $\Diamond p_b \supset p_a$ (from (2.3))

(2.10) $\Box(\Diamond p_b \supset p_a)$ (from (2.9) and the rule of necessitation)

(2.11) $\Box\Diamond p_b \supset \Box p_a$ (from (2.10) with a true inference in **K**)

(2.12) $\Box p_a \supset p_a$ (true in **KT**)

(2.13) $\Box\Diamond p_b \supset p_a$ (from (2.11) and (2.12))

(2.14) $p_b \supset \Box\Diamond p_b$ (true in **KB**)
(2.15) $\Diamond p_c \supset p_b$ (from (2.4))
(2.16) $\Diamond p_c \supset p_a$ (from (2.13) and (2.14) and (2.15)) †
(2.17) $\Diamond p_a \supset p_a . \wedge . \Box p_a \supset \Box p_c . \wedge . \Diamond p_c \supset p_a$ (from (2.5), (2.8) and (2.16))
(2.18) $I^M(\epsilon ac)$ (from (2.17) and Definition 1.1)
(2.19) $I^M(\epsilon ab \wedge \epsilon bc. \supset \epsilon ac)$ (from (∗∗), (2.18) and Definition 1.1).

(Case 3) Lastly we shall proceed to the case of (Ax3). Let a, b and c be name variables. Then we also have the following inferences in **KTB**:

(∗∗∗) $I^M(\epsilon ab \wedge \epsilon bc)$ (Assumption)
(3.1) $I^M(\epsilon ab)$ (from (∗∗∗) and Definition 1.1)
(3.2) $I^M(\epsilon bc)$ (from (∗∗∗) and Definition 1.1)
(3.3) $\Diamond p_a \supset p_a . \wedge . \Box p_a \supset \Box p_b . \wedge . \Diamond p_b \supset p_a$ (from (3.1) and Def 1.1)
(3.4) $\Diamond p_b \supset p_b . \wedge . \Box p_b \supset \Box p_c . \wedge . \Diamond p_c \supset p_b$ (from (3.2) and Def 1.1)
(3.5) $\Diamond p_b \supset p_b$ (from (3.4)) †
(3.6) $\Diamond p_b \supset p_a$ (from (3.3))
(3.7) $\Box(\Diamond p_b \supset p_a)$ (from (3.6) and the rule of necessitation)
(3.8) $\Box\Diamond p_b \supset \Box p_a$ (from (3.7) with a true inference in **K**)
(3.9) $p_b \supset \Box\Diamond p_b$ (true in **KB**)
(3.10) $\Box p_b \supset p_b$ (true in **KT**)
(3.11) $\Box p_b \supset \Box p_a$ (from (3.8) and (3.9) and (3.10)) †
(3.12) $\Diamond p_a \supset p_a$ (from (3.3))
(3.13) $p_a \supset \Box\Diamond p_a$ (true in **KB**)
(3.14) $\Diamond p_a \supset \Box\Diamond p_a$ (from (3.12) and (3.13))
(3.15) $\Box(\Diamond p_a \supset p_a)$ (from (3.12) and the rule of necessitation)
(3.16) $\Box\Diamond p_a \supset \Box p_a$ (from (3.15) with a true inference in **K**)
(3.17) $\Diamond p_a \supset \Box p_a$ (from (3.14) and (3.16))
(3.18) $\Box p_a \supset \Box p_b$ (from (3.3))
(3.19) $\Diamond p_a \supset \Box p_b$ (from (3.17) and (3.18))
(3.20) $\Box p_b \supset p_b$ (true in **KT**)
(3.21) $\Diamond p_a \supset p_b$ (from (3.19) and (3.20)) †

(3.22) $\Diamond p_b \supset p_b . \wedge . \Box p_b \supset \Box p_a . \wedge . \Diamond p_a \supset p_b$

(from (3.5), (3.11) and (3.21))

(3.23) $I^M(\epsilon ba)$ (from (3.22) and Definition 1.1)

(3.24) $I^M(\epsilon ab \wedge \epsilon bc. \supset \epsilon ba)$ (from $(***)$, (3.23) and Definition 1.1).

Induction Steps. The induction step is easily dealt with. Suppose that ϕ and $\phi \supset \psi$ are theorems of $\mathbf{L_1}$. By induction hypthesis, $I^M(\phi)$ and $I^M(\phi \supset \psi)$ ($\leftrightarrow I^M(\phi) \supset I^M(\psi)$) are theorems of **KTB**. By modus ponens, we obtain $\vdash_{\mathbf{KTB}} I^M(\psi)$. Thus this completes the proof the theorem. □

4. Comments

One motive from which I wrote [9] and [10] is that I wished to understand Leśniewski's epsilon ϵ on the basis of my recognition that Leśniewski's epsilon would be a variant of truth-functional equivalence $\equiv$. Namely, my original approach to the interpretation of ϵ was to express the deflection of ϵ from $\equiv$ in terms of Kripke models. Another (hidden) motive of mine for I^M is to interpret $\mathbf{L_1}$ in intuitionistic logic and bi-modal logic. It is well-known that Leśniewski's epsilon can be interpreted by the Russellian-type definite description in classical first-order predicate logic with equality (see [12]). Takano [18] proposed a natural set-theoretic interpretation for the epsilon. To repeat, I do not deny the interpretation using the Russellian-type definite description and a set-theoretic one. I wish to obtain another interpretation of Leśniewski's epsilon having a more propositional character. We have the following direct open problems.

Open problem 1: Is I^M faithful?

Open problem 2: Find the set of other translations and modal logics in which $\mathbf{L_1}$ is embedded. I think that there seems to be many possibilities.

Open problem 3: Can $\mathbf{L_1}$ be embedded in **S4.2**? (See e.g. Hamkins and Löwe [7].)

Open problem 4: Can $\mathbf{L_1}$ be embedded in Grzegorczyk's modal Logic? (See e.g. Savateev and Shamkanov [16])

My conjectures are the following.

Conjecture 4.1. I^M is faithful.

Conjecture 4.2. t seems that $\mathbf{L_1}$ cannot be embedded in intuitionistic propositional logic.

Conjecture 4.3. It seems that $\mathbf{L_1}$ can well be embedded in intuitionistic modal propositional logic.

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ON COMPLETE REPRESENTATIONS AND MINIMAL COMPLETIONS IN ALGEBRAIC LOGIC, BOTH POSITIVE AND NEGATIVE RESULTS

Abstract

Fix a finite ordinal $n \geq 3$ and let α be an arbitrary ordinal. Let CA_n denote the class of cylindric algebras of dimension n and RA denote the class of relation algebras. Let $\mathsf{PA}_\alpha(\mathsf{PEA}_\alpha)$ stand for the class of polyadic (equality) algebras of dimension α. We reprove that the class CRCA_n of completely representable CA_ns, and the class CRRA of completely representable RAs are not elementary, a result of Hirsch and Hodkinson. We extend this result to any variety V between polyadic algebras of dimension n and diagonal free CA_ns. We show that that the class of completely and strongly representable algebras in V is not elementary either, reproving a result of Bulian and Hodkinson. For relation algebras, we can and will, go further. We show the class CRRA is not closed under $\equiv_{\infty,\omega}$. In contrast, we show that given $\alpha \geq \omega$, and an atomic $\mathfrak{A} \in \mathsf{PEA}_\alpha$, then for any $n < \omega$, $\mathsf{Nr}_n\mathfrak{A}$ is a completely representable PEA_n. We show that for any $\alpha \geq \omega$, the class of completely representable algebras in certain reducts of PA_αs, that happen to be varieties, is elementary. We show that for $\alpha \geq \omega$, the the class of polyadic-cylindric algebras dimension α, introduced by Ferenczi, the completely representable algebras (slightly altering representing algebras) coincide with the atomic ones. In the last algebras cylindrifications commute only one way, in a sense weaker than full fledged commutativity of cylindrifications enjoyed by classical cylindric and polyadic algebras. Finally, we address closure under Dedekind–MacNeille completions for cylindric-like algebras of dimension n and PA_αs for α an infinite ordinal, proving negative results for the first and positive ones for the second.

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1. Introduction

Unless otherwise indicated, $2 < n < \omega$. Lately, it has become fashionable for algebras of relations, such as relation algebras, cylindric algebras due mainly to Tarski and polyadic algebras due to Halmos, to study representations that preserve infinitary meets and joins.

This phenomenon is extensively discussed in [24], where it is shown that it has an affinity with the algebraic notion of complete representations for cylindric like algebras and atom-canonicity in varieties of Boolean algebras with operators (BAOs). a prominent persistence property studied in modal logic.

A completely additive variety V of BAOs is atom-canonical, if whenever $\mathfrak{A} \in \mathsf{V}$ is atomic, then its Dedekind–MacNeille completions, namely, the complex algebra of its atom structure, in symbols $\mathfrak{Cm}\mathsf{At}\mathfrak{A}$ is also in V. The Dedekind–MacNeille completion of a CA_n is often referred to as its minimal Monk completion, since Monk showed that the Dedekind–MacNeille completion of a CA_n is again a CA_n. Here we use *minimal Dedekind–MacNeille completions*, or simply *the Dedekind–MacNeille completions*.

As for complete representations, the typical question is: given an algebra and a set of meets, is there a representation that carries this set of meets to set theoretic intersections? (assuming that our semantics is specified by set algebras, with the concrete Boolean operation of intersection among its basic operations.) When the algebra in question is countable, and we have countably many meets; this is an algebraic version of an omitting types theorem; the representation omits the given set of possibly infinitary meets or non-principal types. When it is only one meet consisting of co-atoms, in an atomic algebra, this representation is a complete one. The correlation of atomicity to complete representations has caused a lot of confusion in the past. It was mistakenly thought for a while, among algebraic logicians, that atomic representable relation and cylindric algebras are completely representable, an error attributed to Lyndon and now referred to as Lyndon's error. For Boolean algebras, however, this is true.

Follows is a crash rundown of known results: For Boolean algebras, the class of completely representable algebras is simply the class of atomic ones, hence is elementary. The class of completely representable polyadic algebras coincide with the class of atomic, completely additive algebras in this class, hence is also elementary [26]. The class CRCA_n of completely representable CA_ns is proved not to be elementary by Hirsch and Hodkinson in [9]. For any pair of ordinal $\alpha < \beta$, $\mathsf{Nr}_\alpha\mathsf{CA}_\beta(\subseteq \mathsf{CA}_\alpha)$ denotes the class of neat α-reducts of CA_βs as defined in [7, Definition 2.2.28]. Neat embeddings and complete representations are linked in [25, Theorem 5.3.6] where it is shown that CRCA_n coincides with the class $\mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_\omega$- on atomic algebra having countably many atoms. Here $\mathbf{S}_c$ denotes the operation of forming *complete* subalgebras, that is to say, given a class of algebras K having a Bolean reduct, then $\mathbf{S}_c\mathsf{K} = \{\mathfrak{B} : (\exists\mathfrak{A} \in \mathsf{K})(\forall X \subseteq \mathfrak{A}\sum^{\mathfrak{A}} X = 1 \implies \sum^{\mathfrak{B}} X = 1\}$ where $\sum$ denotes 'supremum' with the superscript specificying the algebra 'the evaluated supremum' exist in. The analogous result for relation algebras is proved in [8]. The latter result on charecterization of completely representable algebra via neat embedings will be extended below to the infinite dimensional case by defining complete representations via so-called *weak set algebras*.

In [17] it is proved that for any pair of ordinals $\alpha < \beta$, the class $\mathsf{Nr}_\alpha\mathsf{CA}_\beta$ is not elementary. A different model theoretic proof for finite α is given in [25, Theorem 5.4.1]. This result is extended to many cylindric like algebras like Halmos' polyadic algebras with and without equality, and Pinter's substitution algebras [18, 21, 19], cf. [20] for an overview. Below we give a single proof to all cases. The analogous result for relation agebras is proved in [22]. The paper is divided to two parts. Part 1 is devoted to cylindric-like algebras, while Part 2 is devoted to polyadic-like algebras. These two paradigms, the cylindric as opposed to the polyadic, often exhibit conflcting behavior.

Cylindric paradigm:

- In Section 2.1, we give the basic definitions of cylindric and relation algebras. Atomic networks and two player deteministic games between two players $\exists$ Ellosie and $\forall$ belard games characterizing neat embeddings, played on such networks, are defined in Section 2.2. Lemma 2.5 is the main result in Section 2.2. In all games used throughout the paper one of th players has a winning strategy thart can be implemented explicitly using a finite or tranfinite number of rounds and a set of 'nodes' usually finite. There are no draws.

- In Section 3 we reprove a classical result of Hirsch and Hodkinson [9]. Let $2 < n < \omega$. In Section 3.1 we show that the class of completely representable cylindric algebras of dimension n, briefly CRCA_n and the class of completely representable relation algebras, briefly CRRA are not elementary. The proof depends on so called Monk-Maddux relation algebras possessing what Maddux calls cylindric basis [16], cf. Lemma 2.6. We highlight the difference between our proof and the orginal first poof of the result (in print at least) in [9]. The two proofs are conceptually 'disjoint' as is illustrated. Using two player determinisc games between $\exists$ and $\forall$ on pebble paired structures, we go further by showing that CRRA is not closed under $\equiv_\infty$ in Theorem 2.8, thus answering a question posed by Hirsch and Hodkinson in [9, 11].

- Fix $2 < n \leq m \leq \omega$. We study locally classic representations, and locally classic complete representations, referred to as m-square representations or m-clique guarded semantics [10, 27] relating it to neat embeddings via existence of m-dilations and games using m nodes.

- We prove that for any variety V between PEA_n and Pinter's substitution algebras of dimension n (a notion to be made precise), the class $\mathsf{Nr}_n\mathsf{V}_m$ is not elementary for any ordinal $m > n > 1$ unifying the proofs of results established in [18, 21, 17, 25], cf. Theorem 3.5. Our new proof is model-theoretic, resorting to a Fraïssé constuction, analogous to the proof in [25] where the result restricted to only cylindric algebras is proved.

Polyadic paradigm:

- We show that given any atomic $\mathfrak{A} \in \mathsf{PEA}_\alpha$, α an infinite ordinal, we can obtain a plethora of completely representable algebras from $\mathfrak{A}$ for each $n < \omega$, by taking the operation of n neat reduct. In more detail, let $\mathfrak{A} \in \mathsf{PEA}_\alpha$ be atomic, then for any $n < \omega$, any complete subalgebra of $\mathsf{Nr}_n\mathfrak{A}$ is completely representable, cf. Theorem 4.1.

- We show that the class of completely representable algebras, of the variety obtained from polyadic algebras of infinite dimension, by discarding infinitary cylindrications while keeping all substitution operators is elementary, and that the class of polyadic cylindric of infinite dimensional algebras introduced by Ferenczi in [4] is also elementary; in fact in the former case the class of completely representable algebras coincide with the atomic completely additive ones, and in the second case the class of completely representable algebras, are like the case of Boolean algebras, simply the atomic ones, cf. the second part Theorem 4.1.

- Let $2 < n < \omega$. Closure under Dedekind–MacNeille completions, often referred to as *minimal completions* (which is the term used in he title) and Sahlqvist axiomatizability for varieties between QEA_n and Sc_n, where the last denotes the class of Pinter's substituition algebras as defined in [13] and also for the polyadic-like algebras addressed above are approached, cf. Theorems 5.5 and 5.6. Again negative results are obtained in the first case for cylindric-like algebras, while positive results prevail in the second polyadic paradigm, where all substitution operations are available in the signature.

Our results further emphasizes the dichotomy existing between the cylindric paradigm and the polyadic one, a phenomena recurrent in the literature of Tarski's cylindric algebras and Halmos' polyadic algebras, with algebras 'in between' such as Ferenzci's cylindric–polyadic algebras with and without equality, aspiring to share only nice desirable properties of both.

Such properties, some of which are thoroughly investigated below, include (not exclusively) finite axiomatizablity of the variety of representable algebras, the canonicity and atom-canonicity of such varieties, decidability of its equational/ or and universal theory, and the first order definability of the notion of complete representability [4, 5, 6].

2. The cylindric paradigm

2.1. The algebras and some basic concepts

For a set V, $\mathcal{B}(V)$ denotes the Boolean set algebra $\langle \wp(V), \cup, \cap, \sim, \emptyset, V\rangle$. Let U be a set and α an ordinal; α will be the dimension of the algebra. For $s, t \in {}^{\alpha}U$ write $s \equiv_i t$ if $s(j) = t(j)$ for all $j \neq i$. For $X \subseteq {}^{\alpha}U$ and $i, j < \alpha$, let

$$\mathsf{C}_i X = \{s \in {}^{\alpha}U : (\exists t \in X)(t \equiv_i s)\}$$

and

$$\mathsf{D}_{ij} = \{s \in {}^{\alpha}U : s_i = s_j\}.$$

The algebra $\langle \mathcal{B}({}^{\alpha}U), \mathsf{C}_i, \mathsf{D}_{ij} \rangle_{i,j<\alpha}$ is called *the full cylindric set algebra of dimension* α with unit (or greatest element) ${}^{\alpha}U$ referred to as a *cartesian square of dimension* α. Here full refers to the fact that the universe of the algebra is all of $\wp({}^{\alpha}U)$.

Fix an ordinal α. A *cylindric set algebra of dimension* α is a subalgebra of a full cylindric set algebra of the same dimension. The class of cylindric set algebras of dimension α is denoted by Cs_{α}. It is known that the variety gernerated by Cs_{α}, in symbols RCA_{α} denoting the class of *representable cylindric algebras of dimension* α, is the class $\mathbf{SP}\mathsf{Cs}_{\alpha}$ where $\mathbf{S}$ denotes the operation of forming subalgebras and $\mathbf{P}$ is the operation of forming products. Thus the class RCA_{α} is closed under $\mathbf{H}$ (forming homomorphic images). Furthermore, it is known that $\mathsf{RCA}_{\alpha} = \mathbf{I}\mathsf{Gs}_{\alpha}$ where $\mathbf{Gs}_{\alpha}$ is the class of *generalized set algebras of dimension* α and $\mathbf{I}$ is the operation of forming isomorphic images.

An algebra $\mathfrak{A} \in \mathbf{Gs}_{\alpha}$ if it has top element a disjoint union of cartesian squares each of dimension α and all of the the cylindric operations are defined like in the class of set algebras of the same dimension. In particular, the Boolean operations of meet, join and complemenation are the set theoretic operations of intersection, union, and taking complements relative to the top element, respectively. Let α be an ordinal. The (equationally defined) CA_{α} class is obtained from cylindric set algebras by a process of abstraction and is defined by a *finite* schema of equations given in [7, Definition 1.1.1] that holds of course in the more concrete (generalized) set algebras of dimension α.

Definition 2.1. Let $n < \omega$. Then $\mathfrak{A} \in \mathsf{CA}_n$ is *completely representable*, if there exists $\mathfrak{B} \in \mathsf{Gs}_n$ and an isomorphism $f : \mathfrak{A} \to \mathfrak{B}$ such for all $X \subseteq \mathfrak{A}$, $f(\prod X) = \bigcap_{x \in X} f(x)$ whenever $\prod X$ exists.

We consider relation algebras as algebras of the form $\mathcal{R} = \langle R, +, \cdot, -, 1', \smile, ;, \rangle$, where $\langle R, +, \cdot, - \rangle$ is a Boolean algebra $1' \in R$, $\smile$ is a unary operation and ; is a binary operation. A relation algebra is *representable* $\Longleftrightarrow$ it is isomorphic to a subalgebra of the form $\langle \wp(X), \cup, \cap, \sim, \smile, \circ, Id \rangle$ where X is an equivalence relation, $1'$ is interpreted as the identity relation, $\smile$ is the operation of forming converses, and the binary operation ; is interpreted as composition of relations. Following standard notation, RA denotes the class of relation algebras. The class RA is a discriminator variety that is finitely axiomatizable, cf. [10, Definition 3.8, Theorems 3.19]. The variety of representable relation algebras is denoted by RRA. It is known that RRA is not finitely axiomatizable; a classical result of Monk using a sequence of called non-representable Lyndon algebars whose ultraproduct is representable. Later this non-finite axiomatizability result was refined considerably by Maddux, Hirsch, Hodkinson and Sagi [12, 16]. We let CRRA and LRRA denote the classes of completely representable RAs, and its elementary closure, namely, the class of RAs satisfying the Lyndon conditions as defined in [10, § 11.3.2], respectively. Complete representability of RAs is defined like the CA case. We denote by CRRA the class of completely representable RAs.

Let α be an ordinal and $\mathfrak{A} \in \mathsf{CA}_\alpha$. For any $i, j, l < \alpha$, let $\mathsf{s}_i^j x = x$ if $i = j$ and $\mathsf{s}_i^j x = \mathsf{c}_j(\mathsf{d}_{ij} \cdot x)$ if $i \neq j$. Let ${}_l\mathsf{s}(i,j)x = \mathsf{s}_i^l \mathsf{s}_j^i \mathsf{s}_l^j x$. In the next definition, in its first item we define the notion of forming α-neat reducts of CA_βs with $\beta > \alpha$, in symbols $\mathfrak{Nr}_\alpha$, and in the second item we define relation algebras obtained from cylindric algebras using the operator $\mathfrak{Nr}_2$.

Definition 2.2.

1. Assume that $\alpha < \beta$ are ordinals and that $\mathfrak{B} \in \mathsf{CA}_\beta$. Then the *$\alpha$–neat reduct* of $\mathfrak{B}$, in symbols $\mathfrak{Nr}_\alpha\mathfrak{B}$, is the algebra obtained from $\mathfrak{B}$, by discarding cylindrifiers and diagonal elements whose indices are in $\beta \setminus \alpha$, and restricting the universe to the set $Nr_\alpha B = \{x \in \mathfrak{B} : \{i \in \beta : \mathsf{c}_i x \neq x\} \subseteq \alpha\}$.
2. Assume that $\alpha \geq 3$. Let $\mathfrak{A} \in \mathsf{CA}_\alpha$. Then $\mathsf{Ra}\mathfrak{A} = \langle Nr_2\mathfrak{A} : +, \cdot, -, ;, \smile, \mathsf{d}_{01}\rangle$ where for any $x, y \in Nr_n\mathfrak{A}$, $x; y = \mathsf{c}_2(\mathsf{s}_2^1 x \cdot \mathsf{s}_2^0 y)$ and $x \smile = {}_2\,\mathsf{s}(0.1)x$

If $\mathfrak{A} \in \mathsf{CA}_3$, $\mathsf{Ra}\mathfrak{A}$, having the same signature as RA may not be a relation algebra as associativiy of the (abstract) composition operation may fail, but for $\alpha \geq 4$, $\mathsf{RaCA}_\beta \subseteq \mathsf{RA}$. relativized to V. By the same token the variety of representable relation algebras is denoted by RRA. It is known that $\mathbf{I}\mathsf{Gs}_\alpha = \mathsf{RCA}_\alpha = \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{\alpha+\omega} = \bigcap_{k\in\omega} \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{\alpha+k}$ and that $\mathsf{RRA} = \mathbf{S}\mathsf{RaCA}_\omega = \bigcap_{\mathsf{k}\in\omega} \mathbf{S}\mathsf{RaCA}_{3+\mathsf{k}}$.

2.2. Neat embeddings and games

From now on, unless otherwise indicated, n is fixed to be a finite ordinal > 2. Let $i < n$. For n–ary sequences $\bar{x}$ and $\bar{y}$, we write $\bar{x} \equiv_i \bar{y} \iff \bar{y}(j) =$

$\bar{x}(j)$ for all $j \neq i$, To define certain games to be used in the sequel, we recall the notions of *atomic networks* and *atomic games* [10, 11]. Let $i < n$. For n–dimensional atomic networks M and N, we write $M \equiv_i N \iff M(\bar{y}) = N(\bar{y})$ for all $\bar{y} \in {}^n(n \sim \{i\})$.

DEFINITION 2.3.

1. Assume that $\mathfrak{A} \in \mathsf{CA}_n$ is atomic and that $m, k \leq \omega$. The *atomic game* $G^m_k(\mathsf{At}\mathfrak{A})$*, or simply* G^m_k, is the game played on atomic networks of $\mathfrak{A}$ using m nodes and having k rounds [11, Definition 3.3.2], where $\forall$ is offered only one move, namely, *a cylindrifier move*:
   Suppose that we are at round $t > 0$. Then $\forall$ picks a previously played network N_t ($\mathsf{nodes}(N_t) \subseteq m$), $i < n$, $a \in \mathsf{At}\mathfrak{A}$, $\bar{x} \in {}^n\mathsf{nodes}(N_t)$, such that $N_t(\bar{x}) \leq \mathsf{c}_i a$. For her response, $\exists$ has to deliver a network M such that $\mathsf{nodes}(M) \subseteq m$, $M \equiv_i N$, and there is $\bar{y} \in {}^n\mathsf{nodes}(M)$ that satisfies $\bar{y} \equiv_i \bar{x}$ and $M(\bar{y}) = a$.

   We write $G_k(\mathsf{At}\mathfrak{A})$, or simply G_k, for $G^m_k(\mathsf{At}\mathfrak{A})$ if $m \geq \omega$.

2. The ω–rounded game $\mathbf{G}^m(\mathsf{At}\mathfrak{A})$ or simply $\mathbf{G}^m$ is like the game $G^m_\omega(\mathsf{At}\mathfrak{A})$ except that $\forall$ has the option to reuse the m nodes in play.

DEFINITION 2.4. Let m be a finite ordinal > 0. An s word is a finite string of substitutions (s^j_i) $(i, j < m)$, a c word is a finite string of cylindrifications $(\mathsf{c}_i), i < m$; an sc word w, is a finite string of both, namely, of substitutions and cylindrifications. An sc word induces a partial map $\hat{w} : m \to m$:

- $\hat{\epsilon} = Id$,
- $\widehat{w^i_j} = \hat{w} \circ [i|j]$,
- $\widehat{w\mathsf{c}_i} = \hat{w} \restriction (m \smallsetminus \{i\})$.

If $\bar{a} \in {}^{<m-1}m$, we write $\mathsf{s}_{\bar{a}}$, or $\mathsf{s}_{a_0 \ldots a_{k-1}}$, where $k = |\bar{a}|$, for an arbitrary chosen sc word w such that $\hat{w} = \bar{a}$. Such a w exists by [10, Definition 5.23 Lemma 13.29].

In the next theorem $\mathbf{S}_c$ stands for the operation of forming complete subalgebras.

LEMMA 2.5. *Fix finite* $n \geq 3$. *If* $\mathfrak{A} \in \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_m$ *is atomic, then* $\exists$ *has a winning strategy in* $\mathbf{G}^m(\mathsf{At}\mathfrak{A})$.

Proof: Fix $2 < n < m$. Assume that $\mathfrak{C} \in \mathsf{CA}_m$, $\mathfrak{A} \subseteq_c \mathfrak{Nr}_n\mathfrak{C}$ is an atomic CA_n and N is an $\mathfrak{A}$–network with $\mathsf{nodes}(N) \subseteq m$. Define $N^+ \in \mathfrak{C}$ by (with notation as introducted in Definition 2.4):

$$N^+ = \prod_{i_0,\ldots,i_{n-1} \in \mathsf{nodes}(N)} \mathsf{s}_{i_0,\ldots,i_{n-1}} N(i_0,\ldots,i_{n-1}).$$

For a network N and function θ, the network $N\theta$ is the complete labelled graph with nodes $\theta^{-1}(\mathsf{nodes}(N)) = \{x \in \mathsf{dom}(\theta) : \theta(x) \in \mathsf{nodes}(N)\}$, and labelling defined by

$$(N\theta)(i_0,\ldots,i_{n-1}) = N(\theta(i_0), \theta(i_1), \ldots, \theta(i_{n-1})),$$

for $i_0,\ldots,i_{n-1} \in \theta^{-1}(\mathsf{nodes}(N))$. Then the following hold:

(1): for all $x \in \mathfrak{C} \setminus \{0\}$ and all $i_0,\ldots,i_{n-1} < m$, there is $a \in \mathsf{At}\mathfrak{A}$, such that $\mathsf{s}_{i_0,\ldots,i_{n-1}} a \,.\, x \neq 0$,

(2): for any $x \in \mathfrak{C} \setminus \{0\}$ and any finite set $I \subseteq m$, there is a network N such that $\mathsf{nodes}(N) = I$ and $x \cdot N^+ \neq 0$. Furthermore, for any networks M, N if $M^+ \cdot N^+ \neq 0$, then $M{\restriction}_{\mathsf{nodes}(M)\cap\mathsf{nodes}(N)} = N{\restriction}_{\mathsf{nodes}(M)\cap\mathsf{nodes}(N)}$,

(3): if θ is any partial, finite map $m \to m$ and if $\mathsf{nodes}(N)$ is a proper subset of m, then $N^+ \neq 0 \to (N\theta)^+ \neq 0$. If $i \notin \mathsf{nodes}(N)$, then $\mathsf{c}_i N^+ = N^+$.

Since $\mathfrak{A} \subseteq_c \mathfrak{Nr}_n\mathfrak{C}$, then $\sum^{\mathfrak{C}} \mathsf{At}\mathfrak{A} = 1$. *For (1), s^i_j is a completely additive operator (any $i, j < m$), hence $\mathsf{s}_{i_0,\ldots,i_{n-1}}$ is, too.* So $\sum^{\mathfrak{C}}\{\mathsf{s}_{i_0\ldots,i_{n-1}} a : a \in \mathsf{At}(\mathfrak{A})\} = \mathsf{s}_{i_0\ldots i_{n-1}} \sum^{\mathfrak{C}} \mathsf{At}\mathfrak{A} = \mathsf{s}_{i_0\ldots,i_{n-1}} 1 = 1$ for any $i_0,\ldots,i_{n-1} < m$. Let $x \in \mathfrak{C} \setminus \{0\}$. Assume for contradiction that $\mathsf{s}_{i_0\ldots,i_{n-1}} a \cdot x = 0$ for all $a \in \mathsf{At}\mathfrak{A}$. Then $1 - x$ will be an upper bound for $\{\mathsf{s}_{i_0\ldots i_{n-1}} a : a \in \mathsf{At}\mathfrak{A}\}$. But this is impossible because $\sum^{\mathfrak{C}}\{\mathsf{s}_{i_0\ldots,i_{n-1}} a : a \in \mathsf{At}\mathfrak{A}\} = 1$.

To prove the first part of (2), we repeatedly use (1). We define the edge labelling of N one edge at a time. Initially, no hyperedges are labelled. Suppose $E \subseteq \mathsf{nodes}(N) \times \mathsf{nodes}(N) \ldots \times \mathsf{nodes}(N)$ is the set of labelled hyperedges of N (initially $E = \emptyset$) and $x \,.\, \prod_{\bar{c} \in E} \mathsf{s}_{\bar{c}} N(\bar{c}) \neq 0$. Pick $\bar{d}$ such that $\bar{d} \notin E$. Then by (1) there is $a \in \mathsf{At}(\mathfrak{A})$ such that $x \,.\, \prod_{\bar{c} \in E} \mathsf{s}_{\bar{c}} N(\bar{c}) \,.\, \mathsf{s}_{\bar{d}} a \neq 0$. Include the hyperedge $\bar{d}$ in E. We keep on doing this until eventually all hyperedges will be labelled, so we obtain a completely labelled graph N with $N^+ \neq 0$. it is easily checked that N is a network.

For the second part of (2), we proceed contrapositively. Assume that there is $\bar{c} \in \mathsf{nodes}(M) \cap \mathsf{nodes}(N)$ such that $M(\bar{c}) \neq N(\bar{c})$. Since edges are labelled by atoms, we have $M(\bar{c}) \cdot N(\bar{c}) = 0$, so $0 = \mathsf{s}_{\bar{c}} 0 = \mathsf{s}_{\bar{c}} M(\bar{c}) \,.\, \mathsf{s}_{\bar{c}} N(\bar{c}) \geq$

$M^+ \cdot N^+$. A piece of notation. For $i < m$, let Id_{-i} be the partial map $\{(k,k) : k \in m \smallsetminus \{i\}\}$. For the first part of (3) (cf. [10, Lemma 13.29] using the notation in *op.cit*), since there is $k \in m \setminus \mathsf{nodes}(N)$, θ can be expressed as a product $\sigma_0\sigma_1 \ldots \sigma_t$ of maps such that, for $s \le t$, we have either $\sigma_s = Id_{-i}$ for some $i < m$ or $\sigma_s = [i/j]$ for some $i, j < m$ and where $i \notin \mathsf{nodes}(N\sigma_0 \ldots \sigma_{s-1})$. But clearly $(NId_{-j})^+ \ge N^+$ and if $i \notin \mathsf{nodes}(N)$ and $j \in \mathsf{nodes}(N)$, then $N^+ \neq 0 \to (N[i/j])^+ \neq 0$. The required now follows. The last part is straightforward. Using the above proven facts, we are now ready to show that $\exists$ has a winning strategy in $\mathbf{G}^m$. She can always play a network N with $\mathsf{nodes}(N) \subseteq m$, such that $N^+ \neq 0$.
In the initial round, let $\forall$ play $a \in \mathsf{At}\mathfrak{A}$. $\exists$ plays a network N with $N(0, \ldots, n-1) = a$. Then $N^+ = a \neq 0$. Recall that here $\forall$ is offered only one (cylindrifier) move. At a later stage, suppose $\forall$ plays the cylindrifier move, which we denote by $(N, \langle f_0, \ldots, f_{n-2}\rangle, k, b, l)$. He picks a previously played network N, $f_i \in \mathsf{nodes}(N)$, $l < n, k \notin \{f_i : i < n-2\}$, such that $b \le \mathsf{c}_l N(f_0, \ldots, f_{i-1}, x, f_{i+1}, \ldots, f_{n-2})$ and $N^+ \neq 0$. Let $\bar{a} = \langle f_0 \ldots f_{i-1}, k, f_{i+1}, \ldots f_{n-2}\rangle$. Then by second part of (3) we have that $\mathsf{c}_l N^+ \cdot \mathsf{s}_{\bar{a}} b \neq 0$ and so by first part of (2), there is a network M such that $M^+ \cdot \mathsf{c}_l N^+ \cdot \mathsf{s}_{\bar{a}} b \neq 0$. Hence $M(f_0, \ldots, f_{i-1}, k, f_{i-2}, \ldots, f_{n-2}) = b$, $\mathsf{nodes}(M) = \mathsf{nodes}(N) \cup \{k\}$, and $M^+ \neq 0$, so this property is maintained. □

2.3. The class of completely representable relation and cylindric algebras is not elementary

Let LRRA be the class of relation algebra whose atom structures satisfy the Lyndon condition, and LCA_n denote the class of CA_ns whose atom structures are in $\mathsf{LCAS_n}$ as defined in [11]; i.e those algebras whose atom structures also satisfy the Lyndon conditions for cylindric algebras.

Lemma 2.6. *For any infinite cardinal κ, there exists an atomless $\mathfrak{C} \in \mathsf{CA}_n$ such that for all $2 < n < \omega$, $\mathfrak{Nr}_n\mathfrak{C}$ and $\mathfrak{Ra}\mathsf{CA}_\omega$ are atomic, with $|\mathsf{At}(\mathfrak{Nr}_n\mathfrak{C})| = |\mathsf{At}(\mathfrak{Ra}\mathfrak{C}| = 2^\kappa$, $\mathfrak{Nr}_n\mathfrak{C} \in \mathsf{LCA}_n$ and $\mathfrak{Ra}\mathfrak{C} \in \mathsf{LRRA}$, but neither $\mathfrak{Nr}_n\mathfrak{C}$ nor $\mathfrak{Ra}\mathfrak{C}$ are completely representable.*

Proof: We use the following uncountable version of Ramsey's theorem due to Erdös and Rado: If $r \ge 2$ is finite, k an infinite cardinal, then $exp_r(k)^+ \to (k^+)^{r+1}_k$, where $exp_0(k) = k$ and inductively $exp_{r+1}(k) = 2^{exp_r(k)}$. The above partition symbol describes the following statement. If

f is a coloring of the $r+1$ element subsets of a set of cardinality $exp_r(k)^+$ in k many colors, then there is a homogeneous set of cardinality k^+ (a set, all whose $r+1$ element subsets get the same f-value). We will construct the required $\mathfrak{C} \in \mathsf{CA}_\omega$ from a relation algebra (to be denoted in a while by $\mathfrak{A}$) having an 'ω-dimensional cylindric basis.' in the sense of Maddux [16] To define the relation algebra, we specify its atoms and forbidden triples. Let κ be the given cardinal in the hypothesis of the Theorem. The atoms are Id, $\mathsf{g}_0^i : i < 2^\kappa$ and $\mathsf{r}_j : 1 \leq j < \kappa$, all symmetric. The forbidden triples of atoms are all permutations of (Id, x, y) for $x \neq y$, $(\mathsf{r}_j, \mathsf{r}_j, \mathsf{r}_j)$ for $1 \leq j < \kappa$ and $(\mathsf{g}_0^i, \mathsf{g}_0^{i'}, \mathsf{g}_0^{i^*})$ for $i, i', i^* < 2^\kappa$. Write g_0 for $\{\mathsf{g}_0^i : i < 2^\kappa\}$ and r_+ for $\{\mathsf{r}_j : 1 \leq j < \kappa\}$. Call this atom structure α. Consider the term algebra $\mathfrak{A}$ defined to be the subalgebra of the complex algebra of this atom structure generated by the atoms. We claim that $\mathfrak{A}$, as a relation algebra, has no complete representation, hence any algebra sharing this atom structure is not completely representable, too. Indeed, it is easy to show that if $\mathfrak{A}$ and $\mathfrak{B}$ are atomic relation algebras sharing the same atom structure, so that $\mathsf{At}\mathfrak{A} = \mathsf{At}\mathfrak{B}$, then $\mathfrak{A}$ is completely representable $\iff$ $\mathfrak{B}$ is completely representable.

Assume for contradiction that $\mathfrak{A}$ has a complete representation with base M. Let x, y be points in the representation with $\mathsf{M} \models \mathsf{r}_1(x, y)$. For each $i < 2^\kappa$, there is a point $z_i \in \mathsf{M}$ such that $\mathsf{M} \models \mathsf{g}_0^i(x, z_i) \wedge \mathsf{r}_1(z_i, y)$. Let $Z = \{z_i : i < 2^\kappa\}$. Within Z, each edge is labelled by one of the κ atoms in r_+. The Erdos-Rado theorem forces the existence of three points $z^1, z^2, z^3 \in Z$ such that $\mathsf{M} \models \mathsf{r}_j(z^1, z^2) \wedge \mathsf{r}_j(z^2, z^3) \wedge \mathsf{r}_j(z^3, z_1)$, for some single $j < \kappa$. This contradicts the definition of composition in $\mathfrak{A}$ (since we avoided monochromatic triangles). Let S be the set of all atomic $\mathfrak{A}$-networks N with nodes ω such that $\{\mathsf{r}_i : 1 \leq i < \kappa : \mathsf{r}_i \text{ is the label of an edge in } N\}$ is finite. Then it is straightforward to show S is an amalgamation class, that is for all $M, N \in S$ if $M \equiv_{ij} N$ then there is $L \in S$ with $M \equiv_i L \equiv_j N$, witness [10, Definition 12.8] for notation. Now let X be the set of finite $\mathfrak{A}$-networks N with nodes $\subseteq \kappa$ such that:

1. each edge of N is either (a) an atom of $\mathfrak{A}$ or (b) a cofinite subset of $\mathsf{r}_+ = \{\mathsf{r}_j : 1 \leq j < \kappa\}$ or (c) a cofinite subset of $\mathsf{g}_0 = \{\mathsf{g}_0^i : i < 2^\kappa\}$ and

2. N is 'triangle-closed', i.e. for all $l, m, n \in \mathsf{nodes}(N)$ we have $N(l, n) \leq N(l, m); N(m, n)$. That means if an edge (l, m) is labelled by Id then $N(l, n) = N(m, n)$ and if $N(l, m), N(m, n) \leq \mathsf{g}_0$ then $N(l, n) \cdot \mathsf{g}_0 = 0$ and if $N(l, m) = N(m, n) = \mathsf{r}_j$ (some $1 \leq j < \omega$) then $N(l, n) \cdot \mathsf{r}_j = 0$.

For $N \in X$ let $\widehat{N} \in \mathfrak{Ca}(S)$ be defined by

$$\{L \in S : L(m,n) \leq N(m,n) \text{ for } m,n \in \mathsf{nodes}(N)\}.$$

For $i \in \omega$, let $N\restriction_{-i}$ be the subgraph of N obtained by deleting the node i.

Then if $N \in X,\ i < \omega$ then $\widehat{\mathsf{c}_i N} = \widehat{N\restriction_{-i}}$. The inclusion $\widehat{\mathsf{c}_i N} \subseteq (\widehat{N\restriction_{-i}})$ is clear.

Conversely, let $L \in (\widehat{N\restriction_{-i}})$. We seek $M \equiv_i L$ with $M \in \widehat{N}$. This will prove that $L \in \widehat{\mathsf{c}_i N}$, as required. Since $L \in S$ the set $T = \{\mathsf{r}_i \notin L\}$ is infinite. Let T be the disjoint union of two infinite sets $Y \cup Y'$, say. To define the ω-network M we must define the labels of all edges involving the node i (other labels are given by $M \equiv_i L$). We define these labels by enumerating the edges and labeling them one at a time. So let $j \neq i < \kappa$. Suppose $j \in \mathsf{nodes}(N)$. We must choose $M(i,j) \leq N(i,j)$. If $N(i,j)$ is an atom then of course $M(i,j) = N(i,j)$. Since N is finite, this defines only finitely many labels of M. If $N(i,j)$ is a cofinite subset of g_0 then we let $M(i,j)$ be an arbitrary atom in $N(i,j)$. And if $N(i,j)$ is a cofinite subset of r_+ then let $M(i,j)$ be an element of $N(i,j) \cap Y$ which has not been used as the label of any edge of M which has already been chosen (possible, since at each stage only finitely many have been chosen so far). If $j \notin \mathsf{nodes}(N)$ then we can let $M(i,j) = \mathsf{r}_k \in Y$ some $1 \leq k < \kappa$ such that no edge of M has already been labelled by r_k. It is not hard to check that each triangle of M is consistent (we have avoided all monochromatic triangles) and clearly $M \in \widehat{N}$ and $M \equiv_i L$. The labeling avoided all but finitely many elements of Y', so $M \in S$. So $(\widehat{N\restriction_{-i}}) \subseteq \widehat{\mathsf{c}_i N}$.

Now let $\widehat{X} = \{\widehat{N} : N \in X\} \subseteq \mathfrak{Ca}(S)$. Then we claim that the subalgebra of $\mathfrak{Ca}(S)$ generated by $\widehat{X}$ is simply obtained from $\widehat{X}$ by closing under finite unions. Clearly all these finite unions are generated by $\widehat{X}$. We must show that the set of finite unions of $\widehat{X}$ is closed under all cylindric operations. Closure under unions is given. For $\widehat{N} \in X$ we have $-\widehat{N} = \bigcup_{m,n \in \mathsf{nodes}(N)} \widehat{N_{mn}}$ where N_{mn} is a network with nodes $\{m,n\}$ and labeling $N_{mn}(m,n) = -N(m,n)$. N_{mn} may not belong to X but it is equivalent to a union of at most finitely many members of $\widehat{X}$. The diagonal $\mathsf{d}_{ij} \in \mathfrak{Ca}(S)$ is equal to $\widehat{N}$ where N is a network with nodes $\{i,j\}$ and labeling $N(i,j) = \mathsf{Id}$. Closure under cylindrification is given. Let $\mathfrak{C}$ be the subalgebra of $\mathfrak{Ca}(S)$ generated by $\widehat{X}$. Then $\mathfrak{A} = \mathfrak{Ra}\mathfrak{C}$.

To see why, each element of $\mathfrak{A}$ is a union of a finite number of atoms, possibly a co–finite subset of g_0 and possibly a co–finite subset of r_+. Clearly $\mathfrak{A} \subseteq \mathfrak{Ra}\mathfrak{C}$. Conversely, each element $z \in \mathfrak{Ra}\mathfrak{C}$ is a finite union $\bigcup_{N\in F} \widehat{N}$, for some finite subset F of X, satisfying $\mathsf{c}_i z = z$, for $i > 1$. Let $i_0, \ldots, i_k$ be an enumeration of all the nodes, other than 0 and 1, that occur as nodes of networks in F. Then, $\mathsf{c}_{i_0} \ldots \mathsf{c}_{i_k} z = \bigcup_{N\in F} \mathsf{c}_{i_0} \ldots \mathsf{c}_{i_k} \widehat{N} = \bigcup_{N\in F} (\widehat{N{\restriction}_{\{0,1\}}}) \in \mathfrak{A}$. So $\mathfrak{Ra}\mathfrak{C} \subseteq \mathfrak{A}$. Thus $\mathfrak{A}$ is the relation algebra reduct of $\mathfrak{C} \in \mathsf{CA}_\omega$, but $\mathfrak{A}$ has no complete representation. Let $n > 2$. Let $\mathfrak{B} = \mathfrak{Nr}_n\mathfrak{C}$. Then $\mathfrak{B} \in \mathsf{Nr}_n\mathsf{CA}_\omega$, is atomic, but has no complete representation for plainly a complete representation of $\mathfrak{B}$ induces one of $\mathfrak{A}$. In fact, because $\mathfrak{B}$ is generated by its two dimensional elements, and its dimension is at least three, its Df reduct is not completely representable.

It remains to show that the ω–dilation $\mathfrak{C}$ is atomless. For any $N \in X$, we can add an extra node extending N to M such that $\emptyset \subsetneq M' \subsetneq N'$, so that N' cannot be an atom in $\mathfrak{C}$. By Lemma 2.5, $\exists$ has a winning strategy in $\mathbf{G}^\omega(\mathsf{At}\mathfrak{B})$. Since infnitely many nodes are in play, then reusing nodes does not make $\mathbf{G}^\omega$ any stronger than the usual ω rounded game G_ω according to [11, Definition 3.3.2]. Thus $\exists$ has a winning strategy in $G_\omega(\mathsf{At}\mathfrak{B})$, *a fortiori*, that $\exists$ has a winning strategy in the k rounded atomic game $G_k(\mathsf{At}\mathfrak{B})$ for all finite $k \in \omega$. By definition; coding winning strategy's in the first order Lyndon conditions, we get $\mathfrak{B} \in \mathsf{LCA}_n$. For relation algebras, we have $\mathfrak{A} \in \mathsf{RaCA}_\omega$ and $\mathfrak{A}$ has no complete representation. The rest is like the CA case, using the Ra analogue of Lemma 2.5, when the dilation is ω-dimensional, namely, $\mathfrak{A} \in \mathbf{S}_c\mathsf{RaCA}_\omega \implies$, and $\exists$ has a winning strategy in $\mathbf{G}^\omega$ with the last notation taken from [8]. $\square$

Corollary 2.7. For $2 < n < \omega$, the classes CRCA_n and CRRA are not elementary.

Proof: $\mathsf{LCA}_n = \mathbf{El}\mathsf{CRCA}_n$, hence $\mathfrak{B} \in \mathbf{El}\mathsf{CRCA}_n \sim \mathsf{CRCA}_n$, so CRCA_n is not elementary. For relation algebras, we use the algebra $\mathfrak{A}$ constructed in the previous Theorem, too. We have $\mathfrak{A} \in \mathsf{RaCA}_\omega$ and $\mathfrak{A}$ has no complete representation. The rest is like the CA case, using the Ra analogue of Lemma 2.5, when the dilation is ω–dimensional, namely, $\mathfrak{A} \in \mathbf{S}_c\mathsf{RaCA}_\omega \implies \exists$ has a winning strategy in F^ω with the last notation taken from [8]. $\square$

The last was proved by Hirsch and Hodkinsdon in [9]. Our proof here is entirely different using so-called Maddux relation algebras by specifying forbidden list of atoms, cf. [16, 10]. These algebars have ω–dimensional cylindric basis. The proof of Hirsch and Hodkinson uses so-called Rainbow constuction. The two proofs are not only distinct but they are conceptually disjoint.

But we can even go further for relation algebras:

THEOREM 2.8. *The class* CRRA *is not closed under* $\equiv_{\infty,\omega}$.

PROOF: Take $\mathfrak{R}$ to be a symmetric, atomic relation algebra with atoms

$$\mathsf{Id}, \mathsf{r}(i), \mathsf{y}(i), \mathsf{b}(i) : i < \omega.$$

Non-identity atoms have colors, r is red, b is blue, and y is yellow. All atoms are self-converse. The composition of atoms is defined by listing the forbidden triples. The forbidden triples are (Peircean transforms) or permutations of (Id, x, y) for $x \neq y$, and

$$(\mathsf{r}(i), \mathsf{r}(i), \mathsf{r}(j)),\ (\mathsf{y}(i), \mathsf{y}(i), \mathsf{y}(j)),\ (\mathsf{b}(i), \mathsf{b}(i), \mathsf{b}(j))\ \ i \leq j < \omega$$

$\mathfrak{R}$ is the complex algebra over this atom structure. Let α be an ordinal. $\mathfrak{R}^\alpha$ is obtained from $\mathfrak{R}$ by splitting the atom $\mathsf{r}(0)$ into α parts $\mathsf{r}^k(0) : k < \alpha$ and then taking the full complex algebra. In more detail, we put red atoms $r^k(0)$ for $k < \alpha$. In the altered algebra the forbidden triples are $(\mathsf{y}(i), \mathsf{y}(i), \mathsf{y}(j)), (\mathsf{b}(i), \mathsf{b}(i), \mathsf{b}(j)),\ \ i \leq j < \omega$, $(\mathsf{r}(i), \mathsf{r}(i), \mathsf{r}(j)),\ \ 0 < i \leq j < \omega$, $(\mathsf{r}^k(0), \mathsf{r}^l(0), \mathsf{r}(j)),\ \ 0 < j < \omega, k, l < \alpha$, $(\mathsf{r}^k(0), \mathsf{r}^l(0), \mathsf{r}^m(0)),\ \ k, l, m < \alpha$. Now let $\mathfrak{B} = \mathfrak{R}^\omega$ and $\mathfrak{A} = \mathfrak{R}^{\mathfrak{n}}$ with $\mathfrak{n} \geq 2^{\aleph_0}$. For an ordinal α, $\mathfrak{R}^\alpha$ is as defined in the previous remark. In $\mathfrak{R}^\alpha$, we use the following abbreviations: $r(0) = \sum_{k<\alpha} \mathsf{r}^k(0)$ $\mathsf{r} = \sum_{i<\omega} \mathsf{r}(i)$ $\mathsf{y} = \sum_{i<\omega} \mathsf{y}(i)$ $\mathsf{b} = \sum_{i<\omega} \mathsf{b}(i)$. These suprema exist because they are taken in the complex algebras which are complete. The *index* of $\mathsf{r}(i), \mathsf{y}(i)$ and $\mathsf{b}(i)$ is i and the index of $\mathsf{r}^k(0)$ is also 0. Now let $\mathfrak{B} = \mathfrak{R}^\omega$ and $\mathfrak{A} = \mathfrak{R}^{\mathfrak{n}}$ with $\mathfrak{n} \geq 2^{\aleph_0}$. We claim that $\mathfrak{B} \in \mathsf{RaCA}_\omega$ and $\mathfrak{A} \equiv \mathfrak{B}$. For the first required, we show that $\mathfrak{B}$ has a cylindric bases by exhibiting a winning strategy for $\exists$ in the cylindric-basis game, which is a simpler version of the hyperbasis game [10, Definition 12.26]. At some stage of the game, let the play so far be $N_0, N_1, \ldots, N_{t-1}$ for some $t < \omega$. We say that an edge (m, n) of an atomic network N is a *diversity edge* if $N(m, n) \cdot \mathsf{Id} = 0$. Each diversity edge of each atomic network in the play has an owner—either $\exists$ or $\forall$, which we will allocate as we define $\exists$'s strategy. If an edge (m, n) belongs to player p then so does the reverse edge (n, m)

and we will only specify one of them. Since our algebra is symmetric, so the label of the reverse edge is equal to the label of the edge, so again need to specify only one. For the next round $\exists$ must define N_t in response to $\forall$'s move. If there is an already played network N_i (some $i < t$) and a finitary map $\sigma : \omega \to \omega$ such that $N_t\sigma$ 'answers' his move, then she lets $N_t = N_i\sigma$. From now on we assume that there is no such N_i and σ. We consider the three types of $\forall$ can make. If he plays an *atom move* by picking an atom a, $\exists$ plays an atomic network N with $N(0, 1) = a$ and for all $x \in \omega \setminus \{1\}$, $N(0, x) = \mathsf{Id}$.

If $\forall$ plays a *triangle move* by picking a previously played N_x (some $x < t$), nodes i, j, k with $k \notin \{i, j\}$ and atoms a, b with $a; b \geq N_x(i, j)$, we know that $a, b \neq 1'$, as we are assuming the $\exists$ cannot play an embedding move (if $a = \mathsf{Id}$, consider N_x and the map $[k/i]$). $\exists$ must play a network $N_t \equiv_k N_x$ such that $N_t(i, k) = a$, $N_t(k, j) = b$. These edges, (i, k) and (k, j), belong to $\forall$ in N_t. All diversity edges not involving k have the same owner in N_t as they did in N_x. And all edges (l, k) for $k \notin \{i, j\}$ belong to $\exists$ in N_x. To label these edges $\exists$ chooses a colour c different than the colours of a, b(we have three colours so this is possible). Then, one at a time, she labels each edge (l, k) by an atom with colour c and a non-zero index which has not yet been used to label any edge of any network played in the game. She does this one edge at a time, each with a new index. There are infinitely many indices to choose, so this can be done.

Finally, $\forall$ can play an amalgamation move by picking M, $N \in \{N_s : s < t\}$, nodes i, j such that $M \equiv_{ij} N$. If there is N_s (some $s < t$) and a map $\sigma : \mathsf{nodes}(N_s) \to \mathsf{nodes}(M) \cup \mathsf{nodes}(N)$ such that $M \equiv_i N_s\sigma \equiv_j N$ then $\exists$ lets $N_t = N_s\sigma$. Ownership of edges is inherited from N_s. If there is no such N_s and σ then there are two cases. If there are three nodes x, y, z in the 'amalgam' such that $M(j, x)$ and $N(x, i)$ are both red and of the same index, $M(j, y)$, $N(y, i)$ are both yellow and of the same index and $M(j, z)$, $N(z, i)$ are both blue and of the same index, then the new edge (i, j) belongs to $\forall$ in N_t. It will be labelled by either $\mathsf{r}^0(0)$, $\mathsf{b}(0)$ or $\mathsf{y}(0)$ and it it is easy to show that at least one of these will be a consistent choice. Otherwise, if there is no such x, y, z then the new edge (i, j) belongs to $\exists$ in N_t. She chooses a colour c such that there is no x with $M(j, x)$ and $N(x, i)$ both having colour c and the same index. And she chooses a non-zero index for $N_t(i, j)$ which is new to the game (as with triangle moves). If $k \neq k' \in M \cap N$ then (j, k) has the same owner in N_t as it does in M, (k, i) has the same owner in N_t as it does in N and (k, k') belongs to $\exists$ in

N_t if it belongs to $\exists$ in *either* M or N, otherwise it belongs to $\forall$ in N_t. Now the only way $\exists$ could lose, is if $\forall$ played an amalgamation move (M, N, i, j) such that there are $x, y, z \in M \cap N$ such that $M(j,x) = \mathsf{r}^k(0)$, $N(x,i) = \mathsf{r}^{k'}(0)$, $M(j,y) = N(y,i) = \mathsf{b}(0)$ and $M(j,z) = N(z,i) = \mathsf{y}(0)$. But according to $\exists$'s strategy, she never chooses atoms with index 0, so all these edges must have been chosen by $\forall$. This contradiction proves the required.

Now, let $\mathcal{H}$ be an ω-dimensional cylindric basis for $\mathfrak{B}$. Then $\mathfrak{Ca}\mathcal{H} \in \mathsf{CA}_\omega$. Consider the cylindric algebra $\mathfrak{C} = \mathfrak{Sg}^{\mathfrak{Ca}\mathcal{H}}\mathfrak{B}$, the subalgebra of $\mathfrak{Ca}\mathcal{H}$ generated by $\mathfrak{B}$. In principal, new two dimensional elements that were not originally in $\mathfrak{B}$, can be created in $\mathfrak{C}$ using the spare dimensions in $\mathfrak{Ca}(\mathcal{H})$. But next we exclude this possibility. We show that $\mathfrak{B}$ exhausts the 2–dimensional elements of $\mathsf{Ra}\mathfrak{C}$, more concisely, we show that $\mathfrak{B} = \mathsf{Ra}\mathfrak{C}$. For this purpose, we want to find out what are the elements of $\mathfrak{Ca}\mathcal{H}$ that are generated by $\mathfrak{B}$. Let M be a (not necessarily atomic) finite network over $\mathfrak{B}$ whose nodes are a finite subset of ω.

- Define (using the same notation in the proof of Theorem 2.6) $\widehat{M} = \{N \in \mathcal{H} : N \leq M\} \in \mathfrak{Ca}\mathcal{H}$. ($N \leq M$ means that for all $i, j \in M$ we have $N(i,j) \leq M(i,j)$.)
- A *block* is an element of the form $\widehat{M}$ for some finite network M such that
  1. M is triangle-closed, i.e. for all $i, j, k \in M$ we have $M(i,k) \leq M(i,j); M(j,k)$
  2. If x is the label of an irreflexive edge of M then $x = \mathsf{Id}$ or $x \leq \mathsf{r}$ or $x \leq \mathsf{y}$ or $x \leq \mathsf{b}$ (we say x is 'monochromatic'), and $|\{i : x \cdot (\mathsf{r}(i) + \mathsf{y}(i) + \mathsf{b}(i)) \neq 0\}|$ is either $0, 1$ or infinite (we say that the number of indices of x is either $0, 1$ or infinite).

We prove:

1. For any block $\widehat{M}$ and $i < \omega$ we have
$$\mathsf{c}_i\widehat{M} = (M{\restriction}_{\mathsf{dom}(M)\setminus\{i\}})^\frown$$
2. The domain of $\mathfrak{C}$ consists of finite sums of blocks.

$\mathsf{c}_i\widehat{M} \subseteq (M{\restriction}_{\mathsf{dom}(M)\setminus\{i\}})^\frown$ is obvious. If $i \notin M$ the equality is trivial. Let $N \in (M{\restriction}_{\mathsf{dom}(M)\setminus\{i\}})^\frown$, i.e. $N \leq M{\restriction}_{\mathsf{dom}(M)\setminus\{i\}}$. We must show that $N \in \mathsf{c}_i\widehat{M}$

and for this we must find $L \equiv_i N$ with $L \in \widehat{M}$. $L \equiv_i N$ determines every edge of L except those involving i. For each $j \in M$, if the number of indices in $M(i,j)$ is just one, say $M(i,j) = \mathsf{r}(k)$, then let $L(i,j)$ be an arbitrary atom below $\mathsf{r}(k)$. There should be no inconsistencies in the labelling so far defined for L, by triangle-closure for M. For all the other edges (i,j) if $j \in M$ there are infinitely many indices in $M(i,j)$ and if $j \notin M$ then we have an unrestricted choice of atoms for the label. These edges are labelled one at a time and each label is given an atom with a new index, thus avoiding any inconsistencies. This defines $L \equiv_i N$ with $L \in \widehat{M}$. For the second part, we already have seen that the set of finite sums of blocks is closed under cylindrification. We'll show that this set is closed under all the cylindric operations and includes $\mathfrak{B}$. For any $x \in \mathfrak{B}$ and $i,j < \omega$, let N^{ij}_x be the $\mathfrak{B}$-network with two nodes $\{i,j\}$ and labelling $N^{ij}_x(i,i) = N^{ij}_x(j,j) = \mathsf{Id}$, and $N^{ij}(i,j) = x$, $N^{ij}_x(j,i) = \breve{x}$. Clearly N^{ij}_x is triangle closed. And $\widehat{N^{01}_x} = x$. For any $x \in \mathfrak{B}$, we have $x = x \cdot \mathsf{Id} + x \cdot \mathsf{r} + x \cdot \mathsf{y} + x \cdot \mathsf{b}$, so $x = \widehat{N^{01}_{x\cdot\mathsf{Id}}} + \widehat{N^{01}_{x\cdot\mathsf{r}}} + \widehat{N^{01}_{x\cdot\mathsf{y}}} + \widehat{N^{01}_{x\cdot\mathsf{b}}}$ and the labels of these four networks are monochromatic. The first network defines a block and for each of the last three, if the number if indices is infinite then it is a block. If the number of indices is finite then it is a finite union of blocks. So every element of $\mathfrak{B}$ is a finite union of blocks.

For the diagonal elements, $\mathsf{d}_{ij} = \widehat{N^{ij}_{\mathsf{Id}}}$. Closure under sums is obvious. For negation, take a block $\widehat{M}$. Then $-\widehat{M} = \sum_{i,j\in M} \widehat{N^{ij}_{-N(i,j)}}$. As before we can replace $\widehat{N^{ij}_{-N(i,j)}}$ by a finite union of blocks. Thus the set of finite sums of blocks includes $\mathfrak{B}$ and the diagonals and is closed under all the cylindric operations. Since every block is clearly generated from $\mathfrak{B}$ using substitutions and intersection only. It remains to show that $\mathfrak{B} = \mathsf{Ra}\mathfrak{C}$. Take a block $\widehat{M} \in \mathsf{Ra}\mathfrak{C}$. Then $\mathsf{c}_i\widehat{M} = \widehat{M}$ for $2 \leq i < \omega$. By the first part of the lemma, $\widehat{M} = \widehat{M\restriction_{\{0,1\}}} \in \mathfrak{B}$.

We finally show that $\exists$ has a winning strategy in an Ehrenfeucht–Fraïssé-game over $(\mathfrak{A}, \mathfrak{B})$ concluding that $\mathfrak{A} \equiv_\infty \mathfrak{B}$. At any stage of the game, if $\forall$ places a pebble on one of $\mathfrak{A}$ or $\mathfrak{B}$, $\exists$ must place a matching pebble, on the other algebra. Let $\bar{a} = \langle a_0, a_1, \ldots, a_{n-1}\rangle$ be the position of the pebbles played so far (by either player) on $\mathfrak{A}$ and let $\bar{b} = \langle b_0, \ldots, b_{n-1}\rangle$ be the the position of the pebbles played on $\mathfrak{B}$. $\exists$ maintains the following properties throughout the game.

- For any atom x (of either algebra) with $x \cdot \mathsf{r}(0) = 0$ then $x \in a_i \iff x \in b_i$.

- $\bar{a}$ induces a finite partition of $\mathsf{r}(0)$ in $\mathfrak{A}$ of 2^n (possibly empty) parts $p_i : i < 2^n$ and $\bar{b}$ induces a partition of $\mathsf{r}(0)$ in $\mathfrak{B}$ of parts $q_i : i < 2^n$. p_i is finite iff q_i is finite and, in this case, $|p_i| = |q_i|$.

Now we show that CRRA is not closed under $\equiv_{\infty,\omega}$. Since $\mathfrak{B} \in \mathsf{RaCA}_\omega$ has countably many atoms, then $\mathfrak{B}$ is completely representable [8, Theorem 29]. For this purpose, we show that $\mathfrak{A}$ is not completely representable. We work with the term algebra, $\mathfrak{TmAt}\mathfrak{A}$, since the latter is completely representable $\iff$ the complex algebra is. Let $\mathsf{r} = \{\mathsf{r}(i) : 1 \leq i < \omega\} \cup \{\mathsf{r}^k(0) : k < 2^{\aleph_0}\}$, $\mathsf{y} = \{\mathsf{y}(i) : i \in \omega\}$, $\mathsf{b}^+ = \{\mathsf{b}(i) : i \in \omega\}$. It is not hard to check every element of $\mathfrak{TmAt}\mathfrak{A} \subseteq \wp(\mathsf{At}\mathfrak{A})$ has the form $F \cup R_0 \cup B_0 \cup Y_0$, where F is a finite set of atoms, R_0 is either empty or a co-finite subset of r, B_0 is either empty or a co–finite subset of b, and Y_0 is either empty or a co–finite subset of y. Using an argument similar to that used in the proof of Lemma 2.6, we show that the existence of a complete representation necessarily forces a monochromatic triangle, that we avoided at the start when defining $\mathfrak{A}$. Let x, y be points in the representation with $M \models \mathsf{y}(0)(x,y)$. For each $i < 2^{\aleph_0}$, there is a point $z_i \in M$ such that $M \models \mathsf{red}(x, z_i) \wedge \mathsf{y}(0)(z_i, y)$ (some red $\mathsf{red} \in \mathsf{r}$). Let $Z = \{z_i : i < 2^{\aleph_0}\}$. Within Z each edge is labelled by one of the ω atoms in y^+ or b^+. The Erdos-Rado theorem forces the existence of three points $z^1, z^2, z^3 \in Z$ such that $M \models \mathsf{y}(j)(z^1, z^2) \wedge \mathsf{y}(j)(z^2, z^3) \wedge \mathsf{y}(j)(z^3, z_1)$, for some single $j < \omega$ or three points $z^1, z^2, z^3 \in Z$ such that $M \models \mathsf{b}(l)(z^1, z^2) \wedge \mathsf{b}(l)(z^2, z^3) \wedge \mathsf{b}(l)(z^3, z_1)$, for some single $l < \omega$. This contradicts the definition of composition in $\mathfrak{A}$ (since we avoided monochromatic triangles). We have proved that CRRA is not closed under $\equiv_{\infty,\omega}$, since $\mathfrak{A} \equiv_{\infty,\omega} \mathfrak{B}$, $\mathfrak{A}$ is not completely representable, but $\mathfrak{B}$ is completely representable. $\square$

3. Other algebras of relations

We shall have the occasion to deal with (in addition to CAs) the following cylindric–like algebras [1]: Df short for diagonal free cylindric algebras, Sc short for Pinter's substitution algebras, QA(QEA) short for quasi–polyadic (equality) algebras, PA(PEA) short for polyadic (equality) algebras. For

K any of these classes and α any ordinal, we write K_α for variety of α–dimensional K algebras which can be axiomatized by a finite schema of equations, and RK_α for the class of representable K_αs, which happens to be a variety too (that cannot be axiomatized by a finite schema of equations for $\alpha > 2$ unless $\mathsf{K} = \mathsf{PA}$ and $\alpha \geq \omega$). The standard reference for all the classes of algebras mentioned previously is [7]. We recall the concrete versions of such algebras. Let $\tau : \alpha \to \alpha$ and $X \subseteq {}^\alpha U$, then

$$\mathsf{S}_\tau X = \{s \in {}^\alpha U : s \circ \tau \in X\}.$$

For $i, j \in \alpha$, $[i|j]$ is the replacement on α that sends i to j and is the identity map on $\alpha \sim \{i\}$ while $[i, j]$ is the transposition on α that interchanges i and j.

- A *diagonal free cylindric set algebra of dimension* α is an algebra of the form $\langle \mathfrak{B}({}^\alpha U), \mathsf{C}_i \rangle_{i,j<\alpha}$.
- A *Pinter's substitution set algebra of dimension* α is an algebra of the form
$\langle \mathfrak{B}({}^\alpha U), \mathsf{C}_i, \mathsf{S}_{[i|j]} \rangle_{i,j<\alpha}$.
- A *quasi-polyadic set algebra of dimension* α is an algebra of the form $\langle \mathfrak{B}({}^\alpha U), \mathsf{C}_i, \mathsf{S}_{[i|j]}, \mathsf{S}_{[i,j]} \rangle_{i,j<\alpha}$.
- A *quasi-polyadic equality set algebra* is an algebra of the form $\langle \mathfrak{B}({}^\alpha U), \mathsf{C}_i, \mathsf{S}_{[i|j]}, \mathsf{S}_{[i,j]}, \mathsf{D}_{ij} \rangle_{i,j<\alpha}$.
- A *polyadic set algebra of dimension* α is an algebra of the form $\langle \mathfrak{B}({}^\alpha U), \mathsf{C}_i, \mathsf{S}_\tau \rangle_{\tau:\alpha\to\alpha}$.
- A *polyadic equality set algebra of dimension* α is an algebra of the form
$\langle \mathfrak{B}({}^\alpha U), \mathsf{C}_i, \mathsf{S}_\tau \rangle_{\tau:\alpha\to\alpha, i,j<\alpha}$

Let α be an ordinal. For any such abstract class of algebras K_α in the above table, RK_α is defined to be the subdirect product of set algebras of dimension α. For $\alpha < \omega$, $\mathsf{PA}_\alpha(\mathsf{PEA}_\alpha)$ is definitionally equivalent to $\mathsf{QA}_\alpha(\mathsf{QEA}_\alpha)$ which is no longer the case for infinite α where the deviation is largely significant. For example a countable QA_ω has a countable signature, while a countable PA_ω has an uncountable signature having the same cardinality as (substitutions in) ${}^\omega\omega$. The class of completely representable

class	extra non-Boolean operators
Df_α	$\mathsf{c}_i : i < \alpha$
Sc_α	$\mathsf{c}_i, \mathsf{s}_i^j : i, j < \alpha$
CA_α	$\mathsf{c}_i, \mathsf{d}_{ij} : i, j < \alpha$
PA_α	$\mathsf{c}_i, \mathsf{s}_\tau : i < n,\ \tau \in {}^\alpha\alpha$
PEA_α	$\mathsf{c}_i, \mathsf{d}_{ij}, \mathsf{s}_\tau : i, j < n,\ \tau \in {}^\alpha\alpha$
QA_α	$\mathsf{c}_i, \mathsf{s}_i^j, \mathsf{s}_{[i,j]} : i, j < \alpha$
QEA_α	$\mathsf{c}_i, \mathsf{d}_{ij}, \mathsf{s}_i^j, \mathsf{s}_{[i,j]} : i, j < \alpha$

Figure 1. Non-Boolean operators for the classes

K_αs (K any of the above classes) is denoted by CRK_α. For a BAO, $\mathfrak{A}$ say, for any ordinal α, $\mathfrak{Rd}_{ca}\mathfrak{A}$ denotes the cylindric reduct of $\mathfrak{A}$ if it has one, $\mathfrak{Rd}_{sc}\mathfrak{A}$ denotes the Sc reduct of $\mathfrak{A}$ if it has one, and $\mathfrak{Rd}_{df}\mathfrak{A}$ denotes the reduct of $\mathfrak{A}$ obtained by discarding all the operations except for cylindrifications. If $\mathfrak{A}$ is any of the above classes, it is always the case that $\mathfrak{Rd}_{df}\mathfrak{A} \in \mathsf{Df}_\alpha$. If $\mathfrak{A} \in \mathsf{CA}_\alpha$, then $\mathfrak{Rd}_{sc}\mathfrak{A} \in \mathsf{Sc}_\alpha$, and if $\mathfrak{A} \in \mathsf{QEA}_\alpha$ then $\mathfrak{Rd}_{ca}\mathfrak{A} \in \mathsf{CA}_\alpha$. Roughly speaking for an ordinal α, CA_αs are not expansions of Sc_αs, but they are *definitionally equivalent* to expansions of Sc_α, because the s_i^js are term definable in CA_αs by $\mathsf{s}_i^j(x) = \mathsf{c}_i(x \cdot -\mathsf{d}_{ij})$ $(i, j < \alpha)$. This operation reflects algebraically the substitution of the variable v_j for v_i in a formula such that the substitution is free; this can be always done by reindexing bounded variables. In such situation, we say that Scs are *generalized reducts* of CAs. However, CA_αs and QA_α are (real)reducts of QEAs (in the universal algebraic sense), simply obtained by discarding the operations in their signature not in the signature of their common expansion QEA_α.

Definition 3.1. Let α be an ordinal. We say that a variety V is a variety between Df_α and QEA_α if the signature of V expands that of Df_α and is contained in the signature of QEA_α. Furthermore, any equation formulated in the signature of Df_α that holds in V also holds in Sc_α and all equations that hold in V holds in QEA_α.

Proper examples include Sc, CA_α and QA_α (meaning strictly between). Analogously we can define varieties between Sc_α and CA_α or QA_α and QEA_α, and more generally between a class K of BAOs and a generalized reduct of it. Notions like neat reducts generalize verbatim to such algebras, namely, to Dfs and QEAs, and in any variety in between. This stems from the observation that for any pair of ordinals $\alpha < \beta$, $\mathfrak{A} \in \mathsf{QEA}_\beta$ and any non-Boolean extra operation in the signature of QEA_β, f say, if $x \in \mathfrak{A}$ and $\Delta x \subseteq \alpha$, then $\Delta(f(x)) \subseteq \alpha$. Here $\Delta x = \{i \in \beta : \mathsf{c}_i x \neq x\}$ (as defined in the introduction) is referred as *the dimension set* of x; it reflects algebraically the essentially free variables occurring in a formula ϕ. A variable is essentially free in a formula $\Psi \iff$ it is free in every formula equivalent to Ψ.[1] Therefore given a variety V between Sc_β and QEA_β, if $\mathfrak{B} \in \mathsf{V}$ then the algebra $\mathfrak{Nr}_\alpha\mathfrak{B}$ having universe $\{x \in \mathfrak{B} : \Delta x \subseteq \alpha\}$ is closed under all operations in the signature of V.

Definition 3.2. Let $2 < n < \omega$. For a variety V between Df_n and QEA_n, a V *set algebra* is a subalgebra of an algebra, having the same signature as V, of the form $\langle \mathfrak{B}({}^nU), f_i^U\rangle$, say, where f_i^U is identical to the interpretation of f_i in the class of quasi-polyadic equality set algebras. Let $\mathfrak{A}$ be an algebra having the same signature of V; then $\mathfrak{A}$ is *a representable* V *algebra*, or simply *representable* $\iff$ $\mathfrak{A}$ is isomorphic to a subdirect product of V set algebras. We write RV for the class of representable V algebras

It can be proved that the class RV, as defined above, is also closed under $\mathbf{H}$, so that it is a variety.

Proposition 3.3. Let $2 < n < \omega$. Let V be a variety between Df_n and QEA_n. Then RV is not a finitely axiomatizable variety.

Proof: In [15] a sequence $\langle \mathfrak{A}_i : i \in \omega\rangle$ of algebras is constructed such that $\mathfrak{A}_i \in \mathsf{QEA}_n$ and $\mathfrak{Rd}_{df}\mathfrak{A}_n \notin \mathsf{RDf}_n$, but $\Pi_{i\in\omega}\mathfrak{A}_i/F \in \mathsf{RQEA}_n$ for any non principal ultrafilter on ω. An application of Los' Theorem, taking the ultraproduct of V reduct of the $\mathfrak{A}_i$s, finishes the proof. In more detail, let $\mathfrak{Rd}_{\mathsf{V}}$ denote restricting the signature to that of V. Then $\mathfrak{Rd}_{\mathsf{V}}\mathfrak{A}_i \notin \mathsf{RV}$ and $\mathfrak{Rd}_{\mathsf{V}}\Pi_{i\in I}(\mathfrak{A}_i/F) \in \mathsf{RV}$. $\square$

The last result generalizes to infinite dimensions replacing finite axiomatization by axiomatized by a finite schema [7, 13].

[1] It can well happen that a variable is free in formula that is equivalent to another formula in which this same variable is not free.

THEOREM 3.4. *Let $2 < n < \omega$. Let V be any variety between Df_n an QEA_n. Then the class of completely representable algebras in V is not elementary.*

PROOF: For a complete labelled graph graph N and function θ, the graph $N\theta$ is the complete labelled graph with nodes $\theta^{-1}(\mathsf{nodes}(N)) = \{x \in \mathsf{dom}(\theta) : \theta(x) \in \mathsf{nodes}(N)\}$, and labelling defined by

$$(N\theta)(i_0, \ldots, i_{n-1}) = N(\theta(i_0), \theta(i_1), \ldots, \theta(i_{n-1})),$$

for $i_0, \ldots, i_{n-1} \in \theta^{-1}(\mathsf{nodes}(N))$. We have S is symmetric, that is, if $N \in S$ and $\theta : \omega \to \omega$ is a finitary function, in the sense that $\{i \in \omega : \theta(i) \neq i\}$ is finite, then $N\theta$ is in S. It follows that the complex algebra $\mathfrak{Ca}(S) \in \mathsf{QEA}_\omega$. Thus the algebra $\mathfrak{B}$ can be expanded into a polyadic algebra of dimension n. Also, generated by two dimensional elements, the Df reduct of $\mathfrak{B}$ is not completely representable by [14, Proposition 4.10]. $\square$

In [9] it is proved that the class CRCA_α, where α is an infinite ordinal, is not elementary either. The proof can be generalized to any variety V between CA and QEA. We do not know whether it generalizes to equality free algebras such as Df, Sc and QA for the proof in the infinite dimensional case of CAs in [9] essentially depens on the presence of diagonal elements, namely, only one diagonal $\mathsf{d}_{0,1}$. Recall that $\mathfrak{Rd}_{ca}$ denote the cylindric reduct. One shows that if $\mathfrak{C} \in \mathsf{QEA}_\omega$ is completely representable and $\mathfrak{C} \models \mathsf{d}_{01} < 1$, then $|\mathsf{At}\mathfrak{C}| \geq 2^\omega$. The argument is as follows: Suppose that $\mathfrak{C} \models \mathsf{d}_{01} < 1$. Then there is $s \in h(-\mathsf{d}_{01})$ so that if $x = s_0$ and $y = s_1$, we have $x \neq y$. Fix such x and y. For any $J \subseteq \omega$ such that $0 \in J$, set a_J to be the sequence with ith co-ordinate is x if $i \in J$, and is y if $i \in \omega \setminus J$. By complete representability every a_J is in $h(1^{\mathfrak{C}})$ and so it is in $h(x)$ for some unique atom x, since the representation is an atomic one. Let $J, J' \subseteq \omega$ be distinct sets containing 0. Then there exists $i < \omega$ such that $i \in J$ and $i \notin J'$. So $a_J \in h(\mathsf{d}_{0i})$ and $a'_J \in h(-\mathsf{d}_{0i})$, hence atoms corresponding to different a_J's with $0 \in J$ are distinct. It now follows that $|\mathsf{At}\mathfrak{C}| = |\{J \subseteq \omega : 0 \in J\}| \geq 2^\omega$. Take $\mathfrak{D} \in \mathsf{Pes}_\omega$ with universe $\wp({}^\omega 2)$. Then $\mathfrak{D} \models \mathsf{d}_{01} < 1$ and plainly $\mathfrak{D}$ is completely representable. Using the downward Löwenheim–Skolem–Tarski theorem, take a countable elementary subalgebra $\mathfrak{B}$ of $\mathfrak{D}$. This is possible because the signature of QEA_ω is countable. Then in $\mathfrak{B}$ we have $\mathfrak{B} \models \mathsf{d}_{01} < 1$ because $\mathfrak{B} \equiv \mathfrak{C}$. But

$\mathfrak{Rd}_{ca}\mathfrak{B}$ cannot be completely representable, because if it were then by the above argument, we get that $|\mathsf{At}\mathfrak{Rd}_{ca}\mathfrak{B}| = |\mathsf{At}\mathfrak{B}| \geq 2^\omega$, which is impossible because $\mathfrak{B}$ is countable.

3.1. For $2 < n < \omega$, the class of neat reducts is not elemenatry for any V between Sc_n and QEA_n

THEOREM 3.5. *For any finite $n > 1$, and any uncountable cardinal $\kappa \geq |\alpha|$, there exist completely representable algebras $\mathfrak{A}, \mathfrak{B} \in \mathsf{QEA}_n$, that are set algebras, such that $|\mathfrak{A}| = |\mathfrak{B}| = \kappa$, $\mathfrak{A} \in \mathsf{Nr}_\alpha\mathsf{QEA}_\omega$, $\mathfrak{Rd}_{sc}\mathfrak{B} \notin \mathsf{Nr}_\alpha\mathsf{Sc}_{n+1}$, $\mathfrak{A} \equiv_{\infty,\omega} \mathfrak{B}$ and $\mathsf{At}\mathfrak{A} \equiv_{\omega,\infty} \mathsf{At}\mathfrak{B}$.*

PROOF: Fix $1 < n < \omega$. Let L be a signature consisting of the unary relation symbols $P_0, P_1, \ldots, P_{n-1}$ and uncountably many n–ary predicate symbols. M is as in [25, Lemma 5.1.3], but the tenary relations are replaced by n–ary ones, and we require that the interpretations of the n–ary relations in M are *pairwise disjoint* not only distinct. This can be fixed. In addition to pairwise disjointness of n–ary relations, we require their symmetry, that is, permuting the variables does not change their semantics. In fact the construction is presented this way in [17]. For $u \in {}^nn$, let χ_u be the formula $\bigwedge_{u \in {}^nn} P_{u_i}(x_i)$. We assume that the n–ary relation symbols are indexed by (an uncountable set) I and that there is a binary operation $+$ on I, such that $(I, +)$ is an abelian group, and for distinct $i \neq j \in I$, we have $R_i \circ R_j = R_{i+j}$. For $n \leq k \leq \omega$, let $\mathfrak{A}_k = \{\phi^{\mathsf{M}} : \phi \in L_k\} (\subseteq \wp({}^k\mathsf{M}))$, where ϕ is taken in the signature L, and $\phi^{\mathsf{M}} = \{s \in {}^k\mathsf{M} : \mathsf{M} \models \phi[s]\}$.

Let $\mathfrak{A} = \mathfrak{A}_n$, then $\mathfrak{A} \in \mathsf{Pes_n}$ by the added symmetry condition. Also $\mathfrak{A} \cong \mathsf{Nr}_n\mathfrak{A}_\omega$; the isomorphism is given by $\phi^{\mathsf{M}} \mapsto \phi^{\mathsf{M}}$. The map is obviously an injective homomorphism; it is surjective, because M (as stipulated in [25, item (1) of lemma 5.1.3]), has quantifier elimination. For $u \in {}^nn$, let $\mathfrak{A}_u = \{x \in \mathfrak{A} : x \leq \chi_u^{\mathsf{M}}\}$. Then $\mathfrak{A}_u$ is an uncountable and atomic Boolean algebra (atomicity follows from the new disjointness condition) and $\mathfrak{A}_u \cong \mathsf{Cof}(|I|)$, the finite–cofinite Boolean algebra on $|I|$. Define a map $f : \mathfrak{B}\restriction\mathfrak{A} \to \mathbf{P}_{u \in {}^nn}\mathfrak{A}_u$, by $f(a) = \langle a \cdot \chi_u \rangle_{u \in {}^nn+1}$. Let $\mathcal{P}$ denote the structure for the signature of Boolean algebras expanded by constant symbols 1_u, $u \in {}^nn$, d_{ij}, and unary relation symbols $\mathsf{s}_{[i,j]}$ for each $i, j \in n$. Then for each $i < j < n$, there are quantifier free formulas $\eta_i(x, y)$ and $\eta_{ij}(x, y)$ such that $\mathcal{P} \models \eta_i(f(a), b) \iff b = f(\mathsf{c}_i^{\mathfrak{A}}a)$, and $\mathcal{P} \models \eta_{ij}(f(a), b) \iff b = f(\mathsf{s}_{[i,j]}a)$. The one corresponding to cylindrifiers is exactly like the CA case [25, pp. 113–114]. For substitutions corresponding to transpositions, it is simply $y = \mathsf{s}_{[i,j]}x$. The diagonal elements and the Boolean operations are easy to interpret. Hence, $\mathcal{P}$ is interpretable in $\mathfrak{A}$, and the interpretation is one dimensional and quantifier free. For $v \in {}^nn$, by the Tarski–Skolem downward theorem, let $\mathfrak{B}_v$ be a countable elementary subalgebra of $\mathfrak{A}_v$. (Here we are using the countable signature of PEA_n). Let $S_n(\subseteq {}^nn)$ be the set of permuations in ${}^nn$.

Take $u_1 = (0,1,0,\ldots,0)$ and $u_2 = (1,0,0,\ldots,0) \in {}^nn$. Let $v = \tau(u_1,u_2)$ where $\tau(x,y) = \mathsf{c}_1(\mathsf{c}_0x \cdot \mathsf{s}^0_1\mathsf{c}_1y)\cdot\mathsf{c}_1x\cdot\mathsf{c}_0y$. We call τ an approximate witness. It is not hard to show that $\tau(u_1,u_2)$ is actually the composition of u_1 and u_2, so that $\tau(u_1,u_2)$ is the constant zero map; which we denote by $\mathbf{0}$; it is also in ${}^nn$. Clearly for every $i<j<n$, ${\mathsf{s}_{[i,j]}}^{{}^nn}\{\mathbf{0}\} = \mathbf{0} \notin \{u_1,u_2\}$. We can assume without loss that the Boolean reduct of $\mathfrak{A}$ is the following product:

$$\mathfrak{A}_{u_1} \times \mathfrak{A}_{u_2} \times \mathfrak{A}_{\mathbf{0}} \times \mathbf{P}_{u\in V\sim J}\mathfrak{A}_u,$$

where $J = \{u_1,u_2,\mathbf{0}\}$. Let

$$\mathfrak{B} = ((\mathfrak{A}_{u_1} \times \mathfrak{A}_{u_2} \times \mathfrak{B}_{\mathbf{0}} \times \mathbf{P}_{u\in V\sim J}\mathfrak{A}_u), 1_u, \mathsf{d}_{ij}, \mathsf{s}_{[i,j]}x)_{i,j<n},$$

recall that $\mathfrak{B}_{\mathbf{0}} \prec \mathfrak{A}_{\mathbf{0}}$ and $|\mathfrak{B}_{\mathbf{0}}| = \omega$, inheriting the same interpretation. Then by the Feferman–Vaught theorem, we get that $\mathfrak{B} \equiv \mathfrak{A}$.

Now assume for contradiction, that $\mathfrak{Rd}_{sc}\mathfrak{B} = \mathsf{Nr}_n\mathfrak{D}$, with $\mathfrak{D} \in \mathsf{Sc}_{n+1}$. Let $\tau_n(x,y)$, which we call an *n–witness*, be defined by $\mathsf{c}_n(\mathsf{s}^1_n\mathsf{c}_nx\cdot\mathsf{s}^0_n\mathsf{c}_ny)$. By a straightforward, but possibly tedious computation, one can obtain $\mathsf{Sc}_{n+1} \models \tau_n(x,y) \le \tau(x,y)$ so that the approximate witness *dominates* the n–witness. The term $\tau(x,y)$ does not use any spare dimensions, and it 'approximates' the term $\tau_n(x,y)$ that uses the spare dimension n. Let $\lambda = |I|$. For brevity, we write $\mathbf{1}_u$ for χ^{M}_u. The algebra $\mathfrak{A}$ can be viewed as splitting the atoms of the atom structure $\mathbf{At} = ({}^nn, \equiv, \equiv_{ij}, D_{ij})_{i,j<n}$ each to uncountably many atoms. We denote $\mathfrak{A}$ by $\mathsf{split}(\mathbf{At}, \mathbf{1}_{\mathbf{0}}, \lambda)$. On the other hand, $\mathfrak{B}$ can be viewed as splitting the same atom structure, each atom – except for one atom that is split into countably many atoms – is also split into uncountably many atoms (the same as in $\mathfrak{A}$). We denote $\mathfrak{B}$ by $\mathsf{split}(\mathbf{At}, \mathbf{1}_{\mathbf{0}}, \omega)$. On the 'global' level, namely, in the complex algebra of the finite (splitted) atom structure ${}^nn$, these two terms are equal, the approximate witness is the n-witness. The complex algebra $\mathfrak{Cm}({}^nn)$ does not 'see' the nth dimension. But in the algebras $\mathfrak{A}$ and $\mathfrak{B}$ (obtained after splitting), the n-witness becomes then a *genuine witness*, not an approximate one. The approximate witness *strictly dominates* the n-witness. The n-witness

using the spare dimension n, detects the cardinality twist that $L_{\infty,\omega}$, *a priori*, first order logic misses out on. If the n-witness were term definable (in the case we have a full neat reduct of an algebra in only one extra dimension), then it takes two uncountable component to an uncountable one, and this is not possible for $\mathfrak{B}$, because in $\mathfrak{B}$, the target component is forced to be countable.

Now for $x \in \mathfrak{B}_{u_1}$ and $y \in \mathfrak{B}_{u_2}$, we have

$$\tau_n^{\mathfrak{D}}(x, y) \le \tau_n^{\mathfrak{D}}(\chi_{u_1}, \chi_{u_2}) \le \tau^{\mathfrak{D}}(\chi_{u_1}, \chi_{u_2}) = \chi_{\tau^{\wp({}^n n)}}(u_1, u_2) = \chi_{\tau(u_1,u_2)} = \chi_{\mathbf{0}}.$$

But for $i \neq j \in I$, $\tau_n^{\mathfrak{D}}(R_i^{\mathsf{M}} \cdot \chi_{u_1}, R_j^{\mathsf{M}} \cdot \chi_{u_2}) = R_{i+j}^{\mathsf{M}} \cdot \chi_v$, and so $\mathfrak{B}_{\mathbf{0}}$ will be uncountable, which is impossible. We now show that $\exists$ has a winning strategy in an Ehrenfeucht–Fraïssé back–and–forth game over the now atomic $(\mathfrak{A}, \mathfrak{B})$. At any stage of the game, if $\forall$ places a pebble on one of $\mathfrak{A}$ or $\mathfrak{B}$, $\exists$ must place a matching pebble on the other algebra. Let $\bar{a} = \langle a_0, a_1, \ldots, a_{m-1}\rangle$ be the position of the pebbles played so far (by either player) on $\mathfrak{A}$ and let $\bar{b} = \langle b_0, \ldots, b_{m-1}\rangle$ be the the position of the pebbles played on $\mathfrak{B}$. Denote χ_u^{M}, by $\mathbf{1}_u$. Then $\exists$ has to maintain the following properties throughout the game:

- for any atom x (of either algebra) with $x \cdot \mathbf{1}_{\mathbf{0}} = 0$, , then $x \in a_i$ iff $x \in b_i$,

- $\bar{a}$ induces a finite partition of $\mathbf{1}_{\mathbf{0}}$ in $\mathfrak{A}$ of 2^m (possibly empty) parts $p_i : i < 2^m$ and the $\bar{b}$ induces a partition of 1_u in $\mathfrak{B}$ of parts $q_i : i < 2^m$ such that p_i is finite iff q_i is finite and, in this case, $|p_i| = |q_i|$.

It is easy to see that $\exists$ can maintain these two properties in every round. In this back–and–forth game, $\exists$ will always find a matching pebble, because the pebbles in play are finite. For each $w \in {}^n n$ the component $\mathfrak{B}_w = \{x \in \mathfrak{B} : x \le \mathbf{1}_v\} (\subseteq \mathfrak{A}_w = \{x \in \mathfrak{A} : x \le \mathbf{1}_v\})$ contains infinitely many atoms. For any $w \in V$, $|\mathsf{At}\mathfrak{A}_w| = |I|$, while for $u \in V \sim \{\mathbf{0}\}$, $\mathsf{At}\mathfrak{A}_u = \mathsf{At}\mathfrak{B}_u$. For $|\mathsf{At}\mathfrak{B}_{\mathbf{0}}| = \omega$, but it is still an infinite set. Therefore $\mathfrak{A} \equiv_\infty \mathfrak{B}$. It is clear that the above argument works for any $\mathfrak{C}$ such that $\mathsf{At}\mathfrak{C} = \mathsf{At}\mathfrak{B}$, hence $\mathfrak{B} \equiv_{\infty,\omega} \mathfrak{C}$. □

Corollary 3.6. For any $2 < n < \omega$, for any variety V between Sc and QEA and any ordinal $m > n$, the variety $\mathsf{Nr}_n\mathsf{V}_m$ is not elementray [18, 21, 17].

4. Polyadic paradigm, positive results

4.1. Halmos' polyadic algebras of infinite dimension with and witout equality

Throughout this section α is an infinite ordinal. Recall that $\mathsf{PA}_\alpha(\mathsf{PEA}_\alpha)$ denotes the class of polydic algebras of dimension α (with equality) as defined in [7, Definition 5.4.1]. Neat reducts for such algebras are defined in [7, Definition 4.4.16]. For a class K of Boolean algebras with operators, we write K^{ad} for the class of completely aditive algebras in K, and we write $\mathsf{K}\cap\mathbf{At}$ for the class of atomic algebras in K.

THEOREM 4.1. *Let α be an infinite ordinal and $n<\omega$. If $\mathfrak{D}\in\mathsf{PEA}_\alpha$ is atomic, then any complete subalgebra of $\mathfrak{Nr}_n\mathfrak{D}$ is completely representable as a PEA_n. If $\mathfrak{D}\in\mathsf{PA}_\alpha$ is atomic and completely additive and $n\leq\alpha$, then $\mathsf{Nr}_n\mathfrak{A}$ is completely representable. In particular, $\mathbf{S}_c\mathsf{PA}^{\mathsf{ad}}_\alpha\cap\mathbf{At}=\mathsf{PA}^{\mathsf{ad}}_\alpha\cap\mathbf{At}=\mathsf{CRPA}_\alpha$ and the class CRPA_α is elementary.*

PROOF: Assume that $\mathfrak{A}\subseteq_c\mathfrak{Nr}_n\mathfrak{D}$, where $\mathfrak{D}\in\mathsf{PEA}_\alpha$ is atomic. Let $c\in\mathfrak{A}$ be non–zero. We will find a homomorphism $f:\mathfrak{A}\to\wp({}^nU)$ such that $f(c)\neq 0$, and preserves infinitary joins. Assume for the moment (to be proved in a while) that $\mathfrak{A}\subseteq_c\mathfrak{D}$. Then by [10, Lemma 2.16] $\mathfrak{A}$ is atomic because $\mathfrak{D}$ is. For brevity, let $X=\mathsf{At}\mathfrak{A}$. Let $\mathfrak{m}$ be the local degree of $\mathfrak{D}$, $\mathfrak{c}$ its effective cardinality and let β be any cardinal such that $\beta\geq\mathfrak{c}$ and $\sum_{s<\mathfrak{m}}\beta^s=\beta$; such notions are defined in [3]. We can assume that $\mathfrak{D}=\mathfrak{Nr}_\alpha\mathfrak{B}$, with $\mathfrak{B}\in\mathsf{PEA}_\beta$ [7, Theorem 5.4.17]. For any ordinal $\mu\in\beta$, and $\tau\in{}^\mu\beta$, write τ^+ for $\tau\cup Id_{\beta\setminus\mu}(\in{}^\beta\beta)$. Consider the following family of joins evaluated in $\mathfrak{B}$, where $p\in\mathfrak{D}$, $\Gamma\subseteq\beta$ and $\tau\in{}^\alpha\beta$: (*) $\mathsf{c}_{(\Gamma)}p=\sum^{\mathfrak{B}}\{\mathsf{s}_{\tau^+}p:\tau\in{}^\omega\beta,\ \ \tau\restriction\alpha\setminus\Gamma=Id\}$, and (**): $\sum\mathsf{s}^{\mathfrak{B}}_{\tau^+}X=1$. The first family of joins exists [3, Proof of Theorem 6.1], and the second exists, because $\sum^{\mathfrak{A}}X=\sum^{\mathfrak{D}}X=\sum^{\mathfrak{B}}X=1$ and τ^+ is completely additive, since $\mathfrak{B}\in\mathsf{PEA}_\beta$. The last equality of suprema follows from the fact that $\mathfrak{D}=\mathfrak{Nr}_\alpha\mathfrak{B}\subseteq_c\mathfrak{B}$ and the first from the fact that $\mathfrak{A}\subseteq_c\mathfrak{D}$. We prove the former, the latter is exactly the same replacing α and β, by n and α, respectivey, proving that $\mathfrak{Nr}_n\mathfrak{D}\subseteq_c\mathfrak{D}$, hence $\mathfrak{A}\subseteq_c\mathfrak{D}$. We prove that $\mathfrak{Nr}_\alpha\mathfrak{B}\subseteq_c\mathfrak{B}$. Assume that $S\subseteq\mathfrak{D}$ and $\sum^{\mathfrak{B}}S=1$, and for contradiction, that there exists $d\in\mathfrak{B}$ such that $s\leq d<1$ for all $s\in S$. Let $J=\Delta d\setminus\omega$ and take $t=-\mathsf{c}_{(J)}(-d)\in\mathfrak{D}$. Then $\mathsf{c}_{(\beta\setminus\alpha)}t=\mathsf{c}_{(\beta\setminus\alpha)}(-\mathsf{c}_{(J)}(-d))=\mathsf{c}_{(\beta\setminus\alpha)}-\mathsf{c}_{(J)}(-d)=\mathsf{c}_{(\beta\setminus\alpha)}-\mathsf{c}_{(\beta\setminus\alpha)}\mathsf{c}_{(J)}(-d)=-\mathsf{c}_{(\beta\setminus\alpha)}\mathsf{c}_{(J)}(-d)=-\mathsf{c}_{(J)}(-d)=t$.

We have proved that $t \in \mathfrak{D}$. We now show that $s \leq t < 1$ for all $s \in S$, which contradicts $\sum^{\mathfrak{D}} S = 1$. If $s \in S$, we show that $s \leq t$. By $s \leq d$, we have $s \cdot -d = 0$. Hence by $\mathsf{c}_{(J)}s = s$, we get $0 = \mathsf{c}_{(J)}(s \cdot -d) = s \cdot \mathsf{c}_{(J)}(-d)$, so $s \leq -\mathsf{c}_{(J)}(-d)$. It follows th at $s \leq t$ as required. Assume for contradiction that $1 = -\mathsf{c}_{(J)}(-d)$. Then $\mathsf{c}_{(J)}(-d) = 0$, so $-d = 0$ which contradicts that $d < 1$. We have proved that $\sum^{\mathfrak{B}} S = 1$, so $\mathfrak{D} \subseteq_c \mathfrak{B}$. Let F be any Boolean ultrafilter of B generated by an atom below a. We show that F will preserve the family of joins in (*) and (**). One forms nowhere dense sets in the Stone space of $\mathfrak{B}$ corresponding to the aforementioned family of joins as follows: The Stone space of (the Boolean reduct of) $\mathfrak{B}$ has underlying set S, the set of all Boolean ultrafilters of $\mathfrak{B}$. For $b \in \mathfrak{B}$, let N_b be the clopen set $\{F \in S : b \in F\}$. The required nowhere dense sets are defined for $\Gamma \subseteq \beta$, $p \in \mathfrak{D}$ and $\tau \in {}^{\alpha}\beta$ via: $A_{\Gamma,p} = N_{\mathsf{c}_{(\Gamma)}p} \setminus \bigcup_{\tau:\alpha\to\beta} N_{\mathsf{s}_{\tau}+p}$, and $A_{\tau} = S \setminus \bigcup_{x \in X} N_{\mathsf{s}_{\tau}+x}$. The principal ultrafilters are isolated points in the Stone topology, so they lie outside the nowhere dense sets defined above. Hence any such ultrafilter preserve the joins in (*) and (**). Fix a principal ultrafilter F with $a \in F$. Define the equivalence relation E (on β) by setting $iEj \iff \mathsf{d}^{\mathfrak{B}}_{ij} \in F (i, j \in \beta)$. Define $f : \mathfrak{A} \to \wp({}^{n}(\beta/E))$, via $x \mapsto \{\bar{t} \in {}^{n}(\beta/E) : \mathsf{s}^{\mathfrak{B}}_{t \cup Id_{\beta \sim n}} x \in F\}$, where $\bar{t}(i/E) = t(i)$ $(i < n)$ and $t \in {}^{n}\beta$. Let $V = {}^{\beta}\beta^{(Id)}$. To show that f is well defined, it suffices to show that for all $\sigma, \tau \in V$, if $(\tau(i), \sigma(i)) \in E$ for all $i \in \beta$, then for any $x \in \mathfrak{A}$, $\mathsf{s}_{\tau}x \in F \iff \mathsf{s}_{\sigma}x \in F$. We proceed by by induction on $|\{i \in \beta : \tau(i) \neq \sigma(i)\}| (< \omega)$. If $J = \{i \in \beta : \tau(i) \neq \sigma(i)\}$ is empty, the result is obvious. Otherwise assume that $k \in J$. We introduce a helpful piece of notation. For $\eta \in V$, let $\eta(k \mapsto l)$ stand for the η' that is the same as η except that $\eta'(k) = l$. Now take any $\lambda \in \{\eta \in \beta : (\sigma)^{-1}\{\eta\} = (\tau)^{-1}\{\eta\} = \{\eta\}\} \smallsetminus \Delta x$. Recall that $\Delta x = \{i \in \beta : \mathsf{c}_i x \neq x\}$ and that $\beta \setminus \Delta x$ is infinite because $\Delta x \subseteq n$, so such a λ exists. Now we freely use properties of substitutions for cylindric algebras. We have by [7, 1.11.11(i)(iv)] (a) $\mathsf{s}_{\sigma}x = \mathsf{s}^{\lambda}_{\sigma k}\mathsf{s}_{\sigma(k \mapsto \lambda)}x$, and (b) $\mathsf{s}^{\lambda}_{\tau k}(\mathsf{d}_{\lambda,\sigma k} \cdot \mathsf{s}_{\sigma}x) = \mathsf{d}_{\tau k,\sigma k}\mathsf{s}_{\sigma}x$, and (c) $\mathsf{s}^{\lambda}_{\tau k}(\mathsf{d}_{\lambda,\sigma k} \cdot \mathsf{s}_{\sigma(k \mapsto \lambda)}x) = \mathsf{d}_{\tau k,\sigma k} \cdot \mathsf{s}_{\sigma(k \mapsto \tau k)}x$, and finally (d) $\mathsf{d}_{\lambda,\sigma k} \cdot \mathsf{s}^{\lambda}_{\sigma k}\mathsf{s}_{\sigma(k \mapsto \lambda)}x = \mathsf{d}_{\lambda,\sigma k} \cdot \mathsf{s}_{\sigma(k \mapsto \lambda)}x$. Then by (b), (a), (d) and (c), we get,

$\mathsf{d}_{\tau k,\sigma k} \cdot \mathsf{s}_{\sigma}x = \mathsf{s}^{\lambda}_{\tau k}(\mathsf{d}_{\lambda,\sigma k} \cdot \mathsf{s}_{\sigma}x) = \mathsf{s}^{\lambda}_{\tau k}(\mathsf{d}_{\lambda,\sigma k} \cdot \mathsf{s}^{\lambda}_{\sigma k}\mathsf{s}_{\sigma(k \mapsto \lambda)}x) = \mathsf{s}^{\lambda}_{\tau k}(\mathsf{d}_{\lambda,\sigma k} \cdot \mathsf{s}_{\sigma(k \mapsto \lambda)}x) = \mathsf{d}_{\tau k,\sigma k} \cdot \mathsf{s}_{\sigma(k \mapsto \tau k)}x$. But F is a filter and $(\tau k, \sigma k) \in E$, we conclude that $\mathsf{s}_{\sigma}x \in F \iff \mathsf{s}_{\sigma(k \mapsto \tau k)}x \in F$. The conclusion follows from the induction hypothesis. We check only cylindrifications since the other operations are entirely straightforward to handle. Let $k < n$ and $a \in A$.

Let $\bar{\sigma} \in \mathsf{c}_k f(a)$. Then for some $\lambda \in \beta$, we have $\bar{\sigma}(k \mapsto \lambda/E) \in f(a)$ hence $\mathsf{s}_{\sigma^+(k\mapsto\lambda)} a \in F$. It follows from the inclusion $a \leq \mathsf{c}_k a$ that $\mathsf{s}_{\sigma^+(k\mapsto\lambda)}\mathsf{c}_k a \in F$, so $\mathsf{s}_{\sigma^+}\mathsf{c}_k a \in F$. Thus $\mathsf{c}_k f(a) \subseteq f(\mathsf{c}_k a.)$ We prove the other more difficult inclusion that uses the condition (*) of eliminating cylindrifiers. Let $a \in A$ and $k < n$. Let $\bar{\sigma}' \in f\mathsf{c}_k a$ and let $\sigma = \sigma' \cup Id_{\beta\sim n}$. Then $\mathsf{s}_\sigma^{\mathfrak{B}}\mathsf{c}_k a = \mathsf{s}_{\sigma'}^{\mathfrak{B}}\mathsf{c}_k a \in F$. Pick $\lambda \in \{\eta \in \beta : \sigma^{-1}\{\eta\} = \{\eta\}\} \smallsetminus \Delta a$, such a λ exists because Δa is finite, and $|\{i \in \beta : \sigma(i) \neq i\}| < \omega$. Let $\tau = \sigma \restriction \mathfrak{n} \smallsetminus \{k, \lambda\} \cup \{(k, \lambda), (\lambda, k)\}$. Then (in $\mathfrak{B}$):

$$\mathsf{c}_\lambda \mathsf{s}_\tau a = \mathsf{s}_\tau \mathsf{c}_k a = \mathsf{s}_\sigma \mathsf{c}_k a \in F.$$

By the construction of F, there is some $u(\notin \Delta(\mathsf{s}_\tau^{\mathfrak{B}} a))$ such that $\mathsf{s}_u^\lambda \mathsf{s}_\tau a \in F$, so $\mathsf{s}_{\sigma(k\mapsto u)} a \in F$. Hence $\sigma(k \mapsto u) \in f(a)$, from which we get that $\bar{\sigma}' \in \mathsf{c}_k f(a)$. By construction, for every $s \in {}^n(\beta/E)$, there exists $x \in X(= \mathsf{At}\mathfrak{A})$, such that $\mathsf{s}^{\mathfrak{B}}_{s\cup Id_{\beta\sim n}} x \in F$, from which we get $\bigcup_{x\in X} f(x) = {}^n(\beta/E)$ hence f is an atomic, thus a complete representation. If $\mathfrak{A} \in \mathsf{PA}_\alpha$, we do not need to bother about diagonal elements and so the base of the representation will be simply β (as defined above for PEA_α), not β/E, and the desired homomorphism, with $n \leq \alpha$, is defined via $g : \mathfrak{A} \to \wp({}^n\beta)$, via $x \mapsto \mathsf{t} \in {}^n\beta : \mathsf{s}^{\mathfrak{B}}_{t\cup Id_{\beta\sim n}} x \in F\}$. Checking that g preserves the operations and that g is atomic, hence complete, is exactly like the PEA case. For PA_α, atomicity can be expressed by a first order sentence, and complete additivity can be captured by the following continuum many first order formulas, that form a single schema. Let $\mathsf{At}(x)$ be the first order formula expressing that x is an atom. That is $\mathsf{At}(x)$ is the formula $x \neq 0 \wedge (\forall y)(y \leq x \to y = 0 \vee y = x)$. For $\tau \in {}^\alpha\alpha$, let ψ_τ be the formula: $y \neq 0 \to \exists x(\mathsf{At}(x) \wedge \mathsf{s}_\tau x \cdot y \neq 0)$. Let Σ be the set of first order formulas obtained by adding all formulas ψ_τ $(\tau \in {}^\alpha\alpha)$ to the polyadic schema. Then it is not hard to show that $\mathsf{CRPA}_\alpha = \mathbf{Mod}(\Sigma)$. The underlying idea here is that the notion of complete additivity on atomic algebras is definable in $L_{\omega,\omega}$. In more detail: Let $\mathfrak{A} \in \mathsf{CRPA}_\alpha$ with set of atoms X. Then, $\sum_{x\in X} \mathsf{s}_\tau x = 1$ for all $\tau \in {}^\alpha\alpha$. Let $\tau \in {}^\alpha\alpha$. Let a be non-zero, then $a \cdot \sum_{x\in X} \mathsf{s}_\tau x = a \neq 0$, hence there exists $x \in X$, such that $a \cdot \mathsf{s}_\tau x \neq 0$, and so $\mathfrak{A} \models \psi_\tau$. Conversely, let $\mathfrak{A} \models \Sigma$. Then for all $\tau \in {}^\alpha\alpha$, $\sum_{x\in X} \mathsf{s}_\tau x = 1$. Indeed, assume that for some τ, $\sum_{x\in X} \mathsf{s}_\tau x \neq 1$. Let $a = 1 - \sum_{x\in X} \mathsf{s}_\tau x$. Then $a \neq 0$. But then, by assumption, there exists $x' \in X$, such that $\mathsf{s}_\tau x' \cdot a = \mathsf{s}_\tau x' \cdot (1 - \sum_{x\in X} \mathsf{s}_\tau x) = \mathsf{s}_\tau x' - \sum_{x\in X} \mathsf{s}_\tau x \neq 0$, which is impossible. $\square$

4.2. Algebras in between the cylindric and polyadic paradigms; Ferenczi's cylindric-polyadic algebras

We recall the definition of certain reducts of polyadic algebras. By $I \subseteq_\omega J$, we undestand that I is a *finite subset of* J.

DEFINITION 4.2. Let α be an ordinal. By a *cylindric polyadic algebra* of dimension α, or a CPA_α for short, we understand an algebra of the form

$$\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, \mathsf{c}_{(\Gamma)}, \mathsf{s}_\tau \rangle_{\Gamma \subseteq_\omega \alpha, \tau \in {}^\alpha\alpha}$$

where $\mathsf{c}_{(\Gamma)}$ $\Gamma \subseteq_\omega \alpha)$ and s_τ $(\tau \in {}^\alpha\alpha)$ are unary operations on A, such that postulates below hold for $x, y \in A$, $\tau, \sigma \in {}^\alpha\alpha$ and $\Gamma, \Delta \subseteq_\omega \alpha$

1. $\langle A, +, \cdot, -, 0, 1 \rangle$ is a Boolean algebra
2. $\mathsf{c}_{(0)} x = x$
3. $\mathsf{c}_{(\Gamma)} 0 = 0$
4. $x \leq \mathsf{c}_{(\Gamma)} x$
5. $\mathsf{c}_{(\Gamma)}(x \cdot \mathsf{c}_{(\Gamma)} y) = \mathsf{c}_{(\Gamma)} x \cdot \mathsf{c}_{(\Gamma)} y$
6. $\mathsf{c}_{(\Gamma)} \mathsf{c}_{(\Delta)} x = \mathsf{c}_{(\Gamma \cup \Delta)} x$
7. s_τ is a Boolean endomorphism
8. $\mathsf{s}_{Id} x = x$
9. $\mathsf{s}_{\sigma \circ \tau} = \mathsf{s}_\sigma \circ \mathsf{s}_\tau$
10. if $\sigma \upharpoonright (\alpha \sim \Gamma) = \tau \upharpoonright (\alpha \sim \Gamma)$, then $\mathsf{s}_\sigma \mathsf{c}_{(\Gamma)} x = \mathsf{s}_\tau \mathsf{c}_{(\Gamma)} x$
11. If $\tau^{-1}\Gamma = \Delta$ and $\tau \upharpoonright \Delta$ is one to one, then $\mathsf{c}_{(\Gamma)} \mathsf{s}_\tau x = \mathsf{s}_\tau \mathsf{c}_{(\Delta)} x$.

The definition of neat reducts for CPA_α is defined as follows: Given any pair of infinite ordinals $\alpha < \beta$ and $\mathfrak{B} \in \mathsf{CPA}_\beta$ then $\mathsf{Nr}_\alpha \mathfrak{B}$ is the CPA_α with domain $Nr_\alpha \mathfrak{B} = \{a \in \mathfrak{B} : \mathsf{c}_i a = a, \forall i \in \beta \sim \alpha\}$ and with all operations except substitutions are those of $\mathfrak{B}$ indexed up to α. As for substitutions, given $\tau \in {}^\alpha\alpha$, and $a \in Nr_\alpha B$, $\mathsf{s}_\tau^{\mathfrak{A}} a = \mathsf{s}_{\bar{\tau}}^{\mathfrak{B}} a$ where $\bar{\tau} = \tau \cup Id \upharpoonright \beta \sim \alpha$.

Next we prove that the class of completely representable CPA_βs, β an infinite ordinal, is elementary. This is in sharp contrst to the CA case. The idea of the proof of the next theorem, is simple and in essence the gist of the idea is analogous to the previous proof. Start with an atomic completely additive $\mathfrak{A} \in \mathsf{CPA}_\alpha$. Then $\mathfrak{A}$ neatly embeds into an algebra $\mathfrak{B} \in \mathsf{CPA}_\beta$, having enough spare dimensions $|\beta| > |\alpha|$, called a β- dilation of $\mathfrak{A}$, that is $\mathfrak{A} = \mathsf{Nr}_\alpha\mathfrak{B}$. As it turns out, $\mathfrak{B}$ is also atomic, and by complete additivity the sum of all substituted versions of the set of atoms is the top element in $\mathfrak{B}$. The desired representation is built from any principal ultrafilter thet preserves this set of infinitary joins as well as some infinitary joins that have to do with eliminating cylindrifiers. A principal ultrafilter preserving these sets of joins can always be found because, on the one hand, the set of principal ultrafilters are dense in the Stone space of the Boolean reduct of $\mathfrak{B}$ since the latter is atomic, and on the other hand, finding an ultrafilter preserving the given set joints amounts to finding a principal ultrafilter outside a nowhere dense set corresponding to the infinitary joins. Any such ultrafilter can be used to build the desired representation. But first a definition:

Definition 4.3. A *transformation system* is a quadruple of the form $(\mathfrak{A}, I, G, \mathsf{S})$ where $\mathfrak{A}$ is an algebra of any similarity type, I is a non empty set (we will only be concerned with infinite sets), G is a subsemigroup of $({}^I I, \circ)$ (the operation $\circ$ denotes composition of maps) and S is a homomorphism from G to the semigroup of endomorphisms of $\mathfrak{A}$. Elements of G are called transformations.

Theorem 4.4. *Let α be an infinite ordinal. Let $\mathfrak{A} \in \mathsf{CPA}_\alpha$ be atomic and completely additive. Then $\mathfrak{A}$ has a complete representation.*

Proof: Let $c \in A$ be non-zero. It suffices to find a set U and a homomorphism from $\mathfrak{A}$ into the set algebra with universe $\wp({}^\alpha U)$ that preserves arbitrary suprema whenever they exist and also satisfies that $f(c) \neq 0$. U is called the base of the set algebra. Let $\mathfrak{m}$ be the local degree of $\mathfrak{A}$, $\mathfrak{c}$ its effective cardinality and $\mathfrak{n}$ be any cardinal such that $\mathfrak{n} \geq \mathfrak{c}$ and $\sum_{s<\mathfrak{m}} \mathfrak{n}^s = \mathfrak{n}$. The cardinal $\mathfrak{n}$ will be the base of our desired representation. Substitutions in $\mathfrak{A}$, induce a homomorphism of semigroups $S : {}^\alpha\alpha \to End(\mathfrak{A})$, via $\tau \mapsto \mathsf{s}_\tau$. The operation on both semigroups is composition of maps; the latter is the semigroup of endomorphisms on $\mathfrak{A}$. For any set X, let $F({}^\alpha X, \mathfrak{A})$ be the set of all functions from ${}^\alpha X$ to $\mathfrak{A}$ endowed with Boolean operations defined pointwise and for $\tau \in {}^\alpha\alpha$ and $f \in F({}^\alpha X, \mathfrak{A})$, put $\mathsf{s}_\tau f(x) = f(x \circ \tau)$. This turns $F({}^\alpha X, \mathfrak{A})$ to a transformation system as well that is completely additive The map $H : \mathfrak{A} \to F({}^\alpha\alpha, \mathfrak{A})$ defined by $H(p)(x) = \mathsf{s}_x p$ is easily checked to be an embedding. Assume that $\beta \supseteq \alpha$. Then $K : F({}^\alpha\alpha, \mathfrak{A}) \to F({}^\beta\alpha, \mathfrak{A})$ defined by $K(f)x = f(x \upharpoonright \alpha)$ is an embedding, too. These facts are straighforward to establish, cf. [3, Theorems 3.1, 3.2]. Call $F({}^\beta\alpha, \mathfrak{A})$ a minimal functional dilation of $F({}^\alpha\alpha, \mathfrak{A})$. Elements of the big algebra, or the (cylindrifier free) functional dilation, are of form $\mathsf{s}_\sigma p$, $p \in F({}^\beta\alpha, \mathfrak{A})$ where σ is one to one on α, cf. [3, Theorems 4.3–4.4].

We can assume that $|\alpha| < \mathfrak{n}$. Let $\mathfrak{B}$ be the algebra obtained from $\mathfrak{A}$, by discarding its cylindrifiers, then dilating it to $\mathfrak{n}$ dimensions, that is, taking a minimal functional dilation in $\mathfrak{n}$ dimensions, and then re-defining cylindrifiers and boxes in the bigger algebra, by setting for each $\Gamma \subseteq \mathfrak{n}$:

$$\mathsf{c}_{(\Gamma)}\mathsf{s}^{\mathfrak{B}}_\sigma p = \mathsf{s}^{\mathfrak{B}}_{\rho^{-1}}\mathsf{c}^{\mathfrak{A}}_{(\rho(\Gamma)\cap\sigma\alpha)}\mathsf{s}^{\mathfrak{A}}_{(\rho\sigma\upharpoonright\alpha)}p.$$

Here ρ is any permutation such that $\rho \circ \sigma(\alpha) \subseteq \sigma(\alpha)$. The definition is sound, that is, it is independent of ρ, σ, p. Moreover, it agrees with the old cylindrifiers and boxes in $\mathfrak{A}$. Identifying algebras with their transformation systems we have $\mathfrak{A}$ is embeddable in $\mathsf{Nr}_\alpha\mathfrak{B}$, via H defined for $f \in \mathfrak{A}$ and $x \in {}^{\mathfrak{n}}\alpha$ by $H(f)x = f(y)$ where $y \in {}^\alpha\alpha$ and $x \upharpoonright \alpha = y$; furthermore H so defined exhausts all α dimensional elements of $\mathfrak{B}$ meaning that $\mathfrak{A} = \mathsf{Nr}_\alpha\mathfrak{B}$, cf. [3, Theorem 3.10]. The local degree of $\mathfrak{B}$ is the same as that of $\mathfrak{A}$, in particular, each $x \in \mathfrak{B}$ admits a support of cardinality $< \mathfrak{n}$. Also $|\mathfrak{n} \sim \alpha| = |\mathfrak{n}|$ and for all $Y \subseteq A$, we have $\mathfrak{Sg}^{\mathfrak{A}}Y = \mathsf{Nr}_\alpha\mathfrak{Sg}^{\mathfrak{B}}Y$. All this can be found in [3], see the proof of Theorem 6.1 therein; in such a proof, $\mathfrak{B}$ is called a minimal dilation of $\mathfrak{A}$, due to the fact that $\mathfrak{B}$ is unique up to isomorphisms that fix $\mathfrak{A}$ pointwise. Clearly $F({}^{\mathfrak{n}}\alpha, \mathfrak{A})$, hence the Boolean reduct of $\mathfrak{B}$, is atomic, because it is isomorphic to a Boolean product of the atomic Boolean reduct of $\mathfrak{A}$. Let $\Gamma \subseteq \alpha$ and $p \in \mathfrak{A}$. Then in $\mathfrak{B}$ we have, see [3, proof of Theorem 6.1]:

$$\mathsf{c}_{(\Gamma)}p = \sum\{\mathsf{s}_{\bar\tau}p : \tau \in {}^\alpha\mathfrak{n},\quad \tau \upharpoonright \alpha \sim \Gamma = Id\}. \tag{4.1}$$

Here, and elsewhere throughout the paper, for a transformation τ with domain α and range included in $\mathfrak{n}$, $\bar\tau = \tau \cup Id_{\mathfrak{n}\sim\alpha}$. Let X be the set of atoms of $\mathfrak{A}$. Since $\mathfrak{A}$ is atomic, then $\sum^{\mathfrak{A}} X = 1$. By $\mathfrak{A} = \mathsf{Nr}_\alpha\mathfrak{B}$, we also have $\sum^{\mathfrak{B}} X = 1$. By complete additivity we have for all $\tau \in {}^\alpha\mathfrak{n}$,

$$\sum \mathsf{s}^{\mathfrak{B}}_{\bar\tau} X = 1. \tag{4.2}$$

Let S be the Stone space of $\mathfrak{B}$, whose underlying set consists of all Boolean ulltrafilters of $\mathfrak{B}$. Let X^* be the set of principal ultrafilters of $\mathfrak{B}$ (those generated by the atoms). These are isolated points in the Stone topology, and they form a dense set in the Stone topology since $\mathfrak{B}$ is atomic. So we have $X^* \cap T = \emptyset$ for every nowhere dense set T (since principal ultrafilters, which are isolated points in the Stone topology, lie outside nowhere dense sets). For $a \in \mathfrak{B}$, let N_a denote the set of all Boolean ultrafilters containing a. Now for all $\Gamma \subseteq \alpha$, $p \in A$ and $\tau \in {}^{\alpha}\mathfrak{n}$, we have, by the suprema, evaluated in (1) and (2):

$$G_{\Gamma,p} = N_{\mathsf{c}_{(\Gamma)}p} \sim \bigcup_{\tau \in {}^{\alpha}\mathfrak{n}} N_{s_{\bar{\tau}}p} \tag{4.3}$$

and

$$G_{X,\tau} = S \sim \bigcup_{x \in X} N_{s_{\bar{\tau}}x}. \tag{4.4}$$

are nowhere dense. Let F be a principal ultrafilter of S containing c. This is possible since $\mathfrak{B}$ is atomic, so there is an atom x below c; just take the ultrafilter generated by x. Then $F \in X^*$, so $F \notin G_{\Gamma,p}$, $F \notin G_{X,\tau}$, for every $\Gamma \subseteq \alpha$, $p \in A$ and $\tau \in {}^{\alpha}\mathfrak{n}$. Now define for $a \in A$

$$f(a) = \{\tau \in {}^{\alpha}\mathfrak{n} : \mathsf{s}_{\bar{\tau}}^{\mathfrak{B}} a \in F\}.$$

Then f is a polyadic homomorphism from $\mathfrak{A}$ to the full set algebra with unit ${}^{\alpha}\mathfrak{n}$, such that $f(c) \neq 0$. We have $f(c) \neq 0$ because $c \in F$, so $Id \in f(c)$. That f is a homomorphism can be proved exactly as in the proof of Theorem 4.1; the preservation of the Boolean operations and substitutions is fairly straightforward. Preservation of cylindrifications is guaranteed by the condition that $F \notin G_{\Gamma,p}$ for all $\Gamma \subseteq \alpha$ and all $p \in A$. (Basically an elimination of cylindrifications, this condition is also used in [3] to prove the main representation result for polyadic algebras.) The proof is complete.

Moreover f is an atomic representation since $F \notin G_{X,\tau}$ for every $\tau \in {}^{\alpha}\mathfrak{n}$, which means that for every $\tau \in {}^{\alpha}\mathfrak{n}$, there exists $x \in X$, such that $\mathsf{s}_{\bar{\tau}}^{\mathfrak{B}} x \in F$, and so $\bigcup_{x \in X} f(x) = {}^{\alpha}\mathfrak{n}$. We conclude that f is a complete representation, since in this case too it can be proved exactly like the CA case that complete and atomic rtepresenations coincide. $\square$

THEOREM 4.5. *The class* CPA_α *is elementary, and it is axiomatizable by a finite schema in first order logic. Furthermore, for any infinite ordinals* $\alpha < \beta$, $\mathsf{Nr}_\alpha \mathsf{CPA}_\beta$ *is elementary.*

PROOF: Like the proof of Theorem 4.1. □

Also the technique used here adapts without much dificulty to prove completely analagous results for the so-called cylindric-polyadic algebras introduced by Ferenczi in [4] and [5, Definition 6.3.7]. We denote the class of such algebras of dimension α by CPEA_α. For CPEA_α diagonal algebras are present in their signature, so that complete additivity holds anyway. The complete representation in this case is not with respect to square Tarskian semantics, as is the case here, but is relativized to units that are (not necessarily disjoint) unions of Cartesian spaces. The class of such concrete representable algebras of dimension α is denoted by Gp_α. Recall that for a calss $\mathbf{K}$ having a Boolean reduct, we write $\mathbf{K} \cap \mathbf{At}$ for the class of atomic algebras in $\mathbf{K}$.

THEOREM 4.6. *The class of completely representable algebras in* CPEA_α *coincides with* $\mathsf{CPEA}_\alpha \cap \mathbf{At}$, *hence is elementary.*

PROOF: We start with the general idea. then follows a more technical proof. If $\mathfrak{A}$ is atomic, and $\mathfrak{B}$ is the minimal dilation of $\mathfrak{A}$, then $\mathfrak{B}$ is also atomic since its Boolean reduct is simply an infinite product of the atomic $\mathfrak{A}$. This can now be used to show that atomic algebras are completely representable. Like in the above proof, start with an atomic $\mathfrak{A} \in \mathsf{CPEA}_\alpha$. Then $\mathfrak{A}$ is completely additive and it neatly embeds into an algebra $\mathfrak{B}$ having enough spare dimensions, the minimal dilation of $\mathfrak{A}$, that is $\mathfrak{A} = \mathsf{Nr}_\alpha\mathfrak{B}$. As it turns out, $\mathfrak{B}$ is also atomic, and by complete additivity the sum of all all substituted versions of the set of atoms is the top element in $\mathfrak{B}$. The desired representation is built from any principal ultrafilter that preserves this set of infinitary joins as well as some infinitary joins that have to do with eliminating cylindrifiers. A principal ultrafilter preserving these sets of joins can always be found because, on the one hand, the set of principal ultrafilters are dense in the Stone space of the Boolean reduct of $\mathfrak{B}$ since the latter is atomic, and on the other hand, finding an ultrafilter preserving the given set of infinitary joins really amounts to finding a a principal ultrafilter outside a nowhere dense set corresponding to the infinitary joins. The hitherto obtained ultrafilter in $\mathfrak{B}$ can be easily modified to give a so-called *perfect ultrafilter*. One such ultrafilter is found for every non-zero element of $a \in A$ in the dilation $\mathfrak{B}$, containing a, giving an atomic simple representation (model) of $\mathfrak{A}$. Taking the subdirect product of these representations, we get the desired complete representation, whose unit is a disjoint union of units of such simple representations.

More technically, let $c \in \mathfrak{A}$ be non-zero. We will find a $\mathfrak{B} \in \mathsf{Gp}_\alpha$ and a homomorphism from $f : \mathfrak{A} \to \mathfrak{B}$ that preserves arbitrary suprema whenever they exist and also satisfies that $f(c) \neq 0$. Now there exists $\mathfrak{B} \in \mathsf{CPEA}_\mathfrak{n}$, $\mathfrak{n}$ a regular cardinal. such that $\mathfrak{A} \subseteq \mathsf{Nr}_\alpha\mathfrak{B}$ and A generates $\mathfrak{B}$ and we can assume that $|\mathfrak{n} \sim \alpha| = |\mathfrak{n}|$. We also have for all $Y \subseteq A$, we have $\mathfrak{Sg}^{\mathfrak{A}}Y = \mathsf{Nr}_\alpha\mathfrak{Sg}^{\mathfrak{B}}Y$. This dilation also has Boolean reduct isomophic to $F({}^{\mathfrak{n}}\alpha, \mathfrak{A})$, in particular, it is atomic because $\mathfrak{A}$ is atomic. Also cylindrifiers are defined on this minimal functional dilation exactly like above by restricting to singletions. Let adm be the set of admissable substitutions. The transformation τ is admissable if $\mathsf{dom}\tau \subseteq \alpha$ and $\mathsf{rng}\tau \cap \alpha = \emptyset$. Then we have for all $i < \mathfrak{n}$ and $\sigma \in adm$,

$$\mathsf{s}_\sigma\mathsf{c}_i p = \sum \mathsf{s}_\sigma\mathsf{s}_i^j p \tag{4.5}$$

This uses that $\mathsf{c}_k = \sum \mathsf{s}_k^i x$, which is proved like the cylindric case; the proof depends on diagonal elements. Let X be the set of atoms of $\mathfrak{A}$. Since $\mathfrak{A}$ is atomic, then $\sum^{\mathfrak{A}} X = 1$. By $\mathfrak{A} = \mathsf{Nr}_\alpha\mathfrak{B}$, we also have $\sum^{\mathfrak{B}} X = 1$. Because substitutions are completely additive we have for all $\tau \in {}^\alpha\mathfrak{n}$

$$\sum \mathsf{s}_{\bar{\tau}}^{\mathfrak{B}} X = 1. \tag{4.6}$$

Let S be the Stone space of $\mathfrak{B}$, whose underlying set consists of all boolean ulltrafilters of $\mathfrak{B}$, and let F be a principal ultrafilter chosen as before. Let $\mathfrak{B}'$ be the minimal completion of $\mathfrak{B}$. Exists by completey additivity. Take the filter G in $\mathfrak{B}'$ generated by the generator of F and let $F = G \cap \mathfrak{B}$. Then F is a perfect ultrafilter. Because our algebras have diagonal algebras, we have to factor our base by a congruence relation that reflects equality. Define an equivalence relation on $\Gamma = \{i \in \beta : \exists j \in \alpha : \mathsf{c}_i\mathsf{d}_{ij} \in F\}$, via $m \sim n$ iff $\mathsf{d}_{mn} \in F$. Then $\Gamma \subset \alpha$ and the desired representation is defined on a Gp_α with base $\Gamma/\sim$. We omit the details which are the same as in the proof of [27, Theorem 3.4, item 3]. $\square$

5. Related results on minimal Dedekind–MacNeille completions

Unless otherwise indicated, we fix $2 < n < \omega$. In our next Theorem we use rainbow constructions following almost verbatim [9, § 4.3] abeit adding a clause for the polyadic accessibility relations as follows: $[a]T_{ij}[b] \iff a \circ [i,j] = b$ where $a : n \to \Delta$ and $b : n \to \Gamma$ are surjections into complete (finite) coloured graphs Δ an Γ. This allows us to construct n dimensional polyadic equaltiy rainbow atom structures. (Everything else is like the CA_n case dealt with in detail in [9]). However, for the polyadic case, networks should be defined as the cylindric case with an additional symmetry condition:

Definition 5.1. An *n–dimensional atomic network* on an atomic algebra $\mathfrak{A} \in \mathsf{QEA}_n$ is a map $N : {}^n\Delta \to \mathsf{At}\mathfrak{A}$, where Δ is a non–empty finite set of *nodes*, denoted by $\mathsf{nodes}(N)$, satisfying the following consistency conditions for all $i < j < n$:

(i) If $\bar{x} \in {}^n\mathsf{nodes}(N)$ then $N(\bar{x}) \leq \mathsf{d}_{ij} \iff \bar{x}_i = \bar{x}_j$,

(ii) If $\bar{x}, \bar{y} \in {}^n\mathsf{nodes}(N)$, $i < n$ and $\bar{x} \equiv_i \bar{y}$, then $N(\bar{x}) \leq \mathsf{c}_i N(\bar{y})$,

(iii) (Symmetry): if $\bar{x} \in {}^n\mathsf{nodes}(N)$, then $\mathsf{s}_{[i,j]}N(\bar{x}) = N(\bar{x} \circ [i,j])$.

We give a detailed description of the rainbow-like construction we use. Let G be a relational structures. Let $2 < n < \omega$. Then we specify a list of colours from which our algebras are to be constructed:

- greens: g_i $(1 \leq i \leq n-2)$, g_0^i, $i \in \mathsf{G}$,
- whites : $\mathsf{w}_i : i \leq n-2$,
- reds: r_{ij} $i < j \in n$,
- shades of yellow : $\mathsf{y}_S : S$ a finite subset of ω or $S = \omega$.

A *coloured graph* is a graph such that each of its edges is labelled by the colours in the above first three items, greens, whites or reds, and some $n-1$ hyperedges are also labelled by the shades of yellow. Certain coloured graphs will deserve special attention.

Definition 5.2. Let $i \in \mathsf{G}$, and let M be a coloured graph consisting of n nodes $x_0, \ldots, x_{n-2}, z$. We call M *an i-cone* if $M(x_0, z) = \mathsf{g}_0^i$ and for every

$1 \leq j \leq n-2$, $M(x_j, z) = \mathsf{g}_j$, and no other edge of M is coloured green. $(x_0, \ldots, x_{n-2})$ is called the *base of the cone*, z the *apex of the cone* and i the *tint of the cone*.

The rainbow algebra depending on G and n from the class $\mathbf{K}$ consisting of all coloured graphs M such that:

1. M is a complete graph and M contains no triangles (called forbidden triples) of the following types:

$$\begin{array}{rl} (\mathsf{g}, \mathsf{g}^{'}, \mathsf{g}^*), (\mathsf{g}_i, \mathsf{g}_i, \mathsf{w}_i) & \text{any } 1 \leq i \leq n-2, \\ (\mathsf{g}_0^j, \mathsf{g}_0^k, \mathsf{w}_0) & \text{any } j, k \in \mathsf{G}, \\ (\mathsf{r}_{ij}, \mathsf{r}_{j'k'}, \mathsf{r}_{i^*k^*}) & \text{unless } |\{(j,k), (j',k'), (j^*, k^*)\}| = 3 \end{array}$$

   and no other triple of atoms is forbidden.

2. If $a_0, \ldots, a_{n-2} \in M$ are distinct, and no edge (a_i, a_j) $i < j < n$ is coloured green, then the sequence $(a_0, \ldots, a_{n-2})$ is coloured a unique shade of yellow. No other $(n-1)$ tuples are coloured shades of yellow. Finally, if $D = \{d_0, \ldots, d_{n-2}, \delta\} \subseteq M$ and $M \upharpoonright D$ is an i cone with apex δ, inducing the order $d_0, \ldots, d_{n-2}$ on its base, and the tuple $(d_0, \ldots, d_{n-2})$ is coloured by a unique shade y_S then $i \in S$.

Let G and n be relational structures as above. Take the set J consisting of all surjective maps $a : n \to \Delta$, where $\Delta \in \mathbf{K}$ and define an equivalence relation $\sim$ on this set relating two such maps iff they essentially define the same graph [9]; the nodes are possibly different but the graph structure is the same. Let At be the atom structure with underlying set $J\sim$. We denote the equivalence class of a by $[a]$. Then define, for $i < j < n$, the accessibility relations corresponding to ijth–diagonal element, and ith–cylindrifier, as follows:

(1) $[a] \in E_{ij}$ iff $a(i) = a(j)$,

(2) $[a]T_i[b]$ iff $a \upharpoonright n \smallsetminus \{i\} = b \upharpoonright n \smallsetminus \{i\}$,

(3) $[a]T_{ij}[b] \iff a \circ [i, j] = b$.

This, as easily checked, defines a QEA_n atom structure. The game $\mathbf{G}^m$ played on networks lifts to a game on coloured graphs like the CA case, that is like the graph games G^m_ω [9], where the number of nodes of graphs played during the ω rounded game does not exceed m, but $\forall$ has the option to re-use nodes. The typical winning strategy for $\forall$ in the graph version of both atomic games is bombarding $\exists$ with cones having a common base and *green* tints until she runs out of (suitable) *reds*, that is to say, reds whose indicies do not match [9, § 4.3].

Let K_n be a variety between Sc_n and QEA_n.

DEFINITION 5.3. A K_n atom structure $\mathbf{At}$ is *weakly representable* if there is an atomic $\mathfrak{A} \in \mathsf{RK}_n$ such that $\mathbf{At} = \mathsf{At}\mathfrak{A}$; it is *strongly representable* if $\mathfrak{Cm}\mathbf{At} \in \mathsf{RK}_n$.

These two notions are distinct, cf. [14] and the following Theorem 5.5. Let $2 < n < m \leq \omega$. The notions of m-square, and m-flat representations are defined and extensively studied in [27, § 5.1]. Let $\mathsf{V} \subseteq \mathsf{W}$ be varieties of Boolean algebras with operators. We say hat V is *atom canonical with respect to* W, if whenever $\mathfrak{A} \in \mathsf{V}$ is atomic, then its Dedekind–MacNeille completion, which is the complex algebra of its atom structure, in symbols $\mathfrak{Cm}\mathsf{At}\mathfrak{A}$ is in W. Let Sc_n denote the class of Pinter's subnstitution algebras as defined in [7] and the appendix of [13] and $\mathfrak{Rd}_{sc}$ denotes the Sc reduct. The following is proved in [27, Lemma 5.7]

LEMMA 5.4. *Let $2 < n < \omega$ and let $\mathfrak{A}$ have signature of CA_n satisfying all axioms except commutativity of cylindrifications. Then $\mathfrak{A}$ has a complete m-square representation $\iff$ $\exists$ has a winning strategy in $G^m_\omega(\mathsf{At}\mathfrak{A})$. The last result extends to any variety V between QEA_n and Sc_n. In particular, $\mathfrak{Rd}_{sc}\mathfrak{A} \notin \mathbf{S}\mathsf{Nr}_n\mathsf{Sc}_m$.*

With these preliminaries out of the way, we are now ready to start digging deeper: The next Theorem generalizes a result proved in [27, Theorem 5.9, Corollary 5.10] for CA_ns to any variety between Sc_n and QEA_n. We use a so called *blow up and blow construction.* This subtle construction may be applied to any two classes $\mathbf{L} \subseteq \mathbf{K}$ of completely additive BAOs. One takes an atomic $\mathfrak{A} \notin \mathbf{K}$ (usually but not always finite), blows it up, by splitting one or more of its atoms each to infinitely many subatoms, obtaining an (infinite) *countable* atomic $\mathfrak{Bb}(\mathfrak{A}) \in \mathbf{L}$, such that $\mathfrak{A}$ *is blurred* in $\mathfrak{Bb}(\mathfrak{A})$ meaning that $\mathfrak{A}$ *does not* embed in $\mathfrak{Bb}(\mathfrak{A})$, but $\mathfrak{A}$ embeds in the Dedekind–MacNeille completion of $\mathfrak{Bb}(\mathfrak{A})$, namely, $\mathfrak{Cm}\mathsf{At}\mathfrak{Bb}(\mathfrak{A})$. Then any class $\mathbf{M}$ say, between $\mathbf{L}$ and $\mathbf{K}$ that is closed under forming subalgebras will not be atom–canonical, for $\mathfrak{Bb}(\mathfrak{A}) \in \mathbf{L}(\subseteq \mathbf{M})$, but $\mathfrak{Cm}\mathsf{At}\mathfrak{Bb}(\mathfrak{A}) \notin \mathbf{K}(\supseteq \mathbf{M})$

because $\mathfrak{A} \notin \mathbf{M}$ and $\mathbf{SM} = \mathbf{M}$. We say, in this case, that $\mathbf{L}$ *is not atom–canonical with respect to* $\mathbf{K}$. This method is applied to $\mathbf{K} = \mathsf{SRaCA}_l$, $l \geq 5$ and $\mathbf{L} = \mathsf{RRA}$ in [10, § 17.7] and to $\mathbf{K} = \mathsf{RRA}$ and $\mathbf{L} = \mathsf{RRA} \cap \mathsf{RaCA}_k$ for all $k \geq 3$ in [2], and will applied now below to $\mathbf{K} = \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{t(n)}$ where $t(n) = n(n+1)/2$.

THEOREM 5.5. *Let $2 < n < \omega$. The following propostions 1, 2, and 3 below are true:*

1. *The variety* RRA *is not atom-canonical with respect to* SRaCA_k*, for any $k \geq 6$,*
2. *Let* K *be any variety between* Sc *and* QEA*. Let $t(n) = n(n+1)/2+1$. Then* RK_n *is not-atom canonical with respect to* $\mathbf{S}\mathsf{Nr}_n\mathsf{K}_{t(n)}$*. In fact, there is a countable atomic simple $\mathfrak{A} \in \mathsf{RQEA}_n$ such that $\mathfrak{Rd}_{sc}\mathfrak{Cm}\mathsf{At}\mathfrak{A}$ does not have an $t(n)$-square,a fortiori $t(n)$-flat, representation.*
3. RDf_n *is not atom-canonical.*

PROOF: For item (1) cf. [11, Lemmata 17.32, 17.34, 17.35, 17.36].

Item (2): The proof is long and uses many ideas in [14]. The proof is divided into four parts:

1. **Blowing up and blurring $\mathfrak{B}_f$ forming a weakly representable atom structure $\mathbf{At}$**: Take the finite rainbow QEA_n, $\mathfrak{B}_f$ where the reds is the complete irreflexive graph n, and the greens are $\{\mathsf{g}_i : 1 \leq i < n-1\} \cup \{\mathsf{g}_0^i : 1 \leq i \leq n(n-1)/2+2\}$, endowed with the quasi-polyadic operations. We will show $\mathfrak{Rd}_{\mathsf{K}}\mathfrak{B}_f$ detects that RK_n is not atom-canonical with respect to $\mathbf{S}\mathsf{Nr}_n\mathsf{K}_{t(n)}$ with $t(n)$ as specified in the statement of the theorem. Denote the finite atom structure of $\mathfrak{B}_f$ by $\mathbf{At}_f$; so that $\mathbf{At}_f = \mathsf{At}(\mathfrak{B}_f)$. One then defines a larger the class of coloured graphs like in [14, Definition 2.5]. Let $2 < n < \omega$. Then the colours used are like above except that each red is 'split' into ω many having 'copies' the form r_{ij}^l with $i < j < n$ and $l \in \omega$, with an additional shade of red ρ such that the consistency conditions for the new reds (in addition to the usual rainbow consistency conditions) are as follows:

- $(r_{jk}^i, r_{j'k'}^i, r_{j^*k^*}^{i^*})$ unless $i = i' = i^*$ and $|\{(j,k), (j',k'), (j^*,k^*)\}| = 3$
- (r, ρ, ρ) and $(\mathsf{r}, \mathsf{r}^*, \rho)$, where r, r^* are any reds.

The consistency conditions can be coded in an $L_{\omega,\omega}$ theory T having signture the reds with ρ together with all other colours like in [11, Definition

3.6.9]. The theory T is only a first order theory (not an $L_{\omega_1,\omega}$ theory) because the number of greens is finite which is not the case with [11] where the number of available greens are countably infinite coded by an infinite disjunction. One construct an n-homogeneous model M is as a countable limit of finite models of T using a game played between $\exists$ and $\forall$like in [14, Theorem 2.16]. In the rainbow game $\forall$ challenges $\exists$ with *cones* having green *tints* (g_0^i), and $\exists$ wins if she can respond to such moves. This is the only way that $\forall$ can force a win. $\exists$ has to respond by labelling *appexes* of two succesive cones, having the *same base* played by $\forall$. By the rules of the game, she has to use a red label. She resorts to ρ whenever she is forced a red while using the rainbow reds will lead to an inconsistent triangle of reds; [14, Proposition 2.6, Lemma 2.7]. The number of greens make [14, Lemma 3.10] work with the same proof using only finitely many green and not infinitely many. The winning strategy implemented by $\exists$ using the red label ρ that comes to her rescue whenever she runs out of 'rainbow reds', so she can always and consistently respond with an extended coloured graph.

2. **Representing a term algebra (and its completion) as (generalized) set algebras**: Having M at hand, one constructs two atomic n–dimensional set algebras based on M, sharing the same atom structure and having the same top element. The atoms of each will be the set of coloured graphs, seeing as how, quoting Hodkinson [14] such coloured graphs are 'literally indivisible'. Now L_n and $L^n_{\infty,\omega}$ are taken in the rainbow signature (without ρ). Continuing like in *op.cit*, deleting the one available red shade, set $W = \{\bar{a} \in {}^n\mathsf{M} : \mathsf{M} \models (\bigwedge_{i<j<n} \neg\rho(x_i, x_j))(\bar{a})\}$, and for $\phi \in L^n_{\infty,\omega}$, let $\phi^W = \{s \in W : \mathsf{M} \models_W \phi[s]\}$. Here W is the set of all n–ary assignments in ${}^n\mathsf{M}$, that have no edge labelled by ρ and $\models_W$ is first order emantics with quantifiers relativized to W, cf. [14, §3.2 and Definition 4.1]. Let $\mathfrak{A}$ be the relativized set algebra with domain $\{\varphi^W : \varphi$ a first-order L_n – formula$\}$ and unit W, endowed with the usual concrete cylindric operations read off the connectives. Classical semantics for L_n *rainbow formulas* and their semantics by relativizing to W coincide [14, Proposition 3.13] *but not with respect to $L^n_{\infty,\omega}$ rainbow formulas.* Hence the set algebra $\mathfrak{A}$ is isomorphic to a cylinric set algebra of dimension n having top element ${}^n\mathsf{M}$, so $\mathfrak{A}$ is simple, in fact its Df reduct is simple. Let $\mathfrak{E} = \{\phi^W : \phi \in L^n_{\infty,\omega}\}$ [14, Definition 4.1] with the operations defined like on $\mathfrak{A}$ the usual way. $\mathfrak{Cm}\mathbf{At}$ is a complete CA_n and, so like in [14, Lemma 5.3] we have an isomorphism from $\mathfrak{Cm}\mathbf{At}$ to $\mathfrak{E}$ defined via $X \mapsto \bigcup X$. Since $\mathsf{At}\mathfrak{A} = \mathsf{At}\mathfrak{Tm}(\mathsf{At}\mathfrak{A})$, which we refer to

only by $\mathbf{At}$, and $\mathfrak{Tm}\mathsf{At}\mathfrak{A} \subseteq \mathfrak{A}$, hence $\mathfrak{Tm}\mathsf{At}\mathfrak{A} = \mathfrak{Tm}\mathbf{At}$ is representable. The atoms of $\mathfrak{A}$, $\mathfrak{Tm}\mathsf{At}\mathfrak{A}$ and $\mathfrak{Cm}\mathsf{At}\mathfrak{A} = \mathfrak{Cm}\mathbf{At}$ are the coloured graphs whose edges are *not labelled* by ρ. These atoms are uniquely determined by the interpretion in M of so-called MCA formulas in the rainbow signature of $\mathbf{At}$ as in [14, Definition 4.3].

3. **Embedding $\mathfrak{B}_f$ into $\mathfrak{Cm}(\mathbf{At})$**: Let CRG_f be the class of coloured graphs on $\mathbf{At}_f$ and CRG be the class of coloured graph on $\mathbf{At}$. We can (and will) assume that $\mathsf{CRG}_f \subseteq \mathsf{CRG}$. Write M_a for the atom that is the (equivalence class of the) surjection $a : n \to M$, $M \in \mathsf{CGR}$. Here we identify a with $[a]$; no harm will ensue. We define the (equivalence) relation $\sim$ on $\mathbf{At}$ by $M_b \sim N_a$, $(M, N \in \mathsf{CGR})$:

- $a(i) = a(j) \iff b(i) = b(j)$,
- $M_a(a(i), a(j)) = \mathsf{r}^l \iff N_b(b(i), b(j)) = \mathsf{r}^k$, for some $l, k \in \omega$,
- $M_a(a(i), a(j)) = N_b(b(i), b(j))$, if they are not red,
- $M_a(a(k_0), \ldots, a(k_{n-2})) = N_b(b(k_0), \ldots, b(k_{n-2}))$, whenever defined.

We say that M_a is a *copy of* N_b if $M_a \sim N_b$ (by symmetry N_b is a copy of M_a.) Indeed, the relation 'copy of' is an equivalence relation on $\mathbf{At}$. An atom M_a is called a *red atom*, if M_a has at least one red edge. Any red atom has ω many copies, that are *cylindrically equivalent*, in the sense that, if $N_a \sim M_b$ with one (equivalently both) red, with $a : n \to N$ and $b : n \to M$, then we can assume that $\mathsf{nodes}(N) = \mathsf{nodes}(M)$ and that for all $i < n$, $a \upharpoonright n \sim \{i\} = b \upharpoonright n \sim \{i\}$. In $\mathfrak{Cm}\mathbf{At}$, we write M_a for $\{M_a\}$ and we denote suprema taken in $\mathfrak{Cm}\mathbf{At}$, possibly finite, by $\sum$. Define the map Θ from $\mathfrak{A}_{n+1,n} = \mathfrak{Cm}\mathbf{At}_f$ to $\mathfrak{Cm}\mathbf{At}$, by specifing first its values on $\mathbf{At}_f$, via $M_a \mapsto \sum_j M_a^{(j)}$ where $M_a^{(j)}$ is a copy of M_a. So each atom maps to the suprema of its copies. This map is well-defined because $\mathfrak{Cm}\mathbf{At}$ is complete. We check that Θ is an injective homomorphim. Injectivity is easy. We check preservation of all the CA_n extra Boolean operations.

- Diagonal elements. Let $l < k < n$. Then:

$$\begin{aligned}
M_x \leq \Theta(\mathsf{d}_{lk}^{\mathfrak{Cm}\mathbf{At}_f}) &\iff M_x \leq \sum_j \bigcup_{a_l = a_k} M_a^{(j)} \\
&\iff M_x \leq \bigcup_{a_l = a_k} \sum_j M_a^{(j)} \\
&\iff M_x = M_a^{(j)} \text{ for some } a : n \to M \text{ such that } a(l) = a(k) \\
&\iff M_x \in \mathsf{d}_{lk}^{\mathfrak{Cm}\mathbf{At}}.
\end{aligned}$$

- Cylindrifiers. Let $i < n$. By additivity of cylindrifiers, we restrict our attention to atoms $M_a \in \mathbf{At}_f$ with $a : n \to M$, and $M \in \mathsf{CRG}_f \subseteq \mathsf{CRG}$. Then:

$$\Theta(\mathsf{c}_i^{\mathfrak{Cm}\mathbf{At}_f} M_a) = f(\bigcup_{[c] \equiv_i [a]} M_c) = \bigcup_{[c] \equiv_i [a]} \Theta(M_c)$$

$$= \bigcup_{[c] \equiv_i [a]} \sum_j M_c^{(j)} = \sum_j \bigcup_{[c] \equiv_i [a]} M_c^{(j)} = \sum_j \mathsf{c}_i^{\mathfrak{Cm}\mathbf{At}} M_a^{(j)}$$

$$= \mathsf{c}_i^{\mathfrak{Cm}\mathbf{At}} (\sum_j M_a^{(j)}) = \mathsf{c}_i^{\mathfrak{Cm}\mathbf{At}} \Theta(M_a).$$

- Substitutions: Let $i, k < n$. By additivity of the $\mathsf{s}_{[i,k]}$s, we again restrict ourselves to atoms of the form M_a as specified in the previous items. Now computing we get: $\Theta(\mathsf{s}_{[i,k]}^{\mathfrak{Cm}\mathbf{At}_f} M_a) = \Theta(M_{a \circ [i,k]}) = \sum_j^{\mathfrak{Cm}\mathbf{At}} (M_{a \circ [i,k]}^{(j)}) = \sum_j \mathsf{s}_{[i,k]}^{\mathfrak{Cm}\mathbf{At}} M_a^{(j)} = \mathsf{s}_{[i,k]}^{\mathfrak{Cm}\mathbf{At}} (\sum_j M_a^{(j)}) = \mathsf{s}_{[i,k]}^{\mathfrak{Cm}\mathbf{At}} \Theta(M_a)$.

4. $\forall$ **has a winning strategy in** $G^{t(n)}\mathsf{At}(Rd\mathfrak{B}_f)$**; and the required result**: It is straightforward to show that $\forall$ has winning strategy first in the Ehrenfeucht–Fraïssé forth private game played between $\exists$ and $\forall$ on the complete irreflexive graphs $n(n-1)/2 + 2)$ and n in $n(n-1)/2 + 2$ rounds $\mathsf{EF}_{n(n-1)+2}^{n(n-1)2+2}(n+1, n)$ [11, Definition 16.2] since $n(n-1)/2 + 2$ is 'longer' than n. Using (any) $p > n$ many pairs of pebbles avalable on the board $\forall$ can win this game in $n + 1$ many rounds. For brevity, write $\mathfrak{D} \in \mathsf{Sc}_n$

instead of $\mathfrak{Rd}_{\mathsf{Sc}}\mathfrak{B}$. $\forall$ lifts his winning strategy from the last private Ehrenfeucht–Fraïssé forth game to the graph game on $\mathbf{At}_f = \mathsf{At}(\mathfrak{D})$ [9, p. 841] forcing a win using $t(n)$ nodes. One uses the $n(n-1)/2+2$ green relations in the usual way to force a red clique C, say with $n(n-1)/2+2$. Pick any point $x \in C$. Then there are $> n(n-1)/2$ points y in $C \setminus \{x\}$. There are only $n(n-1)/2$ red relations. So there must be distinct $y, z \in C \setminus \{x\}$ such that (x, y) and (x, z) both have the same red label (it will be some r_{ij}^m for $i < j < n$). But (y, z) is also red, and this contradicts (1.3) above. In more detail, $\forall$ bombards $\exists$ with cones having common base and distinct green tints until $\exists$ is forced to play an inconsistent red triangle (where indicies of reds do not match). He needs $n-1$ nodes as the base of cones, plus $|P|+2$ more nodes, where $P = \{(i, j) : i < j < n\}$ forming a red clique, triangle with two edges satisfying the same r_p^m for $p \in P$. Calculating, we get $t(n) = n-1+n(n-1)/2+2 = n(n+1)/2+1$. By Lemma 2.5, $\mathfrak{D} \notin \mathbf{S}_c\mathsf{Nr}_n\mathsf{Sc}_{t(n)}^{\mathsf{ad}}$ when $2 < n < \omega$). Since $\mathfrak{D}$ is finite, then $\mathfrak{D} \notin \mathbf{S}\mathsf{Nr}_n\mathsf{Sc}_{t(n)}$, because $\mathfrak{D}$ coincides with its canonical extension and for any $\mathfrak{D} \in \mathsf{Sc}_n$, $\mathfrak{D} \in \mathbf{S}\mathsf{Nr}_n\mathsf{Sc}_{t(n)} \implies \mathfrak{D}^+ \in \mathbf{S}_c\mathsf{Nr}_n\mathsf{Sc}_{t(n)}$. To see why, we could omit the superscrpt ad, abbreviating additivity, assume that $\mathfrak{D} \subseteq \mathfrak{Nr}_n\mathfrak{E}^{\mathsf{ad}}$, $\mathfrak{E} \in \mathsf{Sc}_{n+3}$. Let $\mathfrak{E}' = \mathfrak{Sg}^{\mathfrak{E}}\mathfrak{D}$, then $\mathfrak{E}'$ is finite, hence completely additive and $\mathfrak{D} \subseteq \mathfrak{Nr}_n\mathfrak{E}'$. But $\mathfrak{B}_f$ embeds into $\mathfrak{CmAt}\mathfrak{A}$, hence $\mathfrak{Rd}_{sc}\mathfrak{CmAt}\mathfrak{A}$ is outside the variety $\mathbf{S}\mathsf{Nr}_n\mathsf{Sc}_{t(n)}$, as well. Since $\mathfrak{Rd}_{sc}\mathfrak{A}$ is completely additive because it is a reduct of a QEA_n, then $\mathfrak{CmAt}\mathfrak{Rd}_{Sc}\mathfrak{A}$ is the Dedekind–MacNeille completion of $\mathfrak{Rd}_{sc}\mathfrak{A}$. By Lemma 5.4, the required follows. But $\mathfrak{D}$ embeds into $\mathfrak{Rd}_{sc}\mathfrak{CmAt}\mathfrak{A}$, hence $\mathfrak{CmAt}\mathfrak{Rd}_{sc}\mathfrak{A}$ is outside the variety $\mathbf{S}\mathsf{Nr}_n\mathsf{Sc}_{t(n)}$, as well.

Now we prove the last item, namely, that RDf_n is not atom-canonical. Using essentially the argument in [7, Lemma 5.1.50, Theorem 5.1.51] by considering closure under infinite intersections instead of intersections, it is enough to show that $\mathfrak{CmAt}\mathfrak{A}$ is generated by elements whose dimension sets have cardinality $< n$ using infinite unions. We show that for any rainbow atom $[a]$, $a : n \to \Gamma$, Γ a coloured graph, that $[a] = \prod_{i<n} \mathsf{c}_i[a]$. Clearly $\leq$ holds. Assume that $b : n \to \Delta$, Δ a coloured graph, and $[a] \neq [b]$. We show that $[b] \notin \prod_{i<n} \mathsf{c}_i[a]$ by which we will be done. Because a is not equivalent to b, we have one of two possibilities; either $(\exists i, j < n)(\Delta(b(i), b(j) \neq \Gamma(a(i), a(j))$ or $(\exists i_1, \ldots, i_{n-1} < n)(\Delta(b_{i_1}, \ldots, b_{i_{n-1}}) \neq \Gamma(a_{i_1}, \ldots, a_{i_{n-1}}))$. Assume the first possibility (the second is similar): Choose $k \notin \{i, j\}$. This is possible because $n > 2$. Assume for contradiction that $[b] \in \mathsf{c}_k[a]$. Then

$(\forall i,j\in n\setminus\{k\})(\Delta(b(i),b(j))=\Gamma(a(i)a(j)))$. By assumption and the choice of k, $(\exists i,j\in n\setminus k)(\Delta(b(i),b(j))\neq\Gamma(a(i),a(j)))$, contradiction. $\square$

COROLLARY 5.6. Let $2<n<\omega$, and let $t(n)=n(n+1)/2+1$ and V be any variety between Sc and QEA. Then the following propsitions 1, 2, 3 and 4 are valid:

1. There exists an algebra outside $\mathbf{S}\mathsf{Nr}_n\mathsf{V}_{t(n)}$ with a representable dense subalgebra
2. There exists a countable atomic algebra $\mathfrak{A}\in\mathsf{V}_n$ that is not strongly representable up to $t(n)$.
3. The varieties $\mathbf{S}\mathsf{Nr}_n\mathsf{V}_m$ for any $m\geq t(n)$ are not atom-canonical, *a fortiori* are not closed under Dedekind–MacNeille completions
4. There is an atom structure $\mathbf{At}$ such that $\mathfrak{Tm}\mathbf{At}\in\mathsf{RV}_n$ and $\mathfrak{Cm}\mathbf{At}\notin\mathbf{S}\mathsf{Nr}_n\mathsf{V}_{t(n)}$.

For a class $\mathbf{K}$ of BAOs, let $\mathbf{K}\cap\mathsf{Count}$ denote the class of atomic algebras in $\mathbf{K}$ having countably many atoms.

PROPOSITION 5.7. Let $2<n<\omega$. The following propositions 1,2, and 3 below are valid:

1. For any ordinal $0\leq j$, $\mathsf{RCA}_n\cap\mathsf{Nr}_n\mathsf{CA}_{n+j}\cap\mathsf{Count}$ is not atom-canonical with respect to RCA_n if and only if $j<\omega$,
2. For any ordinal j, $\mathsf{Nr}_n\mathsf{CA}_{n+j}\cap\mathsf{RCA}_n\cap\mathbf{At}\nsubseteq\mathsf{CRCA}_n$,
3. There exists an atomic RCA_n such that its Dedekind–MacNeille (minimal) completion does not embed into its canonical extension.[2]

PROOF: 1. One implication follows from [2] where for each $2<n<l<\omega$ an algebra $\mathfrak{A}_l\in\mathsf{RCA}_n\cap\mathsf{Nr}_n\mathsf{CA}_l$ is constructed such that $\mathfrak{Cm}\mathsf{At}\mathfrak{A}_l\notin\mathsf{RCA}_n$, so $\mathfrak{A}_l$ cannot be completely representable. Conversely, for any infinite ordinal j, $\mathsf{Nr}_n\mathsf{CA}_{n+j}=\mathsf{Nr}_n\mathsf{CA}_\omega$ and if $\mathfrak{A}\in\mathsf{Nr}_n\mathsf{CA}_\omega\cap\mathsf{Count}$, then by [24, Theorem 5.3.6], $\mathfrak{A}\in\mathsf{CRCA}_n$, so $\mathfrak{Cm}\mathsf{At}\mathfrak{A}\in\mathsf{RCA}_n$.

[2] In the CA context, the terminology *minimal completion* is misleading because $\mathfrak{A}^+$ is *another completion* of $\mathfrak{A}$; so supposedly the minimal completion of $\mathfrak{A}$ should embed into $\mathfrak{A}^+$, which is not, as we have already seen in Theorem 5.5, always true. Conversely, for an atomic Boolean algebra $\mathfrak{B}$, $\mathfrak{Cm}\mathsf{At}\mathfrak{B}$ always embeds into $\mathfrak{B}^+$ as it should.

2. The case $j < \omega$, follows from the fact that the algebra $\mathfrak{A}_{n+j}$ used in the previous item is in $\mathsf{Nr}_n\mathsf{CA}_{n+j} \cap \mathsf{RCA}_n$ but has no complete representation. For infinite j one uses the construction in Theorem 2.6.

3. Let $\mathfrak{A} = \mathfrak{TmAt}$ be the CA_n as defined in the proof of Theorem 5.5. Since $\mathfrak{Cm}\mathsf{At}\mathfrak{A} \notin \mathsf{RCA}_n$, it does not embed into $\mathfrak{A}^+$, because $\mathfrak{A}^+ \in \mathsf{RCA}_n$ since $\mathfrak{A} \in \mathsf{RCA}_n$ and RCA_n is a canonical variety. $\square$

The strongest result on first order definability is proved by the present author where it is shown that for any class K such that $\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n \subseteq \mathsf{K} \subseteq \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_{n+3}$, we have K is not elementary. This generalizes to any V between Sc_n and QEA_n. For more on connections between atom-canonicity, complete representations with repercussions on omitting types theorems for modal fragments of $L_{\omega,\omega}$, the reader is referred to [29, 28, 23].

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AN ARITHMETICALLY COMPLETE PREDICATE MODAL LOGIC

Abstract

This paper investigates a first-order extension of GL called ML^3. We outline briefly the history that led to ML^3, its key properties and some of its toolbox: the *conservation theorem*, its cut-free Gentzenisation, the "formulators" tool. Its semantic completeness (with respect to finite reverse well-founded Kripke models) is fully stated in the current paper and the proof is retold here. Applying the Solovay technique to those models the present paper establishes its main result, namely, that ML^3 is arithmetically complete. As expanded below, ML^3 is a first-order modal logic that along with its built-in ability to simulate general classical first-order provability—"$\Box$" simulating the the informal classical "$\vdash$"—is also arithmetically complete in the Solovay sense.

Keywords: Predicate modal logic, arithmetical completeness, logic GL, Solovay's theorem, equational proofs.

1. Introduction

Solovay introduced in [23] the propositional provability logic GL (Gödel-Łöb logic) and proved that it is *arithmetically complete*, meaning that any GL formula is a theorem of GL if all its *arithmetical interpretations* are provable in Peano Arithmetic (PA). This particular version of completeness gives GL the name *provability logic* since it models the behaviour of *provability in PA*.

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There has been a lot of interest in discovering first-order provability logics (cf. [3]). The obvious idea seemed to be defining *extensionally* a "Quantified GL"—or QGL_1[1]—as the set of *theorems* P below, over a first-order *classical* language augmented by the modal $\Box$ that has the property that $\Box A$ has the same free variables as A for all formulae A.

$$QGL_1 = \{P : \vdash_{PA} f(P) \text{ for } \textit{every} \text{ arithmetical interpretation } f\} \quad (1.1)$$

Vardanyan however showed ([29]) that this first-order logic is not recursively axiomatisable; in fact he proved a stronger result: The QGL_1 of (1.1) is Π^0_2-complete.

Thus the idea of taking "QGL" extensionally failed badly as we cannot make it into a tangible axiomatic system that is *usable*.

Another "QGL" was built "forward" rather than "backward", namely, as an already (recursively) axiomatised first-order extension of GL over the same language as (1.1), with the same behaviour of $\Box$ vis a vis free variables, as that of QGL_1.

This *intentional* logic QGL, being necessarily *different* from the QGL_1 above in view of Vardanyan's result, has a minimal (finite) set of modal axioms added on top of the usual first-order classical axioms (cf. [2, 17, 26]). It turns out that this QGL has shortcomings too:

- It has no cut-free *Gentzenisation*, i.e., no cut-free Gentzen-equivalent logic (cf. [2]).

- It is *not complete* with respect to *any* class of Kripke frames and it *is not arithmetically complete*, both of the last two negative results being due to Montagna [17].

In [1, 12] a first-order QGL-like extension of GL is investigated and proved to be arithmetically complete. *However*, while on one hand $\Box$ was still "transparent" to free variables, on the other the "finite" Kripke models were overly restrictive: The *domains* of each world were required to be *finite* as well and to satisfy certain inclusion relations at that.

Later on, Yavorsky [30] modified QGL into QGL^b, where this time the modal operator $\Box$ binds all free variables in a formula making them invisible: every $\Box A$ is a sentence. QGL^b is recursively axiomatised and

[1] There is a QGL_2—our numbering meaning to distinguish the two—that we will simply call QGL.

its primary rules are modus ponens, *strong generalisation* $A \vdash (\forall x)A$ and *strong necessitation* $A \vdash \Box A$. He proved that it is arithmetically complete.

A closely related first-order logic is the ML^3 of [19], with essentially the same set of axioms and also with an "opaque" $\Box$, but for technical reasons this has only the first two of the above rules as primary, "hiding" necessitation in the axioms in the style of [22], thus, ML^3 has an *admissible* rule of *weak* necessitation instead: If $\vdash A$ then $\vdash \Box A$.

It has been long understood by the research community working on provability predicate modal logics that the failure of the attempts to obtain a first-order recursively axiomatised provability logic was due to the insistence on having a "transparent" $\Box$.

Indeed, the concluding remarks in [30] note the detrimental effects of a "transparent" $\Box$ on arithmetical completeness. Thus, while the earlier paper of [1] obtained arithmetical completeness of a predicate modal logic *with* a transparent $\Box$ it did so *on the condition that* such a logic had severely restricted finite Kripke models ("finite" applying to the *domains* of said models as well).

Yavorski [30] successfully experimented with an *opaque* $\Box$ and with the restricted Barkan formula

$$\Box A \to \Box(\forall x)A \tag{1.2}$$

as one of his axioms and showed that QGL^b is a first-order provability logic. His paper does not explain the significance of the choice of (1.2) (see however [27, 28, 26] who chose this axiom for reasons totally unrelated to arithmetical comleteness).

Through a different route, with some interesting intermediate stops, [27, 28, 19] arrived at the logic ML^3 that is the focus of the present paper, while [26] further explored the significance of axiom (1.2) in ML^3 and M^3, in particular proving

- It is independent from the other axioms
- If removed, the resulting logics are arithmetically *incomplete*.

Thus, all other axioms being left *as is*, (1.2) is *essential* for arithmetical completeness.

ML^3 has an interesting and consistent history. [27, 28] introduced M^3 in response to a problem stated in [9]. The authors of the latter noted that formal (classical) equational proofs

$$A_1 \Leftrightarrow {}^{2} A_2 \Leftrightarrow \ldots \Leftrightarrow A_i \qquad \overbrace{\vdash\dashv}^{\text{metatheoretical step}} \qquad (\forall x)A_i \Leftrightarrow \ldots \Leftrightarrow A_n$$

must be necessarily *disconnected* at the step above where we want to state "A_i iff $(\forall x)A_i$". The step is metatheoretical because the formal $A_i \Leftrightarrow (\forall x)A_i$ is invalid, in particular $A_i \to (\forall x)A_i$ is. Thus [9] asked: *Given that* $\vdash A_i$ *iff* $\vdash (\forall x)A_i$ *holds in the metatheory, can we recast the equational proof above* within modal logic *like this*

$$\Box A_1 \Leftrightarrow \Box A_2 \Leftrightarrow \ldots \Leftrightarrow \Box A_i \Leftrightarrow \Box(\forall x)A_i \Leftrightarrow \ldots \Leftrightarrow \Box A_n$$

where $\Box$ *means classical provability* ($\vdash$), *and thus make all classical equational proofs so translated both formal (within modal logic) and also* connected?

[27, 28] answered this question affirmatively, building the first-order modal logic M^3 and proving *semantically* (via Kripke models) their *conservation theorem* which, essentially, states

$$A \vdash B \text{ classically iff } \vdash \Box A \to \Box B \text{ modally} \tag{1.3}$$

M^3 is a first-order extension of the propositional modal logic K4, and was introduced to satisfy (1.3), that is, to be a "provability logic" for *pure classical predicate logic* rather than for PA. Such a provability logic is especially useful in the practice of *equational proofs* of [4, 8, 25].

[27, 28] and the related [13] contain several examples of disconnected classical equational proofs that (1.3) helps to convert into *connected modal translations* of the former proofs.

There were two key design criteria for M^3:

- $\Box$ in M^3 (and later ML^3) *has* to be opaque, that is, $\Box A$ is closed for all formulae A, since for classical first-order strong generalisation logic (cf. [16, 21, 24]) we have $A \vdash (\forall x)A$. In the words of [27, 28],

> *The motivation regarding [free] object variables [in $\Box A$] is our intended intuitive interpretation of $\Box$ as the classical $\vdash$, and therefore as the classical $\models$ as well. When we say "$\models A$" classically, we mean that for all structures where*

[2] Conjunctional formal equivalence. That is, $A \Leftrightarrow B \Leftrightarrow C$ is defined to mean $A \equiv B$ *and* $B \equiv C$.

> *we interpret A, and for all value-assignments to the free object variables of A, the formula is true. Thus the variables in a statement such as "$\models A$" are implicitly universally quantified and are unavailable for substitutions.*

- We have strong generalisation in M^3 (and ML^3), that is $A \vdash (\forall x)A$, and thus we *must* have, by (1.3), the special case $\Box A \to \Box(\forall x)A$—the (1.2) above—as a (modal) theorem. The easy approach to *have* this special case as a theorem was to adopt it as an axiom. It was not known to the authors of [27, 28, 13, 19] at the time whether (1.2) was independent of the remaining axioms. This was established to be the case by one of the authors later [26].

Thus the above (original) interpretation of $\Box$ in M^3 and its extension ML^3 is totally different from the interpretation of the $\Box$ in GL. The box operator of GL is interpreted *arithmetically* as, essentially, $\Theta(x)$, defined below in this paragraph. The interpretation mapping is usually denoted by *. Thus, by induction on the formation of GL formulae, atomic formulae A of GL are mapped to arbitrarily chosen sentences A^* of PA. For the induction step * *commutes* with $\neg$ and $\wedge$, that is, $(\neg A)^*$ is $\neg A^*$ and $(A \wedge B)^*$ is $A^* \wedge B^*$. Finally, $(\Box A)^*$ is interpreted as $\Theta(\ulcorner A^* \urcorner)$—which says "$A^*$" is a PA-theorem—where $\ulcorner X \urcorner$ denotes the Gödel number of X [22, 7]. The Σ_1-formula $\Theta(x)$ stands for $(\exists y)Pr(y, x)$ where $Pr(y, x)$ is true iff the Gödel number y codes a PA-proof of the formula with Gödel number x. Thus $\Theta(x)$ is true iff x is the Gödel number of a theorem of PA.

The logic ML^3 was introduced in [19], adding Löb's axiom $\Box(\Box A \to A) \to \Box A$ to M^3,[3] thus it is a first-order extension of both GL and M^3, and hence can (provably) simulate classical provability $\vdash$ through $\Box$ as well. ML^3 is over the same language as its predecessor M^3, and in particular, $\Box A$ is closed for all A.

[18, 19] developed the proof theory for M^3 and ML^3 by devising cut-free Gentzenisations of each, called GTKS and GLTS respectively. They gave completely detailed proofs of the admissibility of cut in each logic. Using a Gentzen logic as a proxy to study the proof theory of some Hilbert-style logic is a well-known methodology that profits from the *subformula property* of cut-free Gentzen proofs.

[3] While Łöb's axiom can prove the axiom $\Box A \to \Box\Box A$ of M^3, we will retain it here for technical convenience as was done in [19].

In fact one of the results in the aforementioned references was a proof-theoretic (syntactic) proof of (1.3).

We also note in this historical review that [6] devised significantly shorter proofs than those in [18, 19] for the admissibility of cut in each of M^3 and ML^3.

[20] introduced certain formula to formula mappings named *formulators* (*formu*la trans*lators*). Such mappings preserve proofs in logics such as M^3, ML^3, and QGL, that is, if $\Gamma \vdash A$ holds in any one of these logics, then for any well-chosen formulator $\mathfrak{F}$ in each case we can have $\mathfrak{F}(\Gamma) \vdash \mathfrak{F}(A)$. The formulators tool allows one to do metamathematical investigations directly on Hilbert-style proofs without Gentzenisation, bypassing messy cut elimination proofs. Even for QGL, a logic that *provably* does *not* admit cut elimination ([2]), the formulators tool was applied profitably ([20, 26]).

For completeness sake, here is the definition of a formulator mapping $\mathfrak{F}$:

Definition 1.1 (Formulators [20, 26]). A *formula translator* or *formulator* is a mapping, $\mathfrak{F}$, from the set of formulae over a modal language L to itself such that:

1. $\mathfrak{F}(A) = A$ for every *atomic* formula A.

2. $\mathfrak{F}(A \to B) = \mathfrak{F}(A) \to \mathfrak{F}(B)$ for all formulae A, B.

3. $\mathfrak{F}((\forall x)A[x]) = (\forall x)B[x]$, where $B[a] = \mathfrak{F}(A[a])$.

4. The free variables of $\mathfrak{F}(\Box A)$ are among those of $\Box A$.

   Thus $\mathfrak{F}(\Box A)$ can be almost anything, subject to the restriction stated. □

[19] proved the completeness of ML^3 with respect to *finite* reverse well-founded Kripke models, and also its *arithmetical soundness*. Because of this, and looking back at Solovay's proof [23] which heavily hinges on such finite Kripke models, the authors conjectured the arithmetical completeness of ML^3 in the conclusions section (cf. also the introduction section of [26]).

The present paper proves this conjecture, adapting the idea from [30] to work with a finite consistent *extension* of PA rather than PA itself.

Thus ML^3 is a new example of a *predicate provability logic* that can also simulate equational classical proofs.

2. Language and symbols

We will not go over the well known inductive definition of formulae over a first order alphabet (cf. [21, 24, 27]),[4] but we will note our notational conventions.

We use specific bold lower case latin letters, with or without primes or subscripts, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, $\mathbf{m}$, for arbitrary *imported* constants from $\mathbb{N}$ that we will need in the semantics section.

Formulae are denoted by capital latin letters A, B, C (with or without primes or subscripts). The formal logical connectives are $\neg$, $\wedge$, $\forall$, $\Box$. $\forall A$ denotes *the universal closure* of A, that is $(\forall x_1)(\forall x_2)\ldots(\forall x_n)A$, where the list $x_1, x_2, \ldots, x_n$ includes *all* the free variables of A. "*The*" is justified since we can reorder the quantification sequence and also eliminate repetitions without affecting either the meaning or the provability of the closure.

We call a formula $\Box A$ *boxed*. It is always a sentence (cf. [26] for the exact syntax of $\Box A$). A formula is a *classical* formula if it does not contain $\Box$, otherwise it is a *modal* formula.

The connectives $\vee$, $\rightarrow$, $\leftrightarrow$, $\exists$ are introduced *via definitions*. To reduce brackets in informal writing we assume the usual connective priorities and that they are all right-associative. $\Leftrightarrow$ is metatheoretical *conjunctional* equivalence synonymous with "iff". That is, $A \Leftrightarrow B \Leftrightarrow C$ means "$A \Leftrightarrow B$ *and* $B \Leftrightarrow C$".

Capital Greek letters (with or without primes or subscripts) that do not match a Latin capital letter, e.g., Γ, Δ, Φ, etc., denote *sets* of formulae. $\Box\Delta$ denotes $\{\Box A : A \in \Delta\}$, $\forall\Delta$ denotes $\{\forall A : A \in \Delta\}$, and $\Delta\Box$ denotes $\{\forall A : \Box A \in \Delta\}$.

We write $B[x := y]$, $B[z := \mathbf{i}]$ and $B[q := A]$ to denote substitution into targets x, z, q in B. $A(x, u, w)$ coveys that x, u, w are *all* the free variables of A while $A[x, u, w]$ conveys that x, u, w *may* be free in A. In the former case we may write $A(\mathbf{i}, u, w)$, in the latter $A[\mathbf{i}, u, w]$, to indicate the result of $A[x := \mathbf{i}]$.

[4] Note that as a technical convenience towards effecting Gentzenisation, [19] separates object variables into *free* and *bound* types. Here we follow the standard syntactic approach where bound vs. free is determined by how the variable is used syntactically.

3. The Logic ML^3

The language L of ML^3 in the present paper will have predicate symbols but no function symbols or constants. However, the language will later be augmented (cf. 4.2 and 4.5) to include *imported* constants.

DEFINITION 3.1 (Basic Axiom Schemata of ML^3).

A1 All tautologies

A2 $(\forall x)A \to A[y]$ and $(\forall x)A \to A[\mathbf{k}]$, if $\mathbf{k}$ is a *constant* (cf. 4.2 and 4.5 that refer to *imported constants*). The result $A[y]$ is undefined if "y is captured by a quantifier" as in, e.g., [24].

A3 $A \to (\forall x)A$, if x does not occur free in A

A4 $(\forall x)(A \to B) \to (\forall x)A \to (\forall x)B$

A5 $\Box(A \to B) \to \Box A \to \Box B$

A6 $\Box(\Box A \to A) \to \Box A$

A7 $\Box A \to \Box(\forall x)A$

A8 $\Box A \to \Box\Box A$.

The set of *all instances* of the schemata A1–A8 is denoted by Λ. The set of (closed) *axioms* is $\forall\Lambda \cup \Box\Lambda$. The inclusion of $\Box\Lambda$ is the "Smoryński trick" that "hides" weak necessitation in the axioms.

$\Box A \to \Box\Box A$ can be derived in ML^3 from the schema A6, but is included for convenience to avoid also adding $\Box\Box\Lambda$ to the axioms.

[19] has introduced and studied a variant of ML^3 above, *with* function and constant symbols and with equality (and its axioms) included. It is simpler—and customary ([1, 12, 30])—to discuss arithmetical completeness without these features. $\Box$

DEFINITION 3.2. The rules of inference of ML^3 are two, modus ponens (MP) $A, A \to B \vdash B$ and (strong) generalisation $A \vdash (\forall x)A$.[5]

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[5] This is equivalent to "$\Gamma \vdash A$ implies $\Gamma \vdash (\forall x)A$". *Weak* generalisation requires this Γ to contain no formula where x occurs free.

$\Gamma \vdash A$ (resp. $\vdash_\Gamma A$) in ML^3 means that A is derived from axioms and hypotheses Γ (resp. hypotheses $\Gamma \cup \Box\Gamma$). Note that in a classical proof system $\vdash_\Gamma A$ means the same as $\Gamma \vdash A$. $\Box$

Unlike QGL^b where necessitation is postulated as a *strong* primary rule $A \vdash \Box A$, in ML^3 *weak* necessitation is admissible (cf. [27, 19, 26]).

Remark 3.3 (Tautological implication). One writes $A_1, A_2, \ldots, A_n \models_{taut} B$ pronounced "the $A_1, A_2, \ldots, A_n$ tautologically imply B". This means that $A_1 \to A_2 \to \ldots \to A_n \to B$ is a tautology, in symbols, $\models_{taut} A_1 \to A_2 \to \ldots \to A_n \to B$. $\Box$

Axiom group A1 immediately implies

Theorem 3.4 (Proof by tautological implication). *If $A_1, A_2, \ldots, A_n \models_{taut} B$, then $A_1, A_2, \ldots, A_n \vdash_{ML^3} B$.*

For the following see [19, 26].

Theorem 3.5 (Weak Necessitation). *If $\Gamma \vdash_{ML^3} A$, where $\Gamma = \Gamma' \cup \Box\Gamma'$ or $\Gamma = \Box\Gamma'$, then $\Gamma \vdash_{ML^3} \Box A$.*

4. Kripke semantics

Kripke's possible worlds semantics [15] is the standard model theoretic approach to modal logic.

Definition 4.1 (Kripke Frames). A *Kripke frame* is a pair $\mathcal{F} = \langle W, R\rangle$ where W is a non-empty set of *(possible) worlds* and R is a binary relation on W known as the *accessibility relation.*

We are interested in frames where R is transitive, irreflexive and *reverse well-founded* the latter meaning that there is no infinite R-chain $w' R w'' R w''' \ldots$ $\Box$

Definition 4.2 (Pointed Kripke Frames). $\mathcal{F} = \langle W, R, w_0\rangle$ is a *pointed Kripke frame* if $\langle W, R\rangle$ is a Kripke frame and $w_0 \in W$ is a designated "initial" world. w_0 is selected to be *R-minimum*, called *the minimum node*, that is, $(\forall w \in W)(w = w_0 \vee w_0 R w)$. $\Box$

Definition 4.3 (Primary Interpretation Mapping). Let L be a modal language, and let M_w be a non-empty countable set of objects, for each $w \in W$.

I_w is an *interpretation* that *maps* the elements of L to the "concrete" domain M_w. It suffices to take each M_w to be enumerable since so is our alphabet and thus we take $M_w = \mathbb{N}$, for all $w \in W$. The I_w have the properties:

1. $I_w(q) \in \{\mathbf{t}, \mathbf{f}\}$ for every Boolean variable $q \in L$.
2. $I_w(\bot) = \mathbf{f}$ and $I_w(\top) = \mathbf{t}$.
3. $I_w(\phi) \subseteq \mathbb{N}^n$ for every predicate letter $\phi \in L$ of arity $n > 0$. □

We want a Henkin theory for L so rather than *assigning* (constant) values to variables we will *copy* values into variables. Values being *metalogical*, the Henkin trick is to import them into the language L of our logic: Every $k \in M_w$ is imported as a formal constant $\mathbf{k}$. The resulting language is denoted by $L(M_w)$ ([21, 24]).

Definition 4.4. If $A(x_1, \ldots, x_n)$ is over L, then $A(\mathbf{k_1}, \ldots, \mathbf{k_n})$ over $L(\mathbb{N})$ is a sentence with *parameters* from $\mathbb{N}$. □

The extended mapping for all closed formulae with parameters from M_w is defined as follows:

Definition 4.5 (Extended Interpretation; forcing truth in a world.). Firstly, we interpret all the imported constants of $L(\mathbb{N})$:
$I_w(\mathbf{k}) = k \in \mathbb{N}$, for each $\mathbf{k} \in L(\mathbb{N})$.
Next, by induction on *closed* formulae of $L(\mathbb{N})$, for every $w \in W$:

1. $I_w(\phi(\mathbf{k_1}, \ldots, \mathbf{k_n})) = \mathbf{t}$ iff $I_w(\phi)(k_1, \ldots, k_n) = \mathbf{t}$, for any n-ary predicate $\phi \in L$, where the k_i are in $\mathbb{N}$.
2. $I_w(\neg A) = \mathbf{t}$ iff $I_w(A) = \mathbf{f}$ for any closed formula A of $L(\mathbb{N})$.
3. $I_w(A \wedge B) = \mathbf{t}$ iff $I_w(A) = \mathbf{t}$ and $I_w(B) = \mathbf{t}$, for any closed formulae A and B of $L(\mathbb{N})$.
4. $I_w((\forall x)A) = \mathbf{t}$ iff $I_w(A[x := \mathbf{k}]) = \mathbf{t}$ for all $k \in \mathbb{N}$, where $(\forall x)A$ is a sentence of $L(\mathbb{N})$.
5. $I_w(\Box A) = \mathbf{t}$ iff, for all w' such that wRw', we have $I_{w'}(\forall A) = \mathbf{t}$, where A is a formula of $L(\mathbb{N})$, closed or not.

If a sentence A over $L(M_w)$ satisfies $I_w(A) = \mathbf{t}$, then we write $w \Vdash A$. The notation $w \Vdash A$ is pronounced "*w forces* A". □

Definition 4.6 (Kripke Structures). A Kripke structure for the modal language L is a pair $\mathcal{M} = (\mathcal{F}, \{(M_w, I_w) : w \in W\})$ where $\mathcal{F}$, M_w and I_w are defined as above. $\square$

Definition 4.7 (Truth in Kripke Models). For a modal language L and a modal formula A of L, a structure $\mathcal{M} = (\mathcal{F}, \{(M_w, I_w) : w \in W\})$ where $\mathcal{F} = (W, R, w_0)$ is a *Kripke model* of A, iff A is *true* in $\mathcal{M}$ at w_0, meaning $I_{w_0}(\forall A) = \mathbf{t}$, that is, $w_0 \Vdash \forall A$. *We can also write* $\models_{\mathcal{M}} A$ *in this case.*

We will not use the related concept of *validity* in a Kripke structure (defined as truth in *every* world) as it is equivalent to $w_0 \Vdash \Box A \wedge \forall A$.

For a modal language L and a set Γ of formulae of L, a structure $\mathcal{M}$ is a Kripke model of Γ iff $\mathcal{M}$ is a Kripke model of *every* A in Γ, written, *metatheoretically*, as $\models_{\mathcal{M}} \Gamma$.

Semantic implication of X from assumptions Γ, in symbols $\Gamma \models X$, means that every model of Γ is also a model of X; metatheoretically we may indicate this definition by "$(\forall \mathcal{M})\Big(\models_{\mathcal{M}} \Gamma$ implies $\models_{\mathcal{M}} X\Big)$". $\square$

5. Semantic completeness

This section proves the completeness of ML^3 with respect to *finite* Kripke models. It is based on the Kripke-completeness of M^3.

The *soundness* of ML^3 is proved in [19] and will be omitted. It states,

Proposition 5.1. For any given set of modal formulae Γ and any modal formula A, $\Gamma \vdash A$ implies that $\Gamma \models A$, where semantics are over finite transitive and irreflexive Kripke structures.

The Consistency Theorem [21, 22, 24] provides our first step towards proving the *Completeness* of ML^3 with respect to *finite* Kripke models.

The latter states $ML^3 \models A$ implies $ML^3 \vdash A$, where by "$ML^3 \models A$" we mean

$$(\forall \textit{ finite}, \text{irreflexive, transitive } \mathcal{M}^f)(\models_{\mathcal{M}^f} ML^3 \text{ implies } \models_{\mathcal{M}^f} A) \qquad (\ddagger)$$

It turns out that we can obtain (‡) *from the proof of the Completeness of the subtheory* M^3 *via the latter's Consistency Theorem.*

Theorem 5.2 (Consistency Theorem for a $\mathcal{T}$ over the language of M^3). *If a set of modal* sentences $\mathcal{T}$ *over the language of* M^3 *is consistent, then it has a Kripke model* $\mathcal{M}$.

Proof: ([28]) The proof in its entirety can be found in loc. cit. and we will not repeat it here. In outline, let $\mathcal{T}$ be a consistent closed modal *theory* over the language of M^3.[6] For example, if we take $\mathcal{T}$ to be (intentionally) ML^3, then $\mathcal{T} = \forall\Lambda \cup \Box\Lambda$.

Firstly, we construct (loc. cit.) a *maximal* consistent extension of $\mathcal{T}$, called a *completion* of $\mathcal{T}$, following Henkin (for the classical case cf. [21, 24]). Since the language of M^3 is enumerable it is well-known that Henkin's method will work by taking $M_w = \mathbb{N}$, for all $w \in W$, for the sought Kripke model $\mathcal{M} = (\mathcal{F}, \{(M_w, I_w) : w \in W\})$. Of course, W, w_0 and R of $\mathcal{F} = \langle W, R, w_0\rangle$ are yet to be determined.

For any such completion Γ of $\mathcal{T}$, the central lemma is the following

Lemma 5.3 (Main Semantic Lemma for M^3, [21, 24, 28]).
Let $\mathcal{T}$ be a consistent set of modal sentences *over the language of M^3, and let M be an enumerable set (in our case $\mathbb{N}$). Then there is a* completion *Γ of $\mathcal{T}$ over $L(\mathbb{N})$ such that*

(1) $\mathcal{T} \subseteq \Gamma$

(2) Γ *is consistent.*

(3) Maximality*. For any sentence A over $L(\mathbb{N})$, either A or $\neg A$ is in Γ. This implies that Γ is* deductively closed*, i.e., $\Gamma \vdash A$ implies $A \in \Gamma$. The converse trivially holds.*

(4) Henkin Property*. If Γ proves the sentence $(\exists x)A$ over $L(\mathbb{N})$, then it also proves $A[x := \mathbf{m}]$ for some $m \in \mathbb{N}$.*

Now fix any *completion* Γ of $\mathcal{T}$ and call it w_0. Let Δ denote generically any such completion. We *define* (cf. [22, 28]) a relation R on the set of all completions by

$$\Delta R \Delta' \text{ iff } \Delta\Box\,{}^{7} \subseteq \Delta'$$

This R is transitive ([22, 28, 19]). Thus we let $W = \{w_0\} \cup \{w_a : w_0 R w_a\}$, discarding all inaccessible completions. The next lemma (not proved here) is

$$\textit{For all modal sentences } B \textit{ over } L(\mathbb{N}) \textit{ we have } w_a \Vdash B \text{ iff } B \in w_a \qquad (\dagger)$$

[6] A closed theory *extensionally* is just a set of sentences; its closed theorems. *Intensionally* a theory usually is a set of rules and closed axioms intended to *generate* its set of theorems.

[7] $\Delta\Box$ is defined in Section 2.

By (†) we are done with the Consistency Theorem: If $\mathcal{T}$ is consistent, then construct $\mathcal{M}$ as above. But then, if $\mathcal{T} \vdash A$ for some sentence over L, then $w_0 \vdash A$ since $\mathcal{T} \subseteq w_0$. Thus $A \in w_0$ by deductive closure, hence $w_0 \Vdash A$ by (†). Thus $\mathcal{M}$ is a Kripke model of $\mathcal{T}$. $\Box$

We next prove in detail that

THEOREM 5.4. *ML^3 is complete for finite, irreflexive and transitive Kripke models.*

We proceed contrapositively and start here:

$$\text{Assume for the sentence } A \text{ over } L \text{ that } \mathrm{ML}^3 \nvdash A. \tag{¶}$$

By (¶), we have also $\mathrm{M}^3 \nvdash A$ since M^3 is a subtheory of ML^3. Thus by the preceding construction we have a Kripke model $\mathcal{M}$ for $\mathrm{M}^3 \cup \{\neg A\}$.

Using the "trick" of [19] below (5.8 and 5.10) we cut down the $\mathcal{M}$ model into a *finite, irreflexive, transitive* Kripke model, $\mathcal{M}^f$, of $\mathrm{M}^3 \cup \{\neg A\}$. As such $\mathcal{M}^f$ will be also *reverse well-founded* and hence also a model of ML^3 since it will satisfy also Łöb's axiom. The details follow.

Remark 5.5. Note that every modal A can be put into a provably equivalent *normal form* where in each subformula of A of the form $\Box B$ the B can be replaced by $\forall B$. This is due to $\vdash_{M^3} \Box\forall B \leftrightarrow \Box B$ and the equivalence theorem.[8] Indeed, in one direction, note $\vdash_{M^3} \Box\forall B \to \Box B$ using repeated use of axiom A2, followed by weak necessitation and then repeated application of A5. In the other direction note $\vdash_{M^3} \Box B \to \Box\forall B$ by A7 followed by repeated application of axiom A5. $\Box$

"*Adequate sets*" of formulae occur in the literature in the construction of finite Kripke models and countermodels (e.g., [12]).

DEFINITION 5.6 (Adequate set of formulae). An *adequate set of formulae* Φ satisfies

1. It is *subformula-closed*, that is, if $A \in \Phi$, then all subformulae of A are also in Φ.

2. If $A \in \Phi$, then also $\neg A$ is in Φ where we apply recursively the rule of writing X for $\neg\neg X$. $\Box$

[8] Replacing a subformula of a formula by a provably equivalent formula.

Definition 5.7. For any closed formula A in *normal form*—which without loss of generality has the form $\forall B$ for some B—over the language $L(\mathbb{N})$, the *augmemted* set of subformulae of A, denoted by $S(A)$, is the *smallest adequate set that contains* A. Why "augmented"? Because the set of subformulae of A does not necessarily meet requirement 2 above.

Note that not all formulae of $S(A)$ are closed. For example, if $(\forall x)B$ is a closed subformula of A, then B is in $S(A)$ but is *not* a *closed* subformula if $(\forall x)$ is not redundant. $\square$

Trivially, $S(A)$ is a finite set. We next define a set of worlds W^f of the under construction finite Kripke structure and the related accessibility relation $\widehat{R}$. As in [19] we use the set $S(A)$ to help us "flag" the *finite* subset W^f of worlds W that *we intend to keep*. Thus we define:

Definition 5.8. Two worlds w and w' of the Kripke model $\mathcal{M}$ (for $\mathrm{M}^3 \cup \{\neg A\}$) above are said to be *equivalent*, in symbols $w \sim w'$, iff $w \cap S(A) = w' \cap S(A)$.[9] We take w_0 as the start world in W^f and we also select exactly *one* world from each *equivalence class* $[w]_\sim$—where $w \nsim w_0$—to form a finite set of worlds W^f. Therefore the distinct worlds that we keep are the finitely many mutually non-equivalent worlds $w \in W$ as described.

To avoid confusion, if we selected $W^f = \{w_0, w_1, \ldots, w_{n-1}\}$ we *rename each such* w_i as α_i, so $W^f = \{\alpha_0, \alpha_1, \ldots, \alpha_{n-1}\}$.

(1) The accessibility relation $\widehat{R}$ on W^f is defined as follows

$\alpha \widehat{R} \beta$ iff both of the following bullets hold:

- For every subformula $\Box B$ of A, if $\Box B \in \alpha$, then $\{\Box B, \forall B\} \subseteq \beta$
- There is a subformula $\Box C$ of A in β such that $\Box C \notin \alpha$

(2) *Refine* W^f to omit redundant worlds: $W^f \overset{\text{reset}}{=} \left\{\beta : \alpha_0 \widehat{R} \beta\right\}$.

(3) Define $\alpha \Vdash_S F$ for all atomic *closed* $F \in S(A)$ to *mean* $F \in \alpha$. $\square$

Proposition 5.9. $\widehat{R}$ is reverse well-founded being (provably) irreflexive and transitive.

Proof: We verify irreflexivity and transitivity. The consequence of this—reverse well-foundedness—is well known.

[9] By its definition $\sim$ is trivially an equivalence relation.

- *Irreflexivity.* Can we have $\beta\widehat{R}\beta$? If so, then some $\Box C$ (in $S(A)$) that is in the second copy of β will not be in the first copy of β (cf. second bullet of (1) above). Absurd.

- *Transitivity.* Let $\alpha\widehat{R}\beta\widehat{R}\gamma$. To prove $\alpha\widehat{R}\gamma$ let $\Box B$ be a subformula of A and $\Box B \in \alpha$. Then $\Box B$ (and $\forall B$) is in β. But then, by assumption, $\Box B$ and $\forall B$ is in γ. To conclude we check bullet two in condition of (1) above: Let the subformula $\Box C$ of A satisfy $\Box C \in \beta$ but $\Box C \notin \alpha$. But $\beta\widehat{R}\gamma$ implies $\Box C \in \gamma$. We are done. $\Box$

Lemma 5.10. *For A, α_i and $\widehat{R}$ as defined above and, for any* closed *X that is a subformula of A, we have $\alpha_i \Vdash_S X$ iff $X \in \alpha_i$.*

Proof: This is from [19] and is provided here for easy access. Induction on the complexity of X. As in loc. cit. we define the complexity of $\forall B$ to be lower than that of $\Box B$.

1. X is an atomic sentence with parameters from $\mathbb{N}$. Done by Definition 5.8 (3).

   Two cases are more "interesting" than the others:

2. Case where X is $(\forall x)B$.

   - Say, $\alpha_i \Vdash_S (\forall x)B$, that is, $\alpha_i \Vdash_S B[x := \mathbf{k}]$ for all $k \in \mathbb{N}$. By the I.H. all the $B[x := \mathbf{k}]$ are in α_i. Now if $(\forall x)B \notin \alpha_i$ then the sentence $\neg(\forall x)B$ is in α_i by maximality of α_i; that is, $(\exists x)\neg B$ is. But then there is a Henkin witness $\mathbf{m}$ such that $\neg B[x := \mathbf{m}]$ is in α_i contradicting consistency.

   - Say $(\forall x)B \in \alpha_i$, hence $\alpha_i \vdash (\forall x)B$. By axiom A2 and MP we have $\alpha_i \vdash B[x := \mathbf{k}]$, for *all* $k \in \mathbb{N}$. By deductive closure $B[x := \mathbf{k}] \in \alpha_i$—and by the I.H. $\alpha_i \Vdash_S B[x := \mathbf{k}]$—for *all* $k \in \mathbb{N}$. By 4.5, case 5, $\alpha_i \Vdash_S (\forall x)B$.

3. Case where $X = \Box B$.

   - Suppose $\Box B \in \alpha_i$. Thus, using "$\Rightarrow$" conjunctionally (metatheoretically)

$$\begin{aligned}\Box B \in \alpha_i &\overset{5.8(1)}{\Rightarrow} (\forall \alpha_j)(\alpha_i \widehat{R} \alpha_j \to \forall B \in \alpha_j)\\ &\overset{I.H.}{\Rightarrow} (\forall \alpha_j)(\alpha_i \widehat{R} \alpha_j \to \alpha_j \Vdash_S \forall B)\\ &\overset{4.5\ 6.}{\Rightarrow} \alpha_i \Vdash_S \Box B\end{aligned}$$

- For the converse we proceed contrapositively.

  So let
  $$\Box B \notin \alpha_i \tag{5.1}$$
  Let next $T = \{\Box B, \neg\forall B\} \cup \{\Box C \in S(A) : \Box C \in \alpha_i\} \cup \{\forall C \in S(A) : \Box C \in \alpha_i\}$. We write T as
  $$T = \{\Box D_1, \forall D_1, \ldots, \Box D_m, \forall D_m, \Box B, \neg\forall B\} \tag{5.2}$$
  for some m. We claim that T *is consistent*. Proceeding by contradiction, *suppose otherwise*. Then (proof by contradiction, followed by the deduction theorem) $\Box D_1, \forall D_1, \ldots, \Box D_m, \forall D_m \vdash_{ML^3} \Box B \to \forall B$. Thus $\Box D_1, \forall D_1, \ldots, \Box D_m, \forall D_m \vdash_{ML^3} \Box B \to B$ (from $\forall B \to B$) hence $\Box D_1, \ldots, \Box D_m, \forall D_m \vdash_{ML^3} \Box(\Box B \to B)$ by weak necessitation. Now by tautological implication (via Löb's axiom) we get $\Box D_1, \forall D_1, \ldots, \Box D_m, \forall D_m \vdash_{ML^3} \Box B$, which implies $\Box B \in \alpha_i$ since α_i is deductively closed and contains the premises. We have just contradicted the main hypothesis of this bullet.
  $$\text{Let then } \alpha_j \text{ be a } \textit{completion} \text{ of the consistent } T \tag{5.3}$$
  Now, $\forall B \notin \alpha_j$ since $\neg\forall B$ is in α_j (consistency). By the I.H.,
  $$\alpha_j \nVdash_S \forall B \tag{5.4}$$
  If we can argue that we have
  $$\alpha_i \widehat{R} \alpha_j \tag{5.5}$$
  then we are done since (5.4) and (5.5) imply $\alpha_i \nVdash_S \Box B$. So let $\Box C \in \alpha_i \cap S(A)$. Then $\Box C$ and $\forall C$ are in α_j (definition of T). Being subformulae of A we have established "half" of (5.5). For the other half we have (5.1) and also need that $\Box B \in \alpha_j$. This is true by (5.2) and (5.3). □

THEOREM 5.11 ([19]). *ML^3 is complete with respect to* finite *reverse well-founded Kripke models (irreflexive and transitive).*

PROOF: To summarise, start at (¶). Then also $\mathrm{M}^3 \nvdash A$. Let $\mathcal{M} = (\langle W, R, w_0\rangle, \{(\mathbb{N}, I_w)\} : w \in W)$ be a model for M^3, where $w_0 \nVdash A$, as above. The model $\mathcal{M}^f$ for $\mathrm{M}^3 \cup \{\neg A\}$ on the frame $\langle W^f, \widehat{R}, \alpha_0\rangle$ constructed in the preceding discussion and used in 5.10 is a *finite* irreflexive and transitive model for M^3 hence also for ML^3 because of the implied reverse well-foundedness of $\widehat{R}$. Moreover we saw in 5.10 that $\alpha_0 \Vdash_S X$ iff $X \in \alpha_0$ for all $X \in S(A)$. In particular $\alpha_0 \Vdash_S A$ iff $A \in \alpha_0$, thus $\alpha_0 \nVdash_S A$ since $A \notin \alpha_0$. □

6. Arithmetical completeness

The main tool in this section is Solovay's work [23]. We build on [19] but also use two tools from [30], namely, a definition and a lemma in loc. cit., which appear modified below as 6.7 and 6.8 respectively. Our induction in the proof of 6.9 proceeds in its details differently.

THEOREM 6.1 (Main Theorem). *ML^3 is arithmetically complete in some recursive extension $\mathcal{T}$ of PA in the sense that, for any closed A over the language of ML^3, if all arithmetical realisations A^* of A are provable in $\mathcal{T}$, then A is provable in ML^3.*

As in [23] (for GL) we prove 6.1 contrapositively: Thus, assume $\mathrm{ML}^3 \nvdash A$, for some fixed modal *sentence* A over L, and find an arithmetical realisation in $\mathcal{T}$ such that $\nvdash_{\mathcal{T}} A^*$.

The first phase of this plan is to build a finite, irreflexive and transitive Kripke model $\mathcal{M} = \left(\left\langle W^f, \widehat{R}, \alpha_0\right\rangle, \{(\mathbb{N}, \Vdash_S) : \alpha_i \in W^f\}\right)$ for $\mathrm{ML}^3 \cup \{\neg A\}$, therefore one where

$$\alpha_0 \nVdash_S A \tag{§}$$

This was done in 5.11 above.

The second phase is to apply Solovay's technique [23] to embed $\mathcal{M}$ in an appropriate $\mathcal{T}$—*which is a* finite *extension of PA that we define below*—and propose an arithmetical *realisation* * such that $\mathcal{T} \nvdash A^*$.

An *a priori* requirement of the embedding is that the worlds α_i (cf. 5.8) make sense in the language of PA, thus we rename them into numbers.

$$W^f = \{1, 2, 3, \ldots, n\}$$

where "$i + 1$" stands for "α_i".

For technical reasons[10] Solovay adds a new world named 0—in our case with $M_0 = \mathbb{N}$—and modifies $\mathcal{M}$ to $\mathcal{M}^0$, by modifying:

- $\widehat{R}$ into $\widehat{R}^0 = \widehat{R} \cup \{(0, i) : 1 \leq i \leq n\}$
- the forcing relation $\Vdash_S$ into $\Vdash_{S^0}$ by letting $0 \Vdash_{S^0} X$ iff $1 \Vdash_S X$, while $i \Vdash_{S^0} X$ iff $i \Vdash_S X$, for $1 \leq i \leq n$.
- $W^{f,0} = \{0, 1, 2, 3, \ldots, n\}$

It is this $\mathcal{M}^0$ that Solovay embeds into PA (or extension $\mathcal{T}$). Below we list the Solovay lemmata that, interestingly, *can be used here as is* without reference to their complex proofs (not so in [1, 12]). For simplicity of *use* and exposition, many authors ([3, 30, 1]) use the abbreviations S_k or σ_k for the formal sentence (in PA) "$\mathbf{l} = \widetilde{k}$" that is pervasive in [23], where $\widetilde{k}$ is the formal counterpart in PA—a *numeral*—of the number $k \in \mathbb{N}$ and $\mathbf{l}$ denotes a formal term that is the *limit* of Solovay's "function h" whose outputs are in $W^{f,0}$.

Lemma 6.2 (Solovay's Lemmata). *$\mathcal{T}$ is some recursive extension of PA over a finite extension of the PA language. There are sentences S_i, for $0 \leq i \leq n$, of the language, such that*

1. *For all $i \neq j$, $\vdash_{\mathcal{T}} \neg S_i \vee \neg S_j$.*
2. *For $0 \leq i \leq n$, $\mathcal{T} + S_i$ is consistent.*
3. *If $i\widehat{R}^0 j$, then $\vdash_{\mathcal{T}} S_i \to \neg\Theta_{\mathcal{T}}(\ulcorner \neg S_j \urcorner)$, where $\Theta_{\mathcal{T}}$ is the provability predicate for $\mathcal{T}$.*
   Under the given assumptions, [23] formulated this as the equivalent $\vdash_{\mathcal{T}} S_i \to Cons_{\mathcal{T}+S_j}$. In words, $\mathcal{T}$ proves the formalised in $\mathcal{T}$ consistency of $\mathcal{T} + S_j$ from premise S_i.
4. *If $i > 0$, then $\vdash_{PA} S_i \to \Theta_{\mathcal{T}}(\ulcorner \bigvee_{i\widehat{R}^0 j} S_j \urcorner)$.*

As in [30] we will work with a specific finite consistent extension $\mathcal{T}$ of PA rather than PA. Towards obtaining this theory, we build consistent sets of *classical* formulae $\mathcal{C}_i$ (6.4 below) as follows.

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[10] The technical reason is simply that Solovay's Kripke-frame-walking function h must be *total*—in fact, with some care ([12]) h can be proved to be primitive recursive—indeed must be initialised as $h(0) = 0$. We do not use Solovay's S_0 in our proof, nor do we mention S_0 in Lemma 6.2. Incidentally, S_0 is true in the standard model of PA, but not provable in PA. Solovay and [3] use the truth of S_0 in proving arithmetical completeness of GL. [30] and [22] do not. We follow the latter's paradigm here.

We note that while $i \cap S(A)$ is consistent it is *not a maximal* consistent finite subset of $S(A)$ since i contains only sentences. Thus if $X(y)$ is in $S(A)$—as a result of the presence of $(\forall y)X$ as a closed subformula of A—it is not in i $(= \alpha_{i-1})$. On the other hand, if $(\forall y)X$ is consistent with ML^3, then so is $X(y)$ and vice versa by virtue of $\vdash (\forall y)X \to X(y)$ absolutely (axiom A2) and $X(y) \vdash (\forall y)X$. Thus we depart from the worlds i of [19], only using finite parts of them to define (in 6.4 via 6.3) the *finite* classical sets $\mathcal{C}_i$. These sets are needed for Proposition 6.6 that leads to the *finite* extension of PA.

DEFINITION 6.3. For each $1 \le i \le n$, $S^i_{\max}(A)$ denotes a *maximal consistent subset of* $S(A)$ that contains $i \cap S(A)$ $(= \alpha_{i-1} \cap S(A))$.[11] $\square$

Such an $S^i_{\max}(A)$ along with a $\forall X$ that it might contain will also contain all formulae obtained from $\forall X$ by stripping one $(\forall u)$ at a time, from left to right, from the prefix $\forall$ of X (axiom A2).

DEFINITION 6.4. We next define a set of classical formulae $\mathcal{C}_i$, for each $1 \le i \le n$.

(1) If $X \in S^i_{\max}(A)$ is a classical first-order formula, then X is *transformed* into itself (no change), and is added to $\mathcal{C}_i$ under the name $X^{t,i}$.

(2) If $X \in S^i_{\max}(A)$ contains at least one $\square$, then every *top level occurrence* of $\square B$ *in* X is changed to $\top$ iff $\square B \in i$, else it is changed to $\perp$.[12] The transformed formula X—again given the name $X^{t,i}$—is placed in $\mathcal{C}_i$. $\square$

Remark 6.5. "t" is for "transformed" formula. But why the extra superscript i? Because the same X may appear in i and j, for $i \ne j$. But some top level subformula $\square B$ of X may be in i but not in j. This results in having two distinct transforms $X^{t,i}$ and $X^{t,j}$. $\square$

PROPOSITION 6.6. $\mathcal{C}_i$ *is consistent iff* $S^i_{\max}(A)$ *is consistent.*

PROOF: Let $X \in S^i_{\max}(A)$. Note that, if $\square B \in i$, then $i \vdash \square B \equiv \top$[13] while if $\square B \notin i$, then $\neg\square B$ is in i by maximality, hence $i \vdash \square B \equiv \perp$.[14]

[11] Such maximal consistent subsets trivially exist by finiteness of $S(A)$.

[12] Case of $\neg\square X$ being in i. Incidentally, if X contains the subformula $\square(\ldots\square C\ldots)$ at the top level it is clear that there is no point to replace $\square C$ by $\top$ or $\perp$.

[13] $i \vdash \square B$ and tautological implication.

[14] Since $i \vdash \neg\square B$.

Now let $\Box B \in S^i_{\max}(A)$. Then the first $\vdash$-statement above is refined to $S^i_{\max}(A) \vdash \Box B \equiv \top$. In the opposite case $\neg\Box B$ is in i and thus in $S^i_{\max}(A)$ and hence $S^i_{\max}(A) \vdash \Box B \equiv \bot$.

Therefore $S^i_{\max}(A) \vdash X \leftrightarrow X^{t,i}$ since $X^{t,i}$ is obtained by a finite sequence of replacing "equivalents by equivalents" according to the preceding paragraph. Thus, $\mathcal{C}_i$ proves $\bot$ iff $S^i_{\max}(A)$ proves $\bot$. $\Box$

Now, each $S^i_{\max}(A)$ *is* consistent, hence *each* $\mathcal{C}_i$ is also a consistent finite set of (classical) formulae over the language $L(\mathbb{N})$.

Note that the formulae X of the classical sets $\mathcal{C}_i$ with parameters in $\mathbb{N}$ can each be realised in the language of PA (cf. also [11, Vol. II] and [10, 14]) as a true formula in the standard model. Indeed, add all the *finitely* many predicate letters found in $\mathcal{C}_i$ to the language of PA and also replace each parameter $\mathbf{k}$ (imported constant, 5.2) that occurs in every such X into the numeral $\widetilde{k}$ to obtain a formula $re_i(X)$ in the language of PA. We denote by $re_i(\mathcal{C}_i)$ the set $\{re_i(X) : X \in \mathcal{C}_i\}$.

It follows that each set $re_i(\mathcal{C}_i)$ is *consistent with* PA since the latter's standard model is also a model of $re_i(\mathcal{C}_i)$ and thus of PA $+\, re_i(\mathcal{C}_i)$ as well.

Therefore, for each $i = 1, \ldots, n$, we can consistently add to PA the new axiom

$$\mathscr{A}_i \overset{Def}{\leftrightarrow} \left(\bigwedge_{X \in re_i(\mathcal{C}_i)} X \right)$$

We define

$$\mathcal{T} \overset{Def}{=} PA + \{\mathscr{A}_1, \ldots, \mathscr{A}_n\}$$

Now the arithmetical realisation $*$ of modal formulae, as usual, maps all the subformulae X of A into formulae of PA in the standard manner, that is, $*$ commutes with the Boolean connectives and $(\forall x)$, it preserves the free variables of X, and also commutes with substitution of variables for variables, that is if $X(x_1, \ldots, x_m)^* = Y(x_1, \ldots, x_m)$, then $X(y_1, \ldots, y_m)^* = Y(y_1, \ldots, y_m)$. Lastly, $(\Box A)^* = \Theta(\ulcorner A^* \urcorner)$, where here and for the rest of the proof we write just "Θ" for "$\Theta_{\mathcal{T}}$".

Definition 6.7 (Arithmetical realisation; initialisation).

Let B be any *atomic* subformula of A, where A was fixed at the outset of this section (cf. (§)). Being atomic it is classical.

Then for the *basis* of the realisation $*$ we set ([30]),[15]

$$B^* \overset{Def}{\leftrightarrow} \bigvee_{\substack{1 \le j \le n \\ j \Vdash \forall B}} S_j \wedge re_j(B^{t,j}) \tag{6.1}$$

If the $\bigvee$ is empty, then we set B^* to be a simple expression equivalent to $\bot$, say, $\neg \bigwedge_{1 \le i \le m} u_i = u_i$, where $u_1, u_2, \ldots, u_m$ are all the free variables of B and thus of B^*. Of course, $\mathcal{T}$ is a logic *with* equality. □

The following useful lemma is stated in Yavorsky [30] without proof. A proof is the following.

LEMMA 6.8. $\vdash_{\mathcal{T}} S_i \to (B^* \leftrightarrow re_i(B^{t,i}))$ *for any* classical *first-order subformula* B *of* A, *and* $1 \le i \le n$.

PROOF: We do induction on the *classical* complexity of B (number of $\neg, \wedge$ and $\forall$ connectives).

First, since S_i is a sentence, invoking the deduction theorem

$$\text{we need to prove instead } \vdash_{\mathcal{T}+S_i} B^* \leftrightarrow re_i(B^{t,i}) \tag{6.2}$$

We now proceed with our induction on classical formulae B:

1. *B is atomic* (*Basis*): Having S_i as a hypothesis in (2), tautological implication yields from (1),

$$\vdash_{\mathcal{T}+S_i} B^* \leftrightarrow re_i(B^{t,i}) \vee \bigvee_{\substack{j \ne i \\ j \Vdash \forall B}} S_j \wedge re_j(B^{t,j}) \tag{6.3}$$

   Note that by 6.2(1), we have $\vdash_{\mathcal{T}+S_i} \neg S_j$ for $j \ne i$. Thus by tautological implication the "$\bigvee$" part above drops out (is provably equivalent to $\bot$). We have proved the Basis step.

   We omit the cases of Boolean connectives as trivial but sample the equally trivial case of the $\forall$ connective below.

2. B is $(\forall x)D$. By I.H. $\vdash_{\mathcal{T}+S_i} D[x]^* \leftrightarrow re_i(D^{t,i}[x])$. By the equivalence theorem, $\vdash_{\mathcal{T}+S_i} (\forall x)D^* \leftrightarrow (\forall x)re_i(D^{t,i})$. But $((\forall x)D)^*$ is $(\forall x)D^*$

[15]Recall the renaming of α_j as $j+1$, at the beginning of Section 6.

by the definition of * while, by the definition of re_i, $re_i\big((\forall x)D^{t,i}\big)$ is $(\forall x)re_i(D^{t,i})$. $\square$

The proof of the Main Lemma below will use Löb's "derivability conditions" (DC) 1 and 2 which we list below for the record (cf. [24] for their rather lengthy proofs).

DC 1 If $\vdash_{\mathcal{T}} A$, then $\vdash_{\mathcal{T}} \Theta(\ulcorner A \urcorner)$.

DC 2 $\vdash_{\mathcal{T}} \Theta(\ulcorner A \to B \urcorner) \to \Theta(\ulcorner A \urcorner) \to \Theta(\ulcorner B \urcorner)$.

Lemma 6.9 (Main Lemma). *Having got a finite Kripke model of n-nodes such that $1 \nVdash_{S^0} A$ (cf. §), where "1" is α_0 and "n" is α_{n-1} and A is closed, we will prove, for* every closed *subformula X of A, and for all $1 \le i \le n$, that*

(1) *If $i \Vdash_{S^0} X$, then $\vdash_{\mathcal{T}} S_i \to X^*$*

(2) *If $i \nVdash_{S^0} X$, then $\vdash_{\mathcal{T}} S_i \to \neg X^*$*

Proof: Induction on the complexity of the modal *sentence* X. Throughout, by the deduction theorem we routinely replace the tasks "$\vdash_{\mathcal{T}} S_i \to \ldots$" by the tasks "$\vdash_{\mathcal{T}+S_i} \ldots$"

1. X is atomic.

   (a) Verify (1) of the lemma. So we have $i \Vdash_{S^0} X$. Hence (by 6.8) $\vdash_{\mathcal{T}+S_i} X^* \leftrightarrow re_i(X^{t,i})$. But $re_i(X^{t,i})$ is a conjunct of an axiom of $\mathcal{T}$ thus $\vdash_{\mathcal{T}} re_i(X^{t,i})$. By tautological implication, $\vdash_{\mathcal{T}+S_i} X^*$.

   (b) Verify (2) of the lemma. So $i \nVdash_{S^0} X$, thus by (6.1) the disjunct $S_i \wedge re_{i-1}(X^{t,i-1})$ is missing. By item 1. in the proof of 6.8 we have $\vdash_{\mathcal{T}+S_i} X^* \leftrightarrow \bot$, that is, $\vdash_{\mathcal{T}+S_i} \neg X^*$.

   The interesting induction steps are for X of the form $\Box B$ or $(\forall x)B$.

2. X is $\Box B$.

   (1) *of the Lemma.* Assume $i \Vdash_{S^0} \Box B$. Then for all j such that $i\widehat{R}^0 j$ it is $j \Vdash_{S^0} \forall B$. By I.H.[16] and definition by cases,

[16] We remind the reader that as in [19] $\Box B$ is more complex than $\forall B$.

$$\vdash_{\mathcal{T}} \bigvee_{i\widehat{R}^0 j} S_j \to (\forall B)^*$$

Applying DC1 then DC2 followed by modus ponens,

$$\vdash_{\mathcal{T}} \Theta(\ulcorner \bigvee_{i\widehat{R}^0 j} S_j \urcorner) \to \Theta(\ulcorner (\forall B)^* \urcorner) \tag{$*$}$$

By 6.2(4) $\vdash_{\mathcal{T}} S_i \to \Theta(\ulcorner \bigvee_{i\widehat{R}^0 j} S_j \urcorner)$ and hence, by ($*$),

$$\vdash_{\mathcal{T}} S_i \to \Theta(\ulcorner (\forall B)^* \urcorner) \tag{$**$}$$

Now $\vdash \forall B \to B$ (absolutely) and also $\vdash_{\mathcal{T}} (\forall B)^* \to B^*$ since $(\forall B)^*$ is $\forall(B^*)$. Hence, by DC1 and DC2, $\vdash_{\mathcal{T}} \Theta(\ulcorner (\forall B)^* \urcorner) \to \Theta(\ulcorner B^* \urcorner)$.

This and tautological implication from ($**$) yields

$$\vdash_{\mathcal{T}} S_i \to \Theta(\ulcorner B^* \urcorner)$$

Noting that $(\Box B)^*$ is $\Theta(\ulcorner B^* \urcorner)$, this case is done.

(2) *of the Lemma.* Assume $i \not\Vdash_{S^0} \Box B$. Then for *some* j such that $i\widehat{R}^0 j$ it is $j \not\Vdash_{S^0} \forall B$. *We pick one such* j.

By I.H.

$$\vdash_{\mathcal{T}} S_j \to \neg(\forall B)^*$$

hence $\vdash_{\mathcal{T}} (\forall B)^* \to \neg S_j$. By DC1 and DC2, $\vdash_{\mathcal{T}} \Theta(\ulcorner (\forall B)^* \urcorner) \to \Theta(\ulcorner \neg S_j \urcorner)$, hence

$$\vdash_{\mathcal{T}} \neg\Theta(\ulcorner \neg S_j \urcorner) \to \neg\Theta(\ulcorner (\forall B)^* \urcorner) \tag{§§}$$

By 6.2(3), $i\widehat{R}^0 j$ yields $\vdash_{\mathcal{T}} S_i \to \neg\Theta(\ulcorner \neg S_j \urcorner)$. Therefore, a tautological implication using this and (§§) derives

$$\vdash_{\mathcal{T}} S_i \to \neg\Theta(\ulcorner (\forall B)^* \urcorner) \tag{$***$}$$

By successive applications of axiom A7 of ML^3 we obtain $\vdash_{ML^3} \Box B \to \Box\forall B$, hence (by definition of * and arithmetical soundness, not proved in this paper), $\vdash_{\mathcal{T}} (\Box B)^* \to (\Box\forall B)^*$, that is, $\vdash_{\mathcal{T}} \Theta(\ulcorner B^* \urcorner) \to \Theta(\ulcorner (\forall B)^* \urcorner)$. From ($***$) and the preceding we now get $\vdash_{\mathcal{T}} S_i \to \neg\Theta(\ulcorner B^* \urcorner)$, that is, $\vdash_{\mathcal{T}} S_i \to \neg(\Box B)^*$.

3. X is $(\forall x)B$. If the quantification is not redundant, then the subformula B is not a sentence and the I.H. does not apply to it. Thus we proceed using 6.8 instead.

(I) $(\forall x)B$ is classical. Thus

$$\vdash_{\mathcal{T}+S_i} (\forall x)B^* \leftrightarrow re_i((\forall x)B) \tag{6.4}$$

(a) Now, if $i \Vdash_{S^0} (\forall x)B$, then $\vdash_{\mathcal{T}} re_i((\forall x)B)$. Tautological implication and (6.4) yield $\vdash_{\mathcal{T}+S_i} (\forall x)B^*$.

(b) If $i \nVdash_{S^0} (\forall x)B$, then $(\forall x)B$ is false in the world i, hence the true $\neg(\forall x)B$ is in $S^i_{\max}(A)$. Thus $re_i\big(\neg(\forall x)B\big)$ is a conjunct of an axiom of $\mathcal{T}$ and therefore $\vdash_{\mathcal{T}} re_i(\neg(\forall x)B)$, i.e., $\vdash_{\mathcal{T}} \neg re_i((\forall x)B)$. (6.4) now yields $\vdash_{\mathcal{T}+S_i} \neg(\forall x)B^*$.

(II) $(\forall x)B$ is *not* classical.

(a) Assume $i \Vdash_{S^0} (\forall x)B$.

- Let $\Box C$ be a topmost occurrence in $(\forall x)B$ and $\Box C \in S^i_{\max}(A)$.
  Let B' be B with said occurrence of $\Box C$ replaced by $\top$. Since $i \Vdash_{S^0} (\forall x)B$ iff $i \Vdash_{S^0} (\forall x)B'$ the I.H. yields

$$\vdash_{\mathcal{T}+S_i} \big((\forall x)B'\big)^* \tag{6.5}$$

  The I.H. also yields $\vdash_{\mathcal{T}+S_i} \big(\Box C\big)^*$, hence $\vdash_{\mathcal{T}+S_i} \big(\Box C\big)^* \leftrightarrow \top$ (recall that $\top^*$ is by definition $\top$). From the latter and the equivalence theorem we get $\vdash_{\mathcal{T}+S_i} (\forall x)B^* \leftrightarrow \big((\forall x)B'\big)^*$ and we are done by (6.5).
- Let $\Box C$ be a topmost occurrence in $(\forall x)B$ and $(\neg\Box C) \in S^i_{\max}(A)$. This is entirely analogous with the above, but note that we replace here $\Box C$ by $\perp$ on the ML^3 side and by $\perp^*$ on the $\mathcal{T}$ side.

(b) Assume $i \nVdash_{S^0} (\forall x)B$.

- Let $\Box C$ be a topmost occurrence in $(\forall x)B$ and $\Box C \in S^i_{\max}(A)$.

Let B' be B with said occurrence of $\Box C$ replaced by $\top$. Since $i \not\Vdash_{S^0} (\forall x)B$ iff $i \not\Vdash_{S^0} (\forall x)B'$ the I.H. yields

$$\vdash_{\mathcal{T}+S_i} \neg\big((\forall x)B'\big)^* \tag{6.6}$$

The concluding paragraph of this subcase proceeds exactly as in bullet one of (I): we have $\vdash_{\mathcal{T}+S_i} (\forall x)B^* \leftrightarrow \big((\forall x)B'\big)^*$ but this time it is (6.6) that yields $\vdash_{\mathcal{T}+S_i} \neg(\forall x)B^*$.

- The subcase where a topmost occurrence of $\Box C$ in $(\forall x)B$ satisfies $(\neg\Box C) \in S^i_{\max}(A)$ does not offer any new insights. $\Box$

Proof of the main theorem. By 6.9, since A is a subformula of itself and $1 \not\Vdash_{S^0} A$ we have $\vdash_{\mathcal{T}} S_1 \to \neg A^*$. By Lemma 6.2(2) $\mathcal{T}+S_1$ is consistent, hence so is $\mathcal{T} + \neg A^*$.[17] Thus $\nvdash_{\mathcal{T}} A^*$. $\Box$

7. Concluding note

As remarked in [19] and more recently in [26], ML^3, being a first-order extension of GL due to the inclusion of the Löb axiom (A6), was meant to be a possible *candidate* for a modal *first-order* provability logic for (arithmetised provability in) PA.

Secondly, it was deliberately built as an extension of M^3 in order to *remain* a provability logic for *classical pure first-order logic*.

Indeed, the *conservation theorem* was proved (syntactically) for ML^3 (as it was for M^3) in [19] verifying that the second design criterion was met.

Given the establishment of its *semantic completeness* with respect to reverse well-founded *finite* and transitive Kripke structures ([19], and also in this paper), [19, 26] conjectured that the first design criterion ought to be also met. A proof of this has been offered in the present paper.

This paper benefits from the idea in [30] to show arithmetical completeness with respect to a *finite extension of PA* and also from Lemma 6.8 which is only stated in [30] but it is proved here.

[17] If $\vdash_{\mathcal{T}+\neg A^*} \bot$, then $\vdash_{\mathcal{T}} A^*$ thus also $\vdash_{\mathcal{T}+S_1} A^*$ contradicting the consistency of $\mathcal{T} + S_1$.

Unlike QGL^b, the ML^3 does not have *necessitation* as a primary rule and as a result has the added desirable attribute that some of its metatheoretical work be done directly, without Gentzenisation, using formulators to investigate the Hilbert-style axiom system 3.1—[20, 26]. The second of the preceding references shows that in the presence of all the other axioms, the addition of A7 is *essential* for arithmetical completeness, since *all* its arithmetical interpretations *are* provable in PA, but A7 is independent of the other axioms of ML^3 (and M^3).

Moreover, *Craig's Interpolation* holds both for the Gentzenisation GLTS of ML^3 and the GTKS of M^3 ([19]), a property that fails for predicate modal logics in general ([5]).

[30] does not remark on whether QGL^b admits a Gentzenisation (cut-free or otherwise) but more remarkably it does not discuss the central importance of A7 as an axiom towards arithmetical completeness.

The origins of QGL^b and ML^3 are quite distinct, as the former was built to answer "are there arithmetically complete first-order modal logics?" while the origin of ML^3 (via its predecessor M^3) was to build a modal first-order logic that can effectively simulate classical first-order equational proofs. Thus the former chose the "*opaque*" $\Box$ to avoid known negative results—that hinge on the presence of a "transparent" $\Box$—towards arithmetical completeness, while the latter chose this very same feature for a *totally different design reason*: to enable M^3 and ML^3 to simulate, using $\Box$, the classical $\vdash$ of a logic where $A \vdash (\forall x)A$ is an unconstrained rule. This was carefully explained in [27, 28]—see also the quotation from [27, 28] in the present paper, on p. 4, first bullet—where we also explicate the choice of A7 (second bullet) as the modal counterpart of the classical $A \vdash (\forall x)A$. A7 appears to have been adopted without any obvious rationale in [30], mentioned only in passing as an assumption on which the normal form of modal formulae is based (*loc. cit.*, remark below Definition 2.1 on p. 3).

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