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Tarek Sayed Ahmed

OMITTING TYPES IN FRAGMENTS AND EXTENSIONS OF FIRST ORDER LOGIC

Abstract

Fix $2 < n < \omega$. Let L_n denote first order logic restricted to the first n variables. Using the machinery of algebraic logic, positive and negative results on omitting types are obtained for L_n and for infinitary variants and extensions of $L_{\omega,\omega}$.

Keywords: Algebraic logic, multimodal logic, omitting types, completions.

1. Introduction

Let $\mathfrak{L}$ be an extension or reduct or variant of first order logic, like first logic itself possibly without equality, L_n as defined in the abstract with $2 < n < \omega$, $L_{\omega_1,\omega}$, L_ω as defined in [10, § 4.3], ..., etc. An omitting types theorem for $\mathfrak{L}$, briefly an OTT, is typically of the form 'A countable family of non-isolated types in a countable $\mathfrak{L}$ theory T can be omitted in a countable model of T. From this it directly follows, that if a type is realizable in every model of a countable theory T, then there should be a formula consistent with T that isolates this type. A type is simply a set of formulas Γ say. The type Γ is realizable in a model if there is an assignment that satisfies (uniformly) all formulas in Γ. Finally, ϕ isolates Γ means that $T \vdash \phi \to \psi$ for all $\psi \in \Gamma$. What Orey and Henkin proved is that the OTT holds for $L_{\omega,\omega}$ when such types are *finitary* meaning that they all consist of n-variable formulas for some $n < \omega$. For L_n, as defined in the abstract, the situation turns out drastically different. It is known [2] that the OTT fails in the following (strong) sense. For every $2 < n \leq l < \omega$, there is a

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countable and complete L_n theory T, and a type that is realizable in every model of T, but cannot be isolated by a formula using l variables.

In this paper we prove other negative OTTs for L_n when types are required to be omitted with respect to certain generalized semantics.

By imposing extra conditions on theories and / or types required to be omitted (like quantifier elimination and maximality, respectively), we obtain positive OTTs for L_n theories; addressing possibly uncountably many types. Also, we study OTTs for algebraizable extensions of $L_{\omega,\omega}$, namely, the (algebraizable) so–called *infinitary logics of infinitary relations* studied extensively in [10, § 4.3]. In this context, we prove negative results on OTTs. Here semantics are the usual Tarskian semantics respecting commutativity of cylindrifiers. Sometimes such logics are referred to as *typless logics*; the adjective typless pointing out to dropping the arity of relation symbols in their formalism.

Conversely, we prove positive OTTs for logics corresponding to variants of ω–dimensional polyadic algebras with equality (PEA_ωs) with equality studied in [8, 18] by taking reducts and/or weakening the axioms of PEA_ω. In the logics studied in [8], Tarskian semantics are relativized, and consequently we do not have full fledged commutativity of cylindrifiers. The logic studied in [18] can be regarded as a classical algebraizable extension of $L_{\omega,\omega}$ *without equality*; here by classical we understand that Tarskian semantics are preserved in such extensions.

We follow the notation of [1] which is in conformity with the notation in the monograph [10]. In particular, for any pair of ordinal $\alpha < \beta$, CA_α stands for the class of cylindric algebras of dimension α, RCA_α denotes the class of representable CA_αs and $\mathsf{Nr}_\alpha\mathsf{CA}_\beta(\subseteq \mathsf{CA}_\alpha)$ denotes the class of α–neat reducts of CA_βs. The last class is studied extensively in the chapter [20] of [1] as a key notion in the representation theory of cylindric algebras. $\mathbf{S}$ denotes the operation of forming subalgebras and $\mathbf{P}$ denotes the operation of forming direct products. For any ordinal α, Cs_α denotes the class of cylindric set algebras of dimension α whose top elements are α–dimensional cartesian spaces and Gs_α denotes the class of generalized cylindric set algebras of dimension α, whose top elements are generalized α–dimensional cartesian spaces. An α–dimensional cartesian space is a set of the form ${}^\alpha U$ (U a non-empty set) and a generalized α–dimensional cartesian space is a disjoint union of α dimensional cartesian spaces. By definition $\mathsf{RCA}_\alpha = \mathbf{SP}\mathsf{Cs}_\alpha$ and it is known (and indeed not hard to show that) $\mathsf{RCA}_\alpha = \mathbf{I}\mathsf{Gs}_\alpha$ where $\mathbf{I}$ is the operation of forming isomorphic images.

In cylindric–polyadic algebras of dimension α (α an infinite ordinal) studied in [8], units are unions of cartesian spaces that are not necessarily disjoint. We assume familiarity with the basics of duality theory of Boolean algebras with operators BAOs, like *atom structures* and *complex algebras*. A more than an adequate reference is [12, Chapter 2]. Throughout the paper, unless otherwise indicated, we fix $2 < n < \omega$.

Layout

- In § 2 we recall the needed basic concepts to be used in the sequel.
- In § 3 we prove negative results on OTT for L_n algebraically by proving that infintely mjany varities of CA_ns are not atom-canonical (to be defined below).
- In § 4 we prove positive results on OTT for L_n and a multitude of algebraizable versions of $L_{\omega,\omega}$.

2. Some basics

We fix the notation, in the process recalling some basic needed definitions:

Definition 2.1. Let α be an ordinal and λ be a cardinal.

(1) A *weak space of dimension* α is a set V of the form $\{s \in {}^{\alpha}U : |\{i \in \alpha : s_i \neq p_i\}| < \omega\}$ where U is a non-empty set and $p \in {}^{\alpha}U$. We denote V by ${}^{\alpha}U^{(p)}$. We write Gws_α short hand for the class of *generalized weak set algebras* as defined in [10, Definition 3.1.2, item (iv)]. By definition $\mathsf{Gws}_\alpha = \mathbf{SP}\mathsf{Ws}_\alpha$, where Ws_α denotes the class of weak set algebra of dimension α. The top elements of Gws_αs are *generalized weak spaces* of dimension α; these are disjoint unions of weak spaces of the same dimension. Plainly when $\alpha < \omega$, $\mathsf{Ws}_\alpha = \mathsf{Cs}_\alpha$ and $\mathsf{Gws}_\alpha = \mathsf{Gs}_\alpha$, in which case we use the notation Cs_α and Gs_α.

Fix $\mathfrak{A} \in \mathsf{RCA}_\alpha$.

(2) Let $\mathbf{K} \in \{\mathsf{Gs}_\alpha, \mathsf{Gws}_\alpha\}$. If $\mathbf{X} = (X_i : i < \lambda)$ is family of subsets of $\mathfrak{A}$, we say that $\mathbf{X}$ *is omitted with respect to* $\mathbf{K}$ if there exist in $\mathfrak{C} \in \mathbf{K}_\alpha$, and

an isomorphism $f : \mathfrak{A} \to \mathfrak{C}$ such that $\bigcap f(X_i) = \emptyset$ for all $i < \lambda$. When we want to stress the role of f, we say that $\mathbf{X}$ is omitted in $\mathfrak{C}$ via f.

(3) If $X \subseteq \mathfrak{A}$ and $\prod X = 0$, then we refer to X as a *non-principal type* of $\mathfrak{A}$.

(4) If $\mathbf{K} \in \{\mathsf{Gs}_\alpha, \mathsf{Gws}_\alpha\}$, $\mathfrak{A}$ is atomic and the non-principal type consisting of co-atoms (a co-atom is the complement of an atom) omitted in $\mathfrak{C} \in \mathbf{K}$ via f, then we say that $\mathfrak{C}$ is a *complete representation of* $\mathfrak{A}$ *via* f or simply a complete representation of $\mathfrak{A}$, and that $\mathfrak{A}$ is *completely representable with respect to* $\mathbf{K}$.

Let $\mathbf{K} \in \{\mathsf{Gs}_\alpha, \mathsf{Gws}_\alpha\}$. It is known that an atomic $\mathfrak{A} \in \mathsf{CA}_\alpha$ is completely representable with respect to $\mathbf{K}$ via $f \iff$ there exists $\mathfrak{C} \in \mathbf{K}$ such that for all $X \subseteq \mathfrak{A}$, $f(\sum X) = \bigcup_{x \in X} f(x)$, whenever $\sum X$ exists in $\mathfrak{A}$, hence the term *complete representation.* We note that in the last part (after the equivalence) atomicity is redundant, cf. [11].

For some time we fix $2 < n < \omega$. The subtle phenomena of complete representability is closely related to the algebraic notion of atom–canonicity of (certain supervarieties of) RCA_n (like $\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_m$ for $2 < n < m < \omega$), and to the metalogical property of omitting types in n–variable fragments of first order logic [19, Theorems 3.1.1, 3.1.2, p. 211, Theorems 3.2.8, 3.2.9, 3.2.10], when non-principal types are omitted with respect to (relativized) semantics.

Atom–canonicity is an important *persistence property* in various modal logics, that applies to the class of their modal algebras; for example the variety RCA_n viewed as the class of modal algebras of the (modal formalism) of L_n is not atom–canonical, because applying the complex algebra operator to countable atom structures of RCA_ns, can give non-representable CA_ns, more succintly, $\mathfrak{Cm}(\mathsf{AtRCA}_n) \nsubseteq \mathsf{RCA}_n$. The term algebra on any such atom structure $\mathbf{At}$ say, cannot be completely representable, for a complete representation of $\mathfrak{Tm}\mathbf{At}$ (the term algebra) induces a representation of $\mathfrak{Cm}\mathbf{At}$. This implies that OTT fails for L_n as indicated in the introduction when $n = l$. That OTT fails for L_n in the stronger sense indicated also in the introduction when $n < l < \omega$, follows from the fact that for all $2 < n \leq l < \omega$, there exists a countable $\mathfrak{A} \in \mathsf{RCA}_n \cap \mathsf{Nr}_n\mathsf{CA}_l$ that is not completely representable. The last statement is proved in [2]. We start by showing that infinitely many varieties of $\mathsf{CA}_n s$ (containing and including RCA_n) are not atom–canonical. This will imply that OTT fails strongly but in a different way; the OTT fails for L_n with respect to so–called clique guarded semantics [13] which is a form of generalized semantics. Here the class of models allowed to omit non-principal types is broadened considerably. Models can be only $n+3$–*flat* a notion to be defined below. To get an idea of the *how much broadening the permissable models is occuring here*; for $2 < n < m < l \leq \omega$, the notion of l– flatness is *not* finitely axiomatizable over the notion of m–flatness in a precise sense given in theorem 3.9 below, and that ordinary countable models coincide with ω–flat models. We show that even one–non principal type in a complete and countable L_n theory may not be omitted in any $n+k$–flat model when $k \geq 3$.

3. Non-atom–canonicity of $\mathsf{SNr}_n\mathsf{CA}_{n+k}$ for $k \geq 3$ and failure of OTT with respect to clique-guarded semantics

For sequences f, g having the same domain an ordinal α say, and $i \in \mathsf{dom} f$, we write $f \equiv_i g \iff$ *f and g agree off of i*, that is to say $f(x) = g(x)$ for all $x \in \mathsf{dom}(f) \sim \{i\}$.

Definition 3.1. Let $2 < n < \omega$ and assume that $\mathfrak{A} \in \mathsf{CA}_n$ is atomic.

(1) An *n–dimensional atomic network* on an $\mathfrak{A}$ is a map $N : {}^n\Delta \to \mathsf{At}\mathfrak{A}$, where Δ is a non-empty set of *nodes*, denoted by $\mathsf{nodes}(N)$, satisfying the following consistency conditions for all $i < j < n$:

- If $\bar{x} \in {}^n\mathsf{nodes}(N)$ then $N(\bar{x}) \leq \mathsf{d}_{ij} \iff x_i = x_j$,
- If $\bar{x}, \bar{y} \in {}^n\mathsf{nodes}(N)$, $i < n$ and $x \equiv_i y$, then $N(\bar{x}) \leq \mathsf{c}_i N(\bar{y})$,

(2) Assume that $m, k \leq \omega$. The *atomic game $G^m_k(\mathsf{At}\mathfrak{A})$, or simply G^m_k*, is the game played on atomic networks of $\mathfrak{A}$ using m nodes, each node only once, so that any node being used is not alllowed to be reused; and having k rounds [13, Definition 3.3.2], where $\forall$ is offered only one move, namely, *a cylindrifier move*: Suppose that we are at round $t > 0$. Then $\forall$ picks a previously played network N_t ($\mathsf{nodes}(N_t) \subseteq m$), $i < n$, $a \in \mathsf{At}\mathfrak{A}$, $x \in {}^n\mathsf{nodes}(N_t)$, such that $N_t(\bar{x}) \leq \mathsf{c}_i a$. For her response, $\exists$ has to deliver a network M such that $\mathsf{nodes}(M) \subseteq m$, $M \equiv_i N$, and there is $\bar{y} \in {}^n\mathsf{nodes}(M)$ that satisfies $\bar{y} \equiv_i \bar{x}$ and $M(\bar{y}) = a$, cf. [12, Definition 12.5(2)] for the notation $M \equiv_i N$.

(3) We write $G_k(\mathsf{At}\mathfrak{A})$, or simply G_k, for $G_k^m(\mathsf{At}\mathfrak{A})$ if $m \geq \omega$.

(4) The ω–rounded game $\mathbf{G}^m(\mathsf{At}\mathfrak{A})$ or simply $\mathbf{G}^m$ is like the game $G_\omega^m(\mathsf{At}\mathfrak{A})$ except that $\forall$ has the option to reuse the m nodes in play.

For BAOs, $\mathfrak{A}$ and $\mathfrak{B}$ say, having the same signature, we say that $\mathfrak{A}$ is *dense* in $\mathfrak{B}$ if $\mathfrak{A} \subseteq \mathfrak{B}$ and for all non-zero $b \in \mathfrak{B}$, there is a non-zero $a \in A$ such that $a \leq b$. An atom structure will be denoted by $\mathbf{At}$. An atom structure $\mathbf{At}$ has the signature of CA_α, α an ordinal, if $\mathfrak{Cm}\mathbf{At}$ has the signature of CA_α.

Definition 3.2. Let V be a completely additive variety of BAOs. Then V is *atom–canonical* if whenever $\mathfrak{A} \in \mathsf{V}$ and $\mathfrak{A}$ is atomic, then $\mathfrak{Cm}\mathsf{At}\mathfrak{A} \in \mathsf{V}$. The *Dedekind–MacNeille completion* of $\mathfrak{A} \in \mathsf{V}$, is the unique (up to isomorphisms that fix $\mathfrak{A}$ pointwise) complete $\mathfrak{B}$ such that $\mathfrak{A} \subseteq \mathfrak{B}$ and $\mathfrak{A}$ is *dense* in $\mathfrak{B}$.

From now on fix $2 < n < \omega$. If $\mathfrak{A} \in \mathsf{CA}_n$ is atomic, then $\mathfrak{Cm}\mathsf{At}\mathfrak{A}$ is the *Dedekind–MacNeille completion of* $\mathfrak{A}$. If $\mathfrak{A} \in \mathsf{CA}_n$, then its atom structure will be denoted by $\mathsf{At}\mathfrak{A}$ with domain the set of atoms of $\mathfrak{A}$ denoted by $At\mathfrak{A}$.

Lemma 3.3. *Let $2 < n < m < \omega$ and assume that $\mathfrak{A} \in \mathsf{CA}_n$ is atomic. If $\mathfrak{A} \in \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_m$, then $\exists$ has a winning strategy in $\mathbf{G}^m(\mathsf{At}\mathfrak{A})$. In particular, If $\mathfrak{A}$ is finite and $\forall$ has a winning strategy in $\mathbf{G}^m_\omega(\mathsf{At}\mathfrak{A})$, then $\mathfrak{A} \notin \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_m$.*

Proof: [23, Lemma 4.3]. □

In the next theorem 3.5, we show non-atom canonicity of the varieties $\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{n+k}$ for $k \geq 3$. The gist of the idea is a combination of the model–theoretic techniques of Hodkinson's used in [15] conjuncted with a blow up and blur construction in the sense of [2]. The idea of a 'a blow up and blur' construction is simple, but powerful and subtle. We give the general idea. One starts with a finite algebra $\mathfrak{D} \in \mathsf{CA}_n$, blowing its atom structure, by *splitting one or more of its atoms into infinitely many* thereby obtaining a new infinite atom structure, call it $\mathbf{At}$, such that $\mathfrak{D}$ embeds into $\mathfrak{Cm}\mathbf{At}$. If $\mathfrak{D}$ is not representable, or even has only finite representations (representations on finite bases) and $\mathfrak{Tm}\mathbf{At}$ happens to be representable, then the Dedekind–MacNeille completion $\mathfrak{Cm}\mathbf{At}$ of $\mathfrak{Tm}\mathbf{At}$ will not be representable, because a representation of the infinite algebra $\mathfrak{Cm}\mathbf{At}$ necessarily has an infinite base, inducing an infinite representation of $\mathfrak{D}$, since $\mathfrak{D}$ embeds in $\mathfrak{Cm}\mathbf{At}$ and $\mathsf{RCA_n}$ is a variety. So one thereby obtains a weakly representable atom structure $\mathbf{At}$, that is not strongly representable. But this same idea can also be applied to the varieties $\mathsf{V}_k = \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{n+k}$ for $k > 1$, approximating RCA_n. One blows up and blur a finite algebra $\mathfrak{D}$ *outside the (larger)* V_k *(when* $k < \omega$*)*, thereby obtaining a weakly representable atom structure $\mathbf{At}$, such $\mathfrak{Cm}\mathbf{At}$ is outside V_k because $\mathfrak{D}$ embeds into $\mathfrak{Cm}\mathbf{At}$. If for some $k_0 > 1$, the atom structure $\mathbf{At}$ obtained after blowing up and blurring the finite algebra that is outside V_{k_0} is representable, it will readily follow that V_k, for all $k \geq k_0$ is not atom–canonical. The term *blur* refers to the fact that the algebraic structure of $\mathfrak{D}$ is blurred at the level of $\mathfrak{Tm}\mathbf{At}$, it *does not* embed into $\mathfrak{Tm}\mathbf{At}$ prohibiting its representability, but it it is *not blurred* on the 'global level' of $\mathfrak{Cm}\mathbf{At}$, in the sense that $\mathfrak{D}$ embeds into $\mathfrak{Cm}\mathbf{At}$.

One might be tempted to think that our next result can be obtained by 'lifting somehow' to higher dimensions the construction for RAs proving that $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_k$, $k \geq 6$ is not atom–canonical proved in [12] using a blow up and blur construction for relation algebras. In [12], an representable atomic relation algebra $\mathfrak{R}$, whose Dedekind–MacNeille completion is outside $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_6$, is constructed. But this cannot be done with the lifting construction in [18] as it stands, for given an atomic $\mathfrak{R} \in \mathsf{RA}$, it does not necessarily embed in the Ra reduct of the atomic CA_n constructed from the $\mathfrak{R}$ as described in *op.cit* if $n \geq 6$. It can only be done for $n = 3$. We briefly review the blow up and blur construction in [12, 17.32, 17.34, 17.36] for relation algebras proving that $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_k$, for $k \geq 6$ is not atom canonical. We need some preparation. Let $2 \leq n \leq \omega$ and $r \leq \omega$. Let $\mathfrak{R}$ be an atomic relation algebra. Then the r–rounded game $G^n_r(\mathsf{At}\mathfrak{R})$ [12, Definition 12.24] is the (usual) atomic game played on networks of an atomic relation algebra $\mathfrak{R}$ using n nodes.

Let L be a relational signature. Let G (the greens) and R (the reds) be L structures and $p, r \leq \omega$. The game $\mathsf{EF}^p_r(\mathsf{G}, \mathsf{R})$, defined in [12, Definition 16.1.2], is an Ehrenfeucht–Fraïssé forth 'pebble game' with r rounds and p pairs of pebbles. In [12, 16.2], a relation algebra *rainbow atom structure* is associated for relational structures G and R. We denote by $\mathbf{R}_{A,B}$ the (full) complex algebra over this atom structure. The **Rainbow Theorem** [12, Theorem 16.5] states that: *If* G, R *are relational structures and* $p, r \leq \omega$, *then* $\exists$ *has a winning strategy in* $G^{2+p}_{1+r}(\mathbf{R}_{\mathsf{G},\mathfrak{R}}) \iff$ *she has a winning strategy in* $\mathsf{EF}^p_r(\mathsf{G}, \mathsf{R})$.

For $5 \leq l < \omega$, RA_l is the class of relation algebras whose canonical extensions have an l–dimensional relational basis [12, Definition 12.30]. RA_l is a variety containing properly the variety $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_l$. Furthermore, $\exists$ has a winning strategy in $G^n_\omega(\mathsf{At}\mathfrak{R}) \implies \mathfrak{R} \in \mathsf{RA}_l$, cf. [12, Proposition 12.31] and [12, Remark 15.13]. We now show:

THEOREM 3.4. *For any $k \geq 6$, the varieties RA_k and $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_k$ are not atom–canonical.*

PROOF: We follow the notation in [12, lemmas 17.32, 17.34, 17.35, 17.36] with the sole exception that we denote by m (instead of $\mathbf{K}_m$) the complete irreflexive graph on m defined the obvious way; that is we identify this graph with its set of vertices. Fix $2 < n < m < \omega$. Let $\mathfrak{R} = \mathbf{R}_{m,n}$. Then by the rainbow theorem $\forall$ has a winning strategy in $G^{m+2}_{m+1}(\mathsf{At}\mathfrak{R})$, since it clealy has a winning strategy in the Ehrenfeucht–Fraïssé game $\mathsf{EF}^m_m(m, n)$ because m is 'longer' than n. Then $\mathfrak{R} \notin \mathsf{RA}_{m+2}$ by [12, Propsition 12.25, Theorem 13.46 (4) $\iff$ (5)], so $\mathfrak{R} \notin \mathbf{S}\mathfrak{Ra}\mathsf{CA}_{m+2}$. Next one 'splits' every red atom to ω–many copies obtaining the infinite atomic countable (term) relation algebra denoted in *op.cit* by $\mathcal{T}$, with atom structure α, cf. [12, item (4) top of p. 532]. Then $\mathfrak{Cm}\alpha \notin \mathbf{S}\mathfrak{Ra}\mathsf{CA}_{m+2}$ because $\mathfrak{R}$ embeds into $\mathfrak{Cm}\alpha$ by mapping every red to the join of its copies, and $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_{m+2}$ is closed under $\mathbf{S}$. Finally, one (completely) represents (the canonical extension of) $\mathcal{T}$ like in [12]. By taking $m = 4$ and $n = 3$ the required follows. □

We next *blow up and blur a finite rainbow* $\mathsf{CA}_n (2 < n < \omega)$. The proof, otherwise, is presented in a model–theoretic framework as done in [15], where it is proved that RCA_n is not atom–canonical. We briefly review rainbow constructions for CAs [11, 13]. Fix $2 < n < \omega$. Given relational structures G (the greens) and R (the reds) the rainbow atom structure of a CA_n consists of equivalence classes of surjective maps $a : n \to \Delta$, where Δ is a coloured graph. A *coloured graph* is a complete graph labelled by the rainbow colours, the greens $\mathsf{g} \in \mathsf{G}$, reds $\mathsf{r} \in \mathsf{R}$, and whites; and some $n - 1$ tuples are labelled by 'shades of yellow'. In coloured graphs certain triangles are not allowed for example all green triangles are forbidden. Some (but not all) of the red triples are forbidden. cf. [11, 4.3.3]. The equivalence relation relates two such maps $\iff$ they essentially define the same graph [11, 4.3.4]. We let $[a]$ denote the equivalence class containing a. The accessibilty (binary relations) corresponding to cylindric operations are like in [11]. Special coloured graphs typically used by $\forall$ during implementing his winning strategy are called *cones*: *Let $i \in \mathsf{G}$, and let M be a coloured graph consisting of n nodes $x_0, \ldots, x_{n-2}, z$. We call M an i – cone if $M(x_0, z) = \mathsf{g}^i_0$ and for every $1 \leq j \leq n - 2$, $M(x_j, z) = \mathsf{g}_j$, and no other* *edge of M is coloured green.* $(x_0, \ldots, x_{n-2})$ *is called the* **base of the cone**, z *the* **apex of the cone** *and* i *the* **tint of the cone**. For $2 < n < \omega$, we use the graph version of the games $G^m_\omega(\beta)$ and $\mathbf{G}^m(\beta)$ where β is a CA_n rainbow atom structure, cf. [11, 4.3.3]. The (complex) rainbow algebra based on G and R is denoted by $\mathfrak{A}_{\mathsf{G},\mathsf{R}}$. The dimension n will always be clear from context. For relation algebras the relation algebra $\mathbf{R}_{4.3}$ was blown up and blurred, now we blow up and blur $\mathsf{CA}_{n+1,n}$

THEOREM 3.5. *Let* $2 < n < \omega$. *Then there exists* $\mathfrak{B} \in \mathsf{Cs}_n$ *such that* $\mathfrak{Cm}\mathsf{At}\mathfrak{B} \notin \mathsf{SNr}_n\mathsf{CA}_{n+3}$. *In particular,* $\mathsf{SNr}_n\mathsf{CA}_{n+k}$ *is not atom canonical for all* $k \geq 3$

PROOF: We finish off with the second part modulo the first. Then we prove the first part. We have $\mathfrak{B} \in \mathsf{RCA}_n = \bigcap_{m>0} \mathsf{SNr}_n\mathsf{CA}_{n+m}$ but $\mathfrak{Cm}\mathsf{At}\mathfrak{B} \notin \mathsf{SNr}_n\mathsf{CA}_{n+k}$ for all $k \geq 3$.

The proof of the first part is divided to three parts:

(a) Blowing up and blurring a finite rainbow algebra: Take the finite CA rainbow algebra $\mathfrak{D}$ as defined in [13] where the reds R is the complete irreflexive graph n, and the greens are $\mathsf{G} = \{\mathsf{g}_i : 1 \leq i < n-1\} \cup \{\mathsf{g}^i_0 : 1 \leq i \leq n+1\}$, endowed with the polyadic operations. Denote $\mathfrak{D}$ by $\mathsf{CA}_{n+1,n}$ and for the sake of brevity, denote its finite atom structure by $\mathbf{At}_f$; so that $\mathbf{At}_f = \mathsf{At}(\mathsf{CA}_{n+1,n})$. One then replaces the red colours of the finite rainbow algebra of $\mathsf{CA}_{n+1,n}$ each by infinitely many reds (getting their superscripts from ω), obtaining this way a weakly representable atom structure $\mathbf{At}$. The resulting atom structure after 'splitting the reds', namely, $\mathbf{At}$, is like the weakly but not strongly representable atom structure of the atomic, countable and simple algebra $\mathfrak{A}$ constructed in [15], the sole difference is that we have $n+1$ greens and not infinitely many as is the case in [15]. We denote our algebra also by $\mathfrak{A}$. No confusion is likely to ensue. We will go further by showing that $\mathfrak{Cm}\mathsf{At}\mathfrak{A} \notin \mathsf{SNr}_n\mathsf{CA}_{n+3}$. The rainbow signature [13, Definition 3.6.9] L now consists of $\mathsf{g}_i : 1 \leq i < n-1$, $\mathsf{g}^i_0 : 1 \leq i \leq n+1$, $\mathsf{w}_i : i < n-1$, $\mathsf{r}^t_{kl} : k < l < n$, $t \in \omega$, binary relations, and $n-1$ ary relations y_S, $S \subseteq n+1$. There is a shade of red ρ; the latter is a binary relation that is *outside* the rainbow signature, but it labels coloured graphs during a 'rainbow game'. $\exists$ can win the rainbow ω–rounded game and build an n–homogeneous model M by using ρ when she is forced a red; [15, Proposition 2.6, Lemma 2.7]. From now on, forget about ρ; having done its task as a colour to (weakly) represent $\mathfrak{A}$, it will play no further role. Having M at hand, one constructs two atomic n–dimensional set algebras based on M, sharing the same atom structure and having the same top element. The atoms of each will be the set of coloured graphs, seeing as how, quoting Hodkinson [15] such coloured graphs are 'literally indivisible'. Now L_n and $L^n_{\infty,\omega}$ are taken in the rainbow signature (without ρ). Continuing like in *op.cit*, deleting the one available red shade, set $W = \{\bar{a} \in {}^n\mathsf{M} : \mathsf{M} \models (\bigwedge_{i<j<n} \neg\rho(x_i, x_j))(\bar{a})\}$, and for $\phi \in L^n_{\infty,\omega}$, let $\phi^W = \{s \in W : \mathsf{M} \models \phi[s]\}$. Here W is the set of all n–ary assignments in ${}^n\mathsf{M}$, that have no edge labelled by ρ. We note that ρ is used by $\exists$ infinitely many times during the game forming a 'red clique' in M [15]. Let $\mathfrak{A}$ be the relativized set algebra with domain $\{\varphi^W : \varphi$ a first-order $L_n -$ formula$\}$ and unit W, endowed with the usual concrete operations read off the connectives. Classical semantics for L_n *rainbow formulas* and their semantics by relativizing to W coincide [15, Proposition 3.13] *but not with respect to $L^n_{\infty,\omega}$ rainbow formulas*. This depends essentially on [15, Lemma 3.10], which is the heart and soul of the proof in [15], and for what matters this proof. The referred to lemma says that any permutation χ of $\omega \cup \{\rho\}$, Θ^χ as defined in [15, Definitions 3.9, 3.10] is an n back–and–forth system induced by any permutation of $\omega \cup \{\rho\}$. Let χ be such a permutation. Thee system Θ^χ consists of isomorphisms between coloured graphs such that the superscripts of reds are 're-shuffled along' χ in such a way that rainbow red labels are permuted ρ is replaced by a red rainbow colour, and all other colours are preserved. One uses such n-back-and-forth systems mapping a tuple $\bar{b} \in {}^n\mathsf{M} \sim W$ to a tuple $\bar{c} \in W$ preserving any formula in the rainbow signature not containing the non-red symbols that are 'moved' by the system, so if $\bar{b} \in {}^n\mathsf{M}$ refutes the L_n rainbow formula ϕ, then there is a $\bar{c} \in W$ refuting ϕ, as well. The rainbow algebra $\mathfrak{A}$ is then isomorphic to cylindric set algebra having top element ${}^n\mathsf{M}$, so $\mathfrak{A}$ is simple, in fact it can be shown that even its diagonal free reduct is simple. Let $\mathfrak{E} = \{\phi^W : \phi \in L^n_{\infty,\omega}\}$ [15, Definition 4.1] with the operations defined like on $\mathfrak{A}$ the usual way. $\mathfrak{Cm}\mathbf{At}$ is complete and, so like in [15, Lemma 5.3] we have an isomorphism from $\mathfrak{Cm}\mathbf{At}$ to $\mathfrak{E}$ defined via $X \mapsto \bigcup X$. We have $\mathsf{At}\mathfrak{A} = \mathsf{At}\mathfrak{Tm}(\mathsf{At}\mathfrak{A}) = \mathbf{At}$ (where $\mathfrak{Tm}(\mathsf{At}\mathfrak{A})$ denotes the subalgebra of $\mathfrak{Cm}\mathsf{At}\mathfrak{A}$ generated by the atoms; the term algebra) and $\mathfrak{Tm}\mathsf{At}\mathfrak{A} \subseteq \mathfrak{A}$, hence $\mathfrak{Tm}\mathsf{At}\mathfrak{A}$ is representable. The atoms of $\mathfrak{A}$, $\mathfrak{Tm}\mathsf{At}\mathfrak{A}$ and $\mathfrak{Cm}\mathsf{At}\mathfrak{A} = \mathfrak{Cm}\mathbf{At}$ are the coloured graphs whose edges are *not labelled* by ρ. These atoms are uniquely determined (syntactically) by MCA formulas in the rainbow signature of $\mathbf{At}$ as in [15, Definition

4.3]. The expression blow up and blur is an indicative term introduced in [2]. Blowing up means splitting the atoms of a finite algebra; in our context $\mathsf{CA}_{n+1,n}$ each into infinitely many obtaining a new atom structure denoted above by $\mathbf{At}$. Blurring, means that the algebraic structure of $\mathsf{CA}_{n+1,n}$ is *blurred* in $\mathfrak{Tm}\mathbf{At}$, its algebraic structure is disorganized or distorted in such a way that it does not embed into $\mathfrak{Tm}\mathbf{At}$. Nevertheless, it reapperas in the Dedekind–MacNeille completion of $\mathfrak{Tm}\mathbf{At}$, namely, in $\mathfrak{Cm}\mathbf{At}$ as we shall see in a moment; $\mathsf{CA}_{n+1,n}$ embeds into $\mathfrak{Cm}\mathbf{At}$ by mapping every splitted 'red atom' to the suprema of the subatoms into which it was split. This sprema exists because (the Boolean reduct of) $\mathfrak{Cm}\mathbf{At}$ is a complete algebra, which is not the case with $\mathfrak{Tm}\mathbf{At}$. The last is not complete,

(b) Embedding $\mathsf{CA}_{n+1,n}$ into the complex algebra $\mathfrak{Cm}\mathbf{At}$: Now to embed $\mathsf{CA}_{n+1,n}$ into $\mathfrak{Cm}\mathbf{At} = \mathfrak{Cm}\mathsf{At}\mathfrak{A}$, we need some preparing to do. To start with, we Identify r with r^0, so that we consider that $\mathbf{At}_f \subseteq \mathbf{At}$. Let CRG_f be the class of coulored graphs on $\mathbf{At}_f$ and CRG be the class of coloured graph on $\mathbf{At}$. By the above identification, we can assume that $\mathsf{CRG}_f \subseteq \mathsf{CRG}$. Write M_a for the atom that is the (equivalence class of the) surjection $a : n \to M$, $M \in \mathsf{CGR}$. Here we identify a with $[a]$; no harm will ensue. We define the (equivalence) relation $\sim$ on $\mathbf{At}$ by $M_b \sim N_a$, $(M, N \in \mathsf{CGR})$

- $a(i) = a(j) \iff b(i) = b(j)$,
- $M_a(a(i), a(j)) = \mathsf{r}^l \iff N_b(b(i), b(j)) = \mathsf{r}^k$, for some $l, k \in \omega$,
- $M_a(a(i), a(j)) = N_b(b(i), b(j))$, if they are not red,
- $M_a(a(k_0), \ldots, a(k_{n-2})) = N_b(b(k_0), \ldots, b(k_{n-2}))$, whenever defined.

We say that M_a is a *copy of* N_b if $M_a \sim N_b$ (by symmetry N_b is a copy of M_a.) Indeed, the relation 'copy of' is an equivalence relation on $\mathbf{At}$. An atom M_a is called a *red atom*, if M has at least one red edge. Any red atom plainly has ω-many copies (including itself); furthermore (as is the case with *splitting* arguments) all such copies are *cylindrically equivalent*, in the sense that, if $N_a \sim M_b$ with one (equivalently both) red, with $a : n \to N$ and $b : n \to M$, then we can assume that $\mathsf{nodes}(N) = \mathsf{nodes}(M)$ and that for all $i < n$, $a \upharpoonright n \sim \{i\} = b \upharpoonright n \sim \{i\}$. In $\mathfrak{Cm}\mathbf{At}$, we write M_a for $\{M_a\}$ and we denote suprema taken in $\mathfrak{Cm}\mathbf{At}$, possibly finite, by $\sum$. If N_b is a red copy of M_a, then we may denote N_b by $M_a^{(j)}$ $(j \in \omega)$. Note that a red atom M_a has ω many copies forming a countable (infinite) set

$\{M_a^{(j)} : j \in \omega\}$ of red graphs. If M_a is a red atom, then by $\sum_j M_a^{(j)}$ we understand the infinite sum of its copies evaluated in $\mathfrak{Cm}\mathbf{At}$. If M_a is not red, then it has only one copy, namely, itself. Now we define the map Θ from $\mathsf{CA}_{n+1,n} = \mathfrak{Cm}\mathbf{At}_f$ to $\mathfrak{Cm}\mathbf{At}$, by $\Theta(X) = \bigcup_{x \in \mathbf{At_f}} \Theta(x)$ $(X \subseteq \mathbf{At}_f)$, by specifing first its values on At_f, via $M_a \mapsto \sum_j M_a^{(j)}$; each atom maps to the suprema of its copies. If M_a is not red, then by $\sum_j M_a^{(j)}$, we understand M_a. This map is well-defined because $\mathfrak{Cm}\mathbf{At}$ is complete. We check that f is an injective homomorphim. Injectivity follows from $M_a \leq f(M_a)$, hence $f(x) \neq 0$ for every atom $x \in \mathsf{At}(\mathsf{CA}_{n+1,n})$. Now we check presevation of operations. The Boolean join is obvious.

- For complementation: It suffices to check preservation of complementation 'at atoms' of $\mathbf{At}_f$. So let $M_a \in \mathbf{At}_f$ with $a : n \to M$, $M \in \mathsf{CGR_f} \subseteq \mathsf{CGR}$. Then:

$$\begin{aligned}
\Theta(\sim M_a) &= \Theta(\bigcup_{[b]\neq[a]} M_b) = \bigcup_{[b]\neq[a]} f(M_b) = \bigcup_{[b]\neq[a]} \sum_j M_b^{(j)} \\
&= \bigcup_{[b]\neq[a]} \sim \sum_j [\sim (M_a)^{(j)}] = \bigcup_{[b]\neq[a]} \sim \sum_j [(\sim M_b)^j] \\
&= \bigcup_{[b]\neq[a]} \bigwedge_j M_b^{(j)} = \bigwedge_j \bigcup_{[b]\neq[a]} M_b^{(j)} = \bigwedge_j (\sim M_a)^j = \sim (\sum M_a^j) \\
&= \sim \Theta(a).
\end{aligned}$$

- Diagonal elements. Let $l < k < n$. Then:

$$\begin{aligned}
M_x \leq f(\mathsf{d}_{lk}^{\mathfrak{Cm}\mathbf{At_f}}) &\iff M_x \leq \sum_j \bigcup_{a_l = a_k} M_a^{(j)} \\
&\iff M_x \leq \bigcup_{a_l = a_k} \sum_j M_a^{(j)} \\
&\iff M_x = M_a^{(j)} \text{ for some } a : n \to M \text{ such that } a(l) = a(k) \\
&\iff M_x \in \mathsf{d}_{lk}^{\mathfrak{Cm}\mathbf{At}}.
\end{aligned}$$

- Cylindrifiers. Let $i < n$. By additivity of cylindrifiers, we restrict our attention to atoms $M_a \in \mathbf{At_f}$ with $a : n \to M$, and $M \in \mathsf{CRG_f} \subseteq \mathsf{CRG}$.

Then:

$$\begin{aligned} f(\mathsf{c}_i^{\mathfrak{Cm}\mathbf{At_f}} M_a) &= f(\bigcup_{[c]\equiv_i[a]} M_c) = \bigcup_{[c]\equiv_i[a]} f(M_c) \\ &= \bigcup_{[c]\equiv_i[a]} \sum_j M_c^{(j)} = \sum_j \bigcup_{[c]\equiv_i[a]} M_c^{(j)} \\ &= \sum_j \mathsf{c}_i^{\mathfrak{Cm}\mathbf{At}} M_a^{(j)} = \mathsf{c}_i^{\mathfrak{Cm}\mathbf{At}} (\sum_j M_a^{(j)}) \\ &= \mathsf{c}_i^{\mathfrak{Cm}\mathbf{At}} f(M_a). \end{aligned}$$

We have proved that $\mathsf{CA}_{n+1,n}$ embeds into $\mathfrak{Cm}\mathbf{At}$, so that it is not blurred at the level of the last complex algebra.

(c) $\forall$ s winning strategy in $\mathbf{G}^{n+3}(\mathsf{AtCA}_{n+1,n})$: It is straightforward to show that, like in the relation algebra case that $\forall$ has a winning strategy in the Ehrenfeucht–Fraïssé forth private game played between $\exists$ and $\forall$ on the complete irreflexive graphs $n+1$ and n, namely, in $\mathsf{EF}_{n+1}^{n+1}(n+1, n)$ (using $n+1$ pebble pairs in $n+1$ rounds). This game lifts to a graph game [11, pp.841] on $\mathbf{At}_f$ which in this case is equivalent to the graph version of $\mathbf{G}^{n+3}$, but here $\forall$ does not need to re-use pebbles, so that the game is actually G^{n+3} but of course it ends after only finitely many rounds. $\forall$ lifts his winning strategy from the private Ehrenfeucht–Fraïssé forth game, to the graph game on $\mathbf{At}_f = \mathsf{At}(\mathsf{CA}_{n+1,n})$ using the standard rainbow strategy [11]. He bombards $\exists$ with cones having the same base with green tints, demanding that $\exists$ delivers a red label each time for the succesive appexes of the cones he plays. It is not hard to show that he will need two more nodes in the graph game to win. Thus by lemma 3.3, $\mathsf{CA}_{n+1,n} \notin \mathsf{SNr}_n\mathsf{CA}_{n+3}$. Since $\mathsf{CA}_{n+1,n}$ embeds into $\mathfrak{Cm}\mathsf{At}\mathfrak{A}$, hence $\mathfrak{Cm}\mathsf{At}\mathfrak{A}$ is outside $\mathsf{SNr}_n\mathsf{CA}_{n+3}$, too. □

Remark 3.6. One can describe $\mathsf{CA}_{n+1,n}$ differently as a subalgebra of the algebra $\mathfrak{C}$ in [15, Defnition 5.1] as foillows. Let Z be the finite subsignature of L obtained by deletng all r_{jk}^i for $i > 0$ but keeping r_{jk}^0. For each $Z_{\infty\omega}^n$ formulu ϕ, Define the $L_{\infty\omega}$ formula $\hat{(\phi)}$ to be the result of replacing each subformula $r_{jk}^0(x, y)$ in ϕ by $\bigvee_{i\in\omega} r_{jk}^i(x, y)$. It is clearly a finite subagebra of $\mathfrak{C}$ with atoms $\hat{\alpha}^W$ where α is an MCA Z^n formula as defined in [15].

Corollary 3.7. There are infinitely many subvarieties of CA_n containing RCA_n that are not atom-canonical.

Proof: It is known that for any pair of ordinals $\alpha < \beta$, $\mathbf{S}\mathsf{Nr}_\alpha\mathsf{CA}_\beta$ is a variety, and that for $k \geq 1$ an $2 < n < \omega$, $\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{n+k+1} \subsetneq \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{n+k}$ [12, Chapter 15] □

Using the previous algebraic result on non atom canonicity, we adress algebraically a version of the omitting types theorems in the framework of the *clique guarded* n–variable fragments of first order logic. We define the notion of *clique guarded semantics.*

Definition 3.8. Let $2 < n \leq m < \omega$. Let M be the base of a relativized representation of $\mathfrak{A} \in \mathsf{CA}_n$ witnessed by an injective homomorphism $f : \mathfrak{A} \to \wp(V)$, where $V \subseteq {}^n\mathsf{M}$ and $\bigcup_{s\in V} \mathsf{rng}(s) = \mathsf{M}$. We write $\mathsf{M} \models a(s)$ for $s \in f(a)$. Let $\mathfrak{L}(\mathfrak{A})^m$ be the first order signature using m variables and one n–ary relation symbol for each element in A. Let $\mathfrak{L}(\mathfrak{A})^m_{\infty,\omega}$ be the infinitary extension of $\mathfrak{L}(\mathfrak{A})^m$ allowing infinite conjunctions. Then *an n–clique* is a set $C \subseteq \mathsf{M}$ such $(a_1, \ldots, a_n) \in V = 1^{\mathsf{M}}$ for distinct $a_1, \ldots, a_n \in C$.

Let $\mathsf{C}^m(\mathsf{M}) = \{s \in {}^m\mathsf{M} : \mathsf{rng}(s)$ is an n–clique$\}$. $\mathsf{C}^m(\mathsf{M})$ is called the *n–Gaifman hypergraph of* M, with the n–hyperedge relation 1^{M}.
The *clique guarded semantics* $\models_c$ are defined inductively. For atomic formulas and Boolean connectives they are defined like the classical case and for existential quantifiers (cylindrifiers) they are defined as follows: for $\bar{s} \in {}^m\mathsf{M}$, $i < m$, $\mathsf{M}, \bar{s} \models_c \exists x_i\phi \iff$ there is a $\bar{t} \in \mathsf{C}^m(\mathsf{M})$, $\bar{t} \equiv_i \bar{s}$ such that $\mathsf{M}, \bar{t} \models \phi$.

(1) We say that M is *an m–square representation* of $\mathfrak{A}$, if for all $\bar{s} \in \mathsf{C}^m(\mathsf{M}), a \in \mathfrak{A}$, $i < n$, and injective map $l : n \to m$, whenever $\mathsf{M} \models \mathsf{c}_i a(s_{l(0)}, \ldots, s_{l(n-1)})$, then there is a $\bar{t} \in \mathsf{C}^m(\mathsf{M})$ with $\bar{t} \equiv_i \bar{s}$, and $\mathsf{M} \models a(t_{l(0)}, \ldots, t_{l(n-1)})$. M is *a complete m–square representation of* $\mathfrak{A}$ *via* f, or simply a complete representation of $\mathfrak{A}$ if $f(\sum X) = \bigcup_{x\in X} f(x)$, for all $X \subseteq \mathfrak{A}$ for which $\sum X$ exists. (Like in the classical case this is equivalent to that $\mathfrak{A}$ is atomic and that $\bigcup_{x\in\mathsf{At}\mathfrak{A}} f(x) = 1^{\mathsf{M}}$).

(2) We say that M is an *(infinitary) m–flat representation* of $\mathfrak{A}$ if it is m–square and for all $\phi \in (\mathfrak{L}(\mathfrak{A})^m_{\infty,\omega})\mathfrak{L}(\mathfrak{A})^m$, for all $\bar{s} \in \mathsf{C}^m(\mathsf{M})$, for all distinct $i, j < m$, $\mathsf{M} \models_c [\exists x_i\exists x_j\phi \longleftrightarrow \exists x_j\exists x_i\phi](\bar{s})$. Complete representability is defined like for squareness.

The proof of the following lemma can be distilled from its RA analogue [12, Theorem 13.20], by reformulating deep concepts originally introduced by Hirsch and Hodkinson for RAs in the CA context. cf. [12, Definitions 12.1, 12.9, 12.10, 12.25, Propositions 12.25, 12.27].

THEOREM 3.9. *[12, Theorems 13.45, 13.36]. Assume that* $2 < n < m < \omega$ *and let* $\mathfrak{A} \in \mathsf{CA}_n$. *Then* $\mathfrak{A} \in \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_m \iff \mathfrak{A}$ *has an infinitary* m*–flat representation* $\iff \mathfrak{A}$ *has an* m*–flat representation. In particular, the variety of algebras having* $m+1$*-flat representations is not finitely axiomatizable over the variety of algebras having* m*–flat representations.*

PROOF: We give (more than) a glimpse of the ideas used. We prove first that the existence of m–flat representations, implies the existence of m–dilations. Let M be an m–flat representation of $\mathfrak{A}$. We show that $\mathfrak{A} \subseteq \mathsf{Nr}_n\mathfrak{D}$, for some $\mathfrak{D} \in \mathsf{CA}_m$, For $\phi \in \mathfrak{L}(\mathfrak{A})^m$ (as defined above), let $\phi^{\mathsf{M}} = \{\bar{a} \in \mathsf{C}^m(\mathsf{M}) : \mathsf{M} \models_c \phi(\bar{a})\}$, where $\mathsf{C}^m(\mathsf{M})$ is the n–Gaifman hypergraph. Let $\mathfrak{D}$ be the algebra with universe $\{\phi^M : \phi \in \mathfrak{L}(\mathfrak{A})^m\}$ and with cylindric operations induced by the n-clique–guarded (flat) semantics. Recall that for $r \in \mathfrak{A}$, and $\bar{x} \in \mathsf{C}^m(\mathsf{M})$, we identify r with the formula it defines in $\mathfrak{L}(\mathfrak{A})^m$, and we write $r(\bar{x})^{\mathsf{M}} \iff \mathsf{M}, \bar{x} \models_c r$. Then certainly $\mathfrak{D}$ is a subalgebra of the Crs_m (the class of algebras whose units are arbitrary sets of m–ary sequences) with domain $\wp(\mathsf{C}^m(\mathsf{M}))$, so $\mathfrak{D} \in \mathsf{Crs}_m$ with unit $1^{\mathfrak{D}} = \mathsf{C}^m(\mathsf{M})$. Since M is m–flat, then cylindrifiers in $\mathfrak{D}$ commute, and so $\mathfrak{D} \in \mathsf{CA}_m$. Now define $\theta : \mathfrak{A} \to \mathfrak{D}$, via $r \mapsto r(\bar{x})^{\mathsf{M}}$. Then exactly like in the proof of [12, Theorem 13.20], θ is a neat embedding, that is, $\theta(\mathfrak{A}) \subseteq \mathsf{Nr}_n\mathfrak{D}$. It is straightforward to check that θ is a homomorphism. We show that θ is injective. Let $r \in A$ be non-zero. Then M is a relativized representation, so there is $\bar{a} \in M$ with $r(\bar{a})$, hence $\bar{a}$ is a clique in M, and so $M \models r(\bar{x})(\bar{a})$, and $\bar{a} \in \theta(r)$, proving the required. M itself might not be infinitary m–flat, but one can build an infinitary m–flat representation of $\mathfrak{A}$, whose base is an ω–saturated model of the consistent first order theory, stipulating the existence of an m–flat representation, cf. [12, Proposition 13.17, Theorem 13.46 items (6) and (7)]. The inverse implication (existence of m–dilations $\implies$ existence of m–flat represenations) is harder. One constructs from the given m–dilation, an m–dimensional hyperbasis (redeined to adapt to CA_ns without too much difficulty) from which the required m– relativized representation is built. This can be done in a step–by step manner treating the hyperbasis as a ‘saturated set of mosaics’, cf. [12, Proposition 12.37].

The last part follows from [13, §15.1-3] where it is proved that $\mathsf{SNr}_n\mathsf{CA}_{m+1}$ is not finitely axiomatizable over $\mathsf{SNr}_n\mathsf{CA}_m$. $\square$

Lemma 3.10. *Let $2 < n < m < \omega$, and $\mathfrak{A} \in \mathsf{CA}_n$ be an atomic algebra. Then $\mathfrak{A}$ has a complete m-square representation $\iff$ $\exists$ has a winning strategy in $G^m_\omega(\mathsf{At}\mathfrak{A})$.*

Proof: [22, Lemma 5.8]. $\square$

Corollary 3.11. There exists $\mathfrak{A} \in \mathsf{Cs}_n$ such that $\mathfrak{CmAt}\mathfrak{A}$ does not have an $n+3$-square representation.

Proof: This follows from the previous Lemma, together with the proof of (c) in Theorem 3.5 by observing that $\forall$ has a winning strategy in $G^{n+3}_\omega\mathsf{CA}_{n+1,n}$ (in finitely many rounds of course) without the need to reuse nodes. The game $\mathbf{G}^m$ is stronger than what is really needed. $\square$

Lemma 3.12. *if $\mathfrak{A} \in \mathsf{CA}_n$ has a complete m–flat representation, then $\mathfrak{A}$ is atomic and $\mathfrak{CmAt}\mathfrak{A}$ has an m-flat representation. An entirely anhalogous result holds by replacing m-flat by m-square.*

Proof: Atomicity is like the classical case [11]. Now let $f : \mathfrak{A} \to \wp(V)$ be a complete m–flat representation $\mathfrak{A}$ with $V \subseteq {}^n\mathsf{M}$ where M is the base of the representation, so that $\mathsf{M} = \bigcup_{s\in V} \mathsf{rng}(s)$. For $a \in \mathfrak{CmAt}\mathfrak{A}$, let $a\downarrow = \{x \in \mathsf{At}\mathfrak{A} : x \leq a\}$. Define $g : \mathfrak{CmAt}\mathfrak{A} \to \wp(V)$ by $g(a) = \bigcup_{x\in\downarrow a} f(x)$. Then g is a complete m-flat representation of $\mathfrak{CmAt}\mathfrak{A}$ with base M. $\square$

For an L_n theory T, $\mathfrak{Fm}_T$, denotes the Tarski–Lindenbaum quotient RCA_n corresponding to T where the quoitent modulo T is defined semantically. Given an L_n theory T and $m > n$, by an *m–flat model of T*, we understand an m– flat representation of $\mathfrak{Fm}_T$ when $m < \omega$, and an ordinary representation of $\mathfrak{Fm}_T$ if m is infinite. An *atomic L_n* theory T is one for which $\mathfrak{Fm}_T$ is atomic. A *co-atom of T* is a formula ϕ such that $(\neg\phi)_T$ is an atom in $\mathfrak{Fm}_T$.

Corollary 3.13. There is a countable, atomic and complete L_n theory T such that the non-principal type consisting of co–atoms cannot be omitted in an $n+3$-square, *a fortiori* $n+3$-flat model.

Proof: Let $\mathfrak{A} \in \mathsf{Cs}_n$ be countable (and simple) such that its Dedekind–MacNeille completion does not have an $n+3$-square representation. This

$\mathfrak{A}$ exists by Theorem 3.5. By [10, § 4.3], we can (and will) assume that $\mathfrak{A} = \mathfrak{Fm}_T$ for a countable, atomic theory L_n theory T. Let Γ be the n–type consisting of co–atoms of T. Then Γ is a non principal type that cannot be omitted in any $n+3$–square model, for if M is an $n+3$–square model omitting Γ, then M would be the base of a complete $n+3$-square representation of $\mathfrak{A}$, giving, by Lemma 3.12, representation of $\mathfrak{Cm}\mathsf{At}\mathfrak{A}$, which is impossible. □

There exists a countable, complete and atomic L_n first order theory T in a signature L such that the type Γ consisting of co-atoms in the cylindric Tarski–Lindenbaum quotient algebra $\mathfrak{Fm}_T$ is realizable in every m–square model, but Γ cannot be isolated using $\leq l$ variables, where $n \leq l < m \leq \omega$. A co-atom of $\mathfrak{Fm}_T$ is the negation of an atom in $\mathfrak{Fm}_T$, that is to say, is an element of the form $\Psi/\equiv_T$, where $\Psi/\equiv_T = (\neg\phi/\equiv_T) = \sim (\phi/\equiv_T)$ and $\phi/\equiv_T$ is an atom in $\mathfrak{Fm}_T$ (for L-fomrulas, ϕ and ψ). Here the quotient algebra $\mathfrak{Fm}_T$is formed relative to the congruence relaton of semantical equivalence moduol T; for formulas ϕ and θ in the signature L , $\phi \equiv_T \theta \iff T \models \phi \longleftrightarrow \theta$. An m-square model of T is an m-square represenation of $\mathfrak{Fm}_T$. The statement $\Psi(l,m)$, short for Vaught's Theorem (VT*) fails at (the parameters) l and m.* Let $\mathsf{VT}(l,m)$ stand for VT *holds at l and m*, so that by definition $\Psi(l,m) \iff \neg\mathsf{VT}(l,m)$. We also include $l=\omega$ in the equation by defining $\mathsf{VT}(\omega,\omega)$ as VT holds for $L_{\omega,\omega}$: Atomic countable first order theories have atomic countable models. It is well known that $\mathsf{VT}(\omega,\omega)$ is a direct consequence of the Orey-Henkin OTT. Let $2 < n \leq l < m \leq \omega$. Consider the statemens $\Psi(l,m)$ and $\mathsf{VT}(l,m) = \neg\Psi(l,m)$ as defined in the introduction. Recall that $\mathsf{VT}(\omega,\omega)$ is just Vaught's theorem, namely, countable atomic theories have atomic countable models. For $2 < n \leq l < m \leq \omega$ and $l = m = \omega$, it is likely and plausible that (**): $\mathsf{VT}(l,m) \iff l = m = \omega$. In other words: *Vaught's theorem holds only in the limiting case when $l \to \infty$ and $m = \omega$ and not 'before'.* We give sufficient condition for (**) to happen.

THEOREM 3.14. *For $2 < n < \omega$ and $n \leq l < \omega$, $\Psi(n, n+3)$ and $\Psi(l,\omega)$ hold. Furthermore, if for each $n < m < \omega$, there exists a finite relation algebra $\mathfrak{R}_m$ having $m-1$ strong blur and no m-dimensional relational basis, then (**) above for* VT *holds.*

PROOF: We start by the last part. Let $\mathfrak{R}_m$ be as in the hypothesis with strong $m-1$–blur (J,E) and m-dimensional relational basis. We 'blow up and blur' $\mathfrak{R}_m$ in place of the Maddux algebra $\mathfrak{E}_k(2,3)$ blown up and blurred in [2, Lemma 5.1], where $k < \omega$ is the number of non-identity atoms and k depends recursively on l, giving the desired strong l–blurness, cf. [2, Lemmata 4.2, 4.3]. Now take $\mathfrak{A} = \mathfrak{Bb}_n(\mathfrak{R}_m, J, E)$ the term algebra obtained after blowing up and blurring $\mathfrak{R}$ to a weakly representable atom structure [2]. Then $\mathfrak{A} \in \mathsf{RCA}_n \cap \mathsf{Nr}_n\mathsf{CA}_l$ but $\mathfrak{A}$ has no complete m-square representation. For if it did, then a complete m–square representation of an atomic $\mathfrak{B} \in \mathsf{CA}_n$ induces an m–square representation of $\mathfrak{Cm}\mathsf{At}\mathfrak{B}$. But $\mathfrak{Cm}\mathsf{At}\mathfrak{A}$ does not have an m-square representation, because $\mathfrak{R}$ does not have an m-dimensional relational basis, and $\mathfrak{R} \subseteq \mathfrak{Ra}\mathfrak{Cm}\mathsf{At}\mathfrak{A}$. So an m-square representation of $\mathfrak{Cm}\mathsf{At}\mathfrak{A}$ induces one of $\mathfrak{R}$ which by Lemma 3.9 implies that $\mathfrak{R}$ has no m-dimensional relational basis, a contradiction.

We prove $\Psi(m-1, m)$, hence the required, namely, (**). By [10, § 4.3], we can (and will) assume that $\mathfrak{A} = \mathfrak{Fm}_T$ for a countable, simple and atomic theory L_n theory T. Let Γ be the n–type consisting of co–atoms of T. Then Γ is realizable in every m–square model, for if M is an m–square model omitting Γ, then M would be the base of a complete m–square representation of $\mathfrak{A}$, and so by Theorem 3.9 $\mathfrak{A} \in \mathbf{S}_c\mathsf{Nr}_n\mathsf{D}_m$ which is impossible. Suppose for contradiction that ϕ is an $m-1$ witness, so that $T \models \phi \to \alpha$, for all $\alpha \in \Gamma$, where recall that Γ is the set of coatoms. Then since $\mathfrak{A}$ is simple, we can assume without loss that $\mathfrak{A}$ is a set algebra with base M say. Let $\mathsf{M} = (M, R_i)_{i\in\omega}$ be the corresponding model (in a relational signature) to this set algebra in the sense of [10, § 4.3]. Let ϕ^{M} denote the set of all assignments satisfying ϕ in M. We have $\mathsf{M} \models T$ and $\phi^{\mathsf{M}} \in \mathfrak{A}$, because $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_{m-1}$. But $T \models \exists x\phi$, hence $\phi^{\mathsf{M}} \neq 0$, from which it follows that ϕ^{M} must intersect an atom $\alpha \in \mathfrak{A}$ (recall that the latter is atomic). Let ψ be the formula, such that $\psi^{\mathsf{M}} = \alpha$. Then it cannot be the case that $T \models \phi \to \neg\psi$, hence ϕ is not a witness, contradiction and we are done. Finally, $\Psi(n, n+3)$ and $\Psi(l, \omega)$ $(n \leq l < \omega)$ follow from Corollary 3.13 and [2]. □

Corollary 3.15. There exists an atomic $\mathcal{T} \in \mathsf{RRA}$ and an atomic $\mathfrak{A} \in \mathsf{RCA}_n$ such that their Dedekind–MacNeille completions do not embed into their canonical extensions.

Proof: We prove the CA case only. The RA case is entirely analagous. Since RCA_n is canonical [10] and $\mathfrak{A} \in \mathsf{RCA}_n$, then its canonical extension $\mathfrak{A}^+ \in \mathsf{RCA}_n$. But $\mathfrak{Cm}\mathsf{At}\mathfrak{A} \notin \mathsf{RCA}_n$, so it does not embed into $\mathfrak{A}^+$, because RCA_n is a variety, *a fortiori* closed under $\mathbf{S}$. □

Algebraically, so–called *persistence properties* refer to closure of a variety V under passage from a given algebra $\mathfrak{A} \in \mathsf{V}$ to some 'larger' algebra $\mathfrak{A}^*$. Atom–canonicity is concerned with closure under forming Dedekind–MacNeille completions. Atom–canonicity, implies the algebraic property of *single-persistence* which in turn corresponds in modal logic to the notion of a formula being *di-persistent.* A formula is *di-persistent* if whenever it is valid in some *general discrete* frame $(\mathfrak{F}, P)$, that is, P contains all singletions, then is valid in the Kripke frame $\mathfrak{F}$ [4, §5.6]. Sometimes Dedekind–MacNeille completions, investigated for cylindric algebras by Monk, are referred to as *minimal completions*, the name suggesting that Dedekind–MacNeille completion of an algebra $\mathfrak{A}$ is the 'smallest' in the sense that it embeds into other any completion of $\mathfrak{A}$. Here by a completion we understand any complete algebra containing $\mathfrak{A}$. Canonicity, which is the most prominent persistence property in modal logic, the 'large algebra' $\mathfrak{A}^*$ is the canonical embedding algebra (or perfect) extension of $\mathfrak{A}$, a complex algebra based on the *ultrafilter frame* of $\mathfrak{A}$, in symbols $\mathsf{Uf}\mathfrak{A}$, whose underlying set is the set of all Boolean ultrafilters of $\mathfrak{A}$. This is another completion of $\mathfrak{A}$. The Dedekind–MacNeille completion of a BAO and its canonical extension coincide $\iff$ $\mathfrak{A}$ is finite. By the last result formulated in Corollary 3.15 the term *minimal* is misleading. A minimal completion of $\mathfrak{A} \in \mathsf{RCA}_n$s, namely $\mathfrak{CmAt}\mathfrak{A}$, may not embed into its canonical extension $\mathfrak{A}^+ = \mathfrak{Cm}\mathsf{Uf}\mathfrak{A}$.

Canonicity corresponds to the notion of a formula being *dpersistent* [4, Definition 5.65, Proposition 5.85]. A modal formula in L_n is *canonical* if it is validated in the *canonical frame* of every normal modal logic containing ϕ [4, Definition 4.30]. Algebraically, ϕ is canonical $\iff$ ϕ translates to an equation in the signature of RCA_n that is preserved under canonical extensions. An example of formulas that are both di-persistent and canonical (d-persistent) are the so-called *very simple Sahlqvist formulas* [4, Theorem 5.90] which are, as the name suggests, instances of Sahlqvist formulas [12, Definition 3.51].

Sahlqvist formulas are a certain kind of modal formula with remarkable properties. The Sahlqvist correspondence theorem states that every Sahlqvist formula corresponds to a first order definable class of Kripke frames. Sahlqvist's definition characterizes a decidable set of modal formulas with first-order correspondents. Since it is undecidable, by Chagrova's theorem, whether an arbitrary modal formula has a first-order correspondent [4, Theorem 3.56], there are formulas with first-order frame conditions that are not Sahlqvist. But this is not the end of the story, for it might be the case that every modal formula with a first order correspondant is *equivalent* to a Sahlqvist one, which is not the case [4, Example 3.57]. The reader is referred to [4] and [12, 2.7] for more on aspects of duality for BAOs and in particular for Sahlvist axiomatizability in general. By the dualiity theory betwem BAOs and multimodal logic, Sahlqvist formulas in the latter transfrm to Sahlqvist equations in modal algebras. A variety V of BAOs is Sahlqvist if it can be axiomatized by Sahlvist equations.

THEOREM 3.16. *For any* $2 < n < m \leq \omega$ *the variety* $\mathsf{SNr}_n\mathsf{CA}_m$ *is not Sahlqvist. Conversely, for any pair of infinite ordinals* $\alpha < \beta$*, the varieties* $\mathsf{SNr}_\alpha\mathsf{PA}_\beta$ *and* $\mathsf{SNr}_\alpha\mathsf{PEA}_\beta$ *are Sahlqvist, and is closed under Dedekind–MacNeille completions.*

PROOF: Let $\alpha < \beta$ be infinite ordinals. Then $\mathbf{S}\mathsf{Nr}_\alpha\mathsf{PA}_\beta = \mathsf{Nr}_\alpha\mathsf{PA}_\beta = \mathsf{PA}_\alpha$, cf. the remark before [10, Theorem 5.4.17]. The last is axiomatized by positive equations [10, Definition 5.4.1] which are Sahlqvist. Applying [25] we are done. The PEA case is entirely analogous using the axiomatization in the aforementioned definition. □

Let $2 < n < \omega$. We approach the modal version of L_n without equality, namely, $\mathbf{S5}^n$. The corresponding class of modal algebras is the variety RDf_n of *diagonal free* $\mathsf{RCA}_n s$ [10]. Let $\mathfrak{Rd}_{df}$ denote 'diagonal free reduct'.

LEMMA 3.17. *Let* $2 < n < \omega$*. If* $\mathfrak{A} \in \mathsf{CA}_n$ *is such that* $\mathfrak{Rd}_{df}\mathfrak{A} \in \mathsf{RDf}_n$*, and* $\mathfrak{A}$ *is generated by* $\{x \in \mathfrak{A} : \Delta x \neq n\}$ *(with other* CA *operations) using infinite intersections, then* $\mathfrak{A} \in \mathsf{RCA}_n$*.*

PROOF: Easily follows from [10, Lemma 5.1.50, Theorem 5.1.51]. Assume that $\mathfrak{A} \in \mathsf{CA}_n$, $\mathfrak{Rd}_{df}\mathfrak{A}$ is a set algebra (of dimension n) with base U, and $R \subseteq U \times U$ are as in the hypothesis of [10, Theorem 5.1.49]. Let $E = \{x \in A : (\forall x, y \in {}^nU)(\forall i < n)(x_iRy_i \implies (x \in X \iff y \in X))\}$. Then $\{x \in \mathfrak{A} : \Delta x \neq n\} \subseteq E$ and $E \in \mathsf{CA}_n$ is closed under infinite intersections. The required follows. □

THEOREM 3.18. *For* $2 < n < \omega$*,* RDf_n *is not atom–canonical, hence not Sahlqvist.*

PROOF: It is enough to show that $\mathfrak{CmAt}\mathfrak{A}$, where $\mathfrak{A}$ is constructed in Theorem 3.5 is generated by elements whose dimension sets have cardinality $< n$ using infinite unions, for in this case $\mathfrak{Rd}_{df}A$ will be atomic, countable and representable, but having no complete representation. Indeed, by Lemma 3.17 and Theorem 3.5, $\mathfrak{Rd}_{df}\mathfrak{Cm}\mathsf{At}\mathfrak{A} = \mathfrak{Cm}\mathsf{At}\mathfrak{Rd}_{df}\mathfrak{A}$ will not be representable. We show that for any rainbow atom $[a]$, $a : n \to \Gamma$, Γ a coloured graph, that $[a] = \prod_{i<n} \mathsf{c}_i[a]$. Clearly $\leq$ holds. Assume that $b : n \to \Delta$, Δ a coloured graph, and $[a] \neq [b]$. We show that $[b] \notin \prod_{i<n} \mathsf{c}_i[a]$ by which we will be done. Because a is not equivalent to b, we have one of two possibilities; either $(\exists i, j < n)(\Delta(b(i), b(j) \neq \Gamma(a(i), a(j))$ or $(\exists i_1, \dots, i_{n-1} < n)(\Delta(b_{i_1}, \dots, b_{i_{n-1}}) \neq \Gamma(a_{i_1}, \dots, a_{i_{n-1}}))$. Assume the first possibility: Choose $k \notin \{i, j\}$. This is possible because $n > 2$. Assume for contradiction that $[b] \in \mathsf{c}_k[a]$. Then $(\forall i, j \in n \setminus \{k\})(\Delta(b(i), b(j)) = \Gamma(a(i)a(j)))$. By assumption and the choice of k, $(\exists i, j \in n \setminus k)(\Delta(b(i), b(j)) \neq \Gamma(a(i), a(j)))$, contradiction. For the second possibility, one chooses $k \notin \{i_1, \dots i_{n-1}\}$ and proceeds like the first case deriving an analogous contradiction. □

$\mathbf{K}^n$ is the logic of n-ary product frames, of the form $(W_i, R_i)_{i<n}$ where for each $i < n$, R_i is any any relation on W_i. On the other hand, $\mathbf{S5}^n$ can be regarded as the logic of n–ary product frames of the form $(W_i, R_i)_{i<n}$ such that for each $i < n$, R_i is an equivalence relation. It is known that logics between $\mathbf{K}^n$ and $\mathbf{S5}^n$ are quite complicated, cf. [16] for a detailed overview. Theorem 3.19 to be proved in a moment adds to their complexity.

It is known that modal languages can come to grips with a strong fragment of second order logic. Modal formulas translate to second order formulas, *their correspondants* on frames. Some of these formulas can be *genuinely second order*; they are not equivalent to first order formulas. An example is the *McKinsey formula*: $\Box\Diamond p \to \Diamond\Box p$. This can be proved by showing that its correspondant violates the downward Löwenheim- Skolem Theorem. The next proposition bears on the last two issues. For a class $\mathbf{L}$ of frames, let $\mathfrak{L}(\mathbf{L})$ be the class of modal formulas valid in $\mathbf{L}$. It is difficult to find explicity (necessarily) infinite axiomatizations for $\mathbf{S5}^n$ as well:

THEOREM 3.19. *Let $2 < n < \omega$. There is no axiomatization of $\mathbf{S5}^n$ with formulas having first order correspondence. For any canonical logic $\mathfrak{L}$ between $\mathbf{K}^n$ and $\mathbf{S5}^n$, it is undecidable to tell whether a finite frame is a frame for $\mathfrak{L}$, $\mathfrak{L}$ cannot be finitely axiomatized in kth order logic (for any finite k), and $\mathfrak{L}$ cannot be axiomatized by canonical formulas, a fortiori Sahlqvist formulas.*

PROOF: Let $\mathbf{L}$ be the class of square frames for $\mathbf{S5}^n$. Then $\mathfrak{L}(\mathbf{L}) = \mathbf{S5}^n$ [16, p. 192]. But the class of frames $\mathfrak{F}$ valid in $\mathfrak{L}(\mathbf{L})$ coincides with the class of *strongly representable* Df_n *atom structures* which is *not elementary* as proved in [5]. This gives the first required result for $\mathbf{S5}^n$. With lemma 3.17 at our disposal, a slightly different proof can be easily distilled from the construction adressing CAs in [13] or [14]. We adopt the construction in the former reference, using the Monk–like CA_ns $\mathfrak{M}(\Gamma)$, Γ a graph, as defined in [13, Top of p.78]. For a graph $\mathfrak{G}$, let $\chi(\mathfrak{G})$ denote it chromatic number. Then it is proved in *op.cit* that for any graph Γ, $\mathfrak{M}(\Gamma) \in \mathsf{RCA}_n \iff \chi(\Gamma) = \infty$. By lemma 3.17, $\mathfrak{Rd}_{df}\mathfrak{M}(\Gamma) \in \mathsf{RDf}_n \iff \chi(\Gamma) = \infty$, because $\mathfrak{M}(\Gamma)$ is generated by the set $\{x \in \mathfrak{M}(\Gamma) : \Delta x \neq n\}$ using infinite unions.

Now we adopt the argument in [13]. Using Erdos' probabalistic graphs [7], for each finite κ, there is a finite graph G_κ with $\chi(G_\kappa) > \kappa$ and with no cycles of length $< \kappa$. Let Γ_κ be the disjoint union of the G_l for $l > \kappa$. Then $\chi(\Gamma_\kappa) = \infty$, and so $\mathfrak{Rd}_{df}\mathfrak{M}(\Gamma_\kappa)$ is representable. Now let Γ be a non-principal ultraproduct $\Pi_D\Gamma_\kappa$ for the Γ_κs. For $\kappa < \omega$, let σ_κ be a first-order sentence of the signature of the graphs stating that there are no cycles of length less than κ. Then $\Gamma_l \models \sigma_\kappa$ for all $l \geq \kappa$. By Loś's Theorem, $\Gamma \models \sigma_\kappa$ for all κ. So Γ has no cycles, and hence by $\chi(\Gamma) \leq 2$. Thus $\mathfrak{Rd}_{df}\mathfrak{M}(\Gamma)$ is not representable. (Observe that the the term algebra $\mathfrak{Tm}\mathsf{At}(\mathfrak{M}(\Gamma))$ is representable (as a CA_n), because the class of weakly representable atom structures is elementary [12, Theorem 2.84].) Since Sahlqvist formulas have first order correspondants, then $\mathbf{S5}^n$ is not Sahlqvist. In [14], it is proved that it is undecidable to tell whether a finite frame is a frame for $\mathfrak{L}$, and this gives the non-finite axiomatizability result required as indicated in *op. cit*, and obviously implies undecidability. The rest follows by transferring the required results holding for $\mathbf{S5}^n$ [5, 14] to $\mathfrak{L}$ since $\mathbf{S5}^n$ is finitely axiomatizable over $\mathfrak{L}$, and any axiomatization of RDf_n must contain infinitely many non-canonical equations. □

Results involving notions like atom–canonicity, for the infinite dimensional case, are extremely rare in algebraic logic [13, Problem 3.8.3]; in fact, almost non-existent. We present a conditional result (the condition is very likely to be true). For each finite $k \geq 3$, let $\mathfrak{A}(k)$ be an atomic countable simple representable CA_k such that $\mathfrak{B}(k) = \mathfrak{CmAt}\mathfrak{A}(k) \notin \mathsf{SNr}_k\mathsf{CA}_{k+3}$. We know that such algebras exist by Theorem 3.5. *We make the following assumption:* (*) *Assume that* $\mathfrak{B}(m)$ *embeds into* $\mathfrak{Rd}_m\mathfrak{B}(t)$, *whenever* $3 \leq m < t < \omega$. Our next theorem lifts Theorem 3.5 to the transfinite conditionally (modulo (*)).

THEOREM 3.20. *Assume that (*) above holds for the algebras constructed in Theorem 3.5 (or any other algebras). Then for* $k \geq 3$, $\mathsf{SNr}_\omega\mathsf{CA}_{\omega+k}$ *is not atom–canonical. In particular,* RCA_ω *cannot be axiomatized by (a necessarily infinite schema of) Sahlqvist equations.*

PROOF: For each finite $k \geq 3$, let $\mathfrak{A}(k)$ and $\mathfrak{B}(k)$ be the algebras constructed in Theorem 3.5 (of dimension k) and assume further that the assumption abbreviated by (*) preceding the theorem holds for the algebras constructed in *op.cit.* Let $\mathfrak{A}_k$ be an (atomic) algebra having the signature of CA_ω such that $\mathfrak{Rd}_k\mathfrak{A}_k = \mathfrak{A}(k)$. Analogously, let $\mathfrak{B}_k$ be an algebra having the signature of CA_ω such that $\mathfrak{Rd}_k\mathfrak{B}_k = \mathfrak{B}(k)$, and we require in addition that $\mathfrak{B}_k = \mathfrak{Cm}(\mathsf{At}\mathfrak{A}_k)$. We use a lifting argument using ultraproducts. Let $\mathfrak{B} = \Pi_{i\in\omega\setminus 3}\mathfrak{B}_i/F$. It is easy to show that $\mathfrak{A} = \Pi_{i\in\omega\setminus 3}\mathfrak{A}_i/F \in \mathsf{RCA}_\omega$. Furthermore, a direct computation gives:

$$\begin{aligned}\mathfrak{Cm}\mathsf{At}\mathfrak{A} &= \mathfrak{Cm}(\mathsf{At}[\Pi_{i\in\omega\setminus 3}\mathfrak{A}_i/F]) = \mathfrak{Cm}[\Pi_{i\in\omega\setminus 3}(\mathsf{At}\mathfrak{A}_i)/F)] \\ &= \Pi_{i\in\omega\setminus 3}(\mathfrak{Cm}(\mathsf{At}\mathfrak{A}_i)/F) = \Pi_{i\in\omega\setminus 3}\mathfrak{B}_i/F \\ &= \mathfrak{B}.\end{aligned}$$

By the same token, $\mathfrak{B} \in \mathsf{CA}_\omega$. Assume for contradiction that $\mathfrak{B} \in \mathsf{SNr}_\omega\mathsf{CA}_{\omega+3}$. Then $\mathfrak{B} \subseteq \mathfrak{Nr}_\omega\mathfrak{C}$ for some $\mathfrak{C} \in \mathsf{CA}_{\omega+3}$. Let $3 \leq m < \omega$ and let $\lambda : m+3 \to \omega+3$ be the function defined by $\lambda(i) = i$ for $i < m$ and $\lambda(m+i) = \omega + i$ for $i < 3$. Then we get (**): $\mathfrak{Rd}^\lambda\mathfrak{C} \in \mathsf{CA}_{m+3}$ and $\mathfrak{Rd}_m\mathfrak{B} \subseteq \mathfrak{Nr}_m\mathfrak{Rd}^\lambda\mathfrak{C}$. By assumption let $I_t : \mathfrak{B}_m \to \mathfrak{Rd}_m\mathfrak{B}_t$ be an injective homomorphism for $3 \leq m < t < \omega$. Let $\iota(b) = (I_t b : t \geq m)/F$ for $b \in \mathfrak{B}_m$. Then ι is an injective homomorphism that embeds $\mathfrak{B}_m$ into $\mathfrak{Rd}_m\mathfrak{B}$. By (**) we know that $\mathfrak{Rd}_m\mathfrak{B} \in \mathsf{SNr}_m\mathsf{CA}_{m+3}$, hence $\mathfrak{B}_m \in \mathsf{SNr}_m\mathsf{CA}_{m+3}$, too. This is a contradiction, and we are done. □

4. Positive results on omitting types

We start by recalling certain cardinals that play a key role in (positive) omitting types theorems for $L_{\omega,\omega}$. Let covK be the cardinal used in [19, Theorem 3.3.4]. The cardinal $\mathfrak{p}$ satisfies $\omega < \mathfrak{p} \leq 2^\omega$ and has the following property: If $\lambda < \mathfrak{p}$, and $(A_i : i < \lambda)$ is a family of meager subsets of a Polish space X (of which Stone spaces of countable Boolean algebras are examples) then $\bigcup_{i\in\lambda} A_i$ is meager. For the definition and required properties of $\mathfrak{p}$, witness [9, pp. 3, 44–45, Corollary 22c].

It is consistent that $\omega < \mathfrak{p} < \mathsf{covK} \leq 2^\omega$ [9], but it is also consistent that they are equal; equality holds for example in the Cohen real model of Solovay and Cohen. Martin's axiom implies that both cardinals are the continuum. To prove the main result on positive omitting types theorems, we need the following lemma due to Shelah:

LEMMA 4.1. *Assume that λ is an infinite regular cardinal. Suppose that T is a first order theory, $|T| \leq \lambda$ and ϕ is a formula consistent with T, then there exist models $\mathsf{M}_i : i < {}^\lambda 2$, each of cardinality λ, such that ϕ is satisfiable in each, and if $i(1) \neq i(2)$, $\bar{a}_{i(l)} \in M_{i(l)}$, $l = 1, 2,$, $\mathsf{tp}(\bar{a}_{l(1)}) = \mathsf{tp}(\bar{a}_{l(2)})$, then there are $p_i \subseteq \mathsf{tp}(\bar{a}_{l(i)})$, $|p_i| < \lambda$ and $p_i \vdash \mathsf{tp}(\bar{a}_{l(i)})$ ($\mathsf{tp}(\bar{a})$ denotes the complete type realized by the tuple $\bar{a}$)*

PROOF: [24, Theorem 5.16, Chapter IV]. □

In the next theorem $n < \omega$. Furthermore the maximality condition expressed in ultrafilters (which are *maximal* filters) delineates the edge of an independent statement to a provable one. Considering only filters leads to an independent statement, cf. [19, Theorem 3.2.8]:

THEOREM 4.2. *Let μ be a countable or regular uncountable cardinal. Let $\mathfrak{A} \in \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_\omega$ be such that $|A| \leq 2^\mu$. Let $\lambda < 2^\mu$ and let $\mathbf{X} = (X_i : i < \lambda)$ be a family of non-principal types of $\mathfrak{A}$. Then the following hold:*

1. *If $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_\omega$ and the X_is are non-principal ultrafilters, then $\mathbf{X}$ can be omitted in a Gs_n. Furthrmore, the condition of maximality cannot be dispensed with,*
2. *If $\mathfrak{A}$ is countable, then every subfamily of $\mathbf{X}$ of cardinality $< \mathfrak{p}$ can be omitted in a Gs_n; in particular, every countable subfamily of $\mathbf{X}$ can be omitted in a Gs_n, If $\mathfrak{A}$ is simple, then every subfamily of $\mathbf{X}$ of cardinlity $< \mathsf{covK}$ can be omitted in a Cs_n.*

PROOF: For the first item we prove the special case when $\mu\omega$. The general case follows from the fact that (**) below holds for any infinite regular cardinal. We assume that $\mathfrak{A}$ is simple (a condition that can be easily removed). We have $\prod^{\mathfrak{B}} X_i = 0$ for all $i < \kappa$ because, $\mathfrak{A}$ is a complete subalgebra of $\mathfrak{B}$. Since $\mathfrak{B}$ is a locally finite (if not replace $\mathfrak{B}$ by $\mathfrak{Sg}^{\mathfrak{B}}\mathfrak{A}$), we can assume that $\mathfrak{B} = \mathfrak{Fm}_T$ for some countable consistent theory T. For each $i < \kappa$, let $\Gamma_i = \{\phi/T : \phi \in X_i\}$. Let $\mathbf{F} = (\Gamma_j : j < \kappa)$ be the corresponding set of types in T. Then each Γ_j $(j < \kappa)$ is a non-principal and *complete n-type* in T, because each X_j is a maximal filter in $\mathfrak{A} = \mathfrak{Nr}_n\mathfrak{B}$.

(**) Let $(\mathsf{M}_i : i < 2^\omega)$ be a set of countable models for T that overlap only on principal maximal types; these exist by lemma 4.1. Asssume for contradiction that for all $i < 2^\omega$, there exists $\Gamma \in \mathbf{F}$, such that Γ is realized in M_i. Let $\psi : 2^\omega \to \wp(\mathbf{F})$, be defined by $\psi(i) = \{F \in \mathbf{F} : F \text{ is realized in } \mathsf{M}_i\}$. Then for all $i < 2^\omega$, $\psi(i) \neq \emptyset$. Furthermore, for $i \neq j$, $\psi(i) \cap \psi(j) = \emptyset$, for if $F \in \psi(i) \cap \psi(j)$, then it will be realized in M_i and M_j, and so it will be principal. This implies that $|\mathbf{F}| = 2^\omega$ which is impossible. Hence we obtain a model $\mathsf{M} \models T$ omitting $\mathbf{X}$ in which ϕ is satisfiable. The map f defined from $\mathfrak{A} = \mathfrak{Fm}_T$ to $\mathsf{Cs}_n^{\mathsf{M}}$ (the set algebra based on M [10, 4.3.4]) via $\phi_T \mapsto \phi^{\mathsf{M}}$, where the latter is the set of n–ary assignments in M satisfying ϕ, omits $\mathbf{X}$. Injectivity follows from the facts that f is non-zero and $\mathfrak{A}$ is simple. For the second part of (1), we use the construction in [23, Thgeorem 4.5], where an atomic $\mathfrak{B} \in \mathsf{Nr}_n\mathsf{CA}_\omega$ with uncountably many atoms that is not completely representable is constructed. This implies that the maximality condition cannot be dispensed with; else the set of co–atoms of $\mathfrak{B}$ call it X will be a non-principal type that cannot be omitted, because any Gs_n omitting X yields a complete representation of $\mathfrak{B}$, witness the last paragraph in [19].

For (2), we can assume that $\mathfrak{A} \subseteq_c \mathsf{Nr}_n\mathfrak{B}$, $\mathfrak{B} \in \mathsf{Lf}_\omega$. We work in $\mathfrak{B}$. Using the notation on [19, p. 216 of proof of Theorem 3.3.4] replacing $\mathfrak{Fm}_T$ by $\mathfrak{B}$, we have $\mathbf{H} = \bigcup_{i\in\lambda}\bigcup_{\tau\in V}\mathbf{H}_{i,\tau}$ where $\lambda < \mathfrak{p}$, and V is the weak space ${}^\omega\omega^{(Id)}$, can be written as a countable union of nowhere dense sets, and so can the countable union $\mathbf{G} = \bigcup_{j\in\omega}\bigcup_{x\in\mathfrak{B}}\mathbf{G}_{j,x}$. So for any $a \neq 0$, there is an ultrafilter $F \in N_a \cap (S \setminus \mathbf{H} \cup \mathbf{G})$ by the Baire category theorem. This induces a homomorphism $f_a : \mathfrak{A} \to \mathfrak{C}_a$, $\mathfrak{C}_a \in \mathsf{Cs}_n$ that omits the given types, such that $f_a(a) \neq 0$. (First one defines f with domain $\mathfrak{B}$ as on p. 216, then restricts f to $\mathfrak{A}$ obtaining f_a the obvious way.) The map $g : \mathfrak{A} \to \mathbf{P}_{a\in\mathfrak{A}\setminus\{0\}}\mathfrak{C}_a$ defined via $x \mapsto (g_a(x) : a \in \mathfrak{A} \setminus \{0\})(x \in \mathfrak{A})$ is as required. In case $\mathfrak{A}$ is simple, then by properties of covK, $S \setminus (\mathbf{H} \cup \mathbf{G})$ is non-empty, so if $F \in S\setminus(\mathbf{H}\cup\mathbf{G})$, then F induces a non-zero homomorphism f with domain $\mathfrak{A}$ into a Cs_n omitting the given types. By simplicity of $\mathfrak{A}$, f is injective. □

Corollary 4.3.

(1) If T is a countable theory that admits elmination of quantifiers, and λ is a cardinal $< 2^{\aleph_0}$, and $\mathbf{F} = \langle \Gamma_i : i < \lambda\rangle$ is a family of complete non-principal types, then $\mathbf{F}$ can be omitted in a countable model of T.

(2) If T is any countable theory, then $< \mathfrak{p}$ non-principal types can be omitted; if T is complete, we can further replace $\mathfrak{p}$ by covK.

Proof: Let T be as given in a signature L having n variables. Let $\mathfrak{A} = \mathfrak{Fm}_T$, and $\mathbf{G}_i = \{\phi_T : \phi \in \Gamma_i\}$. Then $\mathbf{G}_i$ is a a non-principal ultrafilter; maximality follows fom the completeness of types considered. By completeness of T, $\mathfrak{A}$ is simple. Since T admits elimination of quantifiers, then $\mathfrak{Fm}_T \in \mathsf{Nr}_n\mathsf{CA}_\omega$. Indeed, let T_ω be the theory in the same signature L but using ω many variables. Let $\mathfrak{C} = \mathfrak{Fm}_{T_\omega}$ be the Tarski–Lindenbaum quotient algebra. Then $\mathfrak{C} \in \mathsf{CA}_\omega$; in fact $\mathfrak{C} \in \mathbf{I}\mathsf{Cs}_\omega$, and the map Φ defined from $\mathfrak{A}$ to $\mathfrak{Nr}_n\mathfrak{C}$ via $\phi/\equiv_T \mapsto \phi/\equiv_{T_\omega}$ is injective and bijective, that is to say, Φ having domain $\mathfrak{A}$ and codomain $\mathfrak{Nr}_n\mathfrak{C}$ is in fact *onto* $\mathfrak{Nr}_n\mathfrak{C}$ due to quantifier elimination. An application of Theorem 4.2 finishes the proof. The second part is proved exactly like the proof of [19, Theorem 3.2.4] replacing covK by $\mathfrak{p}$. □

Here we adress omitting types theorems for certain infinitary extensions of first order logic. Our treatment remains to be purely algebraic. For $\alpha \geq \omega$, we let Dc_α denote the class of *dimension complemented* $\mathsf{CA}_\alpha s$, so that $\mathfrak{A} \in \mathsf{Dc}_\alpha \iff \alpha \setminus \Delta x$ is infinite for every $x \in \mathfrak{A}$.

Theorem 4.4. *Let α be a countable infinite ordinal.*

(1) *There exists a countable atomic $\mathfrak{A} \in \mathsf{RCA}_\alpha$ such that the non-principal types of co–atoms cannot be omitted in a Gs_α,*

(2) *If $\mathfrak{A} \in \mathbf{S}_c\mathsf{Nr}_\alpha\mathsf{CA}_{\alpha+\omega}$ is countable, λ a cardinal $< \mathfrak{p}$ and $\mathbf{X} = (X_i : i < \lambda)$ is a family of non-principal types, then $\mathbf{X}$ can be omited in a Gws_α (in the sense of definition 2.1 upon replacing Gs_α by Gws_α).*

(3) *Assume that the assumption (*) formulated before Theorem 3.20 holds. Then there exists an atomic $\mathfrak{A} \in \mathsf{RCA}_\alpha$ such that its Dedekind–MacNeille completion, namely, $\mathfrak{Cm}\mathsf{At}\mathfrak{A}$ is not in $\mathsf{SNr}_\alpha\mathsf{CA}_{\alpha+k}$ for any*

$k \geq 3$. *Furthermore,* $\mathfrak{A}$ *cannot be completely represented by any algebra in* Gws_α*.*

PROOF:

(1) Using exactly the same argument in [11], one shows that if $\mathfrak{C} \in \mathsf{CA}_\omega$ is completely representable $\mathfrak{C} \models \mathsf{d}_{01} < 1$, then $|\mathsf{At}\mathfrak{C}| \geq 2^\omega$. The argument is as follows: Suppose that $\mathfrak{C} \models \mathsf{d}_{01} < 1$. Then there is $s \in h(-\mathsf{d}_{01})$ so that if $x = s_0$ and $y = s_1$, we have $x \neq y$. Fix such x and y. For any $J \subseteq \omega$ such that $0 \in J$, set a_J to be the sequence with ith co-ordinate is x if $i \in J$, and is y if $i \in \omega \setminus J$. By complete representability every a_J is in $h(1^{\mathfrak{C}})$ and so it is in $h(x)$ for some unique atom x, since the representation is an atomic one. Let $J, J' \subseteq \omega$ be distinct sets containing 0. Then there exists $i < \omega$ such that $i \in J$ and $i \notin J'$. So $a_J \in h(\mathsf{d}_{0i})$ and $a'_J \in h(-\mathsf{d}_{0i})$, hence atoms corresponding to different a_J's with $0 \in J$ are distinct. It now follows that $|\mathsf{At}\mathfrak{C}| = |\{J \subseteq \omega : 0 \in J\}| \geq 2^\omega$.
Take $\mathfrak{D} \in \mathsf{Cs}_\omega$ with universe $\wp({}^\omega 2)$. Then $\mathfrak{D} \models \mathsf{d}_{01} < 1$ and plainly $\mathfrak{D}$ is completely representable. Using the downward Löwenheim–Skolem–Tarski theorem, take a countable elementary subalgebra $\mathfrak{B}$ of $\mathfrak{D}$. This is possible because the signature of CA_ω is countable. Then in $\mathfrak{B}$ we have $\mathfrak{B} \models \mathsf{d}_{01} < 1$ because $\mathfrak{B} \equiv \mathfrak{C}$. But $\mathfrak{B}$ cannot be completely representable, because if it were then by the above argument, we get that $|\mathsf{At}\mathfrak{B}| \geq 2^\omega$, which is impossible because $\mathfrak{B}$ is countable.

(2) Now we prove the second item, which is a generalization of [19, Theorem 3.2.4]. Though the generalization is strict, in the sense that $\mathsf{Dc}_\omega \subsetneq \mathbf{S}_\mathsf{c}\mathsf{Nr}_\omega\mathsf{CA}_{\omega+\omega}$ [1] the proof is the same. Without loss, we can take $\alpha = \omega$. Let $\mathfrak{A} \in \mathsf{CA}_\omega$ be as in the hypothesis. For brevity, let $\beta = \omega + \omega$. By hypothesis, we have $\mathfrak{A} \subseteq_c \mathsf{Nr}_\alpha\mathfrak{D}$, with $\mathfrak{D} \in \mathsf{CA}_\beta$. We can also assume that $\mathfrak{D} \in \mathsf{Dc}_\beta$ by replacing, if necessary, $\mathfrak{D}$ by $\mathfrak{Sg}^{\mathfrak{D}}\mathfrak{A}$. Since $\mathfrak{A}$ is a complete sublgebra of $\mathsf{Nr}_\omega\mathfrak{D}$ which in turn is a complete subalgebra of $\mathfrak{D}$, we have $\mathfrak{A} \subseteq_c \mathfrak{D}$. Thus given $< \mathfrak{p}$ non-principal types in $\mathfrak{A}$ they stay non-principal in $\mathfrak{D}$. Next one proceeds like in *op.cit* since $\mathfrak{D} \in \mathsf{Dc}_\beta$ is countable; this way omitting any $\mathbf{X}$ consisting of $< \mathfrak{p}$ non-principal types. For all non-zero $a \in \mathfrak{D}$, there exists $\mathfrak{B} \in \mathsf{Ws}_\beta$ and a homomorphism $f_a : \mathfrak{D} \to \mathfrak{B}$ (not necessarily injective) such that $f_a(a) \neq \emptyset$ and f_a omits $\mathbf{X}$. Let $\mathfrak{C} = \mathbf{P}_{a \in \mathfrak{D}, a \neq 0}\mathfrak{B}_a \in \mathsf{Gws}_\beta$. Define

[1] It is not hard to see that the full set algebra with universe $\wp({}^\omega\omega)$ is in $\mathsf{Nr}_\omega\mathsf{CA}_{\omega+\omega} \subseteq \mathbf{S}_c\mathsf{Nr}_\omega\mathsf{CA}_{\omega+\omega}$ but it is not in Dc_ω because for any $s \in {}^\omega U$, $\Delta\{s\} = \omega$.

$g : \mathfrak{D} \to \mathfrak{C}$ by $g(x) = (f_a(x) : a \in \mathfrak{D} \setminus \{0\})$, and then relativize g to $\mathfrak{A}$ as follows: Let W be the top element of $\mathfrak{C}$. Then $W = \bigcup_{i\in I} {}^{\beta}U_i^{(p_i)}$, where $p_i \in {}^{\beta}U_i$ and ${}^{\beta}U_i^{(p_i)} \cap {}^{\beta}U_j^{(p_j)} = \emptyset$, for $i \neq j \in I$. Let $V = \bigcup_{i\in I} {}^{\alpha}U_i^{(p_i\restriction\alpha)}$. For $s \in V$, $s \in {}^{\alpha}U_i^{(p_i\restriction\alpha)}$ (for a unique i), let $s^+ = s \cup p_i \restriction \beta \setminus \alpha$. Now define $f : \mathfrak{A} \to \wp(V)$, via $a \mapsto \{s \in V : s^+ \in g(a)\}$. Then f is as required.

The proof of (3) is like the proof of Theorem 3.20 □

4.1. Other variants of $L_{\omega,\omega}$

Now we prove an omitting types theorem for a countable version of the so–called ω–dimensional cylindric polyadic algebras with equality, in symbols CPE_ω, as defined in [8]. Consider the semigroup T generated by the set of transformations $\{[i|j], [i,j], i,j \in \omega, \mathsf{suc}, \mathsf{pred}\}$ defined on ω. Then T is a *strongly rich* subsemigroup of $({}^{\omega}\omega, \circ)$ in the sense of [18], where suc and pred are the successor and predecessor functions on ω, respectively. For a set X, let $\mathfrak{B}(X)$ denote the Boolean set algebra $\langle \wp(X), \cup, \cap, \sim\rangle$. Let K_T be the class of set algebras of the form $\langle \mathfrak{B}(V), \mathsf{C}_i, \mathsf{S}_\tau\rangle_{i\in\omega,\tau\in\mathsf{T}}$, where $V \subseteq {}^{\omega}U$, V is a *compressed space*, that is $V = \bigcup_{i\in I} {}^{\alpha}U_i^{(p)}$ where for each $i, j \in I$, $U_i = U_j$ or $U_i \cap U_j = \emptyset$. Let Σ_1 be the set of equations defined in [18] axiomatizing K_T; that is $\mathsf{Mod}\Sigma_1 = \mathsf{K}_\mathsf{T}$. Here we *do not* have diagonal elements in the signature; the corresponding logic is a conservative extension of $L_{\omega,\omega}$ *without* equality, and it is a proper extension.

Let Gp_T be the class of set algebras of the form $\langle \mathfrak{B}(V), \mathsf{C}_i, \mathsf{D}_{ij}, \mathsf{S}_\tau\rangle_{i,j\in\omega,\tau\in\mathsf{T}}$, where $V \subseteq {}^{\omega}U$, V a non-empty union (not necessarily a disjoint one) of cartesian spaces. Here we have diagonal elements in the signature; the corresponding logic is a *variant* of $L_{\omega,\omega}$ where quantifiers do not necessarily commute, so $L_{\omega,\omega}$ does not 'embed' in this logic its (square Tarskian) semantics are different. Let Σ_2 be the set of equations defining CPE_ω in [8, Definition 6.3.7] restricted to the countable signature of Gp_T. In the next theorem complete additivity is given explicitly in the second item only. Any algebra $\mathfrak{A}$ satisifying Σ_2 is completely additive (due to the presence of diagonal elements), cf. [8].

THEOREM 4.5.

(1) *If $\mathfrak{A} \models \Sigma_2$ is countable and $\mathbf{X} = (X_i : i < \lambda)$, $\lambda < \mathfrak{p}$ is a family of subsets of $\mathfrak{A}$, such that $\prod X_i = 0$ for all $i < \lambda$, then there exists*

$\mathfrak{B} \in \mathsf{Gp_T}$ and an isomorphism $f : \mathfrak{A} \to \mathfrak{B}$ such that $\bigcap_{x\in X_i} f(x) = \emptyset$ for all $i < \lambda$.

(2) *If $\mathfrak{A} \models \Sigma_1$ is countable, and completely additive and $\mathbf{X} = (X_i : i < \lambda)$, $\lambda < \mathfrak{p}$ is a family of subsets of $\mathfrak{A}$, such that $\prod X_i = 0$ for all $i < \lambda$, then there exists $\mathfrak{B} \in \mathsf{K_T}$ and an isomorphism $f : \mathfrak{A} \to \mathfrak{B}$ such that $\bigcap_{x\in X_i} f(x) = \emptyset$ for all $i < \lambda$.*

(3) *In particular, for both cases any countable atomic algebra is completely representable.*

Proof: For brevity, throughout the proof of the first two items, let $\alpha = \omega + \omega$. By strong richness of T, it can be proved that $\mathfrak{A} = \mathfrak{Nr}_\omega\mathfrak{B}$ where $\mathfrak{B}$ is an α–dimensional dilation with substitution operators coming from a countable subsemigroup $\mathsf{S} \subseteq ({}^\alpha\alpha, \circ)$ [22]. It suffices to show that for any non-zero $a \in \mathfrak{A}$, there exist a countable $\mathfrak{D} \in \mathsf{Gp_T}$ and a homomorphism (that is not necessarily injective) $f : \mathfrak{A} \to \mathfrak{D}$, such that $\bigcap_{x\in X_i} f(x) = \emptyset$ for all $i \in \omega$ and $f(a) \neq 0$. So fix non-zero $a \in \mathfrak{A}$. For $\tau \in \mathsf{S}$, set $\mathsf{dom}(\tau) = \{i \in \alpha : \tau(i) \neq i\}$ and $\mathsf{rng}(\tau) = \{\tau(i) : i \in \mathsf{dom}(\tau)\}$. Let adm be the set of admissible substitutions in S, where now $\tau \in \mathsf{adm}$ if $\mathsf{dom}\tau \subseteq \omega$ and $\mathsf{rng}\tau \cap \omega = \emptyset$. Since S is countable, we have $|\mathsf{adm}| \leq \omega$; in fact it can be easily shown that $|\mathsf{adm}| = \omega$. Then for all $i < \alpha$, $p \in \mathfrak{B}$ and $\sigma \in \mathsf{adm}$, $\mathsf{s}_\sigma\mathsf{c}_i p = \sum_{j\in\alpha} \mathsf{s}_\sigma\mathsf{s}^i_j p$. By $\mathfrak{A} = \mathfrak{Nr}_\omega\mathfrak{B}$ we also have, for each $i < \omega$, $\prod^{\mathfrak{B}} X_i = 0$, since $\mathfrak{A}$ is a complete subalgebra of $\mathfrak{B}$. Because substitutions are completely additive, for all $\tau \in \mathsf{adm}$ and all $i < \lambda$, $\prod \mathsf{s}^{\mathfrak{B}}_\tau X_i = 0$. For better readability, for each $\tau \in \mathsf{adm}$, for each $i \in \omega$, let $X_{i,\tau} = \{\mathsf{s}_\tau x : x \in X_i\}$. Then by complete additivity, we have: $(\forall\tau \in \mathsf{adm})(\forall i \in \lambda)\prod^{\mathfrak{B}} X_{i,\tau} = 0$. Let S be the Stone space of $\mathfrak{B}$, whose underlying set consists of all Boolean ultrafilters of $\mathfrak{B}$ and for $b \in B$, let N_b denote the clopen set consisting of all ultrafilters containing b. Then from the suprema obtained above, it follows that for $x \in \mathfrak{B}$, $j < \alpha$, $i < \lambda$ and $\tau \in \mathsf{adm}$, the sets $\mathbf{G}_{\tau,j,x} = N_{\mathsf{s}_\tau\mathsf{c}_j x} \setminus \bigcup_i N_{\mathsf{s}_\tau\mathsf{s}^j_i x}$ and $\mathbf{H}_{i,\tau} = \bigcap_{x\in X_i} N_{\mathsf{s}_\tau x}$ are closed nowhere dense sets in S. Also each $\mathbf{H}_{i,\tau}$ is closed and nowhere dense. Like before, we can assume that $\mathfrak{B}$ is countable by assuming that $\mathfrak{A}$ generates $\mathfrak{B}$ is the presence of $|$ $alpha| = (|A| = \omega)$ many operations. Let $\mathbf{G} = \bigcup_{\tau\in\mathsf{adm}}\bigcup_{i\in\alpha}\bigcup_{x\in B}\mathbf{G}_{\tau,i,x}$ and $\mathbf{H} = \bigcup_{i\in\lambda}\bigcup_{\tau\in\mathsf{adm}}\mathbf{H}_{i,\tau}$. Then $\mathbf{H}$ is meager, that is it can be written as a countable union of nowghere dense sets. This follows from the properties of $\mathfrak{p}$ By the Baire Category theorem for compact Hausdorff spaces, we get that $X = S \setminus \mathbf{H} \cup \mathbf{G}$ is dense in S, since $\mathbf{H} \cup \mathbf{G}$ is meager, because

$\mathbf{G}$ is meager, too, since adm, α and $\mathfrak{B}$ are all countable. Accordingly, let F be an ultrafilter in $N_a \cap X$, then by its construction F is a *perfect ultrafilter* [20, p. 128]. Let $\Gamma = \{i \in \alpha : \exists j \in \omega : \mathsf{c}_i\mathsf{d}_{ij} \in F\}$. Since $\mathsf{c}_i\mathsf{d}_{ii} = 1$, then $\omega \subseteq \Gamma$. Furthermore the inclusion is proper, because for every $i \in \omega$, there is a $j \in \alpha \setminus \omega$ such that $\mathsf{d}_{ij} \in F$. Define the relation $\sim$ on Γ via $m \sim n \iff \mathsf{d}_{mn} \in F$. Then $\sim$ is an equivalence relation because for all $i, j, k \in \alpha$, $\mathsf{d}_{ii} = 1 \in F$, $\mathsf{d}_{ij} = \mathsf{d}_{ji}$, $\mathsf{d}_{ik} \cdot \mathsf{d}_{kj} \leq \mathsf{d}_{lk}$ and filters are closed upwards. Now we show that the required representation will be a Gp_T with base $M = \Gamma/\sim$. One defines the homomorphism f using the hitherto obtained perfect ultrafilter F as follows: For $\tau \in {}^\omega\Gamma$, such that $\mathsf{rng}(\tau) \subseteq \Gamma \setminus \omega$ (the last set is non-empty, because $\omega \subsetneq \Gamma$), let $\bar{\tau} : \omega \to M$ be defined by $\bar{\tau}(i) = \tau(i)/\sim$ and write τ^+ for $\tau \cup Id_{\alpha\setminus\omega}$. Then $\tau^+ \in \mathsf{adm}$, because $\tau^+ \upharpoonright \omega = \tau$, $\mathsf{rng}(\tau) \cap \omega = \emptyset$, and $\tau^+(i) = i$ for all $i \in \alpha \setminus \omega$. Let $V = \{\bar{\tau} \in {}^\omega M : \tau : \omega \to \Gamma, \mathsf{rng}(\tau) \cap \omega = \emptyset\}$. Then $V \subseteq {}^\omega M$ is non-empty (because $\omega \subsetneq \Gamma$). Now define f with domain $\mathfrak{A}$ via: $a \mapsto \{\bar{\tau} \in V : \mathsf{s}^{\mathfrak{B}}_{\tau^+} a \in F\}$. Then f is well defined, that is, whenever $\sigma, \tau \in {}^\omega\Gamma$ and $\tau(i) \setminus \sigma(i)$ for all $i \in \omega$, then for any $x \in \mathfrak{A}$, $\mathsf{s}^{\mathfrak{B}}_{\tau^+} x \in F \iff \mathsf{s}^{\mathfrak{B}}_{\sigma^+} x \in F$. Furthermore $f(a) \neq 0$, since $\mathsf{s}_{Id} a = a \in F$ and Id is clearly admissable. The congruence relation just defined on Γ guarantees that the hitherto defined homomorphism respects the diagonal elements. As before, for the other operations, preservation of cylindrifiers is guaranteed by the condition that $F \notin G_{\tau,i,p}$ for all $\tau \in \mathsf{adm}, i \in \alpha$ and all $p \in A$. For omitting the given family of non-principal types, we use that F is outside $\mathbf{H}$, too. This means (by definition) that for each $i < \lambda$ and each $\tau \in \mathsf{adm}$ there exists $x \in X_i$, such that $\mathsf{s}^{\mathfrak{B}}_\tau x \notin F$. Let $i < \lambda$. If $\bar{\tau} \in V \cap \bigcap_{x \in X_i} f(x)$, then $\mathsf{s}^{\mathfrak{B}}_{\tau^+} x \in F$ which is impossible because $\tau^+ \in \mathsf{adm}$. We have shown that for each $i < \omega$, $\bigcap_{x \in X_i} f(x) = \emptyset$.

For the second required one deals with all substitutions in the semigroup S determining the signature of the dilation not just adm, namely, the admissable ones as defined above. More succintly, now *all* substitutions in S are admissable. Other than that, the idea is essentially the same appealing to the Baire category theorem. Let T be as above. Assume that $\mathfrak{A} \models \Sigma_1$ is countable, and fix non-zero $a \in \mathfrak{A}$. Similarly to the first part we will construct a set algebra $\mathfrak{C}$ in K_T and a homomorphism $f : \mathfrak{A} \to \mathfrak{C}$ omitting the given non-principal types and satisfying that $f(a) \neq 0$. By [18], there exists $\mathfrak{B}$ such that $\mathfrak{A} = \mathsf{Nr}_\omega\mathfrak{B}$ and the signature of $\mathfrak{B}$ has, besides all the Boolean operations, all cylindrifiers $\mathsf{c}_i : i \in \alpha$, and the substitutions are determined by a semigroup defined from the rich semigroup T. Substitutions in the signature of $\mathfrak{B}$ are indexed by transformations in S; which we explicitly describe. The semigroup S is the subsemigroup of ${}^{\alpha}\alpha$ generated by the set $\{\bar{\tau} : \tau \in \mathsf{T}\}$ together with all replacements and transpositions on α. Here $\bar{\tau}$ is the transformation that agrees with τ on ω and otherwise is the identity. For all $i < \alpha$, $p \in \mathfrak{B}$, we have $\mathsf{c}_i p = \sum_{j\in\alpha} \mathsf{s}^i_j p$.

By $\mathfrak{A} = \mathsf{Nr}_\omega\mathfrak{B}$ we also have, for each $i < \omega$, $\prod^{\mathfrak{B}} X_i = 0$, since $\mathfrak{A}$ is a complete subalgebra of $\mathfrak{B}$. Let V be the generalized ω-dimensional weak space $\bigcup_{\tau\in S} {}^{\omega}\alpha^{(\tau)}$. Recall that ${}^{\omega}\alpha^{(\tau)} = \{s \in {}^{\omega}\alpha : |\{i \in \omega : \mathsf{s}_i \neq \tau_i\}| < \omega\}$. For each $\tau \in V$ and for each $i \in \lambda$, let $X_{i,\tau} = \{\mathsf{s}^{\mathfrak{B}}_{\bar{\tau}} x : x \in X_i\}$. Here we are using that for any $\tau \in V$, $\bar{\tau} \in \mathsf{S}$. By complete additivity *which is given as an assumption*, it follows that $(\forall \tau \in V)(\forall i \in \kappa) \prod^{\mathfrak{B}} X_{i,\tau} = 0$.

Let S denote the Stone space of the boolean part of $\mathfrak{B}$. Like before, for $p \in B$, let N_p be the clopen set of S consisting of all ultrafilters of the boolean part of $\mathfrak{B}$ containing p. Then for $x \in \mathfrak{B}$, $j < \alpha$, $i < \lambda$, $\tau \in \mathsf{S}$ (using the suprema just established) , the sets $\mathbf{G}_{j,x} = N_{\mathsf{c}_j x} \setminus \bigcup_i N_{\mathsf{s}^j_i x}$ and $\mathbf{H}_{i,\tau} = \bigcap_{x\in X_i} N_{\mathsf{s}_\tau x}$ are closed nowhere dense sets in S. Also each $\mathbf{H}_{i,\tau}$ is closed and nowhere dense.

Let $\mathbf{G} = \bigcup_{i\in\alpha} \bigcup_{x\in B} \mathbf{G}_{i,x}$ and $\mathbf{H} = \bigcup_{i\in\lambda} \bigcup_{\tau\in\mathsf{S}} \mathbf{H}_{i,\tau}$. Then $\mathbf{H}$ is meager, since it is a countable union of nowhere dense sets. Once more by the Baire Category theorem for compact Hausdorff spaces, we get that $X = S \smallsetminus \mathbf{H} \cup \mathbf{G}$ is dense in S, Let F be an ultrafilter in $N_a \cap X$. One builds the required represention from F as follows [18]: Let $\wp(V)$ be the full boolean set algebra with unit V. Let f be the function with domain A such that $f(a) = \{\tau \in V : \mathsf{s}^B_{\bar{\tau}} a \in F\}$. Then f is the desired homorphism from $\mathfrak{A}$ into the set algebra $\langle \wp(V), \mathsf{c}_i, \mathsf{s}_\tau\rangle_{i\in\omega,\tau\in\mathsf{T}}$. In particular, $f(a) \neq 0$, because $Id \in f(a)$. That f omits the given non-principal types is exactly like the first part, modulo replacing adm by (the whole of the semigroup) S.

Given $\mathfrak{A}$ as in the hypothesis, the last required follows by omitting the non-principal type consisting of co-atoms obtaining a complete representation of $\mathfrak{A}$. □

The cylindric reduct of the algebra $\mathfrak{TmAt}$ in the proof of Theorem 3.5 is representable, but not completely representable, for a complete representation of $\mathfrak{TmAt}$ induces an ordinary representation for $\mathfrak{CmAt}$. In fact, it is known that for $2 < n < \omega$ the class CRCA_n is not elementary [11]. We give a short proof. Let $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_\omega$ be an atomic algebra with uncountable many atoms having no complete representation. This algebra exists

[23, Theorem 4.5]. Let LCA_n be the class of CA_ns satifying the Lyndon conditions in the sense of [13]. Then using Lemma 3.3, $\exists$ has a winning strategyin $\mathbf{G}^\omega(\mathsf{At}\mathfrak{A})$, hence she has winning strategyin $G^\omega_\omega(\mathsf{At}\mathfrak{A})$, *a fortiori* in the usual k rounded atomic game $G_k(\mathsf{At}\mathfrak{A})$ for all $k\in\omega$. Thus by definition $\mathfrak{A}\in\mathsf{LCA}_n$. But LCA_n is the elementary closure of CRCA_n and we are done. For a class K, let K^{ad} be the class of completey additive algebras in K. In contrast for polyadic (equality) algebras of infinite dimension, we have the following result proved in [21, 23]. We give a unifted proof.

THEOREM 4.6. *Let α be an infinite ordinal and $n<\omega(\leq\alpha)$. If $\mathfrak{D}\in\mathsf{PEA}_\alpha$ (PA_α is completely additive and) is atomic, then any complete subalgebra of $\mathfrak{Nr}_n\mathfrak{D}$ is completely representable as a PEA_n (PA_n). In particular, $\mathbf{S}_c\mathsf{PA}^{\mathsf{ad}}_\alpha\cap\mathbf{At}=\mathsf{PA}^{\mathsf{ad}}_\alpha\cap\mathbf{At}=\mathsf{CRPA}_\alpha$ and the class CRPA_α is elementary.*

PROOF SKETCH. Assume that $\mathfrak{A}\subseteq_c\mathfrak{Nr}_n\mathfrak{D}$, where $\mathfrak{D}\in\mathsf{PEA}_\alpha$ is atomic. Let $c\in\mathfrak{A}$ be non-zero. We will find a homomorphism $f:\mathfrak{A}\to\wp({}^nU)$ such that $f(c)\neq 0$, and preserves infinitary joins. Assume for the moment (to be proved in a while) that $\mathfrak{A}\subseteq_c\mathfrak{D}$. Then by [12, Lemma 2.16] $\mathfrak{A}$ is atomic because $\mathfrak{D}$ is. For brevity, let $X=\mathsf{At}\mathfrak{A}$. Let $\mathfrak{m}$ be the local degree of $\mathfrak{D}$, $\mathfrak{c}$ its effective cardinality and let β be any cardinal such that $\beta\geq\mathfrak{c}$ and $\sum_{s<\mathfrak{m}}\beta^s=\beta$; such notions are defined in [6]. We can assume that $\mathfrak{D}=\mathfrak{Nr}_\alpha\mathfrak{B}$, with $\mathfrak{B}\in\mathsf{PEA}_\beta$ [10, Theorem 5.4.17]. For any ordinal $\mu\in\beta$, and $\tau\in{}^\mu\beta$, write τ^+ for $\tau\cup Id_{\beta\setminus\mu}(\in{}^\beta\beta)$. Consider the following family of joins evaluated in $\mathfrak{B}$, where $p\in\mathfrak{D}$, $\Gamma\subseteq\beta$ and $\tau\in{}^\alpha\beta$: (*) $\mathsf{c}_{(\Gamma)}p=\sum^{\mathfrak{B}}\{\mathsf{s}_{\tau^+}p:\tau\in{}^\omega\beta,\ \ \tau\upharpoonright\alpha\setminus\Gamma=Id\}$, and (**): $\sum\mathsf{s}^{\mathfrak{B}}_{\tau^+}X=1$. The first family of joins exists [6, Proof of Theorem 6.1], and the second exists, because $\sum^{\mathfrak{A}}X=\sum^{\mathfrak{D}}X=\sum^{\mathfrak{B}}X=1$ and τ^+ is completely additive, since $\mathfrak{B}\in\mathsf{PEA}_\beta$. The last equality of suprema follows from the fact that $\mathfrak{D}=\mathfrak{Nr}_\alpha\mathfrak{B}\subseteq_c\mathfrak{B}$ and the first from the fact that $\mathfrak{A}\subseteq_c\mathfrak{D}$. All this is proved in [23]. Let F be any Boolean ultrafilter of B generated by an atom below a. We show that F will preserve the family of joins in (*) and (**). While in proving a positive a OTT for L_n in item (2) of Theorem 4.2 we resorted to the Baire Category Theorem, now we use a far more basic less sophisticated topological argument. One forms nowhere dense sets in the Stone space of $\mathfrak{B}$ corresponding to the aforementioned family of joins as follows: The Stone space of (the Boolean reduct of) $\mathfrak{B}$ has underlying set S, the set of all Boolean ultrafilters of $\mathfrak{B}$. For $b\in\mathfrak{B}$, let N_b be the clopen set $\{F\in S:b\in F\}$. The required nowhere dense sets are defined for $\Gamma\subseteq\beta$, $p\in\mathfrak{D}$ and $\tau\in{}^\alpha\beta$ via: $A_{\Gamma,p}=N_{\mathsf{c}_{(\Gamma)}p}\setminus\bigcup_{\tau:\alpha\to\beta}N_{\mathsf{s}_{\tau^+}p}$, and $A_\tau = S \setminus \bigcup_{x\in X} N_{\mathsf{s}_\tau + x}$. The principal ultrafilters are isolated points in the Stone topology, so they lie outside the nowhere dense sets defined above. Hence any such ultrafilter preserve the joins in (*) and (**). Fix a principal ultrafilter F with $a \in F$. Define the equivalence relation E (on β) by setting $iEj \iff \mathsf{d}_{ij}^{\mathfrak{B}} \in F (i, j \in \beta)$. Define $f : \mathfrak{A} \to \wp({}^n(\beta/E))$, via $x \mapsto \{\bar{t} \in {}^n(\beta/E) : \mathsf{s}^{\mathfrak{B}}_{t\cup Id_{\beta\sim n}} x \in F\}$, where $\bar{t}(i/E) = t(i)$ $(i < n)$ and $t \in {}^n\beta$. Then f is a well–defined homomorphism; preserving cylindrifiers depends on (*). f defines a complete representation such that Also $f(c) \neq 0$ because $Id \in f(c)$. To show that f is an atomic, hence complete representation, one uses (**) as follows: By construction, for every $s \in {}^n(\beta/E)$, there exists $x \in X(= \mathsf{At}\mathfrak{A})$, such that $\mathsf{s}^{\mathfrak{B}}_{s\cup Id_{\beta\sim n}} x \in F$, from which we get $\bigcup_{x\in X} f(x) = {}^n(\beta/E)$. If $\mathfrak{A} \in \mathsf{PA}_\alpha$, we do not need to bother about diagonal elements and so the base of the representation will be simply β (as defined above for PEA_α), not β/E, and the desired homomorphism, with $n \leq \alpha$, is defined via $g : \mathfrak{A} \to \wp({}^n\beta)$, via $x \mapsto \mathsf{t} \in {}^n\beta : \mathsf{s}^{\mathfrak{B}}_{t\cup Id_{\beta\sim n}} x \in F\}$. Checking that g preserves the operations and that g is atomic, hence complete, is exactly like the PEA case. For PA_α, atomicity can be expressed by a first order sentence, and complete additivity can be captured by continuum many first order formulas [21] □

5. Concluding remarks and related results

(1) A Theorem of Vaught in basic model theory, says that a countable atomic $L_{\omega,\omega}$ theory T has a unique atomic (equivalently in this context prime) model. This can be proved by a direct application of the clssical Orey-Henkin Omitting Types Theorem. The unique atomic atomic model is the 'smallest' models of T, in the sense that it elementary embeds into other models of T. The last theorem says that Keisler's logics which allow formulas of infinite length and quantification on infinitely many variables, enjoys a form of Vaught's theorem. And in Keisler's logics there is the additional advantage that there is no restrictions on the cardinality of atomic theories (algebras) considered. For $L_{\omega,\omega}$, Vaught's theorem is known to fail for theories having uncountable cadinality. If T is an atomic theory in Keisler's logic, and the Tarski–Lindenbaum atomic quotient algebra $\mathfrak{Fm}_T$ happens to be completely additve, then T has an atomic model. In contrast, in Corollary 3.13, we actually showed that Vaught's theorem fails for L_n when we substantially broaden the class of permissable models; it

fails even for '$n+3$-square models.' For $2<n<\omega$, there is a countable atomic L_n theory that lacks even an atomic $n+3$-square model (let alone an ordinary atomic model), i.e a complete $n+3$-square representation of the Tarski–Lindenbaum quotient algebra $\mathfrak{Fm}_T(\in \mathsf{RCA}_n)$.

(2) Let $2<n<l\leq m\leq\omega$. Consider the statemet $\mathsf{notVT}(l,m)$: *There exists a countable, complete and atomic L_n first order theory T in a signature L such that the type Γ consisting of co-atoms in the cylindric Tarski–Lindenbaum quotient algebra $\mathfrak{Fm}_T$ is realizable in every m–square model, but Γ cannot be isolated using $\leq l$ variables, where $n\leq l<m\leq\omega$.* An m-square model of T is an m-square represenation of $\mathfrak{Fm}_T$. The statement $\mathsf{notVT(l,m)}$, short for Vaught's Theorem (VT*) fails at (the parameters) l and m.* Let $\mathsf{VT}(l,m)$ stand for VT *holds at l and m*, so that by definition $\mathsf{notVT}(l,m)\iff\neg\mathsf{VT}(l,m)$. We also include $l=\omega$ in the equation by defining $\mathsf{VT}(\omega,\omega)$ as VT holds for $L_{\omega,\omega}$: Atomic countable first order theories have atomic countable models. For $2<n<l\leq m\leq\omega$ and $l=m=\omega$, it is likely and plausible that *(***):* $\mathsf{VT}(l,m)\iff l=m=\omega$. In other words: *Vaught's theorem holds only in the limiting case when $l\to\infty$ and $m=\omega$ and not 'before'.* We give sufficient condition for (***) to happen. The following definition to be used in the sequel is taken from [2]:

Definition 5.1. [2, Definition 3.1] Let $\mathfrak{R}$ be a relation algebra, with non-identity atoms I and $2<n<\omega$. Assume that $J\subseteq\wp(I)$ and $E\subseteq {}^3\omega$. We say that (J,E) is a *strong n–blur* for $\mathfrak{R}$ if it (J,E) is an n–blur of R in the sense of [2, Definition 3.1], that is to say J is a complex n blur and E is an index blur such that the complex n–blur satisfies:

$$(\forall V_1,\dots V_n,W_2,\dots W_n\in J)(\forall T\in J)(\forall 2\leq i\leq n)\mathsf{safe}(V_i,W_i,T).$$

Theorem 5.2. *For $2<n<\omega$ and $n\leq l<\omega$, $\mathsf{notVT}(n,n+3)$ and $\mathsf{notVT}(l,\omega)$ hold. Furthermore, if for each $n<m<\omega$, there exists a finite relation algebra $\mathfrak{R}_m$ having $m-1$ strong blur and no m-dimensional relational basis, then (***) above for* VT *holds.*

Proof: We start by the last part. Let $\mathfrak{R}_m$ be as in the hypothesis with strong $m-1$–blur (J,E) and m-dimensional relational basis. We 'blow up and blur' $\mathfrak{R}_m$ in place of the Maddux algebra $\mathfrak{E}_k(2,3)$ blown up and blurred in [2, Lemma 5.1], where $k<\omega$ is the number of non-identity atoms and k depends recursively on l, giving the desired'strong' l–blurness, cf. [2, Lemmata 4.2, 4.3]. The relation algebra $\mathfrak{B}b(\mathfrak{R}_m, J, E)$, obtained by blowing up and blurring $\mathfrak{R}_m$ with respect to (J, E), is $\mathfrak{Tm}\mathbf{At}$ (the term algebra). For brevity call it $\mathcal{R}$. Now take $\mathfrak{A} = \mathfrak{Bb}_n(\mathfrak{R}_m, J, E)$ as defined in [2] to be the CA_n obtained after blowing up and blurring $\mathfrak{R}_m$ to a weakly representable relation algebra atom structure, namely, $\mathbf{At} = \mathsf{At}\mathcal{R}$. Here by [2, Theorem 3.2 9(iii)], $\mathsf{Mat}_n\mathsf{At}\mathcal{R}$ (the set of n-basic matrices on $\mathsf{At}\mathcal{R}$) is a CA_n atom structure and $\mathfrak{A}$ is an atomic subalgebra of $\mathfrak{Cm}\mathsf{Mat}_n(\mathsf{At}\mathcal{R})$ containing $\mathfrak{Tm}\mathsf{Mat}_n(\mathsf{At}\mathcal{R})$, cf. [2]. In fact, by [2, item (3) p. 80], $\mathfrak{A} \cong \mathfrak{Nr}_n\mathfrak{B}b_l(\mathfrak{R}_m, J, E)$.The last algebra $\mathfrak{B}b_l(\mathfrak{R}_m, J, E)$ is defined and the isomorphism holds because $\mathfrak{R}_m$ has a strong l-blur. The embedding $h : \mathfrak{Rd}_n\mathfrak{Bb}_l(\mathfrak{R}_m, J, E) \to \mathfrak{A}$ defined via $x \mapsto \{M \upharpoonright n : M \in x\}$ restricted to $\mathfrak{Nr}_n\mathfrak{B}b_l(\mathfrak{R}_m, J, E)$ is an isomorphism onto $\mathfrak{A}$ [2, p. 80]. Surjectiveness uses the displayed condition in Definition 5.1 of *strong l-blurness*. Then $\mathfrak{A} \in \mathsf{RCA}_n \cap \mathsf{Nr}_n\mathsf{CA}_l$, but $\mathfrak{A}$ has no complete m-square representation. For if it did, then this induces an m–square representation of $\mathfrak{Cm}\mathsf{At}\mathfrak{A}$, But $\mathfrak{Cm}\mathsf{At}\mathfrak{A}$ does not have an m-square representation, because $\mathfrak{R}$ does not have an m-dimensional relational basis, and $\mathfrak{R} \subseteq \mathfrak{Ra}\mathfrak{Cm}\mathsf{At}\mathfrak{A}$. So an m-square representation of $\mathfrak{Cm}\mathsf{At}\mathfrak{A}$ induces one of $\mathfrak{R}$ which that $\mathfrak{R}$ has no m-dimensional relational basis, a contradiction. We prove $\mathsf{notVT}(m-1, m)$, hence the required, namely, (***). By [10, § 4.3], we can (and will) assume that $\mathfrak{A} = \mathfrak{Fm}_T$ for a countable, simple and atomic theory L_n theory T. Let Γ be the n–type consisting of co–atoms of T. Then Γ is realizable in every m–square model, for if M is an m–square model omitting Γ, then M would be the base of a complete m–square representation of $\mathfrak{A}$, and so by Theorem 3.9 $\mathfrak{A} \in \mathbf{S}_c\mathsf{Nr}_n\mathsf{D}_m$ which is impossible. Suppose for contradiction that ϕ is an $m-1$ witness, so that $T \models \phi \to \alpha$, for all $\alpha \in \Gamma$, where recall that Γ is the set of coatoms. Then since $\mathfrak{A}$ is simple, we can assume without loss that $\mathfrak{A}$ is a set algebra with base M say. Let $\mathsf{M} = (M, R_i)_{i\in\omega}$ be the corresponding model (in a relational signature) to this set algebra in the sense of [10, § 4.3]. Let ϕ^{M} denote the set of all assignments satisfying ϕ in M. We have $\mathsf{M} \models T$ and $\phi^{\mathsf{M}} \in \mathfrak{A}$, because $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_{m-1}$. But $T \models \exists x\phi$, hence $\phi^{\mathsf{M}} \neq 0$, from which it follows that ϕ^{M} must intersect an atom $\alpha \in \mathfrak{A}$ (recall that the latter is atomic). Let ψ be the formula, such that $\psi^{\mathsf{M}} = \alpha$. Then it cannot be the case that $T \models \phi \to \neg\psi$, hence ϕ is not a witness, contradiction and we are done. Finally, $\mathsf{notVT}(n, n+3)$ and $\mathsf{notVT}(l, \omega)$ $(n \leq l < \omega)$ follow from Theorm 3.5 and [2] using the same reasoning as above. □

(3) Let $2 < n < \omega$. For any $m > n$ there exists an n–variable formula that cannot be proved using $m-1$ variables, but can be proved using m variables [12, Theorem 15.17], using any standard Hilbert style proof system [10, § 4.3]. To prove this, for each $m > n+1$ Hirsch and Hodkinson constructed a finite relation algebra, such that $\mathfrak{R}_m$ has an $m-1$ dimensional hyperbasis, but no m–dimensional hyperbasis [12, § 15.2–15.4]. To prove that VT fails everywhere, as defined above, one needs to construct, for each $n+1 < m < \omega$, a finite relation algebra $\mathfrak{R}_m$ having a strong $m-1$ blur, but no m–dimensional basis. In this case blowing up and blurring $\mathfrak{R}_m$ gives a(n infinite) relation algebra having an $m-1$ dimensional cylindric basis, *whose Dedekind–MacNeille completion* has no m–dimensional basis.

(4) Coming back full circle we reprove strong non-finite axiomatizibility results refining Monk's obtained by Maddux and Biro. Let $2 < n \leq l < m \leq \omega$. In $\mathsf{VT}(l,m)$, while the parameter l measures how close we are to $L_{\omega,\omega}$, m measures the *'degree' of squareness* of permitted models. Using elementary calculas terminology one can view $\mathsf{lim}_{l\to\infty}\mathsf{VT}(l,\omega) = \mathsf{VT}(\omega,\omega)$ algebraically using ultraproducts as follows. Fix $2 < n < \omega$. For each $2 < n \leq l < \omega$, let $\mathfrak{R}_l$ be the finite Maddux algebra $\mathfrak{E}_{f(l)}(2,3)$ with strong l–blur (J_l, E_l) and $f(l) \geq l$ as specified in [2, Lemma 5.1] (denoted by k therein). Let $\mathcal{R}_l = \mathfrak{Bb}(\mathfrak{R}_l, J_l, E_l) \in \mathsf{RRA}$ and let $\mathfrak{A}_l = \mathfrak{Nr}_n\mathfrak{Bb}_l(\mathfrak{R}_l, J_l, E_l) \in \mathsf{RCA}_n$. Then $(\mathsf{At}\mathcal{R}_l : l \in \omega \sim n)$, and $(\mathsf{At}\mathfrak{A}_l : l \in \omega \sim n)$ are sequences of weakly representable atom structures that are not strongly representable with a completely representable ultraproduct.

Corollary 5.3. Let $2 < n < \omega$. Then the varieties RCA_n and RRA, together with any finite first order definable expansion of each, cannot be derived from any finite set of equations valid in the variety [3, 17].

We used a rainbow construction to show ultimatey that the m-clique guraded-fragments of L_n with respect to m square and m flat models, equivalently the m-packed fragments of L_n are not Sahlqvist. We show thay $\mathsf{notVT}(l,m)$ fails on the 'horizontal x axis' and the 'vertical y-axis.' To show that VT*fails everywhere*, that is to prove that $\mathsf{VT(l,m)} \iff l = m = \omega$, we reduced the problem in Theorem 5.2 to finding a finite relation algebra having a strong l blur and no m-dimensional relational basis. Using elemenatary Calculus terminogy, we can express this fact via the following double limit. $\mathsf{lim}_{l\to\omega,m\to\omega}\mathsf{VT}(l,m) = \mathsf{VT}(l\to\omega, m\to\omega) = \mathsf{VT}(\omega,\omega) = \mathsf{VT}$. This notation admittedly may be misleading, since it can be interpretated as that the limit of a constant sequence whose every term is $False$ is $True$. This course is blatantly absurd. What is meant by this double limit is rather the following: For $l < l' \leq \omega$ and $m \leq m'$ with $m < l$ and $m' < l'$, $\mathsf{VT}(l, m)$ and $\mathsf{VT}(l', m')$ are both false, but the last is closer to the truth. At the limit, it becomes actually true. For $2 < n \leq l < m < \omega$, $\mathsf{VT}(l, m)$ is not regarded in this context as $False$ nor $True$, but rather having a 'fuzzy' value if you like, or $\mathsf{VT}(\mathsf{l}, \mathsf{m})$ is a probablity function whose values are between 0 and 1. The fuzziness decreases and the probability increases to reach certainty, namely, probability 1, asserting that *Atomic countable theories have countable models*, namely, that VT holds for $L_{\omega,\omega}$. Having said that, perhaps the more suitable notation would be the (double) $\sum_m \sum_l \mathsf{VT}(l, m) = \mathsf{VT}$.

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A NOTE ON 3 × 3-VALUED ŁUKASIEWICZ ALGEBRAS WITH NEGATION

Abstract

In 2004, C. Sanza, with the purpose of legitimizing the study of $n \times m$-valued Łukasiewicz algebras with negation (or $NS_{n\times m}$-algebras) introduced 3×3-valued Łukasiewicz algebras with negation. Despite the various results obtained about $NS_{n\times m}$-algebras, the structure of the free algebras for this variety has not been determined yet. She only obtained a bound for their cardinal number with a finite number of free generators. In this note we describe the structure of the free finitely generated $NS_{3\times3}$-algebras and we determine a formula to calculate its cardinal number in terms of the number of free generators. Moreover, we obtain the lattice $\Lambda(\boldsymbol{NS_{3\times3}})$ of all subvarieties of $\boldsymbol{NS_{3\times3}}$ and we show that the varieties of Boolean algebras, three-valued Łukasiewicz algebras and four-valued Łukasiewicz algebras are proper subvarieties of $\boldsymbol{NS_{3\times3}}$.

Keywords: n-valued Łukasiewicz–Moisil algebras, $n \times m$-valued Łukasiewicz algebras with negation, free algebras, lattice of subvarieties.

2020 Mathematical Subject Classification: 06D30, 03G10, 08B20, 08B15.

1. Introduction

N. Belnap in [1] introduced four-valued logic, with the purpose of reasoning about incomplete (none) and inconsistent (both) information from different sources. This logical system is well known for the many applications it has found in several fields, for example in the study of deductive data-bases and

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distributed logic programs handling information that may contain conflicts or gaps. Taking into consideration Belnap's four-valued logic, C. Sanza considered an extension from which 3×3-valued Łukasiewicz algebras with negation are obtained as described in [12, 14]. Then in [13] she generalizes this concept defining the $n \times m$-valued Łukasiewicz algebras with negation which constitute a non-trivial generalization of n-valued Łukasiewicz–Moisil algebras ([2, 10, 11]) and a particular case of matrix Łukasiewicz algebras defined by W. Suchoń in [16]. More precisely, $NS_{n\times m}$-algebras rise from matrix Łukasiewicz algebras without the restriction that the endomorphisms be pairwise different and endowed with a De Morgan negation in the following way:

An $n \times m$-valued Łukasiewicz algebra with negation (or $NS_{n\times m}$-algebra), in which n and m are integers, $n \geq 2$, $m \geq 2$, is an algebra $\langle L, \wedge, \vee, \sim, \{\sigma_{ij}\}_{(i,j)\in(n\times m)}, 0, 1\rangle$ where $(n \times m)$ is the cartesian product $\{1, \ldots, n-1\} \times \{1, \ldots, m-1\}$, the reduct $\langle L, \wedge, \vee, \sim, 0, 1\rangle$ is a De Morgan algebra and $\{\sigma_{ij}\}_{(i,j)\in(n\times m)}$ is a family of unary operations on L which fulfills the following conditions:

(T1) $\sigma_{ij}(x \vee y) = \sigma_{ij}x \vee \sigma_{ij}y$,

(T2) $\sigma_{ij}x \wedge \sigma_{(i+1)j}x = \sigma_{ij}x$,

(T3) $\sigma_{ij}x \wedge \sigma_{i(j+1)}x = \sigma_{ij}x$,

(T4) $\sigma_{ij}\sigma_{rs}x = \sigma_{rs}x$,

(T5) $\sigma_{ij} \sim x = \sim \sigma_{(n-i)(m-j)}x$,

(T6) $\sigma_{ij}x \vee \sim \sigma_{ij}x = 1$,

(T7) $$x \wedge \bigwedge_{(i,j)\in(n\times m)} ((\sim \sigma_{ij}x \vee \sigma_{ij}y) \wedge (\sim \sigma_{ij}y \vee \sigma_{ij}x)) = y \wedge \bigwedge_{(i,j)\in(n\times m)} ((\sim \sigma_{ij}x \vee \sigma_{ij}y) \wedge (\sim \sigma_{ij}y \vee \sigma_{ij}x)).$$ ([12])

In what follows, we will indicate by $\boldsymbol{NS_{n\times m}}$ the variety of $NS_{n\times m}$-algebras.

By [14, Remark 3.1] we have that every $NS_{2\times m}$-algebra is isomorphic to an m-valued Łukasiewicz–Moisil algebra. It is worth mention that $\boldsymbol{NS_{n\times m}}$ was widely studied in [13, 12, 14, 15, 7, 8].

The notions and results announced here for $NS_{n\times m}$-algebras will be used throughout this article.

Let L be an $NS_{n\times m}$-algebra. A filter F of L is a Stone filter if and only of the hypothesis $x \in F$ implies $\sigma_{11}x \in F$ ([13, Proposition 3.2]). The lattice of all Stone filters of L will be denoted by $\mathcal{F}_S(L)$.

(T8) Let L be an $NS_{n\times m}$-algebra with more than one element and let $Con(L)$ be the lattice of all congruences on L. Then $Con(L) = \{R(F) : F \in \mathcal{F}_S(L)\}$, where $R(F) = \{(x, y) \in L\times L :$ there exists $f \in F$ such that $x \wedge f = y \wedge f\}$. Besides, the lattices $Con(L)$ and $\mathcal{F}_S(L)$ are isomorphic considering the mappings $\theta \longmapsto [1]_\theta$ and $F \longmapsto R(F)$ which are mutually inverse, where $[x]_\theta$ stands for the equivalence class of x modulo θ ([13, Proposition 3.3 and Theorem 3.6]).

(T9) $\boldsymbol{NS_{n\times m}}$ is a discriminator variety ([15, Theorem 3.1]).

(T10) Let L be a non-trivial $NS_{n\times m}$-algebra. Then L is simple if and only if $B(L) = \{0, 1\}$, where $B(L)$ is the set of all Boolean elements of L, ([14, Theorem 5.1]).

(T11) $\boldsymbol{NS_{n\times m}}$ is locally finite ([14, Theorem 5.2]).

Let B be a non trivial Boolean algebra and $x \in B$, we will write x' the Boolean complement of x. Furthermore, we will denote by $B\uparrow^{(n\times m)}= \{f : (n \times m) \longrightarrow B$ such that for arbitraries i, j, $r \leq s$, implies $f(r, j) \leq f(s, j)$ and $f(i, r) \leq f(i, s)\}$. Then

(T12) $\langle B\uparrow^{(n\times m)}, \wedge, \vee, \sim, \{\sigma_{ij}\}_{(i,j)\in(n\times m)}, O, I\rangle$ is an $NS_{n\times m}$-algebra where for each $f \in B\uparrow^{(n\times m)}$ and for $(i, j) \in (n\times m)$, $(\sim f)(i, j) = (f(n - i, m - j))'$, $(\sigma_{rs}f)(i, j) = f(r, s)$, for all $(r, s) \in (n \times m)$, $O(i, j) = 0$, $I(i, j) = 1$ and the remaining operations are defined componentwise ([14, Proposition 3.2]).

(T13) $\mathbf{S}_{n\times m} = \langle\{0, 1\}\uparrow^{(n\times m)}, \wedge, \vee, \sim, \{\sigma_{ij}\}_{(i,j)\in(n\times m)}, O, I\rangle$ generates the variety $\boldsymbol{NS_{n\times m}}$ ([14, Theorem 5.5])

2. Free $\mathbf{NS_{3\times3}}$-algebras

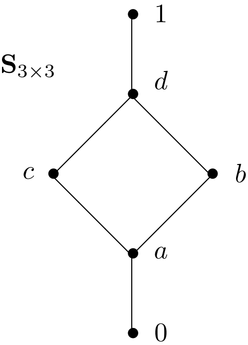
From now on, we will denote by $\mathcal{F}_{3\times3}(t)$ the free $NS_{3\times3}$-algebra with a set G of free generators such that $|G| = t$ where t is a cardinal number,

$0 < t < \omega$. The notion of free $NS_{3\times 3}$-algebra is the usual one and since $NS_{3\times 3}$-algebras are equationally definable, for any cardinal number t, $t > 0$, the free algebra $\mathcal{F}_{3\times 3}(t)$ exists and it is unique up to isomorphism ([3]).

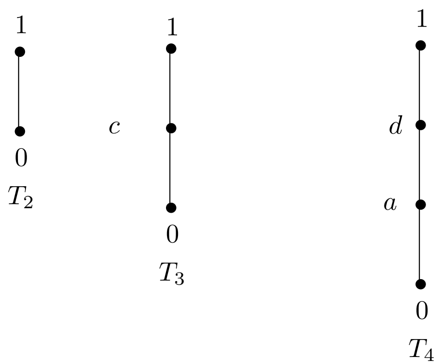
On the other hand, from (T13) we have that $\boldsymbol{NS_{3\times 3}}$ is generated by $\mathbf{S}_{3\times 3}$ described in [14, p. 85] as follows:

x	$\sim x$	$\sigma_{11}x$	$\sigma_{12}x$	$\sigma_{21}x$	$\sigma_{22}x$
0	1	0	0	0	0
a	d	0	0	0	1
b	b	0	1	0	1
c	c	0	0	1	1
d	a	0	1	1	1
1	0	1	1	1	1

$\mathbf{S}_{3\times 3}$



Furthermore, $\mathbf{S}_{3\times 3}$ has four non-isomorphic subalgebras: the chains T_2, T_3 and T_4 with $2, 3$ and 4 elements respectively and T_6 which is the algebra itself.



Hence, from the above results and bearing in mind $(T9)$ and $(T11)$ we know that $\mathcal{F}_{3\times 3}(t)$ is finite. Furthermore, we have that:

$$\mathcal{F}_{3\times 3}(t) \approx T_2^{\alpha_2} \otimes T_3^{\alpha_3} \otimes T_4^{\alpha_4} \otimes T_6^{\alpha_6},$$

where $\alpha_i = |\mathcal{E}_i| = |\{F : F$ is a maximal Stone filter of $\mathcal{F}_{3\times3}(t)$ and $\mathcal{F}_{3\times3}(t)/F \approx T_i\}|$, for $i = 2, 3, 4, 6$.

Let us see that

$$\alpha_i = \frac{|Epi(\mathcal{F}_{3\times3}(t), T_i)|}{|Aut(T_i)|}, \quad i \in \{2, 3, 4, 6\}.$$

where $Epi(\mathcal{F}_{3\times3}(t), T_i)$ is the set of all epimorphisms from $\mathcal{F}_{3\times3}(t)$ onto T_i and $Aut(T_i)$ is the set of all automorphisms of T_i.

Let us consider the function $\alpha : Epi(\mathcal{F}_{3\times3}(t), T_i) \longrightarrow \mathcal{E}_i$ defined by $\alpha(h) = ker(h)$, where $ker(h) = \{x \in \mathcal{F}_{3\times3}(t) : h(x) = 1\}$. Hence, α is onto. Indeed, for each $F \in \mathcal{E}_i$ let us consider the function $f = \gamma_F \circ q_F$, where q_F is the natural map and γ_F is the $NS_{3\times3}$-isomorphism from $\mathcal{F}_{3\times3}(t)/F$ to T_i. Thus, $f \in Epi(\mathcal{F}_{3\times3}(t), T_i)$ and $ker(f) = F$. Consequently $\alpha(f) = F$. Furthermore, for all $F \in \mathcal{E}_i$ there exists $h' \in Epi(\mathcal{F}_{3\times3}(t), T_i)$ such that $\alpha(h') = F$. Besides, let us note that $\alpha^{-1}(F) = \{f \in Epi(\mathcal{F}_{3\times3}(t), T_i) : ker(f) = F\} = \{f \in Epi(\mathcal{F}_{3\times3}(t), T_i) : ker(f) = ker(h')\} = \{f \in Epi(\mathcal{F}_{3\times3}(t), T_i) : f = g \circ h', g \in Aut(T_i)\}$. Then, $|\alpha^{-1}(F)| = |Aut(T_i)|$ for $i = 2, 3, 4, 6$.

Besides, observe that $Epi(\mathcal{F}_{3\times3}(t), T_i)$ and $F^*(G, T_i)$ have the same size, where $F^*(G, T_i)$ is the set of all functions $f : G \longrightarrow T_i$ such that $\overline{f(G)} = T_i$, being $\overline{X}$ the $NS_{3\times3}$-subalgebra of T_i generated by X.

Indeed, let $\beta : Epi(\mathcal{F}_{3\times3}(t), T_i) \longrightarrow F^*(G, T_i)$ be the function defined by $\beta(h) = h|_G$ (i.e. β and h agree on G). It is simple to verify that β is injective. Moreover, for each $f \in F^*(G, T_i)$ there is a unique homomorphism $h_f : \mathcal{F}_{3\times3}(t) \longrightarrow T_i$ such that h_f and f agree on G. Besides, $h_f(\mathcal{F}_{3\times3}(t)) = h_f(\overline{G}) = \overline{f(G)} = T_i$. Therefore, h is onto and so $Epi(\mathcal{F}_{3\times3}(t), T_i) = F^*(G, T_i)$.

On the other hand, suppose that $f, g \in Aut(T_i)$ and that there is $x \in T_i$ such that $f(x) \neq g(x)$. Hence, by [13, Theorem 2.7] there is $(s_0, j_0) \in (3\times3)$ such that $\sigma_{s_0j_0} f(x) \neq \sigma_{s_0j_0} g(x)$ and as T_i is a simple $NS_{3\times3}$-algebra for all $i \in \{2, 3, 4, 6\}$ we have that $\sigma_{sj}(T_i) = B(T_i) = \{0, 1\}$ for all $(s, j) \in (3 \times 3)$. Then, without loss of generality we have that $\sigma_{s_0j_0} f(x) = 0$ and $\sigma_{s_0j_0} g(x) = 1$, so $f(\sigma_{s_0j_0} x) = f(0)$ and $g(\sigma_{s_0j_0} x) = g(1)$. Since f, g are injective we conclude that $\sigma_{s_0j_0} x = 0$ and $\sigma_{s_0j_0} x = 1$, which is a contradiction. Therefore, $|Aut(T_i)| = 1$, $i \in \{2, 3, 4, 6\}$.

Bearing in mind the above results and the fact that T_2, T_3 and T_4 are Łukasiewicz–Moisil algebras of order $n = 2$, $n = 3$ and $n = 4$ respectively, from [4] we have that:

$$\alpha_2 = 2^t, \quad \alpha_3 = 2(3^t - 2^t), \quad \alpha_4 = 4^t - 2^t.$$

Therefore, it only remains to determine α_6. Let us consider the functions $f : \{g_1, g_2, \ldots, g_t\} \longrightarrow T_6$ such that $f(g_i) = b$ and $f(g_j) = c$ for some $i, j \in \{1, \ldots, t\}$, $i \neq j$. If b and c are the image of k generators $1 \leq k \leq t$, then we have that there are $\binom{t}{k} \cdot (2^k - 2) \cdot 4^{t-k}$ different functions f from G to T_6. Hence,

$$\alpha_6 = \sum_{i=1}^{t} \binom{t}{i} \cdot (2^i - 2) \cdot 4^{t-i} = 6^t - 2 \cdot 5^t + 4^t.$$

Then, we have shown

Theorem 2.1. *Let $\mathcal{F}_{3\times 3}(t)$ be the free $NS_{3\times 3}$-algebra with t generators. Then its cardinality is given by the following formula:*

$$|\mathcal{F}_{3\times 3}(t)| = 2^{2^t} \cdot 3^{2(3^t - 2^t)} \cdot 4^{4^t - 2^t} \cdot 6^{6^t - 2\cdot 5^t + 4^t}.$$

Remark 2.2. By Theorem 2.1 we have that for $t = 1$ and $t = 2$,

$$|\mathcal{F}_{3\times 3}(1)| = 2^2 \cdot 3^2 \cdot 4^2 \cdot 6^0 = 576,$$

$$|\mathcal{F}_{3\times 3}(2)| = 2^4 \cdot 3^{10} \cdot 4^{12} \cdot 6^2 = 16836317.$$

We will now compare these values with the following bound that C. Sanza determines in [12]:

$$|\mathcal{F}_{n\times m}(t)| \leq |\mathbf{S}_{n\times m}|^{|\mathbf{S}_{n\times m}|^t \cdot K},$$

where K is the number of simple $NS_{n\times m}$-algebras and $|\mathbf{S}_{n\times m}|$ is given by:

$$|\mathbf{S}_{n\times m}| = \begin{cases} m, & \text{if } n = 2 \\ 1 + \sum_{j=2}^{m} |\mathbf{S}_{(n-1)\times j}|, & \text{if } n > 2. \end{cases}$$

Then, we have that $|\mathcal{F}_{3\times 3}(t)| \leq 6^{6^t \cdot 4}$

$$|\mathcal{F}_{3\times 3}(1)| \leq 6^{24} = 4,7383813 \cdot 10^{18},$$

$$|\mathcal{F}_{3\times 3}(2)| \leq 6^{144} = 1,131827 \cdot 10^{112}$$

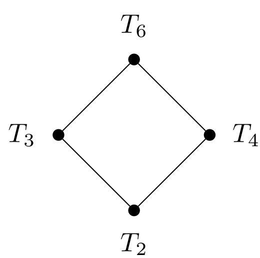
which differ notably from the ones indicated in Remark 2.2.

3. The lattice $\mathbf{\Lambda}(\mathbf{NS_{3\times 3}})$ of all subvarieties of $\mathbf{NS_{3\times 3}}$

If K is a finite set of finite algebras we will denote by $\mathcal{V} = Var(K)$ the variety generated by K. On the other hand, by Jónsson's Lemma ([9]), the lattice $\Lambda(\mathcal{V})$ of all subvarieties of $\mathcal{V}$ is a finite distributive lattice and $\Lambda(\mathcal{V})$ is isomorphic to the lattice $\mathcal{O}(P)$ of order-ideals of the poset P of all join-irreducible elements of $\Lambda(\mathcal{V})$. Again by Jónsson's Lemma, $\mathcal{V}'$ is join-irreducible in $\Lambda(\mathcal{V})$ if and only if there exists some (necessarily finite) subdirectly irreducible algebra $A \in \mathcal{V}$ such that $\mathcal{V}' = Var(\{A\})$. Furthermore, if A and B are subdirectly irreducible algebras of $\mathcal{V}$, then $Var(\{A\}) \subseteq Var(\{B\})$ if and only if $A \in \mathbf{H}(\mathbf{S}(B))$, where $\mathbf{H}(W) = \{C \in \mathcal{V}$: there exists an epimorphism $p : W \to C\}$ and $\mathbf{S}(Z)$ is the set of all subalgebras of Z.

Taking into account (T10) and (T13) we have that $\mathbf{Si}(\mathbf{NS_{3\times 3}}) = \{T_2, T_3, T_4, T_6\}$ where $\mathbf{Si}(S)$ is the set of all finite subdirectly irreducible $NS_{3\times 3}$-algebras. It is not difficult to see that $\mathbf{H}(\mathbf{S}(A)) = \mathbf{S}(A)$, for all $A \in \mathbf{NS_{3\times 3}}$. Then, $\mathbf{H}(\mathbf{S}(T_2)) = \{T_2\}$, $\mathbf{H}(\mathbf{S}(T_3)) = \{T_2, T_3\}$, $\mathbf{H}(\mathbf{S}(T_4)) = \{T_2, T_4\}$ and $\mathbf{H}(\mathbf{S}(T_6)) = \{T_2, T_3, T_4, T_6\}$.

Then, the poset $(\mathbf{Si}(\mathbf{NS_{3\times 3}}), \leq)$ has the following Hasse diagram:



Let us observe that $\mathcal{V}_2 = Var(T_2)$, $\mathcal{V}_3 = Var(T_3)$, $\mathcal{V}_4 = Var(T_4)$, $\mathcal{V}_5 = Var(\{T_2, T_3, T_4\})$. Clearly $\mathcal{V}_2$ is the variety of Boolean algebras, $\mathcal{V}_3$

is the variety of three-valued Łukasiewicz algebras and $\mathcal{V}_4$ is the variety of four-valued Łukasiewicz algebras.

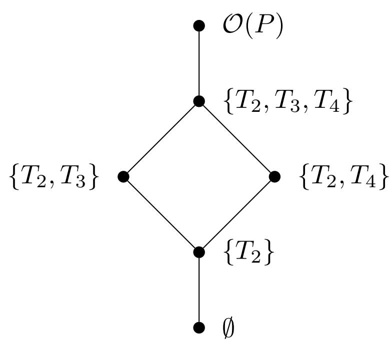
On the other hand, recall that an element x of a complete lattice L is a completely join irreducible (CJI), if $x \leq \bigvee_{i \in I} y_i$ implies $x \leq y_i$ for some $i \in I$. Besides, a finite subdirectly irreducible algebra A in a variety K is a splitting algebra in K if $Var(\{A\})$ is a CJI in $\Lambda(K)$.

Remark 3.1. Taking into account (T9), (T11) and the results established in [5], all finite subdirectly irreducible $NS_{3\times 3}$-algebra is a splitting algebra.

Now, Proposition 3.2 is a direct consequence of Remark 3.1, (T11) and [6, Proposition 2.2].

Proposition 3.2. The natural map from $\Lambda(\mathcal{V})$ to $\mathcal{O}(P)$ is an isomorphism.

Then, we can assert that $\Lambda(\mathbf{NS_{3\times 3}})$ is the following finite distributive lattice:



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TENSE OPERATORS ON BL-ALGEBRAS AND THEIR APPLICATIONS

Abstract

In this paper, the notions of tense operators and tense filters in BL-algebras are introduced and several characterizations of them are obtained. Also, the relation among tense BL-algebras, tense MV-algebras and tense Boolean algebras are investigated. Moreover, it is shown that the set of all tense filters of a BL-algebra is complete sublattice of $F(L)$ of all filters of BL-algebra L. Also, maximal tense filters and simple tense BL-algebras and the relation between them are studied. Finally, the notions of tense congruence relations in tense BL-algebras and strict tense BL-algebras are introduced and an one-to-one correspondence between tense filters and tense congruences relations induced by tense filters are provided.

Keywords: (simple) tense BL-algebra, tense operators, tense filter, tense congruence.

2000 Mathematical Subject Classification: 06E99, 03G25.

1. Introduction

BL-algebras are the algebraic structures for Hájek Basic logic [8], in order to investigate many valued logic by algebraic means. His motivations for introducing BL-algebras were of two kinds. The first one was providing an algebraic counterpart of a propositional logic, called Basic Logic, which embodies a fragment common to some of the most important many-valued logics, namely Lukasiewicz Logic, Gödel Logic and Product Logic. This

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Basic Logic (BL for short) is proposed as "the most general" many-valued logic with truth values in [0,1] and BL-algebras are the corresponding Lindenbaum-Tarski algebras. The second one was to provide an algebraic mean for the study of continuous t-norms (or triangular norms) on [0,1]. Most familiar example of a BL-algebra is the unit interval [0,1] endowed with the structure induced by a continuous t-norm. In 1958, Chang introduced the concept of an MV-algebra which is one of the most classes of BL-algebras. MV-algebras, Gödel algebras and product algebras are the most known classes of BL-algebras. Hájek in [8], introduced the notions of filters and prime filters in BL-algebra and by using the prime filters of BL-algebras, he proved the completeness of basic logic BL. Filter theory play an important rule in studying these algebras. From logical point of view, various filter correspond to various set of provable formulas.
Study of tense operators was originated in 1980's, see e.g. a compendium [2]. The classical tense logic is a logical system obtained from the bivalent logic by adding the tense operators G (it is always going to be the case that) and H (it has always been the case that). Starting with other logical systems (intuitionistic calculus, many-valued logics etc.) and adding appropriate tense operators we arrive to new tense logics. Two other operators F and P are usually defined via G and H by $F(x) = \neg G(\neg x)$ and $P(x) = \neg H(\neg x)$, where $\neg x$ denotes negation of the proposition x. So, G and H can be recognized as tense for all quantifiers and P and F as tense existential quantifiers. Recall that for a classical propositional calculus represented by means of a Boolean algebra $B = (B, \vee, \wedge, \neg, 0, 1)$, tense operators were axiomatized in [2] by the following axioms:

($B1$) $G(1) = 1$, $H(1) = 1$,

($B2$) $G(x \wedge y) = G(x) \wedge G(y)$, $H(x \vee y) = H(x) \vee H(y)$,

($B3$) $\neg G \neg H(x) \leq x$, $\neg H \neg G(x) \leq x$.

For Boolean algebras, the axiom ($B3$) is equivalent to

($B3$') $G(x) \vee y = x \vee H(y)$.

To introduce tense operators in non-classical logics, some more axioms must be added on G and H to express connections with additional operations or logical connectives. Tense operators have been studied by different authors for various classes of algebras. For example, tense operators on Basic algebras and effect algebras, on MV-algebras and Lukasiewicz-Moisil algebras

and on intuitionistic logic (corresponding to Heyting algebras) were studied by Botur et al. [1], Diaconescu et al. [5] and Chajda [3], respectively. This motivated us to introduce tense operators on the structure of BL-algebras as an extension of the tense MV-algebras and because there was an negation on BL-algebras, the operators F and P were introduced as similar to tense operators on MV-algebras with two additional conditions. For other interesting algebras the reader is referred to [4, 7, 6, 9]. This paper is organized as follows:
Section 2 contains some fundamental definitions and results. In Section 3 we introduce the notion of tense operators on BL-algebras and we study relation among tense BL-algebras, tense MV-algebras and tense Boolean algebras. In Section 4 we introduce the notion of tense filters on BL-algebras and we prove that the set of all tense filters of a BL-algebra is complete sublattice of $F(L)$ of all filters of BL-algebra L. Also, we study maximal tense filters and simple tense BL-algebras and the relation between them. In Section 5 we introduce the notions of tense congruence in tense BL-algebras and strict tense BL-algebras and we give some related results.

2. Preliminaries

In this section, we give some fundamental definitions and results. For more details, refer to the references.

Definition 2.1. [8] A BL-algebra is an algebra $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ such that

$(BL1)$ $(L, \vee, \wedge, 0, 1)$ is a bounded lattice,

$(BL2)$ $(L, \odot, 1)$ is a commutative monoid,

$(BL3)$ $z \leq x \rightarrow y$ if and only if $x \odot z \leq y$,

$(BL4)$ $x \wedge y = x \odot (x \rightarrow y)$,

$(BL5)$ $(x \rightarrow y) \vee (y \rightarrow x) = 1$, for all $x, y, z \in L$.

A BL-algebra L is called a Gödel algebra, if $x^2 = x \odot x = x$, for all $x \in L$ and a BL-algebra L is called an MV-algebra, if $(x^-)^- = x$, for all $x \in L$, where $x^- = x \rightarrow 0$. A BL-algebra L is Boolean algebra if and only if $x^2 = x$ and $(x^-)^- = x$, for all $x \in L$.

PROPOSITION 2.2. [11, 12] In any BL-algebra L the following hold:

$(BL6)$ $x \leq y$ if and only if $x \to y = 1$,

$(BL7)$ $x \leq x^{--}$ and $x \to (y \to z) = y \to (x \to z)$,

$(BL8)$ $x \leq y$ implies $x \odot z \leq y \odot z$, $y \to z \leq x \to z$ and $z \to x \leq z \to y$,

$(BL9)$ $y \to x \leq (z \to y) \to (z \to x)$,

$(BL10)$ $x \to (y \to z) = x \odot y \to z$,

$(BL11)$ $x \odot y = 0$ if and only if $x \leq y^-$,

$(BL12)$ $x^{---} = x^-$, $x \leq y \to x$ and $x \odot x^- = 0$,

$(BL13)$ $x \to \bigwedge_{i \in I} y_i = \bigwedge_{i \in I} (x \to y_i)$,

$(BL14)$ $(x \wedge y)^{--} = x^{--} \wedge y^{--}$, $(x \to y)^{--} = x^{--} \to y^{--}$ and $(x \odot y)^{--} = x^{--} \odot y^{--}$, for all $x, y, z, y_i \in L$.

DEFINITION 2.3 ([11, 12]). Let L be a BL-algebra and F be a nonempty subset of L. Then

(i) F is called a *filter* of L if $x \odot y \in F$, for any $x, y \in F$ and if $x \in F$ and $x \leq y$ then $y \in F$, for all $x, y \in L$.

(ii) F is called a *maximal filter* of L if it is a proper filter and is not properly contained in any other proper filter of L.

(iii) L is called a *simple* BL-algebra if L is non-trivial and $\{1\}$ is its only proper filter.

THEOREM 2.4 ([8]). *Let F be a filter of BL-algebra L. Then the binary relation $\equiv_F$ on L which is defined by*

$$x \equiv_F y \quad \text{if and only if } x \to y \in F \quad \text{and } y \to x \in F$$

is a congruence relation on L.(Filters of L and congruence relations $\equiv_F$ on L are in one-to-one correspondence.) Define $\cdot$, $\rightharpoonup$, $\sqcup$, $\sqcap$ on $\frac{L}{F}$, the set of all congruence classes of L, as follows:

$$[x] \cdot [y] = [x \odot y], \ [x] \rightharpoonup [y] = [x \to y], \ [x] \sqcup [y] = [x \vee y], \ [x] \sqcap [y] = [x \wedge y].$$

Then $(\frac{L}{F}, \cdot, \rightharpoonup, \sqcup, \sqcap, [0], [1])$ is a BL-algebra which is called quotient BL-algebra with respect to F.

DEFINITION 2.5. An MV-algebra is an algebra $(L, \oplus, \neg, 0, 1)$ of type $(2, 1, 0)$ satisfying the following axioms for any $x, y, z \in L$:

$(MV1)$ $x \oplus y = y \oplus x$,

$(MV2)$ $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,

$(MV3)$ $x \oplus 0 = x$,

$(MV4)$ $\neg\neg x = x$,

$(MV5)$ $x \oplus 1 = 1$, where $1 := \neg 0$,

$(MV6)$ $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

In any MV-algebra L we can introduce the new operations $\odot$, $\vee$, $\wedge$ and $\rightarrow$ for any $x, y \in L$ as follow:
$x \odot y = (x^- \oplus y^-)^-$, $x \vee y = x \oplus (\neg x \odot y) = y \oplus (\neg y \odot x)$, $x \wedge y = x \odot (\neg x \oplus y) = y \odot (\neg y \oplus x)$ and $x \rightarrow y = \neg x \oplus y$.

DEFINITION 2.6. [5] Let $(L, \oplus, \neg, 0, 1)$ be an MV-algebra and $G, H : L \rightarrow L$, be two unary operations on L. Then the structure $(L; G, H)$ is called a tense MV-algebra if it satisfies in the following conditions for any $x, y \in L$:

$(A0)$ $G(1) = 1$, $H(1) = 1$,

$(A1)$ $G(x \rightarrow y) \leq G(x) \rightarrow G(y)$, $H(x \rightarrow y) \leq H(x) \rightarrow H(y)$,

$(A2)$ $G(x) \oplus G(y) \leq G(x \oplus y), H(x) \oplus H(y) \leq H(x \oplus y)$,

$(A3)$ $G(x \oplus x) \leq G(x) \oplus G(x), H(x \oplus x) \leq H(x) \oplus H(x)$,

$(A4)$ $F(x) \oplus F(x) \leq F(x \oplus x), P(x) \oplus P(x) \leq P(x \oplus x)$,

$(A5)$ $x \leq GP(x)$, $x \leq HF(x)$, where F and P are the unary operations of L defined by $F(x) = (\neg G(\neg x))$, $P(x) = (\neg H(\neg x))$.

3. Tense Operators on BL-algebras

In this section, we introduce the notion of tense operators on BL-algebras and we give some related results.

Definition 3.1. Let $(L, \vee, \wedge, \rightarrow, \odot, 0, 1)$ be a BL-algebra and $G, H : L \rightarrow L$ be two unary operations on L. The structure $(L; G, H)$ is called a *tense BL-algebra* if the following conditions hold:

$(TBL0)$ $G(1) = 1$, $H(1) = 1$.

$(TBL1)$ $G(x \rightarrow y) \leq G(x) \rightarrow G(y)$, $H(x \rightarrow y) \leq H(x) \rightarrow H(y)$.

$(TBL2)$ $x \leq GP(x)$, $x \leq HF(x)$, where F and P are two unary operations of L defined by $F(x) = (G(x^-))^-$ and $P(x) = (H(x^-))^-$, with additional conditions $(G(x^{--}))^{--} = G(x)$ and $(H(x^{--}))^{--} = H(x)$, for all $x, y \in L$.

Note that by additional conditions in Definition 3.1, we conclude that $(F(x^-))^- = (G((x^-)^-)^-)^- = (G(x^{--}))^{--} = G(x)$ and $(P(x^-))^- = (H((x^-)^-)^-)^- = (H(x^{--}))^{--} = H(x)$. Hence F and G, P and H are in some sense equivalent.

Example 3.2. [10] Let $L = \{0, a, b, 1\}$, where $0 < a < b < 1$ and $x \wedge y = \min\{x, y\}$, $x \vee y = \max\{x, y\}$ and operations $\odot$ and $\rightarrow$ are defined as the following tables:

Table 1

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	a	b
1	0	a	b	1

Table 2

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	b	1	1
1	0	a	b	1

Then $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ is a BL-algebra. We define the operations $G = H$ on L as $G(0) = 0, G(a) = a, G(b) = b, G(1) = 1$. It is not difficult to check that G and H are tense operators on L and so $(L; G, H)$ is a tense BL-algebra.

Example 3.3. Every tense MV-algebra is a tense BL-algebra.

Recall that a frame is a pair (X, R), where X is a nonempty set and R is a binary relation on X [2]. The notion of frame allows us to construct the second example of tense BL-algebra. Also, we mention that if L is a BL-algebra and X a set, then L^X the set of all mappings from X into L, together with the operations is a BL-algebra,

- $(f \vee g)(x) = f(x) \vee g(x)$,
- $(f \wedge g)(x) = f(x) \wedge g(x)$,
- $(f \to g)(x) = f(x) \to g(x)$,
- $f(x \odot y) = f(x) \odot f(y)$, $0(x) = 0$, $1(x) = 1$.

Now, we define L_2^X as follow:

$$L_2^X = \{f \in L^X \mid f^{--}(x) = f(x), for\, any\, x \in X\}$$

it is clear by $(BL14)$, L_2^X is a sub BL-algebra of L^X.

LEMMA 3.4. *Let L be a BL-algebra and $a_i, b_i \in L$, for any $i \in I$. Then*

$$\bigwedge_{i\in I}(a_i \to b_i) \odot \bigwedge_{i\in I} a_i \leq \bigwedge_{i\in I} b_i$$

(whenever the arbitrary meets exist.)

PROOF: Let $a_i, b_i \in L$, for any $i \in I$. Then by $(BL13)$,

$$\bigwedge_{i\in I} a_i \to \bigwedge_{i\in I} b_i = \bigwedge_{i\in I}(\bigwedge_{i\in I} a_i \to b_i)$$

Now, since $\bigwedge\limits_{i\in I} a_i \leq a_i$, for any $i \in I$, by $(BL8)$, we get that $a_i \to b_i \leq \bigwedge\limits_{i\in I} a_i \to b_i$, for any $i \in I$ and so $a_i \to b_i \leq \bigwedge\limits_{i\in I}(\bigwedge\limits_{i\in I} a_i \to b_i)$. Hence,

$$\begin{aligned}\bigwedge_{i\in I}(a_i \to b_i) &\leq \bigwedge_{i\in I}(\bigwedge_{i\in I} a_i \to b_i)\\ &= \bigwedge_{i\in I} a_i \to \bigwedge_{i\in I} b_i.\end{aligned}$$

Hence, by $(BL3)$, we conclude that

$$\bigwedge_{i\in I}(a_i \to b_i) \odot \bigwedge_{i\in I} a_i \leq \bigwedge_{i\in I} b_i. \quad \square$$

Theorem 3.5. *Let L be a complete BL-algebra, (X,R) be a frame with R reflexive, G^* and H^* the unary operations on BL-algebra L_2^X defined by*

$$G^*(f)(x) = \bigwedge\{f(y)|y \in X, xRy\}$$
$$H^*(f)(x) = \bigwedge\{f(y)|y \in X, yRx\}$$

for all $f \in L_2^X$ and $x \in X$. Then (L_2^X, G^*, H^*) is a tense BL-algebra.

Proof: Let $x \in X$. Then

$$\begin{aligned} G^*(1)(x) &= \bigwedge\{1(y)|y \in X, xRy\} \\ &= \bigwedge\{1|y \in X, xRy\} \\ &= 1. \end{aligned}$$

Similarly, $H^*(1)(x) = 1$. For $f, g \in L_2^X$ and $x \in X$, we have

$$\begin{aligned} G^*(f \to g)(x) \odot G^*(f)(x) &= \bigwedge\{(f \to g)(y)|y \in X, xRy\} \\ &\quad \odot \bigwedge\{f(y)|y \in X, xRy\} \\ &= \bigwedge\{f(y) \to g(y)|y \in X, xRy\} \\ &\quad \odot \bigwedge\{f(y)|y \in X, xRy\} \\ &\leq \bigwedge\{g(y)|y \in X, xRy\}, \text{ By Lemma 3.4} \\ &= G^*(y)(x) \end{aligned}$$

and so by $(BL3)$, we conclude that $G^*(f \to g)(x) \leq G^*(f)(x) \to G^*(g)(x)$. Hence, $G^*(f \to g) \leq G^*(f) \to G^*(g)$. Similarly, $H^*(f \to g) \leq H^*(f) \to H^*(g)$. Moreover, for $f \in L_2^X$ and $x \in X$, we have

$$\begin{aligned} G^*P^*(f)(x) &= G^*((H(f^-))^-(x)) \\ &= \bigwedge\{(H(f^-))^-(y)|xRy, y \in X\}. \end{aligned}$$

Now, by $(BL7)$, we get that

$$
\begin{aligned}
(H(f^-)(y))^- &= (\bigwedge\{f^-(z)|zRy\})^- \\
&= \bigvee\{f^{--}(z)|zRy\} \\
&= \bigvee\{f(z)|zRy\}.
\end{aligned}
$$

Since xRy, we get that $\bigvee\{f(z)|zRy\} \geq f(x)$. Hence, for any $x \in L$ such that xRy, $(H(f^-))^-(y) \geq f(x)$ and so $\bigwedge\{(H(f^-))^-(y)|xRy\} \geq f(x)$. Hence, $G^*(P^*(f))(x) \geq f(x)$ and so $G^*P^*(f) \geq f$, similarly, $H^*F^*(f) \geq f$. Moreover, for $f \in L_2^X$ and $x \in X$, by $(BL14)$, we get that

$$
\begin{aligned}
(G^*(f^{--})(x))^{--} &= (\bigwedge\{f^{--}(y)|yRx\})^{--} \\
&= \bigwedge\{f^{--}(y)|yRx\} \\
&= \bigwedge\{f(y)|yRx\} \\
&= G^*(f)(x).
\end{aligned}
$$

Hence, $(G^*(f^{--}))^{--} = G^*(f)$ and similarly we have $(H^*(f^{--}))^{--} = H^*(f)$. Therefore, $(L_2^X; G^*, H^*)$ is a tense BL-algebra. □

Proposition 3.6. In any tense BL-algebra $(L; G, H)$, the following statements hold for any $x, y \in L$:

(i) If $x \leq y$, then $G(x) \leq G(y)$, $H(x) \leq H(y)$, $F(x) \leq F(y)$ and $P(x) \leq P(y)$.

(ii) $G(x \to y) \leq F(x) \to F(y)$ and $H(x \to y) \leq P(x) \to P(y)$.

(iii) $x \odot F(y) \leq F(P(x) \odot y)$ and $x \odot P(y) \leq P(F(x) \odot y)$.

(iv) $P \leq PGP$ and $F \leq FHF$.

(v) $PG(x) \leq x^{--}$ and $FH(x) \leq x^{--}$.

(vi) $G(x) \odot G(y) \leq G(x \odot y)$ and $H(x) \odot H(y) \leq H(x \odot y)$.

Proof:

(i) If $x \leq y$, for $x, y \in L$, then by $(BL6)$, $x \to y = 1$. From $(TBL0)$, $G(x \to y) = H(x \to y) = 1$ and from $(TBL1)$, $G(x \to y) \leq G(x) \to G(y)$ and $H(x \to y) \leq H(x) \to H(y)$. Hence, $G(x) \to G(y) = 1$ and $H(x) \to H(y) = 1$. Therefore, $G(x) \leq G(y)$ and $H(x) \leq H(y)$. Moreover, if $x \leq y$, for $x, y \in L$, then by $(BL8)$, $y^- \leq x^-$ and so $G(y^-) \leq G(x^-)$ and $H(y^-) \leq$

$H(x^-)$. Hence, by $(BL8)$, we conclude that $(G(x^-))^- \leq (G(y^-))^-$ and $(H(x^-))^- \leq (H(y^-))^-$ and so $F(x) \leq F(y)$ and $P(x) \leq P(y)$.

(ii) Since by $(BL8)$ and $(BL12)$, $x \to y \leq x \to y^{--} = x \to (y^- \to 0) = y^- \to x^-$, so by (i), $(TBL1)$ and $(BL9)$, we have

$$\begin{aligned} G(x \to y) &\leq G(y^- \to x^-) \\ &\leq G(y^-) \to G(x^-) \\ &\leq (G(x^-) \to 0) \to (G(y^-) \to 0) \\ &= (G(x^-))^- \to (G(y^-))^- \\ &= F(x) \to F(y). \end{aligned}$$

The other inequality for H, is proved analogously.

(iii) Since $x \odot y \leq x \odot y$, by $(BL3)$, we get that $x \leq y \to x \odot y$. Consider $x = P(x)$, so $P(x) \leq y \to P(x) \odot y$. By (i) and (ii),

$$G(P(x)) \leq G(y \to P(x) \odot y) \leq F(y \to (P(x) \odot y)) \leq F(y) \to F(P(x) \odot y).$$

Since by $(TBL3)$, $x \leq GP(x)$, we get that $x \leq F(y) \to F(P(x) \odot y)$ and so by $(BL3)$, $x \odot F(y) \leq F(P(x) \odot y)$. By similar way, $x \odot P(x) \leq P(F(x) \odot y)$.

(iv) From $(TBL3)$, $x \leq GP(x)$ and $x \leq HF(x)$, so by (i), $P(x) \leq PGP(x)$ and $F(x) \leq FHF(x)$. Hence, $P \leq PGP$ and $F \leq FHF$.
(v) From $(TBL3)$, $x^- \leq HF(x^-)$, by $(BL12)$, $x \leq x^{--}$ and by (i), $G(x) \leq G(x^{--})$. By $(BL8)$, $G(x^{--})^- \leq G(x)^-$ and so by (i), $HF(x^-) = H(G(x^{--})^-) \leq H(G(x)^-)$. Hence, $x^- \leq H(G(x)^-)$ and so by $(BL8)$, $(H(G(x)^-))^- \leq x^{--}$. Therefore, $PG(x) \leq x^{--}$. By similar way, $FH(x) \leq x^{--}$.

(v) By $(TBL1)$ and $(BL8)$,

$$\begin{aligned} G(x \to y) \odot G(x) &\leq (G(x) \to G(y)) \odot G(x) \\ &= G(x) \wedge G(y) \\ &\leq G(y) \end{aligned}$$

taking $y = x \odot y$, it follows that $G(x \to x \odot y) \odot G(x) \leq G(x \odot y)$. Since by $(BL10)$, $y \to (x \to x \odot y) = x \odot y \to x \odot y = 1$, we have $y \leq x \to x \odot y$ and so by (i), $G(y) \leq G(x \to x \odot y)$. Hence, by $(BL8)$,

$$\begin{aligned} G(y) \odot G(x) &\leq G(x \to x \odot y) \odot G(x) \\ &\leq G(x \odot y). \end{aligned}$$

Therefore, $G(x) \odot G(y) \leq G(x \odot y)$. The proof for H is similar. □

In the following, we study relation among tense BL-algebras, tense MV-algebras and tense Boolean algebras.

THEOREM 3.7. *Let $(L; G, H)$ be a tense BL-algebra and $x^{--} = x$, $x^2 = x$, for any $x \in L$. Then $(L; G, H)$ is a tense MV-algebra.*

PROOF: Let $(L; G, H)$ be a tense BL-algebra and $x^{--} = x$, $x^2 = x$, for any $x \in L$. Then by Definition 3.1, $(A0)$, $(A1)$ and $(A5)$ are established. We will prove $(A2)$, $(A3)$ and $(A4)$. By $(BL12)$, $x, y \leq y^- \to x = y \oplus x = x \oplus y$ and by Proposition 3.6(i), $G(x), G(y) \leq G(x \oplus y)$ and so $G(x) \oplus G(y) \leq G(x \oplus y) \oplus G(x \oplus y)$. Since $x^{--} = x$, $x^2 = x$, for any $x \in L$, we get that $x \oplus x = (x^- \odot x^-)^- = (x^-)^- = x$, for any $x \in L$. Hence, $G(x) \oplus G(y) \leq G(x \oplus y)$ and by similar way, $H(x) \oplus H(y) \leq H(x \oplus y)$ and so $(A2)$ is established. Since $x \oplus x = x$, for any $x \in L$, we have $G(x \oplus x) = G(x) = G(x) \oplus G(x)$, $H(x \oplus x) = H(x) = H(x) \oplus H(x)$, $F(x \oplus x) = F(x) = F(x) \oplus F(x)$ and $P(x \oplus x) = P(x) = P(x) \oplus P(x)$. Therefore, $(A3)$ and $(A4)$ hold and so $(L; G, H)$ is a tense MV-algebra. □

THEOREM 3.8. *Let $(L; G, H)$ be a tense BL-algebra and $x^{--} = x$, $x^2 = x$, $G(x^-) = G(x)^-$ and $H(x^-) = H(x)^-$ for any $x \in L$. Then $(L; G, H)$ is a tense Boolean algebra.*

PROOF: Let $(L; G, H)$ be a tense BL-algebra and $x^{--} = x$, $x^2 = x$, $G(x^-) = G(x)^-$ and $H(x^-) = H(x)^-$ for any $x \in L$. Then by Definition 2.1, L is a Boolean algebra and by Theorem 3.7, $(L; G, H)$ is a tense MV-algebra. By Definition 2.6, $(B1)$ and $(B3)$ hold. Now, we will prove $(B2)$. Since $x \wedge y \leq x, y$, by Proposition 3.6(i), we get that $G(x \wedge y) \leq G(x), G(y)$ and so $G(x \wedge y) \leq G(x) \wedge G(y)$. Now, by Proposition 3.6(vi) and $(A2)$, for $x, y \in L$, we have

$$\begin{aligned}G(x \wedge y) &= G(x \odot (x^- \oplus y)\\&\geq G(x) \odot (G(x^- \oplus y))\\&\geq G(x) \odot (G(x^-) \oplus G(y))\\&\geq G(x) \odot (G(x)^- \oplus G(y))\\&\geq G(x) \odot (G(x) \to G(y))\\&\geq G(x) \wedge G(y).\end{aligned}$$

Therefore, $G(x \wedge y) = G(x) \wedge G(y)$, by similar way, we conclude $H(x \wedge y) = H(x) \wedge H(y)$. Moreover, for $x, y \in L$,

$$\begin{aligned}G(x \vee y) &= G((x^- \wedge y^-)^-)\\&= (G(x^- \wedge y^-))^-\\&= (G(x^-) \wedge G(y^-))^-\\&= (G(x)^- \wedge G(y)^-)^-\\&= G(x)^{--} \vee G(y)^{--}\\&= G(x) \vee G(y).\end{aligned}$$

Similarly, we conclude $H(x \vee y) = H(x) \vee H(y)$. Therefore, $(B2)$ hold and so $(L; G, H)$ is a tense Boolean algebra. □

Definition 3.9. Let $(L; G, H)$ be a tense BL-algebra. Then we define two unary operations d and ρ on L by $d(x) = x \wedge G(x) \wedge H(x)$ and $\rho(x) = x \odot G(x) \odot H(x)$, for any $x \in L$. We observe that for any $x \in L$, $\rho(x) \leq d(x) \leq x$ and if $(L; G, H)$ is a tense Boolean algebra, then $\rho(x) = d(x)$. Now, we define $d^n(x)$ and $\rho^n(x)$, for any $n \in \mathbb{N}$ and for any $x \in L$, by induction as follow:

$$d^0(x) = \rho^0(x) = x,\ d^{n+1}x = d(d^n(x)),\ \rho^{n+1}(x) = \rho(\rho^n(x)).$$

Moreover, for nonempty subset X of L, $\rho^k(X)$ is define as follow:

$$\rho^0(X) = X,\ \rho(X) = \{\rho(x) | x \in X\},\ \rho^{k+1}(X) = \rho(\rho^k(X)).$$

Lemma 3.10. *In any tense BL-algebra $(L; G, H)$, for any $x, y \in L$ and $n \in \mathbb{N}$, the following statements hold:*

(i) $d^n(0) = 0,\ d^n(1) = 1,\ d^{n+1}(x) \leq d^n(x)$.

(ii) If $x \leq y$, then $d^n(x) \leq d^n(y)$.

(iii) $x = d(x)$ *if and only if* $d^n(x) = x$*, for any* $n \in \mathbb{N}$*.*

(iv) $x \leq d^n(d^n(x^-))^-$*.*

(v) *If* $d(x) = x$*, then* $d(x^-) = x^-$*.*

PROOF:

(i) $d(0) = 0 \wedge G(0) \wedge H(0) = 0$ so $d^2(0) = d(d(0)) = d(0) = 0,\ldots,$ $d^n(0) = d(d^{n-1}(0)) = 0$ and $d(1) = 1 \wedge G(1) \wedge H(1) = 1$ so $d^2(1) = d(d(1)) = d(1) = 1,\ldots,d^n(1) = d(d^{n-1}(1)) = d(1) = 1$ and $d^{n+1}(x) = d(d^n(x)) = d^n(x) \wedge G(d^n(x)) \wedge H(d^n(x)) \leq d^n(x)$.

(ii) If $x \leq y$, then by Proposition 3.6*(i)*, $G(x) \leq G(y)$ and $H(x) \leq H(y)$. Therefore,

$$d(x) = x \wedge G(x) \wedge H(x) \leq y \wedge G(y) \wedge H(y) = d(y)$$

and so $d(d(x)) \leq d(d(y))$. Hence, $d^n(x) \leq d^n(y)$.

(iii) If $x = d(x)$, then

$$\begin{aligned} d^2(x) &= d(d(x)) = d(x) = x \\ d^3(x) &= d(d^2(x)) = d(x) = x \\ &\vdots \\ d^n(x) &= d(d^{n-1}(x)) = d(x) = x. \end{aligned}$$

If $d^n(x) = x$, for any $n \in \mathbb{N}$, then for $n = 1$, $d(x) = x$.

(iv) We prove by induction on n. If $n = 1$, then by $(TBL2)$

$$\begin{aligned} x &\leq x \wedge GP(x) \wedge HF(x) \\ &\leq (x \vee P(x) \vee F(x)) \wedge G(x \vee P(x) \wedge F(x)) \wedge H(x \vee P(x) \vee F(x)) \\ &= d(x \vee P(x) \vee F(x)) \\ &\leq d(x^{--} \vee (H(x^-))^- \vee (G(x^-))^-) \\ &= d((x^- \wedge H(x^-) \wedge G(x^-))^-) \\ &= d(d(x^-))^-. \end{aligned}$$

Suppose that the inequality holds for n, then we show that it is correct for $n+1$. Since $x \leq d(d(x^-))^-$, consider $z = (d^n(x^-))^-$, we have:

$$\begin{aligned}(d^n(x^-))^- &= z\\ &\le d(d(z^-))^-\\ &= d(d(d^n(x^-))^{--})^-\\ &\le d(d(d^n(x^-)))^- && \text{by } (BL8), (BL12) \text{ and } (ii)\\ &= d(d^{n+1}(x^-))^-.\end{aligned}$$

Now by (i), $d^n(d^n(x^-))^- \le d^n(d(d^{n+1}(x^-))^-) = d^{n+1}(d^{n+1}(x^-))^-$ and since $x \le d^n(d^n(x^-))^-$, so we get that $x \le d^{n+1}(d^{n+1}(x^-))^-$. Therefore, (iv) follows by induction.

(v) If $d(x) = x$, then by (iv), $x^- \le d(d(x^{--}))^- \le d(d(x))^- = d(x^-)$. Also, $d(x^-) = x^- \wedge G(x^-) \wedge H(x^-) \le x^-$ and so $d(x^-) = x^-$. □

Proposition 3.11. In any tense BL-algebra $(L; G, H)$, for any $x, y \in L$ and $k, n \in \mathbb{N}$, the following statements hold:

(i) $\rho^n(0) = 0$, $\rho^n(1) = 1$, $\rho^{n+1}(x) \le \rho^n(x)$.

(ii) If $x \le y$, then $\rho^n(x) \le \rho^n(y)$.

(iii) $\rho^k(x) \odot \rho^k(y) \le \rho^k(x \odot y)$.

(iv) $\rho^k(x^n) \ge (\rho^k(x))^n$.

Proof:

(i) $\rho(0) = 0 \odot G(0) \odot H(0) = 0$ so $\rho^2(0) = \rho(\rho(0)) = \rho(0) = 0$,..., $\rho^n(0) = \rho(\rho^{n-1}(0)) = 0$ and $\rho(1) = 1 \odot G(1) \odot H(1) = 1$ and so $\rho^2(1) = \rho(\rho(1)) = \rho(1) = 1$,...,$\rho^n(1) = \rho(\rho^{n-1}(1)) = \rho(1) = 1$. Moreover, for $x \in L$, $\rho^{n+1}(x) = \rho(\rho^n(x)) = \rho^n(x) \odot G(\rho^n(x)) \odot H(\rho^n(x)) \le \rho^n(x)$.

(ii) If $x \le y$, for $x, y \in L$, then by Proposition 3.6(i), $G(x) \le G(y)$ and $H(x) \le H(y)$. Therefore, $\rho(x) = x \odot G(x) \odot H(x) \le y \odot G(y) \odot H(y) = \rho(y)$, and so $\rho(\rho(x)) \le \rho(\rho(y))$. Hence, $\rho^n(x) \le \rho^n(y)$.

(iii) By Proposition 3.6(vi), for $x, y \in L$:

$$\begin{aligned}\rho(x) \odot \rho(y) &= (x \odot G(x) \odot H(x)) \odot (y \odot G(y) \odot H(y))\\ &= (x \odot y) \odot (G(x) \odot G(y)) \odot (H(x) \odot H(y))\\ &\le x \odot y \odot G(x \odot y) \odot H(x \odot y)\\ &= \rho(x \odot y).\end{aligned}$$

By induction, let $\rho^n(x) \odot \rho^n(y) \leq \rho^n(x \odot y)$, for $x, y \in L$. Then by Proposition 3.6(vi),

$$\begin{aligned}
\rho^{n+1}(x) \odot \rho^{n+1}(y) &= \rho(\rho^n(x)) \odot \rho(\rho^n(y)) \\
&= (\rho^n(x) \odot G(\rho^n(x)) \odot H(\rho^n(x))) \\
&\quad \odot (\rho^n(y) \odot G(\rho^n(y)) \odot H(\rho^n(y))) \\
&= (\rho^n(x) \odot \rho^n(y)) \odot (G(\rho^n(x)) \odot G(\rho^n(y))) \\
&\quad \odot (H(\rho^n(x) \odot H(\rho^n(y))) \\
&\leq \rho^n(x \odot y) \odot G(\rho^n(x) \odot \rho^n(y)) \odot H(\rho^n(x) \odot \rho^n(y)) \\
&\leq \rho^n(x \odot y) \odot G(\rho^n(x \odot y) \odot H(\rho^n(x \odot y)) \\
&= \rho(\rho^n(x \odot y)) \\
&= \rho^{n+1}(x \odot y).
\end{aligned}$$

(iv) By (iii), for $x \in L$, we get that $(\rho^k(x))^n = \rho^k(x) \odot \rho^k(x) \odot .... \odot \rho^k(x) \leq \rho^k(x \odot x \odot ... \odot x) = \rho^k(x^n)$. □

4. Tense filters in BL-algebras and simple tense BL-algebras

In this section, we introduce the notions of tense filters in BL-algebras and simple tense BL-algebras and we give some related results.

Definition 4.1. Let $(L; G, H)$ be a tense BL-algebra and F be a filter of L. Then F is called a *tense filter* if $G(x) \in F$ and $H(x) \in F$, for all $x \in F$. Not that if F is a tense filter of tense BL-algebra $(L; G, H)$, then $\rho(x) \in F$ and $d(x) \in F$, for any $x \in F$.

Example 4.2. [10] Let $L = \{0, a, b, 1\}$, where $0 < a < b < 1$ and $x \wedge y = \min\{x, y\}$, $x \vee y = \max\{x, y\}$ and operations $\odot$ and $\rightarrow$ are defined as the following tables:

Table 3

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
1	0	a	b	1

Table 4

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	a	1	1
1	0	a	b	1

Then $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ is a BL-algebra and it is not an MV-algebra. We define the operations $G = H$ on L as $G(0) = 0, G(a) = G(b) = G(1) = 1$. It is not difficult to check that G and H are tense operators on L. Now, let $F_1 = \{1\}$ and $F_2 = \{1, b\}$. Then F_1 and F_2 are tense filters of L.

THEOREM 4.3. *The tense filter $[X)$ of tense BL-algebra $(L; G, H)$ generated by nonempty subset X has the following form:*

$$[X) = \{y \in L | y \geq a_1 \odot ... \odot a_n, a_i \in \rho^{k_i}(X); i = 1, ..., n, k_i \in \mathbb{N}, n \geq 1\}.$$

PROOF: Let $A = \{y \in L | y \geq a_1 \odot ... \odot a_n, a_i \in \rho^{k_i}(X); i = 1, ..., n, k_i \in \mathbb{N}, n \geq 1\}$. Firstly, we prove that A is a tense filter of L. Obviously $1 \in A$. Let $x, y \in A$. Then there exist $a_1, ..., a_n, b_1, ..., b_m \in L$ such that $a_i \in \rho^{k_i}(X), b_j \in \rho^{t_j}(X)$, $k_i, t_j \in \mathbb{N}, m, n \geq 1, 1 \leq i \leq n, 1 \leq j \leq m$ and $x \geq a_1 \odot a_2 \odot ... \odot a_n$, $y \geq b_1 \odot ... \odot b_m$. Hence, $x \odot y \geq a_1 \odot a_2 \odot ... \odot a_n \odot b_1 \odot ... \odot b_m$ and so $x \odot y \in A$. If $x \leq y$ and $x \in A$, then, there exist $a_1, ..., a_p \in L$ such that $a_i \in \rho^{k_i}(X)$ and $a_1 \odot ... \odot a_p \leq x$, since $x \leq y$, we get that $a_1 \odot a_2 \odot ... \odot a_p \leq y$. Hence, $y \in A$. Thus, A is a filter of L. Now, we show that A is a tense filter. If $x \in A$, then there exist $a_1, ..., a_w \in L$, $a_i \in \rho^{k_i}(X)$ and $a_1 \odot a_2 \odot ... \odot a_w \leq x$. Since $a_i \in \rho^{k_i}(X)$, by Definition 3.9, there exist $x_i \in X$, such that $a_i = \rho^{k_i}(x_i)$ for any i $(1 \leq i \leq w)$. Hence, $a_1 \odot a_2 \odot ... \odot a_w = \rho^{k_1}(x_1) \odot \rho^{k_2}(x_2) \odot ... \odot \rho^{k_w}(x_w) \leq x$, by Proposition 3.11$(ii)$, we have

$$\rho(a_1 \odot ... \odot a_w) \leq \rho(x) = x \odot G(x) \odot H(x) \leq G(x), H(x)$$

and since by Proposition 3.11(iii),

$$\rho(a_1) \odot \rho(a_2) \odot ... \odot \rho(a_w) \leq \rho(a_1 \odot ... \odot a_w)$$

we get that $\rho(a_1) \odot \rho(a_2) \odot ... \odot \rho(a_w) \leq G(x), H(x)$. Hence, $\rho^{k_1+1}(x_1) \odot \rho^{k_2+1}(x_2) \odot ... \odot \rho^{k_w+1}(x_w) \leq G(x), H(x)$ and so $G(x), H(x) \in A$. Therefore, A is a tense filter of L. If $x \in X$, since $x \geq \rho(x)$, we conclude $x \in A$. Hence, $X \subseteq A$. Now, let B be a tense filter containing X and $z \in A$, then there exist $a_1, ..., a_n \in L$ such that $a_i \in \rho^{k_i}(X)$ and $a_1 \odot a_2 \odot ... \odot a_n \leq x$, i.e. $\rho^{k_i}(x_1) \odot ... \odot \rho^{k_n}(x_n) \leq x$. Since $x_i \in X \subseteq B$ and B is a tense filter. we get that $\rho^{k_i}(x_i) \in B$ and so $\rho^{k_1}(x_1) \odot ... \odot \rho^{k_n}(x_n) \in B$ and since $\rho^{k_1}(x_1) \odot ... \odot \rho^{k_n}(x_n) \leq x$, we have $x \in B$. Therefore, A is a the least tense filter of L containing X and so $[X) = A$. □

Proposition 4.4. Let $(L; G, H)$ be a tense BL-algebra and $x \in L$. Then

$$[x) = \{y \in L | y \geq (\rho^k(x))^n; for\, some\, n, k \in \mathbb{N}\}.$$

Proof: By Theorem 4.3, $[x) = \{y \in L | y \geq a_1 \odot a_2 \odot ... \odot a_n, a_i \in \rho^{k_i}(x); k_i \in \mathbb{N}, 1 \leq i \leq n, n \in \mathbb{N}\}$. Consider $k = max\{k_1, k_2, ..., k_n\}$ such that $a_i \in \rho^{k_i}(x)$. By Proposition 3.11 (i), we get that $\rho^{k_i}(x) \geq \rho^k(x)$. Now, we have

$$y \geq \rho^{k_1}(x) \odot \rho^{k_2}(x) \odot ... \odot \rho^{k_n}(x) \geq \rho^k(x) \odot \rho^k(x) \odot ... \odot \rho^k(x) = (\rho^k(x))^n.$$

Hence, $y \geq (\rho^k(x))^n$ and so $[x) \subseteq \{y \in L | y \geq (\rho^k(x))^n; for\, some\, n, k \in \mathbb{N}\}$. If $y \in L$, such that $y \geq (\rho^k(x))^n$, then $y \geq \rho^k(x) \odot \rho^k(x) \odot ... \odot \rho^k(x)$ and so by Theorem 4.3, $y \in [x)$. Therefore,

$$[x) = \{y \in L | y \geq (\rho^k(x))^n; for\, some\, n, k \in \mathbb{N}\}. \qquad \square$$

Proposition 4.5. Let F be a tense filter of tense BL-algebra $(L; G, H)$ and $x \in L$. Then the tense filter generated by $F \cup \{x\}$ is characterized as

$$[F \cup \{x\}) = \{y \in L | y \geq a \odot (\rho^k(x))^n; for\, some\, a \in F, k, n \in \mathbb{N}\}$$

Proof: Let $A = \{y \in L | y \geq a \odot (\rho^k(x))^n; for\, some\, a \in F,\ k, n \in \mathbb{N}\}$. We prove that A is the least tense filter of L containing $F \cup \{x\}$. Let $x, y \in A$, then there exist $a, b \in F$, $k, k', n, n' \in \mathbb{N}$ such that $x \geq a \odot (\rho^k(x))^n$ and $y \geq b \odot (\rho^{k'}(x))^{n'}$.
Hence, $x \odot y \geq (a \odot (\rho^k(x))^n) \odot b \odot (\rho^{k'}(x))^{n'} = (a \odot b) \odot (\rho^k(x))^n \odot (\rho^{k'}(x))^{n'}$. Taking $t = Max\{k, k'\}$, then by Proposition 3.11(i), $\rho^k(x) \geq \rho^t(x)$ and $\rho^{k'}(x) \geq \rho^t(x)$ and so $(\rho^k(x))^n \odot (\rho^{k'}(x))^{n'} \geq (\rho^t(x))^{n+n'}$ and so $x \odot y \geq (a \odot b) \odot (\rho^t(x))^{n+n'}$. Therefore, $x \odot y \in A$. If $x \leq y$ and $x \in A$, then there exist $a \in F$ and $k, n \in \mathbb{N}$ such that $x \geq a \odot (\rho^k(x))^n$. Hence, $y \geq a \odot (\rho^k(x))^n$ and so $y \in A$. Therefore, A is a filter of L. If $x \in A$, then there exist $a \in F$ and $k, n \in \mathbb{N}$ $x \geq a \odot (\rho^k(x))^n$, and so by Proposition 3.11(ii), $\rho(x) \geq \rho(a \odot (\rho^k(x))^n)$. From Proposition 3.11$(iii)$, we get that

$$\begin{aligned} \rho(x) \geq \rho(a \odot (\rho^k(x))^n) &\geq \rho(a) \odot \rho((\rho^k(x)^n) \\ &\geq \rho(a) \odot (\rho(\rho^k(x)))^n \\ &= \rho(a) \odot (\rho^{k+1}(x))^n \end{aligned}$$

and since F is a tense filter of L, we get that $\rho(a) \in F$ and since $G(x) \geq \rho(x)$, we have $G(x) \geq \rho(a) \odot (\rho^{k+1}(x))^n$. Hence, $G(x) \in A$ and similarly,

$H(x) \in A$. Therefore, A is a tense filter of L. Now, if B is a tense filter containing $F \cup \{x\}$ and $z \in A$, then there exist $a \in F$ and $k, n \in \mathbb{N}$ such that, $z \geq a \odot (\rho^k(x))^n$. Since $x \in B$ and B is a tense filter we have $((\rho^k(x))^n \in B$ and since $a \in F \subseteq B$, we get that $a \odot (\rho^k(x))^n \in B$. Hence, $z \in B$ and so A is the least tense filter of L containing $F \cup \{x\}$. Thus,

$$[F \cup \{x\}) = \{y \in L | y \geq a \odot (\rho^k(x))^n; for\, some\, a \in F, k, n \in \mathbb{N}\}. \qquad \square$$

As usual, for two filters F_1 and F_2 of BL-algebra L, we let $F_1 \wedge F_2 := F_1 \cap F_2$ and $F_1 \vee F_2 = [F_1 \cup F_2)$ and it is easy to check

$$F_1 \vee F_2 = \{y | y \geq x_1 \odot x_2; for\, some\, x_1 \in F_1, x_2 \in F_2\}$$

Theorem 4.6. *$F_t(L)$ of all tense filter of tense BL-algebra $(L; G, H)$ is a complete sublattice of $F(L)$ of all filter of L.*

Proof: Let F_1 and F_2 be two tense filter and $x \in F_1 \wedge F_2$. Then $x \in F_1$ and $x \in F_2$ so $G(x) \in F_1$ and $G(x) \in F_2$. Hence, $G(x) \in F_1 \wedge F_2$ and by similar way $H(x) \in F_1 \wedge F_2$. Also, if $x \in F_1 \vee F_2$, then there exist $x_1 \in F_1$ and $x_2 \in F_2$ such that $x \geq x_1 \odot x_2$. Now by Proposition 3.6(i) and (vi), we get that $G(x) \geq G(x_1 \odot x_2) \geq G(x_1) \odot G(x)$. Since F_1 and F_2 are tense filters, we conclude that $G(x_1) \in F_1$ and $G(x_2) \in F_2$ and so $G(x) \in [F_1 \cup F_2) = F_1 \vee F_2$. By similar way, $H(x) \in F_1 \vee F_2$. Therefore, $F_1 \vee F_2$ is a tense filter and so $F_t(L)$ is complete sublattice of $F(L)$. $\square$

Theorem 4.7. *Let F be a proper tense filter of tense BL-algebra $(L; G, H)$. Then the following statements are equivalent:*

(i) F is a maximal tense filter of $(L; G, H)$,

(ii) for each $x \in L \backslash F$, there exist $a \in F$ and $k, m \in \mathbb{N}$ such that $a \odot (\rho^k(x))^m = 0$.

Proof:

$(i) \Rightarrow (ii)$ Let F be a maximal tense filter of tense BL-algebra $(L; G, H)$ and $x \in L \backslash F$. Then by $F \subset [F \cup \{x\}) \subseteq L$, we conclude that $[F \cup \{x\}) = L$ and since $0 \in L$, we get that $0 \in [F \cup \{x\})$. From Proposition 4.5, there exist $a \in F$ and $k, m \in \mathbb{N}$ such that $0 \geq a \odot ((\rho^k(x))^m$ and so $a \odot (\rho^k(x))^m = 0$.

$(ii) \Rightarrow (i)$ Let E be a tense filter of L such that $F \subset E \subseteq L$. If there exist $x \in E \backslash F$, then by (ii) there exist $b \in F$ and $k, m \in \mathbb{N}$ such that

$b \odot (\rho^k(x))^m = 0$. Now, Since $b \in F \subseteq E$, $x \in E$ and E is a tense filter, we get that $(\rho^k(x))^m \in E$ and so $0 = b \odot (\rho^k(x))^m \in E$. Hence, $E = L$ and so F is a maximal tense filter of L. □

THEOREM 4.8. *For any tense BL-algebra* $(L; G, H)$*, the following statements are equivalent:*

(i) $(L; G, H)$ *is a simple tense BL-algebra,*

(ii) for any $x \in L\backslash\{1\}$*, there exist* $k, n \in \mathbb{N}$*, such that* $(\rho^k(x))^n = 0$*.*

PROOF:

$(i) \Rightarrow (ii)$ Let $(L; G, H)$ be a simple tense BL-algebra. Then $\{1\}$ is a maximal filter of L and so by Theorem 4.7 for any $x \in L\backslash\{1\}$, there exist $k, n \in \mathbb{N}$ such that $1 \odot (\rho^k(x))^n = 0$. Therefore, $(\rho^k(x))^n = 0$.

$(ii) \Rightarrow (i)$ If for any $x \in L\backslash\{1\}$ there exist $k, n \in \mathbb{N}$ such that $(\rho^k(x))^n = 0$, then by Theorem 4.7, $F = \{1\}$ is a maximal tense filter and so there is not nontrivial tense filter of L and so L is a simple tense BL-algebra. □

THEOREM 4.9. *Let* F *be a proper tens filter of tense BL-algebra* $(L; G, H)$*. Then the following statements are equivalent:*

(i) F *is a maximal tense filter of* $(L; G, H)$*,*

(ii) for each $x \in L$*,* $x \notin F$ *if and only if* $((\rho^k(x))^n)^- \in F$*, for some* $k, n \in \mathbb{N}$*.*

PROOF:

$(i) \Rightarrow (ii)$ Let F be a maximal tense filter of $(L; G, H)$ and $x \in L\backslash F$. Then by Theorem 4.7, there exist $a \in F$ and $n, k \in \mathbb{N}$, such that $a \odot (\rho^k(x))^n = 0$. By $(BL11)$, $a \leq ((\rho^k(x)^n)^-$ and since $a \in F$, we conclude that $((\rho^k(x)^n))^- \in F$. Conversely, let $((\rho^k(x))^n)^- \in F$ for some $k, n \in \mathbb{N}$. If $x \in F$, then $\rho(x) \in F$ and so $(\rho^k(x))^n \in F$. By $(BL12)$, $0 = (\rho^k(x))^n \odot ((\rho^k(x))^n)^- \in F$ and so $F = L$ which is contradiction. Therefore, $x \notin F$.

$(ii) \Rightarrow (i)$ Let $F \subset E \subseteq L$ and E be a tense filter of L. Then there exists $x \in E$ such that $x \notin F$. By (ii) there exist $k, n \in \mathbb{N}$, such that $((\rho^k(x))^n)^- \in F \subseteq E$, since E is a tense filter and $x \in E$, we have $(\rho^k(x))^n \in E$ and so by $(BL12)$, $0 = (\rho^k(x))^n \odot ((\rho^k(x)^n))^- \in E$. Hence $E = L$ and so F is a maximal tense filter of $(L; G, H)$. □

5. Tense congruence relations in tense BL-algebras

In this section, we introduce the notions of tense congruence in tense BL-algebras and strict tense BL-algebras and we give some related results.

Definition 5.1. Let θ be a congruence relation on BL-algebra L and $(L; G, H)$ be a tense BL-algebra. Then θ is called a *tense congruence* if it is compatible with respect to the operations G and H. In fact, if $x\theta y$, then $G(x)\theta G(y)$ and $H(x)\theta H(y)$, for any $x, y \in L$.

Proposition 5.2. Let $(L; G, H)$ be a tense BL-algebra, F be a filter of L and θ_F be a congruence relation induced by F. Then F is a tense filter of L if and only if θ_F is a tense congruence.

Proof: Let θ_F be a tense congruence relation induced by F and $x \in F$. Then $1 \to x \in F$ and $x \to 1 \in F$ and so $1\theta_F x$. Since θ_F is tense congruence, we get that $G(1)\theta_F G(x)$ and $H(1)\theta_F H(x)$ and so $1\theta G(x)$ and $1\theta H(x)$. Hence, $G(x) \in F$ and $H(x) \in F$ and so F is a tense filter of L. Conversely, let F be a tense filter of L and $x\theta_F y$, for $x, y \in L$. Then $x \to y \in F$ and $y \to x \in F$ and since F is a tense filter of L, we have $G(x \to y) \in F$ and $H(x \to y) \in F$ and by $(TBL1)$, $G(x \to y) \leq G(x) \to G(y)$ and $H(x \to y) \leq H(x) \to H(y)$. Now, since F is a filter of L, we conclude that $G(x) \to G(y) \in F$ and $H(x) \to H(y) \in F$. By similar way, we get that $G(y) \to G(x) \in F$ and $H(y) \to H(x) \in F$. Hence, $G(x)\theta_F G(y)$ and $H(x)\theta_F H(y)$. Therefore, θ_F is a tense congruence relation on L. □

Proposition 5.3. Let $(L; G, H)$ be a tense BL-algebra. Then there is an one-to-one correspondence between tense filters of L and tense congruences relations induced by tense filters of L.

Proof: It follows by Theorem 2.4 and Proposition 5.2. □

Theorem 5.4. *Let $(L; G, H)$ be a tense BL-algebra and F be a filter of L. Then F is a tense filter of L if and only if $(\frac{L}{F}; G^*, H^*)$ by the operators $G^*, H^* : \frac{L}{F} \to \frac{L}{F}$ such that*

$$G^*([x]) := [G(x)],\ \ H^*([x]) := [H(x)]$$

and $F^*([x]) := [F(x)],\ P^*([x]) := [P(x)]$ is a tense BL-algebra.

Proof: Let $(L; G, H)$ be a tense BL-algebra and F be a tense filter of L. Then by Theorem 2.4, $(\frac{L}{F}, \cdot, \rightharpoonup, \sqcup, \sqcap, [0], [1])$ is a BL-algebra. Define operators $G^*, H^* : \frac{L}{F} \to \frac{L}{F}$ by

$$G^*([x]) := [G(x)],\ \ H^*([x]) := [H(x)].$$

Now, we prove $(\frac{L}{F}; G^*, H^*)$ is a tense BL-algebra. Firstly, we prove that operations G^* and H^* are well-defined. Let $[x] = [y]$. Then $x \to y, y \to x \in F$. Since F is a tense filter of L, by similar proof of Proposition 5.2, we get that $G(x) \to G(y) \in F$ and $G(y) \to G(x) \in F$. Hence, $[G(x)] = [G(y)]$ and so $G^*([x]) = G^*([y])$. Similarly, we have $H^*([x]) = H^*([y])$ and so operations G^* and H^* are well-defined. By $(TBL0)$ in tense BL-algebra L, $G^*([1]) = [G(1)] = [1]$ and similarly, $H^*([1]) = [H(1)] = [1]$, and so $(TBL0)$ holds in $\frac{L}{F}$. Let $[x], [y] \in \frac{L}{F}$. Then by $(TBL1)$ in tense BL-algebra L,

$$\begin{aligned} G^*([x] \rightharpoonup [y]) &= G^*([x \to y]) \\ &= [G(x \to y)] \\ &\leq [G(x) \to G(y)] \\ &= [G(x)] \rightharpoonup [G(y)] \\ &\leq G^*([x]) \rightharpoonup G^*([y]). \end{aligned}$$

Similarly, we get that $H^*([x] \rightharpoonup [y]) \leq H^*([x]) \rightharpoonup H^*([y])$ and so $(TBL1)$ holds in $\frac{L}{F}$. Finally, By $(TBL3)$ in tense BL-algebra L, we have

$$\begin{aligned} G^*P^*([x]) &= G^*(P^*[x]) \\ &= G^*((H^*[x^-])^-) \\ &= G^*([H(x^-)]^-) \\ &= G^*([(H(x^-))^-]) \\ &= [G((H(x^-))^-] \\ &= [GP(x))] \\ &\geq [x]. \end{aligned}$$

Similarly, we get that $H^*F^*([x]) \geq [x]$ and so $(TBL2)$ holds in $\frac{L}{F}$. Moreover, for $[x] \in \dfrac{L}{F}$,

$$\begin{aligned}(G^*([x]^{--}))^{--} &= (G^*([x^{--}]))^{--}\\ &= ([G(x^{--})])^{--}\\ &= [G(x^{--})^{--}]\\ &= [G(x)]\\ &= G^*([x])\end{aligned}$$

Similarly, $(H^*([x]^{--}))^{--} = H^*([x])$. Therefore, $(\dfrac{L}{F}; G^*, H^*)$ is a tense BL-algebra. Conversely, let F be filter of L, $x \in F$ and $(\dfrac{L}{F}; G^*, H^*)$ is a tense BL-algebra. Then $[x] = [1]$ and so $G^*([x]) = G^*([1])$. Hence, $[G(x)] = [1]$ and so $G(x) \in F$. Similarly, $H(x) \in F$ and so F is a tense filter of L. □

Definition 5.5. Let $(L_1; G_1, H_1)$ and $(L_2; G_2, H_2)$ be two tense BL-algebras and $\phi : L_1 \to L_2$ be a BL-homomorphism. Then ϕ is called a tense BL-homomorphism (or briefly, a TBL-homomorphism) if $G(\phi(x)) = \phi(G(x))$ and $H(\phi(x)) = \phi(H(x))$, for all $x \in L_1$.

Proposition 5.6. Let $\phi : (L_1; G_1, H_1) \to (L_2; G_2, H_2)$ be a TBL-homomorphism. Then the following statements hold:

(i) $\ker\phi$ is a tense filter of L_1.

(ii) If F is a tense filter of L_2, then $\phi^{-1}(F)$ is a tense filter of L_1.

(iii) If $\ker\phi \subseteq F$, ϕ is onto and F is a tense filter of L_1, then $\phi(F)$ is a tense filter of L_2.

Proof:

(i) It is easy to check that $\ker\phi$ is a filter of L_1. Now, let $x \in \ker\phi$. Then $\phi(x) = 1$ and so $1 = G(1) = G(\phi(x)) = \phi(G(x))$. Hence, $G(x) \in \ker\phi$, by similar way, $H(x) \in \ker\phi$ and so $\ker\phi$ is a tense filter of L_1.

(ii) Let F be a tense filter of L_2 and $x \in \phi^{-1}(F)$. Then $\phi^{-1}(F)$ is a filter of L_1 and $\phi(x) \in F$ and so $\phi(G(x)) = G(\phi(x)) \in F$. Hence $G(x) \in \phi^{-1}(F)$, by similar way, $H(x) \in \phi^{-1}(F)$. Therefore, $\phi^{-1}(F)$ is a tense filter of L_1.

(iii) Assume that $\ker\phi \subseteq F$, ϕ is onto and F is a tense filter of L_1. Firstly, we prove $\phi(F)$ is a filter of L_2. Let $a, b \in \phi(F)$. Then there exist

$x, y \in F$, such that $a = \phi(x), b = \phi(y)$ and $a \odot b = \phi(x) \odot \phi(y) = \phi(x \odot y)$. Since $x \odot y \in F$, we get that $a \odot b \in \phi(F)$. Moreover, if $a \leq b$ and $a \in \phi(F)$, then there exists $z \in F$ and $w \in L_1$, such that $a = \phi(z), b = \phi(w)$. Hence, $\phi(z) \leq \phi(w)$ and so $\phi(z \to w) = 1$. Thus, $z \to w \in \ker \phi \subseteq F$ and since $z \in F$, we get that $w \in F$. Therefore, $b = \phi(w) \in \phi(F)$ and so $\phi(F)$ is a filter of L_2. Now, let $x \in \phi(F)$. Then there exists $t \in F$, such that $x = \phi(t)$ and since F is a tense filter of L_1, we have $G(t) \in F$ and so $G(x) = G(\phi(t)) = \phi(G(t)) \in \phi(F)$. By similar way, $H(x) \in \phi(F)$ and so $\phi(F)$ is a tense filter of L_2. □

Definition 5.7. A tense BL-algebra $(L; G, H)$ is called strict if for all $x \in L$, $G(x \odot x) = G(x) \odot G(x)$ and $H(x \odot x) = H(x) \odot H(x)$.

Example 5.8. Let $(L; G, H)$ be tense BL-algebra Example 3.2. Then $(L; G, H)$ is a strict tense BL-algebra.

Proposition 5.9. Let $(L; G, H)$ be a strict tense BL-algebra and F be a tense filter of L. Then $(\frac{L}{F}; G^*, H^*)$ is a strict tense BL-algebra.

Proof: By Theorem 5.4, $(\frac{L}{F}; G^*, H^*)$ is a tense BL-algebra, when F is a tense filter of L. Let $[x], [y] \in \frac{L}{F}$. Since $(L; G, H)$ is a strict tense BL-algebra, we conclude that

$$\begin{aligned} G^*([x].[y]) &= G^*([x \odot y]) \\ &= [G(x \odot y)] \\ &= [G(x) \odot G(y)] \\ &= [G(x)].[G(y)] \\ &= G^*([x]).G^*([y]). \end{aligned}$$

Similarly, $H^*([x].[y]) = H^*([x]).H^*([y])$. Therefore, $(\frac{L}{F}; G^*, H^*)$ is a strict tense BL-algebra. □

Theorem 5.10. *Let $(L; G, H)$ be a strict tense BL-algebra and for any $x \in L$, $x^{--} = x$, $G(x^-) = (G(x))^-$ and $H(x^-) = (H(x))^-$. Then $(L; G, H)$ is a tense MV-algebra.*

Proof: Let $(L; G, H)$ be a strict tense BL-algebra and $x^{--} = x$, for any $x \in L$. Then L is a MV-algebra and by Definition 3.1, $(A0)$, $(A1)$ and

$(A5)$ are hold. Now, we prove $(A2)$, $(A3)$ and $(A4)$. Let $x, y \in L$. Then by Definition 2.5,

$$\begin{aligned} G(x) \oplus G(y) &= (G(x)^- \odot G(y))^- \\ &= (G(x^-) \odot G(y^-))^- \\ &= (G(x^- \odot y^-))^- \\ &= G((x^- \odot y^-)^-) \\ &= G(x \oplus y). \end{aligned}$$

Similarly, $H(x) \oplus H(y) = H(x \oplus y)$ and so $(A2)$ holds. Moreover, if $y = x$, then $G(x) \oplus G(x) = G(x \oplus x)$ and $H(x) \oplus H(x) = H(x \oplus x)$ and so $(A3)$ holds. For $(A4)$, since $(L; G, H)$ is a strict tense BL-algebra, we have

$$\begin{aligned} F(x) \oplus F(x) &= G(x^-)^- \oplus G(x^-)^- \\ &= (G(x^-) \odot G(x^-))^- \\ &= (G(x^- \odot x^-))^- \\ &= (G((x \oplus x)^-))^- \\ &= F(x \oplus x). \end{aligned}$$

Similarly, $P(x) \oplus P(y) = P(x \oplus y)$ and so $(A4)$ holds. Therefore, $(L; G, H)$ is a tense MV-algebra. □

6. Conclusion

The results of this paper will be devoted to study the notion of the tense operators on BL-algebras. We presented a characterization and several important properties of the tense operators on BL-algebras. Moreover, we investigated the relation among tense BL-algebras, tense MV-algebras and tense Boolean algebras. Also, we defined the notions of tense filters and maximal tense filters in BL-algebras and we stated and proved some theorems which determine the relationship between this notions and simple tense BL-algebra and we proved that the set of all tense filters of a BL-algebra is complete sublattice of $F(L)$. Finally, we introduced the notions of tense congruence relations in tense BL-algebras and strict tense BL-algebras and we shown that there is an one-to-one correspondence between tense filters and tense congruences relations induced by tense filters.

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A NOTE ON GÖDEL-DUMMET LOGIC LC

Abstract

Let $A_0, A_1, ..., A_n$ be (possibly) distintict wffs, n being an odd number equal to or greater than 1. Intuitionistic Propositional Logic IPC plus the axiom $(A_0 \to A_1) \vee ... \vee (A_{n-1} \to A_n) \vee (A_n \to A_0)$ is equivalent to Gödel-Dummett logic LC. However, if n is an even number equal to or greater than 2, IPC plus the said axiom is a sublogic of LC.

Keywords: Intermediate logics, Gödel-Dummett logic LC.

1. Introduction

Propositional Intuitionistic Logic IPC can be axiomatized as follows (cf. [5] and references therein):

Axioms:

A1. $A \to (B \to A)$

A2. $[A \to (B \to C)] \to [(A \to B) \to (A \to C)]$

A3. $(A \wedge B) \to A$; $(A \wedge B) \to B$

A4. $A \to [B \to (A \wedge B)]$

*http://grobv.unileon.es.
**https://sites.google.com/site/sefusmendez.

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A5. $A \to (A \vee B)$; $B \to (A \vee B)$

A6. $(A \vee B) \to [(A \to C) \to [(B \to C) \to C]]$

A7. $(A \to \neg B) \to (B \to \neg A)$

A8. $\neg A \to (A \to B)$

Rule of inference:

Modus Ponens (MP): If A and $A \to B$, then B

The following wffs and rule (derivable in IPC) are used in the sequel:

t1. $A \to A$

t2. $(B \to C) \to [(A \to B) \to (A \to C)]$

t3. $(A \to B) \to [(B \to C) \to (A \to C)]$

Transitivity (Trans): If $A \to B$ and $B \to C$, then $A \to C$

In what follows, regardless of a particular order or association of the n implicative wffs $A_1, ..., A_n$ connected by $\vee$ as the sole connective, in general, we simply write $A_1 \vee A_2 ... \vee A_n$.

By IPC$_+$, we refer to the negationless fragment of IPC, axiomatized by A1 through A6 and MP. Well then, in [4] it is noted that Gödel-Dummett logic LC (cf. [2], [3]) can be axiomatized by adding any of the following axiom schemes to IPC:

a1. $(A \to B) \vee (B \to A)$

a2. $(A \to B) \vee [(A \to B) \to A]$

a3. $(A \to B) \vee [(A \to B) \to B]$

a4. $[A \to (B \vee C)] \to [(A \to B) \vee (A \to C)]$

a5. $[(A \wedge B) \to C] \to [(A \to C) \vee (B \to C)]$

a6. $[[(A \to B) \to B] \wedge [(B \to A) \to A]] \to (A \vee B)$

We remark that Dummett's original axiomatization of LC is the result of adding a1 to IPC (cf. [2]). We will occasionally refer to a1 as "Dummett's axiom".

The authors of [4] add: "An even larger number of equivalents [axioms] arises by the fact that in IPC $\vdash A \vee B$ iff $\vdash (A \to B) \wedge (B \to C) \to C$ (**DR**), and, more generally, $\vdash D \to A \vee B$ iff $\vdash D \wedge (A \to C) \wedge (B \to C) \to C$ (**EDR**)" ([2], p. 1).

The aim of this note is to increase the number of equivalent axioms recorded above by showing that, for any odd number n equal to or greater than 1 and (possibly) distinct wffs $A_1, A_2, ..., A_n$, addition of

$$A_0 \to A_1 \vee ... \vee A_{n-1} \to A_n \vee A_n \to A_0$$

to IPC is an axiomatization of LC.

As a by-product of the fact just stated, it also will be shown that if in the preceding wff n is an even number equal to or greater than 2, addition of it to IPC results in an intermediate logic included in (but not including) LC.

To the best of our knowledge, neither of these facts is recorded in the literature.

2. IPC plus $(A \to B) \vee [(B \to C) \vee (C \to A)]$

Let $A_0, A_1, ...A_n, A_{n+1}, A_{n+2}$ be (possibly) distinct wffs, n being an even number equal to or greater than 2. Consider now the following wffs:

α. $A_0 \to A_1 \vee A_1 \to A_2 \vee A_2 \to A_0$

β. $A_0 \to A_1 \vee ... \vee A_{n-1} \to A_n \vee A_n \to A_0$

γ. $A_0 \to A_1 \vee ... \vee A_{n-1} \to A_n \vee A_n \to A_{n+1} \vee A_{n+1} \to A_{n+2} \vee A_{n+2} \to A_0$

We prove:

PROPOSITION 2.1 (IPC$_+$ & β proves α). The wff α is provable in IPC$_+$ plus β.

PROOF:

$$1.\ A_0 \to A_1 \vee ... \vee A_{n-1} \to A_n \vee A_n \to A_0 \qquad \beta$$

By changing in (1), for each $i \geq 3$, A_i by A_1 (resp., A_2) if i is an odd number (resp., even number), we get

$$2.\ A_0 \to A_1 \vee A_1 \to A_2 \vee A_2 \to A_1 \vee A_2 \to A_0$$

or equivalently

$$3.\ A_0 \to A_1 \vee A_1 \to A_2 \vee A_2 \to A_0 \vee A_2 \to A_1$$

Moreover, by changing in (1), for each $i \geq 3$, A_i by A_0 (resp., A_1) if i is an odd number (resp., even number), we get

$$4.\ A_0 \to A_1 \vee A_1 \to A_2 \vee A_2 \to A_0 \vee A_1 \to A_0$$

Next, we proceed as follows. Obviously, we have

$$5.\ (A_2 \to A_0) \to (\alpha)$$

In addition,

$$6.\ (A_1 \to A_0) \to [(A_2 \to A_1) \to (A_2 \to A_0)] \qquad \text{t2}$$
$$7.\ (A_1 \to A_0) \to [(A_2 \to A_1) \to (\alpha)] \qquad \text{t2, Trans, 5, 6}$$
$$8.\ (\alpha) \to [(A_2 \to A_1) \to (\alpha)] \qquad \text{A1}$$

Then,

$$9.\ (A_2 \to A_1) \to (\alpha) \qquad \text{A6, 4, 7, 8}$$

Now, by using

$$10.\ (\alpha) \to (\alpha) \qquad \text{t1}$$

3, 9 and A6, we derive

$$11.\ A_0 \to A_1 \vee A_1 \to A_2 \vee A_2 \to A_0$$

as it was to be proved. □

Proposition 2.2 (IPC$_+$ & α proves β). *The wff β is provable in IPC$_+$ plus α.*

Proof: Firstly, we show,

(I) The wff δ, $A_0 \to A_1 \vee A_1 \to A_2 \vee A_2 \to A_3 \vee A_3 \to A_4 \vee A_4 \to A_0$, is provable in IPC$_+$ plus α:

$$1.\ A_0 \to A_1 \vee A_1 \to A_2 \vee A_2 \to A_0 \qquad \alpha$$
$$2.\ A_2 \to A_3 \vee A_3 \to A_4 \vee A_4 \to A_2 \qquad \alpha$$

We trivially have:

$$3.\ (A_0 \to A_1 \vee A_1 \to A_2) \to (\delta)$$
$$4.\ (A_2 \to A_3 \vee A_3 \to A_4) \to (\delta)$$
$$5.\ (A_4 \to A_0) \to (\delta)$$

Then, we get

$$6.\ [(A_4 \to A_2) \to (\delta)] \to (\delta) \qquad \text{A6, 2, 4}$$

In addition,

$$7.\ (A_2 \to A_0) \to [(A_4 \to A_2) \to (A_4 \to A_0)] \qquad \text{t2}$$
$$8.\ (A_2 \to A_0) \to [(A_4 \to A_2) \to (\delta)] \qquad \text{t2, Trans, 5, 7}$$
$$9.\ (A_2 \to A_0) \to (\delta) \qquad \text{Trans, 6, 8}$$

Finally,

$$10.\ A_0 \to A_1 \vee A_1 \to A_2 \vee A_2 \to A_3 \vee A_3 \to A_4 \vee A_4 \to A_0 \qquad \text{A6, 1, 3, 9}$$

(II) Given (I), the wff ε, $A_0 \to A_1 \vee A_1 \to A_2 \vee A_2 \to A_3 \vee A_3 \to A_4 \vee A_4 \to A_5 \vee A_5 \to A_6 \vee A_6 \to A_0$, is provable in IPC$_+$ plus α similarly as δ has been proved above. We can use δ, α and t2 in the forms $A_4 \to A_5 \vee A_5 \to A_6 \vee A_6 \to A_4$ and $(A_4 \to A_0) \to [(A_6 \to A_4) \to (A_6 \to A_0)]$, respectively.

(III) In this way, the wff γ, displayed at the beginning of the section, can be obtained given β (i.e., $A_0 \to A_1 \vee ... \vee A_{n-1} \to A_n$), and α and t2 in the forms $A_n \to A_{n+1} \vee A_{n+1} \to A_{n+2} \vee A_{n+2} \to A_n$ and $(A_n \to A_0) \to [(A_{n+2} \to A_n) \to (A_{n+2} \to A_0)]$, respectively.

Once (I), (II) and (III) are proved, it is clear that β is derivable from IPC$_+$ plus α. □

Given Propositions 2.1 and 2.2, we have the following corollary.

COROLLARY 2.3 (IPC & α is equivalent to IPC & β). Let $A_0, A_1, ..., A_n$ be (possibly) distinct wffs, n being an even number equivalent to or greater than 2. The systems IPC plus α (i.e., $A_0 \to A_1 \vee A_1 \to A_2 \vee A_2 \to A_0$) and IPC plus β (i.e., $A_0 \to A_1 \vee ... \vee A_{n-1} \to A_n \vee A_n \to A_0$) are deductively equivalent.

The section is ended by proving that Dummett's axiom $(A \to B) \vee (B \to A)$ (a1) is not provable from IPC plus $(A \to B) \vee [(B \to C) \vee (C \to A)]$. Let us provisionally name LC_2 the result of adding $(A \to B) \vee [(B \to C) \vee (C \to A)]$ to IPC. We have:

PROPOSITION 2.4 (Dummett's axiom is not provable in LC_2). Dummett's axiom $(A \to B) \vee (B \to A)$ is not provable in LC_2, that is, the result of adding $(A \to B) \vee [(B \to C) \vee (C \to A)]$ to IPC.

PROOF: Consider the following set of truth-tables (4 is the only designated value):

$\to$	0	1	2	3	4	$\neg$
0	4	4	4	4	4	4
1	2	4	2	4	4	2
2	1	1	4	4	4	1
3	0	1	2	4	4	0
4	0	1	2	3	4	0

$\wedge$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	0	1	1
2	0	0	2	2	2
3	0	1	2	3	3
4	0	1	2	3	4

$\vee$	0	1	2	3	4
0	0	1	2	3	4
1	1	1	3	3	4
2	2	3	2	3	4
3	3	3	3	3	4
4	4	4	4	4	4

This set verifies all axioms of IPC (A1-A8) plus $(A \to B) \vee [(B \to C) \vee (C \to A)]$ and the rule MP, but falsifies Dummett's axiom: let v be any assignment to the propositional variables such that $v(p) = 2$ and $v(q) = 1$, for distinct propositional variables p and q. Then, $v[(p \to q) \vee (q \to p)] = 3$. □

It follows from this proposition that LC is not included in LC_2. Instead, in the following section, it is proved that LC_2 is included in LC.

3. A sequence of axioms equivalent to Dummett's axiom

Let $A_0, A_1, ..., A_n$ be distinct wffs, n being an odd number equal to or greater than 1. Now, consider the following wffs:

ε. $A_0 \to A_1 \vee A_1 \to A_0$

θ. $A_0 \to A_1 \vee ... \vee A_{n-1} \to A_n \vee A_n \to A_0$

We prove:

PROPOSITION 3.1 (IPC$_+$ & θ proves ε). The wff ε is provable from IPC$_+$ plus θ.

PROOF:

$$1.\ A_0 \to A_1 \vee ... \vee A_{n-1} \to A_n \vee A_n \to A_0 \qquad \theta$$

By changing in (1), for each $i \geq 2$, A_i by A_0 (resp., A_1) if i is an even number (resp., odd number), we get

$$2.\ A_0 \to A_1 \vee A_1 \to A_0 \vee ... \vee A_0 \to A_1 \vee A_1 \to A_0$$

that is,

$$3.\ A_0 \to A_1 \vee A_1 \to A_0$$

i.e., the characteristic axiom of LC. □

PROPOSITION 3.2. Consider the following wff η, $A_0 \to A_1 \vee ... \vee A_{n-1} \to A_n \vee A_n \to A_0$, where $A_0, A_1, ..., A_{n-1}, A_n$ are (possibly) distinct wffs. This wff η is provable in LC (notice that n is any natural number equal to or greater than 1).

PROOF:

1. $(A_n \to A_{n-1}) \to [(A_{n-1} \to A_{n-2}) \to (A_n \to A_{n-2})]$ t3
2. $(A_n \to A_{n-2}) \to [(A_{n-2} \to A_{n-3}) \to (A_n \to A_{n-3})]$ t3
3. $(A_n \to A_{n-1}) \to [(A_{n-1} \to A_{n-2}) \to [(A_{n-2} \to A_{n-3}) \to (A_n \to A_{n-3})]]$ t2, Trans, 1, 2

In this way, we have

4. $(A_n \to A_{n-1}) \to [(A_{n-1} \to A_{n-2}) \to [... \to [(A_1 \to A_0) \to (A_n \to A_0)]...]]$

Now, we obviously have

$$5.\ (A_n \to A_0) \to (\eta)$$

and

$$6.\ (A_{n-1} \to A_n) \to (\eta)$$

So, by t2, t3, (4) and (5), we derive

$$7.\ (A_n \to A_{n-1}) \to [(A_{n-1} \to A_{n-2}) \to [... \to [(A_1 \to A_0) \to (\eta)]...]]$$

And by A1, (6) and Trans, we obtain

$$8.\ (A_{n-1} \to A_n) \to [(A_{n-1} \to A_{n-2}) \to [... \to [(A_1 \to A_0) \to (\eta)]...]]$$

Now, by Dummett's axiom, we have

$$9.\ (A_{n-1} \to A_n) \vee (A_n \to A_{n-1})$$

whence

$$10.\ (A_{n-1} \to A_{n-2}) \to [(A_{n-2} \to A_{n-3}) \to [... \to [(A_1 \to A_0) \to (\eta)]...]]$$

follows by A6, (7), (8) and (9).

Next, notice that, for any k $(0 \leq k \leq n-1)$,

$$11.\ (A_k \to A_{k+1}) \to (\eta)$$

is clearly provable.

Finally, proceeding from (10) and (11), similarly as we have proceeded from (4), (7), (8) and (9) to (10), we eventually derive

$$12.\ A_0 \to A_1 \vee ...\ A_{n-1} \to A_n \vee A_n \to A_0$$

that is, the wff η, as it was to be proved. □

Given Propositions 3.1 and 3.2, we immediately have the following corollary.

Corollary 3.3 (IPC & θ is equivalent to LC). Let $A_0, A_1, ..., A_n$ be (possibly) distinct wffs, n being an odd number equivalent to or greater than 1. The result of adding the wff θ (i.e., $A_0 \to A_1 \vee ... A_{n-1} \to A_n \vee A_n \to A_0$) to IPC is a system deductively equivalent to LC.

On the other hand, given Propositions 2.4 and 3.2, the following corollary is immediate.

COROLLARY 3.4 (LC_2 is included in LC). The system LC_2, that is, IPC plus the axiom $(A \to B) \vee [(B \to C) \vee (C \to A)]$ is included in (but does not include) LC.

4. A couple of remarks

This note is ended with a couple of remarks.

1. The proofs of Propositions 2.1, 2.2, 3.1 and 3.2 are given within the context of IPC_+, but it is possible that weaker systems are sufficient. For example, MaGIC (cf. [7]) does not find a set of truth-tables verifying *Ticket Entailment* (cf. [1]) plus Dummett's axiom but falsifying $(A \to B) \vee (B \to C) \vee (C \to D) \vee (D \to A)$.

2. An IPC-model is the following structure $(K, R, \vDash)$, where K is a non-empty set, R is a reflexive and transitive binary relation defined on K and $\vDash$ is a (valuation) relation such that for each $a \in K$, propositional variable p and wffs A, B, the following conditions (clauses) are fulfilled:

(i) $(Rab \;\; \& \;\; a \vDash p) \Rightarrow b \vDash p$

(ii) $a \vDash A \wedge B$ iff $a \vDash A$ and $a \vDash B$

(iii) $a \vDash A \vee B$ iff $a \vDash A$ or $a \vDash B$

(iv) $a \vDash A \to B$ iff for all $b \in K$, $(Rab$ and $b \vDash A) \Rightarrow b \vDash B$

(v) $a \vDash \neg A$ iff for all $b \in K$, $Rab \Rightarrow b \nvDash A$

We have: for any set of wffs Γ and wff A, $\Gamma \vdash_{\mathrm{IPC}} A$ iff $\Gamma \vDash A$ ($\Gamma \vDash A$ iff for any IPC-model $\mathcal{M}$ and $a \in K$, $a \vDash A$ if $a \vDash \Gamma$, where $a \vDash \Gamma$ iff $a \vDash B$ for all $B \in \Gamma$) (cf. [5] or [6] and references therein).

Well then, let us name LC_n the result of adding the axiom

$$A_0 \to A_1 \vee ... \vee A_{n-1} \to A_n \vee A_n \to A_0$$

to IPC; and let LC_n-models be the result of adding the following condition to IPC-models: for any $a_0, a_1, ..., a_n \in K$, if Ra_0a_1 and Ra_0a_2 and ,..., and Ra_0a_n, then, Ra_1a_n or Ra_2a_1 or ,..., or Ra_na_{n-1}. For instance, LC_2-models (i.e., models for IPC plus the axiom $(A \to B) \vee (B \to C) \vee (C \to A)$)

are defined when adding to IPC-models the condition, for any $a, b, c, d \in K$, $(Rab\ \&\ Rac\ \&\ Rad) \Rightarrow (Rbd$ or Rcb or $Rdc)$. It is not difficult to prove that LC_n is (strongly) sound and complete w.r.t. LC_n-models.

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FALLING SHADOW THEORY WITH APPLICATIONS IN HOOPS

Abstract

The falling shadow theory is applied to subhoops and filters in hoops. The notions of falling fuzzy subhoops and falling fuzzy filters in hoops are introduced, and several properties are investigated. Relationship between falling fuzzy subhoops and falling fuzzy filters are discussed, and conditions for a falling fuzzy subhoop to be a falling fuzzy filter are provided. Also conditions for a falling shadow of a random set to be a falling fuzzy filter are displayed.

Keywords: Hoop, fuzzy subhoop, fuzzy filter, falling fuzzy subhoop, falling fuzzy filter.

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1. Introduction

In the study of a unified treatment of uncertainty modelled by meaning of combining probability and fuzzy set theory, Wang and Sanchez [17] introduced the theory of falling shadows which directly relates probability concepts with the membership function of fuzzy sets. Falling shadow representation theory shows us the way of selection relaid on the joint degrees distributions. It is reasonable and convenient approach for the theoretical

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development and the practical applications of fuzzy sets and fuzzy logics. Falling shadow representation theory shows us the way of selection relaid on the joint degrees distributions. It is reasonable and convenient approach for the theoretical development and the practical applications of fuzzy sets and fuzzy logics. The mathematical structure of the theory of falling shadows is formulated in [15]. After that many scholars applied it to (fuzzy) algebraic structures (see [13, 11, 10, 12, 19, 20, 21]). Hoops which are introduced by B. Bosbach in [6, 7] are naturally ordered commutative residuated integral monoids. In [1], Agliáno introduced a continuous t-norm which is a continuous map $*$ from $[0,1]^2$ into $[0,1]$ such that $\langle [0,1], *, 1\rangle$ is a commutative totally ordered monoid. Since the natural ordering on $[0,1]$ is a complete lattice ordering, each continuous t-norm induces naturally a residuation $\to$ and $\langle [0,1], *, \to, 1\rangle$ becomes a commutative naturally ordered residuated monoid, also called a hoop. The variety of basic hoops is precisely the variety generated by all algebras $\langle [0,1], *, \to, 1\rangle$, where $*$ is a continuous t-norm. In [1], they investigated the structure of the variety of basic hoops and some of its subvarieties. In particular, they provided a complete description of the finite subdirectly irreducible basic hoops, and they showed that the variety of basic hoops is generated as a quasivariety by its finite algebras. They extended these results to Hájeks BL-algebras, and gived an alternative proof of the fact that the variety of BL-algebras is generated by all algebras arising from continuous t-norms on $[0,1]$ and their residua. Also, they in [2], overviewed recent results about the lattice of subvarieties of the variety BL of BL-algebras and the equational definition of some families of them. Kondo [14] considered fundamental properties of some types of (implicative, positive implicative and fantastic) filters of hoops, and R. A. Borzooei and M. Aaly Kologani [4] investigated some properties and equivalent definitions of these filters on hoops. Also, they studied the relation between these filters and found that under which conditions they are equivalent. Borzooei et al. studied fuzzy set theory of subhoops and filters in hoops (see [3, 5]).

In this paper, we apply the falling shadow theory to subhoops and filters in hoops. We introduce the notions of falling fuzzy subhoops and falling fuzzy filters in hoops, and investigate several properties. We consider relationship between falling fuzzy subhoops and falling fuzzy filters. We provide conditions for a falling fuzzy subhoop to be a falling fuzzy filter. We also provide conditions for a falling shadow of a random set to be a falling fuzzy filter. Also, we show that every fuzzy filter of a hoop is a falling fuzzy filter and falling fuzzy subhoop and we prove that under which conditions a falling shadow can be a falling fuzzy filter of a hoop.

2. Preliminaries

By a *hoop* we mean an algebra $(H, \odot, \to, 1)$ in which $(H, \odot, 1)$ is a commutative monoid and, for any $x, y, z \in H$, the following assertions are valid.

(H1) $x \to x = 1$,

(H2) $x \odot (x \to y) = y \odot (y \to x)$,

(H3) $x \to (y \to z) = (x \odot y) \to z$.

We define a relation "$\leq$" on a hoop H by

$$(\forall x, y \in H)(x \leq y \;\Leftrightarrow\; x \to y = 1). \tag{2.1}$$

It is easy to see that $(H, \leq)$ is a poset. A nonempty subset S of H is called a *subhoop* of H if it satisfies:

$$(\forall x, y \in S)(x \odot y \in S,\; x \to y \in S). \tag{2.2}$$

Note that every subhoop contains the element 1.

Proposition 2.1 ([8]). Let $(H, \odot, \to, 1)$ be a hoop. For any $x, y, z \in H$, the following conditions hold:

(a1) $(H, \leq)$ is a meet-semilattice with $x \wedge y = x \odot (x \to y)$.

(a2) $x \odot y \leq z$ if and anly if $x \leq y \to z$.

(a3) $x \odot y \leq x, y$ and $x^n \leq x$, for any $n \in \mathbb{N}$.

(a4) $x \leq y \to x$.

(a5) $1 \to x = x$ and $x \to 1 = 1$.

(a6) $x \odot (x \to y) \leq y$, $x \odot y \leq x \wedge y \leq x \to y$.

(a7) $x \to y \leq (y \to z) \to (x \to z)$.

(a8) $x \leq y$ implies $x \odot z \leq y \odot z$, $z \to x \leq z \to y$ and $y \to z \leq x \to z$.

(a9) $x \to (y \to z) = (x \odot y) \to z = y \to (x \to z)$.

A nonempty subset F of a hoop H is called a *filter* of H (see [8]) if the following assertions are valid.

$$(\forall x, y \in H)(x, y \in F \;\Rightarrow x \odot y \in F), \tag{2.3}$$

$$(\forall x, y \in H)(x \in F,\; x \leq y \;\Rightarrow\; y \in F). \tag{2.4}$$

Note that the conditions (2.3) and (2.4) means that F is closed under the operation $\odot$ and F is upward closed, respectively.

Note that a subset F of a hoop H is a filter of H if and only if the following assertions are valid (see [8]):

$$1 \in F, \tag{2.5}$$

$$(\forall x, y \in H)\,(x \rightarrow y \in F,\; x \in F \;\Rightarrow\; y \in F)\,. \tag{2.6}$$

A fuzzy set μ in a hoop H is called a *fuzzy subhoop* of H if it satisfies:

$$\begin{aligned}(\forall x, y \in H)(\mu(x \odot y) \geq \min\{\mu(x), \mu(y)\},\\ \mu(x \rightarrow y) \geq \min\{\mu(x), \mu(y)\}).\end{aligned} \tag{2.7}$$

A fuzzy set μ in a hoop H is called a *fuzzy filter* of H (see [3]) if the following assertions are valid.

$$(\forall x \in H)\,(\mu(x) \leq \mu(1))\,, \tag{2.8}$$

$$(\forall x, y \in H)\,(\mu(y) \geq \min\{\mu(x), \mu(x \rightarrow y))\,. \tag{2.9}$$

Given a fuzzy set μ in H and $t \in [0, 1]$, the set

$$\mu_t := \{x \in H \mid \mu(x) \geq t\} \tag{2.10}$$

is called the *t-level set* of μ in H.

We now display the basic theory on falling shadows. We refer the reader to the papers [9, 15, 16, 18, 17] for further information regarding the theory of falling shadows.

Given a universe of discourse U, let $\mathcal{P}(U)$ denote the power set of U. For each $u \in U$, let

$$\ddot{u} := \{E \mid u \in E \text{ and } E \subseteq U\}, \tag{2.11}$$

and for each $E \in \mathcal{P}(U)$, let

$$\ddot{E} := \{\ddot{u} \mid u \in E\}. \tag{2.12}$$

An ordered pair $(\mathcal{P}(U), \mathcal{B})$ is said to be a hyper-measurable structure on U if $\mathcal{B}$ is a σ-field in $\mathcal{P}(U)$ and $\ddot{U} \subseteq \mathcal{B}$. Given a probability space $(\mho, \mathcal{A}, P)$ and a hyper-measurable structure $(\mathcal{P}(U), \mathcal{B})$ on U, a random set on U is defined to be a mapping $\eta : \mho \to \mathcal{P}(U)$ which is $\mathcal{A}$-$\mathcal{B}$ measurable, that is,

$$(\forall C \in \mathcal{B})\,(\eta^{-1}(C) = \{\varepsilon \mid \varepsilon \in \mho \text{ and } \eta(\varepsilon) \in C\} \in \mathcal{A}). \tag{2.13}$$

Suppose that η is a random set on U. Let

$$\tilde{f}(u) := P(\varepsilon \mid u \in \eta(\varepsilon)) \text{ for each } u \in U.$$

Then $\tilde{f}$ is a kind of fuzzy set in U. We call $\tilde{f}$ a falling shadow of the random set η, and η is called a cloud of $\tilde{f}$.

For example, $(\mho, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$, where $\mathcal{A}$ is a Borel field on $[0,1]$ and m is the usual Lebesgue measure. Let $\tilde{f}$ be a fuzzy set in U and $\tilde{f}_t := \{u \in U \mid \tilde{f}(u) \geq t\}$ be a t-cut of $\tilde{f}$. Then

$$\eta : [0,1] \to \mathcal{P}(U),\ t \mapsto \tilde{f}_t$$

is a random set and η is a cloud of $\tilde{f}$. We shall call η defined above as the cut-cloud of $\tilde{f}$ (see [9]).

3. Falling fuzzy subhoops and filters

In what follows, let H denote a hoop unless otherwise specified.

Definition 3.1. Let $(\mho, \mathcal{A}, P)$ be a probability space, and let

$$\eta : \mho \to \mathcal{P}(H)$$

be a random set. If $\eta(\varepsilon)$ is a filter (resp. a subhoop) of H for any $\varepsilon \in \mho$ with $\eta(\varepsilon) \neq \emptyset$, then the falling shadow $\tilde{f}$ of the random set η, i.e.,

$$\tilde{f}(x) = P(\varepsilon \mid x \in \eta(\varepsilon)) \tag{3.1}$$

is called a *falling fuzzy filter* (resp. *falling fuzzy subhoop*) of H.

Example 3.2. Consider a hoop $(H, \odot, \to, 1)$ in which $H = \{0, a, b, 1\}$ is a set with Cayley tables (Tables 1 and 2). Let $(\mho, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$ and consider a mapping

Table 1. Cayley table for the binary operation "$\odot$"

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

Table 2. Cayley table for the binary operation "$\to$"

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

$$\eta : [0,1] \to \mathcal{P}(H),\ t \mapsto \begin{cases} \{1\} & \text{if } t \in [0, 0.3), \\ \{1, a\} & \text{if } t \in [0.3, 0.7], \\ \{1, b\} & \text{if } t \in (0.7, 1] \end{cases} \tag{3.2}$$

Then $\eta(t)$ is both a subhoop and a filter of H for all $t \in [0,1]$. Thus the falling shadow $\tilde{f}$ of η is both a falling fuzzy subhoop and a falling fuzzy filter of H, and it is given as follows:

$$\tilde{f}(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x = 1, \\ 0.4 & \text{if } x = a, \\ 0.3 & \text{if } x = b. \end{cases} \tag{3.3}$$

Example 3.3. Consider a hoop $(H, \odot, \to, 1)$ in which $H = [0,1]$ is the unit interval in $\mathbb{R}$ and $\odot$ and $\to$ are given by $a \odot b = \min\{a, b\}$ and

$$a \to b = \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{if } a > b \end{cases} \tag{3.4}$$

for all $a, b \in H$. Let $(\mho, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$ and let $\eta : [0,1] \to \mathcal{P}(H)$ be defined by

$$\eta(t) = \begin{cases} [\frac{2}{3}, 1] & \text{if } t \in [0.6, 1], \\ [\frac{1}{2}, 1] & \text{if } t \in [0, 0.6] \end{cases} \tag{3.5}$$

Then $\eta(t)$ is a filter of H for all $t \in [0, 1]$. Thus the falling shadow $\tilde{f}$ of η is a falling fuzzy filter of H, and it is given as follows:

$$\tilde{f}(x) = \begin{cases} 0.4 & \text{if } x \in [\frac{2}{3}, 1], \\ 1 & \text{if } x \in [\frac{1}{2}, 1], \\ 0 & \text{if } x \in [0, \frac{1}{2}). \end{cases} \tag{3.6}$$

Example 3.4. Given a probability space $(\mho, \mathcal{A}, P)$, let $\mathcal{H}$ denote the set of all mappings from $\mho$ to a hoop H, that is,

$$\mathcal{H} := \{h \mid h : \mho \to H \ \text{ is a mapping}\}. \tag{3.7}$$

Let $\boxdot$ and $\twoheadrightarrow$ be binary operations on $\mathcal{H}$ defined by

$$(\forall \varepsilon \in \mho) \begin{pmatrix} (f \boxdot g)(\varepsilon) = f(\varepsilon) \odot g(\varepsilon) \\ (f \twoheadrightarrow g)(\varepsilon) = f(\varepsilon) \to g(\varepsilon) \end{pmatrix} \tag{3.8}$$

for all $f, g \in \mathcal{H}$. Also, we define a mapping

$$\mathbf{1} : \mho \to H, \ \varepsilon \mapsto 1. \tag{3.9}$$

It is routine to verify that $(\mathcal{H}, \boxdot, \twoheadrightarrow, \mathbf{1})$ is a hoop. For any subhoop and/or filter F of H and $h \in \mathcal{H}$, let

$$F_h := \{\varepsilon \in \mho \mid h(\varepsilon) \in F\} \tag{3.10}$$

and

$$\eta : \mho \to \mathcal{P}(\mathcal{H}), \ \varepsilon \mapsto \{h \in \mathcal{H} \mid h(\varepsilon) \in F\}. \tag{3.11}$$

Then $F_h \in \mathcal{A}$ and $\eta(\varepsilon) = \{h \in \mathcal{H} \mid h(\varepsilon) \in F\}$ is a subhoop and/or filter of $\mathcal{H}$. Since

$$\eta^{-1}(\ddot{h}) = \{\varepsilon \in \mho \mid h \in \eta(\varepsilon)\} = \{\varepsilon \in \mho \mid h(\varepsilon) \in F\} = F_h \in \mathcal{A}, \tag{3.12}$$

we know that η is a random set of $\mathcal{H}$. Let

$$\tilde{G}(h) = P(\varepsilon \mid h(\varepsilon) \in F).$$

Then $\tilde{G}$ is a falling fuzzy subhoop and/or filter of $\mathcal{H}$.

A *BE-algebra* is an algebra$(A, \rightsquigarrow, 1)$ of the type $(2, 0)$ such that for all $x, y, z \in A$ the following axioms are fulfilled:

$(BE1)$ $x \rightsquigarrow x = 1$,

$(BE2)$ $x \rightsquigarrow 1 = 1$,

$(BE3)$ $1 \rightsquigarrow x = x$,

$(BE4)$ $x \rightsquigarrow (y \rightsquigarrow z) = y \rightsquigarrow (x \rightsquigarrow z)$.

COROLLARY 3.5. (i) The algebraic structure $(\mathcal{H}, \boxdot, \twoheadrightarrow, \mathbf{1})$ is a BCK-algebra. (ii) The algebraic structure $(\mathcal{H}, \twoheadrightarrow, \mathbf{1})$ is a BE-algebra.

PROOF: The proof is straightforward. □

THEOREM 3.6. *Every fuzzy filter* (resp. *fuzzy subhoop*) *of H is a falling fuzzy filter* (resp. *falling fuzzy subhoop*) *of H.*

PROOF: Let $\tilde{f}$ be a fuzzy filter (resp. fuzzy subhoop) of H. Then $\tilde{f}_t$ is a filter (resp. subhoop) of H for all $t \in [0, 1]$. Define a random set as follows:

$$\eta : [0, 1] \to \mathcal{P}(H),\ t \mapsto \tilde{f}_t.$$

Then $\tilde{f}$ is a falling fuzzy filter (resp. falling fuzzy subhoop) of H. □

The converse of Theorem 3.6 is not true, in general. In fact, the falling fuzzy filter $\tilde{f}$ in Example 3.2 is not a fuzzy filter of H since $\tilde{f}(0) = 0 < 0.3 = \min\{\tilde{f}(a), \tilde{f}(a \to 0)\}$.

THEOREM 3.7. *Every falling fuzzy filter is a falling fuzzy subhoop.*

PROOF: Straightforward. □

COROLLARY 3.8. Every fuzzy filter is a falling fuzzy subhoop.

The following example shows that the converse of Theorem 3.7 and Corollary 3.8 are not true in general.

Example 3.9. Consider a hoop $(H, \odot, \to, 1)$ in which $H = \{0, a, b, 1\}$ is a set with Cayley tables (Tables 3 and 4).

Table 3. Cayley table for the binary operation "$\odot$"

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

Table 4. Cayley table for the binary operation "$\rightarrow$"

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

Let $(\mho, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$ and consider a mapping

$$\eta : [0,1] \rightarrow \mathcal{P}(H),\ t \mapsto \begin{cases} \{1, a, 0\} & \text{if } t \in [0, 0.4), \\ \{1, a\} & \text{if } t \in [0.4, 0.75], \\ \{1, b, 0\} & \text{if } t \in (0.75, 1] \end{cases} \tag{3.13}$$

Then $\eta(t)$ is a subhoop H for all $t \in [0,1]$. Thus the falling shadow $\tilde{f}$ of η is a falling fuzzy subhoop of H which is given as follows.

$$\tilde{f}(x) = \begin{cases} 0.65 & \text{if } x = 0, \\ 1 & \text{if } x = 1, \\ 0.75 & \text{if } x = a, \\ 0.25 & \text{if } x = b. \end{cases} \tag{3.14}$$

But $\eta(t) = \{1, a, 0\}$ is not a filter of H for $t \in [0, 0.4)$ since $a \in \eta(t)$ and $a \rightarrow b = 1 \in \eta(t)$, but $b \notin \eta(t)$. Hence $\tilde{f}$ is not a falling fuzzy filter of H. Since

$$\tilde{f}(b) = 0.25 < 0.75 = \min\{\tilde{f}(a \rightarrow b), \tilde{f}(a)\},$$

we know that $\tilde{f}$ is not a fuzzy filter of H.

We provide a condition for a falling fuzzy subhoop to be a falling fuzzy filter.

THEOREM 3.10. *Given a falling fuzzy subhoop $\tilde{f}$ of H, the following are equivalent.*

(1) *$\tilde{f}$ is a falling fuzzy filter of H.*

(2) *For each $\varepsilon \in \mho$, the following is valid.*

$$(\forall x, y \in H)\,(x \in \eta(\varepsilon), y \in H \setminus \eta(\varepsilon) \;\Rightarrow\; x \to y \in H \setminus \eta(\varepsilon))\,. \tag{3.15}$$

PROOF: Assume that $\tilde{f}$ is a falling fuzzy filter of H. Then $\eta(\varepsilon)$ is a filter of H for all $\varepsilon \in \mho$. Let $x, y \in H$ be such that $x \in \eta(\varepsilon)$ and $y \in H \setminus \eta(\varepsilon)$. If $x \to y \in \eta(\varepsilon)$, then $y \in \eta(\varepsilon)$ which is a contradiction. Hence $x \to y \in H \setminus \eta(\varepsilon)$. Let $\tilde{f}$ be a falling fuzzy subhoop of H in which (2) is true. Then $\eta(\varepsilon)$ is a subhoop of H for all $\varepsilon \in \mho$. Thus $1 \in \eta(\varepsilon)$. Let $x, y \in H$ be such that $x \in \eta(\varepsilon)$ and $x \to y \in \eta(\varepsilon)$. If $y \in H \setminus \eta(\varepsilon)$, then $x \to y \in H \setminus \eta(\varepsilon)$ by (3.15). This is a contradiction, and so $y \in \eta(\varepsilon)$. Therefore $\eta(\varepsilon)$ is a filter of H for all $\varepsilon \in \mho$, and thus $\tilde{f}$ is a falling fuzzy filter of H. □

Given a probability space $(\mho, \mathcal{A}, P)$ and a falling shadow $\tilde{f}$ of a random set η on H, consider the set

$$\mho(x; \eta) := \{\varepsilon \in \mho \mid x \in \eta(\varepsilon)\} \tag{3.16}$$

for $x \in H$. Then $\mho(x; \eta) \in \mathcal{A}$.

PROPOSITION 3.11. If $\tilde{f}$ is a falling fuzzy filter of H, then

$$(\forall x, y \in H)\,(x \leq y \;\Rightarrow\; \mho(x; \eta) \subseteq \mho(y; \eta))\,, \tag{3.17}$$

$$(\forall x, y \in H)\,(\mho(x \to y; \eta) \cap \mho(x; \eta) \subseteq \mho(y; \eta))\,, \tag{3.18}$$

$$(\forall x \in H)\,(\mho(x; \eta) \subseteq \mho(1; \eta))\,, \tag{3.19}$$

$$(\forall x, y \in H)\,(\mho(y; \eta) \subseteq \mho(x \to y; \eta))\,. \tag{3.20}$$

$$(\forall x, y, z \in H)\,(x \odot y \leq z \;\Rightarrow\; \mho(x; \eta) \cap \mho(y; \eta) \subseteq \mho(z; \eta))\,. \tag{3.21}$$

PROOF: Let $\tilde{f}$ be a falling fuzzy filter of H. Then $\eta(\varepsilon)$ is a filter of H for all $\varepsilon \in \mho$. Let $x, y \in H$ be such that $x \leq y$ and let $\varepsilon \in \mho(x; \eta)$. Then $x \to y = 1 \in \eta(\varepsilon)$ and $x \in \eta(\varepsilon)$. Thus $y \in \eta(\varepsilon)$, that is, $\varepsilon \in \mho(y; \eta)$. Hence $\mho(x; \eta) \subseteq \mho(y; \eta)$. Let $\varepsilon \in \mho(x \to y; \eta) \cap \mho(x; \eta)$ for all $x, y \in H$. Then

$x \to y \in \eta(\varepsilon)$ and $x \in \eta(\varepsilon)$. Since $\eta(\varepsilon)$ is a filter of H, we have $y \in \eta(\varepsilon)$, and so $\varepsilon \in \mho(y;\eta)$. This shows that (3.18) is valid. Since $x \leq 1$ for all $x \in H$, it follows from (3.17) that (3.19) holds. Since $y \leq x \to y$ for all $x, y \in H$, it follows from (3.17) that (3.20) holds. Let $x, y, z \in H$ be such that $x \odot y \leq z$. Then $x \leq y \to z$, i.e., $x \to (y \to z) = 1$. It follows from (3.18) and (3.19) that

$$\begin{aligned}\mho(z;\eta) &\supseteq \mho(y \to z;\eta) \cap \mho(y;\eta)\\ &\supseteq \mho(x;\eta) \cap \mho(x \to (y \to z);\eta) \cap \mho(y;\eta)\\ &= \mho(x;\eta) \cap \mho(1;\eta) \cap \mho(y;\eta)\\ &= \mho(x;\eta) \cap \mho(y;\eta).\end{aligned}$$

Hence (3.21) is valid. □

PROPOSITION 3.12. If $\tilde{f}$ is a falling fuzzy subhoop of H, then

$$(\forall x, y \in H) \begin{pmatrix} \mho(x;\eta) \cap \mho(y;\eta) \subseteq \mho(x \odot y;\eta) \\ \mho(x;\eta) \cap \mho(y;\eta) \subseteq \mho(x \to y;\eta) \end{pmatrix}. \tag{3.22}$$

PROOF: If $\tilde{f}$ is a falling fuzzy subhoop of H, then $\eta(\varepsilon)$ is a subhoop of H for all $\varepsilon \in \mho$. Let $\varepsilon \in \mho(x;\eta) \cap \mho(y;\eta)$. Then $x \in \eta(\varepsilon)$ and $y \in \eta(\varepsilon)$. It follows that $x \odot y \in \eta(\varepsilon)$, that is, $\varepsilon \in \mho(x \odot y;\eta)$. Hence $\mho(x;\eta) \cap \mho(y;\eta) \subseteq \mho(x \odot y;\eta)$. Similarly, we get $\mho(x;\eta) \cap \mho(y;\eta) \subseteq \mho(x \to y;\eta)$. □

COROLLARY 3.13. Every falling fuzzy filter $\tilde{f}$ of H satisfies the condition (3.22).

PROPOSITION 3.14. If $\tilde{f}$ is a falling fuzzy filter of H, then

$$(\forall x, y \in H)\,(\mho(x \odot y;\eta) = \mho(x;\eta) \cap \mho(y;\eta)), \tag{3.23}$$

$$(\forall x, y, z \in H)\,(\mho((x \to y) \to z;\eta) \cap \mho(y;\eta) \subseteq \mho(x \to z;\eta)). \tag{3.24}$$

PROOF: Since $x \odot y \leq x$ and $x \odot y \leq y$ for all $x, y \in H$, it follows from (3.17) that $\mho(x \odot y;\eta) \subseteq \mho(x;\eta)$ and $\mho(x \odot y;\eta) \subseteq \mho(y;\eta)$ Hence $\mho(x;\eta) \cap \mho(y;\eta) \supseteq \mho(x \odot y;\eta)$ for all $x, y \in H$. Combining this and Proposition 3.12 induces (3.23). Since

$$y \odot ((x \to y) \to z) \leq y \odot (y \to z) \leq z \leq x \to z$$

for all $x, y, z \in H$, we have

$$\mho(x \to z; \eta) \supseteq \mho(y \odot ((x \to y) \to z); \eta) = \mho(y; \eta) \cap \mho((x \to y) \to z; \eta)$$

by (3.17) and (3.23). □

PROPOSITION 3.15. If $\tilde{f}$ is a falling fuzzy subhoop of H, then

$$(\forall x, y \in H) \left(\tilde{f}(x \odot y) \geq \tilde{f}(x) + \tilde{f}(y) - 1, \ \tilde{f}(x \to y) \geq \tilde{f}(x) + \tilde{f}(y) - 1 \right). \tag{3.25}$$

PROOF: Assume that $\tilde{f}$ is a falling fuzzy subhoop of H. Then $\eta(\varepsilon)$ is a subhoop of H for all $\varepsilon \in \mho$. Hence

$$\{\varepsilon \in \mho \mid x \in \eta(\varepsilon)\} \cap \{\varepsilon \in \mho \mid y \in \eta(\varepsilon)\} \subseteq \{\varepsilon \in \mho \mid x \odot y \in \eta(\varepsilon)\}$$

and

$$\{\varepsilon \in \mho \mid x \in \eta(\varepsilon)\} \cap \{\varepsilon \in \mho \mid y \in \eta(\varepsilon)\} \subseteq \{\varepsilon \in \mho \mid x \to y \in \eta(\varepsilon)\}.$$

for any $x, y \in H$, and so

$$\begin{aligned} \tilde{f}(x \odot y) &= P(\varepsilon \mid x \odot y \in \eta(\varepsilon)) \\ &\geq P(\varepsilon \mid x \in \eta(\varepsilon)) \cap P(\varepsilon \mid y \in \eta(\varepsilon)) \\ &\geq P(\varepsilon \mid x \in \eta(\varepsilon)) + P(\varepsilon \mid y \in \eta(\varepsilon)) - P(\varepsilon \mid x \in \eta(\varepsilon) \ \text{ or } \ y \in \eta(\varepsilon)) \\ &= \tilde{f}(x) + \tilde{f}(y) - 1 \end{aligned}$$

and

$$\begin{aligned} \tilde{f}(x \to y) &= P(\varepsilon \mid x \to y \in \eta(\varepsilon)) \\ &\geq P(\varepsilon \mid x \in \eta(\varepsilon)) \cap P(\varepsilon \mid y \in \eta(\varepsilon)) \\ &\geq P(\varepsilon \mid x \in \eta(\varepsilon)) + P(\varepsilon \mid y \in \eta(\varepsilon)) - P(\varepsilon \mid x \in \eta(\varepsilon) \ \text{ or } \ y \in \eta(\varepsilon)) \\ &= \tilde{f}(x) + \tilde{f}(y) - 1. \end{aligned}$$

This completes the proof. □

PROPOSITION 3.16. If $\tilde{f}$ is a falling fuzzy filter of H, then

$$\tilde{f}(y) \geq \tilde{f}(x \to y) + \tilde{f}(x) - 1 \tag{3.26}$$

for all $x, y \in H$ with $\tilde{f}(x \to y) + \tilde{f}(x) \geq 1$.

PROOF: If $\tilde{f}$ is a falling fuzzy filter of H, then $\eta(\varepsilon)$ is a filter of H for all $\varepsilon \in \mho$. For any $x, y \in H$, if $\tilde{f}(x \to y) + \tilde{f}(x) \geq 1$, then

$$\{\varepsilon \in \mho \mid x \to y \in \eta(\varepsilon)\} \cap \{\varepsilon \in \mho \mid x \in \eta(\varepsilon)\} \subseteq \{\varepsilon \in \mho \mid y \in \eta(\varepsilon)\},$$

and so

$$\begin{aligned}\tilde{f}(y) &= P(\varepsilon \mid y \in \eta(\varepsilon)) \\ &\geq P(\varepsilon \mid x \to y \in \eta(\varepsilon)) \cap P(\varepsilon \mid x \in \eta(\varepsilon)) \\ &\geq P(\varepsilon \mid x \to y \in \eta(\varepsilon)) + P(\varepsilon \mid x \in \eta(\varepsilon)) - P(\varepsilon \mid x \to y \in \eta(\varepsilon) \text{ or } x \in \eta(\varepsilon)) \\ &= \tilde{f}(x \to y) + \tilde{f}(x) - 1.\end{aligned}$$

This completes the proof. □

THEOREM 3.17. *For any falling shadow $\tilde{f}$ of the random set η, if two conditions (3.18) and (3.19) are valid, then $\tilde{f}$ is a falling fuzzy filter of H.*

PROOF: Assume that $\eta(\varepsilon)$ is nonempty for all $\varepsilon \in \mho$. Then there exists $x \in \eta(\varepsilon)$ and so $\varepsilon \in \mho(x; \eta) \subseteq \mho(1; \eta)$. Thus $1 \in \eta(\varepsilon)$. Let $x, y \in H$ be such that $x \to y \in \eta(\varepsilon)$ and $x \in \eta(\varepsilon)$. Then $\varepsilon \in \mho(x \to y; \eta)$ and $\varepsilon \in \mho(x; \eta)$. It follows from (3.18) that

$$\varepsilon \in \mho(x \to y; \eta) \cap \mho(x; \eta) \subseteq \mho(y; \eta).$$

Thus $y \in \eta(\varepsilon)$, and hence $\eta(\varepsilon)$ is a filter of H. Therefore the falling shadow $\tilde{f}$ of the random set η is a falling fuzzy filter of H. □

THEOREM 3.18. *If a falling shadow $\tilde{f}$ of the random set η satisfies (3.17), (3.19) and (3.23), then $\tilde{f}$ is a falling fuzzy filter of H.*

PROOF: Let $x, y \in H$. Since $x \odot (x \to y) \leq y$, it follows from (3.17) and (3.23) that

$$\mho(y; \eta) \supseteq \mho(x \odot (x \to y); \eta) = \mho(x; \eta) \cap \mho(x \to y; \eta).$$

Therefore $\tilde{f}$ is a falling fuzzy filter of H by Theorem 3.17. □

PROPOSITION 3.19. If a falling shadow $\tilde{f}$ of the random set η satisfies (3.17) and (3.23), then

$$(\forall x, y, z \in H)\,(\mho(x \to y; \eta) \cap \mho(y \to z; \eta) \subseteq \mho(x \to z; \eta)), \tag{3.27}$$

$$(\forall x, y, z \in H)\,(\mho(x \odot z; \eta) \cap \mho(x \to y; \eta) \subseteq \mho(y \odot z; \eta)). \tag{3.28}$$

PROOF: Since $(x \to y) \odot (y \to z) \leq x \to z$ for all $x, y, z \in H$, the condition (3.27) is induced by (3.17) and (3.23). Since $(z \odot x) \odot (x \to y) \leq z \odot y$ for all $x, y, z \in H$, the condition (3.28) is induced by (3.17) and (3.23). □

Since every falling fuzzy filter $\tilde{f}$ of H satisfies two conditions (3.17) and (3.23), we have the following corollary.

COROLLARY 3.20. Every falling fuzzy filter $\tilde{f}$ of H satisfies the conditions (3.27) and (3.28).

THEOREM 3.21. *If a falling shadow $\tilde{f}$ of the random set η satisfies* (3.19) *and* (3.21)*, then $\tilde{f}$ is a falling fuzzy filter of H.*

PROOF: Let $\tilde{f}$ be a falling shadow of the random set η satisfying (3.19) and (3.21). Since $x \odot (x \to y) \leq y$ for all $x, y \in H$, we have $\mho(x; \eta) \cap \mho(x \to y; \eta) \subseteq \mho(y; \eta)$. Using Theorem 3.17, we know that $\tilde{f}$ is a falling fuzzy filter of H. □

THEOREM 3.22. *If a falling shadow $\tilde{f}$ of the random set η satisfies* (3.19) *and* (3.24)*, then $\tilde{f}$ is a falling fuzzy filter of H.*

PROOF: Let $\tilde{f}$ be a falling shadow of the random set η satisfying (3.19) and (3.24). Taking $x = 1$, $y = x$ and $z = y$ in (3.24) induces the condition (3.18). Therefore $\tilde{f}$ is a falling fuzzy filter of H by Theorem 3.17. □

THEOREM 3.23. *If a falling shadow $\tilde{f}$ of the random set η satisfies* (3.19) *and* (3.28)*, then $\tilde{f}$ is a falling fuzzy filter of H.*

PROOF: Let $\tilde{f}$ be a falling shadow of the random set η satisfying (3.19) and (3.28). Taking $z = 1$ in (3.28) induces the condition (3.18). Therefore $\tilde{f}$ is a falling fuzzy filter of H by Theorem 3.17. □

4. Conclusions and future work

The falling shadow theory is applied to subhoops and filters in hoops. The notions of falling fuzzy subhoops and falling fuzzy filters in hoops are introduced, and several properties are investigated. Relationship between falling fuzzy subhoops and falling fuzzy filters are discussed, and conditions for a falling fuzzy subhoop to be a falling fuzzy filter are provided. Also conditions for a falling shadow of a random set to be a falling fuzzy filter are displayed. On the basis of these results, we will apply the theory of falling shadows to the another type of ideals and filters in hoops and investigate some properties and equali definition of them and study the relation between them in future study.

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A GENERAL MODEL OF NEUTROSOPHIC IDEALS IN BCK/BCI-ALGEBRAS BASED ON NEUTROSOPHIC POINTS

Abstract

More general form of $(\in, \in\vee q)$-neutrosophic ideal is introduced, and their properties are investigated. Relations between $(\in, \in)$-neutrosophic ideal and $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic ideal are discussed. Characterizations of $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic ideal are discussed, and conditions for a neutrosophic set to be an $(\in, \in\vee q_{(k_T,k_I,k_F)})$-neutrosophic ideal are displayed.

Keywords: Ideal, neutrosophic $\in$-subset, neutrosophic q_k-subset, neutrosophic $\in\vee q_k$-subset, $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic ideal.

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1. Introduction

Smarandache [23, 24] introduced the concept of neutrosophic sets which is a more general platform to extend the notions of the classical set and (intuitionistic, interval valued) fuzzy set. Neutrosophic set theory is applied to several parts which are referred to the site http://fs.gallup.unm.

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edu/neutrosophy.htm. Jun [10] introduced the notion of neutrosophic subalgebras in BCK/BCI-algebras based on neutrosophic points. Borumand and Jun [22] studied several properties of $(\in, \in \vee q)$-neutrosophic subalgebras and $(q, \in \vee q)$-neutrosophic subalgebras in BCK/BCI-algebras. Jun et al. [11] discussed neutrosophic $\mathcal{N}$-structures with an application in BCK/BCI-algebras, and in [13, 14] introduced neutrosophic quadruple numbers based on a set and construct neutrosophic quadruple BCK/BCI-algebras.

Song et al. [25] introduced the notion of commutative $\mathcal{N}$-ideal in BCK-algebras and investigated several properties. Bordbar, Jun and et al. [21] and [17] introduced the notion of $(q, \in \vee q)$-neutrosophic ideal, and $(\in, \in \vee q)$-neutrosophic ideal in BCK/BCI-algebras, and investigated related properties. Also in [7, 26], they discussed the notion of $BMBJ$-neutrosophic sets, subalgebra and ideals, as a generalisation of neutrosophic set, and investigated its application and related properties to BCI/BCK-algebras.

For more information about the mentioned topics, please refer to [3, 4, 8, 12, 16, 18, 19, 20].

In this paper, we introduce a more general form of $(\in, \in \vee q)$-neutrosophic ideal, and investigate their properties. We discuss relations between $(\in, \in)$-neutrosophic ideal and $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic ideal. We consider characterizations of $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic ideal. We investigate conditions for a neutrosophic set to be an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic ideal. We find conditions for an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic ideal to be an $(\in, \in)$-neutrosophic ideal.

2. Preliminaries

By a *BCI-algebra* we mean a set X with a binary operation $*$ and the special element 0 satisfying the axioms:

(a1) $((x * y) * (x * z)) * (z * y) = 0$,

(a2) $(x * (x * y)) * y = 0$,

(a3) $x * x = 0$,

(a4) $x * y = y * x = 0 \ \Rightarrow \ x = y$, for all $x, y, z \in X$. If a BCI-algebra X satisfies the axiom

(a5) $0 * x = 0$ for all $x \in X$,

then we say that X is a BCK-*algebra*. A subset I of a BCK/BCI-algebra X is called an *ideal* of X (see [9, 15]) if it satisfies:

$$0 \in I, \tag{2.1}$$

$$(\forall x, y \in X)\,(x * y \in I,\, y \in I \;\Rightarrow\; x \in I)\,. \tag{2.2}$$

The collection of all BCK-algebras and all BCI-algebras are denoted by $\mathcal{B}_K(X)$ and $\mathcal{B}_I(X)$, respectively. Also $\mathcal{B}(X) := \mathcal{B}_K(X) \cup \mathcal{B}_I(X)$.

We refer the reader to the books [9] and [15] for further information regarding BCK/BCI-algebras.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\bigvee\{a_i \mid i \in \Lambda\} = \sup\{a_i \mid i \in \Lambda\}$$

and

$$\bigwedge\{a_i \mid i \in \Lambda\} = \inf\{a_i \mid i \in \Lambda\}.$$

If $\Lambda = \{1, 2\}$, we will also use $a_1 \vee a_2$ and $a_1 \wedge a_2$ instead of $\bigvee\{a_i \mid i \in \{1, 2\}\}$ and $\bigwedge\{a_i \mid i \in \{1, 2\}\}$, respectively.

Let X be a non-empty set. A *neutrosophic set* (NS) in X (see [23]) is a structure of the form:

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}$$

where $A_T : X \to [0, 1]$ is a truth membership function, $A_I : X \to [0, 1]$ is an indeterminate membership function, and $A_F : X \to [0, 1]$ is a false membership function. For the sake of simplicity, we shall use the symbol $A = (A_T, A_I, A_F)$ for the neutrosophic set

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}.$$

Given a neutrosophic set $A = (A_T, A_I, A_F)$ in a set X, $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$, we consider the following sets (see [10]):

$T_\in(A; \alpha) := \{x \in X \mid A_T(x) \geq \alpha\}$,

$I_\in(A; \beta) := \{x \in X \mid A_I(x) \geq \beta\}$,

$F_\in(A;\gamma) := \{x \in X \mid A_F(x) \leq \gamma\}$.

We say $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are *neutrosophic* $\in$*-subsets*.

3. Generalizations of neutrosophic ideals based on neutrosophic points

In what follows, let k_T, k_I and k_F denote arbitrary elements of $[0,1)$ unless otherwise specified. If k_T, k_I and k_F are the same number in $[0,1)$, then it is denoted by k, i.e., $k = k_T = k_I = k_F$.

Given a neutrosophic set $A = (A_T, A_I, A_F)$ in a set X, $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$, we consider the following sets:

$T_{q_{k_T}}(A;\alpha) := \{x \in X \mid A_T(x) + \alpha + k_T > 1\}$,

$I_{q_{k_I}}(A;\beta) := \{x \in X \mid A_I(x) + \beta + k_I > 1\}$,

$F_{q_{k_F}}(A;\gamma) := \{x \in X \mid A_F(x) + \gamma + k_F < 1\}$,

$T_{\in\vee\, q_{k_T}}(A;\alpha) := \{x \in X \mid A_T(x) \geq \alpha \text{ or } A_T(x) + \alpha + k_T > 1\}$,

$I_{\in\vee\, q_{k_I}}(A;\beta) := \{x \in X \mid A_I(x) \geq \beta \text{ or } A_I(x) + \beta + k_I > 1\}$,

$F_{\in\vee\, q_{k_F}}(A;\gamma) := \{x \in X \mid A_F(x) \leq \gamma \text{ or } A_F(x) + \gamma + k_F < 1\}$.

We say $T_{q_{k_T}}(A;\alpha)$, $I_{q_{k_I}}(A;\beta)$ and $F_{q_{k_F}}(A;\gamma)$ are *neutrosophic* q_k*-subsets*; and $T_{\in\vee\, q_{k_T}}(A;\alpha)$, $I_{\in\vee\, q_{k_I}}(A;\beta)$ and $F_{\in\vee\, q_{k_F}}(A;\gamma)$ are *neutrosophic* $\in\vee\, q_k$*-subsets*. For $\psi \in \{\in, q, q_k, q_{k_T}, q_{k_I}, q_{k_F}, \in\vee\, q, \in\vee\, q_k, \in\vee\, q_{k_T}, \in\vee\, q_{k_I}, \in\vee\, q_{k_F}\}$, the element of $T_\psi(A;\alpha)$ (resp., $I_\psi(A;\beta)$ and $F_\psi(A;\gamma)$) is called a *neutrosophic* T_ψ*-point* (resp., *neutrosophic* I_ψ*-point* and *neutrosophic* F_ψ*-point*) with value α (resp., β and γ).

It is clear that

$$T_{\in\vee\, q_{k_T}}(A;\alpha) = T_\in(A;\alpha) \cup T_{q_{k_T}}(A;\alpha), \tag{3.1}$$

$$I_{\in\vee\, q_{k_I}}(A;\beta) = I_\in(A;\beta) \cup I_{q_{k_I}}(A;\beta), \tag{3.2}$$

$$F_{\in\vee\, q_{k_F}}(A;\gamma) = F_\in(A;\gamma) \cup F_{q_{k_F}}(A;\gamma). \tag{3.3}$$

THEOREM 3.1. *Given a neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$, the following assertions are equivalent.*

(1) *The nonempty neutrosophic $\in$-subsets $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are ideals of X for all $\alpha \in (\frac{1-k_T}{2}, 1]$, $\beta \in (\frac{1-k_I}{2}, 1]$ and $\gamma \in [0, \frac{1-k_F}{2})$.*

(2) *$A = (A_T, A_I, A_F)$ satisfies the following assertion.*

$$(\forall x \in X)\left(\begin{array}{l} A_T(x) \leq A_T(0) \vee \frac{1-k_T}{2} \\ A_I(x) \leq A_I(0) \vee \frac{1-k_I}{2} \\ A_F(x) \geq A_F(0) \wedge \frac{1-k_F}{2} \end{array}\right) \tag{3.4}$$

and

$$(\forall x, y \in X)\left(\begin{array}{l} A_T(x) \vee \frac{1-k_T}{2} \geq A_T(x*y) \wedge A_T(y) \\ A_I(x) \vee \frac{1-k_I}{2} \geq A_I(x*y) \wedge A_I(y) \\ A_F(x) \wedge \frac{1-k_F}{2} \leq A_F(x*y) \vee A_F(y) \end{array}\right) \tag{3.5}$$

Proof: Assume that the nonempty neutrosophic $\in$-subsets $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are ideals of X for all $\alpha \in (\frac{1-k_T}{2}, 1]$, $\beta \in (\frac{1-k_I}{2}, 1]$ and $\gamma \in [0, \frac{1-k_F}{2})$. If there are $a, b \in X$ such that $A_T(a) > A_T(0) \vee \frac{1-k_T}{2}$, then $a \in T_\in(A;\alpha_a)$ and $0 \notin T_\in(A;\alpha_a)$ for $\alpha_a := A_T(a) \in (\frac{1-k_T}{2}, 1]$. This is a contradiction, and so $A_T(x) \leq A_T(0) \vee \frac{1-k_T}{2}$ for all $x \in X$. We also know that $A_I(x) \leq A_I(0) \vee \frac{1-k_I}{2}$ for all $x \in X$ by the similar way. Now, let $x \in X$ be such that $A_F(x) < A_F(0) \wedge \frac{1-k_F}{2}$. If we take $\gamma_x := A_F(x)$, then $\gamma_x \in [0, \frac{1-k_F}{2})$ and so $0 \in F_\in(A;\gamma_x)$ since $F_\in(A;\gamma_x)$ is an ideal of X. Hence $A_F(0) \leq \gamma_x = A_F(x)$, which is a contradiction. Hence $A_F(x) \geq A_F(0) \wedge \frac{1-k_F}{2}$ for all $x \in X$. Suppose that $A_I(x) \vee \frac{1-k_I}{2} < A_I(x*y) \wedge A_I(y)$ for some $x, y \in X$ and take $\beta := A_I(x*y) \wedge A_I(y)$. Then $\beta \in (\frac{1-k_I}{2}, 1]$ and $x*y, y \in I_\in(A;\beta)$. But $x \notin I_\in(A;\beta)$ which is a contradiction. Thus $A_I(x) \vee \frac{1-k_I}{2} \geq A_I(x*y) \wedge A_I(y)$ for all $x, y \in X$. Similarly, we have $A_T(x) \vee \frac{1-k_T}{2} \geq A_T(x*y) \wedge A_T(y)$ for all $x, y \in X$. Suppose that there exist $x, y \in X$ such that $A_F(x) \wedge \frac{1-k_F}{2} > A_F(x*y) \vee A_F(y)$. Taking $\gamma := A_F(x*y) \vee A_F(y)$ implies that $\gamma \in [0, \frac{1-k_F}{2})$, $x*y \in F_\in(A;\gamma)$ and $y \in F_\in(A;\gamma)$, but $x \notin F_\in(A;\gamma)$. This is a contradiction, and so $A_F(x) \wedge \frac{1-k_F}{2} \leq A_F(x*y) \vee A_F(y)$ for all $x, y \in X$.

Conversely, suppose that $A = (A_T, A_I, A_F)$ satisfies two conditions (3.4) and (3.5). Let $\alpha \in (\frac{1-k_T}{2}, 1]$, $\beta \in (\frac{1-k_I}{2}, 1]$ and $\gamma \in [0, \frac{1-k_F}{2})$ be such that $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are nonempty. For any $x \in T_\in(A;\alpha)$, $y \in I_\in(A;\beta)$ and $z \in F_\in(A;\gamma)$, we get

$$A_T(0) \vee \tfrac{1-k_T}{2} \geq A_T(x) \geq \alpha > \tfrac{1-k_T}{2},$$
$$A_I(0) \vee \tfrac{1-k_I}{2} \geq A_I(y) \geq \beta > \tfrac{1-k_I}{2},$$
$$A_F(0) \wedge \tfrac{1-k_F}{2} \leq A_F(z) \leq \gamma < \tfrac{1-k_F}{2},$$

and so $A_T(0) \geq \alpha$, $A_I(0) \geq \beta$ and $A_F(0) \leq \gamma$. Hence $0 \in T_\in(A;\alpha)$, $0 \in I_\in(A;\beta)$ and $0 \in F_\in(A;\gamma)$. Let $a,b,x,y,u,v \in X$ be such that $a*b \in T_\in(A;\alpha)$, $b \in T_\in(A;\alpha)$, $x*y \in I_\in(A;\beta)$, $y \in I_\in(A;\beta)$, $u*v \in F_\in(A;\gamma)$, and $v \in F_\in(A;\gamma)$. It follows from (3.5) that

$$A_T(a) \vee \tfrac{1-k_T}{2} \geq A_T(a*b) \wedge A_T(b) \geq \alpha > \tfrac{1-k_T}{2},$$
$$A_I(x) \vee \tfrac{1-k_I}{2} \geq A_I(x*y) \wedge A_I(y) \geq \beta > \tfrac{1-k_I}{2},$$
$$A_F(u) \wedge \tfrac{1-k_F}{2} \leq A_F(u*v) \vee A_F(v) \leq \gamma < \tfrac{1-k_F}{2}.$$

Hence $A_T(a) \geq \alpha$, $A_I(x) \geq \beta$ and $A_F(u) \leq \gamma$, that is, $a \in T_\in(A;\alpha)$, $x \in I_\in(A;\beta)$ and $u \in F_\in(A;\gamma)$. Therefore $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are ideals of X for all $\alpha \in (\frac{1-k_T}{2}, 1]$, $\beta \in (\frac{1-k_I}{2}, 1]$ and $\gamma \in [0, \frac{1-k_F}{2})$. □

COROLLARY 3.2 ([21]). Given a neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$, the following assertions are equivalent.

(1) The nonempty neutrosophic $\in$-subsets $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are ideals of X for all $\alpha, \beta \in (0.5, 1]$ and $\gamma \in [0, 0.5)$.

(2) $A = (A_T, A_I, A_F)$ satisfies the following assertion.

$$(\forall x \in X) \begin{pmatrix} A_T(x) \leq A_T(0) \vee 0.5 \\ A_I(x) \leq A_I(0) \vee 0.5 \\ A_F(x) \geq A_F(0) \wedge 0.5 \end{pmatrix}$$

and

$$(\forall x, y \in X) \begin{pmatrix} A_T(x) \vee 0.5 \geq A_T(x*y) \wedge A_T(y) \\ A_I(x) \vee 0.5 \geq A_I(x*y) \wedge A_I(y) \\ A_F(x) \wedge 0.5 \leq A_F(x*y) \vee A_F(y) \end{pmatrix}$$

DEFINITION 3.3. A neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$ is called an $(\in, \in \vee\, q_{(k_T,k_I,k_F)})$*-neutrosophic ideal* of X if the following assertions are valid.

$$(\forall x \in X)\begin{pmatrix} x \in T_{\in}(A;\alpha_x) \;\Rightarrow\; 0 \in T_{\in\vee\, q_{k_T}}(A;\alpha_x) \\ x \in I_{\in}(A;\beta_x) \;\Rightarrow\; 0 \in I_{\in\vee\, q_{k_I}}(A;\beta_x) \\ x \in F_{\in}(A;\gamma_x) \;\Rightarrow\; 0 \in F_{\in\vee\, q_{k_F}}(A;\gamma_x) \end{pmatrix}, \tag{3.6}$$

$$(\forall x, y \in X)\begin{pmatrix} x*y \in T_{\in}(A;\alpha_x),\ y \in T_{\in}(A;\alpha_y) \Rightarrow x \in T_{\in\vee\, q_{k_T}}(A;\alpha_x \wedge \alpha_y) \\ x*y \in I_{\in}(A;\beta_x),\ y \in I_{\in}(A;\beta_y) \Rightarrow x \in I_{\in\vee\, q_{k_I}}(A;\beta_x \wedge \beta_y) \\ x*y \in F_{\in}(A;\gamma_x),\ y \in F_{\in}(A;\gamma_y) \Rightarrow x \in F_{\in\vee\, q_{k_F}}(A;\gamma_x \vee \gamma_y) \end{pmatrix} \tag{3.7}$$

for all $\alpha_x, \alpha_y, \beta_x, \beta_y \in (0, 1]$ and $\gamma_x, \gamma_y \in [0, 1)$.

Example 3.4. Let $X = \{0, 1, 2, 3, 4\}$ be a set with the binary operation $*$ which is given in Table 1.

Table 1. Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	2	0
3	3	1	3	0	3
4	4	4	4	4	0

Then $(X, *, 0)$ is a BCK-algebra (see [15]). Consider a neutrosophic set $A = (A_T, A_I, A_F)$ in X which is given by Table 2.

Table 2. Tabular representation of $A = (A_T, A_I, A_F)$

X	$A_T(x)$	$A_I(x)$	$A_F(x)$
0	0.6	0.5	0.45
1	0.5	0.3	0.93
2	0.3	0.7	0.67
3	0.4	0.3	0.93
4	0.1	0.2	0.74

Routine calculations show that $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic ideal of X for $k_T = 0.24$, $k_I = 0.08$ and $k_F = 0.16$.

THEOREM 3.5. *A neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$ is an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic ideal of $X \in \mathcal{B}(X)$ if and only if $A = (A_T, A_I, A_F)$ satisfies the following assertions.*

$$(\forall x \in X) \left(\begin{array}{l} A_T(0) \geq A_T(x) \wedge \frac{1-k_T}{2} \\ A_I(0) \geq A_I(x) \wedge \frac{1-k_I}{2} \\ A_F(0) \leq A_F(x) \vee \frac{1-k_F}{2} \end{array} \right), \tag{3.8}$$

$$(\forall x, y \in X) \left(\begin{array}{l} A_T(x) \geq \bigwedge\{A_T(x*y), A_T(y), \frac{1-k_T}{2}\} \\ A_I(x) \geq \bigwedge\{A_I(x*y), A_I(y), \frac{1-k_I}{2}\} \\ A_F(x) \leq \bigvee\{A_F(x*y), A_F(y), \frac{1-k_F}{2}\} \end{array} \right). \tag{3.9}$$

PROOF: Assume that $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$ is an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic ideal of $X \in \mathcal{B}(X)$. If $A_T(0) < A_T(a) \wedge \frac{1-k_T}{2}$ for some $a \in X$, then there exists $\alpha_a \in (0, 1]$ such that $A_T(0) < \alpha_a \leq A_T(a) \wedge \frac{1-k_T}{2}$. It follows that $\alpha_a \in (0, \frac{1-k_T}{2}]$, $a \in T_\in(A; \alpha_a)$ and $0 \notin T_\in(A; \alpha_a)$. Also, $A_T(0) + \alpha_a + k_T < 2\alpha_a + k_T \leq 1$, i.e., $0 \notin T_{q_{k_T}}(A; \alpha_a)$. Hence $0 \notin T_{\in\vee\, q_{k_T}}(A; \alpha_a)$, a contradiction. Thus $A_T(0) \geq A_T(x) \wedge \frac{1-k_T}{2}$ for all $x \in X$. Similarly, we have $A_I(0) \geq A_I(x) \wedge \frac{1-k_I}{2}$ for all $x \in X$. Suppose that $A_F(0) > A_F(z) \vee \frac{1-k_F}{2}$ for some $z \in X$ and take $\gamma_z := A_F(z) \vee \frac{1-k_F}{2}$. Then $\gamma_z \geq \frac{1-k_F}{2}$, $z \in F_\in(A; \gamma_z)$ and $0 \notin F_\in(A; \gamma_z)$. Also $A_F(0) + \gamma_z + k_F \geq 1$, that is, $0 \notin F_{q_{k_F}}(A; \gamma_z)$. This is a contradiction, and thus $A_F(0) \leq A_F(x) \vee \frac{1-k_F}{2}$ for all $x \in X$. Suppose that $A_I(a) < \bigwedge\{A_I(a*b), A_I(b), \frac{1-k_I}{2}\}$ for some $a, b \in X$ and take $\beta := \bigwedge\{A_I(a * b), A_I(b), \frac{1-k_I}{2}\}$. Then $\beta \leq \frac{1-k_I}{2}$, $a * b \in I_\in(A; \beta)$, $b \in I_\in(A; \beta)$ and $a \notin I_\in(A; \beta)$. Also, we have $A_I(a) + \beta + k_I \leq 1$, i.e., $a \notin I_{q_{k_F}}(A; \beta)$. This is impossible, and therefore $A_I(x) \geq \bigwedge\{A_I(x*y), A_I(y), \frac{1-k_I}{2}\}$ for all $x, y \in X$. By the similar way, we can verify that $A_T(x) \geq \bigwedge\{A_T(x * y), A_T(y), \frac{1-k_T}{2}\}$ for all $x, y \in X$. Now assume that $A_F(a) > \bigvee\{A_F(a * b), A_F(b), \frac{1-k_F}{2}\}$ for some $a, b \in X$. Then there exists $\gamma \in [0, 1)$ such that $A_F(a) > \gamma \geq \bigvee\{A_F(a * b), A_F(b), \frac{1-k_F}{2}\}$. Then $\gamma \geq \frac{1-k_F}{2}$, $a * b \in F_\in(A; \gamma)$, $b \in F_\in(A; \gamma)$ and $a \notin F_\in(A; \gamma)$. Also, $A_F(a)+\gamma+k_F \geq 1$, i.e., $a \notin F_{q_{k_F}}(A; \gamma)$. Thus $a \notin F_{\in\vee\, q_{k_F}}(A; \gamma)$, which is a contradiction. Hence $A_F(x) \leq \bigvee\{A_F(x*y), A_F(y), \frac{1-k_F}{2}\}$ for all $x, y \in X$.

Conversely, suppose that $A = (A_T, A_I, A_F)$ satisfies two conditions (3.8) and (3.9). For any $x, y, z \in X$, let $\alpha_x, \beta_y \in (0, 1]$ and $\gamma_z \in [0, 1)$ be such that $x \in T_\in(A; \alpha_x)$, $y \in I_\in(A; \beta_y)$ and $z \in F_\in(A; \gamma_z)$. Then $A_T(x) \geq \alpha_x$, $A_I(y) \geq \beta_y$ and $A_F(z) \leq \gamma_z$. Assume that $A_T(0) < \alpha_x$, $A_I(0) < \beta_y$ and $A_F(0) > \gamma_z$. If $A_T(x) < \frac{1-k_T}{2}$, then

$$A_T(0) \geq A_T(x) \wedge \tfrac{1-k_T}{2} = A_T(x) \geq \alpha_x,$$

a contradiction. Hence $A_T(x) \geq \frac{1-k_T}{2}$, and so

$$A_T(0) + \alpha_x + k_T > 2A_T(0) + k_T \geq 2\left(A_T(x) \wedge \tfrac{1-k_T}{2}\right) + k_T = 1.$$

Hence $0 \in T_{q_{k_T}}(A; \alpha_x) \subseteq T_{\in\vee\, q_{k_T}}(A; \alpha_x)$. Similarly, we get $0 \in I_{q_{k_I}}(A; \beta_y) \subseteq I_{\in\vee\, q_{k_I}}(A; \beta_y)$. If $A_F(z) > \frac{1-k_F}{2}$, then $A_F(0) \leq A_F(z) \vee \frac{1-k_F}{2} = A_F(z) \leq \gamma_z$ which is a contradiction. Hence $A_F(z) \leq \frac{1-k_F}{2}$, and thus

$$A_F(0) + \gamma_z + k_F < 2A_F(0) + k_F \leq 2\left(A_F(z) \vee \tfrac{1-k_F}{2}\right) + k_F = 1.$$

Hence $0 \in F_{q_{k_F}}(A; \gamma_z) \subseteq F_{\in\vee\, q_{k_F}}(A; \gamma_z)$. For any $a, b, p, q, x, y \in X$, let $\alpha_a, \alpha_b, \beta_p, \beta_q \in (0, 1]$ and $\gamma_x, \gamma_y \in [0, 1)$ be such that $a * b \in T_\in(A; \alpha_a)$, $b \in T_\in(A; \alpha_b)$, $p * q \in I_\in(A; \beta_p)$, $q \in I_\in(A; \beta_q)$, $x * y \in F_\in(A; \gamma_x)$, and $y \in F_\in(A; \gamma_y)$. Then $A_T(a * b) \geq \alpha_a$, $A_T(b) \geq \alpha_b$, $A_I(p * q) \geq \beta_p$, $A_I(q) \geq \beta_q$, $A_F(x * y) \leq \gamma_x$, and $A_F(y) \leq \gamma_y$. Suppose that $a \notin T_\in(A; \alpha_a \wedge \alpha_b)$. Then $A_T(a) < \alpha_a \wedge \alpha_b$. If $A_T(a * b) \wedge A_T(b) < \frac{1-k_T}{2}$, then

$$A_T(a) \geq \bigwedge\{A_T(a * b), A_T(b), \tfrac{1-k_T}{2}\} = A_T(a * b) \wedge A_T(b) \geq \alpha_a \wedge \alpha_b.$$

This is a contradiction, and so $A_T(a * b) \wedge A_T(b) \geq \frac{1-k_T}{2}$. Thus

$$\begin{aligned} A_T(a) + (\alpha_a \wedge \alpha_b) + k_T &> 2A_T(a) + k_T \\ &\geq 2\left(\bigwedge\{A_T(a * b), A_T(b), \tfrac{1-k_T}{2}\}\right) + k_T = 1, \end{aligned}$$

which induces $a \in T_{q_{k_T}}(A; \alpha_a \wedge \alpha_b) \subseteq T_{\in\vee\, q_{k_T}}(A; \alpha_a \wedge \alpha_b)$. By the similarly way, we get $p \in I_{\in\vee\, q_{k_I}}(A; \beta_p \wedge \beta_q)$. Suppose that $x \notin F_\in(A; \gamma_x \vee \gamma_y)$, that is, $A_F(x) > \gamma_x \vee \gamma_y$. If $A_F(x * y) \vee A_F(y) > \frac{1-k_F}{2}$, then

$$A_F(x) \leq \bigvee\{A_F(x * y), A_F(y), \tfrac{1-k_F}{2}\} = A_F(x * y) \vee A_F(y) \leq \gamma_x \vee \gamma_y,$$

which is impossible. Thus $A_F(x * y) \vee A_F(y) \leq \frac{1-k_F}{2}$, and so

$$A_F(x) + (\gamma_x \vee \gamma_y) + k_F < 2A_F(x) \\ \leq 2\left(\bigvee\{A_F(x*y), A_F(y), \tfrac{1-k_F}{2}\}\right) + k_F = 1.$$

This implies that $x \in F_{q_{k_F}}(A; \gamma_x \vee \gamma_y) \subseteq F_{\in\vee\, q_{k_F}}(A; \gamma_x \vee \gamma_y)$. Consequently, $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee\, q_{(k_T,k_I,k_F)})$-neutrosophic ideal of $X \in \mathcal{B}(X)$. □

Corollary 3.6 ([21]). For a neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$, the following are equivalent.

(1) $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee\, q)$-neutrosophic ideal of $X \in \mathcal{B}(X)$.

(2) $A = (A_T, A_I, A_F)$ satisfies the following assertions.

$$(\forall x \in X)\left(\begin{array}{l} A_T(0) \geq A_T(x) \wedge 0.5 \\ A_I(0) \geq A_I(x) \wedge 0.5 \\ A_F(0) \leq A_F(x) \vee 0.5 \end{array}\right),$$

$$(\forall x, y \in X)\left(\begin{array}{l} A_T(x) \geq \bigwedge\{A_T(x*y), A_T(y), 0.5\} \\ A_I(x) \geq \bigwedge\{A_I(x*y), A_I(y), 0.5\} \\ A_F(x) \leq \bigvee\{A_F(x*y), A_F(y), 0.5\} \end{array}\right).$$

Theorem 3.7. *A neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$ is an $(\in, \in \vee\, q_{(k_T,k_I,k_F)})$-neutrosophic ideal of $X \in \mathcal{B}(X)$ if and only if the nonempty neutrosophic $\in$-subsets $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are ideals of X for all $\alpha \in (0, \frac{1-k_T}{2}]$, $\beta \in (0, \frac{1-k_I}{2}]$ and $\gamma \in [\frac{1-k_F}{2}, 1)$.*

Proof: Suppose that $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee\, q_{(k_T,k_I,k_F)})$-neutrosophic ideal of $X \in \mathcal{B}(X)$ and let $\alpha \in (0, \frac{1-k_T}{2}]$, $\beta \in (0, \frac{1-k_I}{2}]$ and $\gamma \in [\frac{1-k_F}{2}, 1)$ be such that $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are nonempty. Using (3.8), we get $A_T(0) \geq A_T(x) \wedge \frac{1-k_T}{2}$, $A_I(0) \geq A_I(y) \wedge \frac{1-k_I}{2}$, and $A_F(0) \leq A_F(z) \vee \frac{1-k_F}{2}$ for all $x \in T_\in(A;\alpha)$, $y \in I_\in(A;\beta)$ and $z \in F_\in(A;\gamma)$. It follows that $A_T(0) \geq \alpha \wedge \frac{1-k_T}{2} = \alpha$, $A_I(0) \geq \beta \wedge \frac{1-k_I}{2} = \beta$, and $A_F(0) \leq \gamma \vee \frac{1-k_F}{2} = \gamma$, that is, $0 \in T_\in(A;\alpha)$, $0 \in I_\in(A;\beta)$ and $0 \in F_\in(A;\gamma)$. Let $x, y, a, b, u, v \in X$ be such that $x*y \in T_\in(A;\alpha)$, $y \in T_\in(A;\alpha)$, $a*b \in I_\in(A;\beta)$, $b \in I_\in(A;\beta)$, $u*v \in F_\in(A;\gamma)$, and $v \in F_\in(A;\gamma)$ for $\alpha \in (0, \frac{1-k_T}{2}]$, $\beta \in (0, \frac{1-k_I}{2}]$ and $\gamma \in [\frac{1-k_F}{2}, 1)$. Then $A_T(x*y) \geq \alpha$, $A_T(y) \geq \alpha$, $A_I(a*b) \geq \beta$, $A_I(b) \geq \beta$, $A_F(u*v) \leq \gamma$, and $A_F(v) \leq \gamma$. It follows from (3.9) that

$A_T(x) \geq \bigwedge\{A_T(x*y), A_T(y), \frac{1-k_T}{2}\} \geq \alpha \wedge \frac{1-k_T}{2} = \alpha,$

$A_I(a) \geq \bigwedge\{A_I(a*b), A_I(b), \frac{1-k_I}{2}\} \geq \beta \wedge \frac{1-k_I}{2} = \beta,$

$A_F(u) \leq \bigvee\{A_F(u*v), A_F(v), \frac{1-k_F}{2}\} \leq \gamma \vee \frac{1-k_F}{2} = \gamma$

and so that $x \in T_\in(A;\alpha)$, $a \in I_\in(A;\beta)$ and $u \in F_\in(A;\gamma)$. Therefore $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are ideals of X for all $\alpha \in (0, \frac{1-k_T}{2}]$, $\beta \in (0, \frac{1-k_I}{2}]$ and $\gamma \in [\frac{1-k_F}{2}, 1)$.

Conversely, let $A = (A_T, A_I, A_F)$ be a neutrosophic set in $X \in \mathcal{B}(X)$ such that the nonempty neutrosophic $\in$-subsets $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are ideals of X for all $\alpha \in (0, \frac{1-k_T}{2}]$, $\beta \in (0, \frac{1-k_I}{2}]$ and $\gamma \in [\frac{1-k_F}{2}, 1)$. If there exist $x, y, z \in X$ such that $A_T(0) < A_T(x) \wedge \frac{1-k_T}{2}$, $A_I(0) < A_I(y) \wedge \frac{1-k_I}{2}$, and $A_F(0) > A_F(z) \vee \frac{1-k_F}{2}$, then $0 \notin T_\in(A;\alpha_x)$, $0 \notin I_\in(A;\beta_y)$ and $0 \notin F_\in(A;\gamma_z)$ by taking $\alpha_x := A_T(x) \wedge \frac{1-k_T}{2}$, $\beta_y := A_I(y) \wedge \frac{1-k_I}{2}$, and $\gamma_z := A_F(z) \vee \frac{1-k_F}{2}$. This is a contradiction, and so $A_T(0) \geq A_T(x) \wedge \frac{1-k_T}{2}$, $A_I(0) \geq A_I(x) \wedge \frac{1-k_I}{2}$, and $A_F(0) \leq A_F(x) \vee \frac{1-k_F}{2}$ for all $x \in X$. Now, suppose that there $x, y, a, b, u, v \in X$ be such that $A_T(x) < \bigwedge\{A_T(x*y), A_T(y), \frac{1-k_T}{2}\}$, $A_I(a) < \bigwedge\{A_I(a*b), A_I(b), \frac{1-k_I}{2}\}$, and $A_F(u) > \bigvee\{A_F(u*v), A_F(v), \frac{1-k_F}{2}\}$. If we take $\alpha := \bigwedge\{A_T(x*y), A_T(y), \frac{1-k_T}{2}\}$, $\beta := \bigwedge\{A_I(a*b), A_I(b), \frac{1-k_I}{2}\}$, and $\gamma := \bigvee\{A_F(u*v), A_F(v), \frac{1-k_F}{2}\}$, then $\alpha \leq \frac{1-k_T}{2}$, $\beta \leq \frac{1-k_I}{2}$, $\gamma \geq \frac{1-k_F}{2}$, $x*y \in T_\in(A;\alpha)$, $y \in T_\in(A;\alpha)$, $a*b \in I_\in(A;\beta)$, $b \in I_\in(A;\beta)$, $u*v \in F_\in(A;\gamma)$, and $v \in F_\in(A;\gamma)$. But $x \notin T_\in(A;\alpha)$, $a \notin I_\in(A;\beta)$ and $u \notin F_\in(A;\gamma)$, which induces a contradiction. Therefore $A_T(x) \geq \bigwedge\{A_T(x*y), A_T(y), \frac{1-k_T}{2}\}$, $A_I(x) \geq \bigwedge\{A_I(x*y), A_I(y), \frac{1-k_I}{2}\}$, and $A_F(x) \leq \bigvee\{A_F(x*y), A_F(y), \frac{1-k_F}{2}\}$ for all $x, y \in X$. Using Theorem 3.5, we conclude that $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee\, q_{(k_T,k_I,k_F)})$-neutrosophic ideal of $X \in \mathcal{B}(X)$. □

Corollary 3.8 ([21]). A neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$ is an $(\in, \in \vee\, q)$-neutrosophic ideal of $X \in \mathcal{B}(X)$ if and only if the nonempty neutrosophic $\in$-subsets $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are ideals of X for all $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$.

It is clear that every $(\in, \in)$-neutrosophic ideal is an $(\in, \in \vee\, q_{(k_T,k_I,k_F)})$-neutrosophic ideal. But the converse is not true in general. For example, the $(\in, \in \vee\, q_{(k_T,k_I,k_F)})$-neutrosophic ideal $A = (A_T, A_I, A_F)$ with $k_T = 0.24$, $k_I = 0.08$ and $k_F = 0.16$ in Example 3.4 is not an $(\in, \in)$-neutrosophic ideal since $2 \in I_\in(A; 0.56)$ and $0 \notin I_\in(A; 0.56)$.

We now consider conditions for an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic ideal to be an $(\in,\in)$-neutrosophic ideal.

THEOREM 3.9. *Let $A = (A_T, A_I, A_F)$ be an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic ideal of $X \in \mathcal{B}(X)$ such that*

$$(\forall x \in X)\left(A_T(x) < \tfrac{1-k_T}{2},\ A_I(x) < \tfrac{1-k_I}{2},\ A_F(x) > \tfrac{1-k_F}{2}\right).$$

Then $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of $X \in \mathcal{B}(X)$.

PROOF: Let $x, y, z \in X$, $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$ be such that $x \in T_\in(A; \alpha)$, $y \in I_\in(A; \beta)$ and $z \in F_\in(A; \gamma)$. Then $A_T(x) \geq \alpha$, $A_I(y) \geq \beta$ and $A_F(z) \leq \gamma$. It follows from (3.8) that

$$A_T(0) \geq A_T(x) \wedge \tfrac{1-k_T}{2} = A_T(x) \geq \alpha,$$

$$A_I(0) \geq A_I(y) \wedge \tfrac{1-k_I}{2} = A_I(y) \geq \beta,$$

$$A_F(0) \leq A_F(z) \vee \tfrac{1-k_F}{2} = A_F(z) \leq \gamma.$$

Hence $0 \in T_\in(A; \alpha)$, $0 \in I_\in(A; \beta)$ and $0 \in F_\in(A; \gamma)$. For any $x, y, a, b, u, v \in X$, let α_x, α_y, β_a, $\beta_b \in (0, 1]$ and γ_u, $\gamma_v \in [0, 1)$ be such that $x * y \in T_\in(A; \alpha_x)$, $y \in T_\in(A; \alpha_y)$, $a * b \in I_\in(A; \beta_a)$, $b \in I_\in(A; \beta_b)$, $u * v \in F_\in(A; \gamma_u)$, and $v \in F_\in(A; \gamma_v)$. Then $A_T(x * y) \geq \alpha_x$, $A_T(y) \geq \alpha_y$, $A_I(a * b) \geq \beta_a$, $A_I(b) \geq \beta_b$, $A_F(u * v) \leq \gamma_u$, and $A_F(v) \leq \gamma_v$. It follows from (3.9) that

$$A_T(x) \geq \bigwedge\{A_T(x * y), A_T(y), \tfrac{1-k_T}{2}\} = A_T(x * y) \wedge A_T(y) \geq \alpha_x \wedge \alpha_y,$$

$$A_I(a) \geq \bigwedge\{A_I(a * b), A_I(b), \tfrac{1-k_I}{2}\} = A_I(a * b) \wedge A_I(b) \geq \beta_a \wedge \beta_b,$$

$$A_F(u) \leq \bigvee\{A_F(u * v), A_F(v), \tfrac{1-k_F}{2}\} = A_F(u * v) \vee A_F(v) \leq \gamma_u \vee \gamma_v.$$

Thus $x \in T_\in(A; \alpha_x \wedge \alpha_y)$, $a \in I_\in(A; \beta_a \wedge \beta_b)$ and $u \in F_\in(A; \gamma_u \vee \gamma_v)$. Therefore $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of $X \in \mathcal{B}(X)$. □

COROLLARY 3.10 ([21]). Let $A = (A_T, A_I, A_F)$ be an $(\in, \in \vee q)$-neutrosophic ideal of $X \in \mathcal{B}(X)$ such that

$$(\forall x \in X)\,(A_T(x) < 0.5,\ A_I(x) < 0.5,\ A_F(x) > 0.5).$$

Then $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of $X \in \mathcal{B}(X)$.

THEOREM 3.11. *Given a neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$, if the nonempty neutrosophic $\in \vee q_k$-subsets $T_{\in\vee q_{k_T}}(A; \alpha)$, $I_{\in\vee q_{k_I}}(A; \beta)$*

and $F_{\in\vee\, q_{k_F}}(A;\gamma)$ *are ideals of* X *for all* $\alpha \in (0, \frac{1-k_T}{2}]$*,* $\beta \in (0, \frac{1-k_I}{2}]$ *and* $\gamma \in [\frac{1-k_F}{2}, 1)$*, then* $A = (A_T, A_I, A_F)$ *is an* $(\in, \in \vee\, q_{(k_T,k_I,k_F)})$*-neutrosophic ideal of* X*.*

Proof: Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in $X \in \mathcal{B}(X)$ such that the nonempty neutrosophic $\in \vee\, q_k$-subsets $T_{\in\vee\, q_{k_T}}(A;\alpha)$, $I_{\in\vee\, q_{k_I}}(A;\beta)$ and $F_{\in\vee\, q_{k_F}}(A;\gamma)$ are ideals of X for all $\alpha \in (0, \frac{1-k_T}{2}]$, $\beta \in (0, \frac{1-k_I}{2}]$ and $\gamma \in [\frac{1-k_F}{2}, 1)$. If $A_T(0) < A_T(x) \wedge \frac{1-k_T}{2} := \alpha_x$, $A_I(0) < A_I(y) \wedge \frac{1-k_I}{2} := \beta_y$ and $A_F(0) > A_F(z) \vee \frac{1-k_F}{2} := \gamma_z$ for some $x, y, z \in X$, then $x \in T_\in(A;\alpha_x) \subseteq T_{\in\vee\, q_{k_T}}(A;\alpha_x)$, $y \in I_\in(A;\beta_y) \subseteq I_{\in\vee\, q_{k_I}}(A;\beta_y)$, $z \in F_\in(A;\gamma_z) \subseteq F_{\in\vee\, q_{k_F}}(A;\gamma_z)$, $0 \notin T_\in(A;\alpha_x)$, $0 \notin I_\in(A;\beta_y)$, and $0 \notin F_\in(A;\gamma_z)$. Also, since $A_T(0) + \alpha_x + k_T < 2\alpha_x + k_T \leq 1$, i.e., $0 \notin T_{q_{k_T}}(A;\alpha_x)$, $A_I(0) + \beta_y + k_I < 2\beta_y + k_I \leq 1$, i.e., $0 \notin I_{q_{k_I}}(A;\beta_Y)$, $A_F(0) + \gamma_z + k_F > 2\gamma_z + k_F \geq 1$, i.e., $0 \notin F_{q_{k_F}}(A;\gamma_z)$, we get $0 \notin T_{\in\vee\, q_{k_T}}(A;\alpha_x)$, $0 \notin I_{\in\vee\, q_{k_I}}(A;\beta_y)$, and $0 \notin F_{\in\vee\, q_{k_F}}(A;\gamma_z)$. This is a contradiction, and thus (3.8) is valid. Suppose that there exist $a, b \in X$ such that $A_I(a) < \bigwedge\{A_I(a * b), A_I(b), \frac{1-k_I}{2}\}$. Taking $\beta := \bigwedge\{A_I(a * b), A_I(b), \frac{1-k_I}{2}\}$ implies that $a * b \in I_\in(A;\beta) \subseteq I_{\in\vee\, q_{k_I}}(A;\beta)$, $b \in I_\in(A;\beta) \subseteq I_{\in\vee\, q_{k_I}}(A;\beta)$. Since $I_{\in\vee\, q_{k_I}}(A;\beta)$ is an ideal of X, it follows that $a \in I_{\in\vee\, q_{k_I}}(A;\beta)$, i.e., $a \in I_\in(A;\beta)$ or $a \in I_{q_{k_I}}(A;\beta)$, and so that $a \in I_{q_{k_I}}(A;\beta)$, i.e., $A_I(a) + \beta + k_I > 1$, since $a \notin I_\in(A;\beta)$. But $A_I(a) + \beta + k_I < 2\beta + k_I \leq 1$, a contradiction. Hence $A_I(x) \geq \bigwedge\{A_I(x * y), A_I(y), \frac{1-k_I}{2}\}$ for all $x, y \in X$. Similarly, we can verify that $A_T(x) \geq \bigwedge\{A_T(x * y), A_T(y), \frac{1-k_T}{2}\}$ for all $x, y \in X$. Assume that $A_F(a) > \bigvee\{A_F(a * b), A_F(b), \frac{1-k_F}{2}\} := \gamma$ for some $a, b \in X$. Then $a \notin F_\in(A;\gamma)$, $a * b \in F_\in(A;\gamma) \subseteq F_{\in\vee\, q_{k_F}}(A;\gamma)$, $b \in F_\in(A;\gamma) \subseteq F_{\in\vee\, q_{k_F}}(A;\gamma)$. Since $F_{\in\vee\, q_{k_F}}(A;\gamma)$ is an ideal of X, we have $a \in F_{\in\vee\, q_{k_F}}(A;\gamma)$. On the other hand, $A_F(a) + \gamma + k_F > 2\gamma + k_F \geq 1$, that is, $a \notin F_{q_{k_F}}(A;\gamma)$. Hence $a \notin F_{\in\vee\, q_{k_F}}(A;\gamma)$, a contradiction. Thus $A_F(x) \leq \bigvee\{A_F(x * y), A_F(y), \frac{1-k_F}{2}\}$ for all $x, y \in X$. Therefore (3.9) is valid, and consequently $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee\, q_{(k_T,k_I,k_F)})$-neutrosophic ideal of X by Theorem 3.5. □

Corollary 3.12 ([21]). Given a neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$, if the nonempty neutrosophic $\in \vee\, q$-subsets $T_{\in\vee\, q}(A;\alpha)$, $I_{\in\vee\, q}(A;\beta)$ and $F_{\in\vee\, q}(A;\gamma)$ are ideals of X for all $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$, then $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee\, q)$-neutrosophic ideal of X.

4. Conclusions

More general form of $(\in, \in \vee q)$-neutrosophic ideal was introduced, and their properties were investigated. Relations between $(\in, \in)$-neutrosophic ideal and $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic ideal were discussed. Characterizations of $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic ideal were discussed, and conditions for a neutrosophic set to be an $(\in, \in\vee q_{(k_T,k_I,k_F)})$-neutrosophic ideal were displayed.

These results can be applied to characterize the neutrosophic ideals in a *BCK*/*BCI*-algebra. In our future research, we will focus on some properties of ideal such as intersections, unions, maximality, primeness and height, and try to find the relations between these properties of ideals and the results of this paper. For instance, how we can define the prime and maximal neutrosophic ideals? Whatis the meaning of height of these types of ideals? For information about the maximality, primeness and height of ideals, please refer to [1, 2, 6, 5].

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NEIGHBOURHOOD SEMANTICS FOR GRADED MODAL LOGIC

Abstract

We introduce a class of neighbourhood frames for graded modal logic embedding Kripke frames into neighbourhood frames. This class of neighbourhood frames is shown to be first-order definable but not modally definable. We also obtain a new definition of graded bisimulation with respect to Kripke frames by modifying the definition of monotonic bisimulation.

Keywords: Graded modal logic, neighbourhood frames, bisimulation.

1. Introduction

Graded modal logic **GrK** is an extension of propositional logic with graded modalities $\Diamond_n (n \in \mathbb{N})$ that count the number of successors of a given state. The interpretation of formula $\Diamond_n \varphi$ in a Kripke model is that the number of successors that satisfy φ is at least n. Originally introduced in Goble [9], the notion of a graded modality is developed so that 'propositions can be distinguished by degrees or grades of necessity or possibility' [9, p. 1]. This

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language was studied in Kaplan [11] as an extension of **S5**. Fine [8], De Caro [6] and Cerrato [2] investigated the completeness of **GrK** and its extensions. Van der Hoek [15] investigated the expressibility, decidability and definability of graded modal logic and also correspondence theory. Cerrato [3] proved the decidability by filtration for graded modal logic.

De Rijke [7] introduced graded tuple bisimulation for graded modal logic. Using this he proved the finite model property (which was first proved in Cerrato [3] via filtration) and that a first-order formula is invariant under graded bisimulation iff it is equivalent to a graded modal formula. Aceto, Ingolfsdottir and Sack [1] showed that resource bisimulation and graded bisimulation coincide over image-finite Kripke frames. Van der Hoek and Meyer [16] proposed a graded modal logic **GrS5**, which is seen as a graded epistemic logic and is able to express 'accepting φ if there are at most n exceptions to φ'. Ma and van Ditmarsch [13] developed dynamic extensions of graded epistemic logics.

Monotonic modal logics are weakenings of normal modal logics in which the *additivity* ($\Diamond\bot \leftrightarrow \bot$ and $\Diamond p \vee \Diamond q \leftrightarrow \Diamond(p \vee q)$) of the diamond modality has been weakened to *monotonicity* ($\Diamond p \vee \Diamond q \leftrightarrow \Diamond(p \vee q)$), which can also be formulated as a derivation rule: from $\vdash \varphi \rightarrow \psi$ infer $\vdash \Diamond\varphi \rightarrow \Diamond\psi$. Monotonic modal logics are interpreted over monotonic neighbourhood frames, that is neighbourhood frames where the collection of neighbourhoods of a point is closed under supersets. There have been many results about monotonic modal logics and monotonic neighbourhood frames [4, 10, 14], including model constructions, definability, correspondence theory, canonical model constructions, algebraic duality, coalgebraic semantics, interpolation, simulations of monotonic modal logics by bimodal normal logics, etc.

In this paper, we propose a neighbourhood semantics for graded modal logic. We define an operation $(.)^\bullet$ (Def. 4.2) to obtain a class of monotonic neighbourhood frames on which graded modal logic is interpreted. This class of neighbourhood frames is shown to be first-order definable in Section 5 and modally undefinable in Section 6. In Section 7 we obtain a new definition of graded bisimulation with respect to Kripke frames by modifying the definition of monotonic bisimulation and show that it is equivalent to the one proposed in [7]. Our results show that techniques for monotonic modal logics can be successfully applied to graded modal logic.

2. Preliminaries

2.1. Graded modal logic

Language. Let Prop be a set of proposition letters. Language $\mathcal{L}_g$ is defined by induction as follows:

$$\mathcal{L}_g \ni \varphi ::= p \mid \neg\varphi \mid (\varphi \vee \varphi) \mid \Diamond_n\varphi$$

where $p \in \mathsf{Prop}$ and $n \in \mathbb{N}$. We recall that $\mathbb{N}$ is the set of natural numbers. The *complexity* of a formula $\varphi \in \mathcal{L}_g$ is the number of connectives occurring in φ. Other propositional connectives $\bot$, $\top$, $\wedge$, $\rightarrow$, $\leftrightarrow$ are defined as usual. The *dual* of $\Diamond_n\varphi$ is defined as $\Box_n\varphi := \neg\Diamond_n\neg\varphi$. Further, define $\Diamond\varphi := \Diamond_1\varphi$ and $\Diamond_{!n}\varphi := \Diamond_n\varphi \wedge \neg\Diamond_{n+1}\varphi$. The interpretation of a formula $\Diamond_n\varphi$ in a Kripke model is that the number of successors that satisfy φ is at least n. The interpretation of formula $\Diamond_{!n}\varphi$ is that the number of successors that satisfy φ is exactly n.

Kripke semantics. A *Kripke frame* is a pair (W, R), denoted $\mathcal{F}$, where W is a set of states and R is a binary relation on W. Denote by F_K the class of all Kripke frames. A *Kripke model* is a pair $\mathcal{M} = (\mathcal{F}, V)$ where $\mathcal{F}$ is a Kripke frame and $V : \mathsf{Prop} \rightarrow \mathcal{P}(W)$ is a *valuation*. For model $\mathcal{M} = (W, R, V)$ and $w \in W$, we call $\mathcal{M}, w$ a *pointed model*.

Given a set X, denote by $|X|$ the *cardinality* of X. Suppose that w is a state in a Kripke model $\mathcal{M} = (W, R, V)$. The *truth* of a $\mathcal{L}_g$-formula φ at w in $\mathcal{M}$, notation $\mathcal{M}, w \Vdash \varphi$, is defined inductively as follows:

$$\begin{array}{lll}
\mathcal{M}, w \Vdash p & \text{iff} & p \in V(p) \\
\mathcal{M}, w \Vdash \neg\psi & \text{iff} & \mathcal{M}, w \nVdash \psi \\
\mathcal{M}, w \Vdash \psi_1 \vee \psi_2 & \text{iff} & \mathcal{M}, w \Vdash \psi_1 \text{ or } \mathcal{M}, w \Vdash \psi_2 \\
\mathcal{M}, w \Vdash \Diamond_n\psi & \text{iff} & \big|R[w] \cap [\![\psi]\!]_{\mathcal{M}}\big| \geq n
\end{array}$$

where $R[w] = \{v \in W : Rwv\}$ is the set of w-successors and $[\![\psi]\!]_{\mathcal{M}} = \{v \in W : \mathcal{M}, v \Vdash \psi\}$ is the *truth set* of φ in $\mathcal{M}$. For a set Γ of $\mathcal{L}_g$-formulas, we write $\mathcal{M}, w \Vdash \Gamma$ if $\mathcal{M}, w \Vdash \varphi$ for all $\varphi \in \Gamma$. Pointed models $\mathcal{M}, w$ and $\mathcal{M}', w'$ are said to be *modally equivalent* (notation: $\mathcal{M}, w \equiv_k \mathcal{M}', w'$) if for all $\mathcal{L}_g$-formulas φ, we have $\mathcal{M}, w \Vdash \varphi$ iff $\mathcal{M}', w' \Vdash \varphi$.

A formula φ is *valid at a state w in a frame $\mathcal{F}$*, notation $\mathcal{F}, w \Vdash \varphi$, if φ is true at w in every model $(\mathcal{F}, V)$ based on $\mathcal{F}$; φ is *valid in a frame $\mathcal{F}$*, notation $\mathcal{F} \Vdash \varphi$, if it is valid at every state in $\mathcal{F}$; φ is *valid in a class of frames* S_K, notation $\Vdash_{S_K} \varphi$, if $\mathcal{F} \Vdash \varphi$ for all $\mathcal{F} \in S_K$.

Let S_K be a class of Kripke frames and $\Gamma \cup \{\varphi\}$ a set of $\mathcal{L}_g$-formulas. We say that φ is a *(local) semantic consequence of* Γ *over* S_K, notation $\Gamma \Vdash_{S_K} \varphi$, if for all models $\mathcal{M}$ based on frames in S_K, and all states in $\mathcal{M}$, if $\mathcal{M}, w \Vdash \Gamma$ then $\mathcal{M}, w \Vdash \varphi$.

Graded semantics. In this subsection, we recall the graded semantics from Ma and van Ditmarsch [13]. The sum operation and the 'greater than or equal to' relation ($\geq$) are defined over natural numbers $\mathbb{N}$ plus ω, the least ordinal number greater than any natural number, i.e., $\forall n \in \mathbb{N}, n < \omega$. Variables n, m, i, j range over the natural numbers $\mathbb{N}$, not over $\mathbb{N} \cup \{\omega\}$.

A *graded frame* is a pair $\mathfrak{f} = (W, \sigma)$, where W is a set of states and $\sigma : W \times W \to \mathbb{N} \cup \{\omega\}$ is a function assigning a natural number or ω to each pair of states. Denote by F_G the class of all graded frames. A *graded model* is a pair $\mathfrak{M} = (\mathfrak{f}, V)$ where $\mathfrak{f}$ is a graded frame and $V : \mathsf{Prop} \to \mathcal{P}(W)$ is a valuation.

For $X \subseteq W$ and $w \in W$, define $\sigma(w, X)$ as $\Sigma_{u \in X} \sigma(w, u)$, the sum of $\sigma(w, u)$ for all $u \in X$. In particular, we define $\sigma(w, \emptyset) = 0$. The notation $X \subseteq_{<\omega} W$ represents that X is a finite subset of W and $\mathcal{P}_{<\omega}(W)$ is the set of finite subsets of W.

Suppose that w is a state in a graded model $\mathfrak{M} = (W, \sigma, V)$. The *truth* of a $\mathcal{L}_g$-formula φ at w in $\mathfrak{M}$, notation $\mathfrak{M}, w \Vdash \varphi$, is defined inductively as follows:

$\mathfrak{M}, w \Vdash p$	iff	$w \in V(p)$
$\mathfrak{M}, w \Vdash \neg\psi$	iff	$\mathfrak{M}, w \nVdash \psi$
$\mathfrak{M}, w \Vdash \psi_1 \vee \psi_2$	iff	$\mathfrak{M}, w \Vdash \psi_1$ or $\mathfrak{M}, w \Vdash \psi_2$
$\mathfrak{M}, w \Vdash \Diamond_n \psi$	iff	$\exists X \subseteq_{<\omega} W \ (\sigma(w, X) \geq n \ \& \ X \subseteq [\![\psi]\!]_{\mathfrak{M}})$

To our knowledge, graded frames first appeared in [6] as an intermediate structure to prove completeness of **GrK** with respect to Kripke frames. They are called *multiframes* in [1]. Graded frames are alternative semantics for graded modal logic, indeed each graded frame can be associated with a Kripke frame validating the same formulas, and vice versa as follows (cf. [13, Proposition 2.12]): Given a Kripke frame $\mathcal{F} = (W, R)$, the associated graded frame $\mathcal{F}^\circ = (W, \sigma)$ is defined by setting $\sigma(w, u) = 1$ if wRu, and $\sigma(w, u) = 0$ otherwise; given a graded frame $\mathcal{F} = (W, \sigma)$, the associated

Kripke frame $\mathcal{F}_\circ = (W_\circ, R)$ is defined by setting $W_\circ = \{(w, i) \mid w \in W \;\&\; i \in \mathbb{N} \cup \{\omega\}\}$ and $(w, i)R(u, j)$ iff $\sigma(w, u) \geq j > 0$.

Axiomatization. The *minimal graded modal logic* **GrK** consists of the following axiom schemas and inference rules:

$(Ax1)$	all instances of propositional tautologies
$(Ax2)$	$\Diamond_0\varphi \leftrightarrow \top$
$(Ax3)$	$\Diamond_n\bot \leftrightarrow \bot \qquad (n > 0)$
$(Ax4)$	$\Diamond_{n+1}\varphi \to \Diamond_n\varphi$
$(Ax5)$	$\Box(\varphi \to \psi) \to (\Diamond_n\varphi \to \Diamond_n\psi)$
$(Ax6)$	$\neg\Diamond(\varphi \wedge \psi) \wedge \Diamond_{!m}\varphi \wedge \Diamond_{!n}\psi \to \Diamond_{!(m+n)}(\varphi \vee \psi)$
(MP)	from φ and $\varphi \to \psi$ infer ψ
(Gen)	from φ infer $\Box\varphi$

The set of theorems derivable in the system **GrK** is also called **GrK**. A *graded modal logic* is a set Λ of $\mathcal{L}_g$-formulas with **Grk** $\subseteq \Lambda$. If $\varphi \in \Lambda$, we write $\vdash_\Lambda \varphi$.

THEOREM 2.1 ([6]). **GrK** is sound and complete with respect to the class of all Kripke frames.

THEOREM 2.2 (Theorem 3.2 of [13]). **GrK** is sound and complete with respect to the class of all graded frames.

2.2. Monotonic modal logic

We consider monotonic modal logic with modalities parametrized by natural numbers, i.e. $\Diamond_n$ and $\Box_n$ with $n \in \mathbb{N}$ instead of the usual single modality. As there is no interaction between different $\Diamond_n$ and $\Diamond_m$, the logic for such modalities is not essentially different from the logic for a single modality $\Diamond$ that was originally proposed.

First, a word on notation. In graded modal logic $\Diamond_n$ denotes the existence of at least n worlds. So in particular $\Diamond$ denotes the existence of at least one world. Whereas in monotonic logic the existence of a neighbourhood is denoted by $\Box$ [4] or ∇ [10]. We prefer to stick to the notation matching usage in graded modal logic. Therefore also in monotonic modal logic write $\Diamond$ (or $\Diamond_n$) to denote the existence of a neighbourhood instead of $\Box$ or ∇ ($\Box_n$ or ∇_n). Consequently, the duals of modalities are also swapped.

Neighbourhood Semantics. A *neighbourhood frame* is a tuple $\mathbb{F} = (W, \{\nu_n\}_{n\in\mathbb{N}})$ where W is a set of states and each $\nu_n : W \to \mathcal{PP}(W)$, called *neighbourhood function*. Denote by F_N the class of all neighbourhood frames. A *neighbourhood model* is a pair $\mathbb{M} = (\mathbb{F}, V)$, where $\mathbb{F}$ is a neighbourhood frame and $V : \mathsf{Prop} \to \mathcal{P}(W)$ is a valuation.

The *truth of a $\mathcal{L}_g$-formula φ at a state w of a neighbourhood model* $\mathbb{M} = (\mathbb{F}, V)$, notation, $\mathbb{M}, w \Vdash \varphi$, is defined inductively as follows, where $n \in \mathbb{N}$:

$$\begin{array}{lll} \mathbb{M}, w \Vdash p & \text{iff} & p \in V(p) \\ \mathbb{M}, w \Vdash \neg\psi & \text{iff} & \mathbb{M}, w \nVdash \psi \\ \mathbb{M}, w \Vdash \psi_1 \vee \psi_2 & \text{iff} & \mathbb{M}, w \Vdash \psi_1 \text{ or } \mathbb{M}, w \Vdash \psi_2 \\ \mathbb{M}, w \Vdash \Diamond_n\psi & \text{iff} & [\![\psi]\!]_{\mathbb{M}} \in \nu_n(w) \end{array}$$

As an example, Figure 1 depicts a Kripke model, graded model and a neighbourhood model which all make $\Diamond_3 p$ true.

A neighbourhood function $\nu : W \to \mathcal{PP}(W)$ is *supplemented* or *closed under supersets* if for all $w \in W$ and $X \subseteq W$, $X \in \nu(w)$ and $X \subseteq Y$ imply $Y \in \nu(w)$. A neighbourhood frame $\mathbb{F} = (W, \{\nu_n\}_{n\in\mathbb{N}})$ is *monotonic* if each ν_n is supplemented. A neighbourhood model $\mathbb{M} = (\mathbb{F}, V)$ is *monotonic* if $\mathbb{F}$ is monotonic. Denote by F_M the class of all monotonic neighbourhood frames. Monotonic pointed models $\mathbb{M}, w$ and $\mathbb{M}', w'$ are said to be *modally equivalent* if for all $\mathcal{L}_g$-formulas φ, we have $\mathbb{M}, w \Vdash \varphi$ iff $\mathbb{M}', w' \Vdash \varphi$. For monotonic model $\mathbb{M}$, we have

$$\mathbb{M}, w \Vdash \Diamond_n\varphi \quad \text{iff} \quad \exists X (X \in \nu_n(w) \ \& \ X \subseteq [\![\varphi]\!]_{\mathbb{M}}).$$

Axiomatization. The *minimal monotonic modal logic* $\mathbf{M}_{\mathbb{N}}$ consists of the following axioms and inference rules, where $n \in \mathbb{N}$:

$(Ax1)$ all instances of propositional tautologies

(MP) from φ and $\varphi \to \psi$ infer ψ

(RM_n) from $\varphi \to \psi$ infer $\Diamond_n\varphi \to \Diamond_n\psi$

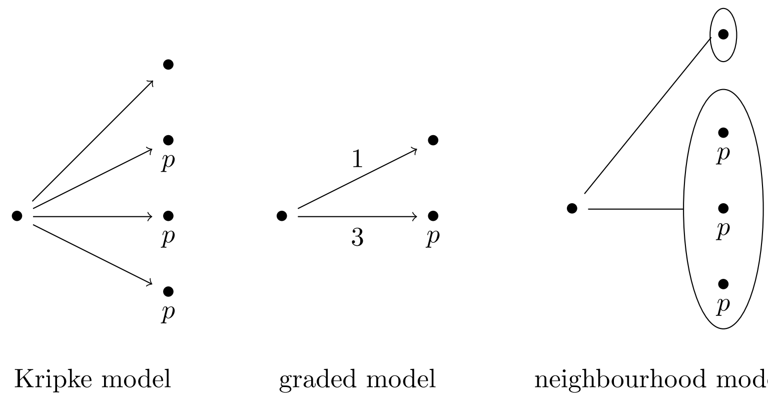

Figure 1. Three different ways to make $\Diamond_3 p$ true

The set of theorems derivable in the system $\mathbf{M}_{\mathbb{N}}$ is also called $\mathbf{M}_{\mathbb{N}}$. A *monotonic modal logic* is a set Λ of $\mathcal{L}_{\mathbb{N}}$-formulas with $\mathbf{M}_{\mathbb{N}} \subseteq \Lambda$. If $\varphi \in \Lambda$, we write $\vdash_\Lambda \varphi$.

THEOREM 2.3 ([14, Thm. 2.41]). $\mathbf{M}_{\mathbb{N}}$ is sound and strongly complete with respect to F_M.

3. Graded modal logics are monotonic modal logics

In this section we show that graded modal logics are monotonic modal logics. Let $\mathbf{G}$ be a graded modal logic.

PROPOSITION 3.1. Graded modal logics are monotonic modal logics.

PROOF: Let $\mathbf{G}$ be a graded modal logic. To show that $\mathbf{G}$ is a monotonic modal logic, it suffices to show that (i) $\mathbf{G}$ is closed under (MP) and (ii) for all $n \in \mathbb{N}$, $\mathbf{G}$ is closed under (RM_n). Item (i) is immediate. We now show item (ii). We distinguish the case $n = 0$ from the case $n > 0$.

Let $n = 0$. Assume that $\mathbf{G} \vdash \varphi \to \psi$. By $(Ax2)$, we have $\Diamond_0\varphi \leftrightarrow \top$ and $\Diamond_0\psi \leftrightarrow \top$ and hence $\Diamond_0\varphi \to \top$ and $\top \to \Diamond_0\psi$. It follows that $\mathbf{G} \vdash \Diamond_0\varphi \to \Diamond_0\psi$.

Let now $n > 0$. Assume that $\mathbf{G} \vdash \varphi \to \psi$. By (Gen), $\mathbf{G} \vdash \Box(\varphi \to \psi)$. Then by $(Ax5)$, $\mathbf{G} \vdash \Box(\varphi \to \psi) \to (\Diamond_n\varphi \to \Diamond_n\psi)$. Finally, by (MP) we get $\mathbf{G} \vdash \Diamond_n\varphi \to \Diamond_n\psi$. $\Box$

Corollary 3.2. **GrK** is a monotonic modal logic.

We now define axiomatization $\mathbf{GrK}_{Mon}$ as the extension of $\mathbf{M}_{\mathbb{N}}$ with $(Ax2)$ – $(Ax6)$ of **GrK** and the novel axiom $(Ax7)$ $\Diamond(\varphi \vee \psi) \leftrightarrow \Diamond\varphi \vee \Diamond\psi$. We show that **GrK** and $\mathbf{GrK}_{Mon}$ derive the same theorems.

Proposition 3.3. For any formula φ, $\mathbf{GrK} \vdash \varphi$ iff $\mathbf{GrK}_{Mon} \vdash \varphi$.

Proof: $(\Leftarrow)$ (Gen) is derivable in $\mathbf{GrK}_{Mon}$ as follows:

1	φ	assumption
2	$\varphi \to (\neg\varphi \to \bot)$	Duns Scotus law
3	$\neg\varphi \to \bot$	1,2 (MP)
4	$\Diamond\neg\varphi \to \Diamond\bot$	3 by (RM_1)
5	$\Diamond\neg\varphi \to \bot$	4 by $(Ax3)$
6	$\top \to \neg\Diamond\neg\varphi$	5 by contraposition
7	$\Box\varphi$	6 by def. of $\Box$ and $(Ax1)$

$(\Rightarrow)$ It suffices to show that $(Ax7)$ is derivable and (RM_n) is admissible rule in **GrK**. The latter follows from Proposition 3.1. $(Ax7)$ is equivalent to (i) $\Diamond\varphi \vee \Diamond\psi \to \Diamond(\varphi \vee \psi)$ and (ii) $\Diamond(\varphi \vee \psi) \to \Diamond\varphi \vee \Diamond\psi$. (i) and (ii) are derivable as follows:

1	$\Box(\varphi \to \varphi \vee \psi)$	by $(Ax1)$ and (Gen)
2	$\Diamond\varphi \to \Diamond(\varphi \vee \psi)$	1 and (Ax5) by (MP)
3	$\Box(\psi \to \varphi \vee \psi)$	by $(Ax1)$ and (Gen)
4	$\Diamond\psi \to \Diamond(\varphi \vee \psi)$	3 and (Ax5) by (MP)
5	$\Diamond\varphi \vee \Diamond\psi \to \Diamond(\varphi \vee \psi)$	2 and 4 by (Ax1)

1	$\neg\Diamond(\varphi \wedge \psi) \wedge \Diamond_0\varphi \wedge \neg\Diamond\varphi \wedge \Diamond_0\psi \wedge \neg\Diamond\psi \rightarrow \Diamond_0(\varphi \vee \psi) \wedge \neg\Diamond(\varphi \vee \psi)$	(Ax6) with $m = n = 0$
2	$\neg\Diamond(\varphi \wedge \psi) \wedge \neg\Diamond\varphi \wedge \neg\Diamond\psi \rightarrow \neg\Diamond(\varphi \vee \psi)$	1 by $(Ax2)$ and $\top \wedge \varphi \leftrightarrow \varphi$
3	$\Diamond(\varphi \vee \psi) \rightarrow \Diamond(\varphi \wedge \psi) \vee \Diamond\varphi \vee \Diamond\psi$	2 by contraposition, De Morgan and double negation
4	$\varphi \wedge \psi \rightarrow \varphi$	classical tautology
5	$\Diamond(\varphi \wedge \psi) \rightarrow \Diamond\varphi$	4, RM_1
6	$\Diamond(\varphi \wedge \psi) \rightarrow \Diamond\varphi \vee \Diamond\psi$	5, property of $\vee$
7	$\Diamond\varphi \rightarrow \Diamond\varphi \vee \Diamond\psi$	classical tautology
8	$\Diamond\psi \rightarrow \Diamond\varphi \vee \Diamond\psi$	classical tautology
9	$\Diamond(\varphi \wedge \psi) \vee \Diamond\varphi \vee \Diamond\psi \rightarrow \Diamond\varphi \vee \Diamond\psi$	6, 7, 8, property of $\vee$
10	$\Diamond(\varphi \vee \psi) \rightarrow \Diamond\varphi \vee \Diamond\psi$	3, 9, hypothetical syllogism

□

Another interesting question is whether there exists a class of neighbourhood frames with respect to which **GrK** is sound and complete. In monotonic neighbourhood frames the class of so-called *KW-formulas* ([10, Def. 5.13]) is elementary ([10, Thm. 5.14] and canonical ([10, Thm. 10.34]). Therefore, a presentation where each axiom is a KW-formula would make it straightforward to prove soundness and strong completeness. Unfortunately, (Ax5) and (Ax6) are not KW-formulas, since they have $\neg$ inside the scope of $\Diamond$, which is forbidden in KW-formulas. Therefore we can not prove completeness of **GrK** indirectly via a reference to KW-formulas.

If we adopt a more direct method to prove the completeness, we need to show that the properties defined by (Ax2)–(Ax7) holds in the canonical frame of monotonic modal logic containing them. Axioms (Ax5) and (Ax6) resp. correspond to the properties:

$$\forall w \forall X \forall Y \Big(X \cap (W \setminus Y) \notin \nu_1(w) \;\&\; X \in \nu_n(w) \Rightarrow Y \in \nu_n(w) \Big)$$

$$\begin{aligned} \forall w \forall X \forall Y \Big(& X \cap Y \notin \nu_1(w) \;\&\; X \in \nu_m(w) \;\&\; X \notin \nu_{m+1}(w) \\ & \;\&\; Y \in \nu_n(w) \;\&\; Y \notin \nu_{n+1}(w) \\ & \Rightarrow X \cup Y \in \nu_{m+n}(w) \;\&\; X \cup Y \notin \nu_{(m+n+1)}(w) \Big) \end{aligned}$$

The difficulty lies at showing that (Ax5) and (Ax6) are valid in the canonical frame of monotonic modal logic containing (Ax5) and (Ax6). For canonical frames of monotonic modal logics, we refer to [4, Def. 9.3], [10, Def. 6.2] and [14, Def. 2.37].

In the next section, we identify a class of complete neighbourhood frames via an operation $(.)^\bullet$, which is shown to be first-order definable in Section 5 and modally undefinable in Section 6.

4. Graded neighbourhood frames

Given a set X, denote by $\mathcal{P}_{\geq n}(X)$ the set of subsets of X such that the cardinality of each subset is at least n, in other words, $\mathcal{P}_{\geq n}(X) = \{X' \subseteq X \mid |X'| \geq n\}$. For $\Gamma \subseteq \mathcal{P}(W)$, define $\Uparrow\Gamma$ to be the up-set generated by Γ, that is, $\Uparrow\Gamma := \{Y \in \mathcal{P}(W) \mid \exists X(X \in \Gamma \ \& \ X \subseteq Y)\}$.

Definition 4.1. A neighbourhood frame $\mathbb{F} = (W, \{\nu_n\}_{n\in\mathbb{N}})$ is a *graded neighbourhood frame* if for all $w \in W$, there exists an $A \subseteq W$ such that for all $n \in \mathbb{N}$, $\nu_n(w) = \uparrow\mathcal{P}_{\geq n}(A)$.

Definition 4.2. For a Kripke frame $\mathcal{F} = (W, R)$, the associated graded neighbourhood frame of $\mathcal{F}$ is $\mathcal{F}^\bullet = (W, \{\nu_n\}_{n\in\mathbb{N}})$, where for $w \in W$ and $n \in \mathbb{N}$, $\nu_n(w) = \uparrow\mathcal{P}_{\geq n}(R[w])$.

That each ν_n in $\mathcal{F}^\bullet = (W, \{\nu_n\}_{n\in\mathbb{N}})$ is monotonic follows directly from the definition. Then we have the following result:

Proposition 4.3. Let $\mathcal{F} = (W, R)$ be a Kripke frame and V a valuation on $\mathcal{F}$. Then for all $w \in W$ and all formulas φ

$$(\mathcal{F}, V), w \Vdash \varphi \quad \text{iff} \quad (\mathcal{F}^\bullet, V), w \Vdash \varphi.$$

Proof: The proof is by induction on φ. The propositional cases follows from the definition and induction hypothesis.

As for the modal case, let φ be $\Diamond_n\psi, n \in \mathbb{N}$, we have

$$\begin{array}{llll} (\mathcal{F}, V), w \Vdash \Diamond_n\psi & \text{iff} & \left|R[w] \cap [\![\psi]\!]_{(\mathcal{F},V)}\right| \geq n & \\ & \text{iff} & \left|R[w] \cap [\![\psi]\!]_{(\mathcal{F}^\bullet,V)}\right| \geq n & \text{(IH)} \\ & \text{iff} & \exists X \subseteq W \ (X \in \nu_n(w) \ \& \ X \subseteq [\![\psi]\!]_{(\mathcal{F}^\bullet,V)}) & (*) \\ & \text{iff} & (\mathcal{F}^\bullet, V), w \Vdash \Diamond_n\psi & \end{array}$$

Here is the proof for the equivalence marked by $(*)$. First assume that $\left|R[w] \cap [\![\psi]\!]_{(\mathcal{F}^\bullet,V)}\right| \geq n$. Then $R[w] \cap [\![\psi]\!]_{(\mathcal{F}^\bullet,V)} \in \mathcal{P}_{\geq n}(R[w])$. By definition,

$\nu_n(w) = \uparrow\mathcal{P}_{\geq n}(R[w])$. Hence, $R[w] \cap [\![\psi]\!]_{(\mathcal{F}^\bullet,V)} \in \nu_n(w)$. We also have $R[w] \cap [\![\psi]\!]_{(\mathcal{F}^\bullet,V)} \subseteq [\![\psi]\!]_{(\mathcal{F}^\bullet,V)}$, which completes the proof of this direction. Now assume that $X \in \nu_n(w)$ and $X \subseteq [\![\psi]\!]_{(\mathcal{F}^\bullet,V)}$. Since $\nu_n(w) = \uparrow\mathcal{P}_{\geq n}(R[w])$, $X \in \uparrow\mathcal{P}_{\geq n}(R[w])$. Then there exists $Y \in \mathcal{P}_{\geq n}(R[w])$ and $Y \subseteq X$. It follows that $Y \subseteq R[w]$ and $|Y| \geq n$. Since $X \subseteq [\![\psi]\!]_{(\mathcal{F}^\bullet,V)}$, $Y \subseteq [\![\psi]\!]_{(\mathcal{F}^\bullet,V)}$. Hence, $Y = Y \cap [\![\psi]\!]_{(\mathcal{F}^\bullet,V)} \subseteq R[w] \cap [\![\psi]\!]_{(\mathcal{F}^\bullet,V)}$ and therefore $\left|R[w] \cap [\![\psi]\!]_{(\mathcal{F}^\bullet,V)}\right| \geq |Y| \geq n$. □

Given a graded neighbourhood frame $\mathbb{F} = (W, \{\nu_n\}_{n\in\mathbb{N}})$ with $\nu_n(w) = \uparrow \mathcal{P}_{\geq n}(A_w)$, we can associate it with a Kripke frame $\mathbb{F}_\bullet = (W, R)$ with $R[w] = A_w$. It follows from definitions that $(\mathbb{F}_\bullet)^\bullet = \mathbb{F}$ and $(\mathcal{F}^\bullet)_\bullet = \mathcal{F}$.

For a class of Kripke frames S_K, let $S_K^\bullet = \{\mathcal{F}^\bullet \mid \mathcal{F} \in S_K\}$. Recall that F_K is the class of all Kripke frames. Since $(\mathbb{F}_\bullet)^\bullet = \mathbb{F}$ for any graded neighbourhood frame $\mathbb{F}$, $\mathsf{F}_K^\bullet$ is equivalent to the class of all graded neighbourhood frames.

Theorem 4.4. **GrK** is sound and strongly complete with respect to the class of graded neighbourhood frames.

Proof: By Theorem 2.1, **GrK** is sound and strongly complete with respect to F_K. By Proposition 4.3, **GrK** is sound and strongly complete with respect to $\mathsf{F}_K^\bullet$. Then the claim follows from the fact that $\mathsf{F}_K^\bullet$ is equivalent to the class of all graded neighbourhood frames. □

5. Graded neighbourhood frames are first-order definable

A class S_N of neighbourhood frames is *first-order definable* if there exists a set of first-order formulas Γ such that $\mathbb{F} \models \Gamma$ iff $\mathbb{F} \in S_N$. In this section, we show that the class of graded neighbourhood frames is(two-sorted) first-order definable in the (two-sorted) first-order language $\mathcal{L}_g^1$ of $\mathcal{L}_g$ defined below.

Each monotonic neighbourhood frame $\mathbb{F} = (W, \{\nu_n\}_{n\in\mathbb{N}})$ can be seen as a two-sorted relational structure $(W, \mathcal{P}(W), \{R_{\nu_n}\}_{n\in\mathbb{N}}, R_\ni)$ where $R_{\nu_n} \subseteq W \times \mathcal{P}(W)$ and $R_\ni \subseteq \mathcal{P}(W) \times W$ such that $wR_{\nu_n}X$ iff $X \in \nu_n(w)$ and $XR_\ni w$ iff $w \in X$. Accordingly, the (two-sorted) first-order language $\mathcal{L}_g^1$ of $\mathcal{L}_g$ has equality $=$, first-order variables $w, u, v, \ldots$ over W, first-order variables $X, Y, Z, \dots$ over $\mathcal{P}(W)$, binary symbols R_{ν_n} for $n \in \mathbb{N}$ and $R_\ni$, and unary relation symbols $P, Q, \dots$ corresponding to $p, q, \dots \in \mathsf{Prop}$.

In other words, given sets of variables Ψ and Φ, formulas in $\mathcal{L}^1_g$ are defined inductively as follows:

$$\mathcal{L}^1_g \ni \chi ::= w = u \mid X = Y \mid Pw \mid R_{\nu_n} wX \mid R_\ni Xw \mid \neg\chi \mid \chi \vee \chi \mid \forall x \chi \mid \forall X \chi$$

where $w, u \in \Psi$, $X, Y \in \Phi$, P corresponds to $p \in \mathsf{Prop}$ and $n \in \mathbb{N}$.

A set A is called *atomic in* $\nu_1(w)$ if for all $a \in A$, $\{a\} \in \nu_1(w)$. Denote by $(\star)$ the following conditions: for all $w \in W$

($\star$1) $\nu_0(w) = \mathcal{P}(W)$.

($\star$2) $\nu_n(w)$ is closed under supersets for $n \in \mathbb{N}$.

($\star$3) $\emptyset \notin \nu_n(w)$ for $n \in \mathbb{N}$.

($\star$4) If $X \in \nu_n(w)$, then there exists a minimal $Y \in \nu_n(w)$ such that $Y \subseteq X$.

($\star$5) If Y is a minimal element in $\nu_n(w)$, then $|Y| = n$ and Y is atomic in $\nu_1(w)$.

($\star$6) If $\{y_1\}, \dots, \{y_n\} \in \nu_1(w)$ and $y_1, \dots, y_n$ are pairwise distinct, then $\bigcup_{1 \le i \le n} \{y_i\}$ is a minimal element in $\nu_n(w)$.

Note that conditions $(\star)$ can be expressed in language $\mathcal{L}^1_g$. For example, $|Y| \geq n$ iff $y_1 \in Y \wedge \dots \wedge y_n \in Y \wedge \bigwedge_{i \neq j} y_i \neq y_j$, and Y is atomic in $\nu_1(w)$ iff $\forall Z (\forall Z' (Z' \subseteq Z \Rightarrow Z' = \emptyset \text{ or } Z' = Z) \,\&\, Z \subseteq Y \Rightarrow Z \in \nu_1(w))$.

Proposition 5.1. Let $\mathbb{F} = (W, \{\nu_n\}_{n \in \mathbb{N}})$ be a neighbourhood frame. Then $\mathbb{F}$ is graded iff $\mathbb{F}$ satisfies $(\star)$.

Proof: For the left-to-right direction, assume that $\mathbb{F} = (W, \{\nu_n\}_{n \in \mathbb{N}})$ is a graded neighbourhood frame, that is, for all $w \in W$, there exists some $A \subseteq W$ such that for all $n \in \mathbb{N}$, $\nu_n(w) = \uparrow \mathcal{P}_{\geq n}(A)$. Since $\uparrow \mathcal{P}_{\geq 0}(A) = \uparrow \mathcal{P}(A) = \mathcal{P}(W)$, item ($\star$1) holds. Item ($\star$2) and ($\star$3) also follow directly.

Now assume that $X \in \nu_n(w)$. Since $\nu_n(w) = \uparrow \mathcal{P}_{\geq n}(A)$, there exists $Y \in \mathcal{P}_{\geq n}(A)$ with $Y \subseteq X$. It follows that $|Y| \geq n$. Let Y' be a subset of Y containing exactly n-elements. Then Y' is a minimal element in $\nu_n(w)$ and $Y' \subseteq X$. Hence, item ($\star$4) follows.

Now assume that Y is a minimal element in $\nu_n(w) = \uparrow\mathcal{P}_{\geq n}(A)$. Then $Y \subseteq A$ and $|Y| = n$. Since $\nu_1(w) = \uparrow\mathcal{P}_{\geq 1}(A)$, for all $a \in A$, $\{a\} \in \nu_1(w)$. It follows that Y is atomic in $\nu_1(w)$. Hence, item ($\star$5) holds. For item ($\star$6), assume that $\{y_1\} \neq \ldots \neq \{y_n\} \in \nu_1(w) = \uparrow\mathcal{P}_{\geq 1}(A)$. Then $\{y_1, \ldots, y_n\} \in \uparrow \mathcal{P}_{\geq n}(A)$. It follows that $\{y_1, \ldots, y_n\}$ is a minimal element in $\nu_n(w)$. Hence, item ($\star$6) holds.

The right-to-left direction follows from Lemma 5.4 and 5.5 below. □

Lemma 5.2. Let $\mathbb{F} = (W, \{\nu_n\}_{n\in\mathbb{N}})$ be a neighbourhood frame satisfying ($\star$). If $X \in \nu_1(w)$, there exists $x \in X$ such that $\{x\} \in \nu_1(w)$.

Proof: Assume that $X \in \nu_1(w)$. By ($\star$4), there exists a minimal $Y \in \nu_1(w)$ such that $Y \subseteq X$. By ($\star$3), $X \neq \emptyset$ and $Y \neq \emptyset$. By ($\star$5), Y is atomic in $\nu_1(w)$, i.e., for all $y \in Y$, $\{y\} \in \nu_1(w)$. It follows that there exists $x \in X$ such that $\{x\} \in \nu_1(w)$. □

Lemma 5.3. Let $\mathbb{F} = (W, \{\nu_n\}_{n\in\mathbb{N}})$ be a neighbourhood frame satisfying ($\star$). If $\nu_1(w) \neq \emptyset$, there exists a set $A \subseteq W$ such that A is the maximum atomic set in $\nu_1(w)$.

Proof: Since $\nu_1(w) \neq \emptyset$, we assume $X \in \nu_1(w)$. By ($\star$3), $X \neq \emptyset$. By ($\star$4), there exists a minimal $X' \in \nu_1(w)$ such that $X' \subseteq A$. By ($\star$5), $|X'| = 1$ and X' is atomic in $\nu_1(w)$. Hence, we can assume $X' = \{a\}$. Let A be the union of all singletons in $\nu_1(w)$. Since $\{a\} \in \nu_1(w)$, $A \neq \emptyset$. Now we show that A is the maximum atomic set in $\nu_1(w)$. Since A is the union of all singletons in $\nu_1(w)$, A is atomic. Let B be an atomic set in $\nu_1(w)$. For any $b \in B$, by atomicity, $\{b\} \in \nu_1(w)$. It follows that $b \in A$. Therefore, $B \subseteq A$. Hence, A is the maximum atomic set in $\nu_1(w)$. □

Lemma 5.4. Let $\mathbb{F} = (W, \{\nu_n\}_{n\in\mathbb{N}})$ be a neighbourhood frame satisfying ($\star$). If $\nu_1(w) \neq \emptyset$, then $\nu_1(w) = \uparrow\mathcal{P}_{\geq 1}(A)$, where A is the maximum atomic set in $\nu_1(w)$.

Proof: If $\nu_1(w) = \emptyset$, then $A = \emptyset$. Then $\nu_1(w) = \uparrow\mathcal{P}_{\geq 1}(A)$. If $\nu_1(w) \neq \emptyset$, assume that $X \in \nu_1(w)$. By Lemma 5.2, there exists an $x \in X$ such that $\{x\} \in \nu_1(w)$. Since A is the maximum atomic set in A, we have $x \in A$. It follows that $\{x\} \in \mathcal{P}_{\geq 1}(A)$. Since $x \in X$, $X \in \uparrow\mathcal{P}_{\geq 1}(A)$.

Assume that $X \in \uparrow\mathcal{P}_{\geq 1}(A)$. Then there exists $Y \in \mathcal{P}_1(A)$ such that $Y \subseteq X$. Since A is atomic in $\nu_1(w)$, for all $y \in Y$, $\{y\} \in \nu_1(w)$. By ($\star$2), $\nu_1(w)$ is monotonic. Therefore, $Y \in \nu_1(w)$. Since $Y \subseteq X$, $X \in \nu_1(w)$. □

LEMMA 5.5. Let $\mathbb{F} = (W, \{\nu_n\}_{n\in\mathbb{N}})$ be a neighbourhood frame satisfying $(\star)$. Then for $w \in W$,

1. If $\nu_1(w) = \emptyset$, then $\nu_n(w) = \emptyset$ for $n > 1$.
2. If $\nu_1(w) \neq \emptyset$, then $\nu_n(w) = \uparrow \mathcal{P}_{\geq n}(A)$ for $n > 1$, where A is the maximum atomic set in $\nu_1(w)$.

PROOF: For item 1, we prove by contradiction. Assume that $\nu_1(w) = \emptyset$ and for some $n > 1$, $X \in \nu_n(w)$. By $(\star 3)$, $X \neq \emptyset$. By $(\star 4)$ and $(\star 5)$, there exists $X' \subseteq X$ such that X' is atomic in $\nu_1(w)$. By $(\star 3)$, $X' \neq \emptyset$. By atomicity of X', $\nu_1(w) \neq \emptyset$, contradiction .

Now we prove item 2 and assume that $X \in \nu_n(w)$. By $(\star 4)$, there exists a minimal element of $\nu_n(w)$ such that $Y \subseteq X$. By $(\star 5)$, $|Y| \geq n$ and Y is atomic in $\nu_1(w)$. Since A is the maximum atomic set of $\nu_1(w)$, $Y \subseteq A$. Since $|Y| \geq n$, $Y \in \mathcal{P}_{\geq n}(A)$. Since $Y \subseteq X$, $X \in \uparrow \mathcal{P}_{\geq n}(A)$.

Assume that $X \in \uparrow \mathcal{P}_{\geq n}(A)$. Then there exists $Y \in \mathcal{P}_{\geq n}(A)$ such that $Y \subseteq X$. It follows that $|Y| \geq n$. Since A is the maximum atomic set of $\nu_1(w)$, Y is atomic in $\nu_1(w)$. Hence, there exist distinct $y_1, \ldots, y_n \in Y$ such that $\{y_1\}, \ldots, \{y_n\} \in \nu_1(w)$ and $y_1 \neq \ldots \neq y_n$. By $(\star 6)$, $\bigcup_{1\leq i\leq n}\{y_i\}$ is a minimal element in $\nu_n(w)$. Since $\bigcup_{1\leq i\leq n}\{y_i\} \subseteq Y \subseteq X$ and $\nu_n(w)$ is monotonic by $(\star 2)$, $X \in \nu(w)$. □

6. Graded neighbourhood frames are not modally definable

A class S_N of neighbourhood frames is *modally definable* if there exists a set of modal formulas Δ such that $\mathbb{F} \Vdash \Delta$ iff $\mathbb{F} \in S_N$. In this section, we show that the class of graded neighbourhood frames is not modally definable. It is well known that if the class of neighbourhood frames is modally definable, then it is closed under bounded morphic images. Below we show that the class of graded neighbourhood frames is not closed under bounded morphic images (by exhibiting a counterexample), so we conclude that it is not modally definable.

Given a function $f : W \to W'$ and $X \subseteq W$, define $f[X] := \{f(x) : x \in X\}$.

DEFINITION 6.1. Let $\mathbb{F} = (W, \{\nu_n\}_{n\in\mathbb{N}})$ and $\mathbb{F}' = (W, \{\nu'_n\}_{n\in\mathbb{N}})$ be neighbourhood frames. A *bounded morphism* from $\mathbb{F}$ to $\mathbb{F}'$ is a function $f : W \to W'$ satisfying for $n \in \mathbb{N}$

$(BM1_n)$ If $X \in \nu_n(w)$, then $f[X] \in \nu'_n(f(w))$.

$(BM2_n)$ If $X' \in \nu'_n(f(w))$, then there exists $X \subseteq W$ such that $f[X] \subseteq X'$ and $X \in \nu(w)$.

If there is a surjective bounded morphism from $\mathbb{F}$ to $\mathbb{F}'$, we say that $\mathbb{F}'$ is a *bounded morphic image* of $\mathbb{F}$.

Proposition 6.2 (Prop. 5.3 of [10]). Let $\mathbb{F}$ and $\mathbb{F}'$ be neighbourhood frames. If $\mathbb{F}'$ is a bounded morphic image of $\mathbb{F}$, then $\mathbb{F} \Vdash \varphi$ implies $\mathbb{F}' \Vdash \varphi$.

Proposition 6.3. If a class of neighbourhood frames is modally definable, then it is closed under bounded morphic images.

Proof: Let S_N be a class of neighbourhood frames defined by a set of formulas Δ, $\mathbb{F} \in S_N$ and $\mathbb{F}'$ a bounded morphic image of $\mathbb{F}$. Since $\mathbb{F} \in S_N$, $\mathbb{F} \Vdash \Delta$. By Proposition 6.2, $\mathbb{F}' \Vdash \Delta$ and therefore $\mathbb{F}' \in S_N$. □

Example 6.4. Consider neighbourhood frames $\mathbb{F} = (\{a,b\}, \{\nu_n\}_{n\in\mathbb{N}})$ such that for $n \in \mathbb{N}$, $\nu_n(a) = \nu_n(b) = \uparrow\mathcal{P}_{\geq n}(\{a,b\})$ and $\mathbb{F}' = (\{c\}, \{\nu'_n\}_{n\in\mathbb{N}})$ such that $\nu'_0(c) = \{\emptyset, \{c\}\}$, $\nu'_1(c) = \nu'_2(c) = \{\{c\}\}$ and $\nu'_k(c) = \emptyset$ for $k > 2$. By Definition 4.1, $\mathbb{F}$ is a graded neighbourhood frame. As for $\mathbb{F}'$, we have $\nu_1(c) = \uparrow\mathcal{P}_{\geq 1}(\{c\})$ while $\nu_2(c) \neq \uparrow\mathcal{P}_{\geq 2}(\{c\})$. Therefore, $\mathbb{F}'$ is not a graded neighbourhood frame. It can be verified that function $f : \{a,b\} \to \{c\}$, with $f(a) = f(b) = c$, is a subjective bounded morphism from $\mathbb{F}$ to $\mathbb{F}'$. Therefore, the class of graded neighbourhood frames is not closed under bounded morphic images.

Proposition 6.5. The class of graded neighbourhood frames is not modally definable.

Proof: It follows from Example 6.4 and the contraposition of Proposition 6.3. □

7. Bisimulation

The notion of graded tuple bisimulation was first proposed in de Rijke [7]. In this section, we obtain a new definition of graded bisimulation by substituting $\nu_n(w)$ with $\uparrow\mathcal{P}_{\geq n}(R[w])$ in the definition of monotonic bisimulation. And we prove that the new definition is equivalent to the old one (cf. Proposition 7.6 and 7.9).

7.1. From monotonic bisimulation to graded bisimulation

Definition 7.1 (Monotonic bisimulation, Def. 4.10 of [10]). Suppose that $\mathbb{M} = (W, \{\nu_n\}_{n\in\mathbb{N}}, V)$ and $\mathbb{M}' = (W', \{\nu'_n\}_{n\in\mathbb{N}}, V')$ are monotonic neighbourhood models. A non-empty relation $Z \subseteq W \times W'$ is a *monotonic bisimulation* (notation: $Z : \mathbb{M} \leftrightarroweq_m \mathbb{M}'$) provided that

- **(Prop)** If wZw', then w and w' satisfy the same proposition letters.
- **(Forth)** If wZw' and $X \in \nu_n(w)$, then there is $X' \subseteq W'$ such that $X' \in \nu'_n(w')$ and $\forall x' \in X' \exists x \in X : xZx'$.
- **(Back)** If wZw' and $X' \in \nu'_n(w')$, then there is $X \subseteq W$ such that $X \in \nu_n(w)$ and $\forall x \in X \exists x' \in X' : xZx'$.

If $w \in \mathbb{M}$ and $w' \in \mathbb{M}'$, then w and w' are *monotonic bisimilar states* (notation: $\mathbb{M}, w \leftrightarroweq_m \mathbb{M}', w'$) if there is a bisimulation $Z : \mathbb{M} \leftrightarroweq_m \mathbb{M}'$ with wZw'.

Proposition 7.2 (Prop. 4.11 of [10]). Let $\mathbb{M} = (W, \{\nu_n\}_{n\in\mathbb{N}}, V)$ and $\mathbb{M}' = (W', \{\nu'_n\}_{n\in\mathbb{N}}, V')$ be monotonic neighbourhood models. If $\mathbb{M}, w \leftrightarroweq_m \mathbb{M}', w'$, then for $\mathcal{L}_g$-formula φ, $\mathbb{M}, w \Vdash \varphi$ iff $\mathbb{M}', w' \Vdash \varphi$.

Substituting $\nu_n(w)$ in Definition 7.1 with $\uparrow\mathcal{P}_{\geq n}(R[w])$, we have:

Definition 7.3 (Graded bisimulation). Suppose that $\mathcal{F} = (W, R, V)$ and $\mathcal{M}' = (W', R', V)$ are Kripke models. A non-empty relation $Z \subseteq W \times W'$ is a *graded bisimulation* (notation: $Z : \mathcal{M} \leftrightarroweq_g \mathcal{M}'$) provided that

- **(Prop)** If wZw', then w and w' satisfy the same proposition letters.
- **(Forth)** If wZw' and $X \in\uparrow\mathcal{P}_{\geq n}(R[w])$, then there is an $X' \subseteq W'$ such that $X' \in\uparrow\mathcal{P}_{\geq n}(R'[w'])$ and $\forall x' \in X' \exists x \in X : xZx'$.
- **(Back)** If wZw' and $X' \in\uparrow\mathcal{P}_{\geq n}(R'[w'])$, then there is an $X \subseteq W$ such that $X \in\uparrow\mathcal{P}_{\geq n}(R[w])$ and $\forall x \in X \exists x' \in X' : xZx'$.

If $w \in \mathcal{M}$ and $w' \in \mathcal{M}'$, then w and w' are *graded bisimilar states* (notation: $\mathcal{M}, w \leftrightarroweq_g \mathcal{M}', w'$) if there is a bisimulation $Z : \mathcal{M} \leftrightarroweq_g \mathcal{M}'$ with wZw'.

Proposition 7.4. Let $\mathcal{M} = (W, R, V)$ and $\mathcal{M}' = (W', R', V')$ be Kripke models. If $\mathcal{M}, u \leftrightarroweq_g \mathcal{M}', u'$, then $\mathcal{M}, u \equiv_k \mathcal{M}', u'$.

Proof: Since $\mathcal{M}, u \leftrightarroweq_g \mathcal{M}', u'$, there exists a non-empty relation $Z \subseteq W \times W'$ such that $Z : \mathcal{M} \leftrightarroweq_g \mathcal{M}'$ and uZu'. For neighbourhood frames

$\mathcal{M}^\bullet = (W, \{\nu_n\}_{n\in\mathbb{N}}, V)$ and $\mathcal{M}'^\bullet = (W, \{\nu'_n\}_{n\in\mathbb{N}}, V')$, by definition, for $w \in W$ and $w' \in W'$, $\nu_n(w) = \uparrow\mathcal{P}_{\geq n}(R[w])$ and $\nu'_n(w') = \uparrow\mathcal{P}_{\geq n}(R'[w'])$. Substituting $\uparrow\mathcal{P}_{\geq n}(R[w])$ with $\nu_n(w)$ and $\uparrow\mathcal{P}_{\geq n}(R'[w'])$ with $\nu'_n(w')$ in the definition of $Z : \mathcal{M} \leftrightarroweq_g \mathcal{M}'$, we have $Z : \mathcal{M}^\bullet, u \leftrightarroweq_m \mathcal{M}'^\bullet, u'$ and uZu'. For all formulas φ, that $\mathcal{M}, u \Vdash \varphi$ iff $\mathcal{M}', u' \Vdash \varphi$ can be proved as follows:

$\mathcal{M}, u \Vdash \varphi$	iff	$\mathcal{M}^\bullet, u \Vdash \varphi$	Proposition 4.3
	iff	$\mathcal{M}'^\bullet, u' \Vdash \varphi$	Proposition 7.2
	iff	$\mathcal{M}', u' \Vdash \varphi$	Proposition 4.3 □

7.2. Graded bisimulation is equivalent to graded tuple bisimulation

In the rest of this section, we recall the definition of graded tuple bisimulation in de Rijke [7] and show that it is equivalent to Definition 7.3. Given a set X, denote by $\mathcal{P}_{<\omega}(X)$ the set of finite subsets of X. We now get:

Definition 7.5 (Graded tuple bisimulation). Let $\mathcal{M} = (W, R, V)$ and $\mathcal{M} = (W', R', V')$ be two Kripke models. A tuple $\mathcal{Z} = (\mathcal{Z}_1, \mathcal{Z}_2, \ldots)$ of relations is called *graded tuple bisimulation* between $\mathcal{M}$ and $\mathcal{M}'$ (notation: $\mathcal{Z} : \mathcal{M} \leftrightarroweq_{gt} \mathcal{M}'$) iff:

(1) $\mathcal{Z}_1$ is non-empty;

(2) for all i, $\mathcal{Z}_i \subseteq \mathcal{P}_{<\omega}(W_1) \times \mathcal{P}_{<\omega}(W_2)$;

(3) if $X\mathcal{Z}_iX'$, then $|X| = |X'| = i$;

(4) if $\{w\}\mathcal{Z}_1\{w'\}$, then w and w' satisfy the same proposition letters;

(5) if $\{w\}\mathcal{Z}_1\{w'\}$, $X \subseteq R[w]$ and $|X| = i \geq 1$, then there exists $X' \in \mathcal{P}_{<\omega}(W')$ with $X' \subseteq R'[w']$ and $X\mathcal{Z}_iX'$;

(6) if $\{w\}\mathcal{Z}_1\{w'\}$, $X' \subseteq R[w']$ and $|X'| = i \geq 1$, then there exists $X \in \mathcal{P}_{<\omega}(W)$ with $X \subseteq R[w]$ and $X\mathcal{Z}_iX'$;

(7) if $X\mathcal{Z}_iX'$, then (a) $\forall x \in X \exists x' \in X' : \{x\}\mathcal{Z}_1\{x'\}$, and (b) $\forall x' \in X' \exists x \in X : \{x\}\mathcal{Z}_1\{x'\}$.

Proposition 7.6. Let $\mathcal{M} = (W, R, V)$ and $\mathcal{M}' = (W', R', V')$ be Kripke models and $\mathcal{Z} = (\mathcal{Z}_1, \mathcal{Z}_2, \ldots)$ a tuple of relations such that $\mathcal{Z} : \mathcal{M} \leftrightarroweq_{gt} \mathcal{M}'$.

Define $Z \subseteq W \times W'$ to be a relation such that wZw' iff $\{w\}\mathcal{Z}_1\{w'\}$. Then $Z : \mathcal{M} \leftrightarrows_g \mathcal{M}'$.

PROOF: (**Prop**) follows from item (4) of Definition 7.5. As for (**Forth**), assume that wZw' and $X \in\uparrow\mathcal{P}_{\geq n}(R[w])$. Then there exists $Y \subseteq R[w]$ such that $Y \subseteq X$ and $|Y| = n$. Since $|Y| = n$ and $\{w\}\mathcal{Z}_1\{w'\}$, by items (5) and (3) there exists $Y' \subseteq R'[w']$, $|Y'| = n$ and $Y\mathcal{Z}_nY'$. It follows that $Y' \in\uparrow\mathcal{P}_{\geq n}(R'[w'])$. By item (7)(b), $\forall y' \in Y' \exists y \in Y : \{y\}\mathcal{Z}_1\{y'\}$. Since $Y \subseteq X$ and xZy iff $\{x\}\mathcal{Z}_1\{y\}$, we have $\forall y' \in Y' \exists x \in X : xZy'$, which completes the proof of that Z satisfies (**Forth**). That Z satisfies (**Back**) can be proved in a similar way. □

Now we show how to construct a graded tuple bisimulation out of a graded bisimulation, with the following lemmas:

LEMMA 7.7. Let $\mathcal{M}$ and $\mathcal{M}'$ be Kripke models and $Z : \mathcal{M}, w \leftrightarrows_g \mathcal{M}', w'$.

(1) If $u \in R[w]$, then there exists $u' \in R'[w']$ with uZu'.

(2) If $u' \in R'[w']$, then there exists $u \in R[w]$ with uZu'.

PROOF: (1) Since $u \in R[w]$, $\{u\} \in\uparrow \mathcal{P}_{\geq 1}(R[w])$. By (**Forth**), there exists $Y' \in \uparrow\mathcal{P}_{\geq 1}(R'[w'])$ such that $\forall y' \in Y' \exists x \in \{u\} : xZy'$. It follows that $\forall y' \in Y' : uZy'$. Since $Y' \in\uparrow\mathcal{P}_{\geq 1}(R'[w'])$, there exists $u' \in R'[w']$ such that $u' \in Y'$. It follows that uZu'.

Claim (2) can be proved in a similar way by using (**Back**). □

Let W and W' be sets, $X \subseteq W$, $X' \subseteq W'$ and $Z \subseteq W \times W'$. Sets X and X' are called a *Z-pair* if $\forall x \in X \exists x' \in X' : xZx'$ and $\forall x' \in X' \exists x \in X : xZx'$.

LEMMA 7.8. Let $\mathcal{M}$ and $\mathcal{M}'$ be Kripke models and $Z : \mathcal{M}, w \leftrightarrows_g \mathcal{M}', w'$.

(1) If $X \subseteq R[w]$ and $|X| = i \geq 1$, then there exists $X' \subseteq R'[w']$ with $|X'| = i$ such that X and X' form a Z-pair.

(2) If $X' \subseteq R'[w']$ and $|X'| = i \geq 1$, then there exists $X \subseteq R[w]$ with $|X| = i$ such that X and X' form a Z-pair.

PROOF: (1) The proof is by induction on i. If $i = 1$, we may assume that $X = \{u\}$. Since $X \subseteq R[w]$, we have $u \in R[w]$. By Lemma 7.7, there exists

$u' \in R'[w']$ with uZu'. Let $X' = \{u'\}$. It follows that $|X'| = 1$ and that X and X' form a Z-pair.

Consider the case that $i > 1$. We may assume that $X = \{u\} \cup Y$, where $Y \subseteq R[w]$ and $u \notin Y$. It follows that $|Y| = i - 1 \geq 1$. By induction hypothesis, there exists an $Y' \subseteq R'[w']$ such that $|Y'| = i - 1$ and that Y and Y' forms a Z-pair. Since $u \in R[w]$, by Lemma 7.7, there exists $u' \in R'[w']$ with uZu'. If $u' \notin Y'$, let $X' = Y' \cup \{u'\}$. Then $|X'| = i$ and X and X' forms a Z-pair.

If $u' \in Y'$, there are two subcases: $\exists y \in Y \exists v' \in R'[w'] \backslash Y' : yZv'$ and for all $y \in Y$ and $v' \in R'[w'] \backslash Y'$, not yZv'.

Consider the case that $\exists y \in Y \exists v' \in R'[w'] \backslash Y' : yZv'$. Let $X' = Y' \cup \{v'\}$. Then $|X'| = i$. Since Y and Y' form a Z-pair, uZu' and yZv', X and X' form a Z-pair.

Consider the case that for all $y \in Y$ and $v' \in R'[w'] \backslash Y'$, not yZv'. Since $X \in \uparrow \mathcal{P}_{\geq i}(R[w])$, by (**Forth**), there exists $B' \in \uparrow \mathcal{P}_{\geq i}(R'[w'])$ such that $\forall b' \in B' \exists x \in X : xZb'$. Since $B' \in \uparrow \mathcal{P}_{\geq i}(R'[w'])$, there exists $B'' \subseteq B'$ such that $B'' \subseteq R'[w']$ and $|B''| \geq i$. Since $|Y'| = i - 1$, there exists $b'' \in B''$ such that $b'' \in R'[w'] \backslash Y'$. Since for all $y \in Y$ and $v' \in R'[w'] \backslash Y'$, not yZv', we have for all $y \in Y$, not yZb''. Since $\forall b' \in B' \exists x \in X : xZb'$ and $X = \{u\} \cup Y$, we have uZb''. Let $X' = Y' \cup \{b''\}$. Then $|X'| = i$. Since Y and Y' form a Z-pair and uZb'', X and X' form a Z-pair.

Claim (2) can be proved in a similar way by using (**Back**). □

Proposition 7.9. Let $\mathcal{M} = (W, R, V)$ and $\mathcal{M}' = (W', R', V')$ be Kripke models and $Z \subseteq W \times W'$ a non-empty relation such that $Z : \mathcal{M} \leftrightarrows_g \mathcal{M}'$. Define a tuple of relations $\mathcal{Z} = (\mathcal{Z}_1, \mathcal{Z}_2, \ldots)$ as: $\mathcal{Z}_1 = \{(\{w\}, \{w'\}) \mid wZw'\}$, and $\mathcal{Z}_n = \{(X, X') \mid |X| = |X'| = n,\ X \text{ and } X' \text{ form a } Z\text{-pair}\}$, for $n > 1$. Then $\mathcal{Z} : \mathcal{M} \leftrightarrows_{gt} \mathcal{M}'$.

Proof: Since Z is non-empty, $\mathcal{Z}_1$ is non-empty. So item (1) in Definition 7.5 is satisfied. Items (2), (3) and (4) are satisfied by the definition of Z. Items (5) and (6) are satisfied by Lemma 7.8. Item (7) is satisfied by the definition of $\mathcal{Z}_i$ and the definition of Z-pairs. □

In summary, we showed how to construct a graded bisimulation out of a graded tuple bisimulation (Prop. 7.6), and vice versa (Prop. 7.9). Hence, graded bisimulation (Def. 7.3) and graded tuple bisimulation (Def. 7.5) are equivalent. Another notion of bisimulation called *resource bisimulation* was proposed in [1], which is very similar to the notion later proposed in

[13]. A precise comparison of graded bisimulation to these notions is left for future research.

8. Conclusion

Inspired by graded models, we proposed a class of graded neighbourhood frames, and we showed that the axiomatiziation **GrK** is sound and strongly complete for this class. We further showed that graded neighbourhood frames are first-order definable but not modally definable. We also obtained a new definition of graded bisimulation building upon the notion of monotonic bisimulation, where some details concerning resource bisimulation are left for further research. Our results show that techniques for monotonic modal logics can be successfully applied to graded modal logics.

There are many options for further research:

(1) Using the approach developed in this paper, updating neighbourhood models [12] can be compared to updated graded models [13].

(2) Building on multi-type display calculi for monotonic logics [5] we plan to introduce multi-type display calculi for graded modal logic.

(3) With yet another notion of bisimulation on graded frames, and algorithms to calculate two-sorted first-order correspondence on neighbourhood frames [10, 5], we plan to get two-sorted first-order correspondence on graded frames.

(4) Finally, given the logic **GrK** in Section 2 for n grades, and given its alternative incarnation as a monotonic modal logic in Section 3, we wish to find the axiomatization of the graded modal logic for one grade. In Proposition 3.1 we showed that (RM_n) is admissible in **GrK**. As **GrK** only has necessitation for $\Box$, this is indeed of some minor interest. We can also pose this question in the other direction: is **GrK** derivable in some extension of $\mathbf{M}_{\mathbb{N}}$, that makes the monotonic character of the logic clearer? Because of the axioms $(Ax4)$, $(Ax5)$ and $(Ax6)$, we should not expect this to be without interaction axioms for different modalities. However, an interesting case is graded modal logic for a single modality $\Diamond_n$: is there a monotonic modal logic axiomatizing this case, without interaction axioms? This logic should contain $\Diamond_n \bot \leftrightarrow \bot$, corresponding to the requirement that for all states w in the domain of a model, $\emptyset \notin \nu_n(w)$. Such a logic should

also contain, for example, $(\Diamond_n\phi \wedge \Diamond_n\neg\phi) \to (\Diamond_n\psi \vee \Diamond_n\neg\psi)$. It is easy to see that this is valid in **GrK**. However, $(\Diamond_n\phi \wedge \Diamond_n\neg\phi) \to (\Diamond_n\psi \vee \Diamond_n\neg\psi)$ is not derivable in monotone modal logic, as there are models of monotone modal logic in which it is false. We leave the axiomatization of single-grade graded modal logic for future research.

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