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David Makinson

THE PHENOMENOLOGY OF SECOND-LEVEL INFERENCE: PERFUMES IN THE DEDUCTIVE GARDEN

Abstract

We comment on certain features that second-level inference rules commonly used in mathematical proof sometimes have, sometimes lack: suppositions, indirectness, goal-simplification, goal-preservation and premise-preservation. The emphasis is on the roles of these features, which we call 'perfumes', in mathematical practice rather than on the space of all formal possibilities, deployment in proof-theory, or conventions for display in systems of natural deduction.

Keywords: Second-level inference, suppositions, indirect inference, goal simplification, goal preservation, wlog, premise preservation.

1. Introduction

In logic, it is commonplace to distinguish between inferences of first and second levels. In broad terms, a first-level inference passes from certain statements serving as premises to a statement taken as conclusion. In concise notation, it is of the type $\Gamma \vdash \gamma$, where γ is a statement and Γ is a finite set of the same, with the sign $\vdash$ indicating passage from one to the other. A second-level inference, on the other hand infers the validity of an entire argument from the validity of one or more other arguments. In the same notation, it is of the form $\Gamma_i \vdash \gamma_i (i \leq n)/\Delta \vdash \delta$ where the slash marks a step from the n subordinate inferences $\Gamma_i \vdash \gamma_i$ on the left to the principal one on the right. The former kind of inference can, of course, be regarded

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as the limiting case of the latter where $n = 0$, but here we will always be considering the principle case that $n \geq 1$.

Not only are these two levels formally distinct, but they also ‘smell’ quite differently. One also senses differences within the second level; for example, each of the rules of conditional proof ($CP, \to^+$), *reductio ad absurdum* (RAA), disjunctive proof ($DP, \vee^+$), universal generalization ($UG, \forall^+$) and existential instantiation ($EI, \exists^-$) has its own intuitive feel or, as we shall say, its ‘perfume’. This article is about such perfumes, which can be defined quite precisely. We will see how a second-level inference-rule may be (or fail to be) suppositional, indirect, goal-simplifying, goal-preserving or premise-preserving, in senses to be defined, discuss why these features can be useful in practice, and draw attention to a connection with without loss of generality (wlog) reasoning.

Some disclaimers may forestall misunderstanding. There are no theorems, no philosophical messages or agenda. We are not interested in the space of all mathematically possible forms of second-level proof, but with steps that, arguably, one finds in everyday mathematical reasoning. Since that proceeds in accord with classical logic, it is the only kind of logic to concern us, not free, intuitionistic, relevance-sensitive, paraconsistent, fuzzy or any other variety. Nor are we concerned with the various display systems that have been devised in textbook presentations of natural deduction (see the overviews in e.g. [10], [3, chapter 2], [11]). The text merely offers an organized review of folklore about inferential practice; in the more pretentious language of our title, it is a foray in the phenomenology of second-level inference.

As is well known, any second-level inference rule $\Gamma_i \vdash \gamma_i (i \leq n) / \Delta \vdash \delta$ may equivalently be presented as what has been called a ‘split-level’ rule $\Delta; \Gamma_i \vdash \gamma_i (i \leq n) / \delta$, where the premise-set Δ of the principal inference is moved to the left of the slash (cf. [8, chapter 10]). In the split-level version the conclusion is thus a proposition δ (as in a first-level rule), which is obtained from a set Δ of propositions (again like the first level) that is, however, accompanied by the inferences $\Gamma_i \vdash \gamma_i$ (as for the second level). Arguably, this is the form that corresponds most closely to inferential practice and to what is done in systems of natural deduction. The present text could be written using either idiom; we have chosen the second-level one because it is a little easier to read and more familiar to readers.

Throughout, we use the terms ‘proof’, ‘inference’ and ‘argument’ interchangeably according to the whims of style; similarly with ‘subordinate inference' and 'sub-proof', 'statement' and 'proposition' (understood in a broad sense, as possibly containing free variables), 'supposition', 'assumption', and 'hypothesis', while recognizing that in other contexts it can be useful, even essential, to make distinctions between them. Table 1, at the end of the paper, keeps track of the discussion.

2. Suppositions

Do second-level proofs have anything in common apart from the general type $\Gamma_i \vdash \gamma_i(i \leq n)/\Delta \vdash \delta$ mentioned above? It is tempting to say that, in every instance, each of the subordinate inferences $\Gamma_i \vdash \gamma_i$ makes a *supposition* (also called assumption or hypothesis); in other words, that for all $i \leq n$, there is a statement α with $\alpha \in \Gamma_i$ but $\alpha \notin \Delta$.

That is almost true but, notoriously, not quite. There is a very important form of second-level inference, with just one subordinate inference, that makes no supposition, namely universal generalization ($UG, \forall^+$):

$$UG : \Gamma \vdash \gamma/\Gamma \vdash \forall x(\gamma), \text{ when } x \text{ has no free occurrences in } \Gamma.$$

Here the premise-set Γ of the unique subordinate inference $\Gamma \vdash \gamma$ is exactly the same as that of the principal inference $\Gamma \vdash \forall x(\gamma)$, nothing more, nothing less.

From a heuristic point of view, the presence of a supposition in a subordinate argument gives it one more item to grab and deploy, so the absence of a supposition means that it foregoes that bonus. But UG has another feature that tends to compensate: it is 'goal-simplifying', in the sense that the conclusion γ of its subordinate argument is simpler, in its logical structure, than the conclusion $\forall x(\gamma)$ of the principal argument. We will return to goal-simplification in section 4.

All other rules for logical connectives in Table 1 are suppositional, as can be seen by inspection. For disjunctive proof,

$$DP, \vee^- : \Gamma \cup \{\alpha\} \vdash \gamma; \Gamma \cup \{\beta\} \vdash \gamma/\Gamma \cup \{\alpha \vee \beta\} \vdash \gamma,$$

the suppositions are α, β. Another form of disjunctive proof widely used in informal practice, sometimes called proof by cases, is $DPx : \Gamma, \alpha \vdash \gamma$; $\Gamma, \neg\alpha \vdash \gamma/\Gamma \vdash \gamma$, where the suppositions are $\alpha, \neg\alpha$. For existential instantiation, the supposition is α:

$$EI, \exists^{-}: \Gamma \cup \{\alpha\} \vdash \gamma / \Gamma \cup \{\exists x(\alpha)\} \vdash \gamma, \text{ when } x \text{ has no free occurrences in } \Gamma, \gamma.$$

For the two forms of conditional proof in Table 1, the suppositions are respectively $\alpha, \neg\gamma$:

$$CPa, \rightarrow^{+}: \Gamma \cup \{\alpha\} \vdash \gamma / \Gamma \vdash \alpha \rightarrow \gamma$$

$$CPb, \rightarrow^{+}: \Gamma \cup \{\neg\gamma\} \vdash \neg\alpha / \Gamma \vdash \alpha \rightarrow \gamma.$$

For *reductio ad absurdum*, the supposition is $\neg\alpha$ in each of the two subordinate arguments of $RAAa$, as also in the unique subordinate argument of $RAAb$:

$$RAAa: \Gamma \cup \{\neg\alpha\} \vdash \gamma; \Gamma \cup \{\neg\alpha\} \vdash \neg\gamma / \Gamma \vdash \alpha$$

$$RAAb: \Gamma \cup \{\neg\alpha\} \vdash \alpha / \Gamma \vdash \alpha.$$

We have distinguished between two forms of CP, and two forms of RAA, because they disagree on other perfumes to be considered later. On the other hand, we do not distinguish between the above forms of RAA and those obtained by inverting the positive and negative occurrences of α. Although the latter distinction is important for intuitionistic logic, we remain in the classical domain and it turns out that the inverted forms agree with the above ones on all perfumes considered.

What about connective-free (sometimes called structural) rules? Reflexivity is connective-free, but first-degree, while both monotony and cumulative transitivity are second-degree:

$$Monotony: \Gamma \vdash \gamma / \Gamma \cup \Delta \vdash \gamma,$$

$$CT: \Gamma \vdash \gamma; \Gamma \cup \{\gamma\} \vdash \delta / \Gamma \vdash \delta.$$

Their status is rather special, in inferential practice as much as in logical theory. They are less visible than the rules for connectives in everyday mathematical reasoning, as well as in systems of natural deduction, because they are rarely rendered explicit when applied. In effect, they are implicit in the very structure of a deduction as it develops, whether in the usual linear fashion or in tree form. Moreover, in the present author's view, their connections with the various perfumes that we are considering are less interesting than in the case of rules for connectives. For these reasons, we settle on a compromise: the behaviour of these two rules with respect to each perfume is recorded in Table 1 (last two rows), but not discussed further in the text.

The following remarks may help put this bare picture in perspective.

History. It is interesting to recall that Jaśkowski, in his seminal paper of 1934 ([5]), appears to have been reluctant to accept that there are second-level inferences without suppositions. The title of his paper is "On the rules of suppositions in formal logic". When presenting UG, he introduces its subordinate inference by writing Tx on a new line, explaining that T "is here a new constant analogous to the symbol of supposition S" (section 5, page 29).

To be sure, choosing an item arbitrarily by declaring a fresh variable does have some resemblance to the act of making a supposition. This is reflected in the language used: in English, at least, both "let x be..." and "suppose that..." are in the imperative rather than the indicative mood. But they are not quite the same action, and a 'semi-supposition' display risks obscuring the difference.

In effect, we can read Jaśkowski's Tx entry in either of two ways. On one reading, which is rather confusing, it articulates the constraint on the variable x, which is a condition in the metalanguage, as if it were an additional premise of the subordinate inference. On another reading, it merely announces that one is about to enter a subordinate argument and that one intends to generalize on the variable x when leaving it – which is what is also done by the informal mathematical phrase "let x be an arbitrary so-and-so". However, giving Tx a line number in a derivation may not be the most transparent way of signalling that reading.

In any case, the Tx notation was not used in the independently conceived work of Gentzen [1], which was much more influential for theoretical investigations of mathematical logicians. And although Jaśkowski's paper was, directly or indirectly, a basic inspiration for textbook accounts of natural deduction for students of philosophy in the second half of the twentieth century, few of them adopted this part of his notation, one well-known exception being, however, [6] (cf. [4]).

Flattening. On the other hand, textbooks presenting systems of natural deduction often streamline displays by "flattening" UG, that is, transforming it into a step that is first-level, but procedural rather than inferential. The rule is articulated as authorizing passage from a *proposition* γ to the corresponding *proposition* $\forall x(\gamma)$ under the proviso that x does not occur free in any of the premises or suppositions on which γ depends in the deduction under construction. That the passage is procedural rather than inferential is evident from the fact that in general $\gamma \not\models \forall x(\gamma)$, irrespective of whether the proviso is satisfied. It is only when we reflect on the rationale for the proviso that we can begin to see the second-level inference underlying the procedural step.

While such flattening simplifies the formal display of a natural deduction, in the present author's view it runs a danger of obscuring what is really going on. Given that we have deduced γ from various premises and suppositions (the Γ in the rule) in which x does not occur free, UG authorizes us to conclude $\forall x(\gamma)$ on the basis of those same propositions. That is second-level and the student should brought to realize it, and not allowed to forget it. In particular, UG should not be confused with the first-level and genuinely inferential step of vacuous generalization $VG : \gamma/\forall x(\gamma)$ whenever x has no free occurrences in γ.

Universal generalization is not the only second-level rule that systems of natural deduction like to flatten. Even more so is existential instantiation which, we recall, tells us: $\Gamma \cup \{\alpha\} \vdash \gamma/\Gamma \cup \{\exists x(\alpha)\} \vdash \gamma$, when x has no free occurrences in Γ, γ. This is treated as authorizing passage (again, procedurally, but not inferentially) from a proposition $\exists x(\alpha)$ to the corresponding proposition α, under a suitable condition. The precise formulation of that condition varies with the conventions of the particular natural deduction system, but its essential content is that x does not occur free in any of the premises or suppositions on which $\exists x(\alpha)$ depends, nor on any conclusion that is subsequently derived from α. This reference to premises and conclusions again alerts us to the fact that that there is a second-level inference underlying the first-level procedural step.

Colloquial mathematical reasoning with EI also flattens it, in a less formal way. In a proof, having reached a proposition $\exists x(\alpha)$, where α contains x free, one simply says "choose any one such x" and works on α, taking care not to use the same variable x for anything else until the proof is complete. The second-level nature of the manoeuvre is thus left implicit, and perhaps corresponds more closely to a variant of EI, namely the rule $\Gamma \vdash \exists x(\alpha), \Gamma \cup \{\alpha\} \vdash \gamma/\Gamma \vdash \gamma$, under the same proviso that x has no free occurrences in Γ, γ. There is more on 'flattening' second-level and split-level rules in [8, section 10.3.3].

Making UG suppositional. It is also possible to reformulate UG to render it suppositional. We may add to the premise-set Γ of the subordinate inference $\Gamma \vdash \gamma$, as a supposition, either $\neg\gamma$, or a tautology of classical propositional logic such as $p \vee \neg p$, or a theorem of first-order logic such as $a = a$, without needing actually to use it. The modified rule remains classically correct and one can carry out the same derivations as before without change. Systems of natural deduction that proceed this way, for classical and non-classical logics, are discussed in [3]. But, in the classical context, why bother unless one thinks that there is something philosophically wrong about second-level inference without a supposition? The manoeuvre does not correspond to mathematical practice and detours through an idle or artificially employed assumption.

The "let x be..." locution. In practice, when one uses a phrase like 'Let x be an arbitrary so-and-so' (say, an arbitrary equilateral triangle) the 'so-and-so' condition almost always identifies a class that is more restricted than the entire universe of discourse under consideration (say, the class of points, lines and figures on a plane). This is because we are doing two things at the same time. We are trying to prove, from given information Γ (say, the axioms of plane geometry), a general conditional $\forall x(\varphi(x) \to \psi(x))$. To that end, we begin by establishing $\psi(x)$ from $\Gamma \cup \{\varphi(x)\}$ and then carry out two second-level steps. The first applies CP to conclude that $\Gamma \vdash \varphi(x) \to \psi(x)$, to which the second applies UG, under the condition that x does not occur free in Γ, to conclude that $\Gamma \vdash \forall x(\varphi(x) \to \psi(x))$. To streamline the argument, the two steps are customarily run together.

Non-classical logics. Although we are concerned only with classical reasoning, we note in passing that supposition-free second-level inference rules also appear in some well-known non-classical logics, notably for introducing the box connective in natural deduction systems for the modal logic $S5$ and some of its sub-logics, as well as for logics of relevance-sensitive conditionals when those conditionals are understood as conveying some kind of necessity (as is the case for the relevance logic E, but not for R).

For $S5$, that is not really anything new since its box can be seen as shorthand for a universal quantifier with a single fixed variable x. Specifically, modal formulae can be translated to classical first-order formulae with monadic predicates: fix a variable x, associate injectively each sentence letter p with a one-place predicate letter P, put $\mathcal{T}(p) = P(x)$ for sentence letters (always with the fixed choice of variable x), translate truth-functional connectives into themselves, and put $\mathcal{T}(\Box\alpha) = \forall x\mathcal{T}(\alpha)$. When the $S5$ rule $\Box^+$ is expressed in the form $\Gamma \vdash \gamma/\Gamma \vdash \Box\gamma$ under the proviso that no sentence letter in γ occurs unmodalized in Γ, it corresponds to classical UG. In this way, UG can be seen as the ultimate source of supposition-free second-level rules in $S5$ and thus also, indirectly, in some of its subsystems.

3. Indirectness

The best-known of the perfumes, used and discussed since Greek antiquity, is *indirectness*. As it has received so much attention, we will be very brief.

Often, *reductio ad absurdum* alone is counted as indirect, but it seems reasonable to include contrapositive conditional proof under this name, as is sometimes done. Accordingly, we define a second-level inference form $\Gamma_i \vdash \gamma_i(i \leq n)/\Delta \vdash \delta$ to be indirect iff each subordinate argument has a supposition that negates (or is negated by) the conclusion (or the consequent of the conclusion) of the principal argument. Indirect inferences are thus by definition a particular type of suppositional inference. Inspection tells us that in this sense $RAAa$, $RAAb$ and CPb are indirect, while the other argument-forms in Table 1 are direct.

Reductio often permits very short and elegant arguments; see e.g. [7, section 2.3] for a collection of examples. In the present author's view, this is not so much due to the manipulation of negation as to the fact, that it is both suppositional and, at least for the more elegant applications of $RAAa$, goal-simplifying in the sense defined in section 2 and discussed further in the section 4. On the negative side, *reductio* proofs for $\forall\exists$ statements are sometimes non-constructive (no witness provided for the existential quantification) which, notoriously, makes them unfriendly to computation; it has also sometimes given rise to philosophical doubts about its legitimacy (see e.g. [12]).

For CPb, where the desired conclusion is of the form $\alpha \to \beta$, the inferential convenience of $\neg\beta$ as a supposition with $\neg\alpha$ as goal in the subordinate inference can be greater, less, or about the same as, that of a corresponding application of CPa where α is supposition and β is goal, depending on the internal logical structure of those propositions.

The extent to which mathematicians use indirect inference is partly a matter of personal style. Reticence about CPb is not common, but some prefer not to bring in the *reductio* artillery except when it provides major benefits. On the other hand, others deploy it routinely. A mixed strategy is to give an indirect inference if it is succinct and, if a witness is missing or computation blocked, accompany it by a longer and perhaps more intricate constructive argument, if one can be found.

Of course, direct second-level inference forms can always be rendered indirect. For example, the first trick mentioned in section 2 for making UG suppositional, adding $\neg\gamma$ to the premises of the subordinate inference $\Gamma \vdash \gamma$, at the same time makes it indirect. However, as mentioned there, this manoeuvre does not correspond to mathematical practice, being artificial with an unnecessary detour. One occasionally sees a less generic, but more interesting, indirect variant of DP that eliminates one of the two disjuncts, namely the rule: $DPy : \Gamma \cup \{\alpha\} \vdash \gamma; \Gamma \cup \{\beta\} \vdash \delta; \Gamma \cup \{\beta\} \vdash \neg\delta / \Gamma, \alpha \vee \beta \vdash \gamma$. However, from a conceptual point of view, we see this as a combination of standard DP with reductio and, for this reason, do not include it in Table 1.

Why, then, have we given CPb a seat at that table, since it too can be seen as conceptually composite, a combination of CPa with a first-level inference? The reason is practical rather than formal: CPb is extremely common in everyday mathematical reasoning, while the indirect form of disjunctive proof is very much less so. But nothing prevents the reader from extending Table 1 with rows for rules such as DPy, DPx (section 2) or any other second-level inference forms that can reasonably claim to be deployed in inferential practice.

4. Goal-simplification

In section 2, we observed in passing that although UG is not suppositional, it is *conclusion-simplifying* (more briefly, *goal-simplifying*), in the sense that the conclusion γ of the subordinate argument is strictly logically simpler than the conclusion $\forall x(\gamma)$ of the principal argument. Evidently, this property facilitates inference, by reducing the complexity of what we have to prove.

Not many patterns of second-level inference in everyday mathematical reasoning have this property. DP and EI fail it, as the conclusions of their subordinate inferences are the same as the conclusions of their respective principal inferences, thus not strictly simpler. On the other hand, conditional proof is a big goal-simplifier. This is patently so in the case of *CPa*, where the conclusion β of the subordinate inference is only the consequent of the conclusion $\alpha \to \beta$ of the principal one. It is less clear for the contrapositive version *CPb*, for the negation sign in the conclusion $\neg\alpha$ of the subordinate inference does not appear in the conclusion $\alpha \to \beta$ of the principal one. However, the elimination of the arrow intuitively compensates for the addition of a negation: we can understand logical complexity in a way that gives arrows more weight than negations (as is sometimes done when defining the notion of a subformula for inductive arguments in proof-theory), thereby also treating *CPb* as goal-simplifying.

RAAa is not in general a goal-simplifier, but it is so in some instances. The conclusions $\gamma, \neg\gamma$ of the two subordinate arguments may have any complexity at all compared to the conclusion α of the principal inference but, in practice, they are often simpler. As we are concerned with practice as well as form, we put 0/1 in this cell of Table 1.

Thus, conditional proof in both forms *CPa*, *CPb* and *RAAa* (in some of its applications) are the only second-level inference rules of Table 1 that are both suppositional and goal-simplifying. The additional power brought by the availability of a supposition, combined with the reduced complexity obtained by goal-chipping, iterated as many times as occasion arises, can transform a complex inferential task into a trivial one. That is surely part of the reason why those three rules are such great work-horses, stars of the second-level stable.

5. Goal-preservation and wlog reasoning

Call a second-level inference rule $\Gamma_i \vdash \gamma_i (i \leq n) / \Delta \vdash \delta$ *conclusion-preserving* (briefly, *goal-preserving*) iff $\gamma_i = \delta$ for all $i \leq n$. By definition, the goal-preserving rules are disjoint from the goal-simplifying ones, which make the conclusion strictly simpler. They are also disjoint from the indirect ones, which radically modify the conclusion. Clearly, *DP*, *EI* and *RAAb* are goal-preserving while *UG*, *CPa*, *CPb* and *RAAa* are not.

Thus, all of the goal-preserving rules for connectives that we consider in Table 1 turn out to be suppositional. To find a non-suppositional goal-preserving rule, one can turn to the connective-free rule of monotony. Indeed, that is essentially the only manner in which the combination can hold for, quite generally, a second-level rule $\Gamma_i \vdash \gamma_i (i \leq n) / \Delta \vdash \delta$ that is non-suppositional must have some $\Gamma_j \subseteq \Delta$ while, if it is goal-preserving, it must put each $\gamma_i = \delta$. Thus, it must be of the form $\Gamma_j \vdash \delta; \Gamma_i \vdash \delta(j \neq i \leq n)/\Gamma_j \cup \Gamma' \vdash \delta$, which is just monotony possibly accompanied by additional, but redundant, subordinate inferences.

Goal-preservation underlies a kind of reasoning that is familiar in mathematical practice but seldom discussed in logic textbooks. It is typically introduced by a such as "assume without loss of generality that...", abbreviated as "assume wlog that...". This tells us that we are making a supposition in a (single) subordinate inference, with conclusion unchanged. In general, the assumption is mathematically substantive.

A well-known example arises when one is working with well-ordered sets. In that context, a standard wlog move is to say, when there is an x with a certain property, that we can choose such an x and assume wlog that it is least among them (or minimal among them, in the more general case of well-founded sets).

Although content-specific, this example has an interesting parallel with the logical rule EI. It may be understood as a second-level inference pattern that is available when Γ tells us, *inter alia*, that the domain of discourse is well-founded by a relation $<$:

$$EI_< : \Gamma, \alpha_< \vdash \gamma / \Gamma, \exists x(\alpha) \vdash \gamma, \text{ when } x \text{ has no free occurrences in } \Gamma, \gamma.$$

Here $\alpha_<$ abbreviates $\alpha \wedge \neg\exists y(y < x \wedge \alpha_{x:=y})$, where $\alpha_{x:=y}$ is the result of substituting a variable y not occurring in α, γ or Γ, for all free occurrences of x in α. The only difference between this and EI as formulated earlier, is that $\alpha_<$ replaces α as premise of the subordinate inference. Trivially, $\alpha_< \models \alpha$, so $EI_<$ is more powerful than plain EI. It can be regarded as running EI together with modus ponens in the context of a well-founded domain, that is, where $\Gamma \models \exists x(\alpha) \to \exists x(\alpha_<)$. The merging can render presentation more elegant; in particular, it often permits us to halve the number of different variables that are needed for clear exposition.

While the above example of wlog reasoning is closely related to EI, there are many others that appear to be less so. For example, if we want to show that every Boolean algebra has a certain property that we know to be preserved under Boolean isomorphisms, then we may consider an arbitrary Boolean algebra B and assume without loss of generality that it is a field of sets. This is because we have a representation theorem telling us that every Boolean algebra is isomorphic to some field of sets. However, care needs to be taken since, notoriously, some interesting properties of Boolean algebras are not preserved under isomorphisms, for example, the property of infinite distributivity (see e.g. [13, section 35]).

In this example, the parallel with EI reappears if the wlog step is made in the course of an indirect proof: supposing that there is a Boolean algebra B that lacks a certain property, we seek a contradiction; if the property is preserved under isomorphism, then we may suppose wlog that B is a field of sets, and continue to contradiction from there.

For examples further away from logic and set theory, see e.g. [2]. For *aficionados* of non-classical logic, there is an interesting use of a wlog procedure in the construction of relevance-sensitive truth-trees. When the arrow connective is understood as relevance-sensitive, then decomposition of $\neg(\varphi \rightarrow \psi)$ into $\varphi, \neg\psi$ on a branch of a decomposition tree is no longer an act of first-level inference, as it is for classical truth-trees; it is a wlog second-level step. See [8, chapter 11], with a more detailed account in [9].

6. Premise-preservation

Evidently, one may define a dual to goal-preservation. Call a second-level inference rules $\Gamma_i \vdash \gamma_i (i \leq n)/\Delta \vdash \delta$ *premise-preserving* iff $\Delta \subseteq \Gamma_i$ for all $i \leq n$. In [8, chapter 10], this property was called "incrementality".

Inspection of the rules in Table 1 shows that UG, CT, CP (both forms) and RAA (both forms) are premise-preserving. On the other hand, EI and DP are not since, in general, $\exists x(\alpha) \notin \Gamma \cup \{\alpha\}$ and $\alpha \vee \beta \notin \Gamma \cup \{\alpha\}, \Gamma \cup \{\beta\}$.

On the other hand, the rather special form of DP that we called DPx (section 2) is premise-preserving. Moreover, since trivially $\alpha \models \alpha \vee \beta$, $\beta \models \alpha \vee \beta$, $\alpha \models \exists x(\alpha)$, both DP and EI may both be formulated in equivalent, but redundant, ways that are premise-preserving:

$$\Gamma \cup \{\alpha \vee \beta\} \cup \{\alpha\} \vdash \gamma; \Gamma \cup \{\alpha \vee \beta\} \cup \{\beta\} \vdash \gamma \,/\, \Gamma \cup \{\alpha \vee \beta\} \vdash \gamma$$

$$\Gamma \cup \{\exists x(\alpha)\} \cup \{\alpha\} \vdash \gamma \,/\, \Gamma \cup \{\exists x(\alpha)\} \vdash \gamma, \text{ when } x \text{ has no free occurrences in } \Gamma, \gamma.$$

To mark the fact that these variant formulations of DP, EI are premise-preserving while the standard forms are not, we write $0, 1$ in the corresponding cells of Table 1.

It is perhaps not immediately obvious, as it was with suppositionality and goal simplification, how premise preservation can be of assistance in

Table 1. Selected perfumes for familiar second-order rules

		Suppositional	**Goal-simplifying**	**Indirect**	**Goal-preserving**	**Premise-preserving**
$UG, \forall^+$		0	1	0	0	1
$DP, \vee^-$		1			1	0, 1
$EI, \exists^-$						0, 1
$CP, \to^+$	CPa		1		0	0
	CPb			1		
RAA	$RAAa$		0/1			
	$RAAb$		0		1	
Monotony		0	0	0	1	0
CT					0	1

The entries 0/1 and 0, 1 in certain cells are explained in the corresponding sections. The acronym EI is for "existential instantiation"; the reader should be warned that some textbook presentations of natural deduction use the same acronym for the first-level rule of "existential introduction".

the business of deduction. We suggest that it can be helpful, illustrating with the premise-preserving rule CPa. Consider a situation where we are working within a mathematical theory that is axiomatized by a set Γ of propositions, from which a considerable number of consequences have been derived by many hands over a long period of time, and that we wish to prove a conditional proposition $\alpha \to \gamma$. We take α as a supposition and seek to get γ from it, making free use of anything that has already been obtained from Γ. If we succeed, then we can apply CPa (along with implicit appeal to cumulative transitivity and monotony) and we are done.

But what would the situation be like if instead of $CPa : \Gamma \cup \{\alpha\} \vdash \gamma / \Gamma \vdash \alpha \rightarrow \gamma$ we had a rule $\Gamma' \cup \{\alpha\} \vdash \gamma / \Gamma \vdash \alpha \rightarrow \gamma$ that is not-premise-preserving, that is, where $\Gamma \nsubseteq \Gamma'$? When carrying out the subordinate inference $\Gamma' \cup \{\alpha\} \vdash \gamma$, we would not know, without re-checking, which of the many theorems already deduced from Γ are also available for use in the sub-proof. In cases where there are elements of Γ that are not trivially implied by Γ', that checking could be arduous indeed.

7. Conclusion

Second-level inference rules come with or without various features that we have dubbed 'perfumes': suppositionality, indirectness, goal-simplification, goal-preservation and premise-preservation. The presence of these perfumes tends to confer practical advantages in the articulation and communication of mathematical inference, each perfume with its own advantage. Familiar rules of mathematical practice, as recorded in Table 1, all have at least one of the perfumes, sometimes more. Conditional proof in both its direct and contraposed forms (CPa, CPb) as well as *reductio ad absurdum* in its standard form ($RAAa$), are particularly well endowed in this respect, which may explain why they are such great work-horses. Goal-preservation is also an essential part of wlog reasoning in mathematics, with some examples having analogies to EI.

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FROM INTUITIONISM TO BROUWER'S MODAL LOGIC

Abstract

We try to translate the intuitionistic propositional logic **INT** into Brouwer's modal logic **KTB**. Our translation is motivated by intuitions behind Brouwer's axiom $p \to \Box\Diamond p$. The main idea is to interpret intuitionistic implication as modal strict implication, whereas variables and other positive sentences remain as they are. The proposed translation preserves fragments of the Rieger-Nishimura lattice which is the Lindenbaum algebra of monadic formulas in **INT**. Unfortunately, **INT** is not embedded by this mapping into **KTB**.

Keywords: Intuitionistic logic, Kripke frames, Brouwer's modal logic.

1. Introduction

Brouwer's modal logic **KTB** is defined as the normal extension of the minimal modal logic **K** with the axioms $T = \Box p \to p$ and $B = p \to \Box\Diamond p$. The set of rules consists of the modus ponens, the rule of uniform substitution and the rule of necessitation. **KTB** is complete with respect to reflexive and symmetric Kripke frames. It has been known since the 1930's when O. Becker [1], and C.I. Lewis and C.H. Langford [5] formulated the strict form of the Brouwerian axiom $p \prec \Box\Diamond p$, and considered the appropriate system of logic. It turned out that the Brouwer system is stronger than the Lewis system **S3** and weaker than **S5**.

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There are some connections between the intuitionistic logic and the axiom B. For instance, let us quote G.E. Hughes and M.J. Cresswell [4, p. 57]:

> As it is known, L. Brouwer is the founder of the intuitionist school of mathematics. The law of double negation does not hold in intuitionistic logic. Exactly it holds that (i) $\vdash_{INT} p \to \neg\neg p$ but (ii) $\nvdash_{INT} \neg\neg p \to p$. Suppose that negation has a stronger meaning – necessarily negative. Hence $\neg p$ may be translated as $\Box\neg p$. The corresponding modal formula to (i) is $p \to \Box\neg\Box\neg p$, which gives us $p \to \Box\Diamond p$ and obviously $\vdash_{KTB} p \to \Box\Diamond p$. If we translate (ii) in this way, we obtain: $\Box\Diamond p \to p$, which is not a thesis even of the system **S5** defined below. Hence $\nvdash_{KTB} \Box\Diamond p \to p$. (..) Thus although the connection with Brouwer is somewhat tenuous, historical usage has continued to associate his name with this formula.

This motivation will be a starting point for our research. We define a function t from the intuitionistic propositional language $\{\to, \wedge, \vee, \bot\}$ into the modal language $\{\to, \wedge, \vee, \Box, \bot\}$. Thus, let us define

Definition 1.1.

$$t(\bot) = \bot, \qquad t(p) = p, \qquad t(\alpha \to \beta) = \Box(t(\alpha) \to t(\beta)),$$
$$t(\alpha \wedge \beta) = t(\alpha) \wedge t(\beta), \qquad t(\alpha \vee \beta) = t(\alpha) \vee t(\beta).$$

The function t will be the desired translation if the equivalence holds:

$$\alpha \in \mathbf{INT} \text{ iff } t(\alpha) \in \mathbf{KTB}.$$

Our translation differs from the standard one (see, for instance, [3, 7]), known as the Gödel-McKinsey-Tarski translation, for which **S4** turns out to be a modal companion of the intuitionistic logic. Note that the Gödel-McKinsey-Tarski translation maps p onto $\Box p$, instead of p, for any propositional variable p. Nevertheless, we have $t(\neg p) = \Box\neg p$ (as $\neg p = p \to \bot$) and $t(\neg\neg p) = \Box\neg\Box\neg p = \Box\Diamond p$.

Suppose a logic **L** is given (in the sequel we deal mainly with **KTB**). We write $\phi =_L \psi$ if both $\phi \to \psi$ and $\psi \to \varphi$ are **L**–valid. We even omit the subscript L, and write $\phi = \psi$ instead of $\phi =_L \psi$, if there is no risk of misunderstanding. It does not mean, however, that we identify

L–equivalent formulas neither we regard any formula as its equivalence class in the the so-called Lindenbaum-Tarski's algebra of **L**.

In our paper we omit definitions of some logical concepts if they can be found in standard text-books on modal logic, e.g., [2, 3]

2. Preliminaries

Our function t translates the intuitionistic law of doubled negation onto Brouwer's axiom:

$$t(p \to \neg\neg p) = p \to \Box\Diamond p.$$

We ask if other intuitionistic theorems are preserved. Let us consider the law of contraposition in the form: $[(p \to q) \wedge \neg q] \to \neg p)$. After applying t we get: $\Box[[\Box(p \to q) \wedge \Box\neg q] \to \Box\neg p]$. We prove that

Lemma 2.1. $\Box[[\Box(p \to q) \wedge \Box\neg q] \to \Box\neg p] \in$ **KTB**.

Proof: Suppose that $\Box[[\Box(p \to q) \wedge \Box\neg q] \to \Box\neg p] \notin$ **KTB**. Then exists a KTB-model $\mathfrak{M} = \langle W, R, V\rangle$ and a point $x_1 \in W$ such that:

$$x_1 \models \Box(p \to q) \wedge \Box\neg q \tag{2.1}$$

$$x_1 \not\models \Box\neg p \tag{2.2}$$

From (2.2) there is another point, say x_2 such that $x_1 R x_2$ and $x_2 \not\models \neg p$, which means that $x_2 \models p$. From (2.1) it follows that for all $x_i \in W$ such that $x_1 R x_i$, we have: $x_i \models p \to q$ and $x_i \models \neg q$. Hence it holds also at the point x_2. Then we obtain:

$$x_2 \models (p \to q), \quad x_2 \models p, \quad x_2 \models \neg q\,. \tag{2.3}$$

This is a contradiction. □

On the other hand, one may notice that this contraposition law in the form : $(p \to q) \to (\neg q \to \neg p)$ after translation is not a theorem of **KTB**.

Lemma 2.2. $\Box[\Box(p \to q) \to \Box(\Box\neg q \to \Box\neg p)] \notin$ **KTB**.

Proof: Let us consider a KTB-model $\mathfrak{M} = \langle W, R, v\rangle$ such that $W = \{x_1, x_2, x_3\}$, $x_i R x_j$ iff $|i - j| \leq 1$ and $v(p) = \{x_3\}$ and $v(q) = \emptyset$. Then we get $x_2 \models \Box\neg q$ and $x_2 \not\models \Box\neg p$. Hence $x_2 \not\models \Box\neg q \to \Box\neg p$ and $x_1 \not\models$

$\Box(\Box\neg q \to \Box\neg p)$. Also $x_i \models p \to q$ for $i = 1, 2$. Then $x_1 \models \Box(p \to q)$. Hence $x_1 \not\models \Box(p \to q) \to \Box(\Box\neg q \to \Box\neg p)$. $\Box$

From the above, it follows that the law of importation: $[p \to (q \to r)] \to [(p \wedge q) \to r]$ is preserved but the exportation $[(p \wedge q) \to r] \to [p \to (q \to r)]$, is not. The negative results for formulas in two and more variables make us study the monadic fragment of intuitionistic logic. At least, the axiom B is a formula in one variable and B turns out to be the translation of the appropriate intutionistic law. Although the deficiency of the modal analog of the exportation law in **KTB** will be an impediment, we might expect that the monadic language is more fit for our translation.

3. Monadic formulas in KTB

As it is known, see for instance [10], intuitionistic formulas containing only one variable, say p, may be enumerated as follows:

Definition 3.1.

$$\begin{array}{lll} \alpha_0 = \bot, & \alpha_1 = p, & \alpha_2 = p \to \bot, \\ \alpha_{2n+1} = \alpha_{2n} \vee \alpha_{2n-1}, & \alpha_{2n+2} = \alpha_{2n} \to \alpha_{2n-1}, & \text{for any } \ n \geq 1 \\ \alpha_\omega = p \to p. & & \end{array}$$

Every monadic formula is equivalent in the intuitionistic logic to one of the α_m's. Therefore, the formulas give rise to the so-called Rieger-Nishimura algebra $\mathcal{R}$, which is a single-generated free Heyting algebra (see Figure 1). The order relation in the algebra may be defined as follows:

$$\alpha \leq \beta \qquad \text{iff} \qquad \alpha \to \beta \in INT.$$

Our aim is to check if the algebra is preserved under the translation t or, more specifically, whether the translations of the formulas α_n give rise to the same algebra in the logic **KTB**.

The translations of α_n's do not cover all monadic modal formulas which means that there are monadic modal formulas, for instance $\neg p$ or $\Diamond p$, which are not equivalent to any $t(\alpha_n)$. It will also turn out that the translation t does not preserve the equivalence of (intuitionistic) formulas. We shall start out, however, our considerations with the observation that the "bottom" fragment of the Rieger-Nishimura algebra, consisting of the formulas

$\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4$, is preserved under the translation. Thus, in **KTB**, all "intuitionistic" relations between the formulas $\alpha_0 - \alpha_4$ are preserved:

OBSERVATION 1. $p \wedge \Box\neg p = \Box\Diamond p \wedge \Box\neg p = \Box\left((p \vee \Box\neg p) \to \bot\right) = \bot$
$\Box\Diamond p \wedge (p \vee \Box\neg p) = p$
$\Box(\Box\Diamond p \to \bot) = \Box(\Box\Diamond p \to \Box\neg p) = \Box(p \to \Box\neg p) = \Box((p \vee \Box\neg p) \to \Box\neg p) = \Box(p \to \bot) = \Box\neg p$
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Adding $t(\alpha_5) = \Box\Diamond p \vee \Box\neg p$ destroys the → structure of the algebra. In **KTB**, we do not have $\Box[(\Box\Diamond p \vee \Box\neg p) \to p] = p$ though $\alpha_5 \to \alpha_1$ is

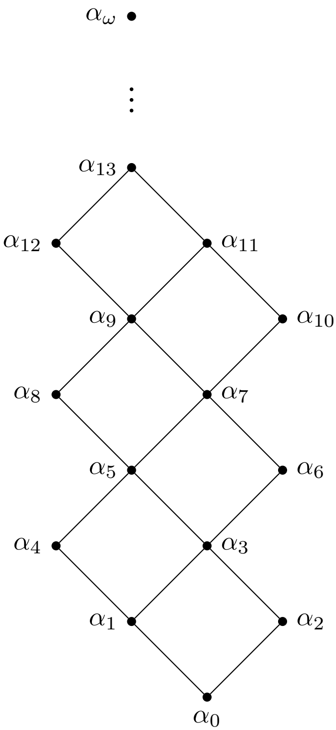

Figure 1

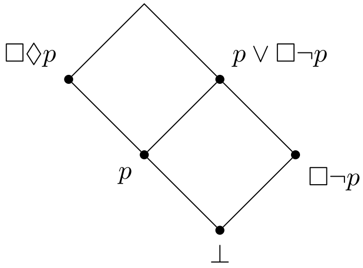

Figure 2

intuitionistically equivalent to α_1. It is clear that we should not expect our translation preserves $\to$. Moreover, it is not true that

$$\varphi =_{INT} \psi \qquad \Rightarrow \qquad t(\varphi) =_{KTB} t(\psi).$$

Let us concentrate on the lattice structure of $\mathcal{R}$ and ask if the Rieger-Nishimura lattice (not Heyting algebra) is preserved under the translation t in **KTB**. Obviously, the fragment of the lattice consisting of $\alpha_0 - \alpha_5$ is preserved. However, even such modified hypothesis turns out to be false as adding $t(\alpha_6) = \Box(\Box\Diamond p \to p)$ to the picture destroys the lattice structure. In the Rieger-Nishimura lattice we have: $\alpha_{2n+3} \wedge \alpha_{2n+4} = \alpha_{2n+1}$, for any $n \geq 0$. We prove that t does not preserve this equation for $n \geq 1$. First, note that:

Lemma 3.2. *$t[(\alpha_{2n+3} \wedge \alpha_{2n+4}) \to \alpha_{2n+1}] \in$ **KTB**, for any $n \geq 1$.*

Proof: We need to show:

$$\{[t(\alpha_{2n+1}) \vee t(\alpha_{2n+2})] \wedge \Box[t(\alpha_{2n+2}) \to t(\alpha_{2n+1})]\} \to t(\alpha_{2n+1}) \in \mathbf{KTB}$$

which is quite obvious. $\Box$

Before we prove that

$$t[\alpha_{2n+1} \to (\alpha_{2n+3} \wedge \alpha_{2n+4})] \notin \mathbf{KTB}, \text{ for any } n \geq 1 \tag{3.1}$$

we shall consider the simplest case when $n = 1$.

LEMMA 3.3. $t[\alpha_3 \to (\alpha_5 \wedge \alpha_6)] \notin \mathbf{KTB}$.

PROOF: We shall prove that $t(\alpha_3) \to \{\Box[t(\alpha_4) \to t(\alpha_3)] \wedge t(\alpha_5)\} \notin \mathbf{KTB}$. Let us take the model $\mathfrak{M}_1 = \langle W, R, v\rangle$ where $W = \{x_0, x_1\}$, and R is the total relation on W, and $x_i \models p$ iff $i = 0$.

Then we have $x_0 \models p$, which gives $x_0 \models p \vee \Box\neg p$ and hence $x_0 \models t(\alpha_3)$. Thus, $x_1 \models \Box\Diamond p$ and this means that $x_1 \models t(\alpha_4)$. We have $x_1 \not\models p$ and $x_1 \not\models \Box\neg p$. Hence we get $x_1 \not\models p \vee \Box\neg p$ which shows $x_1 \not\models t(\alpha_3)$. It means that $x_1 \not\models t(\alpha_4) \to t(\alpha_3)$ and $x_0 \not\models \Box[t(\alpha_4) \to t(\alpha_3)]$. Thus, we proved $x_0 \not\models t(\alpha_3) \to \{\Box[t(\alpha_4) \to t(\alpha_3)] \wedge t(\alpha_5)\}$. $\Box$

For proving (3.1), we shall define some special *KTB*-models which are extensions of the above $\mathfrak{M}_1$. Let

DEFINITION 3.4. $\mathfrak{M}_n = \langle W_n, R_n, v_n\rangle$, for $n \geq 2$, where $W_n = \{x_0, x_1, x_2, ..., x_n\}$, R_n is reflexive and symmetric on W_n, and

$$x_0 R x_i \text{ iff } i \neq 1, \qquad \text{for any } i \leq n; \tag{3.2}$$

$$x_1 R x_i \text{ iff } i \notin \{0, 3\}, \qquad \text{for any } i \leq n; \tag{3.3}$$

$$x_2 R x_i \;, \qquad \text{for any } i \leq n; \tag{3.4}$$

$$x_3 R x_i \text{ iff } i \notin \{1, 4\}, \qquad \text{for any } i \leq n; \tag{3.5}$$

$$\begin{aligned}&\text{if } \; 3 < k < n-1, \; \text{ then } \; x_k R x_i \; \text{ iff } \; i \notin \{k+1, k-1\},\\ &\text{for any } i \leq n;\end{aligned} \tag{3.6}$$

$$\neg x_{n-1} R x_n. \tag{3.7}$$

The valuation v_n is defined: $v_n(p) = \{x_0\}$. See Figure 3.

OBSERVATION 2. If $i \leq n$, then in the model $\mathfrak{M}_n$ it holds that

$$\begin{array}{llllll}
x_i \models \Diamond p & \Leftrightarrow & i \neq 1 & x_i \models t(\alpha_2) & \Leftrightarrow & i = 1;\\
x_i \models t(\alpha_3) & \Leftrightarrow & i = 0, 1 & x_i \models t(\alpha_4) & \Leftrightarrow & i = 0, 3;\\
x_i \models t(\alpha_5) & \Leftrightarrow & i = 0, 1, 3 & x_i \models t(\alpha_6) & \Leftrightarrow & i = 1, 4;\\
x_i \models t(\alpha_7) & \Leftrightarrow & i = 0, 1, 3, 4 & x_i \models t(\alpha_8) & \Leftrightarrow & i = 3, 5.
\end{array}$$

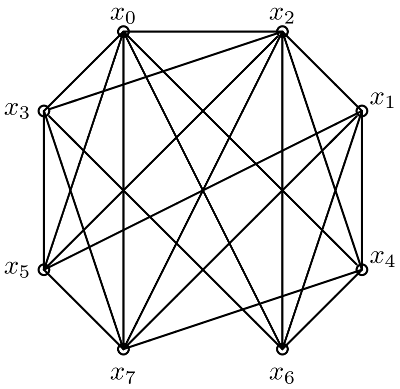

Figure 3. The frame of $\mathfrak{M}_7$

Further:

$$\begin{array}{llllll}
x_i \not\models t(\alpha_4) \to t(\alpha_3) & \Leftrightarrow & i = 3; & x_i \models \Box[t(\alpha_4) \to t(\alpha_3)] & \Leftrightarrow & i = 1, 4; \\
x_i \not\models t(\alpha_6) \to t(\alpha_5) & \Leftrightarrow & i = 4; & x_i \models \Box[t(\alpha_6) \to t(\alpha_5)] & \Leftrightarrow & i = 3, 5; \\
x_i \not\models t(\alpha_8) \to t(\alpha_7) & \Leftrightarrow & i = 5; & x_i \models \Box[t(\alpha_8) \to t(\alpha_7)] & \Leftrightarrow & i = 4, 6.
\end{array}$$

Then we get:

Lemma 3.5. *If* $2 \leq n \leq k$ *and* $i \leq n$ *, then in the model* $\mathfrak{M}_k$ *it holds that*

(i) $x_i \models t(\alpha_{2n+1})$ *iff* $i \leq n + 1$ *and* $i \neq 2$*;*

(ii) $x_i \not\models t(\alpha_{2n}) \to t(\alpha_{2n-1})$ *iff* $i = n + 1$*;*

(iii) $x_i \models \Box[t(\alpha_{2n}) \to t(\alpha_{2n-1})]$ *iff* $i = n$ *or* $i = n + 2$*, (for* $n \geq 3$*).*

Proof: We prove it by induction on n. Let $n = 2$. Then, from Observation 1, we get: $x_i \models t(\alpha_5)$ iff $i = 0, 1, 3$. Also $x_i \not\models t(\alpha_4) \to t(\alpha_3)$ iff $i = 3$. Further $x_i \models \Box[t(\alpha_4) \to t(\alpha_3)]$ iff $i = 1$ or $i = 4$. For $n = 3$, from Observation 1, we get $x_i \models t(\alpha_7)$ iff $i = 0, 1, 3, 4$, and $x_i \not\models t(\alpha_6) \to t(\alpha_5)$ iff $i = 4$, and $x_i \models \Box[t(\alpha_6) \to t(\alpha_5)]$ iff $i = 3$ or $i = 5$.

Assume our lemma holds for n and prove it also holds for $n + 1$. We have $t(\alpha_{2n+3}) = t(\alpha_{2n+2}) \vee t(\alpha_{2n+1})$ and $t(\alpha_{2n+2}) = \Box(t(\alpha_{2n}) \to t(\alpha_{2n-1}))$. From our inductive hypothesis (i) and (ii), we get $x_i \models t(\alpha_{2n+3})$ iff $i \leq n+2$ and $i \neq 2$.

Let us consider $t(\alpha_{2n+2}) \to t(\alpha_{2n+1})$. As $t(\alpha_{2n+2}) = \Box[t(\alpha_{2n}) \to t(\alpha_{2n-1})]$, we get $x_i \not\models t(\alpha_{2n+2}) \to t(\alpha_{2n+1})$ iff $i = n+2$, by our inductive hypothesis (i) and (iii).

From the above and the definition of the relation R_n, it follows that $x_i \models \Box[t(\alpha_{2n+2}) \to t(\alpha_{2n+1})]$ iff $i = n+1$ or $i = n+3$. $\Box$

Then we may prove that (3.1) holds.

Lemma 3.6. *For any $n \geq 2$ we get*

$$t[\alpha_{2n+1} \to (\alpha_{2n+3} \land \alpha_{2n+4})] \notin \mathbf{KTB}$$

Proof: We take advantage of the model $\mathfrak{M}_n$. From Lemma 3.5 we get $x_n \models t(\alpha_{2n+1})$ and $x_n \not\models t(\alpha_{2n+4})$, for any $n \geq 2$. $\Box$

We see that the Rieger-Nishimura lattice loses, after the translation t, some meets of classes of formulas. Since the joins are preserved by the definition of the translation, we conclude that the obtained structure is a join semi-lattice, only. Figure 3 presents the diagram of the (Rieger-Nishimura) join semi-lattice which is preserved under the translation t. Note that the received structure is infinite as from Lemma 3.5 we get

Corollary 3.7. For any $n \geq 1$, we have $t(\alpha_{2n-1}) \to t(\alpha_{2n+1}) \in \mathbf{KTB}$ and $t(\alpha_{2n+1}) \to t(\alpha_{2n-1}) \notin \mathbf{KTB}$.

We also conclude that the function t is a translation for some classes of formulas.

Corollary 3.8. For any n, $k \geq 1$, we have:

1. $\alpha_{2n-1} \to \alpha_{2k-1} \in \mathbf{INT}$ iff $t(\alpha_{2n-1}) \to t(\alpha_{2k-1}) \in \mathbf{KTB}$,
2. $\alpha_{2n-2} \to \alpha_{2k-1} \in \mathbf{INT}$ iff $t(\alpha_{2n-2}) \to t(\alpha_{2k-1}) \in \mathbf{KTB}$.

3.1. Modal counterpart of Glivienko's theorem

Glivienko's theorem says that the double negation of any classically valid propositional formula is intuitionistically valid. Its analog for the modal logics **S5** and **S4** states that $\alpha \in \mathbf{S5}$ iff $\Diamond\Box\alpha \in \mathbf{S4}$, see [6]. There are other results in this subject e.g Rybakov [8] proved that $\Box\Diamond\alpha \to \Box\Diamond\beta \in \mathbf{K4}$ iff $\Diamond\alpha \to \Diamond\beta \in \mathbf{S5}$. Recently Shapirovsky [9] generalizes Glivenko's translation for logics of arbitrary finite height.

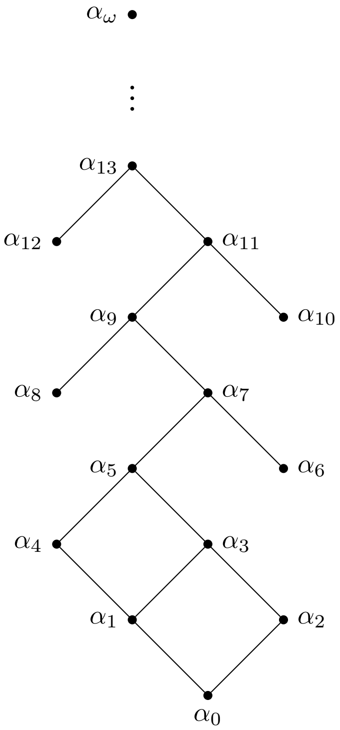

Figure 4

Our approach to Glivienko's theorem is more elementary. The translation t examined in this paper suggests a modal version of this theorem. One could think that it suffices to take $\Box\Diamond\alpha$, instead of the double negation of the classically valid formula α, to obtain the modal version of Glivienko's theorem.

Certainly, it holds for some monadic formulas.

Lemma 3.9. *For any* $n \geq 1$, *we have* $\Box\Diamond t(\alpha_{2n+1}) \in \mathbf{KTB}$.

Proof: By Corollary 3.7, it suffices to show that $\Box\Diamond t(\alpha_3) \in \mathbf{KTB}$ which would be tantamount to prove that $\Diamond(\Box\neg p \vee p) \in \mathbf{KTB}$. But in any modal logic $\Diamond(\Box\neg p \vee p) = \Box\Diamond p \to \Diamond p$ and $\Box\Diamond p \to \Diamond p$ is $\mathbf{KT}$ valid. $\Box$

One could expect that, for any $n \geq 3$, we also have $\Box\Diamond t(\alpha_{2n}) \in \mathbf{KTB}$. But it is not the case. For instance, using the model $\mathfrak{M}_1$ (defined in the proof of Lemma 3.3) one easily shows $\Box\Diamond t(\alpha_6) \notin \mathbf{KTB}$.

3.2. From INT into $KTB.Alt_n$

We may also consider some extensions of the logic **KTB**. Let $\mathbf{KTB.Alt_n}$, for $n \geq 2$, be such an extension with

$$alt_n = \Box p_1 \vee \Box(p_1 \to p_2) \vee ... \vee \Box((p_1 \wedge ... \wedge p_n) \to p_{n+1}).$$

Logics $\mathbf{KTB.Alt_n}$ are characterized by reflexive and symmetric Kripke frames, in which one point has at most n successors (including itself), see [3, p. 82]. We show that there is a simple correlation between the degree of branching and the possibility of falsifying the formula $t(\alpha_{2n+1})$. Namely, we get:

Lemma 3.10. *For each $n \geq 2$, the model $\mathfrak{M}_n$ is a minimal KTB-model falsifying $t(\alpha_{2n+1})$ (which means that any model falsifying this formula contains $\mathfrak{M}_n$ as a submodel).*

Proof: By induction on n. We construct a KTB-model falsifying $t(\alpha_3)$. Because $t(\alpha_3) = p \vee \Box\neg p$ then in some point x falsifying $t(\alpha_3)$ we have

$$x \not\models p, \tag{3.8}$$

$$x \not\models \Box\neg p. \tag{3.9}$$

From (3.9) we get that $x \models \Diamond p$. Then there must exist a point $x^* \in W$ such that xRx^* and $x^* \models p$. By (3.8) we know that $x^* \neq x$. Then we obtain two-point model which is isomorphic to $\mathfrak{M}_1$.

Before we start doing the induction step, we show how the model rises if we want to falsify the formula $t(\alpha_5)$. Because $t(\alpha_5) = p \vee \Box\neg p \vee \Box\Diamond p$ then at the point x we get (3.8), (3.9) and

$$x \not\models \Box\Diamond p. \tag{3.10}$$

From (3.8) and (3.9) we obtain the existence of another point x^* such that xRx^* and $x^* \models p$. Also $x^* \neq x$. From (3.10) we see that there must exist another point, say $x^{**} \in W$ such that xRx^{**} and $x^{**} \not\models \Diamond p$. Hence $x^{**} \not\models p$ and $x \not\models p$. Also $x^{**} \neq x$ and $x^{**} \neq x^*$. We conclude that $\neg\, x^*Rx^{**}$. Then the falsifying model has to have at least three points.

It is not a cluster and the point x sees two others. Since the situation is analogous to the one described in $\mathfrak{M}_2$ we may substitute: $x := x_2$, $x^* := x_0$ and $x^{**} := x_1$. We really have got a minimal model falsifying $t(\alpha_5)$.

Let us try to falsify the formula $t(\alpha_7) = t(\alpha_5) \vee t(\alpha_6)$. For falsifying $t(\alpha_5)$ we need the model $\mathfrak{M}_2$. Then we try to falsify $t(\alpha_6)$ at x_2 which is $x_2 \not\models \Box[t(\alpha_4) \to t(\alpha_3)]$. Then there must exist a point, say x_3, $x_2 R x_3$, such that $x_3 \not\models t(\alpha_4) \to t(\alpha_3)$ what provides to:

$$x_3 \models \Box\Diamond p, \tag{3.11}$$

$$x_3 \not\models p \vee \Box\neg p. \tag{3.12}$$

Because (3.12) holds then $x_3 \neq x_i$ for $i = 0, 1$. Because of (3.11) we get $x_3 \neq x_2$. We need a successor of x_3 in which p is validated. We may take $x_3 R x_0$. Further, we know that $\neg\, x_3 R x_1$. One should remember that the relation R is reflexive and symmetric. Then we see that the minimal model for falsifying $t(\alpha_7)$ has to have four points with the relations and valuation as in $\mathfrak{M}_3$.

Suppose that our thesis holds for n. Then we know that $\mathfrak{M}_n$ is a minimal KTB-model falsifying $t(\alpha_{2n+1})$ and we take advantage of Observation 1 and Lemma 3.5.

We show that the thesis holds for $n + 1$.

We have $t(\alpha_{2n+3}) = t(\alpha_{2n+1}) \vee t(\alpha_{2n+2})$. We want to falsify the formula at the point x_2. For falsifying $t(\alpha_{2n+1})$ the assumption works and we get a model $\mathfrak{M}_n$ such that $(\mathfrak{M}_n, x_2) \not\models t(\alpha_{2n+1})$. Then we want to get $(\mathfrak{M}_n, x_2) \not\models t(\alpha_{2n+2})$ that is $(\mathfrak{M}_n, x_2) \not\models \Box[t(\alpha_{2n}) \to t(\alpha_{2n-1})]$.

There must exist a new point, say x_{n+1} such that $x_2 R x_{n+1}$ and

$$x_{n+1} \not\models t(\alpha_{2n}) \to t(\alpha_{2n-1}), \tag{3.13}$$

From Lemma 3.5 we know that the point x_{n+1} is a new point different from the others. Also $x_2 R x_{n+1}$. Because $x_{n+1} \models t(\alpha_{2n})$ and $x_n \not\models t(\alpha_{2n-2}) \to t(\alpha_{2n-3})$ then $\neg x_n R x_{n+1}$. We also conclude that x_{n+1} sees all other points x_i for $i \neq n$ because we want to have $x_{n+1} \not\models t(\alpha_{2n})$ for $k < n$.

Then, adding a new point x_{n+1} to $\mathfrak{M}_n$, with the suitable relations, we obtain $\mathfrak{M}_{n+1}$. □

A correlation between the degree of branching of a frame and the validity of the formula $t(\alpha_{2n+1})$ is as follows:

Theorem 3.11. *For each* $n \geq 2$, $t(\alpha_{2n+1}) \in \mathbf{KTB.Alt_i}$ *iff* $i \leq n$.

Proof: If $t(\alpha_{2n+1}) \notin \mathbf{KTB.Alt_i}$ then from Lemma 3.10 we conclude that the minimal model falsifying this formula contains the model $\mathfrak{M}_{n+1}$. In this model (see Definition 3.4) the point x_2 sees all other points (including itself), hence the degree of branching of $\mathfrak{M}_{n+1}$ is equal to $n+1$. Then $i > n$. On the other hand, if $i > n$ then among the models for $\mathbf{KTB.Alt_i}$ is the model $\mathfrak{M}_i$, falsifying $t(\alpha_{2n+1})$. $\square$

One may notice that the formulas $t(\alpha_{2n+1})$, $n \geq 1$ written in one variable, have a similar significance as the formulas alt_n, at least in KTB-frames.

Corollary 3.12. $\mathbf{KTB.Alt_i} = \mathbf{KTB} \oplus t(\alpha_{2n+1})$ for any $n \geq 1$.

4. Specific questions

The main problem concerning our translation is the fact that it does not preserve the intuitionistic equivalence of formulas. More specifically, it is not true that

$$\alpha \to \beta \in \mathbf{INT} \qquad \Rightarrow \qquad t(\alpha) \to t(\beta) \in \mathbf{KTB}.$$

We suppose our problem might be solved if we significantly modify our approach. It would be required to opt out from the attempts to define intuitionistic connectives in **KTB** but to translate each formula in its specific way. Technically, it will relay on adding $\Box^k$, for some k, to the predecessor of $t(\alpha) \to t(\beta)$. The number k depends on the difference of modal degrees of the antecedent and consequent of the implication. Let us consider the formula $\Box(t(\alpha_4) \to t(\alpha_3)) \wedge t(\alpha_5) = t(\alpha_3)$ which is not theorem of **KTB** because the reverse implication is not. See Lemma 3.3. The simple implication:

$$(\Box(t(\alpha_4) \to t(\alpha_3)) \wedge t(\alpha_5)) \to t(\alpha_3)$$

which is

$$\{\Box[\Box\Diamond p \to (p \vee \Box\neg p)] \wedge (\Box\Diamond p \vee p \vee \Box\neg p)\} \to (p \vee \Box\neg p)$$

is a theorem of **KTB**. We see that $md\{\Box[\Box\Diamond p \to (p \vee \Box\neg p)] \wedge (\Box\Diamond p \vee p \vee \Box\neg p)\} = 3$ and $md(p \vee \Box\neg p) = 1$. Hence modal degree of the antecedent is larger than the degree of the consequent.

In the reverse implication the situation is opposite and we have $t(\alpha_3) \to [\Box(t(\alpha_4) \to t(\alpha_3)) \wedge t(\alpha_5)] \notin$ **KTB**. We propose the following strengthening of the above formula.

Since $md\{[\Box[t(\alpha_4) \to t(\alpha_3)] \wedge t(\alpha_5)]\} - md(t(\alpha_3)) = 2$ then we consider the formula $\Box^3 t(\alpha_3) \to [\Box(t(\alpha_4) \to t(\alpha_3)) \wedge t(\alpha_5)]$ and obtain:

LEMMA 4.1. *The formula* $\Box^3 t(\alpha_3) \to [\Box(t(\alpha_4) \to t(\alpha_3)) \wedge t(\alpha_5)]$ *is a theorem of* **KTB**.

PROOF: Suppose that $\Box^3 t(\alpha_3) \to [\Box(t(\alpha_4) \to t(\alpha_3)) \wedge t(\alpha_5)] \notin$ **KTB**. Then there exists a model $\mathfrak{M} = \langle W, R, v\rangle$ and a point $x \in W$ such that

$$x \models \Box^3 t(\alpha_3) \tag{4.1}$$

$$x \not\models \Box(t(\alpha_4) \to t(\alpha_3)) \wedge t(\alpha_5) \tag{4.2}$$

From (4.2) we know that $x \not\models \Box(t(\alpha_4) \to t(\alpha_3))$ or $x \not\models t(\alpha_5)$.

I. If $x \not\models \Box(t(\alpha_4) \to t(\alpha_3))$ then there is another point, say x_2, xRx_2 such that $x_2 \not\models t(\alpha_4) \to t(\alpha_3)$ what means that

$$x_2 \models t(\alpha_4) \tag{4.3}$$

$$x_2 \not\models t(\alpha_3) \tag{4.4}$$

But from (4.1) and from reflexivity of R we know that $x_2 \models t(\alpha_3)$. This is a contradiction.

II. If $x \not\models t(\alpha_5)$ then since $\alpha_5 = \alpha_3 \vee \alpha_4$ then $x \not\models t(\alpha_3)$ and $x \not\models t(\alpha_4)$. But $x \not\models t(\alpha_3)$ is in contradiction with (4.1). □

Despite the above example, one should not expect the following holds: if $\alpha \to \beta \in$ **INT**, then

1. if $md(t(\alpha)) > md(t(\beta))$ then $t(\alpha) \to t(\beta) \in$ **KTB**,
2. if $md(t(\beta)) - md(t(\alpha)) = k \geq 0$ then $\Box^{k+1} t(\alpha) \to t(\beta) \in$ **KTB**.

We show that this is false (even for formulas in one variable). The counterexample is the formula $\Box\Diamond t(\alpha_6) = \Box[\Box(t(\alpha_6) \to \bot) \to \bot]$. We see that $\Box[\Box(t(\alpha_6) \to \bot) \to \bot] \in$ **KTB** iff $\Box(t(\alpha_6) \to \bot) \to \bot \in$ **KTB**. Obviously $md(\Box(t(\alpha_6) \to \bot)) > md(\bot)$. Let us take the model $\mathfrak{M}_1$, see Definition 3.4. One may easily obtain that $\mathfrak{M}_1 \not\models \Box\Diamond t(\alpha_6)$. Hence $\Box\Diamond t(\alpha_6) \notin$ **KTB**.

Now, let us consider the implication:

$$t(\alpha) \to t(\beta) \in \mathbf{KTB} \qquad \Rightarrow \qquad \alpha \to \beta \in \mathbf{INT}.$$

We show that it is false. The counterexample is the following formula $\alpha = \neg\neg(T \to p) \to (T \to p)$ which is equivalent to the strong law of doubled negation. Obviously $\alpha \notin \mathbf{INT}$. But we shall prove that:

LEMMA 4.2. $t(\alpha) \in \mathbf{KTB}$.

PROOF: Let us write the formula $t(\alpha) = \Box[\Box\Diamond\Box(T \to p) \to \Box(T \to p)]$. By Brouwer's axiom we have: $\Diamond\Box(T \to p) \to (T \to p) \in \mathbf{KTB}$. Then by the rule of necessitation and the axiom K we obtain $\Box\Diamond\Box(T \to p) \to \Box(T \to p) \in \mathbf{KTB}$. Again by the rule of necessitation we get: $\Box[\Box\Diamond\Box(T \to p) \to \Box(T \to p)] \in \mathbf{KTB}$. $\Box$

5. Conclusions

Since we see that $t(\mathbf{INT}) \not\subseteq \mathbf{KTB}$, we would like to know what is the image of **INT** by the function t.

As it was mentioned above the formula $\Box\Diamond t(\alpha_6) \notin \mathbf{KTB}$ and moreover the model falsifying it is the model $\mathfrak{M}_1$, see Definition 3.4. Actually, $\mathfrak{M}_1$ is a two-element cluster. One easily conclude that $\Box\Diamond t(\alpha_6) \notin \mathbf{S5}$. Hence it must be $\Box\Diamond t(\alpha_6) \in \mathbf{Triv}$. It means that the least modal logic containing $t(\mathbf{INT})$ is **Triv** which is highly unsatisfactory.

Let us add that we do not decide if there is any other translation from **INT** into **KTB**. We leave this problem open. It seems that the intuitionistic logic is too strong for being translated into any intransitive modal logic.

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EMPIRICAL NEGATION, CO-NEGATION AND THE CONTRAPOSITION RULE II: PROOF-THEORETICAL INVESTIGATIONS

Abstract

We continue the investigation of the first paper where we studied logics with various negations including empirical negation and co-negation. We established how such logics can be treated uniformly with R. Sylvan's $\mathbf{CC}_\omega$ as the basis. In this paper we use this result to obtain cut-free labelled sequent calculi for the logics.

Keywords: Empirical negation, co-negation, labelled sequent calculus, intuitionism.

1. Introduction

In the first paper, we semantically investigated how some logics with non-standard negation ($\mathbf{IPC}^{\sim}$ [1, 2], $\mathbf{TCC}_\omega$ [4], **daC** [9] and $\mathbf{CC}_\omega$ [10]) are related to each other. In particular, we noted how the difference between $\mathbf{IPC}^{\sim}$ and $\mathbf{TCC}_\omega$ can be understood as the difference between Kripke and Beth semantics. We also observed, by giving a uniform axiomatisation, how other logics can be captured in the semantics of $\mathbf{CC}_\omega$ in terms of frame conditions. These frames conditions will play an essential role in the proof-theoretical investigation of the present paper.

In this paper, we first re-introduce some definitions and results from the first paper. Then we formulate labelled sequent calculi for the logics, and

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show cut-admissibility and the equivalence of the calculi with Hilbert-style formulations. Finally we have discussions about some topics related to the contents of the two papers.

2. Preliminaries

Let us use the following notations for metavariables.

- $p, q, r, \ldots$ for propositional variables.
- $A, B, C, \ldots$ for formulae.

In this paper, we shall consider the following propositional language

$$\mathcal{L} ::= p \mid (A \wedge B) \mid (A \vee B) \mid (A \to B) \mid {\sim}A.$$

Parentheses will be omitted if there is no fear of ambiguity. We shall use the convention $A \leftrightarrow B := (A \to B) \wedge (B \to A)$.

We shall give Hilbert-style proof systems for $\mathbf{CC}_\omega$, **daC**, $\mathbf{TCC}_\omega$ and $\mathbf{IPC}^{\sim}$, using the uniform axiomatisations we established in the first paper. We identify the systems with the logics themselves for convenience, and denote them simply as $\mathbf{CC}_\omega$, **daC** etc..

Definition 2.1. We will consider following axiom schemata and rules.

<u>Axioms</u>

[Ax1] $A \to (B \to A)$

[Ax2] $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

[Ax3] $(A \wedge B) \to A$

[Ax4] $(A \wedge B) \to B$

[Ax5] $(C \to A) \to ((C \to B) \to (C \to (A \wedge B)))$

[Ax6] $A \to (A \vee B)$

[Ax7] $B \to (A \vee B)$

[Ax8] $(A \to C) \to ((B \to C) \to ((A \vee B) \to C))$

[Ax9] $A \vee {\sim}A$

[Ax10] ${\sim}A \to ({\sim}{\sim}A \to B)$

[Ax11] ${\sim}{\sim}A \to A$

[Ax12] $\sim(\sim(A \vee B) \vee A) \to B$

[Ax13] $(\sim A \wedge \sim B) \to \sim(A \vee B)$

Rules

[MP] $\dfrac{A \qquad A \to B}{B}$ [RC] $\dfrac{A \to B}{\sim B \to \sim A}$

The logics $\mathbf{CC}_\omega$, **daC**, $\mathbf{TCC}_\omega$ and $\mathbf{IPC}^{\sim}$ are each defined by [MP], [RC] and the following axioms.

$\mathbf{CC}_\omega$: [Ax1]–[Ax9] and [Ax11].
daC: [Ax1]–[Ax9] and [Ax12].
$\mathbf{TCC}_\omega$: [Ax1]–[Ax10].
$\mathbf{IPC}^{\sim}$: [Ax1]–[Ax10] and [Ax13].

The derivability in $\mathbf{CC}_\omega$ is denoted by $\vdash_c$. We recall that $\mathbf{IPC}^{\sim}$ and $\mathbf{CC}_\omega$ are the strongest and the weakest of the four systems, and **daC** and $\mathbf{TCC}_\omega$ lie somewhere between the two, while being mutually incomparable.

Semantically, the Kripke semantics of $\mathbf{CC}_\omega$ gives the basis for our inquiry.

DEFINITION 2.2 (Semantics of $\mathbf{CC}_\omega$). A *Kripke frame* $\mathcal{F}^c_{\mathcal{K}}$ for $\mathbf{CC}_\omega$ is a triple $(W, \leq, S)$, where $S \subseteq W \times W$ is a reflexive and symmetric (accessibility) relation such that $u \leq v$ and uSw implies vSw, i.e. S is upward closed. A *Kripke model* $\mathcal{M}^c_{\mathcal{K}}$ for $\mathbf{CC}_\omega$ a pair $(\mathcal{F}^c_{\mathcal{K}}, \mathcal{V})$, where $\mathcal{V}$ is a mapping that assigns a set of worlds $\mathcal{V}(p) \subseteq W$ to each propositional variable p. We assume $\mathcal{V}$ to be *monotone*, viz. $w \in \mathcal{V}(p)$ and $w' \geq w$ implies $w' \in \mathcal{V}(p)$. To denote a model, we shall use both $\mathcal{M}^c_{\mathcal{K}}$ and $(\mathcal{F}^c_{\mathcal{K}}, \mathcal{V})$ interchangeably.

Given $\mathcal{M}^c_{\mathcal{K}}$, the *forcing* (or *valuation*) of a formula in a world, denoted $\mathcal{M}^c_{\mathcal{K}}, w \Vdash_{\mathcal{K}c} A$, is inductively defined as follows.

$$
\begin{array}{lcl}
\mathcal{M}^c_{\mathcal{K}}, w \Vdash_{\mathcal{K}c} p & \Longleftrightarrow & w \in \mathcal{V}(p). \\
\mathcal{M}^c_{\mathcal{K}}, w \Vdash_{\mathcal{K}c} A \wedge B & \Longleftrightarrow & \mathcal{M}^c_{\mathcal{K}}, w \Vdash_{\mathcal{K}c} A \text{ and } \mathcal{M}^c_{\mathcal{K}}, w \Vdash_{\mathcal{K}c} B. \\
\mathcal{M}^c_{\mathcal{K}}, w \Vdash_{\mathcal{K}c} A \vee B & \Longleftrightarrow & \mathcal{M}^c_{\mathcal{K}}, w \Vdash_{\mathcal{K}c} A \text{ or } \mathcal{M}^c_{\mathcal{K}}, w \Vdash_{\mathcal{K}c} B. \\
\mathcal{M}^c_{\mathcal{K}}, w \Vdash_{\mathcal{K}c} A \to B & \Longleftrightarrow & \text{for all } w' \geq w, \text{ if } \mathcal{M}^c_{\mathcal{K}}, w' \Vdash_{\mathcal{K}c} A, \\
 & & \text{then } \mathcal{M}^c_{\mathcal{K}}, w' \Vdash_{\mathcal{K}c} B. \\
\mathcal{M}^c_{\mathcal{K}}, w \Vdash_{\mathcal{K}c} \sim A & \Longleftrightarrow & \mathcal{M}^c_{\mathcal{K}}, w' \nVdash_{\mathcal{K}c} A \text{ for some } w' \text{ such that } wSw'.
\end{array}
$$

We shall occasionally denote uSv also by $vS^{-1}u$. As S is symmetric in $\mathbf{CC}_\omega$, this distinction is not quite necessary. This however clarifies appeals to symmetry in proofs, which becomes significant in a broader context.

Now the following gives the summary of semantical characterisations for our logics.

THEOREM 2.3. *Let $\mathcal{F}^c_{\mathcal{K}}$ be a $\mathbf{CC}_\omega$-frame. Then the following conditions are equivalent:*

(i) $\mathbf{CC}_\omega$ *is sound and weakly complete with respect to the class of all* $\mathbf{CC}_\omega$*-frames.*

(ii) $\mathbf{daC}$ *is sound and weakly complete with respect to the class of* $\mathbf{CC}_\omega$*-frames satisfying the condition*

$$\forall u, v(uSv \text{ implies } \exists wS^{-1}v(w \leq u \text{ and } w \leq v)).$$

(iii) $\mathbf{TCC}_\omega$ *is sound and weakly complete with respect to* $\mathbf{CC}_\omega$*-frames where S is transitive.*

(iv) $\mathbf{IPC}^{\sim}$ *is sound and strongly complete with respect to* $\mathbf{CC}_\omega$*-frames where S is transitive and satisfying the condition*

$$\forall u, v, w(uSv \text{ and } uSw \text{ implies } \exists xS^{-1}u(v \geq x \text{ and } w \geq x).$$

PROOF: (i) and (iii) are established in [10] and [4], respectively. (ii) is Corollary 5.2 of the first paper, and (iv) is a consequence of Proposition 5.6 of the same paper, as discussed thereafter. □

3. Labelled sequent calculus

In this section, we define a labelled sequent calculus for some of the logics we have treated ($\mathbf{CC}_\omega, \mathbf{TCC}_\omega, \mathbf{daC}, \mathbf{IPC}^{\sim}$), with the aid of the insights obtained in the last section regarding their relationship. We shall show the admissibility of cut and the correspondence with the Hilbert-style system.

A *labelled formula* is an expression of the form $x : A$, where A is a formula and x is a *label.* We shall use $x, y, z \ldots$ for labels. We shall additionally consider *relational atoms*, which either have the form xSy, or $x \leq y$. An *item* is either a labelled formula or a relational atom. We denote items by $\alpha, \beta, \ldots$. A *sequent* has the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite multisets of items.

We shall consider the following calculus $\mathbf{G3cc}_\omega$.

Definition 3.1 ($\mathbf{G3cc}_\omega$).

$$x \leq y, x : p, \Gamma \Rightarrow \Delta, y : p \ (\mathrm{Ax1}) \quad x \leq y, \Gamma \Rightarrow \Delta, x \leq y \ (\mathrm{Ax2})$$

$$xSy, \Gamma \Rightarrow \Delta, xSy \ (\mathrm{Ax3})$$

$$\frac{x : A, x : B, \Gamma \Rightarrow \Delta}{x : A \wedge B, \Gamma \Rightarrow \Delta} \ (\mathrm{L}\wedge) \qquad \frac{\Gamma \Rightarrow \Delta, x : A \qquad \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A \wedge B} \ (\mathrm{R}\wedge)$$

$$\frac{x : A, \Gamma \Rightarrow \Delta \qquad x : B, \Gamma \Rightarrow \Delta}{x : A \vee B, \Gamma \Rightarrow \Delta} \ (\mathrm{L}\vee) \ \frac{\Gamma \Rightarrow \Delta, x : A, x : B}{\Gamma \Rightarrow \Delta, x : A \vee B} \ (\mathrm{R}\vee)$$

$$\frac{x \leq y, x : A \to B, \Gamma \Rightarrow \Delta, y : A \qquad x \leq y, x : A \to B, y : B, \Gamma \Rightarrow \Delta}{x \leq y, x : A \to B, \Gamma \Rightarrow \Delta} \ (\mathrm{L}\to)$$

$$\frac{x \leq y^*, y^* : A, \Gamma \Rightarrow \Delta, y^* : B}{\Gamma \Rightarrow \Delta, x : A \to B} \ (\mathrm{R}\to)$$

$$\frac{xSy^*, \Gamma \Rightarrow \Delta, y^* : A}{x : {\sim}A, \Gamma \Rightarrow \Delta} \ (\mathrm{L}{\sim}) \qquad \frac{xSy, y : A, \Gamma \Rightarrow \Delta, x : {\sim}A}{xSy, \Gamma \Rightarrow \Delta, x : {\sim}A} \ (\mathrm{R}{\sim})$$

$$\frac{x \leq x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \ (\mathrm{Ref}) \qquad \frac{x \leq z, x \leq y, y \leq z, \Gamma \Rightarrow \Delta}{x \leq y, y \leq z, \Gamma \Rightarrow \Delta} \ (\mathrm{Trans})$$

$$\frac{xSx, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \ (\mathrm{Ref}_S) \qquad \frac{xSy, ySx, \Gamma \Rightarrow \Delta}{xSy, \Gamma \Rightarrow \Delta} \ (\mathrm{Sym}_S)$$

$$\frac{x \leq y, xSz, ySz, \Gamma \Rightarrow \Delta}{x \leq y, xSz, \Gamma \Rightarrow \Delta} \ (\mathrm{Up})$$

A proof (derivation/deduction) of a sequent $\Gamma \Rightarrow \Delta$ in $\mathbf{G3cc}_\omega$ (to be denoted $\vdash_{Gc} \Gamma \Rightarrow \Delta$) is a tree whose root is the sequent, whose nodes are applications of rules, and whose leaves are axioms (0-premise rules).

In the rules, variables indicated by * are *eigenvariables*, meaning that they cannot occur in the conclusion of the rules. Γ, Δ are called *contexts*, and non-context items in the conclusion are called *principal.* The calculus is in a large part an amalgamation of the labelled calculus for modal logic [6] and intuitionistic logic [7]. It has the rules (Ref_S) and (Sym_S) corresponding to the reflexivity and symmetry in $\mathbf{CC}_\omega$. Additionally, it has the rule (Up) corresponding to the condition for upward closure in $\mathbf{CC}_\omega$.

We shall also consider the following additional rules, corresponding to the additional frame conditions for **daC**, $\mathbf{TCC}_\omega$ and $\mathbf{IPC}^{\sim}$, to $\mathbf{G3cc}_\omega$.

Definition 3.2.

$$\frac{xSy, ySz^*, z^* \leq x, z^* \leq y, \Gamma \Rightarrow \Delta}{xSy, \Gamma \Rightarrow \Delta}\,(\mathrm{Pr}) \qquad \frac{xSz, xSy, ySz, \Gamma \Rightarrow \Delta}{xSy, ySz, \Gamma \Rightarrow \Delta}\,(\mathrm{Trans}_S)$$

$$\frac{xSy, xSz, t^* \leq y, t^* \leq z, xSt^*, \Gamma \Rightarrow \Delta}{xSy, xSz, \Gamma \Rightarrow \Delta}\,(\mathrm{De})$$

Where, as before, labels indicted with * are eigenvariables. The intention is that the addition of (Pr) should correspond to **daC**, (Trans_S) to $\mathbf{TCC}_\omega$, and (Trans_S) and (De) to $\mathbf{IPC}^{\sim}$. We shall denote the addition of some of these rules to $\mathbf{G3cc}_\omega$ by $\mathbf{G3cc}_\omega^+$, and the deducibility is denoted by $\vdash_{Gc+}$.

We shall later observe how the sequent calculi correspond to the Hilbert-style systems. We now proceed with checking some standard properties of the calculi.

Proposition 3.3. $\vdash_{Gc} x \leq y, x : A, \Gamma \Rightarrow \Delta, y : A$

Proof: By [8, Lemma 12.25], we only have to consider the case for $\sim$. When $A \equiv {\sim}B$,

$$\dfrac{\dfrac{\dfrac{\dfrac{z \leq z, z : B, xSz, x \leq y, ySz, \Gamma \Rightarrow \Delta, y : {\sim}B, z : B}{z : B, xSz, x \leq y, ySz, \Gamma \Rightarrow \Delta, y : {\sim}B, z : B}\,(\mathrm{Ref})}{xSz, x \leq y, ySz, \Gamma \Rightarrow \Delta, y : {\sim}B, z : B}\,(\mathrm{R}{\sim})}{xSz, x \leq y, \Gamma \Rightarrow \Delta, y : {\sim}B, z : B}\,(\mathrm{Up})}{x \leq y, x : {\sim}B, \Gamma \Rightarrow \Delta, y : {\sim}B}\,(\mathrm{L}{\sim})$$

where the first line is obtained from the inductive hypothesis. □

Definition 3.4 (substitution of labels). We define the substitution of a label by another label $x[z/w]$, substitution for an item $\alpha[z/w]$ and for a multiset $\Gamma[z/w]$ by the following clauses. ($\circ \in \{\leq, S\}$)

$$x[z/w] \equiv w \text{ if } x \equiv z.$$
$$x[z/w] \equiv x \text{ if } x \not\equiv z.$$
$$\alpha[z/w] \equiv x[z/w] \circ y[z/w] \text{ if } \alpha \equiv x \circ y.$$
$$\alpha[z/w] \equiv x[z/w] : A \text{ if } \alpha \equiv x : A.$$
$$\Gamma[z/w] \equiv \{\alpha[z/w] : \alpha \in \Gamma\}$$

We shall denote instances of substitution by (Sub). In addition, we shall write $\mathbf{G3cc}_\omega^+ \vdash_n \Gamma \Rightarrow \Delta$ if the sequent has a derivation whose depth is less than n. We say a rule is *depth-preserving admissible* (*dp-admissible*) if and only if: if there are derivations of the premises of the rule each with the depth less than n, then there exists a derivation of the conclusion with the depth less than n. If the depth is not preserved, we just say the rule is *admissible.* We shall indicate an application of an admissible rule by a dashed line.

Proposition 3.5 (dp-admissibility of substitution).
The rule $\dfrac{\Gamma \Rightarrow \Delta}{\Gamma[z/w] \Rightarrow \Delta[z/w]}$ (Sub) is dp-admissible in $\mathbf{G3cc}_\omega^+$.

Proof: We argue by induction on the depth of deduction. The intuitionistic rules are already treated in [8, Lemma 12.26]. For others, the case for (Ax2) is immediate, since the result of the substitution is also an instance of (Ax2). The case for (L$\sim$) and (R$\sim$) are similar to those of R$\Box$ and L$\Box$ in modal calculi, respectively; c.f. [8, Lemma 11.4]. The other rules are instances of either *the scheme for mathematical rules* or *the geometric rule scheme* [8, pp. 98, 134], so can be dealt with by the methodology of [8, Lemma 11.4]. □

We shall now move on to consider structural rules.

Definition 3.6 (structural rules).

$$\frac{\Gamma \Rightarrow \Delta}{\alpha, \Gamma \Rightarrow \Delta}\ (\mathrm{LW}) \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \alpha}\ (\mathrm{RW})$$

$$\frac{\alpha, \alpha, \Gamma \Rightarrow \Delta}{\alpha, \Gamma \Rightarrow \Delta}\ (\mathrm{LC}) \quad \frac{\Gamma \Rightarrow \Delta, \alpha, \alpha}{\Gamma \Rightarrow \Delta, \alpha}\ (\mathrm{RC})$$

$$\frac{\Gamma \Rightarrow \Delta, \alpha \qquad \alpha, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ (\mathrm{Cut})$$

Our goal is to prove that (Cut) is admissible. For this purpose we check that the rules of Weakening (LW,RW) and Contraction (LC,RC) are dp-admissible. We start with Weakening.

Proposition 3.7 (dp-admissibility of Weakening). (LW) and (RW) are dp-admissible in $\mathbf{G3cc}_\omega^+$.

Proof: The proof is by induction on the depth of deduction. In cases of applications of (Ax1)-(Ax3), the result of Weakening is again an instance of the axiom. For other rules, we apply the inductive hypothesis to the premises of the rule, and thereafter apply the rule to obtain the desired sequent; however for rules involving eigenvariables, we first need to apply dp-admissible substitution (Proposition 3.5) to substitute the eigenvariable with a fresh variable, so as to avoid the clash of variables. Then we apply the above procedure. □

For Contraction, we first need to demonstrate that the rules of $\mathbf{G3cc}^{+}_{\omega}$ are *dp-invertible*: that is, given a derivation of the conclusion of a rule, we can find a depth-preserving derivation of the premises.

Lemma 3.8. *The rules of* $\mathbf{G3cc}^{+}_{\omega}$ *are dp-invertible.*

Proof: We argue by induction on the depth of deduction. For the intuitionistic rules, we refer to [8, Theorem 12.28]. For the rules (R∼), (Up), (Ref$_S$), (Sym$_S$), (Trans$_S$), (Pr) and (De), we can invert the sequent by dp-admissible weakening.

The case for (L∼) is quite similar to that of R□ for modal logic [8, Lemma 11.7]. If $\vdash_0 x : {\sim}A, \Gamma \Rightarrow \Delta$, then the derivation is an instance of (Ax1), (Ax2) or (Ax3). In each case, $xSy, \Gamma \Rightarrow \Delta, y : A$ is also an instance of the same axiom. If $\vdash_{n+1} x : {\sim}A, \Gamma \Rightarrow \Delta$, then if it is obtained by (L∼) with $x : {\sim}A$ principal, i.e. it is of the form

$$\frac{\vdash_n xSz, \Gamma \Rightarrow \Delta, z : A}{\vdash_{n+1} x : {\sim}A, \Gamma \Rightarrow \Delta}\,(\mathrm{L}{\sim})$$

where z does not occur in the conclusion; then by dp-admissible substitution, $\vdash_n xSy, \Gamma \Rightarrow \Delta, y : A$ (where y is a fresh variable).

We exemplify with (R→) the case where $x : {\sim}A$ is obtained by a rule with eigenvalue condition in which it is not principal

$$\frac{\vdash_n z \leq t, t : C, x : {\sim}A, \Gamma' \Rightarrow \Delta', t : D}{\vdash_{n+1} x : {\sim}A, \Gamma' \Rightarrow \Delta', z : C \to D}\,(\mathrm{R}{\to})$$

then by dp-substitution, $\vdash_n z \leq t', t' : C, x : {\sim}A, \Gamma' \Rightarrow \Delta', t' : D$, where $t' \not\equiv y$. (Note that y is fixed beforehand.) By I.H., $\vdash_n z \leq t', t' : C, xSy, \Gamma' \Rightarrow \Delta', y : A, t' : D$. So by (R→), $\vdash_{n+1} xSy, \Gamma' \Rightarrow \Delta', y : A, z : C \to D$.

If it is obtained by a rule without eigenvariable condition, then apply I.H. to the premise and apply the same rule. □

Proposition 3.9 (dp-admissibility of Contraction). (LC) and (RC) are dp-admissible in $\mathbf{G3cc}^{+}_{\omega}$.

Proof: We argue by simultaneous induction ((LC),(RC)) on the depth of the deduction. General outline is as in [8, Theorem 12.28]. As an example, suppose $\vdash_{n+1} x : {\sim}A, x : {\sim}A, \Gamma \Rightarrow \Delta$ and the last step is an instance of (L$\sim$) with $x : {\sim}A$ principal.

$$\frac{\vdash_n xSy, x : {\sim}A, \Gamma \Rightarrow \Delta, y : A}{\vdash_{n+1} x : {\sim}A, x : {\sim}A, \Gamma \Rightarrow \Delta}\ (\mathrm{L}{\sim})$$

Then by dp-admissible invertibility of (L$\sim$),

$$\vdash_n xSy, xSy, \Gamma \Rightarrow \Delta, y : A, y : A$$

So by I.H.

$$\vdash_n xSy, \Gamma \Rightarrow \Delta, y : A$$

Thus by (L$\sim$)

$$\vdash_{n+1} x : {\sim}A, \Gamma \Rightarrow \Delta$$

□

We are now ready to prove the admissibility of (Cut). We shall call the item to be eliminated in (Cut) the *cut-item.*

Theorem 3.10 (admissibility of Cut). *(Cut) is admissible in* $\mathbf{G3cc}^{+}_{\omega}$*.*

Proof: We argue by induction on the complexity of cut-items, with a subinduction on the *level* (the sum of the depths of the deductions of the premises) of (Cut). Again the outline is the same as that of the intuitionistic case [8, Theorem 12.30]. In particular, rules that are mathematical or geometric are treated similarly to those of intermediate axioms.

Here we shall consider the case where the cut-item is principal in both of the premises, and has the form $x : {\sim}A$. We have

$$\frac{\dfrac{\vdash_{m-1} xSy, y : A, \Gamma \Rightarrow \Delta, x : {\sim}A}{\vdash_m xSy, \Gamma \Rightarrow \Delta, x : {\sim}A}\ (\mathrm{R}{\sim}) \qquad \dfrac{\vdash_{n-1} xSz, \Gamma' \Rightarrow \Delta', z : A}{\vdash_n x : {\sim}A, \Gamma' \Rightarrow \Delta'}\ (\mathrm{L}{\sim})}{\vdash xSy, \Gamma\Gamma' \Rightarrow \Delta\Delta'}\ (\mathrm{Cut})$$

Then

$$\begin{array}{c} \vdash_{n-1} xSz, \Gamma' \Rightarrow \Delta', z : A \\ \hdashline \vdash_{n-1} xSy, \Gamma' \Rightarrow \Delta', y : A \end{array} \ (\text{Sub})$$

(note that z is an eigenvariable, so cannot occur in Γ', Δ'). Moreover, by I.H. the following cut of a lower level ($m + n - 1 < m + n$) is admissible:

$$\begin{array}{c} \vdash_{m-1} xSy, y : A, \Gamma \Rightarrow \Delta, x : {\sim}A \qquad \vdash_{n} x : {\sim}A, \Gamma' \Rightarrow \Delta' \\ \hdashline \vdash xSy, y : A, \Gamma\Gamma' \Rightarrow \Delta\Delta' \end{array} \ (\text{I.H.})$$

From these, and a cut of lower complexity (admissible by I.H.), we obtain

$$\begin{array}{c} \begin{array}{c} \vdash_{n-1} xSy, \Gamma' \Rightarrow \Delta', y : A \qquad \vdash xSy, y : A, \Gamma\Gamma' \Rightarrow \Delta\Delta' \\ \hdashline \vdash xSy, xSy, \Gamma\Gamma'\Gamma' \Rightarrow \Delta\Delta'\Delta' \end{array} \ (\text{I.H.}) \\ \hdashline \vdash xSy, \Gamma\Gamma' \Rightarrow \Delta\Delta' \end{array} \ (\text{Contraction}) \qquad \square$$

Next we observe that $\mathbf{G3cc}_\omega$ and calculi in $\mathbf{G3cc}_\omega^+$ indeed correspond to $\mathbf{CC}_\omega$, $\mathbf{daC}$, $\mathbf{TCC}_\omega$ and $\mathbf{IPC}^\sim$.

Proposition 3.11.

(i) $\vdash_c A$ implies $\vdash_{Gc} \Rightarrow x : A$.

(ii) If we add (Pr)/(trans$_S$) to the calculus, then the axioms of $\mathbf{daC}/\mathbf{TCC}_\omega$ become derivable. If we add (trans$_S$) and (De), the axioms of $\mathbf{IPC}^\sim$ become derivable.

Proof:

(i) For $\mathbf{CC}_\omega$, the positive axioms can be shown to be derivable as in the intuitionistic case. We need to check [Ax9],[Ax10] and [RC].

[Ax9]

$$\dfrac{\dfrac{\dfrac{\dfrac{x \leq x, xSx, x : A \Rightarrow x : A, x : {\sim}A}{xSx, x : A \Rightarrow x : A, x : {\sim}A}\ (\text{Ref})}{xSx \Rightarrow x : A, x : {\sim}A}\ (\text{R}{\sim})}{\Rightarrow x : A, x : {\sim}A}\ (\text{Ref}_S)}{\Rightarrow x : A \vee {\sim}A}\ (\text{R}\vee)$$

[Ax10]

$$\begin{array}{cl}
y \leq y, x \leq y, ySz, zSy, y : A \Rightarrow z : {\sim}A, y : A & \\ \hline
x \leq y, ySz, zSy, y : A \Rightarrow z : {\sim}A, y : A & \text{(Ref)} \\ \hline
x \leq y, ySz, zSy \Rightarrow z : {\sim}A, y : A & \text{(R}{\sim}\text{)} \\ \hline
x \leq y, ySz \Rightarrow z : {\sim}A, y : A & (\text{Sym}_S) \\ \hline
x \leq y, y : {\sim}{\sim}A \Rightarrow y : A & \text{(L}{\sim}\text{)} \\ \hline
\Rightarrow x : {\sim}{\sim}A \rightarrow A & (\text{R}\rightarrow)
\end{array}$$

[RC]

We first observe that $x : A \rightarrow B, x : A \Rightarrow x : B$ is derivable:

$$\begin{array}{cl}
x \leq x, x : A \rightarrow B, x : A \Rightarrow x : A \quad x \leq x, x : A \rightarrow B, x : B \Rightarrow x : B & \\ \hline
x \leq x, x : A \rightarrow B, x : A \Rightarrow x : B & (\text{L}\rightarrow) \\ \hline
x : A \rightarrow B, x : A \Rightarrow x : B & \text{(Ref)}
\end{array}$$

Then

$$\begin{array}{cl}
\Rightarrow x : A \rightarrow B \qquad x : A \rightarrow B, x : A \Rightarrow x : B & \\ \hdashline
x : A \Rightarrow x : B & \text{(Cut)} \\ \hdashline
y \leq z, zSx, x : A \Rightarrow x : B, z : {\sim}A & \text{(Weakening)} \\ \hline
y \leq z, zSx \Rightarrow x : B, z : {\sim}A & \text{(R}{\sim}\text{)} \\ \hline
y \leq z, z : {\sim}B \Rightarrow z : {\sim}A & \text{(L}{\sim}\text{)} \\ \hline
\Rightarrow y : {\sim}B \rightarrow {\sim}A & (\text{R}\rightarrow) \\ \hdashline
\Rightarrow x : {\sim}B \rightarrow {\sim}A & \text{(Sub)}
\end{array}$$

(ii) We need to check each of the additional axioms are derivable in the corresponding calculi.

daC

First we apply (L$\vee$) to

$$x \leq y, ySz, zSt, t \leq y, t \leq z, t : A \Rightarrow z : {\sim}(A \vee B), z : A, y : B;$$
$$x \leq y, ySz, zSt, t \leq y, t \leq z, t : B \Rightarrow z : {\sim}(A \vee B), z : A, y : B$$

to obtain

$$x \le y, ySz, zSt, t \le y, t \le z, t : A \vee B \Rightarrow z : {\sim}(A \vee B), z : A, y : B$$

Then,

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x \le y, ySz, zSt, t \le y, t \le z, t : A \vee B \Rightarrow z : {\sim}(A \vee B), z : A, y : B}{x \le y, ySz, zSt, t \le y, t \le z \Rightarrow z : {\sim}(A \vee B), z : A, y : B}\,(\mathrm{R}{\sim})}{x \le y, ySz \Rightarrow z : {\sim}(A \vee B), z : A, y : B}\,(\mathrm{Pr})}{x \le y, ySz \Rightarrow z : {\sim}(A \vee B) \vee A, y : B}\,(\mathrm{R}\vee)}{x \le y, y : {\sim}({\sim}(A \vee B) \vee A) \Rightarrow y : B}\,(\mathrm{L}{\sim})}{\Rightarrow x : {\sim}({\sim}(A \vee B) \vee {\sim}A) \to B}\,(\mathrm{R}{\to})$$

<u>TCC$_\omega$</u>

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{u \le u, y \le z, ySu, zSv, zSu, vSz, vSu, u : A \Rightarrow u : A, v : {\sim}A}{y \le z, ySu, zSv, zSu, vSz, vSu, u : A \Rightarrow u : A, v : {\sim}A}\,(\mathrm{Ref})}{y \le z, ySu, zSv, zSu, vSz, vSu \Rightarrow u : A, v : {\sim}A}\,(\mathrm{R}{\sim})}{y \le z, ySu, zSv, zSu, vSz \Rightarrow u : A, v : {\sim}A}\,(\mathrm{Trans}_S)}{y \le z, ySu, zSv, zSu \Rightarrow u : A, v : {\sim}A}\,(\mathrm{Sym}_S)}{y \le z, ySu, zSv \Rightarrow u : A, v : {\sim}A}\,(\mathrm{Up})}{y \le z, ySu, z : {\sim}{\sim}A \Rightarrow u : A}\,(\mathrm{L}{\sim})}{\begin{array}{c} y \le z, y : {\sim}A, z : {\sim}{\sim}A \Rightarrow \\ \hdashline x \le y, y \le z, y : {\sim}A, z : {\sim}{\sim}A \Rightarrow z : B \end{array}\,(\mathrm{Weakening})}\,(\mathrm{L}{\sim})}{x \le y, y : {\sim}A \Rightarrow y : {\sim}{\sim}A \to B}\,(\mathrm{R}{\to})}{x : {\sim}A \to ({\sim}{\sim}A \to B)}\,(\mathrm{R}{\to})$$

<u>IPC$^{\sim}$</u>

First we apply (L∨) to

$$ySz, ySt, w : A, w \le z, w \le t, ySw \Rightarrow z : A, t : B, y : {\sim}(A \vee B);$$
$$ySz, ySt, w : B, w \le z, w \le t, ySw \Rightarrow z : A, t : B, y : {\sim}(A \vee B)$$

to obtain

$$ySz, ySt, w : A \vee B, w \le z, w \le t, ySw \Rightarrow z : A, t : B, y : {\sim}(A \vee B)$$

Then

$$
\begin{array}{cl}
ySz, ySt, w : A \vee B, w \leq z, w \leq t, ySw \Rightarrow z : A, t : B, y : {\sim}(A \vee B) & \\ \hline
ySz, ySt, w \leq z, w \leq t, ySw \Rightarrow z : A, t : B, y : {\sim}(A \vee B) & (\mathrm{R}{\sim}) \\ \hline
ySz, ySt \Rightarrow z : A, t : B, y : {\sim}(A \vee B) & (\mathrm{De}) \\ \hline
ySz, y : {\sim}B \Rightarrow z : A, y : {\sim}(A \vee B) & (\mathrm{L}{\sim}) \\ \hline
y : {\sim}A, y : {\sim}B \Rightarrow y : {\sim}(A \vee B) & (\mathrm{L}{\sim}) \\ \hline
y : {\sim}A \wedge {\sim}B \Rightarrow y : {\sim}(A \vee B) & (\mathrm{L}\wedge) \\ \hdashline
x \leq y, y : {\sim}A \wedge {\sim}B \Rightarrow y : {\sim}(A \vee B) & (\text{Weakening}) \\ \hline
\Rightarrow x : ({\sim}A \wedge {\sim}B) \rightarrow {\sim}(A \vee B) & (\mathrm{R}{\rightarrow})
\end{array}
$$

□

For the converse direction, we argue via completeness of the Hilbert-style systems with respect to Kripke semantics. For this purpose the notion of valuation has to be modified in the style of [8, Definition 11.25-26], to accommodate labelled formulae and relational atoms.

Definition 3.12 (modified valuation for $\mathbf{CC}_\omega$-model). Let $\mathcal{F}^c_{\mathcal{K}}$ be a $\mathbf{CC}_\omega$-frame. A *modified valuation* $\mathcal{V}_m$ is a pair $(\mathcal{V}, l)$, where $\mathcal{V}$ is the ordinary valuation for $\mathcal{F}^c_{\mathcal{K}}$ introduced before, and l maps each label x into to a world $l(x)$of $\mathcal{F}^c_{\mathcal{K}}$.

We say an item α is valid in a *modified model* $\mathcal{M}^{mc}_{\mathcal{K}} = (\mathcal{F}^c_{\mathcal{K}}, \mathcal{V}_m)$ (denoted $\mathcal{M}^{mc}_{\mathcal{K}} \Vdash^m_{\mathcal{K}c} \alpha$), when

- $l(x) \leq l(y)$ (or $l(x)Sl(y)$) in $\mathcal{F}^c_{\mathcal{K}}$, if $\alpha \equiv x \leq y$ (or xSy).
- $(\mathcal{F}^c_{\mathcal{K}}, \mathcal{V}), l(x) \Vdash_{\mathcal{K}c} A$, if $\alpha \equiv x : A$.

We say a sequent $\Gamma \Rightarrow \Delta$ is valid in $\mathcal{M}^{mc}_{\mathcal{K}}$ (denoted $\mathcal{M}^{mc}_{\mathcal{K}} \vDash^m_{\mathcal{K}c} \Gamma \Rightarrow \Delta$), if $\mathcal{M}^{mc}_{\mathcal{K}} \Vdash^m_{\mathcal{K}c} \alpha$ for all $\alpha \in \Gamma$ implies $\mathcal{M}^{mc}_{\mathcal{K}} \Vdash^m_{\mathcal{K}c} \beta$ for some $\beta \in \Delta$. If $\mathcal{M}^{mc}_{\mathcal{K}}$ is arbitrary, we say $\Gamma \Rightarrow \Delta$ is valid and write $\vDash^m_{\mathcal{K}c} \Gamma \Rightarrow \Delta$.

Note that $\vDash^m_{\mathcal{K}c} \Rightarrow x : A$ is valid if and only if $\vDash_{\mathcal{K}c} A$. Similar statements hold when we restrict the class of frames. We now wish to demonstrate the following.

Proposition 3.13.

(i) $\vdash_{Gc} \Gamma \Rightarrow \Delta$ implies $\vDash^m_{\mathcal{K}c} \Gamma \Rightarrow \Delta$.

(ii) (Pr)/(Trans$_S$)/(De) become sound if we restrict consideration to the corresponding classes of frames.

Proof:

(i) We argue by induction on the depth of deductions. The cases for the intuitionistic rules follow straightforwardly from the definition of intuitionistic Kripke models. We shall look at the cases for (L$\sim$), (R$\sim$). The other cases are straightforward. In each case, we consider an arbitrary modified model $\mathcal{M}^{mc}_{\mathcal{K}} = (\mathcal{F}^{c}_{\mathcal{K}}, \mathcal{V}_m)$ with $\mathcal{V}_m = (\mathcal{V}, l)$.

(L$\sim$) Suppose $\mathcal{M}^{mc}_{\mathcal{K}} \Vdash^{m}_{\mathcal{K}c} \alpha$ for all $\alpha \in \{x : {\sim}A\} \cup \Gamma$. Then $(\mathcal{F}^{c}_{\mathcal{K}}, \mathcal{V}), l(x) \Vdash_{\mathcal{K}c} {\sim}A$. So there is w such that $l(x)Sw$ and $(\mathcal{F}^{c}_{\mathcal{K}}, \mathcal{V}), w \nVdash_{\mathcal{K}c} A$. Take $\mathcal{V}'_m = (\mathcal{V}, l')$ where $l' = l$ except $l'(y) = w$. Note, since y does not occur in Γ and Δ, l and l' evaluate them in the same way. Thus $(\mathcal{F}^{c}_{\mathcal{K}}, \mathcal{V}'_m) \Vdash^{m}_{\mathcal{K}c} \alpha$ for all $\alpha \in \{xSy\} \cup \Gamma$. So by I.H., $(\mathcal{F}^{c}_{\mathcal{K}}, \mathcal{V}'_m) \Vdash^{m}_{\mathcal{K}c} \beta$ for some $\beta \in \{y : A\} \cup \Delta$. If it validates $y : A$, however, then $(\mathcal{F}^{c}_{\mathcal{K}}, \mathcal{V}), l'(y) \Vdash_{\mathcal{K}c} A$, a contradiction. Therefore $(\mathcal{F}^{c}_{\mathcal{K}}, \mathcal{V}_m) \Vdash^{m}_{\mathcal{K}c} \beta$ for some $\beta \in \Delta$. Since $\mathcal{M}^{mc}_{\mathcal{K}}$ is arbitrary, $\vDash^{m}_{\mathcal{K}c} x : {\sim}A, \Gamma \Rightarrow \Delta$.

(R$\sim$) Suppose $\mathcal{M}^{mc}_{\mathcal{K}} \Vdash^{m}_{\mathcal{K}c} \alpha$ for all $\alpha \in \{xSy\} \cup \Gamma$. If $l(y) \nVdash_{\mathcal{K}c} A$, then $l(x) \Vdash_{\mathcal{K}c} {\sim}A$. Otherwise, $l(y) \Vdash_{\mathcal{K}c} A$, so by I.H., $\mathcal{M}^{mc}_{\mathcal{K}} \Vdash^{m}_{\mathcal{K}c} \alpha$ for all $\alpha \in \{xSy, y : A\} \cup \Gamma$. So in either case (the latter with I.H.), $\mathcal{M}^{mc}_{\mathcal{K}} \Vdash^{m}_{\mathcal{K}c} \beta$ for some $\beta \in \Delta \cup \{x : {\sim}A\}$.

(ii) The case for (Trans$_S$) is straightforward and (Pr), (De) are similar to the case for (L$\sim$); one needs to appeal to the frame condition to pick out a world satisfying the desired order relation; then define a new modified valuation which is identical to the original except it assigns the world to the eigenvariable; then the rest follows as in the case for (L$\sim$). □

This allows us to conclude the other direction.

Corollary 3.14.

(i) $\vdash_{Gc} \Rightarrow x : A$ implies $\vdash_c A$.

(ii) $\vdash_{Gc+}$ is sound with respect to the corresponding logics (**daC**,**TCC**$_\omega$,**IPC**$^{\sim}$).

Proof:

(i) If $\vdash_{Gc} \Rightarrow x : A$, then by the previous proposition, $\vDash^{m}_{\mathcal{K}c} \Rightarrow x : A$. Then as we remarked before, $\vDash_{\mathcal{K}c} A$. Thus by the completeness of **CC**$_\omega$, $\vdash_c A$.

(ii) Similar. □

4. Discussion

We have seen that $\mathbf{TCC}_\omega$ can be regarded as the logic of empirical and co-negation for Beth semantics, which differs from $\mathbf{IPC}^{\sim}$ and **daC** for Kripke semantics. According to the interpretation in [11, p.679], the difference between Kripke and Beth semantics is the treatment of time. A node in Kripke models signifies a state of information, whereas in Beth models it signifies a moment in time. So for instance, to decide the forcing of a disjunction in a Kripke model, one can stay in a world as much as one likes, until one learns which of the disjuncts is true. In comparison, in Beth models this waiting time is expressed by posterior nodes, so we need to refer to those other worlds to decide the forcing of the disjunction in the original world. The two kinds of empirical and co-negation can be interpreted similarly.

The question remains, however, which empirical (or co-) negation one actually means in an assertion of negation. For example, if one says "There is no proof of P=NP", does it mean there is no proof at the present state of information, or there is no proof at the present moment?

Changing perspective, from an intuitionistic viewpoint there is a certain advantage in considering Beth semantics. There is a relatively simple proof of intuitionistic completeness (proving completeness with only intuitionistically accepted principles) for intuitionistic logic [3, 11]. The intuitionistic completeness proof for Kripke semantics [12] gives a more refined result, but is comparatively more involved. A possible future direction is to show the intuitionistic completeness for $\mathbf{TCC}_\omega$. An obstacle would be the treatment of excluded middle, but classical logic also has an intuitionistic completeness proof [5], so possibly this may be overcome. An intuitionistic completeness would be desirable if one is a full-fledged intuitionist, especially when the logic is motivated from the semantics, rather than from the syntax.

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LENGTH NEUTROSOPHIC SUBALGEBRAS OF BCK/BCI-ALGEBRAS[1]

Abstract

Given $i, j, k \in \{1, 2, 3, 4\}$, the notion of (i, j, k)-length neutrosophic subalgebras in BCK/BCI-algebras is introduced, and their properties are investigated. Characterizations of length neutrosophic subalgebras are discussed by using level sets of interval neutrosophic sets. Conditions for level sets of interval neutrosophic sets to be subalgebras are provided.

Keywords: Interval neutrosophic set, interval neutrosophic length, length neutrosophic subalgebra.

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1. Introduction

The intuitionistic fuzzy set, which has been introduced by Atanassov [1], consider both truth-membership and falsity membership. The neutrosophic set developed by Smarandache [6, 7, 8] is a formal framework which generalizes the concept of the classic set, fuzzy set, interval valued fuzzy set, intuitionistic fuzzy set, interval valued intuitionistic fuzzy set and paraconsistent set etc. Neutrosophic set theory is applied to various part, including algebra, topology, control theory, decision making problems, medicines and in many real life problems. Wang et al. [9, 11] presented the concept of interval neutrosophic sets, which is more precise and more flexible than the single-valued neutrosophic set. An interval-valued neutrosophic set is a generalization of the concept of single-valued neutrosophic set, in which three membership (t, i, f) functions are independent, and their values belong to the unit interval $[0, 1]$. The interval neutrosophic set can represent uncertain, imprecise, incomplete and inconsistent information which exists in real world. Jun et al. [4] discussed interval neutrosophic sets in BCK/BCI-algebras. They introduced the notion of $(T(i, j), I(k, l), F(m, n))$-interval neutrosophic subalgebras in BCK/BCI-algebras for $i, j, k, l, m, n \in \{1, 2, 3, 4\}$, and investigated several properties and relations. They also introduced the notion of interval neutrosophic length of an interval neutrosophic set, and investigated related properties.

In this paper, we introduce the notion of (i, j, k)-length neutrosophic subalgebras in BCK/BCI-algebras for $i, j, k \in \{1, 2, 3, 4\}$, and investigate several properties. We consider relations of (i, j, k)-length neutrosophic subalgebras, and discuss characterizations of (i, j, k)-length neutrosophic subalgebras. Using subalgebras of a BCK-algebra, we construct (i, j, k)-length neutrosophic subalgebras for $i, j, k \in \{1, 4\}$. We consider conditions for level sets of interval neutrosophic set to be subalgebras of a BCK/BCI-algebra.

2. Preliminaries

By a *BCI-algebra* we mean a system $X := (X, *, 0) \in K(\tau)$ in which the following axioms hold:

(I) $((x * y) * (x * z)) * (z * y) = 0,$

(II) $(x * (x * y)) * y = 0,$

(III) $x * x = 0$,

(IV) $x * y = y * x = 0 \;\Rightarrow\; x = y$

for all $x, y, z \in X$. If a BCI-algebra X satisfies $0 * x = 0$ for all $x \in X$, then we say that X is a *BCK-algebra*.

A non-empty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$.

The collection of all BCK-algebras and all BCI-algebras are denoted by $\mathcal{B}_K(X)$ and $\mathcal{B}_I(X)$, respectively. Also $\mathcal{B}(X) := \mathcal{B}_K(X) \cup \mathcal{B}_I(X)$.

We refer the reader to the books [2] and [5] for further information regarding BCK/BCI-algebras.

By a *fuzzy structure* over a nonempty set X we mean an ordered pair (X, ρ) of X and a fuzzy set ρ on X.

DEFINITION 2.1 ([3]). For any $(X, *, 0) \in \mathcal{B}(X)$, a fuzzy structure (X, μ) over $(X, *, 0)$ is called a

- *fuzzy subalgebra* of $(X, *, 0)$ with type 1 (briefly, 1-*fuzzy subalgebra* of $(X, *, 0)$) if

$$(\forall x, y \in X)\,(\mu(x * y) \geq \min\{\mu(x), \mu(y)\})\,, \tag{2.1}$$

- *fuzzy subalgebra* of $(X, *, 0)$ with type 2 (briefly, 2-*fuzzy subalgebra* of $(X, *, 0)$) if

$$(\forall x, y \in X)\,(\mu(x * y) \leq \min\{\mu(x), \mu(y)\})\,, \tag{2.2}$$

- *fuzzy subalgebra* of $(X, *, 0)$ with type 3 (briefly, 3-*fuzzy subalgebra* of $(X, *, 0)$) if

$$(\forall x, y \in X)\,(\mu(x * y) \geq \max\{\mu(x), \mu(y)\})\,, \tag{2.3}$$

- *fuzzy subalgebra* of $(X, *, 0)$ with type 4 (briefly, 4-*fuzzy subalgebra* of $(X, *, 0)$) if

$$(\forall x, y \in X)\,(\mu(x * y) \leq \max\{\mu(x), \mu(y)\})\,. \tag{2.4}$$

Let X be a non-empty set. A neutrosophic set (NS) in X (see [7]) is a structure of the form:

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}$$

where $A_T : X \to [0, 1]$ is a truth membership function, $A_I : X \to [0, 1]$ is an indeterminate membership function, and $A_F : X \to [0, 1]$ is a false membership function.

An interval neutrosophic set (INS) A in X is characterized by truth-membership function T_A, indeterminacy membership function I_A and falsity-membership function F_A. For each point x in X, $T_A(x)$, $I_A(x)$, $F_A(x) \in [0, 1]$ (see [11, 10]).

In what follows, let $(X, *, 0) \in \mathcal{B}(X)$ and $\mathcal{P}^*([0, 1])$ be the family of all subintervals of $[0, 1]$ unless otherwise specified.

Definition 2.2 ([11, 10]). An *interval neutrosophic set* in a nonempty set X is a structure of the form:

$$\mathcal{I} := \{\langle x, \mathcal{I}[T](x), \mathcal{I}[I](x), \mathcal{I}[F](x)\rangle \mid x \in X\}$$

where

$$\mathcal{I}[T] : X \to \mathcal{P}^*([0, 1])$$

which is called *interval truth-membership function*,

$$\mathcal{I}[I] : X \to \mathcal{P}^*([0, 1])$$

which is called *interval indeterminacy-membership function*, and

$$\mathcal{I}[F] : X \to \mathcal{P}^*([0, 1])$$

which is called *interval falsity-membership function.*

For the sake of simplicity, we will use the notation $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ for the interval neutrosophic set

$$\mathcal{I} := \{\langle x, \mathcal{I}[T](x), \mathcal{I}[I](x), \mathcal{I}[F](x)\rangle \mid x \in X\}.$$

Given an interval neutrosophic set $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ in X, we consider the following functions (see [4]):

$$\begin{aligned}\mathcal{I}[T]_{\inf} &: X \to [0,1],\ x \mapsto \inf\{\mathcal{I}[T](x)\}\\ \mathcal{I}[I]_{\inf} &: X \to [0,1],\ x \mapsto \inf\{\mathcal{I}[I](x)\}\\ \mathcal{I}[F]_{\inf} &: X \to [0,1],\ x \mapsto \inf\{\mathcal{I}[F](x)\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{I}[T]_{\sup} &: X \to [0,1],\ x \mapsto \sup\{\mathcal{I}[T](x)\}\\ \mathcal{I}[I]_{\sup} &: X \to [0,1],\ x \mapsto \sup\{\mathcal{I}[I](x)\}\\ \mathcal{I}[F]_{\sup} &: X \to [0,1],\ x \mapsto \sup\{\mathcal{I}[F](x)\}.\end{aligned}$$

Definition 2.3 ([4]). Given an interval neutrosophic set $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ in X, we define the *interval neutrosophic length* of $\mathcal{I}$ as an ordered triple $\mathcal{I}_\ell := (\mathcal{I}[T]_\ell, \mathcal{I}[I]_\ell, \mathcal{I}[F]_\ell)$ where

$$\mathcal{I}[T]_\ell : X \to [0,1],\ x \mapsto \mathcal{I}[T]_{\sup}(x) - \mathcal{I}[T]_{\inf}(x),$$

$$\mathcal{I}[I]_\ell : X \to [0,1],\ x \mapsto \mathcal{I}[I]_{\sup}(x) - \mathcal{I}[I]_{\inf}(x),$$

and

$$\mathcal{I}[F]_\ell : X \to [0,1],\ x \mapsto \mathcal{I}[F]_{\sup}(x) - \mathcal{I}[F]_{\inf}(x),$$

which are called *interval neutrosophic T-length*, *interval neutrosophic I-length* and *interval neutrosophic F-length* of $\mathcal{I}$, respectively.

3. Length neutrosophic subalgebras

Definition 3.1. Given $i, j, k \in \{1, 2, 3, 4\}$, an interval neutrosophic set $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ in X is called an *(i, j, k)-length neutrosophic subalgebra* of $(X, *, 0)$ if the interval neutrosophic T-length of $\mathcal{I}$ is an i-fuzzy subalgebra of $(X, *, 0)$, the interval neutrosophic I-length of $\mathcal{I}$ is a j-fuzzy subalgebra of $(X, *, 0)$, and the interval neutrosophic F-length of $\mathcal{I}$ is a k-fuzzy subalgebra of $(X, *, 0)$.

Example 3.2. Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the binary operation $*$ which is given in Table 1 (see [5]).
Let $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ be an interval neutrosophic set in $(X, *, 0)$ where $\mathcal{I}[T]$, $\mathcal{I}[I]$ and $\mathcal{I}[F]$ are given as follows:

Table 1. Cayley table for the binary operation “$*$”

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	3	4	1	0

$$\mathcal{I}[T] : X \to \mathcal{P}^*([0,1]), \quad x \mapsto \begin{cases} [0.1, 0.8) & \text{if } x = 0, \\ (0.3, 0.7] & \text{if } x = 1, \\ [0.0, 0.6] & \text{if } x = 2, \\ [0.4, 0.8] & \text{if } x = 3, \\ [0.2, 0.5] & \text{if } x = 4, \end{cases}$$

$$\mathcal{I}[I] : X \to \mathcal{P}^*([0,1]), \quad x \mapsto \begin{cases} [0.2, 0.8) & \text{if } x = 0, \\ (0.4, 0.8] & \text{if } x = 1, \\ [0.1, 0.6] & \text{if } x = 2, \\ [0.6, 0.9] & \text{if } x = 3, \\ [0.3, 0.5] & \text{if } x = 4, \end{cases}$$

and

$$\mathcal{I}[F] : X \to \mathcal{P}^*([0,1]), \quad x \mapsto \begin{cases} [0.1, 0.4) & \text{if } x = 0, \\ (0.4, 0.8] & \text{if } x = 1, \\ [0.1, 0.5] & \text{if } x = 2, \\ [0.2, 0.7) & \text{if } x = 3, \\ [0.3, 0.9] & \text{if } x = 4. \end{cases}$$

Then the interval neutrosophic length $\mathcal{I}_\ell := (\mathcal{I}[T]_\ell, \mathcal{I}[I]_\ell, \mathcal{I}[F]_\ell)$ of $\mathcal{I}$ is given by Table 2.
It is routine to verify that $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(1, 1, 4)$-length neutrosophic subalgebra of $(X, *, 0)$.

Table 2. Interval neutrosophic length of $\mathcal{I}$

X	$\mathcal{I}[T]_\ell$	$\mathcal{I}[I]_\ell$	$\mathcal{I}[F]_\ell$
0	0.7	0.6	0.3
1	0.4	0.4	0.4
2	0.6	0.5	0.4
3	0.4	0.3	0.5
4	0.3	0.2	0.6

PROPOSITION 3.3. Given an (i, j, k)-length neutrosophic subalgebra $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ of $(X, *, 0)$, we have the following assertions.

(1) If $i, j, k \in \{1, 3\}$, then

$$(\forall x \in X)(\mathcal{I}[T]_\ell(0) \geq \mathcal{I}[T]_\ell(x), \mathcal{I}[I]_\ell(0) \geq \mathcal{I}[I]_\ell(x), \mathcal{I}[F]_\ell(0) \geq \mathcal{I}[F]_\ell(x)). \tag{3.1}$$

(2) If $i, j, k \in \{2, 4\}$, then

$$(\forall x \in X)(\mathcal{I}[T]_\ell(0) \leq \mathcal{I}[T]_\ell(x), \mathcal{I}[I]_\ell(0) \leq \mathcal{I}[I]_\ell(x), \mathcal{I}[F]_\ell(0) \leq \mathcal{I}[F]_\ell(x)). \tag{3.2}$$

(3) If $i, j \in \{1, 3\}$ and $k \in \{2, 4\}$, then

$$(\forall x \in X)(\mathcal{I}[T]_\ell(0) \geq \mathcal{I}[T]_\ell(x), \mathcal{I}[I]_\ell(0) \geq \mathcal{I}[I]_\ell(x), \mathcal{I}[F]_\ell(0) \leq \mathcal{I}[F]_\ell(x)). \tag{3.3}$$

(4) If $i, j \in \{2, 4\}$ and $k \in \{1, 3\}$, then

$$(\forall x \in X)(\mathcal{I}[T]_\ell(0) \leq \mathcal{I}[T]_\ell(x), \mathcal{I}[I]_\ell(0) \leq \mathcal{I}[I]_\ell(x), \mathcal{I}[F]_\ell(0) \geq \mathcal{I}[F]_\ell(x)). \tag{3.4}$$

PROOF: Let $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ be an (i, j, k)-length neutrosophic subalgebra of $(X, *, 0)$. If $(i, j, k) = (1, 3, 1)$, then

$$\mathcal{I}[T]_\ell(0) = \mathcal{I}[T]_\ell(x * x) \geq \min\{\mathcal{I}[T]_\ell(x), \mathcal{I}[T]_\ell(x)\} = \mathcal{I}[T]_\ell(x)$$

$$\mathcal{I}[I]_\ell(0) = \mathcal{I}[I]_\ell(x * x) \geq \max\{\mathcal{I}[I]_\ell(x), \mathcal{I}[I]_\ell(x)\} = \mathcal{I}[I]_\ell(x)$$

$$\mathcal{I}[F]_\ell(0) = \mathcal{I}[F]_\ell(x * x) \geq \min\{\mathcal{I}[F]_\ell(x), \mathcal{I}[F]_\ell(x)\} = \mathcal{I}[F]_\ell(x)$$

for all $x \in X$. Similarly, we can verify that (3.1) is true for other cases of (i, j, k). Using the similar way to the proof of (1), we can prove that (2), (3) and (4) hold. □

THEOREM 3.4. *Given a subalgebra* S *of* $(X, *, 0)$ *and* $A_1, A_2, B_1, B_2, C_1, C_2 \in \mathcal{P}^*([0, 1])$*, let* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *be an interval neutrosophic set in* $(X, *, 0)$ *given by*

$$\mathcal{I}[T] : X \to \mathcal{P}^*([0, 1]),\ x \mapsto \begin{cases} A_2 & \text{if } x \in S, \\ A_1 & \text{otherwise,} \end{cases} \tag{3.5}$$

$$\mathcal{I}[I] : X \to \mathcal{P}^*([0, 1]),\ x \mapsto \begin{cases} B_2 & \text{if } x \in S, \\ B_1 & \text{otherwise,} \end{cases} \tag{3.6}$$

$$\mathcal{I}[F] : X \to \mathcal{P}^*([0, 1]),\ x \mapsto \begin{cases} C_2 & \text{if } x \in S, \\ C_1 & \text{otherwise.} \end{cases} \tag{3.7}$$

(1) *If* $A_1 \subsetneq A_2$, $B_1 \subsetneq B_2$ *and* $C_1 \subsetneq C_2$, *then* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(1, 1, 1)$*-length neutrosophic subalgebra of* $(X, *, 0)$.

(2) *If* $A_1 \supsetneq A_2$, $B_1 \supsetneq B_2$ *and* $C_1 \supsetneq C_2$, *then* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(4, 4, 4)$*-length neutrosophic subalgebra of* $(X, *, 0)$.

(3) *If* $A_1 \subsetneq A_2$, $B_1 \supsetneq B_2$ *and* $C_1 \subsetneq C_2$, *then* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(1, 4, 1)$*-length neutrosophic subalgebra of* $(X, *, 0)$.

(4) *If* $A_1 \supsetneq A_2$, $B_1 \subsetneq B_2$ *and* $C_1 \supsetneq C_2$, *then* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(4, 1, 4)$*-length neutrosophic subalgebra of* $(X, *, 0)$.

(5) *If* $A_1 \subsetneq A_2$, $B_1 \subsetneq B_2$ *and* $C_1 \supsetneq C_2$, *then* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(1, 1, 4)$*-length neutrosophic subalgebra of* $(X, *, 0)$.

(6) *If* $A_1 \supsetneq A_2$, $B_1 \supsetneq B_2$ *and* $C_1 \subsetneq C_2$, *then* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(4, 4, 1)$*-length neutrosophic subalgebra of* $(X, *, 0)$.

PROOF: We will prove (3) only, and others can be obtained by the similar way. Assume that $A_1 \subsetneq A_2$, $B_1 \supsetneq B_2$ and $C_1 \subsetneq C_2$. If $x \in S$, then $\mathcal{I}[T](x) = A_2$, $\mathcal{I}[I](x) = B_2$ and $\mathcal{I}[F](x) = C_2$. Hence

$$\begin{aligned}
\mathcal{I}[T]_\ell(x) &= \mathcal{I}[T]_{\sup}(x) - \mathcal{I}[T]_{\inf}(x) = \sup\{A_2\} - \inf\{A_2\},\\
\mathcal{I}[I]_\ell(x) &= \mathcal{I}[I]_{\sup}(x) - \mathcal{I}[I]_{\inf}(x) = \sup\{B_2\} - \inf\{B_2\},\\
\mathcal{I}[F]_\ell(x) &= \mathcal{I}[F]_{\sup}(x) - \mathcal{I}[F]_{\inf}(x) = \sup\{C_2\} - \inf\{C_2\}.
\end{aligned}$$

If $x \notin S$, then $\mathcal{I}[T](x) = A_1$, $\mathcal{I}[I](x) = B_1$ and $\mathcal{I}[F](x) = C_1$, and so

$$\begin{aligned}
\mathcal{I}[T]_\ell(x) &= \mathcal{I}[T]_{\sup}(x) - \mathcal{I}[T]_{\inf}(x) = \sup\{A_1\} - \inf\{A_1\},\\
\mathcal{I}[I]_\ell(x) &= \mathcal{I}[I]_{\sup}(x) - \mathcal{I}[I]_{\inf}(x) = \sup\{B_1\} - \inf\{B_1\},\\
\mathcal{I}[F]_\ell(x) &= \mathcal{I}[F]_{\sup}(x) - \mathcal{I}[F]_{\inf}(x) = \sup\{C_1\} - \inf\{C_1\}.
\end{aligned}$$

Since $A_1 \subsetneqq A_2$, $B_1 \supsetneqq B_2$ and $C_1 \subsetneqq C_2$, we have

$$\begin{aligned}
\sup\{A_2\} - \inf\{A_2\} &\geq \sup\{A_1\} - \inf\{A_1\},\\
\sup\{B_2\} - \inf\{B_2\} &\leq \sup\{B_1\} - \inf\{B_1\},\\
\sup\{C_2\} - \inf\{C_2\} &\geq \sup\{C_1\} - \inf\{C_1\}.
\end{aligned}$$

Let $x, y \in X$. If $x, y \in S$, then $x * y \in S$ and so

$$\begin{aligned}
\mathcal{I}[T]_\ell(x * y) &= \sup\{A_2\} - \inf\{A_2\} = \min\{\mathcal{I}[T]_\ell(x), \mathcal{I}[T]_\ell(y)\},\\
\mathcal{I}[I]_\ell(x * y) &= \sup\{B_2\} - \inf\{B_2\} = \max\{\mathcal{I}[I]_\ell(x), \mathcal{I}[I]_\ell(y)\},\\
\mathcal{I}[F]_\ell(x * y) &= \sup\{C_2\} - \inf\{C_2\} = \min\{\mathcal{I}[F]_\ell(x), \mathcal{I}[F]_\ell(y)\}.
\end{aligned}$$

If $x, y \notin S$, then

$$\begin{aligned}
\mathcal{I}[T]_\ell(x * y) &\geq \sup\{A_1\} - \inf\{A_1\} = \min\{\mathcal{I}[T]_\ell(x), \mathcal{I}[T]_\ell(y)\},\\
\mathcal{I}[I]_\ell(x * y) &\leq \sup\{B_1\} - \inf\{B_1\} = \max\{\mathcal{I}[I]_\ell(x), \mathcal{I}[I]_\ell(y)\},\\
\mathcal{I}[F]_\ell(x * y) &\geq \sup\{C_1\} - \inf\{C_1\} = \min\{\mathcal{I}[F]_\ell(x), \mathcal{I}[F]_\ell(y)\}.
\end{aligned}$$

Assume that $x \in S$ and $y \notin S$ (or, $x \notin S$ and $y \in S$). Then

$$\begin{aligned}
\mathcal{I}[T]_\ell(x * y) &\geq \sup\{A_1\} - \inf\{A_1\} = \min\{\mathcal{I}[T]_\ell(x), \mathcal{I}[T]_\ell(y)\},\\
\mathcal{I}[I]_\ell(x * y) &\leq \sup\{B_1\} - \inf\{B_1\} = \max\{\mathcal{I}[I]_\ell(x), \mathcal{I}[I]_\ell(y)\},\\
\mathcal{I}[F]_\ell(x * y) &\geq \sup\{C_1\} - \inf\{C_1\} = \min\{\mathcal{I}[F]_\ell(x), \mathcal{I}[F]_\ell(y)\}.
\end{aligned}$$

Therefore $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(1, 4, 1)$-length neutrosophic subalgebra of $(X, *, 0)$. □

Remark 3.5. We have the following relations.

(1) Every (i, j, k)-length neutrosophic subalgebra of $(X, *, 0)$ for $i, j, k \in \{1, 3\}$ is a $(1, 1, 1)$-length neutrosophic subalgebra of $(X, *, 0)$.

(2) Every (i, j, k)-length neutrosophic subalgebra of $(X, *, 0)$ for $i, j, k \in \{2, 4\}$ is a $(4, 4, 4)$-length neutrosophic subalgebra of $(X, *, 0)$.

(3) Every (i, j, k)-length neutrosophic subalgebra of $(X, *, 0)$ for $i, j \in \{1, 3\}$ and $k \in \{2, 4\}$ is a $(1, 1, 4)$-length neutrosophic subalgebra of $(X, *, 0)$.

(4) Every (i, j, k)-length neutrosophic subalgebra of $(X, *, 0)$ for $i, j \in \{2, 4\}$ and $k \in \{1, 3\}$ is a $(4, 4, 1)$-length neutrosophic subalgebra of $(X, *, 0)$.

(5) Every (i, j, k)-length neutrosophic subalgebra of $(X, *, 0)$ for $i, k \in \{2, 4\}$ and $j \in \{1, 3\}$ is a $(4, 1, 4)$-length neutrosophic subalgebra of $(X, *, 0)$.

(6) Every (i, j, k)-length neutrosophic subalgebra of $(X, *, 0)$ for $i, k \in \{1, 3\}$ and $j \in \{2, 4\}$ is a $(1, 4, 1)$-length neutrosophic subalgebra of $(X, *, 0)$.

The following example shows that the converse in Remark 3.5 is not true in general. We consider the cases (5) and (6) only in Remark 3.5.

Example 3.6. Consider the BCK-algebra $(X, *, 0)$ in Example 3.2. Given a subalgebra $S = \{0, 1, 2\}$ of $(X, *, 0)$, let $\mathcal{I} := (\mathcal{I}[T],\ \mathcal{I}[I],\ \mathcal{I}[F])$ be an interval neutrosophic set in $(X, *, 0)$ given by

$$\mathcal{I}[T] : X \to \mathcal{P}^*([0,1]),\ x \mapsto \begin{cases} [0.2, 0.7) & \text{if } x \in S, \\ (0.1, 0.8] & \text{otherwise,} \end{cases}$$

$$\mathcal{I}[I] : X \to \mathcal{P}^*([0,1]),\ x \mapsto \begin{cases} [0.2, 0.9) & \text{if } x \in S, \\ (0.3, 0.7] & \text{otherwise,} \end{cases}$$

and

$$\mathcal{I}[F] : X \to \mathcal{P}^*([0,1]),\ x \mapsto \begin{cases} [0.4, 0.5) & \text{if } x \in S, \\ (0.3, 0.6] & \text{otherwise.} \end{cases}$$

Then $\mathcal{I} := (\mathcal{I}[T],\ \mathcal{I}[I],\ \mathcal{I}[F])$ is a $(4, 1, 4)$-length neutrosophic subalgebra of $(X, *, 0)$ by Theorem 3.4(4). Since

$$\mathcal{I}[I]_\ell(2) = \mathcal{I}[I]_{\sup}(2) - \mathcal{I}[I]_{\inf}(2) = 0.9 - 0.2 = 0.7$$

and

$$\mathcal{I}[I]_\ell(3 * 2) = \mathcal{I}[I]_\ell(3) = \mathcal{I}[I]_{\sup}(3) - \mathcal{I}[I]_{\inf}(3) = 0.7 - 0.3 = 0.4,$$

we have $\mathcal{I}[I]_\ell(3 * 2) = 0.4 < 0.7 = \max\{\mathcal{I}[I]_\ell(3), \mathcal{I}[I]_\ell(2)\}$. Hence $\mathcal{I} := (\mathcal{I}[T],\ \mathcal{I}[I],\ \mathcal{I}[F])$ is not an $(i, 3, k)$-length neutrosophic subalgebra of $(X, *, 0)$ for $i, k \in \{2, 4\}$. Given a subalgebra $S = \{0, 1, 2, 3\}$ of $(X, *, 0)$, let $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ be an interval neutrosophic set in $(X, *, 0)$ given by

$$\mathcal{I}[T] : X \to \mathcal{P}^*([0, 1]),\ x \mapsto \begin{cases} [0.2, 0.7) & \text{if } x \in S, \\ (0.3, 0.5] & \text{otherwise,} \end{cases}$$

$$\mathcal{I}[I] : X \to \mathcal{P}^*([0, 1]),\ x \mapsto \begin{cases} [0.4, 0.6) & \text{if } x \in S, \\ (0.3, 0.8] & \text{otherwise,} \end{cases}$$

and

$$\mathcal{I}[F] : X \to \mathcal{P}^*([0, 1]),\ x \mapsto \begin{cases} [0.2, 0.8) & \text{if } x \in S, \\ (0.3, 0.6] & \text{otherwise.} \end{cases}$$

Then $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(1, 4, 1)$-length neutrosophic subalgebra of $(X, *, 0)$ by Theorem 3.4(3). But it is not an $(i, 2, k)$-length neutrosophic subalgebra of $(X, *, 0)$ for $i, k \in \{1, 3\}$ since

$$\mathcal{I}[I]_\ell(4 * 2) = \mathcal{I}[I]_\ell(4) = 0.5 > 0.2 = \min\{\mathcal{I}[I]_\ell(4), \mathcal{I}[I]_\ell(2)\}.$$

Given an interval neutrosophic set $\mathcal{I} := (\mathcal{I}[T],\ \mathcal{I}[I],\ \mathcal{I}[F])$ in $(X, *, 0)$, we consider the following level sets:

$$\begin{aligned} U_\ell(\mathcal{I}[T]; \alpha_T) &:= \{x \in X \mid \mathcal{I}[T]_\ell(x) \geq \alpha_T\}, \\ U_\ell(\mathcal{I}[I]; \alpha_I) &:= \{x \in X \mid \mathcal{I}[I]_\ell(x) \geq \alpha_I\}, \\ U_\ell(\mathcal{I}[F]; \alpha_F) &:= \{x \in X \mid \mathcal{I}[F]_\ell(x) \geq \alpha_F\}, \end{aligned}$$

and

$$\begin{aligned} L_\ell(\mathcal{I}[T]; \beta_T) &:= \{x \in X \mid \mathcal{I}[T]_\ell(x) \leq \beta_T\}, \\ L_\ell(\mathcal{I}[I]; \beta_I) &:= \{x \in X \mid \mathcal{I}[I]_\ell(x) \leq \beta_I\}, \\ L_\ell(\mathcal{I}[F]; \beta_F) &:= \{x \in X \mid \mathcal{I}[F]_\ell(x) \leq \beta_F\}. \end{aligned}$$

Theorem 3.7. *Given an interval neutrosophic set* $\mathcal{I} := (\mathcal{I}[T],\, \mathcal{I}[I],\, \mathcal{I}[F])$ *in* $(X, *, 0)$ *and for any* $\alpha_T,\ \alpha_I,\ \alpha_F \in [0,1]$*, the following assertions are equivalent.*

(1) $\mathcal{I} := (\mathcal{I}[T],\, \mathcal{I}[I],\, \mathcal{I}[F])$ *is a* $(1,1,1)$*-length neutrosophic subalgebra of* $(X, *, 0)$.

(2) $U_\ell(\mathcal{I}[T]; \alpha_T)$*,* $U_\ell(\mathcal{I}[I]; \alpha_I)$ *and* $U_\ell(\mathcal{I}[F]; \alpha_F)$ *are subalgebras of* $(X, *, 0)$ *whenever they are nonempty.*

Proof: Assume that $\mathcal{I} := (\mathcal{I}[T],\, \mathcal{I}[I],\, \mathcal{I}[F])$ is a $(1,1,1)$-length neutrosophic subalgebra of $(X, *, 0)$ and let $\alpha_T,\ \alpha_I,\ \alpha_F \in [0,1]$ be such that $U_\ell(\mathcal{I}[T]; \alpha_T)$, $U_\ell(\mathcal{I}[I]; \alpha_I)$ and $U_\ell(\mathcal{I}[F]; \alpha_F)$ are nonempty. If $x, y \in U_\ell(\mathcal{I}[T]; \alpha_T)$, then $\mathcal{I}[T]_\ell(x) \geq \alpha_T$ and $\mathcal{I}[T]_\ell(y) \geq \alpha_T$. Hence

$$\mathcal{I}[T]_\ell(x * y) \geq \min\{\mathcal{I}[T]_\ell(x), \mathcal{I}[T]_\ell(y)\} \geq \alpha_T,$$

that is, $x * y \in U_\ell(\mathcal{I}[T]; \alpha_T)$. Similarly, we can see that if $x, y \in U_\ell(\mathcal{I}[I]; \alpha_I)$, then $x * y \in U_\ell(\mathcal{I}[I]; \alpha_I)$, and if $x, y \in U_\ell(\mathcal{I}[F]; \alpha_F)$, then $x * y \in U_\ell(\mathcal{I}[F]; \alpha_F)$. Therefore $U_\ell(\mathcal{I}[T]; \alpha_T)$, $U_\ell(\mathcal{I}[I]; \alpha_I)$ and $U_\ell(\mathcal{I}[F]; \alpha_F)$ are subalgebras of $(X, *, 0)$.

Conversely, suppose that (2) is valid. If there exist $a, b \in X$ such that

$$\mathcal{I}[T]_\ell(a * b) < \min\{\mathcal{I}[T]_\ell(a), \mathcal{I}[T]_\ell(b)\},$$

then $a, b \in U_\ell(\mathcal{I}[T]; \alpha_T)$ by taking $\alpha_T = \min\{\mathcal{I}[T]_\ell(a), \mathcal{I}[T]_\ell(b)\}$, and so $a * b \in U_\ell(\mathcal{I}[T]; \alpha_T)$. It follows that $\mathcal{I}[T]_\ell(a * b) \geq \alpha_T$, a contradiction. Hence

$$\mathcal{I}[T]_\ell(x * y) \geq \min\{\mathcal{I}[T]_\ell(x), \mathcal{I}[T]_\ell(y)\}$$

for all $x, y \in X$. Similarly, we can check that

$$\mathcal{I}[I]_\ell(x * y) \geq \min\{\mathcal{I}[I]_\ell(x), \mathcal{I}[I]_\ell(y)\}$$

and

$$\mathcal{I}[F]_\ell(x * y) \geq \min\{\mathcal{I}[F]_\ell(x), \mathcal{I}[F]_\ell(y)\}$$

for all $x, y \in X$. Thus $\mathcal{I} := (\mathcal{I}[T],\, \mathcal{I}[I],\, \mathcal{I}[F])$ is a $(1,1,1)$-length neutrosophic subalgebra of $(X, *, 0)$. $\square$

Corollary 3.8. If $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is an (i, j, k)-length neutrosophic subalgebra of $(X, *, 0)$ for $i, j, k \in \{1, 3\}$, then $U_\ell(\mathcal{I}[T]; \alpha_T)$, $U_\ell(\mathcal{I}[I]; \alpha_I)$ and $U_\ell(\mathcal{I}[F]; \alpha_F)$ are subalgebras of $(X, *, 0)$ whenever they are nonempty for all α_T, α_I, $\alpha_F \in [0, 1]$.

The following example shows that the converse of Corollary 3.8 is not true.

Example 3.9. Consider a BCI-algebra $X = \{0, 1, 2, a, b\}$ with the binary operation $*$ which is given in Table 3 (see [5]).

Table 3. Cayley table for the binary operation "$*$"

$*$	0	1	2	a	b
0	0	0	0	a	a
1	1	0	1	b	a
2	2	2	0	a	a
a	a	a	a	0	0
b	b	a	b	1	0

Let $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ be an interval neutrosophic set in $(X, *, 0)$ given by

$$\mathcal{I}[T] : X \to \mathcal{P}^*([0,1]),\ x \mapsto \begin{cases} [0.3, 0.9) & \text{if } x = 0, \\ (0.5, 0.7] & \text{if } x = 1, \\ [0.1, 0.6] & \text{if } x = 2, \\ [0.4, 0.7] & \text{if } x = a, \\ (0.3, 0.5] & \text{if } x = b, \end{cases}$$

$$\mathcal{I}[I] : X \to \mathcal{P}^*([0,1]),\ x \mapsto \begin{cases} [0.2, 0.9) & \text{if } x = 0, \\ (0.1, 0.8] & \text{if } x = 1, \\ [0.5, 0.9] & \text{if } x = 2, \\ [0.4, 0.7] & \text{if } x = a, \\ (0.4, 0.7] & \text{if } x = b, \end{cases}$$

and

$$\mathcal{I}[F] : X \to \mathcal{P}^*([0,1]),\ x \mapsto \begin{cases} [0.1, 0.6) & \text{if } x = 0, \\ (0.6, 0.9) & \text{if } x = 1, \\ (0.4, 0.8] & \text{if } x = 2, \\ [0.5, 0.7] & \text{if } x = a, \\ (0.5, 0.7] & \text{if } x = b. \end{cases}$$

Then the interval neutrosophic length $\mathcal{I}_\ell := (\mathcal{I}[T]_\ell,\ \mathcal{I}[I]_\ell,\ \mathcal{I}[F]_\ell)$ of $\mathcal{I}$ is given by Table 4.

Table 4. Interval neutrosophic length of $\mathcal{I}$

X	$\mathcal{I}[T]_\ell$	$\mathcal{I}[I]_\ell$	$\mathcal{I}[F]_\ell$
0	0.6	0.7	0.5
1	0.2	0.7	0.3
2	0.5	0.4	0.4
a	0.3	0.3	0.2
b	0.2	0.3	0.2

Hence we have

$$U_\ell(\mathcal{I}[T]; \alpha_T) = \begin{cases} \emptyset & \text{if } \alpha_T \in (0.6, 1], \\ \{0\} & \text{if } \alpha_T \in (0.5, 0.6], \\ \{0, 2\} & \text{if } \alpha_T \in (0.3, 0.5], \\ \{0, 2, a\} & \text{if } \alpha_T \in (0.2, 0.3], \\ X & \text{if } \alpha_T \in [0, 0.2], \end{cases}$$

$$U_\ell(\mathcal{I}[I]; \alpha_I) = \begin{cases} \emptyset & \text{if } \alpha_I \in (0.7, 1], \\ \{0, 1\} & \text{if } \alpha_I \in (0.4, 0.7], \\ \{0, 1, 2\} & \text{if } \alpha_I \in (0.3, 0.4], \\ X & \text{if } \alpha_I \in [0, 0.3], \end{cases}$$

and

$$U_\ell(\mathcal{I}[F];\alpha_F)=\begin{cases}\emptyset & \text{if } \alpha_F\in(0.5,1],\\ \{0\} & \text{if } \alpha_F\in(0.4,0.5],\\ \{0,2\} & \text{if } \alpha_F\in(0.3,0.4],\\ \{0,1,2\} & \text{if } \alpha_F\in(0.2,0.3],\\ X & \text{if } \alpha_F\in[0,0.2],\end{cases}$$

and so $U_\ell(\mathcal{I}[T];\alpha_T)$, $U_\ell(\mathcal{I}[I];\alpha_I)$ and $U_\ell(\mathcal{I}[F];\alpha_F)$ are subalgebras of $(X,*,0)$ for all α_T, α_I, $\alpha_F\in[0,1]$ such that $U_\ell(\mathcal{I}[T];\alpha_T)$, $U_\ell(\mathcal{I}[I];\alpha_I)$ and $U_\ell(\mathcal{I}[F];\alpha_F)$ are nonempty. But $\mathcal{I}:=(\mathcal{I}[T],\,\mathcal{I}[I],\,\mathcal{I}[F])$ is not an (i,j,k)-length neutrosophic subalgebra of $(X,*,0)$ for $i,j,k\in\{1,3\}$ with $(i,j,k)\neq(1,1,1)$ since

$$\mathcal{I}[T]_\ell(b*2)=\mathcal{I}[T]_\ell(b)=0.2\ngeq 0.5=\max\{\mathcal{I}[T]_\ell(b),\mathcal{I}[T]_\ell(2)\},$$

$$\mathcal{I}[I]_\ell(a*1)=\mathcal{I}[I]_\ell(a)=0.3\ngeq 0.7=\max\{\mathcal{I}[I]_\ell(a),\mathcal{I}[I]_\ell(1)\},$$

and/or

$$\mathcal{I}[F]_\ell(b*1)=\mathcal{I}[F]_\ell(a)=0.2\ngeq 0.3=\max\{\mathcal{I}[F]_\ell(b),\mathcal{I}[F]_\ell(1)\}.$$

THEOREM 3.10. *Given an interval neutrosophic set* $\mathcal{I}:=(\mathcal{I}[T],\,\mathcal{I}[I],\,\mathcal{I}[F])$ *in* $(X,*,0)$ *and for any* β_T, β_I, $\beta_F\in[0,1]$, *the following assertions are equivalent.*

(1) $\mathcal{I}:=(\mathcal{I}[T],\,\mathcal{I}[I],\,\mathcal{I}[F])$ *is a* $(4,4,4)$*-length neutrosophic subalgebra of* $(X,*,0)$.

(2) $L_\ell(\mathcal{I}[T];\beta_T)$, $L_\ell(\mathcal{I}[I];\beta_I)$ *and* $L_\ell(\mathcal{I}[F];\beta_F)$ *are subalgebras of* $(X,*,0)$ *whenever they are nonempty.*

PROOF: Suppose that $\mathcal{I}:=(\mathcal{I}[T],\,\mathcal{I}[I],\,\mathcal{I}[F])$ is a $(4,4,4)$-length neutrosophic subalgebra of $(X,*,0)$ and let β_T, β_I, $\beta_F\in[0,1]$ be such that $L_\ell(\mathcal{I}[T];\beta_T)$, $L_\ell(\mathcal{I}[I];\beta_I)$ and $L_\ell(\mathcal{I}[F];\beta_F)$ are nonempty. For any $x,y\in X$, if $x,y\in L_\ell(\mathcal{I}[T];\beta_T)$, then $\mathcal{I}[T]_\ell(x)\le\beta_T$ and $\mathcal{I}[T]_\ell(y)\le\beta_T$. It follows that

$$\mathcal{I}[T]_\ell(x*y)\le\max\{\mathcal{I}[T]_\ell(x),\mathcal{I}[T]_\ell(y)\}\le\beta_T$$

and so that $x*y\in L_\ell(\mathcal{I}[T];\beta_T)$. Similarly, if $x,y\in L_\ell(\mathcal{I}[I];\beta_I)$, then $x*y\in L_\ell(\mathcal{I}[I];\beta_I)$, and if $x,y\in L_\ell(\mathcal{I}[F];\beta_F)$, then $x*y\in L_\ell(\mathcal{I}[F];\beta_F)$.

Therefore (2) is valid.

Conversely, assume that $L_\ell(\mathcal{I}[T];\beta_T)$, $L_\ell(\mathcal{I}[I];\beta_I)$ and $L_\ell(\mathcal{I}[F];\beta_F)$ are subalgebras of $(X,*,0)$ whenever they are nonempty for all β_T, β_I, $\beta_F \in [0,1]$. If there are $a,b \in X$ such that

$$\mathcal{I}[F]_\ell(a*b) > \max\{\mathcal{I}[F]_\ell(a),\mathcal{I}[F]_\ell(b)\},$$

then $a,b \in L_\ell(\mathcal{I}[F];\beta_F)$ by taking $\beta_F = \max\{\mathcal{I}[F]_\ell(a),\mathcal{I}[F]_\ell(b)\}$. Thus $a*b \in L_\ell(\mathcal{I}[F];\beta_F)$, which implies that $\mathcal{I}[F]_\ell(a*b) \leq \beta_F$. This is a contradiction, and so

$$\mathcal{I}[F]_\ell(x*y) \leq \max\{\mathcal{I}[F]_\ell(x),\mathcal{I}[F]_\ell(y)\}$$

for all $x,y \in X$. Similarly, we get

$$\mathcal{I}[T]_\ell(x*y) \leq \max\{\mathcal{I}[T]_\ell(x),\mathcal{I}[T]_\ell(y)\}$$

and

$$\mathcal{I}[I]_\ell(x*y) \leq \max\{\mathcal{I}[I]_\ell(x),\mathcal{I}[I]_\ell(y)\}$$

for all $x,y \in X$. Consequently, $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(4,4,4)$-length neutrosophic subalgebra of $(X,*,0)$. □

Corollary 3.11. If $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is an (i,j,k)-length neutrosophic subalgebra of $(X,*,0)$ for $i,j,k \in \{2,4\}$, then $L_\ell(\mathcal{I}[T];\beta_T)$, $L_\ell(\mathcal{I}[I];\beta_I)$ and $L_\ell(\mathcal{I}[F];\beta_F)$ are subalgebras of $(X,*,0)$ whenever they are nonempty for all β_T, β_I, $\beta_F \in [0,1]$.

The following example shows that the converse of Corollary 3.11 is not true.

Example 3.12. Consider the BCI-algebra $X = \{0,1,2,a,b\}$ in Example 3.9 and let $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ be an interval neutrosophic set in $(X,*,0)$ given by

$$\mathcal{I}[T] : X \to \mathcal{P}^*([0,1]),\ x \mapsto \begin{cases} [0.5, 0.7) & \text{if } x = 0, \\ (0.2, 0.6] & \text{if } x = 1, \\ [0.3, 0.6] & \text{if } x = 2, \\ [0.1, 0.7] & \text{if } x = a, \\ (0.2, 0.8] & \text{if } x = b, \end{cases}$$

$$\mathcal{I}[I]: X \to \mathcal{P}^*([0,1]),\ x \mapsto \begin{cases} [0.66, 0.99) & \text{if } x = 0, \\ (0.15, 0.59] & \text{if } x = 1, \\ [0.22, 0.88) & \text{if } x = 2, \\ (0.35, 0.90] & \text{if } x = a, \\ (0.20, 0.75) & \text{if } x = b, \end{cases}$$

and

$$\mathcal{I}[F]: X \to \mathcal{P}^*([0,1]),\ x \mapsto \begin{cases} [0.75, 0.90) & \text{if } x = 0, \\ (0.45, 0.90) & \text{if } x = 1, \\ (0.25, 0.50] & \text{if } x = 2, \\ [0.50, 0.85] & \text{if } x = a, \\ (0.15, 0.60] & \text{if } x = b. \end{cases}$$

Then the interval neutrosophic length $\mathcal{I}_\ell := (\mathcal{I}[T]_\ell, \mathcal{I}[I]_\ell, \mathcal{I}[F]_\ell)$ of $\mathcal{I}$ is given by Table 5.

Table 5. Interval neutrosophic length of $\mathcal{I}$

X	$\mathcal{I}[T]_\ell$	$\mathcal{I}[I]_\ell$	$\mathcal{I}[F]_\ell$
0	0.2	0.33	0.15
1	0.4	0.44	0.45
2	0.3	0.66	0.25
a	0.6	0.55	0.35
b	0.6	0.55	0.45

Hence we have

$$L_\ell(\mathcal{I}[T]; \beta_T) = \begin{cases} \emptyset & \text{if } \beta_T \in [0, 0.2), \\ \{0\} & \text{if } \beta_T \in [0.2, 0.3), \\ \{0, 2\} & \text{if } \beta_T \in [0.3, 0.4), \\ \{0, 1, 2\} & \text{if } \beta_T \in [0.4, 0.6), \\ X & \text{if } \beta_T \in [0.6, 1], \end{cases}$$

$$L_\ell(\mathcal{I}[I]; \beta_I) = \begin{cases} \emptyset & \text{if } \beta_I \in [0, 0.33), \\ \{0\} & \text{if } \beta_I \in [0.33, 0.44), \\ \{0, 1\} & \text{if } \beta_I \in [0.44, 0.55), \\ \{0, 1, a, b\} & \text{if } \beta_I \in [0.55, 0.66), \\ X & \text{if } \beta_I \in [0.66, 1], \end{cases}$$

and

$$L_\ell(\mathcal{I}[F];\beta_F)=\begin{cases}\emptyset & \text{if } \beta_F\in[0,0.15),\\ \{0\} & \text{if } \beta_F\in[0.15,0.25),\\ \{0,2\} & \text{if } \beta_F\in[0.25,0.35),\\ \{0,2,a\} & \text{if } \beta_F\in[0.35,0.45),\\ X & \text{if } \beta_F\in[0.45,1],\end{cases}$$

which are subalgebras of $(X,*,0)$ for all β_T, β_I, $\beta_F \in [0,1]$ such that $L_\ell(\mathcal{I}[T];\beta_T)$, $L_\ell(\mathcal{I}[I];\beta_I)$ and $L_\ell(\mathcal{I}[F];\beta_F)$ are nonempty. But $\mathcal{I}:=(\mathcal{I}[T]$, $\mathcal{I}[I]$, $\mathcal{I}[F])$ is not an (i,j,k)-length neutrosophic subalgebra of $(X,*,0)$ for $i,j,k\in\{2,4\}$ with $(i,j,k)\neq(4,4,4)$ since

$$\mathcal{I}[T]_\ell(a*1)=0.6\not\leq 0.4=\min\{\mathcal{I}[T]_\ell(a),\mathcal{I}[T]_\ell(1)\},$$

$$\mathcal{I}[I]_\ell(a*0)=0.55\not\leq 0.33=\min\{\mathcal{I}[I]_\ell(a),\mathcal{I}[I]_\ell(0)\},$$

and/or

$$\mathcal{I}[F]_\ell(2*a)=0.35\not\leq 0.25=\min\{\mathcal{I}[F]_\ell(2),\mathcal{I}[F]_\ell(a)\}.$$

Using the similar way to the proofs of Theorems 3.7 and 3.10, we have the following theorem.

Theorem 3.13. *Given an (i,j,k)-length neutrosophic subalgebra $\mathcal{I}:=(\mathcal{I}[T]$, $\mathcal{I}[I]$, $\mathcal{I}[F])$ of $(X,*,0)$ for $i,j,k\in\{1,2,3,4\}$, the following assertions are valid.*

(1) *If $i,j\in\{1,3\}$ and $k\in\{2,4\}$, then $U_\ell(\mathcal{I}[T];\alpha_T)$, $U_\ell(\mathcal{I}[I];\alpha_I)$ and $L_\ell(\mathcal{I}[F];\beta_F)$ are subalgebras of $(X,*,0)$ whenever they are nonempty.*

(2) *If $i,k\in\{1,3\}$ and $j\in\{2,4\}$, then $U_\ell(\mathcal{I}[T];\alpha_T)$, $L_\ell(\mathcal{I}[I];\beta_I)$ and $U_\ell(\mathcal{I}[F];\alpha_F)$ are subalgebras of $(X,*,0)$ whenever they are nonempty.*

(3) *If $i\in\{2,4\}$ and $j,k\in\{1,3\}$, then $L_\ell(\mathcal{I}[T];\beta_T)$, $U_\ell(\mathcal{I}[I];\alpha_I)$ and $U_\ell(\mathcal{I}[F];\alpha_F)$ are subalgebras of $(X,*,0)$ whenever they are nonempty.*

(4) *If $i,j\in\{2,4\}$ and $k\in\{1,3\}$, then $L_\ell(\mathcal{I}[T];\beta_T)$, $L_\ell(\mathcal{I}[I];\beta_I)$ and $U_\ell(\mathcal{I}[F];\alpha_F)$ are subalgebras of $(X,*,0)$ whenever they are nonempty.*

(5) *If* $i, k \in \{2, 4\}$ *and* $j \in \{1, 3\}$, *then* $L_\ell(\mathcal{I}[T]; \beta_T)$, $U_\ell(\mathcal{I}[I]; \alpha_I)$ *and* $L_\ell(\mathcal{I}[F]; \beta_F)$ *are subalgebras of* $(X, *, 0)$ *whenever they are nonempty.*

(6) *If* $i \in \{1, 3\}$ *and* $j, k \in \{2, 4\}$, *then* $U_\ell(\mathcal{I}[T]; \alpha_T)$, $L_\ell(\mathcal{I}[I]; \beta_I)$ *and* $L_\ell(\mathcal{I}[F]; \beta_F)$ *are subalgebras of* $(X, *, 0)$ *whenever they are nonempty.*

THEOREM 3.14. *If an interval neutrosophic set* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(2, 3, 2)$*-length neutrosophic subalgebra of* $(X, *, 0)$, *then* $U_\ell(\mathcal{I}[T]; \alpha_T)^c$, $L_\ell(\mathcal{I}[I]; \beta_I)^c$ *and* $U_\ell(\mathcal{I}[F]; \alpha_F)^c$ *are subalgebras of* $(X, *, 0)$ *whenever they are nonempty for all* α_T, β_I, $\alpha_F \in [0, 1]$.

PROOF: Assume that $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(2, 3, 2)$-length neutrosophic subalgebra of $(X, *, 0)$. Let α_T, β_I, $\alpha_F \in [0, 1]$ be such that $U_\ell(\mathcal{I}[T]; \alpha_T)^c$, $L_\ell(\mathcal{I}[I]; \beta_I)^c$ and $U_\ell(\mathcal{I}[F]; \alpha_F)^c$ are nonempty. If $x, y \in U_\ell(\mathcal{I}[T]; \alpha_T)^c$, then $\mathcal{I}[T]_\ell(x) < \alpha_T$ and $\mathcal{I}[T]_\ell(y) < \alpha_T$. Hence

$$\mathcal{I}[T]_\ell(x * y) \leq \min\{\mathcal{I}[T]_\ell(x), \mathcal{I}[T]_\ell(y)\} < \alpha_T,$$

and so $x * y \in U_\ell(\mathcal{I}[T]; \alpha_T)^c$. If $x, y \in L_\ell(\mathcal{I}[I]; \beta_I)^c$, then $\mathcal{I}[I]_\ell(x) > \beta_I$ and $\mathcal{I}[I]_\ell(y) > \beta_I$. Thus

$$\mathcal{I}[I]_\ell(x * y) \geq \max\{\mathcal{I}[I]_\ell(x), \mathcal{I}[I]_\ell(y)\} > \beta_I,$$

which implies that $x * y \in L_\ell(\mathcal{I}[I]; \beta_I)^c$. Let $x, y \in U_\ell(\mathcal{I}[F]; \alpha_F)^c$. Then $\mathcal{I}[F]_\ell(x) < \alpha_F$ and $\mathcal{I}[F]_\ell(y) < \alpha_F$. Hence

$$\mathcal{I}[F]_\ell(x * y) \leq \min\{\mathcal{I}[F]_\ell(x), \mathcal{I}[F]_\ell(y)\} < \alpha_F,$$

and so $x * y \in U_\ell(\mathcal{I}[F]; \alpha_F)^c$. Therefore $U_\ell(\mathcal{I}[T]; \alpha_T)^c$, $L_\ell(\mathcal{I}[I]; \beta_I)^c$ and $U_\ell(\mathcal{I}[F]; \alpha_F)^c$ are subalgebras of $(X, *, 0)$ for all α_T, β_I, $\alpha_F \in [0, 1]$. $\square$

The converse of Theorem 3.14 is not true in general as seen in the following example.

Example 3.15. Consider a *BCI*-algebra $X = \{0, 1, a, b, c\}$ with the binary operation $*$ which is given in Table 6 (see [5]).
Let $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ be an interval neutrosophic set in $(X, *, 0)$ given by

Table 6. Cayley table for the binary operation “$*$”

$*$	0	1	a	b	c
0	0	0	a	b	c
1	1	0	a	b	c
a	a	a	0	c	b
b	b	b	c	0	a
c	c	c	b	a	0

$$\mathcal{I}[T] : X \to \mathcal{P}^*([0,1]),\ x \mapsto \begin{cases} [0.50, 0.75) & \text{if } x = 0, \\ (0.25, 0.70] & \text{if } x = 1, \\ [0.10, 0.65] & \text{if } x = a, \\ [0.05, 0.70) & \text{if } x = b, \\ (0.10, 0.75] & \text{if } x = c, \end{cases}$$

$$\mathcal{I}[I] : X \to \mathcal{P}^*([0,1]),\ x \mapsto \begin{cases} [0.05, 0.80] & \text{if } x = 0, \\ (0.10, 0.80) & \text{if } x = 1, \\ [0.26, 0.89] & \text{if } x = a, \\ (0.16, 0.79) & \text{if } x = b, \\ (0.07, 0.75] & \text{if } x = c, \end{cases}$$

and

$$\mathcal{I}[F] : X \to \mathcal{P}^*([0,1]),\ x \mapsto \begin{cases} [0.23, 0.67) & \text{if } x = 0, \\ (0.03, 0.58] & \text{if } x = 1, \\ (0.18, 0.73) & \text{if } x = a, \\ [0.14, 0.80] & \text{if } x = b, \\ (0.07, 0.73] & \text{if } x = c. \end{cases}$$

Then the interval neutrosophic length $\mathcal{I}_\ell := (\mathcal{I}[T]_\ell,\, \mathcal{I}[I]_\ell,\, \mathcal{I}[F]_\ell)$ of $\mathcal{I}$ is given by Table 7.

Then

$$U_\ell(\mathcal{I}[T]; \alpha_T)^c = \begin{cases} \emptyset & \text{if } \alpha_T \in [0, 0.25], \\ \{0\} & \text{if } \alpha_T \in (0.25, 0.45], \\ \{0, 1\} & \text{if } \alpha_T \in (0.45, 0.55], \\ \{0, 1, a\} & \text{if } \alpha_T \in (0.55, 0.65], \\ X & \text{if } \alpha_T \in (0.65, 1], \end{cases}$$

Table 7. Interval neutrosophic length of $\mathcal{I}$

X	$\mathcal{I}[T]_\ell$	$\mathcal{I}[I]_\ell$	$\mathcal{I}[F]_\ell$
0	0.25	0.75	0.44
1	0.45	0.70	0.55
a	0.55	0.63	0.55
b	0.65	0.63	0.66
c	0.65	0.68	0.66

$$L_\ell(\mathcal{I}[I];\beta_I)^c = \begin{cases} \emptyset & \text{if } \beta_I \in [0.75, 1], \\ \{0\} & \text{if } \beta_I \in [0.70, 0.75), \\ \{0,1\} & \text{if } \beta_I \in [0.68, 0.70), \\ \{0,1,c\} & \text{if } \beta_I \in [0.63, 0.68), \\ X & \text{if } \beta_I \in [0, 0.63), \end{cases}$$

and

$$U_\ell(\mathcal{I}[F];\alpha_F)^c = \begin{cases} \emptyset & \text{if } \alpha_F \in [0, 0.44], \\ \{0\} & \text{if } \alpha_F \in (0.44, 0.55], \\ \{0,1,a\} & \text{if } \alpha_F \in (0.55, 0.66], \\ X & \text{if } \alpha_F \in (0.66, 1] \end{cases}$$

are subalgebras of $(X, *, 0)$ whenever they are nonempty for all α_T, β_I, $\alpha_F \in [0, 1]$. But $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is not a $(2, 3, 2)$-length neutrosophic subalgebra of $(X, *, 0)$ since

$$\mathcal{I}[T]_\ell(b * a) = \mathcal{I}[T]_\ell(c) = 0.65 > 0.55 = \min\{\mathcal{I}[T]_\ell(b), \mathcal{I}[T]_\ell(a)\},$$

$$\mathcal{I}[I]_\ell(b * c) = \mathcal{I}[I]_\ell(a) = 0.63 < 0.68 = \max\{\mathcal{I}[I]_\ell(b), \mathcal{I}[I]_\ell(c)\},$$

and/or

$$\mathcal{I}[F]_\ell(b * a) = \mathcal{I}[F]_\ell(c) = 0.66 > 0.55 = \min\{\mathcal{I}[F]_\ell(b), \mathcal{I}[F]_\ell(a)\}.$$

By the similar way to the proof of Theorem 3.14, we have the following theorem.

THEOREM 3.16. *Given an (i, j, k)-length neutrosophic subalgebra $\mathcal{I} := (\mathcal{I}[T]$, $\mathcal{I}[I]$, $\mathcal{I}[F])$ of $(X, *, 0)$, the following assertions are valid.*

(1) *If* $(i,j,k) = (2,2,2)$, *then* $U_\ell(\mathcal{I}[T];\alpha_T)^c$, $U_\ell(\mathcal{I}[I];\alpha_I)^c$ *and* $U_\ell(\mathcal{I}[F];\alpha_F)^c$ *are subalgebras of* $(X,*,0)$ *whenever they are nonempty for all* α_T, α_I, $\alpha_F \in [0,1]$.

(2) *If* $(i,j,k) = (2,2,3)$, *then* $U_\ell(\mathcal{I}[T];\alpha_T)^c$, $U_\ell(\mathcal{I}[I];\alpha_I)^c$ *and* $L_\ell(\mathcal{I}[F];\beta_F)^c$ *are subalgebras of* $(X,*,0)$ *whenever they are nonempty for all* α_T, α_I, $\beta_F \in [0,1]$.

(3) *If* $(i,j,k) = (2,3,3)$, *then* $U_\ell(\mathcal{I}[T];\alpha_T)^c$, $L_\ell(\mathcal{I}[I];\beta_I)^c$ *and* $L_\ell(\mathcal{I}[F];\beta_F)^c$ *are subalgebras of* $(X,*,0)$ *whenever they are nonempty for all* α_T, β_I, $\beta_F \in [0,1]$.

(4) *If* $(i,j,k) = (3,2,2)$, *then* $L_\ell(\mathcal{I}[T];\beta_T)^c$, $U_\ell(\mathcal{I}[I];\alpha_I)^c$ *and* $U_\ell(\mathcal{I}[F];\alpha_F)^c$ *are subalgebras of* $(X,*,0)$ *whenever they are nonempty for all* β_T, α_I, $\alpha_F \in [0,1]$.

(5) *If* $(i,j,k) = (3,2,3)$, *then* $L_\ell(\mathcal{I}[T];\beta_T)^c$, $U_\ell(\mathcal{I}[I];\alpha_I)^c$ *and* $L_\ell(\mathcal{I}[F];\beta_F)^c$ *are subalgebras of* $(X,*,0)$ *whenever they are nonempty for all* β_T, α_I, $\beta_F \in [0,1]$.

(6) *If* $(i,j,k) = (3,3,2)$, *then* $L_\ell(\mathcal{I}[T];\beta_T)^c$, $L_\ell(\mathcal{I}[I];\beta_I)^c$ *and* $U_\ell(\mathcal{I}[F];\alpha_F)^c$ *are subalgebras of* $(X,*,0)$ *whenever they are nonempty for all* β_T, β_I, $\alpha_F \in [0,1]$.

(7) *If* $(i,j,k) = (3,3,3)$, *then* $L_\ell(\mathcal{I}[T];\beta_T)^c$, $L_\ell(\mathcal{I}[I];\beta_I)^c$ *and* $L_\ell(\mathcal{I}[F];\beta_F)^c$ *are subalgebras of* $(X,*,0)$ *whenever they are nonempty for all* β_T, β_I, $\beta_F \in [0,1]$.

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FOUR-VALUED EXPANSIONS OF DUNN-BELNAP'S LOGIC (I): BASIC CHARACTERIZATIONS

Abstract

Basic results of the paper are that any four-valued expansion L_4 of Dunn-Belnap's logic DB_4 is defined by a unique (up to isomorphism) conjunctive matrix $\mathcal{M}_4$ with exactly two distinguished values over an expansion $\mathfrak{A}_4$ of a De Morgan non-Boolean four-valued diamond, but by no matrix with either less than four values or a single [non-]distinguished value, and has no proper extension satisfying Variable Sharing Property (VSP). We then characterize L_4's having a theorem / inconsistent formula, satisfying VSP and being [inferentially] maximal / subclassical / maximally paraconsistent, in particular, algebraically through $\mathcal{M}_4|\mathfrak{A}_4$'s (not) having certain submatrices|subalgebras.

Likewise, [providing $\mathfrak{A}_4$ is regular / has no three-element subalgebra] L_4 has a proper consistent axiomatic extension if[f] $\mathcal{M}_4$ has a proper paraconsistent / two-valued submatrix [in which case the logic of this submatrix is the only proper consistent axiomatic extension of L_4 and is relatively axiomatized by the *Excluded Middle law* axiom]. As a generic tool (applicable, in particular, to both classically-negative and implicative expansions of DB_4), we also prove that the lattice of axiomatic extensions of the logic of an implicative matrix $\mathcal{M}$ with equality determinant is dual to the distributive lattice of lower cones of the set of all submatrices of $\mathcal{M}$ with non-distinguished values.

Keywords: Propositional logic, logical matrix, Dunn-Belnap's logic, expansion, [bounded] distributive/De Morgan lattice, equality determinant.

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1. Introduction

Dunn-Belnap's four-valued logic (cf. [5] and [3]) arising as the logic of *first-degree entailment* (*FDE*, for short) in relevance logic *R* has been naturally expanded by additional connectives in [11]. The present paper, equally belonging to General Logic, pursues this line of research in the following *generic* respects in addition to those of functional completeness and both sequential and equational axiomatizations comprehensively explored therein.

First of all, the most natural way of expanding FDE consists in expanding the matrix $\mathcal{DM}_4$ defining FDE by additional connectives. This inevitably raises the question which exactly expansions of FDE are covered by such approach. As we argue here, these are exactly *all four-valued* ones (that excludes *E* and *R*). And what is more, any four-valued expansion of FDE is defined by a *unique* expansion of $\mathcal{DM}_4$.

In addition, as a by-product of auxiliary results, we prove that any four-valued expansion of FDE is defined by no matrix with either a unique (non-)distinguished value or less than four values and has no proper extension satisfying *Variable Sharing Property* (*VSP*, for short; cf. [1]), according to which any entailment $\phi \to \psi$ holds only if ϕ and ψ have a propositional variable in common, that is one of the most fundamental peculiarities of FDE, quite independently from whether the expansion itself satisfies VSP. The latter result has been proved for FDE alone in [9] and means, perhaps, a principal maximality of expansions of FDE. In this connection, we find purely algebraic criteria of a FDE expansion's satisfying VSP, being [inferentially] maximal in the sense of not having a proper [inferentially] consistent extension,[1] being a sublogic of a definitional copy of the classical logic and being *maximally* paraconsistent in the sense of [10] (viz., having no proper paraconsistent extension).

After all, we study the issue of axiomatic extensions within the framework of FDE expansions.

The rest of the paper is as follows. The exposition of the material of the paper is entirely self-contained (of course, modulo very basic issues concerning Set Theory, Lattice Theory, Universal Algebra, Model Theory and Mathematical Logic not specified here explicitly, to be found, e.g., in

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[1] It is the absence of theorems in FDE, being an inevitable consequence of VSP, that makes "inferential" versions of standard conceptions of consistency and maximality acute within the framework of FDE expansions to be equally void of theorems.

standard mathematical handbooks like [2] and [7]). Section 2 is a concise summary of basic issues underlying the paper, most of which have actually become a part of logical and algebraic folklore. Section 3 is devoted to certain key preliminary issues concerning equality determinants (in the sense of [13]), implicative matrices and De Morgan lattices. In Section 4 we formulate and prove main results of the paper described above. Then, in Section 5 we apply general results of previous two sections to three generic – classically-negative, bilattice and implicative – classes of FDE expansions.

2. Basic issues

Standard notations like img, dom, ker, hom, π_i, Con, et. al., as well as related notions are supposed to be clear.

2.1. Set-theoretical background

We follow the standard convention (among other things, contracting cumbersome finite sequence notations), according to which natural numbers (including 0) are treated as finite ordinals (viz., sets of lesser natural numbers), the ordinal of all them being denoted by ω. The proper class of all ordinals is denoted by ∞. Likewise, functions are viewed as binary relations. In addition, singletons are often identified with their unique elements, unless any confusion is possible.

Given a set S, the set of all subsets of S [of cardinality $\in K \subseteq \infty$][2] is denoted by $\wp_{[K]}(S)$. A subset $T \subseteq S$ is said to be *proper*, if $T \neq S$. Further, given any equivalence relation θ on S, as usual, by ν_θ we denote the function with domain S defined by $\nu_\theta(a) \triangleq \theta[\{a\}]$, for all $a \in S$, in which case $\ker \nu_\theta = \theta$, whereas we set $(T/\theta) \triangleq \nu_\theta[T]$, for every $T \subseteq S$. Next, S-tuples (viz., functions with domain S) are often written in either sequence $\bar{t}$ or vector $\vec{t}$ forms, its s-th component (viz., the value under argument s), where $s \in S$, being written as either t_s or t^s. Given two more sets A and B, any relation $R \subseteq (A \times B)$ (in particular, a mapping $R : A \to B$) determines the equally-denoted relation $R \subseteq (A^S \times B^S)$ (resp., mapping $R : A^S \to B^S$) point-wise, that is, $R \triangleq \{\langle \bar{a}, \bar{b} \rangle \in (A^S \times B^S) \mid \forall s \in S : a_s \; R \; b_s\}$. Likewise, given a set A, an S-tuple $\overline{B}$ of sets and any

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[2] As usual, parentheses as well as both square, figure and angle brackets are often used for surrounding a (possibly, multiple) optional content.

$\bar{f} \in (\prod_{s\in S} B_s^A)$, put $(\prod \bar{f}) : A \to (\prod \overline{B}), a \mapsto \langle f_s(a)\rangle_{s\in S}$. (In case $I = 2$, $f_0 \times f_1$ stands for $(\prod \bar{f})$.) Further, a *lower cone of* a $T \subseteq \wp(S)$ is any $L \subseteq T$ such that, for each $X \in L$, $(\wp(X) \cap T) \subseteq L$. Likewise, an *anti-chain of* T is any $A \subseteq T$ such that $\max(A) = A$. (Clearly, in case S is finite, the unary operations $A \mapsto (T \cap \bigcup\{\wp(X) \mid X \in A\})$ and $L \mapsto \max(L)$ on $\wp(\wp(S))$ form inverse to one another bijections between the sets of all anti-chains and all lower cones of T.) Furthermore, set $\Delta_S \triangleq \{\langle a, a\rangle | a \in S\}$, functions of such a kind being referred to as *diagonal*. Finally, given any $R \subseteq S^2$, $\mathrm{Tr}(R) \triangleq \{\langle \pi_0(\pi_0(\bar{r})), \pi_1(\pi_{l-1}(\bar{r}))\rangle | \bar{r} \in R^l, l \in (\omega \setminus 1)\}$ is the least transitive binary relation on S including R, known as the *transitive closure of* R.

2.2. Algebraic background

Unless otherwise specified, abstract algebras are denoted by Fraktur letters (possibly, with indices/prefixes/suffixes), their carriers (viz., underlying sets) being denoted by corresponding Italic letters (with same indices/prefixes/suffixes, if any).

A *(propositional/sentential) language/signature* is any algebraic (viz., functional) signature Σ (to be dealt with by default throughout the paper) constituted by function (viz., operation) symbols of finite arity to be treated as *(propositional/sentential) connectives*. Given any $\alpha \in \wp_{\infty\setminus 1}(\omega)$, put $V_\alpha \triangleq \{x_\beta | \beta \in \alpha\}$, elements of which being viewed as *(propositional/sentential) variables of rank* α. Then, we have the absolutely-free Σ-algebra $\mathfrak{Fm}_\Sigma^\alpha$ freely-generated by the set V_α, referred to as the *formula* Σ*-algebra of rank* α, its endomorphisms/elements of its carrier $\mathrm{Fm}_\Sigma^\alpha$ (viz., Σ-terms of rank α) being called *(propositional/sentential)* Σ*-substitutions/-formulas of rank* α. (In general, the reservation "of rank α" is normally omitted, whenever $\alpha = \omega$.) Given a Σ-formula φ, $\mathrm{Var}(\varphi)$ denotes the set of all variables *actually* occurring in φ.

Recall the following useful well-known algebraic fact:

Lemma 2.1. *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be* Σ*-algebras and* $h \in \hom(\mathfrak{A}, \mathfrak{B})$*. [Suppose* $(\operatorname{img} h) = B$*.] Then, for every* $\vartheta \in \mathrm{Con}(\mathfrak{B})$*,* $h^{-1}[\vartheta] \in \{\theta \in \mathrm{Con}(\mathfrak{A}) \mid (\ker h) \subseteq \theta\}$ *[whereas* $h[h^{-1}[\vartheta]] = \vartheta$*, while, conversely, for every* $\theta \in \mathrm{Con}(\mathfrak{A})$ *such that* $(\ker h) \subseteq \theta$*,* $h[\theta] \in \mathrm{Con}(\mathfrak{B})$*, whereas* $h^{-1}[h[\theta]] = \theta$*].*

2.3. Propositional logics and matrices

A *[finitary]* Σ*-rule* is any couple $\langle\Gamma,\varphi\rangle$, where $(\Gamma\cup\{\varphi\})\in\wp_{[\omega]}(\mathrm{Fm}^{\omega}_{\Sigma})$, normally written in the standard sequent form $\Gamma\vdash\varphi$, φ/any element of Γ being referred to as the/a *conclusion/premise of* it. A *(substitutional)* Σ*-instance* of it is then any Σ-rule of the form $\sigma(\Gamma\vdash\varphi)\triangleq(\sigma[\Gamma]\vdash\sigma(\varphi))$, where σ is a Σ-substitution. As usual, Σ-rules without premises are called Σ*-axioms* and are identified with their conclusions. A[n] *[axiomatic] (finitary)* Σ*-calculus* is any set of (finitary) Σ-rules[-axioms].

A *(propositional/sentential)* Σ*-logic* (cf., e.g., [6]) is any closure operator C over $\mathrm{Fm}^{\omega}_{\Sigma}$ that is *structural* in the sense that $\sigma[C(X)]\subseteq C(\sigma[X])$, for all $X\subseteq\mathrm{Fm}^{\omega}_{\Sigma}$ and all $\sigma\in\hom(\mathfrak{Fm}^{\omega}_{\Sigma},\mathfrak{Fm}^{\omega}_{\Sigma})$. A(n) *(in)consistent set of* C is any $X\subseteq\mathrm{Fm}^{\omega}_{\Sigma}$ such that $C(X)\neq(=)\,\mathrm{Fm}^{\omega}_{\Sigma}$. Then, C is said to be *[inferentially] (in)consistent*, provided $\varnothing[\cup\{x_0\}]$ is a(n in)consistent set of C or, equivalently, in view of the structurality of C, $x_1\notin(\in)C(\varnothing[\cup\{x_0\}])$. A Σ-rule $\Gamma\vdash\varphi$ is said to be *satisfied in* C, provided $\varphi\in C(\Gamma)$, Σ-axioms satisfied in C being called its *theorems*. A *[proper] extension of* C is any Σ-logic $C'\supseteq C$ [distinct from C], in which case C is said to be a *[proper] sublogic of* C'. Then, an extension C' of C is said to be *axiomatized by* a Σ-calculus $\mathcal{C}$ *relatively to* C, provided it is the least extension of C satisfying each rule of $\mathcal{C}$. Furthermore, an extension C' of C is said to be *axiomatic*, whenever it is relatively axiomatized by an axiomatic Σ-calculus. Next, C is said to be *[inferentially] maximal(ly consistent)*, whenever it is [inferentially] consistent and has no proper [inferentially] consistent extension. Further, C is said to be $\diamond$*-conjunctive*, where $\diamond$ is a (possibly, secondary) binary connective of Σ, provided $C(\phi\diamond\psi)=C(\{\phi,\psi\})$, for all $\phi,\psi\in\mathrm{Fm}^{\omega}_{\Sigma}$, in which case any extension of C is so. Likewise, C is said to be *[maximally]* $\wr$*-paraconsistent*, where $\wr$ is a unary connective of Σ, provided $x_1\notin C(\{x_0,\wr x_0\})$ [and C has no proper $\wr$-paraconsistent extension]. In addition, C is said to be *theoremless*, provided $C(\varnothing)=\varnothing$. Finally, *Variable Sharing Property* (*VSP*, for short; cf. [1]) is said to *hold/be satisfied for* C, provided, for all $\phi\in\mathrm{Fm}^{\omega}_{\Sigma}$ and all $\psi\in C(\phi)$, it holds that $(\mathrm{Var}(\phi)\cap\mathrm{Var}(\psi))\neq\varnothing$, in which case C has neither a theorem nor an inconsistent formula, in view of the finiteness of the set $\mathrm{Var}(\varphi)$, where $\varphi\in\mathrm{Fm}^{\omega}_{\Sigma}$.

A *(logical)* Σ*-matrix* (cf. [6]) is any couple of the form $\mathcal{A}=\langle\mathfrak{A},D^{\mathcal{A}}\rangle$, where $\mathfrak{A}$ is a Σ-algebra, called the *underlying algebra of* $\mathcal{A}$, while $D^{\mathcal{A}}\subseteq A$ is called the *truth predicate of* $\mathcal{A}$, elements of which being referred to as *distinguished values of* $\mathcal{A}$. (In general, matrices are denoted by Calligraphic letters [possibly, with indices/prefixes/suffixes], their underlying algebras being denoted by corresponding Fraktur letters [with same indices/prefixes/suffixes, if any].) This is said to be *n-valued/truth[-non]-empty/(in)consistent/false-singular/truth-singular*, where $n \in \omega$, provided $|A| = n/D^{\mathcal{A}} = [\neq]\varnothing/D^{\mathcal{A}} \neq (=)A/|A \setminus D^{\mathcal{A}}| \in 2/|D^{\mathcal{A}}| \in 2$. Next, given any $\Sigma' \subseteq \Sigma$, put $(\mathcal{A}{\upharpoonright}\Sigma') \triangleq \langle \mathfrak{A}{\upharpoonright}\Sigma', D^{\mathcal{A}} \rangle$, in which case $\mathcal{A}$ is said to be a *(Σ-)expansion of* $\mathcal{A}{\upharpoonright}\Sigma'$. (Any notation, being specified for single matrices, is supposed to be extended to classes of matrices member-wise.)

A Σ-matrix $\mathcal{A}$ is said to be *finite/finitely-generated/generated by* a $B \subseteq A$, whenever $\mathfrak{A}$ is so. Then, $\mathcal{A}$ is said to be *K-generated*, where $K \subseteq \infty$, whenever it is generated by a $B \in \wp_K(A)$.

As usual, Σ-matrices are treated as first-order model structures (viz., algebraic systems; cf. [7]) of the first-order signature $\Sigma \cup \{D\}$ with unary predicate D, any [finitary] Σ-rule $\Gamma \vdash \phi$ being viewed as the [first-order] Horn formula $(\bigwedge \Gamma) \to \phi$ under the standard identification of any propositional Σ-formula ψ with the first-order atomic formula $D(\psi)$. Then, the class of all models of a Σ-calculus $\mathcal{C}$ is denoted by $\mathrm{Mod}(\mathcal{C})$. In that case, given any class of Σ-matrices M, $\mathcal{C}$ is said to *axiomatize* $\mathsf{M} \cap \mathrm{Mod}(\mathcal{C})$ *relatively to* M.

Given any $\alpha \in \wp_{\infty\setminus 1}(\omega)$ and any class M of Σ-matrices, we have the closure operator $\mathrm{Cn}^{\alpha}_{\mathsf{M}}$ over $\mathrm{Fm}^{\alpha}_{\Sigma}$ defined by $\mathrm{Cn}^{\alpha}_{\mathsf{M}}(X) \triangleq (\mathrm{Fm}^{\alpha}_{\Sigma} \cap \bigcap\{h^{-1}[D^{\mathcal{A}}] | \mathcal{A} \in \mathsf{M}, h \in \hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{A}), h[X] \subseteq D^{\mathcal{A}}\}$, for all $X \subseteq \mathrm{Fm}^{\alpha}_{\Sigma}$, in which case we have:

$$\mathrm{Cn}^{\alpha}_{\mathsf{M}}(X) = (\mathrm{Fm}^{\alpha}_{\Sigma} \cap \mathrm{Cn}^{\omega}_{\mathsf{M}}(X)), \tag{2.1}$$

because $\hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{A}) = \{h{\upharpoonright}\mathrm{Fm}^{\alpha}_{\Sigma} | h \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})\}$, for any Σ-algebra $\mathfrak{A}$, as $A \neq \varnothing$. (Note that $\mathrm{Cn}^{\alpha}_{\mathsf{M}}(\varnothing) = \varnothing$, whenever M has a truth-empty member.) Then, $\mathrm{Cn}^{\omega}_{\mathsf{M}}$ is a Σ-logic called the one *of* M. Next, a Σ-logic C is said to be *K-defined by* M, where $K \subseteq \infty$, if $(C{\upharpoonright}\wp_K(\mathrm{Fm}^{\omega}_{\Sigma})) = (\mathrm{Cn}^{\omega}_{\mathsf{M}}{\upharpoonright}\wp_K(\mathrm{Fm}^{\omega}_{\Sigma}))$. (As usual, "finitely-" stands for "ω-". Likewise, "∞-" is normally omitted, whenever no confusion is possible.) A Σ-logic C is said to be *[minimally] n-valued*, where $n \in \omega$, whenever it is defined by an n-valued Σ-matrix [but by no m-valued one, where $m \in n$], in which case C is finitary (cf. [6]). A Σ-matrix $\mathcal{A}$ is said to be *$\wr$-paraconsistent*, where $\wr$ is a unary connective of Σ, whenever the logic of $\mathcal{A}$ is so. (Clearly, the logic of any class of matrices is [inferentially] consistent iff the class contains a consistent [truth-non-empty] member.)

Let $\mathcal{A}$ and $\mathcal{B}$ be two Σ-matrices. A *(strict) [surjective] homomorphism from* $\mathcal{A}$ *[on]to* $\mathcal{B}$ is any $h \in \hom(\mathfrak{A}, \mathfrak{B})$ such that $[h[A] = B$ and$]$ $D^{\mathcal{A}} \subseteq (=) h^{-1}[D^{\mathcal{B}}]$, the set of all them being denoted by $\hom^{[\mathrm{S}]}_{(\mathrm{S})}(\mathcal{A}, \mathcal{B})$. Recall that $\forall h \in \hom(\mathfrak{A}, \mathfrak{B}) : [((\operatorname{img} h) = B) \Rightarrow](\hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{B}) \supseteq [=]\{h \circ g | g \in \hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{A})\})$, and so we have:

$$(\exists h \in \hom^{[\mathrm{S}]}_{\mathrm{S}}(\mathcal{A}, \mathcal{B})) \Rightarrow (\mathrm{Cn}^{\alpha}_{\mathcal{B}} \subseteq [=] \, \mathrm{Cn}^{\alpha}_{\mathcal{A}}), \tag{2.2}$$

$$(\exists h \in \hom^{\mathrm{S}}(\mathcal{A}, \mathcal{B})) \Rightarrow (\mathrm{Cn}^{\alpha}_{\mathcal{A}}(\varnothing) \subseteq \mathrm{Cn}^{\alpha}_{\mathcal{B}}(\varnothing)), \tag{2.3}$$

for all $\alpha \in \wp_{\infty \setminus 1}(\omega)$. Then, $\mathcal{A}$ is said to be a *[proper] submatrix of* $\mathcal{B}$, whenever $\Delta_A \in \hom_{\mathrm{S}}(\mathcal{A}, \mathcal{B})$ [and $\mathcal{A} \neq \mathcal{B}$], in which case we set $(\mathcal{B} {\restriction} A) \triangleq \mathcal{A}$. Injective/bijective strict homomorphisms from $\mathcal{A}$ to $\mathcal{B}$ are referred to as *embeddings/isomorphisms of/from* $\mathcal{A}$ *into/onto* $\mathcal{B}$, in case of existence of which $\mathcal{A}$ is said to be *embeddable/isomorphic into/to* $\mathcal{B}$.

Let $\mathcal{A}$ be a Σ-matrix. Elements of $\mathrm{Con}(\mathcal{A}) \triangleq \{\theta \in \mathrm{Con}(\mathfrak{A}) | \theta[D^{\mathcal{A}}] \subseteq D^{\mathcal{A}}\} \ni \Delta_A$ are called *congruences of* $\mathcal{A}$. Given any $\varnothing \neq \Theta \subseteq \mathrm{Con}(\mathcal{A}) \subseteq \mathrm{Con}(\mathfrak{A})$, $\mathrm{Tr}(\bigcup \Theta)$, being well-known to be a congruence of $\mathfrak{A}$, is then easily seen to be a congruence of $\mathcal{A}$. Therefore, $\Game(\mathcal{A}) \triangleq (\bigcup \mathrm{Con}(\mathcal{A})) \in \mathrm{Con}(\mathcal{A})$, in which case this is the greatest congruence of $\mathcal{A}$ (it is this fact that justifies using the symbol $\Game$). Then, $\mathcal{A}$ is said to be *simple/irreducible*, provided $\Game(\mathcal{A}) = \Delta_A$. Given any $\theta \in \mathrm{Con}(\mathfrak{A}[\mathcal{A}])$, we have the *quotient* Σ-matrix $(\mathcal{A}/\theta) \triangleq \langle \mathfrak{A}/\theta, D^{\mathcal{A}}/\theta \rangle$, in which case $\nu_{\theta} \in \hom^{\mathrm{S}}_{[\mathrm{S}]}(\mathcal{A}, \mathcal{A}/\theta)$. The quotient $\Re(\mathcal{A}) \triangleq (\mathcal{A}/\Game(\mathcal{A}))$ is called the *reduction of* $\mathcal{A}$.

A Σ-matrix $\mathcal{A}$ is said to be a *model of* a Σ-logic C, provided $C \subseteq \mathrm{Cn}^{\omega}_{\mathcal{A}}$, the class of all [irreducible of] them being denoted by $\mathrm{Mod}_{[\Im]}(C)$. Next, $\mathcal{A}$ is said to be $\diamond$*-conjunctive*, where $\diamond$ is a (possibly, secondary) binary connective of Σ, provided $(\{a, b\} \subseteq D^{\mathcal{A}}) \Leftrightarrow ((a \diamond^{\mathfrak{A}} b) \in D^{\mathcal{A}})$, for all $a, b \in A$, that is, $\mathrm{Cn}^{\omega}_{\mathcal{A}}$ is $\diamond$-conjunctive.

Remark 2.2. As an immediate consequence of Lemma 2.1, given any Σ-matrices $\mathcal{A}$ and $\mathcal{B}$ and any $h \in \hom^{[\mathrm{S}]}_{\mathrm{S}}(\mathcal{A}, \mathcal{B})$, for every $\vartheta \in \mathrm{Con}(\mathcal{B})$, $h^{-1}[\vartheta] \in \{\theta \in \mathrm{Con}(\mathcal{A}) \mid (\ker h) \subseteq \theta\}$ [whereas $h[h^{-1}[\vartheta]] = \vartheta$, while, conversely, for every $\theta \in \mathrm{Con}(\mathcal{A})$ such that $(\ker h) \subseteq \theta$, $h[\theta] \in \mathrm{Con}(\mathcal{B})$, whereas $h^{-1}[h[\theta]] = \theta$]. ☐

By Remark 2.2, we immediately have:

Corollary 2.3. Let $\mathcal{A}$ and $\mathcal{B}$ be Σ-matrices and $h \in \hom_{\mathrm{S}}(\mathcal{A}, \mathcal{B})$. Suppose $\mathcal{A}$ is simple. Then, h is injective.

Remark 2.4 (Matrix Homomorphism Theorem). As an immediate consequence of the Algebra Homomorphism Theorem, given any Σ-matrices $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, any $f \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{A},\mathcal{B})$ and any $g \in \hom_{[\mathrm{S}]}^{(\mathrm{S})}(\mathcal{A},\mathcal{C})$ such that $(\ker f) \subseteq \{=\}(\ker g)$, it holds that $(g \circ f^{-1}) \in \hom_{[\mathrm{S}]}^{(\mathrm{S})}(\mathcal{B},\mathcal{C})$ {is injective}. □

Proposition 2.5. Let $\mathcal{A}$ and $\mathcal{B}$ be two Σ-matrices and $h \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{A},\mathcal{B})$. Then, $\Game(\mathcal{A}) = h^{-1}[\Game(\mathcal{B})]$ and $\Game(\mathcal{B}) = h[\Game(\mathcal{A})]$.

Proof: As $\Delta_B \in \mathrm{Con}(\mathcal{B})$, by Remark 2.2, we have $\ker h = h^{-1}[\Delta_B] \in \mathrm{Con}(\mathcal{A})$, and so $\ker h \subseteq \Game(\mathcal{A})$, in which case, by Remark 2.2, we get:

$$\begin{array}{rcl} h^{-1}[\Game(\mathcal{B})] & \subseteq & \Game(\mathcal{A}), \\ h[h^{-1}[\Game(\mathcal{B})]] & = & \Game(\mathcal{B}), \\ h[\Game(\mathcal{A})] & \subseteq & \Game(\mathcal{B}), \\ h^{-1}[h[\Game(\mathcal{A})]] & = & \Game(\mathcal{A}). \end{array}$$

These collectively imply the equalities to be proved, as required. □

Since, for any equivalence θ on any set A, it holds that $\nu_\theta[\theta] = \Delta_{A/\theta}$, as an immediate consequence of Proposition 2.5, we also have:

Corollary 2.6. Let $\mathcal{A}$ be a Σ-matrix. Then, $\mathcal{A}/\Game(\mathcal{A})$ is simple.

Given a set I and an I-tuple $\overline{\mathcal{A}}$ of Σ-matrices, the Σ-matrix $(\prod_{i\in I} \mathcal{A}_i) \triangleq \langle \prod_{i\in I} \mathfrak{A}_i, (\prod_{i\in I} A_i) \cap \bigcap_{i\in I} \pi_i^{-1}[D^{\mathcal{A}_i}]\rangle$ is called the *direct product of* $\overline{\mathcal{A}}$. (As usual, when $I = 2$, $\mathcal{A}_0 \times \mathcal{A}_1$ stands for the direct product involved. Likewise, if $(\mathrm{img}\,\overline{\mathcal{A}}) \subseteq \{\mathcal{A}\}$, where $\mathcal{A}$ is a Σ-matrix, $\mathcal{A}^I$ stands for the direct product involved.) Any submatrix $\mathcal{B}$ of the direct product involved is referred to as a *subdirect product of* $\overline{\mathcal{A}}$, whenever, for each $i \in I$, $\pi_i[B] = A_i$.

Lemma 2.7 (Subdirect Product Lemma). *Let* M *be a [finite] class of [finite]* Σ*-matrices and* $\mathcal{A}$ *a {truth-non-empty} (simple)* $([\omega\cap](\omega+1))$*-generated model of the logic of* M*. Then, there is some strict surjective homomorphism from a subdirect product of a [finite] tuple constituted by members of* $\mathbf{S}_*^{\{*\}}(\mathsf{M})$ *onto* $\mathcal{A}/\Game(\mathcal{A})$ *(resp., onto* $\mathcal{A}$ *itself).*

Proof: Take any $A' \in \wp_{[\omega\cap](\omega+1)}(A)$ generating $\mathfrak{A}$ and any $a \in A \neq \varnothing$, in which case $A'' \triangleq (A' \cup \{a\}) \in \wp_{([\omega\cap](\omega+1))\setminus 1}(A)$ generates $\mathfrak{A}$, and so $\alpha \triangleq |A''| \in (([\omega\cap](\omega+1)) \setminus 1) \subseteq \wp_{\infty\setminus 1}(\omega)$. Next, take any bijection from

V_α onto A'' to be extended to a surjective $h \in \hom(\mathrm{Fm}^\alpha_\Sigma, \mathfrak{A})$, in which case it is a surjective strict homomorphism from $\mathcal{B} \triangleq \langle \mathrm{Fm}^\alpha_\Sigma, X\rangle$, where $\{\varnothing \neq\}X \triangleq h^{-1}[D^{\mathcal{A}}]$, onto $\mathcal{A}$, and so, by (2.2), $\mathcal{B}$ is a {truth-non-empty} model of the logic of M. Then, applying (2.1) twice, we get $\mathrm{Cn}^\alpha_{\mathsf{M}}(X) \subseteq \mathrm{Cn}^\alpha_{\mathcal{B}}(X) \subseteq X \subseteq \mathrm{Cn}^\alpha_{\mathsf{M}}(X)$. Furthermore, we have the [finite] set $I \triangleq \{\langle h', \mathcal{D}\rangle \mid h' \in \hom(\mathcal{B}, \mathcal{D}), \mathcal{D} \in \mathsf{M}, (\mathrm{img}\, h') \nsubseteq D^{\mathcal{D}}\}$, in which case, for every $i \in I$, we set $h_i \triangleq \pi_0(i)$, and so $\mathcal{C}_i \triangleq (\pi_1(i){\restriction}(\mathrm{img}\, h_i))$ is a consistent {truth-non-empty} submatrix of $\pi_1(i) \in \mathsf{M}$. Clearly, $X = \mathrm{Cn}^\alpha_{\mathsf{M}}(X) = (\mathrm{Fm}^\alpha_\Sigma \cap \bigcap_{i\in I} h_i^{-1}[D^{\mathcal{C}_i}])$. Therefore, the mapping $g \triangleq (\prod_{i\in I} h_i) : \mathrm{Fm}^\alpha_\Sigma \to (\prod_{i\in I} C_i)$ is a strict homomorphism from $\mathcal{B}$ to $\prod_{i\in I} \mathcal{C}_i$ such that, for each $i \in I$, $(\pi_i \circ g) = h_i$, in which case $\pi_i[g[\mathrm{Fm}^\alpha_\Sigma]] = h_i[\mathrm{Fm}^\alpha_\Sigma] = C_i$, and so g is a surjective strict homomorphism from $\mathcal{B}$ onto the subdirect product $\mathcal{E} \triangleq ((\prod_{i\in I} \mathcal{C}_i){\restriction}(\mathrm{img}\, g))$ of $\overline{\mathcal{C}}$. Put $\theta \triangleq \eth(\mathcal{A})(= \Delta_A)$ and $\mathcal{F} \triangleq (\mathcal{A}/\theta)$. Then, $f \triangleq (\nu_\theta \circ h) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{B}, \mathcal{F})$. Therefore, by Remark 2.2, Proposition 2.5 and Corollary 2.6, we have $(\ker g) = g^{-1}[\Delta_E] \subseteq \eth(\mathcal{B}) = f^{-1}[\Delta_F] = (\ker f)$, in which case, by Remark 2.4, $e \triangleq (f \circ g^{-1}) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{E}, \mathcal{F})$ (and so $(\nu_\theta^{-1} \circ e) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{E}, \mathcal{A})$), as required. □

Given a class M of Σ-matrices, the class of all (truth-non-empty) [consistent] submatrices of members of M is denoted by $\mathbf{S}^{(*)}_{[*]}(\mathsf{M})$. Likewise, the class of all [sub]direct products of tuples (of cardinality $\in K \subseteq \infty$) constituted by members of M is denoted by $\mathbf{P}^{[\mathrm{SD}]}_{(K)}(\mathsf{M})$. Clearly, model classes are closed under $\mathbf{P}$.

THEOREM 2.8. *Let* K *and* M *be classes of* Σ*-matrices,* C *the logic of* M *and* C' *an extension of* C*. Suppose (both* M *and all members of it are finite and)* $[\Re](\mathbf{P}^{\mathrm{SD}}_{(\omega)}(\mathbf{S}_*(\mathsf{M}))) \subseteq \mathsf{K}$ *{in particular,* $[\Re](\mathbf{S}(\mathbf{P}_{(\omega)}(\mathsf{M}))) \subseteq \mathsf{K}$ *⟨in particular,* $\mathsf{K} \supseteq \mathsf{M}$ *is closed under both* $\mathbf{S}$ *and* $\mathbf{P}_{(\omega)}$ *[as well as* $\Re$*]⟩}. Then,* C' *is (finitely-)defined by* $\mathsf{S} \triangleq (\mathrm{Mod}_{[\Im]}(C') \cap \mathsf{K})$*.*

PROOF: Clearly, $C' \subseteq \mathrm{Cn}^\omega_{\mathsf{S}}$, for $\mathsf{S} \subseteq \mathrm{Mod}(C')$. Conversely, consider any $(\Gamma \cup \{\varphi\}) \in \wp_{(\omega)}(\mathrm{Fm}^\omega_\Sigma)$, in which case (there is some $\alpha' \in (\omega \setminus 1)$ such that $(\Gamma \cup \{\varphi\}) \subseteq \mathrm{Fm}^{\alpha'}_\Sigma$, and so) $(\Gamma \cup \{\varphi\}) \subseteq \mathrm{Fm}^\alpha_\Sigma$, where $\alpha \triangleq ((\alpha' \cap)\omega) \in \wp_{\infty\setminus 1}(\omega)$, such that $\varphi \notin C'(\Gamma)$. Then, by the structurality of C', $\langle \mathfrak{Fm}^\omega_\Sigma, C'(\Gamma)\rangle$ is a model of C' {in particular, of C}, and so is its $(\alpha+1)$-generated (and so ω-generated) submatrix $\mathcal{A} \triangleq \langle \mathfrak{Fm}^\alpha_\Sigma, C'(\Gamma) \cap \mathrm{Fm}^\alpha_\Sigma\rangle$, in view of (2.2), in which case $\varphi \notin \mathrm{Cn}^\alpha_{\mathcal{A}}(\Gamma)$, and so $\varphi \notin \mathrm{Cn}^\omega_{\mathcal{A}}(\Gamma)$, in view of (2.1). Therefore, by Lemma 2.7, there are some $\mathcal{B} \in \mathbf{P}^{\mathrm{SD}}_{(\omega)}(\mathbf{S}_*(\mathsf{M}))$, in which case $\mathcal{D} \triangleq [\Re](\mathcal{B}) \in [\Re](\mathbf{P}^{\mathrm{SD}}_{(\omega)}(\mathbf{S}_*(\mathsf{M}))) \subseteq \mathsf{K}$, and some $g \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{B}, \mathcal{A}/\partial(\mathcal{A}))$. Then, by (2.2), $\mathrm{Cn}^{\omega}_{\mathcal{D}} = \mathrm{Cn}^{\omega}_{\mathcal{A}}$, in which case [by Corollary 2.6] $\mathcal{D} \in \mathsf{S}$, and so $\varphi \notin \mathrm{Cn}^{\omega}_{\mathsf{S}}(\Gamma)$, as required. □

Corollary 2.9. Let M be a class of Σ-matrices and $\mathcal{A}$ an axiomatic Σ-calculus. Then, the axiomatic extension C' of the logic C of M relatively axiomatized by $\mathcal{A}$ is defined by $\mathbf{S}_*(\mathsf{M}) \cap \mathrm{Mod}(\mathcal{A})$.

Proof: Then, $\mathrm{Mod}(C') = (\mathrm{Mod}(C) \cap \mathrm{Mod}(\mathcal{A}))$, and so (2.2), (2.3) and Theorem 2.8 with $\mathsf{K} \triangleq \mathbf{P}^{\mathrm{SD}}_{(\omega)}(\mathbf{S}_*(\mathsf{M})) \subseteq \mathrm{Mod}(C)$, in which case $(\mathrm{Mod}(C') \cap \mathsf{K}) = (\mathrm{Mod}(\mathcal{A}) \cap \mathsf{K}) = \mathbf{P}^{\mathrm{SD}}_{(\omega)}(\mathbf{S}_*(\mathsf{M}) \cap \mathrm{Mod}(\mathcal{A}))$, complete the argument. □

Given any Σ-logic C and any $\Sigma' \subseteq \Sigma$, in which case $\mathrm{Fm}^{\alpha}_{\Sigma} \subseteq \mathrm{Fm}^{\alpha}_{\Sigma'}$ and $\hom(\mathfrak{Fm}^{\alpha}_{\Sigma'}, \mathfrak{Fm}^{\alpha}_{\Sigma'}) = \{h{\restriction}\mathrm{Fm}^{\alpha}_{\Sigma'} \mid h \in \hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{Fm}^{\alpha}_{\Sigma}), h[\mathrm{Fm}^{\alpha}_{\Sigma'}] \subseteq \mathrm{Fm}^{\alpha}_{\Sigma'}\}$, for all $\alpha \in \wp_{\infty\setminus 1}(\omega)$, we have the Σ'-logic C', defined by $C'(X) \triangleq (\mathrm{Fm}^{\omega}_{\Sigma'} \cap C(X))$, for all $X \subseteq \mathrm{Fm}^{\omega}_{\Sigma'}$, called the *$\Sigma'$-fragment of* C, in which case C is said to be a *(Σ-)expansion of* C'. In that case, given also any class M of Σ-matrices defining C, C' is, in its turn, defined by $\mathsf{M}{\restriction}\Sigma'$.

2.3.1. Classical matrices and logics

Let $\wr \in \Sigma$ be unary.

A two-valued consistent Σ-matrix $\mathcal{A}$ is said to be *$\wr$-classical*, provided, for all $a \in A$, $(a \in D^{\mathcal{A}}) \Leftrightarrow (\wr^{\mathfrak{A}} a \notin D^{\mathcal{A}})$, in which case it is truth-non-empty, and so both false- and truth-singular, but is not $\wr$-paraconsistent.

A Σ-logic is said to be *$\wr$-[sub]classical*, whenever it is [a sublogic of] the logic of a $\wr$-classical Σ-matrix.

3. Preliminary key issues

3.1. Equality determinants

According to [13], an *equality determinant for* a Σ-matrix $\mathcal{A}$ is any $\Upsilon \subseteq \mathrm{Fm}^{1}_{\Sigma}$ such that any $a, b \in A$ are equal, whenever, for all $\upsilon \in \Upsilon$, $\upsilon^{\mathfrak{A}}(a) \in D^{\mathcal{A}}$ iff $\upsilon^{\mathfrak{A}}(b) \in D^{\mathcal{A}}$.

Example 3.1. $\{x_0\}$ is an equality determinant for any consistent truth-non-empty two-valued (in particular, classical) matrix. □

Lemma 3.2. *Let $\mathcal{A}$ be a Σ-matrix and Υ an equality determinant for $\mathcal{A}$. Then, $\mathcal{A}$ is simple.*

Proof: Consider any $\theta \in \mathrm{Con}(\mathcal{A})$ and any $\langle a, b\rangle \in \theta$. Then, for each $v \in \Upsilon$, $v^{\mathfrak{A}}(a)\ \theta\ v^{\mathfrak{A}}(b)$, in which case $(v^{\mathfrak{A}}(a) \in D^{\mathcal{A}}) \Leftrightarrow (v^{\mathfrak{A}}(b) \in D^{\mathcal{A}})$, and so $a = b$, as required. $\square$

Lemma 3.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be Σ-matrices, Υ an equality determinant for $\mathcal{B}$ and $e \in \mathrm{hom}_{\mathsf{S}}(\mathcal{A}, \mathcal{B})$. Suppose e is injective. Then, Υ is an equality determinant for $\mathcal{A}$.*

Proof: In that case, for all $a \in A$ and every $v \in \Upsilon$, it holds that $v^{\mathfrak{A}}(a) \in D^{\mathcal{A}}$ iff $v^{\mathfrak{B}}(e(a)) = e(v^{\mathfrak{A}}(a)) \in D^{\mathcal{B}}$, and so the injectivity of e completes the argument. $\square$

3.2. Implicative matrices with equality determinant

Let $\diamond$ and $\veebar$ be (possibly, secondary) binary connectives of Σ.

A Σ-matrix $\mathcal{A}$ is said to be *$\diamond$-implicative/-disjunctive*, provided, for all $a, b \in A$, it holds that $((a \in / \notin D^{\mathcal{A}}) \Rightarrow (b \in D^{\mathcal{A}})) \Leftrightarrow ((a \diamond^{\mathfrak{A}} b) \in D^{\mathcal{A}})$, in which case it is $\vee_{\diamond}$-disjunctive, where $(x_0 \vee_{\diamond} x_1) \triangleq ((x_0 \diamond x_1) \diamond x_1)$.

Lemma 3.4. *Let $\mathcal{A}$ be a finite $\diamond$-implicative and $\veebar$-disjunctive (in particular, $\veebar = \vee_{\diamond}$) Σ-matrix with equality determinant Υ, $\mathsf{S} \subseteq \mathbf{S}(\mathcal{A})$, $n \triangleq |\mathsf{S}|$ and $\mathcal{B} \in \mathbf{S}_*(\mathcal{A})$. Suppose $\mathcal{B} \notin \mathbf{S}(\mathsf{S})$. Then, there is some Σ-axiom in $\mathrm{Fm}^{n+1}_{\Sigma}$, which is true in S but is not true in $\mathcal{B}$.*

Proof: Take any bijection $\overline{\mathcal{C}} : n \to \mathsf{S}$. Consider any $i \in n$, in which case $B \nsubseteq C_i$, and so there is some $a_i \in (B \setminus C_i) \neq \varnothing$. Define a $\psi_i \in \mathrm{Fm}^2_{\Sigma}$ as follows. Take any bijection $\bar{c} : m \triangleq |C_i| \to C_i$. By induction on any $j \in (m+1)$, define a $\phi_j \in \mathrm{Fm}^2_{\Sigma}$ such that, for all $b \in (A \setminus D^{\mathcal{A}})$, it holds that $\phi^{\mathfrak{A}}_j[x_0/a_i, x_1/b] \notin D^{\mathcal{A}}$, while, providing $x_1 \in \mathrm{Var}(\phi_j)$, for all $a \in A$ and all $d \in D^{\mathcal{A}}$, it holds that $\phi^{\mathfrak{A}}_j[x_0/a, x_1/d] \in D^{\mathcal{A}}$, whereas, for all $k \in j$ and all $a \in A$, it holds that $\phi^{\mathfrak{A}}_j[x_0/c_k, x_1/a] \in D^{\mathcal{A}}$, as follows. First, put $\phi_j \triangleq x_1$, if $j = 0$. Otherwise, $(j-1) \in m \subseteq (m+1)$, in which case $c_{j-1} \neq a_i$, for $c_{j-1} \in C_i \not\ni a_i$, and so there is some $v \in \Upsilon$ such that $v^{\mathfrak{A}}(a_i) \in D^{\mathcal{A}}$ iff $v^{\mathfrak{A}}(c_{j-1}) \notin D^{\mathcal{A}}$. Then, set:

$$\phi_j \triangleq \begin{cases} \upsilon \diamond \phi_{j-1} & \text{if } \upsilon^{\mathfrak{A}}(a_i) \in D^{\mathcal{A}}, \\ & \exists a \in A : \phi^{\mathfrak{A}}_{j-1}[x_0/c_{j-1}, x_1/a] \notin D^{\mathcal{A}}, \\ \upsilon \veebar \phi_{j-1} & \text{if } x_1 \notin \mathrm{Var}(\phi_{j-1}), \upsilon^{\mathfrak{A}}(a_i) \notin D^{\mathcal{A}}, \\ & \exists a \in A : \phi^{\mathfrak{A}}_{j-1}[x_0/c_{j-1}, x_1/a] \notin D^{\mathcal{A}}, \\ \varphi_{j-1}[x_1/\upsilon] & \text{if } x_1 \in \mathrm{Var}(\phi_{j-1}), \upsilon^{\mathfrak{A}}(a_i) \notin D^{\mathcal{A}}, \\ & \exists a \in A : \phi^{\mathfrak{A}}_{j-1}[x_0/c_{j-1}, x_1/a] \notin D^{\mathcal{A}}, \\ \phi_{j-1} & \text{otherwise.} \end{cases}$$

In this way, $\psi_i \triangleq \phi_m \in \mathrm{Fm}^2_\Sigma$ is true in $\mathcal{C}_i$, while, for all $b \in (A \setminus D^{\mathcal{A}})$, it holds that $\psi^{\mathfrak{A}}_i[x_0/a_i, x_1/b] \notin D^{\mathcal{A}}$, whereas, providing $x_1 \in \mathrm{Var}(\psi_i)$, for all $a \in A$ and all $d \in D^{\mathcal{A}}$, it holds that $\psi^{\mathfrak{A}}_i[x_0/a, x_1/d] \in D^{\mathcal{A}}$. Finally, by induction on any $l \in (n+1)$, define a $\varphi_l \in \mathrm{Fm}^{l+1}_\Sigma$ such that for all $b \in (A \setminus D^{\mathcal{A}})$, it holds that $\varphi^{\mathfrak{A}}_l[x_{k+1}/a_k, x_0/b]_{k \in l} \notin D^{\mathcal{A}}$, while, providing $x_0 \in \mathrm{Var}(\varphi_l)$, for all $\bar{c} \in A^l$ and all $d \in D^{\mathcal{A}}$, it holds that $\varphi^{\mathfrak{A}}_l[x_0/d, x_{k+1}/c_k]_{k \in l} \in D^{\mathcal{A}}$, whereas, for all $k \in l$, $\mathcal{C}_k \models \varphi_l$, as follows. First, put $\varphi_l \triangleq x_0$, if $l = 0$. Otherwise, $(l-1) \in n \subseteq (n+1)$, so set:

$$\varphi_l \triangleq \begin{cases} \psi_{l-1}[x_1/\varphi_{l-1}, x_0/x_l] & \text{if } x_1 \in \mathrm{Var}(\psi_{l-1}), \mathcal{C}_{l-1} \not\models \varphi_{l-1}, \\ \varphi_{l-1}[x_0/(\psi_{l-1}[x_0/x_l])] & \text{if } x_0 \in \mathrm{Var}(\varphi_{l-1}), \\ & x_1 \notin \mathrm{Var}(\psi_{l-1}), \mathcal{C}_{l-1} \not\models \varphi_{l-1}, \\ \varphi_{l-1} \veebar (\psi_{l-1}[x_0/x_l]) & \text{if } x_0 \notin \mathrm{Var}(\varphi_{l-1}), \\ & x_1 \notin \mathrm{Var}(\psi_{l-1}), \mathcal{C}_{l-1} \not\models \varphi_{l-1}, \\ \varphi_{l-1} & \text{otherwise.} \end{cases}$$

Thus, $\varphi_n \in \mathrm{Fm}^{n+1}_\Sigma$ is true in S but $\mathcal{B} \not\models \varphi_n[x_{i+1}/a_i; x_0/b]_{i \in n}$, where $b \in (B \setminus D^{\mathcal{A}}) \neq \varnothing$, for $\mathcal{B}$ is consistent, as required. ☐

Since model classes are closed under $\mathbf{S}$ (cf. (2.2)), while any axiomatic extension of a logic is relatively axiomatized by the set of all its theorems, whereas lower cones sets are closed under intersections and unions, combining Corollary 2.9 and Lemma 3.4, we eventually get:

THEOREM 3.5. *Let $\mathcal{A}$ be a finite $\diamond$-implicative Σ-matrix with equality determinant and $\mathsf{S} \triangleq \mathbf{S}_*(\mathcal{A})$. Then, the mappings:*

$$\begin{aligned} C &\mapsto (\mathrm{Mod}(C) \cap \mathsf{S}) = (\mathrm{Mod}(C(\varnothing)) \cap \mathsf{S}), \\ \mathsf{C} &\mapsto \mathrm{Cn}^{\omega}_{\mathsf{C}} \end{aligned}$$

are inverse to one another dual isomorphisms between the lattices of all axiomatic extensions of the logic of $\mathcal{A}$ and of all lower cones of S (under identification of submatrices of $\mathcal{A}$ with the carriers of their underlying algebras), corresponding axiomatic extensions of the logic of $\mathcal{A}$ and lower cones of S having same axiomatic relative axiomatizations, both lattices being distributive. Moreover, for every $\mathsf{M} \subseteq \mathsf{S}$, the logic of M is the axiomatic extension of the logic of $\mathcal{A}$ corresponding to $\mathbf{S}_*(\mathsf{M})$.

It is remarkable that the proof of Lemma 3.4 is constructive, so, in case Σ is finite, it collectively with Theorem 3.5 yield an effective procedure of finding the lattice of axiomatic extensions of the logic of $\mathcal{A}$ collectively with their finite relative axiomatizations and finite anti-chain matrix semantics. In this connection, we should like to highlight that the effective procedure of finding relative axiomatizations of axiomatic extensions to be extracted from the constructive proof of Lemma 3.4 is definitely and obviously much less computationally complex than the straightforward one of direct search among all finite sets of formulas.

3.3. Distributive and De Morgan lattices

Let $\Sigma_{+[01]} \triangleq (\{\wedge, \vee\}[\cup\{\bot, \top\}])$ be the [bounded] lattice signature with binary $\wedge$ (conjunction) and $\vee$ (disjunction) [as well as nullary $\bot$ and $\top$ (falsehood/zero and truth/unit constants, respectively)].

Then, given any Σ-algebra $\mathfrak{A}$ such that $\Sigma_+ \subseteq \Sigma$ and $\mathfrak{A}{\restriction}\Sigma_+$ is a lattice, the partial ordering of $\mathfrak{A}{\restriction}\Sigma_+$ is denoted by $\leqslant^{\mathfrak{A}}$.

Given any $n \in (\omega \setminus 1)$, by $\mathfrak{D}_{n[01]}$ we denote the [bounded] distributive lattice given by the chain n ordered by the natural ordering.

We also deal with the signature $\Sigma_{\sim[01]} \triangleq (\Sigma_{+[01]} \cup \{\sim\})$ with unary $\sim$ (weak negation).

A *[bounded] De Morgan lattice* (cf. [11]; bounded De Morgan lattices are also traditionally called *De Morgan algebras* - cf., e.g., [2]) is any $\Sigma_{\sim[01]}$-algebra $\mathfrak{A}$ such that $\mathfrak{A}{\restriction}\Sigma_{+[01]}$ is a [bounded] distributive lattice (cf. [2]) and the following $\Sigma_\sim$-identities are true in $\mathfrak{A}$:

$$\sim\sim x_0 \approx x_0, \tag{3.1}$$

$$\sim(x_0 \vee x_1) \approx \sim x_0 \wedge \sim x_1, \tag{3.2}$$

the variety of all them being denoted by $[\mathsf{B}]\mathsf{DML}$.

By $\mathfrak{DM}_{4[01]}$ we denote the [bounded] De Morgan lattice such that $(\mathfrak{DM}_{4[01]}{\restriction}\Sigma_{+[01]}) \triangleq \mathfrak{D}^2_{2[01]}$ and $\sim^{\mathfrak{DM}_{4[01]}}\vec{a} \triangleq \langle 1 - a_{1-i}\rangle_{i\in 2}$, for all $\vec{a} \in 2^2$. In this connection, we use the following abbreviations going back to [3]:

$$\mathsf{t} \triangleq \langle 1,1\rangle, \qquad \mathsf{f} \triangleq \langle 0,0\rangle, \qquad \mathsf{b} \triangleq \langle 1,0\rangle, \qquad \mathsf{n} \triangleq \langle 0,1\rangle.$$

In addition, set $\mu : 2^2 \to 2^2, \langle a,b\rangle \mapsto \langle b,a\rangle$. Finally, an n-ary operation f on $B \subseteq 2^2$, where $n \in \omega$, is said to be *regular*, provided it is monotonic with respect to the partial ordering $\sqsubseteq$ on 2^2 defined by $(\vec{a} \sqsubseteq \vec{b}) \overset{\text{def}}{\Longleftrightarrow} ((a_0 \leqslant b_0)\&(b_1 \leqslant a_1))$, for all $\vec{a}, \vec{b} \in 2^2$, in the sense that, for all $\bar{a}, \bar{b} \in B^n$ such that $a_i \sqsubseteq b_i$, for each $i \in n$, it holds that $f(\bar{a}) \sqsubseteq f(\bar{b})$.

Remark 3.6. Clearly, $\{\mathsf{b},\mathsf{t}\}$ is a prime filter of $\mathfrak{D}^2_2$, in which case, in particular, $\mathcal{DM}_{4[01]} \triangleq \langle \mathfrak{DM}_{4[01]}, \{\mathsf{b},\mathsf{t}\}\rangle$ is $\wedge$-conjunctive and $\vee$-disjunctive. Moreover, $\{x_0, \sim x_0\}$ is an equality determinant for it. □

Recall also the following well-known algebraic fact:

LEMMA 3.7. *Let $\mathfrak{B}$ be a subalgebra of $\mathfrak{DM}_4$. Then,* $\mathrm{Con}(\mathfrak{B}) \subseteq \{\Delta_B, B^2\}$.

THEOREM 3.8. *Let $\mathfrak{A}$ be a $\Sigma_\sim$-algebra and $(\mathcal{H} \cup \{h\}) \in \wp_\omega(\hom(\mathfrak{A}, \mathfrak{DM}_4))$. Suppose $(\bigcap\{\ker g \mid g \in \mathcal{H}\}) \subseteq (\ker h) \neq A^2$. Then,* $(\ker h) = (\ker g)$*, for some $g \in \mathcal{H}$.*

PROOF: In that case, combining Lemma 11 and Claim on p. 300 (inside the proof of Lemma 10) of [13] with Remark 3.6, we first conclude that $(\ker g) \subseteq (\ker h)$, for some $g \in \mathcal{H}$, in which case g is a surjective homomorphism from $\mathfrak{A}$ onto the subalgebra $\mathfrak{B} \triangleq (\mathfrak{DM}_4{\restriction}(\mathrm{img}\, g))$ of $\mathfrak{DM}_4$, and so, by the Algebra Homomorphism Theorem, $f \triangleq (h \circ g^{-1}) \in \hom(\mathfrak{B}, \mathfrak{DM}_4)$. Hence, by Lemma 2.1, $(\ker f) \in \mathrm{Con}(\mathfrak{B})$. Moreover, $(\ker f) \neq B^2$, for $(\ker h) \neq A^2$. Therefore, by Lemma 3.7, f is injective. Thus, $(\ker h) \subseteq (\ker g)$, as required. □

4. Main results

Fix any language $\Sigma \supseteq \Sigma_{\sim[01]}$ such that either $\{\bot,\top\} \subseteq \Sigma$ or $(\{\bot,\top\} \cap \Sigma) = \varnothing$ and any Σ-algebra $\mathfrak{A}$ such that $(\mathfrak{A}{\restriction}\Sigma_{\sim[01]}) = \mathfrak{DM}_{4[01]}$. Put $\mathcal{A} \triangleq \langle \mathfrak{A}, \{\mathsf{b},\mathsf{t}\}]\rangle$. Since [the bounded version of] Dunn-Belnap's four-valued logic [5] (cf. [3]), denoted by $C_{\text{[B]DB}}$ from now on, is defined by $\mathcal{DM}_{4[01]} = (\mathcal{A}{\restriction}\Sigma_{\sim[01]})$ (cf. [9]), the logic C of $\mathcal{A}$ is a four-valued expansion of $C_{\text{[B]DB}}$.

A subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ is said to be *specular*, whenever $(\mu \restriction B) \in \hom(\mathfrak{B}, \mathfrak{A})$. Likewise, it is said to be *regular*, whenever its primary operations are so, in which case its secondary ones are so as well. (Clearly, $\mathfrak{B}$ is specular/regular, whenever $\mathfrak{A}$ is so. Moreover, $\mathfrak{DM}_{4[01]}$ is both specular and regular.)

4.1. Characteristic matrix expansions

Lemma 4.1. *Let I be a set, $\overline{\mathcal{C}} \in \mathbf{S}(\mathcal{A})^I$, $\mathcal{B}$ a Σ-matrix and e an embedding of $\mathcal{B}$ into $\prod_{i \in I} \mathcal{C}_i$. Suppose $\{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, $\{I \times \{d\} \mid d \in \{\mathsf{f}, \mathsf{t}\}\} \subseteq e[B]$ and, for each $i \in I$, both $\{\mathsf{f}, \mathsf{b}, \mathsf{t}\} \cup C_i$ forms a regular subalgebra of $\mathfrak{A}$ and either $\mathsf{n} \notin C_i$ or $\mathfrak{A} \restriction \{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$ is specular. Then, $(B \dot{+} 2) \triangleq ((B \times \{\mathsf{b}\}) \cup \{\langle e^{-1}(I \times \{d\}), d\rangle \mid d \in \{\mathsf{f}, \mathsf{t}\}\})$ forms a subalgebra of $\mathfrak{B} \times (\mathfrak{A} \restriction \{\mathsf{f}, \mathsf{b}, \mathsf{t}\})$, in which case $\pi_0 \restriction (B \dot{+} 2)$ is a surjective strict homomorphism from $(\mathcal{B} \dot{+} 2) \triangleq ((\mathcal{B} \times (\mathcal{A} \restriction \{\mathsf{f}, \mathsf{b}, \mathsf{t}\})) \restriction (B \dot{+} 2))$ onto $\mathcal{B}$.*

Proof: Consider any $\varsigma \in \Sigma$ of arity $n \in \omega$ and any $\bar{b} \in (B \dot{+} 2)^n$. In case $\varsigma^{\mathfrak{A}}(\bar{a}) = \mathsf{b}$, where $\bar{a} \triangleq (\pi_1 \circ \bar{b}) \in \{\mathsf{f}, \mathsf{b}, \mathsf{t}\}^n$, we clearly have $\varsigma^{\mathfrak{B} \times \mathfrak{A}}(\bar{b}) = \langle \varsigma^{\mathfrak{B}}(\pi_0 \circ \bar{b}), \varsigma^{\mathfrak{A}}(\bar{a})\rangle = \langle \varsigma^{\mathfrak{B}}(\pi_0 \circ \bar{b}), \mathsf{b}\rangle \in (B \times \{\mathsf{b}\}) \subseteq (B \dot{+} 2)$. Otherwise, since $\{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, we have $\varsigma^{\mathfrak{A}}(\bar{a}) \in \{\mathsf{f}, \mathsf{t}\}$. Put $N \triangleq \{k \in n \mid a_k = \mathsf{b}\}$. Consider any $i \in I$. Put $\bar{c} \triangleq (\pi_i \circ e \circ \pi_0 \circ \bar{b}) \in C_i^n$. Then, for every $j \in (n \setminus N)$, it holds that $c_j = a_j \in \{\mathsf{f}, \mathsf{t}\}$. Hence, $c_j \sqsubseteq a_j$, for all $j \in n$. Therefore, by the regularity of $\mathfrak{A} \restriction (\{\mathsf{f}, \mathsf{b}, \mathsf{t}\} \cup C_i)$, we have $\varsigma^{\mathfrak{A}}(\bar{c}) \sqsubseteq \varsigma^{\mathfrak{A}}(\bar{a})$. Consider the following complementary cases:

1. $\mathsf{n} \in C_i$.
   Then, $\mu(a_j) \sqsubseteq c_j$, for all $j \in n$. Therefore, as, in that case, $\mathfrak{A} \restriction \{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$ is specular, by the regularity of $(\mathfrak{A} \restriction (\{\mathsf{f}, \mathsf{b}, \mathsf{t}\} \cup C_i)) = \mathfrak{A}$, we have $\varsigma^{\mathfrak{A}}(\bar{a}) = \mu(\varsigma^{\mathfrak{A}}(\bar{a})) = \varsigma^{\mathfrak{A}}(\mu \circ \bar{a}) \sqsubseteq \varsigma^{\mathfrak{A}}(\bar{c})$, and so we get $\varsigma^{\mathfrak{A}}(\bar{c}) = \varsigma^{\mathfrak{A}}(\bar{a})$.

2. $\mathsf{n} \notin C_i$.
   Then, $\varsigma^{\mathfrak{A}}(\bar{c}) \in C_i \subseteq \{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$. Therefore, since both f and t are minimal elements of the poset $\{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$ ordered by $\sqsubseteq$, we get $\varsigma^{\mathfrak{A}}(\bar{c}) = \varsigma^{\mathfrak{A}}(\bar{a})$.

Thus, in any case, we have $\varsigma^{\mathfrak{A}}(\bar{c}) = \varsigma^{\mathfrak{A}}(\bar{a})$, and so, since e is an embedding of $\mathfrak{B}$ into $\prod_{i \in I} \mathfrak{C}_i$, we get $\varsigma^{\mathfrak{B} \times \mathfrak{A}}(\bar{b}) = \langle e^{-1}(I \times \{\varsigma^{\mathfrak{A}}(\bar{a})\}), \varsigma^{\mathfrak{A}}(\bar{a})\rangle \in \{\langle e^{-1}(I \times \{d\}), d\rangle \mid d \in \{\mathsf{f}, \mathsf{t}\}\} \subseteq (B \dot{+} 2)$, as required. □

Lemma 4.2. *Let $\mathcal{B}$ be a model of C. Suppose either $\{\mathsf{b}\}$ forms a subalgebra of $\mathfrak{A}$ or both $\mathfrak{A}$ is regular and $\{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$ forms a specular subalgebra of $\mathfrak{A}$ (in particular, $\Sigma = \Sigma_{\sim[01]}$), while the rule:*

$$\{x_0, {\sim}x_0\} \vdash (x_1 \vee {\sim}x_1) \tag{4.1}$$

is not true in $\mathcal{B}$. Then, there is some submatrix $\mathcal{D}$ of $\mathcal{B}$ such that $\mathcal{A}$ is isomorphic to $\Re(\mathcal{D})$.

Proof: In that case, there are some $a, b \in B$ such that (4.1) is not true in $\mathcal{B}$ under $[x_0/a, x_1/b]$. Then, in view of (2.2), the submatrix $\mathcal{E}$ of $\mathcal{B}$ generated by $\{a, b\}$ is a finitely-generated model of C, in which (4.1) is not true under $[x_0/a, x_1/b]$ as well. Hence, by Lemma 2.7 with $\mathsf{M} = \{\mathcal{A}\}$, there are some set I, some I-tuple $\overline{\mathcal{C}}$ constituted by submatrices of $\mathcal{A}$, some subdirect product $\mathcal{F}$ of $\overline{\mathcal{C}}$, in which case $(\mathfrak{F}{\restriction}\Sigma_\sim) \in \mathsf{DML}$, for $\mathsf{DML} \ni \mathfrak{DM}_4$ is a variety, and some $g \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{F}, \Re(\mathcal{E}))$, in which case, by (2.2), $\mathcal{F}$ is a model of C, in which case it is $\wedge$-conjunctive, for $\mathcal{A}$ is so (cf. Remark 3.6), but is not a model of (4.1), in which case there are some $c, d \in F$ such that $\{c, {\sim}^{\mathfrak{F}}c\} \subseteq D^{\mathcal{F}} \not\ni d \geqslant^{\mathfrak{F}} {\sim}^{\mathfrak{F}}d$. Then, $c = (I \times \{\mathsf{b}\})$, in which case ${\sim}^{\mathfrak{F}}c = c$, and so $(F \setminus D^{\mathcal{F}}) \ni e \triangleq ((c \wedge^{\mathfrak{F}} d) \vee^{\mathfrak{F}} {\sim}^{\mathfrak{F}}d) = {\sim}^{\mathfrak{F}}e \leqslant^{\mathfrak{F}} d$. Hence, $e \in \{\mathsf{b}, \mathsf{n}\}^I$, while $J \triangleq \{i \in I \mid \pi_i(e) = \mathsf{n}\} \neq \varnothing$. Given any $\bar{a} \in A^2$, set $(a_0|a_1) \triangleq ((J \times \{a_0\}) \cup ((I \setminus J) \times \{a_1\})) \in A^I$. In this way, we have:

$$F \ni c = (\mathsf{b}|\mathsf{b}), \tag{4.2}$$

$$F \ni e = (\mathsf{n}|\mathsf{b}), \tag{4.3}$$

$$F \ni (c \wedge^{\mathfrak{F}} e) = (\mathsf{f}|\mathsf{b}), \tag{4.4}$$

$$F \ni (c \vee^{\mathfrak{F}} e) = (\mathsf{t}|\mathsf{b}). \tag{4.5}$$

Consider the following complementary cases:

1. either $\{\mathsf{b}\}$ forms a subalgebra of $\mathfrak{A}$ or $J = I$.
   Then, by (4.2), (4.3), (4.4) and (4.5), $f \triangleq \{\langle x, (x|\mathsf{b})\rangle \mid x \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{F}$, in which case $g' \triangleq (g \circ f) \in \hom_{\mathrm{S}}(\mathcal{A}, \Re(\mathcal{E}))$, and so, by Corollary 2.3, Lemma 3.2 and Remark 3.6, g' is injective. In this way, g' is an isomorphism from $\mathcal{A}$ onto the submatrix $\mathcal{G} \triangleq (\Re(\mathcal{E}){\restriction}(\operatorname{img} g'))$ of $\Re(\mathcal{E})$, and so $h \triangleq g'^{-1} \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{G}, \mathcal{A})$.

2. $\{\mathsf{b}\}$ does not form a subalgebra of $\mathfrak{A}$ and $J \neq I$.
   Then, there is some $\varphi \in \mathrm{Fm}^1_\Sigma$ such that $\varphi^{\mathfrak{A}}(\mathsf{b}) \neq \mathsf{b}$, in which case

$\phi^{\mathfrak{A}}(\mathsf{b}) = \mathsf{f}$ and $\psi^{\mathfrak{A}}(\mathsf{b}) = \mathsf{t}$, where $\phi \triangleq (x_0 \wedge (\varphi \wedge {\sim}\varphi))$ and $\psi \triangleq (x_0 \vee (\varphi \vee {\sim}\varphi))$, and so, by (4.2), we get:

$$F \ni \phi^{\mathfrak{F}}(c) = (\mathsf{f}|\mathsf{f}), \tag{4.6}$$

$$F \ni \psi^{\mathfrak{F}}(c) = (\mathsf{t}|\mathsf{t}). \tag{4.7}$$

Moreover, in that case, both $\mathfrak{A}$ is regular and $\{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$ forms a specular subalgebra of $\mathfrak{A}$. And what is more, $e' \triangleq \{\langle a', 1 \times \{a'\}\rangle \mid a' \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{A}^1$ such that $\{1 \times \{x\} \mid x \in \{\mathsf{f}, \mathsf{t}\}\} = e'[\{\mathsf{f}, \mathsf{t}\}] \subseteq e'[A]$. In this way, Lemma 4.1 with 1, $\mathcal{A}$ and e' instead of I, $\mathcal{B}$ and e, respectively, used tacitly throughout the rest of the proof, is well-applicable to $\mathcal{A}$. Then, since $J \neq \varnothing \neq (I \setminus J)$, by (4.2), (4.3), (4.4), (4.5), (4.6) and (4.7), we see that $f \triangleq \{\langle\langle x, y\rangle, (x|y)\rangle \mid \langle x, y\rangle \in (A \dotplus 2)\}$ is an embedding of $\mathcal{H} \triangleq (\mathcal{A} \dotplus 2)$ into $\mathcal{F}$, while $h' \triangleq (\pi_0 \upharpoonright (A \dotplus 2)) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{H}, \mathcal{A})$. Then, $g' \triangleq (g \circ f) \in \hom_{\mathrm{S}}(\mathcal{H}, \Re(\mathcal{E}))$, and so g' is a surjective strict homomorphism from $\mathcal{H}$ onto the submatrix $\mathcal{G} \triangleq (\Re(\mathcal{E}) \upharpoonright (\mathrm{img}\, g'))$ of $\Re(\mathcal{E})$. And what is more, by Lemma 3.2 and Remark 3.6, $\mathcal{A}$ is simple. Hence, by Remark 2.2 and Proposition 2.5, we get $(\ker g') \subseteq \eth(\mathcal{H}) = (\ker h')$. Therefore, by Remark 2.4, $h \triangleq (h' \circ g'^{-1}) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{G}, \mathcal{A})$.

Thus, in any case, there are some submatrix $\mathcal{G}$ of $\mathcal{E}/\theta$, where $\theta \triangleq \eth(\mathcal{E})$, and some $h \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{G}, \mathcal{A})$. Then, $\mathcal{D} \triangleq (\mathcal{E} \upharpoonright \nu_\theta^{-1}[G])$, being a submatrix of $\mathcal{E}$, is so of $\mathcal{B}$, in which case $h'' \triangleq (\nu_\theta \upharpoonright D) \in \hom_{\mathrm{S}}(\mathcal{D}, \mathcal{G})$ is surjective, and so is $h''' \triangleq (h \circ h'') \in \hom_{\mathrm{S}}(\mathcal{D}, \mathcal{A})$. On the other hand, by Lemma 3.2 and Remark 3.6, $\mathcal{A}$ is simple. Hence, by Proposition 2.5, $\vartheta \triangleq \eth(\mathcal{D}) = (\ker h''')$. Therefore, by Remark 2.4, $\nu_\vartheta \circ h'''^{-1}$ is an isomorphism from $\mathcal{A}$ onto $\Re(\mathcal{D})$, as required. □

Corollary 4.3. Let C' be an extension of C. Suppose either $\{\mathsf{b}\}$ forms a subalgebra of $\mathfrak{A}$ or both $\mathfrak{A}$ is regular and $\{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$ forms a specular subalgebra of $\mathfrak{A}$ (in particular, $\Sigma = \Sigma_{\sim[01]}$), while the rule (4.1) is not satisfied in C'. Then, $C' = C$.

Proof: In that case, $(x_1 \vee {\sim}x_1) \notin T \triangleq C'(\{x_0, {\sim}x_0\})$, so, by the structurality of C', $\langle \mathfrak{Fm}^{\omega}_{\Sigma}, T\rangle$ is a model of C' (in particular, of C), in which (4.1) is not true under the diagonal Σ-substitution. In this way, (2.2) and Lemma 4.2 complete the argument. □

Proposition 4.4. Let M be a class of $\Sigma_\sim$-matrices. Suppose C_{DB} is defined by M. Then, there are some $\mathcal{B} \in \mathsf{M}$ and some submatrix $\mathcal{D}$ of $\mathcal{B}$ such that $\mathcal{DM}_4$ is isomorphic to $\mathcal{D}/\eth(\mathcal{D})$.

Proof: Note that the rule (4.1) is not satisfied in C_{DB}, because it is not true in $\mathcal{DM}_4$ under $[x_0/\mathsf{b}, x_1/\mathsf{n}]$. Therefore, as C_{DB} is defined by M, there is some model $\mathcal{B} \in \mathsf{M}$ of C_{DB} not being a model of (4.1), in which case Lemma 4.2 completes the argument. $\square$

Now, we are in a position to argue several interesting corollaries of Proposition 4.4:

Corollary 4.5. Let M be a class of Σ-matrices. Suppose the logic of M is an expansion of C_{DB} (in particular, $\Sigma = \Sigma_\sim$ and the logic of M is C_{DB} itself). Then, some $\mathcal{B} \in \mathsf{M}$ is not truth-/false-singular. In particular, any four-valued expansion of C_{DB} (including C_{DB} itself) is defined by no truth-/false-singular matrix.

Proof: By contradiction. For suppose every member of M is truth-/false-singular. Then, $\mathsf{M}{\upharpoonright}\Sigma_\sim$ is a class of truth-/false-singular $\Sigma_\sim$-matrices defining C_{DB}. Then, by Proposition 4.4, there are some $\mathcal{B} \in (\mathsf{M}{\upharpoonright}\Sigma_\sim)$ and some submatrix $\mathcal{D}$ of $\mathcal{B}$ such that $\mathcal{DM}_4$ is isomorphic to $\mathcal{E} \triangleq (\mathcal{D}/\theta)$, where $\theta \triangleq \eth(\mathcal{D})$, in which case $\mathcal{E}$ is truth-/false-singular, for $\mathcal{D}$ is so, because $\mathcal{B}$ is so/, while $((D/\theta) \setminus (D^{\mathcal{D}}/\theta)) \subseteq ((D \setminus D^{\mathcal{D}})/\theta)$, and so is $\mathcal{DM}_4$. This contradiction completes the argument. $\square$

Corollary 4.6. Any four-valued $\Sigma_\sim$-matrix $\mathcal{B}$ defining C_{DB} is isomorphic to $\mathcal{DM}_4$.

Proof: By Proposition 4.4, there are then some submatrix $\mathcal{D}$ of $\mathcal{B}$ and some isomorphism e from $\mathcal{DM}_4$ onto $\mathcal{D}/\theta$, where $\theta \triangleq \eth(\mathcal{D})$, in which case $4 = |DM_4| = |D/\theta| \leqslant |D| \leqslant |B| = 4$, in which case $4 = |D/\theta| = |D| = |B|$, and so ν_θ is injective, while $D = B$. In this way, $e^{-1} \circ \nu_\theta$ is an isomorphism from $\mathcal{B}$ onto $\mathcal{DM}_4$, as required. $\square$

This, in its turn, enables us to prove:

Theorem 4.7. *Any four-valued Σ-expansion of C_{DB} is defined by a Σ-expansion of $\mathcal{DM}_4$.*

Proof: Let $\mathcal{B}$ be a four-valued Σ-matrix defining an expansion of C_{DB}. Then, $\mathcal{B}{\upharpoonright}\Sigma_\sim$ is a four-valued $\Sigma_\sim$-matrix defining C_{DB} itself. Hence, by

Corollary 4.6, there is an isomorphism e from $\mathcal{B}{\restriction}\Sigma_{\sim}$ onto $\mathcal{DM}_4$. In that case, e is an isomorphism from $\mathcal{B}$ onto the Σ-expansion $\langle e[\mathfrak{B}], e[D^{\mathcal{B}}]\rangle$ of $\mathcal{DM}_4$. In this way, (2.2) completes the argument. $\square$

Thus, the natural way of construction of four-valued expansions chosen above does exhaust *all* of them. And what is more, any of them is defined by a *unique* expansion of $\mathcal{DM}_4$, as it follows from:

THEOREM 4.8. *Let $\mathcal{B}$ be a Σ-matrix. Suppose $(\mathcal{B}{\restriction}\Sigma_{\sim}) = \mathcal{DM}_4$ and $\mathcal{B}$ is a model of C (in particular, C is defined by $\mathcal{B}$). Then, $\mathcal{B} = \mathcal{A}$.*

PROOF: In that case, $\mathcal{B}$, being finite, is finitely-generated. In addition, by Lemma 3.2 and Remark 3.6, it is simple. Therefore, as $\mathcal{A}$ is finite, by Lemma 2.7 with $\mathsf{M} = \{\mathcal{A}\}$, there are some finite set I, some I-tuple $\overline{\mathcal{C}}$ constituted by submatrices of $\mathcal{A}$, some subdirect product $\mathcal{D}$ of $\overline{\mathcal{C}}$ and some $g \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D}, \mathcal{B}) \subseteq \hom(\mathfrak{D}{\restriction}\Sigma_{\sim}, \mathfrak{DM}_4)$, in which case, as $|\operatorname{img} g| = |B| = 4 \neq 1$, $(\bigcap_{i \in I} \ker(\pi_i{\restriction}D)) = \Delta_D \subseteq (\ker g) \neq D^2$, while $\{\pi_i{\restriction}D \mid i \in I\} \in \wp_{\omega}(\hom(\mathfrak{D}{\restriction}\Sigma_{\sim}, \mathfrak{DM}_4))$, and so, by Theorem 3.8, there is some $i \in I$ such that $\ker(\pi_i{\restriction}D) = (\ker g)$. Hence, as $(\pi_i{\restriction}D) \in \hom(\mathcal{D}, \mathcal{C}_i)$, by Remark 2.4, $e \triangleq ((\pi_i{\restriction}D) \circ g^{-1}) \in \hom(\mathcal{B}, \mathcal{C}_i) \subseteq \hom(\mathcal{B}, \mathcal{A})$ is injective, in which case $e[\{\mathsf{n}, \mathsf{b}\}] \subseteq \{\mathsf{n}, \mathsf{b}\}$ and $e[\{\mathsf{f}, \mathsf{t}\}] \subseteq \{\mathsf{f}, \mathsf{t}\}$, because ${\sim}^{\mathfrak{DM}_4} a = a$ iff $a \in \{\mathsf{n}, \mathsf{b}\}$, for all $a \in DM_4$, and so e is diagonal, for $(D^{\mathcal{DM}_4} \cap \{\mathsf{n}, \mathsf{b}\}) = \{\mathsf{b}\}$ and $(D^{\mathcal{DM}_4} \cap \{\mathsf{f}, \mathsf{t}\}) = \{\mathsf{t}\}$. In this way, $\mathcal{B} = \mathcal{A}$, for $B = A$ and $D^{\mathcal{B}} = D^{\mathcal{A}}$, as required. $\square$

In view of Theorem 4.8, $\mathcal{A}$ is said to be *characteristic for/of C*.

COROLLARY 4.9. Let $\Sigma' \supseteq \Sigma$ be a signature and C' a four-valued Σ'-expansion of C. Then, C' is defined by a unique Σ'-expansion of $\mathcal{A}$.

PROOF: Then, by Theorem 4.7, C' is defined by a Σ'-expansion $\mathcal{A}'$ of $\mathcal{DM}_4$, in which case C is defined by the Σ-expansion $\mathcal{A}'{\restriction}\Sigma$ of $\mathcal{DM}_4$, and so $(\mathcal{A}'{\restriction}\Sigma) = \mathcal{A}$, in view of Theorem 4.8. In this way, Theorem 4.8 completes the argument. $\square$

4.1.1. Minimal four-valuedness

As a one more interesting consequence of Proposition 4.4, we have:

THEOREM 4.10. *Let M be a class of Σ-matrices. Suppose the logic of M is an expansion of C_{DB} (in particular, $\Sigma = \Sigma_{\sim}$ and the logic of M is C_{DB}*

itself). Then, $4 \leqslant |B|$, *for some* $\mathcal{B} \in \mathsf{M}$. *In particular, any four-valued expansion of* C_{DB} *(including* C_{DB} *itself) is minimally four-valued.*

Proof: In that case, C_{DB} is defined by $\mathsf{M}{\upharpoonright}\Sigma_{\sim}$, and so, by Proposition 4.4, there are some $\mathcal{B} \in \mathsf{M}$ and some submatrix $\mathcal{D}$ of $\mathcal{B}{\upharpoonright}\Sigma_{\sim}$ such that $\mathcal{DM}_4$ is isomorphic to $\mathcal{D}/\theta$, where $\theta \triangleq \partial(\mathcal{D})$. In this way, $4 = |DM_4| = |D/\theta| \leqslant |D| \leqslant |B|$, as required. $\square$

4.2. Variable sharing property

Lemma 4.11. *C is theorem-less iff* $\{\mathsf{n}\}$ *forms a subalgebra of* $\mathfrak{A}$.

Proof: First, assume $\{\mathsf{n}\}$ forms a subalgebra of $\mathcal{A}$, in which case $\mathcal{A}{\upharpoonright}\{\mathsf{n}\}$ is a truth-empty submatrix of $\mathcal{A}$, and so C is theorem-less, in view of (2.2).

Conversely, assume $\{\mathsf{n}\}$ does not form a subalgebra of $\mathfrak{A}$. Then, there is some $\varphi \in \mathrm{Fm}^1_\Sigma$ such that $\varphi^{\mathfrak{A}}(\mathsf{n}) \neq \mathsf{n}$, in which case $(\varphi^{\mathfrak{A}}(\mathsf{n}) \vee^{\mathfrak{A}} \sim^{\mathfrak{A}} \varphi^{\mathfrak{A}}(\mathsf{n})) \in D^{\mathcal{A}}$, and so $((x_0 \vee \sim x_0) \vee (\varphi \vee \sim\varphi)) \in C(\varnothing)$, as required. $\square$

Lemma 4.12. *C has no inconsistent formula iff* $\{\mathsf{b}\}$ *forms a subalgebra of* $\mathfrak{A}$.

Proof: First, assume $\{\mathsf{b}\}$ does not form a subalgebra of $\mathfrak{A}$. Then, there is some $\varphi \in \mathrm{Fm}^1_\Sigma$ such that $\varphi^{\mathfrak{A}}(\mathsf{b}) \neq \mathsf{b}$, in which case $(\varphi^{\mathfrak{A}}(\mathsf{b}) \wedge^{\mathfrak{A}} \sim^{\mathfrak{A}} \varphi^{\mathfrak{A}}(\mathsf{b})) \notin D^{\mathcal{A}}$, and so $((x_0 \wedge \sim x_0) \wedge (\varphi \wedge \sim\varphi))$ is an inconsistent formula of C.

Conversely, assume $\{\mathsf{b}\}$ forms a subalgebra of $\mathcal{A}$. Let us prove, by contradiction, that C has no inconsistent formula. For suppose some $\varphi \in \mathrm{Fm}^\omega_\Sigma$ is an inconsistent formula of C, in which case $\varphi \in \mathrm{Fm}^\alpha_\Sigma$, for some $\alpha \in (\omega \setminus 1)$, while $x_\alpha \in C(\varphi)$. Let $h \in \hom(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{A})$ extend $(V_\alpha \times \{\mathsf{b}\}) \cup (V_{\omega\setminus\alpha} \times \{\mathsf{f}\})$. Then, $h(\varphi) = \mathsf{b} \in D^{\mathcal{A}}$, whereas $h(x_\alpha) = \mathsf{f} \notin D^{\mathcal{A}}$. This contradiction completes the argument. $\square$

Theorem 4.13. *The following are equivalent:*

(i) *C satisfies VSP;*

(ii) *C has neither a theorem nor an inconsistent formula;*

(iii) *both* $\{\mathsf{n}\}$ *and* $\{\mathsf{b}\}$ *form subalgebras of* $\mathfrak{A}$.

Proof: First, (ii) is a particular case of (i). Next, (ii)⇒(iii) is by Lemmas 4.11 and 4.12.

Finally, assume (iii) holds. Consider any $\phi, \psi \in \mathrm{Fm}^{\omega}_{\Sigma}$ such that $V \triangleq \mathrm{Var}(\phi)$ and $\mathrm{Var}(\psi)$ are disjoint. Let $h \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$ extend $(V \times \{\mathsf{b}\}) \cup ((V_\omega \setminus V) \times \{\mathsf{n}\})$. Then, $h(\phi) = \mathsf{b} \in D^{\mathcal{A}}$, whereas $h(\psi) = \mathsf{n} \notin D^{\mathcal{A}}$. Thus, $\notin C(\phi)$, and so (i) holds, as required. ☐

Corollary 4.14 (cf. Theorem 4.2 of [9] for the case $\Sigma = \Sigma_{\sim}$). *C has no proper extension satisfying VSP.*

Proof: Consider any extension C' of C satisfying VSP, in which case C, being a sublogic of C', does so as well, and so, by Theorem 4.13(i)⇒(iii), $\{\mathsf{b}\}$ forms a subalgebra of $\mathfrak{A}$. Moreover, as C' is $\wedge$-conjunctive, for $\mathcal{A}$ is so (cf. Remark 3.6), (4.1) is not satisfied in C', for $\mathrm{Var}(x_0 \wedge {\sim} x_0) = \{x_0\}$ and $\mathrm{Var}(x_1 \vee {\sim} x_1) = \{x_1\}$ are disjoint. In this way, Corollary 4.3 completes the argument. ☐

Perhaps, this is the principal *specific* maximality of C in addition to the standard one studied in the next subsection.

4.3. Maximality

Lemma 4.15. *Any proper submatrix $\mathcal{B}$ of $\mathcal{A}$ defines a proper extension C' of C.*

Proof: For consider the following complementary cases:

1. $\mathsf{b} \in B$.
   Then, $\mathsf{n} \notin B$, for $B \neq A$, while $(\mathsf{n} \wedge^{\mathfrak{B}} \mathsf{b}) = \mathsf{f}$, whereas $(\mathsf{n} \vee^{\mathfrak{B}} \mathsf{b}) = \mathsf{t}$. In that case, $(x_0 \vee {\sim} x_0) \in (C'(\varnothing) \setminus C(\varnothing))$.
2. $\mathsf{b} \notin B$.
   Then, $\mathcal{B}$ is not $\sim$-paraconsistent, as opposed to $\mathcal{A}$, and so is C', as opposed to C.

Thus, in any case, $C' \neq C$, as required, in view of (2.2). ☐

Clearly, $\mathcal{A}$ is consistent (and truth-non-empty), and so C is (inferentially) consistent. In this connection, we have:

Theorem 4.16. *C is [inferentially] maximal iff $\mathcal{A}$ has no proper consistent [truth-non-empty] submatrix.*

Proof: First, consider any proper consistent [truth-non-empty] submatrix $\mathcal{B}$ of $\mathcal{A}$. Then, by Lemma 4.15, the logic C' of $\mathcal{B}$ is a[n inferentially] consistent proper extension of C, and so C is not [inferentially] maximal.

Conversely, assume $\mathcal{A}$ has no proper consistent [truth-non-empty] submatrix. Consider any [inferentially] consistent extension C' of C. Then, $x_0 \notin T \triangleq C'(\varnothing[\cup\{x_1\}])[\ni x_1]$, while, by the structurality of C', $\langle \mathfrak{Fm}^{\omega}_{\Sigma}, T\rangle$ is a model of C' (in particular, of C), and so is its consistent [truth-non-empty] finitely-generated submatrix $\mathcal{B} = \langle \mathfrak{Fm}^2_{\Sigma}, \mathrm{Fm}^2_{\Sigma} \cap T\rangle$, in view of (2.2). Hence, by Lemma 2.7 with $\mathsf{M} = \{\mathcal{A}\}$, there are some finite set I, some I-tuple $\overline{\mathcal{C}}$ constituted by consistent [truth-non-empty] submatrices of $\mathcal{A}$, some subdirect product $\mathcal{D}$ of $\overline{\mathcal{C}}$, and some $g \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D}, \mathcal{B}/\partial(\mathcal{B}))$, in which case, by (2.2), $\mathcal{D}$ is a consistent model of C', and so, in particular, $I \neq \varnothing$. Moreover, for any $i \in I$, as $\mathcal{C}_i$ is consistent [and truth-non-empty] submatrix of $\mathcal{A}$, $\mathcal{C}_i = \mathcal{A}$ is truth non-empty anyway. Hence, by the following claim, both $D \ni a \triangleq (I \times \{\mathsf{f}\})$ and $D \ni b \triangleq (I \times \{\mathsf{t}\})$:

Claim 4.17. Let I be a finite set, $\overline{\mathcal{C}} \in \mathbf{S}^*_*(\mathcal{A})^I$ and $\mathcal{B}$ a subdirect product of $\overline{\mathcal{C}}$. Then, $\{I \times \{\mathsf{f}\}, I \times \{\mathsf{t}\}\} \subseteq B$.

PROOF: In that case, $\mathfrak{B}{\restriction}\Sigma_+$ is a finite lattice, so it has both a zero a and a unit b. Consider any $i \in I$. Then, as $\mathcal{C}_i$ is both consistent and truth-non-empty, by the following claim, we have $\{\mathsf{f},\mathsf{t}\} \subseteq C_i$:

Claim 4.18. Let $\mathcal{D} \in \mathbf{S}^*_*(\mathcal{A})$. Then, $\{\mathsf{f},\mathsf{t}\} \subseteq D$.

PROOF: In that case, we have $(\{\mathsf{f},\mathsf{n}\} \cap D) \neq \varnothing \neq (\{\mathsf{b},\mathsf{t}\} \cap D)$. In this way, the fact that $(\mathsf{n} \wedge^{\mathfrak{A}} \mathsf{b}) = \mathsf{f}$, while ${\sim}^{\mathfrak{A}}\mathsf{f} = \mathsf{t}$, whereas ${\sim}^{\mathfrak{A}}\mathsf{t} = \mathsf{f}$, completes the argument. □

Therefore, since $\pi_i[B] = C_i$, there are some $c, d \in B$, such that $\pi_i(c) = \mathsf{f}$ and $\pi_i(d) = \mathsf{t}$, in which case we have $(c \wedge^{\mathfrak{B}} a) = a$ and $(d \vee^{\mathfrak{D}} b) = b$, and so, as $(\pi_i{\restriction}B) \in \hom(\mathfrak{B}{\restriction}\Sigma_+, \mathfrak{C}_i{\restriction}\Sigma_+)$, we eventually get $\pi_i(a) = (\mathsf{f} \wedge^{\mathfrak{A}} \pi_i(a)) = \mathsf{f}$ and $\pi_i(b) = (\mathsf{t} \vee^{\mathfrak{A}} \pi_i(b)) = \mathsf{t}$. Thus, $B \ni a = (I \times \{\mathsf{f}\})$ and $B \ni b = (I \times \{\mathsf{t}\})$, as required. □

Next, if $\{\mathsf{f},\mathsf{t}\} \subsetneq A$ [distinct from $\{\mathsf{n}\}$] did form a subalgebra of $\mathfrak{A}$, $\mathcal{A}{\restriction}\{\mathsf{f},\mathsf{t}\}$ would be a proper consistent [truth-non-empty] submatrix of $\mathcal{A}$. Therefore, there are some $\phi \in \mathrm{Fm}^2_{\Sigma}$ and $j \in 2$ such that $\phi^{\mathfrak{A}}(\mathsf{f},\mathsf{t}) = \langle j, 1-j\rangle$. Likewise, if $\{\mathsf{f}, \langle j, 1-j\rangle, \mathsf{t}\} \subsetneq A$ [distinct from $\{\mathsf{n}\}$] did form a subalgebra of $\mathfrak{A}$, $\mathcal{A}{\restriction}\{\mathsf{f}, \langle j, 1-j\rangle, \mathsf{t}\}$ would be a proper consistent [truth-non-empty] submatrix of $\mathcal{A}$. Therefore, there is some $\psi \in \mathrm{Fm}^3_{\Sigma}$ such that $\psi^{\mathfrak{A}}(\mathsf{f}, \langle j, 1-j\rangle, \mathsf{t}) = \langle 1-j, j\rangle$. In this way, $\{\phi^{\mathfrak{A}}(\mathsf{f},\mathsf{t}), \psi^{\mathfrak{A}}(\mathsf{f}, \phi^{\mathfrak{A}}(\mathsf{f},\mathsf{t}), \mathsf{t})\} = \{\mathsf{n},\mathsf{b}\}$. Then, $D \supseteq \{\phi^{\mathfrak{D}}(a,b), \psi^{\mathfrak{D}}(a, \phi^{\mathfrak{D}}(a,b), b)\} = \{I \times \{\mathsf{n}\}, I \times \{\mathsf{b}\}\}$. Thus,

$\{I \times \{c\} \mid c \in A\} \subseteq D$. Hence, as $I \neq \varnothing$, $\{\langle c, I \times \{c\}\rangle \mid c \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{D}$, in which case, by (2.2), C is an extension of C', and so $C' = C$, as required. $\Box$

4.4. Subclassical expansions

LEMMA 4.19. *Let $\mathcal{B}$ be a (simple) finitely-generated consistent truth-non-empty model of C. Then, the following hold:*

(i) *$\mathcal{B}$ is $\sim$-paraconsistent, if $\sim(x_0 \wedge \sim x_0)$ is true in $\mathcal{B}$ and $\{\mathsf{f}, \mathsf{t}\}$ does not form a subalgebra of $\mathfrak{A}$;*

(ii) *$\mathcal{A} {\restriction} \{\mathsf{f}, \mathsf{t}\}$ is embeddable into $\mathcal{B}/\partial(\mathcal{B})$ (resp., into $\mathcal{B}$ itself), if $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$.*

PROOF: Put $\mathcal{E} \triangleq (\mathcal{B}/\partial(\mathcal{B}))$ (resp., $\mathcal{E} \triangleq \mathcal{B}$). Then, by Lemma 2.7 with $\mathsf{M} = \{\mathcal{A}\}$, there are some finite set I, some I-tuple $\overline{\mathcal{C}}$ constituted by consistent truth-non-empty submatrices of $\mathcal{A}$, some subdirect product $\mathcal{D}$ of $\overline{\mathcal{C}}$ and some $g \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D}, \mathcal{E})$, in which case, by (2.2), $\mathcal{D}$ is consistent, and so, in particular, $I \neq \varnothing$. Hence, by Claim 4.17, both $D \ni a \triangleq (I \times \{\mathsf{f}\})$ and $D \ni b \triangleq (I \times \{\mathsf{t}\})$. Consider the following respective cases:

(i) $\sim(x_0 \wedge \sim x_0)$ is true in $\mathcal{B}$ and $\{\mathsf{f}, \mathsf{t}\}$ does not form a subalgebra of $\mathfrak{A}$. Then, there is some $\varphi \in \mathrm{Fm}^2_\Sigma$ such that $\varphi^{\mathfrak{A}}(\mathsf{f}, \mathsf{t}) \in \{\mathsf{n}, \mathsf{b}\}$. Take any $i \in I \neq \varnothing$. Then, $\{\mathsf{f}, \mathsf{t}\} = \pi_i[\{a, b\}] \subseteq C_i$. Moreover, $(\pi_i {\restriction} D) \in \hom^{\mathrm{S}}(\mathcal{D}, \mathcal{C}_i)$, in which case, by (2.2) and (2.3), $\mathcal{C}_i$ is a model of $\sim(x_0 \wedge \sim x_0)$, and so $\mathsf{n} \notin C_i$, for $\sim^{\mathfrak{A}}(\mathsf{n} \wedge^{\mathfrak{A}} \sim^{\mathfrak{A}} \mathsf{n}) = \mathsf{n} \notin D^{\mathcal{A}}$. And what is more, $\mathfrak{C}_i$ is a subalgebra of $\mathfrak{A}$. Hence, $\varphi^{\mathfrak{A}}(\mathsf{f}, \mathsf{t}) \in C_i$, and so $\varphi^{\mathfrak{A}}(\mathsf{f}, \mathsf{t}) = \mathsf{b}$, for $\mathsf{n} \notin C_i$. Then, $D \ni c \triangleq \varphi^{\mathfrak{D}}(a, b) = (I \times \{\mathsf{b}\})$, in which case $\sim^{\mathfrak{D}} c = c \in D^{\mathcal{D}}$, and so $\mathcal{D}$, being consistent, is $\sim$-paraconsistent, and so is $\mathcal{B}$, in view of (2.2), as required.

(ii) $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$.
Then, $\mathcal{F} \triangleq (\mathcal{A} {\restriction} \{\mathsf{f}, \mathsf{t}\})$ is $\sim$-classical, and so simple, in view of Example 3.1 and Lemma 3.2. Finally, as $\{I \times \{d\} \mid d \in F\} \subseteq D$ and $I \neq \varnothing$, $e \triangleq \{\langle d, I \times \{d\}\rangle \mid d \in F\}$ is an embedding of $\mathcal{F}$ into $\mathcal{D}$, in which case, $(g \circ e) \in \hom_{\mathrm{S}}(\mathcal{F}, \mathcal{E})$, and so Corollary 2.3 completes the argument. $\Box$

THEOREM 4.20. *C is $\sim$-subclassical iff* $\{\mathsf{f},\mathsf{t}\}$ *forms a subalgebra of* $\mathfrak{A}$*, in which case* $\mathcal{A}{\restriction}\{\mathsf{f},\mathsf{t}\}$ *is isomorphic to any $\sim$-classical model of C, and so its logic is the only $\sim$-classical extension of C.*

PROOF: Let $\mathcal{B}$ be a $\sim$-classical model of C, in which case it is simple (cf. Example 3.1 and Lemma 3.2) and finite (in particular, finitely-generated) but is not $\sim$-paraconsistent.

First, consider any $a \in B$. Then, $\{a, \sim^{\mathfrak{B}} a\} \nsubseteq D^{\mathcal{B}}$, for $\mathcal{B}$ is $\sim$-classical, in which case $(a \wedge^{\mathfrak{B}} \sim^{\mathfrak{B}} a) \notin D^{\mathcal{B}}$, for $\mathcal{B}$ is $\wedge$-conjunctive, because C is so, since $\mathcal{A}$ is so (cf. Remark 3.6), and so $\sim^{\mathfrak{B}}(a \wedge^{\mathfrak{B}} \sim^{\mathfrak{B}} a) \in D^{\mathcal{B}}$, for $\mathcal{B}$ is $\sim$-classical. Thus, $\sim(x_0 \wedge \sim x_0)$ is true in $\mathcal{B}$. Hence, by Lemma 4.19(i), $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$.

Conversely, assume $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, in which case $\mathcal{D} \triangleq (\mathcal{A}{\restriction}\{\mathsf{f},\mathsf{t}\})$ is a $\sim$-classical model of C, by (2.2), and embeddable into $\mathcal{B}$, by Lemma 4.19(ii), so is isomorphic to $\mathcal{B}$, for $|D| = 2 = |B|$. Then, (2.2) completes the argument. □

In view of Theorem 4.20, the unique $\sim$-classical extension of a $\sim$-subclassical four-valued expansion $\boldsymbol{C}$ of C_{DB} is said to be *characteristic for $\boldsymbol{C}$* and denoted by $\boldsymbol{C}^{\mathrm{PC}}$. Its *specific* maximality feature is as follows:

THEOREM 4.21. *Let C' be an inferentially consistent extension of C. Suppose* $\{\mathsf{f},\mathsf{t}\}$ *forms a subalgebra of* $\mathfrak{A}$*. Then,* $\mathcal{A}{\restriction}\{\mathsf{f},\mathsf{t}\}$ *is a model of C'.*

PROOF: Then, $x_1 \notin C'(x_0) \ni x_0$, while, by the structurality of C', $\langle \mathfrak{Fm}^{\omega}_{\Sigma}, C'(x_0)\rangle$ is a model of C' (in particular, of C), and so is its consistent truth-non-empty finitely-generated submatrix $\langle \mathfrak{Fm}^{2}_{\Sigma}, \mathrm{Fm}^{2}_{\Sigma} \cap C'(x_0)\rangle$, in view of (2.2). In this way, (2.2) and Lemma 4.19(ii) complete the argument. □

On the other hand, the reservation "inferentially" cannot, generally speaking, be omitted in the formulation of Theorem 4.21, as it ensues from:

Example 4.22. When $\Sigma = \Sigma_{\sim}$, $\{\mathsf{n}\}$ forms a subalgebra of $\mathfrak{A}$, in which case $\mathcal{B} \triangleq (\mathcal{A}{\restriction}\{\mathsf{n}\})$ is a consistent truth-empty submatrix of $\mathcal{A}$, and so, by (2.2), the logic C' of $\mathcal{B}$ is a consistent but inferentially inconsistent extension of C. Then, C' is not subclassical, because any classical logic is inferentially consistent, for any classical matrix is both consistent and truth-non-empty. □

4.5. Axiomatic extensions

Lemma 4.23. *Suppose $\mathfrak{A}$ is regular and $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of it. Then, so does $\{\mathsf{f},\mathsf{b},\mathsf{t}\}$.*

Proof: By contradiction. For suppose $\{\mathsf{f},\mathsf{b},\mathsf{t}\}$ does not form a subalgebra of $\mathfrak{A}$, in which case there is some $\varphi \in \mathrm{Fm}^3_\Sigma$ such that $\varphi^{\mathfrak{A}}(\mathsf{f},\mathsf{b},\mathsf{t}) = \mathsf{n}$. Therefore, as $\mathsf{t} \sqsubseteq \mathsf{b}$, by the regularity of $\mathfrak{A}$ and the reflexivity of $\sqsubseteq$, we get $\varphi^{\mathfrak{A}}(\mathsf{f},\mathsf{t},\mathsf{t}) \sqsubseteq \mathsf{n}$. Hence, $\varphi^{\mathfrak{A}}(\mathsf{f},\mathsf{t},\mathsf{t}) = \mathsf{n} \notin \{\mathsf{f},\mathsf{t}\}$. This contradicts to the assumption that $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, as required. □

Lemma 4.24 (cf. Lemma 4.14 of [12] for the case $B = \{\mathsf{f},\mathsf{t}\}$ and $\Sigma = \Sigma_\sim$). *Let $\mathcal{B} \in \mathbf{S}(\mathcal{A})$. Suppose $B \cup \{\mathsf{b}\}$ forms a regular subalgebra of $\mathfrak{A}$. Then, any Σ-axiom, being true in $\mathcal{B}$, is so in $\mathcal{A}{\restriction}(B \cup \{\mathsf{b}\})$.*

Proof: Consider any $\varphi \in \mathrm{Fm}_\Sigma$ not true in $\mathcal{A}{\restriction}(B \cup \{\mathsf{b}\})$, in which case there is some $h \in \hom(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{A}{\restriction}(B \cup \{\mathsf{b}\}))$ such that $h(\varphi) \in \{\mathsf{f},\mathsf{n}\}$, and so $h(\varphi) \sqsubseteq \mathsf{f}$. Take any $b \in B \neq \varnothing$. Define a $g : V_\omega \to B$ by setting:

$$g(v) \triangleq \begin{cases} b & \text{if } h(v) = \mathsf{b}, \\ h(v) & \text{otherwise}, \end{cases}$$

for all $v \in V_\omega$. Let $e \in \hom(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{B}) \subseteq \hom(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{A}{\restriction}(B \cup \{\mathsf{b}\}))$ extend g. Then, $e(v) = g(v) \sqsubseteq h(v)$, for all $v \in V_\omega$, in which case, by the regularity of $\mathfrak{A}{\restriction}(B \cup \{\mathsf{b}\})$, we have $e(\varphi) \sqsubseteq h(\varphi) \sqsubseteq \mathsf{f}$, and so we eventually get $e(\varphi) \in \{\mathsf{f},\mathsf{n}\}$, as required. □

Lemma 4.25 (cf. Corollary 5.3 of [9] for the case $\Sigma = \Sigma_\sim$). *Suppose $\{\mathsf{f},\mathsf{b},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$/ $\{\mathsf{f},\mathsf{t}\}[\cup\{\mathsf{b}\}]$ does [not] form a subalgebra of $\mathfrak{A}$. Then, the logic of $\mathcal{A}_{\not\mathsf{n}/\not\mathsf{n}\not\mathsf{b}} \triangleq (\mathcal{A}{\restriction}(\{\mathsf{f},\mathsf{b},\mathsf{t}\}/\{\mathsf{f},\mathsf{t}\}))$ is the proper consistent axiomatic extension of C relatively axiomatized by*

$$x_1 \vee {\sim} x_1. \tag{4.8}$$

Proof: In that case, $(\mathrm{Mod}\,(4.8) \cap \mathbf{S}_*(\mathcal{A})) = \mathbf{S}_*(\mathcal{A}_{\not\mathsf{n}/\not\mathsf{n}\not\mathsf{b}})$. In this way, (2.2), Corollary 2.9, the consistency of $\mathcal{A}_{\not\mathsf{n}/\not\mathsf{n}\not\mathsf{b}}$ and the fact that (4.8) is not true in $\mathcal{A}$ under $[x_1/\mathsf{n}]$ complete the argument. □

Theorem 4.26. *[Providing $\mathfrak{A}$ is regular/has no three-element subalgebra] C has a proper consistent axiomatic extension if[f] $\{\mathsf{f},\mathsf{b},\mathsf{t}\}/\{\mathsf{f},\mathsf{t}\}$ forms a* *subalgebra of* $\mathfrak{A}$ *[in which case the logic of* $\mathcal{A}_{\not{\mathsf{n}}/\not{\mathsf{n}}\not{\mathsf{b}}}$ *is the only proper consistent axiomatic extension of* C *and is relatively axiomatized by* (4.8)*].*

Proof: The "if" part is by Lemma 4.25. [Conversely, assume $\mathfrak{A}$ is regular/has no three-element subalgebra. Consider any $\mathcal{A} \subseteq \mathrm{Fm}_\Sigma$ such that the axiomatic extension C' of C relatively axiomatized by $\mathcal{A}$ is both proper, in which case $\mathcal{A} \neq \varnothing$, and consistent, in which case, by Corollary 2.9, C' is the logic of $\mathsf{S} \triangleq (\mathrm{Mod}(\mathcal{A}) \cap \mathbf{S}_*(\mathcal{A}))$, and so $\mathcal{A} \notin \mathsf{S} \neq \varnothing$. Take any $\mathcal{B} \in \mathsf{S}$, in which case it is both consistent and, as $\mathcal{A} \neq \varnothing$, truth-non-empty. Hence, by Claim 4.18, we have $\{\mathsf{f}, \mathsf{t}\} \subseteq B$. Therefore, if n was in B, then $(B \cup \{\mathsf{b}\})$ would be equal to A/B would belong to $\{\{\mathsf{f}, \mathsf{n}, \mathsf{t}\}, A\}$, in which case, by Lemma 4.24/the fact that $\{\mathsf{f}, \mathsf{n}, \mathsf{t}\}$, being three-element, does not form a subalgebra of $\mathfrak{A}$, $\mathcal{A}$ would belong to S. Thus, $B \in \{\{\mathsf{f}, \mathsf{t}\}, \{\mathsf{f}, \mathsf{b}, \mathsf{t}\}\}$. Then, by Lemma 4.23/the fact that $\{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$, being three-element, does not form a subalgebra of $\mathfrak{A}$, we conclude that $\{\mathsf{f}, \mathsf{b}, \mathsf{t}\}/\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$. And what is more, in that case, by Lemma 4.24/the fact that $\{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$, being three-element, does not form a subalgebra of $\mathfrak{A}$, we have $\mathcal{A}_{\not{\mathsf{n}}/\not{\mathsf{n}}\not{\mathsf{b}}} \in \mathsf{S} \subseteq \mathbf{S}_*(\mathcal{A}_{\not{\mathsf{n}}/\not{\mathsf{n}}\not{\mathsf{b}}})$, and so, by (2.2), C' is equal to the logic of $\mathcal{A}_{\not{\mathsf{n}}/\not{\mathsf{n}}\not{\mathsf{b}}}$. In this way, Lemma 4.25 completes the argument.] □

The logic of $\mathcal{DM}_{4[01],\not{\mathsf{n}}}$ is [the bounded version of] the logic of paradox $LP_{[01]}$ [8] (cf. [10]; viz., in the "unbounded" case, the implication-less fragment of any *paraconsistent* Dunn's $RM\{(2 \cdot n) + 3\}$ {where $n \in \omega$} – cf. [4] and the proof of Corollary 4.15 of [12]). Therefore, in view of the regularity of $\mathfrak{DM}_{4[01]}$, Theorem 4.26 immediately yields:

Corollary 4.27. $LP_{[01]}$ is the only proper consistent axiomatic extension of $C_{[\mathrm{B}]\mathrm{DB}}$ and is relatively axiomatized by (4.8).

In Section 5 we consider more classes of expansions of FDE in this connection.

4.6. Maximal paraconsistency versus paracompleteness

The axiomatic extension of C relatively axiomatized by (4.8) is denoted by C^{EM}. An/A extension/model of C is said to be *paracomplete*, provided it is not that of C^{EM}. Clearly, a submatrix $\mathcal{B}$ of $\mathcal{A}$ is paracomplete/$\sim$-paraconsistent iff $\mathsf{n} \in B$/both $\mathsf{b} \in B$ and $(B \cap \{\mathsf{n}, \mathsf{f}\}) \neq \varnothing$. In particular, $\mathcal{A}$ is both $\sim$-paraconsistent and paracomplete, and so is C.

By $\mathcal{A}_{-\mathsf{n}}$ we denote the submatrix of $\mathcal{A}$ generated by $\{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$ — this the least $\sim$-paraconsistent submatrix of $\mathcal{A}$, the logic of it being denoted by $C^{-\mathsf{n}}$. (Clearly, $\mathcal{A}_{-\mathsf{n}} = \mathcal{A}_{\not\mathsf{n}}$, whenever $\{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, and $\mathcal{A}_{-\mathsf{n}} = \mathcal{A}$, otherwise.)

Lemma 4.28. *Let $\mathcal{B}$ be a $\sim$-paraconsistent model of C. Then, there is some submatrix $\mathcal{D}$ of $\mathcal{B}$ such that $\mathcal{A}_{-\mathsf{n}}$ is embeddable into $\mathcal{D}/\partial(\mathcal{D})$.*

Proof: In that case, there are some $a \in D^{\mathcal{B}}$ such that $\sim^{\mathfrak{B}} a \in D^{\mathcal{B}}$ and some $b \in (B \setminus D^{\mathcal{B}})$. Then, in view of (2.2), the submatrix $\mathcal{D}$ of $\mathcal{B}$ generated by $\{a, b\}$ is a $\sim$-paraconsistent finitely-generated model of C. Hence, by Lemma 2.7 with $\mathsf{M} = \{\mathcal{A}\}$, there are some finite set I, some I-tuple $\overline{\mathcal{C}}$ constituted by consistent submatrices of $\mathcal{A}$, some subdirect product $\mathcal{E}$ of $\overline{\mathcal{C}}$ and some $g \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{E}, \mathcal{D}/\partial(\mathcal{D}))$. Hence, by (2.2), $\mathcal{E}$ is $\sim$-paraconsistent, in which case it is consistent, and so $I \neq \varnothing$. Take any $a \in D^{\mathcal{E}}$ such that $\sim^{\mathfrak{E}} a \in D^{\mathcal{E}}$. Then, $E \ni a = (I \times \{\mathsf{b}\})$, in which case, for each $i \in I$, $D^{\mathcal{C}_i} \ni \pi_i(a)$, and so $\mathcal{C}_i$ is truth-non-empty. Therefore, by Claim 4.17, we also have both $E \ni b \triangleq (I \times \{\mathsf{f}\})$ and $E \ni c \triangleq (I \times \{\mathsf{t}\})$. Consider the following complementary cases:

1. $\{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$ does not form a subalgebra of $\mathfrak{A}$.
   Then, $A_{-\mathsf{n}} = A$ and there is some $\varphi \in \mathrm{Fm}_{\Sigma}^{3}$ such that $\varphi^{\mathfrak{A}}(\mathsf{f}, \mathsf{b}, \mathsf{t}) = \mathsf{n}$, in which case $E \ni \varphi^{\mathfrak{E}}(b, a, c) = (I \times \{\varphi^{\mathfrak{A}}(\mathsf{f}, \mathsf{b}, \mathsf{t})\}) = (I \times \{\mathsf{n}\})$, and so $\{I \times \{d\} \mid d \in A_{-\mathsf{n}}\} \subseteq E$.

2. $\{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$.
   Then, $A_{-\mathsf{n}} = \{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$, and so $\{I \times \{d\} \mid d \in A_{-\mathsf{n}}\} \subseteq E$.

Thus, in any case, $\{I \times \{d\} \mid d \in A_{-\mathsf{n}}\} \subseteq E$. Then, as $I \neq \varnothing$, $e \triangleq \{\langle d, I \times \{d\}\rangle \mid d \in A_{-\mathsf{n}}\}$ is an embedding of $\mathcal{A}_{-\mathsf{n}}$ into $\mathcal{E}$, in which case $(g \circ e) \in \hom_{\mathrm{S}}(\mathcal{A}_{-\mathsf{n}}, \mathcal{D}/\partial(\mathcal{D}))$, and so Corollary 2.3, Lemmas 3.2, 3.3 and Remark 3.6 complete the argument. □

Theorem 4.29. *$\mathcal{A}_{-\mathsf{n}}$ is a model of any $\sim$-paraconsistent extension of C. In particular, $C^{-\mathsf{n}}$ is the greatest $\sim$-paraconsistent extension of C, and so maximally $\sim$-paraconsistent, in which case an extension of C is $\sim$-paraconsistent iff it is a sublogic of $C^{-\mathsf{n}}$.*

Proof: Consider any $\sim$-paraconsistent extension C' of C, in which case $x_1 \notin T \triangleq C'(\{x_0, \sim x_0\})$, and so, by the structurality of C', $\langle \mathfrak{Fm}^{\omega}_{\Sigma}, T\rangle$ is a $\sim$-paraconsistent model of C', and so of C. Then, (2.2) and Lemma 4.28 complete the argument. □

Corollary 4.30 (cf. the reference [Pyn 95b] of [10]). Let $\mathcal{B}$ be a Σ-expansion of $\mathcal{DM}_{4,\not\mathsf{n}}$. Then, the logic of $\mathcal{B}$ is maximally $\sim$-paraconsistent.

Proof: In that case, there is clearly a Σ-expansion $\mathcal{A}'$ of $\mathcal{DM}_4$ such that $\mathcal{B}$ is a submatrix of $\mathcal{A}'$, so Theorem 4.29 completes the argument. □

Corollary 4.30 [with $\Sigma = \Sigma_{\sim}$] covers Dunn's *RM*3 [4] [subsumes Theorem 2.1 of [10]].

Theorem 4.31. *The following are equivalent:*

(i) *C is maximally $\sim$-paraconsistent;*

(ii) $C = C^{-\mathsf{n}}$*;*

(iii) $C^{\mathrm{EM}} \neq C^{-\mathsf{n}}$*;*

(iv) $\{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$ *does not form a subalgebra of* $\mathfrak{A}$*;*

(v) C^{EM} *is not $\sim$-paraconsistent;*

(vi) C^{EM} *is not maximally $\sim$-paraconsistent;*

(vii) *any $\sim$-paraconsistent extension of C is paracomplete;*

(viii) *no expansion of LP is an extension of C;*

(ix) C^{EM} *is not an expansion of LP;*

(x) $C^{-\mathsf{n}}$ *is paracomplete;*

(xi) $\mathcal{A}$ *has no proper $\sim$-paraconsistent submatrix;*

(xii) *any $\sim$-paraconsistent submatrix of $\mathcal{A}$ is paracomplete;*

(xiii) C^{EM} *is either $\sim$-classical, if C is $\sim$-subclassical, or inconsistent, otherwise;*

(xiv) *any consistent non-$\sim$-classical extension of C is paracomplete.*

PROOF: First, (i)⇒(ii) is by (2.2). Next, both (ii)⇒(i), (vi)⇒(iii) and (x)⇒(vii) are by Theorem 4.29. Moreover, (ii)⇒(x) is by the paracompleteness of C. In addition, (xiii)⇒(xiv) is by Theorems 4.20 and 4.21, because any consistent logic with theorems is inferentially consistent.

Further, assume $\{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, in which case $A_{-\mathsf{n}} = A_{\not\mathsf{n}}$, and so, by Lemma 4.25, $C^{\mathrm{EM}} = C^{-\mathsf{n}}$ is an expansion of LP. Thus, both (iii)⇒(iv) and (ix)⇒(iv) hold.

Conversely, assume (iv) holds. Let S be the set of all non-paracomplete consistent submatrices of $\mathcal{A}$, in which case, by Corollary 2.9, C^{EM} is defined by S. Consider any $\mathcal{B} \in \mathsf{S}$. Since it is not paracomplete, we have $\mathsf{n} \notin B$, in which case $\mathsf{f} \in B$, for it is consistent, and so $\mathsf{t} = {\sim}^{\mathfrak{A}}\mathsf{f} \in B$. Therefore, by (iv), $\mathsf{b} \notin B$, for $\{\mathsf{f}, \mathsf{t}\} \subseteq B \not\ni \mathsf{n}$. Thus, $B = \{\mathsf{f}, \mathsf{t}\}$. In this way, by Theorem 4.20, either $\mathsf{S} = \{\mathcal{B}\}$, in which case C^{EM} is $\sim$-classical, if C is $\sim$-subclassical, or $\mathsf{S} = \varnothing$, in which case C^{EM} is inconsistent, otherwise. Thus, (xiii) holds.

Furthermore, (xii)⇔(xi)⇔(x)⇔(iv)⇒(ii) are immediate.

Finally, (ix/viii) is a particular case of (viii/vii). Likewise, (vi) is a particular case of (v), while (v) is a particular case of (vii), whereas (vii) is a particular case of (xiv), as required. □

It is Theorem 4.31(i)⇔(iv) that provides a quite useful algebraic criterion of the maximal $\sim$-paraconsistency of C inherited by its four-valued expansions, in view of Corollary 4.9, applications of which are demonstrated in Section 5.

Combining Lemmas 4.23, 4.24, Theorems 4.20, 4.31 and (2.2), we immediately get:

COROLLARY 4.32. Suppose C is $\sim$-subclassical and $\mathfrak{A}$ is regular. Then, C is not maximally $\sim$-paraconsistent and $C^{\mathrm{PC}}(\varnothing) = C^{\mathrm{EM}}(\varnothing)$.

Concluding this subsection, we explore the least non-$\sim$-paraconsistent extension $C^{\mathrm{EM+NP}}$ of C^{EM}, viz., that which is relatively axiomatized by the *Ex Contradictione Quodlibet* rule:

$$\{x_0, {\sim}x_0\} \vdash x_1. \tag{4.9}$$

LEMMA 4.33. *Let I be a finite set, $\overline{\mathcal{C}} \in \{\mathcal{A}, \langle\mathfrak{A}, \{\mathsf{t}, \mathsf{n}\}\rangle, \langle\mathfrak{A}, \{\mathsf{t}\}\rangle\}^I$ and $\mathcal{B}$ a consistent non-$\sim$-paraconsistent submatrix of $\prod_{i\in I} \mathcal{C}_i$. Then, $\hom(\mathcal{B}, \langle\mathfrak{A}, \{\mathsf{t}\}\rangle) \neq \varnothing$.*

Proof: Consider the following complementary cases:

· $\mathcal{B}$ is truth-empty.
Take any $i \in I \neq \varnothing$, for $\mathcal{B}$ is consistent. Then, $h \triangleq (\pi_i \restriction B) \in \hom(\mathfrak{B}, \mathfrak{A})$. Moreover, $D^{\mathcal{B}} = \varnothing \subseteq h^{-1}[\{\mathsf{t}\}]$. Hence, $h \in \hom(\mathcal{B}, \langle \mathfrak{A}, \{\mathsf{t}\} \rangle)$, as required.

· $\mathcal{B}$ is truth-non-empty.
Then, $B \subseteq A^I$ is finite, for both I and A are so, and so is $D^{\mathcal{B}} \subseteq B$. Hence, as $\mathfrak{B} \restriction \Sigma_+$ is a lattice, $D^{\mathcal{B}}$, being non-empty, has a least element a, in which case, as $\mathcal{B}$ is consistent but not $\sim$-paraconsistent, $\sim^{\mathfrak{B}} a \notin D^{\mathcal{B}}$, and so there is some $i \in I$, in which case $h \triangleq (\pi_i \restriction B) \in \hom(\mathcal{B}, \mathcal{C}_i)$, such that $h(\sim^{\mathfrak{B}} a) \notin D^{\mathcal{C}_i}$. If there was some $b \in D^{\mathcal{B}}$ such that $h(b) \neq \mathsf{t}$, we would have $\mathcal{C}_i \in \{\mathcal{A}, \langle \mathfrak{A}, \{\mathsf{t}, \mathsf{n}\} \rangle\}$ and $(\{\mathsf{b}, \mathsf{n}\} \cap D^{\mathcal{C}_i}) \ni h(b) \leqslant^{\mathfrak{A}} h(a) \leqslant^{\mathfrak{A}} h(b)$, for $D^{\mathcal{B}} \ni a \leqslant^{\mathfrak{B}} b$, in which case we would get $h(a) = h(b)$, and so $h(\sim^{\mathfrak{B}} a) = \sim^{\mathfrak{A}} h(a) = \sim^{\mathfrak{A}} h(b) = h(b) \in D^{\mathcal{C}_i}$. Thus, $h \in \hom(\mathcal{B}, \langle \mathfrak{A}, \{\mathsf{t}\} \rangle)$, as required. ☐

Corollary 4.34. Let I be a finite set, $\overline{\mathcal{C}} \in \{\mathcal{A}, \langle \mathfrak{A}, \{\mathsf{t}, \mathsf{n}\} \rangle, \langle \mathfrak{A}, \{\mathsf{t}\} \rangle\}^I$ and $\mathcal{B}$ a consistent non-$\sim$-paraconsistent non-paracomplete submatrix of $\prod_{i \in I} \mathcal{C}_i$. Then, $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$ and $\hom(\mathcal{B}, \mathcal{A}_{\not\mathsf{n}\not\mathsf{b}}) \neq \varnothing$.

Proof: Then, by Lemma 4.33, there is some $h \in \hom(\mathcal{B}, \langle \mathfrak{A}, \{\mathsf{t}\} \rangle) \neq \varnothing$, in which case $D \triangleq (\operatorname{img} h)$ forms a subalgebra of $\mathfrak{A}$, and so $h \in \hom^{\mathrm{S}}(\mathcal{B}, \mathcal{D})$, where $\mathcal{D} \triangleq (\langle \mathfrak{A}, \{\mathsf{t}\} \rangle \restriction D)$. Hence, by (2.3), $\mathcal{D}$ is not paracomplete. Therefore, as (4.8) is true in $\langle \mathfrak{A}, \{\mathsf{t}\} \rangle$ under neither $[x_1/\mathsf{b}]$ nor $[x_1/\mathsf{n}]$, we have $(D \cap \{\mathsf{b}, \mathsf{n}\}) = \varnothing$. On the other hand, $\mathcal{D}$, being non-paracomplete, is truth-non-empty, for $D \neq \varnothing$. Therefore, $\mathsf{t} \in D$, in which case $\mathsf{f} = \sim^{\mathfrak{A}} \mathsf{t} \in D$, and so $D = \{\mathsf{f}, \mathsf{t}\}$, in which case $\mathcal{D} = (\mathcal{A} \restriction D) = \mathcal{A}_{\not\mathsf{n}\not\mathsf{b}}$, as required. ☐

Theorem 4.35. *Suppose C is [not] maximally $\sim$-paraconsistent. Then, $C^{\mathrm{EM+NP}}$ is consistent iff C is $\sim$-subclassical, in which case $C^{\mathrm{EM+NP}}$ is defined by $[\mathcal{A}_{\not\mathsf{n}} \times] \mathcal{A}_{\not\mathsf{n}\not\mathsf{b}}$.*

Proof: First, assume $C^{\mathrm{EM+NP}}$ is consistent, in which case $x_0 \notin T \triangleq C^{\mathrm{EM+NP}}(\varnothing)$, while, by the structurality of $C^{\mathrm{EM+NP}}$, $\langle \mathfrak{Fm}^{\omega}_{\Sigma}, T \rangle$ is a model of $C^{\mathrm{EM+NP}}$ (in particular, of C), and so is its consistent finitely-generated submatrix $\mathcal{B} \triangleq \langle \mathfrak{Fm}^{1}_{\Sigma}, T \cap \mathrm{Fm}^{1}_{\Sigma} \rangle$, in view of (2.2). Hence, by Lemma 2.7, there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}(\mathcal{A})^I$, some subdirect product $\mathcal{D}$ of it, in which case this is a submatrix of $\mathcal{A}^I$, and some $h \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D}, \Re(\mathcal{B}))$, in which case, by (2.2), $\mathcal{D}$ is a consistent model of $C^{\mathrm{EM+NP}}$, so it is neither

$\sim$-paraconsistent nor paracomplete. Thus, by Corollary 4.34 and Theorem 4.20, C is $\sim$-subclassical.

Conversely, assume C is $\sim$-subclassical. Consider the following complementary cases:

· C is maximally $\sim$-paraconsistent.
Then, by Theorems 4.20 and 4.31(i)$\Rightarrow$(v,xiii) $C^{\mathrm{EM+NP}} = C^{\mathrm{EM}} = C^{\mathrm{PC}}$ is defined by the consistent $\mathcal{A}_{\not{\mathsf{n}}\not{\mathsf{b}}}$, and so, in particular, is consistent, as required.

· C is not maximally $\sim$-paraconsistent.
Then, by Theorem 4.31(iii/iv)$\Rightarrow$(i), C^{EM} is defined by $\mathcal{A}_{-\mathsf{n}} = \mathcal{A}_{\not{\mathsf{n}}}$. Moreover, by Theorem 4.20, $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, and so of $\mathfrak{A}_{\not{\mathsf{n}}}$, in which case $\mathcal{A}_{\not{\mathsf{n}}\not{\mathsf{b}}}$ is a submatrix of $\mathcal{A}_{\not{\mathsf{n}}}$, and so, by (2.2), $\mathcal{B} \triangleq (\mathcal{A}_{\not{\mathsf{n}}} \times \mathcal{A}_{\not{\mathsf{n}}\not{\mathsf{b}}})$ is a model of C^{EM}. Moreover, $\{a, \sim^{\mathfrak{A}} a\} \subseteq \{\mathsf{t}\}$, for no $a \in \{\mathsf{f},\mathsf{t}\}$. Therefore, $\mathcal{B}$ is not $\sim$-paraconsistent, so it is a model of $C^{\mathrm{EM+NP}}$. Conversely, consider any finite set I, any $\overline{\mathcal{C}} \in \mathbf{S}(\mathcal{A}_{\not{\mathsf{n}}})^I$ and any subdirect product $\mathcal{D} \in \mathrm{Mod}(C^{\mathrm{EM+NP}})$ of $\overline{\mathcal{C}}$, in which case $\mathcal{D}$ is a non-$\sim$-paraconsistent non-paracomplete submatrix of $\mathcal{A}^I$. Put $J \triangleq \hom(\mathcal{D},\mathcal{B})$. Consider any $a \in (D \setminus D^{\mathcal{D}})$, in which case $\mathcal{D}$ is consistent, and so, by Corollary 4.34, there is some $g \in \hom(\mathcal{D},\mathcal{A}_{\not{\mathsf{n}}\not{\mathsf{b}}}) \neq \varnothing$. Moreover, there is some $i \in I$, in which case $f \triangleq (\pi_i \restriction D) \in \hom(\mathcal{D},\mathcal{A}_{\not{\mathsf{n}}})$, such that $f(a) \notin D^{\mathcal{A}_{\not{\mathsf{n}}}}$. Then, $h \triangleq (f \times g) \in J$ and $h(a) \notin D^{\mathcal{B}}$. In this way, $(\prod \Delta_J) \in \hom_{\mathrm{S}}(\mathcal{D},\mathcal{B}^J)$. Thus, by (2.2) and Theorem 2.8, $C^{\mathrm{EM+NP}}$ is finitely-defined by the consistent six-valued $\mathcal{B}$, and so is consistent and, being finitary, for both (4.8) and (4.9) are finitary, while the four-valued C is finitary, is defined by $\mathcal{B}$, as required. □

Corollary 4.36 (cf. the last assertion of Theorem 4.13 of [12] for the case $\Sigma = \Sigma_{\sim}$). Let $\mathcal{B}$ be a Σ-expansion of $\mathcal{DM}_{4,\not{\mathsf{n}}}$. Suppose $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{B}$. Then, the extension of the logic of $\mathcal{B}$ relatively axiomatized by (4.9) is defined by $\mathcal{B} \times (\mathcal{B} \restriction \{\mathsf{f},\mathsf{t}\})$.

Proof: In that case, there is clearly a Σ-expansion $\mathcal{A}'$ of $\mathcal{DM}_4$ such that $\mathcal{B}$ is a submatrix of $\mathcal{A}'$, so Theorems 4.20, 4.31 and 4.35 complete the argument. □

This is equally applicable to, in particular, *RM*3 [4] and subsumes specific results concerning purely-implicative expansions of $C_{[\mathrm{B}]\mathrm{DB}}$ obtained *ad hoc* in [14] (cf. the last paragraph of Subsection 5.3).

5. Miscellaneous examples

We entirely follow notations of the previous sections.

5.1. Classically-negative expansions

Here, it is supposed that Σ contains a unary connective $\neg$ (classical negation), while $\neg^{\mathfrak{A}}\langle i, j\rangle \triangleq \langle 1-i, 1-j\rangle$, for all $i, j \in 2$, in which case $\neg^{\mathfrak{A}}\langle k, 1-k\rangle = \langle 1-k, k\rangle$, for each $k \in 2$, and so $\neg^{\mathfrak{A}}$ is not regular, for $\mathsf{b} \not\sqsubseteq \mathsf{n} \sqsubseteq \mathsf{b}$. Then, $\{\mathsf{f}, \mathsf{t}\}$ is the only proper subset of A which may form a subalgebra of $\mathfrak{A}$. Thus, by Theorems 4.16, 4.20, 4.26 and 4.31, we have:

Corollary 5.1. *C*:

(i) has no, if it is not $\sim$-subclassical, in which case it is maximal, and, otherwise (in particular, when $\Sigma = (\Sigma_{\sim[01]} \cup \{\neg\})$), a unique proper consistent axiomatic extension, in which case this is equal to $C^{\mathrm{PC}} = C^{\mathrm{EM}}$;

(ii) is maximally $\sim$-paraconsistent.

This provides an application of the "non-regular" particular case of Theorem 4.26. (Another one is provided by the next subsection.) On the other hand, $\mathcal{A}$ is $(\neg x_0 \vee x_1)$-implicative. Therefore, in view of Remark 3.6, Corollary 5.1(i) (but the maximality reservation) equally ensues from Theorem 3.5. After all, Corollary 5.1(ii) provides examples of maximally paraconsistent *four*-valued logics. (Others are provided by the next subsection.)

5.2. Bilattice expansions

Here, it is supposed that Σ contains binary connectives $\sqcap$ and $\sqcup$ (*knowledge* conjunction and disjunction, respectively), while

$$(\langle i, j\rangle(\sqcap/\sqcup)^{\mathfrak{A}}\langle k, l\rangle) \triangleq \langle(\min/\max)(i,k), (\max/\min)(j,l)\rangle,$$

for all $i, j, k, l \in 2$ (cf., e.g., [11]), in which case $(\mathsf{f}(\sqcap/\sqcup)^{\mathfrak{A}}\mathsf{t}) = (\mathsf{n}/\mathsf{b})$, and so, since any non-one-element subalgebra of $\mathfrak{DM}_4$ contains both f and t, $\mathfrak{A}$ has no proper non-one-element subalgebra. Hence, by Theorems 4.16, 4.26 and 4.31, we have:

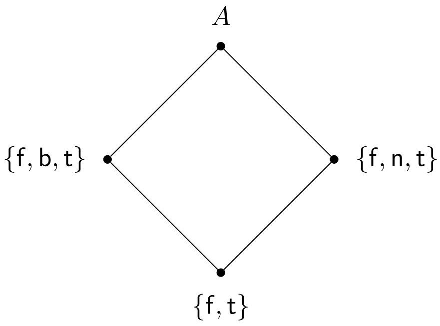

Figure 1. The poset $\mathbf{S}_*(\mathcal{A})$.

Corollary 5.2. C is inferentially maximal, and so both has no proper consistent axiomatic extension and is maximally $\sim$-paraconsistent.

This provides both a one more application of the "non-regular" particular case of Theorem 4.26 and more examples of maximally paraconsistent *four*-valued logics. Moreover, it is bilattice expansions that justify studying the maximality issue within the framework of FDE expansions.

5.3. Implicative expansions

Here, it is supposed that Σ contains a binary connective $\supset$ (implication), while:

$$(\vec{a} \supset^{\mathfrak{A}} \vec{b}) \triangleq \begin{cases} \vec{b} & \text{if } a_0 = 1, \\ \mathsf{t} & \text{otherwise,} \end{cases}$$

for all $\vec{a}, \vec{b} \in 2^2$ (cf. [11]), in which case $\mathcal{A}$ is $\supset$-implicative, while $(\mathsf{f} \supset^{\mathfrak{A}} \mathsf{f}) = \mathsf{t}$, whereas $(\mathsf{b} \supset^{\mathfrak{A}} \mathsf{f}) = \mathsf{f}$, and so $\supset^{\mathfrak{A}}$ is not regular, for $\mathsf{t} \not\sqsubseteq \mathsf{f} \sqsubseteq \mathsf{b}$. From now on, it is supposed that $\Sigma = (\Sigma_{\sim[01]} \cup \{\supset\})$ (the opposite case is considered in a similar way *ad hoc*, depending upon which of the four subsets of A depicted at Figure 1 form subalgebras of $\mathfrak{A}$). Moreover, submatrices of $\mathcal{A}$ are identified with the carriers of their underlying algebras. Then, since $\mathcal{DM}_4 \restriction \{\mathsf{b}\}$ is not consistent, while $(\mathsf{n} \supset^{\mathfrak{A}} \mathsf{n}) = \mathsf{t} \neq \mathsf{n}$, in which case $\{\mathsf{n}\}$ does not form a subalgebra of $\mathfrak{A}$, the poset $\mathbf{S}_*(\mathcal{A})$ forms the diamond depicted at Figure 1, so, in particular, by Theorems 4.16, 4.20 and 4.31, we have:

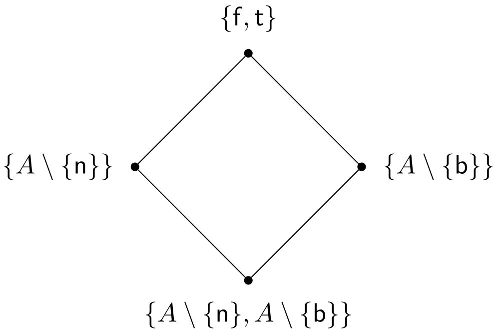

Figure 2. Proper consistent axiomatic extensions of C.

Corollary 5.3. C is $\sim$-subclassical but not maximal(ly $\sim$-paraconsistent).

Note that

$$\sim x_1 \supset (x_1 \supset (x_2 \vee \sim x_2)) \tag{5.1}$$

is true in $\{\{\mathsf{f},\mathsf{n},\mathsf{t}\},\{\mathsf{f},\mathsf{b},\mathsf{t}\}\}$ but is not true in $\mathcal{A}$ under $[x_1/\mathsf{b},x_2/\mathsf{n}]$. Moreover,

$$\sim x_1 \supset (x_1 \supset x_0) \tag{5.2}$$

is true in $\{\mathsf{f},\mathsf{n},\mathsf{t}\}$ but is not true in $\{\mathsf{f},\mathsf{b},\mathsf{t}\}$ under $[x_1/\mathsf{b},x_0/\mathsf{f}]$. Finally, (4.8) is satisfied in $\{\mathsf{f},\mathsf{b},\mathsf{t}\}$ but is not satisfied in $\{\mathsf{f},\mathsf{n},\mathsf{t}\}$ under $[x_1/\mathsf{n}]$. In this way, by Theorem 3.5 and Remark 3.6, we eventually get:

Corollary 5.4. Proper consistent axiomatic extensions of C (given by defining matrix anti-chains) form the diamond depicted at Figure 2 and are relatively axiomatized as follows (actually, according to the constructive proof of Lemma 3.4):

$$\begin{array}{rcl}
\{A\setminus\{\mathsf{n}\},A\setminus\{\mathsf{b}\}\} & : & (5.1),\\
\{A\setminus\{\mathsf{b}\}\} & : & (5.2),\\
\{A\setminus\{\mathsf{n}\}\} & : & (4.8),\\
\{\{\mathsf{f},\mathsf{t}\}\} & : & \{(5.2),(4.8)\}.
\end{array}$$

This, in particular, shows that the optional precondition in the formulation of Theorem 4.26 is essential for the uniqueness of a proper consistent axiomatic extension of C.

Concluding this discussion, recall that the [four-element chain] lattice of *all* extensions of $C^{[\mathrm{EM}]}$ [being a definitional copy of Dunn's $RM3$ [4] in the "unbounded" case] has been found in [14] – taking the general preliminary part of [12] into account – with using an equally automated method but as for merely defining matrices. However, the mentioned study does not at all subsume Corollary 5.4 because of not implying the fact that there is no more proper consistent axiomatic extension of C other than the four ones depicted at Figure 2. This goes without saying that the present study has provided relative axiomatizations *quite effectively.*

6. Conclusions

Aside from the general results and their numerous *generic* illustrative applications, the present paper demonstrates a special value of the conception of equality determinant studied in [13].

And what is more, the methodological algebraic result of Theorem 3.8, in its turn, based upon the apparatus of equality determinant well-advanced in [13], has found more applications within the general topic of FDE expansions, being however beyond the scopes of the present paper and going to be discussed elsewhere.

In general, the topic of [extensions of] expansions of Dunn-Belnap's four-valued logic is too inexhaustible to be studied within a single paper *comprehensively.* The present paper constitutes just a first part of it. Others are going to be presented elsewhere.

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