UNIVERSITY OF ŁÓDŹ
DEPARTMENT OF LOGIC

BULLETIN
OF THE SECTION OF LOGIC

VOLUME 50, NUMBER 1

ŁÓDŹ, MARCH 2021

C O P E
Member since 2018
JM13714

Layout
Michał Zawidzki

Initiating Editor
Katarzyna Smyczek

Printed directly from camera-ready materials provided
to the Łódź University Press

Published by Łódź University Press
First edition. W.10287.21.0.C

Printing sheets 8.125

Łódź University Press
90-131 Łódź, 8 Lindleya St.
www.wydawnictwo.uni.lodz.pl
e-mail: ksiegarnia@uni.lodz.pl
tel. +48 42 665 58 63

The **Bulletin of the Section of Logic** (*BSL*) is a quarterly peer-reviewed journal published with the support from the University of Łódź. Its aim is to act as a forum for a wide and timely dissemination of new and significant results in logic through rapid publication of short papers. *BSL* publishes contributions on topics dealing directly with logical calculi, their methodology, and algebraic interpretation.

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Volume 50/1 (2021), pp. 1–34
https://doi.org/10.18778/0138-0680.2021.02

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SUPER-STRICT IMPLICATIONS[1]

Abstract

This paper introduces the logics of super-strict implications, where a super-strict implication is a strengthening of C.I. Lewis' strict implication that avoids not only the paradoxes of material implication but also those of strict implication. The semantics of super-strict implications is obtained by strengthening the (normal) relational semantics for strict implication. We consider all logics of super-strict implications that are based on relational frames for modal logics in the modal cube. it is shown that all logics of super-strict implications are connexive logics in that they validate Aristotle's Theses and (weak) Boethius's Theses. A proof-theoretic characterisation of logics of super-strict implications is given by means of G3-style labelled calculi, and it is proved that the structural rules of inference are admissible in these calculi. It is also shown that validity in the $\mathsf{S5}$-based logic of super-strict implications is equivalent to validity in G. Priest's negation-as-cancellation-based logic. Hence, we also give a cut-free calculus for Priest's logic.

Keywords: Strict implication, paradoxes of implication, connexive implication, sequent calculi, structural rules.

2020 Mathematical Subject Classification: 03B45, 03B60, 03F05.

[1]We are grateful to two anonymous reviewers for their many helpful remarks.

Presented by: Stéphane Demri
Received: July 1, 2020
Published online: January 20, 2021

1. Introduction

Given its centrality in deductive reasoning, "implication seems to be the most important connective" [18, p. 167]. Nevertheless there are different competing formal explications of implication. Just to mention some cases, we have *material implication* ($\supset$); its modalised versions such as *strict implication* ($\strictif$) and *variably strict implication* ($\Box\!\!\rightarrow$); *relevant implications*; and *connexive implications*. This paper proposes a new pair of modalised versions of material implication which avoid the paradoxes of strict implication and generate a family of connexive logics with a simple relational semantics.

Strict implication has been introduced by C.I. Lewis [6] to overcome the paradoxes of material implication:

$$\neg A \supset (A \supset B) \tag{MI1}$$

$$B \supset (A \supset B) \tag{MI2}$$

But, as connexivists [13] and relevantists [1] argued, strict implication is plagued by its own paradoxes:

$$\bot \strictif B \qquad [\text{and} \qquad \Box\neg A \supset (A \strictif B)] \tag{SI1}$$

$$A \strictif \top \qquad [\text{and} \qquad \Box B \supset (A \strictif B)] \tag{SI2}$$

We agree with connexivists and relevantists in taking the paradoxes of strict implication to be as questionable as the paradoxes of material implication. If, following C.I. Lewis, it is not maintained that 'A implies B' follows from the hypothesis that A is false *simpliciter* (MI1) or from the hypothesis that B is true *simpliciter* (MI2), then it is unnatural to maintain that it follows from the hypothesis that A is inconsistent/impossible (SI1) or from the hypothesis that B is a logical/necessary truth (SI2).

This is not to say that we have quarrels with material implication, as is witnessed by the fact that we will consider logics containing material implication: we maintain that there are uses of 'A implies B' for which material implication is not appropriate, and that for these uses Lewis' strict implication fares no better than material implication. If 'A implies B' means that the truth of B *depends*, in one way or another, on that of

A, in all the aforementioned cases we should accept that 'A does not imply B' – i.e., among the validities governing implication we do not find (SI1) and (SI2) but, instead, their negations. We find no reasonable sense of 'depending' under which the truth of some B might depend on the truth of a sentence A that cannot be true (apart, possibly, the degenerate case in which B itself cannot be true that will be disregarded in this paper). Analogously, the truth of some B that cannot be false does not seem to depend on the truth of any other sentence.[2]

Another way of making the same point goes as follows. If 'A implies B' means that establishing the truth of A is a good way of establishing the truth of B – i.e., if the key role of an implication is its use in *Modus Ponens* – then we immediately see that '$\bot$ implies B' is useless for inquiring into the truth of B. Analogously, 'A implies $\top$' is a roundabout way for reaching the truth of $\top$ (from this perspective (SI1) is way more problematic than (SI2)).

This paper studies two strengthenings of $\strictif$ – called *super-strict implications* (SSI) – that are designed to validate 'A does not imply B' when A is logically impossible or B is logically necessary. *Weak SSI* ($\rhd$) does not validate (SI1) nor (SI2), but it validates the negation of (SI1). *Strong SSI* ($\blacktriangleright$) does not validate (SI1) nor (SI2), but it validates their negations.

Semantically, SSI are obtained by modifying the truth condition for $\strictif$ in such a way that (SI1) and (SI2) no longer hold. As is well known, C.I. Lewis' strict implication can be defined in terms of the modal operator $\Box$ as:

$$A \strictif B \quad \equiv \quad \Box(A \supset B)$$

If we express this in terms of the relational semantics for normal modalities (for C.I. Lewis' stronger calculi S4 and S5), we obtain the following semantic clause:

> $A \strictif B$ holds at a world w if and only if B holds at every w-accessible world satisfying A.

Needless to say, if A is semantically equivalent to $\bot$ then this semantic clause has an unsatisfiable antecedent and, hence, (SI1) is valid. Analo-

[2] Observe that intuitionistic implication ($\to$) with its BHK-interpretation – "a proof of $A \to B$ is a construction which permits us to transform any proof of A into a proof of B" [21, p. 9] – might seem to express an appropriate notion of dependency, but it doesn't fare better than material and strict implcations since it allows for the dummy transformation when $A \equiv \bot$ or $B \equiv \top$.

gously, if B is equivalent to $\top$ then the consequent of this semantic clause is always satisfied and (SI2) is valid.

Weak SSI avoids the first paradox since

> $A \rhd B$ holds at a world w whenever not only B holds at every w-accessible world satisfying A but also A holds at some w-accessible world.

Hence, $\rhd$ does not validate the paradoxes of strict implication: it validates the negation of (SI1). Strong SSI does better in this respect in that it validates also the negation of the second paradox (SI2):

> $A \blacktriangleright B$ holds at a world w iff the two clauses for $\rhd$ are satisfied and, moreover, B does not hold at some w-accessible world.

Having sketched the semantical interpretation of the notions of truth-value dependency and SSI that we subscribe, the next step will be that of introducing the class of logics that we are going to consider. Logics of SSI will be defined semantically as sets of formulas that are valid in some class of relational frames. In particular, we consider the logics determined by the classes of frames defined *via* the following well-known properties of the accessibility relation: seriality, reflexivity, transitivity, symmetry, and Euclideaness (see Tables 1 and 2). Proof-theoretically, we will characterise each logic by means of a G3-style labelled calculus [12, Chapter 11] where all structural rules will be shown to be admissible (both syntactically and semantically).

By means of these calculi it will be shown that all logics of SSI are connexive logics – see [14] for a recent introduction to connexive logics – in that they include both *Aristotle's Theses* ($\multimap$ stands for either $\rhd$ or $\blacktriangleright$):

$$\neg(A \multimap \neg A) \qquad \neg(\neg A \multimap A) \qquad \text{(AT1 \& AT2)}$$

and *(weak) Boethius' Theses*:

$$(A \multimap B) \supset \neg(A \multimap \neg B) \qquad (A \multimap \neg B) \supset \neg(A \multimap B) \qquad \text{(BT1 \& BT2)}$$

Observe that the validity of Aristotle's and Boethius' Theses is a natural outcome for an implication that expresses a notion of 'truth-value dependency'. Even if we have not tried to give a precise meaning to this notion of truth-value dependency, one minimal property that we can ascribe is the impossibility for both a proposition and its negation to depend on the same proposition. Accordingly, SSI validate Boethius' Theses. Another property that we can ascribe to this notion of truth-value dependency is that no proposition depends on its own negation. Whence, SSI validate Aristotle's Theses. SSI are connexive implications (though they do not validate the *strong Boethius' Theses* that are obtained from the weak ones by replacing $\supset$ with $\multimap$). To sum up, SSI are simple modality-based non-paradoxical and connexive implications where the connexion between antecedent and consequent of a true implication is some kind of 'truth-value dependency'.

Outline Section 2 discusses some related works and gives some motivations for SSI. Section 3 introduces the syntax and semantics of logics of SSI. Section 4 introduces labelled calculi for them and Section 5 proves the admissibility of the structural rules of inference. In Section 6 it is proved that these calculi are sound and complete with respect to their intended semantics. In Section 7 the semantics of the $\mathsf{S5}$-based logic of SSI is shown to be equivalent to the semantics for a connexive logic considered by G. Priest [16]. Section 8 sketches some future works.

2. Super-strict implications and related works

SSI are interesting for at least two different but related reasons. First, they explicate the idea of implication as expressing some kind of truth-value dependency: 'A implies B' is taken to mean that establishing A is a way to establish B. Under this reading of implication the paradoxes of both material and strict implication do not hold. SSI validate instead the negation of the first paradox of strict implication (SI1); strong SSI validates also the negation of the second paradox (SI2). The second reason for studying SSI is that they generate a family of connexive logics having a very simple relational semantics. SSI show that we can achieve connexivity by adding to the truth conditions for strict implication the condition that the antecedent of a true implication has to be possible. Moreover, this additional condition is not just a formal machinery to achieve connexivity: it can be philosophically justified by taking implication to express some kind of truth-value dependency. In this section we compare SSI with some related approaches in order to highlight the main novelties of SSI.

The idea of strengthening C.I. Lewis' $\strictif$ to obtain a more apt formal explication of implication is not new. The most famous example of this phenomenon is D. Lewis' and Stalnaker's logic of *variably strict implication* $\Box\!\!\rightarrow$ (a.k.a. the logic of counterfactual). We will briefly return to variably strict implication in Section 8; here we just highlight that the goal of proponents of variably strict implication is orthogonal to our own, since $\Box\!\!\rightarrow$ is designed to avoid the transitivity of $-\!\!3$ and not its paradoxes (SI1) and (SI2).

A proposal that looks similar to SSI is made by E. J. Lowe in a series of papers [8, 9] where he proposes to capture indicative conditionals by explicating 'A implies B' in the modal language as $\Box(A \supset B) \land (\Diamond A \lor \Box B)$. Lowe explicitly rejects a modal analogous of weak SSI because of the following counterexample: "If n were the greatest natural number, then there would be a natural number greater than n" [9, p. 48]. We are not particularly moved by this counterexample since our intuitions about its truth value differ from Lowe's ones. While Lowe maintains that a conditional with a necessary consequent must be true even if its antecedent is impossible, we believe, to the opposite, that no formula is implied by an impossible formula. This is motivated by our understanding of implication as expressing some kind of truth-value dependency. 'A implies B' means that establishing A allows us to establish B. If A is impossible then it cannot be used to establish the truth of some B, not even when B is necessarily true as in Lowe's example: we might have many different ways to establish B, but A is not among them.[3] Hence, when A is impossible we take 'A implies B' to be (logically) false. Notice also that under Lowe's explication of implication the second paradox of strict implication (SI2) turns out to be valid. Even if Lowe's proposal were able to explicate indicative conditionals, it would not provide a non-paradoxical implication. SSI are instead meant to provide simple non-paradoxical modalised implication that need not be formal explications of indicative conditionals in natural language. For this reason we are not moved by the counterexamples given in [4] either.

Another proposal that is quite similar in spirit to the present one is Hitchcock's [5] *enthymematic consequence*: $A \Vdash B$ iff it is impossible that A is true and B is false, but both A and $\neg B$ are possibly true. Obviously, strong SSI can be seen as an object language representation of Hitchcock's consequence relation; and, analogously, weak SSI can be seen as an ob-

[3] If we consider strong SSI then Lowe's example turn out to be false for the very same reason used by Lowe to argue that it is true.

ject language representation of B. Bolzano's consequence relation since he required the premisses be consistent, see [2].

A connexive implication that is somehow similar to SSI is C. Pizzi's *consequential implication* [15]: A consequentially implies B if and only if $A \dashv B$ and, moreover, A and B cannot have incompatible modal status. If the underlying modal logic is at least the serial logic KD, consequential implication validates Aristotle's Theses and weak Boethius' Theses. Moreover, connexive consequential implication does not validate strong Boethius' Theses and, as shown in [15], a consequential implication validating Strong Boethius' would be a commutative operator. One of the main differences between Pizzi's consequential implication and SSI is that the former is semantically analysed only via a translation in the modal language and its formal definition makes essential use of modal notions. The definition of SSI, instead, does not depend on modalities and, as will be shown in Section 3, the (normal) modal operators $\Box$ and $\Diamond$ can be defined in terms of weak SSI in a non-circular way. Another striking difference – which can be taken to show that SSI generate a better behaved family of connexive logics than consequential implication – is that SSI validate Aristotle's and Boethius' Theses even on structures falsifying the seriality axiom $D := \Box A \supset \Diamond A$ (see Example 4.1), whereas consequential implication is connexive only in the presence of the seriality axiom. All in all, SSI seem better behaved than consequential implication for introducing connexive implications with a simple relational semantics: modal notions should not be involved in the definition of a connexive implication (but they might be defined in terms of connexive implication) and, moreover, it might be interesting to consider modality-based connexive logics where the seriality axiom fails (even if seriality expresses the connexive idea that A and $\neg A$ express incompatible propositions).

Finally, another formal semantics for a connexive logic that is very similar to the one for SSI is the one adopted by G. Priest [16] to model the cancellation account of negation [19]. Section 7 will compare our proposal with Priest's one. Here we just anticipate that Priest's motivation is completely independent from ours. Priest's aim is that of making precise the cancellation account of negation and showing how this "gives rise to a semantics for a simple connexivist logic" [16, p. 141]. As we will see in Proposition 7.1, validity (but not consequence) in Priest's connexive logic can be seen as a particular case of our more general approach obtained by considering the $\mathsf{S5}$-based logic of SSI. Hence, in providing proof systems for logics of SSI, we will provide also a proof system for validities in Priest's cancellation account of negation. As we will argue in Section 7, the cancellation account of negation cannot be formalised suitably by the given formal semantics, which is instead appropriate to capture the idea of implication as expressing some kind of truth-value dependency.

3. Syntax and semantics

The language ($\mathcal{L}$) of SSI is determined by the following grammar, where p ranges over a denumerable set of propositional variables $\mathcal{P}$:

$$A \ ::= \ p \mid \bot \mid A \wedge A \mid A \vee A \mid A \supset A \mid A \rhd A \mid A \blacktriangleright A \qquad (\mathcal{L})$$

Parentheses follow the usual conventions (where SSI bind lighter than all other operators). We will use p, q, r as metavariables for propositional variables and A, B, C for arbitrary formulas. $\top$ is short for $\bot \supset \bot$ and $\neg A$ is short for $A \supset \bot$. Moreover, $A \multimap B$ will be short for both $A \rhd B$ and $A \blacktriangleright B$. Given a formula A, its *weight* $\mathtt{w}(A)$ is thus defined: $\mathtt{w}(p) = \mathtt{w}(\bot) = 0$ and $\mathtt{w}(A \circ B) = \mathtt{w}(A) + \mathtt{w}(B) + 1$ for $\circ \in \{\wedge, \vee, \supset, \rhd, \blacktriangleright\}$.

As in relational semantics for modal logics, a *frame* $\mathcal{F}$ is a pair $\langle W, R\rangle$ composed of a non-empty set of worlds and an accessibility relation; a *model* $\mathcal{M}$ is a triple $\langle W, R, V\rangle$ where V is a valuation function mapping each propositional variable in $\mathcal{P}$ to a subset of W. If $\mathcal{M} = \langle W, R, V\rangle$ and $\mathcal{F} = \langle W, R\rangle$, we say that the model $\mathcal{M}$ is *based* on the frame $\mathcal{F}$ ($\mathcal{F}$*-model*, for short).

The truth of a formula A at a world w of a model $\mathcal{M}$ (in symbols $\models^{\mathcal{M}}_{w} A$) is defined as usual for the classical operators and for the SSI is defined as follows:

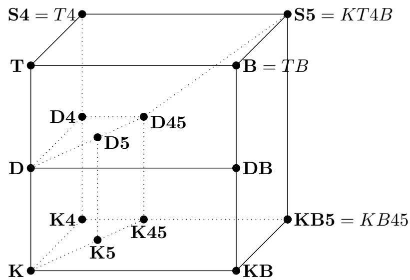
$$\begin{array}{lll} \models^{\mathcal{M}}_{w} A \rhd B & \text{iff} & \text{for all } v \in W, \text{ if } wRv \text{ and } \models^{\mathcal{M}}_{v} A, \text{ then } \models^{\mathcal{M}}_{v} B \quad \& \\ & & \text{some } v \in W \text{ is such that } wRv \text{ and } \models^{\mathcal{M}}_{v} A \\[1em] \models^{\mathcal{M}}_{w} A \blacktriangleright B & \text{iff} & \text{for all } v \in W, \text{ if } wRv \text{ and } \models^{\mathcal{M}}_{v} A, \text{ then } \models^{\mathcal{M}}_{v} B \quad \& \\ & & \text{some } v \in W \text{ is such that } wRv \text{ and } \models^{\mathcal{M}}_{v} A \quad \& \\ & & \text{some } v \in W \text{ is such that } wRv \text{ and } \not\models^{\mathcal{M}}_{v} B \end{array}$$

Truth in a model ($\models^{\mathcal{M}} A$) and *validity* in a frame ($\mathcal{F} \models A$) or in a class $\mathcal{C}$ of frames ($\mathcal{C} \models A$) are defined in the usual way. A formula is *valid* if it

Table 1. Modal correspondence results

Name	Property	Modal axiom
(D)	*seriality* $\forall w \exists v(wRv)$	$\Box A \supset \Diamond A$
(T)	*reflexivity* $\forall w(wRw)$	$\Box A \supset A$
(4)	*transitivity* $\forall w, v, u(wRv \wedge vRu \supset wRu)$	$\Box A \supset \Box\Box A$
(B)	*symmetry* $\forall w, v(wRv \supset vRw)$	$A \supset \Box\Diamond A$
(5)	*Euclideaness* $\forall w, v, u(wRv \wedge wRu \supset vRu)$	$\Diamond A \supset \Box\Diamond A$

Table 2. Cube of normal modalities



is valid in the class of all frames. Given a class $\mathcal{C}$ of frames (a $\mathcal{C}$-model is a model based on a frame in $\mathcal{C}$ and) its *(local) consequence relation* is thus defined:

$$\Gamma \models^{\mathcal{C}} A \quad \text{iff} \quad \text{for each world } w \text{ of a } \mathcal{C}\text{-model, if all formulas in } \Gamma \text{ are true at } w \text{ then } A \text{ is true at } w$$

A *logic* of SSI is defined as the set of $\mathcal{L}$-formulas that are valid in a class of frames. In particular, we consider the logics determined by the classes of frames defined *via* the well-known properties of the accessibility relation given in Table 1. We follow the standard naming conventions for $\mathcal{L}^{\Box}$-logics, see Table 2.

Now we present some important properties of logics of SSI.

Proposition 3.1.

1. None of the paradoxes of material implication – i.e., (MI1) and (MI2) – is valid for $\multimap$.
2. None of the paradoxes of strict implication is valid for $\multimap$.
3. The negation of (SI1) is valid for $\multimap$; the negation of (SI2) is valid for $\blacktriangleright$.
4. The connexive principles (AT1 & AT2) and (BT1 & BT2) are valid for SSI.

5. Strong Boethius' thesis: $(A \multimap B) \multimap \neg(A \multimap \neg B)$ is not valid.

6. SSI are not reflexive: in general $A \multimap A$ is not valid.

7. Contraposition is valid for $\blacktriangleright$ $(A \blacktriangleright B \models \neg B \blacktriangleright \neg A)$ but not for $\triangleright$ $(A \triangleright B \not\models \neg B \triangleright \neg A)$.

PROOF: The proof of items 1, 2, 5, 6, and 7 is left to the reader. Example 4.1 gives a syntactic proof of some of the claims made in items 3 and 4. $\square$

We finish this section by presenting the relationship between SSI and modal operators. It is immediate to notice that $\Box$ and $\Diamond$ can be defined in terms of weak SSI as follows:

$$\Diamond A \equiv A \triangleright \top \qquad \Box A \equiv \neg(\neg A \triangleright \top) \qquad \text{(Def. } \Box/\Diamond)$$

Moreover, since $A \prec B \equiv \Box(A \supset B)$, we can express strict implication in terms of weak SSI as follows:

$$A \prec B \equiv \neg((A \wedge \neg B) \triangleright \top) \qquad \text{(Def. } \prec)$$

Finally, we can define strong SSI in terms of the weak one as follows:

$$A \blacktriangleright B \equiv ((A \triangleright B) \wedge (\neg B \triangleright \top)) \qquad \text{(Def. } \blacktriangleright)$$

Neither the unary modalities, nor strict/weak SSI can be expressed in terms of strong SSI. This can be seen as a reason to prefer $\triangleright$ over $\blacktriangleright$ despite the fact that $\triangleright$ does not validate the negation of the paradox (SI2). Observe also that over serial frames the translation of $\Box A$ can be simplified into $\top \triangleright A$.

It is also possible to express SSI in terms of the standard modal language $\mathcal{L}^{\Box}$: $A \triangleright B$ is expressed by $\Diamond A \wedge \Box(A \supset B)$ and $A \blacktriangleright B$ by $\Diamond A \wedge \Box(A \supset B) \wedge \Diamond \neg B$. Thus, the logics of SSI are embeddable in the corresponding modal logics. Formally, we have the following translation $\mathtt{t}$ mapping $\mathcal{L}$-formulas into $\mathcal{L}^{\Box}$-formulas.

DEFINITION 3.2. Let A be an $\mathcal{L}$-formula, the $\mathcal{L}^{\Box}$-formula $\mathtt{t}(A)$ is thus defined:

$$
\begin{aligned}
\mathtt{t}(A) &\equiv A && \text{iff} && A \text{ is a propositional variable or } \bot \\
\mathtt{t}(B \circ C) &\equiv \mathtt{t}(B) \circ \mathtt{t}(C) && \text{iff} && \circ \in \{\wedge, \vee, \supset\} \\
\mathtt{t}(B \rhd C) &\equiv \Diamond \mathtt{t}(B) \wedge \Box(\mathtt{t}(B) \supset \mathtt{t}(C)) \\
\mathtt{t}(B \blacktriangleright C) &\equiv \Diamond \mathtt{t}(B) \wedge \Box(\mathtt{t}(B) \supset \mathtt{t}(C)) \wedge \Diamond(\neg \mathtt{t}(C))
\end{aligned}
$$

It is immediate to see that the following theorem holds.

THEOREM 3.3. *Let $\mathcal{C}$ be a class of frames. The $\mathcal{L}$-formula A is valid in $\mathcal{C}$ if and only if the $\mathcal{L}^{\Box}$-formula $\mathtt{t}(A)$ is valid in $\mathcal{C}$ (w.r.t. $\mathcal{C}$-models for the modal language).*

This entails that all the logics of SSI we consider are decidable. In particular, [3] shows that the satisfiability problem is NP-complete for the modal logics $\mathsf{S5}$ and $\mathsf{KD45}$ and PSPACE-complete for the logics K, D, T, and $\mathsf{S4}$. In translating an $\mathcal{L}$-formula A into the corresponding $\mathcal{L}^{\Box}$-formula $\mathtt{t}(A)$ we have an exponential blowup in the weight of the formula since, e.g., $\mathtt{t}(B)$ occurs twice in $\mathtt{t}(B \rhd C)$. Nevertheless, if $|A|$ stands for the number of subformulas of A, the translation $\mathtt{t}$ has linear complexity. Given that the complexity algorithms in [3] depend on $|A|$ rather than on $\mathtt{w}(A)$, we conclude that the satisfiability problem is PSPACE-complete for the K-, D-, T-, and $\mathsf{S4}$-based logics of SSI and it is NP-complete for the $\mathsf{S5}$- and $\mathsf{KD45}$-based ones.[4]

4. Labelled calculi

We introduce G3-style labelled calculi for logics of SSI. We assume the reader is familiar with labelled calculi for modal logics, see [12, Chapter 11]. First of all, we introduce a set LAB of fresh variables, called *labels*. Labels will be denoted by $w, v, u, \dots$ and will be used to represent worlds. Then, we extend the set of formulas by adding *relational atoms* of shape $w\mathcal{R}v$ – expressing that v is accessible from w. Moreover, we replace each $\mathcal{L}$-formula A with the *labelled formulas* $w : A$ – expressing that A holds at w. Finally, in analogy with [10, 11], we add *(existential* and *universal)* *forcing* formulas of shape $\Vdash^{\exists} A$ and $\Vdash^{\forall} A$ ($A \in \mathcal{L}$) – expressing that A holds at some/all worlds accessible from w.[5] The *weight* of a formula E

[4] Thanks are due to an anonymous reviewer for spotting a mistake in our original argument.

[5] As is explained in Fn.6, forcing formulas are needed to obtain complete rules for $\blacktriangleright$.

of the extended language, $\mathtt{w}(E)$, is a pair $\langle n, \ell \rangle$ (ordered lexicographically) where n is the weight of the $\mathcal{L}$-formula used to construct E or 0 if E is a relational atom and ℓ is 1 if E is a forcing formula, else it is 0. This definition is designed to have $\mathtt{w}(w : A_1 \blacktriangleright A_2) > \mathtt{w}(w \Vdash^{\exists/\forall} A_i) > \mathtt{w}(w : A_i)$, see the proof of Lemma 5.1. Given a formula E of this extended language, $E[w/v]$ is the formula obtained by substituting each occurrence of v in E with an occurrence of w. A *labelled sequent* is an expression:

$$\Gamma \Rightarrow \Delta$$

where Γ is a finite multiset composed of labelled formulas, forcing formulas, and relational atoms, and Δ is a finite multiset of labelled and forcing formulas only. Substitution of labels is extended to sequents by applying it componentwise.

The rules of the calculus $\mathsf{G3SS.L}$ for the logic of SSI over frames for the normal modal logic L are given in Table 3. The calculus $\mathsf{G3SS.K}$ contains all the rules in Table 3 but the non-logical ones. If L is an extension of K from Table 2, the calculus $\mathsf{G3SS.L}$ extends $\mathsf{G3SS.K}$ with the non-logical rules expressing the semantic properties of frames for L (a calculus contains the contracted rule instance *Euclid*^{c} iff it contains *Euclid*). To illustrate, the calculus $\mathsf{G3SS.S4}$ contains all rules of Table 3 but the non-logical rules *Ser*, *Sym*, *Euclid*, and *Euclid*^{c}. Observe that in a derivation there can be at most one instance of one of the rules $L \rhd'$ and $L \blacktriangleright'$ (some relational atom will occur in all nodes of the tree above this rule instance); moreover, as will be shown in Corollary 5.4, these rules are eliminable from calculi where rule *Ser* is admissible. We allow ourselves to use the following admissible rules:

$$\frac{\Gamma \Rightarrow \Delta, w : A}{w : \neg A, \Gamma \Rightarrow \Delta}\, {\scriptstyle L\neg} \quad \text{and} \quad \frac{w : A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, w : \neg A}\, {\scriptstyle R\neg} \quad \text{and} \quad \frac{}{\Gamma \Rightarrow \Delta, w : \top}\, {\scriptstyle R\top}$$

A $\mathsf{G3SS.L}$-*derivation* of a sequent $\Gamma \Rightarrow \Delta$ is a tree of sequents, whose leaves are initial sequents, whose root is $\Gamma \Rightarrow \Delta$, and which grows according to the rules of $\mathsf{G3SS.L}$. The *height* of a $\mathsf{G3SS.L}$-derivation is the number of nodes of its longest branch. We say that $\Gamma \Rightarrow \Delta$ is $\mathsf{G3SS.L}$-derivable (with height n), and we write $\mathsf{G3SS.L} \vdash^{(n)} \Gamma \Rightarrow \Delta$, if there is a $\mathsf{G3SS.L}$-derivation (of height at most n) of $\Gamma \Rightarrow \Delta$. A rule is said to be *(height-preserving) admissible* in $\mathsf{G3SS.L}$, if, whenever its premisses are $\mathsf{G3SS.L}$-derivable (with

Table 3. Rules of the calculi for logics of SSI

initial sequents: $w : p, \Gamma \Rightarrow \Delta, w : p$, with p atomic

logical rules:

$$\frac{}{w : \bot, \Gamma \Rightarrow \Delta}\; L\bot$$

$$\frac{w : A, w : B, \Gamma \Rightarrow \Delta}{w : A \wedge B, \Gamma \Rightarrow \Delta}\; L\wedge \qquad \frac{\Gamma \Rightarrow \Delta, w : A \quad \Gamma \Rightarrow \Delta, w : B}{\Gamma \Rightarrow \Delta, w : A \wedge B}\; R\wedge$$

$$\frac{w : A, \Gamma \Rightarrow \Delta \quad w : B, \Gamma \Rightarrow \Delta}{w : A \vee B, \Gamma \Rightarrow \Delta}\; L\vee \qquad \frac{\Gamma \Rightarrow \Delta, w : A, w : B}{\Gamma \Rightarrow \Delta, w : A \vee B}\; R\vee$$

$$\frac{\Gamma \Rightarrow \Delta, w : A \quad w : B, \Gamma \Rightarrow \Delta}{w : A \supset B, \Gamma \Rightarrow \Delta}\; L\supset \qquad \frac{w : A, \Gamma \Rightarrow \Delta, w : B}{\Gamma \Rightarrow \Delta, w : A \supset B}\; R\supset$$

$$\frac{w\mathcal{R}u, w\mathcal{R}v, u : A, w : A \rhd B, \Gamma \Rightarrow \Delta, v : A \qquad w\mathcal{R}u, w\mathcal{R}v, v : B, u : A, w : A \rhd B, \Gamma \Rightarrow \Delta}{w\mathcal{R}v, w : A \rhd B, \Gamma \Rightarrow \Delta}\; L\rhd, u \text{ fresh}$$

$$\frac{w\mathcal{R}u, u : A, w : A \rhd B, \Gamma \Rightarrow \Delta}{w : A \rhd B, \Gamma \Rightarrow \Delta}\; L\rhd', u \text{ fresh, no relational atom in } \Gamma$$

$$\frac{w\mathcal{R}u, w\mathcal{R}v, u : A, \Gamma \Rightarrow \Delta, u : B \qquad w\mathcal{R}v, \Gamma \Rightarrow \Delta, w : A \rhd B, v : A}{w\mathcal{R}v, \Gamma \Rightarrow \Delta, w : A \rhd B}\; R\rhd, u \text{ fresh}$$

$$\frac{w \Vdash^{\exists} A, w\mathcal{R}v, w : A \blacktriangleright B, \Gamma \Rightarrow \Delta, v : A, w \Vdash^{\forall} B \qquad w\mathcal{R}v, v : B, w \Vdash^{\exists} A, w : A \blacktriangleright B, \Gamma \Rightarrow \Delta, w \Vdash^{\forall} B}{w\mathcal{R}v, w : A \blacktriangleright B, \Gamma \Rightarrow \Delta}\; L\blacktriangleright$$

$$\frac{w \Vdash^{\exists} A, w : A \blacktriangleright B, \Gamma \Rightarrow \Delta, w \Vdash^{\forall} B}{w : A \blacktriangleright B, \Gamma \Rightarrow \Delta}\; L\blacktriangleright', \text{ no relational atom in } \Gamma$$

$$\frac{w\mathcal{R}u, u : A, \Gamma \Rightarrow \Delta, u : B \qquad \Gamma \Rightarrow \Delta, w \Vdash^{\exists} A \qquad w \Vdash^{\forall} B, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, w : A \blacktriangleright B}\; R\blacktriangleright, u \text{ fresh}$$

forcing rules:

$$\frac{w\mathcal{R}u, u : A, \Gamma \Rightarrow \Delta}{w \Vdash^{\exists} A, \Gamma \Rightarrow \Delta}\; L\Vdash^{\exists}, u \text{ fresh} \qquad \frac{w\mathcal{R}v, \Gamma \Rightarrow \Delta, w \Vdash^{\exists} A, v : A}{w\mathcal{R}v, \Gamma \Rightarrow \Delta, w \Vdash^{\exists} A}\; R\Vdash^{\exists}$$

$$\frac{v : A, w\mathcal{R}v, w \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}{w\mathcal{R}v, w \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}\; L\Vdash^{\forall} \qquad \frac{w\mathcal{R}u, \Gamma \Rightarrow \Delta, u : A}{\Gamma \Rightarrow \Delta, w \Vdash^{\forall} A}\; L\Vdash^{\forall}, u \text{ fresh}$$

non-logical rules:

$$\frac{w\mathcal{R}u, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\; Ser, u \text{ fresh} \qquad \frac{w\mathcal{R}w, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\; Ref \qquad \frac{w\mathcal{R}u, w\mathcal{R}v, v\mathcal{R}u, \Gamma \Rightarrow \Delta}{w\mathcal{R}v, v\mathcal{R}u\Gamma \Rightarrow \Delta}\; Trans$$

$$\frac{v\mathcal{R}w, w\mathcal{R}v, \Gamma \Rightarrow \Delta}{w\mathcal{R}v, \Gamma \Rightarrow \Delta}\; Sym \qquad \frac{v\mathcal{R}u, w\mathcal{R}v, w\mathcal{R}u, \Gamma \Rightarrow \Delta}{w\mathcal{R}v, w\mathcal{R}u, \Gamma \Rightarrow \Delta}\; Euclid \qquad \frac{v\mathcal{R}v, w\mathcal{R}v, \Gamma \Rightarrow \Delta}{w\mathcal{R}v, \Gamma \Rightarrow \Delta}\; Euclid^{c}$$

height at most n), also its conclusion is $\mathsf{G3SS.L}$-derivable (with height at most n). In each rule depicted in Table 3, Γ and Δ are called *contexts*, the formulas occurring in the conclusion are called *principal*, and those occurring in the premisses only are called *active*.

Example 4.1. The following sequents are $\mathsf{G3SS.L}$-derivable:

1. The negation of (SI1): $\Rightarrow w : \neg(\bot \multimap A)$
2. The negation of (SI2): $\Rightarrow w : \neg(A \blacktriangleright \top)$
3. First Aristotle's Thesis (AT1): $\Rightarrow w : \neg(A \multimap \neg A)$
4. Second Aristotle's Thesis (AT2): $\Rightarrow w : \neg(\neg A \multimap A)$
5. First Boethius' Thesis (BT1): $\Rightarrow w : (A \multimap B) \supset \neg(A \multimap \neg B)$
6. Second Boethius' Thesis (BT2): $\Rightarrow w : (A \multimap \neg B) \supset \neg(A \multimap B)$

Proof: For simplicity, we assume $\multimap$ is $\rhd$.

1.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{$L\bot$}
\UnaryInfC{$w\mathcal{R}u, u : \bot, w : \bot \rhd A \Rightarrow$}
\RightLabel{$L\rhd'$}
\UnaryInfC{$w : \bot \rhd A \Rightarrow$}
\RightLabel{$R\neg$}
\UnaryInfC{$\Rightarrow w : \neg(\bot \rhd A)$}
\end{prooftree}
$$

2.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{$R\top$}
\UnaryInfC{$w\mathcal{R}u, w \Vdash^{\exists} A, w : A \blacktriangleright \top \Rightarrow u : \top$}
\RightLabel{$R\Vdash^{\forall}$}
\UnaryInfC{$w \Vdash^{\exists} A, w : A \blacktriangleright \top \Rightarrow w \Vdash^{\forall} \top$}
\RightLabel{$L\blacktriangleright'$}
\UnaryInfC{$w : A \blacktriangleright \top \Rightarrow$}
\RightLabel{$R\neg$}
\UnaryInfC{$\Rightarrow w : \neg(A \blacktriangleright \top)$}
\end{prooftree}
$$

3.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{$L.5.1$}
\UnaryInfC{$w\mathcal{R}v, v : A, w\mathcal{R}u, u : A, w : A \rhd \neg A \Rightarrow u : A$}
\AxiomC{}
\RightLabel{$L.5.1$}
\UnaryInfC{$w\mathcal{R}v, v : A, w\mathcal{R}u, u : A, w : A \rhd \neg A \Rightarrow u : A$}
\RightLabel{$L\neg$}
\UnaryInfC{$w\mathcal{R}v, v : A, u : \neg A, w\mathcal{R}u, u : A, w : A \rhd \neg A \Rightarrow$}
\RightLabel{$L\rhd$}
\BinaryInfC{$w\mathcal{R}u, u : A, w : A \rhd \neg A \Rightarrow$}
\RightLabel{$L\rhd'$}
\UnaryInfC{$w : A \rhd \neg A \Rightarrow$}
\RightLabel{$R\neg$}
\UnaryInfC{$\Rightarrow w : \neg(A \rhd \neg A)$}
\end{prooftree}
$$

4. Analogous to the derivation of AT1.

5.

$$
\dfrac{\dfrac{}{w\mathcal{R}u, u:A, w:A \rhd \neg B, v:A, w:A \rhd B \Rightarrow v:A}\,{\scriptstyle L.5.1} \qquad \dfrac{\dfrac{}{u:A \Rightarrow u:A}\,{\scriptstyle L.5.1} \qquad \dfrac{\dfrac{}{u:A \Rightarrow u:A}\,{\scriptstyle L.5.1} \qquad \dfrac{\dfrac{}{u:B \Rightarrow u:B}\,{\scriptstyle L.5.1}}{u:B, u:\neg B \Rightarrow}\,{\scriptstyle L\neg}}{u:B, w\mathcal{R}u, u:A, w:A \rhd \neg B \Rightarrow}\,{\scriptstyle L\rhd}}{w\mathcal{R}u, u:A, v:B, w:A \rhd \neg B, v:A, w:A \rhd B \Rightarrow}\,{\scriptstyle L\rhd}}{\dfrac{w\mathcal{R}v, v:A, w:A \rhd \neg B, w:A \rhd B \Rightarrow}{\dfrac{w:A \rhd \neg B, w:A \rhd B \Rightarrow}{\dfrac{w:A \rhd B \Rightarrow w:\neg(A \rhd \neg B)}{\Rightarrow w:(A \rhd B) \supset \neg(A \rhd \neg B)}\,{\scriptstyle R\supset}}\,{\scriptstyle R\neg}}\,{\scriptstyle L\rhd'}}\,{\scriptstyle L\rhd}
$$

6. Analogous to the derivation of BT1. □

5. Structural rules of inference

In this section we prove the admissibility of the structural rules of inference: weakening and contraction will be shown to be height-preserving admissible and cut will be shown to be admissible. Moreover, we'll show that all rules are height-preserving invertible. All proofs of this section will be based on (the non-modal fragment of) those in [10, 12] for the labelled calculi for normal and non-normal modal logics. In particular, in the proof of all lemmas/theorems we will not consider the propositional and non-logical cases since their proof can be found in [12] nor the forcing ones whose proof is in [10].

LEMMA 5.1. *$w : A, \Gamma \Rightarrow \Delta, w : A$ and $w \Vdash^{\exists/\forall} A, \Gamma \Rightarrow \Delta, w \Vdash^{\exists/\forall} A$ – with A arbitrary $\mathcal{L}$-formula – are derivable in* G3SS.L.

PROOF: By simultaneous induction on the weight of $w : A$ and $w \Vdash^{\exists/\forall} A$. We consider only the case when $A \equiv B \rhd C$ and, without loss of generality, we assume $\Gamma \equiv w\mathcal{R}v$ (if no relational atom is in Γ, we use $L \rhd'$ instead of $L \rhd$) and $\Delta \equiv \emptyset$. We have the following derivation (omitting all formulas that, bottom-up, become useless):

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{u':B \Rightarrow u':B}\,{\scriptstyle IH} \qquad \dfrac{}{u':C \Rightarrow u':C}\,{\scriptstyle IH}}{w\mathcal{R}u', u':B, w\mathcal{R}u, u:B, w:B \rhd C \Rightarrow u':C}\,{\scriptstyle L\rhd} \qquad \dfrac{}{u:B \Rightarrow u:B}\,{\scriptstyle IH}}{w\mathcal{R}u, u:B, w:B \rhd C \Rightarrow w:B \rhd C, v:B}\,{\scriptstyle R\rhd} \qquad \dfrac{\dfrac{\dfrac{}{u':B \Rightarrow u':B}\,{\scriptstyle IH} \qquad \dfrac{}{u':C \Rightarrow u':C}\,{\scriptstyle IH}}{w\mathcal{R}u', u':B, w:B \rhd C \Rightarrow u':C}\,{\scriptstyle L\rhd} \qquad \dfrac{}{u:B \Rightarrow u:B}\,{\scriptstyle IH}}{w\mathcal{R}u, v:C, u:B, w:B \rhd C \Rightarrow w:B \rhd C}\,{\scriptstyle R\rhd}}{w\mathcal{R}v, w:B \rhd C \Rightarrow w:B \rhd C}\,{\scriptstyle L\rhd}
$$

To prove the lemma for $w : A_1 \blacktriangleright A_2, \Gamma \Rightarrow \Delta, w : A_1 \blacktriangleright A_2$ it is essential that $\mathtt{w}(w : A_1 \blacktriangleright A_2) > \mathtt{w}(\Vdash^{\forall/\exists} A_i)$; to prove it for $w \Vdash^{\forall/\exists} A, \Gamma \Rightarrow \Delta, w \vdash^{\forall/\exists} A$ it is essential that $\mathtt{w}(\Vdash^{\forall/\exists} A) > \mathtt{w}(w : A)$. □

LEMMA 5.2. *If* $\mathsf{G3SS.L} \vdash^h \Gamma \Rightarrow \Delta$ *then* $\mathsf{G3SS.L} \vdash^h \Gamma[u_2/u_1] \Rightarrow \Delta[u_2/u_1]$.

PROOF: The proof is by induction on height h of the derivation $\mathcal{D}$ of $\Gamma \Rightarrow \Delta$. Suppose the last step of $\mathcal{D}$ is by the following instance of $R \rhd$:

$$\dfrac{w\mathcal{R}v_2, w\mathcal{R}v_1, v_2 : A, \Gamma \Rightarrow \Delta, v_2 : B \qquad w\mathcal{R}v_1, \Gamma \Rightarrow \Delta, w\ A \rhd B, v_1 : A}{w\mathcal{R}v_1, \Gamma \Rightarrow \Delta, w : A \rhd B}\ {\scriptstyle R\rhd,\ v_2 \text{ fresh}}$$

Let us consider a label v_3 that is new to $\mathcal{D}$ and not in $\{u_1, u_2\}$. We transform $\mathcal{D}$ into the following derivation $\mathcal{D}[u_2/u_1]$ having same height as $\mathcal{D}$:

$$\dfrac{\dfrac{\dfrac{w\mathcal{R}v_2, w\mathcal{R}v_1, v_2 : A, \Gamma \Rightarrow \Delta, v_2 : B}{w\mathcal{R}v_3, w\mathcal{R}v_1, v_3 : A, \Gamma \Rightarrow \Delta, v_3 : B}\ {\scriptstyle IH[v_3/v_2]}}{w[u_2/u_1]\mathcal{R}v_3, v_3 : A, (w\mathcal{R}v_1, \Gamma \Rightarrow \Delta)[u_2/u_1], v_3 : B}\ {\scriptstyle IH[u_2/u_1]} \quad \dfrac{w\mathcal{R}v_1, \Gamma \Rightarrow \Delta, w\ A \rhd B, v_1 : A}{(w\mathcal{R}v_1, \Gamma \Rightarrow \Delta, wA \rhd B, v_1 : A)[u_2/u_1]}\ {\scriptstyle IH[u_2/u_1]}}{(w\mathcal{R}v_1, \Gamma \Rightarrow \Delta, w : A \rhd B)[u_2/u_1]}\ {\scriptstyle R\rhd}$$

The transformations for rules $L \rhd^{(\prime)}$, $L \blacktriangleright^{(\prime)}$, and $R \blacktriangleright$ are similar and can thus be omitted. □

Next is height-preserving admissibility of weakening.

THEOREM 5.3 (Weakening). *If* $\mathsf{G3SS.L} \vdash^h \Gamma \Rightarrow \Delta$ *then* $\mathsf{G3SS.L} \vdash^h \Pi, \Gamma \Rightarrow \Delta, \Sigma$.

PROOF: By induction on the height h of $\mathcal{D}$. If the last step of $\mathcal{D}$ is by a rule *Rule* for weak or strong SSI, then we start by applying Lemma 5.2 to the derivation of its premisses in order to replace its *eigenvariable*, if any, with variables new to Π, Σ and to $\mathcal{D}$. Next, we apply the inductive hypothesis, and an instance of *Rule*. □

The following corollary of Theorem 5.3 shows that rules $L \rhd'$ and $L \blacktriangleright'$ are needed only for logics of SSI defined by non-serial classes of frames.

COROLLARY 5.4. Rules $L \rhd'$ and $L \blacktriangleright'$ are eliminable from $\mathsf{G3SS.L}$ when $\mathsf{L} \supseteq \mathsf{D}$.

PROOF: Suppose the last step of $\mathcal{D}$ is by the following instance of $L \rhd'$:

$$\dfrac{w\mathcal{R}u, u : A, w : A \rhd B, \Gamma \Rightarrow \Delta}{w : A \rhd B, \Gamma \Rightarrow \Delta}\ {\scriptstyle L\rhd'}$$

We transform $\mathcal{D}$ into the following $L \rhd'$-free derivation (v new to $\mathcal{D}$):

$$\dfrac{\dfrac{\dfrac{w\mathcal{R}u, u : A, w : A \rhd B, \Gamma \Rightarrow \Delta}{w\mathcal{R}v, w\mathcal{R}u, u : A, w : A \rhd B, \Gamma \Rightarrow \Delta, v : A}\ \textit{Thm.5.3} \qquad \dfrac{w\mathcal{R}u, u : A, w : A \rhd B, \Gamma \Rightarrow \Delta}{w\mathcal{R}v, v : B, w\mathcal{R}u, u : A, w : A \rhd B, \Gamma \Rightarrow \Delta}\ \textit{Thm.5.3}}{w\mathcal{R}v, w : A \rhd B, \Gamma \Rightarrow \Delta}\ L\rhd}{w : A \rhd B, \Gamma \Rightarrow \Delta}\ \textit{Ser}$$

The transformation for rule $L \blacktriangleright'$ is analogous. $\square$

To prove that contraction is height-preserving admissible, we need height-preserving invertibility of each rule – i.e., the derivability (with height n) of a sequent that can be the conclusion of a rule instance entails the derivability (with height n) of the premisses of that rule instance.

Lemma 5.5 (Inversion). *Each rule of* G3SS.L *is height-preserving invertible.*

Proof: The height-preserving invertibility of the rules for SSI with respect to premisses with repetition of all principal formulas follows from Theorem 5.3 (we have cases of 'Kleene-invertibility'). Hence, we have to consider only the invertibility of rule $R \rhd$ with respect to its leftmost premiss, and that of $R \blacktriangleright$ with respect to each one of its premisses. Let's consider inversion or $R \rhd$ w.r.t. its leftmost premiss (the proof is similar for the leftmost premiss of $R \blacktriangleright$). Suppose the conclusion of $\mathcal{D}$ is:

$$w\mathcal{R}v, \Gamma \Rightarrow \Delta, w : A \rhd B$$

If the last step of $\mathcal{D}$ is by an instance of $R \rhd$ there is nothing to prove. Else, the last step of $\mathcal{D}$ is by a rule *Rule* with either one or two premisses $w\mathcal{R}v, \Gamma' \Rightarrow \Delta', w : A \rhd B$ and $w\mathcal{R}v, \Gamma'' \Rightarrow \Delta'', w : A \rhd B$. Let u be some fresh variable, we apply the inductive hypothesis to the derivations of the premisses in order to obtain derivations (having same height) of the sequents $w\mathcal{R}u, w\mathcal{R}v, u : A, \Gamma' \Rightarrow \Delta', u : B$ and $w\mathcal{R}u, w\mathcal{R}v, u : A, \Gamma'' \Rightarrow \Delta'', u : B$. By applying an instance of *Rule* we obtain a derivation (having same height of $\mathcal{D}$) of:

$$w\mathcal{R}u, w\mathcal{R}v, u : A, \Gamma \Rightarrow \Delta, u : B$$

Next, we prove hp-inversion of $R \blacktriangleright$ w.r.t. its second premiss (the proof is similar for the third one). Let's suppose $w : A \blacktriangleright B$ is not principal in the last step of $\mathcal{D}$, which is by *Rule* and has premiss(es) $\Gamma' \Rightarrow \Delta', w : A \blacktriangleright B$ (and $\Gamma'' \Rightarrow \Delta'', w : A \blacktriangleright B$). We apply the inductive hypothesis to the premiss(es) and then an instance of *Rule* to conclude $\Gamma \Rightarrow \Delta, w \Vdash^{\exists} A$. $\square$

Now we can prove height-preserving admissibility of contraction.

Theorem 5.6 (Contraction). *If* $\mathsf{G3SS.L} \vdash^h \Gamma, \Gamma \Rightarrow \Delta, \Delta$ *then* $\mathsf{G3SS.L} \vdash^h \Gamma \Rightarrow \Delta$.

Proof: The proof is by induction on the height h of the derivation $\mathcal{D}$ of the premiss. We assume, w.l.o.g., we are contracting a single formula (occurring in the antecedent or in the consequent). We consider only the cases where the last step of $\mathcal{D}$ is by a rule *Rule* for $\multimap$. If the contraction formula is not principal in *Rule*, we apply the inductive hypothesis to the premiss(es) of *Rule* and then an instance of the same rule. If, instead, the contraction formula is principal in *Rule* then we apply Lemmas 5.5 and 5.2 to its premiss(es) without repetition of the principal formulas; next, we apply the inductive hypothesis to each (modified) premiss and we conclude by an instance of *Rule*. To illustrate, if the last step of $\mathcal{D}$ is:

$$\dfrac{w\mathcal{R}u, w\mathcal{R}v, u : A, \Gamma \Rightarrow \Delta, w : A \rhd B, u : B \qquad w\mathcal{R}v, \Gamma \Rightarrow \Delta, w : A \rhd B, w : A \rhd B, v : A}{w\mathcal{R}v, \Gamma \Rightarrow \Delta, w : A \rhd B, w : A \rhd B}\ R\rhd$$

we transform $\mathcal{D}$ as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{w\mathcal{R}u, w\mathcal{R}v, u : A, \Gamma \Rightarrow \Delta, w : A \rhd B, u : B}{w\mathcal{R}u_1, w\mathcal{R}u, w\mathcal{R}v, u_1 : A, u : A, \Gamma \Rightarrow \Delta, u_1 : B, u : B}\ \mathit{Lem.5.5}}{w\mathcal{R}u, w\mathcal{R}u, w\mathcal{R}v, u : A, u : A, \Gamma \Rightarrow \Delta, u : B, u : B}\ \mathit{Lem.5.2}\,[u/u_1]}{w\mathcal{R}u, w\mathcal{R}v, u : A, \Gamma \Rightarrow \Delta, u : B}\ \mathit{IH} \qquad \dfrac{w\mathcal{R}v, \Gamma \Rightarrow \Delta, w : A \rhd B, w : A \rhd B, v : A}{w\mathcal{R}v, \Gamma \Rightarrow \Delta, w : A \rhd B, v : A}\ \mathit{IH}}{w\mathcal{R}v, \Gamma \Rightarrow \Delta, w : A \rhd B}\ R\rhd$$

□

We are now ready to prove cut elimination.

Theorem 5.7 (Cut). *Let C be either a labelled or forcing formula. If* $\mathsf{G3SS.L} \vdash \Gamma \Rightarrow \Delta, C$ *and* $\mathsf{G3SS.L} \vdash C, \Pi \Rightarrow \Sigma$ *then* $\mathsf{G3SS.L} \vdash \Gamma, \Pi \Rightarrow \Delta, \Sigma$.

Proof: As usual for G3-style calculi, see [12], the proof considers an uppermost instance of *Cut* and proceeds by lexicographical induction on the pair $\langle$weight of the cut formula, cut-height$\rangle$, where the *cut-height* is the sum of the height of the derivations of the two premisses of cut.

It is useful to organise the proof in three cases: (i) at least one premiss has a derivation of height 0; (ii) the cut formula is not principal in at least one of the premisses; (iii) the cut formula is principal in both premisses. We consider only cases (ii) and (iii) where the last step of the derivation of some premiss is by a rule for $\multimap$.

In case (ii) we can permute the cut upwards in the derivation of a premiss where the cut formula is not principal in the last step (if needed, we use Lemma 5.2 to replace *eigenvariables* with the appropriate labels). Suppose, e.g., the cut formula C is not principal in the last step of the left premiss, which is by an instance of $L\rhd$. We transform:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\mathcal{D}_{11}\\ w\mathcal{R}u, w\mathcal{R}v, u:A, w:A\rhd B,\Gamma\Rightarrow\Delta, C, v:A\end{array}\quad\begin{array}{c}\vdots\mathcal{D}_{12}\\ w\mathcal{R}u, w\mathcal{R}v, v:B, u:A, w:A\rhd B,\Gamma\Rightarrow\Delta, C\end{array}}{w\mathcal{R}v, w:A\rhd B,\Gamma\Rightarrow\Delta, C}{\scriptstyle L\rhd}\qquad\begin{array}{c}\vdots\mathcal{D}_{2}\\ C,\Pi\Rightarrow\Sigma\end{array}}{w\mathcal{R}v, w:A\rhd B,\Gamma,\Pi\Rightarrow\Delta,\Sigma}{\scriptstyle Cut}$$

into the following derivation (where u_1 is a fresh label):

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\mathcal{D}_{11}\\ w\mathcal{R}u, w\mathcal{R}v, u:A, w:A\rhd B,\Gamma\Rightarrow\Delta, v:A, C\end{array}}{w\mathcal{R}u_1, w\mathcal{R}v, u_1:A, w:A\rhd B,\Gamma\Rightarrow\Delta, v:A, C}{\scriptstyle L.5.2}\quad\begin{array}{c}\vdots\mathcal{D}_{2}\\ C,\Pi\Rightarrow\Sigma\end{array}}{w\mathcal{R}u_1, w\mathcal{R}v, u_1:A, w:A\rhd B,\Gamma,\Pi\Rightarrow\Delta,\Sigma, v:A}{\scriptstyle Cut}\quad\dfrac{\dfrac{\begin{array}{c}\vdots\mathcal{D}_{12}\\ w\mathcal{R}u, w\mathcal{R}v, v:B, u:A, w:A\rhd B,\Gamma\Rightarrow\Delta, C\end{array}}{w\mathcal{R}u_1, w\mathcal{R}v, v:B, u_1:A, w:A\rhd B,\Gamma\Rightarrow\Delta, C}{\scriptstyle L.5.2}\quad\begin{array}{c}\vdots\mathcal{D}_{2}\\ C,\Pi\Rightarrow\Sigma\end{array}}{w\mathcal{R}u_1, w\mathcal{R}v, v:B, u_1:A, w:A\rhd B,\Gamma,\Pi\Rightarrow\Delta,\Sigma}{\scriptstyle Cut}}{w\mathcal{R}v, w:A\rhd B,\Gamma,\Pi\Rightarrow\Delta,\Sigma}{\scriptstyle L\rhd}$$

where we have two instances of *Cut* that are admissible having lesser cut-height.

If we are in Case (iii) and the cut formula is not of shape $w:A\multimap B$, see [10, Theorem 4.9]. If, instead, the cut formula has shape $w:A\rhd B$ and the right premiss is by rule $L\rhd$ we have the following derivation:

$$\dfrac{\dfrac{\vdots\mathcal{D}_{11}\qquad\vdots\mathcal{D}_{12}}{w\mathcal{R}v,\Gamma\Rightarrow\Delta, w:A\rhd B}{\scriptstyle R\rhd}\qquad\dfrac{\vdots\mathcal{D}_{21}\qquad\vdots\mathcal{D}_{22}}{w:A\rhd B, w\mathcal{R}u,\Pi\Rightarrow\Sigma}{\scriptstyle L\rhd}}{w\mathcal{R}v,\Gamma, w\mathcal{R}u,\Pi\Rightarrow\Delta,\Sigma}{\scriptstyle Cut}$$

Where:

- $\mathcal{D}_{11}$ is: $\begin{array}{c}\vdots\\ w\mathcal{R}v_1, v_1:A, w\mathcal{R}v,\Gamma\Rightarrow\Delta, v_1:B\end{array}$
- $\mathcal{D}_{12}$ is: $\begin{array}{c}\vdots\\ w\mathcal{R}v,\Gamma\Rightarrow\Delta, w:A\rhd B, v:A\end{array}$
- $\mathcal{D}_{21}$ is: $\begin{array}{c}\vdots\\ w\mathcal{R}u_1, u_1:A, w\mathcal{R}u, w:A\rhd B,\Pi\Rightarrow\Sigma, u:A\end{array}$
- $\mathcal{D}_{22}$ is: $\begin{array}{c}\vdots\\ w\mathcal{R}u_1, u:B, u_1:A, w\mathcal{R}u, w:A\rhd B,\Pi\Rightarrow\Sigma\end{array}$

We transform it into the following derivation containing some instances of *Cut* that are admissible by inductive hypothesis (here and in the following derivations, instances of cut marked with (†) have lesser cut-height, and those marked with (‡) have a cut formula of lower weight; moreover Γ^k stands for k copies of Γ):

$$\dfrac{\dfrac{\begin{array}{c}\vdots\mathcal{D}_\alpha\\ (w\mathcal{R}v)^3,(w\mathcal{R}u)^2,\Gamma^2,\Pi^2\Rightarrow\Delta^2,\Sigma^2,u:A\end{array}\qquad \begin{array}{c}\vdots\mathcal{D}_\beta\\ u:A,w\mathcal{R}v,w\mathcal{R}u,\Gamma,\Pi\Rightarrow\Delta,\Sigma\end{array}}{(w\mathcal{R}v)^4,(w\mathcal{R}u)^3,\Gamma^3,\Pi^3\Rightarrow\Delta^3,\Sigma^3}\ \mathit{Cut}\ (\ddagger)}{w\mathcal{R}v,w\mathcal{R}u,\Gamma,\Pi\Rightarrow\Delta,\Sigma}\ \mathit{Thm.}5.6$$

Where:

- $\mathcal{D}_\alpha$ is the following derivation:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\mathcal{D}_{12}\\ w\mathcal{R}v,\Gamma\Rightarrow\Delta,v:A,w:A\rhd B\end{array}\quad \begin{array}{c}\vdots\mathcal{D}_{2}\\ w:A\rhd B,w\mathcal{R}u,\Pi\Rightarrow\Sigma\end{array}}{w\mathcal{R}u,w\mathcal{R}v,\Gamma,\Pi\Rightarrow\Delta,\Sigma,v:A}\ (\dagger)\qquad \dfrac{\begin{array}{c}\vdots\mathcal{D}_{1}\\ w\mathcal{R}v,\Gamma\Rightarrow\Delta,w:A\rhd B\end{array}\quad \dfrac{\begin{array}{c}\vdots\mathcal{D}_{21}\\ w:A\rhd B,u_1:A,w\mathcal{R}u_1,w\mathcal{R}u,\Pi\Rightarrow\Sigma,u:A\end{array}}{w:A\rhd B,v:A,w\mathcal{R}v,w\mathcal{R}u,\Pi\Rightarrow\Sigma,u:A}\ L.5.2}{v:A,w\mathcal{R}u,(w\mathcal{R}v)^2,\Gamma,\Pi\Rightarrow\Delta,\Sigma,u:A}\ (\dagger)}{(w\mathcal{R}u)^2,(w\mathcal{R}v)^3,\Gamma^2,\Pi^2\Rightarrow\Delta^2,\Sigma^2,u:A}\ (\ddagger)$$

- $\mathcal{D}_\beta$ is the following derivation:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\mathcal{D}_{11}\\ w\mathcal{R}v_1,w\mathcal{R}v,v_1:A,\Gamma\Rightarrow\Delta,v_1:B\end{array}}{w\mathcal{R}u,w\mathcal{R}v,u:A,\Gamma\Rightarrow\Delta,u:B}\ L.5.2\qquad \dfrac{\begin{array}{c}\vdots\mathcal{D}_{1}\\ w\mathcal{R}v,\Gamma\Rightarrow\Delta,w:A\rhd B\end{array}\quad \dfrac{\begin{array}{c}\vdots\mathcal{D}_{22}\\ w:A\rhd B,u:B,w\mathcal{R}u_1,w\mathcal{R}u,u_1:A,\Pi\Rightarrow\Sigma\end{array}}{w:A\rhd B,u:B,(w\mathcal{R}u)^2,u:A,\Pi\Rightarrow\Sigma}\ L.5.2}{u:B,w\mathcal{R}v,(w\mathcal{R}u)^2,u:A,\Gamma,\Pi\Rightarrow\Delta,\Sigma}\ (\dagger)}{(u:A)^2,(w\mathcal{R}v)^2,(w\mathcal{R}u)^3,\Gamma^2,\Pi\Rightarrow\Delta^2,\Sigma}\ \mathit{Cut}\ (\ddagger)}{u:A,w\mathcal{R}v,w\mathcal{R}u,\Gamma,\Pi\Rightarrow\Delta,\Sigma}\ \mathit{Thm.}5.6$$

If the cut formula has shape $w : A \rhd B$ and the right premiss is by rule $L\rhd'$, we transform the following derivation (the conclusion of $\mathcal{D}_{11}$ is $w\mathcal{R}v_1, v_1 : A, \Gamma \Rightarrow \Delta, v_1 : B$):

$$\dfrac{\dfrac{\begin{array}{c}\vdots\mathcal{D}_{11}\\ \mathcal{S}\end{array}\quad \begin{array}{c}\vdots\mathcal{D}_{12}\\ w\mathcal{R}v,\Gamma\Rightarrow\Delta,w:A\rhd B,v:A\end{array}}{w\mathcal{R}v,\Gamma\Rightarrow\Delta,w:A\rhd B}\ R{\rhd}\qquad \dfrac{\begin{array}{c}\vdots\mathcal{D}_{21}\\ w:A\rhd B,u:A,w\mathcal{R}u,\Pi\Rightarrow\Sigma\end{array}}{w:A\rhd B,\Pi\Rightarrow\Sigma}\ L{\rhd}'}{w\mathcal{R}v,\Gamma,\Pi\Rightarrow\Delta,\Sigma}\ \mathit{Cut}$$

into the following one where we have some admissible cuts:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\mathcal{D}_{12}\\ w\mathcal{R}v,\Gamma\Rightarrow\Delta,w:A\rhd B,v:A\end{array}\quad \begin{array}{c}\vdots\mathcal{D}_{2}\\ w:A\rhd B,\Pi\Rightarrow\Sigma\end{array}}{w\mathcal{R}v,\Gamma,\Pi\Rightarrow\Delta,\Sigma,v:A}\ (\dagger)\qquad \dfrac{\begin{array}{c}\vdots\mathcal{D}_{1}\\ w\mathcal{R}v,\Gamma\Rightarrow\Delta,w:A\rhd B\end{array}\quad \dfrac{\begin{array}{c}\vdots\mathcal{D}_{21}\\ w:A\rhd B,u:A,w\mathcal{R}u,\Pi\Rightarrow\Sigma\end{array}}{w:A\rhd B,v:A,w\mathcal{R}v,\Pi\Rightarrow\Sigma}\ L.5.2}{v:A,(w\mathcal{R}v)^2,\Gamma,\Pi\Rightarrow\Delta,\Sigma}\ (\dagger)}{(w\mathcal{R}v)^3,\Gamma^2,\Pi^2\Rightarrow\Delta^2,\Sigma^2}\ (\ddagger)}{w\mathcal{R}v,\Gamma,\Pi\Rightarrow\Delta,\Sigma}\ \mathit{Thm.}5.6$$

If the cut formula has shape $w : A \blacktriangleright B$ and the right premiss is by rule $L\blacktriangleright$, we have the following derivation:

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\mathcal{D}_{11}\\ u:A, w\mathcal{R}u, \Gamma \Rightarrow \Delta, u:B\end{matrix} \qquad \begin{matrix}\vdots\,\mathcal{D}_{12}\\ \Gamma \Rightarrow \Delta, w \Vdash^{\exists} A\end{matrix} \qquad \begin{matrix}\vdots\,\mathcal{D}_{13}\\ w \Vdash^{\forall} B, \Gamma \Rightarrow \Delta\end{matrix}}{\Gamma \Rightarrow \Delta, w : A \blacktriangleright B}\,R\blacktriangleright \qquad \vdots\,\mathcal{D}_2}{w\mathcal{R}v, \Gamma, \Pi \Rightarrow \Delta, \Sigma}\,Cut$$

Where $\mathcal{D}_2$ is the following derivation:

$$\dfrac{\begin{matrix}\vdots\,\mathcal{D}_{21}\\ w \Vdash^{\exists} A, w : A \blacktriangleright B, w\mathcal{R}v, \Pi \Rightarrow \Sigma, w \Vdash^{\forall} B, v : A\end{matrix} \qquad \begin{matrix}\vdots\,\mathcal{D}_{22}\\ v : B, w \Vdash^{\exists} A, w : A \blacktriangleright B, w\mathcal{R}v, \Pi \Rightarrow \Sigma, w \Vdash^{\forall} B\end{matrix}}{w : A \blacktriangleright B, w\mathcal{R}v, \Pi \Rightarrow \Sigma}\,L\blacktriangleright$$

We transform it into the following derivation containing some admissible instances of cut:

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\mathcal{D}_{\alpha}\\ (w\mathcal{R}v)^2, \Gamma^4, \Pi \Rightarrow \Delta^4, \Sigma, v : B\end{matrix} \qquad \begin{matrix}\vdots\,\mathcal{D}_{\beta}\\ v : B, w\mathcal{R}v, \Gamma^3, \Pi \Rightarrow \Delta^3, \Sigma\end{matrix}}{(w\mathcal{R}v)^3, \Gamma^7, \Pi^2 \Rightarrow \Delta^7, \Sigma^2}\,(\ddagger)}{w\mathcal{R}v, \Gamma, \Pi \Rightarrow \Delta, \Sigma}\,Thm.5.6$$

Where:

- $\mathcal{D}_{\alpha}$ is the following derivation:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\,\mathcal{D}_{12}\\ \Gamma \Rightarrow \Delta, w \Vdash^{\exists} A\end{matrix} \qquad \dfrac{\begin{matrix}\vdots\,\mathcal{D}_{1}\\ \Gamma \Rightarrow \Delta, w : A \blacktriangleright B\end{matrix} \quad \begin{matrix}\vdots\,\mathcal{D}_{21}\\ w : A \blacktriangleright B, w \Vdash^{\exists} A, w\mathcal{R}v, \Pi \Rightarrow \Sigma, v : A, w \Vdash^{\forall} B\end{matrix}}{w \Vdash^{\exists} A, w\mathcal{R}v, \Gamma, \Pi \Rightarrow \Delta, \Sigma, v : A, w \Vdash^{\forall} B}\,(\dagger)}{w\mathcal{R}v, \Gamma^2, \Pi \Rightarrow \Delta^2, \Sigma, v : A, w \Vdash^{\forall} B}\,(\ddagger) \qquad \begin{matrix}\vdots\,\mathcal{D}_{13}\\ w \Vdash^{\forall} B, \Gamma \Rightarrow \Delta\end{matrix}}{w\mathcal{R}v, \Gamma^3, \Pi \Rightarrow \Delta^3, \Sigma, v : A}\,(\ddagger) \qquad \dfrac{\begin{matrix}\vdots\,\mathcal{D}_{11}\\ u : A, w\mathcal{R}u, \Gamma \Rightarrow \Delta, u : B\end{matrix}}{v : A, w\mathcal{R}v, \Gamma \Rightarrow \Delta, v : B}\,L.5.2}{(w\mathcal{R}v)^2, \Gamma^4, \Pi \Rightarrow \Delta^4, \Sigma, v : B}\,(\ddagger)$$

- $\mathcal{D}_{\beta}$ is the following derivation:

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\mathcal{D}_{12}\\ \Gamma \Rightarrow \Delta, w \Vdash^{\exists} A\end{matrix} \qquad \dfrac{\begin{matrix}\vdots\,\mathcal{D}_{1}\\ \Gamma \Rightarrow \Delta, w : A \blacktriangleright B\end{matrix} \quad \begin{matrix}\vdots\,\mathcal{D}_{22}\\ w : A \blacktriangleright B, w \Vdash^{\exists} A, v : B, w\mathcal{R}v, \Pi \Rightarrow \Sigma, w \Vdash^{\forall} B\end{matrix}}{w \Vdash^{\exists} A, v : B, w\mathcal{R}v, \Gamma, \Pi \Rightarrow \Delta, \Sigma, w \Vdash^{\forall} B}\,(\dagger)}{w\mathcal{R}v, v : B, \Gamma^2, \Pi \Rightarrow \Delta^2, \Sigma, w \Vdash^{\forall} B}\,(\ddagger) \qquad \begin{matrix}\vdots\,\mathcal{D}_{13}\\ w \Vdash^{\forall} B, \Gamma \Rightarrow \Delta\end{matrix}}{v : B, w\mathcal{R}v, \Gamma^3, \Pi \Rightarrow \Delta^3, \Sigma}\,(\ddagger)$$

Finally, if the cut formula has shape $w : A \blacktriangleright B$ and the right premiss is by rule $L\blacktriangleright'$, we transform the following derivation (the conclusion of $\mathcal{D}_{11}$ is $w\mathcal{R}u, u : A, \Gamma \Rightarrow \Delta, u : B$):

$$\dfrac{\dfrac{\overset{\vdots\,\mathcal{D}_{11}}{\mathcal{S}} \quad \overset{\vdots\,\mathcal{D}_{12}}{\Gamma \Rightarrow \Delta, w \Vdash^{\exists} A} \quad \overset{\vdots\,\mathcal{D}_{13}}{w \Vdash^{\forall} B, \Gamma \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta, w : A \blacktriangleright B}\,R\blacktriangleright \qquad \dfrac{\overset{\vdots\,\mathcal{D}_{21}}{w : A \blacktriangleright B, w \Vdash^{\exists} A, \Pi \Rightarrow \Sigma, w \Vdash^{\forall} B}}{w : A \blacktriangleright B, \Pi \Rightarrow \Sigma}\,L\blacktriangleright'}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\,Cut$$

into the following derivation containing some admissible instances of cut:

$$\dfrac{\dfrac{\dfrac{\overset{\vdots\,\mathcal{D}_{12}}{\Gamma \Rightarrow \Delta, w \Vdash^{\exists} A} \quad \dfrac{\overset{\vdots\,\mathcal{D}_{1}}{\Gamma \Rightarrow \Delta, w : A \blacktriangleright B} \quad \overset{\vdots\,\mathcal{D}_{21}}{w : A \blacktriangleright B, w \Vdash^{\exists} A, \Pi \Rightarrow \Sigma, w \Vdash^{\forall} B}}{w \Vdash^{\exists} A, \Gamma, \Pi \Rightarrow \Delta, \Sigma, w \Vdash^{\forall} B}\,(\dagger)}{\Gamma^2, \Pi \Rightarrow \Delta^2, \Sigma, w \Vdash^{\forall} B}\,(\ddagger) \quad \overset{\vdots\,\mathcal{D}_{13}}{w \Vdash^{\forall} B, \Gamma \Rightarrow \Delta}}{\dfrac{\Gamma^3, \Pi \Rightarrow \Delta^3, \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\,T.5.6}\,(\ddagger)$$

□

6. Soundness and completeness

In this section it is proved that the calculus G3SS.L is sound and complete with respect to L-frames. In particular, the proof of completeness will be a modular Tait-Shütte-Takeuti style proof: we define an exhaustive proof-search procedure that either outputs a G3SS.L-derivation or allows to build a countermodel based on an L-frame.

6.1. Soundness

In order to show that G3SS.L is sound with respect to L-frames we extend the notion of validity to sequents. We begin with some preliminary definitions.

Let $\mathcal{M} = \langle W, R, V\rangle$ be a model and let $\sigma : LAB \longrightarrow W$ be a function mapping labels to worlds of the model $\mathcal{M}$. We say that:

- $\mathcal{M}, \sigma$ *satisfies* the relational atom $w\mathcal{R}v$ iff $\sigma(w)R\sigma(v)$;
- $\mathcal{M}, \sigma$ *satisfies* the forcing formula $w \Vdash^{\forall/\exists} A$ iff each/some v such that $\sigma(w)R\sigma(v)$ is such that $\models^{\mathcal{M}}_{\sigma(v)} A$;

- $\mathcal{M}, \sigma$ *satisfies* the labelled formula $w : A$ iff $\models^{\mathcal{M}}_{\sigma(w)} A$.

A sequent $\Gamma \Rightarrow \Delta$ is (L-)*valid* iff each pair $\mathcal{M}, \sigma$ (with $\mathcal{M}$ based on an L-frame) satisfying all formulas in Γ satisfies some formula in Δ.

Theorem 6.1 (Soundness). *If a sequent $\mathcal{S}$ is derivable in* G3SS.L*, then it is* L*-valid.*

Proof: The proof is by induction on the height of the derivation $\mathcal{D}$ of $\mathcal{S}$. The base cases hold trivially. For the inductive steps, we have to check that each rule of G3SS.L preserves validity over L-frames. First, we prove that the logical rules preserve validity. We consider only rules for $\rhd$ and for $\blacktriangleright$ (the other cases are as in [10, Theorem 5.3]).

For rule $L\rhd$, assume the last step of $\mathcal{D}$ has shape (u not in the conclusion):

$$\frac{w\mathcal{R}u, w\mathcal{R}v, u : A, w : A \rhd B, \Gamma \Rightarrow \Delta, v : A \qquad w\mathcal{R}u, w\mathcal{R}v, v : B, u : A, w : A \rhd B, \Gamma \Rightarrow \Delta}{w\mathcal{R}v, w : A \rhd B, \Gamma \Rightarrow \Delta}\ {\scriptstyle L\rhd}$$

We need prove that each pair $\mathcal{M}, \sigma$ satisfying (all formulas in) $w\mathcal{R}v, w : A \rhd B, \Gamma$ (Π, for short) satisfies some formula in Δ. Let $\mathcal{M}, \sigma'$ be a generic pair satisfying all of $w\mathcal{R}u, u : A, \Pi$. By induction on the left premiss, we know that $\mathcal{M}, \sigma'$ satisfies also some formula in Δ or $v : A$. In the first case we are done (u is not in Π, Δ and, hence, $\mathcal{M}, \sigma'$ is a generic pair satisfying Π). In the second case $\mathcal{M}, \sigma'$ satisfies $v : A \supset B$, and, hence, it satisfies $v : B$. By induction on the second premiss, we obtain that $\mathcal{M}, \sigma'$ satisfies some formula in Δ and we are done.

For rule $L\rhd'$, assume the last step of $\mathcal{D}$ has shape (u not in the conclusion):

$$\frac{w\mathcal{R}u, u : A, w : A \rhd B, \Gamma \Rightarrow \Delta}{w : A \rhd B, \Gamma \Rightarrow \Delta}\ {\scriptstyle L\rhd'}$$

Let us consider a generic pair $\mathcal{M}, \sigma$ satisfying $w : A \rhd B$ and all formulas in Γ. Since $w : A \rhd B$ is satisfied, we know there is a world of the model $\mathcal{M}$ accessible from $\sigma(w)$ where A is true. Let σ' be like σ except for the label u that is mapped on that world. The pair $\mathcal{M}, \sigma'$ satisfies $w\mathcal{R}u, u : A, w : A \rhd B, \Gamma$ and, by inductive hypothesis, it satisfies also some formula in Δ. We conclude that $\mathcal{M}, \sigma$ satisfies some formula in Δ (since u is not in $w : A \rhd B, \Gamma \Rightarrow \Delta$).

For rule $R \rhd$, assume the last step of $\mathcal{D}$ has shape (u not in the conclusion):

$$\dfrac{w\mathcal{R}u, w\mathcal{R}v, u : A, \Gamma \Rightarrow \Delta, u : B \qquad w\mathcal{R}v, \Gamma \Rightarrow \Delta, w : A \rhd B, v : A}{w\mathcal{R}v, \Gamma \Rightarrow \Delta, w : A \rhd B}\ {\scriptstyle R\rhd,\ u \text{ fresh}}$$

We have to prove that each pair $\mathcal{M}, \sigma$ satisfying all formulas in $w\mathcal{R}v, \Gamma$ (Π, for short) satisfies the formula $w : A \rhd B$ or some formula in Δ. By induction hypothesis applied to the second premiss, we know that $\mathcal{M}, \sigma$ satisfies some formula in Δ or $w : A \rhd B$ or $v : A$. The non trivial case happens when $\mathcal{M}, \sigma$ satisfies no formula in Δ but it satisfies $v : A$. In this case, we know that for the world $\sigma(v)$ both $\sigma(w)\mathcal{R}\sigma(v)$ and $\models^{\mathcal{M}}_{\sigma(v)} A$ hold, and hence there exists a pair $\mathcal{M}, \sigma'$ such that σ' differs from σ possibly only on u and such that it satisfies $w\mathcal{R}u, u : A, \Pi$. Let us consider then any generic pair of this kind. Observe that also $\mathcal{M}, \sigma'$ satisfies no formula in Δ, since u does not occur in Δ. Hence, by induction on the left premiss, $\mathcal{M}, \sigma'$ satisfies $u : B$. Therefore, we have seen that A is true in some world related to $\sigma(w)$ and, by genericity of σ', that any world related to $\sigma(w)$ satisfies $A \supset B$. We conclude that every pair $\mathcal{M}, \sigma$ satisfying all of Π but none of Δ must satisfy $w : A \rhd B$.

For rule $L \blacktriangleright$, assume the last step of $\mathcal{D}$ has shape:

$$\dfrac{w \Vdash^{\exists} A, w\mathcal{R}v, w : A \blacktriangleright B, \Gamma \Rightarrow \Delta, v : A, w \Vdash^{\forall} B \qquad w\mathcal{R}v, v : B, w \Vdash^{\exists} A, w : A \blacktriangleright B, \Gamma \Rightarrow \Delta, w \Vdash^{\forall} B}{w\mathcal{R}v, w : A \blacktriangleright B, \Gamma \Rightarrow \Delta}\ {\scriptstyle L\blacktriangleright}$$

We have to prove that each pair $\mathcal{M}, \sigma$ satisfying all formulas in $w\mathcal{R}v, w : A \blacktriangleright B$ satisfies some formula in Δ. The satisfaction of $w : A \blacktriangleright B$ guarantees the satisfaction of $w \Vdash^{\exists} A$ and the non satisfaction of $w \Vdash^{\forall} B$. Finally, it guarantees that $\not\models^{\mathcal{M}}_{\sigma(v)} A$ or $\models^{\mathcal{M}}_{\sigma(v)} B$. In the second case, the induction hypothesis applied to the second premiss will show that $\mathcal{M}, \sigma$ satisfies some formula in Δ. In the first case, the induction hypothesis applied to the first premiss will show again that $\mathcal{M}, \sigma$ satisfies some formula in Δ.

The proof for rule $L \blacktriangleright'$ is straightforward. Assume the last step of $\mathcal{D}$ has shape (no relational atom in Γ):

$$\dfrac{w \Vdash^{\exists} A, w : A \blacktriangleright B, \Gamma \Rightarrow \Delta, w \Vdash^{\forall} B}{w : A \blacktriangleright B, \Gamma \Rightarrow \Delta}\ {\scriptstyle L\blacktriangleright'}$$

Suppose that $\mathcal{M}, \sigma$ satisfies all of $w : A \blacktriangleright B, \Gamma$. The satisfiability of $w : A \blacktriangleright B$ entails that there is a world accessible from w where A is true. Thanks to the inductive hypothesis the pair $\mathcal{M}, \sigma$ satisfies some formula in Δ or $w \Vdash^{\forall} B$. The latter is impossible since $\mathcal{M}$ contains a world accessible from w where B is false (otherwise $\mathcal{M}, \sigma$ would not satisfy $w : A \blacktriangleright B$). We conclude that $\mathcal{M}, \sigma$ satisfies some formula in Δ.

For rule $R\blacktriangleright$, assume the last step of $\mathcal{D}$ has shape (u not in the conclusion):

$$\frac{w\mathcal{R}u, u : A, \Gamma \Rightarrow \Delta, u : B \qquad \Gamma \Rightarrow \Delta, w \Vdash^{\exists} A \qquad w \Vdash^{\forall} B, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, w : A \blacktriangleright B}\ R\blacktriangleright$$

We have to prove that each pair $\mathcal{M}, \sigma$ satisfying all formulas in Γ satisfies the formula $w : A \blacktriangleright B$ or some formula in Δ. By applying the induction hypothesis to the second premiss, we know that $\mathcal{M}, \sigma$ satisfies some formula in Δ or $w \Vdash^{\exists} A$. In the first case we are done. Otherwise, suppose that $\mathcal{M}, \sigma$ satisfies no formula in Δ but it satisfies $w \Vdash^{\exists} A$. There exists then a world related to $\sigma(w)$ in which A is true. Moreover, the induction hypothesis applied to the third premiss will show that $\mathcal{M}, \sigma$ cannot satisfy $w \Vdash^{\forall} B$, for otherwise we would infer the satisfaction of some formula in Δ. Therefore there must exist some world related to $\sigma(w)$ that falsifies B. Finally, an argument analogous to that used for $R\triangleright$ will show that every world related to $\sigma(w)$ satisfies $A \supset B$. Overall, we conclude that $\mathcal{M}, \sigma$ satifies $w : A \blacktriangleright B$.

Each non-logical rule preserves validity over frames satisfying the corresponding semantic properties; cf. [12, Thm. 12.13]. □

6.2. Completeness

Definition 6.2 (Saturation). A branch $\mathcal{B}$ of a G3SS.L-proof-search tree for a sequent $\mathcal{S}$ (see procedure 6.5) is L-*saturated* if it satisfies the following conditions, where $\mathbf{\Gamma}$ ($\mathbf{\Delta}$) is the union of the antecedents (succedents) occurring in that branch,

1. no $w : p$ occurs in $\mathbf{\Gamma} \cap \mathbf{\Delta}$;

2. no $w : \bot$ occurs in $\mathbf{\Gamma}$;

3. if $w : A \wedge B$ is in $\mathbf{\Gamma}$ ($w : A \vee B \in \mathbf{\Delta}/w : A \supset B \in \mathbf{\Delta}$), then both $w : A$ and $w : B$ are in $\mathbf{\Gamma}$ ($w : A, w : B \in \mathbf{\Gamma}/w : A \in \mathbf{\Gamma}$ and $w : B \in \mathbf{\Delta}$, resp.);
4. if $w : A \wedge B$ is in $\mathbf{\Delta}$ ($w : A \vee B$ is in $\mathbf{\Gamma}/w : A \supset B$ is in $\mathbf{\Gamma}$), then at least one of $w : A$ and $w : B$ is in $\mathbf{\Delta}$ ($w : A \in \mathbf{\Gamma}$ or $w : B \in \mathbf{\Gamma}/w : A \in \mathbf{\Delta}$ or $w : B \in \mathbf{\Gamma}$, resp.);
5. if $w : A \rhd B$ is in $\mathbf{\Gamma}$, then
   (a) for some u, both $w\mathcal{R}u$ and $u : A$ are in $\mathbf{\Gamma}$; and
   (b) $v : A$ is in $\mathbf{\Delta}$ or $v : B$ is in $\mathbf{\Gamma}$, for any v such that $w\mathcal{R}v$ is in $\mathbf{\Gamma}$;
6. if $w : A \rhd B$ is in $\mathbf{\Delta}$, then,
   (a) for some u, both $w\mathcal{R}u$ and $u : A$ are in $\mathbf{\Gamma}$ and $u : B$ is in $\mathbf{\Delta}$; or
   (b) $v : A$ is in $\mathbf{\Delta}$ for any v such that $w\mathcal{R}v$ is in $\mathbf{\Gamma}$;
7. if $w : A \blacktriangleright B$ is in $\mathbf{\Gamma}$, then
   (a) for some u_1, u_2, all of $w\mathcal{R}u_1$ and $w\mathcal{R}u_2$ and $u_1 : A$ are in $\mathbf{\Gamma}$, moreover $u_2 : B$ is in $\mathbf{\Delta}$; and
   (b) $v : A$ is in $\mathbf{\Delta}$ or $v : B$ is in $\mathbf{\Gamma}$, for any v such that $w\mathcal{R}v$ is in $\mathbf{\Gamma}$;
8. if $w : A \blacktriangleright B$ is in $\mathbf{\Delta}$, then
   (a) for some u, $u : A$ in $\mathbf{\Gamma}$ and $u : B$ is in $\mathbf{\Delta}$; or
   (b) $v : A$ is in $\mathbf{\Delta}$ for any v such that $w\mathcal{R}v$ is in $\mathbf{\Gamma}$; or
   (c) $v : B$ is in $\mathbf{\Gamma}$, for any v such that $w\mathcal{R}v$ is in $\mathbf{\Gamma}$;

9_R. if *Rule* is a non-logical rule of $\mathsf{G3SS.L}$, then for any set of principal formulas of *Rule* that are in $\mathbf{\Gamma}$ also the corresponding active formulas are in $\mathbf{\Gamma}$.

DEFINITION 6.3. Let $\mathcal{B}$ be L-saturated. The model $\mathcal{M}^{\mathcal{B}} = \langle W, R, V\rangle$ is thus defined:

1. W is the set of labels occurring in $\mathbf{\Gamma} \cup \mathbf{\Delta}$;
2. for each w, $v \in W$, wRv iff the formula $w\mathcal{R}v$ is in $\mathbf{\Gamma}$;
3. $V(p)$ is the set of all w such that $w : p$ is in $\mathbf{\Gamma}$.

Moreover, $\sigma_=$ denotes the identity function on LAB.

Observe that $\mathcal{M}^{\mathcal{B}}$ is well-defined thanks to clauses 1 and 2 of Def. 6.2.

LEMMA 6.4. *Let $\mathcal{B}$ be an* L*-saturated branch. Then*

1. *for any $\mathcal{L}$-formula A occurring in $\mathcal{B}$ we have that $\models^{\mathcal{M}^{\mathcal{B}}}_{\sigma_{=}(w)} A$ iff $w : A$ is in $\mathbf{\Gamma}$; and*
2. *$\mathcal{M}^{\mathcal{B}}$ is based on a frame for* L.

PROOF: Claim (2) follows by Definition 6.2.9_R and by construction of $\mathcal{M}^{\mathcal{B}}$ (Definition 6.3) and of $\sigma_{=}$.

The proof of claim (1) is by induction on the construction of A. The base case holds by construction of $\mathcal{M}^{\mathcal{B}}$ and of $\sigma_{=}$, and the inductive cases depend on Definition 6.2.3–8.

To illustrate, we consider the case when A has shape $w : B \rhd C$ and it occurs in $\mathcal{B}$.

If $w : B \rhd C$ is in $\mathbf{\Gamma}$, then we have to show that $\models^{\mathcal{M}^{\mathcal{B}}}_{\sigma_{=}(w)} B \rhd C$. By Definition 6.2.5(a), we know that, for some u, $w\mathcal{R}u$ and $u : B$ are in $\mathbf{\Gamma}$. Hence, $\sigma_{=}(w)R\sigma_{=}(u)$ and, by induction, $\models^{\mathcal{M}^{\mathcal{B}}}_{\sigma_{=}(u)} B$. Moreover, by Definition 6.2.5(b), we have that all v such that $w\mathcal{R}v$ is in $\mathbf{\Gamma}$ are such that either $v : B \in \mathbf{\Delta}$ or $v : C \in \mathbf{\Gamma}$. In both cases we have that if $\sigma_{=}(w)R\sigma_{=}(v)$ then $\models^{\mathcal{M}^{\mathcal{B}}}_{\sigma_{=}(v)} B \supset C$. We conclude that $\models^{\mathcal{M}^{\mathcal{B}}}_{\sigma_{=}(w)} B \rhd C$.

If, instead, $w : B \rhd C$ is in $\mathbf{\Delta}$, then we have to show that $\not\models^{\mathcal{M}^{\mathcal{B}}}_{\sigma_{=}(w)} B \supset C$. By Definition 6.2.6, we know that, either for some u, $w\mathcal{R}u$ and $u : A$ are in $\mathbf{\Gamma}$ and $u : C \in \mathbf{\Delta}$, or $v : B \in \mathbf{\Delta}$ for all v such that $w\mathcal{R}v \in \mathbf{\Gamma}$. In the first case we have that $\sigma_{=}(w)R\sigma_{=}(u)$ and, by induction, $\not\models^{\mathcal{M}^{\mathcal{B}}}_{\sigma_{=}(u)} B \supset C$. In the latter case we have that $\not\models^{\mathcal{M}^{\mathcal{B}}}_{\sigma_{=}(v)} B$ for all v such that $\sigma_{=}(w)R\sigma_{=}(v)$. In both cases we conclude that $\not\models^{\mathcal{M}^{\mathcal{B}}}_{\sigma_{=}(w)} B \rhd C$.

The case $w : B \blacktriangleright C \in \mathbf{\Gamma}$ is analogous to the corresponding one for $\rhd$ with the only further requirement of proving the existence of some world related to $\sigma_{=}(w)$ that falsifies B. But this is guaranteed by Definition 6.2.7(a), according to which there is some u_2 such that $w\mathcal{R}u_2$ is in $\mathbf{\Gamma}$ and $u_2 : B$ is in $\mathbf{\Delta}$. This means that $\sigma(w)R\sigma(u_2)$ and, by induction, that $\not\models^{\mathcal{M}^{\mathcal{B}}}_{\sigma_{=}(u_2)} B$.

For $w : B \blacktriangleright C \in \mathbf{\Delta}$, to prove that $\not\models^{\mathcal{M}^{\mathcal{B}}}_{\sigma_{=}(w)} B \blacktriangleright C$ one has to prove that at least one of the following conditions holds: (i) the existence of a world related to w that falsifies $B \supset C$, (ii) the falsity of B in every world related to $\sigma(w)$, (iii) the truth of C in every world related to $\sigma(w)$. For (i), we argue analogously to the case $w : B \rhd C \in \mathbf{\Delta}$. For (ii), one has to consider Definition 6.2.8(b), according to which for every v such that

$w\mathcal{R}v \in \mathbf{\Gamma}$, it holds $v : B \in \mathbf{\Delta}$. By induction, this means $\not\models^{\mathcal{M}^{\mathcal{B}}}_{\sigma=(v)} B$. Since by construction of $\mathcal{M}^{\mathcal{B}}$, all worlds related to w are of type $\sigma(v)$ for $w\mathcal{R}v \in \mathbf{\Gamma}$, we are done. The proof for (iii) runs analogously to (ii) through Definition 6.2.8(c). □

Procedure 6.5. A G3SS.L-proof-search tree for a sequent $\mathcal{S}$ is a tree of sequents that has $\mathcal{S}$ as root and whose branches grow according to the following procedure: if the leaf is an initial sequent or an instance of rule $L\bot$ the branch stops growing, else either no instance of rules of G3SS.L is applicable root first to it, or k instances are (where rules *Ser* and *Ref* are applied once w.r.t. each label occurring in the branch). In the first case, the branch stops growing; in this case it is immediate to see that we have a finite L-saturated branch. In the second case, we apply the k rule instances that are applicable in some order (each one will be applied to all end-sequents that are generated at the previous step). If the tree never stops growing then, by König's Lemma, it has an infinite branch which, as the reader can easily check, is L-saturated.[6]

Theorem 6.6 (Completeness). *If a sequent $\mathcal{S}$ is* L*-valid then it is derivable in* G3SS.L.

Proof: The proof is in three steps. First, in Def. 6.2, we define a notion of L-saturated branch of a proof-search for a sequent $\mathcal{S}$, where, intuitively, a branch is L-saturated when all applicable instances of G3SS.L-rules have been applied. Then, with Def. 6.3 and Lemma 6.4, we show that an L-saturated branch allows us to define a countermodel for $\mathcal{S}$ that is based on a frame for L. Finally, we give a root first G3SS.L-proof-search procedure, Prop. 6.5, that either outputs a G3SS.L-derivation of $\mathcal{S}$ – and, by Thm. 6.1, $\mathcal{S}$ is L-valid – or it outputs an L-saturated branch – and, therefore, $\mathcal{S}$ has a countermodel based on an appropriate frame. □

Corollary 6.7. The structural rules of inference – i.e., (left) weakening, (left) contraction, and cut – are semantically admissible in G3SS.L.

[6] Forcing formulas are needed to ensure that, when applying root-first all possible instances of rule $R\blacktriangleright$ (see Table 3), we obtain a saturated branch (see Def. 6.2.8). If in the second and third premisses of $R\blacktriangleright$ we had $v : A$ and $v : B$ (as it happens for rule $R\triangleright$) in place of the forcing formulas, we would obtain a completed proof search with an open branch that is not saturated because for each v such that $w\mathcal{R}v \in \mathbf{\Gamma}$ we would have that either $v : A \in \mathbf{\Delta}$ or $v : B \in \mathbf{\Gamma}$.

7. Priest's cancellation account of negation and SSI

In [16] G. Priest has considered the cancellation account of negation – i.e. the idea that "$\neg A$ deletes, neutralizes, erases, cancels $A[\ldots]$ so that $\neg A$ together with A leaves nothing" [19, p. 205] – and he modeled it by means of a relational semantics that is almost identical to the one we gave for the S5-based logic of SSI. If we represent Priest's non-symmetric and symmetric implication by $\rhd$ and $\blacktriangleright$, respectively, we have that a $\mathcal{L}$-formula is valid in Priest's semantics if and only if it is valid in the class of S5-frames for logics of SSI.

Priest considers a language containing one of $\rhd$ and $\blacktriangleright$. Priest's models are relational models without an accessibility relation – i.e., $\mathfrak{M} = \langle W, V\rangle$ – and truth of a formula in a world of a model is defined as in Section 3 save that in the truth-clauses for $\rhd$ and $\blacktriangleright$ the quantifiers are not restricted to accessible worlds. To illustrate, $\rhd$ is thus defined:

$$\models^{\mathfrak{M}}_{w} A \rhd B \quad \text{iff} \quad \exists u \in W(\models^{\mathfrak{M}}_{u} A) \text{ and } \forall v \in W(\models^{\mathfrak{M}}_{v} A \text{ implies } \models^{\mathfrak{M}}_{v} B)$$

The notions of truth in a model and validity are defined as in Section 3, but the definition of logical consequence differs since in Priest's approach it is a metatheoretical version of $\rhd$ or $\blacktriangleright$: for A to be a consequence of Γ, Γ must have a model and, possibly, $\neg A$ must have a model. This gives an highly non-Tarskian consequence relation in that, in both cases, it is neither reflexive, nor monotone, nor transitive.

This is not the place to discuss in full details the account of negation as cancellation (see [23] for some criticisms) nor the way Priest models it. We just note that Priest's semantics is not apt to model the cancellation account of negation since it adhere to a perfectly classical definition of negation. The non-classical elements are only the definition of implication and that of logical consequence. It follows that Priest's semantics has only a partial overlap with the cancellation account of negation: they share the claim that nothing follows from a contradiction (*alone*). But the cancellation account of negation cannot be reduced to this claim. If $\neg A$ deletes A so that Γ is extensionally identical to $\Gamma \cup \{A, \neg A\}$, then, if B is a consequence of some set of sentences Γ, it must be the case that B is a consequence of the set $\Gamma \cup \{A, \neg A\}$. Nevertheless in Priest's semantics nothing follows from a set containing contradictory formulas.

Priest's notion of validity is easily seen to be equivalent to that for the S5-based logic of super strict implication (analogously to what happens in standard modal logics, an accessibility relation that is an equivalence relation is equivalent to a universal accessibility relation). Formally we have the following result:

Proposition 7.1. Let A be any $\mathcal{L}$-formula containing only one of $\rhd$ and $\blacktriangleright$, $\mathcal{C}^{\mathfrak{M}}$ the class of all Priest's models and $\mathcal{C}^{\mathsf{S5}}$ the class of all frames from Section 3 where R is an equivalence relation, then $\mathcal{C}^{\mathfrak{M}} \models A$ iff $\mathcal{C}^{\mathsf{S5}} \models A$.

We find this result interesting since it entails that the logics of SSI are connexive logics having the same set of S5-validities of Priest's one, that are obtained without having to tamper with the Tarskian notion of consequence relation nor with the classical explosion model of negation [19]. Moreover, this entails that in Section 4 we have introduced a cut-free sequent calculus that characterizes validity in Priest's formal semantics.

All in all, the logics of SSI can be seen as modal extensions of classical logic that satisfies

> the central concern of connexive logic [which] consists of developing connexive systems that are naturally motivated conceptually or in terms of applications, that admit of a simple and plausible semantics, and that can be equipped with proof systems possessing nice proof-theoretical properties, such as the eliminability of the cut-rule. [14, p. 381]

8. Future works

For brevity, we have considered only the logics of SSI based on frames for normal modal logics. It should be possible to consider also the logics of SSI based on frames for C.I. Lewis's non-normal systems S1, S2 and S3. In particular, it would be interesting to present the S2-based logic of SSI since C.I. Lewis believed it the more likely correct logic of strict implication. It would also be interesting to see if for these weaker systems we can still use labelled calculi like the one we gave in Section 4, or if we have to introduce calculi more akin to those for non-normal modal logics presented in [10]. It might also be interesting to study some modifications of SSI, e.g., their reflexivizations (which should be very similar to Pizzi's consequential implication) and their constructive analogous. It would also be interesting to see if it is possible to have a SSI validating strong Boethius' Thesis without making the implication commutative as for consequential implications. Another problem that remains open is to give a complete axiomatisation of logics of SSI; we conjecture that this can be done by using the transfer methodology used in [17]. Finally, it would be extremely interesting to check if it is possible to give a proof-theoretical characterisation of the logics of SSI by means of some internal calculus such as hypersequents or nested sequents.

In this paper we have argued that the logics of SSI are connexive logics. In the future we plan to investigate whether SSI have some feature of the other main family of implications that avoid the paradoxes of strict implications, namely relevant implications. The philosophical motivation behind the introduction of SSI is quite similar to that behind relevant implications. Very roughly, we wanted 'A implies B' to be true just in case the truth of B *depends* on the truth of A and relevant logicians just in case the truth of A *is relevant* to the truth of B. It might well be that our notion of *dependance* is nothing but some notion of *relevance*. It is difficult to find a precise formal explication of what relevance is. Apart from the falsification of the paradoxes of material and strict implication, one typical condition for an implication being relevant is its having a *variable-sharing property*: if 'A implies B' is true then there must be some (all) propositional variables that are common to A and B [1, p. 33]. Hence, to see whether SSI are relevant implications we have to check whether they satisfy some variable-sharing property.

In the introduction we have claimed that the motivation for introducing SSI is orthogonal to D. Lewis' [7] and Stalnaker's [20] one for introducing variably strict implications. SSI are designed to avoid the paradoxes of strict implication (SI1) and (SI2). Variably strict implications, instead, are designed to avoid other properties of strict implications: monotonicity, contraposition and, last but not least, transitivity. Even if there is a partial overlap between them in that monotonicity fails also for SSI and contraposition fails for weak SSI, no one of the two approaches can be seen as a subsystem of the other in that SSI validate transitivity and variably strict implications validate the paradoxes of strict implication. Just as we tweaked strict implication to overcome its paradoxes, it should be possible to tweak variably strict implications in order to obtain variably SSI. Weak variably SSI has been considered and discharged by D. Lewis under the name of 'would counterfactual', cf [7, p. 25]; a connexive and variably strict implication has been considered in [22]. If variably strict implications approximate the logic of counterfactual conditionals, it might well be that variably SSI approximate the logic of indicative conditionals.

Acknowledgements. Partially supported by the Academy of Finland, research project no. 1308664.

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Bulletin of the **S**ection of **L**ogic
Volume 50/1 (2021), pp. 35–53
https://doi.org/10.18778/0138-0680.2020.24

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A SEMI-LATTICE OF FOUR-VALUED LITERAL-PARACONSISTENT-PARACOMPLETE LOGICS

Abstract

In this paper, we consider the class of four-valued literal-paraconsistent-paracomplete logics constructed by combination of isomorphs of classical logic CPC. These logics form a 10-element upper semi-lattice with respect to the functional embeddinig one logic into another. The mechanism of variation of paraconsistency and paracompleteness properties in logics is demonstrated on the example of two four-element lattices included in the upper semi-lattice. Functional properties and sets of tautologies of corresponding literal-paraconsistent-paracomplete matrices are investigated. Among the considered matrices there are the matrix of Puga and da Costa's logic V and the matrix of paranormal logic P^1I^1, which is the part of a sequence of paranormal matrices proposed by V. Fernández.

Keywords: Four-valued logics, paraconsistent logics, paracomplete logics, isomorphisms, literal-paraconsistent-paracomplete logics, semi-lattice of logics.

1. Introduction

Literal-paraconsistent-paracomplete logics (or LPP logics) are logics in which paraconsistency and/or paracompleteness occurs only at the level of literals, that is, formulas that are propositional letters or their iterated negations [13, p. 478].

*I want to thank the referee for the helpful comments on an earlier draft of this paper.

Presented by: Janusz Ciuciura
Received: May 22, 2020
Published online: November 13, 2020

The class of LPP logics is well studied. Among the LPP logics considered in this paper, there are the logics described in [19], [18], [21], [16], [7], [12], [5], [15].

There are several algorithms of constructing classes of such logics, for example, we can note the following ones: (1) construction of LPP logics by combination of isomorphs of classical logic CPC [11]; (2) construction of LPP logics by using literal-paraconsistent-paracomplete matrices (or LPP-matrices) [13]. But the classes of LPP logics constructed by these two methods are not equal: the class obtained by (1) is a subclass of the class obtained by (2).

It is known that Sette's three-valued paraconsistent logic P^1 [19] and three-valued paracomplete logic I^1 [21] can be represented as combinations of two three-valued isomorphs of classical propositional logic, contained in three-valued Bochvar's logic B_3 [11].

The paper [23] briefly describes the result of the application of the first method of constructing LPP logics to the four-valued case. So, the sixteen LPP logics form the upper semi-lattice with respect to the functional inclusion.

This paper is devoted to the study of four-valued propositional LPP logics that form the above-mentioned upper semi-lattice. Some properties of the class of four-valued paranormal logics constituting the supremum of the said semi-lattice were regarded in paper [22]. In this paper, we consider two interesting lattices of LPP logics that are included in the upper semi-lattice.

As a result, it allows us to demonstrate some properties of the negation operation in LPP logics, to compare LPP logics by functional properties and classes of tautologies.

The paper is stuctured as follows.

In the next section, we introduce some basic definitions. In the third section, we present the upper semi-lattice of four-valued LPP logics. In the next, we select two four-element lattices of LPP logics included in that semi-lattice, and consequently consider the properties of the logics that constitute these lattices.

2. Basic definitions

There are several approaches to the representation and analysis of logical systems. In this paper, logical systems are represented by means of logical matrices. Let us introduce some basic definitions.

Let $\mathcal{L}$ be a sentential language, i.e. $\mathcal{L} = \langle For, F_1, \ldots, F_m \rangle$ is an algebra generated by a set of variables $Var = \{p, q, r \ldots\}$. Elements of For are generated from variables with the use of operations $F_1, \ldots, F_m$, representing sentential conectives.

Let $\mathcal{A} = \langle V, f_1, \ldots, f_m \rangle$ be an algebra similar to $\mathcal{L}$, where V is the set of truth-values and each f_i is a function on V with the same arity as F_i.

Definition 2.1. A structure $\mathfrak{M} = \langle \mathcal{A}, D \rangle$ with $\mathcal{A}$ being an algebra similar to a propositional language $\mathcal{L}$ and $D \subseteq V$ — a non-empty subset of the universe of $\mathcal{A}$ is called a *logical matrix* for $\mathcal{L}$. Elements of D are called *designated elements* of $\mathfrak{M}$.

Throughout the paper we use the same symbols both for the propositional connective and the corresponding function on V.

Definition 2.2. A *valuation* v of the formula A in the matrix $\mathfrak{M}$ for the language $\mathcal{L}$ is a homomorphism from $\mathcal{L}$ into $\mathcal{A} = \langle V, f_1, \ldots, f_m \rangle$, such that

1. if p is a propositional variable, then $v(p) \in V$;
2. if $A_1, A_2, \ldots, A_n$ are formulas and F^n is an n-ary connective of language $\mathcal{L}$, then $v(F^n(A_1, A_2, \ldots, A_n)) = f^n(v(A_1), v(A_2), \cdots, v(A_n))$, where f^n is a function on V corresponding to F^n.

Definition 2.3. Some formula A is a *tautology* in $\mathfrak{M}$ (abbreviated to $\models_{\mathfrak{M}} A$), iff for every valuation v in $\mathfrak{M}$ it is true that $v(A) \in D$.

Definition 2.4. The theory generated by $\mathfrak{M}$ is the set of all tautologies in $\mathfrak{M}$. It is denoted by $E(\mathfrak{M})$.

Definition 2.5. The formula B logically follows from the set of formulas $\Gamma = \{A_1, A_2, \ldots, A_n\}$ in $\mathfrak{M}$ (abbreviated to $\Gamma \models_{\mathfrak{M}} B$), iff there is no such valuation v in $\mathfrak{M}$, such that $v(A_i) \in D$ for each $A_i \in \Gamma$ and $v(B) \notin D$.

Definition 2.6. The consequence relation generated by $\mathfrak{M}$ is the set $Cn(\mathfrak{M})$ of ordered pairs $\langle \Gamma, B \rangle$, such that for every valuation v in $\mathfrak{M}$ if $v(\Gamma) \subseteq D$, then $v(B) \in D$.

Let L_1 be a logic represented by matrix $\mathfrak{M}_1$ with the set of function F_1 and L_2 a logic represented by matrix $\mathfrak{M}_2$ with the set of functions F_2.

Definition 2.7. A logic L_1 is *functionally included* in a logic L_2 iff every function of F_1 can be expressed by a superposition of functions of F_2.

Definition 2.8. A logic L_1 is *functionally equivalent* to a logic L_2 iff

(1) L_1 is *functionally included* in L_2 and

(2) L_2 is *functionally included* in L_1.

Definition 2.9. A logic L_1 is a *fragment* of a logic L_2 iff L_1 is functionally included in L_2, but L_2 is not functionally included in L_1, i.e., the opposite does not hold.

Definition 2.10. Some fragment of a logic L is said to be an *isomorph of classical propositional logic* iff L has the classical set of tautologies and the classical consequence relation.

Different formal criteria may be used for the construction of paralogics. Jaśkowski's criteria for constructing paraconsistent logic is considered in some detail in [10]. In our investigation we use its "implicative-negative" part:

Definition 2.11. In a system of *paraconsistent* logic, *the Duns Scotus law* $A \supset (\neg A \supset B)$[1] is not valid, for some formulas A, B.

Definition 2.12. In a *paracomplete* logic system, *the Clavius law* $(\neg A \supset A) \supset A)$ is not valid, for some formula A (see [4]).

Definition 2.13. Logics, which are simultaneously paraconsistent and paracomplete, are called *paranormal* logics.

If logical systems are represented as theories (as classes of tautologies), this criteria best fits the scope.

In terms of logical consequence, logic is *paraconsistent*, iff its consequence relation is not explosive (principle of explosion: $A, \neg A \vDash B$, see [17]). The logic is *paracomplete*, iff there is a set of formulas Γ and formulas A and B, such that $\Gamma, A \vDash B$ and $\Gamma, \neg A \vDash B$, but $\Gamma \nvDash B$ (see [1, p. 1092]).

[1]The implicational law of over-completeness in Jaśkowski's notation (see [8]).

3. An upper semi-lattice of LPP logics

In the book [12, pp. 56–79] a class of four-valued LPP logics obtained by combining isomorphs of classical logic CPC is presented. These four four-valued CPC isomorphs are the fragments of Bochvar's four-valued logic B_4 [2, p. 289], which is determined by the matrix

$\mathfrak{M}_4^B = \langle \{0, {}^1\!/\!_3, {}^2\!/\!_3, 1\}, \sim, \cap, \cup, J_0, J_{1/3}, J_{2/3}, J_1, \{1\} \rangle$,

where $\sim x = 1 - x$, and J-operators, $\cap$ and $\cup$ are defined by the following truth-tables (cf. [2, p. 294]):

x	$J_0(x)$	$J_{1/3}(x)$	$J_{2/3}(x)$	$J_1(x)$
1	0	0	0	1
2/3	0	0	1	0
1/3	0	1	0	0
0	1	0	0	0

$\cap$	1	2/3	1/3	0
1	1	2/3	1/3	0
2/3	2/3	2/3	1/3	1/3
1/3	1/3	1/3	1/3	1/3
0	0	1/3	1/3	0

$\cup$	1	2/3	1/3	0
1	1	2/3	2/3	1
2/3	2/3	2/3	2/3	2/3
1/3	2/3	2/3	1/3	1/3
0	1	2/3	1/3	0

Functional properties of Bochvar's logic B_3 are determined by the union of two types of connectives – internal and external[2]. In the three-valued case internal connectives can be translated into external ones in two different ways [9, pp. 212–213]. These two translations provide construction of two fragments of B_3 isomorphic with CPC. In the one isomorph the truth-value ${}^1\!/\!_2$ is identified with 0 and in the other – with 1.

In the four-valued case there are four translation functions: $f_1(x)$, $f_2(x)$, $f_3(x)$ and $f_4(x)$. They have the following properties:

[2] A function f on V into V with arity n is called *external* iff for any values $x_1 \dots x_n$ we have either $f(x_1, \dots, x_n) = 0$ or $f(x_1, \dots, x_n) = 1$.

(1) $f_1(x)$ is $J_1(x)$ and takes the truth-values $^2/_3$ and $^1/_3$ to 0;

(2) $f_2(x)$ is $\sim J_0(x)$ and takes $^2/_3$ and $^1/_3$ to 1;

(3) $f_3(x)$ is $J_1(x) \cup J_{2/3}(x)$ and takes $^2/_3$ to 1 and $^1/_3$ to 0;

(4) $f_4(x)$ is $J_1(x) \cup J_{1/3}(x)$ and takes $^2/_3$ to 0 and $^1/_3$ to 1.

x	$f_1(x)$	$f_2(x)$	$f_3(x)$	$f_4(x)$
1	1	1	1	1
$^2/_3$	0	1	1	0
$^1/_3$	0	1	0	1
0	0	0	0	0

Using $f_1(x)$, $f_2(x)$, $f_3(x)$ and $f_4(x)$ analogously as it was done for B_3, we can construct four external negations and four external implications:

$\neg_i x := \sim f_i(x)$ and $x \to_i y := \neg_i x \cup f_i(y)$ $(i \in \{1, 2, 3, 4\})$,

which are defined by the following truth-tables:

x	$\neg_1 x$	$\neg_2 x$	$\neg_3 x$	$\neg_4 x$
1	0	0	0	0
$^2/_3$	1	0	0	1
$^1/_3$	1	0	1	0
0	1	1	1	1

$\to_1$	1	$^2/_3$	$^1/_3$	0
1	1	0	0	0
$^2/_3$	1	1	1	1
$^1/_3$	1	1	1	1
0	1	1	1	1

$\to_2$	1	$^2/_3$	$^1/_3$	0
1	1	1	1	0
$^2/_3$	1	1	1	0
$^1/_3$	1	1	1	0
0	1	1	1	1

$\to_3$	1	$^2/_3$	$^1/_3$	0
1	1	1	0	0
$^2/_3$	1	1	0	0
$^1/_3$	1	1	1	1
0	1	1	1	1

$\to_4$	1	$^2/_3$	$^1/_3$	0
1	1	0	1	0
$^2/_3$	1	1	1	1
$^1/_3$	1	0	1	0
0	1	1	1	1

Four-valued CPC isomorphs are determined by the following matrices:

$$\mathfrak{M}_1 = \langle\{0, {}^1/_3, {}^2/_3, 1\}, \neg_1, \to_1, \{1\}\rangle,$$
$$\mathfrak{M}_2 = \langle\{0, {}^1/_3, {}^2/_3, 1\}, \neg_2, \to_2, \{1, {}^2/_3, {}^1/_3\}\rangle,$$
$$\mathfrak{M}_3 = \langle\{0, {}^1/_3, {}^2/_3, 1\}, \neg_3, \to_3, \{1, {}^2/_3\}\rangle,$$
$$\mathfrak{M}_4 = \langle\{0, {}^1/_3, {}^2/_3, 1\}, \neg_4, \to_4, \{1, {}^1/_3\}\rangle.$$

Combining the operations $\neg_i$, $\to_j$ $(i, j \in \{1, 2, 3, 4\})$ of the isomorphs we construct the class of four-valued literal LPP logics. Let us present the corresponding matrices:

paraconsistent

$$\mathfrak{M}_5 = \langle\{0, {}^1/_3, {}^2/_3, 1\}, \neg_1, \to_2, \{1, {}^2/_3, {}^1/_3\}\rangle,$$
$$\mathfrak{M}_6 = \langle\{0, {}^1/_3, {}^2/_3, 1\}, \neg_3, \to_2, \{1, {}^2/_3, {}^1/_3\}\rangle,$$
$$\mathfrak{M}_7 = \langle\{0, {}^1/_3, {}^2/_3, 1\}, \neg_4, \to_2, \{1, {}^2/_3, {}^1/_3\}\rangle,$$
$$\mathfrak{M}_8 = \langle\{0, {}^1/_3, {}^2/_3, 1\}, \neg_1, \to_3, \{1, {}^2/_3\}\rangle,$$
$$\mathfrak{M}_9 = \langle\{0, {}^1/_3, {}^2/_3, 1\}, \neg_1, \to_4, \{1, {}^1/_3\}\rangle.$$

paracomplete

$$\mathfrak{M}_{10} = \langle\{0, {}^1/_3, {}^2/_3, 1\}, \neg_2, \to_1, \{1\}\rangle,$$
$$\mathfrak{M}_{11} = \langle\{0, {}^1/_3, {}^2/_3, 1\}, \neg_3, \to_1, \{1\}\rangle,$$
$$\mathfrak{M}_{12} = \langle\{0, {}^1/_3, {}^2/_3, 1\}, \neg_4, \to_1, \{1\}\rangle,$$
$$\mathfrak{M}_{13} = \langle\{0, {}^1/_3, {}^2/_3, 1\}, \neg_2, \to_3, \{1, {}^2/_3\}\rangle,$$
$$\mathfrak{M}_{14} = \langle\{0, {}^1/_3, {}^2/_3, 1\}, \neg_2, \to_4, \{1, {}^1/_3\}\rangle.$$

paranormal

$\mathfrak{M}_{15} = \langle\{0, {}^1\!/_3, {}^2\!/_3, 1\}, \neg_4, \rightarrow_3, \{1, {}^2\!/_3\}\rangle,$

$\mathfrak{M}_{16} = \langle\{0, {}^1\!/_3, {}^2\!/_3, 1\}, \neg_3, \rightarrow_4, \{1, {}^1\!/_3\}\rangle.$

As a result, a ten-element upper semi-lattice (see Figure 1) is constructed with respect to the functional embedding of matrices that define literal LPP logics and the isomorphs themselves[3].

The question about the functional inclusion one LPP logic to another was solved by A. Nepeivoda (see [23]).

The resulting semi-lattice allows us to build visualization for constructing LPP logics by the combination of *CPC* isomorphs. Note that the isomorphs themselves are included in our class of LPP logics as a degenerate case. The four isomorphs differ by functional properties and have the least expressive power.

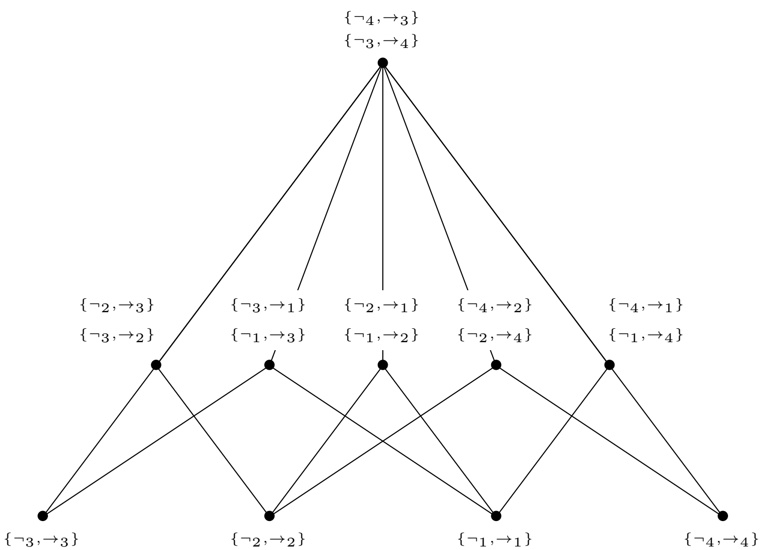

Figure 1. An upper semi-lattice

[3]The sets of basic operations of the corresponding logical matrices are indicated as semilattice elements.

The above structure is indeed an upper semi-lattice, since there is a supremum for any pair of its elements. In some cases, this is clearly seen in the construction of upper semi-lattices, in other cases it requires proof. Let us give the corresponding proof. To do this, it is sufficient to prove the following proposition:

PROPOSITION 3.1. The operations of the set $\{\to_3, \neg_4\}$ are definable by the sets of operations:

(1) $\{\neg_2, \to_3\}$ and $\{\neg_3, \to_1\}$;
(2) $\{\neg_3, \to_1\}$ and $\{\neg_2, \to_1\}$;
(3) $\{\neg_2, \to_1\}$ and $\{\neg_4, \to_2\}$;
(4) $\{\neg_4, \to_2\}$ and $\{\neg_4, \to_1\}$.

PROOF: For (1), it is sufficient to define $\neg_4$ by the sets of functions $\{\neg_2, \to_3\}$ and $\{\neg_3, \to_1\}$. The function $\wedge_3$ can be defined by $\{\neg_2, \to_3\}$ the following way:

$$x \wedge_3 y := \neg_2(x \to_3 \neg_2 y).$$

Further, since the sets $\{\neg_3, \to_1\}$ and $\{\neg_2, \to_3\}$ are functionally equivalent (Fact 1), we have:

$$\neg_4 x := (\neg_3 x \to_3 \neg_2 x) \wedge_3 (x \to_3 \neg_1 x).$$

For (2), it is sufficient to define $\neg_4$ and $\to_3$ by the sets of functions $\{\neg_3, \to_1\}$ and $\{\neg_2, \to_1\}$. Due to Fact 1, it is obvious that the function $\to_3$ is definable. The function $\neg_4$ could be defined in the same way as it was done in the proof of (1).

For (3), it is sufficient to define $\neg_3$ and $\to_4$ by the sets of functions $\{\neg_2, \to_1\}$ and $\{\neg_4, \to_2\}$. Since the sets $\{\neg_4, \to_2\}$ and $\{\neg_2, \to_4\}$ are functionally equivalent (Fact 2), it is obvious that the function $\to_4$ is definable. The function $\neg_3$ could be defined in the following way. Since the function $\wedge_1$ is defined by $\{\neg_2, \to_1\}$:

$$x \wedge_1 y := \neg_2(x \to_1 \neg_2 y),$$

and the sets of functions $\{\neg_2, \to_1\}$ and $\{\neg_1, \to_2\}$ are functionally equivalent, we have:

$$\neg_3 x := (\neg_4 x \to_1 \neg_2 x) \wedge_1 \neg_1 x.$$

For (4), it is sufficient to define $\neg_3$ and $\to_4$ by the sets of functions $\{\neg_4, \to_2\}$ and $\{\neg_4, \to_1\}$. Due to Fact 2, it is obvious that the function $\to_4$ is definable. Since the function $\wedge_2$ is defined by $\{\neg_4, \to_2\}$:

$$x \wedge_2 y := \neg_4(x \to_2 \neg_4 y),$$

and the sets of functions $\{\neg_4, \to_1\}$ and $\{\neg_1, \to_4\}$ are functionally equivalent, and that Fact 2 takes place, we have:

$$\neg_3 x := (\neg_4 x \to_2 \neg_2 x) \wedge_2 \neg_1 x. \qquad \Box$$

In paper [9] we consruct a four-element lattice of three-valued literal LPP logics with respect to the possesion of paraconsistency and paracompleteness properties. And the theorem on the functional equivalence of the LPP logics that form this lattice was proved (p. 230).

The four-valued case is more complicated. The upper semi-lattice of four-valued litearal LPP logics contains a number of four-element lattices with respect to the functional embeddinig one logic into another, on one hand, and possesion of paraconsistency and paracompleteness properties, on the other. In the next section, we consider two interesting lattices.

4. Four-element lattices of four-valued LPP logics

Let us consider the matrices: $\mathfrak{M}_3$, $\mathfrak{M}_6$, $\mathfrak{M}_8$, $\mathfrak{M}_{11}$, $\mathfrak{M}_{13}$, $\mathfrak{M}_{15}$, $\mathfrak{M}_{16}$. The LPP logics determined by the foregoing matrices form two four-element lattices (see Fugure 2 and Fugure 3) with respect to the possesion of paraconsistency and paracompleteness properties, on one hand, and with respect to the functional embedding of logics (corresponding classes of matrix's operations), on the other.

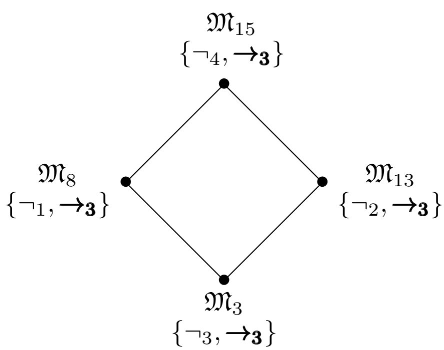

Figure 2. Lattice 1

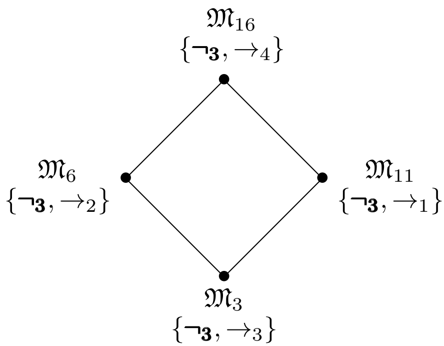

Figure 3. Lattice 2

Lattices in Figure 2 and Figure 3 are included in the upper semi-lattice in Figure 1.

Let us consider the properties of the logics that constitute these lattices.

(1) Logics introduced by the matrices $\mathfrak{M}_6$ and $\mathfrak{M}_8$ are *paraconsistent*.

(2) Logics introduced by the matrices $\mathfrak{M}_{11}$ and $\mathfrak{M}_{13}$ are *paracomplete*.

(3) Logics introduced by the matrices $\mathfrak{M}_{15}$ and $\mathfrak{M}_{16}$ are *paranormal*.

4.1. Functional properties

By the construction of the upper semi-lattice (see Fugure 1), it is obvious that paraconsistent logics with operations $\{\neg_3, \to_2\}$ and $\{\neg_1, \to_3\}$ (matrices $\mathfrak{M}_6$ and $\mathfrak{M}_8$) are different in functional properties. Similarly that about paracomplete logics with operations $\{\neg_3, \to_1\}$ and $\{\neg_2, \to_3\}$ (matrices $\mathfrak{M}_{11}$ and $\mathfrak{M}_{13}$). But the matrices $\mathfrak{M}_6$ and $\mathfrak{M}_{13}$ are functionally equivalent, and the same situation takes place for matrices $\mathfrak{M}_8$ and $\mathfrak{M}_{11}$. Notice that we have a similar property for three-valued logics: the matrices corresponding to paraconsistent logic P^1[19] and paracomplete logic I^1 [21] are also functionally equivalent [12, p. 222].

Paranormal logics with operations $\{\neg_4, \to_3\}$ and $\{\neg_3, \to_4\}$ (matrices $\mathfrak{M}_{15}$ and $\mathfrak{M}_{16}$) are functionally equivalent. In paper [22, p. 81–82] it is proved that these logical matrices correspond to the class of all external four-valued functions.

Also, there are well-known four-valued logics that are functionally equivalent to $\mathfrak{M}_{15}$ ($\mathfrak{M}_{16}$). Logic I^1P^1, which is the part of a hierarchy of paranormal logics called I^nP^k, introduced by V. Fernández [6]. A sound and complete axiomatization for each I^nP^k using the techniques of Rosser-Turquette was defined in [6]. In [5, p. 88] L. Devyatkin construct the four-valued matrices for the logics P^1 and I^1. And he shows that the matrix of I^1P^1 constitutes a functional extension of P^1 and I^1 and this entails that I^1P^1 is a linguistic variant of a common linguistic extension of P^1 and I^1.

The matrix of I^1P^1 coincides with the matrix $\mathfrak{M}_{15}$.

Logic V (see [18, p. 208] for the corresponding matrix) was introdused by L.Z. Puga and N.C.A. da Costa after ideas on the "imaginary logic" by N.A. Vasiliev. Corresponding matrix is functionally equivalent to the matrix $\mathfrak{M}_{15}$ ($\mathfrak{M}_{16}$).

In [16, p. 89] V.M. Popov introduced the matrix $\mathfrak{M}_0$ (this matrix coincides with the matrix of logic V), where truth-tables for $\neg$ and $\rightarrow$ ($\neg_4$ and $\rightarrow_3$ in our notation), may be viewed as four-valued generalizations of P^1's and I^1's tables.

4.2. Classes of tautologies

In this section, we analyze the theories (sets of tautologies) generated by the foregoing matrices.

Paraconsistent logic P^1 [19] and paracomplete logic I^1 [21] play a significant role in our analysis. The calculi P^1 and I^1 are expressed in a language using negation and implication as a primitives.

P^1 is axiomatized by the following axiom schemata:

(A1) $A \rightarrow (B \rightarrow A)$

(A2) $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$

(A3) $(\neg A \rightarrow \neg B) \rightarrow ((\neg A \rightarrow \neg\neg B) \rightarrow A)$

(A4) $(A \rightarrow B) \rightarrow \neg\neg(A \rightarrow B)$

Inference rule: *modus ponens* [20].

The matrix $\mathfrak{M}^{P1} = \langle \{1, {}^1\!/\!_2, 0\}, \neg_{P1}, \rightarrow_{P1}, \{1, {}^1\!/\!_2\} \rangle$, where $\neg_{P1}$ and $\rightarrow_{P1}$ are defined by the tables

x	$\neg_{P1} x$
1	0
$1/2$	1
0	1

$\to_{P1}$	1	$1/2$	0
1	1	1	0
$1/2$	1	1	0
0	1	1	1

gives us a strongly adequate matricial semantics for P^1.

The axioms of I^1 are given by the following schemas:

(A1) $A \to (B \to A)$

(A2) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

(A3') $(\neg\neg A \to \neg B) \to ((\neg\neg A \to B) \to \neg A)$

(A4') $\neg\neg(A \to B) \to (A \to B)$

Inference rule: *modus ponens* [21].

I^1 is complete relative to the matrix $\mathfrak{M}^{I1} = \langle \{1, 1/2, 0\}, \neg_{I1}, \to_{I1}, \{1\} \rangle$, where $\neg_{I1}$ and $\to_{I1}$ are defined by the tables

x	$\neg_{I1} x$
1	0
$1/2$	0
0	1

$\to_{I1}$	1	$1/2$	0
1	1	0	0
$1/2$	1	1	1
0	1	1	1

For ease of comparison, let's also give the axiomatization of classical propositional logic in a language using negation and implication as a primitives:

(A1) $A \to (B \to A)$

(A2) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

(A3") $(\neg B \to \neg A) \to ((\neg B \to A) \to B)$

Inference rule: *modus ponens* [14, p. 35].

Let us consider paraconsistent matrix $\mathfrak{M}_8$:
$\mathfrak{M}_8 = \langle \{0, 1/3, 2/3, 1\}, \neg_1, \to_3, \{1, 2/3\} \rangle$.

The matrix $\mathfrak{M}_8$ can be regarded as four-valued generalizations of the three-valued matrix, introdused by Sette in [19].

The matrix $\mathfrak{M}_8$ generates the same theory as the well-known Sette's paraconsistent logic P^1. It follows from the paper [5, pp. 86–87], where the four-valued matrix $\mathcal{P}^{1f}$ for logic P^1 is constructed. Matrix $\mathcal{P}^{\mathbf{1f}}$ is $\mathfrak{M}_8$ in our notation. It is shown that matrix $\mathcal{P}^{\mathbf{1f}}$ is a homomorphic image of the matrix $\mathfrak{M}^{P1}$ with respect to the mapping h: $h(1/3) = 0$ and $h(x) = x$, if $x \neq 1/3$. As a consequence, matrix $\mathcal{P}^{\mathbf{1f}}$ ($\mathfrak{M}_8$) induces the logic P^1 (i.e. $\mathcal{P}^{\mathbf{1f}}$ ($\mathfrak{M}_8$) is a characteristic matrix[4] for calculus P^1). Moreover,

Proposition 4.1. All paraconsistent matrices, included in the upper semi-lattice in Figure 1 are characteristic for P^1.

Proof: The proof follows from the facts:

(1) if $\mathfrak{M}$ is a homomorphic image of $\mathfrak{N}$ then $E(\mathfrak{M}) = E(\mathfrak{N})$ [3, p. 21].

(2) matrix $\mathfrak{M}_5$ is a homomorphic image of the matrix $\mathfrak{M}^{P1}$ with respect to the mapping h: $h(1/3) = 2/3$ and $h(x) = x$, if $x \neq 1/3$.

(3) matrix $\mathfrak{M}_6$ is a homomorphic image of the matrix $\mathfrak{M}^{P1}$ with respect to the mapping h: $h(2/3) = 1$ and $h(x) = x$, if $x \neq 2/3$.

(4) matrix $\mathfrak{M}_7$ is a homomorphic image of the matrix $\mathfrak{M}^{P1}$ with respect to the mapping h: $h(1/3) = 1$ and $h(x) = x$, if $x \neq 1/3$.

(5) matrix $\mathfrak{M}_9$ is a homomorphic image of the matrix $\mathfrak{M}^{P1}$ with respect to the mapping h: $h(2/3) = 0$ and $h(x) = x$, if $x \neq 2/3$. □

Let us consider the paracomplete matrix $\mathfrak{M}_{13}$:

$\mathfrak{M}_{13} = \langle \{0, 1/3, 2/3, 1\}, \neg_2, \rightarrow_3, \{1, 2/3\} \rangle$.

The matrix $\mathfrak{M}_{13}$ can be regarded as four-valued generalization of three-valued matrix, introdused by Sette and Carnielli in [21].

In paper [5, p. 87] L. Devyatkin construct the four-valued matrix $\mathcal{I}^{\mathbf{1t}}$, which is a homomorphic image of the matrix $\mathfrak{M}^{I1}$ with respect to the mapping h: $h(2/3) = 1$ and $h(x) = x$, if $x \neq 2/3$. The matrix $\mathcal{I}^{\mathbf{1t}}$ is $\mathfrak{M}_{13}$ in our notation. It follows that the matrix $\mathfrak{M}_{13}$ generates the same theory as the paracomplete logic I^1.

[4] Matrix $\mathfrak{M}$ is characteristic for calculus L, if $\vDash_{\mathfrak{M}} A$ iff $\vdash_L A$.

The following proposition takes place:

PROPOSITION 4.2. All paracomplete matrices, included in the upper semi-lattice in Figure 1 are characteristic for I^1.

PROOF: The proof follows from the facts:

(1) if $\mathfrak{M}$ is a homomorphic image of $\mathfrak{N}$ then $E(\mathfrak{M}) = E(\mathfrak{N})$ [3, p. 21].

(2) matrix $\mathfrak{M}_{10}$ is a homomorphic image of the matrix $\mathfrak{M}^{I1}$ with respect to the mapping h: $h(2/3) = 1/3$ and $h(x) = x$, if $x \neq 2/3$.

(3) matrix $\mathfrak{M}_{11}$ is a homomorphic image of the matrix $\mathfrak{M}^{I1}$ with respect to the mapping h: $h(1/3) = 0$ and $h(x) = x$, if $x \neq 1/3$.

(4) matrix $\mathfrak{M}_{12}$ is a homomorphic image of the matrix $\mathfrak{M}^{I1}$ with respect to the mapping h: $h(2/3) = 0$ and $h(x) = x$, if $x \neq 2/3$.

(5) matrix $\mathfrak{M}_{14}$ is a homomorphic image of the matrix $\mathfrak{M}^{I1}$ with respect to the mapping h: $h(1/3) = 1$ and $h(x) = x$, if $x \neq 1/3$. □

The question about the classes of tautologies generated by the matrices $\mathfrak{M}_{15}$ and $\mathfrak{M}_{16}$ is considered in paper [22]. It is proved that the theories generated by these matrices are equivalent.

The analysis of the application of the algorithm for constructing classes of literal LPP logics by combination of isomorphs of classical logic CPC to three-valued and four-valued cases allows us to make two more general assumptions:

1. All paraconsistent (and not paracomplete) matrices constructed by combination of isomorphs of classical logic CPC generate the same theory as Sette's paraconsistent logic P^1 [19].

2. All paracomplete (and not paraconsistent) matrices constructed by combination of isomorphs of classical logic CPC generate the same theory as paracomplete logic I^1 introdused by Sette and Carnielli in [21].

4.3. Some properties of lattices

For our analysis, we have chosen the lattices, presented in Figures 2 and 3, because it helps us to demonstrate, how it is possible to vary paraconsistency and paracompleteness properties in logics.

Let us consider the lattice in Figure 2. Martices $\mathfrak{M}_8$, $\mathfrak{M}_{13}$, $\mathfrak{M}_{15}$, corresponding to LPP logics, differ only in negation operation. It's obvious that negation $\neg_1$ is paraconsistent in the sense that classical negation allows explosity, but the negation $\neg_1$ does not, that is A and $\neg_1 A$ can be true at the same time. Negation $\neg_2$ has the property of paracompleteness, in the sense that A and $\neg_2 A$ can be false at the same time. Thus, by varying the negation operation, we can obtain LPP logics with different properties. And herein the implication operation and the class of designated values in the matrices remain the same. Taking this into account, it is clearly seen that it is natural to axiomatize the LPP logics, varying the axioms for negation. We see this on the example of the foregoing axiomatizations of P^1, I^1 and CPC. The matrices $\mathfrak{M}_8$ and $\mathfrak{M}_{13}$ are four-valued characteristic matrices for the known calculi P^1 and I^1, and differ only in the negation operation.

Let us turn to the lattice in Figure 3. Here, the matrices of the corresponding LPP logics differ in the implication operations and in the class of the designated values. Herein the negation operation is defined by same truth-table in all these matrices ($\mathfrak{M}_3$, $\mathfrak{M}_6$, $\mathfrak{M}_{11}$, $\mathfrak{M}_{16}$) and properties of negation operation (and corresponding LPP logics) are directly dependent on the choice of the designated values class.

The paraconsistent logic corresponding to the matrix $\mathfrak{M}_6$ in Lattice 1 is functionally eqiuvalent to the paracomplete logic corresponding to the matrix $\mathfrak{M}_{13}$ in Lattice 2; and the paraconsistent logic corresponding to the matrix $\mathfrak{M}_{11}$ in Lattice 2 is functionally eqiuvalent to the paracomplete logic corresponding to the matrix $\mathfrak{M}_8$ in Lattice 1. Both paraconsistent logics generate the same theory as well-known Sette's paraconsistent logic [19], and paracomplete logics have the same set of tautologies as the paracomplete logic I^1 [21].

5. Concluding remarks

We have analized the application of the method of constructing LPP logics by combinating isomorphs of classical logic to the four-valued case. Recall that in the case of three-valued logics Sette's paraconsistent logic P^1 and paracomplete logic I^1 can be obtained by using this method. As a result we get four-valued generalizations of these logics.

This method preserves all essential properties of these LPP logics, i.e. allows to construct paraconsistent and paracomplete matrices, which are functionally equivalent, on the one hand, and generate theories equivalent to P^1 and I^1, on the other. In the three-valued case, combination of isomorphs of CPC leads to two LPP logics, in four-valued case, we can obtain[5] four isomorphs of CPC, combinating which allows to obtain five paraconsistent, five paracomplete and two paranormal logics. At that, these LPP logics form the 10-element upper semi-lattice with respect to the functional embeddinig one logic into another.

And the foregoing upper semi-lattice includes several four-element lattices with respect to the functional embeddinig one logic into another, on the one hand, and with respect to the possesion of paraconsistency and paracompleteness properties, on the other. Two such four-element lattices of LPP logics were considered. Functional properties and sets of tautologies of corresponding LPP logics were investigated. On the example of these two lattices the mechanism of variation of paraconsistency and paracompleteness properties in logics is clearly seen.

As a result, the analysis allows us to make an assumption that all n-valued literal paraconsistent matrices (and not paracomplete) and all n-valued literal paracomplete matrices (and not paraconsistent) constructed by combinating isomorphs of classical logic generate the same theories as P^1 and I^1.

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Bulletin of the **S**ection of **L**ogic
Volume 50/1 (2021), pp. 55–80
https://doi.org/10.18778/0138-0680.2020.25

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ONE-SIDED SEQUENT SYSTEMS FOR NONASSOCIATIVE BILINEAR LOGIC: CUT ELIMINATION AND COMPLEXITY

Abstract

Bilinear Logic of Lambek amounts to Noncommutative MALL of Abrusci. Lambek proves the cut–elimination theorem for a one-sided (in fact, left-sided) sequent system for this logic. Here we prove an analogous result for the nonassociative version of this logic. Like Lambek, we consider a left-sided system, but the result also holds for its right-sided version, by a natural symmetry. The treatment of nonassociative sequent systems involves some subtleties, not appearing in associative logics. We also prove the PTime complexity of the multiplicative fragment of NBL.

Keywords: Substructural logic, Lambek calculus, nonassociative linear logic, sequent system, PTime complexity.

1. Introduction

Multiplicative-Additive Linear Logic (MALL) was introduced by Girard [8]. Noncommutative MALL (where product $\otimes$ is noncommutative) is due to Abrusci [1]. This logic, presented as a one-sided (precisely: left-sided[1]) sequent system was studied by Lambek [10] under the name: Classical Bilinear Logic. Lambek proved the cut–elimination theorem for this system in a syntactic way.

[1] Two-sided systems admit sequents $\Gamma \Rightarrow \Delta$, right-sided $\Rightarrow \Delta$, left-sided $\Gamma \Rightarrow$

Presented by: Andrzej Indrzejczak
Received: July 28, 2020
Published online: November 13, 2020

The present paper studies an analogous system for Nonassociative Bilinear Logic (NBL), being a version of Bilinear Logic with nonassociative $\otimes$. Some related logics, restricted to multiplicative connectives and not admitting multiplicative constant (nor the corresponding unit elements in algebraic models), were studied in [5, 3] under the name: Classical Nonassociative Lambek Calculus (CNL). CNL contains one (cyclic) negation $^{\sim}$, satisfying $a^{\sim\sim} = a$ in algebras. Buszkowski [4] considers a weaker logic, called Involutive Nonassociative Lambek Calculus (InNL), with two negations $^{\sim}, ^{-}$, satisfying $a^{-\sim} = a = a^{\sim-}$.

Here we provide a syntactic proof of the cut–elimination theorem for one-sided systems of NBL in the language $\otimes, \oplus, ^{\sim}, ^{-}, \wedge, \vee, 0, 1$ (also $\bot, \top$). Our notation is different from that of [8, 1]. In particular, we write $\oplus$ for coproduct (par), $\vee$ for additive disjunction and 0 for $\bot$, following standards of substructural logics [6]. $\otimes, \oplus, ^{\sim}, ^{-}, 0, 1$ are reffered to as multiplicative connectives and contants, while $\wedge, \vee, \bot, \top$ as additive ones. In algebras, 1 (resp. 0) is interpreted as the unit element for product (resp. coproduct), $\wedge$ (resp. $\vee$) as meet (resp. join) in a lattice and $\bot$ (resp. $\top$) as the least (resp. the greatest) element.

We follow the path presented in [4]. NBL is InNL extended by the multiplicative constants and additive connectives. All the statements and proofs in this paper are similar to these in [4], so we skip the parts that are identical and focus on the differences. The crucial difference is that Buszkowski [4] considers only sequents consisting of at least two formulas, which makes the proofs much simpler. Here we consider all nonempty sequents.

We write a complete proof of the cut–elimination theorem for a left-sided system (a nonassociative version of the system from [10]) without $\bot, \top$ (these constants are added in the subsection 4.1). In subsection 4.2 we obtain an analogous result for a right-sided system, using a natural symmetry of both systems.

InNL is a conservative extension of Nonassociative Lambek Calculus (NL), due to Lambek [9]; see [5, 3]. It can be shown that NBL is a conservative extension of NL with 1 (NL1). These logics have applications in linguistics as type logics for categorial grammars [10, 5, 3] and seem quite natural from the perspective of modal logics, where $\otimes$ can be regarded as a binary possibility operator.

NL and NL1 are usually presented as intuitionistic systems with sequents $\Gamma \Rightarrow A$; in NL Γ must be nonempty. The syntax of the left-sided system for NBL is quite similar to that of NL1 (in a richer language).

The proof of cut elimination for nonassociative logics is roughly similar to those for associative linear logic [1, 8, 9], but the nonassociative framework involves some new subtleties. For instance, the rule (r-shift), expressing the algebraic compatibility condition (see below), must be replaced by weaker rules. In the resulting system (r-shift) and two rules for $^{\sim\sim}$ and $^{--}$ are shown to be admissible (Lemmas 2.2 and 2.4), which is essentially used in the final proof (Theorem 1). Our proof partially follows that from [4] for InNL, but the richer language makes it more complicated.

In our sequent systems, negations appear at variables only (so we consider formulas in negation normal form). Negations of arbitrary formulas are defined in metalanguage. Some systems with negations of formulas in the language were considered in [2] (right-sided) and [6] (two-sided). The system from [2] does not have the subformula property. That from [6] uses sequents $\Gamma \Rightarrow \Delta$. The cut–elimination theorem for this system is proved in [6] by algebraic methods.

Having the cut–elimination theorem, we can prove the decidability of NBL. In the last section we show that the multiplicative fragment of NBL (MNBL) is PTime. The algorithm essentially uses cut elimination. An analogous result for InNL is given in [4].

By atoms in NBL-language we mean two constants: 0 and 1, and $p^{(n)}$, where p is any variable and $n \in \mathbb{Z}$. By $p^{(n)}$ we denote $p^{\sim\sim\cdots\sim}$, where $^\sim$ is iterated n times, if $n \geq 0$, and $p^{--\cdots-}$, where $^-$ is iterated $-n$ times, if $n < 0$. $^\sim$ and $^-$ are involutive negations in NBL, but we do not consider them as connectives, because they occur only with non-constant atoms. It means that the formulas are in negation normal form. The connectives are: $\otimes$ (product), $\oplus$ (coproduct), $\wedge$ (meet) and $\vee$ (join).

We define metalanguage negations for every NBL-formula:

$$0^\sim = 1 \qquad 0^- = 1 \qquad 1^\sim = 0 \qquad 1^- = 0$$

$$(p^{(n)})^\sim = p^{(n+1)} \qquad (p^{(n)})^- = p^{(n-1)}$$

$$(A \otimes B)^\sim = B^\sim \oplus A^\sim \qquad (A \otimes B)^- = B^- \oplus A^-$$

$$(A \oplus B)^\sim = B^\sim \otimes A^\sim \qquad (A \oplus B)^- = B^- \otimes A^-$$

$$(A \wedge B)^{\sim} = A^{\sim} \vee B^{\sim} \qquad (A \wedge B)^{-} = A^{-} \vee B^{-}$$
$$(A \vee B)^{\sim} = A^{\sim} \wedge B^{\sim} \qquad (A \vee B)^{-} = A^{-} \wedge B^{-}$$

One shows: $A^{\sim -} = A^{- \sim} = A$ by induction on formulas.

Definition 1.1. We define *bunches*:

(i) The empty bunch ϵ is a bunch.

(ii) Every formula is a bunch.

(iii) If Γ and Δ are bunches, then (Γ, Δ) is also a bunch.

We assume: $(\Gamma, \epsilon) = \Gamma = (\epsilon, \Gamma)$. A *sequent* in NBL is every nonempty bunch. We often omit outer parentheses in sequents and formulas.

A context is a bunch containing a special atomic formula x. Contexts are denoted by capital Greek letters and square brackets, e.g. $\Gamma[\,], \Delta[\,]$, etc. By $\Gamma[\Delta]$ we mean the substitution of Δ for x in $\Gamma[\,]$.

Now we briefly describe the algebraic models of NBL.

Definition 1.2. An algebra $\mathbf{M} = (M, \otimes, \wedge, \vee, {}^{\sim}, {}^{-}, 1)$, where $\otimes, \wedge, \vee$ are binary operators, ${}^{\sim}, {}^{-}$ are unary operators and 1 is a constant, is called a *lattice ordered (l.o.) involutive unital groupoid*, if:

(i) $(M, \wedge, \vee)$ is a lattice;

(ii) $(M, \otimes, 1)$ is a unital groupoid;

(iii) if $a \otimes b \leq c$, then $c^{-} \otimes a \leq b^{-}$ and $b \otimes c^{\sim} \leq a^{\sim}$, for all $a, b, c \in M$;

(iv) $a^{\sim -} = a^{- \sim} = a$, for all $a \in M$.

In the above definition $\leq$ stands for the lattice order. We can prove for all $a, b \in M$ that $(b^{\sim} \otimes a^{\sim})^{-} = (b^{-} \otimes a^{-})^{\sim}$, so we define $a \oplus b = (b^{\sim} \otimes a^{\sim})^{-}$. One proves that $1^{\sim} = 1^{-}$, hence we define $0 = 1^{\sim}$. One can define residuals (implications) $a \setminus b = a^{\sim} \oplus b$, $a/b = a \oplus b^{-}$, satisfying the residuation laws: $a \otimes b \leq c$ iff $a \leq c/b$ iff $b \leq a \setminus c$. One also gets $a^{\sim} = 0 \setminus a$, $a^{-} = 0/a$.

The condition (iii) is referred to as the compatibility condition. One also proves the implications coverse to (iii).

$$\text{if } c^{-} \otimes a \leq b^{-}, \text{ then } a \otimes b \leq c$$
$$\text{if } b \otimes c^{\sim} \leq a^{\sim}, \text{ then } a \otimes b \leq c$$

It follows from (iii) that negations are antitone: if $a \leq b$, then $b^{-} \leq a^{-}$ and $b^{\sim} \leq a^{\sim}$.

There hold De Morgan laws.

$$(a \otimes b)^{-} = b^{-} \oplus a^{-} \qquad (a \wedge b)^{-} = a^{-} \vee b^{-}$$

$$(a \otimes b)^{\sim} = b^{\sim} \oplus a^{\sim} \qquad (a \wedge b)^{\sim} = a^{\sim} \vee b^{\sim}$$

$$(a \oplus b)^{-} = b^{-} \otimes a^{-} \qquad (a \vee b)^{-} = a^{-} \wedge b^{-}$$

$$(a \oplus b)^{\sim} = b^{\sim} \otimes a^{\sim} \qquad (a \vee b)^{\sim} = a^{\sim} \wedge b^{\sim}$$

The following laws will be useful.

$$a^{-} \otimes (a \oplus b) \leq b$$

$$(a \oplus b) \otimes b^{\sim} \leq a$$

$$b \leq a^{\sim} \oplus (a \otimes b)$$

$$a \leq (a \otimes b) \oplus b^{-}$$

$$\text{if } a \leq b \text{, then } a \otimes c \leq b \otimes c \text{ and } a \oplus c \leq b \oplus c$$

$$\text{if } a \leq b \text{, then } c \otimes a \leq c \otimes b \text{ and } c \oplus a \leq c \oplus b$$

We define a valuation μ as a homomophism of the algebra of formulas into a l.o. involutive unital groupoid. We extend it to sequents by: $\mu((\Gamma, \Delta)) = \mu(\Gamma) \otimes \mu(\Delta)$ and $\mu(\epsilon) = 1$.

We say that a sequent Γ is true in $\mathbf{M}$ for a valuation μ, if $\mu(\Gamma) \leq 0$; we write $\mathbf{M}, \mu \models \Gamma$. A sequent is said to be valid, if it is true in all algebras of this kind for all valuations.

2. Nonassociative Bilinear Logic

Now we present a one-sided sequent system for Nonassociative Bilinear Logic.

We admit axioms:

$$\text{(a-id)} \qquad p^{(n)}, p^{(n+1)} \quad \text{for any variable } p \text{ and any } n \in \mathbb{Z}$$

$$\text{(a-0)} \qquad 0$$

The rules of the cut-free NBL are:

$$\text{(r-}\otimes\text{)}\quad \frac{\Gamma[(A,B)]}{\Gamma[A\otimes B]}$$

$$\text{(r-}\oplus 1\text{)}\quad \frac{\Gamma[B]\quad \Delta, A}{\Gamma[(\Delta, A\oplus B)]}\qquad \text{(r-}\oplus 2\text{)}\quad \frac{\Gamma[A]\quad B,\Delta}{\Gamma[(A\oplus B,\Delta)]}$$

$$\text{(r-1)}\quad \frac{\Gamma[\Delta]}{\Gamma[(1,\Delta)]}\quad \frac{\Gamma[\Delta]}{\Gamma[(\Delta,1)]}$$

$$\text{(r-}\wedge\text{)}\quad \frac{\Gamma[A]}{\Gamma[A\wedge B]}\quad \frac{\Gamma[A]}{\Gamma[B\wedge A]}\qquad \text{(r-}\vee\text{)}\quad \frac{\Gamma[A]\quad \Gamma[B]}{\Gamma[A\vee B]}$$

$$\text{(r-shift)}\quad \frac{(\Gamma,\Delta),\Theta}{\overline{\Gamma,(\Delta,\Theta)}}$$

In this paper we show that the cut rules:

$$(\text{cut}^{\sim})\quad \frac{\Gamma[A]\quad \Delta, A^{\sim}}{\Gamma[\Delta]}\qquad (\text{cut}^{-})\quad \frac{\Gamma[A]\quad A^{-},\Delta}{\Gamma[\Delta]}$$

are admissible in the cut-free NBL.

These axioms and rules are valid. By reflexivity of the lattice order we have $0\leq 0$, which is (a-0) and $p^{(n)}\leq p^{(n)}$, which can be easily transformed into $p^{(n)}\otimes p^{(n+1)}\leq 0$ by the compatibility condition; (r-$\otimes$) is sound by definition of the valuation; (r-$\wedge$) and (r-$\vee$) express the lattice order properties; (r-1) is valid, because 1 is a neutral element of $\otimes$.

Rules (r-$\oplus$1) and (r-$\oplus$2) are sound because of the propeties $a^{-}\otimes(a\oplus b)\leq b$ and $(a\oplus b)\otimes b^{\sim}\leq a$.

We prove that (r-shift) is sound. The following are equivalent:

$$(\mu(\Gamma)\otimes\mu(\Delta))\otimes\mu(\Theta)\leq 0$$
$$0^{-}\otimes(\mu(\Gamma)\otimes\mu(\Delta))\leq(\mu(\Theta))^{-}$$
$$1\otimes(\mu(\Gamma)\otimes\mu(\Delta))\leq(\mu(\Theta))^{-}$$
$$\mu(\Gamma)\otimes\mu(\Delta)\leq(\mu(\Theta))^{-}$$
$$\mu(\Delta)\otimes(\mu(\Theta))^{-\sim}\leq(\mu(\Gamma))^{\sim}$$

$$\mu(\Delta) \otimes \mu(\Theta) \leq (\mu(\Gamma))^{\sim}$$
$$(\mu(\Delta) \otimes \mu(\Theta)) \otimes 0^{\sim} \leq (\mu(\Gamma))^{\sim}$$
$$\mu(\Gamma) \otimes (\mu(\Delta) \otimes \mu(\Theta)) \leq 0$$

The system with the cut-rules is strongly complete with respect to l.o. involutive unital groupoids. We omit a routine proof, using Lindenbaum-Tarski algebras.

Definition 2.1. By the *active formula* (resp. *active bunch*) of a rule we denote the new formula (bunch) intruduced by this rule.

The rule (r-shift) would complicate our syntactic proof of cut elimination. In order to avoid that, we define an equivalent cut-free system, where (r-shift) is replaced by the following rules:

$$(\text{r-}\oplus 3)\quad \frac{A, \Gamma \quad B, \Delta}{A \oplus B, (\Delta, \Gamma)} \qquad (\text{r-}\oplus 4)\quad \frac{\Gamma, A \quad \Delta, B}{(\Delta, \Gamma), A \oplus B}$$

We assume that both Γ and Δ are nonempty. Otherwise (r-$\oplus$3) and (r-$\oplus$4) are special cases of (r-$\oplus$2) and (r-$\oplus$1). One can notice that these rules are just instances of (r-$\oplus$2) and (r-$\oplus$1) with (r-shift) applied to the conclusions. We define the cut-free NBL_0 as the cut-free NBL without (r-shift), but with (r-$\oplus$3) and (r-$\oplus$4). $\vdash_{\mathrm{NBL}_0}$ stands for the provability in the cut-free NBL_0.

Lemma 2.2. *The rule (r-shift) is admissible in NBL_0, i.e.* $\vdash_{NBL_0} (\Gamma, \Delta), \Theta$, *if and only if* $\vdash_{NBL_0} \Gamma, (\Delta, \Theta)$.

Proof: We show only the left-to-right implication. The converse implication is proved analogously. We assume $\vdash (\Gamma_1, \Gamma_2), \Gamma_3$ and prove $\vdash \Gamma_1, (\Gamma_2, \Gamma_3)$ (for better readability, we skip the subscript NBL_0, unless it is necessary). We also assume that none of Γ_i, $(i = 1, 2, 3)$ is empty. Otherwise the claim is trivial. We run induction on the proof of $(\Gamma_1, \Gamma_2), \Gamma_3$.

Firstly, one can easily notice that $(\Gamma_1, \Gamma_2), \Gamma_3$ cannot be an axiom. Hence it is the conclusion of a rule.

Let us consider (r-$\otimes$), (r-$\wedge$) and (r-$\vee$). All but the last one has only one premise. The last one has two premises with the same context. The active formula must occur in one of Γ_i, $(i = 1, 2, 3)$. We apply the induction hypothesis to the premise(s) and use again the same rule.

We consider (r-1). If the active bunch occurs in one of Γ_i we proceed as above. We have only to consider the case when one of Γ_i equals 1. We consider the following instances:

$$\frac{\Gamma_2, \Gamma_3}{(1, \Gamma_2), \Gamma_3} \qquad \frac{\Gamma_1, \Gamma_3}{(\Gamma_1, 1), \Gamma_3} \qquad \frac{\Gamma_1, \Gamma_2}{(\Gamma_1, \Gamma_2), 1}$$

We replace the above instances by the following ones respectively, using (r-1) in different variant if necessary:

$$\frac{\Gamma_2, \Gamma_3}{1, (\Gamma_2, \Gamma_3)} \qquad \frac{\Gamma_1, \Gamma_3}{\Gamma_1, (1, \Gamma_3)} \qquad \frac{\Gamma_1, \Gamma_2}{\Gamma_1, (\Gamma_2, 1)}$$

We consider (r-$\oplus$1). All possible instances with conclusion $(\Gamma_1, \Gamma_2), \Gamma_3$ are the following:

$$(1) \quad \frac{(\Theta_1[B], \Theta_2), \Theta_3 \quad \Delta, A}{(\Theta_1[(\Delta, A \oplus B)], \Theta_2), \Theta_3} \qquad (2) \quad \frac{(\Theta_1, \Theta_2[B]), \Theta_3 \quad \Delta, A}{(\Theta_1, \Theta_2[(\Delta, A \oplus B)]), \Theta_3}$$

$$(3) \quad \frac{(\Theta_1, \Theta_2), \Theta_3[B] \quad \Delta, A}{(\Theta_1, \Theta_2), \Theta_3[(\Delta, A \oplus B)]} \qquad (4) \quad \frac{B, \Theta_3 \quad \Theta_1, A}{(\Theta_1, A \oplus B), \Theta_3}$$

$$(5) \quad \frac{B \quad (\Theta_1, \Theta_2), A}{(\Theta_1, \Theta_2), A \oplus B}$$

(1), (2) and (3) are similar. The active bunch occurs in one of Γ_i. We apply the induction hypothesis to the first premise and use the same rule.

For (4), we use (r-$\oplus$2) with the same premises (interchanged). For (5), we apply the induction hypothesis to the second premise and use (r-$\oplus$2).

We consider (r-$\oplus$2). We have the following instances:

$$\frac{(\Theta_1[A], \Theta_2), \Theta_3 \quad B, \Delta}{(\Theta_1[(A \oplus B, \Delta)], \Theta_2), \Theta_3} \qquad \frac{(\Theta_1, \Theta_2[A]), \Theta_3 \quad B, \Delta}{(\Theta_1, \Theta_2[(A \oplus B, \Delta)]), \Theta_3}$$

$$\frac{(\Theta_1, \Theta_2), \Theta_3[A] \quad B, \Delta}{(\Theta_1, \Theta_2), \Theta_3[(A \oplus B, \Delta)]} \qquad \frac{A, \Theta_3 \quad B, \Theta_2}{(A \oplus B, \Theta_2), \Theta_3}$$

For the first three cases we proceed like for (1)-(3) above. For the last case we use (r-$\oplus$3) with the same premises.[2]

By (r-$\oplus$3) it is not possible to obtain $(\Gamma_1, \Gamma_2), \Gamma_3$.

We consider (r-$\oplus$4). There are three cases:

[2] Here one sees an application of (r-$\oplus$3). (r-$\oplus$4) is used in the skipped part of the proof.

$$\frac{\Theta_2, A \quad \Theta_1, B}{(\Theta_1, \Theta_2), A \oplus B} \qquad \frac{A \quad (\Theta_1, \Theta_2), B}{(\Theta_1, \Theta_2), A \oplus B}$$

$$\frac{(\Theta_1, \Theta_2), A \quad B}{(\Theta_1, \Theta_2), A \oplus B}$$

For the first case we use (r-$\oplus$1) with the same premises (interchanged). For the second case we apply the induction hypothesis to the second premise and use (r-$\oplus$1). For the last case we apply the induction hypothesis to the first premise and use (r-$\oplus$2). □

COROLLARY 2.3. The cut-free NBL and NBL_0 are equivalent, i.e. they have the same theorems.

We can use NBL_0 to prove further properties of NBL.

We need the following rules (called *double negation rules*):

$$(\text{r-}^{\sim\sim}) \quad \frac{A, \Gamma}{\Gamma, A^{\sim\sim}} \qquad (\text{r-}^{--}) \quad \frac{\Gamma, A}{A^{--}, \Gamma}$$

LEMMA 2.4. *The double negation rules are admissible in the cut-free NBL_0.*

PROOF: We prove only the admissibility of (r-$^{\sim\sim}$). The proof for the second rule is similar. We assume $\vdash C, \Theta$ and show $\vdash \Theta, C^{\sim\sim}$. We use outer induction on the number of connectives in C and inner induction on the proof of C, Θ.

Let $C = p^{(n)}$. We run the inner induction. Let $p^{(n)}, \Theta$ be an axiom. Hence $\Theta = p^{(n+1)}$. Then $(\Theta, C^{\sim\sim}) = (p^{(n+1)}, (p^{(n)})^{\sim\sim}) = (p^{(n+1)}, p^{(n+2)})$, which is an axiom, too.

Now we assume that $p^{(n)}, \Theta$ is obtained by a rule. $p^{(n)}$ cannot be the active formula. Then it has to occur in one of the premises. In all but the following cases we just apply the inner induction hypothesis to the premise(s) with $p^{(n)}$ and use the same rule.

We consider the following cases:

$$\frac{B \quad p^{(n)}, A}{p^{(n)}, A \oplus B} \qquad \frac{p^{(n)}, B \quad A}{p^{(n)}, A \oplus B}$$

The first one is an instance of (r-$\oplus$1) or (r-$\oplus$2). We apply the inner induction hypothesis to the premise with $p^{(n)}$ and apply (r-$\oplus$2). The second case is an instance of (r-$\oplus$1). We apply the inner induction hypothesis to the premise with $p^{(n)}$ and use (r-$\oplus$1).

Let $C = 0$. We run the inner induction. Let $0, \Theta$ be an axiom. Then $\Theta = \epsilon$, hence $(\Theta, 0^{\sim\sim}) = 0$, which is an axiom too. Now we assume that $0, \Theta$ is obtained by a rule. Since $C = 0$ cannot the active formula, we proceed as for $p^{(n)}$.

$C = 1$. We run the inner induction. $1, \Theta$ cannot be an axiom. We assume that $1, \Theta$ is the conclusion of a rule. We have $(\Theta, C^{\sim\sim}) = (\Theta, 1)$. If 1 is not the active formula, we proceed as above. Otherwise we have only one rule to consider – (r-1) of the form:

$$\frac{\Theta}{1, \Theta}$$

We just use the other variant of (r-1).

We assume that C is not an atom. We run inner induction. Clearly, C, Θ is not an axiom. So it is obtained by a rule. If C is not the active formula, we proceed as above. Now we assume that C is the active formula.

Let $C = A \otimes B$. Hence $C^{\sim\sim} = A^{\sim\sim} \otimes B^{\sim\sim}$. The only possible rule is (r-$\otimes$) of the form:

$$\frac{(A, B), \Theta}{C, \Theta}$$

We apply Lemma 2.2 to the premise and obtain $A, (B, \Theta)$. A and B each have less connectives than C. By the outer induction hypothesis we get $(B, \Theta), A^{\sim\sim}$. By Lemma 2.2, we get $B, (\Theta, A^{\sim\sim})$, hence by the outer induction hypothesis: $(\Theta, A^{\sim\sim}), B^{\sim\sim}$. Lemma 2.2 yields $\Theta, (A^{\sim\sim}, B^{\sim\sim})$. So $\Theta, C^{\sim\sim}$ arises by (r-$\otimes$).

Let $C = A \oplus B$. Then $C^{\sim\sim} = A^{\sim\sim} \oplus B^{\sim\sim}$. The only possible rules are (r-$\oplus$1) (or (r-$\oplus$2)), (r-$\oplus$2) and (r-$\oplus$3) of the following form:

$$\frac{A \quad B, \Theta}{A \oplus B, \Theta} \qquad \frac{A, \Theta \quad B}{A \oplus B, \Theta} \qquad \frac{A, \Theta_2 \quad B, \Theta_1}{A \oplus B, (\Theta_1, \Theta_2)} \quad (\Theta = (\Theta_1, \Theta_2))$$

For the first case, we apply the outer induction hypothesis to both premises and use (r-$\oplus$1) as below:

$$\frac{A^{\sim\sim} \quad \Theta, B^{\sim\sim}}{\Theta, C^{\sim\sim}}$$

For the second case we apply the outer induction hypothesis to both premises and use (r-$\oplus$2). For the third case we apply the outer induction hypothesis for both premises and use (r-$\oplus$4).

Let $C = A \wedge B$. Then $C^{\sim\sim} = A^{\sim\sim} \wedge B^{\sim\sim}$. We have the following instances if (r-$\wedge$):

$$\frac{A, \Theta}{A \wedge B, \Theta} \qquad \frac{B, \Theta}{A \wedge B, \Theta}$$

In both cases we apply the outer induction hypothesis to the premise and use the same rule.

The last case is $C = A \vee B$. Hence $C^{\sim\sim} = A^{\sim\sim} \vee B^{\sim\sim}$. We have the following instance of (r-$\vee$):

$$\frac{A, \Theta \quad B, \Theta}{A \vee B, \Theta}$$

We apply the outer induction hypothesis to both premises and use the same rule. □

One can easily conclude the following:

Corollary 2.5. $\vdash A^{-}, \Gamma$ if and only if $\vdash \Gamma, A^{\sim}$.

3. Cut elimination

Now we are ready to prove the cut–elimination theorem. The lemmas we have already proved are very useful and with them the proof is much simpler.

Theorem 3.1. *The cut rules are admissible in the cut-free NBL_0 (NBL).*

Proof: We have to show:

(1) if $\vdash \Theta[C]$ and $\vdash \Psi, C^{\sim}$, then $\vdash \Theta[\Psi]$;

(2) if $\vdash \Theta[C]$ and $\vdash C^{-}, \Psi$, then $\vdash \Theta[\Psi]$.

By Corollary 2.5 it suffices to show (1), because (2) follows (1) immediately. As above, $\vdash$ we denote provability in the cut-free NBL_0.

The proof proceeds by the outer induction on the number of connectives in C, the intermediate induction on the proof of $\Theta[C]$ and the inner induction on the proof of $\Psi, C^{\sim}$.

We run the outer induction.

1°. $C = p^{(n)}$. Then $C^{\sim} = p^{(n+1)}$. We run the intermediate induction.

Let $\Theta[C]$ be an axiom. We have two possibilities: $p^{(n)}, p^{(n+1)}$ and $p^{(n-1)}, p^{(n)}$. We run the inner induction.

If $\Psi, C^{\sim}$ is an axiom, then $\Psi = p^{(n)} = C$. Now let $\Psi, C^{\sim}$ be the conclusion of a rule. $C^{\sim}$ cannot be the active formula of any rule. We apply the inner induction hypothesis to the premise(s) with $C^{\sim}$ and use the same rule.

We consider the following special case:

$$\frac{A \quad B, C^{\sim}}{A \oplus B, C^{\sim}},$$

with $\Psi = A \oplus B$. This may be obtained by (r-$\oplus$1) or (r-$\oplus$2). We apply the inner induction hypothesis to the premise $B, C^{\sim}$ and use (r-$\oplus$1).

We assume that $\Theta[C]$ is not an axiom, hence it is obtained by a rule. C cannot be the active formula of any rule. Hence it occurs in at least one premise, so we apply the intermediate induction hypothesis to the premise(s) with C and use the same rule.

2°. $C = 0$. Then $C^{\sim} = 1$. We run the intermediate induction.

Let $\Theta[0]$ be an axiom, then $\Theta[C] = C = 0$ and $\Theta[\Psi] = \Psi$. We run the inner induction. $\Psi, 1$ cannot be an axiom, hence it is obtained by a rule. If $C^{\sim} = 1$ is not the active formula of a rule, we proceed as for $C = p^{(n)}$. If 1 is the active formula, then the rule is (r-1) of the form:

$$\frac{\Psi}{\Psi, 1},$$

The premise is $\Psi = \Theta[\Psi]$.

Now let $\Theta[C]$ be the conclusion of a rule. $C = 0$ cannot be the active formula of any rule. We apply the intermediate induction hypothesis to the premise(s) with $C = 0$ and use the same rule.

3°. $C = 1$. Then $C^{\sim} = 0$. We run the intermediate induction.

$\Theta[1]$ cannot be an axiom, hence it is obtained by a rule. If $C = 1$ is an active formula, then $\Theta[1]$ is obtained by (r-1) admitting $\Delta = \epsilon$ in $\Theta[\Delta]$ as the premise. We run the inner induction. If $\Psi, 0$ is an axiom, then $\Psi = \epsilon$ and $\Theta[\Psi] = \Theta[\epsilon]$. Let $\Psi, 0$ be obtained by a rule. $C^{\sim} = 0$ cannot be the active formula of any rule, so we proceed as for $C = p^{(n)}$.

4°. C is not an atomic formula. We run the intermediate induction.

Since C is not atomic, $\Theta[C]$ cannot be an axiom, hence it has to be the conclusion of a rule. If C is not the active formula, we apply the intermediate induction hypothesis to the premise(s) with C and use the same rule. We assume that C is the active formula.

4.1°. $C = A \otimes B$. So $C^{\sim} = B^{\sim} \oplus A^{\sim}$ and $\Theta[C]$ arises by (r-$\otimes$):

$$\frac{\Theta[(A, B)]}{\Theta[A \otimes B]}$$

We run the inner induction. $\Psi, C^{\sim}$ is not an axiom, hence it is the conclusion of a rule.

In the cases when $C^{\sim}$ does not occur in the active bunch, we apply the inner induction hypothesis to $\Theta[C]$ and the premise(s) with $C^{\sim}$, and use the same rule.

For example:

$$\frac{\Gamma[(D, E)], C^{\sim}}{\Gamma[D \otimes E], C^{\sim}}$$

changes into:

$$\frac{\Theta[\Gamma[(D, E)]]}{\Theta[\Gamma[D \otimes E]]},$$

where $\Psi = \Gamma[D \otimes E]$.

We consider cases when $C^{\sim}$ occurs in the active bunch, but is not the active formula.

$$\frac{D \quad E, C^{\sim}}{D \oplus E, C^{\sim}} \qquad \frac{D, C^{\sim} \quad E}{D \oplus E, C^{\sim}}$$

We apply the inner induction hypothesis to the premise with $C^{\sim}$ and use (r-$\oplus$1).

Let $C^{\sim}$ be the active formula:

$$\frac{\Psi, A^{\sim} \quad B^{\sim}}{\Psi, C^{\sim}} \qquad \frac{\Psi, B^{\sim} \quad A^{\sim}}{\Psi, C^{\sim}} \qquad \frac{\Psi_2, B^{\sim} \quad \Psi_1, A^{\sim}}{(\Psi_1, \Psi_2), C^{\sim}}$$

The first case is obtained by (r-$\oplus$1). We apply the outer induction hypothesis to $\Theta[(A, B)]$ and $\Psi, A^{\sim}$ and then to $\Theta[(\Psi, B)]$ and $B^{\sim}$, obtaining $\Theta[\Psi]$. The second one is obtained by (r-$\oplus$1) or (r-$\oplus$2). We proceed as above: we apply twice the outer induction hypothesis to both premises.

The third case is obtained by (r-$\oplus$4), where $\Psi = (\Psi_1, \Psi_2)$. We apply the outer induction hypothesis twice, obtaining $\Theta[(\Psi_1, \Psi_2)] = \Theta[\Psi]$.

4.2°. $C = A \oplus B$, then $C^{\sim} = B^{\sim} \otimes A^{\sim}$. We have to consider four cases, one for each (r-$\oplus$i).

(1) $\dfrac{\Gamma[B] \quad \Delta, A}{\Gamma[(\Delta, A \oplus B)]}$

We run the inner induction. $\Psi, C^{\sim}$ is not an axiom. We skip cases when $C^{\sim}$ is not the active formula of a rule (in these cases we proceed as above). We consider (r-$\otimes$) as the only possibility:

$$\frac{\Psi, (B^{\sim}, A^{\sim})}{\Psi, C^{\sim}}$$

We apply Lemma 2.2 (admissibility of (r-shift)) to $\Psi, (B^{\sim}, A^{\sim})$, then we apply the outer induction hypothesis to Δ, A and $(\Psi, B^{\sim}), A^{\sim}$ and obtain: $\Delta, (\Psi, B^{\sim})$. By Lemma 2.2 and the outer induction hypothesis applied to $\Theta[B]$ and $(\Delta, \Psi), B^{\sim}$ we obtain $\Gamma[(\Delta, \Psi)] = \Theta[\Psi]$.

(2) $\dfrac{\Gamma[A] \quad B, \Delta}{\Gamma[(A \oplus B, \Delta)]}$

We run the inner induction and consider the same instance as above. We apply Lemma 2.2 to $\Psi, (B^{\sim}, A^{\sim})$, obtaining $(\Psi, B^{\sim}), A^{\sim}$. By Corollary 2.5 we get $A^{-}, (\Psi, B^{\sim})$. We use Lemma 2.2 and apply the outer induction hypothesis to $(A^{-}, \Psi), B^{\sim}$ and B, Δ. We obtain $(A^{-}, \Psi), \Delta$ and apply Lemma 2.2 and Corollary 2.5. We use the outer induction hypothesis with $(\Psi, \Delta), A^{\sim}$ and $\Gamma[A]$, obtaining $\Gamma[(\Psi, \Delta)] = \Theta[\Psi]$.

(3) $\dfrac{A, \Gamma \quad B, \Delta}{A \oplus B, (\Delta, \Gamma)}$

We run the inner induction and consider the same instance as above. We apply Lemma 2.2 to $\Psi, (B^{\sim}, A^{\sim})$ and obtain $(\Psi, B^{\sim}), A^{\sim}$. We apply Corollary 2.5 and get $A^{-}, (\Psi, B^{\sim})$. We use Lemma 2.2 and apply the outer induction hypothesis to $(A^{-}, \Psi), B^{\sim}$ and B, Δ. We have $(A^{-}, \Psi), \Delta$. We apply Lemma 2.2 and Corollary 2.5. We use the outer induction hypothesis to $(\Psi, \Delta), A^{\sim}$ and A, Γ, obtaining $(\Psi, \Delta), \Gamma$. We use Lemma 2.2.

(4) $\dfrac{\Gamma, A \quad \Delta, B}{(\Delta, \Gamma), A \oplus B}$

We run the inner induction and consider the same instance as above. We apply Lemma 2.2 to $\Psi, (B^\sim, A^\sim)$, obtaining $(\Psi, B^\sim), A^\sim$. We apply the outer induction hypothesis to $(\Psi, B^\sim), A^\sim$ and Γ, A. We get $\Gamma, (\Psi, B^\sim)$. We use Lemma 2.2 and apply the outer induction hypothesis to $(\Gamma, \Psi), B^\sim$ and Δ, B. We obtain $\Delta, (\Gamma, \Psi)$ and use Lemma 2.2.

4.3°. $C = A \wedge B$. So $C^\sim = A^\sim \vee B^\sim$. We have the following instances:

$$\frac{\Theta[A]}{\Theta[C]} \qquad \frac{\Theta[B]}{\Theta[C]}$$

We run the inner induction. $\Psi, C^\sim$ is not an axiom. We skip the cases with $C^\sim$ not being the active formula. We have only one possibility:

$$\frac{\Psi, A^\sim \quad \Psi, B^\sim}{\Psi, C^\sim}$$

We apply the outer induction hypothesis to $\Theta[A]$ and $\Psi, A^\sim$ or to $\Theta[B]$ and $\Psi, B^\sim$, depending on the proof of $\Theta[C]$. In both cases we obtain $\Theta[\Psi]$.

4.4°. $C = A \vee B$. So $C^\sim = A^\sim \wedge B^\sim$. We have the following case:

$$\frac{\Theta[A] \quad \Theta[B]}{\Theta[C]}$$

We run the inner induction. $\Psi, C^\sim$ cannot be an axiom. We consider only the cases with $C^\sim$ as the active formula:

$$\frac{\Psi, A^\sim}{\Psi, C^\sim} \qquad \frac{\Psi, B^\sim}{\Psi, C^\sim}$$

In the first case we apply the outer induction hypothesis to $\Theta[A]$ and $\Psi, A^\sim$ and in the second case to $\Theta[B]$ and $\Psi, B^\sim$. $\square$

One can easily prove the strong completness of NBL (with the cut rules) with respect to l.o. involutive unital groupoids. Let X be any set of bunches. We say that formulas A, B are equivalent ($A \simeq B$) if and only if $X \vdash A, B^\sim$ and $X \vdash A^-, B$. It is easy to check that $\simeq$ is a congruence. The quotient algebra is l.o. involutive unital groupoid. We define $\mu(p^{(0)}) = [p]_\simeq$, $\mu(p^{(n+1)}) = \mu(p^{(n)})^\sim$ for $n \geq 0$ and $\mu(p^{(n-1)}) = \mu(p^{(n)})^-$ for $n \leq 0$. One proves $\mu(A) = [A]_\simeq$, hence $\mu((\Gamma, \Delta)) = [\Gamma]_\simeq \otimes [\Delta]_\simeq$. If $X \not\vdash \Gamma$, then $\mu(\Gamma) \not\leq 0$.

4. Other systems

4.1. The additive constants

The presented Nonassociative Bilinear Logic admits only multiplicative constants. We can extend this system by additive constants $\top$ and $\bot$. In the sense of algebraic models, they are the greatest and the smallest elements in the l.o. unital groupoids, respectively, i.e. for all a:

$$\bot \leq a, \qquad a \leq \top$$

In particular $\bot \leq 0$, hence it should be a theorem.

We extend NBL-language by two constants: $\top$ and $\bot$. We also add an axiom:

$$(\text{a-}\bot) \quad \Gamma[\bot]$$

which is valid because one proves $a \otimes \bot = \bot$ and $\bot \otimes a = \bot$.

This is the only axiom we add. We do not extend NBL with any new rules. It is interesting that $\top$ does not appear explicitly in any axiom nor any rule, but it is still in the language.

In the metalanguage we also add the following:

$$\bot^{\sim} = \bot^{-} = \top, \qquad \top^{\sim} = \top^{-} = \bot$$

All results presented in this paper are also true for NBL with these constants. $\top$ and $\bot$ cannot be the active formulas of any rule, so the presented reasoning remains valid. In every proof we need to add an additional case for the new axiom.

In the proof of Lemma 2.2 we consider the case when $(\Gamma_1, \Gamma_2), \Gamma_3$ is an instance of (a-$\bot$). In this case $\bot$ occurs in one of Γ_i. Then $\Gamma_1, (\Gamma_2, \Gamma_3)$ is an instance of (r-$\bot$), too.

In the proof of Lemma 2.4 we have the outer induction and the inner induction. In the outer one we consider the case when $C = \bot$ or $C = \top$. Neither can be the active formula of a rule, hence we proceed as in similar cases for other atomic formulas. In the inner induction we have an additional axiom to consider. We assume that $A, \Gamma[\bot]$ is an axiom, hence $\Gamma[\bot], A^{\sim\sim}$ is an axiom, too. Similarly if $\Gamma[\bot], A$ is an axiom, then $A^{--}, \Gamma[\bot]$ is an axiom.

Our main result is the cut–elimination theorem. In its proof we use three inductions: the outer, the intermediate and the inner. We consider three cases.

1°. $C = \bot$. Then $C^{\sim} = \top$. We run the intermediate induction. Let $\Theta[\bot]$ be an axiom. We run the inner induction. If $\Psi, \top$ is an axiom, then $\Theta[\Psi]$ is also an axiom.

If $\Theta[\bot]$ or $\Psi, \top$ is not an axiom, we proceed as for $C = p^{(n)}$.

2°. $C = \top$. Then $C^{\sim} = \bot$. We run the intermediate induction. If $\Theta[\top]$ is an axiom, then $\Theta[\Psi]$ is an axiom, too. Let $\Theta[\top]$ be obtained by a rule. $\top$ is not the active formula of any rule. We apply the intermediate induction hypothesis to the premise(s) with $C = \top$ and use the same rule.

3°. $C \neq \bot$ and $C \neq \top$. We notice that if $\Theta[C]$ is an instance of (a-$\bot$), then $\Theta[\Psi]$ is an axiom. Also, if $\Psi, C^{\sim}$ is an axiom, then $\Theta[\Psi]$ is an axiom, too.

Corollary 4.1. NBL with the additive constants is a conservative extension of NBL.

4.2. The right-sided system

The presented system is left-sided, but we can consider right-sided and two-sided systems of that logic. A two-sided system for NBL was considered in [6]. It is denoted InGL – Involutive Groupoid Logic. InGL treats negations as connectives, so the logic is much more complex than our system. It is also non-standard, because in the language there is no coproduct and the right side of the sequent serves only for technical purposes.

Our system for NBL can be easily translated into right-sided system. It is dual in the sense that product and coproduct are exchanged, similary meet and join or 1 and 0. All the results proved here can be translated into the right-sided system, remaining true.

The language and models remain the same as for the left-sided system. We modify the definition of valuation. Now $\mu((\Gamma, \Delta)) = \mu(\Gamma) \oplus \mu(\Delta)$ and $\mu(\epsilon) = 0$. We say that the sequent Γ is true in $\mathbf{M}$ for valuation μ, if $1 \leq \mu(\Gamma)$.

We admit axioms:

$$\text{(a*-id)} \qquad p^{(n+1)}, p^{(n)} \quad \text{for any variable } p \text{ and any } n \in \mathbb{Z}$$

$$\text{(a*-1)} \qquad 1$$

The rules are:

$$\text{(r*-}\oplus\text{)} \quad \frac{\Gamma[(A, B)]}{\Gamma[A \oplus B]}$$

$$\text{(r*-}\otimes 1\text{)} \quad \frac{\Gamma[B] \quad \Delta, A}{\Gamma[(\Delta, A \otimes B)]} \qquad \text{(r*-}\otimes 2\text{)} \quad \frac{\Gamma[A] \quad B, \Delta}{\Gamma[(A \otimes B, \Delta)]}$$

$$\text{(r*-0)} \quad \frac{\Gamma[\Delta]}{\Gamma[(0, \Delta)]} \quad \frac{\Gamma[\Delta]}{\Gamma[(\Delta, 0)]}$$

$$\text{(r*-}\vee\text{)} \quad \frac{\Gamma[A]}{\Gamma[A \vee B]} \quad \frac{\Gamma[A]}{\Gamma[B \vee A]} \qquad \text{(r*-}\wedge\text{)} \quad \frac{\Gamma[A] \quad \Gamma[B]}{\Gamma[A \wedge B]}$$

$$\text{(r*-shift)} \quad \frac{(\Gamma, \Delta), \Theta}{\overline{\Gamma, (\Delta, \Theta)}}$$

The cut rules obtain the form:

$$\text{(cut*}^{\sim}\text{)} \quad \frac{\Gamma[A] \quad A^{\sim}, \Delta}{\Gamma[\Delta]} \qquad \text{(cut*}^{-}\text{)} \quad \frac{\Gamma[A] \quad \Delta, A^{-}}{\Gamma[\Delta]}$$

Now we show the way of translating the left-sided system for NBL to the right-sided system. We extend the metalanguage negations $^{\sim}, ^{-}$ to bunches by $(\Gamma, \Delta)^{\sim} = (\Delta^{\sim}, \Gamma^{\sim})$, $\epsilon^{\sim} = \epsilon$ and similarly for $^{-}$. Clearly $(\Gamma^{\sim})^{-} = \Gamma = (\Gamma^{-})^{\sim}$. We also extend these negations for contexts by setting: $x^{\sim} = x^{-} = x$. We obtain $\Gamma[\Delta]^{\sim} = \Gamma^{\sim}[\Delta^{\sim}]$ and similarly for $^{-}$.

Lemma 4.2. *The sequent Θ is provable in the left-sided system if and only if $\Theta^{\sim}$ (resp. Θ^{-}) is provable in the right-sided system.*

Theorem 4.3. *The cut rules (cut$*^{\sim}$) and (cut$*^{-}$) are admissible in the cut-free right-sided system for NBL.*

We prove the theorem for $(\mathrm{cut}^{*\sim})$. The proof for (cut^{*-}) is similar.

PROOF: Let $\vdash_L, \vdash_R$ denote the provability in the left-sided system and in the right-sided system, respectively. Assume $\vdash_R \Gamma[A]$ and $\vdash_L A^{\sim}, \Delta$. By Lemma 4.2, $\vdash_L \Gamma^{\sim}[A^{\sim}]$ and $\vdash_L \Delta^{\sim}, A^{\sim\sim}$. By Theorem 1, $\vdash_L \Gamma^{\sim}[\Delta^{\sim}]$. So $\vdash_R (\Gamma^{\sim}[\Delta^{\sim}])^{-}$, which yields $\vdash_R \Gamma[\Delta]$. $\square$

We can also extend the right-sided system with the additive constants $\bot$ and $\top$. We add the axiom (a*-$\top$) $\Gamma[\top]$ and we define $\bot^{\sim} = \top = \bot^{-}$ and $\top^{\sim} = \bot = \top^{-}$. One proves the lemma above for that extended system.

Since NBL is a conservative extension of NL1 (Nonassociative Lambek Calculus with 1), InNL1 (Involutive Nonassociative Lambek Calculus with 1, i.e. the multiplicative fragment of NBL) and FNL1 (Full Nonassociative Lambek Calculus with 1), all the results remain true for these weaker logics.

5. PTIME complexity

In this section we prove the PTIME complexity of the multiplicative fragment of NBL, i.e. MNBL. This system is denoted InNL1 in [4], which proves the PTIME complexity of InNL and claims the same for InNL1. We provide a proof.

By MNBL we mean NBL without $\wedge, \vee, \bot$ and $\top$ in the language and without the corresponding axioms and rules: (a-$\bot$), (r-$\wedge$), (r-$\vee$) (resp. (a*-$\top$), (r*-$\wedge$), (r*-$\vee$) for right-sided system). All results proved before remain true for MNBL, because NBL is a conservative extension. We focus on the left-sided system. Since we consider only the multiplicative fragment of NBL, we define MNBL_0 as NBL_0 without additive connectives and constants.

DEFINITION 5.1. Let T be a set of formulas. Any sequent built from formulas of T is called *T–sequent*. A *T–proof* is a proof consisting only of T–sequents.

In NBL-language we do not treat the negations as connectives, but all formulas of the form $p^{(n)}$ are atoms. Hence p is not a subformula of $p^{\sim}$ or p^{-} etc. By theorem 3.1 one obtains the following corollary:

Corollary 5.2. For every sequent Γ, if Γ is provable in cut-free NBL, then it has T–proof, where T is the subformula closure of the set of all formulas in Γ.

The above corollary is called *subformula property.* Because NBL is a conservative extension of MNBL, MNBL possess the subformula property. Hence we can consider only T–proofs for any sequent Γ, where T is a set of all subformulas of formulas in Γ. In order to prove the PTime complexity we consider restricted sequents. A sequent is called *restricted* if it consists of at most three formulas. The restricted sequents are of the form: $A; (A, B); (A, (B, C)); ((A, B), C)$.

We define $c(T) = T \cup T^{\sim} \cup T^{-}$, where $T^{\sim} = \{A^{\sim} : A \in T\}, T^{-} = \{A^{-} : A \in T\}$. Now let T be any subformula closed set of formulas, containing 0 and 1.

By MNBL_1^T we mean a new system, defined as follows. The axioms are 0 and all sequents A^{-}, A and $A, A^{\sim}$ for $A \in T$. The inference rules are all rules of MNBL_0 limited to restricted $c(T)$–sequents with the active formula in T and the cut rules (cut$^{\sim}$), (cut^{-}) limited to $c(T)$–sequents. We assume that $\Delta \neq \epsilon$ in the cut rules. Notice that we do not limit cut rules to restricted sequents.

Since T is fixed, we write MNBL_1 for MNBL_1^T. The provability in MNBL_1 is denoted by $\vdash_1$. The system MNBL_1 posseses an interpolation property.

Lemma 5.3. *If* $\vdash_1 \Theta[\Psi]$, $\Theta[\Psi] \neq \Psi$ *and* $\Psi \neq \epsilon$, *then there exists* $D \in c(T)$ *such that* $\vdash_1 \Theta[D]$ *and either* $\vdash_1 D^{-}, \Psi$ *or* $\vdash_1 \Psi, D^{\sim}$.

Proof: We proceed by induction on proofs of $\Theta[\Psi]$ in MNBL_1.

Let Ψ be a formula. We put $D = \Psi$. Clearly $\vdash_1 \Theta[D]$. If $\Psi \in T$, then $\Psi, \Psi^{\sim}$ is an axiom. If $\Psi \in T^{-}$, then $\Psi^{\sim} \in T$, hence $\Psi^{\sim -}, \Psi^{\sim}$ is an axiom. The case for $\Psi \in T^{\sim}$ is analogous.

We assume Ψ is not a formula. $\Theta[\Psi]$ cannot be an axiom. We consider a case for each rule of MNBL_1.

(r-$\otimes$). The only possibilities are:

$$\frac{A, B}{A \otimes B} \qquad \frac{A, (B, C)}{A, B \otimes C} \qquad \frac{(A, B), C}{A \otimes B, C}$$

In all cases all bunches properly contained in the conclusion are formulas.

(r-$\oplus$1). We have the following cases:

(1) $\dfrac{B \quad A}{A \oplus B}$ (2) $\dfrac{B \quad C_1, A}{C_1, A \oplus B}$ (3) $\dfrac{C_1, B \quad A}{C_1, A \oplus B}$

(4) $\dfrac{B, C_1 \quad A}{A \oplus B, C_1}$ (5) $\dfrac{B \quad (C_1, C_2), A}{(C_1, C_2), A \oplus B}$ (6) $\dfrac{C_1, B \quad C_2, A}{C_1, (C_2, A \oplus B)}$

(7) $\dfrac{B, C_1 \quad C_2, A}{(C_2, A \oplus B), C_1}$ (8) $\dfrac{(C_1, C_2), B \quad A}{(C_1, C_2), A \oplus B}$ (9) $\dfrac{C_1, (C_2, B) \quad A}{C_1, (C_2, A \oplus B)}$

(10) $\dfrac{(C_1, B), C_2 \quad A}{(C_1, A \oplus B), C_2}$ (11) $\dfrac{C_1, (B, C_2) \quad A}{C_1, (A \oplus B, C_2)}$ (12) $\dfrac{(B, C_1), C_2 \quad A}{(A \oplus B, C_1), C_2}$

(13) $\dfrac{B, (C_1, C_2) \quad A}{A \oplus B, (C_1, C_2)}$

In cases (1)–(4) all bunches properly contained in the conslusion are formulas.

We consider (5). $\Psi = (C_1, C_2)$. We put $D = A^-$, hence $\Psi, D^\sim$ equals the second premise. Since $A \in T$, A^-, A is an axiom. By (r-$\oplus$1) we obtain $A^-, A \oplus B (= \Theta[D])$ from this axiom and the first premise.

In (6) and (7) $\Psi = (C_2, A \oplus B)$. We put $D = B$, so $\Theta[D]$ equals the first premise. We use (r-$\oplus$1) to obtain $\Psi, D^\sim$ from the second premise and the axiom $B, B^\sim$, since $B \in T$.

In (8) $\Psi = (C_1, C_2)$. We put $D = B^-$ and proceed in the similar way as in (5).

We consider (9). Here $\Psi = (C_2, A \oplus B)$. By the induction hypothesis there is a formula E, such that $\vdash_1 C_1, E$ and $\vdash_1 E^-, (C_2, B)$ or $\vdash_1 (C_2, B), E^\sim$. We put $D = E$. By (r-$\oplus$1) with the second premise A we obtain $E^-, (C_2, A \oplus B)$ or $(C_2, A \oplus B), E^\sim$. Note that $\Theta[D] = (C_1, E)$.

In (10)–(12) we proceed analogously as in (9). In (13) we proceed as in (8).

(r-$\oplus$2). The cases are symmetrical and the arguments are similar to those of (r-$\oplus$1).

(r-$\oplus$3). We have only one possibility:

$$\frac{A, C_1 \quad B, C_2}{A \oplus B, (C_2, C_1)}$$

Hence $\Psi = (C_2, C_1)$. Since $A \oplus B \in T$, then $A \oplus B, (A \oplus B)^{\sim}$ is an axiom. We put $D = (A \oplus B)^{\sim}$, hence $\Theta[D]$ is the axiom and D^{-}, Ψ is the conclusion.

(r-$\oplus$4). We have only one possibility:

$$\frac{C_1, A \quad C_2, B}{(C_2, C_1), A \oplus B}$$

Hence $\Psi = (C_2, C_1)$. Since $A \oplus B \in T$, so $(A \oplus B)^{-}, A \oplus B$ is an axiom. We put $D = (A \oplus B)^{-}$, hence $\Theta[D]$ is this axiom and $\Psi, D^{\sim}$ is the conclusion.

(r-1). We consider the following cases:

$$(1)\ \frac{C}{1, C} \qquad (2)\ \frac{C}{C, 1} \qquad (3)\ \frac{C_1, C_2}{1, (C_1, C_2)}$$

$$(4)\ \frac{C_1, C_2}{(1, C_1), C_2} \qquad (5)\ \frac{C_1, C_2}{C_1, (1, C_2)} \qquad (6)\ \frac{C_1, C_2}{(C_1, 1), C_2}$$

$$(7)\ \frac{C_1, C_2}{(C_1, C_2), 1} \qquad (8)\ \frac{C_1, C_2}{C_1, (C_2, 1)}$$

In the first two cases conclusions have only formulas as properly contained bunches.

In (3) and (7) we have $\Psi = (C_1, C_2)$. We put $D = 0$, hence $\Theta[D]$ can be obtained from the axiom 0 by (r-1) and D^{-}, Ψ is the conclusion.

In (4)–(6) we put $D = C_1$, hence $\Theta[D]$ is the premise. If $C_1 \in T$, then $C_1, C_1^{\sim}$ is an axiom; if $C_1 \in T^{-}$, then $C_1, C_1^{\sim}$ is an axiom; if $C_1 \in T^{\sim}$, then C_1^{-}, C_1 is an axiom. We apply (r-1) to one of those axiom (depending on C_1) and we obtain $\Psi, D^{\sim}$ or D^{-}, Ψ.

In (8) we put $D = C_2$ and proceed analogously.

(cut$^{\sim}$).

$$\frac{\Gamma[A] \quad \Delta, A^{\sim}}{\Gamma[\Delta]}$$

We have $\Theta[\Psi] = \Gamma[\Delta]$. If Ψ occurs in $\Gamma[\ ]$, then $\Gamma[A] = \Xi[\Psi][A]$ and, by the induction hypothesis, there exists E, such that $\vdash_1 \Xi[E][A]$ and $\vdash_1 E^{-}, \Psi$ or $\vdash_1 \Psi, E^{\sim}$. We put $D = E$, then $\Theta[D] = \Xi[D][\Delta]$, which we obtain by (cut$^{\sim}$) from $\Xi[D][A]$ and $\Delta, A^{\sim}$.

If Ψ occurs in Δ, the we use the induction hypothesis for $\Delta, A^{\sim}$ and proceed as above.

Now let $\Gamma[\Delta] = \Gamma_1[\Gamma_2[\Delta]]$ and $\Psi = \Gamma_2[\Delta]$ and $\Psi \neq \Delta$. Hence $\Gamma[A] = \Gamma_1[\Gamma_2[A]]$ and $\Gamma_2[A] \neq \Gamma[A]$. We use the induction hypothesis for $\Gamma_2[A]$ in $\Gamma_1[\Gamma_2[A]]$ to obtain D. We have $\vdash_1 \Gamma_1[D]$ and $\vdash_1 D^-, \Gamma_2[D]$ or $\vdash_1 \Gamma_2[D], D^{\sim}$. We obtain $\vdash_1 D^-, \Gamma_2[\Delta]$ or $\vdash_1 \Gamma_2[\Delta], D^{\sim}$ by (cut$^{\sim}$) applied with $\Delta, D^{\sim}$.

(cut$^-$). We proceed analogously as for (cut$^{\sim}$). □

For a sequent Γ we take T as the subformula closure of the set of all formulas appearing in Γ, also containing 0 and 1. We define MNBL_1^T as above.

Lemma 5.4. *Γ is provable in MNBL if and only if $\vdash_1 \Gamma$.*

Proof: All axioms of MNBL_1 are provable in MNBL, also all rules are valid in MNBL, since they are the instances of original MNBL rules or are admissible. Hence if $\vdash_1 \Gamma$, then $\vdash \Gamma$ in MNBL.

Now assume that Γ is provable in MNBL. We show that it is provable in MNBL_1. By Corollary 2.3, Γ is provable in MNBL_0. Also, because of the subformula property, Γ has a T–proof in MNBL_0. It suffices to show that all T–axioms (axioms which are T–sequents) of MNBL_0 are provable in MNBL_1 and all rules of MNBL_0 limited to T–sequents are admissible in MNBL_1.

If $p^{(n)} \in T$, then $p^{(n)}, p^{(n+1)}$ is the axiom $A, A^{\sim}$ of MNBL_1. Also 0 is an axiom of MNBL_1.

(r-$\otimes$). We assume that $\vdash_1 \Gamma[(A, B)]$ and $\Gamma[A \otimes B]$ is a T–sequent. If $\Gamma[(A, B)] = (A, B)$, then $\vdash_1 \Gamma[A \otimes B]$ ($\Gamma[A \otimes B] = A \otimes B$). We assume $\Gamma[(A, B)] \neq (A, B)$. By Lemma 5.3, there exists $D \in c(T)$, such that $\vdash_1 \Gamma[D]$ and $\vdash_1 D^-, (A, B)$ or $\vdash_1 (A, B), D^{\sim}$. We apply (r-$\otimes$) (in MNBL_1) and obtain $D^-, A \otimes B$ or $A \otimes B, D^{\sim}$. And by one of the cut rules: $\vdash_1 \Gamma[A \otimes B]$.

(r-$\oplus$1). We assume that $\vdash_1 \Gamma[B]$ and $\vdash_1 \Delta, A$ and $\Gamma[(\Delta, A \oplus B)]$ is a T–sequent. We consider two cases:

$\Delta = \epsilon$. From A and the axiom $B, B^{\sim}$ we obtain $A \oplus B, B^{\sim}$ by (r-$\oplus$1) in MNBL_1. By (cut$^{\sim}$) we get $\Gamma[A \oplus B]$.

$\Delta \neq \epsilon$. By Lemma 5.3 we have $D \in c(T)$, such that $\vdash_1 D, A$ and $\vdash_1 D^-, \Delta$ or $\vdash_1 \Delta, D^{\sim}$. From D, A and the axiom $B, B^{\sim}$ we obtain $(D, A \oplus$

$B), B^{\sim}$ by (r-$\oplus$1) in MNBL_1. By one of the cut rules we obtain $(\Delta, A \oplus B), B^{\sim}$ and by (cut$^{\sim}$) we get $\Gamma[(\Delta, A \oplus B)]$.

(r-$\oplus$2). The argument is similar as for (r-$\oplus$1).

(r-$\oplus$3). We assume that $\vdash_1 A, \Gamma$ and $\vdash_1 B, \Delta$ and $A \oplus B, (\Delta, \Gamma)$ is a T–sequent. We apply Lemma 5.3 twice, obtaining D_1 for Γ in A, Γ and D_2 for Δ in B, Δ. We have $\vdash_1 A, D_1$ and $\vdash_1 B, D_2$. We use (r-$\oplus$3) in MNBL_1 and get $A \oplus B, (D_2, D_1)$. We apply appropriate cut rules for both D_1 and D_2 and get $A \oplus B, (\Delta, \Gamma)$.

(r-$\oplus$4). The argument is analogous to that for (r-$\oplus$3).

(r-1). We assume that $\vdash_1 \Gamma[\Delta]$ and $\Gamma[(1, \Delta)]$ is a T–sequent. By Lemma 5.3 there is $D \in c(T)$, such that $\vdash_1 \Gamma[D]$ and $\vdash_1 D^{-}, \Delta$ or $\vdash_1 \Delta, D^{\sim}$. We assume that $\vdash D^{-}, \Delta$. The other case is analogous. By Lemma 5.3 we have $E \in c(T)$, such that $\vdash_1 D^{-}, E$ and $\vdash_1 E^{-}, \Delta$ or $\vdash_1 \Delta, E^{\sim}$. We apply (r-1) to D^{-}, E and get $D^{-}, (1, E)$. Now we use one of the cut rules and obtain $D^{-}, (1, \Delta)$. We use (cut^{-}) and obtain $\Gamma[(1, \Delta)]$. The argument for the other variant of (r-1) is the same. $\square$

We notice that in MNBL_1 if the conclusion is restricted, then the premises are also restricted. Hence every restricted sequent Γ provable in MNBL has a proof in MNBL_1^T, where T is defined above.

For every sequent Γ we define $f(\Gamma)$ as follows: $f(A) = A$, if A is a formula and $f((\Gamma_1, \Gamma_2)) = f(\Gamma_1) \otimes f(\Gamma_2)$. It is clear, that $\vdash \Gamma$ if and only if $\vdash f(\Gamma)$. We see that $f(\Gamma)$ is a restricted sequent.

Let Γ be a restricted sequent. We define *the size of* Γ as follows: $s(p^{(n)}) = |n| + 1, s(0) = s(1) = 1, s(A \otimes B) = s(A) + s(B) + 1, s(A \oplus B) = s(A) + s(B) + 1, s((\Gamma_1, \Gamma_2)) = s(\Gamma_1) + s(\Gamma_2)$. By $|n|$ we can take either the absolute value of n or the length of its binary representation.

We provide an algorithm verifying the provability of this sequent. If we put $n = s(\Gamma)$, the complexity is polynomial with respect to n.

First, we compute T in $O(n^2)$ time and then $c(T)$ in $O(n)$ time. Notice that $c(T)$ has at most $3n$ elements, and hence there are $O(n^3)$ restricted $c(T)$–sequents.

Now we compute the list of all provable sequents of MNBL_1. We put $\Gamma_0 = 0, \Gamma_1, \Gamma_2, \ldots, \Gamma_k$, being the sequence of all axioms of MNBL_1.

We iterate over $i = 1, 2, \ldots$. For every i we extend the list with the new sequents, being the conclusions of of (r-$\oplus$1), (r-$\oplus$2), (r-$\oplus$3), (r-$\oplus$4) and the cut rules with the premises Γ_i and Γ_j, if applicable, where $j < i$. We do not add the sequents, which are already in the list. We also apply (r-$\otimes$) and (r-1) to every sequent, if applicable, and extend the list with the conclusions. Since there are $O(n^3)$ restricted $c(T)$–sequents, the procedure always stops. Assuming that one rule is executed in time $O(n)$, we have the time $O(ni)$ for every i.

The rough estimation of the complexity is $O(n^7)$.

THEOREM 5.5. *MNBL is* PTIME.

If we add the additive connectives $\wedge, \vee$ to MNBL, we obtain NBL, which is PSPACE. We don't know the lower bound of complexity. Pentus [11] proves that MBL (associative MNBL) is NP-complete. BL is PSPACE-complete. Since MNBL is a conservative extension of NL1 (Nonassociative Lambek Calculus with 1), theorem 5.5 remains true for NL1.

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Bulletin of the **S**ection of **L**ogic
Volume 50/1 (2021), pp. 81–96
https://doi.org/10.18778/0138-0680.2020.20

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ON GE-ALGEBRAS

Abstract

Hilbert algebras are important tools for certain investigations in intuitionistic logic and other non-classical logic and as a generalization of Hilbert algebra a new algebraic structure, called a GE-algebra (generalized exchange algebra), is introduced and studied its properties. We consider filters, upper sets and congruence kernels in a GE-algebra. We also characterize congruence kernels of transitive GE-algebras.

Keywords: (transitive) GE-algebra, filter, upper set, congruence kernel.

2010 Mathematical Subject Classification: 06F35, 03G25, 08A30.

1. Introduction

A. Monteiro in ([15]) (see also [16]) called Hilbert algebra a triple $(X, *, 1)$ where X is a non-empty set, $*$ is a binary operation on X, 1 is an element of X such that the following properties are satisfied for every $x, y, z \in X$:

$(H1)$ $x * (y * x) = 1$

$(H2)$ $(x * (y * z)) * ((x * y) * (x * z)) = 1$

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Presented by: Janusz Ciuciura
Received: September 16, 2019
Published online: August 30, 2020

$(H3)$ $x * 1 = 1$

$(H4)$ $x * y = 1$ and $y * x = 1$ imply $x = y$.

In 1960, L. Iturrioz proved that $(H3)$ follows from $(H1)$ and $(H4)$ and that $(H1)$,$(H2)$ and $(H4)$ are independent. In the same year A. Diego, answering a problem posed by A. Monteiro, obtained an equational definition of these algebras. The literature on Hilbert algebras can be seen in ([8],[5, 4],[6],[11],[13],[10, 9]). Kim and Kim ([14]) introduced the notion of a BE-algebra as a generalization of a dual BCK-algebra. R. A. Borzooei and J. Shohani ([3]) introduced the notion of a generalized Hilbert algebra and studied its properties. J. C. Abbott ([1]) introduced a concept of implication algebra in the sake to formalize the logical connective implication in the classical propositional logic. R. A. Borzooei and S. K. Shoar ([2]) have shown that the implication algebras are equivalent to dual implicative BCK-algebras.

In this paper, we introduce the concept of GE-algebra which is a generalization of Hilbert algebra and study its properties. We define the notion of transitive and of commutative GE-algebra and observe that every commutative GE-algebra is a transitive GE-algebra. Also we give a condition under which a GE-algebra to become an Implication algebra. We give the relation between GE-algebra and other algebras (Hilbert algebra, dual implicative BCK-algebra, g-Hilbert algebra and BE-algebra). We consider filters, upper sets and congruence kernels in a GE-algebra and characterize congruence kernels whenever a GE-algebra is transitive.

2. Preliminaries

First, we recall certain definitions from [1],[2],[5],[7], [12] and [14] that are required in the paper.

Definition 2.1. ([1]) An implication algebra is a set X with a binary operation $*$ which satisfies the following conditions:

(I_1) $(x * y) * x = x$,

(I_2) $(x * y) * y = (y * x) * x$,

(I_3) $x * (y * z) = y * (x * z)$, for all $x, y, z \in X$.

Theorem 2.2. ([7]) *In any implication algebra* $(X, *)$, *the following conditions hold:*

(1) $x * (x * y) = x * y$.

(2) $x * x = y * y$.

(3) *There exists a unique element* 1 *in* X *such that*

(a) $x * x = 1$, $1 * x = x$ *and* $x * 1 = 1$.

(b) *if* $x * y = 1$ *and* $y * x = 1$, *then* $x = y$, *for all* $x, y \in X$.

Definition 2.3. ([2]) A dual BCK-algebra is a triple $(X, *, 1)$ where X is a non-empty set with a binary operation $*$ and a constant 1 satisfying the following axioms for all x, y, z in X :

$(DBCK_1)$ $(y * z) * [(z * x) * (y * x)] = 1$,

$(DBCK_2)$ $y * [(y * x) * x] = 1$,

$(DBCK_3)$ $x * x = 1$,

$(DBCK_4)$ $x * y = 1$ and $y * x = 1$ imply $x = y$,

$(DBCK_5)$ $x * 1 = 1$.

Definition 2.4. ([5]) A Hilbert algebra is an algebra $(X, *, 1)$ of type $(2, 0)$ such that the following axioms hold, for all $x, y, z \in X$:

$(H1)$ $x * (y * x) = 1$,

$(H2)$ $(x * (y * z)) * ((x * y) * (x * z)) = 1$,

$(H3)$ if $x * y = y * x = 1$, then $x = y$.

It is proved that the above definition is equivalent to the system $\{H4, H5, H6, H7\}$ where:

$(H4)$ $x * x = 1$.

$(H5)$ $1 * x = x$.

$(H6)$ $x * (y * z) = (x * y) * (x * z)$.

$(H7)$ $(x * y) * ((y * x) * x) = (y * x) * ((x * y) * y)$.

A Hilbert algebra X is said to be commutative if it satisfies (I_2).

Definition 2.5. ([3]) A generalized Hilbert algebra (or biefly, g-Hilbert algebra) is an algebra $(X, *, 1)$ of type $(2, 0)$ such that the following axioms hold, for all $x, y, z \in X$:

$(GH1)$ $1 * x = x$,

$(GH2)$ $x * x = 1$,

$(GH3)$ $x * (y * z) = y * (x * z)$,

$(GH4)$ $x * (y * z) = (x * y) * (x * z)$.

Definition 2.6. ([14]) A BE-algebra is an algebra $(X, *, 1)$ of type $(2, 0)$ such that the following axioms hold, for all $x, y, z \in X$:

$(BE1)$ $x * x = 1$,

$(BE2)$ $1 * x = 1$,

$(BE3)$ $x * 1 = 1$,

$(BE4)$ $x * (y * z) = y * (x * z)$.

A BE-algebra X is said to be commutative and self-distributive if it satisfies (I_2) and $(H6)$.

3. GE-algebras

In this section, we define the notion of a GE-algebra which is a generalization of the notion of Hilbert algebra and study its properties. Also, we define the notions of transitive GE-algebra and of commutative GE-algebra and give conditions under which a GE-algebra to become Implication algebra, dual implicative BCK-algebra and commutative Hilbert algebra.

Definition 3.1. A GE-algebra is a non-empty set X with a constant 1 and a binary operation $*$ satisfying axioms:

$(GE1)$ $x * x = 1$,

$(GE2)$ $1 * x = x$,

$(GE3)$ $x * (y * z) = x * (y * (x * z))$, for all $x, y, z \in X$.

We can observe that every Hilbert algebra/implication algebra is a GE-algebra but the converse need not be true.

Example 3.2. Let $X = \{1, a, b, c, d\}$ be a set with the following table.

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	b	1
b	1	a	1	1	d
c	1	a	1	1	d
d	1	1	c	c	1

Then $(X, *, 1)$ is a GE-algebra but not a Hilbert algebra since $b*c = c*b = 1$ but $b \neq c$.

Let $(X, *, 1)$ be a GE-algebra. Define a binary relation " $\leq$ " on X by

$$x \leq y \text{ if and only if } x * y = 1.$$

There are no hidden difficulties to prove the following theorem hence we omit its proof.

THEOREM 3.3. *In a GE-algebra X, for all $x, y, z \in X$, the following conditions hold:*

(1) $x * 1 = 1$.

(2) $x * (x * y) = x * y$.

(3) $1 \leq x$ *implies* $x = 1$.

(4) $x \leq y * x$.

(5) $x \leq (x * y) * y$.

(6) $x \leq (y * x) * x$.

(7) $x \leq (x * y) * x$.

(8) $x \leq y * (y * x)$.

(9) $x * (y * z) \leq y * (x * z)$.

(10) $x \leq y * z$ *if and only if* $y \leq x * z$.

The following theorem can be proved easily.

THEOREM 3.4. *Let $(X,*,1)$ be a GE-algebra. Then, for all $x,y,z \in X$, the following are equivalent.*

(1) $x*y \leq (z*x)*(z*y)$,

(2) $x*y \leq (y*z)*(x*z)$.

DEFINITION 3.5. A GE-algebra $(X,*,1)$ is said to be transitive if it satisfies

$$x*y \leq (z*x)*(z*y)$$

for all $x,y,z \in X$.

Example 3.6. Let $X = \{1,a,b,c,d\}$ be a set with the following table.

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	c	c
b	1	a	1	d	d
c	1	a	1	1	1
d	1	a	1	1	1

Then $(X,*,1)$ is a transitive GE-algebra but not Hilbert algebra/dual BCK-algebra/BE-algebra since $a*(b*c) = a*d = c \neq d = b*c = b*(a*c)$.

The following theorem can be proved easily.

THEOREM 3.7. *In a transitive GE-algebra (X,*,1), for all $x,y,z \in X$, the following conditions hold:*

(1) $x \leq y$ *implies* $z*x \leq z*y$.

(2) $x*y \leq (y*z)*(x*z)$.

(3) $x \leq y$ *implies* $y*z \leq x*z$.

(4) $((x*y)*y)*z \leq x*z$.

(5) $x \leq y$ *and* $y \leq z$ *imply* $x \leq z$.

DEFINITION 3.8. A GE-algebra $(X,*,1)$ is said to be commutative if it satisfies $(x*y)*y = (y*x)*x$, for all $x,y \in X$.

We can observe that every commutative GE-algebra is a transitive GE-algebra. But converse need not be true. From Example 3.6, we can observe that

$$(b * c) * c = 1 * c = c \neq b = 1 * b = (c * b) * b.$$

The following theorem shows that every commutative GE-algebra is a Hilbert algebra.

THEOREM 3.9. *If $(X, *, 1)$ is a commutative GE-algebra then X is a Hilbert algebra.*

PROOF: Let X be a commutative GE-algebra and $x, y, z \in X$. Then (i) $x*(y*x) = x*(y*(x*x)) = x*(y*1) = x*1 = 1$. (ii) Let $x*y = 1$ and $y*x = 1$. Then $(x*y)*y = y$ and hence $(y*x)*x = x$ which implies $x = y$. (iii) We know that $x*(y*z) = y*(x*z) \leq (x*y)*(x*(x*z)) = (x*y)*(x*z)$. Hence $(x*(y*z))*[(x*y)*(x*z)] = 1$. Thus X is a Hilbert algebra □

The converse of the above theorem need not be true.

Example 3.10. Let $X = \{1, a, b, c\}$ be a set with the following table.

$*$	1	a	b	c
1	1	a	b	c
a	1	1	1	1
b	1	a	1	1
c	1	a	b	1

Then $(X, *, 1)$ is a Hilbert algebra which is not a commutative GE-algebra, since

$$(a * b) * b = 1 * b = b \neq 1 = a * a = (b * a) * a.$$

THEOREM 3.11. *Every generalized Hilbert algebra is a GE-algebra.*

PROOF: Let $(X, *, 1)$ be a generalized Hilbert algebra and $x, y, z \in X$. Then $x*(y*(x*z)) = y*(x*(x*z)) = y*((x*x)*(x*z)) = y*(1*(x*z)) = y*(x*z) = x*(y*z)$. Hence $(X, *, 1)$ is a GE-algebra. □

The converse of the above theorem need not be true. From Example 3.2, we can observe that X is a GE-algebra but not a generalized Hilbert algebra since

$$d * (a * b) = d * b = c \neq b = a * c = a * (d * b).$$

Theorem 3.12. *Every commutative GE-algebra is a generalized Hilbert algebra.*

Proof: Let $(X, *, 1)$ be a commutative GE-algebra. Then X is a Hilbert algebra and hence a generalized Hilbert algebra. □

The converse of the above theorem need not be true. From Example 3.10, we can observe that X is a generalized Hilbert algebra but not a commutative GE-algebra.

Theorem 3.13. *Every self-distributive BE-algebra is a GE-algebra.*

Proof: Let $(X, *, 1)$ be a self-distributive BE-algebra and $x, y, z \in X$. Then $x * (x * y) = (x * x) * (x * y) = 1 * (x * y) = x * y$ and $x * (y * z) = x * (x * (y * z)) = x * (y * (x * z))$. Hence X is a GE-algebra. □

The converse of the above theorem need not be true. From Example 3.2, we can observe that X is a GE-algebra but not a self-distributive BE-algebra.

Theorem 3.14. *Let $(X, *, 1)$ be a BE-algebra satisfying the property $x * (x * y) = x * y$, for all $x, y \in X$. Then X is a GE-algebra.*

All non-existential results known for BE-algebras apply to GE-algebras.

Theorem 3.15. *Let X be a GE-algebra. Then X is commutative if and only if X is an implication algebra.*

Example 3.16. Let $X = \{1, a, b, c\}$ be a set with the following table.

$*$	1	a	b	c
1	1	a	b	c
a	1	1	1	1
b	1	1	1	1
c	1	1	1	1

Then $(X, *, 1)$ is a GE-algebra which is not an Implication algebra, since $(b * c) * c \neq (c * b) * b$. Hence, a commutative condition is necessary in the last theorem.

PROPOSITION 3.17. Let $(X, *, 1)$ be a GE-algebra. Then the following are equivalent.

(i) X is commutative,

(ii) X is implication algebra,

(iii) X is dual implicative BCK-algebra,

(iv) X is commutative Hilbert algebra.

4. Filters and upper sets

In this section, we introduce filters and upper sets in a GE-algebra and study their properties. We characterize filters in terms of upper sets.

DEFINITION 4.1. A subset F of X is called a filter of X if it satisfies the following:

$(GEF1)$ $1 \in F$

$(GEF2)$ if $x * y \in F$ and $x \in F$ then $y \in F$.

Obviously, $\{1\}$ and X are filters of X. A filter F is said to be proper if $F \neq X$.

Example 4.2.

(a) In Example 3.2, we can see that $I_1 = \{1, a, d\}$ and $I_2 = \{1, b, c\}$ are filters of X.

(b) In Example 3.6, we can see that $I_1 = \{1, a\}, I_2 = \{1, a, d\}, I_3 = \{1, b, c\}$ and $I_4 = \{1, d\}$ are filters of X.

We denote the set of all filters of X by $F(X)$.
The proof of the following lemma is straightforward and hence we omit the proof.

LEMMA 4.3. *If $\{F_i\}_{i\in\Lambda}$ is a family of filters of X, then $\bigcap_{i\in\Lambda} F_i$ is a filter of X.*

Since the set $F(X)$ is closed under arbitrary intersections, we have the following theorem.

Theorem 4.4. *$(F(X), \subseteq)$ is a complete lattice.*

Proposition 4.5. Let F be a filter of X. If $a \in F$ and $a \leq x$, then $x \in F$.

Theorem 4.6. *Let X be a GE-algebra and F a non-empty subset of X satisfying the following conditions:*

$(GEF3)$ $x \in X$ and $y \in F$ imply $x * y \in F$

$(GEF4)$ $x \in X, a, b \in F$ imply $(a * (b * x)) * x \in F$.

Then F is a filter of X.

Proof: Let F be a non-empty subset of X satisfying $(GEF3)$ and $(GEF4)$. Then $1 \in F$. Hence $(GEF1)$ holds. Let $x \in F$ and $x * y \in F$. Then $y = 1 * y = [(x * y) * (x * y)] * y \in F$ and hence $(GEF2)$ holds. Therefore F is a filter of X. □

Theorem 4.7. *If X is a GE-algebra and F is a filter of X, then F satisfies $(GEF3)$ and $(GEF4)$.*

Proof: Let F be a filter of X and $a \in F, x \in X$. Then $a * (x * a) = 1 \in F$ and hence, by $(GEF2)$, $x*a \in F$. Let $a, b \in F$. Since $a*[(a*(b*x))*(b*x)] = 1 \in F$, we have $(a * (b * x)) * (b * x) \in F$. Hence $b * [(a * (b * x)) * x] = b * [(a * (b * x)) * (b * x)] \in F$. Thus $(a * (b * x)) * x \in F$. □

Theorem 4.8. *Let F be a non-empty subset X. Then F is a filter of X if and only if for every $a, b \in F$ and $x \in X$, $a * (b * x) = 1$ implies $x \in F$.*

Proof: Suppose F is a filter of X and $a, b \in F, x \in X$ such that $a*(b*x) = 1$. By $(GEF1)$, we have $a*(b*x) \in F$. Then, by $(GEF2)$, we obtain $x \in F$. Conversely, assume that the condition holds. Let $a \in F$. Then $a*(a*1) = 1$ implies $1 \in F$. Suppose $x * y \in F$ and $x \in F$. Then $x * [(x * y) * y] = 1$ implies $y \in F$. Hence F is a filter of X. □

Corollary 4.9. Let F be a non-empty subset of X. Then F is a filter of X if and only if for every $a_i \in F (i \in \mathbb{N})$ and $x \in X, a_n * (\cdots * (a_1 * x) \cdots) = 1$ implies $x \in F$.

Lemma 4.10. *Let F be a filter of X. Then $(a * x) * x \in F$ for all $a \in F$ and $x \in X$.*

Proposition 4.11. A non-empty subset F of a GE-algebra X is a filter of X if and only if it satisfies $(i) 1 \in F$ (ii) $x * (y * z) \in F, y \in F$ implies $x * z \in F$ for all $x, y, z \in X$.

Let X be a GE-algebra and $x, y \in X$. Define

$$U(x) = \{z \in X \mid x * z = 1\} \text{ and } U(x, y) = \{z \in \mid x * (y * z) = 1\}$$

The set $U(x)$ (resp. $U(x, y)$) is called an upper set of x (resp. of x and y). We can observe that $1, x \in U(x)$ and $1, x, y \in U(x, y)$. Also, $U(1) = \{1\}$ is always a filter of X.

The following theorem can be proved easily.

THEOREM 4.12. *Let X be a GE-algebra. Then, for any $x, y \in X$,*

(*i*) *$U(x, y)$ is a subalgebra of X.*

(*ii*) $U(x) = \bigcap_{y \in X} U(x, y)$.

(*iii*) $U(x, y) = U(y, x)$.

COROLLARY 4.13. Let F be a non-empty subset of X. Then F is a filter of X if and only if $U(x, y) \subseteq F$ for all $x, y \in F$.

PROOF: Let F be a filter of X and $x, y \in F, z \in U(x, y)$. Then $x * (y * z) = 1 \in F$ and hence $z \in F$. So that $U(x, y) \subseteq F$. Conversely, assume that $U(x, y) \subseteq F$ for all $x, y \in F$. Since F is non-empty, we have $z \in F$ such that $1 \in U(z, z) \subseteq F$. Hence $(GEF1)$ holds. Let $x * y \in F$ and $x \in F$. Then $y \in U(x * y, x) \subseteq F$. Thus $(GEF2)$ holds. Therefore F is a filter of X. □

PROPOSITION 4.14. Let X be a GE-algebra and F a filter of X. Then, for any $x, y \in F$,

(*i*) $U(x) \subseteq F$.

(*ii*) $F = \bigcup_{x, y \in F} U(x, y)$.

THEOREM 4.15. *Let X be a transitive GE-algebra and $x, y \in X$. Then, for any $x, y \in X$,*

(*i*) *$U(x, y)$ is a filter of X.*

(*ii*) *$U(x)$ is a filter of X.*

(iii) $x \leq y$ *if and only if* $U(y) \subseteq U(x)$.

(iv) $x \leq y$ *and* $y \leq x$ *if and only if* $U(x) = U(y)$.

Finally, we conclude this section with the following theorem.

THEOREM 4.16. *Let* X *be a transitive GE-algebra and* $x, y \in X$. *Then* $y \in U(x)$ *if and only if* $U(x) = U(x, y)$.

5. Congruence kernels

In this section, we give a characterization of congruence kernels in a transitive GE-algebra. Let θ be a binary relation on a GE-algebra $(X, *, 1)$. We denote $\{x \in X \mid (x, 1) \in \theta\}$ by $[1]_\theta$. If θ is a congruence relation on X then $[1]_\theta$ is called a congruence kernel.

LEMMA 5.1. *If* θ *is a congruence relation on* X *then kernel* $[1]_\theta$ *is a filter of* X.

PROOF: Clearly $1 \in [1]_\theta$. Suppose $x \in [1]_\theta$ and $x * y \in [1]_\theta$. Then $(x, 1), (x * y, 1) \in \theta$ and hence $(x * y, y) = (x * y, 1 * y) \in \theta$. By symmetry of θ, $(y, x * y) \in \theta$. Therefore, by transitivity of θ, we obtain $(y, 1) \in \theta$ proving $y \in [1]_\theta$. □

THEOREM 5.2. *Let* $(X, *, 1)$ *be a transitive GE-algebra. Then every filter* F *of* X *is a kernel of a congruence* θ_F *given by*

$$(x, y) \in \theta_F \text{ if and only if } x * y \in F \text{ and } y * x \in F.$$

Moreover, θ_F *is the greatest congruence on* X *having the kernel* F.

PROOF: Let F be a filter of X. Since $1 \in F$, we have θ_F is reflexive. Clearly θ_F is symmetric. We prove transitivity of θ_F. Let $(x, y) \in \theta_F$ and $(y, z) \in \theta_F$. Then $x * y, y * x, y * z, z * y \in F$. Hence, by Theorem 3.7(2) and by Proposition 4.5, $(y * z) * (x * z) \in F$. Therefore $x * z \in F$. Similarly, we can prove that $z * x \in F$. Thus $(x, z) \in \theta_F$. Now, we prove the substitution property of θ_F. Let $(x, y) \in \theta_F$ and $(u, v) \in \theta_F$. Then $x*y, y*x, u*v, v*u \in F$ and hence, by Theorem 3.7(2) and by Proposition 4.5, $(x*u)*(y*u) \in F$ and $(y*u)*(x*u) \in F$. Therefore, $(x*u, y*u) \in \theta_F$. Since X is transitive, we have, by Proposition 4.5, $(y * u) * (y * v) \in F$ and

$(y * v) * (y * u) \in F$. Hence $(y * u, y * v) \in \theta_F$. By transitivity of θ_F, we conclude $(x * u, y * v) \in \theta_F$. Thus θ_F is a congruence relation on X.

If $x \in F$ then $1 * x = x \in F$ and $x * 1 = 1 \in F$. Therefore $(x, 1) \in \theta_F$, i.e., $x \in [1]_{\theta_F}$. Conversely, let $x \in [1]_{\theta_F}$. Then $(x, 1) \in \theta_F$ and hence $x = 1 * x \in F$ which shows that $F = [1]_{\theta_F}$. Thus F is the kernel of the congruence θ_F.

Finally, if ψ is a congruence relation on X such that $[1]_\psi = F$ then for $(x, y) \in \psi$ we have $(x*y, 1) = (x*y, y*y) \in \psi$ and $(y*x, 1) = (y*x, y*y) \in \psi$ thus $x * y \in F$ and $y * x \in F$ which gives $(x, y) \in \theta_F$. Hence $\psi \subseteq \theta_F$ i.e., θ_F is the greatest congruence relation of X having the kernel F. $\square$

The following example shows that filters need not be congruence kernels in a GE-algebra

Example 5.3. Let $X = \{1, a, b, c\}$ be a set with the following table.

$*$	1	a	b	c
1	1	a	b	c
a	1	1	1	c
b	1	a	1	1
c	1	a	b	1

Then $(X, *, 1)$ is a GE-algebra. But it is not transitive since $(b*c)*[(a*b)*(a*c)] = c \neq 1$. Clearly, $F = \{1, a, b\}$ is a filter of X. Let $(b, a) \in \theta$ for some congruence relation θ on X. Then $(1, c) \in \theta$ and hence $c \in [1]_\theta \neq \{1, a, b\}$. Thus F is not a congruence kernel.

Finally, we conclude this section with the following theorem.

Theorem 5.4. *Let $(X, *, 1)$ be a transitive GE-algebra. Then filters of X coincide with congruence kernels.*

6. Conclusion and future work

Hilbert algebras represent the algebraic counterpart of the implicative fragment of intuitionistic propositional logic. In fact, Hilbert algebras are an algebraic counterpart of positive implicational calculus. Various type of generalization of algebraic structures were defined in the literature.

In this paper, we have introduced the concept of GE-algebras as a generalization of Hilbert algebras and studied their properties. In addition, we have considered filters and upper sets in a GE-algebra and characterized filters in terms of upper sets. We characterized congruence kernels in a transitive GE-algebra. Finally, we show that filters and congruence kernels coincide in a transitive GE-algebra.

We hope this work would serve as a foundation for further studies on the structure of GE-algebras like fuzzy GE-algebras, soft GE-algebras and hyper GE-algebras.

Acknowledgements. The authors are grateful to the referee for his valuable comments and suggestions for the improvement of the paper.

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Bulletin of the **S**ection of **L**ogic
Volume 50/1 (2021), pp. 97–123
https://doi.org/10.18778/0138-0680.2020.28

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SOJU FILTERS IN HOOP ALGEBRAS

Abstract

The notions of (implicative) soju filters in a hoop algebra are introduced, and related properties are investigated. Relations between a soju sub-hoop, a soju filter and an implicative soju filter are discussed. Conditions for a soju filter to be implicative are displayed, and characterizations of an implicative soju filters are considered. The extension property of an implicative soju filter is established.

Keywords: Soju sub-hoop, soju filter, implicative soju filter.

2010 Mathematical Subject Classification: 06F35, 03G25, 08A72.

1. Introduction

It is well-known that an intuitionistic fuzzy set is a generalization of a fuzzy set, and it is introduced by Attanassov [1]. Molodtsov [15] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties. For more information on intuitionistic fuzzy sets and soft sets, see [1, 2, 14, 15]. By combining intuitionistic fuzzy set and soft set, Jun et al. [11] introduced a new structure, so called soju structure, and they applied it to BCK/BCI-algebras. Xin et al. [16] introduced the notion of positive implicative soju ideal in BCK-algebra, and investigate related properties. We discuss relations between soju ideal and positive implicative soju ideal,

Presented by: Jie Fang
Received: July 27, 2019
Published online: December 30, 2020

and established characterizations of positive implicative soju ideal. They constructed extension property for positive implicative soju ideal.

In this article, we apply the soju structure to hoop algebras, which is introduced by Bosbach in [8, 9] and studied [12, 4, 3, 5, 6, 7, 16]. We introduce the concepts of soju sub-hoops and (implicative) soju filters in a hoop algebra, and investigate related properties. We discuss the homomorphic preimage and image of soju sub-hoops. We consider relations between a soju sub-hoop, a soju filter and an implicative soju filter. We provide conditions for a soju filter to be implicative, and characterize an implicative soju filter. We establish the extension property of an implicative soju filter.

2. Preliminaries

By a *hoop* (or *hoop algebra*) we mean an algebra $(H, \odot, \to, 1)$ in which $(H, \odot, 1)$ is a commutative monoid and the following assertions are valid.

(H1) $x \to x = 1$,

(H2) $x \odot (x \to y) = y \odot (y \to x)$,

(H3) $x \to (y \to z) = (x \odot y) \to z$

for all $x, y, z \in H$. We define a relation "$\leq$" on a hoop H by

$$(\forall x, y \in H)(x \leq y \;\Leftrightarrow\; x \to y = 1). \tag{2.1}$$

It is easy to see that $(H, \leq)$ is a poset.

A nonempty subset S of a hoop algebra H is called a *sub-hoop* of H if it satisfies:

$$(\forall x, y \in S)(x \odot y \in S,\; x \to y \in S). \tag{2.2}$$

Note that every sub-hoop contains the element 1.

Proposition 2.1 ([10]). Let $(H, \odot, \to, 1)$ be a hoop algebra. For any $x, y, z \in H$, the following conditions hold:

$(a1)$ $(H, \leq)$ is a meet-semilattice with $x \wedge y = x \odot (x \to y)$,

$(a2)$ $x \odot y \leq z$ if and anly if $x \leq y \to z$,

$(a3)$ $x \odot y \leq x, y$ and $x^n \leq x$ for any $n \in \mathbb{N}$,

$(a4)$ $x \leq y \to x$,

$(a5)$ $1 \to x = x$ and $x \to 1 = 1$,

$(a6)$ $x \odot (x \to y) \leq y$, $x \odot y \leq x \wedge y \leq x \to y$,

$(a7)$ $x \to y \leq (y \to z) \to (x \to z)$,

$(a8)$ $x \leq y$ implies $x \odot z \leq y \odot z$, $z \to x \leq z \to y$ and $y \to z \leq x \to z$,

$(a9)$ $x \to (y \to z) = y \to (x \to z)$.

A nonempty subset F of a hoop algebra H is called

- a *filter* of H (see [10]) if the following assertions are valid.

$$(\forall x, y \in H)(x, y \in F \;\Rightarrow x \odot y \in F), \tag{2.3}$$
$$(\forall x, y \in H)(x \in F,\; x \leq y \;\Rightarrow\; y \in F). \tag{2.4}$$

- an *implicative filter* of H (see [13]) if the following assertions are valid.

$$1 \in F, \tag{2.5}$$
$$(\forall x, y, z \in H)(x \to ((y \to z) \to y) \in F,\; x \in F \;\Rightarrow\; y \in F). \tag{2.6}$$

Note that the conditions (2.3) and (2.4) means that F is closed under the operation $\odot$ and F is upward closed, respectively.

Note that a subset F of a hoop algebra H is a filter of H if and only if F satisfies the condition (2.5) and

$$(\forall x, y \in H)\,(x \to y \in F,\; x \in F \;\Rightarrow\; y \in F)\,. \tag{2.7}$$

For more information on intuitionistic fuzzy sets and soft sets, see [4], [3] and [16].

3. Soju sub-hoops and soju filters

In what follows, let U be an initial universe set unless otherwise specified.

Definition 3.1 ([11]). Let E be a set of parameters. For any subset A of E, let $\sigma := (\mu_\sigma, \gamma_\sigma)$ be an intuitionistic fuzzy set in A and $(\tilde{F}, A)$ be a soft set over U. Then a pair $(A, \langle\sigma; \tilde{F}\rangle)$ is called a *soju structure* over $([0,1], U)$.

Given a soju structure $(E, \langle\sigma; \tilde{F}\rangle)$ over $([0,1], U)$, $\alpha \in 2^U$ and $(t,s) \in [0,1] \times [0,1]$ with $t + s \leq 1$, consider the following sets:

$$U(\mu_\sigma; t) := \{x \in E \mid \mu_\sigma(x) \geq t\},$$
$$L(\gamma_\sigma; s) := \{x \in E \mid \gamma_\sigma(x) \leq s\},$$
$$i(\tilde{F}; \alpha) := \{x \in E \mid \tilde{F}(x) \supseteq \alpha\},$$

which are called *soju level sets* of $(E, \langle\sigma; \tilde{F}\rangle)$.

Definition 3.2. Let A be a subset of a hoop algebra E. A soju structure $(A, \langle\sigma; \tilde{F}\rangle)$ over $([0,1], U)$ is called a *soju sub-hoop* based on A (briefly, *soju A-sub-hoop*) of E if the following condition is valid.

$$(\forall x, y \in A)\left(x \bullet y \in A \Rightarrow \begin{cases} \mu_\sigma(x \bullet y) \geq \min\{\mu_\sigma(x), \mu_\sigma(y)\} \\ \gamma_\sigma(x \bullet y) \leq \max\{\gamma_\sigma(x), \gamma_\sigma(y)\} \\ \tilde{F}(x \bullet y) \supseteq \tilde{F}(x) \cap \tilde{F}(y) \end{cases} \right) \tag{3.1}$$

for $\bullet \in \{\odot, \rightarrow\}$.

Example 3.3. Consider a hoop algebra $(E, \odot, \rightarrow, 1)$ in which $E = \{0, a, b, 1\}$ with binary operations $\rightarrow$ and $\odot$ which are given as follows:

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

(1) The set $A = \{0, a, 1\}$ is a sub-hoop of E. Define a soju structure $(A, \langle\sigma; \tilde{F}\rangle)$ over $([0,1], U) = ([0,1], \mathbb{Z})$ by Table 1.
By routine calculations, we know that $(A, \langle\sigma; \tilde{F}\rangle)$ is a soju A-sub-hoop of E.

(2) Define a soju structure $(E, \langle\sigma; \tilde{F}\rangle)$ over $([0,1], U) = ([0,1], \mathbb{Z})$ by Table 2.
It is routine to verify that $(E, \langle\sigma; \tilde{F}\rangle)$ is a soju sub-hoop of E.

Proposition 3.4. Let A be a sub-hoop of a hoop algebra E. Every soju A-sub-hoop $(A, \langle\sigma; \tilde{F}\rangle)$ of E satisfies the following condition.

Table 1. Tabular representation of $(A, \langle\sigma; \tilde{F}\rangle)$

A	$\sigma(x) = (\mu_\sigma(x), \gamma_\sigma(x))$	$\tilde{F}(x)$
0	$(0.75, 0.2)$	$4\mathbb{Z}$
a	$(0.45, 0.5)$	$4\mathbb{N}$
1	$(0.85, 0.1)$	$2\mathbb{Z}$

Table 2. Tabular representation of $(E, \langle\sigma; \tilde{F}\rangle)$

E	$\sigma(x) = (\mu_\sigma(x), \gamma_\sigma(x))$	$\tilde{F}(x)$
0	$(0.4, 0.6)$	$2\mathbb{Z}$
a	$(0.4, 0.5)$	$6\mathbb{N}$
b	$(0.6, 0.3)$	$3\mathbb{Z}$
1	$(0.8, 0.1)$	$\mathbb{Z}$

$$(\forall x \in A)\left(\mu_\sigma(1) \geq \mu_\sigma(x), \gamma_\sigma(1) \leq \gamma_\sigma(x), \tilde{F}(1) \supseteq \tilde{F}(x)\right), \tag{3.2}$$

Proof: Since $x \to x = 1$ for all $x \in E$, it is straightforward by (3.1). □

Theorem 3.5. *Given a hoop algebra E, the soju structure $(E, \langle\sigma; \tilde{F}\rangle)$ over $([0,1], U)$ is a soju sub-hoop of E if and only if its nonempty soju level sets $U(\mu_\sigma; t)$, $L(\gamma_\sigma; s)$ and $i(\tilde{F}; \alpha)$ are sub-hoops of E for all $\alpha \in 2^U$ and $(t, s) \in [0,1] \times [0,1]$ with $t + s \leq 1$.*

Proof: Assume that $(E, \langle\sigma; \tilde{F}\rangle)$ is a soju sub-hoop of E and let $\alpha \in 2^U$ and $(t, s) \in [0,1] \times [0,1]$ be such that $t + s \leq 1$, and $U(\mu_\sigma; t)$, $L(\gamma_\sigma; s)$ and $i(\tilde{F}; \alpha)$ are non-empty. Let $x, y \in E$ be such that $x, y \in U(\mu_\sigma; t) \cap L(\gamma_\sigma; s) \cap i(\tilde{F}; \alpha)$. Then $\mu_\sigma(x) \geq t$, $\mu_\sigma(y) \geq t$, $\gamma_\sigma(x) \leq s$, $\gamma_\sigma(y) \leq s$, $\tilde{F}(x) \supseteq \alpha$ and $\tilde{F}(y) \supseteq \alpha$. It follows from (3.1) that

$$\mu_\sigma(x \bullet y) \geq \min\{\mu_\sigma(x), \mu_\sigma(y)\} \geq t,$$
$$\gamma_\sigma(x \bullet y) \leq \max\{\gamma_\sigma(x), \gamma_\sigma(y)\} \leq s,$$
$$\tilde{F}(x \bullet y) \supseteq \tilde{F}(x) \cap \tilde{F}(y) \supseteq \alpha$$

for $\bullet \in \{\odot, \to\}$. Hence $x \bullet y \in U(\mu_\sigma; t) \cap L(\gamma_\sigma; s) \cap i(\tilde{F}; \alpha)$, and therefore $U(\mu_\sigma; t)$, $L(\gamma_\sigma; s)$ and $i(\tilde{F}; \alpha)$ are sub-hoops of E.

Conversely, suppose that the nonempty soju level sets $U(\mu_\sigma;t)$, $L(\gamma_\sigma;s)$ and $i(\tilde{F};\alpha)$ of $(E,\,\langle\sigma;\tilde{F}\rangle)$ are sub-hoops of E for all $\alpha\in 2^U$ and $(t,s)\in[0,1]\times[0,1]$ with $t+s\le 1$. For any $x,y\in E$, let $t_x,t_y,\ s_x,s_y\in[0,1]$ and $\alpha_x,\alpha_y\in 2^U$ be such that $\mu_\sigma(x)=t_x$, $\mu_\sigma(y)=t_y$, $\gamma_\sigma(x)=s_x$, $\gamma_\sigma(y)=s_y$, $\tilde{F}(x)=\alpha_x$ and $\tilde{F}(y)=\alpha_y$. If we take $t:=\min\{t_x,t_y\}$, $s:=\max\{s_x,s_y\}$ and $\alpha:=\alpha_x\cap\alpha_y$, then $x,y\in U(\mu_\sigma;t)\cap L(\gamma_\sigma;s)\cap i(\tilde{F};\alpha)$. Thus $x\bullet y\in U(\mu_\sigma;t)\cap L(\gamma_\sigma;s)\cap i(\tilde{F};\alpha)$, and so

$$\begin{aligned}
&\mu_\sigma(x\bullet y)\ge t=\min\{t_x,t_y\}=\min\{\mu_\sigma(x),\mu_\sigma(y)\},\\
&\gamma_\sigma(x\bullet y)\le s=\max\{s_x,s_y\}=\max\{\gamma_\sigma(x),\gamma_\sigma(y)\},\\
&\tilde{F}(x\bullet y)\supseteq\alpha=\alpha_x\cap\alpha_y=\tilde{F}(x)\cap\tilde{F}(y)
\end{aligned}$$

for $\bullet\in\{\odot,\rightarrow\}$. Therefore $(E,\,\langle\sigma;\tilde{F}\rangle)$ is a soju sub-hoop of E. □

We define the image and preimage of soju structures. Let E and K be nonempty sets and $f:E\to K$ be a mapping.

Definition 3.6. (1) If $(K,\langle\tau;\tilde{G}\rangle)$ is a soju structure over $([0,1],U)$, then the *preimage* of $(K,\langle\tau;\tilde{G}\rangle)$ under f is denoted by $f^{-1}(K,\,\langle\tau;\tilde{G}\rangle)$ and is defined to be a soju structure $(E,\,\langle\sigma;\tilde{F}\rangle)$ over $([0,1],U)$ with $\mu_\sigma=f^{-1}(\mu_\tau)$, $\gamma_\sigma=f^{-1}(\gamma_\tau)$ and

$$\tilde{F}=f^{-1}(\tilde{G})=\{(x,f^{-1}(\tilde{G})(x))\mid x\in E,f^{-1}(\tilde{G})(x)\in 2^U\}$$

where $\mu_\sigma(x)=f^{-1}(\mu_\tau)(x)=\mu_\tau(f(x))$, $\gamma_\sigma(x)=f^{-1}(\gamma_\tau)(x)=\gamma_\tau(f(x))$, and $f^{-1}(\tilde{G})(x)=\tilde{G}(f(x))$ for all $x\in E$.

(2) If $(E,\,\langle\sigma;\tilde{F}\rangle)$ is a soju structure over $([0,1],U)$, then the *image* of $(E,\,\langle\sigma;\tilde{F}\rangle)$ under f is denoted by $f(E,\,\langle\sigma;\tilde{F}\rangle)$ and is defined to be a soju structure $(K,\langle\tau;\tilde{G}\rangle)$ over $([0,1],U)$ with $\mu_\tau=f(\mu_\sigma)$, $\gamma_\tau=f(\gamma_\sigma)$ and $\tilde{G}=f(\tilde{F})$ where

$$\mu_\tau(y)=f(\mu_\sigma)(y)=\begin{cases}\sup\limits_{x\in f^{-1}(y)}\mu_\sigma(x) & \text{if } f^{-1}(y)\neq\emptyset,\\ 0 & \text{otherwise,}\end{cases}$$

$$\gamma_\tau(y)=f(\gamma_\sigma)(y)=\begin{cases}\inf\limits_{x\in f^{-1}(y)}\gamma_\sigma(x) & \text{if } f^{-1}(y)\neq\emptyset,\\ 1 & \text{otherwise,}\end{cases}$$

and

$$\tilde{G}(y) = f(\tilde{F})(y) = \begin{cases} \bigcup_{x \in f^{-1}(y)} \tilde{F}(x) & \text{if } f^{-1}(y) \neq \emptyset, \\ \emptyset & \text{otherwise} \end{cases}$$

for all $y \in K$.

THEOREM 3.7. *Let $f : E \to K$ be a homomorphism of hoop algebras and let $(K, \langle\tau; \tilde{G}\rangle)$ be a soju structure over $([0, 1], U)$. If $(K, \langle\tau; \tilde{G}\rangle)$ is a soju sub-hoop of K, then the preimage of $(K, \langle\tau; \tilde{G}\rangle)$ under f is a soju sub-hoop of E.*

PROOF: For any $\bullet \in \{\odot, \to\}$ and any $x_1, x_2 \in E$, we have

$$\begin{aligned} \mu_\sigma(x_1 \bullet x_2) &= f^{-1}(\mu_\tau)(x_1 \bullet x_2) = \mu_\tau(f(x_1 \bullet x_2)) \\ &= \mu_\tau(f(x_1) \bullet f(x_2)) \\ &\geq \min\{\mu_\tau(f(x_1)), \mu_\tau(f(x_2))\} \\ &= \min\{f^{-1}(\mu_\tau)(x_1), f^{-1}(\mu_\tau)(x_2)\} \\ &= \min\{\mu_\sigma(x_1), \mu_\sigma(x_2)\}, \end{aligned}$$

$$\begin{aligned} \gamma_\sigma(x_1 \bullet x_2) &= f^{-1}(\gamma_\tau)(x_1 \bullet x_2) = \gamma_\tau(f(x_1 \bullet x_2)) \\ &= \gamma_\tau(f(x_1) \bullet f(x_2)) \\ &\leq \max\{\gamma_\tau(f(x_1)), \gamma_\tau(f(x_2))\} \\ &= \max\{f^{-1}(\gamma_\tau)(x_1), f^{-1}(\gamma_\tau)(x_2)\} \\ &= \max\{\gamma_\sigma(x_1), \gamma_\sigma(x_2)\}, \end{aligned}$$

and

$$\begin{aligned} \tilde{F}(x_1 \bullet x_2) &= f^{-1}(\tilde{G})(x_1 \bullet x_2) = \tilde{G}(f(x_1 \bullet x_2)) \\ &= \tilde{G}(f(x_1) \bullet f(x_2)) \\ &\supseteq \tilde{G}(f(x_1)) \cap \tilde{G}(f(x_2)) \\ &= f^{-1}(\tilde{G})(x_1) \cap f^{-1}(\tilde{G})(x_2) \\ &= \tilde{F}(x_1) \cap \tilde{F}(x_2). \end{aligned}$$

Therefore $(E, \langle\sigma; \tilde{F}\rangle) = f^{-1}(K, \langle\tau; \tilde{G}\rangle)$ is a soju sub-hoop of E. □

THEOREM 3.8. *Let $f : E \to K$ be a homomorphism of hoop algebras and let $(E, \langle\sigma; \tilde{F}\rangle)$ be a soju structure over $([0, 1], U)$. If $(E, \langle\sigma; \tilde{F}\rangle)$ is a soju sub-hoop of E and f is injective, then the image of $(E, \langle\sigma; \tilde{F}\rangle)$ under f is a soju sub-hoop of K.*

Proof: Let $y_1, y_2 \in K$ and $\bullet \in \{\odot, \rightarrow\}$. If at least one of $f^{-1}(y_1)$ and $f^{-1}(y_2)$ is empty, then it is clear that

$$\mu_\tau(y_1 \bullet y_2) \geq \min\{\mu_\tau(y_1), \mu_\tau(y_2)\},$$

$$\gamma_\tau(y_1 \bullet y_2) \leq \max\{\gamma_\tau(y_1), \gamma_\tau(y_2)\},$$

and $\tilde{G}(y_1 \bullet y_2) \supseteq \tilde{G}(y_1) \cap \tilde{G}(y_2)$. Assume that $f^{-1}(y_1)$ and $f^{-1}(y_2)$ are nonempty. Then

$$\begin{aligned}
\min\{\mu_\tau(y_1), \mu_\tau(y_2)\} &= \min\{f(\mu_\sigma)(y_1), f(\mu_\sigma)(y_2)\} \\
&= \min\left\{ \sup_{x_1 \in f^{-1}(y_1)} \mu_\sigma(x_1), \sup_{x_2 \in f^{-1}(y_2)} \mu_\sigma(x_2) \right\} \\
&= \sup_{\substack{x_1 \in f^{-1}(y_1) \\ x_2 \in f^{-1}(y_2)}} \min\{\mu_\sigma(x_1), \mu_\sigma(x_2)\} \\
&\leq \sup_{\substack{x_1 \in f^{-1}(y_1) \\ x_2 \in f^{-1}(y_2)}} \mu_\sigma(x_1 \bullet x_2) \\
&= \sup_{x \in f^{-1}(y_1 \bullet y_2)} \mu_\sigma(x) \\
&= f(\mu_\sigma)(y_1 \bullet y_2) = \mu_\tau(y_1 \bullet y_2),
\end{aligned}$$

$$\begin{aligned}
\max\{\gamma_\tau(y_1), \gamma_\tau(y_2)\} &= \max\{f(\gamma_\sigma)(y_1), f(\gamma_\sigma)(y_2)\} \\
&= \max\left\{ \inf_{x_1 \in f^{-1}(y_1)} \gamma_\sigma(x_1), \inf_{x_2 \in f^{-1}(y_2)} \gamma_\sigma(x_2) \right\} \\
&= \inf_{\substack{x_1 \in f^{-1}(y_1) \\ x_2 \in f^{-1}(y_2)}} \max\{\gamma_\sigma(x_1), \gamma_\sigma(x_2)\} \\
&\geq \inf_{\substack{x_1 \in f^{-1}(y_1) \\ x_2 \in f^{-1}(y_2)}} \gamma_\sigma(x_1 \bullet x_2) \\
&= \inf_{x \in f^{-1}(y_1 \bullet y_2)} \gamma_\sigma(x) \\
&= f(\gamma_\sigma)(y_1 \bullet y_2) = \gamma_\tau(y_1 \bullet y_2),
\end{aligned}$$

and

$$\begin{aligned}
\tilde{G}(y_1) \cap \tilde{G}(y_2) &= f(\tilde{F}(y_1)) \cap f(\tilde{F}(y_2)) \\
&= \left(\bigcup_{x_1 \in f^{-1}(y_1)} \tilde{F}(x_1) \right) \cap \left(\bigcup_{x_2 \in f^{-1}(y_2)} \tilde{F}(x_2) \right) \\
&= \bigcup_{\substack{x_1 \in f^{-1}(y_1) \\ x_2 \in f^{-1}(y_2)}} (\tilde{F}(x_1) \cap \tilde{F}(x_2)) \\
&\subseteq \bigcup_{\substack{x_1 \in f^{-1}(y_1) \\ x_2 \in f^{-1}(y_2)}} \tilde{F}(x_1 \bullet x_2) \\
&= \bigcup_{x \in f^{-1}(y_1 \bullet y_2)} \tilde{F}(x) \\
&= f(\tilde{F})(y_1 \bullet y_2) = \tilde{G}(y_1 \bullet y_2).
\end{aligned}$$

Therefore $(K, \langle \tau; \tilde{G} \rangle)$, the image of $(E, \langle \sigma; \tilde{F} \rangle)$ under f, is a soju sub-hoop of K. □

Definition 3.9. Let A be a subset of a hoop algebra E. A soju structure $(A, \langle \sigma; \tilde{F} \rangle)$ over $([0,1], U)$ is called a *soju filter* based on A (briefly, *soju A-filter*) of E if the following condition is valid.

$$(\forall x, y \in A) \left(x \odot y \in A \Rightarrow \begin{cases} \mu_\sigma(x \odot y) \geq \min\{\mu_\sigma(x), \mu_\sigma(y)\} \\ \gamma_\sigma(x \odot y) \leq \max\{\gamma_\sigma(x), \gamma_\sigma(y)\} \\ \tilde{F}(x \odot y) \supseteq \tilde{F}(x) \cap \tilde{F}(y) \end{cases} \right) \tag{3.3}$$

$$(\forall x, y \in A) \left(x \leq y \Rightarrow \begin{cases} \mu_\sigma(x) \leq \mu_\sigma(y), \\ \gamma_\sigma(x) \geq \gamma_\sigma(y) \\ \tilde{F}(x) \subseteq \tilde{F}(y) \end{cases} \right). \tag{3.4}$$

A soju E-filter is simply called a *soju filter*.

Example 3.10. Consider the hoop algebra $(E, \odot, \rightarrow, 1)$ in Example 3.3. Define a soju structure $(E, \langle \sigma; \tilde{F} \rangle)$ over $([0,1], U) = ([0,1], \mathbb{Z})$ by Table 3. It is routine to verify that $(E, \langle \sigma; \tilde{F} \rangle)$ is a soju filter of E.

Table 3. Tabular representation of $(E, \langle\sigma; \tilde{F}\rangle)$

E	$\sigma(x) = (\mu_\sigma(x), \gamma_\sigma(x))$	$\tilde{F}(x)$
0	$(0.3, 0.6)$	$3\mathbb{N}$
a	$(0.3, 0.5)$	$3\mathbb{N}$
b	$(0.6, 0.3)$	$3\mathbb{Z}$
1	$(0.7, 0.1)$	$\mathbb{Z}$

Theorem 3.11. *Let A be a sub-hoop of a hoop algebra E. Then a soju structure $(A, \langle\sigma; \tilde{F}\rangle)$ over $([0,1], U)$ is a soju A-filter of E if and only if it satisfies* (3.2) *and*

$$(\forall x, y \in A)\left(\begin{array}{l} \mu_\sigma(y) \geq \min\{\mu_\sigma(x), \mu_\sigma(x \to y)\} \\ \gamma_\sigma(y) \leq \max\{\gamma_\sigma(x), \gamma_\sigma(x \to y)\} \\ \tilde{F}(y) \supseteq \tilde{F}(x) \cap \tilde{F}(x \to y) \end{array}\right). \tag{3.5}$$

Proof: Let $(A, \langle\sigma; \tilde{F}\rangle)$ be a soju A-filter of E. Since $x \leq 1$ for all $x \in E$, it follows from (3.4) that we have (3.2). For any $x, y \in A$, we get $x \odot (x \to y) \leq y$. Using (3.3) and (3.4), we have

$$\begin{aligned} &\mu_\sigma(y) \geq \mu_\sigma(x \odot (x \to y)) \geq \min\{\mu_\sigma(x), \mu_\sigma(x \to y)\}, \\ &\gamma_\sigma(y) \leq \gamma_\sigma(x \odot (x \to y)) \leq \max\{\gamma_\sigma(x), \gamma_\sigma(x \to y)\}, \\ &\tilde{F}(y) \supseteq \tilde{F}(x \odot (x \to y)) \supseteq \tilde{F}(x) \cap \tilde{F}(x \to y) \end{aligned}$$

which proves (3.5).

Conversely, suppose that a soju structure $(A, \langle\sigma; \tilde{F}\rangle)$ over $([0,1], U)$ satisfies (3.2) and (3.5). Let $x, y \in A$. Then $x \odot y \in A$ since A is a sub-hoop of E. Since

$$x \to (y \to (x \odot y)) = (x \odot y) \to (x \odot y) = 1 \in A,$$

it follows from (3.2) and (3.5) that

$$\begin{aligned} \mu_\sigma(x \odot y) &\geq \min\{\mu_\sigma(y), \mu_\sigma(y \to (x \odot y))\} \\ &\geq \min\{\mu_\sigma(y), \min\{\mu_\sigma(x), \mu_\sigma(x \to (y \to (x \odot y)))\}\} \\ &= \min\{\mu_\sigma(y), \min\{\mu_\sigma(x), \mu_\sigma(1)\}\} \\ &= \min\{\mu_\sigma(y), \mu_\sigma(x)\}, \end{aligned}$$

$$\begin{aligned}\gamma_\sigma(x \odot y) &\le \max\{\gamma_\sigma(y), \gamma_\sigma(y \to (x \odot y))\}\\ &\le \max\{\gamma_\sigma(y), \max\{\gamma_\sigma(x), \gamma_\sigma(x \to (y \to (x \odot y)))\}\}\\ &= \max\{\gamma_\sigma(y), \max\{\gamma_\sigma(x), \gamma_\sigma(1)\}\}\\ &= \max\{\gamma_\sigma(y), \gamma_\sigma(x)\},\end{aligned}$$

and

$$\begin{aligned}\tilde{F}(x \odot y) &\supseteq \tilde{F}(y) \cap \tilde{F}(y \to (x \odot y))\\ &\supseteq \tilde{F}(y) \cap (\tilde{F}(x) \cap \tilde{F}(x \to (y \to (x \odot y))))\\ &= \tilde{F}(y) \cap (\tilde{F}(x) \cap \tilde{F}(1))\\ &= \tilde{F}(y) \cap \tilde{F}(x).\end{aligned}$$

Let $x, y \in A$ be such that $x \le y$. Then $x \to y = 1 \in A$, and so

$$\begin{aligned}&\mu_\sigma(y) \ge \min\{\mu_\sigma(x), \mu_\sigma(x \to y)\} = \min\{\mu_\sigma(x), \mu_\sigma(1)\} = \mu_\sigma(x),\\ &\gamma_\sigma(y) \le \max\{\gamma_\sigma(x), \gamma_\sigma(x \to y)\} = \max\{\gamma_\sigma(x), \gamma_\sigma(1)\} = \gamma_\sigma(x),\\ &\tilde{F}(y) \supseteq \tilde{F}(x) \cap \tilde{F}(x \to y) = \tilde{F}(x) \cap \tilde{F}(1) = \tilde{F}(x).\end{aligned}$$

Therefore $(A, \langle\sigma; \tilde{F}\rangle)$ is a soju A-filter of E. □

THEOREM 3.12. *For any sub-hoop A of a hoop algebra E, every soju A-filter is a soju A-sub-hoop.*

PROOF: Straightforward. □

The following example shows that the converse of Theorem 3.12 is not true in general.

Example 3.13. Consider a hoop algebra $(E, \odot, \to, 1)$ in which $E = \{0, a, b, 1\}$ with binary operations $\to$ and $\odot$ which are given as follows:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	a	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	b	1	1
1	0	a	b	1

Define a soju structure $(E, \langle\sigma; \tilde{F}\rangle)$ over $([0, 1], U) = ([0, 1], \mathbb{Z})$ by Table 4.

Table 4. Tabular representation of $(A, \langle\sigma;\tilde{F}\rangle)$

A	$\sigma(x)=(\mu_\sigma(x),\gamma_\sigma(x))$	$\tilde{F}(x)$
0	$(0.65, 0.30)$	$8\mathbb{N}$
a	$(0.45, 0.25)$	$4\mathbb{Z}$
b	$(0.25, 0.45)$	$4\mathbb{N}$
1	$(0.75, 0.15)$	$2\mathbb{Z}$

It is routine to verify that $(E, \langle\sigma;\tilde{F}\rangle)$ is a soju sub-hoop of E. But it is not a soju filter of E since $\gamma_\sigma(b)=0.45>0.25=\max\{\gamma_\sigma(a),\gamma_\sigma(a\to b)\}$.

Proposition 3.14. For any hoop algebra E, every soju E-filter $(E, \langle\sigma;\tilde{F}\rangle)$ of E satisfies:

$$(\forall x,y\in E)\left(x\le y\ \Rightarrow \mu_\sigma(x)\le\mu_\sigma(y),\gamma_\sigma(x)\ge\gamma_\sigma(y),\tilde{F}(x)\subseteq\tilde{F}(y)\right)\quad(3.6)$$

Proof: Let $x,y\in E$ be such that $x\le y$. Then $x\to y=1$, and so

$$\begin{aligned}
&\mu_\sigma(y)\ge\min\{\mu_\sigma(x),\mu_\sigma(x\to y)\}=\min\{\mu_\sigma(x),\mu_\sigma(1)\}=\mu_\sigma(x),\\
&\gamma_\sigma(y)\le\max\{\gamma_\sigma(x),\gamma_\sigma(x\to y)\}=\max\{\gamma_\sigma(x),\gamma_\sigma(1)\}=\gamma_\sigma(x),\\
&\tilde{F}(y)\supseteq\tilde{F}(x)\cap\tilde{F}(x\to y)=\tilde{F}(x)\cap\tilde{F}(1)=\tilde{F}(x)
\end{aligned}$$

by (3.2) and (3.5). □

Theorem 3.15. *Given a hoop algebra E, the soju structure $(E, \langle\sigma;\tilde{F}\rangle)$ over $([0,1],U)$ is a soju filter of E if and only if its nonempty soju level sets $U(\mu_\sigma;t)$, $L(\gamma_\sigma;s)$ and $i(\tilde{F};\alpha)$ are filters of E for all $\alpha\in 2^U$ and $(t,s)\in[0,1]\times[0,1]$ with $t+s\le 1$.*

Proof: Assume that $(E, \langle\sigma;\tilde{F}\rangle)$ is a soju filter of E and let $\alpha\in 2^U$ and $(t,s)\in[0,1]\times[0,1]$ be such that $t+s\le 1$, and $U(\mu_\sigma;t)$, $L(\gamma_\sigma;s)$ and $i(\tilde{F};\alpha)$ are non-empty. It is clear that $1\in U(\mu_\sigma;t)\cap L(\gamma_\sigma;s)\cap i(\tilde{F};\alpha)$. Let $x,y\in E$ be such that $x\in U(\mu_\sigma;t)\cap L(\gamma_\sigma;s)\cap i(\tilde{F};\alpha)$ and $x\to y\in U(\mu_\sigma;t)\cap L(\gamma_\sigma;s)\cap i(\tilde{F};\alpha)$. Then $\mu_\sigma(x)\ge t$, $\gamma_\sigma(x)\le s$, $\tilde{F}(x)\supseteq\alpha$, $\mu_\sigma(x\to y)\ge t$, $\gamma_\sigma(x\to y)\le s$, $\tilde{F}(x\to y)\supseteq\alpha$. It follows from (3.5) that

$$\mu_\sigma(y) \geq \min\{\mu_\sigma(x), \mu_\sigma(x \to y)\} \geq t,$$
$$\gamma_\sigma(y) \leq \max\{\gamma_\sigma(x), \gamma_\sigma(x \to y)\} \leq s,$$
$$\tilde{F}(y) \supseteq \tilde{F}(x) \cap \tilde{F}(x \to y) \supseteq \alpha.$$

Hence $y \in U(\mu_\sigma; t) \cap L(\gamma_\sigma; s) \cap i(\tilde{F}; \alpha)$, and therefore $U(\mu_\sigma; t)$, $L(\gamma_\sigma; s)$ and $i(\tilde{F}; \alpha)$ are filters of E.

Conversely, suppose that the nonempty soju level sets $U(\mu_\sigma; t)$, $L(\gamma_\sigma; s)$ and $i(\tilde{F}; \alpha)$ of $(E, \langle\sigma; \tilde{F}\rangle)$ are filters of E for all $\alpha \in 2^U$ and $(t, s) \in [0, 1] \times [0, 1]$ with $t + s \leq 1$. For any $x \in E$, let $\mu_\sigma(x) = t_x$, $\gamma_\sigma(x) = s_x$ and $\tilde{F}(x) = \alpha_x$. Then $x \in U(\mu_\sigma; t_x) \cap L(\gamma_\sigma; s_x) \cap i(\tilde{F}; \alpha_x)$. Since $1 \in U(\mu_\sigma; t_x) \cap L(\gamma_\sigma; s_x) \cap i(\tilde{F}; \alpha_x)$, we have $\mu_\sigma(1) \geq t_x = \mu_\sigma(x)$, $\gamma_\sigma(y) \leq s_x = \gamma_\sigma(x)$ and $\tilde{F}(y) \supseteq \alpha_x = \tilde{F}(x)$. For any $x, y \in E$, let t_x, t_y, $s_x, s_y \in [0, 1]$ and $\alpha_x, \alpha_y \in 2^U$ be such that $\mu_\sigma(x) = t_x$, $\mu_\sigma(x \to y) = t_y$, $\gamma_\sigma(x) = s_x$, $\gamma_\sigma(x \to y) = s_y$, $\tilde{F}(x) = \alpha_x$ and $\tilde{F}(x \to y) = \alpha_y$. If we take $t := \min\{t_x, t_{x\to y}\}$, $s := \max\{s_x, s_{x\to y}\}$ and $\alpha := \alpha_x \cap \alpha_{x\to y}$, then $x, x \to y \in U(\mu_\sigma; t) \cap L(\gamma_\sigma; s) \cap i(\tilde{F}; \alpha)$. Thus $y \in U(\mu_\sigma; t) \cap L(\gamma_\sigma; s) \cap i(\tilde{F}; \alpha)$, and so

$$\mu_\sigma(y) \geq t = \min\{t_x, t_y\} = \min\{\mu_\sigma(x), \mu_\sigma(x \to y)\},$$
$$\gamma_\sigma(y) \leq s = \max\{s_x, s_y\} = \max\{\gamma_\sigma(x), \gamma_\sigma(x \to y)\},$$
$$\tilde{F}(y) \supseteq \alpha = \alpha_x \cap \alpha_y = \tilde{F}(x) \cap \tilde{F}(x \to y).$$

Therefore $(E, \langle\sigma; \tilde{F}\rangle)$ is a soju filter of E. □

Theorem 3.16. *For any sub-hoop A of a hoop algebra E, a soju structure $(A, \langle\sigma; \tilde{F}\rangle)$ over $([0, 1], U)$ is a soju A-filter of E if and only if it satisfies* (3.2) *and*

$$(\forall x, y \in A) \left(\begin{array}{l} \mu_\sigma(x \odot y) = \min\{\mu_\sigma(x), \mu_\sigma(y)\} \\ \gamma_\sigma(x \odot y) = \max\{\gamma_\sigma(x), \gamma_\sigma(y)\} \\ \tilde{F}(x \odot y) = \tilde{F}(x) \cap \tilde{F}(y) \end{array} \right). \tag{3.7}$$

Proof: Assume that $(A, \langle\sigma; \tilde{F}\rangle)$ is a soju A-filter of E and let $x, y \in A$. Since $x \odot y \leq x$ and $x \odot y \leq y$, it follows from (3.4) that

$$\mu_\sigma(x \odot y) \leq \min\{\mu_\sigma(x), \mu_\sigma(y)\},$$
$$\gamma_\sigma(x \odot y) \geq \max\{\gamma_\sigma(x), \gamma_\sigma(y)\},$$
$$\tilde{F}(x \odot y) \subseteq \tilde{F}(x) \cap \tilde{F}(y).$$

Since $x \leq y \rightarrow (x \odot y)$, we have

$$\begin{aligned} \mu_\sigma(x \odot y) &\geq \min\{\mu_\sigma(y), \mu_\sigma(y \rightarrow (x \odot y))\} \\ &\geq \min\{\mu_\sigma(x), \mu_\sigma(y)\} \end{aligned}$$

$$\begin{aligned} \gamma_\sigma(x \odot y) &\leq \max\{\gamma_\sigma(y), \gamma_\sigma(y \rightarrow (x \odot y))\} \\ &\leq \max\{\gamma_\sigma(x), \gamma_\sigma(y)\} \end{aligned}$$

and

$$\tilde{F}(x \odot y) \supseteq \tilde{F}(y) \cap \tilde{F}(y \rightarrow (x \odot y)) \supseteq \tilde{F}(x) \cap \tilde{F}(y)$$

by (3.5) and (3.4). This proves (3.7).

Conversely, suppose that $(A, \langle\sigma; \tilde{F}\rangle)$ satisfies (3.2) and (3.7). Since $x \odot (x \rightarrow y) \leq y$ for all $x, y \in A$, it follows from (3.2) and (3.7) that

$$\begin{aligned} &\mu_\sigma(y) \geq \mu_\sigma(x \odot (x \rightarrow y)) = \min\{\mu_\sigma(x), \mu_\sigma(x \rightarrow y)\}, \\ &\gamma_\sigma(y) \leq \gamma_\sigma(x \odot (x \rightarrow y)) = \max\{\gamma_\sigma(x), \gamma_\sigma(x \rightarrow y)\}, \\ &\tilde{F}(y) \supseteq \tilde{F}(x \odot (x \rightarrow y)) = \tilde{F}(x) \cap \tilde{F}(x \rightarrow y). \end{aligned}$$

Therefore $(A, \langle\sigma; \tilde{F}\rangle)$ is a soju A-filter of E by Theorem 3.11. □

THEOREM 3.17. *For any sub-hoop A of a hoop algebra E, a soju structure $(A, \langle\sigma; \tilde{F}\rangle)$ over $([0,1], U)$ is a soju A-filter of E if and only if it satisfies* (3.2) *and*

$$(\forall x, y, z \in A) \left(\begin{array}{l} \mu_\sigma(x \rightarrow z) \geq \min\{\mu_\sigma(x \rightarrow y), \mu_\sigma(y \rightarrow z)\} \\ \gamma_\sigma(x \rightarrow z) \leq \max\{\gamma_\sigma(x \rightarrow y), \gamma_\sigma(y \rightarrow z)\} \\ \tilde{F}(x \rightarrow z) \supseteq \tilde{F}(x \rightarrow y) \cap \tilde{F}(y \rightarrow z) \end{array} \right). \tag{3.8}$$

PROOF: Assume that $(A, \langle\sigma; \tilde{F}\rangle)$ is a soju A-filter of E and let $x, y, z \in A$. Since

$$(x \rightarrow y) \odot (y \rightarrow z) \leq x \rightarrow z,$$

we have

$$\begin{aligned} &\mu_\sigma(x \rightarrow z) \geq \mu_\sigma((x \rightarrow y) \odot (y \rightarrow z)) = \min\{\mu_\sigma(x \rightarrow y), \mu_\sigma(y \rightarrow z)\}, \\ &\gamma_\sigma(x \rightarrow z) \leq \gamma_\sigma((x \rightarrow y) \odot (y \rightarrow z)) = \max\{\gamma_\sigma(x \rightarrow y), \gamma_\sigma(y \rightarrow z)\}, \\ &\tilde{F}(x \rightarrow z) \supseteq \tilde{F}((x \rightarrow y) \odot (y \rightarrow z)) = \tilde{F}(x \rightarrow y) \cap \tilde{F}(y \rightarrow z) \end{aligned}$$

by (3.4) and (3.7).

Conversely, suppose that $(A, \langle\sigma; \tilde{F}\rangle)$ satisfies (3.2) and (3.8). If we take $x = 1$ in (3.8), then we have (3.5). Therefore $(A, \langle\sigma; \tilde{F}\rangle)$ is a soju A-filter of E by Theorem 3.11. □

THEOREM 3.18. *For any sub-hoop A of a hoop algebra E, a soju structure $(A, \langle\sigma; \tilde{F}\rangle)$ over $([0, 1], U)$ is a soju A-filter of E if and only if it satisfies (3.2) and*

$$(\forall x, y, z \in A)\left(\begin{array}{l}\mu_\sigma(y \odot z) \geq \min\{\mu_\sigma(x \odot z), \mu_\sigma(x \to y)\}\\ \gamma_\sigma(y \odot z) \leq \max\{\gamma_\sigma(x \odot z), \gamma_\sigma(x \to y)\}\\ \tilde{F}(y \odot z) \supseteq \tilde{F}(x \odot z) \cap \tilde{F}(x \to y)\end{array}\right). \tag{3.9}$$

PROOF: Assume that $(A, \langle\sigma; \tilde{F}\rangle)$ is a soju A-filter of E and let $x, y, z \in A$. Note that $(z \odot x) \odot (x \to y) = z \odot (x \odot (x \to y)) \leq z \odot y$. Using (3.4) and (3.7), we get

$$\begin{aligned}&\mu_\sigma(z \odot y) \geq \mu_\sigma((z \odot x) \odot (x \to y)) = \min\{\mu_\sigma(z \odot x), \mu_\sigma(x \to y)\},\\ &\gamma_\sigma(z \odot y) \leq \gamma_\sigma((z \odot x) \odot (x \to y)) = \max\{\gamma_\sigma(z \odot x), \gamma_\sigma(x \to y)\},\\ &\tilde{F}(z \odot y) \supseteq \tilde{F}((z \odot x) \odot (x \to y)) = \tilde{F}(z \odot x) \cap \tilde{F}(x \to y).\end{aligned}$$

Conversely, suppose that $(A, \langle\sigma; \tilde{F}\rangle)$ satisfies (3.2) and (3.9). If we take $z = 1$ in (3.9), then we have (3.5). Therefore $(A, \langle\sigma; \tilde{F}\rangle)$ is a soju A-filter of E by Theorem 3.11. □

THEOREM 3.19. *For any sub-hoop A of a hoop algebra E, a soju structure $(A, \langle\sigma; \tilde{F}\rangle)$ over $([0, 1], U)$ is a soju A-filter of E if and only if it satisfies*

$$(\forall x, y, z \in A)\left(x \leq y \to z \ \Rightarrow \left\{\begin{array}{l}\mu_\sigma(z) \geq \min\{\mu_\sigma(x), \mu_\sigma(y)\}\\ \gamma_\sigma(z) \leq \max\{\gamma_\sigma(x), \gamma_\sigma(y)\}\\ \tilde{F}(z) \supseteq \tilde{F}(x) \cap \tilde{F}(y)\end{array}\right.\right) \tag{3.10}$$

PROOF: Assume that $(A, \langle\sigma; \tilde{F}\rangle)$ is a soju A-filter of E and let $x, y, z \in A$ be such that $x \leq y \to z$. Then $x \to (y \to z) = 1$, and so

$$\begin{aligned}&\mu_\sigma(y \to z) \geq \min\{\mu_\sigma(x), \mu_\sigma(x \to (y \to z))\} = \min\{\mu_\sigma(x), \mu_\sigma(1)\} = \mu_\sigma(x),\\ &\gamma_\sigma(y \to z) \leq \max\{\gamma_\sigma(x), \gamma_\sigma(x \to (y \to z))\} = \max\{\gamma_\sigma(x), \gamma_\sigma(1)\} = \gamma_\sigma(x),\\ &\tilde{F}(y \to z) \supseteq \tilde{F}(x) \cap \tilde{F}(x \to (y \to z)) = \tilde{F}(x) \cap \tilde{F}(1) = \tilde{F}(x)\end{aligned}$$

by (3.5). It follows that

$$\begin{aligned}&\mu_\sigma(z) \geq \min\{\mu_\sigma(y), \mu_\sigma(y \to z)\} \geq \min\{\mu_\sigma(x), \mu_\sigma(y)\},\\ &\gamma_\sigma(z) \leq \max\{\gamma_\sigma(y), \gamma_\sigma(y \to z)\} \leq \max\{\gamma_\sigma(x), \gamma_\sigma(y)\},\\ &\tilde{F}(z) \supseteq \tilde{F}(y) \cap \tilde{F}(y \to z)\} \supseteq \tilde{F}(x) \cap \tilde{F}(y).\end{aligned}$$

Conversely, suppose that $(A, \langle\sigma; \tilde{F}\rangle)$ satisfies (3.2) and (3.10). Since $x \leq (x \to y) \to y$ for all $x, y \in A$, it follows from (3.10) that

$$\begin{aligned}&\mu_\sigma(y) \geq \min\{\mu_\sigma(x), \mu_\sigma(x \to y)\},\\ &\gamma_\sigma(y) \leq \max\{\gamma_\sigma(x), \gamma_\sigma(x \to y)\},\\ &\tilde{F}(y) \supseteq \tilde{F}(x) \cap \tilde{F}(x \to y).\end{aligned}$$

Hence $(A, \langle\sigma; \tilde{F}\rangle)$ is a soju A-filter of E by Theorem 3.11. □

Theorem 3.20. *For any sub-hoop A of a hoop algebra E, a soju structure $(A, \langle\sigma; \tilde{F}\rangle)$ over $([0,1], U)$ is a soju A-filter of E if and only if it satisfies (3.2) and*

$$(\forall x, y, z \in A)\left(\begin{array}{l}\mu_\sigma(x \to z) \geq \min\{\mu_\sigma((x \to y) \to z), \mu_\sigma(y)\}\\ \gamma_\sigma(x \to z) \leq \max\{\gamma_\sigma((x \to y) \to z), \gamma_\sigma(y)\}\\ \tilde{F}(x \to z) \supseteq \tilde{F}((x \to y) \to z) \cap \tilde{F}(y)\end{array}\right). \tag{3.11}$$

Proof: Suppose that $(A, \langle\sigma; \tilde{F}\rangle)$ is a soju A-filter of E. Since $(x \to y) \to z \leq y \to z$ and

$$y \odot ((x \to y) \to z) \leq y \odot (y \to z) \leq z \leq x \to z$$

for all $x, y, z \in A$, we get

$$\begin{aligned}\mu_\sigma(x \to z) \geq \mu_\sigma(z) &\geq \mu_\sigma(y \odot (y \to z))\\ &= \min\{\mu_\sigma(y), \mu_\sigma(y \to z)\}\\ &\geq \min\{\mu_\sigma(y), \mu_\sigma((x \to y) \to z)\},\end{aligned}$$

$$\begin{aligned}\gamma_\sigma(x \to z) \leq \gamma_\sigma(z) &\leq \gamma_\sigma(y \odot (y \to z))\\ &= \max\{\gamma_\sigma(y), \gamma_\sigma(y \to z)\}\\ &\leq \max\{\gamma_\sigma(y), \gamma_\sigma((x \to y) \to z)\},\end{aligned}$$

and

$$\begin{aligned}\tilde{F}(x \to z) \supseteq \tilde{F}(z) &\supseteq \tilde{F}(y \odot (y \to z))\\ &= \tilde{F}(y) \cap \tilde{F}(y \to z)\\ &\supseteq \tilde{F}(y) \cap \tilde{F}((x \to y) \to z).\end{aligned}$$

Conversely, assume that $(A, \langle\sigma; \tilde{F}\rangle)$ satisfies (3.2) and (3.11). If we take $x = 1$ in (3.11), then we have (3.5). Therefore $(A, \langle\sigma; \tilde{F}\rangle)$ is a soju A-filter of E by Theorem 3.11. □

DEFINITION 3.21. Let A be a subset of a hoop algebra E. A soju structure $(A, \langle\sigma; \tilde{F}\rangle)$ over $([0, 1], U)$ is called an *implicative soju filter* based on A (briefly, *implicative soju A-filter*) of E if it satisfies the condition (3.2) and

$$(\forall x, y, z \in A) \begin{pmatrix} \mu_\sigma(y) \geq \min\{\mu_\sigma(x), \mu_\sigma(x \to ((y \to z) \to y))\} \\ \gamma_\sigma(y) \leq \max\{\gamma_\sigma(x), \gamma_\sigma(x \to ((y \to z) \to y))\} \\ \tilde{F}(y) \supseteq \tilde{F}(x) \cap \tilde{F}(x \to ((y \to z) \to y)) \end{pmatrix}. \quad (3.12)$$

Example 3.22. Consider a hoop $(H, \odot, \to, 1)$ in which $H = \{0, a, b, c, 1\}$ with binary operations $\odot$ and $\to$ which are given as follows:

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	c	c	a
b	0	c	b	c	b
c	0	c	c	c	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	b	b	1
b	0	a	1	a	1
c	0	1	1	1	1
1	0	a	b	c	1

Define a soju structure $(E, \langle\sigma; \tilde{F}\rangle)$ over $([0, 1], U) = ([0, 1], \mathbb{Z})$ by Table 5. It is routine to check that $(E, \langle\sigma; \tilde{F}\rangle)$ is an implicative soju filter of E.

THEOREM 3.23. *Given a sub-hoop A of a hoop algebra E, every implicative soju A-filter is a soju A-filter.*

PROOF: Let $(A, \langle\sigma; \tilde{F}\rangle)$ be an implicative soju A-filter of E. If we take $z = 1$ in (3.12) and use $(a5)$, then we have (3.5). Therefore $(A, \langle\sigma; \tilde{F}\rangle)$ is a soju A-filter of E by Theorem 3.11. □

The converse of Theorem 3.23 is not true in general as seen in the following example.

Table 5. Tabular representation of $(E, \langle\sigma; \tilde{F}\rangle)$

E	$\sigma(x) = (\mu_\sigma(x), \gamma_\sigma(x))$	$\tilde{F}(x)$
0	$(0.3, 0.6)$	$3\mathbb{N}$
a	$(0.6, 0.2)$	$3\mathbb{Z}$
b	$(0.6, 0.2)$	$3\mathbb{Z}$
c	$(0.6, 0.2)$	$3\mathbb{Z}$
1	$(0.6, 0.2)$	$3\mathbb{Z}$

Example 3.24. Consider the hoop algebra E in Example 3.22 and let $(E, \langle\sigma; \tilde{F}\rangle)$ be a soju structure over $([0,1], U) = ([0,1], \mathbb{Z})$ defined by Table 6.

Table 6. Tabular representation of $(E, \langle\sigma; \tilde{F}\rangle)$

E	$\sigma(x) = (\mu_\sigma(x), \gamma_\sigma(x))$	$\tilde{F}(x)$
0	$(0.1, 0.7)$	$8\mathbb{N}$
a	$(0.6, 0.3)$	$4\mathbb{Z}$
b	$(0.3, 0.5)$	$4\mathbb{N}$
c	$(0.3, 0.5)$	$4\mathbb{N}$
1	$(0.7, 0.2)$	$2\mathbb{Z}$

It is routine to check that $(E, \langle\sigma; \tilde{F}\rangle)$ is a soju filter of E. But it is not an implicative soju filter of E since $\mu_\sigma(b) = 0.3 < 0.6 = \min\{\mu_\sigma(a), \mu_\sigma(a \to ((b \to 0) \to b))\}$.

Proposition 3.25. Given a sub-hoop A of a hoop algebra E, every implicative soju A-filter $(A, \langle\sigma; \tilde{F}\rangle)$ of E satisfies the following assertions.

$$(\forall x, y \in A)\begin{pmatrix} \mu_\sigma((x \to y) \to x) \leq \mu_\sigma(x) \\ \gamma_\sigma((x \to y) \to x) \geq \gamma_\sigma(x) \\ \tilde{F}((x \to y) \to x) \subseteq \tilde{F}(x) \end{pmatrix}. \tag{3.13}$$

$$(\forall x, y \in A)\begin{pmatrix} \mu_\sigma(((x \to y) \to x) \to x) = \mu_\sigma(1) \\ \gamma_\sigma(((x \to y) \to x) \to x) = \gamma_\sigma(1) \\ \tilde{F}(((x \to y) \to x) \to x) = \tilde{F}(1) \end{pmatrix}. \tag{3.14}$$

Proof: Let A be a sub-hoop and $(A, \langle\sigma; \tilde{F}\rangle)$ an implicative soju A-filter of a hoop algebra E. If we put $y = x$, $x = 1$ and $z = y$ in (3.12) and use ($a5$) and (3.2), then we have (3.13). Using (3.13), (H1), ($a5$), ($a7$), ($a9$) and (3.4), we have

$$\begin{aligned}
\mu_\sigma(((x\to y)\to x)\to x) &\geq \mu_\sigma(((((x\to y)\to x)\to x)\to y)\to(((x\to y)\to x)\to x)) \\
&= \mu_\sigma(((x\to y)\to x)\to(((((x\to y)\to x)\to x)\to y)\to x)) \\
&\geq \mu_\sigma(((((x\to y)\to x)\to x)\to y)\to(x\to y)) \\
&\geq \mu_\sigma(x\to(((x\to y)\to x)\to x)) \\
&= \mu_\sigma(((x\to y)\to x)\to(x\to x)) \\
&= \mu_\sigma(((x\to y)\to x)\to 1) \\
&= \mu_\sigma(1),
\end{aligned}$$

$$\begin{aligned}
\gamma_\sigma(((x\to y)\to x)\to x) &\leq \gamma_\sigma(((((x\to y)\to x)\to x)\to y)\to(((x\to y)\to x)\to x)) \\
&= \gamma_\sigma(((x\to y)\to x)\to(((((x\to y)\to x)\to x)\to y)\to x)) \\
&\leq \gamma_\sigma(((((x\to y)\to x)\to x)\to y)\to(x\to y)) \\
&\leq \gamma_\sigma(x\to(((x\to y)\to x)\to x)) \\
&= \gamma_\sigma(((x\to y)\to x)\to(x\to x)) \\
&= \gamma_\sigma(((x\to y)\to x)\to 1) \\
&= \gamma_\sigma(1),
\end{aligned}$$

$$\begin{aligned}
\tilde{F}(((x\to y)\to x)\to x) &\supseteq \tilde{F}(((((x\to y)\to x)\to x)\to y)\to(((x\to y)\to x)\to x)) \\
&= \tilde{F}(((x\to y)\to x)\to(((((x\to y)\to x)\to x)\to y)\to x)) \\
&\supseteq \tilde{F}(((((x\to y)\to x)\to x)\to y)\to(x\to y)) \\
&\supseteq \tilde{F}(x\to(((x\to y)\to x)\to x)) \\
&= \tilde{F}(((x\to y)\to x)\to(x\to x)) \\
&= \tilde{F}(((x\to y)\to x)\to 1) \\
&= \tilde{F}(1)
\end{aligned}$$

for all $x, y \in A$. It follows from (3.2) that we have (3.14). □

Proposition 3.26. Given a sub-hoop A of a bounded hoop algebra E, every implicative soju A-filter $(A, \langle\sigma; \tilde{F}\rangle)$ of E satisfies the following assertions.

$$(\forall x \in A)\begin{pmatrix} \mu_\sigma((x' \to x) \to x) = \mu_\sigma(1) \\ \gamma_\sigma((x' \to x) \to x) = \gamma_\sigma(1) \\ \tilde{F}((x' \to x) \to x) = \tilde{F}(1) \end{pmatrix}. \tag{3.15}$$

$$(\forall x, y \in A)\begin{pmatrix} \mu_\sigma(x \to y) \geq \mu_\sigma((x \odot y') \to y) \\ \gamma_\sigma(x \to y) \leq \gamma_\sigma((x \odot y') \to y) \\ \tilde{F}(x \to y) \supseteq \tilde{F}((x \odot y') \to y) \end{pmatrix}. \tag{3.16}$$

$$(\forall x, y, z \in A)\begin{pmatrix} \mu_\sigma(x \to z) \geq \min\{\mu_\sigma(y \to z), \mu_\sigma(x \to (z' \to y))\} \\ \gamma_\sigma(x \to z) \leq \max\{\gamma_\sigma(y \to z), \gamma_\sigma(x \to (z' \to y))\} \\ \tilde{F}(x \to z) \supseteq \tilde{F}(y \to z) \cap \tilde{F}(x \to (z' \to y)) \end{pmatrix}. \tag{3.17}$$

Proof: Let A be a sub-hoop and $(A, \langle\sigma; \tilde{F}\rangle)$ an implicative soju A-filter of a bounded hoop algebra E. Then $(A, \langle\sigma; \tilde{F}\rangle)$ is a soju A-filter of E (see Theorem 3.23). If we take $y = 0$ in (3.14), then

$$\begin{aligned} &\mu_\sigma((x' \to x) \to x) = \mu_\sigma(((x \to 0) \to x) \to x) = \mu_\sigma(1), \\ &\gamma_\sigma((x' \to x) \to x) = \gamma_\sigma(((x \to 0) \to x) \to x) = \gamma_\sigma(1), \\ &\tilde{F}((x' \to x) \to x) = \tilde{F}(((x \to 0) \to x) \to x) = \tilde{F}(1) \end{aligned}$$

for all $x \in E$. Note that $y' \to (x \to y) \leq (x \to y)' \to (x \to y)$ for all $x, y \in E$. It follows from (H3) and (3.6) that

$$\begin{aligned} &\mu_\sigma((x \odot y') \to y) = \mu_\sigma(y' \to (x \to y)) \leq \mu_\sigma((x \to y)' \to (x \to y)), \\ &\gamma_\sigma((x \odot y') \to y) = \gamma_\sigma(y' \to (x \to y)) \geq \gamma_\sigma((x \to y)' \to (x \to y)), \\ &\tilde{F}((x \odot y') \to y) = \tilde{F}(y' \to (x \to y)) \subseteq \tilde{F}((x \to y)' \to (x \to y)). \end{aligned} \tag{3.18}$$

Combining $(a5)$, (3.2), (3.12) and (3.18) induce

$$\begin{aligned}\mu_\sigma(x \to y) &\geq \min\{\mu_\sigma(1), \mu_\sigma(1 \to (((x \to y) \to 0) \to (x \to y)))\} \\ &= \mu_\sigma(((x \to y) \to 0) \to (x \to y)) \\ &= \mu_\sigma((x \to y)' \to (x \to y)) \\ &\geq \mu_\sigma((x \odot y') \to y),\end{aligned}$$

$$\begin{aligned}\gamma_\sigma(x \to y) &\leq \max\{\gamma_\sigma(1), \gamma_\sigma(1 \to (((x \to y) \to 0) \to (x \to y)))\} \\ &= \gamma_\sigma(((x \to y) \to 0) \to (x \to y)) \\ &= \gamma_\sigma((x \to y)' \to (x \to y)) \\ &\leq \gamma_\sigma((x \odot y') \to y),\end{aligned}$$

and

$$\begin{aligned}\tilde{F}(x \to y) &\supseteq \mu_\sigma(1) \cap \mu_\sigma(1 \to (((x \to y) \to 0) \to (x \to y))) \\ &= \tilde{F}(((x \to y) \to 0) \to (x \to y)) \\ &= \tilde{F}((x \to y)' \to (x \to y)) \\ &\supseteq \tilde{F}((x \odot y') \to y).\end{aligned}$$

This proves (3.16). Using (HP3), (3.8) and (3.16) impliy that

$$\begin{aligned}\mu_\sigma(x \to z) &\geq \mu_\sigma((x \odot z') \to z) \geq \min\{\mu_\sigma((x \odot z') \to y), \mu_\sigma(y \to z)\} \\ &= \min\{\mu_\sigma(x \to (z' \to y)), \mu_\sigma(y \to z)\}, \\ \gamma_\sigma(x \to z) &\leq \gamma_\sigma((x \odot z') \to z) \leq \max\{\gamma_\sigma((x \odot z') \to y), \gamma_\sigma(y \to z)\} \\ &= \max\{\gamma_\sigma(x \to (z' \to y)), \gamma_\sigma(y \to z)\},\end{aligned}$$

and

$$\begin{aligned}\tilde{F}(x \to z) &\supseteq \mu_\sigma((x \odot z') \to z) \supseteq \tilde{F}((x \odot z') \to y) \cap \tilde{F}(y \to z) \\ &= \tilde{F}(x \to (z' \to y)) \cap \tilde{F}(y \to z),\end{aligned}$$

which proves (3.17). □

THEOREM 3.27. *Given a hoop algebra E, the soju structure $(E, \langle\sigma; \tilde{F}\rangle)$ over $([0,1], U)$ is an implicative soju filter of E if and only if its nonempty soju level sets $U(\mu_\sigma; t)$, $L(\gamma_\sigma; s)$ and $i(\tilde{F}; \alpha)$ are implicative filters of E for all $\alpha \in 2^U$ and $(t, s) \in [0,1] \times [0,1]$ with $t + s \leq 1$.*

PROOF: It is similar to the proof of Theorem 3.15. □

We provide conditions for a soju A-filter to be an implicative soju A-filter.

THEOREM 3.28. *Let A be a sub-hoop of a hoop algebra E. If a soju A-filter $(A, \langle\sigma; \tilde{F}\rangle)$ of E satisfies the condition (3.14), then it is an implicative soju A-filter of E.*

PROOF: Let $x, y, z \in A$. Then

$$\begin{aligned}\mu_\sigma(y) &\geq \min\{\mu_\sigma(((y \to z) \to y) \to y), \mu_\sigma((y \to z) \to y)\}\\ &= \min\{\mu_\sigma(1), \mu_\sigma((y \to z) \to y)\} = \mu_\sigma((y \to z) \to y)\\ &\geq \min\{\mu_\sigma(x), \mu_\sigma(x \to ((y \to z) \to y))\},\end{aligned}$$

$$\begin{aligned}\gamma_\sigma(y) &\leq \max\{\gamma_\sigma(((y \to z) \to y) \to y), \gamma_\sigma((y \to z) \to y)\}\\ &= \max\{\gamma_\sigma(1), \gamma_\sigma((y \to z) \to y)\} = \gamma_\sigma((y \to z) \to y)\\ &\leq \max\{\gamma_\sigma(x), \gamma_\sigma(x \to ((y \to z) \to y))\},\end{aligned}$$

and

$$\begin{aligned}\tilde{F}(y) &\supseteq \tilde{F}(((y \to z) \to y) \to y) \cap \tilde{F}((y \to z) \to y)\\ &= \tilde{F}(1) \cap \tilde{F}((y \to z) \to y) = \tilde{F}((y \to z) \to y)\\ &\supseteq \tilde{F}(x) \cap \tilde{F}(x \to ((y \to z) \to y))\end{aligned}$$

by (3.2), (3.5) and (3.14). Therefore $(A, \langle\sigma; \tilde{F}\rangle)$ is an implicative soju A-filter of E. □

THEOREM 3.29. *Let A be a sub-hoop of a hoop algebra E. If a soju A-filter $(A, \langle\sigma; \tilde{F}\rangle)$ of E satisfies the condition (3.13), then it is an implicative soju A-filter of E.*

PROOF: Let $(A, \langle\sigma; \tilde{F}\rangle)$ be a soju A-filter of E which satisfies (3.13). We have shown that (3.13) implies (3.14) in Proposition 3.25. Therefore $(A, \langle\sigma; \tilde{F}\rangle)$ is an implicative soju A-filter of E by Theorem 3.28. □

THEOREM 3.30. *Let A be a sub-hoop of a hoop algebra E. If a soju A-filter $(A, \langle\sigma; \tilde{F}\rangle)$ of E satisfies the condition (3.15), then it is an implicative soju A-filter of E.*

PROOF: Let $(A, \langle\sigma; \tilde{F}\rangle)$ be a soju A-filter of E which satisfies (3.15). Note that

$$(x' \to x) \to x \le ((x \to y) \to x) \to x$$

for all $x, y \in E$. It follows from (3.2), (3.6) and (3.15) that

$$\mu_\sigma(((x \to y) \to x) \to x) \le \mu_\sigma(1) = \mu_\sigma((x' \to x) \to x) \le \mu_\sigma(((x \to y) \to x) \to x),$$
$$\gamma_\sigma(((x \to y) \to x) \to x) \ge \gamma_\sigma(1) = \gamma_\sigma((x' \to x) \to x) \ge \gamma_\sigma(((x \to y) \to x) \to x),$$
$$\tilde{F}(((x \to y) \to x) \to x) \subseteq \tilde{F}(1) = \tilde{F}((x' \to x) \to x) \subseteq \tilde{F}(((x \to y) \to x) \to x).$$

Thus (3.14) is valid, and therefore $(A, \langle\sigma; \tilde{F}\rangle)$ is an implicative soju A-filter of E by Theorem 3.28. □

Theorem 3.31. *Let A be a sub-hoop of a hoop algebra E. If a soju A-filter $(A, \langle\sigma; \tilde{F}\rangle)$ of E satisfies the condition (3.16), then it is an implicative soju A-filter of E.*

Proof: Let $(A, \langle\sigma; \tilde{F}\rangle)$ be a soju A-filter of E which satisfies (3.16). For any $x, y \in A$, we have

$$\begin{aligned}\mu_\sigma((x' \to x) \to x) &\le \mu_\sigma(1) = \mu_\sigma((x' \to x) \to (x' \to x)) \\ &= \mu_\sigma((x' \odot (x' \to x)) \to x) \le \mu_\sigma((x' \to x) \to x),\end{aligned}$$

$$\begin{aligned}\gamma_\sigma((x' \to x) \to x) &\ge \gamma_\sigma(1) = \gamma_\sigma((x' \to x) \to (x' \to x)) \\ &= \gamma_\sigma((x' \odot (x' \to x)) \to x) \ge \gamma_\sigma((x' \to x) \to x),\end{aligned}$$

and

$$\begin{aligned}\tilde{F}((x' \to x) \to x) &\subseteq \tilde{F}(1) = \tilde{F}((x' \to x) \to (x' \to x)) \\ &= \tilde{F}((x' \odot (x' \to x)) \to x) \subseteq \tilde{F}((x' \to x) \to x)\end{aligned}$$

by (H1), (H3), (3.2) and (3.16). Thus (3.15) is valid, and therefore $(A, \langle\sigma; \tilde{F}\rangle)$ is an implicative soju A-filter of E by Theorem 3.30. □

Theorem 3.32. *Let A be a sub-hoop of a hoop algebra E. If a soju A-filter $(A, \langle\sigma; \tilde{F}\rangle)$ of E satisfies the condition (3.17), then it is an implicative soju A-filter of E.*

Proof: Let $(A, \langle\sigma; \tilde{F}\rangle)$ be a soju A-filter of E which satisfies (3.17). The condition (3.17) implies that

$$\begin{aligned}\mu_\sigma(x \to y) &\geq \min\{\mu_\sigma(y \to y), \mu_\sigma(x \to (y' \to y))\}\\ &= \min\{\mu_\sigma(1), \mu_\sigma((x \odot y') \to y)\}\\ &= \mu_\sigma((x \odot y') \to y),\\ \gamma_\sigma(x \to y) &\leq \max\{\gamma_\sigma(y \to y), \gamma_\sigma(x \to (y' \to y))\}\\ &= \max\{\gamma_\sigma(1), \gamma_\sigma((x \odot y') \to y)\}\\ &= \gamma_\sigma((x \odot y') \to y),\end{aligned}$$

and

$$\begin{aligned}\tilde{F}(x \to y) &\supseteq \tilde{F}(y \to y) \cap \tilde{F}(x \to (y' \to y))\\ &= \tilde{F}(1) \cap \tilde{F}((x \odot y') \to y)\\ &= \tilde{F}((x \odot y') \to y)\end{aligned}$$

for all $x, y \in A$. Hence (3.16) is valid, and therefore $(A, \langle\sigma; \tilde{F}\rangle)$ is an implicative soju A-filter of E by Theorem 3.31. □

Theorem 3.33. (Extension property) *Let $(E, \langle\sigma; \tilde{F}\rangle)$ and $(E, \langle\rho; \tilde{G}\rangle)$ be soju filters of a hoop algebra E such that*

$$\mu_\sigma(1) = \mu_\rho(1),\ \gamma_\sigma(1) = \gamma_\rho(1),\ \tilde{F}(1) = \tilde{G}(1) \tag{3.19}$$

and $(E, \langle\sigma; \tilde{F}\rangle) \Subset (E, \langle\rho; \tilde{G}\rangle)$, that is,

$$(\forall x \in E)(\mu_\sigma(x) \leq \mu_\rho(x),\ \gamma_\sigma(x) \geq \gamma_\rho(x),\ \tilde{F}(x) \subseteq \tilde{G}(x)). \tag{3.20}$$

If $(E, \langle\sigma; \tilde{F}\rangle)$ is an impliative soju filter of E, then so is $(E, \langle\rho; \tilde{G}\rangle)$.

Proof: Assume that $(E, \langle\sigma; \tilde{F}\rangle)$ is an impliative soju filter of E. Then $(E, \langle\sigma; \tilde{F}\rangle)$ is a soju filter of E (see Theorem 3.23). For any $x, y \in E$, we have

$$\begin{aligned}&\mu_\rho((x' \to x) \to x) \geq \mu_\sigma((x' \to x) \to x) = \mu_\sigma(1) = \mu_\rho(1),\\ &\gamma_\rho((x' \to x) \to x) \leq \gamma_\sigma((x' \to x) \to x) = \gamma_\sigma(1) = \gamma_\rho(1),\\ &\tilde{G}((x' \to x) \to x) \supseteq \tilde{F}((x' \to x) \to x) = \tilde{F}(1) = \tilde{G}(1)\end{aligned}$$

by (3.19), (3.20) and (3.15). Since $(E, \langle\sigma; \tilde{F}\rangle)$ is a soju filter of E, it follows from (3.2) that $\mu_\rho((x' \to x) \to x) \leq \mu_\rho(1)$, $\gamma_\rho((x' \to x) \to x) \geq \gamma_\rho(1)$ and $\tilde{G}((x' \to x) \to x) \subseteq \tilde{G}(1)$. Hence $\mu_\rho((x' \to x) \to x) = \mu_\rho(1)$, $\gamma_\rho((x' \to x) \to x) = \gamma_\rho(1)$ and $\tilde{G}((x' \to x) \to x) = \tilde{G}(1)$ for all $x, y \in E$. Therefore $(E, \langle\sigma; \tilde{F}\rangle)$ is an implicative soju filter of E by Theorem 3.30. □

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