UNIVERSITY OF ŁÓDŹ
DEPARTMENT OF LOGIC

BULLETIN
OF THE SECTION OF LOGIC

VOLUME 50, NUMBER 2

ŁÓDŹ, JUNE 2021

UNIVERSITY OF ŁÓDŹ
DEPARTMENT OF LOGIC

BULLETIN
OF THE SECTION OF LOGIC

VOLUME 50, NUMBER 2

ŁÓDŹ, JUNE 2021

C O P E
Member since 2018
JM13714

Layout
Michał Zawidzki

Initiating Editor
Katarzyna Smyczek

Printed directly from camera-ready materials provided
to the Łódź University Press

Published by Łódź University Press
First edition. W.10288.21.0.C

Printing sheets 8.125

Łódź University Press
90-131 Łódź, 8 Lindleya St.
www.wydawnictwo.uni.lodz.pl
e-mail: ksiegarnia@uni.lodz.pl
tel. +48 42 665 58 63

The **Bulletin of the Section of Logic** (*BSL*) is a quarterly peer-reviewed journal published with the support from the University of Łódź. Its aim is to act as a forum for a wide and timely dissemination of new and significant results in logic through rapid publication of short papers. *BSL* publishes contributions on topics dealing directly with logical calculi, their methodology, and algebraic interpretation.

Papers may be submitted to the Editor-in-Chief or to any of the Collecting Editors. While preparing the munuscripts for publication please consult Submission Information.

* * *

Editorial Office: Department of Logic, University of Łódź
ul. Lindleya 3/5, 90–131 Łódź, Poland
e-mail: *bulletin@uni.lodz.pl*

Homepage: *http://czasopisma.uni.lodz.pl/bulletin*

TABLE OF CONTENTS

Bulletin of the Section of Logic
Volume 50/2 (2021), pp. 125–129
https://doi.org/10.18778/0138-0680.2021.14

REASONING ABOUT SOCIAL PHENOMENA

The initiative of this Special Issue of the Bulletin of the Section of Logic was born during one of the Chinese-Polish Workshops on Applied Logic and is connected with the developing collaboration between Chinese and Polish logicians. On the one hand, Poland has a long and strong presence in Western logic, which is witnessed by many influential works in the field from the rise of modern logic until today. On the other hand, China has its own logical tradition that can be traced back to ancient times. The study of it is actually a research field in present China. Somehow independently of it, Chinese scholars also make their input to the modern, Western-rooted logic. Among other sources of inspiration, some works of Polish logicians influenced the development of logic in modern China. One example of this is Łukasiewicz's book 'Aristotle's Syllogistic from the Standpoint of Modern Formal Logic', which was published originally in 1951 and was translated to Chinese in 1981 as a part of the well-known series Chinese Translation of World Academic Classics [1]. Another example is Ziembiński's book 'Practical Logic', which was published originally in 1959 and was translated into Chinese in 1988 [2]. This book played an important role in the study of legal logic in China.

The community of Polish logicians is quite active, while the community of Chinese logicians has got bigger in recent years. However, the interaction between the two communities could hardly be observed until quite recent times. The series of Chinese-Polish Workshops on Applied Logic was meant to change this situation. The first event took place at Beijing Normal University in 2017. The second one took place at Nicolaus Copernicus University in Toruń in 2018. The third one took place at Zhejiang University in 2019.

The distinguished topic of the third Workshop was 'Reasoning about social phenomena', which is also the core subject of the present Special Issue. The choice of the topic reflects the shift of the perspective from the point of view of a single, abstract, ideal subject to the more realistic perspective of multiple, imperfect, interacting subjects, that can be observed in recent decades in logic. The workshop gathered, apart from the Chinese and Polish participants, a group of researchers from other Asian and European countries and gave the participants an oportunity to have very interesting discussions. The program is listed on the workshop webpage: www.xixilogic.org/events/3rd-chinese-polish-workshop/

The call for papers for this Special Issue, following the workshop, encouraged contributions on the following subjects: philosophical logic (deontic, epistemic, causal, probabilistic, etc.) within social context; multi agent logics; non-monotonic reasoning (particularly in cognitive science); formal social sciences; formal ethics.

We received nine submissions and after a careful reviewing process, five of them that fulfilled high scientiffic standards and were relevant to the subject. Let us now briefly summarise their motivations and main contributions.

Federico L. G. Faroldi's paper *Towards a Logic of Value and Disagreement via Imprecise Measures* (pp. 131–149) provides a novel way to formalize how we value things. The solution is based on imprecise measures of values. Then it proposes a logic, called Hyperintensional Value Logic, to make sense of value disagreement among people.

Daniela Glavaničová and Matteo Pascucci's article *The Good, the Bad, and the Right: Formal Reductions Among Deontic Concepts* (pp. 151–176) provides a logical analysis of two important notions in deontic logic: normative ideality and normative awfulness. Then, it studies how to define obligation, explicit permission and Hohfeldian relations by the two notions.

The paper *Tableau Systems for Epistemic Positional Logics* by Mateusz Klonowski, Krzysztof Aleksander Krawczyk, and Bożena Pięta (pp. 177–204) presents a set of logics for analyzing some important epistemic modalities such as knowing and believing, which are based on positional semantics. Sound and complete tableau systems are provided for these logics.

The article *Extended MR with Nesting of Predicate Expressions as a Basic Logic for Social Phenomena* by Aleksander Parol, Krzysztof Pietrowicz, and Joanna Szalacha-Jarmużek (pp. 205–227) aims to develop new perspectives on the applications of positional logic to issues of social sciences such as sociology. It presents a positional logic which extends the MR logic with new expressions for describing complex social situations.

Richmond H. Thomason's article *Common Knowledge, Common Attitudes, and Social Reasoning* (pp. 229–247) discusses arguments against the thesis that people can acquire common knowledge and that common attitudes are needed in social reasoning. The author argues that this skepticism is based on implausible assumptions and thinks that there is enough room for common knowledge in social discourse.

This Special Issue completes a chapter of the research activities on logic supporting reasoning about social phenomena. Most of the papers included in it provide clear guidelines for further research. As the editors of the Special Issue, we are commited to continue the common academic activities of logicians from China and Poland. We believe that the activities succesfully support the research in the field, contribute to the development of logic, and serve the international community of logicians. We are interested in extending our research topics to some emerging areas, espacially logic in new generation artificial intelligence.

Acknowledgements. We want to thank all authors for their contributions and the reviewers, who in this difficult pandemic times responded to our requests for comments and in many cases allowed the authors to improve their works. There would not have been this special issue without their contribution. Special thanks go to the participants of Zhejiang University 2019 meeting for providing inspiration, and—last but not least—Editor-in-Chief of BSL, prof. Andrzej Indrzejczak who encouraged us to prepare this Special Issue of the Bulletin of the Section of Logic.

Tomasz Jarmużek
Fengkui Ju
Piotr Kulicki
Beishui Liao

References

[1] J. Łukasiewicz, **Aristotle's Syllogistic from the Standpoint of Modern Formal Logic**, The Commercial Press (1981), translated to Chinese by Zhen Li and Xiankun Li.

[2] Z. Ziembiński, **Practical Logic**, Qunzhong Press (1988), translated to Chinese by Shengen Liu.

Tomasz Jarmużek
Nicolaus Copernicus University in Toruń
Chair of Logic
ul. Stanisława Moniuszki 16/20
87-100 Toruń, Poland
e-mail: tomasz.jarmuzek@umk.pl

Fengkui Ju
Beijing Normal University
School of Philosophy
19 Xinjiekouwai Street
100875 Beijing, China
The John Paul II Catholic University of Lublin
Institute of Philosophy
Aleje Racławickie 14
20-950 Lublin, Poland
e-mail: fengkui.ju@bnu.edu.cn

Piotr Kulicki
The John Paul II Catholic University of Lublin
Institute of Philosophy
Aleje Racławickie 14
20-950 Lublin, Poland
e-mail: piotr.kulicki@kul.pl

Beishui Liao
Zhejiang University
Institute of Logic and Cognition
148 Tianmushan Road
310058 Hangzhou, China
e-mail: baiseliao@zju.edu.cn

Bulletin of the **S**ection of **L**ogic
Volume 50/2 (2021), pp. 131–149
https://doi.org/10.18778/0138-0680.2021.07

Federico L. G. Faroldi

TOWARDS A LOGIC OF VALUE AND DISAGREEMENT VIA IMPRECISE MEASURES

Abstract

After putting forward a formal account of value disagreement via imprecise measures, I develop a logic of value attribution and of (dis)agreement based on (exact) truthmaker semantics.

Keywords: Value, disagreement, truthmaker semantics, hyperintensionality.

1. Introduction

Suppose you and a friend have different value judgments on a proposition or a state of affairs, where all other relevant considerations are on a par (you both have the same information, same reasoning power, perhaps endorse the same moral theory, etc.): we call such a situation a value-based *peer* disagreement. Think of two act-utilitarians A and B, with the same background information. A thinks that improving an elephant's life from a circus by closing it down is worth firing all the circus' employees; B doesn't. Both agree on the facts; both agree on the factual consequences, they even agree on the value of improving the elephant's life; their only disagreement is on its value (as measured by the (dis)value if some of the consequences, in this case).

Modulo some minor points, modeling *epistemic* peer disagreement seems conceivably routine, at least as soon as we *agree* on how to represent beliefs. This usually involves some probability distribution: for agents A and B, and proposition p, agents A and B disagree over p when $\mathcal{B}^A(p) \neq \mathcal{B}^B(p)$, where

Presented by: Tomasz Jarmużek, Fengkui Ju, Piotr Kulicki, Beishui Liao
Received: August 31, 2020
Published online: May 27, 2021

$\mathcal{B} : Agents \times Prop \rightarrow [0, 1]$ is a belief function over propositions indexed to agents. However, there is no such agreement on value disagreement.

The received view, in the contemporary literature, is usually to take a domain of objects (outcomes, events, individuals, etc.) and to order them via agent-based preference relations. After that, one can either compare them directly or represent these qualitative relations as functions into more or less standard (quantitative) structures preserving the ordering. Given certain structural assumptions, one may then perform more complicated operations, like aggregation, taking averages, etc. These accounts are ridden with very well-known problems. Let me mention a few. Preference orderings are almost always used, and they are transitive and complete relations. Both properties are dubious: first, several people have argued that preference (or goodness) is not transitive, as it is shown by case of value parity and sweetening;[1] second, completeness, at least in the case of value, seems completely unrealistic, if anything, at least for the very fact that there is, reasonably, an infinite number of possibilities, and agent with finite resources will hardly be able to have a definite preference ordering on an infinite number of options.[2] This objection is plausibly weakened if one adopts a theory according to which value assignments are completely objective. Even in this case, however, a plausible case for incompleteness can be built, given some assumption of uncertainty or intrinsic indeterminacy. Third, not all moral theories are expressible via preference relations (cf. [10]), thus making an account which does not depend on preference relations preferable.

Another concern against preference approaches has to do with the fact that often a quantitative representation into certain number structures, like the real numbers, is sought. This poses two kinds of problems: first, in order to guarantee the existence of a quantitative representation, one often needs to require further structural conditions on the qualitative properties, most of which are unnatural (for an example, think of a "continuity" or "solvability" condition); second, one can think that numbers (or in general, other quantitative structures) are philosophically ill-suited to represent, model or understand values.

In the next section, I will sketch a way to solve some of the aforementioned issues.

[1] See for instance [28].

[2] For some work in decision theory without completeness, see [22, 23, 24].

2. Values and imprecise measures, informally

In [12], I put forward and defended a formal account of value disagreement via imprecise measure theory. Very briefly, each agent has associated a set of measures that determine an interval (not necessarily numeric). Philosophical consideration in favor of this approach, besides the fact that it bypasses the criticism of standard preference-based approaches, are to be found when one takes into account (i) considerations of normative uncertainty, (ii) considerations of normative indeterminacy, (iii) issues summarized by the label of "transformative experiences".[3]

As for (i), normative uncertainty, suppose you are about to leave for an imminent trip, and your insurance company instructs you to indicate an amount of money to insure your luggage, in case of loss, theft, and so on. What is the right amount of money (in this case) at which you value your belongings? Modulo considerations of *factual* uncertainty, which will plausibly be taken into account by a probability distribution, you might be unsure how to value different things: that particular sweater your grandmother made for you last Christmas before she died, for instance, is not worth much as a sweater, but is priceless. Or, in a different but possible context, say, if your grandmother were still to live, or after her death you discovered she was a horrible human being, you could instead attach a more definite value. Or again, you might not have any opinion about how valuable something is, at least in come contexts, but not in others, which is again different to say that something has no value (or value 0, if we use money as linear utilities). An interval or imprecise approach can take care of these considerations.

As for (ii), normative indeterminacy, suppose it is indeterminate that an action is permissible or impermissible, plausibly for Sorites-like reasons. For instance it is permissible to interrupt the focus of a bus driver to ask a question, but it is impermissible to harass them for two hours; however, it seems indeterminate whether asking three or four questions is permissible or not (similar examples abound in the literature). It seems that an interval or imprecise approach can take care of these considerations. In this case, the indeterminacy does not depend on subjective extra-normative circumstances (different people having different informations, for example),

[3] A referee points out that this account may even be in agreement with research into the mental processes of estimating cardinalities. Cf. [19, 20].

but on a genuine normative gap, as it were. Of course whether this gap is subject-dependent or subject-independent depends on the normative theory in question. Related considerations can be made with regard to *parity*, a fourth value relation beside 'better than', 'worse than' and 'equal' that some philosophers argue exist.[4] An interval-based approach can arguably take care of this notion.[5]

As for (iii), "transformative experiences", one can increase or decrease the value of something as a result of thinking ahead. Suppose you value immensely philosophy, or abstract thinking. However, you know that, as you age, your intellectual powers will likely decrease, and as a result you will value much more more practical tasks or human companionship, of the kind that just a family or old friends can provide. A family or old friends cannot be obtained overnight, and this perhaps makes you value abstract thinking a bit less *even now*, for valuing it so highly will be detrimental to other activities beneficial to your future value set. This can be understood as a case of interpersonal disagreement in the sense of there being disagreement between myself at the present time and myself in the future. Using a diachronic approach (i.e. indexing one's function to time) might not be the right choice in this context; having intervals of value, rather than sharp values, seems a good way to go, for you can adjust upwards or downwards your current value, singly considered, given this kind of updates.

In the present paper, I briefly go through such an account of value disagreement via imprecise measure theory in Sect. 3. In Sect. 4 I develop some ideas leading to a logic of value attribution and of disagreement, based on truthmaker semantics.

3. Values and imprecise measures, formally

We suggest that every agent $\alpha \in A$ has associated a set of partial functions from sets of states of affairs to their value. Partiality take care that some things may be incomparable or incommensurable, even intrapersonally.

[4] See for one [6, 5, 7].

[5] For a similar approach to parity, without intervals but with multiple relations, see [25, 26].

3.1. Values and imprecise measures: first pass

We now define these intuitions more precisely.

Definition 1 (Value space). Let $(\Omega, \mathcal{F}, \mathcal{M}, A)$ be a value space, where

1. Ω is a set of (partial) states ordered by a partial order $\sqsupseteq$;
2. $\mathcal{F}$ is a suitably generated structure of its subsets;
3. $\mathcal{M}$ is a (finite) set of partial indexed signed measures $\mu : \mathcal{F} \times A \times I \to \mathbb{R}$ which we write μ_i^α, for each agent $\alpha \in A$, and $i \in I$ where I is an index set.
4. A is a finite set of agents.

Given the set of states Ω, where the partial order represents parthood, a suitably generated structure $\mathcal{F}$ of its subsets could be its power set if Ω is finite, a sigma-algebra if not, or a less rich structure. We do not take a stance on this point, except to note that we showed how to construct a measure on a weak structure such as a (join) semilattice in Theorem 13 of [13]. The intuition (which will be made precise later when the logic part is introduced) is that some propositions can be identified with sets of states of affairs (namely, those which verify the proposition).

Although the definition of a value space is quite general, for the immediate purposes of this paper we can ignore point 2, given that for the family $\mathcal{M}^\alpha$ of indexed signed measures μ_i,[6] a set $i \in I$ (used to distinguish measures) for each agent α, no further requirement is imposed. In particular, the measure can take negative values and it is not normalized. Additionally it is not required to be (either finitely or sigma-) additive, in order to cash out a substantial plausible feature of value structures: *first*, value judgments may not be additive, for adding something extraneous to a positively valued state of affairs may decrease its value.[7] One can impose monotonicity as a separate requirement, although the consensus seems to be that value is not monotonic.[8] *Second*, we use $\mathbb{R}$ out of convenience. As

[6] To avoid clutter, I write μ or μ_i suppressing the agent or the index, when it is clear from the context.

[7] For non-additive measures, see [29].

[8] By monotonicity in this context we mean the following: for any $s, t \in \Omega$ and $\mu \in \mathcal{M}$, if $s \sqsupseteq t$ then $\mu(s) \geq \mu(t)$.

an alternative, it has been suggested to use $^*\mathbb{R}$ in order to represent normative reasons (see [13] and [1]), although the use of non-standard structures in utility theory is well-known. *Third*, we have not specified a range: for instance one can assign infinite values to the functions in order to account for some particular phenomena, thus adjoining $\{-\infty; \infty\}$ to the codomain. *Fourth*, in case one thinks that the set of states Ω is a set of partial states that can be joined arbitrarily, thus eschewing a possibility requirement and admitting of impossible state of affairs, one would be at loss with a positive and normalized measure. Moreover, one could very well assign positive value to impossible states. *Fifth*, and finally, one could very well require that $\mu(\emptyset) = 0$, although it is not clear whether there is consensus on the fact that the empty event (or state, etc.) has null value, or there may be situations where it may have a non-null value. Thus the term 'measure' is used pretty loosely.

The first advantage to such a set-up, w.r.t. more traditional ones, is that whatever order structure the set of propositions/states of affairs may have, it is *not* reflected in the value judgments, since the function is not required to be e.g. monotonic. A simple example to illustrate the point at hand. I take measures to be precise. Suppose s_1 = "Mary gives the first six months of her salary to charity", s_2 = "gives the last six months of her salary to charity" and $s_{1+2} = s_1 \sqcup s_2$ = "Mary gives the first and last six months of her salary to charity". Further suppose that $\mu(s_1) = 5; \mu(s_2) = 5$. If measures were monotonic, we should e.g. expect that $\mu(s_{1+2}) \geq \mu(s_1)$ and $\mu(s_{1+2}) \geq \mu(s_2)$. But it is plausible to hold that $\mu(s_{1+2})$ can even get a negative value, for Mary would remain without income that year. A "bigger" state can have a value which is e.g. smaller of the values of its parts.

To jump a bit ahead, before having defined the formal means to account for value disagreement, let's have an informal pass of one possible such notion we the example just given. Suppose agents A and B have two measures each: $\mu_1^A(s_1) = 5; \mu_2^A(s_1) = 6; \mu_1^B(s_1) = 1; \mu_2^B(s_1) = -3$. In this simple example, A values state $s_1 = [5; 6]$, whereas B values state $s_1 = [-3; 1]$: there's no overlapping, so A and B completely disagree over s_1.

Note that these measures, for each agent, can be arbitrary. This means that, on a more philosophical level, such an approach can account for the following two questions: first, value disagreement can be explained via different mechanisms to assign values[9], rather than presupposing the values are assigned just in one way. A consequentialist, for instance, may want the value functions to be additive, whereas a deontologist will presumably reject additivity. Such a flexibility seems possible, for instance, within the very sophisticated framework developed by [10]: where (almost) all normative theories are characterized in a unitary framework, with various combinations of how the options are understood and which properties the preference functions enjoy. If we accept such an approach, then within our framework we can directly compare the deontologist and the consequentialist, the particularist and the utilitarian. But it does not stop here. Second, in fact, we can very easily extend the present account to the *higher-order* to model disagreement *on* value theories themselves, insofar as they can be captured by measure functions, which in the higher-order setting will become the elements of the domain themselves: instead of having first-order measures which takes as arguments states of affairs, like in the framework presented so far, we can have second- (and higher-, potentially) order measures which takes as arguments first-order measures. This makes the present account highly general, but we leave the exploration of this issue to further work.

Let's define a couple of other notions necessary to get a grip on different kinds of disagreement.

With an abuse of notation, for every "proposition"[10] p, we define:

Definition 2 (Lower value). Let the *lower value* of any p be $\underline{M}(p) = inf\{\mu_i^\alpha(p) : \mu_i^\alpha \in \mathcal{M}$ for all $\alpha \in A$ and $i \in I\}$.

Definition 3 (Upper value). Let the *upper value* of any p be $\overline{M}(p) = sup\{\mu_i^\alpha(p) : \mu_i^\alpha \in \mathcal{M}$ for all $\alpha \in A$ and $i \in I\}$.

Definition 4 (Agent-relative Value). Let the agent-relative lower *value* of any p for agent $\alpha \in A$ be $\overline{M}^\alpha(p) = inf\{\mu_i^\alpha(p) : \mu_i^\alpha \in \mathcal{M}$ for $\alpha \in A$ and all $i \in I\}$. Similarly for the upper value.

It is important to note that contrary to upper and lower probabilities from imprecise probability theory, we do *not* require that lower and upper value be conjugate, i.e. we do *not* require that $\underline{M}(p) = 1 - \overline{M}(p)$.

[9]When I use the phrase "assign value", one need not read in it a Nietzschean, as it were, value creation. Such a phrase is meant to be neutral, and can be read also as a merely passive endeavor, that is, simply understanding or feeling what the values already are.

[10]For the purposes of this work, we call proposition or state of affairs any set of states, with all due caveats that should be made precise.

Apart from substantial considerations about values, this follows among other things from the fact that no normalization of measures was imposed.

If we want to keep the analogy with imprecise probability theory, each agent has a precise value measure, but the value of a proposition is an interval. However, I find it much more congenial to the issue of value disagreement to associate to each agent an interval. Technically speaking, this can be done by simply considering that each agent has a family of measures associate to it and by taking the lower and upper value for each agent. With an abuse of notation we can now say that each (second-order, as it were) measure outputs an interval for each agent.[11]

We can now define the following notions.

Definition 5 (Imprecise value space). Let $(\Omega, \mathcal{F}, \mathcal{M}, \underline{M}, \overline{M}, A)$ be an imprecise value space.

Then we can make precise the usual understanding of (sharp) agreement and disagreement in the obvious way, depending on whether the (dis)agreement is just on some propositions (and it is therefore partial) or on all propositions (and it is therefore total).

We define more precisely only one of the several notions one can focus on, because it will be useful later in the logic part:

Definition 6 (Partial Imprecise Peer weak disagreement). Let $(\Omega, \mathcal{F}, \mathcal{M}, \underline{M}, \overline{M}, A)$ be an imprecise value space. Agents $\alpha_1, ... \alpha_n \in A$ are in *partial imprecise weak disagreement* if for some $p \in \mathcal{F}$, $\bigcap_{\alpha_i \in A, i \in \mathbb{N}} [\underline{M}^{\alpha_i}(p), \overline{M}^{\alpha_i}(p)] \neq \emptyset$.

We can now define, for every proposition/state of affairs p:

Definition 7 (Imprecise interval value). Let the imprecise interval value $V(p)$ of any p be
$V(p) = [\underline{M}(p); \overline{M}(p)]$.

This way of modeling value-based peer disagreement is in some continuity with some recent approaches to epistemic peer disagreement (cf. [11]), although the proposed solution to epistemic peer disagreement does not seem to transfer well to the axiological domain, mostly because certain

[11] Somewhat in line with similar ideas in the literature: with intervals (in a variety of manners [16], later revised in [17] and [5]), vectors ([3]), and sets of functions ([18]).

structural properties of belief functions cannot be assumed for value functions, and these properties are crucial in the epistemic domain and in the solutions developed.

4. Logics of value and disagreement

4.1. Generalizing beyond numbers

Suppose you do not like, for various philosophical reasons, the idea that values of states of affairs, actions, propositions are to be modeled or expressed with numbers (regardless of whether the target set is something very simple like the natural numbers moral philosophers often employ, or something more usual like the real numbers economist usually employ, or more exotic number structures like hyperreal numbers).

As I hinted in a preceding section, the exact nature of the target set is immaterial for the purposes of this paper; but more than that: it turns out one can generalize the ideas we sketched above to something which should be adequate to a number of different conceptualizations. In particular, we are going to generalize the notion of interval by employing the notion of open set.

Definition 8 (General value space). Let $(\Omega, \mathcal{F}, \mathcal{M}, X)$ be a general value space, where

1. Ω is a set of (partial) states;
2. $\mathcal{F}$ is a suitably generated structure of its subsets;
3. $\mathcal{M}$ is a family of partial indexed signed measures $\mu_i, i \in I$, s.t. $\mu_i : \mathcal{F} \to T$

where $(\Omega, \mathcal{F})$ is like above, and T is a family of subsets of X such that

1. $X \in T, \emptyset \in T$;
2. $\{O_i\}_{i \in I} \subseteq T \Rightarrow \bigcup_{i \in I} O_i \in T$ and
3. $\{O_i^n\}_{i=1} \subseteq T \Rightarrow \bigcap_{i=1}^n O_i \in T$.

In other words T is a topology on X, which can be for instance a set of actions (we leave open at this point whether X is a subset of Ω).

The underlying philosophical intuitions is that the measure of much you value something is given by the actions you are prepared to perform to preserve or promote it, or some such notion provided by your background metaethical theory, or the things you are prepared give up for it. We can now reformulate all the preceding definitions in terms of open sets, perhaps adjusting one requirement in the following way: $\mu\emptyset = \emptyset$, if the target set is not $\mathbb{R}$.

We can now reformulate all the preceding definitions in terms of open sets, perhaps adjusting one requirement in the following way: $\mu\emptyset = \emptyset$, if the target set is not $\mathbb{R}$. The obvious next step is to use the same topological ideas as above to define a logic of value attribution.

4.2. First pass: topological semantics

Given the generalization of the preceding section to open sets, we can now define the topological semantics for the modal logic of value in the obvious way, i.e. where $\Box$ is in the interior operator on the topological space. This modal logic, which corresponds to the modal logic **S4**, is sound and complete w.r.t. a dense-in-itself metric space (these are well-known standard results).

Definition 9. Let $\mathcal{M}$ be a topological value model $\mathcal{M} = (\tau, v)$ where τ is a topology and $v : Prop \to \mathcal{P}(\mathbb{R})$ is a valuation function from atoms in the language to sets of real numbers. We can define a notion of verification starting from $x \models$ iff $x \in v(p)$ and extending it to boolean cases as usual.

Definition 10. Given v, let $[\phi]$ be $\{x : x \in \mathbb{R} \text{ and } x \models \phi\}$. Then $\Box_\alpha\phi$ (with $\alpha \in A$ an agent) is true if it is in the interior of the set defined by p (via the measure μ_α):
$x \models \Box_\alpha\phi$ iff $\exists U \in \tau, x \in U$ and $\forall y \in U, y \models \phi$, i.e. $[\Box_\alpha\phi] = int[\phi]$.

I propose to informally interpret in such a logic formulas like $\Box_\alpha p$ as "α values that p", since the underlying topological intuition is that p is true in all the open sets, which in our case are the collection of the open intervals of values.

However, such an approach is not only cumbersome, but also not quite satisfactory, as it forgets, as it were, the original measures, and flattens the distinctions, the availability of a metric notwithstanding.

It is clear that this is not a logic of disagreement, but rather of value attribution.

Be as it may, let's check the plausibility of S4 axioms in the intended interpretation.

Axiom K: $\Box(p \to q) \to \Box p \to \Box q$ is intuitively adequate, modulo doubts we may have about the material conditional.

Axiom 4: $\Box p \to \Box\Box p$, i.e. that one values one's valuation seems if not immediate at least acceptable, although with some caution.

Axiom T: $\Box p \to p$ seems completely unjustifiable: there is no reason for the fact that if something has value also needs to be the case.

However, given the topological facts on the interior operator and on open sets, this is unavoidable in all topological semantics.

Topological semantics (and the corresponding modal logic), therefore, does not seem adequate to model properly our intuitions about the logic of value (and *a fortiori*, about the logic of value disagreement).

We now go on to define a hyperintensional logic of value on general value spaces that improves on these topological ideas.

4.3. Logics of value and agreement: hyperintensional logics and semantics

In this section we present hyperintensional logics of value and agreement. Roughly, we can understand a context as hyperintensional if it draws distinction which are finer-grained than simple logical or necessary equivalence.[12] For instance, the above S4 logic is not hyperintensional, in that if agent A values that p, and q is logically equivalent to p, then automatically agent A also values that q.

More generally, we can split a definition of hyperintensional by taking into account either necessity or logical equivalence. So we can say that a sentential context **C** is *non-intensional* iff for every sentence α and β:

$$\text{NON-INTENSIONAL: } \nvdash \Box(\alpha \equiv \beta) \supset \Box(\mathbf{C}(\alpha) \equiv \mathbf{C}(\beta)).$$

A different concept is that of *congruentiality.* It is obtained when we substitute material equivalence with logical equivalence. In general, we can say that a sentential context **C** is *congruential* iff for every sentence α and β:

[12]The term 'hyperintensionality' was proposed by Cresswell some forty years ago (cf. [9]), with reference to logical equivalence. For other early contributions to the topic, see e.g. [27]. [4] is a good introduction to the topic.

CONGRUENTIAL: If $\vdash \alpha \equiv \beta$ then $\vdash \mathbf{C}(\alpha) \equiv \mathbf{C}(\beta)$.

For the purposes of this paper, we take hyperintensionality to cover congruentiality.

Why go the hyperintensional route when it comes to logics of value, then? There are at least two reasons why the background logic here proposed is hyperintensional: first and more specifically, there is the thought that in order to really see whether there is genuine *value* disagreement, we must have a grip on the exact meaning of what we are considering. While classical logic is just good up to (classical) logical equivalence, it is widely agreed that meaning is finer-grained than just truth, or even necessary truth. There is partial consensus that hyperintensionality is a good way to track meaning, and exact truthmaker semantics a good way to to make these hyperintensional ideas precise.[13] Second and more generally, such a logic usually come with the option of being paracomplete, paraconsistent, or both. Having all these options open, i.e. depending on the concrete assignement of verifiers and falsifiers, seems to be the right choice for a logical approach, insofar as it should be general enough to be compatible with different philosophical options to model value incomparability. One prominent position on value incomparability, for instance, takes it to be the case that it should be understood and modeled as a failure of bivalence.[14]

In what follows I sketch a hyperintensional logic of value and disagreement based on truthmaker semantics.

Definition 11 (Value space). Let $(\Omega, S, \mathcal{M})$ be a value space, where

1. Ω is a set of (partial) states;
2. S is a suitably generated structure of its subsets;
3. $\mathcal{M}$ is a (finite) family of partial indexed signed measures $\mu_i^\alpha, i \in I$ is an index, $\alpha \in A$ is an agent, s.t. $\mu_i^\alpha : S \to 2^S$.

The measures are not required to be complete. We can build a model based on a value space in the following way:

[13] I defend a general hyperintensional approach to normative and evaluative phenomena at length in [13]. However, even if one does not find the hyperintensionality route appealing, truthmaker semantics is flexible enough to work with coarser-grained logics, as shown in [30] and [21], for instance.

[14] For a recap on this debate, see [8].

DEFINITION 12 (Value space model). Let $(\Omega, S, \preceq, [\![]\!], \mathcal{M})$ be a value space model, where

1. Ω and $\mathcal{M}$ are as above;
2. $(S, \preceq, [\![]\!])$ is a state space in the exact truthmaker sense, that is:
   (a) S is the non-empty set of states;
   (b) $\preceq$ is a partial order to be interpreted intuitively as the parthood relation, ie reflexive, transitive, and antisymmetric, and it's up-complete: we can therefore get a join operation in the usual way: $a \sqcup b = b$ iff $a \preceq b$;
   (c) $[\![]\!]$ is the valuation function such that $[\![p]\!]^+$ is the set of verifiers of p, and $[\![p]\!]^-$ is the set of falsifiers of p, i.e. partial functions from atoms to non-empty subsets of S.

Here are the standard clauses for an arbitrary formula to be exactly verified (falsified) by a certain state defined by simultaneous double induction:

DEFINITION 4.1 (Exact verification (falsification)).

1. $s \Vdash p$ iff $s \in [\![p]\!]^+$;
2. $s \dashv\Vert p$ iff $s \in [\![p]\!]^-$;
3. $s \Vdash \neg A$ iff $s \dashv\Vert A$;
4. $s \dashv\Vert \neg A$ iff $s \Vdash A$;
5. $s \Vdash A \wedge B$ iff for some s' and s'', $s = s' \sqcup s''$, $s' \Vdash A$ and $s'' \Vdash B$;
6. $s \dashv\Vert A \wedge B$ iff $s \dashv\Vert A$ or $s \dashv\Vert B$;
7. $s \Vdash A \vee B$ iff $s \Vdash A$ or $s \Vdash B$;
8. $s \dashv\Vert A \vee B$ iff for some s' and s'', $s = s' \sqcup s''$, $s' \dashv\Vert A$ and $s'' \dashv\Vert B$

A formula $A \approx_H B$ holds in a model when $[\![A]\!]^+ = [\![B]\!]^+$, i.e. when A and B have the same verifiers. A formula B is a consequence of a formula A, i.e. $A \models_{HDL} B$, iff $[\![A]\!]^+ \succeq [\![B]\!]^+$.

The usual notions are pretty standard and can be found, along with axiomatic systems and proofs of soundness and completeness, in [14, 15], and [2], for instance.

We now move to accounting for the value part of the system. The philosophical underlying intuition is that, for each state s, if it exists, μ picks out the best (positive or negative) states agent α is prepared to bring about to preserve or preclude s.

The imprecise value assigned by an agent α to a proposition is now the collection of the states picked out by all the measures of the states exactly verifying that proposition.

We now define the imprecise value of a proposition p, for agent k, i.e. $V_k(p)$ in the following way:

Definition 13 (Imprecise value). Let $V_k(p)$, the imprecise value an agent k attributes to a proposition p, be $V_k(p) = \bigcup_{s \in S}\{\bigcup_{i \in I} \mu_i^k(s) : s \Vdash p\}$, where $\mu_i^k \in \mathcal{M}$ are the measures for an agent $k \in A$, with $i \in I$ an index.

Not only situations that occur can be evaluated: not all states are actual, yet, they can still verify or falsify propositions (cf. possible worlds). In fact, given the clauses for verification and falsification of a negated formula, it is possible to account for negative value, or the values of falsifiers.

Novel are the following clauses for a 'it is valued by agent i at state s' operator, that intend to capture the intuition that a proposition is valued at a state s in case there's a state s' that is valued from the original state which exactly makes true that proposition:

$$s \Vdash \nabla_i \phi \text{ iff there is an } s', \text{ s.t. } s' \Vdash \phi \text{ and } s' \in V_i(\phi).$$

$$s \dashv\vert\vert\, \nabla_i \phi \text{ iff for all } s' \Vdash \phi, s' \notin V_i(\phi), \text{ or for some } s' \dashv\vert\vert\, \phi, s' \in V_i(\phi).$$

We call the present logic HVL (hyperintensional value logic). We now point out some results.

The resulting axiomatic system is one of those relatively well-known to track hyperintensionality, with the substitution of hyperintensional equivalents and with the addition of an axiom for the value operator that distributes over disjunction.

The axioms and rules are the following:

1. $A \approx_H A$
2. $A \approx_H (A \wedge A)$

3. $A \approx_H (A \vee A)$
4. $A \vee B \approx_H (B \vee A)$
5. $A \wedge B \approx_H (B \wedge A)$
6. $A \wedge (B \wedge C) \approx_H (A \wedge B) \wedge C$
7. $A \vee (B \vee C) \approx_H (A \vee B) \vee C$
8. $A \wedge (B \vee C) \approx_H (A \wedge B) \vee (A \wedge C)$
9. $A \approx_H \neg\neg A$
10. $(\neg A \wedge \neg B) \approx_H \neg(A \vee B)$
11. $(\neg A \vee \neg B) \approx_H \neg(A \wedge B)$
12. $(\neg A \vee \neg B) \approx_H \neg(A \wedge B)$
13. $\nabla_i(A) \vee \nabla_i(B) \approx_H \nabla_i(A \vee B)$

Rule:

1. $A \approx_H B, C(A)\ /\ C(B)$

THEOREM 4.2. *HVL is sound and complete w.r.t. value space semantics.*

PROOF. The proof is similar to the soundness and completeness proof of [13, 2] with obvious variations. □

We now highlight three useful facts.

FACT 4.3. It is not the case that if $\nabla_i(A) \wedge \nabla_i(B)$ then $\nabla_i(A \wedge B)$.

FACT 4.4. It is not the case that if $\nabla_i(A \wedge B)$ then $\nabla_i(A) \wedge \nabla_i(B)$.

Both results seem in line with a plausible notion of value: valuing A and valuing B does not imply valuing A and B, for A and B, as we have discussed in the informal part, can interact in non-aggregative ways, perhaps decreasing the overall value. Conversely, valuing A and B does not imply valuing A and valuing B for an analogous reasoning.

Moreover, “conflicts of value” are not ruled out, consistently with the common occurrence of valuing opposite things:

FACT 4.5. $\not\models \neg(\nabla_i(A) \wedge \nabla_i(\neg A))$.

Logics of (dis)agreement We now introduce an operator that describe the notion of *partial imprecise weak (dis)agreement* among agents $i, ..., j \in I$, namely an operator for weak agreement $\Delta_{i,...,j}$, with $i, ..., j \in I$ and $|I| > 1$ as follows:

> $s \Vdash \Delta_{i,...,j}\phi$ iff for all s' s.t. $s' \Vdash \phi, s' \in V_{i,...,j}(\phi)$, where $V_{i,...,j}(\phi) =_{df} V_i(\phi) \cap ... \cap V_j(\phi)$
>
> $s \dashv\!| \Delta_{i,...,j}\phi$ iff for some $s' \Vdash \phi, s' \notin V_{i,...,j}(\phi)$, or for some $s' \dashv\!| \phi, s' \in V_{i,...,j}(\phi)$.

We can now replace the axiom for value in HVL with the following axiom to obtain HLA, the hyperintensional logic of value agreement:

14. $\Delta_{i,...,j}(A \vee B) \approx_H \Delta_{i,...,j}A \wedge \Delta_{i,...,j}B$

THEOREM 4.6. *HLA is sound and complete w.r.t. the above semantics.*

PROOF. The proof is similar to the soundness and completeness proof of [13, 2] with obvious variations. □

We have an obvious bridge principle between the "it is valued" operator and the (dis)agreement operator:

BP $\Delta_{i,...,j}(\phi) \rightarrow_H \nabla_i\phi \wedge ... \wedge \nabla_j\phi$

where $\rightarrow_H$ is just one half of $\approx_H$. The converse of course does not hold, given the semantic clauses for conjunction and the (dis)agreement operator.

5. Conclusion, limitations, and further work

Building on the idea of understanding value and value disagreement using imprecise measures, a very essential sketch of a hyperintensional logic of value and disagreement, based on truthmaker semantics, has been given. Except for the conciseness of the sketch offered, there are some limitations and room for further research. First, more meta-theoretic results need to be proved. Second, more notions of disagreement needs to be defined precisely and formally modeled. Third, more bridge principles need to be studied and discussed with an eye to their philosophical significance.

Acknowledgements. Many thanks to an anonymous reviewer and to many people who discussed some of the ideas behind this paper over the years in multiple places around the world, but first to Alberto Bardi, who patiently listened to very confused ramblings during a bike ride in the torrid summer of 2017. This work has been done while a postdoctoral research fellow of the Research Foundation, Flanders (FWO) and a Lise Meitner fellow (FWF), grant number M 2527 – G32.

References

[1] H. Andréka, M. Ryan, P.-Y. Schobbens, *Operators and Laws for Combining Preference Relations*, **Journal of Logic and Computation**, vol. 12(1) (2002), pp. 13–53, DOI: https://doi.org/10.1093/logcom/12.1.13.

[2] A. Anglberger, F. L. G. Faroldi, J. Korbmacher, *An Exact Truthmaker Semantics for Obligation and Permission*, [in:] O. Roy, A. Tamminga, M. Willer (eds.), **Deontic Logic and Normative Systems**, DEON16, College Publications, London (2016), pp. 16–31.

[3] R. J. Aumann, *Utility Theory without the Completeness Axiom*, **Econometrica**, vol. 30(3) (1962), pp. 445–462, DOI: https://doi.org/10.2307/1909888.

[4] F. Berto, D. Nolan, *Hyperintensionality*, [in:] E. N. Zalta (ed.), **The Stanford Encyclopedia of Philosophy**, spring 2021 ed., Metaphysics Research Lab, Stanford University (2021).

[5] R. Chang, *Parity, Interval Value, and Choice*, **Ethics**, vol. 115(2) (2005), pp. 331–350, DOI: https://doi.org/10.1086/426307.

[6] R. Chang, *Value Incomparability and Incommensurability*, [in:] I. Hirose, J. Olson (eds.), **The Oxford Handbook of Value Theory**, Oxford University Press (2015), DOI: https://doi.org/10.1093/oxfordhb/9780199959303.013.0012.

[7] R. Chang, *Parity: An Intuitive Case*, **Ratio**, vol. 29(4) (2016), pp. 395–411.

[8] C. Constantinescu, *Value Incomparability and Indeterminacy*, **Ethical Theory and Moral Practice**, vol. 15(1) (2012), pp. 57–70, DOI: https://doi.org/10.1007/s10677-011-9269-8.

[9] M. J. Cresswell, *Hyperintensional Logic*, **Studia Logica**, vol. 34(1) (1975), pp. 25–38, DOI: https://doi.org/10.1007/BF02314421.

[10] F. Dietrich, C. List, *What Matters and How It Matters: A Choice-Theoretic Representation of Moral Theories*, **The Philosophical Review**, (2017), pp. 421–479, DOI: https://doi.org/10.1215/00318108-4173412.

[11] L. Elkin, G. Wheeler, *Resolving Peer Disagreements Through Imprecise Probabilities*, **Nous**, vol. LII(2) (2018), pp. 260–278, DOI: https://doi.org/10.1111/nous.12143.

[12] F. L. G. Faroldi, *Modeling Value Disagreement via Imprecise Measures*, 2017.

[13] F. L. G. Faroldi, **Hyperintensionality and Normativity**, Springer, Dordrecht (2019), DOI: https://doi.org/10.1007%2F978-3-030-03487-0.

[14] K. Fine, *Angellic Content*, **Journal of Philosophical Logic**, vol. 45(2) (2016), pp. 199–226, DOI: https://doi.org/10.1007/s10992-015-9371-9.

[15] K. Fine, *Truthmaker Semantics*, [in:] B. Hale, C. Wright, A. Miller (eds.), **A Companion to the Philosophy of Language**, 2nd ed., Blackwell, London (2017), pp. 556–577.

[16] J. Gert, *Value and Parity*, **Ethics**, vol. 114(3) (2004), pp. 492–510, DOI: https://doi.org/10.1086/381697.

[17] J. Gert, *Parity, Preference and Puzzlement*, **Theoria**, vol. 81(3) (2015), pp. 249–271, DOI: https://doi.org/10.1111/theo.12069.

[18] C. Hare, *Take the Sugar*, **Analysis**, vol. 70(2) (2010), pp. 237–247, DOI: https://doi.org/10.1093/analys/anp174.

[19] W. Krysztofiak, *Algebraic Models of Mental Number Axes: Part II*, **Axiomathes**, vol. 26(2) (2016), pp. 123–155, DOI: https://doi.org/10.1007/s10516-015-9270-2.

[20] W. Krysztofiak, *Representational Structures of Arithmetical Thinking: Part I*, **Axiomathes**, vol. 26(1) (2016), pp. 1–40, DOI: https://doi.org/10.1007/s10516-015-9271-1.

[21] H. Leitgeb, *HYPE: A System of Hyperintensional Logic (with an Application to Semantic Paradoxes)*, **Journal of Philosophical Logic**, vol. 48(2) (2019), pp. 305–405, DOI: https://doi.org/10.1007/s10992-018-9467-0.

[22] D. McCarthy, K. Mikkola, T. Thomas, *Aggregation for General Populations Without Continuity or Completeness*, [in:] **MPRA Paper No. 80820**, University Library of Munich, Germany (2017).

[23] D. McCarthy, K. Mikkola, T. Thomas, *Representation of Strongly Independent Preorders by Sets of Scalar-Valued Functions*, [in:] **MPRA Paper No. 79284**, University Library of Munich, Germany (2017).

[24] D. McCarthy, K. Mikkola, T. Thomas, *Representation of Strongly Independent Preorders by Vector-Valued Functions*, [in:] **MPRA Paper No. 80806**, University Library of Munich, Germany (2017).

[25] W. Rabinowicz, *Value Relations*, **Theoria**, vol. 74(1) (2008), pp. 18–49, DOI: https://doi.org/10.1111/j.1755-2567.2008.00008.x.

[26] W. Rabinowicz, *I—Wlodek Rabinowicz: Incommensurability and Vagueness*, **Aristotelian Society Supplementary Volume**, vol. 83(1) (2009), pp. 71–94, DOI: https://doi.org/10.1111/j.1467-8349.2009.00173.x.

[27] R. Suszko, *An essay in the formal theory of extension and of intension*, **Studia Logica**, vol. 20(1) (1967), pp. 7–34, DOI: https://doi.org/10.1007/BF02340023.

[28] L. S. Temkin, **Rethinking the Good: Moral Ideals and the Nature of Practical Reasoning**, Oxford Ethics, Oxford University Press (2012), DOI: https://doi.org/10.1093/acprof:oso/9780199759446.001.0001.

[29] V. Torra, Y. Narukawa, M. Sugeno (eds.), **Non-Additive Measures. Theory and Applications**, Springer (2014), DOI: https://doi.org/10.1007/978-3-319-03155-2.

[30] B. C. van Fraassen, *Facts and Tautological Entailment*, **Journal of Philosophy**, vol. 66(15) (1969), pp. 477–487, DOI: https://doi.org/10.2307/2024563.

Federico L. G. Faroldi
Research Foundation, Flanders (FWO)/Ghent University
Department of Philosophy
Blandijnberg 2 #9000
Ghent, Belgium
e-mail: federico.faroldi@ugent.be

Bulletin of the **S**ection of **L**ogic
Volume 50/2 (2021), pp. 151–176
https://doi.org/10.18778/0138-0680.2021.05

Daniela Glavaničová
Matteo Pascucci

THE GOOD, THE BAD, AND THE RIGHT: FORMAL REDUCTIONS AMONG DEONTIC CONCEPTS[1]

Abstract

The present article provides a taxonomic analysis of bimodal logics of normative ideality and normative awfulness, two notions whose meaning is here explained in terms of the moral values pursued by a given community. Furthermore, the article addresses the traditional problem of a reduction among deontic concepts: we explore the possibility of defining other relevant normative notions, such as obligation, explicit permission and Hohfeldian relations, in terms of ideality and awfulness. Some proposals in this respect, which have been formulated in the literature over the years, are here improved and discussed with reference to the various logics that we will introduce.

Keywords: Awfulness, explicit permission, Hohfeldian relations, ideality, moral values, obligation.

2020 Mathematical Subject Classification: 03B45, 03B60, 03B80.

[1]Daniela Glavaničová was supported by the Slovak Research and Development Agency under the contract no. APVV-17–0057, VEGA 1/0197/20, and VEGA 2/0117/19. Matteo Pascucci was supported by the *Štefan Schwarz Fund* for the project "A fine-grained analysis of Hohfeldian concepts" and by the VEGA grant no. 2/0117/19.

Presented by: Tomasz Jarmużek, Fengkui Ju, Piotr Kulicki, Beishui Liao
Received: February 11, 2020
Published online: April 1, 2021

1. Moral values and moral principles: a philosophical preamble

There are many sorts of concepts that play an important role in normative reasoning, some of which have been extensively analysed also within formal logic: obligation, prohibition, and permission; right, duty, privilege, power, and immunity; responsibility, liability, blame and praise; good, bad, supererogatory, ideal, awful, and so forth (see [8] for a survey). In the present paper, we will assume that the range of application of these notions is not absolute, but relative to the moral values and normative systems of a given community; however, this is not a necessary stance.[2] To give an example, blame depends on actions of normative parties (individuals or groups), as well as on prospective responsibilities they had at the time of acting (or not acting, for that matter); see [3]. Our work focuses on the notions of normative ideality and awfulness, which are related to what is taken to be "the (highest) good" and "the (lowest) bad", respectively. Arguably, these two notions ultimately depend on the values of a given community: an industrialized country will likely have a different ordering of moral values than a small isolated tribe, and, consequently, they will have a different understanding of the notions of ideality and awfulness. Therefore, here normative ideality is equated with "what is ideal according to the moral values supported by a given community" and normative awfulness is equated with "what is awful according to the moral values supported by a given community".

Of course, a lot has been said about moral values, and they have been understood in different ways. To begin with, it is difficult to distinguish moral values from values of other kind. As Quine puts it in [15], p. 473:

> There are easy extremes: the value that one places on his neighbor's welfare is moral, and the value of peanut brittle is not. The value of decency in speech and dress is moral or ethical in the etymological sense, resting as it does on social custom;

[2] We would also like to note that our assumption leaves it open how large overlaps there are between the moral values of different communities and to which extent these overlaps are relevant when one defines ideality and awfulness. For instance, if one generates the sets of propositions $P_1, P_2, P_3,...$ that hold if the norms of communities c_1, c_2, c_3,... are respectively met, one can say that the propositions that are in the set $P = P_1 \cap P_2 \cap P_3 \cap ...$ (that is, the propositions that are in the "common base") are ideal absolutely. A similar idea can be employed for the notion of awfulness.

> and similarly for observance of the Jewish dietary laws. On the other hand the eschewing of unrefrigerated oysters in the summer, though it is likewise a renunciation of immediate fleshly pleasure, is a case rather of prudence than morality.

Now, the question is: to which extent can moral values and principles be the objects of a formal analysis? More often than not, moral values are understood as notions of fairness, justice, trust, respect, responsibility, privacy, sharing, loyalty, and so forth (see, for instance [16], [17], and [19]). These, if taken straightforwardly, are neither truth-apt, nor can be consistent or inconsistent, nor have consequences. However, it is also very common to equate moral values with the corresponding moral principles or norms (for instance, trying to "teach" moral values to a robot may mean providing it with instructions encoding certain moral principles). We will follow the former common practice and occasionally speak about moral values understood as the corresponding principles.

Moral principles, by contrast, can be regarded as propositions. When moral values are equated with these, they become truth-apt, can have consequences and be either consistent or inconsistent (for instance, the moral value of fairness can be transformed into the moral principle "according to our moral values, everyone is treated fairly"). Naturally, this is just one option how to respond to a variant of Jørgensen dilemma; see [12]. Perceived in this manner, the value of sharing information can go against the value of privacy (see [19]), and the value of loyalty to an authority (like a family member) can very well go against the values of justice and fairness.

Another preliminary remark on the philosophical ground that will be used to support our work is needed: the notion of ideality employed in the present paper is taken to an extreme, since it describes (portions of) perfect situations (all moral values of a given community are realised, or more colloquially, all good things are done). This approach follows certain semantic intuitions at the basis of the traditional interpretation of deontic logic, as discussed in [11]. Similarly, the notion of awfulness employed here describes (portions of) the worst situations possible (no moral values of a given community are realised, or more colloquially, no good things are done). To better illustrate this point, it is useful to offer a comparison with how the notions of ideality and awfulness are used in everyday reasoning: we ordinarily regard a scenario as awful as soon as *one* terrible thing blatantly against the accepted moral values occurs (e.g., a situation where, within a group of accused people, one innocent person is convicted, while all the others are judged in an appropriate way). Similarly, it is common to say that a situation where some relevant set of great things happens is ideal (for instance, when there is perfect justice with respect to important issues), even though other, less relevant, bad things happen (people are occasionally dishonest in minor issues).

In the present article we will provide a taxonomy of bimodal logics to represent the notions of normative ideality and normative awfulness; furthermore, we will assess ideas to formally define other relevant normative notions in terms of these two. In particular, we will examine the case of obligation, of explicit permission, and of some Hohfeldian concepts, improving and extending some proposals available in the literature.

2. Related approaches

A natural starting point of our work is a logical system called **DL** introduced by A. Jones and I. Pörn in [11] in order to represent deductive reasoning with normative ideality and normative sub-ideality. The aim of **DL** is to address some of the criticisms traditionally raised against what is known as the *standard system of deontic logic*, namely **SDL**. The latter is a variant of the alethic modal system **KD** based on a language with a primitive operator of obligation, O.[3] Jones and Pörn claim that the main problem of **SDL** is a semantic one: the meaning of the operator of obligation is explained in terms of a set of normatively ideal situations (or *worlds*), according to the following truth-conditions, where ϕ is a formula denoting an arbitrary proposition, w an arbitrary situation and "iff" abbreviates "if and only if":

- $O\phi$ holds in w iff ϕ holds in every situation v that is normatively ideal with respect to w.

The argument for their criticism is very simple: *what is obligatory* cannot be equated with *what holds in all normatively ideal situations*, since tautologous propositions (e.g., that either it rains or it does not rain) clearly hold in all such situations, while being normally regarded as normatively neutral (that is, as neither obligatory, nor forbidden, nor permitted). In order for

[3] For the criteria followed in the present article for naming alethic modal systems, see, e.g., [4].

a proposition to represent an obligation, there must be some normatively sub-ideal situation in which it fails to hold.[4]

Here we will adopt this view for the sake of exploring its formal consequences, while noting also that the question whether tautologous propositions are permitted heavily depends on how one reads the term "permitted". If the permission of ϕ is merely the lack of an obligation of $\neg\phi$ (i.e., what is sometimes referred to as an *implicit permission*), then it is plausible to say that tautologies are permitted. Natural languages, however, suggest stronger readings of "permitted", such as *explicit permission* or *free choice permission.* Explicit permission corresponds to a permission explicitly given; free choice permission, in turn, corresponds to a permission that allows one to infer "ϕ is permitted" and "ψ is permitted" from "$\phi \vee \psi$ is permitted"; see [6] on the main notions of permission we use in natural languages.

Even if one does not engage with truth conditions for $O\phi$ that involve a complete description of normatively ideal situations, what is ideal with respect to the moral values of a certain community need not correspond to what is obligatory with respect to these values. Nevertheless, the former has a certain impact on the latter. For instance, if a community aims at an equal distribution of the goods available, this has consequences on what an individual is expected to do; however, no single individual is expected to produce *on her own* a situation in which all goods are equally distributed: therefore, the latter situation is normatively ideal according to the moral values of the community, while not constituting *directly* an obligation for anybody. One is rather obliged to perform specific actions (or bring about states-of-affairs) that contribute to its realization.

Looking for a formal distinction between what is normatively ideal and what is obligatory, Jones and Pörn propose to adopt a language where, in the place of the operator O, there are two primitive modal operators that we will here represent as $\square$ and $\blacksquare$. They suggest the following reading:

- $\square\phi$ holds in w iff ϕ holds in every situation v that is normatively ideal with respect to w;
- $\blacksquare\phi$ holds in w iff ϕ holds in every situation v that is normatively sub-ideal with respect to w.

[4] This problem is already pointed out in [1] by A. R. Anderson, who suggests a restriction of deontic discourse to contingent propositions.

Therefore, $\Box$ has in **DL** the same interpretation that O has in **SDL**; however, it is said to be an operator for normative ideality, rather than for obligation. Furthermore, as the truth-conditions indicate, $\blacksquare$ is said to be an operator for normative sub-ideality. Subsequently, they define an operator of obligation in the following manner (we will use the label `Ob1` for this definition):[5]

`Ob1` $O\phi =_{def} \Box\phi \wedge \neg\blacksquare\phi$.

The meaning of this definition is that ϕ is obligatory in a situation w iff ϕ holds in all normatively ideal situations (with respect to w) and fails to hold in some normatively sub-ideal situations (with respect to w).

Jones and Pörn do not engage with the task of axiomatizing their logic, which is rather specified in terms of a list of valid formulas in a given class of relational models. They just observe that **DL** is at least as powerful as a bimodal copy of **SDL**. A complete axiomatization is indicated in [5]: one needs to extend a suitable basis for bimodal **SDL** with the axiom-schema $(\Box\phi \wedge \blacksquare\phi) \to \phi$, which captures the idea that the current situation is either ideal or sub-ideal.

The idea of defining O in terms of $\Box$ and $\blacksquare$ as in `Ob1` looks like a simple and elegant solution; however, it encounters some obstacles when one attempts to deal with deontic paradoxes, as well as to represent contrary-to-duty reasoning (see the discussion in [14] and [13]). The move from a formal analysis of ideality and sub-ideality to a formal analysis of ideality and awfulness is suggested by an alternative definition of O in terms of the same bimodal language that can be found in [5]:

`Ob2` $O\phi =_{def} \Box\phi \wedge \blacksquare\neg\phi$.

On the one hand, `Ob2` is able to overcome some of the problems of `Ob1`, (for instance, by solving some traditional deontic paradoxes). On the other hand, it cannot be easily reconciled with the reading of $\blacksquare$ as "in all normatively sub-ideal situations": even if ϕ is obligatory, it can very well hold in a sub-ideal situation (and thus there can be a sub-ideal situation where $\neg\phi$ does not hold). This is possible because a situation can be sub-ideal due to a violation of some other obligation than ϕ. For example, if everyone in a village pays their debts (ϕ), the situation will still be sub-ideal

[5] We point out that there are some relevant notational differences between this article and [11]: for instance, there O stands for normative ideality, O' for normative sub-ideality, and *Ought* for obligation.

if there is a serial killer murdering people in the neighbourhood. Thus, in [13] it is suggested to adopt `Ob2` and, at the same time, change the reading of $\blacksquare$ to "in all normatively *awful* situations". Such a move allows one to get rid of the axiom $(\Box\phi \wedge \blacksquare\phi) \to \phi$, since the current situation (i.e., the situation in which things are evaluated) might be neither ideal nor awful. Furthermore, explicit permission is defined in [13] as follows, adhering to the original proposal in [11] (though, under a different reading of $\blacklozenge$, which shortens $\neg\blacksquare\neg$):

`Pm1` $P\phi =_{def} \Diamond\phi \wedge \blacklozenge\neg\phi$.

Contrary to **SDL**, the operator of explicit permission in this case is not the dual of the operator of obligation.

3. A general representation of conditional norms

A comprehensive appraisal of the results obtained so far within attempts to reduce deontic concepts to normative ideality and related notions shows several limitations. For instance, it seems that approaches of this kind need additional devices (e.g., reference to levels of ideality) for a proper treatment of contrary-to-duty reasoning. Nevertheless, apart from these limitations, several aspects of the reductionist project have just not been addressed yet. For instance, consider the problem of defining simple conditional obligations, like "ϕ is obligatory under condition ψ", in terms of ideality and awfulness. According to `Ob2`, one could say that each of these obligations corresponds to a formula of the kind $\Box(\psi \to \phi) \wedge \blacksquare\neg(\psi \to \phi)$, which constitutes the *definiens* of $O(\psi \to \phi)$. Yet, this approach is not satisfactory, since in many logical systems it commits one to the claim that the antecedent ψ holds in every normatively awful situation and the consequent ϕ in none: in fact, as soon as it is possible to replace formulas that are provably equivalent in the Propositional Calculus within the scope of $\blacksquare$, one is entitled to infer $\blacksquare(\psi \wedge \neg\phi)$ from $\blacksquare\neg(\psi \to \phi)$.

Here we propose to improve the solutions available by adopting a general definition that applies both to conditional and unconditional obligations. First, we generalize the meaning of $\Box$ and $\blacksquare$, in order to avoid making reference to a complete description of the set of normatively ideal situations and the set of normatively awful situations, differently from [11]. This is due to the fact that reference to complete descriptions of ideal/awful situations commits one to certain forms of logical inference that are not available in the weakest systems that we are going to introduce. For instance, if no other restriction is specified, when one says that in all normative ideal situations ϕ is the case, then one is very likely committed to say that in all such situations $\phi \vee \psi$ is the case too. Thus, in order to avoid similar inferences, a different and more general reading of $\Box$ and $\blacksquare$ is needed.[6] We will say that $\Box\phi$ means that ϕ is a *consequence* of the fact that all the moral values of a given community are pursued, and that $\blacksquare\phi$ means that ϕ is a *consequence* of the fact that no moral values of a given community are pursued. The notion of consequence here involved is intentionally left unspecified; indeed, as we will see, the plausibility of a more precise characterization will depend on the formal system analysed. Accordingly, the intended interpretation of $\Diamond\phi$ (which shortens $\neg\Box\neg\phi$), is that ϕ is *compatible* with the fact that all moral values of a given community are pursued; the intended interpretation of $\blacklozenge\phi$ is that ϕ is *compatible* with the fact that no moral value of a given community is pursued.

Conditional obligations will be taken to represent the most general case, and unconditional obligations will be defined in terms of them. More precisely, an unconditional obligation $O\phi$ will be treated as a shorthand for an obligation trivially depending on a tautologous condition, as it is often done in the literature on dyadic deontic logic (see, e.g., [2]), and will be represented as $O(\phi/\top)$. The general definitional schema that we will adopt is the following:

$$\texttt{Ob}* \;\; O(\phi/\psi) =_{def} \Diamond\psi \wedge \Box(\psi \to \phi) \wedge (\blacklozenge\psi \to \neg\blacklozenge(\psi \wedge \phi)).$$

The meaning of $\texttt{Ob}*$ is that ϕ is obligatory under condition ψ if and only if (i) ψ is compatible with normative ideality, (ii) it is normatively ideal that ψ entails ϕ, and (iii) if ψ is compatible with normative awfulness, then the conjunction of ψ and ϕ is incompatible with normative awfulness. Thus, in the case of unconditional obligations we will get the *definiens*: $\Diamond\top \wedge \Box(\top \to \phi) \wedge (\blacklozenge\top \to \neg\blacklozenge(\top \wedge \phi))$. We will see that in many classes of systems such a *definiens* can be simplified. Looking at the range of application of $\texttt{Ob}*$, we

[6] For a parallel, consider the inadequacy (in general) of reading the modal operator of alethic necessity as "it is the case in all possible worlds that" when one wants do deal with classes of systems weaker than **K**, especially systems not closed under the replacement of provable equivalents. Also in this case, a broader reading of the operator at issue is needed.

need to clarify that this definition works for conditional obligations that do *not* instantiate a form of contrary-to-duty reasoning.[7]

Moreover, we can define *implicit permission* in terms of `Ob*` and negation:

$$\texttt{IPm*}\ \neg O(\neg\phi/\psi) =_{def} \neg((\Diamond\psi \wedge \Box(\psi \rightarrow \neg\phi)) \wedge (\blacklozenge\psi \rightarrow \neg\blacklozenge(\psi \wedge \neg\phi))).$$

In systems where replacement of formulas that are provably equivalent in the Propositional Calculus is available (we will later call this $\texttt{RRPE}_{PC}$), the above *definiens* can be further transformed, so as to get the schema $(\Diamond\psi \rightarrow \Diamond(\psi \wedge \phi)) \vee (\blacklozenge\psi \wedge \blacklozenge(\psi \wedge \neg\phi))$. The latter reads as follows: if the antecedent is compatible with normative ideality, then the antecedent and the consequent are jointly compatible with normative ideality; otherwise the antecedent is compatible with normative awfulness and so are, jointly, the antecedent and the negation of the consequent. The first disjunct of this simplified *definiens* appears plausible with respect to the intended interpretation: if our moral values allow for the condition of a permission to hold, so they allow for this condition along with the permitted formula. The second disjunct can be justified with respect to the intended interpretation too: the condition is compatible with awfulness and so is the condition along with the negation of the permitted formula. A further look at the role played by $\blacklozenge(\psi \wedge \neg\phi)$: for example, one does not get tested for coronavirus even though satisfying all conditions for getting tested, then getting tested for coronavirus (when satisfying all conditions) cannot be prohibited. This issue connects to the intuition that what is not prohibited is implicitly permitted.

By contrast, explicit permission can be defined as follows:

$$\texttt{EPm*}\ P(\phi/\psi) =_{def} (\Diamond\psi \rightarrow \Diamond(\psi \wedge \phi)) \wedge (\blacklozenge\neg\psi \vee \blacklozenge(\psi \wedge \neg\phi))$$

---

[7] As we mentioned in section 2, the representation of contrary-to-duty norms is a very challenging issue in a framework for normative ideality and normative awfulness, and it very likely requires the addition of levels of ideality. This problem, pointed out also by a reviewer, is left open for future research. The reason why `Ob*` is not adequate in this regard is that its "ideality component", that is, the conjunction $\Diamond\psi \wedge \Box(\psi \rightarrow \phi)$, commits one to the claim that ψ is compatible with normative ideality, whereas this is not the case in contrary-to-duty reasoning. However, even weakening this component to $\Diamond\psi \rightarrow \Box(\psi \rightarrow \phi)$ seems to be problematic, because one loses the dependence of ϕ from ψ in ideal situations (namely, the truth of $\Box(\psi \rightarrow \phi)$) in cases in which $\Diamond\psi$ does not hold.

The reading of $\mathtt{EPm}*$ is: "ϕ is permitted under condition ψ iff (i) if ψ is compatible with normative ideality, then so is $\psi \wedge \phi$, and (ii) $\neg\psi$ is compatible with normative awfulness, or $\psi \wedge \neg\phi$ is compatible with normative awfulness." Unconditional explicit permissions are then defined as follows, exploiting the usual strategy of equating $P\phi$ with the conditional permission $P(\phi/\top)$: $(\Diamond\top \rightarrow \Diamond(\top \wedge \phi)) \wedge (\blacklozenge\neg\top \vee \blacklozenge(\top \wedge \neg\phi))$. Notice that, in general, explicit permission entails implicit permission (due to laws of the Propositional Calculus) but not *vice versa*. In this article we will mainly focus on obligations. Furthermore, we will explore a variation of the bimodal language with indexed operators in which it is possible to express basic *Hohfeldian concepts*, such as duties and rights, involving two normative parties [9, 10].

4. Formal language

In the present exposition we will extend the bimodal language used in [11] with a propositional constant $\mathfrak{c}$, called *ideality witness* and meaning "all moral values of the community are pursued". This constant will allow us to ensure that the description of what is normatively ideal and the description of what is normatively awful according to a system are always distinct, unless the set of moral values gives rise to inconsistencies on its own in a given situation. Our language will be simply called $\mathcal{L}$.

Definition 4.1 (Vocabulary). The language $\mathcal{L}$ includes the following primitive symbols:

- a countable set of propositional variables Var, denoted by p, q, r, etc.;
- the propositional constant $\mathfrak{c}$ (ideality witness);
- the modal operators $\Box$ (normative ideality) and $\blacksquare$ (normative awfulness);
- the Boolean connectives $\neg$ (negation) and $\rightarrow$ (material implication);
- round brackets.

Definition 4.2 (Well-formed formulas). The set WFF of well-formed formulas over $\mathcal{L}$ is defined by the grammar below, where $p \in \mathsf{Var}$:

$$\phi ::= p \mid \mathfrak{c} \mid \neg\phi \mid \phi \to \phi \mid \Box\phi \mid \blacksquare\phi$$

Additional Boolean operators ($\wedge$, $\vee$ and $\equiv$) and modal operators ($\Diamond$ and $\blacklozenge$) can be defined in terms of the primitive ones according to the usual conventions. For instance, $\Diamond\phi =_{def} \neg\Box\neg\phi$. We adopt standard conventions also for the definition of the logical constants $\top$ (*verum*) and $\bot$ (*falsum*). Furthermore, we take the dyadic operators O (obligation) and P (explicit permission) to be defined in accordance with `Ob*` and `EPm*`. We use $\mathsf{Var}(\phi)$ to denote the set of propositional variables having some occurrences in ϕ. A formula ϕ is a *substitution instance* of a formula ψ iff ϕ is obtained from ψ by uniformly substituting the occurrences of some elements of $\mathsf{Var}(\psi)$ with a possibly different formula. For instance, $(\Box r \to q) \to (\Box r \to q)$ is a substitution instance of $p \to p$. Furthermore, trivially, according to this definition, every formula is a substitution instance of itself. We will denote by WFF^b the subset of WFF including all formulas with no occurrence of a modal operator (i.e., the set of purely Boolean formulas).

5. Deductive systems

In this section we will describe a series of nine modal systems that can be used for formal reasoning on the notions of normative ideality and normative awfulness. Our aim is giving some examples of a wide range of logical possibilities that can be exploited for various applications. Furthermore, we will see how the definitions of obligation and explicit permission mentioned in the previous part of the article, namely `Ob*` and `EPm*`, behave within these systems. First, we introduce some preliminary notions that will be used in the axiomatic bases of these systems. We will denote the classical Propositional Calculus as PC. The symbol $\vdash$ will indicate derivability in a system (that can be either specified by the context, or arbitrary); the symbol $\vdash_{PC}$ derivability in PC. We start with the notion of a *transformation group*, which will be used to define a form of restricted replacement for provable equivalents.

Definition 5.1 (Transformation group). A transformation group is a set of formulas $g = \{\phi_1, \phi_2, \phi_3, ...\}$ where, for $1 \leq i, j$, we have that $\phi_i, \phi_j \in \mathsf{WFF}^b$ and $\vdash_{PC} \phi_i \equiv \phi_j$.

Transformation groups thus concern only purely Boolean formulas. We will use $\mathcal{C}$ to denote a set of transformation groups. Examples of transformation groups are $\{p, \neg\neg p\}$, $\{p \wedge q, q \wedge p, (p \wedge q) \wedge (q \wedge p)\}$, $\{\neg(\neg p \wedge \neg q), p \vee q, \neg p \rightarrow q\}$, $\{q \rightarrow q, r \rightarrow r\}$, etc. We will restrict our attention to transformation groups in which all formulas are distinct, and to sets of transformation groups where every formula occurs at most in one group.

Furthermore, we need to introduce a notion of *analogous substitution* on which we will rely, in combination with transformation groups, when defining an axiom of Modal Dependence.

Definition 5.2 (Analogous substitution instances). Formulas $\phi_1, ..., \phi_n$ are said to be *analogous substitution instances* of formulas $\psi_1, ..., \psi_n$ iff (i) ϕ_i, for $1 \leq i \leq n$, is a substitution instance of ψ_i, and (ii) any propositional variable occurring in $\psi_1, ..., \psi_n$ is substituted by the same formula in $\phi_1, ..., \phi_n$.

Finally, we provide a standard notion of *mirror-relation* between two formulas of a bimodal language.

Definition 5.3 (Mirror image). The mirror image of a formula ϕ, denoted by $\mathsf{mi}(\phi)$, is the result of replacing in ϕ each occurrence of an "ideality" modal operator with an occurrence of the corresponding "awfulness" modal operator, and vice versa.

For instance, $\mathsf{mi}(\Diamond p \rightarrow \blacksquare q) = \blacklozenge p \rightarrow \Box q$. An immediate consequence of Definition 5.3 is that, for every $\phi \in \mathsf{WFF}$, $\mathsf{mi}(\mathsf{mi}(\phi)) = \phi$.

Next, we provide a list of deductive principles that will be taken into account in the axiomatic bases of the logical systems discussed in the present article. We will call such a list Θ. All principles in Θ are either axioms or rules.[8] The first principle in Θ varies with one's choice of a set of transformation groups $\mathcal{C}$.

[8]We will speak of "axioms" with reference to principles that ultimately represent "axiom-schemata", as far as no ambiguity arises.

$\mathtt{MD}_{\mathcal{C}}$	$\Box\phi \to \Box\phi'$, provided that ϕ and ϕ' are analogous substitution instances of two formulas ψ and ψ', both occurring in one and the same transformation group $g \in \mathcal{C}$;
$\mathtt{RM}_{PC}$	if $\vdash_{PC} \phi \to \psi$, then $\vdash \Box\phi \to \Box\psi$;
RM	if $\vdash \phi \to \psi$, then $\vdash \Box\phi \to \Box\psi$;
K	$\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$;
D	$\Box\phi \to \neg\Box\neg\phi$;
BR1	$\Box\mathfrak{c} \wedge \blacksquare\neg\mathfrak{c}$;
BR2	$\Box\blacksquare\phi \equiv \blacksquare\phi$;
BR3	$\blacksquare\Box\phi \equiv \Box\phi$;
N	$\Box\top$;
$\mathtt{T}^*$	$\Box(\Box\phi \to \phi)$;
4	$\Box\phi \to \Box\Box\phi$.

A brief remark on some labels used: $\mathtt{MD}_{\mathcal{C}}$ denotes an axiom-schema for Modal Dependence modulo a set of transformation groups $\mathcal{C}$; $\mathtt{RM}_{PC}$ denotes monotony of the operator $\Box$ with respect to provable implication in the Propositional Calculus. Rule RM denotes monotony of $\Box$ without restrictions. K, D, N, $\mathtt{T}^*$ and 4 are standard modal axioms. BR1, BR2 and BR3 denote various bridge-axioms connecting ideality and awfulness. Axiom BR1 says that the constant $\mathfrak{c}$ always distinguishes what is normatively ideal from what is normatively awful. Axiom BR2 says that "ideal awfulness" collapses to awfulness (this can be understood as a meta-level approval of what the community morally disapproves; if, according to our moral values, it is good that the slavery is wrong, then according to our values, slavery is wrong). Similarly, axiom BR3 says that "awful ideality" collapses to ideality (this axiom can be understood as a meta-level disapproval of what the community morally approves; for instance, a butcher from a future world can say that it is awful that a community morally approves a vegetarian diet only, and this would be naturally read as implying that the community does so).

If one allows the set of transformation groups $\mathcal{C}$ to be infinite, then the rule $\mathtt{RE}_{PC}$ described below (which indicates congruence of the operator $\Box$ with respect to provable equivalence in PC) can be obtained from $\mathtt{MD}_{\mathcal{C}}$ —for instance, by trivially defining $\mathcal{C}$ as the partition of WFF^b under provable equivalence in PC. However, imposing the restriction that $\mathcal{C}$ is a finite set, and that each transformation group within it is finite as well, can be seen as a way of simulating actual reasoning procedures applied by an agent with bounded rationality. Furthermore, $\mathtt{RE}_{PC}$ is clearly derivable in a system where either $\mathtt{RM}_{PC}$ or $\mathtt{RM}$ is available.

$\mathtt{RE}_{PC}$ if $\vdash_{PC} \phi \equiv \psi$, then $\vdash \Box\phi \equiv \Box\psi$.

Note also that, conversely, $\mathtt{MD}_{\mathcal{C}}$, for any choice of $\mathcal{C}$, can be derived in any system closed under $\mathtt{RE}_{PC}$.

With a little abuse of terminology, we will also speak of the mirror image of an axiom $\mathtt{X}$ and of a rule $\mathtt{RX}$. Our notation for these will be $\mathsf{mi}(\mathtt{X})$ and $\mathsf{mi}(\mathtt{RX})$. The meaning of the former expression is that we replace all ideality operators explicitly mentioned in the general formulation of axiom $\mathtt{X}$ with the corresponding awfulness operators, and vice versa. For instance, $\mathsf{mi}(\mathtt{T}^*) = \blacksquare(\blacksquare\phi \to \phi)$. The meaning of the latter expression is that we replace all ideality operators explicitly mentioned in the general formulation of $\mathtt{RX}$ with the corresponding awfulness operators, and vice versa. For instance, $\mathsf{mi}(\mathtt{RM}) =$ "if $\vdash \phi \to \psi$, then $\vdash \blacksquare\phi \to \blacksquare\psi$".

All systems to be developed here are extensions of PC and are closed under *Modus Ponens* (denoted by $\mathtt{MP}$); we take the latter to be formulated in a way which allows one to reason under assumptions:

$\mathtt{MP}$ from the set of assumptions $\{\phi, \phi \to \psi\}$ infer ψ.

A few informal remarks on the derivations used in this article: a derivation D will be a finite sequence of lines labelled with natural numbers 1,...,n, each including exactly one formula that either (i) is a hypothesis (we regard claims on the set $\mathcal{C}$ as hypotheses too), or (ii) is an instance of an axiom, or (iii) is obtained from other formulas in the previous lines by applying one of the rules available. Axioms and rules used in derivations will depend on the systems under analysis, which will be clarified by the context. When a line l of a derivation includes a formula ϕ that is obtained only via applications of axioms and rules whose mirror images are available in the system, we can add a further line $l + 1$ with the formula $\mathsf{mi}(\phi)$ and use "$l \times$ Mirror Images" as a justification.

Axiomatic bases of systems are here ordered sequences of deductive principles. For each axiomatic basis presented, some initial sub-sequence of deductive principles is closed under mirror images.

Definition 5.4 (Pre-axiomatic basis). A pre-axiomatic basis for a system S over the language $\mathcal{L}$ is an ordered list $\sigma = \langle X_1, ..., X_n \rangle$, where $X_i \in \Theta$, for $1 \leq i \leq n$.

Definition 5.5 (Axiomatic basis). Given a pre-axiomatic basis σ for a system S, the result of putting the symbol $\circ$, called a *mirror image bookmark*, over one of the items in σ is an axiomatic basis for S.

A mirror image bookmark occurring on top of an item X_i in a list σ says that all principles X_j in σ, s.t. $1 \leq j \leq i$, are closed under mirror images.

Now we have all ingredients needed to introduce axiomatic bases of systems. The first group of systems that we are going to analyse (α-systems) allow for a very restricted form of $\Box$-congruence and $\blacksquare$-congruence: it applies only to pairs of formulas that are in a relation of analogous substitution with some pair of formulas in a transformation group of the set $\mathcal{C}$. In all of these systems ideality and awfulness are at least characterized as *contrary notions*, due to the fundamental axiom BR1.

Definition 5.6 (System $S_{\alpha 1}$). The axiomatic basis of system $S_{\alpha 1}$ is specified by the following ordered list of deductive principles: $\langle \overset{\circ}{\mathtt{MD}}_{\mathcal{C}}, \mathtt{BR1} \rangle$.

For any choice of a *finite* set $\mathcal{C}$, $S_{\alpha 1}$ can be regarded as the minimal $\mathcal{C}$-based non-congruential system. In such a system it is possible to derive a rule of restricted replacement of provable equivalents, namely the rule $\mathtt{RRPE}_{\mathcal{C}}$ described below.

$\mathtt{RRPE}_{\mathcal{C}}$ if ϕ and ψ are analogous substitution instances of two formulas ϕ' and ψ' both occurring in one and the same transformation group $g \in \mathcal{C}$, and χ_2 is obtained from χ_1 by replacing some occurrence of ϕ with ψ, then $\vdash \chi_1$ entails $\vdash \chi_2$.

When $\mathtt{RE}_{PC}$ and $\mathsf{mi}(\mathtt{RE}_{PC})$ are derivable (due to the way in which $\mathcal{C}$ is defined) one gets replacement for all formulas that are provably equivalent in PC, which we can denote as $\mathtt{RRPE}_{PC}$.

An example of deductive argument that can be represented within system $S_{\alpha 1}$ is the following, provided that $\mathcal{C}$ includes a transformation group where both $\neg r \to \neg s$ and $r \vee \neg s$, for some $r, s \in \mathsf{Var}$, occur:

> Ideally, if citizens are not tested for coronavirus with a negative result ($\neg p$), they do not go to work ($\neg q$). Therefore, ideally, either citizens. are tested for coronavirus with a negative result, or they do not go to work.

1	$(\neg r \to \neg s), (r \vee \neg s) \in g$, for some $g \in \mathcal{C}$	Hyp.
2	$\Box(\neg p \to \neg q)$	Hyp.
3	$\Box(\neg p \to \neg q) \to \Box(p \vee \neg q)$	1 $\times$ $\mathtt{MD}_{\mathcal{C}}$
4	$\Box(p \vee \neg q)$	2,3 $\times$ $\mathtt{MP}$

System $S_{\alpha 1}$ can be interpreted, for instance, in the $\mathcal{L}$-models described below.

DEFINITION 5.7. An $\mathcal{L}$-model is a tuple $\mathfrak{M} = \langle W, \mathbb{C}, f, h_1, h_2, V \rangle$, where:

- W is a set of possible worlds, or *situations*, denoted by w, v, u, etc.;
- $\mathbb{C}$ is a set of *semantic contents*, denoted by c, d, e, etc.;[9]
- f is a function mapping WFF to $\mathbb{C}$ s.t. $f(\phi)$ is the *semantic content of* ϕ;
- h_1 is a function mapping W to $\wp(\mathbb{C})$ s.t. $h_1(w)$ is the set of *ideal semantic contents* at w;
- h_2 is a function mapping W to $\wp(\mathbb{C})$ s.t. $h_2(w)$ is the set of *awful semantic contents* at w;
- V is a function mapping $\mathsf{Var} \cup \{\mathfrak{c}\}$ to $\wp(W)$ s.t. $V(x)$ is the *valuation* of x.[10]

Truth-conditions with reference to a situation w in an $\mathcal{L}$-model $\mathfrak{M}$ are as usual, except for the following clauses:

- $\mathfrak{M}, w \vDash \mathfrak{c}$ iff $w \in V(\mathfrak{c})$;
- $\mathfrak{M}, w \vDash \Box\phi$ iff $f(\phi) \in h_1(w)$;
- $\mathfrak{M}, w \vDash \blacksquare\phi$ iff $f(\phi) \in h_2(w)$.

[9] A semantic content can be informally interpreted as a fine-grained meaning expressed by a formula, namely something more informative than the set of possible worlds where the formula turns out to be true.

[10] We use x to denote a member of the set $\mathsf{Var} \cup \{\mathfrak{c}\}$.

An $\mathcal{L}$-model $\mathfrak{M}$ for $S_{\alpha 1}$ needs to satisfy the following properties, for a given choice of $\mathcal{C}$ and any situation $w \in W$:

- $f(\mathfrak{c}) \in h_1(w)$ and $f(\neg\mathfrak{c}) \in h_2(w)$;
- if ϕ and ϕ' are analogous substitution instances of two formulas ψ and ψ', both occurring in one and the same transformation group $g \in \mathcal{C}$, then $f(\phi) \in h_i(w)$ only if $f(\phi') \in h_i(w)$, for $i \in \{1, 2\}$.

The soundness of $S_{\alpha 1}$ with respect to this class of models can be easily checked by looking at the correspondence between the deductive principles in its axiomatic basis and the list of model properties. The same holds for the other classes of models that will be presented later and the associated formal systems. We leave open the problem of building a completeness proof in terms of the various classes of $\mathcal{L}$-models.

DEFINITION 5.8 (System $S_{\alpha 2}$). The axiomatic basis of system $S_{\alpha 2}$ is specified by the following ordered list of deductive principles: $\langle \mathtt{MD}_{\mathcal{C}}, \mathring{\mathtt{K}}, \mathtt{BR1} \rangle$.

$S_{\alpha 1}$ is too weak to capture many relevant deductive inferences. For instance, in $S_{\alpha 2}$, but not in $S_{\alpha 1}$, it is possible to represent arguments like the following, under the assumption that $\mathcal{C}$ includes a transformation group where both $r \to s$ and $\neg s \to \neg r$, for some $r, s \in \mathsf{Var}$, occur:

> Ideally, if taxes are evaded (p) a fine applies (q). However, ideally, fines do not apply. Therefore, ideally, taxes are not evaded.

Indeed, such an argument can be encoded as follows:

1	$(r \to s), (\neg s \to \neg r) \in g$, for some $g \in \mathcal{C}$	Hyp.
2	$\Box(p \to q)$	Hyp.
3	$\Box(p \to q) \to \Box(\neg q \to \neg p)$	$1 \times \mathtt{MD}_{\mathcal{C}}$
4	$\Box(\neg q \to \neg p)$	$2,3 \times \mathtt{MP}$
5	$\Box\neg q$	Hyp.
6	$\Box(\neg q \to \neg p) \to (\Box\neg q \to \Box\neg p)$	Axiom $\mathtt{K}$
7	$\Box\neg q \to \Box\neg p$	$4,6 \times \mathtt{MP}$
8	$\Box\neg p$	$5,7 \times \mathtt{MP}$

An $\mathcal{L}$-model for $S_{\alpha 2}$ needs to satisfy all properties of $\mathcal{L}$-models for $S_{\alpha 1}$, plus the following:

- if $f(\phi \to \psi), f(\phi) \in h_i(w)$, then $f(\psi) \in h_i(w)$, for $i \in \{1, 2\}$.

Definition 5.9 (System $S_{\alpha 3}$). The axiomatic basis of system $S_{\alpha 3}$ is specified by the following ordered list of deductive principles: $\langle \mathtt{MD}_{\mathcal{C}}, \mathtt{K}, \mathtt{T}^*, \mathring{\mathtt{4}}, \mathtt{BR1} \rangle$.

In $S_{\alpha 3}$ talk about iterated ideality or about iterated awfulness can be reduced to talk about simple ideality and simple awfulness, respectively, as the following derivation shows:

1	$\Box\phi \to \Box\Box\phi$	Axiom $\mathtt{4}$
2	$\Box(\Box\phi \to \phi)$	Axiom $\mathtt{T}^*$
3	$\Box(\Box\phi \to \phi) \to (\Box\Box\phi \to \Box\phi)$	Axiom $\mathtt{K}$
4	$\Box\Box\phi \to \Box\phi$	2,3 $\times$ $\mathtt{MP}$
5	$\Box\Box\phi \equiv \Box\phi$	1,4 $\times$ PC
6	$\blacksquare\blacksquare\phi \equiv \blacksquare\phi$	5 $\times$ Mirror Images

An $\mathcal{L}$-model for $S_{\alpha 3}$ needs to satisfy all properties of $\mathcal{L}$-models for $S_{\alpha 2}$, plus the following:

- if $f(\phi) \in h_1(w)$, then $f(\Box\phi) \in h_1(w)$;
- if $f(\phi) \in h_2(w)$, then $f(\blacksquare\phi) \in h_2(w)$;
- $f(\Box\phi \to \phi) \in h_1(w)$;
- $f(\blacksquare\phi \to \phi) \in h_2(w)$.

The second group of systems that we are going to analyse (β-systems) satisfy $\Box$-monotony and $\blacksquare$-monotony over the set of theorems of the Propositional Calculus (PC).

Definition 5.10 (System $S_{\beta 1}$). The axiomatic basis of system $S_{\beta 1}$ is specified by the following ordered list of deductive principles: $\langle \mathtt{RM}_{\mathtt{PC}}, \mathring{\mathtt{K}}, \mathtt{BR1} \rangle$.

An $\mathcal{L}$-model for $S_{\beta 1}$ needs to satisfy the following properties:

- $f(\mathfrak{c}) \in h_1(w)$ and $f(\neg\mathfrak{c}) \in h_2(w)$;
- if $\vdash_{PC} \phi \to \psi$, then $f(\phi) \in h_i(w)$ only if $f(\psi) \in h_i(w)$, for $i \in \{1, 2\}$;
- if $f(\phi \to \psi), f(\phi) \in h_i(w)$, then $f(\psi) \in h_i(w)$, for $i \in \{1, 2\}$.

Definition 5.11 (System $S_{\beta 2}$). The axiomatic basis of system $S_{\beta 2}$ is specified by the following ordered list of deductive principles: $\langle \mathtt{RM_{PC}}, \mathtt{K}, \mathtt{T}^*, \mathring{\mathtt{4}}, \mathtt{BR1} \rangle$.

An $\mathcal{L}$-model for $S_{\beta 2}$ needs to satisfy all properties of $\mathcal{L}$-models for $S_{\beta 1}$, plus the following:

- if $f(\phi) \in h_1(w)$, then $f(\Box\phi) \in h_1(w)$;
- if $f(\phi) \in h_2(w)$, then $f(\blacksquare\phi) \in h_2(w)$;
- $f(\Box\phi \to \phi) \in h_1(w)$;
- $f(\blacksquare\phi \to \phi) \in h_2(w)$.

Definition 5.12 (System $S_{\beta 3}$). The axiomatic basis of system $S_{\beta 3}$ is specified by the following ordered list of deductive principles: $\langle \mathtt{RM_{PC}}, \mathtt{K}, \mathtt{T}^*, \mathtt{4}, \mathring{\mathtt{D}}, \mathtt{BR1} \rangle$.

An $\mathcal{L}$-model for $S_{\beta 3}$ needs to satisfy all properties of $\mathcal{L}$-models for $S_{\beta 2}$, plus the following:

- if $f(\phi) \in h_i(w)$, then $f(\neg\phi) \notin h_i(w)$, for $i \in \{1, 2\}$.

In β-systems, as well as in all the α-systems previously introduced, no formula of the form $m(\Box\mathfrak{c} \land \blacksquare\neg\mathfrak{c})$, where m is a finite and non-empty sequence of occurrences of the operators $\Box$ and/or $\blacksquare$, is derivable. Moreover, already in $S_{\beta 1}$ the definition of an unconditional obligation $O\phi$ —that we obtained from the general schema $\mathtt{Ob}*$ by equating $O\phi$ with $O(\phi/\top)$— can be simplified as follows:

$$\mathtt{Ob}*' \quad O\phi =_{def} \Diamond\top \land \Box\phi \land (\blacklozenge\top \to \neg\blacklozenge\phi).$$

Indeed, in $S_{\beta 1}$ the rule $\mathtt{RE}_{PC}$ is derivable. This entails that $\mathtt{RRPE}_{PC}$ is available. Due to the fact that $(\top \to \phi) \equiv \phi$ and $(\top \wedge \phi) \equiv \phi$ are derivable in PC, the intended simplification of $\mathtt{Ob}*$ to $\mathtt{Ob}*'$ for unconditional obligations follows by applying $\mathtt{RRPE}_{PC}$ to the former.

Systems $S_{\beta 2}$ and $S_{\beta 3}$ allow for the already mentioned reduction of iterated ideality and iterated awfulness to simple ideality and simple awfulness, due to $\mathtt{T}^*$, $\mathtt{4}$ and their mirror images. In system $S_{\beta 3}$ the definition of unconditional obligations can be further simplified so as to become identical with the one employed in [5], i.e., $\mathtt{Ob2}$. Indeed, $\Diamond\top$ and $\blacklozenge\top$ are theorems of $S_{\beta 3}$, as the following derivation shows (where we exploit the interdefinability of $\bot$ and $\top$ and the definitions of $\Diamond$ and $\blacklozenge$):

1	$\Box\bot \to \neg\Box\top$	Axiom $\mathtt{D}$
2	$\bot \to \top$	PC
3	$\Box\bot \to \Box\top$	$2 \times \mathtt{RM}_{PC}$
4	$\Box\bot \to \bot$	$1{,}3 \times PC$
5	$\neg\Box\neg\top$	$4 \times PC$
6	$\Diamond\top$	$5 \times$ Def $(\Diamond)$
7	$\blacklozenge\top$	$6 \times$ Mirror Images

This derivation also points out that, as long as $P\phi$ is concerned, $\mathtt{EPm}*$ becomes equivalent to $\mathtt{Pm1}$.

The third group of systems that we present here (γ-systems) satisfy unrestricted $\Box$-monotony and unrestricted $\blacksquare$-monotony.

Definition 5.13 (System $S_{\gamma 1}$). The axiomatic basis of system $S_{\gamma 1}$ is specified by the following ordered list of deductive principles: $\langle \mathtt{RM}, \mathtt{K}, \mathtt{T}^*, \mathtt{4}, \mathring{\mathtt{D}}, \mathtt{BR1} \rangle$.

An $\mathcal{L}$-model for $S_{\gamma 1}$ needs to satisfy all properties of $\mathcal{L}$-models for $S_{\beta 3}$, plus the following one, associated to rule $\mathtt{RM}$ (which entails the one associated to rule $\mathtt{RM}_{PC}$ in $\mathcal{L}$-models for $S_{\beta 3}$):

- if $\vdash_{S_{\gamma 1}} \phi \to \psi$, then $f(\phi) \in h_i(w)$ only if $f(\psi) \in h_i(w)$, for $i \in \{1, 2\}$.

In system $S_{\gamma 1}$ it is already possible to derive all formulas of the form $m(\Box\mathfrak{c} \wedge \blacksquare\neg\mathfrak{c})$, where m is a finite sequence of operators $\Box$ and/or $\blacksquare$. Indeed, we know that, if m has length 0, then $m(\Box\mathfrak{c} \wedge \blacksquare\neg\mathfrak{c})$ is $\mathtt{BR1}$. Furthermore, the derivation below shows how to inductively move from a sequence m of length n to a sequence m' of length $n+1$.

1	$\Box(\Box\phi \to \phi)$	Axiom $\mathtt{T}^*$
2	$\Box(\Box\phi \to \phi) \to \Box\Box(\Box\phi \to \phi)$	Axiom $\mathtt{4}$
3	$\Box\Box(\Box\phi \to \phi)$	1,2 $\times$ $\mathtt{MP}$
4	$m(\Box\mathfrak{c} \wedge \blacksquare\neg\mathfrak{c})$	Induction hypothesis
5	$\Box(\Box\phi \to \phi) \to m(\Box\mathfrak{c} \wedge \blacksquare\neg\mathfrak{c})$	4 $\times$ *PC*
6	$\Box\Box(\Box\phi \to \phi) \to \Box m(\Box\mathfrak{c} \wedge \blacksquare\neg\mathfrak{c})$	5 $\times$ $\mathtt{RM}$
7	$\Box m(\Box\mathfrak{c} \wedge \blacksquare\neg\mathfrak{c})$	3,6 $\times$ $\mathtt{MP}$
8	$\blacksquare m(\Box\mathfrak{c} \wedge \blacksquare\neg\mathfrak{c})$	7 $\times$ Mirror Images

In the following systems, due to $\mathtt{RM}$, $\mathtt{N}$ and their mirror images, the following rule and its mirror image are derivable:

$\mathtt{RN}$ if $\vdash \phi$, then $\vdash \Box\phi$.

Thus, these systems can be also interpreted in standard relational models for multimodal logic.

Definition 5.14 (System $S_{\gamma 2}$). The axiomatic basis of system $S_{\gamma 2}$ is specified by the following ordered list of deductive principles: $\langle \mathtt{RM}, \mathtt{N}, \mathtt{K}, \mathtt{T}^*, \mathtt{4}, \mathring{\mathtt{D}}, \mathtt{BR1}\rangle$.

An $\mathcal{L}$-model for $S_{\gamma 2}$ needs to satisfy all properties of $\mathcal{L}$-models for $S_{\gamma 1}$, plus the following:

- $f(\top) \in h_i(w)$, for $i \in \{1, 2\}$.

Definition 5.15 (System $S_{\gamma 3}$). The axiomatic basis of system $S_{\gamma 3}$ is specified by the following ordered list of deductive principles: $\langle \mathtt{RM}, \mathtt{N}, \mathtt{K}, \mathtt{T}^*, \mathtt{4}, \mathtt{D}, \overset{\circ}{\mathtt{BR2}}, \mathtt{BR1}\rangle$.

An $\mathcal{L}$-model for $S_{\gamma 3}$ needs to satisfy all properties of $\mathcal{L}$-models for $S_{\gamma 2}$, plus the following:

- $f(\blacksquare\phi) \in h_1(w)$ iff $f(\phi) \in h_2(w)$;
- $f(\Box\phi) \in h_2(w)$ iff $f(\phi) \in h_1(w)$.

In system $S_{\gamma 3}$ the principle $\mathtt{BR3}$ is derivable, since $\mathtt{BR3} = \mathsf{mi}(\mathtt{BR2})$. Due to this fact and the other axioms available (in particular, the interaction among $\mathtt{K}$, $\mathtt{T}^*$, $\mathtt{4}$ and their mirror images, as illustrated in a derivation above to get $\Box\Box\phi \equiv \Box\phi$ and its mirror image), in $S_{\gamma 3}$ it is possible to reduce any finite and non-empty sequence m of operators $\Box$ and/or $\blacksquare$ to either a single occurrence of $\Box$ or a single occurrence of $\blacksquare$, and the schema $m(\phi) \rightarrow (\Box\phi \vee \blacksquare\phi)$ is derivable. However, in principle, one can formulate systems in which only one among BR2 and BR3 is derivable, in order to represent an asymmetry between "awful ideality" and "ideal awfulness".[11]

6. Representing Hohfeldian concepts

We conclude this work with a concise discussion of a way in which basic Hohfeldian concepts, such as duty and right, can be represented within an extension of our bimodal language. First, we spend a few words on these concepts. The meaning of the terms "right" and "duty" has been debated at length over the last century — at least since the foundational work by W. N. Hohfeld in [9] and [10]. Hohfeld showed that there are four fundamental concepts that can be expressed by using the term "right" in the legal context: claim-right, privilege, power and immunity. Furthermore, he argued that rights and duties are to be regarded as *correlatives*: saying that a normative party x has a duty towards a normative party y to bring about ϕ is the same as saying that y has a right against x that ϕ be brought about. According to Hohfeld, two normative parties play a central role in descriptions of rights and duties, one of which can be labelled as the *bearer* (of the right/duty) and the other can be labelled as the *counterpart* (of the right/duty).

Hohfeldian concepts involving two normative parties can be captured via a variation of our bimodal language for normative ideality/awfulness including parametric operators, along the lines of [7]. The new language will be called $\mathcal{L}_{\mathsf{Agt}}$. We take a set of agent-constants Agt and, for any $x, y \in \mathsf{Agt} \cup \{0\}$, there will be a pair of primitive modal operators $\Box[x, y]$ and $\blacksquare[x, y]$, in the place of the simple operators $\Box$ and $\blacksquare$. We will say that

---

[11] We stress that axiom D (as well as its mirror image) is independent from the rest of the axioms and rules for $S_{\gamma 3}$ (whence, from the rest of axioms and rules of any γ-system). Indeed, according to relational semantics for normal multimodal logic (see, e.g., [4]), RM, N, K, T*, 4, BR2 and their mirror images, together with BR1, are all valid in a a frame with a single world w that has no access to itself (whence, to any world), whereas D (and its mirror image) can only be valid in frames where accessibility is a serial relation: for all w there is v s.t. w has access to v.

x and y in $\Box[x, y]$ and $\blacksquare[x, y]$ are *parameters*. In normal systems, these two kinds of operators read as follows:

- $\Box[x, y]\phi$ means "in all normatively ideal situations x brings about ϕ for/against y";
- $\blacksquare[x, y]\phi$ means "in all normatively awful situations x brings about ϕ for/against y".

The constant symbol 0 is used to denote "no agent"; for instance, we read $\Box[x, 0]\phi$ as "in all normatively ideal situations x brings about ϕ" (e.g., in all normatively ideal situations, Peter pays his debts/sees to it that his debts are paid); $\Box[0, y]\phi$ as "in all normatively ideal situations ϕ is the case for/against y" (e.g., in all normatively ideal situations, Peter's human rights are secured) and $\Box[0, 0]\phi$ as "in all normatively ideal situations ϕ is the case" (e.g., in all normatively ideal situations, good deeds are rewarded).

Logical systems over the extended language can be supplemented with the following two bridge-schemata (for any $x, y \in \mathsf{Agt}$):

`BS1` $\Box[x, y]\phi \to (\Box[x, 0]\phi \land \Box[0, y]\phi \land \Box[0, 0]\phi)$;
`BS2` $\blacksquare[x, y]\phi \to (\blacksquare[x, 0]\phi \land \blacksquare[0, y]\phi \land \blacksquare[0, 0]\phi)$.

In normal systems, `BS1` reads as follows: If in all normatively ideal situations x brings about ϕ for/against y, then in all normatively ideal situations x brings about ϕ, in all normatively ideal situations ϕ is the case for/against y, and in all normatively ideal situations ϕ is the case. For example (and with a bit of simplification), if in all ideal situations Xavier (x), the gardener, plants roses (ϕ) for Yvonne (y) in her garden, then in all those situations he plants roses, she has the roses planted in her garden, and the roses are planted in her garden. `BS2` reads analogously, but is concerned with normatively awful situations. Thanks to the schemata `BS1` and `BS2`, if one considers systems where the formula $\Diamond[x, y]\top \to \Diamond[0, 0]\top$ and its mirror image are derivable, then the definition `Ob`$*$ allows one to get: $O[x, y]\phi \to O[0, 0]\phi$. However, it is generally not possible to derive the converse implication.

Future research in this direction may explore the possibility of expressing more refined distinctions within Hohfeldian concepts involving two or more parties, as well as other related concepts (see, e.g., [18]). Moreover, it may aim at further assessing the advantages and disadvantages of the project of reducing normative concepts to the notions of normative ideality and normative awfulness, by identifying general expressive limits of the proposed language.

Contributions

The contents of the article are the result of a joint research work of the two authors.

References

[1] A. Anderson, *The formal analysis of normative systems*, [in:] N. Rescher (ed.), **The Logic of Decision and Action**, University of Pittsburgh Press (1967), pp. 147–213.

[2] L. Åqvist, *Deontic logic*, [in:] **Handbook of Philosophical Logic**, vol. 8, Springer Netherlands (2002), pp. 147–264, DOI: https://doi.org/10.1007/978-94-010-0387-2_3.

[3] P. Cane, **Responsibility in Law and Morality**, Hart Publishing (2002).

[4] W. Carnielli, C. Pizzi, **Modalities and Multimodalities**, Springer (2008), DOI: https://doi.org/10.1007/978-1-4020-8590-1.

[5] M. de Boer, D. M. Gabbay, X. Parent, M. Slavkovic, *Two dimensional Standard Deontic Logic [including a detailed analysis of the 1985 Jones–Pörn deontic logic system]*, **Synthese**, vol. 187 (2012), pp. 623–660, DOI: https://doi.org/10.1007/s11229-010-9866-4.

[6] S. Hansson, *The varieties of permission*, [in:] D. Gabbay, J. Horty, X. Parent, R. van der Meyden, L. van der Torre (eds.), **Handbook of Deontic Logic and Normative Systems**, College Publications (2013), pp. 195–240.

[7] H. Herrestad, C. Krogh, *Obligations directed from bearers to counterparties*, [in:] **Proceedings of ICAIL 1995** (1995), pp. 210–218, DOI: https://doi.org/10.1145/222092.222243.

[8] R. Hilpinen, P. McNamara, *Deontic logic: a historical survey and introduction*, [in:] D. Gabbay, J. Horty, X. Parent, R. van der Meyden, L. van der Torre (eds.), **Handbook of Deontic Logic and Normative Systems**, College Publications (2013), pp. 3–136.

[9] W. N. Hohfeld, *Some fundamental legal conceptions as applied in legal reasoning*, **Yale Law Journal**, vol. 23 (1913), pp. 16–59.

[10] W. N. Hohfeld, *Fundamental legal conceptions as applied in judicial reasoning*, **Yale Law Journal**, vol. 26 (1917), pp. 710–770.

[11] A. Jones, I. Pörn, *Ideality, sub-ideality and deontic logic*, **Synthese**, vol. 65 (1985), pp. 275–290, DOI: https://doi.org/10.1007/BF00869304.

[12] J. Jørgensen, *Imperative and logic*, **Erkenntnis**, vol. 7 (1937–1938), pp. 288–296.

[13] T. Libal, M. Pascucci, *Automated reasoning in normative detachment structures with ideal conditions*, [in:] **Proceedings of ICAIL 2019** (2019), pp. 63–72, DOI: https://doi.org/10.1145/3322640.3326707.

[14] H. Prakken, M. Sergot, *Contrary to duty obligations*, **Studia Logica**, vol. 57 (1996), pp. 91–105, DOI: https://doi.org/10.1007/BF00370671.

[15] W. V. O. Quine, *On the nature of moral values*, [in:] A. I. Goldman, J. Kim (eds.), **Values and morals**, Springer, Dordrecht (1978), pp. 37–45, DOI: https://doi.org/10.1007/978-94-015-7634-5_3.

[16] J. Raz, R. J. Wallace, **The Practice of Value**, Oxford University Press (2005), DOI: https://doi.org/10.1093/acprof:oso/9780199278466.001.0001.

[17] M. S. Schwartz, *Universal moral values for corporate codes of ethics*, **Journal of Business Ethics**, vol. 59 (2005), pp. 27–44, DOI: https://doi.org/10.1007/s10551-005-3403-2.

[18] M. Sergot, *Normative positions*, [in:] D. Gabbay, J. Horty, X. Parent, R. van der Meyden, L. van der Torre (eds.), **Handbook of Deontic Logic and Normative Systems**, College Publications (2013), pp. 353–406.

[19] J. Sullins, *Information Technology and Moral Values*, [in:] E. N. Zalta (ed.), **The Stanford Encyclopedia of Philosophy**, spring 2021 ed., Metaphysics Research Lab, Stanford University (2021), https://plato.stanford.edu/archives/spr2021/entries/it-moral-values/.

Daniela Glavaničová

Comenius University in Bratislava
Faculty of Arts
Department of Logic and Methodology of Sciences
Bratislava, Slovak Republic

Slovak Academy of Sciences
Institute of Philosophy
Department of Analytic Philosophy
Bratislava, Slovak Republic

e-mail: daniela.glavanicova@gmail.com

Matteo Pascucci
Slovak Academy of Sciences
Institute of Philosophy
Department of Analytic Philosophy
Bratislava, Slovak Republic
e-mail: matteopascucci.academia@gmail.com

Bulletin of the **S**ection of **L**ogic
Volume 50/2 (2021), pp. 177–204
https://doi.org/10.18778/0138-0680.2021.06

Mateusz Klonowski
Krzysztof Aleksander Krawczyk
Bożena Pięta

TABLEAU SYSTEMS FOR EPISTEMIC POSITIONAL LOGICS

Abstract

The goal of the article is twofold. The first one is to provide logics based on positional semantics which will be suitable for the analysis of epistemic modalities such as 'agent . . . knows/beliefs that . . . '. The second one is to define tableau systems for such logics. Firstly, we present the minimal positional logic MR. Then, we change the notion of formulas and semantics in order to consider iterations of the operator of realization and "free" classical formulas. After that, we move on to weaker logics in order to avoid the well known problem of logical omniscience. At the same time, we keep the positional counterparts of modal axioms (T), (4) and (5). For all of the considered logics we present sound and complete tableau systems.

Keywords: Epistemic logic, logical omniscience, positional logic, tableau system.

1. Introduction

Sentences like 'It is raining' can realize in certain points. Those points can be treated:

- temporally as moments or certain intervals: 'It is raining now', 'It has been raining since Monday',

Presented by: Tomasz Jarmużek, Fengkui Ju, Piotr Kulicki, Beishui Liao
Received: February 5, 2020
Published online: April 1, 2021

- spatially as points or certain parts of space: 'It is raining in Toruń',
- epistemically as (ir)rational agents: 'John knows that it is raining'.

Clearly, the amount of possible interpretations is much richer including alethic, deontic, etc. The goal of introducing positional logics is to enable one the expression of such relativized sentences. The difference between positional and modal logic, which in a sense is also about such relativization, is that the first one introduces points in the object language while the latter treats them implicitly as only semantic entities that are talked about in the metalanguage. On the other hand, the difference between hybrid and positional logics is that the points (worlds) can be treated as independent expressions in hybrid logic, whilst—in the case of positional logics—they can only be used to form more complex formulas.

The origin of positional logic is mainly associated with the emergence of temporal logic. The founder of positional logic, and at the same time of temporal logic, was Jerzy Łoś. His aim was to provide a logical tool for formalizing empirical sentences such as 'it is sunny in Warsaw on 26th July 2019'. The expression 'at ... it is the case that ... ' which Łoś analyzed may be called the connective of temporal realization. Nonetheless, the temporal interpretation of realization is not the only possible one.

In [11] Łoś used the realization operator, i.e. the sentence-forming connective from naming and sentential arguments, to express epistemic modality, while in [10] the temporal interpretation of such operator is considered. The letter used by Łoś for the realization operator in his investigations was U, but due to Rescher [15] it shall be denoted as $\mathcal{R}$ (cf. [17], [16]). Generally, the formula $\mathcal{R}_\alpha A$ can be read in the following manner: A is realised/realizes in α. In temporal understanding such formula would be read as: A takes place/happens in moment α. In epistemic context $\mathcal{R}_\alpha A$ means: agent α knows that A.

The work of Łoś was continued by Jarmużek and Pietruszczak in [6] where the minimal system of positional logic MR was introduced (cf. [1]). Logic MR is the minimal logic among positional logics that are closed under the law of distribution of $\mathcal{R}$ over standard connectives. In [9] Karczewska proved that MR is the maximal logic with respect to so-called single-index rules. Weaker logics than MR, for which the problem of distributivity of $\mathcal{R}$ is discussed, were considered by Tkaczyk in [20], [18], [19]. In [3, pp. 209–224] the discussion of the applications of the $\mathcal{R}$ operator for the analysis of the Master Argument was presented. In [8] an attempt to reduce unary modalities to $\mathcal{R}$-structures was considered. In [7] Jarmużek and Tkaczyk collected the main results and introduced some new ideas concerning positional logic. In fact, they considered normal logics, i.e. logics such that their connectives have the same standard meaning inside and outside the scope of the $\mathcal{R}$ operator.

In [12] the possible application of positional logic for the analysis of the reasoning concerning the social phenomena was presented. In this case the $\mathcal{R}$ operator was modified by replacing the individual constant by a tuple of individual constants. In [5] one can find investigations concerning the extended version of positional logic's language obtained by adding the predicate symbols.

In this paper we start with the presentation of logic MR. After that, we define logic MR^+ in which we can consider the iterations of $\mathcal{R}$. Such a system is the basis for the epistemic interpretation of positional operator. We consider three systems of epistemic positional logic. The first one is the minimal one that contains the counterpart of (T). Then we consider its extensions that contain the counterparts of schemata (4) and (5) respectively. In none of the epistemic logics the logical omniscience problem appears. For all of the considered logics we define sound and complete tableau systems.

2. Logic MR

2.1. Language and semantics of logic MR

The language of MR consists of propositional variables $p_0, p_1, p_2, \ldots$ (we will use letters p, q, r); standard connectives $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$; the operator of realization $\mathcal{R}$; individual constants $a_1, a_2, a_3, \ldots$ (we will use letters a, b, c) and parentheses $), ($. Let VAR (resp. IC) be the set of propositional variables (resp. individual constants). By F we denote the set of formulas of Classical Propositional Logic (for short: CPL) defined in the standard way. The *set of* MR *formulas*, i.e. the set For, is the smallest set X meeting the following conditions:

- if $A \in \mathsf{F}$ than $\mathcal{R}_\alpha A \in X$, where $\alpha \in \mathsf{IC}$,
- if $A \in X$ than $\neg A \in X$,
- if $A, B \in X$ than $(A * B) \in X$, where $* \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$.

As we can see, in the language of MR there are no iterations of $\mathcal{R}$ operator and no "free" CPL formulas outside the scope of $\mathcal{R}$ operator.

By the *complexity of a formula* A we mean number $c(A)$, where $c\colon \mathsf{For} \longrightarrow \mathbb{N}$ is a function such that: $c(A) = 1$, if $A = \mathcal{R}_\alpha B$; $c(A) = c(B) + 1$, if $A = \neg B$; $c(A) = c(B) + c(C) + 1$, if $A = B * C$, where $* \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$. Note that the complexity of the $\mathcal{R}_\alpha A$ formula equals 1, regardless of the formula A. In the proofs we present below, we will also use induction on the complexity of CPL formulas defined similarly to complexity of MR formulas as $o(A)$, where $o\colon \mathsf{F} \longrightarrow \mathbb{N}$ is a function defined as c except instead of $A = \mathcal{R}_\alpha B$ we have $A \in \mathsf{VAR}$ and we put 1.

A *model of* MR (a MR-*model*) is a triple $\langle W, f, v\rangle$ such that:

- W is the non-empty set,
- $f\colon \mathsf{IC} \longrightarrow W$ is the denotation function,
- $v\colon W \times \mathsf{F} \longrightarrow \{0, 1\}$ is a valuation such that for any $w \in W$, for any $A, B \in \mathsf{F}$:

$$v(\langle w, \neg A\rangle) = 1 \quad \text{iff} \quad v(\langle w, A\rangle) = 0 \tag{v$_1$}$$
$$v(\langle w, A \wedge B\rangle) = 1 \quad \text{iff} \quad v(\langle w, A\rangle) = v(\langle w, B\rangle) = 1 \tag{v$_2$}$$
$$v(\langle w, A \vee B\rangle) = 1 \quad \text{iff} \quad v(\langle w, A\rangle) = 1 \text{ or } v(\langle w, B\rangle) = 1 \tag{v$_3$}$$
$$v(\langle w, A \rightarrow B\rangle) = 1 \quad \text{iff} \quad v(\langle w, A\rangle) = 0 \text{ or } v(\langle w, B\rangle) = 1 \tag{v$_4$}$$
$$v(\langle w, A \leftrightarrow B\rangle) = 1 \quad \text{iff} \quad v(\langle w, A\rangle) = v(\langle w, B\rangle). \tag{v$_5$}$$

We have the following truth-conditions for any $A \in \mathsf{F}$ and any $B, C \in \mathsf{For}$:

$$\mathfrak{M} \vDash \mathcal{R}_\alpha A \quad \text{iff} \quad v(\langle f(\alpha), A\rangle) = 1 \tag{m$_1$}$$
$$\mathfrak{M} \vDash \neg B \quad \text{iff} \quad \mathfrak{M} \nvDash B \tag{m$_2$}$$
$$\mathfrak{M} \vDash B \wedge C \quad \text{iff} \quad \mathfrak{M} \vDash B \text{ and } \mathfrak{M} \vDash C \tag{m$_3$}$$
$$\mathfrak{M} \vDash B \vee C \quad \text{iff} \quad \mathfrak{M} \vDash B \text{ or } \mathfrak{M} \vDash C \tag{m$_4$}$$
$$\mathfrak{M} \vDash B \rightarrow C \quad \text{iff} \quad \mathfrak{M} \nvDash B \text{ or } \mathfrak{M} \vDash C \tag{m$_5$}$$
$$\mathfrak{M} \vDash B \leftrightarrow C \quad \text{iff} \quad \mathfrak{M} \vDash B \text{ iff } \mathfrak{M} \vDash C. \tag{m$_6$}$$

The notions of the relation of semantic consequence $\vDash_{\mathsf{MR}}$ and the validity in MR are defined in the standard way. Logic MR might be identified with the relation $\vDash_{\mathsf{MR}}$. In the subsequent sections we will similarly define and denote other relations of semantic consequence and identify certain logics with them.

For any $A, B \in \mathsf{For}$:

$$\mathrm{R}_{\wedge}: \frac{A \wedge B}{A, B} \qquad \mathrm{R}_{\vee}: \frac{A \vee B}{A|B} \qquad \mathrm{R}_{\rightarrow}: \frac{A \rightarrow B}{\neg A|B} \qquad \mathrm{R}_{\leftrightarrow}: \frac{A \leftrightarrow B}{A, B|\neg A, \neg B}$$

$$\mathrm{R}_{\neg\neg}: \frac{\neg\neg A}{A} \qquad \mathrm{R}_{\neg\wedge}: \frac{\neg(A \wedge B)}{\neg A|\neg B} \qquad \mathrm{R}_{\neg\vee}: \frac{\neg(A \vee B)}{\neg A, \neg B}$$

$$\mathrm{R}_{\neg\rightarrow}: \frac{\neg(A \rightarrow B)}{A, \neg B} \qquad \mathrm{R}_{\neg\leftrightarrow}: \frac{\neg(A \leftrightarrow B)}{A, \neg B|\neg A, B}$$

Figure 1. Elimination rules for standard connectives outside the scope of $\mathcal{R}$ operator

Let us notice that by [6, p. 150, p. 155] we have that, for any $A \in \mathsf{F}$, for any $\alpha \in \mathsf{IC}$:

$$\text{if } \vDash_{\mathsf{CPL}} A \text{ then } \vDash_{\mathsf{MR}} \mathcal{R}_\alpha A. \tag{$\dagger$}$$

2.2. Tableau system for logic MR

In this and subsequent sections, in our analysis of tableau systems, we will adopt an approach described in [4] for relating logics that is based on the metatheoretical approach to tableau presented in [2] originally developed for modal logics. Let us start with a general definition of a t-inconsistent (tableau inconsistent) set of formulas. Let X be a set of formulas:

- X is *t-inconsistent* iff there is a formula A, such that $A, \neg A \in X$,
- X is *t-consistent* iff it is not t-inconsistent.

Let us present the tableau rules for logic MR. We are going to follow the index-free approach presented in [7, pp. 128–131]. Firstly, we assume classical rules for connectives outside the scope of $\mathcal{R}$ operator (see Figure 1). These are standard elimination rules for boolean connectives. Secondly, we have specific elimination rules for connectives within the range of $\mathcal{R}$ which are based on a kind of distribution of $\mathcal{R}$ operator over other connectives (see Figure 2).

The set of all tableau rules for logic MR will be denoted as $\mathbf{R}$. For any rules from $\mathbf{R}$ formulas in numerator will be called input, while formulas from denominator will be called output. Let us take as an example the rule $\mathrm{R}_{\mathcal{R}\wedge}$. The input of $\mathrm{R}_{\mathcal{R}\wedge}$ is $\{\mathcal{R}_\alpha(A \wedge B)\}$ and the output is set $\{\mathcal{R}_\alpha A, \mathcal{R}_\alpha B\}$. Notice that this rule is a non-branching one, i.e. it has only one output (one set of formulas). On the other hand, $\mathrm{R}_{\neg\mathcal{R}\wedge}$ is a branching rule which means that we have two outputs: $\{\mathcal{R}_\alpha \neg A\}$ and $\{\mathcal{R}_\alpha \neg B\}$. Once we have a notion of input we can define the notion of applicability of a rule. Let $\mathrm{R} \in \mathbf{R}$ and $X \subseteq \mathsf{For}$. R is *applicable to* X iff for any A from the input of the R, $A \in X$.

We define the relation of tableau consequence by referring to the concept of closure under tableau rules, similarly as in [4]. Our general definition enables one to define a notion of tableau consequence for MR but also for logics considered in subsequent sections. Let $\mathbf{Q}$ be a set of tableau rules and X, Y be sets of formulas. X is a *closure of* Y *under tableau rules from* $\mathbf{Q}$ (for short: $\mathbf{Q}$-*closure of* Y) iff there exists such a subset of natural numbers K that:

- $K = \mathbb{N}$ or $K = \{1, \ldots, n\}$ for some $n \in \mathbb{N}$,
- there exists such an injective string $f \colon K \longrightarrow \{\mathsf{Z} : \mathsf{Z}$ is a subset of formulas$\}$ that:
  - $\mathsf{Z}_1 = Y$,
  - for all $i, i+1 \in K$ there exists such a tableau rule $\mathrm{R} \in \mathbf{Q}$ that its input is included in Z_i, while one of its outputs is equal to $\mathsf{Z}_{i+1} \setminus \mathsf{Z}_i$,
  - for all $i, i+1 \in K$ for any tableau rule $\mathrm{R} \in \mathbf{Q}$ if the input of R is included in Z_i and one of outputs of R is equal to $\mathsf{Z}_{i+1} \setminus \mathsf{Z}_i$, then for no j such that $i < j, j+1 \in K$ one of the remaining outputs of R is equal to $\mathsf{Z}_{j+1} \setminus \mathsf{Z}_j$,
  - for any tableau rule $\mathrm{R} \in \mathbf{Q}$ if the input of R is included in $\bigcup_{i \in K} \mathsf{Z}_i$, then one of outputs of R is in $\bigcup_{i \in K} \mathsf{Z}_i$,
- $X = \bigcup_{i \in K} \mathsf{Z}_i$.

Clearly, a set X is closed under applications of rules from $\mathbf{Q}$ (for short: $\mathbf{Q}$-closed) if X is a $\mathbf{Q}$-closure of some set Y. In practice, we can treat the closure in the presented sense as the so-called complete branch. In fact, it is a union of all elements that are on a complete branch.

For any $A \in \mathsf{F}$:

$$\mathrm{R}_{\mathcal{R}\neg}\colon \frac{\mathcal{R}_\alpha \neg A}{\neg \mathcal{R}_\alpha A} \qquad \mathrm{R}_{\mathcal{R}\wedge}\colon \frac{\mathcal{R}_\alpha (A \wedge B)}{\mathcal{R}_\alpha A, \mathcal{R}_\alpha B} \qquad \mathrm{R}_{\mathcal{R}\vee}\colon \frac{\mathcal{R}_\alpha (A \vee B)}{\mathcal{R}_\alpha A | \mathcal{R}_\alpha B}$$

$$\mathrm{R}_{\mathcal{R}\rightarrow}\colon \frac{\mathcal{R}_\alpha (A \rightarrow B)}{\mathcal{R}_\alpha \neg A | \mathcal{R}_\alpha B} \qquad \mathrm{R}_{\mathcal{R}\leftrightarrow}\colon \frac{\mathcal{R}_\alpha (A \leftrightarrow B)}{\mathcal{R}_\alpha A, \mathcal{R}_\alpha B | \mathcal{R}_\alpha \neg A, \mathcal{R}_\alpha \neg B}$$

$$\mathrm{R}_{\neg\mathcal{R}\neg}\colon \frac{\neg \mathcal{R}_\alpha \neg A}{\neg\neg \mathcal{R}_\alpha A} \qquad \mathrm{R}_{\neg\mathcal{R}\wedge}\colon \frac{\neg \mathcal{R}_\alpha (A \wedge B)}{\mathcal{R}_\alpha \neg A | \mathcal{R}_\alpha \neg B} \qquad \mathrm{R}_{\neg\mathcal{R}\vee}\colon \frac{\neg \mathcal{R}_\alpha (A \vee B)}{\mathcal{R}_\alpha \neg A, \mathcal{R}_\alpha \neg B}$$

$$\mathrm{R}_{\neg\mathcal{R}\rightarrow}\colon \frac{\neg \mathcal{R}_\alpha (A \rightarrow B)}{\mathcal{R}_\alpha A, \mathcal{R}_\alpha \neg B} \qquad \mathrm{R}_{\neg\mathcal{R}\leftrightarrow}\colon \frac{\neg \mathcal{R}_\alpha (A \leftrightarrow B)}{\mathcal{R}_\alpha A, \mathcal{R}_\alpha \neg B | \mathcal{R}_\alpha \neg A, \mathcal{R}_\alpha B}$$

Figure 2. Elimination rules for standard connectives inside the scope of $\mathcal{R}$ operator

A tableau consequence relation in logic MR is defined with respect to **R**-closed sets. A formula A is a *tableau consequence of* X *in* MR (in symb.: $X \rhd_{\mathsf{MR}} A$) iff there is a finite set $Y \subseteq X$ such that any **R**-closure of $Y \cup \{\neg A\}$ is t-inconsistent. And A is a *thesis in* MR (in symb.: $\rhd_{\mathsf{MR}} A$) iff $\emptyset \rhd_{\mathsf{MR}} A$.

2.3. Soundness and completeness of tableau system for MR

In order to prove the soundness and completeness of system MR and other system considered in the subsequent sections, we need to introduce some additional notions. Let $\mathfrak{M}$ be a model and X the set of formulas. We say that $\mathfrak{M}$ is *suitable for* X iff for any formula A, if $A \in X$ then $\mathfrak{M} \models A$.

The following lemma shows that by the applications of the rules from **R** from satisfiable formulas we receive some satisfiable formulas.

LEMMA 2.1. *Let* $X \subseteq \mathsf{For}$ *and* $\mathfrak{M} = \langle W, f, v\rangle$ *be a* MR*-model suitable for* X*. If any rule from* **R** *has been applied to* X*, then* $\mathfrak{M}$ *is suitable for the union of* X *and at least one output obtained by application of that rule.*

PROOF: For the cases of applications of the elimination rules for standard connectives outside the scope of $\mathcal{R}$ operator, i.e. rules R_*, $\mathrm{R}_{\neg *}$, where $*$ is a propositional connective, the proof is standard (cf. [14]).

Suppose $\mathrm{R}_{\mathcal{R}\neg}$ has been applied to X. Then $\mathcal{R}_\alpha\neg A \in X$. Since model $\mathfrak{M}$ is suitable for X, then $\mathfrak{M} \vDash \mathcal{R}_\alpha\neg A$. Thus, by the truth-condition (m_1), $v(\langle f(\alpha), \neg A\rangle) = 1$. Hence, by the condition (v_1), $v(\langle f(\alpha), A\rangle) \neq 1$. Thus, by the truth-conditions (m_1) and (m_2), $\mathfrak{M} \vDash \neg\mathcal{R}_\alpha A$.

Suppose $\mathrm{R}_{\neg\mathcal{R}\neg}$ has been applied to X. Then $\neg\mathcal{R}_\alpha\neg A \in X$. Since model $\mathfrak{M}$ is suitable for X, then $\mathfrak{M} \nvDash \mathcal{R}_\alpha\neg A$. Thus, by the truth-condition (m_1), $v(\langle f(\alpha), \neg A\rangle) = 0$. Hence, by the condition (v_1), $v(\langle f(\alpha), A\rangle) = 1$. Thus, by the truth-conditions (m_1) and (m_2), $\mathfrak{M} \vDash \neg\neg\mathcal{R}_\alpha A$.

Suppose $\mathrm{R}_{\mathcal{R}\wedge}$ has been applied to X. Then $\mathcal{R}_\alpha(A \wedge B) \in X$. Since model $\mathfrak{M}$ is suitable for X, then $\mathfrak{M} \vDash \mathcal{R}_\alpha(A \wedge B)$. Thus, by the truth-condition (m_3), $v(\langle f(\alpha), A \wedge B\rangle) = 1$. Hence, by the condition (v_2), $v(\langle f(\alpha), A\rangle) = v(\langle f(\alpha), B\rangle) = 1$. Thus, by the truth-conditions (m_1), $\mathfrak{M} \vDash \mathcal{R}_\alpha A$ and $\mathfrak{M} \vDash \mathcal{R}_\alpha B$.

Suppose $\mathrm{R}_{\neg\mathcal{R}\wedge}$ has been applied to X. Then $\mathcal{R}_\alpha\neg(A \wedge B) \in X$. Since model $\mathfrak{M}$ is suitable for X, then $\mathfrak{M} \vDash \mathcal{R}_\alpha\neg(A \wedge B)$. Thus, by the truth-condition (m_1), $v(\langle f(\alpha), \neg(A \wedge B)\rangle) = 1$. Hence, by the conditions (v_1) and (v_2), either $v(\langle f(\alpha), \neg A\rangle) = 1$ or $v(\langle f(\alpha), \neg B\rangle) = 1$. Thus, by the truth-condition (m_1), either $\mathfrak{M} \vDash \mathcal{R}_\alpha\neg A$ or $\mathfrak{M} \vDash \mathcal{R}_\alpha\neg B$.

For the remaining cases, we reason in the similar way. □

Let us now introduce the notion of a model generated by a t-consistent **R**-closed set. Let X be the t-consistent **R**-closed set and $\mathsf{IC}_X := \{\alpha \in \mathsf{IC} : \mathcal{R}_\alpha A \in X\}$. A MR-*model generated by* X (for short: MR-X-*model*) is a MR-model $\langle W, f, v\rangle$ such that:

- $W = \mathsf{IC}_X$,
- for any $\alpha \in \mathsf{IC}$ we put:

$$f(\alpha) = \begin{cases} \alpha, & \text{if } \alpha \in \mathsf{IC}_X \\ a_{\min\{n \in \mathbb{N} \,:\, a_n \in \mathsf{IC}_X\}}, & \text{if } \alpha \notin \mathsf{IC}_X \end{cases}$$

- for any $\alpha \in W$ and any $A \in X \cap \mathsf{VAR}$ we put:

$$v(\langle \alpha, A\rangle) = \begin{cases} 1, & \text{if } \mathcal{R}_\alpha A \in X \\ 0, & \text{if } \mathcal{R}_\alpha A \notin X \end{cases}$$

  we extend v on $W \times \mathsf{F}$ by means of conditions (v_1)–(v_5).

We have the following fact:

FACT 2.2. Let $X \subseteq \mathsf{For}$ be the t-consistent $\mathbf{R}$-closed set, $\mathfrak{M} = \langle W, f, v\rangle$ be a MR-X-model and $\alpha \in \mathsf{IC}$. Then, for any $A \in \mathsf{F}$:

- if $\mathcal{R}_\alpha A \in X$ then $v(\langle\alpha, A\rangle) = 1$,
- if $\neg\mathcal{R}_\alpha A \in X$ then $v(\langle\alpha, A\rangle) = 0$.

PROOF: *Base case.* We obtain the result by the definition of a MR-X-model and since X is t-consistent.

Inductive hypothesis. Let $n \in \mathbb{N}$. Suppose that for any $A \in \mathsf{F}$ such that $o(A) \leq n$:

- if $\mathcal{R}_\alpha A \in X$ then $v(\langle\alpha, A\rangle) = 1$,
- if $\neg\mathcal{R}_\alpha A \in X$ then $v(\langle\alpha, A\rangle) = 0$.

Inductive step. Let $A \in \mathsf{F}$ and $o(A) = n + 1$.

Let $A = \neg B$. Suppose $\mathcal{R}_\alpha\neg B \in X$. Since X is a $\mathbf{R}$-closed set, by the application of the rule $\mathrm{R}_{\mathcal{R}\neg}$ $\neg\mathcal{R}_\alpha B \in X$. By the inductive hypothesis $v(\langle\alpha, B\rangle) = 0$. Thus, by the condition (v_1), $v(\langle\alpha, \neg B\rangle) = 1$. Suppose $\neg\mathcal{R}_\alpha\neg B \in X$. Since X is a $\mathbf{R}$-closed set, by the application of the rule $\mathrm{R}_{\neg\mathcal{R}\neg}$ $\neg\neg\mathcal{R}_\alpha B \in X$. Hence, by the application of the rule $\mathrm{R}_{\neg\neg}$, $\mathcal{R}_\alpha B \in X$. By the inductive hypothesis $v(\langle\alpha, B\rangle) = 1$. Thus, by the condition (v_1), $v(\langle\alpha, \neg B\rangle) = 0$.

Let $A = B * C$, where $* \in \{\wedge, \vee, \to, \leftrightarrow\}$. Suppose $* = \wedge$, for other cases we reason in the similar way. Let us assume that $\mathcal{R}_\alpha(B \wedge C) \in X$. Since X is an $\mathbf{R}$-closed set, by the application of the rule $\mathrm{R}_{\mathcal{R}\wedge}$ $\mathcal{R}_\alpha B, \mathcal{R}_\alpha C \in X$. Hence, by the inductive hypothesis, $v(\langle\alpha, B\rangle) = 1$ and $v(\langle\alpha, C\rangle) = 1$. Thus, by the condition (v_2), $v(\langle\alpha, B \wedge C\rangle) = 1$. Suppose $\neg\mathcal{R}_\alpha(B \wedge C) \in X$. Since X is a $\mathbf{R}$-closed set, by the application of the rule $\mathrm{R}_{\neg\mathcal{R}\wedge}$ either $\mathcal{R}_\alpha\neg B$, or $\mathcal{R}_\alpha\neg C \in X$. Thus, by the application of the rule $\mathrm{R}_{\mathcal{R}\neg}$ either $\neg\mathcal{R}_\alpha B \in X$ or $\neg\mathcal{R}_\alpha C \in X$. Hence, by the inductive hypothesis, either $v(\langle\alpha, B\rangle) = 0$ or $v(\langle\alpha, C\rangle) = 0$. Therefore, by the condition (v_2), $v(\langle\alpha, B \wedge C\rangle) = 0$.

For the remaining cases we reason in the similar way. $\square$

By means of fact 2.2 we can prove the following lemma:

LEMMA 2.3. *Let* $X \subseteq \mathsf{For}$ *be a t-consistent* $\mathbf{R}$*-closed set and* $\mathfrak{M} = \langle W, f, v\rangle$ *be an* MR-X*-model. Then, for any* $A \in \mathsf{For}$*:*

- *if* $A \in X$ *then* $\mathfrak{M} \vDash A$*,*
- *if* $\neg A \in X$ *then* $\mathfrak{M} \nvDash A$*.*

PROOF: *Base case.* Let $A \in \mathsf{For}$ and $c(A) = 1$. Thus $A = \mathcal{R}_\alpha B$, where $B \in \mathsf{F}$. Suppose $\mathcal{R}_\alpha B \in X$. Then, by fact 2.2 (1), $v(\langle\alpha, B\rangle) = 1$. Therefore, by the truth-condition (m_1), $\mathfrak{M} \vDash \mathcal{R}_\alpha B$. Suppose $\neg\mathcal{R}_\alpha B \in X$. Hence, by fact 2.2 (2), $v(\langle\alpha, B\rangle) = 0$. Thus, by the truth-conditions (m_1), $\mathfrak{M} \nvDash \mathcal{R}_\alpha B$.

Inductive hypothesis. Let $n \in \mathbb{N}$. Suppose that for any $A \in \mathsf{For}$ such that $c(A) \leq n$:

- if $A \in X$ then $\mathfrak{M} \vDash A$,
- if $\neg A \in X$ then $\mathfrak{M} \nvDash A$.

Inductive step. Let $A \in \mathsf{For}$ and $c(A) = n + 1$.

Let $A = \neg\neg B$. Suppose $\neg\neg B \in X$. Hence, by the application of the rule $\mathrm{R}_{\neg\neg}$, $B \in X$. Thus, by the inductive hypothesis, $\mathfrak{M} \vDash B$. Therefore, by the truth-condition (m_2), $\mathfrak{M} \vDash \neg\neg B$.

Let $A = B * C$, where $* \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$. Suppose $* = \wedge$, for other cases we reason in the similar way. Let us assume that $B \wedge C \in X$. Since X is a $\mathbf{R}$-closed set, by the application of the rule $\mathrm{R}_\wedge$ $B, C \in X$. Hence, by the inductive hypothesis, $\mathfrak{M} \vDash B$ and $\mathfrak{M} \vDash C$. Therefore, by the truth-condition (m_3), $\mathfrak{M} \vDash B \wedge C$.

Let $A = \neg(B * C)$, where $* \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$. Suppose $* = \wedge$, for the other cases we reason in a similar way. Let us assume that $\neg(B \wedge C) \in X$. Hence, by the application of the rule $\mathrm{R}_{\neg\wedge}$, either $\neg B \in X$ or $\neg C \in X$. Thus, by the inductive hypothesis, either $\mathfrak{M} \vDash \neg B$ or $\mathfrak{M} \vDash \neg C$. Therefore, by the truth-conditions (m_2) and (m_3), $\mathfrak{M} \vDash \neg(B \wedge C)$. □

Having proven the introduced facts, we can easily receive the soundness and completeness of our tableau system.

THEOREM 2.4. *Let* $X \cup \{A\} \subseteq \mathsf{For}$*. Then,* $X \rhd_{\mathsf{MR}} A$ *iff* $X \vDash_{\mathsf{MR}} A$*.*

PROOF: Suppose there is finite $Y \subseteq X$ such that any closure of $Y \cup \{\neg A\}$ on rules from $\mathbf{R}$ is t-inconsistent. Let us assume that there is a MR-model

$\mathfrak{M}$ such that $\mathfrak{M} \models X \cup \{\neg A\}$. Hence $\mathfrak{M}$ is suitable to $X \cup \{\neg A\}$, so also to $Y \cup \{\neg A\}$. By lemma 2.1 there is an **R**-closure of $X \cup \{\neg A\}$ to which $\mathfrak{M}$ is suitable. But such closure is t-inconsistent. Hence, there is $A \in \mathsf{For}$ such that $\mathfrak{M} \models A$ and $\mathfrak{M} \not\models A$. Therefore, for any MR-model $\mathfrak{M}$, if $\mathfrak{M} \models X$ then $\mathfrak{M} \models A$, and so $X \models_{\mathsf{MR}} A$.

Suppose $X \models_{\mathsf{MR}} A$. Let us assume that for any finite $Y \subseteq X$ there is t-consistent **R**-closure of $Y \cup \{\neg A\}$. Hence, there is a t-consistent **R**-closure Z such that $X \cup \{\neg A\} \subseteq Z$. Otherwise, any of such a closure would consist some t-inconsistency. But by the definition of a **R**-closure of a set, this would mean that for some finite $Y \subseteq X$ no **R**-closure of $Y \cup \{\neg A\}$ is t-consistent. As a consequence, by lemma 2.3, $\mathfrak{M} \models X \cup \{\neg A\}$, where $\mathfrak{M}$ is a MR-Z-model. Therefore $X \not\models_{\mathsf{MR}} A$. □

3. Logic MR^+

As we noticed, in the language of MR there are no "free" CPL formulas outside the scope of the $\mathcal{R}$ operator. Whereas on the ground of epistemic logic, it is important to be able to refer both to sentences stating that a given agent knows a given thing and to sentences simply expressing states of affairs not propositional attitudes. Furthermore, in the language of MR there are no iterations of the $\mathcal{R}$ operator. But iterations matter in epistemic contexts, especially if we want to consider so-called positive and negative introspection. For this reason, we introduce a modification of the language and semantics of MR.

3.1. Language and semantics of logic MR^+

The language of MR^+ is an extension of the language of MR. The *set of* MR^+ *formulas*, i.e. the set For^+, is defined in the usual way as the smallest set X meeting the following conditions:

- $\mathsf{VAR} \subseteq X$,
- if $A \in X$ than $\mathcal{R}_\alpha A \in X$, where $\alpha \in \mathsf{IC}$,
- if $A \in X$ than $\neg A \in X$,
- if $A, B \in X$ than $(A * B) \in X$, where $* \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$.

Obviously $\mathsf{F}, \mathsf{For} \subset \mathsf{For}^+$.

Let us modify the notion of the complexity of a formula. We define function $c^+ \colon \mathsf{For}^+ \longrightarrow \mathbb{N}$ in the standard way, i.e.: $c^+(A) = 1$, if $A \in \mathsf{VAR}$; $c^+(A) = c^+(B) + 1$, if $A = \neg B$ or $A = \mathcal{R}_\alpha B$; $c^+(A) = c^+(B) + c^+(C) + 1$, if $A = B * C$, where $* \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$.

In this section we also employ the function assigning to a formula its subformulas, i.e. a function $s \colon \mathsf{For}^+ \longrightarrow \mathcal{P}(\mathsf{For}^+)$ such that: $s(A) = \{A\}$, if $A \in \mathsf{VAR}$; $s(A) = \{A\} \cup s(B)$, if $A = \neg B$ or $A = \mathcal{R}_\alpha B$; $s(A) = \{B * C\} \cup s(B) \cup s(C)$, if $A = B * C$, where $* \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$. Let $s(X) := \{s(A) : A \in X\}$.

Let W be a non-empty set. By $\overrightarrow{W}$ we denote the set of all finite strings of elements from W. We have $(w_1, \ldots, w_n) \in \overrightarrow{W}$ iff $n \in \mathbb{N}$ and $w_i \in W$, for any i such that $1 \leqslant i \leqslant n$. If a string has one element w we write w instead of (w). A *model of* MR^+ (a MR^+-*model*) is an ordered triple $\langle W, f, v\rangle$ such that:

- W, f are the same as in the case of MR-model,
- $v \colon (\overrightarrow{W} \times \mathsf{For}^+) \cup \mathsf{F} \longrightarrow \{0, 1\}$ is such that:
  - $v{\restriction}_{\overrightarrow{W} \times \mathsf{For}^+}$ is a valuation such that for any $\overrightarrow{w} = (w_1, \ldots, w_n) \in \overrightarrow{W}$ and any $A, B \in \mathsf{For}^+$:

$$v(\langle \overrightarrow{w}, \neg A\rangle) = 1 \text{ iff } v(\langle \overrightarrow{w}, A\rangle) = 0 \qquad (\mathrm{v}_1^+)$$

$$v(\langle \overrightarrow{w}, A \wedge B\rangle) = 1 \text{ iff } v(\langle \overrightarrow{w}, A\rangle) = v(\langle \overrightarrow{w}, B\rangle) = 1 \qquad (\mathrm{v}_2^+)$$

$$v(\langle \overrightarrow{w}, A \vee B\rangle) = 1 \text{ iff } v(\langle \overrightarrow{w}, A\rangle) = 1 \text{ or } v(\langle \overrightarrow{w}, B\rangle) = 1 \qquad (\mathrm{v}_3^+)$$

$$v(\langle \overrightarrow{w}, A \rightarrow B\rangle) = 1 \text{ iff } v(\langle \overrightarrow{w}, A\rangle) = 0 \text{ or } v(\langle w, B\rangle) = 1 \qquad (\mathrm{v}_4^+)$$

$$v(\langle \overrightarrow{w}, A \leftrightarrow B\rangle) = 1 \text{ iff } v(\langle \overrightarrow{w}, A\rangle) = v(\langle \overrightarrow{w}, B\rangle) \qquad (\mathrm{v}_5^+)$$

$$v(\langle (w_1, \ldots, w_n),\ \mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_m} A\rangle) = 1 \text{ iff } v(\langle (w_1, \ldots, w_n, f(\alpha_1), \ldots, f(\alpha_m)), A\rangle) = 1 \qquad (\mathrm{v}_6^+)$$

  - $v{\restriction}_{\mathsf{F}}$ is the classical CPL valuation.

The truth-conditions (m_1)–(m_6) are now determined for formulas from For^+. Notice that in (m_1) we now have a one element string $(f(\alpha))$ not a point $f(\alpha)$. Moreover, we add the following truth-condition, for any $A \in \mathsf{VAR}$:

$$\mathfrak{M} \models A \quad \text{iff} \quad v(A) = 1. \qquad (\mathrm{m}_7)$$

Thus we get that for any $A \in \mathsf{F}$, $\mathfrak{M} \vDash A$ iff $v(A) = 1$. And so CPL is a proper sublogic of MR^+, i.e. $\vDash_{\mathsf{CPL}} \subset \vDash_{\mathsf{MR}^+}$. Let us also state that MR must be the proper sublogic of MR^+.

Let us notice that for MR^+ we have the counterpart of the property (†). For any $\alpha \in \mathsf{IC}$, for any $A \in \mathsf{For}^+$:

$$\text{if } \vDash_{\mathsf{MR}^+} A \text{ then } \vDash_{\mathsf{MR}^+} \mathcal{R}_\alpha A. \tag{‡}$$

In order to prove that we define a notion of α-model.

Let $\mathfrak{M} = \langle W, f, v\rangle$ be a MR^+- model and $\alpha \in \mathsf{IC}$. An *α-model received from* $\mathfrak{M}$ (for short: *α-model*) is a MR^+-model $\mathfrak{N} = \langle W, f, u\rangle$ where $u\colon (\overrightarrow{W} \times \mathsf{For}^+) \cup \mathsf{F} \longrightarrow \{1, 0\}$ is such that, for any $A \in \mathsf{VAR}$, for any $(w_1, \ldots, w_n) \in \overrightarrow{W}$ we put:

$$u(\langle(w_1, \ldots, w_n), A\rangle) = \begin{cases} 1, & \text{if } v(\langle(f(\alpha), w_1, \ldots, w_n), A\rangle) = 1 \\ 0, & \text{if } v(\langle(f(\alpha), w_1, \ldots, w_n), A\rangle) = 0 \end{cases}$$

$$u(A) = \begin{cases} 1, & \text{if } v(\langle f(\alpha), A\rangle) = 1 \\ 0, & \text{if } v(\langle f(\alpha), A\rangle) = 0 \end{cases}$$

we extend u on $(\overrightarrow{W} \times \mathsf{For}^+) \cup \mathsf{F}$ by means of standard conditions for CPL formulas and conditions (v_1^+)–(v_6^+).

We have the following fact:

Fact 3.1. Let $\mathfrak{M} = \langle W, f, v\rangle$ be a MR^+- model, $\alpha \in \mathsf{IC}$ and $\mathfrak{N} = \langle W, f, u\rangle$ be an α-model received from $\mathfrak{M}$. Then, for any $A \in \mathsf{For}^+$, for any $(w_1, \ldots, w_n) \in \overrightarrow{W}$, $v((f(\alpha), w_1, \ldots, w_n), A) = 1$ iff $u((w_1, \ldots, w_n), A) = 1$.

Proof: *Base case.* By the definition of an α-model.

Inductive hypothesis. Let $m \in \mathbb{N}$. Suppose that for any $A \in \mathsf{For}^+$ such that $c^+(A) \leq m$, for any $(w_1, \ldots, w_n) \in \overrightarrow{W}$, $v((f(\alpha), w_1, \ldots, w_n), A) = 1$ iff $w((w_1, \ldots, w_n), A) = 1$.

Inductive step. Let $A \in \mathsf{For}^+$ and $c^+(A) = m + 1$.

Let $A = \neg B$. Then: $v((f(\alpha), w_1, \ldots, w_n), \neg B) = 1$, by the condition ($\mathrm{v}_1^+$), iff $v((f(\alpha), w_1, \ldots, w_n), B) = 0$, by the inductive hypothesis, iff $u((w_1, \ldots, w_n), B) = 0$, by the condition ($\mathrm{v}_1^+$), iff $u((w_1, \ldots, w_n), \neg B) = 1$.

Let $A = B * C$, where $* \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$. We consider only case for $* = \wedge$. For other cases we reason in the similar way. We have: $v((f(\alpha), w_1, \ldots, w_n), B \wedge C) = 1$, by the condition ($\mathrm{v}_2^+$), iff $v((f(\alpha), w_1, \ldots,$

$w_n), B) = v((f(\alpha), w_1, \ldots, w_n), C) = 1$, by the inductive hypothesis, iff $u((w_1, \ldots, w_n), B) = u((w_1, \ldots, w_n), C) = 1$, by the condition ($\mathrm{v}_1^+$), iff $u((w_1, \ldots, w_n), B \wedge C) = 1$.

Let $A = \mathcal{R}_\beta B$. Then: $v((f(\alpha), w_1, \ldots, w_n), \mathcal{R}_\beta B) = 1$, by the condition ($\mathrm{v}_6^+$), iff $v((f(\alpha), w_1, \ldots, w_n, f(\beta)), B) = 1$, by the inductive hypothesis, iff $u((w_1, \ldots, w_n, f(\beta)), B) = 1$, by the condition ($\mathrm{v}_6^+$), iff $u((w_1, \ldots, w_n), \mathcal{R}_\beta B) = 1$. $\square$

By fact 3.1 we receive the following corollary:

Fact 3.2. Let $\mathfrak{M} = \langle W, f, v\rangle$ be a MR^+- model, $\alpha \in \mathsf{IC}$ and $\mathfrak{N} = \langle W, f, w\rangle$ be an α-model. Then, for any $A \in \mathsf{For}^+$, $\mathfrak{M} \vDash \mathcal{R}_\alpha A$ iff $\mathfrak{N} \vDash A$.

Proof: *Base case.* By the definition of an α-model.

Inductive hypothesis. Let $n \in \mathbb{N}$. Suppose that for any $A \in \mathsf{For}^+$ such that $c^+(A) \leq n$, $\mathfrak{M} \vDash \mathcal{R}_\alpha A$ iff $\mathfrak{N} \vDash A$.

Inductive step. Let $A \in \mathsf{For}^+$ and $c^+(A) = m + 1$.

Let $A = \neg B$. Then: $\mathfrak{M} \vDash \mathcal{R}_\alpha \neg B$, by the truth-condition (m_1), iff $v((f(\alpha), \neg B) = 1$, by the condition ($\mathrm{v}_1^+$), iff $v((f(\alpha), B) = 0$, by the truth-condition (m_1), iff $\mathfrak{M} \nvDash \mathcal{R}_\alpha B$, by the inductive hypothesis iff $\mathfrak{N} \nvDash B$, by the truth-condition (m_2), iff $\mathfrak{N} \nvDash \neg B$.

Let $A = B * C$, where $* \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$. We consider only case for $* = \wedge$. For other cases we reason in the similar way. We have: $\mathfrak{M} \vDash \mathcal{R}_\alpha B \wedge C$, by the truth-condition (m_1), iff $v((f(\alpha), B \wedge C) = 1$, by the condition ($\mathrm{v}_2^+$), iff $v((f(\alpha), B) = v((f(\alpha), C) = 1$, by the truth-condition (m_1), iff $\mathfrak{M} \nvDash \mathcal{R}_\alpha B$ and $\mathfrak{M} \nvDash \mathcal{R}_\alpha C$, by the inductive hypothesis iff $\mathfrak{N} \nvDash B$ and $\mathfrak{N} \nvDash C$, by the truth-condition (m_3), iff $\mathfrak{N} \nvDash B \wedge C$.

Let $A = \mathcal{R}_\beta B$. Then: $\mathfrak{M} \vDash \mathcal{R}_\alpha \mathcal{R}_\beta B$, by the truth-condition (m_1), iff $v(f(\alpha), \mathcal{R}_\beta B) = 1$, by the condition ($\mathrm{v}_6^+$), iff $v((f(\alpha), f(\beta)), B) = 1$, by fact 3.1, iff $u(f(\beta), B) = 1$, by the truth-condition (m_1), iff $\mathfrak{N} \nvDash \mathcal{R}_\beta B$. $\square$

By fact 3.2, if there is a MR^+- model $\mathfrak{M}$ such that $\mathfrak{M} \nvDash \mathcal{R}_\alpha A$, for some $\alpha \in \mathsf{IC}$, then there is MR^+- model $\mathfrak{N}$ such that $\mathfrak{N} \nvDash A$. Therefore (‡) holds.

3.2. Tableau system for logic MR^+

In the case of the elimination rules for standard connectives inside the scope of $\mathcal{R}$ operator (cf. Figure 1), the tableau rules for MR^+ are of the same form as rules for MR. The only difference is that the formulas in the numerator and denominator vary over For^+ instead of just For. The rest of tableau

For any $A, B \in \mathsf{For}^+$:

$$\mathrm{R}_{\mathcal{R}\neg} : \frac{\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} \neg A}{\neg \mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} A} \qquad \mathrm{R}_{\mathcal{R}\wedge} : \frac{\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} (A \wedge B)}{\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} A, \mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha n} B}$$

$$\mathrm{R}_{\mathcal{R}\vee} : \frac{\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} (A \vee B)}{\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha n} A | \mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} B}$$

$$\mathrm{R}_{\mathcal{R}\rightarrow} : \frac{\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} (A \rightarrow B)}{\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} \neg A | \mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} B}$$

$$\mathrm{R}_{\mathcal{R}\leftrightarrow} : \frac{\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} (A \leftrightarrow B)}{\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} A, \mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} B | \mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} \neg A, \mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} \neg B}$$

$$\mathrm{R}_{\neg\mathcal{R}\neg} : \frac{\neg \mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} \neg A}{\neg\neg \mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} A} \qquad \mathrm{R}_{\neg\mathcal{R}\wedge} : \frac{\neg \mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} (A \wedge B)}{\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} \neg A | \mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} \neg B}$$

$$\mathrm{R}_{\neg\mathcal{R}\vee} : \frac{\neg \mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} (A \vee B)}{\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} \neg A, \mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} \neg B}$$

$$\mathrm{R}_{\neg\mathcal{R}\rightarrow} : \frac{\neg \mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} (A \rightarrow B)}{\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} A, \mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} \neg B}$$

$$\mathrm{R}_{\neg\mathcal{R}\leftrightarrow} : \frac{\neg \mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} (A \leftrightarrow B)}{\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} A, \mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} \neg B | \mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} \neg A, \mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} B}$$

Figure 3. Elimination rules for standard connectives inside the scope of $\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n}$, for some $n \in \mathbb{N}$.

rules, i.e. elimination rules for standard connectives inside the scope of operator $\mathcal{R}$, are presented in Figure 3. We will use, however, the same notion for rules. The set of tableau rules for logic MR^+ is denoted as $\mathbf{R}^+$. The notions of thesis and tableau consequence in MR^+ are defined as for MR but with respect to $\mathbf{R}^+$.

3.3. Soundness and completeness of tableau system for MR^+

In order to prove the soundness of our system we use the counterpart of lemma 2.1 for rules from $\mathbf{R}^+$. Notice that the first nine tableau rules in Figure 1 work just fine for formulas of CPL, for any $A, B \in \mathsf{F}$.

LEMMA 3.3. *Let $X \subseteq \mathsf{For}^+$ and $\mathfrak{M} = \langle W, f, v\rangle$ be a MR^+- model suitable for X. If any rule from $\mathbf{R}^+$ has been applied to X, then $\mathfrak{M}$ is suitable for the union of X and at least one output obtained by application of that rule.*

PROOF: Similarly as for lemma 2.1. $\square$

In order to prove completeness we use the same argument as before but with respect to the modified version of the generated model. Let $X \in \mathsf{For}^+$ be a $\mathbf{R}^+$-closed set and $\mathsf{IC}_X = \{\alpha \in \mathsf{IC} : \mathcal{R}_\alpha A \in s(X)\}$. A MR^+*-model generated by* X (for short: MR^+-X*-model*) is a MR^+-model $\langle W, f, v\rangle$ such that:

- W, f are as in the previous case,
- for any $A \in X \cap \mathsf{VAR}$, for any $(\alpha_1, \dots, \alpha_n) \in \overrightarrow{W}$ we put:

$$v(\langle(\alpha_1, \dots, \alpha_n), A\rangle) = \begin{cases} 1, & \text{if } \mathcal{R}_{\alpha_1} \dots \mathcal{R}_{\alpha_n} A \in X \\ 0, & \text{if } \mathcal{R}_{\alpha_1} \dots \mathcal{R}_{\alpha_n} A \notin X \end{cases}$$

$$v(A) = \begin{cases} 1, & \text{if } A \in X \\ 0, & \text{if } A \notin X \end{cases}$$

  we extend v on $(W \times \mathsf{For}^+) \cup \mathsf{F}$ by means of standard conditions for CPL formulas and conditions (v_1^+)–(v_6^+).

First we prove the following fact:

FACT 3.4. Let $X \subseteq \mathsf{For}^+$ be a t-consistent $\mathbf{R}^+$-closed set and $\mathfrak{M} = \langle W, f, v\rangle$ be a MR^+ X-model. Then, for any $A \in \mathsf{For}^+$, for any $\alpha_1, \dots, \alpha_n \in \mathsf{IC}$:

- if $\mathcal{R}_{\alpha_1} \dots \mathcal{R}_{\alpha_n} A \in X$ then $v(\langle(\alpha_1, \dots, \alpha_n), A\rangle) = 1$,
- if $\neg\mathcal{R}_{\alpha_1} \dots \mathcal{R}_{\alpha_n} A \in X$ then $v(\langle(\alpha_1, \dots, \alpha_n), A\rangle) = 0$.

PROOF: *Base case.* By the definition of the MR^+ X-model and since X is t-consistent.

Inductive hypothesis. Let $m \in \mathbb{N}$. Suppose that for any $A \in \mathsf{For}^+$ such that $c^+(A) \leq m$, for any $\alpha_1, \ldots, \alpha_n \in \mathsf{IC}$:

- if $\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} A \in X$ then $v(\langle(\alpha_1, \ldots, \alpha_n), A\rangle) = 1$,
- if $\neg\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} A \in X$ then $v(\langle(\alpha_1, \ldots, \alpha_n), A\rangle) = 0$.

Inductive step. Let $A \in \mathsf{For}^+$ and $c^+(A) = m + 1$.

Let $A = \neg B$. Suppose $\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} \neg B \in X$. Hence, by the application of the rule $\mathrm{R}_{\neg\mathcal{R}} \in \mathbf{R}^+$, $\neg\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} B \in X$. By the inductive hypothesis $v(\langle(f(\alpha_1), \ldots, f(\alpha_n)), B\rangle) = 0$. By the condition ($\mathrm{v}_1^+$) $v(\langle f(\alpha_1), \ldots, f(\alpha_n)), \neg B\rangle) = 1$. Suppose $\neg\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} \neg B \in X$. Hence, by the application of the rule $\mathrm{R}_{\neg\mathcal{R}\neg} \in \mathbf{R}^+$, $\neg\neg\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} B \in X$. By the application of the rule $\mathrm{R}_{\neg\neg} \in \mathbf{R}^+$, $\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} B \in X$. By the inductive hypothesis $v(\langle(f(\alpha_1), \ldots, f(\alpha_n)), B\rangle) = 1$. Thus, by the condition (v_1^+), $v(\langle(f(\alpha_1), \ldots, f(\alpha_n)), \neg B\rangle) = 0$.

Let $A = B * C$, where $* \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$. We consider only case for $* = \wedge$. For other cases we reason in the similar way. Suppose $\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n}(B \wedge C) \in X$. Hence, by the application of the rule $\mathrm{R}_{\mathcal{R}\wedge} \in \mathbf{R}^+$, $\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} B \in X$ and $\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} C \in X$. Thus, by the inductive hypothesis, $v(\langle(f(\alpha_1), \ldots, f(\alpha_n)), B\rangle) = 1$ and $v(\langle(f(\alpha_1), \ldots, f(\alpha_n)), C\rangle) = 1$. Thus, by the condition (v_2^+), $v(\langle(f(\alpha_1), \ldots, f(\alpha_n)), B \wedge C\rangle) = 1$. Suppose $\neg\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n}(B \wedge C) \in X$. Hence, by the application of the rule $\mathrm{R}_{\neg\mathcal{R}\wedge} \in \mathbf{R}^+$, $\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} \neg B \in X$ or $\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} \neg C \in X$. By the application of the rule $\mathrm{R}_{\mathcal{R}\neg} \in \mathbf{R}^+$, $\neg\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} B \in X$ or $\neg\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} C \in X$. Thus, by the inductive hypothesis, either $v(\langle(f(\alpha_1), \ldots, f(\alpha_n)), B\rangle) = 0$ or $v(\langle(f(\alpha_1), \ldots, f(\alpha_n)), C\rangle) = 0$. Thus, by the condition (v_2^+), $v(\langle(f(\alpha_1), \ldots, f(\alpha_n)) B \wedge C\rangle) = 0$.

Let $A = \mathcal{R}_\beta B$. Suppose $\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} A \in X$ (resp. $\neg\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} A \in X$), so $\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} \mathcal{R}_\beta B \in X$ (resp. $\neg\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} \mathcal{R}_\beta B \in X$). Let us assume that that $\mathcal{R}_\beta$ is the longest iteration of $\mathcal{R}$ which appears right after $\mathcal{R}_{\alpha_n}$. If it is not then we can consider the longest since formula is a finite string of symbols. Hence B is a propositional variable or is of the form $\neg C$ or $C * D$, where $* \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$. Thus we can reason as in the previous cases. □

The following lemma enables one to prove completeness theorem:

Lemma 3.5. *Let $X \subseteq \mathsf{For}^+$ be a t-consistent $\mathbf{R}^+$-closed set, $\mathfrak{M}$ be an MR^+ X-model. Then, for any $A \in \mathsf{For}^+$*

- *if $A \in X$ then $\mathfrak{M} \models A$,*
- *if $\neg A \in X$ then $\mathfrak{M} \not\models A$.*

Proof: We reason similarly as in the case of lemma 2.3. In the base case we consider a propositional variable and receive the required result by the definition of the MR^+ X-model. In the inductive step we additionally consider formulas of the form $\mathcal{R}_\alpha B$ and $\neg\mathcal{R}_\alpha B$. In such cases we receive the required result by fact 3.4. □

As in the case for MR, by lemmas 3.3, 3.5 we get the following theorem:

Theorem 3.6. *Let $X \cup \{A\} \subseteq \mathsf{For}^+$. Then, $X \rhd_{\mathsf{MR}^+} A$ iff $X \vDash_{\mathsf{MR}^+} A$.*

4. Modal paradigm and logical omniscience

The standard approach to epistemic logic is based on modal logic, where the necessitation operator $\Box$ is rewritten as K. Formal interpretation of $\Box$ and K is the same if we consider modal logic at least as strong as logic T. Operator $\Box$ supposed to express a kind of necessity, very often called metaphysical or alethic one, while operator K supposed to enable one to express a propositional attitudes, that an agent knows this or that (see for instance [13]). The distinguishing feature of the standard epistemic logic is that it contains the schema (T): $\mathrm{K}A \rightarrow A$. By (T) the classical property of knowledge is expressed, i.e. what is known is true. Other interesting properties that are often considered on the ground of modal epistemic logic are so-called positive and negative introspection. The former means that if an agent knows that A, then he knows that he knows that A. On the formal ground it is expressed by the schema (4): $\mathrm{K}A \rightarrow \mathrm{KK}A$. The latter means that if an agent does not know that A, then he knows that he does not know that A. Such a property is expressed by the schema (5): $\neg\mathrm{K}A \rightarrow \mathrm{K}\neg\mathrm{K}A$.

One of the big questions with respect to propositional attitudes is the logical omniscience, i.e. a problem of the deductive closure of agent's knowledge and a problem of knowing by an agent all thesis of a given logic. On the formal ground the schema (K) $\mathrm{K}(A \rightarrow B) \rightarrow (\mathrm{K}A \rightarrow \mathrm{K}B)$ and the Necessitation Rule (RN): if A is a thesis than $\mathrm{K}A$ is a thesis, also known as the Gödel's Rule, enable one to prove the Monotonicity Rule (RM): if $A \rightarrow B$ is a thesis, then $\mathrm{K}A \rightarrow \mathrm{K}B$ is a thesis. The rule (RM) simply says that an agent's knowledge is deductively closed. And let us remember that metavariables A and B represent formulas of arbitrary complexity. While the bigger the complexity of formula is, the harder the reasoning to perform. The deductive closure, however, makes sure that no matter how hard the reasoning is, the agent is able to derive the consequence. It seems highly unintuitive with regard to empirical agents like human beings. From this perspective even sole (K) and (RN) might seem to be unintuitive. For (K) says that the agent's knowledge is closed under the Modus Ponens and no matter what formulas are taken into account. And (RN) says that agent knows each thesis of a given logic which is rather impossible.

5. Epistemic positional logics

Let us stick to the language of MR^+. Formulas of the form $\mathcal{R}_\alpha A$ might be read: agent α knows that A. By counterparts of modal schemata (K), (T), (4) and (5) in the positional language we mean the following schemata:

$$\mathcal{R}_\alpha(A \to B) \to (\mathcal{R}_\alpha A \to \mathcal{R}_\alpha B) \tag{$\mathcal{R}$K}$$

$$\mathcal{R}_\alpha A \to A \tag{$\mathcal{R}$T}$$

$$\mathcal{R}_\alpha A \to \mathcal{R}_\alpha \mathcal{R}_\alpha A \tag{$\mathcal{R}$4}$$

$$\neg\mathcal{R}_\alpha A \to \mathcal{R}_\alpha \neg\mathcal{R}_\alpha A. \tag{$\mathcal{R}$5}$$

The counterpart of (RN) is of the following form:

$$\text{if } A \text{ is valid, then } \mathcal{R}_\alpha A \text{ is valid.} \tag{$\mathcal{R}$RN}$$

The rule ($\mathcal{R}$RN) is not only positional but also a semantic counterpart of (RN).

Our main goal is to obtain epistemic logic based on positional logic such that:

- it contains the positional counterpart of (T),
- it does not contain the positional counterpart of (K),
- the positional semantic counterpart of (RN) is not satisfied,
- some of its extensions contain counterparts of (4) and (5).

5.1. Semantics

Notice that MR^+ contains ($\mathcal{R}$K). Suppose $\mathfrak{M} \vDash \mathcal{R}_\alpha(A \to B)$ and $\mathfrak{M} \vDash \mathcal{R}_\alpha A$. Then, by the truth-condition (m_1), $v(f(\alpha), A \to B) = 1$ and $v(f(\alpha), A) = 1$. Hence, by the condition (v_4^+), $v(f(\alpha), B) = 1$. Thus, by the truth-condition (m$_1$), $\mathfrak{M} \models \mathcal{R}_\alpha B$. Moreover, by (‡) for MR^+ ($\mathcal{R}$RN) is satisfied. Thus we have to change the notion of a MR^+-model.

A *non-standard* MR^+*-model* (for short: a *non-standard model*) is a triple $\langle W, f, v\rangle$ such that:

- W, f are as in the previous cases,
- $v(\overrightarrow{W} \times \mathsf{For}^+) \cup \mathsf{F} \longrightarrow \{0, 1\}$ is such that:
  - $v\restriction_{\overrightarrow{W}\times\mathsf{For}^+}$ is such that (v_6^+) is satisfied and for other cases is arbitrary,
  - $v\restriction_{\mathsf{F}}$ is a classical CPL valuation.

The truth-conditions are the same as in the case of MR^+-models.

By means of non-standard models we avoid the problem of logical omniscience. For instance, we still have that $p\vee\neg p$ is valid but since the valuation of a non-standard model is arbitrary on $\langle w, p \vee \neg p\rangle$ it does not have to be the case that for any $\alpha \in \mathsf{IC}$, $\mathcal{R}_\alpha p \vee \neg p$. By means of such models we also can falsify ($\mathcal{R}$K). Consider a non-standard model $\mathfrak{M} = \langle W, f, v\rangle$ such that $W = \{w\}$, $f(\mathsf{IC}) = \{w\}$ and v is such that, for any $(w_1, \dots, w_n) \in \overrightarrow{W}$:

- $v(\langle(w_1 \dots, w_n), q\rangle) = 0$, if $n = 1$,
- $v(\langle(w_1 \dots, w_n), q\rangle) = 1$, if $n > 1$,
- $v(\langle(w_1 \dots, w_n), A\rangle) = 1$, for any $A \in \mathsf{For}^+ \setminus \{q\}$,
- $v(\mathsf{VAR}) = \{1\}$ and is extended on F in the standard way.

Then $\mathfrak{M} \vDash \mathcal{R}_\alpha(p \to q)$ and $\mathfrak{M} \vDash \mathcal{R}_\alpha p$ but $\mathfrak{M} \nvDash \mathcal{R}_\alpha q$.

In order to validate formulas of the schema ($\mathcal{R}$T) we need to stipulate some additional restrictions on non-standard models. Let us consider the following condition, for any $\alpha \in \mathsf{IC}$, for any $A \in \mathsf{For}^+$:

$$\text{if } v(\langle f(\alpha), A\rangle) = 1 \text{ then } \mathfrak{M} \vDash A. \qquad (\star)$$

In order to validate formulas of the schema ($\mathcal{R}$4) and ($\mathcal{R}$5) we use the following conditions, for any $\alpha \in \mathsf{IC}$, for any $A \in \mathsf{For}^+$:

$$\text{if } v(\langle f(\alpha), A\rangle) = 1 \text{ then } v(\langle f(\alpha), \mathcal{R}_\alpha A\rangle) = 1 \qquad (\star\star)$$

$$\text{if } v(\langle f(\alpha), A\rangle) = 0 \text{ then } v(\langle f(\alpha), \neg\mathcal{R}_\alpha A\rangle) = 1. \qquad (\star\star\star)$$

We receive the following fact:

FACT 5.1. Let $\mathfrak{M} = \langle W, f, v\rangle$ be a non-standard model, $A \in \mathsf{For}^+$ and $\alpha \in \mathsf{IC}$. Then:

(1) $(\star)$ is satisfied iff $\mathfrak{M} \models \mathcal{R}_\alpha A \to A$,

(2) $(\star\star)$ is satisfied iff $\mathfrak{M} \models \mathcal{R}_\alpha A \to \mathcal{R}_\alpha\mathcal{R}_\alpha A$,

(3) $(\star\star\star)$ is satisfied iff $\mathfrak{M} \models \neg\mathcal{R}_\alpha A \to \mathcal{R}_\alpha\neg\mathcal{R}_\alpha A$.

PROOF: Ad. (1). Suppose that $(\star)$ holds and $\mathfrak{M} \models \mathcal{R}_\alpha A$. Hence $v(f(\alpha), A)$ $= 1$. By $(\star)$ we get $\mathfrak{M} \models A$, hence $\mathfrak{M} \models A$. For the other direction suppose $\mathfrak{M} \models \mathcal{R}_\alpha A \to A$ and $v(\langle f(\alpha), A\rangle) = 1$. This means $\mathfrak{M} \models \mathcal{R}_\alpha A$, which gives us $\mathfrak{M} \models A$.

Ad. (2). Suppose that $(\star\star)$ holds and $\mathfrak{M} \models \mathcal{R}_\alpha A$. Thus $c(\langle f(\alpha), A\rangle) =$ 1. By $(\star\star)$ we get $v(\langle f(\alpha), \mathcal{R}_\alpha A\rangle) = 1$, hence $\mathfrak{M} \models \mathcal{R}_\alpha\mathcal{R}_\alpha A$. For the other direction suppose $\mathfrak{M} \models \mathcal{R}_\alpha A \to \mathcal{R}_\alpha\mathcal{R}_\alpha A$ and $v(\langle f(\alpha), A\rangle) = 1$. Hence $\mathfrak{M} \models \mathcal{R}_\alpha\mathcal{R}_\alpha A$. By ($\mathrm{v}_6^+$) we obtain $v(\langle f(\alpha), \mathcal{R}_\alpha A\rangle) = 1$.

Ad. (3). Suppose that $(\star\star\star)$ holds and $\mathfrak{M} \models \neg\mathcal{R}_\alpha A$. Thus $v(\langle f(\alpha), A\rangle)$ $= 0$. By $(\star\star\star)$ $v(\langle f(\alpha), \neg\mathcal{R}_\alpha A\rangle) = 1$, so $\mathfrak{M} \models \mathcal{R}_\alpha\neg\mathcal{R}_\alpha A$. For the other direction suppose $\mathfrak{M} \models \neg\mathcal{R}_\alpha A \to \mathcal{R}_\alpha\neg\mathcal{R}_\alpha A$ and $v(\langle f(\alpha), A\rangle) = 0$. Hence $\mathfrak{M} \models \neg\mathcal{R}_\alpha A$, so $\mathfrak{M} \models \mathcal{R}_\alpha\neg\mathcal{R}_\alpha A$ which means $v(\langle f(\alpha), \neg\mathcal{R}_\alpha A\rangle) = 1$. □

Any non-standard model such that $(\star)$ is satisfied shall be called a *model of* ER (for short: an ER-*model*). Any ER-model such that $(\star\star)$ (resp. $(\star\star\star)$) is satisfied shall be called a *model of* ER4 (resp. a *model of* ER5) (for short: an ER4-*model*, resp. an ER5-*model*). A logic ER might be considered the minimal epistemic positional logic based in non-standard models. Logics ER4 and ER5 are the minimal epistemic positional logics based on non-standard models that contain ($\mathcal{R}4$) and ($\mathcal{R}5$) respectively.

5.2. Tableau systems of ER, ER4 and ER5

For logics ER, ER4 and ER5 the elimination rules for standard connectives outside the scope of operator $\mathcal{R}$ are the same as in the case of MR^+. For our logics we also have to include the rule $\mathrm{R}_{\mathcal{R}\neg}$ from Figure 3 and the rule $\mathrm{R}_{\mathcal{R}\mathrm{T}}$ from Figure 4. In the case of logic ER4 (resp. ER5) we additionally include $\mathrm{R}_{\mathcal{R}4}$ (resp. $\mathrm{R}_{\mathcal{R}5}$) from Figure 4. In the case of logic ER we assume all the specific rules. The sets of tableau rules for ER (resp. ER4, ER5) shall be denoted as $\mathbf{R}_{\mathsf{ER}}$ (resp. $\mathbf{R}_{\mathsf{ER4}}$, $\mathbf{R}_{\mathsf{ER5}}$).

For any $A, B \in \mathsf{For}^+$:

$$\mathrm{R}_{\mathcal{R}\mathrm{T}} : \frac{\mathcal{R}_\alpha A}{A} \qquad \mathrm{R}_{\mathcal{R}4} : \frac{\neg\mathcal{R}_\alpha\mathcal{R}_\alpha A}{\neg\mathcal{R}_\alpha A} \qquad \mathrm{R}_{\mathcal{R}5} : \frac{\neg\mathcal{R}_\alpha\neg\mathcal{R}_\alpha A}{\mathcal{R}_\alpha A}$$

Figure 4. Specific rules for $\mathcal{R}$ operator

Let us notice an interesting dependence. By means of rules $\mathrm{R}_{\mathcal{R}\mathrm{T}}$, $\mathrm{R}_{\neg\neg}$ and $\mathrm{R}_{\neg\mathcal{R}\neg}$ we can easily derive the rule $\mathrm{R}_{\mathcal{R}5}$.

1. $\neg\mathcal{R}_\alpha\neg\mathcal{R}_\alpha A$
2. $\neg\neg\mathcal{R}_\alpha\mathcal{R}_\alpha A$ — by the rule $\mathrm{R}_{\neg\mathcal{R}\neg}$ and 1
3. $\mathcal{R}_\alpha\mathcal{R}_\alpha A$ — by the rule $\mathrm{R}_{\neg\neg}$ and 2
4. $\mathcal{R}_\alpha A$ — by the rule $\mathrm{R}_{\mathcal{R}\mathrm{T}}$ and 3 $\Box$

Clearly the rule $\mathrm{R}_{\neg\mathcal{R}\neg}$ corresponds with the condition (v_1^+). We have that the logic determined by ER-models such that the condition (v_1^+) is satisfied contains ($\mathcal{R}5$). Suppose $\mathfrak{M} \vDash \neg\mathcal{R}_\alpha\neg\mathcal{R}_\alpha A$. Thus $v(f(\alpha), \neg\mathcal{R}_\alpha A) = 0$, by the condition ($\mathrm{v}_1^+$) $v(f(\alpha), \mathcal{R}_\alpha A) = 1$. By the condition ($\star$) $\mathfrak{M} \vDash \mathcal{R}_\alpha A$.

5.3. Soundness and completeness of tableau systems for ER, ER4 and ER5

A soundness theorem might be proved similarly as in the case of MR^+ and MR. Let us notice that by fact 5.1 by applications of new rules $\mathrm{R}_{\mathcal{R}\mathrm{T}}$, $\mathrm{R}_{\mathcal{R}4}$ and $\mathrm{R}_{\mathcal{R}5}$ from satisfiable formulas we receive some satisfiable formulas.

LEMMA 5.2. *Let $\Lambda \in \{\mathsf{ER}, \mathsf{ER4}, \mathsf{ER5}\}$, $X \subseteq \mathsf{For}^+$ and $\mathfrak{M} = \langle W, f, v\rangle$ be a Λ-model suitable for X. If any rule from $\mathbf{R}_\Lambda$ has been applied to X, then $\mathfrak{M}$ is suitable for the union of X and at least one output obtained by application of that rule.*

PROOF: Similarly as for lemma 2.1.

Suppose that $\mathrm{R}_{\mathcal{R}\mathrm{T}}$ has been applied to X. Hence $\mathcal{R}_\alpha A \in X$. Since $\mathfrak{M}$ is suitable for X $\mathfrak{M} \vDash \mathcal{R}_\alpha A$. By 5.1 (1), we obtain $\mathfrak{M} \vDash A$.

Let $\Lambda = \mathsf{ER4}$. Suppose that $\mathrm{R}_{\mathcal{R}4}$ has been applied to X. Hence $\neg\mathcal{R}_\alpha\mathcal{R}_\alpha A \in X$. Since $\mathfrak{M}$ is suitable for X $\mathfrak{M} \models \neg\mathcal{R}_\alpha\mathcal{R}_\alpha A$. By 5.1 (2) we obtain $\mathfrak{M} \models \neg\mathcal{R}_\alpha A$.

Let $\Lambda = \mathsf{ER5}$. Suppose that $\mathrm{R}_{\mathcal{R}5}$ has been applied to X. Hence $\neg\mathcal{R}_\alpha\neg\mathcal{R}_\alpha A \in X$. Hence $\mathfrak{M} \models \neg\mathcal{R}_\alpha\neg\mathcal{R}_\alpha A$. Since $\mathfrak{M}$ is suitable for X $\mathfrak{M} \models \neg\mathcal{R}_\alpha\neg\mathcal{R}_\alpha A$. By 5.1 (3) and the truth-condition (m_2) we obtain $\mathfrak{M} \models \mathcal{R}_\alpha A$. □

In order to prove completeness we use the same argument as before but with respect to modified version of the generated model. Let us first present special extensions of sets closed under tableau rules. Let $\Lambda \in \{\mathsf{ER}, \mathsf{ER4}, \mathsf{ER5}\}$ and X be a $\mathbf{R}_\Lambda$-closed set. By X_Λ we shall denote a set such that:

- if $\Lambda = \mathsf{ER}$ then $X_\Lambda = X$,
- if $\Lambda = \mathsf{ER4}$ then X_Λ is the smallest set $Y \subseteq \mathsf{For}^+$ such that $X \subseteq Y$ and if $\mathcal{R}_\alpha A \in Y$ then $\mathcal{R}_\alpha\mathcal{R}_\alpha A \in Y$,
- if $\Lambda = \mathsf{ER5}$ then X_Λ is the smallest set $Y \subseteq \mathsf{For}^+$ such that $X \subseteq Y$ and if either $\neg\mathcal{R}_\alpha A \in X$ or $\mathcal{R}_\alpha A \notin Y$ then $\mathcal{R}_\alpha\neg\mathcal{R}_\alpha A \in Y$.

We have the following fact:

Fact 5.3. Let $\Lambda \in \{\mathsf{ER}, \mathsf{ER4}, \mathsf{ER5}\}$ and X be a $\mathbf{R}_\Lambda$-closed set. If X is t-consistent then X_Λ is t-consistent.

Proof: Let $\Lambda = \mathsf{ER4}$. Assume that X is t-consistent and X_{ER4} is t-inconsistent. Note that there are no formulas of the form $\neg A$ in $X_{\mathsf{ER4}} \setminus X$ – there are formulas preceded by an $\mathcal{R}$ operator only. For this reason X_{ER4} can be t-inconsistent only when $\neg\underbrace{\mathcal{R}_\alpha \ldots \mathcal{R}_\alpha}_{n} A \in X$ and $\underbrace{\mathcal{R}_\alpha \ldots \mathcal{R}_\alpha}_{n} A \in X_{\mathsf{ER4}} \setminus X$, for some $n \geq 1$. Hence by definition of X_{ER4}, $\underbrace{\mathcal{R}_\alpha \ldots \mathcal{R}_\alpha}_{k} A \in X$, for some $k < n$. But by application of rule $\mathrm{R}_{\mathcal{R}4}$ $n - k$ times we obtain $\neg\underbrace{\mathcal{R}_\alpha \ldots \mathcal{R}_\alpha}_{k} A \in X$, so X is t-inconsistent which gives us contradiction with the assumption.

Let $\Lambda = \mathsf{ER5}$. Reasoning in the same manner, we assume that X is t-consistent and X_{ER5} is t-inconsistent. Hence $\mathcal{R}_\alpha\neg\mathcal{R}_\alpha A \in X_{\mathsf{ER5}} \setminus X$ and $\neg\mathcal{R}_\alpha \neg\mathcal{R}_\alpha A \in X$. By the definition of X_{ER5}, either $\neg\mathcal{R}_\alpha A \in X$ or $\mathcal{R}_\alpha A \notin X_{\mathsf{ER5}}$.

In the second case we get $\mathcal{R}_\alpha A \notin X \subseteq X_{\mathsf{ER5}}$. By the application of the rule $\mathrm{R}_{\mathcal{R}5}$ we obtain $\mathcal{R}_\alpha A \in X$. In both cases we get a contradiction. □

Let $\Lambda \in \{\mathsf{ER}, \mathsf{ER4}, \mathsf{ER5}\}$ and X be a $\mathbf{R}_\Lambda$-closed set. A Λ-*model generated by* X_Λ (for short: a X_Λ-*model*) is a Λ-model $\langle W, f, v\rangle$ such that:

- W, f are as in the previous case,
- for any for any $(\alpha_1, \ldots, \alpha_n) \in \overrightarrow{W}$ and any $A \in \mathsf{For}^+$:

$$v(\langle(\alpha_1, \ldots, \alpha_n), A\rangle) = \begin{cases} 1, & \text{if } \mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} A \in X_\Lambda \\ 0, & \text{if } \mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} A \notin X_\Lambda \end{cases}$$

- for any $A \in \mathsf{VAR}$ we stipulate:

$$v(A) = \begin{cases} 1, & \text{if } A \in X_\Lambda \\ 0, & \text{if } A \notin X_\Lambda \end{cases}$$

  we extend v^E on F in the standard way.

Because of the definition of valuation from X_Λ-model, implications of the fact 3.4 are obvious. The implications obviously hold if in the antecedents we change X on X_Λ.

LEMMA 5.4. *Let* $\Lambda \in \{\mathsf{ER}, \mathsf{ER4}, \mathsf{ER5}\}$, X *be a* $\mathbf{R}_\Lambda$*-closed set and* $\mathfrak{M}$ *be a* X_Λ*-model. Then, for any* $A \in \mathsf{For}^+$*:*

- *if* $A \in X_\Lambda$ *then* $\mathfrak{M} \vDash A$,
- *if* $\neg A \in X_\Lambda$ *then* $\mathfrak{M} \nvDash A$.

PROOF: *Base case.* Let $A \in \mathsf{For}^+$ and $c^+(A) = 1$. Thus $A \in \mathsf{VAR}$. Suppose $A \in X$. Then, by the definition of v, $v(A) = 1$.

Inductive hypothesis. Let $n \in \mathbb{N}$. Suppose that for any $A \in \mathsf{For}^+$ such that $c^+(A) \leq n$, if $A \in X$ then $\mathfrak{M} \vDash A$.

Inductive step. Let $A \in \mathsf{For}^+$ and $c^+(A) = n + 1$. We consider the following cases, the others are considered in a similar way.

Let $A = \neg\neg B$. Suppose $\neg\neg B \in X_\Lambda$. Thus, by the definition of X_Λ, $\neg\neg B \in X$. Hence, by the application of the rule $\mathrm{R}_{\neg\neg}$, $B \in X$. Thus, by the inductive hypothesis, $\mathfrak{M} \vDash B$.

Let $A = B * C$, where $* \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$. We consider only case for $* = \wedge$. For other cases we reason in the similar way. Suppose $B \wedge C \in X_\Lambda$.

Thus, by the definition of X_Λ, $B \wedge C \in X$. Since X is $\mathbf{R}_\Lambda$-closed set, by the application of the rule $\mathrm{R}_\wedge$, $B, C \in X$. Hence, by the inductive hypothesis, $\mathfrak{M} \models B$ and $\mathfrak{M} \models C$. Therefore, by the truth-condition (m_3), $\mathfrak{M} \models B \wedge C$.

Let $A = \neg(B * C)$, where $* \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$. We consider only case for $* = \wedge$. For other cases we reason in the similar way. Suppose $\neg(B \wedge C) \in X_\Lambda$. Thus, by the definition of X_Λ, $\neg(B \wedge C) \in X$. Since X is $\mathbf{R}_\Lambda$-closed set, by the application of the rule $\mathrm{R}_{\neg\wedge}$, either $\neg B \in X$ or $\neg C \in X$. Hence, by the inductive hypothesis and the truth-condition (m_2), either $\mathfrak{M} \not\models B$ or $\mathfrak{M} \not\models C$. Therefore, by the truth-condition (m_3), $\mathfrak{M} \models B \wedge C$.

Let $A = \mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} B$. Suppose $\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} B \in X_\Lambda$. By the definition of a X_Λ-model, the truth-condition (m_1) and the condition (v_6^+) $\mathfrak{M} \models \mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} B$.

Let $A = \neg\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} B$. Suppose $\neg\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} B \in X_\Lambda$. By the definition of a X_Λ-model the truth-condition (m_1) and the condition (v_6^+) $\mathfrak{M} \not\models \mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} B$. □

The following fact shows that generated models satisfy the proper conditions.

Fact 5.5. Let $\Lambda \in \{\mathsf{ER}, \mathsf{ER4}, \mathsf{ER5}\}$, $X \subseteq \mathsf{For}^+$ be a t-consistent $\mathbf{R}_\Lambda$-closed set and $\mathfrak{M} = \langle W, f, v\rangle$ be a X_Λ-model. Then:

(1) the condition ($\star$) is satisfied,

(2) if $\Lambda = \mathsf{ER4}$ then the condition ($\star\star$) is satisfied,

(2) if $\Lambda = \mathsf{ER5}$ then the condition ($\star\star\star$) is satisfied.

Proof: Ad (1). Suppose $v(\langle f(\alpha), A\rangle) = 1$. Hence, by the definition of a X_Λ-model, $\mathcal{R}_\alpha A \in X_\Lambda$. Assume $\mathcal{R}_\alpha A \in X$. Thus, by the rule $\mathrm{R}_{\mathcal{R}\mathrm{T}}$, $A \in X$. By lemma 5.4 (1) $\mathfrak{M} \models A$. Assume $\mathcal{R}_\alpha A \notin X$. We have two possible cases. Let $\mathcal{R}_\alpha A = \mathcal{R}_\alpha \mathcal{R}_\alpha \ldots \mathcal{R}_\alpha B$. Thus, by the definition of a X_{ER4}-model, $\mathcal{R}_\alpha \ldots \mathcal{R}_\alpha B \in X_{\mathsf{ER4}}$. By lemma 5.4 (1) $\mathfrak{M} \models \mathcal{R}_\alpha \ldots \mathcal{R}_\alpha B$. Let $\mathcal{R}_\alpha A = \mathcal{R}_\alpha \neg\mathcal{R}_\alpha B$. Thus $A = \neg\mathcal{R}_\alpha B$ and either (a) $\neg\mathcal{R}_\alpha B \in X \subseteq X_{\mathsf{ER5}}$ or (b) $\mathcal{R}_\alpha B \notin X_{\mathsf{ER5}}$. If (a), then by lemma 5.4 (1) $\mathfrak{M} \models \neg\mathcal{R}_\alpha B$. If (b), then by the definition of X_{ER5}-model, $v(\langle f(\alpha), B\rangle) = 0$. Hence, by truth-conditions (m_1) and (m_2), $\mathfrak{M} \models \neg\mathcal{R}_\alpha B$.

Ad (2). Suppose $v(\langle f(\alpha), A\rangle) = 1$. Hence, by the definition of a X_{ER4}-model, $\mathcal{R}_\alpha A \in X_{\mathsf{ER4}}$. Thus $\mathcal{R}_\alpha \mathcal{R}_\alpha A \in X_{\mathsf{ER4}}$. By lemma 5.4 (1) $\mathfrak{M} \models \mathcal{R}_\alpha \mathcal{R}_\alpha A$. Hence, by the truth-condition (m_1), $v(\langle f(\alpha), \mathcal{R}_\alpha A\rangle) = 1$.

Ad (3). Suppose $v(\langle f(\alpha), A\rangle) = 0$. Thus, by truth-conditions (m_1) and (m_2), $\mathfrak{M} \nvDash \mathcal{R}_\alpha A$. By lemma 5.4 (1) $\mathcal{R}_\alpha A \notin X_{\mathsf{ER5}}$. Hence, by the definition of X_{ER5}, $\mathcal{R}_\alpha \neg \mathcal{R}_\alpha A \in X_{\mathsf{ER5}}$. By lemma 5.4 (1) $\mathfrak{M} \vDash \mathcal{R}_\alpha A \neg \mathcal{R}_\alpha A$. Hence, by the truth-condition (m_1), $v(\langle f(\alpha), \neg \mathcal{R}_\alpha A\rangle) = 1$. □

As before, by lemmas 5.2, 5.4, we get the following theorem:

THEOREM 5.6. *Let* $X \cup \{A\} \subseteq \mathsf{For}^+$. *Then:*

(1) $X \rhd_{\mathsf{ER}} A$ *iff* $X \vDash_{\mathsf{ER}} A$,

(2) $X \rhd_{\mathsf{ER4}} A$ *iff* $X \vDash_{\mathsf{ER4}} A$,

(3) $X \rhd_{\mathsf{ER5}} A$ *iff* $X \vDash_{\mathsf{ER5}} A$.

Acknowledgements. This work was supported by the National Science Centre, Poland, under Grants UMO-2015/19/N/HS1/02401 and UMO-2015/19/B/HS1/02478. The authors would like to thank Tomasz Jarmużek for all his ideas and valuable suggestions.

References

[1] T. Jarmużek, *Minimal Logical Systems With R-operator: Their Metalogical Properties and Ways of Extensions*, [in:] J. Bézieau, A. Costa-Leite (eds.), **Perspectives on Universal Logic**, Polimetrica Publisher (2007), pp. 319–333.

[2] T. Jarmużek, *Tableau Metatheorem for Modal Logics*, [in:] R. Ciuni, H. Wansing, C. Willkommen (eds.), **Trends in Logic Vol. 41. Recent Trends in Philosophical Logic**, Springer International Publishing (2014), pp. 103–126, DOI: https://doi.org/10.1007/978-3-319-06080-4_8.

[3] T. Jarmużek, **On the Sea Battle Tomorrow That May Not Happen**, Peter Lang, Berlin (2018), DOI: https://doi.org/10.3726/b14343.

[4] T. Jarmużek, M. Klonowski, *Some Intensional Logics Defined by Relating Semantics and Tableau Systems*, [in:] A. Giordani, J. Malinowski (eds.), **Logic in High Definition. Current Issues in Logical Semantics**, Springer International Publishing (2021), pp. 31–48, DOI: https://doi.org/10.1007/978-3-030-53487-5_3.

[5] T. Jarmużek, A. Parol, *On Some Language Extension of Logic MR: A Semantic and Tableau Approach*, **Roczniki Filozoficzne**, vol. 68 (2021), pp. 345–366, DOI: https://doi.org/10.18290/rf20684-16.

[6] T. Jarmużek, A. Pietruszczak, *Completeness of Minimal Positional Calculus*, **Logic and Logical Philosophy**, vol. 13 (2004), pp. 147–162, DOI: https://doi.org/10.12775/LLP.2004.009.

[7] T. Jarmużek, M. Tkaczyk, **Normalne logiki pozycyjne (Normal Positional Logics)**, Wydawnictwo KUL, Lublin (2015).

[8] T. Jarmużek, M. Tkaczyk, *Expressive Power of the Positional Operator R: a Case Study in Modal Logic and Modal Philosophy*, **Ruch Filozoficzny**, vol. 75 (2019), pp. 93–107, DOI: https://doi.org/10.12775/RF.2019.022.

[9] A. Karczewska, *Maximality of the Minimal R-Logic*, **Logic and Logical Philosophy**, vol. 27 (2017), pp. 193–203, DOI: https://doi.org/10.12775/LLP.2017.008.

[10] J. Łoś, *Podstawy analizy metodologicznej kanonów Milla (The Foundations of Methodological Analysis of Mill's Canons)*, **Annales Universitatis Mariae Curie-Skłodowska**, vol. 2 (1947), pp. 269–301.

[11] J. Łoś, *Logiki wielowartościowe a formalizacja funkcji intensjonalnych (Many-Valued Logics and Formalization of Intensional Functions)*, **Kwartalnik Filozoficzny**, vol. 17(1) (1948), pp. 59–68.

[12] J. Malinowski, K. Pietrowicz, J. Szalacha-Jarmużek, *Logic of Social Ontology and Łoś's Operator*, **Logic and Logical Philosophy**, vol. 29 (2020), pp. 239–258, DOI: https://doi.org/10.12775/LLP.2020.005.

[13] J. Meyer, *Epistemic Logic*, [in:] L. Goble (ed.), **The Blackwell Guide to Philosophical Logic**, Blackwell (2001), pp. 183–202, DOI: https://doi.org/10.1002/9781405164801.

[14] G. Priest, **An Introduction to Non-Classical Logic. From If to Is. 2nd ed**, Cambridge University Press, Cambridge (2008), DOI: https://doi.org/10.1017/CBO9780511801174.

[15] N. Rescher, *On the Logic of Chronological Propositions*, **Mind**, vol. 75(297) (1966), pp. 75–96, DOI: https://doi.org/10.2307/2251711.

[16] N. Rescher, *Topological Logic*, [in:] **Topics in Philosophical Logic**, Reidel Publishing Company, Dordrecht (1968), pp. 229–249, DOI: https://doi.org/10.1007/978-94-017-3546-9_13.

[17] N. Rescher, A. Urquhart, **Temporal Logic**, Springer Verlag, Vienna–New York (1971).

[18] M. Tkaczyk, **Logika czasu empirycznego: funktor realizacji czasowej w językach teorii fizykalnych** (**Logic of Empirical Time: Functor

of Temporal Realisation in the Languages of Phisical Theories), Wydawnictwo KUL, Lublin (2009).

[19] M. Tkaczyk, *Negation in Weak Positional Calculi*, **Logic and Logical Philosophy**, vol. 22 (2013), pp. 3–19, DOI: https://doi.org/10.12775/LLP.2013.001.

[20] M. Tkaczyk, *Distribution Laws in Weak Positional Logics*, **Roczniki Filozoficzne**, vol. 66 (2018), pp. 163–179, DOI: https://doi.org/10.18290/rf.2018.66.3-8.

Mateusz Klonowski
Nicolaus Copernicus University
Department of Logic
ul. Stanisława Moniuszki 16/20
87-100 Toruń, Poland
e-mail: mateusz.klonowski@umk.pl

Krzysztof Aleksander Krawczyk
Nicolaus Copernicus University
Department of Logic
ul. Stanisława Moniuszki 16/20
87-100 Toruń, Poland
e-mail: krawczyk@doktorant.umk.pl

Bożena Pięta
Nicolaus Copernicus University
Department of Logic
ul. Stanisława Moniuszki 16/20
87-100 Toruń, Poland
e-mail: b.pieta@doktorant.umk.pl

Bulletin of the **S**ection of **L**ogic
Volume 50/2 (2021), pp. 205–228
https://doi.org/10.18778/0138-0680.2021.09

Aleksander Parol
Krzysztof Pietrowicz
Joanna Szalacha-Jarmużek

EXTENDED **MR** WITH NESTING OF PREDICATE EXPRESSIONS AS A BASIC LOGIC FOR SOCIAL PHENOMENA

Abstract

In this article, we present the positional logic that is suitable for the formalisation of reasoning about social phenomena. It is the effect of extending the Minimal Realisation (**MR**) logic with new expressions. These expressions allow, inter alia, to consider different points of view of social entities (humanistic coefficient). In the article, we perform a metalogical analysis of this logic. Finally, we present some simple examples of its application.

Keywords: Logic for social sciences, positional logic, realisation operator, social phenomena.

1. Introduction: Quality vs quantity

Our work aims to develop a new perspective on the possibility of applying positional logic to social sciences issues. This paper can also be perceived as an attempt to build a bridge between philosophical and logical concepts and the specific needs of sociology. However, this analysis does not only refer to classic philosophical theories (as sometimes sociologist did in the past), but also presents the proposal of extension and usage of Minimal Realisation (**MR**) logic for solving an important methodological problem: How to combine qualitative and quantitative perspectives in sociology. The

Presented by: Tomasz Jarmużek, Fengkui Ju, Piotr Kulicki, Beishui Liao
Received: March 17, 2021
Published online: May 28, 2021

problem discussed in this paper is very similar to the issue that could recently be found in [6], concerning: how to build a bridge between big data and thick data in the sociology of the Internet. However, our answer is completely different. This work treats tradition (in this case Jerzy Łoś concepts, which are the foundation for **MR** logic) not only as an important point of reference, but also as a practical 'tool' for contemporary research and vital methodological issues of sociology.

In contemporary sociology, there is a clear division between quantitative and qualitative researchers. Quantitative researchers seek to explain social phenomena in the manner of natural science. The emphasis is therefore on the formalisation, validity, reliability, and looking for cause–effect relationships. The qualitatively oriented researchers focus on meanings, understanding (Verstehen), local descriptions, interpretations and reconstructions of collective ways of perceiving the world.

Our proposal is a continuation of the attempt to build a bridge between the two kinds of research orientations. It seems that the grammatical constructions typical for positional logic, especially Minimal Realisation, allow combining the quantitative formalisation with the humanistic coefficient. The humanistic coefficient concept was developed one hundred years ago by Florian Znaniecki [10], who postulated the need not to limit researchers' observation only to their own direct experience of the data, but to reconstruct the experience of the people who are the subject of the research. Thus, it is a kind of qualitative perspective.

This paper is inspired by [7]. In our article, we develop the programme described there. We extend the **MR** logic with new means of expressions. While in [4] the **MR** logic was extended with multiple positions in the range of operator $\mathcal{R}$ and expressions with predicates, here we take another step forward. We add the expressions with nested predicate expressions in the range of operator $\mathcal{R}$ to the language. It allows us to talk about relations and properties 'from some point of view' which is typical for a qualitative description of social phenomena. Although this is not the final level of extension of **MR**, the logic we propose already permits the description of quite complex social situations. Some examples are provided in section 6.

2. Language and semantics

The purpose of the logical part of this paper is to develop a formal framework for social sciences. Our approach to achieving this goal is to follow the programme article [7]. The cited paper established a strategy of extending the system of Minimal Realisation, in a way suitable for our purpose. We attempt to partially execute this task in the following sections. We start by providing the syntactic and semantic base.

We will use the minimal system for $\mathcal{R}$–operator as a basis for further extensions. This logic, abbreviated as **MR**, was presented for the first time in a paper by [5] as the general system of positional logic. Its minimalism is a result of both, semantic and syntactic weaknesses. Indeed, in the context of the systems preceding it—systems constructed by Łoś, Prior and Rescher—**MR** is characterized by the minimal number of assumptions and the poorest language to express them.

The mentioned weaknesses of the system lead to some unfavorable consequences. Among other things, the poor language reduces the expressive power of the theory built upon it. Such theory may not be sufficient to express facts regarding complex phenomena. On the other hand, the minimalism of the system makes it easy to extend.

In our investigations, we follow the design of extension partially outlined in ([7], pp. 13–16). It requires addition of predicate symbols to the alphabet. By doing so, the language of our logic will consists of: logical connectives $\mathsf{Con} = \{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$, variables $\mathsf{Var} = \{p_i : i \in \mathbb{N}\}$, positional letters $\mathsf{PL} = \{a_i : i \in \mathbb{N}\}$, predicates $\mathsf{PS} = \{P^i_n : i, n \in \mathbb{N}\}$, realisation operator $\mathcal{R}$ and brackets:), (, where $\mathbb{N}$ denotes the set of natural numbers. For the definitions and theorems ahead, let us denote the set of predictate expressions: $\mathsf{PE} = \{P^{\mathsf{i}}_n(\alpha_1, ..., \alpha_i) : P^{\mathsf{i}}_n \in \mathsf{PS},\ \alpha_1, ..., \alpha_i \in \mathsf{PL}$, for some $i, n \in \mathbb{N}\}$. Additional changes are carried out on the level of the grammatical rules. We prefer to extend the class of expressions in a way that allows speaking about the context in the manner of a more complex structure. Therefore, expressions consisting of the $\mathcal{R}$–operator will not contain one positional letter, but a sequence of positional letters of any length. From the semantic point of view, it will allow accounting for more than one context factor, considering the truth value of a given expression (notice that both changes were examined in [4]).

However, the crucial new modification is also on the level of grammatical rules. What is new is that we add to the language the expressions with nested predicate expressions in the range of operator $\mathcal{R}$. The new expressions allow us to talk about relations and properties 'from some point of view' which is characteristic of social phenomena.

Let us start with the introduction of basic syntactic notions.

Definition 2.1 (Auxiliary Expressions). The set of auxiliary expressions AE is the smallest set satisfying the conditions stated as follows:

1. $\mathsf{Var} \subseteq \mathsf{AE}$,
2. $\mathsf{PE} \subseteq \mathsf{AE}$,
3. $\neg A \in \mathsf{AE}$, where $A \in \mathsf{AE}$,
4. $A * B \in \mathsf{AE}$, where $A, B \in \mathsf{AE}$, and $* \in \mathsf{Con} \setminus \{\neg\}$.

Those expressions are in relation to the expressions constructed using the $\mathcal{R}$–operator. That is, the elements of AE are the only expressions that can be in a range of the $\mathcal{R}$–operator. This fact is outlined in the next definition.

Definition 2.2 (Formulas). The set of formulas For is the smallest set satisfying conditions stated as follows:

1. $\mathcal{R}_{\alpha_1,\ldots,\alpha_i}(A) \in \mathsf{For}$, where $A \in \mathsf{AE}$ and $\alpha_1, \ldots, \alpha_i \in \mathsf{PL}$ for some $i \in \mathbb{N}$,
2. $\mathsf{PE} \subseteq \mathsf{For}$,
3. $\neg\phi \in \mathsf{For}$, where $\phi \in \mathsf{For}$,
4. $\phi * \psi \in \mathsf{For}$, where $\phi, \psi \in \mathsf{For}$ and $* \in \mathsf{Con} \setminus \{\neg\}$.

From the set of all formulas, the subset of all formulas that do not contain standard logical connectives outside the range of the $\mathcal{R}$–operator or belong to PE, can be distinguished. We will denote it by $\mathsf{For}_{\mathsf{AT}}$.

To simplify the notation, let us abbreviate $\Gamma, \Gamma_1, \Gamma_2, \ldots$ for any sequences of positional letters $\alpha_1, \ldots \alpha_n$, for some $n \in \mathbb{N}$. The set of all finite sequences of positional letters will be denoted by SE. We can formally construct this set as follows:

$$\mathsf{SE} = \{\alpha_1, \ldots, \alpha_i : \exists_{i \in \mathbb{N}} \forall_{n \in \{1,\ldots,i\}} \alpha_n \in \mathsf{PL}\}.$$

To express the information of the length of a sequence, we will use an upper index. Therefore a sequence of positional letters of a length $i \in \mathbb{N}$

will be denoted by Γ^{i}. Similarly, we will denote the set of all sequences of a given length $i \in \mathbb{N}$ by adding an upper index to the name of this set. For example, the symbol for the set of all sequences of positional letters of the length $i \in \mathbb{N}$, would be SE^{i}. Futher conventions are that any non-empty set of objects will be symbolized by W. Further, $w, w_1, w_2, \ldots$ will denote its elements and by $\mathbf{w}, \mathbf{w}_1, \mathbf{w}_2, \ldots$ we will denote sequences of elements from W of any length. By $\mathbf{W}$ will denote the class of those sequences. Of course, the previous conventions are applicable.

Based on the notions defined above, we present semantics. First, we define the notion of a model for our language. Its definition will be an extension of a corresponding definition provided for Minimal Realisation given in ([5], p. 9).

Definition 2.3 (Model). A model $\mathfrak{M}$ for the set For is any quintuple $\langle W, \mathsf{d}, \delta, \{\delta_{\mathbf{w}}\}_{\mathbf{w}\in\mathbf{W}}, \mathsf{v}\rangle$, where:

- W is a non–empty set of objects,
- $\mathsf{d}\colon \mathsf{SE} \longrightarrow \mathbf{W}$ is such a function that $\forall_{i\in\mathbb{N}}\, \mathsf{d}(\Gamma^{\mathsf{i}}) \in \mathbf{W}^{\mathsf{i}}$,
- $\delta : \mathsf{PS} \longrightarrow \mathcal{P}(\mathbf{W})$ is such a function that $\forall_{i,n\in\mathbb{N}}\, \delta(P^{\mathsf{i}}_n) \subseteq \mathbf{W}^{\mathsf{i}}$,
- $\{\delta_{\mathbf{w}}\}_{\mathbf{w}\in\mathbf{W}}$ is a family of functions $\delta_{\mathbf{w}}$ that fulfil the condition given for δ,
- $\mathsf{v}\colon \mathbf{W} \times \mathsf{AE} \longrightarrow \{0, 1\}$ is a function that for any $\mathbf{w} \in \mathbf{W}$, any $i, n \in \mathbb{N}$, $P^{\mathsf{i}}_n \in \mathsf{PS}$, and $A, B \in \mathsf{AE}$ satisfies the following conditions:
  1. $\mathsf{v}(\mathbf{w}, P^{\mathsf{i}}_n(\Gamma^{\mathsf{i}})) = 1$ iff $\mathsf{d}(\Gamma^{\mathsf{i}}) \in \delta_{\mathbf{w}}(P^{\mathsf{i}}_n)$,
  2. $\mathsf{v}(\mathbf{w}, \neg A) = 1$ iff $\mathsf{v}(\mathbf{w}, A) = 0$,
  3. $\mathsf{v}(\mathbf{w}, A \wedge B) = 1$ iff $\mathsf{v}(\mathbf{w}, A) = 1$ and $\mathsf{v}(\mathbf{w}, B) = 1$,
  4. $\mathsf{v}(\mathbf{w}, A \vee B) = 1$ iff $\mathsf{v}(\mathbf{w}, A) = 1$ or $\mathsf{v}(\mathbf{w}, B) = 1$,
  5. $\mathsf{v}(\mathbf{w}, A \rightarrow B) = 1$ iff $\mathsf{v}(\mathbf{w}, A) = 0$ or $\mathsf{v}(\mathbf{w}, B) = 1$,
  6. $\mathsf{v}(\mathbf{w}, A \leftrightarrow B) = 1$ iff $\mathsf{v}(\mathbf{w}, A) = \mathsf{v}(\mathbf{w}, B)$.

It is worth pointing out three facts. First, in our definition, the arguments of the function d, are sequences of positional letters, not the letters themselves. Therefore, the consistency requires that the function returns a value from the set $\mathbf{W}$ of the length corresponding to the length of the interpreted positional sequence. Second, the same restriction must be imposed on the function δ which ranges across the set of all predicates. Finally, the interpretation of predicates must enable us to treat predicate expressions differently, depending on whether one is in the range of the $\mathcal{R}$–operator or not. In the case of the latter, such expressions could be interpreted by a standard δ function. However, in the case of the former, the interpretation of the expression should be related to the interpretation of the positional sequence bounded by the $\mathcal{R}$–operator. This condition is satisfied by creating a family of $\delta_{\mathbf{w}}$ functions which are defined in the same manner as δ but depending on the $\mathbf{w} \in \mathbf{W}$.

The class of all models satisfying the conditions stated above, will be denoted by $\mathbf{M}$. Considering any model of this class, we would like to evaluate the truth value for any formula in this model. The relation constructed in the next definition, enables us to do so.

Definition 2.4 (Truth in a Model). Let $\mathfrak{M} = \langle W, \mathsf{d}, \delta, \{\delta_{\mathbf{w}}\}_{\mathbf{w}\in\mathbf{W}}, \mathsf{v}\rangle$ and $\mathfrak{M} \in \mathbf{M}$, $\phi \in \mathsf{For}$. A formula ϕ is *true in* $\mathfrak{M}$ (in short: $\mathfrak{M} \vDash \phi$) iff it satisfies the following conditions:

1. if $\phi = \mathcal{R}_\Gamma(A)$ for some $\Gamma \in \mathsf{SE}$ and $A \in \mathsf{AE}$, then $\mathsf{v}(\mathsf{d}(\Gamma), A) = 1$,
2. if $\phi = P^{\mathsf{i}}_n(\Gamma^{\mathsf{i}})$ for some $i, n \in \mathbb{N}$, $\Gamma^{\mathsf{i}}_n \in \mathsf{SE}$ and $P^{\mathsf{i}}_n \in \mathsf{PS}$ then $\mathsf{d}(\Gamma^{\mathsf{i}}) \in \delta(P^{\mathsf{i}}_n)$,
3. if $\phi = \neg\psi$ for some ψ, then it is not that $\mathfrak{M} \vDash \phi$ (in short: $\mathfrak{M} \nvDash \psi$),
4. if $\phi = \psi \wedge \chi$ for some ψ, χ, then $\mathfrak{M} \vDash \psi$ and $\mathfrak{M} \vDash \chi$,
5. if $\phi = \psi \vee \chi$ for some ψ, χ, then $\mathfrak{M} \vDash \psi$ or $\mathfrak{M} \vDash \chi$,
6. if $\phi = \psi \rightarrow \chi$ for some ψ, χ, then $\mathfrak{M} \nvDash \psi$ or $\mathfrak{M} \vDash \chi$,
7. if $\phi = \psi \leftrightarrow \chi$ for some ψ, χ, then $\mathfrak{M} \vDash \psi$ and $\mathfrak{M} \vDash \chi$ or $\mathfrak{M} \nvDash \psi$ and $\mathfrak{M} \nvDash \chi$.

Definition 2.5 (Semantic Consequence Relation). Let $\Lambda \cup \{\phi\} \subseteq \mathsf{For}$. The formula ϕ *follows from the set* Λ *with respect to the set of models* $\mathbf{M}$ (in short: $\Lambda \vDash_{\mathbf{M}} \phi$) iff for any $\mathfrak{M} \in \mathbf{M}$, if for all $\psi \in \Lambda$, $\mathfrak{M} \vDash \psi$ (in short: $\mathfrak{M} \vDash \Lambda$), then $\mathfrak{M} \vDash \phi$. When $\emptyset \vDash_{\mathbf{M}} \phi$, the formula ϕ is called a *tautology* of **M**.

However, as the set of models **M** will be the only one considered here, we will omit its symbol in such contexts. The logic that is determined by the class of models **M**, will be denoted by $\mathbf{MR_{np}}$ as an abbreviation for Minimal Realisation with Nested Predicates.

3. Axiomatic system

In this section, the relation of the syntactic consequence for $\mathbf{MR_{np}}$ is defined. This will be achieved by providing a set of axioms and syntactic rules for the logic. Since **MR** was already presented as an axiomatic system, we take advantage of this fact as it is possible to reuse some of the results concerning the original version of our system.

For this purpose, four axiom schemes are used. To introduce the first one, let us denote the set of all formulas of Classical Propositional Logic (CPL) by $\mathsf{For}_{\mathsf{CPL}}$ and the set of all its tautologies by $\mathbf{Taut}_{\mathsf{CPL}}$. Additionally, we will define the notion of substitution function.

Definition 3.1 (Substitution Function). Substitution function for CPL formulas is any function $\mathfrak{s}\colon \mathsf{For}_{\mathsf{CPL}} \longrightarrow \mathsf{For}$ that for any $\phi, \psi \in \mathsf{For}_{\mathsf{CPL}}$ and $* \in \mathsf{Con} \setminus \{\neg\}$ satisfies following conditions:

1. $\mathfrak{s}(\neg\phi) = \neg\mathfrak{s}(\phi)$,
2. $\mathfrak{s}(\phi * \psi) = \mathfrak{s}(\phi) * \mathfrak{s}(\psi)$.

The first axiom scheme is restricted to CPL tautologies in our language – namely, each substitution of a CPL tautology is an axiom of our logic. The formulation of this scheme in the formal languages looks identical to its formulation in the original version.

Axiom 3.1. $\mathfrak{s}(\phi)$, if $\phi \in \mathbf{Taut}_{\mathsf{CPL}}$ and $\mathfrak{s}$ is a substitution function.

The next two axiom schemes differ in formulation from the corresponding axiom schemes formulations in the original system. Specifically, the $\mathcal{R}$–operator does not bind a positional constant. In our version, it binds a finite sequence of any given length of positional letters. For any expressions $A, B \in \mathsf{AE}$ and any sequence of positional letters $\Gamma \in \mathsf{SE}$ those schemes appear as follows:

Axiom 3.2. $\neg\mathcal{R}_\Gamma A \leftrightarrow \mathcal{R}_\Gamma \neg A$.

Axiom 3.3. $(\mathcal{R}_\Gamma A \wedge \mathcal{R}_\Gamma B) \rightarrow \mathcal{R}_\Gamma(A \wedge B)$.

The last axiom scheme presents the idea that a CPL tautology is true in any given context.

Axiom 3.4. $\mathcal{R}_\Gamma A$, if $A \in \mathbf{Taut}_{\mathsf{CPL}}$.

Besides the aforementioned schemes, we assume the Modus Ponens rule (in short: MP).

$$\frac{\phi, \phi \rightarrow \psi}{\psi}$$

The set of axioms we will denote by $\mathbf{MR^{ax}}$. Having $\mathbf{MR^{ax}}$, we accept the standard notion of syntactic consequence relation.

Definition 3.2 (Syntactic Consequence Relation). Let $\Lambda \cup \{\phi\} \subseteq \mathsf{For}$. The formula ϕ *is provable based on the set* Λ *with respect to* $\mathbf{MR^{ax}}$ (in short: $\Lambda \vdash_{\mathbf{MR^{ax}}} \phi$) iff there is such a sequence of formulas: $\psi_1, \ldots, \psi_n$ that $\psi_n = \phi$ and for all $1 \leq i \leq n$ if at least one of the below conditions is fulfilled:

1. $\psi_i \in \Lambda$
2. $\psi_i \in \mathbf{MR^{ax}}$
3. for some $j, k < i$ there exist such ψ_j, ψ_k that $\psi_k = \psi_j \rightarrow \psi_i$.

When $\emptyset \vdash_{\mathbf{MR^{ax}}} \phi$, the formula ϕ is called a *thesis*.

Since we consider only one axiomatic system, we will write $\vdash$ rather than $\vdash_{\mathbf{MR^{ax}}}$ to simplify the notation. Using those concepts, we will introduce the notion of a maximal consistent set.

Definition 3.3 ($\mathbf{MR^{ax}}$–consistent | $\mathbf{MR^{ax}}$–inconsistent Set of Formulas). Let $\Delta \subseteq \mathsf{For}$. Then:

- Δ is called an $\mathbf{MR^{ax}}$–consistent set of formulas iff $\Delta \nvdash \phi$, for some $\phi \in \mathsf{For}$,

- Δ is called an $\mathbf{MR^{ax}}$–inconsistent set of formulas iff it is not $\mathbf{MR^{ax}}$–consistent.

For any $\mathbf{MR^{ax}}$–consistent set the standard facts about consistent sets hold. This is a consequence of the fact that our logic is founded on CPL as it contains axiom 3.1 and MP. Basing on the previous definition, we can construct the notion of a maximal $\mathbf{MR^{ax}}$–consistent set of formulas.

Definition 3.4 (Maximal $\mathbf{MR^{ax}}$–consistent Set). Let $\Delta \subseteq \mathsf{For}$. We call Δ a maximal $\mathbf{MR^{ax}}$–consistent iff both:

1. Δ is $\mathbf{MR^{ax}}$–consistent,
2. for any $\Lambda \subseteq \mathsf{For}$ if $\Delta \subset \Lambda$, then Λ is $\mathbf{MR^{ax}}$–inconsistent.

Using the symbol $\mathbf{Max_{MR^{ax}}}$, we will denote the class of all maximal $\mathbf{MR^{ax}}$–consistent sets. Some intuitions about the properties of those sets are expressed by the next three facts.

The first fact says that such sets are closed under the syntactic consequence relation. Therefore any formula $\phi \in \mathsf{For}$, for which there exists proof based on the maximal $\mathbf{MR^{ax}}$–consistent set, has to be a element of such set. And conversely, if a formula is an element of a maximal $\mathbf{MR^{ax}}$–consistent set, there is a proof of the formula on the ground of this set.

Fact 3.5. Let $\Delta \in \mathbf{Max_{MR^{ax}}}$ and $\phi \in \mathsf{For}$. Then $\Delta \vdash \phi$ iff $\phi \in \Delta$.

The next fact expresses the relation between the set $\mathbf{MR^{ax}}$ and a maximal $\mathbf{MR^{ax}}$–consistent set. More specifically it states that all formulas from $\mathbf{MR^{ax}}$ are contained in such a set.

Fact 3.6. Let $\Delta \in \mathbf{Max_{MR^{ax}}}$. Then $\mathbf{MR^{ax}} \subseteq \Delta$.

The last of the aforementioned facts states that any maximal $\mathbf{MR^{ax}}$ –consistent set is closed under the listed conditions.

Fact 3.7. Let $\Delta \in \mathbf{Max_{MR^{ax}}}$. Then for any $\phi, \psi \in \mathsf{For}$ it is true that:

- $\neg\phi \in \Delta$ iff $\phi \notin \Delta$,
- $\phi \wedge \psi \in \Delta$ iff $\phi \in \Delta$ and $\psi \in \Delta$,
- $\phi \vee \psi \in \Delta$ iff $\phi \in \Delta$ or $\psi \in \Delta$,

- $\phi \to \psi \in \Delta$ iff $\phi \notin \Delta$ or $\psi \in \Delta$,
- $\phi \leftrightarrow \psi \in \Delta$ iff $\phi \in \Delta$ and $\psi \in \Delta$ or $\phi \notin \Delta$ and $\psi \notin \Delta$.

The most important theorem concerning the maximal $\mathbf{MR^{ax}}$–consistent sets is the so–called Lindenbaum's Lemma. It states that any $\mathbf{MR^{ax}}$–consistent set is a subset of some maximal $\mathbf{MR^{ax}}$–consistent set.

LEMMA 3.8. *Let $\Lambda \subseteq \mathsf{For}$. Then if Λ is $\mathbf{MR^{ax}}$–consistent, there is such $\Delta \subseteq \mathsf{For}$ that $\Lambda \subseteq \Delta$ and $\Delta \in \mathbf{Max}_{\mathbf{MR^{ax}}}$.*

4. Soudness and completeness

In the previous sections, we established the relations of semantic and syntactic consequences. With that in mind, in this section, we investigate a relationship between those two relations and provide a list of theorems and facts regarding this relationship. Two of the main results that we want to present in this section are soundness and completeness of our logic.

To obtain the former result, we will need to first prove the following lemma.

LEMMA 4.1. *For any formula $\phi \in \mathbf{MR^{ax}}$, it is also a tautology.*

PROOF: Of course, the substitution of any tautology of CPL is a tautology of our logic by the notion of the substitution function defined in 3.1 and the truth conditions 2.4. Therefore, any formula that is an instance of an axiom 3.1 is a tautology of our system.

Now let us assume that for any $\mathfrak{M} \in \mathbf{M}$, $\mathfrak{M} \models \neg\mathcal{R}_\Gamma(A)$, for some $\Gamma \in \mathsf{SE}$ and $A \in \mathsf{AE}$. Then according to definition 2.4, it is the case iff $\mathfrak{M} \not\models \mathcal{R}_\Gamma(A)$ and thus $\mathsf{v}(\mathsf{d}(\Gamma), A) = 0$. By definition 2.3, it is equivalent to the $\mathsf{v}(\mathsf{d}(\Gamma), \neg A) = 1$ and thus $\mathfrak{M} \models \mathcal{R}_\Gamma(\neg A)$.

To prove that the axiom scheme 3.3 is tautological, let us assume that for a $\mathfrak{M} \in \mathbf{M}$, $\mathfrak{M} \models (\mathcal{R}_\Gamma A \wedge \mathcal{R}_\Gamma B)$. Then, based on the definition 2.4, it is the case iff $\mathfrak{M} \models \mathcal{R}_\Gamma A$ and $\mathfrak{M} \models \mathcal{R}_\Gamma B$. According to the same definition, by equivalence we obtain $\mathsf{v}(\mathsf{d}(\Gamma), A) = 1$ and $\mathsf{v}(\mathsf{d}(\Gamma), B) = 1$ and using definition 2.3, it is the case iff $\mathsf{v}(\mathsf{d}(\Gamma), A \wedge B) = 1$. And thus, equivalently, $\mathfrak{M} \models \mathcal{R}_\Gamma(A \wedge B)$.

Further, let us assume that $A \in \mathbf{Taut}_{\mathsf{CPL}}$ and $\mathfrak{M} \not\models \mathcal{R}_\Gamma A$, for some $\mathfrak{M} \in \mathbf{M}$. According to the definition 2.4, $\mathfrak{M} \not\models \mathcal{R}_\Gamma A$ iff $\mathsf{v}(\mathsf{d}(\Gamma), A) = 0$ for some valuation function v. Then by definition 2.3, valuation v falsifies formula A. However, as A is a tautology, it leads to an immediate contradiction. $\square$

THEOREM 4.2 (Soundness). *Let $\Lambda \cup \{\phi\} \subseteq \mathsf{For}$. If $\Lambda \vdash \phi$, then $\Lambda \vDash \phi$.*

PROOF: Let us assume that $\Lambda \vdash \phi$, $\mathfrak{M} \in \mathbf{M}$, and that all elements of Λ are true in the model $\mathfrak{M} \in \mathbf{M}$. Thus, according to definition 3.2 there exists such a sequence of formulas $\psi_1, ..., \psi_n$ that $\psi_n = \phi$, for some $n \in \mathbb{N}$. We prove that $\mathfrak{M} \vDash \psi_i$, for $1 \leq i \leq n$, and thus $\mathfrak{M} \vDash \phi$, since $\psi_n = \phi$.

If we assume that $n = 1$, there are two possible cases — $\psi_1 \in \Lambda$ or $\psi_1 \in \mathbf{MR^{ax}}$. Consider the first one according to the assumption $\mathfrak{M} \vDash \psi_1$. In the second case, due to lemma 4.1, $\mathfrak{M} \vDash \psi_1$. Since the sequence is of the length one, we obtain $\mathfrak{M} \vDash \phi$.

Let us assume that $n > 1$. We make an induction, based on the length of the assumed sequence. The initial step is similar to the case when $n = 1$. So, for the inductive step we assume that for some $1 \leq k < n$, if $j \leq k$, then $\mathfrak{M} \vDash \psi_j$. Now, let us consider the formula ψ_{k+1}. There are the following three possibilities:

1. $\psi_{k+1} \in \mathbf{MR^{ax}}$,
2. $\psi_{k+1} \in \Lambda$,
3. there exists ψ_l, ψ_m, such that $\psi_m = \psi_l \rightarrow \psi_{k+1}$, for some $l, m \leq k$.

The first two cases are similar to the case when $n = 1$. Now consider the third one. As $l, m \leq k$, so by the inductive hypothesis, $\mathfrak{M} \vDash \psi_l$ and $\mathfrak{M} \vDash \psi_l \rightarrow \psi_{k+1}$. Now, according to definition 2.4, we get that $\mathfrak{M} \nvDash \psi_l$ or $\mathfrak{M} \vDash \psi_{k+1}$. Thus, $\mathfrak{M} \vDash \psi_{k+1}$. $\square$

Having thus proved the soundness of $\mathbf{MR_{np}}$, we will prove the converse implication—that is the completeness theorem. It requires us to define the notion of a canonical model. It is a special structure that is also a model for our logic interpreted within the set of formulas. However, first we define the notion of canonical quasi-model.

DEFINITION 4.3 (Canonical Quasi-Model). Let $\Delta \in \mathbf{Max_{MR^{ax}}}$. *A canonical quasi-model* is a quintuple $\langle W_\Delta, \mathsf{d}_\Delta, \delta_\Delta, \{\delta_{\Delta_\Gamma}\}_{\Gamma \in W_\Delta}, \mathsf{v}_\Delta \rangle$ such that:

- $W_\Delta = \mathsf{SE}$,
- $\mathsf{d}_\Delta \colon \mathsf{SE} \longrightarrow W_\Delta$ such that $\forall_{i \in \mathbb{N}}\, \mathsf{d}_\Delta(\Gamma^{\mathsf{i}}) = \Gamma^{\mathsf{i}}$,

- $\delta_\Delta\colon \mathsf{PS} \longrightarrow \mathcal{P}(W_\Delta)$ such that $\forall_{i,n\in\mathbb{N}}\, \delta_\Delta(P^{\mathsf{i}}_n) = \{\Gamma^{\mathsf{i}}\colon P^{\mathsf{i}}_n(\Gamma^{\mathsf{i}}) \in \Delta\}$,
- $\{\delta_{\Delta_\Gamma}\}_{\Gamma\in W_\Delta}$ is the family of functions δ_{Δ_Γ} such that $\forall_{i,n\in\mathbb{N}}\, \delta_{\Delta_\Gamma}(P^{\mathsf{i}}_n) = \{\Gamma^{\mathsf{i}}\colon \mathcal{R}_\Gamma(P^{\mathsf{i}}_n(\Gamma^{\mathsf{i}})) \in \Delta\}$,
- $v_\Delta\colon W_\Delta \times (\mathsf{Var} \cup \mathsf{PS}) \longrightarrow \{0,1\}$ such that:
  1. for any $A \in \mathsf{Var}$, $v_\Delta(\Gamma, A) = 1$ iff $\mathcal{R}_\Gamma(A) \in \Delta$,
  2. for any $P^{\mathsf{i}}_n \in \mathsf{PS}$, $i, n \in \mathbb{N}$ and $\Gamma_1 \in \mathsf{SE}$, $v_\Delta(\Gamma, P^{\mathsf{i}}_n(\Gamma_1)) = 1$ iff $\mathsf{d}_\Delta(\Gamma_1) \in \delta_{\Delta_\Gamma}(P^{\mathsf{i}}_n)$.

The above definition presents a structure that does not fully correspond to the definition of a model for $\mathbf{MR_{np}}$. The conditions for the function of valuation in definition 2.3 contain cases of complex expressions formed with logical connectives. In the latter definition, only the primitive cases are considered explicitly. The following lemma will prove that a structure satisfying conditions given in definition 4.3 satisfies the conditions from definition 2.3—for complex expressions within the $\mathcal{R}$–operator.

Fact 4.4. Let $\Delta \in \mathbf{Max_{MR^{ax}}}$ and $\mathfrak{M} = \langle W_\Delta, \mathsf{d}_\Delta, \delta_\Delta, \{\delta_{\Delta_\Gamma}\}_{\Gamma\in W_\Delta}, \mathsf{v}_\Delta\rangle$ be a canonical quasi-model. Then it can be extended to a canonical model (in short: Δ–model).

Proof: Assume all the hypotheses. The fact that $\mathfrak{M}$ can be extended to a Δ–model is equivalent to the fact that the function v_Δ can be extended to range over $W_\Delta \times \mathsf{AE}$. Thus, that it satisfies the following conditions:

1. for any $A \in \mathsf{Var}$, $v_\Delta(\Gamma, A) = 1$ iff $\mathcal{R}_\Gamma(A) \in \Delta$,
2. for any $P^{\mathsf{i}}_n \in \mathsf{PS}$, $i, n \in \mathbb{N}$ and $\Gamma_1 \in \mathsf{SE}$, $v_\Delta(\Gamma, P^{\mathsf{i}}_n(\Gamma_1)) = 1$ iff $\mathsf{d}_\Delta(\Gamma_1) \in \delta_{\Delta_\Gamma}(P^{\mathsf{i}}_n)$,
3. $\mathsf{v}_\Delta(\Gamma, \neg A) = 1$ iff $\mathsf{v}_\Delta(\Gamma, A) = 0$,
4. $\mathsf{v}_\Delta(\Gamma, A \wedge B) = 1$ iff $\mathsf{v}_\Delta(\Gamma, A) = 1$ and $\mathsf{v}_\Delta(\Gamma, B) = 1$,
5. $\mathsf{v}_\Delta(\Gamma, A \vee B) = 1$ iff $\mathsf{v}_\Delta(\Gamma, A) = 1$ or $\mathsf{v}_\Delta(\Gamma, B) = 1$,
6. $\mathsf{v}_\Delta(\Gamma, A \rightarrow B) = 1$ iff $\mathsf{v}_\Delta(\Gamma, A) = 0$ or $\mathsf{v}_\Delta(\Gamma, B) = 1$,
7. $\mathsf{v}_\Delta(\Gamma, A \leftrightarrow B) = 1$ iff $\mathsf{v}_\Delta(\Gamma, A) = \mathsf{v}_\Delta(\Gamma, B)$.

Assume that $C \in \mathsf{AE}$. Let us provide a proof for this fact by induction over complexity of the considered expression. Thus if $C \in \mathsf{Var} \cup \mathsf{PS}$ the theorem is fulfilled due to the fact that $\mathfrak{M}$ is a canonical quasi-model.

Suppose that the theorem is true for the expressions of the complexity equal to n. Let us consider an expression C of the complexity equal to $n+1$. Then it is the case that for some $A, B \in \mathsf{AE}$, C is one of the listed forms $\neg A, A \wedge B, A \vee B, A \to B, A \leftrightarrow B$. Therefore, basing on the definition 4.3, distributivity laws for $\mathcal{R}$ and the classical connectives (see [5], pp. 151–153) and facts 3.7 and 3.5, we obtain the thesis. $\square$

A structure named as the Δ–model should possess a certain property. Namely, any formula that is true in that model should be also an element of the maximal $\mathbf{MR^{ax}}$–consistent set on which the canonical model is based. The fact of possessing this mentioned property is expressed by the next lemma.

Lemma 4.5. *Let* $\Delta \in \mathbf{Max}_{\mathbf{MR^{ax}}}$, $\mathfrak{M}$ *be the* Δ*–model. Then for any* $\phi \in \mathsf{For}$ *it is the case that* $\mathfrak{M} \models \phi$ *iff* $\phi \in \Delta$.

Proof: We will present the proof by induction over the complexity of the formulas. Consider the initial case where $\phi \in \mathsf{For}_{\mathsf{AT}}$. Then the formula ϕ is an expression created with the $\mathcal{R}$–operator, a positional sequence and an expression of the set AE or ϕ is a predicate expression.

Let us assume the former case, that is $\phi = \mathcal{R}_\Gamma(A)$ for some $\Gamma \in \mathsf{SE}$ and $A \in \mathsf{AE}$. We will use here fact 4.4. We have $\mathfrak{M} \models \mathcal{R}_\Gamma(A)$ iff $\mathcal{R}_\Gamma(A) \in \Delta$, if $A \in \mathsf{Var}$. If $A \in \mathsf{PE}$, then let us notice that the condition $\mathsf{d}_\Delta(\Gamma_1) \in \delta_{\Delta_\Gamma}(P^{\mathsf{i}}_n)$, for some $P^{\mathsf{i}}_n \in \mathsf{PS}$, $i, n \in \mathbb{N}$ and $\Gamma_1 \in \mathsf{SE}$, is equivalent to the statement $\mathcal{R}_\Gamma(P^{\mathsf{i}}_n(\Gamma_1)) \in \Delta$. Cases for non-atomic expressions follows also from the fact 4.4. Consider the latter case that ϕ is a predicate expression. Then the condition $\mathsf{d}_\Delta(\Gamma_1) \in \delta_\Delta(P^{\mathsf{i}}_n)$, for all $P^{\mathsf{i}}_n \in \mathsf{PS}$, $i, n \in \mathbb{N}$ and $\Gamma_1 \in \mathsf{SE}$, is equivalent to the statement $P^{\mathsf{i}}_n(\Gamma_1) \in \Delta$.

Assume that the hypothesis is satisfied for all formulas ϕ with complexity equal or lesser than n, for some $n \in \mathbb{N}$. We will prove also that it is satisfied for expressions of the complexity equal to $n+1$. In such case, we should consider the following formulas: $\neg\psi$, $\psi \wedge \chi$, $\psi \vee \chi$, $\psi \to \chi$ or $\psi \leftrightarrow \chi$ for some ψ and χ with the complexities equal or lesser than n. We analyse the cases for $\neg$ and $\wedge$. For the rest of them, the proof could be carried out analogously.

Assume the former case first. Then $\mathfrak{M} \models \neg\psi$ is equivalent to $\mathfrak{M} \not\models \psi$. From the inductive hypothesis, it is equivalent to the fact that $\psi \notin \Delta$. From the assumption that Δ is the maximal $\mathbf{MR^{ax}}$–consistent, we get the next equivalent fact $\neg\psi \in \Delta$.

Assume the second case. Then $\mathfrak{M} \vDash \psi \wedge \chi$ is equivalent to $\mathfrak{M} \vDash \psi$ and $\mathfrak{M} \vDash \chi$. From the fact that complexity of both formulas is lesser or equal to n and the inductive hypothesis, we obtain by equivalence $\psi, \chi \in \Delta$. As Δ is the maximal $\mathbf{MR^{ax}}$–consistent, it is equivalent to the fact that $\psi \wedge \chi \in \Delta$. □

This lemma is crucial for the next theorem. It expresses the fact that for any $\mathbf{MR^{ax}}$–consistent set of formulas, there exists a canonical model in which all the formulas from the set are true.

Theorem 4.6. *Let $\Lambda \subseteq \mathsf{For}$ be a $\mathbf{MR^{ax}}$–consistent set. Then there exists such $\Delta \in \mathbf{Max}_{\mathbf{MR^{ax}}}$ that $\mathfrak{M}$ is a Δ–model and $\mathfrak{M} \vDash \Lambda$.*

Proof: Assume the hypothesis. As Λ is a $\mathbf{MR^{ax}}$–consistent set of formulas, from Lindenbaum's Lemma (lemma 3.8), there exists such a set of formulas $\Delta \in \mathbf{Max}_{\mathbf{MR^{ax}}}$ that $\Lambda \subseteq \Delta$. Then according to the definition 4.3 and the fact 4.4 there exists a Δ–model $\mathfrak{M}$. According to the lemma 4.5 – for any formula $\phi \in \Delta$ it is the case that $\mathfrak{M} \vDash \phi$, where $\mathfrak{M}$ is Δ-model. It leads to the conclusion that $\mathfrak{M} \vDash \Lambda$. □

Theorem 4.7 (Completness Theorem). *Let $\Lambda \cup \{\phi\} \subseteq \mathsf{For}$. If $\Lambda \vDash \phi$, then $\Lambda \vdash \phi$.*

Proof: Assume all hypotheses. Moreover, suppose that $\Lambda \nvdash \phi$. We know that $\Lambda \cup \{\neg\phi\}$ is $\mathbf{MR^{ax}}$–consistent. According to the lemma 3.8, there exists such a maximal $\mathbf{MR^{ax}}$–consistent set $\Delta \in \mathbf{Max}_{\mathbf{MR^{ax}}}$ that $\Lambda \cup \{\neg\phi\} \subseteq \Delta$. Therefore, according to the theorem 4.6, $\mathfrak{M} \vDash \Lambda \cup \{\neg\phi\}$ where $\mathfrak{M}$ is the Δ–model. As a result, we obtain $\Lambda \nvDash \phi$. □

5. Expressive power of presented logic

As we outlined in the previous sections, our system emerges from the Minimal Realisation logic by expanding its alphabet and grammatical rules. The axioms schemes for $\mathbf{MR_{np}}$ have a similar form to the corresponding axiom schemes of the original system. Therefore, in this paper, we expanded the language of our logic, preserving the theses of the base system.

Knowing this, we show, that there exists a mapping from the set of $\mathbf{MR_{np}}$ formulas (in short: $\mathsf{For}_{\mathbf{MR_{np}}}$) into the set of $\mathbf{MR}$ formulas (in short: $\mathsf{For}_{\mathbf{MR}}$). Using this mapping, we prove that any expression of the former system is a thesis if and only if it can be mapped into a corresponding thesis of the latter. The mentioned proof will be provided in a manner similar as in [4] (see: [4], p. 361).

To construct such a mapping, first, we will focus on the expressions that are in the range of the $\mathcal{R}$–operator, that is expressions from the class AE. As this class was extended by the addition of predicate expressions, in a comparison to the system of Minimal Realisation, the mapping must take into account such cases. To distinguish the mentioned classes of expressions for the two systems, we consider in the paper, we add a system name in the lower index, similar to the classes of formulas.

Definition 5.1. $\mu : \mathsf{AE}_{\mathbf{MR_{np}}} \longrightarrow \mathsf{AE}_{\mathbf{MR}}$:

- $\mu(A) = A$ if $A \in \mathsf{Var}$,
- $\mu(A) = p$ if $A \in \mathsf{PE}$, for some $p \in \mathsf{Var}$,
- $\mu(\neg A) = \neg\mu(A)$,
- $\mu(A * B) = \mu(A) * \mu(B)$ for $* \in \mathsf{Con} \setminus \{\neg\}$.

Of course, we have the continuum of such mappings. Any of them serves as the identity function for all such expressions, that do not contain predicates. Otherwise, it translates them into their version in which every occurrence of a predicate expression is replaced by a given sentential variable. Thus, we will define it as a mapping ranging over the class of formulas.

Definition 5.2. $\sigma\colon \mathsf{For}_{\mathbf{MR_{np}}} \longrightarrow \mathsf{For}_{\mathbf{MR}}$:

- $\sigma(\mathcal{R}_{\alpha_1,\ldots,\alpha_n}(A)) = \mathcal{R}_{\alpha_i}(\mu(A))$, for some $1 \leq i \leq n$, any $n \in \mathbb{N}$, $A \in \mathsf{AE}$, and some μ,
- $\sigma(\phi) = \mathcal{R}_{\alpha}(\mu(\phi))$, if $\phi \in \mathsf{PE}$, for some $\alpha \in \mathsf{PL}$, and for some μ,
- $\sigma(\neg\phi) = \neg\sigma(\phi)$,
- $\sigma(\phi * \psi) = \sigma(\phi) * \sigma(\psi)$ for $* \in \mathsf{Con} \setminus \{\neg\}$.

As $\mathbf{MR_{np}}$ is extended by allowing the $\mathcal{R}$–operator to bind sequences of positional letters, defined mapping must accordingly translate them into the expressions of **MR**. This is done by mapping the positional sequence

$\alpha_1, \ldots, \alpha_n$ onto one of its elements. Another case that was not present in the original system is the validity of the formula containing a predicate expression outside the $\mathcal{R}$—operator. This case is considered in the second point of the above definition. A predicate expression is translated into an $\mathcal{R}$—operator expression in the context symbolized by some arbitrarily chosen positional letter. The expression within the $\mathcal{R}$—operator is a translation of the predicate by the previously defined mapping μ.

With the notions defined above, we will attempt to set a certain correspondence between both systems. The following theorem will present this relationship.

THEOREM 5.3. *Let $\phi \in \mathsf{For}$. Then $\vDash_{\mathbf{MR_{np}}} \phi$ iff for any σ, $\vDash_{\mathbf{MR}} \sigma(\phi)$.*

PROOF: Assume the hypothesis. Moreover, let us assume that $\vDash_{\mathbf{MR_{np}}} \phi$. Then, according to theorem 4.7 $\vdash_{\mathbf{MR_{np}}} \phi$. Thus, there exists a proof of a formula ϕ within our system. As the class of axioms does not contain any specific axiom for predicate expressions, it suffices that the proof will be repeated for $\sigma(\phi)$ just by mapping all its elements into the class $\mathsf{For}_{\mathbf{MR}}$. This is since any instance of an $\mathbf{MR_{np}}$ axiom scheme χ is an instance of the $\mathbf{MR}$ axiom scheme after translating $\sigma(\chi)$ for any mapping σ. Hence, if $\psi_1, ..., \psi_i$ for some $i \in \mathbb{N}$ is a proof of ϕ in $\mathbf{MR_{np}}$, then $\sigma(\psi_1), ..., \sigma(\psi_i)$ is a proof of $\sigma(\phi)$ in $\mathbf{MR}$. By the correctness theorem for $\mathbf{MR}$ ([5], p. 155), we obtain $\vDash_{\mathbf{MR}} \sigma(\phi)$.

Now let us assume that $\vDash_{\mathbf{MR}} \sigma(\phi)$ for any σ. If it is the case for any function σ, then especially it is the case for all injective functions. Let us consider such a function. After restricting, the function would satisfy conditions for a bijection $\sigma_{\mathbf{bi}}$. Thus, there exists an inverse function $\sigma_{\mathbf{bi}}^{-1}$. According to the completness theorem for $\mathbf{MR}$ ([5], p. 159), we obtain $\vdash_{\mathbf{MR}} \sigma_{\mathbf{bi}}(\phi)$. It is equivalent to the fact that there exists a proof $\psi_1, ..., \psi_n$ for some $n \in \mathbb{N}$ and for $1 \leq i \leq n$, $\psi_i \in \mathsf{For}_{\mathbf{MR}}$, where $\psi_n = \sigma_{\mathbf{bi}}(\phi)$. Due to the fact, that $\sigma_{\mathbf{bi}}^{-1}$ maps axioms of $\mathbf{MR}$ into some form of $\mathbf{MR_{np}}$ axioms and the MP preserves its properties after such mapping, we get that there exists such a proof $\sigma_{\mathbf{bi}}^{-1}(\psi_1), ..., \sigma_{\mathbf{bi}}^{-1}(\psi_n)$ for some $n \in \mathbb{N}$ and for $1 \leq i \leq n$, $\psi_i \in \mathsf{For}_{\mathbf{MR_{np}}}$ where $\psi_n = \sigma_{\mathbf{bi}}^{-1}(\sigma_{\mathbf{bi}}(\phi))$. It is equivalent to the fact that $\vdash_{\mathbf{MR_{np}}} \phi$. According to the theorem 4.7, we obtain $\vDash_{\mathbf{MR_{np}}} \phi$. □

6. Applications in social sciences

This part of the article is devoted to the presentation of an example of the application of $\mathbf{MR_{np}}$ logic in the sociological perspective. As was stated in the introduction (section 1), of all social sciences we identifies sociology as the science with greater methodological challenges.

Sociology functions as a conglomerate of various theoretical approaches, paradigms, and at the same time it operates in a multitude of sub–disciplines that often intersect with other social sciences (e.g. sociology of politics with political science or sociology of knowledge with philosophy). Among the classic distinction between ways of conducting sociological research, there is also a permanent division into macro, meso and micro–sociology. Each of these subdivisions focuses on other objects and dimensions of social life – from long–term global or national processes, through analyzes of the life of an organization, to sociometric analyzes of interpersonal relations created at the crossroad of sociology and psychology. Whether we are talking about macrosociology or the analysis of small social groups, at each level there is an attempt to find repetitive patterns in complex, multi–layered interpersonal relations. At each of these levels, three issues are also present as follows:

(1) The problem of the complexity of the analyzed world (including individual and collective actors, interactions, cultural patterns, power mechanisms, material resources, etc.),

(2) The problem of actors' self–awareness, and

(3) How this awareness influences the course of the analyzed processes over time and space.

The application of extensions of **MR** logic to sociological concepts and theories is a vast task considering how vast and complex is the field of sociological theory. Therefore, we propose that such a task must start with a reference to basic sociological concepts such as behaviour or interaction. The proposal stated below should be perceived as a kind of a 'sample' in the wider project of application of $\mathbf{MR_{np}}$ logic to the sociological perspective. For this, we decided to use the classical, behavioral postulates by George C. Homans. There are two reasons for this choice. First, there have been attempts in sociology to present Homans' concepts using the language of logic [8]. So our proposal fits into historical research. Second, in our opinion since this is specifically the first case of application of $\mathbf{MR_{np}}$ logic to the social sciences – starting with postulates about basic forms of social relations is the right way to do. In his works Homans presented a very reductionist perspective on human relations and famously formulated 5 postulates (laws) of human interactions. In Homans theory, each social process starts with specific human behaviour and each social interaction starts with human contact. The general concept is that there are patterns of human behaviour that influence interactions and therefore have an impact on the shape of the whole society. Homans' theory is an example of sociologists' attempts to formulate theorems that would have a general range, as much as possible. However it is also an example of a theory that lacks a humanistic approach and is blind to the issue of providing an insight into deeper meanings and understandings of social situations [3].

Therefore we have picked two postulates from Homans' 5 laws of interaction (given below as (P1) and (P2)). These postulates are formulated from a rather 'objective' perspective and are already in quite a formal manner. This means that they lack the humanistic coefficient, or a kind of an in–sight into a subjective perception of human behaviour.

(P1) The more often a particular action of a person is rewarded, the more likely the person is to perform that action.

(P2) If in the past the occurrence of a particular stimulus, or set of stimuli, has been the occasion on which a person's action was rewarded, then the more similar the present stimuli are to the past ones, the more likely the person is to perform the action, or some similar action.

To formalize (P1) and (P2) we need a theory built upon $\mathbf{MR_{np}}$. In the language we distinguish three predicate constants that are read as given on the right:

$\mathsf{Fr}(x, y)$	*x is a smaller frequency than y*
$\mathsf{Prob}(x, y)$	*x is a less probability than y*
$\mathsf{Sim}(x, y, z)$	*y is a stimulus less similar to the stimulus x than the stimulus z is.*

We assume that predicates $\mathsf{Fr}(x, y)$ and $\mathsf{Prob}(x, y)$ are among others irreflexive and transitive (so, also asymmetric):

(Irreflexivity Fr) $\neg\mathsf{Fr}(\alpha_1, \alpha_1)$

(Transitivity Fr) $\mathsf{Fr}(\alpha_1, \alpha_2) \wedge \mathsf{Fr}(\alpha_2, \alpha_3) \to \mathsf{Fr}(\alpha_1, \alpha_3)$

(Irreflexivity Prob) $\neg\mathsf{Prob}(\alpha_1, \alpha_1)$

(Transitivity Prob) $\mathsf{Prob}(\alpha_1, \alpha_2) \wedge \mathsf{Prob}(\alpha_2, \alpha_3) \to \mathsf{Prob}(\alpha_1, \alpha_3)$

Instead of single-argument predicates *being a reward* and *perform*, to simplify a description, we introduce positional letters $\mathsf{perform}$ and reward. Further an agent will be denoted by positional letter a. In the end, we should add that with metavariable A, B with the set of values AE we will understand objects of an agents's activity in a specific context. Thus such activities can be quite complex.

(P1 form) $$\mathsf{Fr}(b_1, b_2) \wedge \mathsf{Prob}(c_1, c_2) \wedge \mathcal{R}_{a,b_1,\mathsf{reward}}(A) \wedge \mathcal{R}_{a,b_2,\mathsf{reward}}(B) \to \\ \to \mathcal{R}_{a,c_1,\mathsf{perform}}(A) \wedge \mathcal{R}_{a,c_2,\mathsf{perform}}(B)$$

Homans' (P1) has been written above using the language of our logic, highlighted predicates and positional letters. It must be mentioned here, that our formalisation reveals a hidden comparison that is stated in the original postulate (formulated in the English language). The same happens in the formalisation of (P2). The expressions used there, namely, 'the more often...' and 'the more likely...' express a comparison of the level of the rewarding degree of a subject's action and the probability of taking on this action, with the level of rewarding and the probability of taking up another, different action, from what was presented in our formalisation. We compare actions A and B. The formula $\mathcal{R}_{a,b_1,\mathsf{reward}}(A)$ says, that an agent a is rewarded for activity A with a frequency b_1, and the formula $\mathcal{R}_{a,b_2,\mathsf{reward}}(B)$ states, that an agent a is rewarded for activity B with a frequency b_2. Because the frequency b_1 is smaller than b_2 (the formula $\mathsf{Fr}(b_1, b_2)$), the agent a will take on activity B (formula $\mathcal{R}_{a,c_2,\mathsf{perform}}(B)$) with greater probability (formula $\mathsf{Prob}(c_1, c_2)$) than activity A (formula $\mathcal{R}_{a,c_1,\mathsf{perform}}(A)$).

Let us address the formalisation of Homans' (P2):

(P2 form) $$\mathcal{R}_{a,s_1,\mathsf{perform},\mathsf{reward}}(A) \wedge \mathsf{Sim}(s_1, s_2, s_3) \wedge \mathsf{Prob}(c_1, c_2) \to \\ \to \mathcal{R}_{a,s_2,c_1,\mathsf{perform}}(A) \wedge \mathcal{R}_{a,s_3,c_2,\mathsf{perform}}(A)$$

Formula $\mathcal{R}_{a,s_1,\mathsf{perform},\mathsf{reward}}(A)$ says, that an agent a performed an activity A because of an stimulus s_1, which later has been rewarded. If this formula is true and a stimulus s_2 is less likely than stimulus s_1 or stimulus s_3 (formula $\mathsf{Sim}(s_1, s_2, s_3)$), then it is less likely that an agent a will take on an activity A because of the stimulus s_2 (formula $\mathcal{R}_{a,s_2,c_1,\mathsf{perform}}(A)$) than because of the stimulus s_3 (formula $\mathcal{R}_{a,s_3,c_2,\mathsf{perform}}(A)$), where $\mathsf{Prob}(c_1, c_2)$ says, that c_1 is a smaller probability than c_2.

Homans' postulates are formulated from the point of view of an objective observer, a scientist who studies human interactions and behaviours, for instance a biologist who observes and studies interactions between animals in a laboratory. Therefore to use Homans' postulates to $\mathbf{MR_{np}}$ logic application, we introduce some changes to his original statements. Below we present them with the addition of the aspect of individual beliefs. So, instead of general statements about human behaviour, we provide statements that contain agents' beliefs about some aspects of the nature of social interactions ((P1h) and (P2h)).

(P1h) If an agent **beliefs** that the more often a particular action of a person is rewarded, the more likely the person is to perform that action.

(P2h) If in the past the occurrence of a particular stimulus, or set of stimuli, was the occasion on which a person's action was rewarded and he **beliefs** that the more similar the present stimuli are to the past ones, the more likely the person is to perform the action, or some similar action.

The reformulation of Homans' postulates has been made to add the humanistic coefficient to the statements about human behaviour. Intending to formalize (P1) and (P2), instead of introducing a single-argument predicate *being a belief*, and to simplify a record, we introduce a positional letter belief.

(P1h form)
$$\mathcal{R}_{a,\mathsf{believe}}(\mathsf{Fr}(b_1, b_2)) \wedge \mathsf{Prob}(c_1, c_2) \wedge \mathcal{R}_{a,b_1,\mathsf{reward}}(A) \wedge \\ \wedge \mathcal{R}_{a,b_2,\mathsf{reward}}(B) \to \mathcal{R}_{a,c_1,\mathsf{perform}}(A) \wedge \mathcal{R}_{a,c_2,\mathsf{perform}}(B)$$

(P2h form)
$$\mathcal{R}_{a,s_1,\mathsf{perform},\mathsf{reward}}(A) \wedge \mathcal{R}_{a,\mathsf{believe}}(\mathsf{Sim}(s_1, s_2, s_3)) \wedge \mathsf{Prob}(c_1, c_2) \to \\ \to \mathcal{R}_{a,s_2,c_1,\mathsf{perform}}(A) \wedge \mathcal{R}_{a,s_3,c_2,\mathsf{perform}}(A)$$

In the formulas (P1h form) and (P2h form) the subformulas $\mathcal{R}_{a,\mathsf{believe}}(\mathsf{Fr}(b_1, b_2))$ and $\mathcal{R}_{a,\mathsf{believe}}(\mathsf{Sim}(s_1, s_2, s_3))$ express a subjective point of view of the agent a. Therefore to conclude from two Homans' postulates one needs something more than $\mathsf{Fr}(b_1, b_2)$ or $\mathsf{Sim}(s_1, s_2, s_3)$ as premises.

Our examples and the whole proposal is of course rather 'modest' and simple. However, they show that we need to develop this project with the addition of the nesting of the $\mathcal{R}$ operator and quantifiers.

7. Further developments

One of our hopes for $\mathbf{MR_{np}}$ logic and the Łoś $\mathcal{R}$–operator is that they will contribute to sociology by connecting the qualitative perspective with the quantitative one. Our proposal for applying Homans' postulates is however, just the beginning, as stated before. Sociological theories that seek to describe complex social phenomena need more accurate modelling of contexts, in which many agents participate in a collective action. Computational sociology with the references and usage of agent–based models (ABMs) is trying to achieve this goal as well. ABMs are considered to be especially instrumental:

> "...when the macro patterns of sociological interest are not the simple aggregation of individual attributes but the result of bottom–up processes at a relational level" [1].

However, one of the many conclusions resulting from the studies on ABMs and sociology, is that this type of modelling of social phenomena has many features typical of methodological individualism [2]. Therefore it presents rather a individualistic point of view and still stumbles upon an issue of 'strong commitment to minimal behavioural complexity' [9].

Nevertheless, our goals are not entirely different from those formulated for ABMs, we try to achieve (as for now) less pragmatic, more theoretical results. Our proposal of extension of **MR** logic is a further step for the programme that was laid out in [7]. The need to combine the humanistic coefficient with the formalisation that was expressed there, can be fulfilled with the language of the **MR** system. The application to sociological theorems and postulates has shown that it is possible to grasp not only one's behavior, but also a set of beliefs as separate variables. The idea of changing George Homans' postulates from an 'objective' style to a more 'subjective' one (with visible convictions of the agent), makes us suppose that it will be possible to represent other quantitative, more formalised sociological theses more qualitatively. We need to add quantifiers and nesting of the $\mathcal{R}$-operator expressions to the language of $\mathbf{MR_{np}}$.

References

[1] F. Bianchi, F. Squazzoni, *Agent–based Models in Sociology*, **WIREs Computational Statistics**, vol. 7 (2015), pp. 284–306, DOI: https://doi.org/10.1002/wics.1356.

[2] M. Gianluca, *Agent–based Models and Methodological Individualism: Are They Fundamentally Linked?*, **L'Année Sociologique**, vol. 70(1) (2020), pp. 197–229, DOI: https://doi.org/10.3917/anso.201.0197.

[3] G. C. Homans, **Social Behavior: Its Elementary Forms**, no. 40 in Social Forces, Harcourt, Brace and World, Inc., New York (1961), DOI: https://doi.org/10.2307/2574301.

[4] T. Jarmużek, A. Parol, *On Some Language Extension of Logic MR: A Semantic and Tableau Approach*, **Roczniki Filozoficzne**, vol. 68(4) (2020), pp. 345–366, DOI: https://doi.org/10.18290/rf20684-16.

[5] T. Jarmużek, A. Pietruszczak, *Completenes of Minimal Positional Calculus*, **Logic and Logical Philosophy**, vol. 13 (2004), pp. 147–162, DOI: https://doi.org/10.12775/LLP.2004.009.

[6] D. Jemielniak, *Socjologia 2.0: o Potrzebie Łączenia Big Data z Etnografią Cyfrową, Wyzwaniach Jakościowej Socjologii Cyfrowej i Systematyzacji Pojęć (Sociology 2.0: On the Need to Combine Big Data with Digital Ethnography, the Challenges of Qualitative Digital Sociology, and Definitions' Systematization)*, **Studia Socjologiczne**, vol. 229(2) (2018), pp. 7–29, DOI: https://doi.org/10.24425/122461.

[7] J. Malinowski, K. Pietrowicz, J. Szalacha-Jarmużek, *Logic of Social Ontology and Łoś Operator*, **Logic and Logical Philosophy**, vol. 29 (2020), pp. 239–258, DOI: https://doi.org/10.12775/LLP.2020.005.

[8] R. Maris, *The Logical Adequacy of Homans' Social Theory*, **American Sociological Review**, vol. 35(6) (1970), pp. 1069–1081, DOI: https://doi.org/10.2307/2093383.

[9] D. O'Sullivan, M. Haklay, *Agent–Based Models and Individualism: Is the World Agent–Based?*, **Environment and Planning A: Economy and Space**, vol. 32(8) (2000), pp. 1409–1425, DOI: https://doi.org/10.1068/a32140.

[10] F. Znaniecki, *The Object Matter of Sociology*, **American Journal of Sociology**, vol. 32 (1927), pp. 529–584, DOI: https://doi.org/10.1086/214184.

Aleksander Parol
Cardinal Stefan Wyszyński University in Warsaw
Department of Philosophy
ul. Wóycickiego 1/3
01-938 Warsaw, Poland
e-mail: a.parol@student.uksw.edu.pl

Krzysztof Pietrowicz
Nicolaus Copernicus University
Institute of Sociology
ul. Fosa Staromiejska 1a
87-100 Toruń, Poland
e-mail: krzysztof.pietrowicz@umk.pl

Joanna Szalacha-Jarmużek
Nicolaus Copernicus University
Institute of Sociology
ul. Fosa Staromiejska 1a
87-100 Toruń, Poland
e-mail: joanna.szalacha@umk.pl

Bulletin of the **S**ection of **L**ogic
Volume 50/2 (2021), pp. 229–248
https://doi.org/10.18778/0138-0680.2021.04

C O P E
Member since 2018
JM13714

Richmond H. Thomason*

COMMON KNOWLEDGE, COMMON ATTITUDES, AND SOCIAL REASONING

Abstract

For as long as there have been theories about common knowledge, they have been exposed to a certain amount of skepticism. Recent more sophisticated arguments question whether agents can acquire common attitudes and whether they are needed in social reasoning. I argue that this skepticism arises from assumptions about practical reasoning that, considered in themselves, are at worst implausible and at best controversial. A proper approach to the acquisition of attitudes and their deployment in decision making leaves room for common attitudes. Postulating them is no worse off than similar idealizations that are usefully made in logic and economics.

Keywords: Common knowledge, belief, nonmonotonic logic, practical reasoning.

1. Preliminaries

Many approaches to coordinated practical reasoning rely on common knowledge—reciprocal and iterative knowledge of a proposition by a group.[1]

*The author wishes to thank two anonymous referees for helpful comments on this paper.

[1] *Warning:* Here and in the title to this paper, I speak of "knowledge" and "common knowledge." This is because that is the familiar term, and it will be easier for readers to identify the topic of this paper if I begin with what is familiar. But what I really have in mind—and this is crucially important—is common belief, or better, common supposition for some particular practical purpose.

Presented by: Tomasz Jarmużek, Fengkui Ju, Piotr Kulicki, Beishui Liao
Received: April 17, 2020
Published online: April 1, 2021

These include game theory, bargaining theory, the theory of communications protocols, the theory of distributed computing, the theory of multiagent systems, the analysis of convention, the theory of grounding in human and machine communication, and various specific applications.[2]

All of these applications have to do with practical reasoning in group situations. The issue of whether common knowledge and similar attitudes are legitimate in social reasoning is a special case of the question of how propositional attitudes figure in practical reasoning of any sort. I believe that it can't be properly understood without situating it in this more general arena.

Despite its acceptance in many theoretical circles, second thoughts about common knowledge come readily to mind. We can understand what it means to ask whether someone knows that we know that they know something, but only with a certain amount of difficulty. And, with further iterations, the difficulty rapidly increases beyond anyone's intuitive capability. Perhaps this is why [20][p. 246] says "as most commentators would agree, mutual knowledge* is from the point of view of psychological reality at best problematic."[3]

Schiffer doesn't say what commentators he has in mind, and may be thinking of personal communications. The only published examples I'm aware of prior to 2017 are [22, 23, 24], which criticize the appropriateness of common knowledge in accounting for conversational common ground. [7] asks how children could acquire the concept. This is a legitimate and perhaps challenging question, but is hardly a criticism.

[23][p. 18] raises perhaps the first objection that would occur to a critic: to establish common knowledge, speakers "would have, in principle, to perform an infinite series of checks." If the point is that a generalization involving infinitely many instances can't be concluded without considering

[2] For game and bargaining theory, see [1, 6]. For distributed computing, see [16]. For multiagent systems, see [13, 2]. For convention, see [11]. For the theory of communication, see [3, 25]. For an application, see [15].

[3] Authors have been slow to coordinate on a terminology; some use "common" others use "mutual." Schiffer adds an asterisk to indicate, apparently, that he finds the notion artificial, although his formalization doesn't deviate in important respects from others. More recently, some economists have added to the confusion by using "mutual knowledge" for knowledge simply shared by a group. According to this terminology, common knowledge would be the limit of all finite iterations of mutual knowledge. In this paper, I myself will use "common" for this limiting notion. I'll avoid the term "mutual," except in quoting authors who use it.

each instance separately, it is clearly wrong: we can know that for each number there is a larger prime number without having to think about each case. Construed this way, the objection does raise the technical challenge of showing that conclusions about common knowledge can be derived from a finite axiomatic basis: [11, 19], for instance, address this issue.

If the point is that it is psychologically difficult to think about even moderately complex iterations of the knowledge operator, the prime number theorem also provides a counterexample, because it is quite difficult to show that large numbers are prime.

But there remains a more problematic version of this objection. The proof of the prime number theorem invokes a uniform method, applicable to any number. But with common knowledge, each finite iteration is defeasible and can depend on new and different evidence. Take the example of a simple public announcement: Ann says to Bob, "I'll be at home this afternoon." To know after the announcement that Bob knows she'll be at home, Ann needs to assume Bob heard and understood her. For Bob to know that Ann knows that Bob knows she'll be at home, Bob needs to assume that Ann knows that he heard and understood what she said; and so forth. In principle, this series of knowledge claims could hold up to any n but fail at $n+1$; this circumstance is particularly salient in the *coordinated attack scenario*, which we'll consider in Section 6.1. This version of the objection raises a technical challenge that has not been adequately addressed in the literature. I'll argue in Section 4 that using a nonmonotonic epistemic logic will solve the problem.

2. Lederman's challenge

Sperber and Wilson raise another, slightly different objection: that "the assumption of mutual knowledge may always be mistaken," [24][p. 19]. The objection amounts to this: in many situations calling for mutual knowledge, the conditions for knowledge simply don't exist. This objection is skeptical, and calls for a philosophical remedy rather than a technical response. But it will be instructive to frame the response in connection with more philosophically sophisticated versions of the objection, presented by Harvey Lederman in two recent articles,[4]

[4] [9, 10]. And [8] provides a useful survey of the relevant issues and literature.

Lederman concentrates on practical decision-making rather than conversational common ground, and questions the value of common attitudes in accounting for publicity and in modeling many cases of interpersonal reasoning. His arguments purport to show that it is impossible in practice for a rational agent to acquire such attitudes. If this were so, then it certainly would make no sense for interpersonal deliberation to require common attitudes. Moreover, Lederman points to experimental evidence that seems to show, in some cases at least, that humans arrive at decisions without obtaining common beliefs, much less common knowledge.

3. Common attitudes and practical reasoning

Before addressing Lederman's arguments in detail, we need to consider the much more general issue of the relation between propositional attitudes and decision making. These strategic considerations matter: some models of belief acquisition make it hard to see how common beliefs could be acquired. And some views about reasons for action make it hard to see how mutual beliefs can serve in this capacity.

Many authors—especially computer scientists—who work with common attitudes speak somewhat recklessly in terms of "knowledge" when, if they were more philosophically minded, they would use "belief." Both knowledge and belief figure in decision making—but knowledge is more appropriate for evaluating decisions once they have been made. An agent who is criticizing another's or her own earlier decision is in a position to separate knowledge from belief. But for a deliberating agent, it is practical conviction that supports action, and it doesn't matter for the decision-making process whether this conviction actually counts as knowledge when the decision is evaluated.

So far, this should not be very controversial. More controversial, perhaps, is the idea that practical acceptance is not a matter of bringing a general-purpose, previously prepared attitude to bear, but is tailored *ad hoc* to the decision at hand, and depends not only on purely evidential factors but on risk.

As far as I know, the idea that agents tailor their practical attitudes to the specific decision-making context first appears in [21]. There, Simon proposed that the standards for an acceptable solution can depend may be adjusted during the deliberative process. In [26, 30], I claim this is also true of practical beliefs; thresholds for activating beliefs in a deliberative context are adjusted according to estimates of the risk of acting on the belief.

The most familiar examples of this phenomenon are cases of *subtractive risk sensitivity*, where a belief disappears upon the realization that it would be risky to act on it. For instance, I normally believe that people receive email messages I send, and because nothing is unusual about a message I sent to my accountant, I believe she received it. But when I realize I may miss a deadline and will be fined a large amount if my accountant didn't get the message, the belief evaporates, and I ask for an acknowledgment. An important ingredient of the coordinated attack scenario, discussed below in Section 6.1, is enhanced risk.

Also, there are cases of *additive risk sensitivity*, in which a belief is created, not because of any relevant evidence, but because of adjusted assessments of risk. To continue the above example, suppose I now learn that the deadline has been extended. The belief that the message was received springs to life again, and I don't bother to ask for confirmation.[5] Such phenomena may seem more plausible to some readers if we recall that we are talking about practical belief—suppositions created in a deliberative context and on which we are willing to act in that context.

Additive adjustments to belief can depend on adjustments to risk tolerance, as well as on new evidence. The following example is from [30].

> Consider a nervous driver at a stop sign at a busy intersection on a dark night. He needs to drive across the intersection. He looks left. A car zooms by from that direction. He looks right. It's clear. He looks left, it's clear. But wait—he can't see what's going on to the right, and doesn't believe it's clear anymore. So he looks right. He repeats the process until he realizes that he'll never get across this way. Time is pressing. But he can't move unless the road is clear. So he lowers his standards, saying to himself "If it was clear to the right a second ago it's clear now." And he hits the gas.

[5] I realize that there is an alternative explanation of this example, in terms of probability and expected utility. But that is beside the point. People engage in belief-based practical reasoning far more commonly than utility calculations. If you like, you can think of risk-sensitive belief as a qualitative way of taking expected utility into account when doing reasoning of that sort.

This passage continues: “Sometimes, of course, there may be no intention to cross the intersection, and no belief—just a sort of desperate hope.” People can act—perhaps out of desperation or frustration—without a supporting belief, simply in the hope that the action will have the desired outcome. And occasionally people may act like good Bayesian decision makers, acting on judgments about probability rather than on beliefs.

It may be difficult in practice to tell whether an agent was acting on a hope, a belief, or a probability, particularly since after the fact people tend to rationalize their decisions. But there are plausible examples of all three of these decision-making mechanisms.

4. Achieving commonality

Attitudes held commonly by a group are iterated versions of individual attitudes. The formation of a common attitude can be illustrated with the simple case of a public announcement. In this case, the members of the group are presented with an announcement. Each is member is sufficiently familiar with the others to know they are capable of understanding the language of the announcement, and each member can observe not only the announcement but the others observing it.

Authors like David Lewis and Steven Schiffer formalize the inference of common attitudes in similar cases by showing how a finite set of axioms can guarantee commonality. This idea comes close to a solution, but falls short in at least two respects: (1) it doesn't explicitly address the defeasibility and even fragility of the assumptions that support commonality conclusions, or issues having to do with common belief revision and (2) it addresses cases where commonality follows from shared perceptions of the environment but has little to say about other cases.

Problem (1) can be addressed by using a nonmonotonic epistemic logic to formalize the construction of mutual attitudes. This is done in [27, 28], and explained with more attention to philosophical issues in [29].

The following axiom exemplifies the idea:

$$([a_1]\phi \wedge \neg Ab(a_1, a_i, \phi)) \rightarrow [a][a_i]\phi.$$

This axiom says that—assuming that the proposition p expressed by ϕ is not abnormal in the relevant way—if agent A_1 supposes[6] p then A_1 also supposes that agent A_i supposes p. If such an axiom is adopted for all formulas ϕ having the form $\texttt{[}a_1\texttt{]} \ldots \texttt{[}a_n\texttt{]}\psi$, where $a_j \in \mathcal{G}$, and all a_i, then $\texttt{[}a_i\texttt{]}\texttt{[}\mathcal{G}\texttt{]}\psi$ is implied if there are no relevant abnormalities, where $\texttt{[}\mathcal{G}\texttt{]}$ is the common attitude induced by $\texttt{[]}$ and the group $\mathcal{G}$.

The result is not quite commonality, but something more attainable and just as good—that if there are no abnormalities for ϕ, $\texttt{[}a_j\texttt{]}\phi$, ..., then agent A_1 will *suppose* that the proposition p expressed by ϕ is a common supposition. This is a nonmonotonic logic, so abnormalities will be minimized—they will be assumed false unless there is some reason to suppose otherwise. Without such reasons, abnormalities do not need to be examined.

Ad hoc attitudes provide a simpler formalization. The members of a group may construct an *ad hoc* G-supposition attitude, expecting it to be common for the purposes at hand. An agent will not G-suppose a proposition unless there is reason to think that it is G-supposed in common. The following axiom is appropriate for this attitude.

$$\forall x \forall y \in \mathcal{G}\, \forall p((\texttt{[}x\texttt{]}p \wedge \neg Ab(x, y, p)) \rightarrow \texttt{[}x\texttt{]}\texttt{[}y\texttt{]}p).$$

In particular applications, such axioms would need to be supplemented by an abnormality theory of the sort described in [12]; such a theory would also provide guidance about the revision of common belief in the presence of new evidence.

In the case of communicative suppositions, for instance, the relevant attitude would be conversational common ground (or supposition for the sake of the conversation), and cues indicating that an interlocutor hasn't heard or understood an utterance would count as abnormalities.

Problem (2) can be addressed by combining ideas of Herbert Clark and his co-authors with G-attitudes. [4] distinguishes personal and communal sources for common ground. Personal sources include features of the common environment. Communal sources use information about shared social background. For instance, speakers will assume knowledge of the nearby geography when talking to others from the same locality, and professional information when talking to someone in the same line of work,

[6] I am using 'suppose' here as a placeholder for whatever attitude is appropriate.

This can be formalized by assuming that declarative memory is not just a collection of stored propositions, but includes metadata, and in particular information about the circumstances in which an item was learned. A modality is a set of propositions. In possible worlds semantics, this would be a set of sets of worlds, but for our present purposes we only need to notice that metadata features classify propositions, so that a boolean combination of these features will determine a set of propositions, i.e. a modality.

For instance, if I was raised in Chicago, my memory may include a feature *Chicago-Native*, indicating that I learned it as a native of Chicago. If I'm an opera fan, it might also include a feature Opera-Fan. Then if I begin a conversation with a stranger, knowing she is a native of Chicago and an opera fan, I can use these two features to define a new modality, $\ulcorner$Chicago-Native & Opera-Fan$\urcorner$, and use this to initialize a conversation.

The idea that social institutions, as well as shared environment, can be a source of common attitudes can of course be applied in other domains. For instance, it can be used to explain the commonality of the rules of a game, of the regulations governing a market, and of social conventions.

5. Belief and decision-making

Although Lederman explicitly considers both common knowledge and belief, I myself, as I explained in Section 3, will be concerned only with belief, on the understanding that this is belief for the sake of some particular decision. We avoid having to deal with largely irrelevant philosophical issues about knowledge skepticism by confining our attention to belief.

Issues having to do with probability are yet another distraction. Sometimes people gamble, basing an action on probabilities. When an agent plays the odds, her actions rest on the hope, rather than the belief, that the outcome will be favorable. To the extent that the probability judgments are sound, such hopes will be well founded. But even the race-track gambler's decision is in part belief-based—she takes it for granted, among other things, that the track will pay for a winning ticket.

Belief is a matter of excluding possibilities, of taking some things for granted in a decision-making context. Which possibilities may sensibly and safely be excluded depends on the context, and in particular on the purposes to which the beliefs are to be put. We may exclude possibilities for various reasons: because we deem them irrelevant, or because it would be complicated and time-consuming to take them into account, or because we think it does no harm to exclude them, or even because they are unpleasant or because we are too impatient to bother with them. Such reasons have little to do with the ideal rationality of classical economics. Some may be deliberative fallacies, but others are hard to classify. A factor like frustration doesn't seem rational, but there are times when it can be useful to act out of frustration, if that is the only way to escape a deliberative quandary.

Very frequently in daily life, when appropriate beliefs are deployed in a practical context, the need for probability is eliminated because nothing is left to chance. For instance, when someone who regularly drives to work plans her day, she automatically believes her car will start when she turn on the ignition. She doesn't bother to calculate the probability of failure. And she makes many other similar assumptions.

According to this picture of practical beliefs, they will have unit probability. This is the approach taken, for instance, in [31]. Of course, this idea makes sense only if both beliefs and probability spaces are relativized to the decision-making context. And of course it relies on the availability of efficient methods for choosing the alternatives that are appropriate for a given decision problem.

Since common beliefs are beliefs, they too will have unit probability. Such common beliefs must have been constructed constructed independently of the deployment of probabilities; e.g., in the process of framing a decision problem.

Some authors, such as Stephen Morris and Cédric Paternotte, propose a probabilistic account of common belief, based on the idea that belief is a matter of high, but not necessarily unit probability; see [14, 17]. This conception of common belief, and of belief in general, belongs to an entirely different approach to decision making, differing fundamentally from the one I've just sketched. This is one of several points at which more general issues in epistemology affect the treatment of common attitudes. I myself doubt that a successful probabilistic account of common attitudes can even be developed. If it can, I don't know whether it could be defended against skeptical arguments.

6. Lederman's worries

Lederman raises two main objections to the use of common attitudes in theories of cooperative activity: [10] argues that the assumption of common knowledge (or belief) yields paradoxical results about two scenarios that have been discussed in the technical literature. And [9] argues for the more radical conclusion that it is impossible to achieve common knowledge and belief about perceptual matters, and indeed about any substantive claim. Both papers concentrate on *practical attitudes*—on knowledge and belief as they figure in decision making.

6.1. Coordinated attack

The coordinated attack scenario involves generals A_1 and A_2 who can only communicate by sending insecure messages across enemy lines, who will win (or are very likely to win) if they both attack, and will lose (or are very likely to lose) if only one attacks. Its main purpose is to illustrate the impossibility of obtaining common knowledge by message passing; but it's plausible that similar situations can occur in real life.

An analysis of this situation in terms of expected utility is rendered problematic by the difficulty A_1 will have in arriving at a probability that A_2 will attack. Suppose for simplicity that the agents are identical decision-making twins, and know this. Then A_1 can determine the probability that A_2 will attack by imitating A_2's reasoning, and asking if in A_2's circumstances, A_1 would attack. This depends on A_2's probability that A_1 will attack—but this is A_1's probability that A_2 will attack, which is what A_1 is trying to estimate in the first place. Fortunately, we do not have to address this problem, because Lederman takes a belief-based approach to the scenario, without making explicit assumptions about the relation between belief and probability.

In discussing the example, Lederman invokes a characterization of commonsense rationality according to which a rational action "makes sense" or is "explainable." But explainability is not at all the same thing as rationality: irrational factors such as wishful thinking and carelessness can be used to explain naturally occurring human decisions, but are hardly rational. I am not confident that there is a robust and useful commonsense notion of rationality, but if we wish to appeal to this notion we will need a better characterization. For present purposes, I'll assume that an action in a set of alternatives is CS-rational if in commonsense terms it is about as good as any alternative action.

Lederman takes it to be CS-rational for A_1 to attack after a finite number of messages have passed, but he doesn't say what the least such finite number is. And this is a problem, because it is hard to see how he or anyone could fix such a number. In fact, risk is a crucial feature of the CA scenario. We can't begin to say what it would be sensible for the generals to do without an idea of of the risks at stake in their decisions—but Lederman ignores risk entirely in his formulation of the scenario.

For the sake of definiteness, assume in what follows that it is about as likely for a message to fail as for it to succeed.

If losing the battle would be catastrophic, while winning would merely be moderately good, it seems pretty clear that doing nothing at all is CS-optimal for both generals. But if losing the battle would be a minor inconvenience, whereas winning would be outstandingly good, then attacking without sending any messages at all seems to be CS-optimal.

Neither of these extreme cases raises special problems about common belief. Consider, then, an intermediate case where risk is significant but not overwhelming, and assume it was CS-rational for A_1 to send a first message rather than to attack without sending a message. Then it doesn't seem as if any number of subsequent messages can produce a situation where it would be CS-rational for A_1 to attack. By assumption, it is not CS-rational for A_1 to attack after sending 0 messages. But after sending n messages, A_1's decision-making context looks the same as it does after 0 messages, because there is no more reason now than at first to believe that A_2 will attack.

What does this show? Is it a paradox? No: it merely shows that, in some versions of the CA scenario—for some values of likelihood and utility—the agents will be perplexed about what to do. Since people do occasionally find themselves in real quandaries—especially in war—this is neither surprising nor paradoxical.

Lederman's intuition that it is CS-rational for A_1 to attack after, say, 13 messages, and the experimental results to which he alludes, in fact have nothing in particular to do with CS-rationality or common attitudes, but with what we can expect of human agents who find themselves in quandaries. Consider the dithering driver scenario that was mentioned in Section 3. After several iterations, it's plausible to think that many human agents will pull out into the intersection, either hoping not to crash or (for no rational reason) choosing to believe that no car is approaching on the blind side. Others might decide to turn around and try another route.

Some factors that are appropriate in the commonsense evaluation of decisions have little to do with rationality. Consider, for instance, the tradeoff between sticking to previously made plans and willingness to abandon those plans to take advantage of apparent opportunities. We can recognize a spectrum of approaches, corresponding to obstinate and opportunistic personalities. We are familiar with individual differences along this spectrum, but it doesn't seem helpful to critique these differences in terms of rationality. Much the same can be said for impatience.

The dithering driver's decision has nothing to do with any useful conception of rationality, and everything to do with impatience and frustration in a difficult decision context. Though of course emotions and frustration can play a role in decision-making behavior, we're disposed to set aside such influences when we speak of rationality, even CS-rationality. The same considerations apply to the coordinated attack scenario, and it is hasty at best to conclude from human behavior in these cases that CS-rationality does not require mutual belief.

Even if there were a sensible policy for the CA scenario that would recommend attacking after, say, 2 successful iterations of message and acknowledgment, this would not show that mutual belief isn't required for coordinated action. This is because, as we mentioned in Section 3, belief thresholds can be adjusted in the course of deliberation, allowing the beliefs that support a choice to spring into existence.

Perhaps the distracted driver, after two left-right iterations, pulls out in the hopeful belief (belief for the sake of the decision) that a car isn't speeding toward him from the blind side. Perhaps the general who attacks after one or more messages does so in the hopeful belief that mutual belief has been achieved.

6.2. Rubinstein's electronic mail game

[10] also discusses the electronic mail game, a scenario due to [18]. Rubinstein provides a game-theoretic formulation of the scenario, so that—unlike informal presentations of the coordinated attack problem—a quantitative formal analysis is available. But in other important respects, the electronic mail game is like the coordinated attack scenario.

Rubinstein showed that the strategy recommended by game theory in his example is independent of the number of messages that have been passed: no amount of message passing can affect the decisions of a game-theoretic agent. Lederman's point about this much-discussed example is only that the game-theoretic results don't match intuitions about CS-rationality.

It is hard to see what to make of this, because the relevant intuitions are far from robust. I myself do not think that Rubinstein's example raises any issues that differ significantly from versions of the coordinated attack scenario with precisely specified utilities and probabilities. Like the message-passing generals, humans who play this game and begin to pass messages will become frustrated after a while, and make a choice. But it isn't clear that these choices are conditioned by anything that could be called rational. I would claim that again, Lederman is misapplying rational criteria to what agents in a hard and perplexing decision context and influenced by human emotions might be expected to do.

If I suppress emotional factors, by imagining that the agents in this scenario are utility-optimizing computer programs, Rubinstein's result doesn't strike me as counterintuitive.

7. Lederman's sailboat scenario

[9] uses attitudes about the value ranges of continuous quantities to argue for the much stronger conclusion that our minds are not in fact "open to each other," which "casts doubt on the idea that people ever have common knowledge or its relatives." As I said, I am not concerned here with knowledge. But Lederman thinks that belief is susceptible to similar arguments.

Lederman's scenario has nothing important to do with sailboats. Two players, visible to each other, observe a long, thin object—the "mast." The mast is then replaced with another that has a randomly selected length (presumably, within a range fairly close to the first mast). Each agent is assigned a button, which they must then decide whether to press. If the new mast is taller than the old one, the reward is +1000. If it is not taller, and neither player presses a button, the reward is 0. Otherwise the reward is -100. On the classical account, and using belief as the relevant attitude, a player A_1 should press the button if and only if A_1 believes that it's mutually believed that the new mast is taller.

Lederman presents his argument using knowledge. A version of it for belief runs as follows.

1. The masts will look to have a certain height to the players; Lederman uses "looks r centimeters high to A," formally LOOKS(A,r), in a slightly peculiar way according to which, for instance, LOOKS(A,100) and LOOKS(A,100.01) are consistent—that is, LOOKS(A,x) will be true over an interval (r_1, r_2). This interval will have a midpoint, say r. Let $\text{LOOKS}'(A, r, \rho)$ hold iff r is the midpoint of the open interval with radius ρ over which LOOKS(A,x) holds; then there will be at most one x such that $\text{LOOKS}'(A, x, \rho)$ is true.

2. Interpersonal estimates of perceptions about continuous quantities have a margin of error. Suppose with Lederman that for estimating the height of the mast this is .03, that the margin is known by the agents, and—crucially—that it is fixed at .03 in all epistemically accessible worlds. In terms of agent beliefs, this means that $\text{LOOKS}'(A_1, x, \rho)$ holds, then so does $\langle A_1 \rangle \text{LOOKS}'(A_2, y, \rho)$, for all y such that $|x - y| \leq .03x$. (Here, angle brackets are labeled possibility signs or diamonds, so that $\langle A_1 \rangle$ stands for "for all A_1 believes.") And likewise for A_2's opinions about how things look to A_1. Not only do these things hold, but both agents are aware that they hold.

3. In terms of the possible worlds semantics for belief, this means that if $\text{LOOKS}'(A_1, 100, \rho)$ holds in world w then for all x such that $.97 \leq x \leq 1.03$, there is a world $w(A_1,x)$ that is A_1-accessible from w, such that $\text{LOOKS}'(A_2, x, \rho)$ holds in $w(A_1,x)$; the range of such worlds is the interval $I_1 = (.97, 1.03)$ with diameter .06. For similar reasons, the interval of worlds A_2-accessible from worlds in I_1 in which $\text{LOOKS}'(A_1, 100, \rho)$ holds is approximately $I_2 = (.941, 1.061)$, with diameter .12. In general, each iteration of the sequence

   $$\langle A_1 \rangle \langle A_2 \rangle \langle A_1 \rangle \langle A_2 \rangle \ldots \text{LOOKS}(A_2, k)$$

   increases the diameter of the error interval by more than .06. Lederman concludes that the agents can't commonly believe that the height is greater than any positive height ϵ, because there will be a point in the attitude iteration including a world where the error interval includes ϵ.

If we accept this sort of argument, we may get more than we bargained for, A similar argument would show that if the mast looks to have height r for a single agent A_1, then it must look to A_1 to have any other height. If the mast looks to A_1 to have height x, then for some fixed ϵ, less than the perceptual threshold for height, it looks to A_1 to have heights $x \pm \epsilon$. By iterating this argument, it looks to have any height whatever. This in fact is Michael Dummett's perceptual version of the sorites paradox, [5].

This similarity to a paradox makes Lederman's argument dubious. So I will consider an improved version, suggested by a referee of this paper.

> Two agents A_2 and A_3 are told that each will be assigned an integer greater than 1, and that the integers will be consecutive. A_2 is given the number 2 and A_3 the number 3. Consider a set $\{w_{i,j} \mid i, j > 0\}$ of alternatives, with the understanding that in $w_{i,j}$ A_2 is given i and A_3 is given j, and let R_2 and R_3 be the epistemic accessibility relations for A_2's and A_3's beliefs, respectively. Then $w_0 R_2 w_1 R_3 w_2 \ldots R_3 w_n$ iff $w_{2n} = w_{2,i}$ for some $i \in [1, n+3]$, and likewise $w_0 R_3 w_1 R_2 w_2 \ldots R_2 w_n$ iff $w_{2n} = w_{2,i}$ for some $i \in [1, n+3]$.
>
> No matter how large n may be, it seems that the agents can't have a common belief that n bounds both agents' numbers.
>
> We can make this practical by probing the agents with a positive integer n and asking them to press a button, with rewards and penalties like those of the sailboat example.

This scenario doesn't use continuous quantities and is not at all similar to the sorites paradox. But now is very like the coordinated attack problem and can be treated in much the same way. It presents the contestants with a practical problem that may well have no solution in terms of economic rationality. But people have other ways of making decisions, some better than others. In this case, a participant may arrive at the belief that, say, it's commonly believed that 10 is an upper bound by eliminating some alternatives. This could be because 10 is a salient number, or it could be a matter of jumping to a conclusion for no very good reason. Or a participant might press a button as a hopeful gamble, without forming any relevant belief at all.

8. Conclusion

Early skeptical doubts about common attitudes emerged in the 1980s. These can be addressed by concentrating on common belief and using non-monotonic logics to respond to the technical challenge of accounting for how agents can arrive at and reason with these attitudes. Lederman's more recent doubts about the appropriateness of common attitudes hinge on idiosyncratic intuitions about "commonsense rationality," which don't provide clear guidance when applied to the behavior of human agents in challenging circumstances.

It is less interesting then, to confront Lederman's conclusions with opposing intuitions about rationality than to ask if common attitudes such as conversational common ground and the common beliefs at stake in a market or a game can be situated within a sensible theoretical approach to practical reasoning. I have argued that a framework based on defaults and rough estimates of likelihoods and risks can account for how common beliefs can originate and how, like other beliefs, they can play a part in decision making. On this picture, common beliefs are readily inferrable, not by any extraordinary and unusual feat of reasoning, but by methods that are constantly in play in our everyday life.

Probably the main source for skepticism about common attitudes is a misunderstanding about the scope and proper place of economic rationality in deliberative contexts. Calculation of rational optima makes good sense when a problem can be framed in terms that enable such calculations to be made. But in real life we are often confronted with problems that can't be framed that way—and this includes the problem of modeling a decision problem. In these cases commonsense reasoning mechanisms such as rough assessments, intuitions about relevance, and defaulr reasoning can come into play. And these mechanisms can support common attitudes—for instance, by justifying the assumption that public announcements create common beliefs.

But this response to philosophical doubts about common attitudes—however successful—doesn't suffice to justify invoking these attitudes in game theory, protocol analysis, the theory of conversations, any of the other areas where they may seem theoretically appropriate. That has to be done on a case-by-case basis, using the methods of the relevant discipline. For theoreticians who postulate common attitudes, then, the philosophical part of this paper is not directly relevant, although the more technical ideas in [28] about the reasoning that supports common attitudes may be useful.

References

[1] R. J. Aumann, *Agreeing to Disagree*, **Annals of Statistics**, vol. 4(6) (1976), pp. 1236–1239, DOI: https://doi.org/10.1214/aos/1176343654.

[2] S. Bucheli, R. Kuznets, T. Studer, *Justifications for Common Knowledge*, **Journal of Applied Non-Classical Logics**, vol. 21(1) (2011), pp. 35–60, DOI: https://doi.org/10.3166/jancl.21.35-60.

[3] H. H. Clark, C. R. Marshall, *Definite Reference and Mutual Knowledge*, [in:] A. Joshi, B. Webber, I. Sag (eds.), **Elements of Discourse Understanding**, Cambridge University Press, Cambridge, England (1981), pp. 10–63.

[4] H. H. Clark, M. Schober, *Understanding by Addressees and Overhearers*, **Cognitive Psychology**, vol. 21(2) (1989), pp. 211–232.

[5] M. A. Dummett, *Wang's Paradox*, [in:] R. Keefe, P. Smith (eds.), **Vagueness: A Reader**, The MIT Press, Cambridge, Massachusetts (1997), pp. 99–118.

[6] J. Geanakoplos, *Common Knowledge*, [in:] R. Aumann, S. Hart (eds.), **Handbook of Game Theory, with Economic Applications**, chap. 40, vol. 2, North-Holland, Amsterdam (1994), pp. 1437–1496, DOI: https://doi.org/10.1016/S1574-0005(05)80072-4.

[7] P. N. Johnson-Laird, *Mutual Ignorance: Comments on Clark and Carlson's Paper*, [in:] **Mutual Knowledge**, Academic Press, London (1982), pp. 40–45.

[8] H. Lederman, *Common Knowledge*, [in:] M. Jankovic, K. Ludwig (eds.), **The Routledge Handbook of Collective Intentionality**, Routledge, New York (2017), pp. 181–195.

[9] H. Lederman, *Uncommon Knowledge*, **Mind**, vol. 127(508) (2017), pp. 1069–1105, DOI: https://doi.org/10.1093/mind/fzw072.

[10] H. Lederman, *Two Paradoxes of Common Knowledge: Coordinated Attack and Electronic Mail*, **Noûs**, vol. 52(4) (2018), pp. 921–945, DOI: https://doi.org/10.1111/nous.12186.

[11] D. K. Lewis, **Convention: A Philosophical Study**, Harvard University Press, Cambridge, Massachusetts (1969), DOI: https://doi.org/10.1002/9780470693711.

[12] V. Lifschitz, *Circumscriptive Theories: A Logic-Based Framework for Knowledge Representation*, **Journal of Philosophical Logic**, vol. 17(3) (1988), pp. 391–441, DOI: https://doi.org/10.1007/BF00297512.

[13] M. Merritt, G. Taubenfeld, *Knowledge in Shared Memory Systems*, **Distributed Computing Archive**, vol. 7 (1991), pp. 99–109, DOI: https://doi.org/10.1007/BF02280839.

[14] S. Morris, H. S. Shin, *Approximate Common Knowledge and Co-ordination: Recent Lessons from Game Theory*, **Journal of Logic, Language, and Information**, vol. 6(2) (1997), pp. 171–190, DOI: https://doi.org/10.1023/A:1008270519000.

[15] D. G. Novick, K. Ward, *Mutual Beliefs of Multiple Conversants: a Computational Model of Collaboration in Air Traffic Control*, [in:] R. Fikes, W. Lehnert (eds.), **Proceedings of the Eleventh National Conference on Artificial Intelligence**, American Association for Artificial Intelligence, AAAI Press, Menlo Park, California (1993), pp. 196–201.

[16] P. Panangaden, K. Taylor, *Concurrent Common Knowledge: Defining Agreement for Asynchronous Systems*, **Distributed Computing**, vol. 6(2) (1992), pp. 73–94, DOI: https://doi.org/10.1007/BF02252679.

[17] C. Paternotte, *The Fragility of Common Knowledge*, **Erkenntnis**, vol. 82 (2017), pp. 451–472, DOI: https://doi.org/10.1007/s10670-016-9828-4.

[18] A. Rubinstein, *The Electronic Mail Game: Strategic Behavior Under 'Almost Common Knowledge'*, **American Economic Review**, vol. 79(3) (1989), pp. 385–391.

[19] S. Schiffer, **Meaning**, Oxford University Press, Oxford (1972).

[20] S. Schiffer, **Remnants of Meaning**, The MIT Press, Cambridge, Massachusetts (1987).

[21] H. A. Simon, *Rational Choice and the Structure of the Environment*, **Psychologicial Review**, vol. 63(2) (1956), pp. 129–138, DOI: https://doi.org/10.1037/h0042769.

[22] D. Sperber, D. Wilson, *Mutual Knowledge and Relevance in Theories of Comprehension*, [in:] N. Smith (ed.), **Mutual Knowledge**, Academic Press, London (1982), pp. 61–85.

[23] D. Sperber, D. Wilson, **Relevance: Communication and Cognition**, 1st ed., Harvard University Press, Cambridge, Massachusetts (1986).

[24] D. Sperber, D. Wilson, **Relevance: Communication and Cognition**, 2nd ed., Blackwell, Oxford (1995).

[25] R. Stalnaker, *Common Ground*, **Linguistics and Philosophy**, vol. 25(5–6) (2002), pp. 701–721, DOI: https://doi.org/10.1023%2FA%3A1020867916902.

[26] R. H. Thomason, *The Multiplicity of Belief and Desire*, [in:] M. P. Georgeff, A. Lansky (eds.), **Reasoning about Actions and Plans**, Morgan Kaufmann, Los Altos, California (1987), pp. 341–360.

[27] R. H. Thomason, *Propagating Epistemic Coordination Through Mutual Defaults I*, [in:] R. Parikh (ed.), **Theoretical Aspects of Reasoning about Knowledge: Proceedings of the Third Conference (TARK 1990)**, Morgan Kaufmann, Los Altos, California (1990), pp. 29–39, DOI: https://doi.org/10.5555/1027014.1027022.

[28] R. H. Thomason, *Modeling the Beliefs of Other Agents*, [in:] J. Minker (ed.), **Logic-Based Artificial Intelligence**, Kluwer Academic Publishers, Dordrecht (2000), pp. 375–473, DOI: https://doi.org/10.1007/978-1-4615-1567-8_17.

[29] R. H. Thomason, *The Beliefs of Other Agents* (2001), http://web.eecs.umich.edu/~rthomaso/documents/nmk/otheragents.pdf.

[30] R. H. Thomason, *Belief, Intention, and Practicality: Loosening Up Agents and Their Propositional Attitudes*, [in:] F. Lihoreau, M. Rebuschi (eds.), **Epistemology, Context, Formalism**, Springer Verlag, Berlin (2013), pp. 167–184, DOI: https://doi.org/10.1007/978-3-319-02943-6_10.

[31] B. C. van Fraassen, *Representation of Conditional Probabilities*, **Journal of Philosophical Logic**, vol. 5(3) (1976), pp. 417–430, DOI: https://doi.org/10.1007/BF00649400.

Richmond H. Thomason
University of Michigan
Philosophy Department
2251 Angell Hall
48109-1003 Ann Arbor
Michigan, USA
e-mail: rthomaso@umich.edu