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Lew Gordeev*
Edward Hermann Haeusler

PROOF COMPRESSION AND NP VERSUS PSPACE II[1]

Abstract

We upgrade [3] to a complete proof of the conjecture NP = PSPACE that is known as one of the fundamental open problems in the mathematical theory of computational complexity; this proof is based on [2]. Since minimal propositional logic is known to be PSPACE complete, while PSPACE to include NP, it suffices to show that every valid purely implicational formula ρ has a proof whose weight (= total number of symbols) and time complexity of the provability involved are both polynomial in the weight of ρ. As is [3], we use proof theoretic approach. Recall that in [3] we considered any valid ρ in question that had (by the definition of validity) a "short" tree-like proof π in the Hudelmaier-style cutfree sequent calculus for minimal logic. The "shortness" means that the height of π and the total weight of different formulas occurring in it are both polynomial in the weight of ρ. However, the size (= total number of nodes), and hence also the weight, of π could be exponential in that of ρ. To overcome this trouble we embedded π into Prawitz's proof system of natural deductions containing single formulas, instead of sequents. As in π, the height and the total weight of different formulas

*Corresponding author.

[1]**Editorial remark.** The subeditor dealing with this paper (Peter Schroeder-Heister) and the two reviewers were not able to check proofs in all detail and therefore cannot fully confirm their correctness. However, in view of the importance of the results claimed and the originality of the logical proof methods employed, and in accordance with the aim of the journal as a forum for the wide dissemination of original results by rapid publication, they agree that the paper should be available to the scientific community in published form to enable further discussion.

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of the resulting tree-like natural deduction ∂_1 were polynomial, although the size of ∂_1 still could be exponential, in the weight of ρ. In our next, crucial move, ∂_1 was deterministically compressed into a "small", although multipremise, dag-like deduction ∂ whose horizontal levels contained only mutually different formulas, which made the whole weight polynomial in that of ρ. However, ∂ required a more complicated verification of the underlying provability of ρ. In this paper we present a nondeterministic compression of ∂ into a desired standard dag-like deduction ∂_0 that deterministically proves ρ in time and space polynomial in the weight of ρ.[2] Together with [3] this completes the proof of NP = PSPACE.

Natural deductions are essential for our proof. Tree-to-dag horizontal compression of π merging equal sequents, instead of formulas, is (possible but) not sufficient, since the total number of different sequents in π might be exponential in the weight of ρ – even assuming that all formulas occurring in sequents are subformulas of ρ. On the other hand, we need Hudelmaier's cutfree sequent calculus in order to control both the height and total weight of different formulas of the initial tree-like proof π, since standard Prawitz's normalization although providing natural deductions with the subformula property does not preserve polynomial heights. It is not clear yet if we can omit references to π even in the proof of the weaker result NP = coNP.

Keywords: Natural deduction, sequent calculus, minimal logic, computational complexity.

1. Introduction

In [3] we presented a dag-like version of Prawitz's [9] tree-like natural deduction calculus for minimal logic, $NM_{\rightarrow}$, and left open a problem of computational complexity of the dag-like provability involved ([3, Problem 22]). In this paper we show a solution that proves the conjecture **NP = PSPACE**. To explain it briefly first consider standard notion of provability. Recall that our basic deduction calculus $NM_{\rightarrow}$ includes two basic inferences

$$\boxed{(\rightarrow I) : \dfrac{\begin{array}{c}[\alpha]\\ \vdots\\ \beta\end{array}}{\alpha \rightarrow \beta}}\ , \quad \boxed{(\rightarrow E) : \dfrac{\alpha \qquad \alpha \rightarrow \beta}{\beta}}$$

[2]It is doubtful that ∂ is convertible into ∂_0 by a polynomial-time deterministic TM.

and one auxiliary repetition rule $\boxed{(R) : \dfrac{\alpha}{\alpha}}$, where $[\alpha]$ in $(\rightarrow I)$ indicates that all α-leaves occurring above β-node exposed are *discharged* assumptions.

DEFINITION 1.1. A given (whether tree- or dag-like) $\mathrm{NM}_{\rightarrow}$-deduction ∂ *proves* its root-formula ρ (abbr.: $\partial \vdash \rho$) iff every maximal thread connecting the root with a leaf labeled α is closed (= discharged), i.e. it contains a $(\rightarrow I)$ with conclusion $\alpha \rightarrow \beta$, for some β. A purely implicational formula ρ is *valid in minimal logic* iff there exists a tree-like $\mathrm{NM}_{\rightarrow}$-deduction ∂ that proves ρ;[3] such ∂ is called a *proof* of ρ.

Remark 1.2. Tree-like constraint in the definition of validity is inessential.

That is, for any dag-like $\partial \in \mathrm{NM}_{\rightarrow}$ with root-formula ρ, if $\partial \vdash \rho$ then ρ is valid in minimal logic. Because any given dag-like ∂ can be unfolded into a tree-like deduction ∂' by straightforward thread-preserving bottom-up recursion. To this end every node $x \in \partial$ with $n > 1$ distinct conclusions has to be replaced by n distinct nodes $x_1, \cdots, x_n \in \partial'$ with corresponding single-node conclusions and identical premises of x. This operation obviously preserves the closure of threads, i.e. $\partial \vdash \rho$ infers $\partial' \vdash \rho$.

Formal verification of the assertion $\partial \vdash \rho$ is simple, as follows – whether for tree-like or generally dag-like ∂. Every node $x \in \partial$ is assigned, by descending recursion, a set of assumptions $A(x)$ such that:

1. $A(x) := \{\alpha\}$ if x is a leaf labeled α,
2. $A(x) := A(y)$ if x is the conclusion of (R) with premise y,
3. $A(x) := A(y) \setminus \{\alpha\}$ if x is the conclusion of $(\rightarrow I)$ with label $\alpha \rightarrow \beta$ and premise y,
4. $A(x) := A(y) \cup A(z)$ if x is the conclusion of $(\rightarrow E)$ with premises y, z.

This easily yields

LEMMA 1.3. *Let* $\partial \in \mathrm{NM}_{\rightarrow}$ *(whether tree- or dag-like). Then* $\partial \vdash \rho \Leftrightarrow A(r) = \emptyset$ *holds with respect to standard set-theoretic interpretations of* $\cup$

[3]Equivalently: $\rho$is *valid in minimal logic* iff it is deducible in Hilbert-style calculus with axioms $\alpha \rightarrow (\beta \rightarrow \alpha)$, $(\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma))$ and inference $(\rightarrow E)$, also known as *modus (ponendo) ponens*; the equivalence follows from corresponding deduction theorem.

and $\setminus$ *in* $A(r)$*, where* r *and* ρ *are the root and root-formula of* ∂*, respectively. Moreover,* $A(r) \stackrel{?}{=} \emptyset$ *is verifiable by a deterministic TM in* $|\partial|$*-polynomial time, where by* $|\partial|$ *we denote the weight of (i.e. total number of symbols occurring in)* ∂*.*[4]

Now let us upgrade $\mathrm{NM}_{\to}$ to $\mathrm{NM}^{\flat}_{\to}$ by adding a new *separation* rule (S)

$$\boxed{(\to S): \frac{\overbrace{\alpha \quad \cdots \quad \alpha}^{n\ times}}{\alpha} \ (n \text{ arbitrary})}$$

whose identical premises are understood disjunctively: "*if at least one premise is proved then so is the conclusion*" (in contrast to ordinary conjunctive inference: "*if all premises are proved then so is the conclusion*"). Note that in dag-like deductions the nodes might have several conclusions (unlike in tree-like ones). The modified assignment A in $\mathrm{NM}^{\flat}_{\to}$ (that works in both tree-like and dag-like cases) is defined by adding to old recursive clauses 1–4 (see above) a new clause 5 with new separation symbol Ⓢ:

5. $A(x) = Ⓢ(A(y_1), \cdots, A(y_n))$ if x is the conclusion of (S) with premises $y_1, \cdots, y_n$.

Claim 1.4. For any dag-like deduction $\partial \in \mathrm{NM}^{\flat}_{\to}$ whose root r is labeled ρ, ρ is valid in minimal logic, provided that $A(r)$ reduces to $\emptyset$ (abbr.: $A(r) \rhd \emptyset$) by standard set-theoretic interpretations of $\cup$, $\setminus$ and nondeterministic disjunctive valuation $Ⓢ(t_1, \cdots, t_n) := t_i$, for any chosen $i \in \{1, \cdots, n\}$. Moreover, the assertion $A(r) \rhd \emptyset$ (that is also referred to as '∂ *proves* ρ') can be confirmed by a nondeterministic TM in $|\partial|$-polynomial time.

This claim reduces to its trivial $\mathrm{NM}_{\to}$ case (see above). For suppose that $A(r) \rhd \emptyset$ holds with respect to a successive nondeterministic valuation of the occurrences Ⓢ. This reduction determines a successive ascending (i.e. bottom-up) thinning of ∂ that results in a "cleansed" (S)-free dag-like deduction $\partial_0 \in \mathrm{NM}^{\flat}_{\to}$, while $A(r) \rhd \emptyset$ in ∂ implies $A(r) = \emptyset$ in ∂_0. Since (S) does not occur in ∂_0 anymore, we have $\partial_0 \in \mathrm{NM}_{\to}$. By previous considerations with regard to $\mathrm{NM}_{\to}$ we conclude that ρ is valid in minimal logic, which can be confirmed in $|\partial|$-polynomial time, as required.

[4]The latter is completely analogous to the well-known polynomial-time decidability of the circuit value problem (see also Appendix).

Since minimal logic is PSPACE complete ([11, 12]), in order to arrive at the desired conclusion **NP = PSPACE** it will suffice to show that for any valid ρ there is a modified dag-like deduction $\partial \in \mathrm{NM}^{\flat}_{\rightarrow}$ of ρ satisfying $A(r) \rhd \emptyset$, and hence a dag-like deduction $\partial_0 \in \mathrm{NM}_{\rightarrow}$ satisfying $A(r) = \emptyset$, whose size and maximal formula weight are polynomial in $|\rho|$. But this is a consequence of [3] that formalized basic theory of dag-like deducibility in question (elaborated by the first author). For in [3] we presented a deterministic tree-to-dag *horizontal compression* of a given "short" tree-like deduction of ρ in $\mathrm{NM}_{\rightarrow}$ that is obtained by embedding a derivation of ρ in a Hudelmaier-style [5] cutfree sequent calculus. It resulted in a suitable $|\rho|$-polynomial dag-like *deduction frame* together with a $|\rho|$-exponential *locally coherent* set of maximal threads, in the multipremise expansion of $\mathrm{NM}_{\rightarrow}$ (called $\mathrm{NM}^{*}_{\rightarrow}$; multiple premises involved arise by merging equal conclusions of different rules). In this paper we observe that such a pair determines a deduction in $\mathrm{NM}^{\flat}_{\rightarrow}$ that admits a *fundamental set of chains* (see below). Moreover, we show that such $\mathrm{NM}^{\flat}_{\rightarrow}$-deduction is convertible by the appropriate nondeterministic dag-to-dag *horizontal cleansing* into the required $\mathrm{NM}_{\rightarrow}$ deduction satisfying $A(r) = \emptyset$.[5]

1.1. Recollection of [3]

Recall that ρ is called *dag-like provable* in $\mathrm{NM}^{*}_{\rightarrow}$ iff there is a *locally correct* (with respect to inferences of $\mathrm{NM}^{*}_{\rightarrow}$) labeled regular dag $\widetilde{D} = \langle D, \mathrm{S}, \ell^{\mathrm{F}} \rangle$ (that may have arbitrary many premises and/or conclusions) with root-formula ρ, together with a *locally coherent* mapping $G : \overrightarrow{\mathrm{E}}(D) \rightarrow \{0, 1\}$ that determines a set of threads that confirms alleged validity of ρ, where $\overrightarrow{\mathrm{E}}(D)$ denotes the set of edge-chains in D (see reference in Lemma 5 below). Such $\widetilde{D}$ and a pair $\partial = \left\langle \widetilde{D}, G \right\rangle$ are called respectively a *deduction frame* (or just $\mathrm{NM}^{*}_{\rightarrow}$*-deduction*) and a *dag-like proof* of ρ in $\mathrm{NM}^{*}_{\rightarrow}$.[6] In [3] we proved that the latter notion of dag-like provability of ρ is equivalent to the validity of ρ in minimal logic (cf. Definition 1). Without loss of generality we assume that $\widetilde{D}$ is horizontally compressed, i.e. $\ell^{\mathrm{F}}(x) \neq \ell^{\mathrm{F}}(y)$ for all $x \neq y$ on the same level in D, and the weight of $\widetilde{D}$ is polynomial in $|\rho|$ (see [3] and below). Such compression runs by bottom-up recursion

[5]This yields a "short" certificate for the local coherence statement that itself requires exponentially many bits to even *describe* (cf. [1, 4.3.2]).

[6]Here and below basic notions and notations are imported from [3].

on the height of a given "short" tree-like deduction with root-formula ρ by successively merging all nodes with identical formulas occurring in the corresponding horizontal sections; thus the weight of resulting dag-like deduction is polynomial in $|\rho|$, since so are the height of, and total weight of different formulas occurring in, the "short" tree-like $\mathrm{NM}_{\rightarrow}$-input in question ([3]: Ch. 3). We noticed that the local correctness of $\widetilde{D}$ is verifiable in $|\rho|$-polynomial time, whereas the local coherence of G has no obvious low-complexity upper bound, as $\overrightarrow{\mathrm{E}}(D)$ is generally exponential (cf. footnote 5). The currently proposed upgrade is based on the *fundamental sets of threads*, instead of G and $\overrightarrow{\mathrm{E}}(D)$, as follows.

1.2. Upgrade in $\mathrm{NM}^{*}_{\rightarrow}$

Let $\widetilde{D} = \langle D, \mathrm{S}, \ell^{\mathrm{F}} \rangle$ be a given locally correct deduction frame with root-formula $\rho = \ell^{\mathrm{F}}(r)$, $\mathrm{K}(D)$ be the set of maximal ascending chains (also called threads) consisting of nodes (vertices) $u \in \mathrm{V}(D)$ connecting root r with leaves. A given set $\mathcal{F} \subset \mathrm{K}(D)$ is a *fundamental set of threads* (abbr.: *fst*) in $\widetilde{D}$ if the following three conditions are satisfied, where for any $\Theta = \left[r = x_0, \cdots, x_{h(D)}\right] \in \mathrm{K}(D)$ and $i \leq h(D)$ we let $\Theta\restriction_{x_i} := [x_0, \cdots, x_i]$.

1. $\mathcal{F}$ is dense in D, i.e. $(\forall u \in \mathrm{V}(D))\,(\exists \Theta \in \mathcal{F})\,(u \in \Theta)$.

2. Every $\Theta \in \mathcal{F}$ is closed, i.e. its leaf-formula $\ell^{\mathrm{F}}\left(x_{h(D)}\right)$ is discharged in Θ.

3. $\mathcal{F}$ preserves $(\rightarrow E)$, i.e.

$$\begin{array}{l}(\forall \Theta \in \mathcal{F})\,(\forall u \in \Theta)\,(\forall v \neq w \in \mathrm{V}(D)\colon \langle u, v\rangle, \langle u, w\rangle \in \mathrm{E}(D) \wedge v \in \Theta)\\ (\exists \Theta' \in \mathcal{F})\,(w \in \Theta' \wedge \Theta\restriction_u = \Theta'\restriction_u) \qquad\qquad\qquad\qquad .\end{array}$$

Lemma 1.5. *Let $\widetilde{D}$ be as above and suppose that there exists a* fst $\mathcal{F}$ *in $\widetilde{D}$. Then ρ is dag-like provable in $\mathrm{NM}^{*}_{\rightarrow}$.*

Proof: Define $G : \overrightarrow{\mathrm{E}}(D) \rightarrow \{0, 1\}$ by $G(\overrightarrow{e}) := 1$ iff $\left(\exists \overrightarrow{f} \supseteq \overrightarrow{e}\right) \Theta\left[\overrightarrow{f}\right] \in \mathrm{K}(D) \cap \mathcal{F}$, where $\Theta\left[\overrightarrow{f}\right]$ contains all nodes occurring in the canonical thread-expansion of $\overrightarrow{f}$. Then $\partial = \left\langle \widetilde{D}, G \right\rangle$ is a dag-like proof of ρ. The local coherence conditions 1, 2, 4, 5 (cf. [3]: Definition 6) are easily verified. In particular, 4 follows from the third *fst* condition with respect to $\mathcal{F}$. $\square$

LEMMA 1.6. *For any dag-like proof* $\left\langle \widetilde{D}, G \right\rangle$ *of* ρ *there are* $D_0 \subseteq D$, $G_0 : \overrightarrow{\mathrm{E}}(D_0) \to \{0,1\}$, $\mathcal{F} \subset \mathrm{K}(D_0)$ *and a dag-like proof* $\left\langle \widetilde{D_0}, G_0 \right\rangle$ *of* ρ *such that* $\mathcal{F}$ *is a* fst *in* $\widetilde{D_0}$.

PROOF: Let $\mathcal{F} := \{\Theta \in \mathrm{K}(D) : G(\overrightarrow{e}[\Theta]) = 1\}$ for $\overrightarrow{e}[\Theta] := \overrightarrow{e_m} \in \overrightarrow{\mathrm{E}}(D)$ determined by Θ as specified in [3]: Definition 8. It is readily seen that such $\mathcal{F}$ is a *fst* in $\widetilde{D}$. The crucial condition 3 follows directly from the corresponding local coherence condition 4 (cf. [3]: Definition 6). Let $D_0 \subseteq D$ be the minimum sub-dag containing every edge occurring in $\bigcup_{\Theta \in \mathcal{F}} \Theta$ and let $\widetilde{D_0} = \langle D_0, \mathrm{S}, \ell^{\mathrm{F}} \rangle$ be the corresponding sub-frame of $\widetilde{D}$. Obviously $\widetilde{D_0}$ is locally correct. Define $G_0 : \overrightarrow{\mathrm{E}}(D_0) \to \{0,1\}$ as in the previous lema with respect to D_0, instead of D. Then $\partial = \left\langle \widetilde{D_0}, G_0 \right\rangle$ is a dag-like proof of ρ. The crucial density of $\mathcal{F}$ in D_0 obviously follows from definitions of D_0 and G_0, as every edge in D_0 occurs in some thread from $\mathcal{F}$, while for any $\overrightarrow{e} \in \overrightarrow{\mathrm{E}}(D_0)$ we have $G_0(\overrightarrow{e}) = 1$ iff $\Theta[\overrightarrow{e}] \in \mathcal{F}$. $\square$

Together with [3]: Corollaries 15, 20 these lemmata yield

COROLLARY 1.7. Any given ρ is valid in minimal logic iff there exists a pair $\left\langle \widetilde{D}, \mathcal{F} \right\rangle$ such that $\widetilde{D}$ is a locally correct deduction frame with root-formula $\rho = \ell^{\mathrm{F}}(r)$ and $\mathcal{F}$ being a *fst* in $\widetilde{D}$. We can just as well assume that $\widetilde{D}$ is horizontally compressed and its weight is polynomial in that of ρ.

Remark 1.8. We can't afford $\mathcal{F}$ to be polynomial in ρ. However, the existence of $\mathcal{F}$ enables a nondeterministic polytime verification of $A(r) \rhd \emptyset$ in the corresponding modified dag-like formalism, as follows. This collapsing makes the trick.

2. Modified dag-like calculus $\mathrm{NM}^{\flat}_{\to}$

As mentioned above, our modified dag-like deduction calculus, $\mathrm{NM}^{\flat}_{\to}$, includes inference rules $(\to I)$, $(\to E)$, (R), (S) (see Introduction). $(\to I)$, (R) and $(\to E)$ have one and two premises, respectively, whereas (S) has two or more ones. $\mathrm{NM}^{\flat}_{\rightarrow}$-deductions are graphically interpreted as labeled rooted regular dags (abbr.: *redags*, cf. [3]) $\partial = \langle \mathrm{V}(\partial), \mathrm{E}(\partial) \rangle$, whose nodes may have arbitrary many parents (conclusions) – and children (premises), just in the case (S), – if any at all. The nodes $(x, y, z, ...)$ are labeled by ℓ^{F} with purely implicational formulas $(\alpha, \beta, \gamma, \rho, ...)$. For the sake of brevity we'll assume that nodes x are supplied with auxiliary *height numbers* $h(x) \in \mathbb{N}$, while all inner nodes also have special labels $\ell^{\mathrm{N}}(x) \in \{\mathrm{I}, \mathrm{E}, \mathrm{R}, \mathrm{S}\}$ showing the names of the inference rules $(\to I)$, $(\to E)$, (R), (S) with conclusion x. The *roots* and *root-formulas* are always designated r and $\rho := \ell^{\mathrm{F}}(r)$, respectively. The *edges* $\langle x, y \rangle \in \mathrm{E}(\partial) \subset \mathrm{V}(\partial)^2$ are directed upwards (thus r is the lowest node in ∂) in which x and y are called *parents* and *children* of each other, respectively. The *leaves* $\mathrm{L}(\partial) \subseteq \mathrm{V}(\partial)$ are the nodes without children. Tree-like $\mathrm{NM}^{\flat}_{\rightarrow}$-deductions are those ones whose redags are trees (whose nodes have at most one parent).

Definition 2.1. A given $\mathrm{NM}^{\flat}_{\rightarrow}$-deduction ∂ is *locally correct* if conditions 1–2 are satisfied, for arbitrary nodes x, y, z, u.

1. ∂ is regular (cf. [3]), i.e.
   (a) if $\langle x, y \rangle \in \mathrm{E}(\partial)$ then $x \notin \mathrm{L}(\partial)$ and $y \neq r$,
   (b) $h(r) = 0$,
   (c) if $\langle x, y \rangle, \langle x, z \rangle \in \mathrm{E}(\partial)$ then $h(y) = h(z) = h(x) + 1$.
2. ∂ formalizes the inference rules, i.e.
   (a) if $\ell^{\mathrm{N}}(x) = \mathrm{R}$ and $\langle x, y \rangle, \langle x, z \rangle \in \mathrm{E}(\partial)$ then $y = z$ and $\ell^{\mathrm{F}}(y) = \ell^{\mathrm{F}}(x)$ [: rule (R)],
   (b) if $\ell^{\mathrm{N}}(x) = \mathrm{I}$ and $\langle x, y \rangle, \langle x, z \rangle \in \mathrm{E}(\partial)$ then $y = z$ and $\ell^{\mathrm{F}}(x) = \alpha \to \ell^{\mathrm{F}}(y)$ for some (uniquely determined) α [: rule $(\to I)$],
   (c) if $\ell^{\mathrm{N}}(x) = \mathrm{E}$ and $\langle x, y \rangle, \langle x, z \rangle, \langle x, u \rangle \in \mathrm{E}(\partial)$ then $|\{y, z, u\}| = 2$ and if $y \neq z$ then either $\ell^{\mathrm{F}}(z) = \ell^{\mathrm{F}}(y) \to \ell^{\mathrm{F}}(x)$ or else $\ell^{\mathrm{F}}(y) = \ell^{\mathrm{F}}(z) \to \ell^{\mathrm{F}}(x)$ [: rule $(\to E)$],
   (d) if $\ell^{\mathrm{N}}(x) = \mathrm{S}$ and $\langle x, y \rangle \in \mathrm{E}(\partial)$ then $\ell^{\mathrm{F}}(y) = \ell^{\mathrm{F}}(x)$ and $\ell^{\mathrm{N}}(y) \neq \mathrm{S}$ [: rule (S)].

$\mathrm{NM}^{*}_{\rightarrow}$ is easily embeddable into $\mathrm{NM}^{\flat}_{\rightarrow}$. Namely, consider a locally correct $\mathrm{NM}^{*}_{\rightarrow}$-deduction frame $\widetilde{D} = \langle D, \mathrm{S}, \ell^{\mathrm{F}} \rangle$.[7] The corresponding locally

[7] For brevity we omit h, as every $h(x)$ is uniquely determined by x.

correct dag-like $\mathrm{NM}^{\flat}_{\rightarrow}$-deduction ∂ arises from D by ascending recursion on the height. The root and basic configurations of types $(\rightarrow I)$, $(\rightarrow E)$, (R) in $\widetilde{D}$ should remain unchanged. Furthermore, if x has several groups of premises in D, i.e. $|\mathrm{s}(x, D)| > 1$ (cf. [3]) then in ∂ we separate these groups via (S) with $|\mathrm{s}(x, D)|$ identical premises; for example this multipremise $\mathrm{NM}^{*}_{\rightarrow}$-configuration in $\widetilde{D}$

$$\boxed{\dfrac{\dfrac{\beta \qquad \gamma \qquad \gamma \rightarrow (\alpha \rightarrow \beta)}{\alpha \rightarrow \beta}}{\gamma \rightarrow (\alpha \rightarrow \beta)}}$$

goes to this $\mathrm{NM}^{\flat}_{\rightarrow}$ -configuration in ∂

$$\boxed{(\rightarrow I)\ \dfrac{(S)\ \dfrac{(\rightarrow I)\ \dfrac{\beta}{\alpha \rightarrow \beta} \qquad (\rightarrow E)\ \dfrac{\gamma \qquad \gamma \rightarrow (\alpha \rightarrow \beta)}{\alpha \rightarrow \beta}}{\alpha \rightarrow \beta}}{\gamma \rightarrow (\alpha \rightarrow \beta)}}\,.$$

Corresponding ℓ^{F}- and ℓ^{N}-labels are induced in an obvious way. Note that the weight of ∂ is linear in that of $\widetilde{D}$.[8]

Now suppose that there is a *fst* $\mathcal{F}$ in a chosen $\mathrm{NM}^{*}_{\rightarrow}$-deduction frame $\widetilde{D}$, and let $\mathcal{F}^{\flat}$ be the image of $\mathcal{F}$ in ∂. It is readily seen that $\mathcal{F}^{\flat}$ is also a dense and $(\rightarrow E)$ preserving set of closed threads in ∂ (see $\mathrm{NM}^{*}_{\rightarrow}$-clauses 1–3 in Ch. 1.2). That is, $\mathcal{F}^{\flat}$ is a dense set of closed threads in ∂ such that for every $\Theta \in \mathcal{F}^{\flat}$ and $(\rightarrow E)$-conclusion $x \in \Theta$, $\ell^{\mathrm{N}}(x) = \mathrm{E}_{\rightarrow}$, with premises y and z, if $y \in \Theta$ then there is a $\Theta' \in \mathcal{F}^{\flat}$ such that $z \in \Theta'$ and Θ coincides with Θ' below x.

2.1. Modified dag-like provability

We formalize in $\mathrm{NM}^{\flat}_{\rightarrow}$ the modified assignment $\mathcal{A} : \partial \ni x \hookrightarrow A(x) \subseteq \mathrm{FOR}(\partial)$.

Definition 2.2 (Assignment $\mathcal{A}$). Let ∂ be any locally correct dag-like $\mathrm{NM}^{\flat}_{\rightarrow}$-deduction. We assign nodes $x \in \partial$ with terms $A(x)$ by descending recursion 1–5.

[8] Recall that according to [3] we can just as well assume that $\widetilde{D}$ is horizontally compressed and its weight is polynomial in that of ρ.

1. $A(x) := \{\alpha\}$ if x is a leaf and $\ell^{\mathrm{F}}(x) = \alpha$.

2. $A(x) := A(y)$ if $\ell^{\mathrm{N}}(x) = \mathrm{R}$ and $\langle x, y\rangle \in \mathrm{E}(\partial)$.

3. $A(x) := A(y) \backslash \{\alpha\}$ if $\ell^{\mathrm{N}}(x) = \mathrm{I}$, $\langle x, y\rangle \in \mathrm{E}(\partial)$ and $\ell^{\mathrm{F}}(x) = \alpha \to \ell^{\mathrm{F}}(y)$.

4. $A(x) := A(y) \cup A(z)$ if $\ell^{\mathrm{N}}(x) = \mathrm{E}$ and $\langle x, y\rangle, \langle x, z\rangle \in \mathrm{E}(\partial)$.

5. $A(x) := \textcircled{S}(A(y_1), \cdots, A(y_n))$ if $\ell^{\mathrm{N}}(x) = \mathrm{S}$ and $(\forall i \in [1, n])\, \langle x, y_i\rangle \in \mathrm{E}(\partial)$.

Definition 2.3 (Nondeterministic reduction). Let ∂ and $\mathcal{A}$ be as above, r the root of ∂, S a set of formulas occurring in ∂. We say that $A(r)$ *reduces* to S (abbr.: $A(r) \rhd S$) if S arises from $A(r)$ by successive (in a left-to-right direction) substitutions $A(u) = \textcircled{S}(A(v_1), \cdots, A(v_n)) := A(v_i)$, for a fixed chosen $i \in \{1, \cdots, n\}$ and for any occurrence $A(u)$ in $A(w)$ and in $A(w')$, for every w' below w, provided that u is a premise of w such that $\ell^{\mathrm{N}}(u) = \mathrm{S}$,[9] while using ordinary set-theoretic interpretations of $\cup$ and $\backslash$. We call ∂ a *modified dag-like proof* of $\rho = \ell^{\mathrm{F}}(r)$ (abbr.: $\partial \vdash \rho$) if $A(r) \rhd \emptyset$ holds.[10]

Example 2.4. Previously shown configuration yields a ∂ such that $\partial \nvdash \rho$:

$$\dfrac{\dfrac{\dfrac{\beta\,;A=\{\beta\}}{\alpha\to\beta:\mathrm{I}\,;A=\{\beta\}} \qquad \dfrac{\gamma\,;A=\{\gamma\} \quad \gamma\to(\alpha\to\beta)\,;A=\{\gamma\to(\alpha\to\beta)\}}{\alpha\to\beta:\mathrm{E}\,;A=\{\gamma,\gamma\to(\alpha\to\beta)\}}}{\alpha\to\beta\ :\mathrm{S}\,;A=\textcircled{S}(\{\beta\},\{\gamma,\gamma\to(\alpha\to\beta)\})}}{\gamma\to(\alpha\to\beta):\mathrm{I}\,;A=\textcircled{S}(\{\beta\},\{\gamma\to(\alpha\to\beta)\})}$$

where $\ell^{\mathrm{N}}(r) = \mathrm{I}$, $\ell^{\mathrm{F}}(r) = \rho = \gamma \to (\alpha \to \beta)$ and $A(r) = \textcircled{S}(\{\beta\}, \{\gamma \to (\alpha \to \beta)\})$. Note that $A(r) \rhd \{\beta\}$ and $A(r) \rhd \{\gamma \to (\alpha \to \beta)\}$, although $A(r) \not\rhd \emptyset$.

To obtain an analogous dag-like proof of (say) $\rho' := \beta \to (\gamma \to (\alpha \to \beta))$ we'll upgrade ∂ to such ∂' :

[9] This operation is graphically interpreted by deleting u along with v_j for all $j \neq i$.

[10] The nondeterminism in question is encoded in $\textcircled{S}$ of Clause 5.

$$\dfrac{\dfrac{\dfrac{\dfrac{\beta\,; A=\{\beta\}}{\alpha\to\beta : \mathrm{I}\,; A=\{\beta\}} \qquad \dfrac{\gamma\,; A=\{\gamma\} \quad \gamma\to(\alpha\to\beta)\,; A=\{\gamma\to(\alpha\to\beta)\}}{\alpha\to\beta : \mathrm{E}\,; A=\{\gamma,\gamma\to(\alpha\to\beta)\}}}{\alpha\to\beta\ : \mathrm{S}\,; A=\circledS\left(\{\beta\},\{\gamma,\gamma\to(\alpha\to\beta)\}\right)}}{\gamma\to(\alpha\to\beta) : \mathrm{I}\,; A=\circledS\left(\{\beta\},\{\gamma\to(\alpha\to\beta)\}\right)\setminus\{\gamma\}}}{\beta\to(\gamma\to(\alpha\to\beta)) : \mathrm{I}\,; A=\circledS\left(\{\beta\},\{\gamma\to(\alpha\to\beta)\}\right)\setminus\{\gamma\}\setminus\{\beta\}}$$

and let $\circledS\left(\{\beta\},\{\gamma,\gamma\to(\alpha\to\beta)\}\right) := \{\beta\}$. Then $A(r) \rhd \emptyset$, i.e. $\partial' \vdash \rho'$ holds.

Lemma 2.5. *Every modified dag-like proof of ρ is convertible to a dag-like* $\mathrm{NM}_{\to}$*-proof of ρ.*

Proof: Let ∂ be a given $\mathrm{NM}^{\flat}_{\to}$-proof of ρ. Its $\mathrm{NM}_{\to}$-conversion is defined by a simple ascending recursion, as follows. Each time we arrive at a w whose premise u is a conclusion of (S), we replace u by its premise that is “guessed” by a given nondeterministic reduction leading to $A(r) \rhd \emptyset$ – alternatively, we can replace this (S) by the corresponding repetition (R). It is readily seen that the resulting dag-like deduction ∂_0 with the same root-formula ρ is locally correct and (S)-free, and hence it belongs to $\mathrm{NM}_{\to}$. Obviously $A(r) \rhd \emptyset$ in ∂ infers $A(r) = \emptyset$ in ∂_0, and hence ∂_0 proves ρ in $\mathrm{NM}_{\to}$. $\square$

This lemma is generalized by

Lemma 2.6. *Let $\widetilde{D}$ be any locally correct deduction frame in* $\mathrm{NM}^{*}_{\to}$ *with root-formula ρ that admits some* fst. *There exists a dag-like* $\mathrm{NM}_{\to}$*-proof of ρ whose weight does not exceed that of $\widetilde{D}$.*

Proof: Let ∂ be the $\mathrm{NM}^{\flat}_{\to}$-deduction of ρ induced by $\widetilde{D}$ and $\mathcal{F}$ any *fst* in $\widetilde{D}$. Furthermore, let $\mathcal{F}^{\flat}$ be the image of $\mathcal{F}$ in ∂ (see above). We will show that $\mathcal{F}^{\flat}$ determines successive left-to-right $\circledS$-eliminations $\circledS\left(A(y_1),\cdots, A(y_n)\right) \hookrightarrow A(y_i)$ inside $A(r)$ leading to a desired reduction $A(r) \rhd \emptyset$. These eliminations together with a suitable sub-*fst* $\mathcal{F}^{\flat}_0 \subseteq \mathcal{F}^{\flat}$ arise as follows by ascending recursion along $\mathcal{F}^{\flat}$. Let x with $\ell^{\mathrm{N}}(x) = \mathrm{E}$ be a chosen lowest conclusion of $(\to E)$ in ∂, if any exists. By the density of $\mathcal{F}^{\flat}$, there exists $\Theta \in \mathcal{F}^{\flat}$ with $x \in \Theta$; we let $\Theta \in \mathcal{F}^{\flat}_0$. Let y and z be the two premises of x and suppose that $y \in \Theta$. By the third *fst* condition there exists a $\Theta' \in \mathcal{F}^{\flat}$ with $z \in \Theta'$ and $\Theta\upharpoonright_x = \Theta'\upharpoonright_x$; so let $\Theta' \in \mathcal{F}^{\flat}_0$ be the corresponding “upgrade”of Θ. In the case $z \in \Theta$ we let $\Theta' := \Theta$. Note that $\Theta\upharpoonright_x$

determines substitutions $A(u) = \textcircled{S}(A(v_1), \cdots, A(v_n)) := A(v_i)$ in all parents of (S)-conclusions u occurring in both Θ and Θ' below x (cf. Definitions 10, 11), if any exist, and hence also $\textcircled{S}$-eliminations $A(u) \hookrightarrow A(v_i)$ in the corresponding subterms of $A(r)$. The same procedure is applied to the nodes occurring in Θ and Θ' between x and the next lowest conclusions of $(\to E)$; this yields new "upgraded" threads $\Theta'', \Theta''', \cdots \in \mathcal{F}_0^\flat$ and $\textcircled{S}$-eliminations in the corresponding initial fragments of $A(r)$. We keep doing this recursively until the list of remaining $\textcircled{S}$-occurrences in $\Theta \in \mathcal{F}_0^\flat$ is empty. The final "cleansed" $\textcircled{S}$-free form of $A(r)$ is represented by a set of formulas that easily reduces to $\emptyset$ by ordinary set-theoretic interpretation of the remaining operations $\cup$ and $\setminus$, since every $\Theta \in \mathcal{F}_0^\flat$ involved is closed. That is, the correlated "cleansed" deduction ∂_0 is a locally correct dag-like deduction of ρ in the (S)-free fragment of $\mathrm{NM}_\to^\flat$, and hence it belongs to $\mathrm{NM}_\to$; moreover the set of ascending threads in ∂_0 is uniquely determined by the remaining rules (R), $(\to I)$, $(\to E)$ (cf. analogous passage in the previous proof). Now by the definition these "cleansed" ascending threads are all included in $\mathcal{F}_0^\flat$ and hence closed with respect to $(\to I)$.[11] This yields a desired reduction $A(r) \rhd \emptyset$, i.e. $A(r) = \emptyset$, in ∂_0. Hence ∂_0 proves ρ in $\mathrm{NM}_\to$. Obviously the weight of ∂_0 does not exceed the weight of $\widetilde{D}$. $\square$

Operation $\partial \hookrightarrow \partial_0$ is referred to as *horizontal cleansing* (cf. Introduction). Together with Remark 2 and Corollary 7 this yields

Corollary 2.7. Any given ρ is valid in minimal logic iff it is provable in $\mathrm{NM}_\to$ by a dag-like deduction ∂_0 whose weight is polynomial in $|\rho|$ and such that $\partial_0 \vdash \rho$ can be confirmed by a deterministic TM in $|\rho|$-polynomial time.[12]

Theorem 2.8. **PSPACE** $\subseteq$ **NP** *and hence* **NP** $=$ **PSPACE**.

Proof: Minimal propositional logic is PSPACE-complete (cf. e.g. [7, 11, 12]). Hence **PSPACE** $\subseteq$ **NP** directly follows from Corollary 15. Note that in contrast to [3] here we use nondeterministic arguments twice. First we "guess" the existence of "short" Hudelmaier-style cutfree sequential deduction of ρ that leads (by deterministic compression) to a "small" natural deduction frame $\widetilde{D}$ that is supposed to have a *fst* $\mathcal{F}$. Then we "guess"

[11] These threads may be exponential in number, but our nondeterministic algorithm runs on the polynomial set of nodes.

[12] See Appendix for a more exhaustive presentation.

the existence of a "cleansed" modified subdeduction that confirms in $|\rho|$-polynomial time the provability of ρ with regard to $\left\langle \widetilde{D}, \mathcal{F} \right\rangle$. □

Corollary 2.9. **NP** = **coNP** and hence the polynomial hierarchy collapses to the first level.

Proof: **NP** = **PSPACE** implies **coNP** = **coPSPACE** = **PSPACE** = **NP** (see also [8, 1]). □

Corollary 2.10. PSPACE (in particular NP) problems are nondeterministically decidable in polynomial time. To put it more precisely, for any given PSPACE language $L \subseteq \{0,1\}^*$ there exists a polynomial $p : \mathbb{N} \to \mathbb{N}$ and a polynomial-time TM M such that for every $x \in \{0,1\}^*$ there exists $u \in \{0,1\}^{p(|x|)}$ satisfying $x \in L \Leftrightarrow M(x,u) = 1$ (i.e.: "*u provides a polynomial test for $x \in^? L$*").[13]

Proof: By theorem 16, it suffices to deal with the NP-complete problem of boolean satisfiability. Let $\varphi(\overrightarrow{v})$ be a given boolean formula, where $\overrightarrow{v}$ is a list of propositional variables that is encoded by $x \in \{0,1\}^*$. Let $x \in L$ abbreviate $\varphi(\overrightarrow{v}) \in$ SAT, then $x \notin L \Leftrightarrow \neg\varphi(\overrightarrow{v}) \in$ VAL. By Corollary 17, SAT and VAL are both in NP. This yields the result by an obvious nondeterministic combination of standard NP-verifications of both conjectures $x \in L$ and $x \notin L$. □

Remark 2.11 ("Hilbert-paradise" of PSPACE world). Corollary 18 yields a following broad conclusion. PSPACE problems are closed under propositional operations and provability (by Savitch's theorem) while being (nondeterministic) decidable in polynomial time (: "in PSPACE there is no polytime ignorabimus").

[13] That is, we rewrite **NP** condition

$$(\forall x \in \{0,1\}^*) \Big(x \in L \Leftrightarrow \Big(\exists u \in \{0,1\}^{p(|x|)} \Big) M(x,u) = 1 \Big)$$

(cf. e.g. [1, 2.1]) to

$$(\forall x \in \{0,1\}^*) \Big(\exists u \in \{0,1\}^{p(|x|)} \Big) (x \in L \Leftrightarrow M(x,u) = 1)$$

or, more precisely, to

$$(\forall x \in \{0,1\}^*) \Big(\neg\neg\exists u \in \{0,1\}^{p(|x|)} \Big) (x \in L \Leftrightarrow M(x,u) = 1) \,.$$

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Appendix: rough complexity estimate

Dag-like proof system $\mathrm{NM}_{\rightarrow}$

We regard $\mathrm{NM}_{\rightarrow}$ as $\mathrm{NM}^{\flat}_{\rightarrow}$ without separation rule (S). Moreover, without loss of generality we suppose that dag-like $\mathrm{NM}_{\rightarrow}$-deductions ∂ of root-formulas ρ have polynomial total number of vertices $|\mathrm{V}(\partial)| = \mathcal{O}\left(|\rho|^4\right)$ while the weights of formulas and the height numbers involved are bounded by $2|\rho|$ and $|\mathrm{V}(\partial)|$, respectively (cf. [3]).

Let $\mathrm{LC}(\partial)$ and $\mathrm{PROV}(\partial)$ be abbreviations for '∂ *is locally correct*' and '∂ *proves* ρ', respectively, and let $\mathrm{PROOF}(\partial) := \mathrm{LC}(\partial) \,\&\, \mathrm{PROV}(\partial)$. We wish to validate the assertion $\mathrm{PROOF}(\partial)$ in polynomial time (and space) by a suitable deterministic TM M. For technical reasons we choose a formalization of ∂ in which edges are redefined as pairs $\langle parent, child\rangle$. Let ρ, $\chi_\rho \in \{\mathrm{I}, \mathrm{E}\}$, $a = 2|\rho|$ and $0 < r < b = \mathcal{O}\left(|\rho|^4\right)$ be fixed.

Input of M**:** List $\mathbf{t}$ consisting of tuples $t(x) = [x, y_1, y_2, h, h_1, h_2, \chi, \gamma, \beta_1, \beta_2]$, for all $0 < x \leq b$, where $\chi \in \{\mathrm{R}, \mathrm{I}, \mathrm{E}, \mathrm{L}\}$ (L stands for 'leaf'), while $x, y_1, y_2 \leq b$, $h, h_1, h_2 \leq b$ and $\gamma, \beta_1, \beta_2 \leq a$ are natural numbers (in binary) which are thought to encode nodes, nodes' heights and formulas, respectively (0 encodes $\emptyset$).

The weight of $\mathbf{t}$ is $\mathcal{O}\left(|\rho|^4 \log|\rho|\right) < \mathcal{O}\left(|\rho|^5\right)$. $\mathrm{LC}(\partial)$ and $\mathrm{PROV}(\partial)$ are verified by M as follows while assuming that: x are parents of $y_i > 0$, $h := h(x)$, $h_i := h(y_i)$, $\gamma := \ell^{\mathrm{F}}(x)$, $\beta_i := \ell^{\mathrm{F}}(y_i)$ ($i \in \{1, 2\}$) and $\chi := \ell^{\mathrm{N}}(x)$ if x is not a leaf, else $\chi := \mathrm{L}$.

Local correctness

$\mathrm{LC}(\partial)$ is equivalent to conjunction of the following conditions 1–8 on $\mathbf{t}$ that (according to above assumptions) uniquely determines the underlying locally correct $\mathrm{NM}_{\rightarrow}$-deduction ∂ by ascending induction on h.

1. If $x = x'$ then $t(x) = t(x')$.
2. If $t(x) = [x, y_1, y_2, h, h_1, h_2, \chi, \gamma, \beta_1, \beta_2]$ and $x' = y_i > 0$ $(i \in \{1, 2\})$ for $t(x') = [x', y_1', y_2', h', h_1', h_2', \chi', \chi_1', \chi_2', \gamma', \beta_1', \beta_2']$, then $h' = h_i$ and $\gamma' = \beta_i$.
3. If $x = r$ then $h = 0$, $\gamma = \rho$ and $\chi = \chi_\rho$.
4. If $\chi = \text{L}$ then $y_1 = y_2 = \beta_1 = \beta_2 = 0$ [: case $x \in \text{L}(\partial)$].
5. If $\chi \neq \text{L}$ then $y_1 + y_2 > 0$ and $h_1 = h_2 = h + 1$.
6. If $\chi = \text{R}$ then $y_2 + \beta_2 = 0 < y_1$ and $\gamma = \beta_1$ [: rule (R)].
7. If $\chi = \text{I}$ then $\gamma = \alpha \to \beta_1$ (for some α) [: rule $(\to I)$].
8. If $\chi = \text{E}$ then $\beta_2 = \beta_1 \to \gamma$ [: rule $(\to E)$].

The verification of conditions 1–8 requires $\mathcal{O}\left(|\rho|^5\right)$ iterations of basic queries $\chi \stackrel{?}{=} \chi'$, $u \stackrel{?}{=} v$, $\delta \stackrel{?}{=} \sigma$, $(\exists ?\alpha)\, \gamma = \alpha \to \beta$ for $\chi, \chi' \in \{\text{R, I, E, L}\}$, $u, v \leq b$ and $\beta, \gamma, \delta, \sigma \leq a$ that are solvable in $\mathcal{O}(|\rho|)$ time (note that $\alpha \to \beta = \to \alpha\beta$ in the Łukasiewicz prefix notation). Summing up there is a deterministic TM M that verifies LC(∂) in $\mathcal{O}\left(|\rho|^5 \cdot |\rho|\right) = \mathcal{O}\left(|\rho|^6\right)$ time and $\mathcal{O}\left(|\rho|^5\right)$ space.

Assignment $\mathcal{A}$

A given locally correct $\text{NM}_\to$-deduction ∂ determines an assignment

$$\mathcal{A} : 0 < x \leq b \hookrightarrow A(x) \subseteq \text{FOR}(\partial)$$

that is defined by the following recursive clauses 1–4 for input $\mathbf{t}$ satisfying above conditions 1–8, where as above $t(x) = [x, y_1, y_2, h, h_1, h_2, \chi, \gamma, \beta_1, \beta_2]$, for all $0 < x \leq b$.

1. $A(x) := \{\gamma\}$ if $\chi = \text{L}$.
2. $A(x) := A(y_1)$ if $\chi = \text{R}$.
3. $A(x) := A(y_1) \setminus \{\alpha\}$ if $\chi = \text{I}$ and $\gamma = \alpha \to \beta_1$.
4. $A(x) := A(y_1) \cup A(y_2)$ if $\chi = \text{E}$.

The length of recursion 1–4 is $b = \mathcal{O}\left(|\rho|^4\right)$. Recursion steps produce (say, sorted) lists of formulas $A(x)$, $|A(x)| \leq b$ using set-theoretic unions $A \cup B$ and subtractions $A \setminus \{\alpha\}$. Each recursion step requires $\mathcal{O}(b \cdot |\rho|) = \mathcal{O}\left(|\rho|^5\right)$ steps of computation. This yields upper bound $\mathcal{O}\left(|\rho|^4 \cdot |\rho|^5\right) = \mathcal{O}\left(|\rho|^9\right)$ for $A(r) \stackrel{?}{=} \emptyset$. Thus PROV$(\partial)$ is verifiable in $\mathcal{O}\left(|\rho|^9\right)$ time and $\mathcal{O}(|\rho|)$ space. Hence by the above estimate of LC(∂) we can safely assume that PROOF(∂) is verifiable by a deterministic TM M in $\mathcal{O}\left(|\rho|^9\right)$ time and $\mathcal{O}\left(|\rho|^5\right)$ space.

Conclusion 2.12. There exist polynomials p, q, r of degrees $5, 9, 5$, respectively, and a deterministic boolean-valued TM M such that for any purely implicational formula ρ the following holds: *ρ is valid in minimal logic iff there exists a $u \in \{0,1\}^{p(|\rho|)}$ such that $M(\rho, u)$ yields 1 after $q(|\rho| + |u|)$ steps of computation in space $r(|\rho| + |u|)$.* Analogous polynomial estimates of the intuitionistic and/or classical propositional and even quantified boolean validity are easily obtained by familiar syntactic interpretations within minimal logic (cf. e.g. [6, 10, 12]).

Remark 2.13. Recall that PROV(∂) is equivalent to the assertion that maximal threads in ∂ are closed. This in turn is equivalent to a variant of non-reachability assertion: '*r is not connected to any leaf z in a subgraph of ∂ that is obtained by deleting all edges $\langle x, y\rangle$ with $\ell^{\mathrm{N}}(x) = \mathrm{I}$ and $\ell^{\mathrm{F}}(x) = \ell^{\mathrm{F}}(z) \to \ell^{\mathrm{F}}(y)$*', which we'll abbreviate by PROV$_1(\partial)$. Now PROV$_1(\partial)$ is verifiable by a deterministic TM in $\mathcal{O}(|\mathrm{V}(\partial)| \cdot |\mathrm{E}(\partial)|) = \mathcal{O}\left(|\rho|^{12}\right)$ time and $\mathcal{O}(|\rho| \cdot |\mathrm{V}(\partial)|) = \mathcal{O}\left(|\rho|^5\right)$ space (cf. e.g. [8]). However this does not improve our upper bound for PROOF(∂). Actually there are known much better estimates of the reachability problem (cf. e.g. [13, 4]), but at this stage we are not interested in a more precise analysis.

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EMPIRICAL NEGATION, CO-NEGATION AND THE CONTRAPOSITION RULE I: SEMANTICAL INVESTIGATIONS

Abstract

We investigate the relationship between M. De's empirical negation in Kripke and Beth Semantics. It turns out empirical negation, as well as co-negation, corresponds to different logics under different semantics. We then establish the relationship between logics related to these negations under unified syntax and semantics based on R. Sylvan's $\mathbf{CC}_{\omega}$.

Keywords: Empirical negation, co-negation, Beth semantics, Kripke semantics, intuitionism.

1. Introduction

The philosophy of Intuitionism has long acknowledged that there is more to negation than the customary, *reduction to absurdity*. Brouwer [1] has already introduced the notion of *apartness* as a positive version of inequality, such that from two apart objects (e.g. points, sequences) one can learn not only they are unequal, but also how much or where they are different. (cf. [19, pp.319–320]). He also introduced the notion of *weak counterexample*, in which a statement is reduced to a constructively unacceptable principle, to conclude we cannot expect to prove the statement [17].

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Another type of negation was discussed in the dialogue of Heyting [8, pp. 17–19]. In it *mathematical negation* characterised by reduction to absurdity is distinguished from *factual negation*, which concerns the present state of our knowledge. In the dialogue it is emphasised that only the former type of negation has a part in mathematics, on the ground that the latter does not have the form of a mathematical assertion, i.e. assertion of a mental construction. Nevertheless it remains the case that factual negation has a place in his theoretical framework.

One formalisation of logic with this "negation at the present stage of knowledge" was given by De [3] and axiomatised by De and Omori [4], under the name of *empirical negation*. The central idea of $\mathbf{IPC}^{\sim}$ is semantic: the Kripke semantics of $\mathbf{IPC}^{\sim}$ is taken to be rooted, with the root being understood as representing the present moment. Then the empirical negation $\sim A$ is defined to be forced at a world, if A is not forced at the root.

Yet another type of negation in the intuitionistic framework is *co-negation* introduced by Rauszer [12, 13]. Seen from Kripke semantics, a co-negation $\sim A$ is forced at a world, if there is a preceding world in which A is not forced. This is dual to the forcing of intuitionistic negation $\neg A$, which requires A not being forced at all succeeding nodes. Co-negation was originally defined in terms of co-implication, but the co-negative fragment was extracted by Priest [11], to define a logic named **daC**.

In both empirical and co- negation, the semantic formulation arguably gives a more fundamental motivation than the syntactic formulation. In particular, in case of empirical negation, it is of essential importance that a Kripke frame can be understood as giving the progression of growth of knowledge. It may be noted, however, that Kripke semantics is not the only semantics to give this kind of picture. *Beth semantics* is another semantics whose frames represent the growth of knowledge. It then appears a natural question to ask, whether the same forcing condition of empirical/co- negation gives rise to the same logic. That is to say, whether $\mathbf{IPC}^{\sim}$ and **daC** will be sound and complete with respect to Beth semantics. Indeed, for co-implication, a similar question was asked by Restall [14]. There it was found out that one needs to alter the forcing condition to get a complete semantics.

In this paper, we shall observe that another logic called $\mathbf{TCC}_{\omega}$, introduced by Gordienko [7], becomes sound and complete with Beth models with the forcing conditions of empirical and co- negation (which turn out to coincide). This is of significant interest for those who advocate empirical or co- negation from a semantic motivation, as it will provide a choice in the logic to which they should adhere.

This is followed by another observation about the axiomatisation of $\mathbf{IPC}^{\sim}$ and **daC**, which employ the disjunctive syllogism rule [RP]. In contrast, the axiomatisation of of $\mathbf{TCC}_{\omega}$ and a related system $\mathbf{CC}_{\omega}$ of Sylvan [15], which is a subsystem of the other three, use the contraposition rule [RC]. We shall observe that this difference in rules can be eliminated, by replacing [RP] with [RC] and an additional axiom. This will give a completeness proof of **daC** with respect to the semantics of $\mathbf{CC}_{\omega}$, and thus the semantics of Došen [5]. It will also provide a more unified viewpoint of the logics related to $\mathbf{CC}_{\omega}$ as defined by extra axioms with no change in rules.

We shall continue our investigation proof-theoretically in a sequel. In the second paper, using the obtained frame properties we shall formulate labelled sequent calculi for the logics considered so far ($\mathbf{CC}_{\omega}$, **daC**, $\mathbf{TCC}_{\omega}$ and $\mathbf{IPC}^{\sim}$). We shall prove the admissibility of structural rules including cut, and then show the correspondence with Hilbert-style calculi.

2. Preliminaries

We shall employ the following notations (taken from [17]) for sequences and related notions.

- $\alpha, \beta, \ldots$: infinite sequences of the form $\langle b_1, b_2, \ldots \rangle$ of natural numbers.
- $\langle\rangle$: the empty sequence.
- $b, b', \ldots$: finite sequences of the form $\langle b_1, \ldots, b_n \rangle$ of natural numbers.
- $b * b'$: b concatenated with b'.
- $lh(b)$: the length of b.
- $b \preceq b'$: $b * b'' = b'$ for some b''.
- $b \prec b'$: $b \preceq b'$ and $b \neq b'$.
- $\bar{\alpha}n$: α's initial segment up to the nth element.
- $\alpha \in b$: b is α's initial segment.

We define a *tree* to be a set T of finite sequences of natural number such that $\langle\rangle \in T$, $b \in T \vee b \notin T$ and $b \in T \wedge b' \prec b \rightarrow b' \in T$. We call each finite sequence in T a *node* and $\langle\rangle$ the *root*. A *successor* of a node b is a node of the form $b * \langle x\rangle$. By *leaves* of T, we mean the nodes of T which do not have a successor, i.e. nodes b such that $\neg\exists x(b * \langle x\rangle) \in T$. A *spread* then is a tree whose nodes always have a successor, i.e. $\forall b \in T\exists x(b * \langle x\rangle \in T)$.

A clarification: whilst $\langle b, b, \ldots\rangle$ denotes an *infinite* sequence consisting just of bs, $\langle b, \ldots, b\rangle$ denotes a *finite* sequence consisting just of bs.

3. Empirical negation in Kripke Semantics

Let us use the following notations for metavariables.

- $p, q, r, \ldots$ for propositional variables.

- $A, B, C, \ldots$ for formulae.

In this paper, we shall consider the following propositional language

$$\mathcal{L} ::= p \mid (A \wedge B) \mid (A \vee B) \mid (A \to B) \mid {\sim}A.$$

Parentheses will be omitted if there is no fear of ambiguity. We shall use the convention $A \leftrightarrow B := (A \to B) \wedge (B \to A)$.

To begin with, we look at the Kripke semantics for the intuitionistic logic with empirical negation $\mathbf{IPC}^{\sim}$ given in [4]. Recall that a reflexive, anti-symmetric and transitive ordering is called a *partial order*.

Definition 3.1 (Kripke model for $\mathbf{IPC}^{\sim}$). A *Kripke Frame* $\mathcal{F}^{\sim}_{\mathcal{K}}$ for $\mathbf{IPC}^{\sim}$ is a partially ordered set $(W, \leq)$ with a *root* $r \in W$ such that $r \leq w$ for all $w \in W$. We shall call each $w \in W$ a *world*. A *Kripke model* $\mathcal{M}^{\sim}_{\mathcal{K}}$ for $\mathbf{IPC}^{\sim}$ is a pair $(\mathcal{F}^{\sim}_{\mathcal{K}}, \mathcal{V})$, where $\mathcal{V}$ is a mapping that assigns a set of worlds $\mathcal{V}(p) \subseteq W$ to each propositional variable p. We assume $\mathcal{V}$ to be *monotone*, viz. $w \in \mathcal{V}(p)$ and $w' \geq w$ implies $w' \in \mathcal{V}(p)$. To denote a model, we shall use both $\mathcal{M}^{\sim}_{\mathcal{K}}$ and $(\mathcal{F}^{\sim}_{\mathcal{K}}, \mathcal{V})$ interchangeably. Similar remarks apply to different notions of model in the later sections.

Given $\mathcal{M}^{\sim}_{\mathcal{K}}$, the *forcing* (or *valuation*) of a formula in a world, denoted $\mathcal{M}^{\sim}_{\mathcal{K}}, w \Vdash_{\mathcal{K}} A$, is inductively defined as follows.

$$
\begin{array}{lll}
\mathcal{M}^{\sim}_{\mathcal{K}}, w \Vdash_{\mathcal{K}} p & \iff & w \in \mathcal{V}(p). \\
\mathcal{M}^{\sim}_{\mathcal{K}}, w \Vdash_{\mathcal{K}} A \wedge B & \iff & \mathcal{M}^{\sim}_{\mathcal{K}}, w \Vdash_{\mathcal{K}} A \text{ and } \mathcal{M}^{\sim}_{\mathcal{K}}, w \Vdash_{\mathcal{K}} B. \\
\mathcal{M}^{\sim}_{\mathcal{K}}, w \Vdash_{\mathcal{K}} A \vee B & \iff & \mathcal{M}^{\sim}_{\mathcal{K}}, w \Vdash_{\mathcal{K}} A \text{ or } \mathcal{M}^{\sim}_{\mathcal{K}}, w \Vdash_{\mathcal{K}} B. \\
\mathcal{M}^{\sim}_{\mathcal{K}}, w \Vdash_{\mathcal{K}} A \to B & \iff & \text{for all } w' \geq w, \text{ if } \mathcal{M}^{\sim}_{\mathcal{K}}, w' \Vdash_{\mathcal{K}} A, \\
& & \text{then } \mathcal{M}^{\sim}_{\mathcal{K}}, w' \Vdash_{\mathcal{K}} B. \\
\mathcal{M}^{\sim}_{\mathcal{K}}, w \Vdash_{\mathcal{K}} {\sim}A & \iff & \mathcal{M}^{\sim}_{\mathcal{K}}, r \nVdash_{\mathcal{K}} A.
\end{array}
$$

We shall occasionally avoid denoting models explicitly when it is apparent from the context. If $\mathcal{M}^{\sim}_{\mathcal{K}}, w \Vdash_{\mathcal{K}} A$ for all $w \in W$, we write $\mathcal{M}^{\sim}_{\mathcal{K}} \vDash_{\mathcal{K}} A$ and say A *is valid in* $\mathcal{M}^{\sim}_{\mathcal{K}}$. For a set of formulae Γ, if $\mathcal{M}^{\sim}_{\mathcal{K}} \vDash_{\mathcal{K}} C$ for all $C \in \Gamma$ implies $\mathcal{M}^{\sim}_{\mathcal{K}} \vDash_{\mathcal{K}} A$, then we write $\Gamma \vDash_{\mathcal{K}} A$ and say A *is a consequence of* Γ. If Γ is empty, we simply write $\vDash_{\mathcal{K}} A$ and say A *is valid* (in $\mathbf{IPC}^{\sim}$).

A Hilbert-style proof system for $\mathbf{IPC}^{\sim}$ is established in [4], which we identify here with the logic itself for convenience, and denote it simply as $\mathbf{IPC}^{\sim}$. We shall apply the same convention to other logics in later sections.

Definition 3.2 ($\mathbf{IPC}^{\sim}$).
The logic $\mathbf{IPC}^{\sim}$ is defined by the following axiom schemata and rules.

Axioms

[Ax1]	$A \to (B \to A)$
[Ax2]	$(A \to (B \to C)) \to ((A \to B) \to (A \to C))$
[Ax3]	$(A \wedge B) \to A$
[Ax4]	$(A \wedge B) \to B$
[Ax5]	$(C \to A) \to ((C \to B) \to (C \to (A \wedge B)))$
[Ax6]	$A \to (A \vee B)$
[Ax7]	$B \to (A \vee B)$
[Ax8]	$(A \to C) \to ((B \to C) \to ((A \vee B) \to C))$
[Ax9]	$A \vee {\sim}A$
[Ax10]	${\sim}A \to ({\sim}{\sim}A \to B)$

Rules

[MP] $\dfrac{A \qquad A \to B}{B}$ [RP] $\dfrac{A \vee B}{{\sim}A \to B}$

We followed [4] in the labelling of the axioms and the rules. A *proof* (or *deduction/derivation*) of A from a (possibly infinite) set of formulae Γ (which we denote by $\Gamma \vdash_{\sim} A$) in $\mathbf{IPC}^{\sim}$ is a finite tree with the number of branching at each node less than or equal two, and whose nodes are labelled by formulae of $\mathcal{L}$ such that

- The formulae in the leaves are either instances of axioms, or from a specified finite subset Γ' of Γ.
- Each formula in non-leaf nodes is obtained from the formulae in the successor nodes by an application of a rule.
- The root of the tree is A.

Then it has been shown by De and Omori that $\mathbf{IPC}^{\sim}$ is sound and complete with the Kripke semantics.

THEOREM 3.3 (Kripke completeness of $\mathbf{IPC}^{\sim}$). *$\Gamma \vdash_{\sim} A \Longleftrightarrow \Gamma \vDash_{\mathcal{K}} A$.*

PROOF: Cf. [4]. $\square$

4. Empirical negation in Beth Semantics

4.1. Beth semantics and $\mathbf{IPC}^{\sim}$

Let us turn our attention to Beth models in this section. Our formalisation will be based on that of [16, 18]. If we apply to the forcing of $\sim$ the same criterion as to the Kripke semantics above, then we obtain the following semantics.

DEFINITION 4.1 (Beth model). A *Beth frame* $\mathcal{F}_{\mathcal{B}}$ is a pair $(W, \preceq)$ that defines a spread. Then A *Beth model* $\mathcal{M}_{\mathcal{B}}$ is a pair $(\mathcal{F}_{\mathcal{B}}, \mathcal{V})$, where $\mathcal{V}$ is an assignment of propositional variables to the nodes such that:

$$b \in \mathcal{V}(p) \Leftrightarrow \forall \alpha \in b \exists m (\bar{\alpha} m \in \mathcal{V}(p)). \text{ [covering]}$$

(The left-to-right direction is trivial, and it is straightforward to see that a covering assignment is monotone.)

The forcing relation $\Vdash_{\mathcal{B}} A$ for a Beth model is defined by the following clauses.

$$\begin{array}{ll}
\mathcal{M}_{\mathcal{B}}, b \Vdash_{\mathcal{B}} p & \iff b \in \mathcal{V}(p). \\
\mathcal{M}_{\mathcal{B}}, b \Vdash_{\mathcal{B}} A \wedge B & \iff \mathcal{M}_{\mathcal{B}}, b \Vdash_{\mathcal{B}} A \text{ and } \mathcal{M}_{\mathcal{B}}, b \Vdash_{\mathcal{B}} B. \\
\mathcal{M}_{\mathcal{B}}, b \Vdash_{\mathcal{B}} A \vee B & \iff \forall \alpha \in b \exists n (\mathcal{M}_{\mathcal{B}}, \bar{\alpha} n \Vdash_{\mathcal{B}} A \text{ or } \mathcal{M}_{\mathcal{B}}, \bar{\alpha} n \Vdash_{\mathcal{B}} B). \\
\mathcal{M}_{\mathcal{B}}, b \Vdash_{\mathcal{B}} A \to B & \iff \text{ for all } b' \succeq b, \text{ if } \mathcal{M}_{\mathcal{B}}, b' \Vdash_{\mathcal{B}} A, \text{ then } \mathcal{M}_{\mathcal{B}}, b' \Vdash_{\mathcal{B}} B. \\
\mathcal{M}_{\mathcal{B}}, b \Vdash_{\mathcal{B}} {\sim} A & \iff \mathcal{M}_{\mathcal{B}}, \langle\rangle \nVdash_{\mathcal{B}} A.
\end{array}$$

Proposition 4.2.

(i) $b \Vdash_{\mathcal{B}} A$ if and only if $\forall \alpha \in b \exists n (\bar{\alpha} n \Vdash_{\mathcal{B}} A)$. (covering property)

(ii) $b' \succeq b$ and $b \Vdash_{\mathcal{B}} A$ implies $b' \Vdash_{\mathcal{B}} A$. (monotonicity)

Proof: We prove (i) by induction on the complexity of formulae. If $b \Vdash_{\mathcal{B}} A$, then trivially $\forall \alpha \in b \exists n (\bar{\alpha} n \Vdash_{\mathcal{B}} A)$. For the converse direction, we show by induction on the complexity of A. Because (i) holds in Beth models for intuitionistic logic, it suffices to check the case where $A \equiv {\sim} B$. If $\forall \alpha \in b \exists n (\bar{\alpha} n \Vdash_{\mathcal{B}} {\sim} B)$, then by definition $\forall \alpha \in b \exists n (\langle\rangle \nVdash_{\mathcal{B}} B)$; i.e. $\langle\rangle \nVdash_{\mathcal{B}} B$. Thus by definition again, $b \Vdash_{\mathcal{B}} {\sim} B$.

(ii) is an immediate consequence of (i). □

How does this semantics relate to $\mathbf{IPC}^{\sim}$? In considering this question, we first look at how to embed Kripke models into Beth models, in accordance with the method outlined in [18].

Given a Kripke model $\mathcal{M}_{\mathcal{K}}^{\sim} = (W_K, \leq, \mathcal{V}_K)$ for $\mathbf{IPC}^{\sim}$, we construct a corresponding Beth model $\mathcal{M}_{\mathcal{B}} = (W_B, \preceq, \mathcal{V}_B)$ with the following stipulations.

- W_B is the set of finite nondecreasing sequences of worlds (i.e. each w in a sequence is followed by w' s.t. $w \leq w'$) from the root r in $(W_K, \leq)$ with length > 0.
- $\preceq$ is defined accordingly.
- $\langle w_0, \ldots, w_n \rangle \in \mathcal{V}_B(p)$ if and only if $w_n \in \mathcal{V}_K(p)$.

The resulting W_B is a spread, because the reflexivity of $\leq$ assures that $\langle w_0, \ldots, w_n \rangle \in W_B$ implies $\langle w_0, \ldots, w_n, w_n \rangle \in W_B$. Note that w_0 is always the root r in $\mathcal{M}_{\mathcal{K}}^{\sim}$, and $\langle w_0 \rangle$ is the root of $\mathcal{M}_{\mathcal{B}}$. The latter slightly differs from our definition of Beth model: we can fit the model to the definition if we reinterpret the sequences as mere labels for the tree, and the actual tree is constructed in such a way that $\langle w_0 \rangle$ is the label for the node $\langle\rangle$, $\langle w_0, w_1, \ldots, w_n \rangle$ is the label for the node $\langle w_1, \ldots, w_n \rangle$. We can also adopt a different embedding, which we shall see later.

For any Kripke model, because we can concatenate the same element indefinitely many times, we can also consider infinite nondecreasing sequences of worlds. This fact will be used in the next lemma.

LEMMA 4.3 (embeddability of Kripke models for **IPC**$^{\sim}$).

(i) *$\mathcal{M}_{\mathcal{B}}$ is indeed a Beth model.*

(ii) *$\mathcal{M}^{\sim}_{\mathcal{K}} \vDash_{\mathcal{K}} A$ if and only $\mathcal{M}_{\mathcal{B}} \vDash_{\mathcal{B}} A$.*

PROOF: For (i), we need to check that $\mathcal{V}_B$ is a covering assignment. If $\forall \alpha \in \langle w_0, \ldots, w_n \rangle \exists m(\bar{\alpha} m \in \mathcal{V}_B(p))$, then in particular, $\alpha_0 := \langle w_0, \ldots, w_n \rangle * \langle w_n, w_n, \ldots \rangle \in \langle w_0, \ldots, w_n \rangle$. So there is an m such that $\bar{\alpha_0} m \in \mathcal{V}_B(p)$. If $m \leq n+1 = lh(\langle w_0, \ldots, w_n \rangle)$, then by the monotonicity of $\mathcal{V}_B$ (which follows from that of $\mathcal{V}_K$, and the fact that $\mathcal{V}_B$ only looks at the last element of a sequence) we have $\langle w_0, \ldots, w_n \rangle \in \mathcal{V}_B(p)$. Otherwise, by definition of $\mathcal{V}_B$, $w_n \in \mathcal{V}_K(p)$; hence $\langle w_0, \ldots, w_n \rangle \in \mathcal{V}_B(p)$.

For (ii), it suffices to show $w_n \Vdash_{\mathcal{K}} A \Leftrightarrow \langle w_0, \ldots, w_n \rangle \Vdash_{\mathcal{B}} A$. We prove this by induction on the complexity of formulae. Given the result for intuitionistic logic, we only need to check for $A \equiv {\sim} B$. In this case, $w_n \Vdash_{\mathcal{K}} {\sim} B \Leftrightarrow w_0 \nVdash_{\mathcal{K}} B \Leftrightarrow \langle w_0 \rangle \nVdash_{\mathcal{B}} B \Leftrightarrow \langle w_0, \ldots, w_n \rangle \Vdash_{\mathcal{B}} {\sim} B$. $\square$

Let Q be the class of Beth models obtained by the above embedding. We shall denote Beth validity with respect to Q as $\vDash_{\mathsf{Q}}$.

THEOREM 4.4 (Beth completeness of **IPC**$^{\sim}$ with respect to Q). *$\Gamma \vdash_{\sim} A$ if and only if $\Gamma \vDash_{\mathsf{Q}} A$.*

PROOF: Because of Theorem 3.3, $\Gamma \vdash_{\sim} A$ if and only if $\Gamma \vDash_{\mathcal{K}} A$. Also by the preceding lemma, $\Gamma \vDash_{\mathcal{K}} A$ if and only if $\Gamma \vDash_{\mathsf{Q}} A$. $\square$

4.2. Beth Semantics and TCC$_{\omega}$

The above theorem shows that **IPC**$^{\sim}$ is sound and complete with respect to a certain class of Beth models. The question remains, however, of whether it is sound and complete with respect to all Beth models. A problem lies in the soundness direction, of the validity of [RP]. In a Beth model, it is possible that a disjunction is forced at a world whilst neither of the disjuncts is.

This is contrastable with an admissible [4] rule [RC] $\dfrac{A \to B}{\sim B \to \sim A}$ of $\mathbf{IPC}^{\sim}$. Given any Beth model and assuming $A \to B$ is valid, if $\sim B$ is forced at a node $b' \succeq b$ given an arbitrary b, then $\langle\rangle$ does not force B, so $\langle\rangle$ cannot force A either; thus we can conclude b' forces $\sim A$ and so b forces $\sim B \to \sim A$, i.e. $\sim B \to \sim A$ is valid.

This admissibility of [RC] in Beth models motivates us to consider a variant of $\mathbf{IPC}^{\sim}$ in which [RP] is replaced with [RC]. As already mentioned in [4], such a logic is known under the name $\mathbf{TCC}_\omega$, formulated by Gordienko in [7].

Definition 4.5 ($\mathbf{TCC}_\omega$). $\mathbf{TCC}_\omega$ is defined by axioms [Ax1] to [Ax10], and rules [MP] and [RC] $\dfrac{A \to B}{\sim B \to \sim A}$.

We shall denote the provability in $\mathbf{TCC}_\omega$ by $\vdash_t$. We shall prove the soundness and completeness of $\mathbf{TCC}_\omega$ with respect to all Beth models. Again we want to embed Kripke models into Beth models; but as we see below, the Kripke models for $\mathbf{TCC}_\omega$ are not necessarily rooted. So we shall embed models in a slightly different way.

Definition 4.6 (Kripke model for $\mathbf{TCC}_\omega$). A *Kripke Frame* $\mathcal{F}^t_{\mathcal{K}} = (W, \leq)$ for $\mathbf{TCC}_\omega$ is a non-empty partially ordered set. A *Kripke model* $\mathcal{M}^t_{\mathcal{K}}$ for $\mathbf{TCC}_\omega$ is a pair $(\mathcal{F}^t_{\mathcal{K}}, \mathcal{V})$, where $\mathcal{V}$ is a monotone mapping that assigns a set of worlds $\mathcal{V}(p) \subseteq W$ for each propositional variable p.

Given $\mathcal{M}^t_{\mathcal{K}}$, The *forcing* of a formula in a world, denoted $\mathcal{M}^t_{\mathcal{K}}, w \Vdash_{\mathcal{K}t} A$, is inductively defined as follows.

$$
\begin{array}{lll}
\mathcal{M}^t_{\mathcal{K}}, w \Vdash_{\mathcal{K}t} p & \iff & w \in \mathcal{V}(p). \\
\mathcal{M}^t_{\mathcal{K}}, w \Vdash_{\mathcal{K}t} A \wedge B & \iff & \mathcal{M}^t_{\mathcal{K}}, w \Vdash_{\mathcal{K}t} A \text{ and } \mathcal{M}^t_{\mathcal{K}}, w \Vdash_{\mathcal{K}t} B. \\
\mathcal{M}^t_{\mathcal{K}}, w \Vdash_{\mathcal{K}t} A \vee B & \iff & \mathcal{M}^t_{\mathcal{K}}, w \Vdash_{\mathcal{K}t} A \text{ or } \mathcal{M}^t_{\mathcal{K}}, w \Vdash_{\mathcal{K}t} B. \\
\mathcal{M}^t_{\mathcal{K}}, w \Vdash_{\mathcal{K}t} A \to B & \iff & \text{for all } w' \geq w, \text{ if } \mathcal{M}^t_{\mathcal{K}}, w' \Vdash_{\mathcal{K}t} A, \\
 & & \text{then } \mathcal{M}^t_{\mathcal{K}}, w' \Vdash_{\mathcal{K}t} B. \\
\mathcal{M}^t_{\mathcal{K}}, w \Vdash_{\mathcal{K}t} \sim A & \iff & \mathcal{M}^t_{\mathcal{K}}, w' \nVdash_{\mathcal{K}t} A \text{ for some } w'.
\end{array}
$$

Theorem 4.7 (Kripke completeness for $\mathbf{TCC}_\omega$). $\vdash_t A$ *if and only if* $\models_{\mathcal{K}t} A$.

Proof: Cf. [7] □

Given a Kripke model $\mathcal{M}^t_{\mathcal{K}} = (W_K, \leq, \mathcal{V}_K)$ for $\mathbf{TCC}_\omega$, we construct a corresponding Beth model $\mathcal{M}_{\mathcal{B}} = (W_B, \preceq, \mathcal{V}_B)$ with the following stipulation.

- W_B is the set of finite nondecreasing sequences in $(W_K, \leq)$ of length ≥ 0.
- $\preceq$ is defined accordingly.
- Define an auxiliary valuation $\bar{\mathcal{V}}_B(p)$ s.t. $\langle w_0, \ldots w_n\rangle \in \bar{\mathcal{V}}_B(p)$ if and only if $w_n \in \mathcal{V}_K(p)$.
- Then $\mathcal{V}_B(p) = \bar{\mathcal{V}}_B(p) \cup \{\langle\rangle\}$ if $\mathcal{V}_K(p) = W_K$; otherwise $\mathcal{V}_B(p) = \bar{\mathcal{V}}_B(p)$.

LEMMA 4.8 (embeddability of Kripke models for $\mathbf{TCC}_\omega$).

(i) *$\mathcal{M}_{\mathcal{B}}$ is indeed a Beth model.*

(ii) *$\mathcal{M}^t_{\mathcal{K}} \vDash_{\mathcal{K}t} A$ if and only $\mathcal{M}_{\mathcal{B}} \vDash_{\mathcal{B}} A$.*

PROOF: In the following, we shall occasionally write $\langle b_0, \ldots, b_{-1}\rangle$ to mean $\langle\rangle$. (This is purely a conventional notation to simplify the exposition, and should not be confused with the notation in the definition of $\bar{\mathcal{V}}_B(p)$, in which n cannot be -1.)

(i) We need to show that the assignment is covering. Suppose $\langle b_0, \ldots, b_n\rangle \in \mathcal{V}_B(p)$. If $n = -1$, then $\langle\rangle \in \mathcal{V}_B(p)$. So by definition of $\mathcal{V}_B$, $w \in \mathcal{V}_K(p)$ for all $w \in W_K$. Hence for each $\alpha = \langle w, \ldots\rangle \in \langle\rangle$, $\langle w\rangle \in \mathcal{V}_B(p)$; so $\exists m(\bar{\alpha}m \in \mathcal{V}_B(p))$. If $n > -1$, then $\langle b_0, \ldots, b_n\rangle \in \mathcal{V}_B(p)$ immediately implies $\forall\alpha \in \langle b_0, \ldots, b_n\rangle \exists m(\bar{\alpha}m \in \mathcal{V}_B(p))$.

Conversely, suppose $\forall\alpha \in \langle b_0, \ldots, b_n\rangle \exists m(\bar{\alpha}m \in \mathcal{V}_B(p))$. If $n = -1$, then for any $w \in W_K$, $\langle w, w, \ldots\rangle \in \langle\rangle$. By our supposition, either $\langle\rangle \in \mathcal{V}_B(p)$ or $\langle w, w, \ldots, w\rangle \in \mathcal{V}_B(p)$. In both cases, $w \in \mathcal{V}_K(p)$. Hence $W_K = \mathcal{V}_K(p)$. Thus $\langle\rangle \in \mathcal{V}_B(p)$, as required. If $n > -1$, then $\langle b_0, \ldots, b_n, b_n, \ldots\rangle \in \langle b_0, \ldots, b_n\rangle$. So either $\langle\rangle \in \mathcal{V}_B(p)$, $\langle b_0, \ldots, b_i\rangle \in \mathcal{V}_B(p)$ for $i < n$, or $\langle b_0, \ldots, b_n, b_n, \ldots, b_n\rangle \in \mathcal{V}_B(p)$. In the first case, $b_n \in \mathcal{V}_K(p)$. In the second case, $b_i \in \mathcal{V}_K(p)$, so by the monotonicity of $\mathcal{V}_K$, $b_n \in \mathcal{V}_K$. In the last case, $b_n \in \mathcal{V}_K(p)$. So in any case, $\langle b_0, \ldots, b_n\rangle \in \mathcal{V}_B(p)$.

(ii) It suffices to show:

(a) $\langle\rangle \Vdash_{\mathcal{B}} A$ if and only if $\mathcal{M}^t_{\mathcal{K}} \vDash_{\mathcal{K}t} A$.

(b) $\langle b_0, \ldots, b_n\rangle \Vdash_{\mathcal{B}} A$ if and only if $b_n \Vdash_{\mathcal{K}t} A$. (where $n > -1$)

We prove these by simultaneous induction on the complexity of A.

If $A \equiv p$, then 1. and 2. follow by definition.

If $A \equiv A_1 \wedge A_2$, then for 1. $\langle\rangle \Vdash_{\mathcal{B}} A_1 \wedge A_2$ if and only if $\langle\rangle \Vdash_{\mathcal{B}} A_1$ and $\langle\rangle \Vdash_{\mathcal{B}} A_2$ By I.H. this is equivalent to $\mathcal{M}^t_{\mathcal{K}} \vDash_{\mathcal{K}t} A_1$ and $\mathcal{M}^t_{\mathcal{K}} \vDash_{\mathcal{K}t} A_2$, which in turn is equivalent to $\mathcal{M}^t_{\mathcal{K}} \vDash_{\mathcal{K}t} A_1 \wedge A_2$. For 2., $\langle b_0, \dots, b_n\rangle \Vdash_{\mathcal{B}} A_1 \wedge A_2$ if and only if $\langle b_0, \dots, b_n\rangle \Vdash_{\mathcal{B}} A_1$ and $\langle b_0, \dots, b_n\rangle \Vdash_{\mathcal{B}} A_2$. By I.H. this is equivalent to $b_n \Vdash_{\mathcal{K}t} A_1$ and $b_n \Vdash_{\mathcal{K}t} A_2$, which in turn is equivalent to $b_n \Vdash_{\mathcal{K}t} A_1 \wedge A_2$.

If $A \equiv A_1 \vee A_2$, then for 1., $\langle\rangle \Vdash_{\mathcal{B}} A_1 \vee A_2$ if and only if $\forall \alpha \in \langle\rangle \exists m(\bar{\alpha}m \Vdash_{\mathcal{B}} A_1$ or $\bar{\alpha}m \Vdash_{\mathcal{B}} A_2)$. For each $w \in W_K$, $\langle w, w, \dots\rangle \in \langle\rangle$, so either $\langle\rangle \Vdash_{\mathcal{B}} A_1$, $\langle\rangle \Vdash_{\mathcal{B}} A_2$, $\langle w, \dots, w\rangle \Vdash_{\mathcal{B}} A_1$ or $\langle w, \dots, w\rangle \Vdash_{\mathcal{B}} A_2$. If one of the former two cases holds, then by I.H. $\mathcal{M}^t_{\mathcal{K}} \vDash_{\mathcal{K}t} A_i$, for one of $i \in \{1, 2\}$; so $w \Vdash_{\mathcal{K}t} A_1 \vee A_2$. If one of the latter two cases hold, then by I.H. $w \Vdash_{\mathcal{K}t} A_i$ for one of $i \in \{1, 2\}$; so $w \Vdash_{\mathcal{K}t} A_1 \vee A_2$. Hence we conclude $w \Vdash_{\mathcal{K}t} A_1 \vee A_2$ for all $w \in W_K$, i.e. $\mathcal{M}^t_{\mathcal{K}} \vDash_{\mathcal{K}t} A_1 \vee A_2$. For the converse direction, assume $\mathcal{M}^t_{\mathcal{K}} \vDash_{\mathcal{K}t} A_1 \vee A_2$ and let $\alpha = \langle w, \dots\rangle \in \langle\rangle$. Then since $w \Vdash_{\mathcal{K}t} A_1$ or $w \Vdash_{\mathcal{K}t} A_2$, $\langle w\rangle \Vdash_{\mathcal{B}} A_1$ or $\langle w\rangle \Vdash_{\mathcal{B}} A_2$ by I.H.. Thus $\forall \alpha \in \langle\rangle \exists m(\bar{\alpha}m \Vdash_{\mathcal{B}} A_1$ or $\bar{\alpha}m \Vdash_{\mathcal{B}} A_2)$. Hence $\langle\rangle \Vdash_{\mathcal{B}} A_1 \vee A_2$.

For 2. If $\langle b_0, \dots, b_n\rangle \Vdash_{\mathcal{B}} A_1 \vee A_2$, then for all $\alpha \in \langle b_0, \dots, b_n\rangle$ there exists m s.t. $\bar{\alpha}m \Vdash_{\mathcal{B}} A_1$ or $\bar{\alpha}m \Vdash_{\mathcal{B}} A_2$. As $\langle b_0, \dots, b_n, b_n, \dots\rangle \in \langle b_0, \dots, b_n\rangle$, we have, for $i \in \{1, 2\}$, either $\langle\rangle \Vdash_{\mathcal{B}} A_i$, $\langle b_0, \dots, b_l\rangle \Vdash_{\mathcal{B}} A_i$ for $l \leq n$, or $\langle b_0, \dots, b_n, b_n, \dots, b_n\rangle \Vdash_{\mathcal{B}} A_i$. In each case $b_n \Vdash_{\mathcal{K}t} A_i$ by I.H.; so $b_n \Vdash_{\mathcal{K}t} A_1 \vee A_2$. Conversely, if $b_n \Vdash_{\mathcal{K}t} A_1 \vee A_2$, then $b_n \Vdash_{\mathcal{K}t} A_1$ or $b_n \Vdash_{\mathcal{K}t} A_2$. So by I.H. $\langle b_0, \dots, b_n\rangle \Vdash_{\mathcal{B}} A_1$ or $\langle b_0, \dots, b_n\rangle \Vdash_{\mathcal{B}} A_2$. Hence immediately $\forall \alpha \in \langle b_0, \dots, b_n\rangle \exists m(\bar{\alpha}m \Vdash_{\mathcal{B}} A_1$ or $\bar{\alpha}m \Vdash_{\mathcal{B}} A_2)$, i.e. $\langle b_0, \dots, b_n\rangle \Vdash_{\mathcal{B}} A_1 \vee A_2$.

If $A \equiv A_1 \to A_2$, then for 1., suppose $\langle\rangle \Vdash_{\mathcal{B}} A_1 \to A_2$. Let $w \in W_K$ and $w' \geq w$. If $w' \Vdash_{\mathcal{K}t} A_1$, then $\langle w'\rangle \Vdash_{\mathcal{B}} A_1$ by I.H.. So $\langle w'\rangle \Vdash_{\mathcal{B}} A_2$ and thus $w' \Vdash_{\mathcal{K}t} A_2$. Consequently $w \Vdash_{\mathcal{K}t} A_1 \to A_2$ and so $\mathcal{M}^t_{\mathcal{K}} \vDash_{\mathcal{K}t} A_1 \to A_2$. Conversely, suppose $\mathcal{M}^t_{\mathcal{K}} \vDash_{\mathcal{K}t} A_1 \to A_2$. Let $\langle b_0, \dots, b_n\rangle \Vdash_{\mathcal{B}} A_1$. If $n = -1$, then by I.H. $\mathcal{M}^t_{\mathcal{K}} \vDash_{\mathcal{K}t} A_1$, so $\mathcal{M}^t_{\mathcal{K}} \vDash_{\mathcal{K}t} A_2$. Hence $\langle b_0, \dots, b_n\rangle \Vdash_{\mathcal{B}} A_2$ again by I.H.. If $n > -1$, then $b_n \Vdash_{\mathcal{K}t} A_1$, so $b_n \Vdash_{\mathcal{K}t} A_2$. Hence $\langle b_0, \dots, b_n\rangle \Vdash_{\mathcal{B}} A_2$. Thus $\langle\rangle \Vdash_{\mathcal{B}} A_1 \to A_2$.

For 2., suppose $\langle b_0, \ldots, b_n \rangle \Vdash_{\mathcal{B}} A_1 \to A_2$ and let $b_{n'} \geq b_n$. If $b_{n'} \Vdash_{\mathcal{K}t} A_1$, then by I.H. $\langle b_0, \ldots, b_n, b_{n'} \rangle \Vdash_{\mathcal{B}} A_1$; so $\langle b_0, \ldots, b_n, b_{n'} \rangle \Vdash_{\mathcal{B}} A_2$. Thus $b_{n'} \Vdash_{\mathcal{K}t} A_2$. Hence $b_n \Vdash_{\mathcal{K}t} A_1 \to A_2$. Conversely, suppose $b_n \Vdash_{\mathcal{K}t} A_1 \to A_2$. Assume $\langle b_0, \ldots, b_n, \ldots, b_{n'} \rangle \Vdash_{\mathcal{B}} A_1$. Then $b_n \leq b_{n'}$ and $b_{n'} \Vdash_{\mathcal{K}t} A_1$. So $b_{n'} \Vdash_{\mathcal{K}t} A_2$. Thus $\langle b_0, \ldots, b_n, \ldots, b_{n'} \rangle \Vdash_{\mathcal{B}} A_2$. Therefore $\langle b_0, \ldots, b_n \rangle \Vdash_{\mathcal{B}} A_1 \to A_2$.

If $A \equiv {\sim}A_1$, then for 1., suppose $\langle\rangle \Vdash_{\mathcal{B}} {\sim}A_1$. Then $\langle\rangle \nVdash_{\mathcal{B}} A_1$. So $\mathcal{M}^t_{\mathcal{K}} \nvDash_{\mathcal{K}t} A_1$ by I.H.. Hence $w \nVdash_{\mathcal{K}t} A_1$ for some $w \in W_K$. Thus $u \Vdash_{\mathcal{K}t} {\sim}A$ for all $u \in W_K$. Thus $\mathcal{M}^t_{\mathcal{K}} \vDash_{\mathcal{K}t} {\sim}A$. Conversely, suppose $\mathcal{M}^t_{\mathcal{K}} \vDash_{\mathcal{K}t} {\sim}A$. Take $w \in W_K$. Then $w \Vdash_{\mathcal{K}t} {\sim}A$, so $u \nVdash_{\mathcal{K}t} A$ for some $u \in W_K$. Hence $\mathcal{M}^t_{\mathcal{K}} \nvDash_{\mathcal{K}t} A$, so $\langle\rangle \nVdash_{\mathcal{B}} A$ by I.H.. Therefore $\langle\rangle \Vdash_{\mathcal{B}} {\sim}A$.

For 2., suppose $\langle b_0, \ldots, b_n \rangle \Vdash_{\mathcal{B}} {\sim}A$. Then $\langle\rangle \nVdash_{\mathcal{B}} A$. So $\mathcal{M}^t_{\mathcal{K}} \nvDash_{\mathcal{K}t} A$. Hence for some $w \in W_K$, $w \nVdash_{\mathcal{K}t} A$. Therefore $b_n \Vdash_{\mathcal{K}t} {\sim}A$. Conversely, if $b_n \Vdash_{\mathcal{K}t} {\sim}A$, then $w \nVdash_{\mathcal{K}t} A$ for some $w \in W_K$. By I.H. $\langle w \rangle \nVdash_{\mathcal{B}} A$. Thus $\langle\rangle \nVdash_{\mathcal{B}} A$. Therefore $\langle b_0, \ldots, b_n \rangle \Vdash_{\mathcal{B}} {\sim}A$. □

THEOREM 4.9 (soundness and weak completeness of $\mathbf{TCC}_\omega$ with Beth semantics). $\vdash_t A$ *if and only if* $\vDash_{\mathcal{B}} A$.

PROOF: We first show the soundness by induction on the depth of deductions. We check [Ax9],[Ax10] and [RC]. Let $\mathcal{M}_{\mathcal{B}} = (W_B, \preceq, \mathcal{V}_B)$ be a Beth model. By monotonicity, it suffices to check the root. For [Ax9], either $\langle\rangle \Vdash_{\mathcal{B}} A$ or $\langle\rangle \nVdash_{\mathcal{B}} A$. If the latter, $\langle\rangle \Vdash_{\mathcal{B}} {\sim}A$. So in either case, $\langle\rangle \Vdash_{\mathcal{B}} A \vee {\sim}A$. For [Ax10], if $b \Vdash_{\mathcal{B}} {\sim}A$ for $b \succeq \langle\rangle$, then if $b' \Vdash_{\mathcal{B}} {\sim}{\sim}A$ for $b' \succeq b$, then $\langle\rangle \nVdash_{\mathcal{B}} {\sim}A$ and $\langle\rangle \nVdash_{\mathcal{B}} A$. But the former implies $\langle\rangle \Vdash_{\mathcal{B}} A$, a contradiction. Therefore $b' \Vdash_{\mathcal{B}} B$; so $\langle\rangle \Vdash_{\mathcal{B}} {\sim}A \to ({\sim}{\sim}A \to B)$. For [RC], by I.H., $\vDash_{\mathcal{B}} A \to B$ and in particular, $\mathcal{M}_{\mathcal{B}} \vDash_{\mathcal{B}} A \to B$. If for $b \succeq \langle\rangle$ we have $b \Vdash_{\mathcal{B}} {\sim}B$, then $\langle\rangle \nVdash_{\mathcal{B}} B$. Now if $\langle\rangle \Vdash_{\mathcal{B}} A$, then as $\langle\rangle \Vdash_{\mathcal{B}} A \to B$, $\langle\rangle \Vdash_{\mathcal{B}} B$, a contradiction. Thus $\langle\rangle \nVdash_{\mathcal{B}} A$; hence $b \Vdash_{\mathcal{B}} {\sim}A$. So $\langle\rangle \Vdash_{\mathcal{B}} {\sim}B \to {\sim}A$.

The completeness follows from the previous lemma and the Kripke completeness of $\mathbf{TCC}_\omega$ [7, Theorem 4.5]. □

4.3. Classical Logic and $\mathbf{TCC}_\omega$

The fact that Kripke and Beth semantics differ on the forcing of disjunction is well-reflected in the following translation of classical logic (**CPC**) into $\mathbf{TCC}_\omega$.

Definition 4.10 (**CPC**). **CPC** is defined by Axioms [Ax1]-[Ax9] and $\sim A \to (A \to B)$ ([Ax10']), plus the rule [MP].

We denote the derivability in **CPC** by $\vdash_{CL}$

Definition 4.11 ($()^t$). We inductively define $()^t$ to be a mapping between formulae in $\mathcal{L}$.

$$
\begin{aligned}
p^t &\equiv p \\
(A \wedge B)^t &\equiv A^t \wedge B^t. \\
(A \vee B)^t &\equiv {\sim\sim} A^t \vee {\sim\sim} B^t. \\
(A \to B)^t &\equiv {\sim\sim} A^t \to {\sim\sim} B^t. \\
({\sim} A)^t &\equiv {\sim} A^t.
\end{aligned}
$$

Beth-semantically speaking, $()^t$ restricts our attention to the root world, when it comes to disjunction and implication. This is related to the connection between empirical negation (of $\mathbf{IPC}^{\sim}$) and classical negation, as observed in [3] and [4]. A new point for $\mathbf{TCC}_\omega$ is that the restriction applies not only to implication but also to disjunction. This corresponds to the fact that in Beth semantics, both disjunction and implication look at other worlds, whereas in Kripke semantics, only the latter does so.

In the following, we make a heavy use of easily checkable equivalences in Beth semantics.

- $b \Vdash_{\mathcal{B}} {\sim\sim} A \iff \langle\rangle \Vdash_{\mathcal{B}} A$.
- $b \Vdash_{\mathcal{B}} {\sim\sim} A \vee {\sim\sim} B \iff \langle\rangle \Vdash_{\mathcal{B}} A$ or $\langle\rangle \Vdash_{\mathcal{B}} B$.
- $b \Vdash_{\mathcal{B}} {\sim\sim} A \to {\sim\sim} B \iff \langle\rangle \Vdash_{\mathcal{B}} A$ implies $\langle\rangle \Vdash_{\mathcal{B}} B$.

Let us use the notation $\Gamma^t := \{B^t : B \in \Gamma\}$. We shall henceforth abbreviate ${\sim\sim} A$ as $\approx A$. Metalinguistic 'implies' ($\Rightarrow$) should not be confused with $\to$ in the proof below.

Proposition 4.12 (faithful embedding of **CPC** into $\mathbf{TCC}_\omega$). $\Gamma \vdash_{CL} A$ if and only if $\Gamma^t \vdash_t A^t$.

Proof: The left-to-right direction is shown by induction on the depth of deductions. If A is an assumption, then correspondingly $A^t \in \Gamma^t$.

If A is an axiom, we exemplify by the case for the axiom $(A \to C) \to ((B \to C) \to (A \vee B \to C))$. $((A \to C) \to ((B \to C) \to (A \vee B \to C)))^t$ is

$$\approx(\approx A^t \to \approx C^t) \to \approx(\approx(\approx B^t \to \approx C^t) \to \approx(\approx(\approx A^t \vee \approx B^t) \to \approx C^t)).$$

Using Beth completeness, it is sufficient to show,

$$b \Vdash_{\mathcal{B}} \approx(\approx A^t \to \approx C^t) \to \approx(\approx(\approx B^t \to \approx C^t) \to \approx(\approx(\approx A^t \vee \approx B^t \to \approx C^t)))$$

holds for any b in an arbitrary Beth model. This is equivalent to

$$\begin{array}{c}\langle\rangle \Vdash_{\mathcal{B}} \approx A^t \to \approx C^t \\ \text{implies } \langle\rangle \Vdash_{\mathcal{B}} \approx(\approx B^t \to \approx C^t) \to \approx(\approx(\approx A^t \vee \approx B^t) \to \approx C^t)\end{array}$$

by one of the above equivalences; this is further equivalent to

$$\begin{array}{c}\langle\rangle \Vdash_{\mathcal{B}} A^t \Rightarrow \langle\rangle \Vdash_{\mathcal{B}} C^t \\ \text{implies } (\langle\rangle \Vdash_{\mathcal{B}} \approx B^t \to \approx C^t) \Rightarrow (\langle\rangle \Vdash_{\mathcal{B}} \approx(\approx(\approx A^t \vee \approx B^t) \to \approx C^t))\end{array}$$

and to

$$\begin{array}{c}\langle\rangle \Vdash_{\mathcal{B}} A^t \Rightarrow \langle\rangle \Vdash_{\mathcal{B}} C^t \\ \text{implies } (\langle\rangle \Vdash_{\mathcal{B}} B^t \Rightarrow \langle\rangle \Vdash_{\mathcal{B}} C^t) \Rightarrow (\langle\rangle \Vdash_{\mathcal{B}} \approx A^t \vee \approx B^t \Rightarrow \langle\rangle \Vdash_{\mathcal{B}} C^t)\end{array}$$

and to

$$\begin{array}{c}(\langle\rangle \Vdash_{\mathcal{B}} A^t \Rightarrow \langle\rangle \Vdash_{\mathcal{B}} C^t) \text{ and } (\langle\rangle \Vdash_{\mathcal{B}} B^t \Rightarrow \langle\rangle \Vdash_{\mathcal{B}} C^t) \\ \text{implies } ((\langle\rangle \Vdash_{\mathcal{B}} A^t \text{ or } \langle\rangle \Vdash_{\mathcal{B}} B^t) \Rightarrow \langle\rangle \Vdash_{\mathcal{B}} C^t))\end{array}$$

and this holds. Here, if it were the case that $(A \vee B)^t \equiv (A^t \vee B^t)$, then we would get $\langle\rangle \Vdash_{\mathcal{B}} A^t \vee B^t$ instead of $\langle\rangle \Vdash_{\mathcal{B}} \approx A^t \vee \approx B^t$, and the formula fails to hold.

If the deduction ends with an application of [MP] $\dfrac{B \qquad B \to A}{A}$, then by I.H., $\Gamma^t \vdash_t B^t$ and $\Gamma^t \vdash_t {\sim\sim} B^t \to {\sim\sim} A^t$. In [4, Lemma 2.8] the rule $\dfrac{A}{\sim\sim A}$ [RD] is shown to be derivable from [RC] in $\mathbf{IPC}^{\sim}$. The proof appeals to [RP] only non-essentially (it is used to derive ${\sim\sim} A \to A$, which is obtainable from [Ax9] and [Ax10] alone), and so [RD] is also derivable in $\mathbf{TCC}_\omega$. Thus we obtain $\Gamma^t \vdash_t {\sim\sim} B^t$. So by [MP], $\Gamma^t \vdash_t {\sim\sim} A^t$; hence $\Gamma^t \vdash_t A^t$ by double negation elimination.

The right-to-left direction follows from the easily noticeable equivalence that $\vdash_{CL} A \leftrightarrow A^t$. $\square$

Before moving on, we shall mention that there exists another reading of the negation in the Beth semantics for $\mathbf{TCC}_\omega$. Because the models are rooted, for any b, $\exists b' \leq b(b' \nVdash A) \Leftrightarrow \langle\rangle \nVdash A$. From this viewpoint the negation of $\mathbf{TCC}_\omega$ can be understood as co-negation as well. For Kripke semantics, the logic of co-negation is the logic **daC** of Priest [11]. A Hilbert-style axiomatisation of **daC** was first formulated by Castiglioni et al. [2]. This axiomatisation is obtained from that of $\mathbf{IPC}^{\sim}$ by removing [Ax10]. If we further replace [RP] with [RC], and add an axiom $\sim\sim A \to A$ (a theorem of **daC**), we obtain the logic $\mathbf{CC}_\omega$ of Sylvan [15]. Note $\mathbf{CC}_\omega$ can be strengthened to $\mathbf{TCC}_\omega$ by adding [Ax10] and dropping $\sim\sim A \to A$, which becomes redundant.

5. Eliminating [RP]

The last section made clear that the negations of $\mathbf{IPC}^{\sim}$ and $\mathbf{TCC}_\omega$ are characterised by the same valuation, but with respect to different semantics: Kripke and Beth. We may understand them as representing different types of experience, and thus different empirical negations. We can make an analogous remark for co-negation. This case is perhaps more interesting, for $\mathbf{TCC}_\omega$ and **daC** are not comparable [10]. In any case, these curious effects of "same forcing-condition in two similar semantics" encourage a further analysis.

Proof-theoretically, however, there is an obstacle in comparing the logics, in that $\mathbf{TCC}_\omega$ and $\mathbf{CC}_\omega$ employ the rule [RC], whereas **daC** and $\mathbf{IPC}^{\sim}$ employ the stronger [RP].

We would like, therefore, to have a new axiomatisation of $\mathbf{IPC}^{\sim}$ and **daC** with [RC], rather than [RP]. We can expect such conversion would allow us to analyse and understand the logics from a more unified perspective.

We shall start such an attempt with $\mathbf{IPC}^{\sim}$, using a provable formula of $\mathbf{IPC}^{\sim}$, $(\sim A \wedge \sim B) \to \sim(A \vee B)$ [4, Proposition 2.14].

Proposition 5.1. The addition of $(\sim A \wedge \sim B) \to \sim(A \vee B)$ to $\mathbf{TCC}_\omega$ derives [RP].

Proof: In $\mathbf{TCC}_\omega$, assuming $(A \vee B)$ we can derive $\sim\sim(A \vee B)$ by [RD]. So we have $\sim B \to (\sim A \to \sim\sim(A \vee B))$. Also we infer from $\sim B \to (\sim A \to (\sim A \wedge \sim B))$ and $(\sim A \wedge \sim B) \to \sim(A \vee B)$ that $\sim B \to (\sim A \to \sim(A \vee B))$. Thus $\sim B \to (\sim A \to (\sim(A \vee B) \wedge \sim\sim(A \vee B)))$. Also by [Ax10], $\sim(A \vee B) \to (\sim\sim(A \vee B) \to B)$. Combine the two and we obtain $\sim B \to (\sim A \to B)$. Then as $B \to (\sim A \to B)$ follows from [Ax1], and $B \vee \sim B$ follows from [Ax9], we conclude $\sim A \to B$. $\square$

Hence we have obtained an alternative axiomatisation of $\mathbf{IPC}^{\sim}$ with [RC].

It is stated in [4] that $\mathbf{TCC}_\omega$ is a strict subsystem of $\mathbf{IPC}^{\sim}$, but no specific example is shown. As a side remark, we can use $(\sim A \wedge \sim B) \to \sim(A \vee B)$ to observe the following.

Proposition 5.2. $(\sim A \wedge \sim B) \to \sim(A \vee B)$ is underivable in $\mathbf{TCC}_\omega$.

Proof: We prove it via Beth completeness. Let $\mathcal{F}_{\mathcal{B}} = (W, \preceq)$ be the set of finite binary sequences ordered by the initial segment relation. Let $\mathcal{M}_{\mathcal{B}} = (\mathcal{F}_{\mathcal{B}}, \mathcal{V})$ be a model such that $b \in \mathcal{V}(p) \Leftrightarrow \langle 0 \rangle \preceq b$ and $b \in \mathcal{V}(q) \Leftrightarrow \langle 1 \rangle \preceq b$. Then it is straightforward to see that this assignment is covering: e.g. if $\forall \alpha \in b \exists m (\bar{\alpha} m \Vdash_{\mathcal{B}} p)$, then clearly $\langle 0 \rangle \preceq b$. Now $\mathcal{M}_{\mathcal{B}}, \langle\rangle \nVdash_{\mathcal{B}} p$ and $\mathcal{M}_{\mathcal{B}}, \langle\rangle \nVdash_{\mathcal{B}} q$, so $\mathcal{M}_{\mathcal{B}}, \langle\rangle \Vdash_{\mathcal{B}} \sim p \wedge \sim q$; but since $\forall \alpha \in \langle\rangle (\bar{\alpha} 1 \Vdash_{\mathcal{B}} p$ or $\bar{\alpha} 1 \Vdash_{\mathcal{B}} q)$, we have $\mathcal{M}_{\mathcal{B}}, \langle\rangle \Vdash_{\mathcal{B}} p \vee q$, i.e. $\mathcal{M}_{\mathcal{B}}, \langle\rangle \nVdash_{\mathcal{B}} \sim(p \vee q)$. Therefore $\mathcal{M}_{\mathcal{B}}, \langle\rangle \nVdash_{\mathcal{B}} (\sim p \wedge \sim q) \to \sim(p \vee q)$. $\square$

Corollary 5.3 (failure of soundness for $\mathbf{IPC}^{\sim}$ with all Beth models). $\vdash_{\sim} A \not\Rightarrow \vDash_{\mathcal{B}} A$.

Proof: Otherwise $\vdash_{\sim} A \Rightarrow \vDash_{\mathcal{B}} A \Leftrightarrow \vdash_t A$, which is absurd. $\square$

Ferguson [6, Theorem 2.3] gives the frame property of $(\sim A \wedge \sim B) \to \sim(A \vee B)$ with respect to $\mathbf{daC}$. We just mention a quite similar observation can be made for the Kripke models for $\mathbf{CC}_\omega$.

Definition 5.4 (Semantics of $\mathbf{CC}_\omega$). A *Kripke frame* $\mathcal{F}^{c}_{\mathcal{K}}$ for $\mathbf{CC}_\omega$ is a triple $(W, \leq, S)$, where $S \subset W \times W$ is a reflexive and symmetric (accessibility) relation such that $u \leq v$ and uSw implies vSw, i.e. S is upward closed. A *Kripke model* $\mathcal{M}^c_{\mathcal{K}}$ for $\mathbf{CC}_\omega$ is defined as usual, except for the forcing condition ($\Vdash_{\mathcal{K}c}$) of negation, which is

$$\mathcal{M}^c_{\mathcal{K}}, w \Vdash_{\mathcal{K}c} {\sim}A \Longleftrightarrow \mathcal{M}^c_{\mathcal{K}}, w' \nVdash_{\mathcal{K}c} A \text{ for some } w' \text{ such that } wSw'.$$

Note if $S = W \times W$, then a $\mathbf{CC}_\omega$-frame (model) is a $\mathbf{TCC}_\omega$-frame (model) [7]. Indeed, what is shown in [7] is that $\mathbf{TCC}_\omega$ is sound and complete with the class of $\mathbf{CC}_\omega$-frames where S is transitive, and in particular the frames with $S = W \times W$ is sufficient for this. We shall occasionally denote uSv also by $vS^{-1}u$. As S is symmetric in $\mathbf{CC}_\omega$, this distinction is not quite necessary. This however clarifies appeals to symmetry in proofs, which becomes significant in a broader context.

Proposition 5.5. Let $\mathcal{F}^c_{\mathcal{K}}$ be a $\mathbf{CC}_\omega$-frame. Then the following conditions are equivalent:

(i) $\mathcal{F}^c_{\mathcal{K}} \vDash_{\mathcal{K}c} ({\sim}A \wedge {\sim}B) \to {\sim}(A \vee B)$ for all A, B.

(ii) $\mathcal{F}^c_{\mathcal{K}}$ satisfies $\forall u, v, w(uSv$ and uSw implies $\exists xS^{-1}u(v \geq x$ and $w \geq x)$.

Proof: We shall first see (i) implies (ii). Suppose uSv and uSw. Let $\mathcal{V}(p) = \{x : v \ngeq x\}$ and $\mathcal{V}(q) = \{x : w \ngeq x\}$. Now if $w \in \mathcal{V}(p)$ and $x' \geq x$, then $v \geq x'$ implies $v \geq x$, a contradiction. So $v \ngeq x'$, and thus $x' \in \mathcal{V}(p)$. Hence $\mathcal{V}(p)$ is upward closed. Similarly $\mathcal{V}(q)$ is upward closed. Now since $v \geq v$ and $w \geq w$, $v \nVdash_{\mathcal{K}c} p$ and $w \nVdash_{\mathcal{K}c} q$. So $u \Vdash_{\mathcal{K}c} {\sim}p \wedge {\sim}q$. Hence by assumption $u \Vdash_{\mathcal{K}c} {\sim}(p \vee q)$. So there is an $xS^{-1}u$ such that $x \nVdash_{\mathcal{K}c} p$ (i.e. $v \geq x$) and $x \nVdash_{\mathcal{K}c} q$ (i.e. $w \geq x$), as we desired.

Next we shall see (ii) implies (i). Assume $\mathcal{F}^c_{\mathcal{K}}$ satisfies (ii) and $\mathcal{V}$, u_0 be arbitrary. If $(\mathcal{F}^c_{\mathcal{K}}, \mathcal{V}), u \Vdash_{\mathcal{K}c} {\sim}A \wedge {\sim}B$ for $u \geq u_0$, then there are $vS^{-1}u$ and $wS^{-1}u$ such that $v \nVdash_{\mathcal{K}c} A$ and $w \nVdash_{\mathcal{K}c} B$. By (ii), there is $xS^{-1}u$ such that $v \geq x$ and $w \geq x$. Now $x \nVdash_{\mathcal{K}c} A \vee B$. Hence $u \Vdash_{\mathcal{K}c} {\sim}(A \vee B)$. So $(\mathcal{F}^c_{\mathcal{K}}, \mathcal{V}), u_0 \Vdash_{\mathcal{K}c} ({\sim}A \wedge {\sim}B) \to {\sim}(A \vee B)$. Since w and $\mathcal{V}$ are arbitrary, $\mathcal{F}^c_{\mathcal{K}} \vDash_{\mathcal{K}c} ({\sim}A \wedge {\sim}B) \to {\sim}(A \vee B)$. $\square$

Given a Kripke frame for $\mathbf{IPC}^{\sim}$, we can regard it as a frame of $\mathbf{TCC}_\omega$ with $S = W \times W$; i.e. there is an embedding. Then it is immediately seen that such a frame satisfies the above condition, because it is rooted. This means the class of Kripke frames for $\mathbf{TCC}_\omega$ satisfying the above condition is complete with respect to $\mathbf{IPC}^{\sim}$, for if a formula is validated by each such frame, then it must be validated by each frame of $\mathbf{IPC}^{\sim}$.

Next we consider **daC**. The formula $\sim A \wedge \sim B \to \sim(A \vee B)$ used for $\mathbf{IPC}^{\sim}$ cannot be used for **daC**, because it is not a theorem of **daC** [9, Table 3]. We instead have to look at another formula $\sim(\sim(A \vee B) \vee A) \to B$.

PROPOSITION 5.6. $\mathbf{CC}_\omega + \sim(\sim(A \vee B) \vee A) \to B = \mathbf{daC}$.

PROOF: It has been observed in [9, Theorem 3.13] that $\sim(\sim(A \vee B) \vee A) \to B$ is a theorem of **daC**. So we only have to check [RP] is admissible in $\mathbf{CC}_\omega + \sim(\sim(A \vee B) \vee A) \to B$. We first note $\dfrac{A}{\sim A \to B}$ is derivable in $\mathbf{CC}_\omega$ by the same argument as in [10, Theorem 4.3]. Assuming $A \vee B$ is derivable, from this we see $\sim(A \vee B) \to A$ is derivable. By [Ax8], we infer $(\sim(A \vee B) \vee A) \to A$, and then by [RC], $\sim A \to \sim(\sim(A \vee B) \vee A)$. On the other hand, $\sim(\sim(A \vee B) \vee A) \to B$ is the added axiom. Thus we conclude $\sim A \to B$. □

$\sim(\sim(A \vee B) \vee A) \to B$ is used in [9, theorem 3.13] to establish that **daC** strictly contains another logic **daC'**, axiomatised by replacing [RP] with a weaker rule $\dfrac{A \vee \sim B}{\sim A \to \sim B}$ [wRP] . We shall note [wRP] in $\mathbf{daC}'$ is similarly reducible to an axiom $\sim(\sim(A \vee \sim B) \vee A) \to \sim B$.

PROPOSITION 5.7. $\mathbf{CC}_\omega + \sim(\sim(A \vee \sim B) \vee A) \to \sim B = \mathbf{daC}'$

PROOF: It has been observed in [10, Lemma 3.2] that $\sim(\sim(A \vee \sim B) \vee A) \to \sim B$ is a theorem of $\mathbf{daC}'$. So we only have to check [wRP] is admissible in $\mathbf{CC}_\omega + \sim(\sim(A \vee \sim B) \vee A) \to \sim B$. This is proved as in the previous proposition, except that we infer $\sim A \to \sim(\sim(A \vee \sim B) \vee A)$ and $\sim(\sim(A \vee \sim B) \vee A) \to \sim B$ to conclude $\sim A \to \sim B$. □

Next, we turn our attention to the semantic side. Our goal will be to establish a connection between the Kripke semantics of $\mathbf{CC}_\omega$ and **daC**. For this we shall first consider the frame condition for $\sim(\sim(A \vee B) \vee A) \to B$.

PROPOSITION 5.8. Let $\mathcal{F}^c_{\mathcal{K}}$ be a $\mathbf{CC}_\omega$-frame. Then the following conditions are equivalent:

(i) $\mathcal{F}^c_{\mathcal{K}} \vDash_{\mathcal{K}c} \sim(\sim(A \vee B) \vee A) \to B$ for all A, B.

(ii) $\mathcal{F}^c_{\mathcal{K}}$ satisfies $\forall u, v(uSv \to \exists wS^{-1}v(w \leq u \text{ and } w \leq v))$.

PROOF: We shall first see (i) implies (ii). We shall show the contrapositive. So suppose for some u and v, uSv holds but $\neg\exists wS^{-1}v(w \leq u$ and $w \leq$

v). Choose $\mathcal{V}$ s.t. $\mathcal{V}(p) = \{w : w \not\leq v\}$ and $\mathcal{V}(q) = \{w : w \not\leq u\}$. It is straightforward to see $\mathcal{V}(p)$ and $\mathcal{V}(q)$ are upward closed. Now since $\forall w S^{-1} v(w \not\leq u$ or $w \not\leq v)$, we have $\forall w S^{-1} v(w \Vdash_{\mathcal{K}c} p$ or $w \Vdash_{\mathcal{K}c} q)$. So $v \nVdash_{\mathcal{K}c} {\sim}(p \vee q)$. In addition, $v \leq v$ means $v \nVdash_{\mathcal{K}c} p$. Thus $u \Vdash_{\mathcal{K}c} {\sim}({\sim}(p \vee q) \vee p)$. On the other hand, $u \leq u$ implies $u \nVdash_{\mathcal{K}c} q$. Thus $u \nVdash_{\mathcal{K}c} {\sim}({\sim}(p \vee q) \vee p) \to q$. Therefore $\mathcal{F}^c_{\mathcal{K}} \nvDash_{\mathcal{K}c} {\sim}({\sim}(p \vee q) \vee p) \to q$.

Next we shall see (ii) implies (i). Assume $\forall u, v(uSv \to \exists w S^{-1} v(w \leq u$ and $w \leq v))$. Let $\mathcal{V}$ and u be arbitrary, and for $v \geq u$, suppose $(\mathcal{F}^c_{\mathcal{K}}, \mathcal{V}), v \Vdash_{\mathcal{K}c} {\sim}({\sim}(A \vee B) \vee A)$. Then for some $w S^{-1} v$, $w \nVdash_{\mathcal{K}c} {\sim}(A \vee B) \vee A$. Thus $w \nVdash_{\mathcal{K}c} A$ and $\forall x S^{-1} w(x \Vdash_{\mathcal{K}c} A \vee B)$. Now by assumption, from vSw we infer $\exists y S^{-1} w(y \leq v$ and $y \leq w)$. From our observation above, we know $y \Vdash_{\mathcal{K}c} A \vee B$. If $y \Vdash_{\mathcal{K}c} A$, then $y \leq w$ implies $w \Vdash_{\mathcal{K}c} A$, a contradiction. So $y \Vdash_{\mathcal{K}c} B$, which with $y \leq v$ implies $v \Vdash_{\mathcal{K}c} B$. Thus $(\mathcal{F}^c_{\mathcal{K}}, \mathcal{V}), u \Vdash_{\mathcal{K}c} {\sim}({\sim}(A \vee B) \vee A) \to B$. Since $\mathcal{V}$ and u are arbitrary, $\mathcal{F}^c_{\mathcal{K}} \vDash_{\mathcal{K}c} {\sim}({\sim}(A \vee B) \vee A) \to B$. □

Note that in the proof no appeal is made to neither the reflexivity nor symmetry of S. Thus we see the correspondence holds for a weaker setting of one of Došen's systems in [5, p.81–83] (under what he calls *condensed* frames). It has the same forcing condition, but the accessibility relation there is not assumed to be reflexive nor symmetric.

With the frame condition at hand, we can now translate back and forth the frames of $\mathbf{CC}_\omega$ and $\mathbf{daC}$.

Definition 5.9 (semantics of $\mathbf{daC}$). A *Kripke frame* $\mathcal{F}^d_{\mathcal{K}}$ for $\mathbf{daC}$ is a pair $(W, \leq)$, and a *Kripke model* $\mathcal{M}^d_{\mathcal{K}}$ for $\mathbf{daC}$ is defined as usual, except for the forcing condition ($\Vdash_{\mathcal{K}c}$) of negation, which is

$$\mathcal{M}^d_{\mathcal{K}}, w \Vdash_{\mathcal{K}d} {\sim}A \Longleftrightarrow \mathcal{M}^d_{\mathcal{K}}, w' \nVdash_{\mathcal{K}d} A \text{ for some } w' \leq w.$$

Proposition 5.10.

(i) Let $\mathcal{F}^c_{\mathcal{K}} = (W, \leq, S)$ be a frame of $\mathbf{CC}_\omega$ satisfying $\forall u, v(uSv \to \exists w S^{-1} v(w \leq u$ and $w \leq v))$. Define $\Phi(\mathcal{F}^c_{\mathcal{K}}) = (W, \leq)$. Then for any $\mathcal{V}$ and w, $(\mathcal{F}^c_{\mathcal{K}}, \mathcal{V}), w \Vdash_{\mathcal{K}c} A \Leftrightarrow (\Phi(\mathcal{F}^c_{\mathcal{K}}), \mathcal{V}), w \Vdash_{\mathcal{K}d} A$.

(ii) Let $\mathcal{F}^d_{\mathcal{K}}$ be a frame of $\mathbf{daC}$. Define $S = \{(u, v) : \exists w(w \leq u$ and $w \leq v)\}$. and $\Psi(\mathcal{F}^d_{\mathcal{K}}) = (W, \leq, S)$. Then for any $\mathcal{V}$ and w, $(\mathcal{F}^d_{\mathcal{K}}, \mathcal{V}), w \Vdash_{\mathcal{K}d} A \Leftrightarrow (\Psi(\mathcal{F}^d_{\mathcal{K}}), \mathcal{V}), w \Vdash_{\mathcal{K}c} A$.

(iii) $\Psi = \Phi^{-1}$ for the above Φ and Ψ.

Note the S defined in (ii) is well-defined: it is easy to check it is reflexive, symmetric and satisfies $\forall u, v(uSv \to \exists wS^{-1}v(w \leq u \text{ and } w \leq v))$.

Proof: In (i) and (ii), we only have to consider the case for negation.

For (i), if $(\mathcal{F}^c_{\mathcal{K}}, \mathcal{V}), w \Vdash_{\mathcal{K}c} {\sim}A$, then for some $w'S^{-1}w$, $(\mathcal{F}^c_{\mathcal{K}}, \mathcal{V}), w' \nVdash_{\mathcal{K}c} A$. By the frame condition, there is $xS^{-1}w$ such that $x \leq w$ and $x \leq w'$. Because of the latter, $(\mathcal{F}^c_{\mathcal{K}}, \mathcal{V}), x \nVdash_{\mathcal{K}c} A$. By I.H., $(\Phi(\mathcal{F}^c_{\mathcal{K}}), \mathcal{V}), x \nVdash_{\mathcal{K}d} A$. Since $x \leq w$, $(\Phi(\mathcal{F}^c_{\mathcal{K}}), \mathcal{V}), w \Vdash_{\mathcal{K}d} {\sim}A$. For the converse direction, if $(\Phi(\mathcal{F}^c_{\mathcal{K}}), \mathcal{V}), w \Vdash_{\mathcal{K}d} {\sim}A$ then for some $w' \leq w$, $(\Phi(\mathcal{F}^c_{\mathcal{K}}), \mathcal{V}), w' \nVdash_{\mathcal{K}d} A$. By I.H., $(\mathcal{F}^c_{\mathcal{K}}, \mathcal{V}), w' \nVdash_{\mathcal{K}c} A$. Here, since $w'Sw'$ by reflexivity and $w' \leq w$, we have $w'Sw$, so by symmetry wSw'. Thus $(\mathcal{F}^c_{\mathcal{K}}, \mathcal{V}), w \Vdash_{\mathcal{K}c} {\sim}A$.

For (ii), if $(\mathcal{F}^d_{\mathcal{K}}, \mathcal{V}), w \Vdash_{\mathcal{K}d} {\sim}A$, then for some $w' \leq w$, $(\mathcal{F}^d_{\mathcal{K}}, \mathcal{V}), w' \nVdash_{\mathcal{K}d} A$. By I.H., $(\Psi(\mathcal{F}^d_{\mathcal{K}}), \mathcal{V}), w' \nVdash_{\mathcal{K}c} A$. Now as $w' \leq w$ and $w'Sw'$, wSw'. So $(\Psi(\mathcal{F}^d_{\mathcal{K}}), \mathcal{V}), w \Vdash_{\mathcal{K}c} {\sim}A$. For the converse direction, if $(\Psi(\mathcal{F}^d_{\mathcal{K}}), \mathcal{V}), w \Vdash_{\mathcal{K}c} {\sim}A$, then for some $w'S^{-1}w$, $(\Psi(\mathcal{F}^d_{\mathcal{K}}), \mathcal{V}), w' \nVdash_{\mathcal{K}c} A$. Thus there is an x such that $x \leq w$ and $x \leq w'$. We have $(\Psi(\mathcal{F}^d_{\mathcal{K}}), \mathcal{V}), x \nVdash_{\mathcal{K}c} A$ by the latter. By I.H., $(\mathcal{F}^d_{\mathcal{K}}, \mathcal{V}), x \nVdash_{\mathcal{K}d} A$. Therefore $(\mathcal{F}^d_{\mathcal{K}}, \mathcal{V}), w \Vdash_{\mathcal{K}d} {\sim}A$.

For (iii), it is immediate to see that $\Phi(\Psi(\mathcal{F}^d_{\mathcal{K}})) = \mathcal{F}^d_{\mathcal{K}}$, as the mappings do not alter $(W, \leq)$. As for $\Psi(\Phi(\mathcal{F}^c_{\mathcal{K}})) = \mathcal{F}^c_{\mathcal{K}}$, we need to check the original S in $\mathcal{F}^c_{\mathcal{K}}$ and the defined S' in $\Psi(\Phi(\mathcal{F}^c_{\mathcal{K}}))$. It is easy from the frame condition that $S \subseteq S'$. Further, if $\exists x(x \leq w \text{ and } x \leq w')$, then xSw' by reflexivity, symmetry and upward closure of S. Thus again by upward closure of S, wSw'; so $S \supseteq S'$. □

This allows us to conclude the following completeness of **daC** with respect to the frames of $\mathbf{CC}_\omega$: let us denote the derivability in **daC** by $\vdash_d$.

Corollary 5.11. $\vdash_d A$ if and only if $\mathcal{F}^c_{\mathcal{K}} \models_{\mathcal{K}c} A$ for all $\mathcal{F}^c_{\mathcal{K}}$ satisfying $\forall u, v(uSv \to \exists wS^{-1}v(w \leq u \text{ and } w \leq v))$.

Proof: The last proposition established a bijection of frames agreeing in forcing. Thus the statement follows from the completeness of **daC** with respect to its models [11]. □

We now look at the frame condition for ${\sim}({\sim}(A \vee {\sim}B) \vee A) \to {\sim}B$.

Proposition 5.12. Let $\mathcal{F}$ be a $\mathbf{CC}_\omega$-frame. Then the following conditions are equivalent.

(i) $\mathcal{F} \models_{\mathcal{K}c} {\sim}({\sim}(A \vee {\sim}B) \vee A) \to {\sim}B$ for all A, B.

(ii) $\mathcal{F}$ satisfies $\forall u, v(uSv \to \exists wS^{-1}v(w \leq v \text{ and } \forall x(wSx \to uSx)))$.

Proof: We shall first see (i) implies (ii). We show this by contraposition. Assume uSv but $\neg\exists wS^{-1}v(w \leq v \text{ and } \forall x(wSx \to uSx))$. Choose $\mathcal{V}$ such that $\mathcal{V}(p) = \{w : w \not\leq v\}$ and $\mathcal{V}(q) = \{w : uSw\}$. Again the former set is upward closed, and the latter set is upward closed because of symmetry and upward closure of S. Now since $\forall wS^{-1}v(w \not\leq v \text{ or } \neg\forall x(wSx \to uSx))$, if the former disjunct holds then $w \in \mathcal{V}(p)$. And if the latter disjunct holds, then $\exists x(wSx \text{ and } \neg uSx)$. So if $x \Vdash_{\mathcal{K}c} q$, then uSx, a contradiction. Thus $x \nVdash_{\mathcal{K}c} q$ and consequently, $w \Vdash_{\mathcal{K}c} {\sim}q$. Thus $\forall wS^{-1}v(w \Vdash_{\mathcal{K}c} p \text{ or } w \Vdash_{\mathcal{K}c} {\sim}q)$. Also if $v \Vdash_{\mathcal{K}c} p$, then $v \not\leq v$, a contradiction. So $v \nVdash_{\mathcal{K}c} p$; hence $u \Vdash_{\mathcal{K}c} {\sim}({\sim}(p \vee {\sim}q) \vee p)$. But if $u \Vdash_{\mathcal{K}c} {\sim}q$, then $\exists xS^{-1}u(x \nVdash_{\mathcal{K}c} q)$. So $\neg uSx$, a contradiction. Hence $u \nVdash_{\mathcal{K}c} {\sim}q$. Thus $u \nVdash_{\mathcal{K}c} {\sim}({\sim}(p \vee {\sim}q) \vee p) \to {\sim}q$. Therefore $\nvDash_{\mathcal{K}c} {\sim}({\sim}(p \vee {\sim}q) \vee p) \to {\sim}q$.

To see (ii) implies (i), let $v \geq u$ for arbitrary and assume $v \Vdash_{\mathcal{K}c} {\sim}({\sim}(A \vee {\sim}B) \vee A)$. We want to show $v \Vdash_{\mathcal{K}c} {\sim}B$. By definition, $\exists wS^{-1}v(w \nVdash_{\mathcal{K}c} {\sim}(A \vee {\sim}B) \vee A)$. So $\forall xS^{-1}w(x \Vdash_{\mathcal{K}c} A \vee {\sim}B)$ (*) and $w \nVdash_{\mathcal{K}c} A$. By the frame condition, there is $xS^{-1}w$ such that $x \leq w$ and $\forall y(xSy \to vSy)$. From (*) we infer $x \Vdash_{\mathcal{K}c} A$ or $x \Vdash_{\mathcal{K}c} {\sim}B$. If the former, then $w \Vdash_{\mathcal{K}c} A$, a contradiction. So $x \Vdash_{\mathcal{K}c} {\sim}B$. But then for some $yS^{-1}x$, $y \nVdash_{\mathcal{K}c} B$. Thus vSy by the frame condition. So $v \Vdash_{\mathcal{K}c} {\sim}B$. Hence $u \Vdash_{\mathcal{K}c} {\sim}({\sim}(A \vee {\sim}B) \vee A) \to {\sim}B$. Since u is arbitrary, $\vDash_{\mathcal{K}c} {\sim}({\sim}(A \vee {\sim}B) \vee A) \to {\sim}B$. $\square$

Note that contrary to the last case, in this proof we appealed to the symmetry of S in $\mathbf{CC}_\omega$.

6. Conclusion

We have looked at a family of logics related to $\mathbf{IPC}^{\sim}$. In the fourth section we observed how Kripke and Beth semantics respectively reflected the (empirical) negations of $\mathbf{IPC}^{\sim}$ and $\mathbf{TCC}_\omega$, and a translation of classical logic into the latter which highlights the difference. In the fifth section, we clarified how we can eliminate the rule [RP] in $\mathbf{IPC}^{\sim}$ and $\mathbf{daC}$, and how we can capture the latter logic in the setting of $\mathbf{CC}_\omega$. This result is further developed in the sequel, where we formulate labelled sequent calculi for the systems treated in this paper.

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NEW MODIFICATION OF THE SUBFORMULA PROPERTY FOR A MODAL LOGIC

Abstract

A modified subformula property for the modal logic KD with the additional axiom $\Box\Diamond(A \vee B) \supset \Box\Diamond A \vee \Box\Diamond B$ is shown. A new modification of the notion of subformula is proposed for this purpose. This modification forms a natural extension of our former one on which modified subformula property for the modal logics K5, K5D and S4.2 has been shown ([2] and [4]). The finite model property as well as decidability for the logic follows from this.

Keywords: Subformula property, modal logic, scope of $\Box$, sequent calculus.

1. Introduction

The modal logic KD ($=$ K $+ \Box A \supset \Diamond A$) is characterized by the class of the serial frames, where a serial frame is a (relational) frame $\langle W, R\rangle$ that satisfies the condition $(\forall x)(\exists y)\, xRy$, that is, *each world can see at least one world* (Hughes-Cresswell [1, p. 45]).

Our target is the modal logic

$$\mathrm{KD\#} = \mathrm{KD} + \Box\Diamond(A \vee B) \supset \Box\Diamond A \vee \Box\Diamond B,$$

which is characterized by the class of the frames that enjoy the property

$$(\forall x)(\exists y)[xRy\ \&\ (\forall x')(\forall y')(xRx'\ \&\ yRy' \Longrightarrow x'Ry')], \qquad (\#)$$

that is, each world x can see at least one world y such that any world that can be seen by x can see any world that can be seen by y.

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The purpose of this article is to show a modified subformula property for this logic. Precisely, a sequent calculus for the logic and the new modification of the notion of subformula which we call *nested K5-subformula* are introduced, and it is shown that in that calculus, every provable sequent has a proof in which only nested K5-subformulas of some formula in the sequent occur. The finite model property as well as decidability of KD# follows from this.

The notion of nested K5-subformula forms a natural extension of our former one, called *K5-subformula*, on which modified subformula property for the modal logics K5, K5D and S4.2 has been shown (Takano [2], [4]). As an example of the modifications, think of the subformulas of $\Box\Box\Box p$, where p is a propositional letter.

- The subformulas (in the original sense) are $\Box\Box\Box p$, $\Box\Box p$, $\Box p$ and p.

- The K5-subformulas are $\Box\neg\Box\Box p$, $\neg\Box\Box p$ $\Box\neg\Box p$ and $\neg\Box p$ as well as the subformulas above. The reason why the first two (the last two, resp.) formulas are incorporated is that $\Box\Box p$ ($\Box p$, resp.) is in the scope of the necessity symbol $\Box$ in $\Box\Box\Box p$.

- The nested K5-subformulas are $\Box\neg\Box\neg\Box p$ and $\neg\Box\neg\Box p$ as well as the K5-subformulas above. The reason why these two formulas are incorporated is that $\Box p$ is in the scope of two occurrences of $\Box$; one is the leftmost occurrence of $\Box$ in $\Box\Box\Box p$ whose scope is $\Box\Box p$, and another is the second occurrence whose scope is $\Box p$ itself. If $\Box p$ were in the scope of three occurrences of $\Box$ moreover, $\Box\neg\Box\neg\Box\neg\Box p$ and $\neg\Box\neg\Box\neg\Box p$ would be incorporated as well.

Formulas are constructed from propositional letters by means of the logical symbols $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\supset$ (implication) and $\Box$ (necessity). The possibility symbol $\Diamond$ is considered as an abbreviation of the concatenation $\neg\Box\neg$, and $(\Box\neg)^n$ designates n successions of $\Box\neg$. Propositional letters and formulas are denoted by $p, q, r, \ldots$ and $A, B, C, \ldots$, respectively. A sequent is an expression of the form $\Gamma \to \Theta$, where the antecedent Γ and the succedent Θ are finite sequences of formulas. But, for convenience, the antecedent and succedent of the sequent are recognized as sets also. Finite sequences (sets) of formulas are denoted by $\Gamma, \Theta, \Delta, \Lambda, \ldots$. We mean by $\Box\Gamma$ the sequence (set) $\{\Box A \mid A \in \Gamma\}$, and similarly for $\Box\neg\Box\Gamma$. In describing formal proofs in sequent calculi, applications of the structural rules except the cut-rule are frequently neglected, and consecutive applications of logical rules are often combined into one.

In the next section, the sequent calculus GKD# for the logic KD# is presented, and it is exemplified that the subformula property (in the original sense) fails to hold for GKD#, and so it is necessary to modify the notion of subformula to get a kind of subformula property. In accordance with this situation, the new modification of the notion of subformula, *nested K5-subformula*, is proposed in Section 3. In the succeeding section, our theorem which asserts the modified subformula property for the calculus GKD# (and so for the logic KD#) on the nested K5-subformulas is stated, and is turned into the lemma for the convenience of proof. The simpler parts of the lemma are demonstrated in the same section, while Sections 5 and 6 are devoted to the proof of the remainder.

2. Sequent calculus GKD#

This section is devoted to present the sequent calculus GKD# for our target logic KD#, which is KD added by the additional axiom $\Box\Diamond(A \vee B) \supset \Box\Diamond A \vee \Box\Diamond B$, and to exemplify that modification of the notion of subformula is necessary to get a kind of subformula property for GKD#.

It is well-known that the modal logic KD is formulated as the sequent calculus, say GKD, which is obtained from the calculus LK for the classical propositional logic by adding the following two inference rules:

$$(\mathrm{K})\ \frac{\Gamma \to A}{\Box\Gamma \to \Box A} \qquad (\mathrm{D})\ \frac{\Gamma \to}{\Box\Gamma \to}$$

Our sequent calculus GKD# is obtained from GKD by modifying the rule (D) into the following one:

$$(\mathrm{D})_{\#}\ \frac{\Box\Delta, \Gamma \to}{\Box\Gamma \to \Box\neg\Box\Delta}$$

By the following proposition, GKD# certainly is a sequent calculus for KD#, that is, a sequent $\Gamma \to \Theta$ is GKD#-provable iff the corresponding formula $\bigwedge\Gamma \supset \bigvee\Theta$ is provable in KD#.

Proposition 2.1. A sequent is GKD#-provable iff it is GKD$^+$-provable, where GKD$^+$ is GKD added by the initial sequent of the form $\Box\Diamond(A\vee B) \to \Box\Diamond A, \Box\Diamond B$.

Proof: *The 'if' part:* It suffices to show that the additional initial sequent is GKD#-provable.

$$
\dfrac{\dfrac{\dfrac{A \to A \quad B \to B}{\neg A, \neg B \to \neg(A \vee B)}}{\Box\neg A, \Box\neg B \to \Box\neg(A \vee B)}\ (\mathrm{K})}{\dfrac{\Box\neg A, \Box\neg B, \Diamond(A \vee B) \to}{\Box\Diamond(A \vee B) \to \Box\Diamond A, \Box\Diamond B}\ (\mathrm{D})_{\#}}
$$

The 'only if' part: It suffices to show that GKD$^+$-provability of the upper sequent $\Box A_1, \dots, \Box A_n, \Gamma \to$ of the rule $(\mathrm{D})_{\#}$ implies that of the lower sequent $\Box\Gamma \to \Box\neg\Box A_1, \dots, \Box\neg\Box A_n$. When $n = 0, 1$, this is justified by the following GKD$^+$-proofs:

$$
\dfrac{\begin{array}{c}\vdots\ \text{GKD}^+\text{-proof} \\ \Gamma \to\end{array}}{\Box\Gamma \to}\ (\mathrm{D})
\qquad
\dfrac{\dfrac{\begin{array}{c}\vdots\ \text{GKD}^+\text{-proof} \\ \Box A_1, \Gamma \to\end{array}}{\Gamma \to \neg\Box A_1}}{\Box\Gamma \to \Box\neg\Box A_1}\ (\mathrm{K})
$$

On the other hand, when $n \geq 2$, it is certified by applying (cut)'s to the following proofs $\boldsymbol{P}$, $\boldsymbol{Q}$ and $\boldsymbol{R}_i$ $(i = 1, \dots, n)$.

$$
\dfrac{\left\{\dfrac{\dfrac{A_i \to A_i}{\neg\bigvee_{k=1}^{n}\neg A_k \to A_i}}{\Box\neg\bigvee_{k=1}^{n}\neg A_k \to \Box A_i}\ (\mathrm{K})\right\}_{i=1,\dots,n} \quad \begin{array}{c}\vdots\ \text{GKD}^+\text{-proof} \\ \Box A_1, \dots, \Box A_n, \Gamma \to\end{array}}{\dfrac{\dfrac{\Box\neg\bigvee_{k=1}^{n}\neg A_k, \Gamma \to}{\Gamma \to \Diamond\bigvee_{k=1}^{n}\neg A_k}}{\Box\Gamma \to \Box\Diamond\bigvee_{k=1}^{n}\neg A_k}\ (\mathrm{K})}\ (\mathrm{cut})\text{'s}
$$

Figure 1. GKD$^+$-proof $\boldsymbol{P}$

$$\dfrac{\left\{\begin{array}{c}\text{additional initial sequent}\\ \Box\Diamond\bigvee_{k=1}^{i+1}\neg A_k \to \Box\Diamond\bigvee_{k=1}^{i}\neg A_k, \Box\Diamond\neg A_{i+1}\end{array}\right\}_{i=1,\ldots,n-1}}{\Box\Diamond\bigvee_{k=1}^{n}\neg A_k \to \Box\Diamond\neg A_1, \ldots, \Box\Diamond\neg A_n}\ \text{(cut)'s}$$

Figure 2. GKD$^+$-proof $\boldsymbol{Q}$

$$\dfrac{\dfrac{\dfrac{\dfrac{A_i \to A_i}{A_i \to \neg\neg A_i}}{\Box A_i \to \Box\neg\neg A_i}\,(\mathrm{K})}{\Diamond\neg A_i \to \neg\Box A_i}}{\Box\Diamond\neg A_i \to \Box\neg\Box A_i}\,(\mathrm{K})$$

Figure 3. GKD$^+$-proof $\boldsymbol{R}_i$ $(i = 1, \ldots, n)$

$\Box$

Though the calculus GKD admits cut-elimination and so enjoys the subformula property, our GKD# lacks both of these properties. In fact, the end-sequent of the following GKD#-proof, for example, has neither a cut-free one nor a proof that consists solely of subformulas of some formula in the sequent.

$$\dfrac{\dfrac{\dfrac{\dfrac{p \to p}{\neg p, p \to}}{\Box\neg p, \Box p \to}\,(\mathrm{D})_{\#}}{\to \Box\neg\Box\neg p, \Box\neg\Box p}\,(\mathrm{D})_{\#} \qquad \dfrac{\dfrac{\Box p \to \Box p}{\neg\Box p \to \Box p \supset q}}{\Box\neg\Box p \to \Box(\Box p \supset q)}\,(\mathrm{K})}{\to \Box\neg\Box\neg p, \Box(\Box p \supset q)}\,(\mathrm{cut})$$

So, it is inevitable to modify the notion of subformula to get a kind of subformula property for GKD#.

3. Nested K5-subformulas

In this section, our new modification of the notion of subformula is proposed, and it is shown that the new notion is (not only reflexive but) transitive.

The followings are our new and former modifications, respectively.

Definition 3.1.

(1) A *nested internal subformula of depth* n of A is a formula which has an occurrence in A that lies in the scope of exactly n occurrences of the necessity symbol $\Box$.

(2) A *nested K5-subformula* of A is either a subformula of A or the formula of the form $(\Box\neg)^n\Box B$ or $\neg(\Box\neg)^{n-1}\Box B$, where $\Box B$ is a nested internal subformula of depth $\geq n$ of A, and $n \geq 1$.

Definition 3.2 ([2, Definition 1]).

(1) An *internal subformula* of A is a subformula of some formula C such that $\Box C$ is a subformula of A.

(2) A *K5-subformula* of A is either a subformula of A or the formula of the form $\Box\neg\Box B$ or $\neg\Box B$, where $\Box B$ is an internal subformula of A.

Obviously, the internal subformulas are nothing but the nested internal subformulas of depth ≥ 1, and the K5-subformulas are the nested K5-subformulas which are restricted to the case $n = 1$. So, it seems that the notion of nested K5-subformula forms a natural extension of that of K5-subformula. Furthermore, the number of the nested K5-subformulas of a formula is finite.

The sets of all the subformulas, all the nested internal subformulas of depth $\geq n$ and all the nested K5-subformulas of A are denoted by $\mathrm{Sf}(A)$, $\mathrm{InSf}^n(A)$ and $\mathrm{Sf}_{\mathrm{N.K5}}(A)$, respectively. Moreover, put $\mathrm{Sf}(\Gamma) = \bigcup\{\mathrm{Sf}(A) \mid A \in \Gamma\}$, and similarly for $\mathrm{InSf}^n(\Gamma)$ and $\mathrm{Sf}_{\mathrm{N.K5}}(\Gamma)$.

Evidently, the relation '*being a nested K5-subformula of*' between formulas is reflexive; besides it is transitive too, as the following proposition shows.

Proposition 3.3.

(1) Suppose $n, k \geq 1$. Then, $\Box B \in \mathrm{InSf}^n(A)$ and $\Box C \in \mathrm{InSf}^k((\Box\neg)^n\Box B)$ imply $(\Box\neg)^k\Box C$, $\neg(\Box\neg)^{k-1}\Box C \in \mathrm{Sf}_{\mathrm{N.K5}}(A)$.

(2) $B \in \mathrm{Sf}_{\mathrm{N.K5}}(A)$ and $C \in \mathrm{Sf}_{\mathrm{N.K5}}(B)$ imply $C \in \mathrm{Sf}_{\mathrm{N.K5}}(A)$.

Proof:

(1) Suppose that $\Box C$ be a nested internal subformula of depth k' of $(\Box\neg)^n\Box B$. Then $k' \geq k$. *The case where* $k' \leq n$: $\Box C$ is $(\Box\neg)^{n-k'}\Box B$.

From $n \geq k+(n-k')$, it follows $\Box B \in \mathrm{InSf}^n(A) \subseteq \mathrm{InSf}^{k+(n-k')}(A)$, and so both $(\Box\neg)^{k+(n-k')}\Box B$ and $\neg(\Box\neg)^{(k-1)+(n-k')}\Box B$, namely $(\Box\neg)^k\Box C$ and $\neg(\Box\neg)^{k-1}\Box C$, are in $\mathrm{Sf}_{\mathrm{N.K5}}(A)$. *The case where $k' > n$:* $\Box C$ is a nested internal subformula of depth $k'-n$ of $\Box B$, So, $\Box C \in \mathrm{InSf}^{k'-n}(\Box B) \subseteq \mathrm{InSf}^{k'-n}(\mathrm{InSf}^n(A)) \subseteq \mathrm{InSf}^{k'}(A) \subseteq \mathrm{InSf}^k(A)$, and so both $(\Box\neg)^k\Box C$ and $\neg(\Box\neg)^{k-1}\Box C$ are in $\mathrm{Sf}_{\mathrm{N.K5}}(A)$.

(2) By the assumption, either $(B1)$ $B \in \mathrm{Sf}(A)$ or $(B2)$ B is $(\Box\neg)^n\Box B'$ or $\neg(\Box\neg)^{n-1}\Box B'$ and $\Box B' \in \mathrm{InSf}^n(A)$ for some B' and $n \geq 1$, and either $(C1)$ $C \in \mathrm{Sf}(B)$ or $(C2)$ C is $(\Box\neg)^k\Box C'$ or $\neg(\Box\neg)^{k-1}\Box C'$ and $\Box C' \in \mathrm{InSf}^k(B)$ for some C' and $k \geq 1$. *The case where $(B1)$ and $(C1)$ hold:* $C \in \mathrm{Sf}(\mathrm{Sf}(A)) \subseteq \mathrm{Sf}(A) \subseteq \mathrm{Sf}_{\mathrm{N.K5}}(A)$. *The case where $(B1)$ and $(C2)$ hold:* $C \in \mathrm{Sf}_{\mathrm{N.K5}}(A)$ follows from $\Box C' \in \mathrm{InSf}^k(\mathrm{Sf}(A)) \subseteq \mathrm{InSf}^k(A)$. *The case where $(B2)$ and $(C1)$ hold:* Either C is $(\Box\neg)^m\Box B'$ or $\neg(\Box\neg)^{m-1}\Box B'$ for some m such that $1 \leq m \leq n$, or $C \in \mathrm{Sf}(\Box B')$. In the former case, $C \in \mathrm{Sf}_{\mathrm{N.K5}}(A)$ follows from $\Box B' \in \mathrm{InSf}^n(A) \subseteq \mathrm{InSf}^m(A)$. In the latter case, on the other hand, $C \in \mathrm{Sf}(\mathrm{InSf}^n(A)) \subseteq \mathrm{Sf}(A) \subseteq \mathrm{Sf}_{\mathrm{N.K5}}(A)$. *The case where $(B2)$ and $(C2)$ hold:* If B is $(\Box\neg)^n\Box B'$, then $C \in \mathrm{Sf}_{\mathrm{N.K5}}(A)$ by (1). So, suppose that B is $\neg(\Box\neg)^{n-1}\Box B'$. If $n \geq 2$, then $C \in \mathrm{Sf}_{\mathrm{N.K5}}(A)$ follows from $\Box B' \in \mathrm{InSf}^n(A) \subseteq \mathrm{InSf}^{n-1}(A)$ and $\Box C' \in \mathrm{InSf}^k(B) = \mathrm{InSf}^k((\Box\neg)^{n-1}\Box B')$ by (1). If $n = 1$, then $C \in \mathrm{Sf}_{\mathrm{N.K5}}(A)$ follows from $\Box C' \in \mathrm{InSf}^k(\neg\Box B') \subseteq \mathrm{InSf}^k(\mathrm{InSf}^1(A)) \subseteq \mathrm{InSf}^k(A)$. $\Box$

Though the following proposition is useless for this article, it shows a characteristic property of the nested K5-subformulas (cf. Corollary 5.4 below).

Proposition 3.4. $\Box A \in \mathrm{Sf}_{\mathrm{N.K5}}(\mathrm{InSf}^n(\Gamma))$ implies $(\Box\neg)^n\Box A \in \mathrm{Sf}_{\mathrm{N.K5}}(\Gamma)$, where $n \geq 1$.

Proof: $\Box A \in \mathrm{Sf}_{\mathrm{N.K5}}(B)$ for some $B \in \mathrm{InSf}^n(\Gamma)$ by the assumption. *The case where $\Box A \in \mathrm{Sf}(B)$:* It follows $\Box A \in \mathrm{InSf}^n(\Gamma)$, and so $(\Box\neg)^n\Box A \in \mathrm{Sf}_{\mathrm{N.K5}}(\Gamma)$. *The case where $\Box A$ is $(\Box\neg)^k\Box A'$ and $\Box A' \in \mathrm{InSf}^k(B)$ for some A' and $k \geq 1$:* It follows $\Box A' \in \mathrm{InSf}^{n+k}(\Gamma)$, and so $(\Box\neg)^{n+k}\Box A'$, namely $(\Box\neg)^n\Box A$, is in $\mathrm{Sf}_{\mathrm{N.K5}}(\Gamma)$. $\Box$

4. Statements of Theorem and Lemma

In this section, our theorem, which forms a modified subformula property for GKD# is stated, and is turned into the lemma which is convenient for proof.

THEOREM 4.1. *Every GKD#-provable sequent* $\Gamma \to \Theta$ *has a GKD#-proof that consists solely of the nested K5-subformulas of some formula in* $\Gamma \cup \Theta$.

This theorem is proved through Lemma 4.2 below.

For the convenience of proof, our sequent calculus GKD# is adjusted by the following two changes.

- To restrict the cut-rule to the following one:

$$(\mathrm{cut})_{\mathrm{N.K5}} \ \frac{\Gamma \to \Theta, (\Box\neg)^n\Box A \quad (\Box\neg)^n\Box A, \Delta \to \Lambda}{\Gamma, \Delta \to \Theta, \Lambda},$$
$$\textit{where } \Box A \in \mathrm{InSf}^n(\Gamma \cup \Theta \cup \Delta \cup \Lambda) \textit{ and } n \geq 1.$$

- To modify the rule (K) into the following one:

$$(\mathrm{K})_{\#} \ \frac{\{\Box\Delta', \Gamma \to \Box(\Delta\backslash\Delta'), A\}_{\Delta'\subseteq\Delta}}{\Box\Gamma \to \Box\neg\Box\Delta, \Box A}$$

Let's call this adjusted calculus as aGKD#. The rule $(\mathrm{cut})_{\mathrm{N.K5}}$ restricted to the case $n = 1$ is the rule $(\mathrm{cut})_{\mathrm{K5}}$, which was utilized in Takano [4] to show the modified subformula property for the logic S4.2 on the K5-subformulas.

Remark that the rule $(\mathrm{K})_{\#}$ can be seen as an abbreviation for the following inference:

$$\cfrac{\cfrac{\cfrac{\{\Box\Delta', \Gamma \to \Box(\Delta\backslash\Delta'), A\}_{\Delta'\subseteq\Delta}}{\Gamma \to A}\ (\mathrm{cut})\text{'s}}{\Box\Gamma \to \Box A}\ (\mathrm{K})}{\Box\Gamma \to \Box\neg\Box\Delta, \Box A}$$

So, aGKD#-provable sequents are GKD#-provable. Moreover, since the relation '*being a nested K5-subformula of*' is reflexive and transitive (Proposition 3.3 (2)), every formula occurring in an aGKD#-proof is a nested K5-subformula of some formula occurring in the end-sequent. Hence, it suffices to show that GKD#-provability implies aGKD#-provability, for the proof of Theorem 4.1. We will prove this in the following form.

LEMMA 4.2. *The following three conditions on a sequent are mutually equivalent.*

(i) *It is GKD#-provable.*

(ii) *It is aGKD#-provable.*

(iii) *It is valid on every frame with the property (#).*

The '*(ii) implies (i)*'-part of this lemma has been remarked above, the '*(i) implies (iii)*'-part is shown as Proposition 4.3 below, and the '*(iii) implies (ii)*'-part will be shown as Proposition 6.7 after necessary preliminaries.

PROPOSITION 4.3. GKD#-provable sequents are valid on every frame with the property (#).

PROOF: It suffices to show that the rule $(\mathrm{D})_\#$ preserves validity. Let $\models$ be the satisfaction relation derived from a model $\langle W, R, V\rangle$ with (#). Suppose $x\,(\in W)$ rejects the lower sequent $\Box\Gamma \to \Box\neg\Box\Delta$ of $(\mathrm{D})_\#$; that is, $x \models \Box A$ for every $A \in \Gamma$, while $x \not\models \Box\neg\Box B$ for every $B \in \Delta$. By (#), $(\forall x')(\forall y')(xRx' \;\&\; yRy' \implies x'Ry')$ for some y such that xRy. We will show that this y rejects the upper sequent $\Box\Delta, \Gamma \to$, and this concludes the proof that $(\mathrm{D})_\#$ preserves validity. First, $y \models A$ for every $A \in \Gamma$, since this follows from $x \models \Box A$ and xRy. On the other hand, let $B \in \Delta$. From $x \not\models \Box\neg\Box B$, it follows $x_B \models \Box B$ for some x_B such that xRx_B. Then, for every y' such that yRy', it follows x_BRy' and so $y' \models B$; hence $y \models \Box B$. So y rejects $\Box\Delta, \Gamma \to$. $\Box$

5. N.K5-analytically saturated sequents

In this section, preparatory to the proof of the '*(iii) implies (ii)*'-part of Lemma 4.2, the notion of N.K5-analytically saturated sequent is introduced.

It is to be remembered that in this section, (un)provability means aGKD#-(un)provability.

Definition 5.1. A sequent $\Gamma \to \Theta$ is *N.K5-analytically saturated*, iff the following properties hold (cf. Takano [3, Definition 1.1]):

(5.1-a) $\Gamma \to \Theta$ is unprovable.

(5.1-b) Suppose $A \in \mathrm{Sf}_{\mathrm{N.K5}}(\Gamma \cup \Theta)$. If $A, \Gamma \to \Theta$ is unprovable then $A \in \Gamma$; while if $\Gamma \to \Theta, A$ is unprovable then $A \in \Theta$.

N.K5-analytically saturated sequents are denoted by $u, v, w, \ldots$; besides, $\mathrm{a}(u)$ and $\mathrm{s}(u)$ denote the antecedent and succedent of u, respectively.

Owing to the initial sequents and the inference rules for the propositional connectives, the following proposition holds.

Proposition 5.2. For every u, A and B, the following properties hold:

(1) $\mathrm{a}(u) \cap \mathrm{s}(u) = \emptyset$.

(2) $\neg A \in \mathrm{a}(u)$ implies $A \in \mathrm{s}(u)$; $\neg A \in \mathrm{s}(u)$ implies $A \in \mathrm{a}(u)$.

(3) $A \wedge B \in \mathrm{a}(u)$ implies $A, B \in \mathrm{a}(u)$; $A \wedge B \in \mathrm{s}(u)$ implies $A \in \mathrm{s}(u)$ or $B \in \mathrm{s}(u)$.

(4) $A \vee B \in \mathrm{a}(u)$ implies $A \in \mathrm{a}(u)$ or $B \in \mathrm{a}(u)$; $A \vee B \in \mathrm{s}(u)$ implies $A, B \in \mathrm{s}(u)$.

(5) $A \supset B \in \mathrm{a}(u)$ implies $A \in \mathrm{s}(u)$ or $B \in \mathrm{a}(u)$; $A \supset B \in \mathrm{s}(u)$ implies $A \in \mathrm{a}(u)$ and $B \in \mathrm{s}(u)$. □

Similarly, thanks to the rule $(\mathrm{cut})_{\mathrm{N.K5}}$, the following proposition holds too.

Proposition 5.3. $\Box A \in \mathrm{InSf}^n(\mathrm{a}(u) \cup \mathrm{s}(u))$ implies $(\Box\neg)^n \Box A \in \mathrm{a}(u) \cup \mathrm{s}(u)$, where $n \geq 1$. □

Corollary 5.4. $\Box A \in \mathrm{Sf}_{\mathrm{N.K5}}(\mathrm{InSf}^n(\mathrm{a}(u) \cup \mathrm{s}(u)))$ implies $(\Box\neg)^n \Box A \in \mathrm{a}(u) \cup \mathrm{s}(u)$, where $n \geq 1$.

Proof: $\Box A \in \mathrm{Sf}_{\mathrm{N.K5}}(B)$ for some $B \in \mathrm{InSf}^n(\mathrm{a}(u) \cup \mathrm{s}(u))$ by the assumption. *The case where* $\Box A \in \mathrm{Sf}(B)$*:* It follows $\Box A \in \mathrm{InSf}^n(\mathrm{a}(u) \cup \mathrm{s}(u))$, and so $(\Box\neg)^n \Box A \in \mathrm{a}(u) \cup \mathrm{s}(u)$ by the proposition. *The case where* $\Box A$ *is* $(\Box\neg)^k \Box A'$ *and* $\Box A' \in \mathrm{InSf}^k(B)$ *for some* A' *and* $k \geq 1$*:* It follows $\Box A' \in \mathrm{InSf}^{n+k}(\mathrm{a}(u) \cup \mathrm{s}(u))$, and so $(\Box\neg)^{n+k} \Box A'$, namely $(\Box\neg)^n \Box A$, is in $\mathrm{a}(u) \cup \mathrm{s}(u)$ by the proposition again. □

Proposition 5.5. If $\Gamma \to \Theta$ is unprovable, then $\Gamma \subseteq \mathrm{a}(u)$, $\Theta \subseteq \mathrm{s}(u)$ and $\mathrm{a}(u) \cup \mathrm{s}(u) \subseteq \mathrm{Sf}_{\mathrm{N.K5}}(\Gamma \cup \Theta)$ for some u.

Proof: Let $A_1, \ldots, A_n$ be an enumeration of all the formulas of $\mathrm{Sf}_{\mathrm{N.K5}}(\Gamma \cup \Theta)$. Put $\Gamma_1 = \Gamma$ and $\Theta_1 = \Theta$. Suppose that Γ_k and Θ_k have been defined $(1 \leq k \leq n)$. If $\Gamma_k \to \Theta_k, A_k$ is unprovable, then put $\Gamma_{k+1} = \Gamma_k$ and $\Theta_{k+1} = \Theta_k \cup \{A_k\}$; if $\Gamma_k \to \Theta_k, A_k$ is provable but $A_k, \Gamma_k \to \Theta_k$ is unprovable, then put $\Gamma_{k+1} = \Gamma_k \cup \{A_k\}$ and $\Theta_{k+1} = \Theta_k$; if both $\Gamma_k \to \Theta_k, A_k$ and $A_k, \Gamma_k \to \Theta_k$ are provable, then put $\Gamma_{k+1} = \Gamma_k$ and $\Theta_{k+1} = \Theta_k$.

Then it is easily shown that $\Gamma_{n+1} \to \Theta_{n+1}$ is the desired u. (See the proof of Takano [3, Lemma 1.3].) $\square$

6. Canonical model

The '*(iii) implies (ii)*'-part of Lemma 4.2 is shown in this section. For this purpose, the canonical model for GKD# is introduced.

Definition 6.1 (Canonical model $\langle W, R, V\rangle$). W is the set of all the N.K5-analytically saturated sequents, the binary relation R on W is defined by: uRv iff the following properties hold for every B,

(6.1-a) $\Box B \in \mathrm{a}(u)$ implies $B \in \mathrm{a}(v)$,

(6.1-b) $\Box B \in \mathrm{a}(v) \cup \mathrm{s}(v)$ implies $\Box\neg\Box B \in \mathrm{a}(u) \cup \mathrm{s}(u)$, and conversely,

and V is the function of the propositional letters to the subsets of W such that $V(p) = \{u \in W \mid p \in \mathrm{a}(u)\}$ for every p.

Remark 6.2. For an GKD#-unprovable sequent $\Gamma \to \Theta$, if W is restricted to those u's such that $\mathrm{a}(u) \cup \mathrm{s}(u) \subseteq \mathrm{Sf}_{\mathrm{N.K5}}(\Gamma \cup \Theta)$, the following argument remains valid. So, the finite model property as well as decidability for KD# follows, since the restricted W is finite.

We have the following three propositions which concern the canonical frame $\langle W, R\rangle$.

Proposition 6.3. $\Box A \in \mathrm{a}(u)$ implies $(\forall v)(uRv \Longrightarrow A \in \mathrm{a}(v))$.

Proof: Immediate by (6.1-a). $\square$

Proposition 6.4. $\Box A \in \mathrm{s}(u)$ implies $(\exists v)(uRv \;\&\; A \in \mathrm{s}(v))$.

Proof: Suppose $\Box A \in \mathrm{s}(u)$. Put $\Gamma = \{B \mid \Box B \in \mathrm{a}(u)\}$ and $\Delta = \{B \mid \Box\neg\Box B \in \mathrm{s}(u)\}$. Then $\Box\Gamma \to \Box\neg\Box\Delta, \Box A$ is aGKD#-unprovable, since

$\Box\Gamma \subseteq \mathrm{a}(u)$ and $\Box\neg\Box\Delta \cup \{\Box A\} \subseteq \mathrm{s}(u)$. According to the rule $(\mathrm{K})_{\#}$, the sequent $\Box\Delta', \Gamma \to \Box(\Delta\backslash\Delta'), A$ is unprovable for some $\Delta' \subseteq \Delta$. Then by Proposition 5.5, $\Box\Delta' \cup \Gamma \subseteq \mathrm{a}(v)$, $\Box(\Delta\backslash\Delta') \cup \{A\} \subseteq \mathrm{s}(v)$ and $\mathrm{a}(v) \cup \mathrm{s}(v) \subseteq \mathrm{Sf}_{\mathrm{N.K5}}(\Box\Delta' \cup \Gamma \cup \Box(\Delta\backslash\Delta') \cup \{A\})$ for some v. Since $A \in \mathrm{s}(v)$, it suffices to show uRv, which will be shown by checking (6.1-a) and (6.1-b). *Check of (6.1-a):* If $\Box B \in \mathrm{a}(u)$, then $B \in \Gamma \subseteq \mathrm{a}(v)$. *Check of (6.1-b):* Suppose $\Box B \in \mathrm{a}(v) \cup \mathrm{s}(v)$. Since $\Box\Delta' \cup \Gamma \cup \Box(\Delta\backslash\Delta') \cup \{A\} \subseteq \mathrm{InSf}^1(\mathrm{a}(u) \cup \mathrm{s}(u))$, it follows $\Box B \in \mathrm{a}(v) \cup \mathrm{s}(v) \subseteq \mathrm{Sf}_{\mathrm{N.K5}}(\mathrm{InSf}^1(\mathrm{a}(u) \cup \mathrm{s}(u)))$, and so $\Box\neg\Box B \in \mathrm{a}(u) \cup \mathrm{s}(u)$ by Corollary 5.4. Conversely, suppose $\Box\neg\Box B \in \mathrm{a}(u) \cup \mathrm{s}(u)$. If $\Box\neg\Box B \in \mathrm{a}(u)$, then $\neg\Box B \in \Gamma \subseteq \mathrm{a}(v)$, and so $\Box B \in \mathrm{s}(v) \subseteq \mathrm{a}(v) \cup \mathrm{s}(v)$ by Proposition 5.2 (2). If $\Box\neg\Box B \in \mathrm{s}(u)$, on the other hand, $\Box B \in \Box\Delta = \Box\Delta' \cup \Box(\Delta\backslash\Delta') \subseteq \mathrm{a}(v) \cup \mathrm{s}(v)$. $\Box$

Proposition 6.5. The canonical frame $\langle W, R\rangle$ enjoys the property $(\#)$, that is, $(\forall u)(\exists v)[uRv \ \&\ (\forall u')(\forall v')(uRu' \ \&\ vRv' \Longrightarrow u'Rv')]$.

Proof: Suppose that u is given. Put $\Gamma = \{B \mid \Box B \in \mathrm{a}(u)\}$ and $\Delta = \{B \mid \Box\neg\Box B \in \mathrm{s}(u)\}$. Since $\Box\Gamma \to \Box\neg\Box\Delta$ is aGKD#-unprovable, $\Box\Delta, \Gamma \to$ is unprovable too by the rule $(\mathrm{D})_{\#}$. So by Proposition 5.5, $\Box\Delta \cup \Gamma \subseteq \mathrm{a}(v)$ and $\mathrm{a}(v) \cup \mathrm{s}(v) \subseteq \mathrm{Sf}_{\mathrm{N.K5}}(\Box\Delta \cup \Gamma)$ for some v.

Since uRv can be shown similarly to the proof of Proposition 6.4, it is left to show the property that uRu' and vRv' imply $u'Rv'$. So, suppose uRu' and vRv'. We will infer $u'Rv'$ by checking (6.1-a) and (6.1-b). *Check of (6.1-a) for $u'Rv'$:* Suppose $\Box B \in \mathrm{a}(u')$. By (6.1-b) for uRu', it follows $\Box\neg\Box B \in \mathrm{a}(u) \cup \mathrm{s}(u)$. But if $\Box\neg\Box B \in \mathrm{a}(u)$, then $\neg\Box B \in \mathrm{a}(u')$ by (6.1-a) for uRu', which contradicts $\Box B \in \mathrm{a}(u')$; so $\Box\neg\Box B \in \mathrm{s}(u)$. Then $\Box B \in \Box\Delta \subseteq \mathrm{a}(v)$, and so $B \in \mathrm{a}(v')$ by (6.1-a) for vRv'. *Check of (6.1-b) for $u'Rv'$:* $\Box B \in \mathrm{a}(v') \cup \mathrm{s}(v')$ iff $\Box\neg\Box B \in \mathrm{a}(v) \cup \mathrm{s}(v)$ iff $\Box\neg\Box\neg\Box B \in \mathrm{a}(u) \cup \mathrm{s}(u)$ iff $\Box\neg\Box B \in \mathrm{a}(u') \cup \mathrm{s}(u')$ by (6.1-b) for vRv', uRv and uRu', respectively. $\Box$

Thanks to Propositions 6.3 and 6.4 as well as Proposition 5.2, the following proposition is shown by induction on the construction of formulas.

Proposition 6.6. Let $\models$ be the satisfaction relation derived from the canonical model $\langle W, R, V\rangle$. Then, $A \in \mathrm{a}(u)$ implies $u \models A$, while $A \in \mathrm{s}(u)$ implies $u \not\models A$, for every u and A. $\Box$

Finally, we are ready to show the following proposition which forms the '*(iii) implies (ii)*'-part of Lemma 4.2.

Proposition 6.7. Those sequents that are valid on every frame with the property (#) are aGKD#-provable.

Proof: Suppose that a sequent $\Gamma \to \Theta$ is valid on every frame with (#), but is aGKD#-unprovable. Then by Proposition 5.5, $\Gamma \subseteq \mathrm{a}(u)$ and $\Theta \subseteq \mathrm{s}(u)$ for some u. It follows by Proposition 6.6 that, this u rejects $\Gamma \to \Theta$ on the canonical model $\langle W, R, V\rangle$. This together with Proposition 6.5 contradicts the assumption. □

7. Concluding remarks

To get a kind of subformula property for the modal logic KD#, we proposed a new modification of the notion of subformula, *nested K5-subformula*, which forms a natural extension of our former modification, *K5-subformula*. Then we showed by means of the sequential version GKD# that, the nested K5-subformulas suffice though the subformulas (in the original sense) do not.

But the author wonders whether the nested K5-subformulas are really necessary. Possibly the K5-subformulas suffice. These problems are left for further consideration.

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MODULE STRUCTURE ON EFFECT ALGEBRAS

Abstract

In this paper, by considering the notions of effect algebra and product effect algebra, we define the concept of effect module. Then we investigate some properties of effect modules, and we present some examples on them. Finally, we introduce some topologies on effect modules.

Keywords: Effect algebra, product effect algebra, effect module, topology.

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1. Introduction

In 1994, Foulis and Bennett [16] introduced the concept of effect algebras with a partially defined addition "+" in order to axiomatize some quantum measurements. They are additive counterparts to D-posets introduced by Kôpka and Chovanec (1994), where the subtraction of comparable elements is a primary notion. They met interest of mathematicians physicits while they give a common base for algebraic as well as fuzzy set properties of the system $\varepsilon(H)$ of all effects of a Hilbert space H, i.e., of all Hermitian operators A on H such that $O \leq A \leq I$, where O and I are the null and the identity operators on H. In many cases, effect algebras are intervals in unital po-groups, e.g., $\varepsilon(H)$ is the interval in the po-group $\beta(H)$ of all Hermitian operators on H; this group is of great importance for physics.

Effect algebras generalize many examples of quantum structures, like Boolean algebras, orthomodular lattices or posets, orthoalgebras, MV-algebras and etc. We recall that MV-algebras are algebraic counterparts

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of the many-valued reasoning, and they appeared in mathematics under many different names, situations and motivations. Even in the theory of effect algebras, they were defined in an equivalent way as phi-symmetric effect algebras [1]. The monograph [2, 11] can serve as a basic source of information about effect algebras. Product effect algebras, were introduced by Anatolij Dvurecenskij [12]. He proved every product effect algebra with the Riesz decomposition property (RDP) is an interval in an Abelian unital interpolation po-ring, and he showed that the category of product effect algebras with the RDP is categorically equivalent with the category of unital Abelian interpolation po-rings. Recently, some researchers worked on modular structures (see, for instance, [3, 4, 9, 10, 17]). Effect modules have been introduced in theoretical physics in the mid-1990 for quantum probability. These structures are effect algebras with a scalar multiplication, with scalars from $[0,1]$, i.e., an effect module M is an effect algebra with an action $[0,1] \times M \longrightarrow M$ that it is an special case. In this paper, we try to present more complete definition than the previous definition. We define effect modules on product effect algebras as an extension of effect algebras.

In the study of effect algebras (or more general, quantum structures) as carriers of states and probability measures, an important tool is the study of topologies on them. In fact, algebra and topology, the two fundamental domains of mathematics, play complementary roles. Topology studies continuity and convergence, and it provides a general framework to study the concept of a limit. Algebra studies all kinds of operations and provides a basis for algorithms and calculations. Because of this difference in nature, algebra and topology to have a strong tendency to develop independently, not in direct contact with each other. However, in applications, in higher level domains of mathematics, such as functional analysis, dynamical systems, representation theory and others, topology and algebra come in contact most naturally. Recently, many mathematicians have studied properties of some algebraic structures endowed with a topology (see, for instance, [5, 6, 7, 15, 18]). We have studied and try to introduce some topologies on effect modules. In fact, we wish to open new fields to anyone that is interested to studying and development of effect algebras and effect modules.

2. Preliminaries

In this section, we review some definitions and related lemmas and theorems that we use in the next sections.

Definition 2.1 ([16]). An *effect algebra* is a partial algebra $E = (E; +, 0, 1)$ with a partially defined operation " $+$ " and two constant elements 0 and 1 such that, for all $a, b, c \in E$,

$(E1)$ Commutative Law: if $a + b$ is defined in E, then $b + a$ is defined in E, and in such the case $a + b = b + a$;

$(E2)$ Associative Law: if $a+(b+c)$ and $b+c$ are defined in E, then $a+b$ and $(a + b) + c$ are defined in E, and in such the case $a + (b + c) = (a + b) + c$;

$(E3)$ Orthocomplementation Law: for any $a \in E$, there exists a unique element $a' \in E$ such that $a + a' = 1$;

$(E4)$ Zero-Unit Law: if $a + 1$ is defined in E, then $a = 0$.

The algebraic structure $(E; +, 0)$ is called an extended effect algebra if

$(GE1)$ E is a partial commutative monoid;

$(GE2)$ $x + z = x + y$ implies $z = y$;

$(GE3)$ $x + y = 0$ implies $x = y = 0$, for every $x, y, z \in E$ (see [11]).

Let E be an effect algebra. If we define $a \leq b$ if and only if there exists an element $c \in E$ such that $a + c = b$, then $\leq$ is a partial ordering, and we write $c := b - a$. A nonempty subset I of E is said to be an *ideal* of E if the following conditions are satisfied: (Id_1) If $x \in I$ and $y \leq x$, then $y \in I$, (Id_2) if $x - y \in I$ and $y \in I$, then $x \in I$, for any $x, y \in E$. Recall that a set $Q \subseteq E$ is called a *sub-effect algebra* of the effect algebra E, if $1 \in Q$ and if out of elements $a, b, c \in E$ with $a + b = c$ two are in Q, then $a, b, c \in Q$. Let F be another effect algebra. A mapping $h : E \longrightarrow F$ is said to be a *homomorphism* of effect algebras (or E-homomorphism) if $h(1) = 1$ and $h(a + b) = h(a) + h(b)$, for any $a, b \in E$ whenever $a + b$ is defined in E.

We say E fulfills the strong Riesz Decomposition Property, $(RDP2)$ for short, if $a_1,\ a_2,\ b_1,\ b_2 \in P$ such that $a_1 + a_2 = b_1 + b_2$, then there are $d_1,\ d_2,\ d_3,\ d_4 \in P$ such that (i) $d_1 + d_2 = a_1$, $d_3 + d_4 = a_2$, $d_1 + d_3 = b_1$, $d_2 + d_4 = b_2$, and (ii) $d_2 \wedge d_3 = 0$ (see [13]).

Definition 2.2 ([12]). A product on effect algebra $E = (E; +, 0, 1)$ is any total binary operation "." on E such that for all $a, b, c \in E$, the following holds:
If $a + b$ is defined in E, then $a.c + b.c$ and $c.a + c.b$ exist in E and

$$(a + b).c = a.c + b.c, \quad c.(a + b) = c.a + c.b.$$

Now, an effect algebra E with a product "." is called a *product effect algebra*.

The product "." on E is associative if $(a.b).c = a.(b.c)$, for every $a, b, c \in E$.

A mapping $h : E \longrightarrow F$ is said to be a homomorphism of product effect algebras (or P-homomorphism) if h ia an E-homomorphism and $h(a.b) = h(a).h(b)$, for every $a, b \in P$.

Proposition 2.3 ([16]). The following properties hold for any effect algebra E:

(i) $a'' = a$,

(ii) $1' = 0$ and $0' = 1$,

(iii) $0 \leq a \leq 1$,

(iv) $a + 0 = a$,

(v) $a + b = 0 \Rightarrow a = b = 0$,

(vi) $a \leq a + b$,

(vii) $a \leq b \Rightarrow b' \leq a'$,

(viii) $b - a = (a + b')'$,

(ix) $a + b' = (b - a)'$,

(x) $a = a - 0$,

(xi) $a - a = 0$,

(xii) $a' = 1 - a$ and $a = 1 - a'$, for every $a, b \in E$.

Definition 2.4 ([8]). An *MV-algebra* is a structure $M = (M, \oplus, ', 0)$ of type $(2, 1, 0)$ that satisfies the following axioms:

$(MV1)$ $(M, \oplus, 0)$ is an Abelian monoid,

$(MV2)$ $(a')' = a$,

$(MV3)$ $0' \oplus a = 0'$,

$(MV4)$ $(a' \oplus b)' \oplus b = (b' \oplus a)' \oplus a$.

An l-group is an algebra $(G, +, -, 0, \vee, \wedge)$, where the following properties hold:

(a) $(G, +, -, 0)$ is a group,

(b) $(G, \vee, \wedge)$ is a lattice,

(c) $x \leq y$ implies that $b + x + a \leq b + y + a$, for any $x, y, a, b \in G$.

A strong unit $u > 0$ is a positive element with property that for any $g \in G$ there exits $n \in \omega$ such that $g \leq nu$. The Abelian l-groups with a strong unit will be simply called lu-groups.

The category whose objects are MV-algebras and whose homomorphisms are MV-homomorphisms is denoted by MV. The category whose objects are pairs (G, u), where G is an Abelian l-group and u is a strong unit of G and whose homomorphisms are l-group homomorphisms is denoted by Ug. The functor that establishes the categorical equivalence between MV and Ug is

$$\Gamma : Ug \longrightarrow MV,$$

where $\Gamma(G, u) = [0, u]_G$, for every lu-group (G, u) and $\Gamma(h) = h|_{[0,u]}$, for every lu-group homomorphism h. The above results allows us to consider an MV-algebra, when necessary, as an interval in the positive cone of an l-group.

DEFINITION 2.5 ([9]). A *product MV-algebra* (or PMV-algebra, for short) is a structure $A = (A, \oplus, ., ', 0)$, where $(A, \oplus, ', 0)$ is an MV-algebra and "." is a binary associative operation on A such that the following property is satisfied: if $x + y$ is defined, then $x.z + y.z$ and $z.x + z.y$ are defined and $(x + y).z = x.z + y.z$, $z.(x + y) = z.x + z.y$, for every $x, y, z \in A$, where " + " is the partial addition on A.

Let $A = (A, \oplus, ., ', 0)$ be a PMV-algebra, $M = (M, \oplus, ', 0)$ be an MV-algebra and the operation $\Phi : A \times M \longrightarrow M$ be defined by $\Phi(a, x) =: ax$, which satisfies the following axioms:

$(AM1)$ If $x+y$ is defined in M, then $ax+ay$ is defined in M and $a(x+y) = ax + ay$,

$(AM2)$ If $a+b$ is defined in A, then $ax+bx$ is defined in M and $(a+b)x = ax + bx$,

$(AM3)$ $(a.b)x = a(bx)$, for every $a, b \in A$ and $x, y \in M$.

Then M is called a (left) *MV-module* over A or briefly an A-module.

We say that M is a *unitary* MV-module if A has a unity for the product and

$(AM4)$ $1_A x = x$, for every $x \in M$.

3. Effect modules

In this section, we present the definition of an effect module in effect algebras and state some results on them.

DEFINITION 3.1. Let $P = (P; +, ., 0, 1)$ be a product effect algebra and $E = (E; +, 0, 1)$ be an effect algebra. Then we say that E is an *effect module* over P or *P-module* if there is an external operation $\varphi : P \times E \longrightarrow E$, with $\varphi(a, x) =: ax$ such that for any $x, y \in E$ and $a, b \in P$, the following properties hold:

$(PE1)$ If $a+b$ is defined, then $ax+bx$ is defined and $(a+b)x = ax+bx$.

$(PE2)$ If $x+y$ is defined, then $ax+ay$ is defined and $a(x+y) = ax+ay$.

$(PE3)$ $(a.b)x = a(bx)$.

Moreover, if $\varphi(1, x) = 1x = x$, for every $x \in E$, then E is called a unitary P-module.

Example 3.2.

(i) Let P be a product effect algebra and E be an effect algebra. If we define $\varphi(a, x) = 0$, for any $a \in P$ and $x \in E$, then E becomes a P-module.

(ii) Consider the real unit interval $[0, 1]$. Let $x \oplus y = min\{x+y, 1\}$, for all $x, y \in [0, 1]$. Then $([0, 1], \oplus, 0, 1)$ is an effect algebra, where "$+$" and "$-$" are the ordinary operations in $\mathbb{R}$. Moreover, consider $ab = a.b$, for every $a, b \in [0, 1]$, where "." is the ordinary operation in $\mathbb{R}$. Then $[0, 1]$ is a $[0, 1]$-module.

(iii) Let $E = \{0, 1, 2, 3\}$ and the operation "$+$" is defined on P as follows:

$+$	0	1	2	3
0	0	1	2	3
1	1	$-$	3	$-$
2	2	3	$-$	$-$
3	3	$-$	$-$	$-$

Then $(E; +, 0, 3)$ is an effect algebra. If we define operation "." by

$.$	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

then $P = (E; +, ., 0, 3)$ is a product effect algebra. Consider $\phi(a, x) = a.x$, for every $a, x \in E$. Then E is a P-module.

(iv) Let $L = \{0, x, 1\}$, $P = \{0, 1\}$ and operations $+$ and $+'$ is defined on L and P, respectively, as follows:

$+$	0	x	1
0	0	x	1
x	x	1	$-$
1	1	$-$	$-$

$+'$	0	1
0	0	1
1	1	$-$

Then $(L; +, 0, 1)$ is an effect algebra and $(P; +', ., 0, 1)$ is a product effect algebra, where "." is the ordinary operation in $\mathbb{R}$. Consider $E = L \times L$. Then $(E; \oplus, (0, 0), (1, 1))$ is an effect algebra, where $(e_1, e_2) \oplus (b_1, b_2) = (e_1 + b_1, e_2 + b_2)$, for every $e_1, e_2, b_1, b_2 \in L$. Now, for any $a \in P$ and $(e_1, e_2) \in E$, we consider $\varphi(a, (e_1, e_2)) = (ae_1, ae_2)$, where for every $e \in L$,

$$ae = \begin{cases} 0 & a = 0 \\ e & a = 1 \end{cases}$$

Then E is a P-module.

LEMMA 3.3. *Every associative product effect algebra* $(P, +, ., 0, 1)$ *is a* P-*module.*

Proof: If we define $\phi(a,b) = a.b$, for every $a, b \in P$, then it is easy to see that P is a P-module. $\square$

Proposition 3.4. Let E be an effect algebra such that for every $a, b \in E$, $a' + (b + a')' = b' + (a + b')'$. Then E can become an E-module.

Proof: If we define $a.b = (a * b)'$, where $a * b = a' + (b + a')'$, then in a straightforward way, E is an associative product effect algebra and so by Lemma 3.3, E is an E-module. $\square$

Note. Let $E = (E; +, 0, 1)$ be an effect algebra. Then for any $a, b \in E$, $a \leq b'$ if and only if $a + b$ is defined in E.

Lemma 3.5. *Let $E = (E; +, 0, 1)$ be an effect algebra. Then the set of*

$$End(E) = \{f : E \to E : f \text{ is an } E\text{-homomorphism}\}$$

is a product effect algebra.

Proof: We consider $f + g : E \to E$, by $(f + g)(x) = f(x) + g(x)$, where $f(x) + g(x)$ is defined in E and $(f + g)(x) = 0$, where $f(x) + g(x)$ is not defined in E. Also, we consider $f \circ g : E \to E$, by $(f \circ g)(x) = f(g(x))$. Let $x + y$ be defined in E. Since f, g are E-homomorphism, $f(x) + f(y)$ and $g(x) + g(y)$ are defined and so it is easy to see that $f + g$ and $f \circ g$ are E-homomorphism. Thus, it is routine to see that $(End(E), +, \circ, I, O)$ is an effect algebra, where $I : E \to E$ and $O : E \to E$ are identity E-homomorphism and zero E-homomorphism, respectively. $\square$

Theorem 3.6. *Let $E = (E; +, 0, 1)$ be an effect algebra and P be a product effect algebra. Then E is a unitary P-module if and only if there exists a P-homomorphism $\varphi : P \to End(E)$.*

Proof: Let E be a unitary P-module with module multiplication $\psi : P \times E \to E$, by $\psi(a, x) = ax$, for every $a \in P$ and $x \in E$. By Lemma 3.5, $End(E)$ is a product effect algebra. We consider the function $\varphi : P \to End(E)$, by $a \to \varphi(a)$, where $\varphi(a) : E \to E$ is defined by $\varphi(a)(x) = ax$, for every $a \in P$ and $x \in E$. We show that φ is a homomorphism of product effect algebras. Let $a + b$ be defined in P, for any $a, b \in P$. Then we have

$$\varphi(a + b)(x) = (a + b)x = ax + bx = \varphi(a)(x) + \varphi(b)(x) = (\varphi(a) + \varphi(b))(x)$$

for every $x \in E$. It results that $\varphi(a + b) = \varphi(a) + \varphi(b)$. Now, for every $a, b \in P$, since

$$\varphi(a.b)(x) = (a.b)x = a(bx) = a(\varphi(b)(x)) = \varphi(a)(\varphi(b)(x)) = (\varphi(a)\circ\varphi(b))(x)$$

for every $x \in E$, we have $\varphi(a.b) = \varphi(a) \circ \varphi(b)$. Also, $\varphi(1)(x) = 1x = x$, for every $x \in E$ and so $\varphi(1) = I$.
Conversely, let there is a P-homomorphism $\varphi : P \to End(E)$. We define $\psi : P \times E \to E$, by $\psi(a, x) = ax = \varphi(a)(x)$, for every $a \in P$ and $x \in E$. It is easy to see that ψ is well defined.

$(PE1)$ Let $a + b$ be defined in P. Then $a \leq b'$ and so $ax \leq b'x$. We must show that $ax + bx$ is defined in E. The first, we show that $b'x \leq (bx)'$. Since $x \leq x$, hence $x' + x$ is defined and so

$$b(x + x') = \psi(b)(x' + x) = \psi(b)(x') + \psi(b)(x) = bx' + bx$$

Then

$$bx' + bx = b(x + x') = b1 \leq 1 = (bx)' + bx$$

and so $b'x \leq (bx)'$. Thus, $ax \leq b'x \leq (bx)'$ and so $ax + bx$ is defined. Hence, it is easy to see that $(a + b)x = ax + bx$.

$(PE2)$ Let $x+y$ be defined in E. Similar to $(PE1)$, we can show that $a(x + y) = ax + ay$.

$(PE3)$ Let $a, b \in P$ and $x \in E$. Then

$$\begin{aligned}(a.b)x &= \psi(a.b)(x) = (\psi(a) \circ \psi(b))(x) = \psi(a)(\psi(b)(x)) = \psi(a)(bx) \\ &= a(bx)\end{aligned}$$

Moreover, $1x = \psi(1)(x) = x$, for every $x \in E$. Therefore, E is a unitary P-module. □

THEOREM 3.7.

(i) *Every MV-module can be transformed into an effect module.*

(ii) *Every effect module satisfying $(RDP2)$ can be transformed into an MV-module.*

PROOF:

(i) Let M be an A-module, where A is a PMV-algebra. We can consider $M = \Gamma(G, u)$, where G is an Abelian l-group and u is a strong unit of G. Define " + " to be a partial operation on M that is defined for elements $a, b \in M$ if and only if $a \leq b'$, and in that case let $a+b := a \oplus b$. Then $(M, +, 0, 1)$ is an effect algebra. Similarly, A can be transformed into a product effect algebra. Now, by MV-module multiplication, M will be an effect module.

(ii) Let E be a P-module satisfying $(RDP2)$. By ([14], Theorem 8.8), E and P are MV-algebras. If we consider $a \bullet b = a.b$, for every $a, b \in P$, where "." is the product operation in P, then P is a PMV-algebra. Now, by effect module multiplication, E can be transformed into an MV-module. $\square$

In the rest of this paper, we let P be a product effect algebra and E be an effect algebra, unless otherwise specified. Also, if we are not sure that $a + b$ is defined in effect algebra E, then we denote $a \oplus b$ instead of $a + b$, for any $a, b \in E$.

4. Some topologies on effect modules

In this section, we introduce five topologies on effect modules.

DEFINITION 4.1. Let E be a P-module. Then $\emptyset \neq I \subseteq E$ is called a *submodule* of E if it satisfies the following conditions, for every $a \in P$ and $x, y \in E$:

(I_1) If $x, y \in I$ and $x + y$ is defined in E, then $x + y \in I$.

(I_2) If $x \leq y$ and $y \in I$, then $x \in I$.

(I_3) If $x \in I$, then $ax \in I$.

$I \subseteq E$ is called a W-submodule (weak submodule) of E if it satisfying (I_3). $I \subseteq E$ is called an E-ideal of E if it satisfying (I_1) and (I_2).

We denote by $Sb_P(E)$ and $WSb_P(E)$, respectively, the set of all submodules of P-module E and the set of all W-submodules of P-module E.

Example 4.2.

(i) For every effect module E, $\{0\}$ and E are trivial submodules of E.

(ii) In Example 3.2 (iii), $I = \{0, 1\}$ and $J = \{0, 2\}$ are submodules of E.

(iii) Every submodule of E is a W-submodule (an E-ideal) of E.

For every subset I of E, we denote

$$U_I = \{\varphi(a, x) : a \in (I : E) \textit{ or } x \in I\},$$

where $(I : E) = \{x \in P : xE \subseteq I\}$.

Proposition 4.3. Let E be a P-module. Then

(i) If $I \subseteq J$, then $(I : E) \subseteq (J : E)$, where I, J be subsets of E.

(ii) $\bigcap_{i\in I}(J_i : E) = (\bigcap_{i\in I} J_i : E)$, where J_i is a subset of E, for every $i \in I$.

(iii) If I is a submodule of E, then $(I : E)$ is an ideal of P.

If I is a W-submodule of P as P-module, where P is an associative product effect algebra, then

(iv) $I \subseteq (I : P)$.

(v) (v) $(I : P)P \subseteq I$.

Proof: The proof is easy. □

Theorem 4.4. *Let E be a unitary P-module and $a.a = a$, for every $a \in P$. Then $\Gamma = \{U_I : I \in WSb_P(E)\}$ is a topology on E.*

Proof: Let E be a P-module, $a.a = a$, for every $a \in P$ and $I, J \in WSb_P(E)$. First we prove that:

(i) $U_\emptyset = \emptyset$ and $U_E = E$.

(ii) $U_I \cap U_J = U_{I\cap J}$.

(iii) $U_I \cup U_J = U_{I\cup J}$.

The proof of (i) is clear. For the proof of (ii), since $I\cap J \subseteq I$ and $I\cap J \subseteq J$, it is easy to see that $U_{I\cap J} \subseteq U_I \cap U_J$. Let $\phi(a, x) \in U_I \cap U_J$. Then $ax \in U_I$ and $ax \in U_J$. It results that $a \in (I : E)$ *or* $x \in I$ and $a \in (J : E)$ *or* $x \in J$. There are four possible cases:

(1) If $a \in (I : E)$ and $a \in (J : E)$, then it is easy to see that $a \in (I\cap J : E)$ and so $\phi(a, x) = ax \in U_{I\cap J}$.

(2) If $x \in I$ and $x \in J$, then $x \in I \cap J$ and so $ax \in U_{I\cap J}$.

(3) Let $a \in (I : E)$ and $x \in J$. Then $aE \subseteq I$ and so $ax \in I$. Since J is a W-submodule of E, $ax \in J$, too. Hence $\phi(a, x) = ax \in I \cap J$ and so $a(ax) = \phi(a, ax) \in U_{I\cap J}$. It results that by $(PE3)$,

$$\phi(a, x) = ax = (a.a)x = a(ax) \in U_{I\cap J}$$

(4) If $x \in I$ and $a \in (J : E)$, then similar to (3), we have $ax \in U_{I\cap J}$. Therefore, $U_I \cap U_J \subseteq U_{I\cap J}$.

(iii) Since $I \subseteq I \cup J$ and $J \subseteq I \cup J$, it is easy to show that $U_I \cup U_J \subseteq U_{I\cup J}$. Let $\phi(a, x) \in U_{I\cup J}$. Then $a \in (I \cup J : E)$ or $x \in I \cup J$. If $a \in (I \cup J : E)$, then $aE \subseteq I \cup J$ and so $ax \in I \cup J$. Thus, $ax \in I$ or $ax \in J$ and so $ax = a(ax) \in U_I$ or $ax = a(ax) \in U_J$. It follows that $ax \in U_I \cup U_J$. Now, let $x \in I \cup J$. Then $x \in I$ or $x \in J$. It results that $ax \in U_I$ or $ax \in U_J$ and so $ax \in U_I \cup U_J$. Hence $U_I \cup U_J = U_{I\cup J}$.

Therefore, by (i), (ii) and (iii), we obtain that Γ is a topology on E. □

Next, we present definition of linear submodules of an effect module and introduce another topology on E.

Definition 4.5. Let I be an E-ideal of E. Then I is called a linear E-ideal of E if I is also a total order set.

Example 4.6. In Example 4.2 (ii), I and J are linear E-ideals of E.

For every subset I of E, we denote

$$L_I = \{(x, y) \in E \times E : x + y \text{ is defined and } \exists c \in I \text{ that } x + c = y \text{ or } \exists d \in I \text{ that } y + d = x\}.$$

Let $L, K \subseteq E \times E$ such that $x + y$ be defined in them, for every x, y in them. Then we denote

$$L^{-1} = \{(y, x) : (x, y) \in L\}\ ,\ L(y) = \{x : (y, x) \in L,\ \text{for every } y \in E\}$$

and

$$L \circ K = \{(x, z) : \exists y \in E \text{ such that } (x, y) \in L \text{ and } (y, z) \in K\}.$$

LEMMA 4.7. *Let I and J be subsets of E.*

(i) *If $I \subseteq J$, then $L_I \subseteq L_J$.*

If E is an extended effect algebra and I, J are E-ideals of E, then

(ii) $L_I \cap L_J = L_{I\cap J}$.

(iii) $L_I \cup L_J \subseteq L_I \circ L_J$.

If E is an extended effect algebra and I, J are linear E-ideals of E, then

(iv) *$I \cap J$ is a linear E-ideal of E;*

(v) $L_I \circ L_I = L_I$.

PROOF:

(i) The proof is clear.

(ii) Let $(x, y) \in L_I \cap L_J$. Then $(x, y) \in L_I$ and $(x, y) \in L_J$ and so $(x + c_1 = y$ or $y + d_1 = x)$ and $(x + c_2 = y$ or $y + d_2 = x)$, for $c_1, d_1 \in I$ and $c_2, d_2 \in J$. There are four possible cases:

Case (1): Let $x + c_1 = y$ and $x + c_2 = y$. Then $x + c_1 = x + c_2$. Since " $+$ " is cancellative, we have $c_1 = c_2 \in I \cap J$ and so $(x, y) \in L_{I\cap J}$.

Case (2): Let $x + c_1 = y$ and $y + d_2 = x$. Then $x \leq y$ and $y \leq x$ and so $x = y$. It means that $c_1 = d_2 = 0 \in I \cap J$ and so $(x, y) \in L_{I\cap J}$.

Case (3): Let $y + d_1 = x$ and $y + d_2 = x$. The proof of this case is similar to the case (1).

Case (4): Let $y + d_1 = x$ and $x + c_2 = y$. The proof of this case is similar to the case (2). Hence $L_I \cap L_J \subseteq L_{I\cap J}$. It is easy to show that $L_{I\cap J} \subseteq L_I \cap L_J$. Therefore, $L_I \cap L_J = L_{I\cap J}$.

(iii) Let $(x, y) \in L_I \cup L_J$. Then $(x, y) \in L_I$ or $(x, y) \in L_J$. Let $(x, y) \in L_I$. Since $(y, y) \in L_{\{0\}} \subseteq L_J$, we have $(x, y) \in L_I \circ L_J$. Similarly, if $(x, y) \in L_J$, then $(x, y) \in L_I \circ L_J$. Thus, $L_I \cup L_J \subseteq L_I \circ L_J$.

(iv) The proof is clear.

(v) Let $(x, z) \in L_I \circ L_I$. Then there is $y \in E$ such that $(x, y) \in L_I$ and $(y, z) \in L_I$. Thus, $(x + c_1 = y$ or $y + d_1 = x)$ and $(x + c_2 = z$ or $z + d_2 = y)$, for $c_1, d_1, c_2, d_2 \in I$. There are four possible cases:

(1) Let $x + c_1 = y$ and $y + c_2 = z$. Then $x + c_1 + c_2 = z$. Since $c_1 + c_2 \in I$, we have $(x, z) \in L_I$.

(2) Let $x + c_1 = y$ and $z + d_2 = y$. Since I is a linear set, we have $c_1 \leq d_2$ or $d_2 \leq c_1$. If $c_1 \leq d_2$, then there is $e \in E$ such that $c_1 + e = d_2$, thus $e \leq d_2 \in I$, so $e \in I$. Also, we have $z + c_1 + e = x + c_1$. So $z + e = x$ and so $(z, x) \in L_I$. Then $(x, z) \in (L_I)^{-1} = L_I$.

(3) Let $y + d_1 = x$ and $y + c_2 = z$. Then similar to (2), we can prove that $(x, z) \in (L_I)^{-1} = L_I$.

(4) $y + d_1 = x$ and $z + d_2 = y$. Similar to (1), we prove that $L_I \circ L_I \subseteq L_I$.
On the other hand, by (iii), it is clear that $L_I \subseteq L_I \circ L_I$. Therefore, $L_I \circ L_I = L_I$. □

THEOREM 4.8. *Let E be an extended effect algebra, $\mathcal{I}$ be a family of all linear E-ideals of E, $K_0 = \{L_I : I \in \mathcal{I}\}$ and*

$$K = \{V \subseteq E \times E : x + y \text{ is defined for every } (x, y) \in V \text{ and } \exists L_I \in K_0 \text{ such that } L_I \subseteq V\}.$$

Then

(1) *If $V \in K$, then $V^{-1} \in K$.*

(2) *For every $V \in K$, there is $L \in K_0$ such that $L \circ L \subseteq V$.*

(3) *For every $V, L \in K$, we have $L \cap V \in K$.*

(4) *If $L \in K$ and $L \subseteq V \subseteq E \times E$ such that for any $(x, y) \in V$, $x + y$ is defined, then $V \in K$.*

PROOF: By Lemma 4.7, the result can obtain immediately. □

COROLLARY 4.9. Consider the set K in Theorem 4.8 and $T = \{L_I(x) : I \in \mathcal{I},\ x \in E\}$. Then

(i) K is a base of a topology of $E \times E$.

(ii) T is a base of a topology of E.

Proof:

(i) We should proof that (1) $E \times E = \bigcup_{V \in K} V$; (2) for any $V_1, V_2 \in K$ and $x \in V_1 \cap V_2$, there exists $V \in K$ such that $x \in V \subseteq V_1 \cap V_2$.

  (1) Let $(x, y) \in E \times E$. Then we can consider $V = L_I(x) \times L_I(y) = \{(a, b) : (x, a) \in L_I(x) \textit{ and } (y, b) \in L_I(y)\}$. Since $(x, x) \in L_I(x)$ and $(y, y) \in L_I(y)$, we have $(x, y) \in V$ and so $E \times E \subseteq \bigcup_{V \in K} V$. Hence $E \times E = \bigcup_{V \in K} V$.

  (2) Let $V_1, V_2 \in K$ and $x \in V_1 \cap V_2$. Then by Theorem 4.8 (3), we have $V_1 \cap V_2 \in K$ and so we consider $V = V_1 \cap V_2$. Therefore, K is a base of a topology of $E \times E$.

(ii) Similar to proof (i), we should prove that $E = \bigcup_{x \in E} L_I(x)$ and there exists $V \in T$ with similar condition (2) in proof (i). Let $x \in E$. Since $x = x + 0$, we have $x \in L_I(x)$. Then $E \subseteq \bigcup_{x \in E} L_I(x)$ and so $E = \bigcup_{x \in E} L_I(x)$. Also, for $L_I(x), L_I(y) \in T$ and $x \in L_I(x) \cap L_I(y)$, by Lemma 4.7 (ii), we have $L_{I \cap J} = L_I(x) \cap L_I(y)$ and so we consider $V = L_I(x) \cap L_I(y)$. Therefore, T is a base of a topology of E. $\square$

In following, we present definition of effect topological modules and we give a general example about them.

Definition 4.10. Let E be a P-module. If $f : E \times E \longrightarrow E$ (defined by $f(e, e') = e + e'$, for every $e, e' \in E$, where $E \times E$ is multiplicative topology in E) and $\mu_x : E \longrightarrow E$ (defined by $\mu_x(e) = xe$, for every $e \in E$ and $x \in P$) are continuous under some topology τ, then (E, τ) is called a topological effect module.

Example 4.11. Let E be a P-module and $\{E_n : E_n \supseteq E_{n+1},\ n \in \mathbf{N}\}$ be a decreasing sequence of proper submodules of E. Then it is routine to see that the collection

$$\tau = \{V \subseteq E : \forall v \in V\ \exists n \in \mathbf{N} \text{ such that } v + E_n \subseteq V\},$$

where $V + E_n = \{v + e : v + e \text{ is defined in } E\}$ forms a topology on E. Also, $\mathcal{B}_\tau = \{x + E_n : x \in E,\ n \in \mathbf{N}\}$ forms a base for τ. Now, we show that the addition "+" and the effect module multiplication are continuous under topology τ. Consider $f : E \times E \longrightarrow E$ defined by $f(e, e') = e + e'$, for $e, e' \in E$ and a basic open set $e + E_n \in \mathcal{B}_\tau$. If $f^{-1}(e + E_n) = \emptyset$, then result holds trivially. If $f^{-1}(e+E_n) \neq \emptyset$, then it is easy to prove that $f^{-1}(e+E_n)$ is open and so f is continuous. Finally, it is easy to show that the mapping $\mu_x : E \longrightarrow E$ defined by $\mu_x(e) = xe$ is continuous, for every $x \in P$. Therefore, τ force E to be a topological effect module.

Next, we present definition of prime submodules in effect modules and we present two topology on them.

Definition 4.12. Let E be a P-module and I be a proper submodule of E. Then I is called a prime submodule of E if it satisfies in the following condition:

If $ax \in I$, then $a \in (I : E)$ or $x \in I$, for any $a \in P$ and $x \in E$.

The set of all prime submodules of E is denoted by $Spec_P(E)$.

Example 4.13. In Example 3.2 (iii), $I = \{0, 1\}$ and $J = \{0, 2\}$ are prime submodules of E and $\{0\}$ is not a prime submodule of E. Note that $Spec_P(E) = \{I, J\}$.

Definition 4.14. Let E be a P-module and $\mathcal{T}(E) = \{\mathcal{V}(I) : I \in Sb_P(E)\}$, where $\mathcal{V}(I) = \{P \in Spec_P(E) : I \subseteq P\}$. If $\mathcal{T}(E)$ is closed under finite union, then E is called a *Top* P-module.

Example 4.15.

(i) If E is a P-module and $Spec_P(E) = \emptyset$, then E is a Top P-module.

(ii) By Example 4.13, $Spec_P(E)=\{I, J\}$ and $\mathcal{T}(E)=\{\emptyset,\{I\},\{J\},\{I, J\}\}$. It is easy to see that E is a Top P-module.

(iii) By Example 3.2 (iv), It is easy to see that $I = \{(0, 0)\}$, $J = \{(0, 0), (0, x), (0, 1)\}$ and $K = \{(0, 0), (x, 0), (1, 0)\}$ are prime submodules of E. We have $\mathcal{V}(I) = \{I\}$, $\mathcal{V}(J) = \{J\}$ and $\mathcal{V}(K) = \{K\}$. It is routine to see that E is not a Top P-module.

Proposition 4.16. Let E be a Top P-module. Then $\mathcal{T}(E)$ satisfies the axioms for closed sets in a topological space.

Proof: Clearly, $\mathcal{V}(E) = \emptyset$ and $\mathcal{V}(\{0\}) = Spec_P(E)$. It is enough to show that $\bigcap_{i\in I} \mathcal{V}(I_i) = \mathcal{V}(\bigvee_{i\in I} I_i)$, where $\bigvee_{i\in I} I_i = Sup\{I_i : i \in I\}$. Let $P \in \bigcap_{i\in I} \mathcal{V}(I_i)$. Then $P \in \mathcal{V}(I_i)$ and so $I_i \subseteq P$, for every $i \in I$. Hence $\bigvee_{i\in I} I_i \subseteq P$ and so $P \in \mathcal{V}(\bigvee_{i\in I} I_i)$. Thus $\bigcap_{i\in I} \mathcal{V}(I_i) \subseteq \mathcal{V}(\bigvee_{i\in I} I_i)$. On the other hand, we have $\mathcal{V}(\bigvee_{i\in I} I_i) \subseteq \bigcap_{i\in I} \mathcal{V}(I_i)$. Therefore, $\bigcap_{i\in I} \mathcal{V}(I_i) = \mathcal{V}(\bigvee_{i\in I} I_i)$. $\square$

Remark 4.17. Let E be a Top P-module. Then By Proposition 4.16, $\mathcal{T}_E = \{\mathcal{V}(I)^c : I \in Sb_P(E)\}$ is a topology on $Spec_P(E)$.

DEFINITION 4.18. Let E be a P-module and K be a submodule of E. If K is an intersection of some prime submodules of E, then K is called a *semiprime* submodule of E.

DEFINITION 4.19. Let E and F be two effect algebras. A mapping $f : E \longrightarrow F$ is said to be a P-homomorphism if (i) f is a homomorphism; (ii) $f(ax) = af(x)$, for any $a \in P$ and $x \in E$. If f is one to one (onto), then f is called a P-monomorphism (P-epimorphism) and if f is onto and one to one, then f is called a P-isomorphism.

LEMMA 4.20.

(i) *E is a Top P-module if and only if for every prime submodule K of E, $N \cap L \subseteq K$ implies that $N \subseteq K$ or $L \subseteq K$ $(*)$, where N, L are semiprime submodules of E.*

(ii) *Let E and F be two P-modules, $f : E \longrightarrow F$ be a P-isomorphism and G be a prime submodule of F satisfying $(*)$. Then $f^{-1}(G)$ is a prime submodule of E satisfying $(*)$.*

PROOF:

(i) Let K be a prime submodule of E, N and L be semiprime submodules of K such that $N \cap L \subseteq K$. Since E is a Top P-module, there exists a submodule J of E such that $\mathcal{V}(N) \cup \mathcal{V}(L) = \mathcal{V}(J)$. Since N is a semiprime submodule of E, $N = \bigcap_{i\in I} P_i$, where $\{P_i\}_{i\in I}$ is a family of prime submodules of E. Then $P_i \in \mathcal{V}(N)$, for any $i \in I$. Since $\mathcal{V}(N) \subseteq \mathcal{V}(J)$, we have $P_i \in \mathcal{V}(J)$. Hence $J \subseteq N$ and $J \subseteq L$ and so $J \subseteq N \cap L$. It follows that $\mathcal{V}(N \cap L) \subseteq \mathcal{V}(J)$. Now, we have $\mathcal{V}(N) \cup \mathcal{V}(L) \subseteq \mathcal{V}(N \cap L) \subseteq \mathcal{V}(J) = \mathcal{V}(N) \cup \mathcal{V}(L)$ and so $\mathcal{V}(N) \cup \mathcal{V}(L) = \mathcal{V}(N \cap L)$. It means that $K \in \mathcal{V}(N)$ or $K \in \mathcal{V}(L)$ and so $N \subseteq K$ or $L \subseteq K$. The proof of converse is routine.

(ii) The proof is routine. $\square$

Theorem 4.21. *Let E and F be two P-modules and $f : E \longrightarrow F$ be a P-isomorphism. If $\mathcal{T}_F$ is a topology on $Spec_P(F)$, then $\mathcal{T}_E^{-1} = \{V(N)^c : N \in Spec_P(E)\}$ is a topology on $Spec_P(E)$, where*

$$V(I) = \{f^{-1}(K) : K \in Spec_P(F) \text{ and } f(I) \subseteq K\},$$

for every $I \subseteq F$.

Proof: Since $\mathcal{T}_F$ is a topology on $Spec_P(F)$, $\mathcal{T}(F)$ is closed under finite union and so by Lemma 4.20 (i), $N \cap L \subseteq K$ implies that $N \subseteq K$ or $L \subseteq K$, for every prime submodule K of F, where N, L are semiprime submodules of F. We claim that $\mathcal{T}^{-1}(E) = \{V(N) : N \in Spec_P(E)\}$ is closed under finite unions. By Lemma 4.20 (ii), $f^{-1}(K)$ is a prime submodule of E, for every $K \in Spec_P(F)$. The first, we prove that $f(G) \in Spec_P(F)$, for every $G \in Spec_P(E)$. Let $x, y \in f(G)$ and $x + y$ be defined in F. Clearly, there are $m, n \in G$ such that $x = f(m),\ y = f(n)$ and $f(m) + f(n)$ is defined in F. Since f^{-1} is a P-homomorphism and $f(m) + f(n)$ is defined in F, we result that $f^{-1}(f(m)) + f^{-1}(f(n))$ is defined in E and so $m + n$ is defined in E. It means that

$$x + y = f(m) + f(n) = f(m + n) \in f(G).$$

Now, let $x \leq y$ and $y \in f(G)$, for any $x, y \in F$. Then there are $m \in G$ and $n \in E$ such that $x = f(m)$ *and* $y = f(n)$. Since $f(m) \leq f(n)$, there is $f(r) = c \in F$ such that $f(m) + f(r) = f(n)$, for $r \in E$ and so $f(m+r) = f(n)$. Hence $m+r = n$ and so $m \leq n \in G$. It means that $m \in G$ and so $x = f(m) \in G$. Thus, $f(G)$ is a submodule of F. It is routine to show that $f(G)$ is a prime submodule of F, for every $G \in Spec_P(E)$. Then $f(N) = \bigcap_{G \in Spec_P(E)} f(G)$ and $f(L) = \bigcap_{G' \in Spec_P(E)} f(G')$ are semiprime submodules of F. Hence by Lemma 4.20, $N \cap L \subseteq f^{-1}(G)$ implies that $N \subseteq f^{-1}(G)$ or $L \subseteq f^{-1}(G)$. Now, it is routine to see that $V(N) \cup V(L) = V(N \cap L)$, for every semiprime submodules of E and so by a straightforward way, we conclud that $\mathcal{T}^{-1}(E)$ is closed under finite unions. Therefore,

$$\mathcal{T}_E^{-1} = \{V(N)^c : N \in Spec_P(E)\}$$

is a topology on $Spec_P(E)$. □

In following, we present topology on $Spec_P(E)$ that is coarser than $\mathcal{T}_E$. Let E be a P-module, N be a submodule of E and $J \subseteq P$. Then we denote:

$W(N) = \{P \in Spec_P(E) : (N : E) \subseteq (P : E)\}$,
$\Upsilon_E = \{W(N)^c : N \in Sb_P(E)\}$
$JE = \{x \in E : x \leq a_1x_1 + \cdots + a_nx_n, \text{ s.t. } \exists a_1 \cdots, a_n \in J, x_1, \cdots, x_n \in E :$
$a_1x_1 + \cdots + a_nx_n \text{ is defined in } E\}$

LEMMA 4.22. *Let E be a P-module and N be a submodule of E. Then $a.b \in (N : E)$, for every $a \in P$ and $b \in (N : E)$.*

PROOF: Let $a \in P$ and $b \in (N : E)$. Then $bE \subseteq N$ and so $be \in N$, for every $e \in N$. Hence $(a.b)e = a(be) \in N$ and so $a.b \in (N : E)$. □

THEOREM 4.23. *Let E be a P-module. Then Υ_E is a topology on $Spec_P(E)$.*

PROOF: It is clear that $W(N) = \emptyset$ and $W(\{0\}) = Spec_P(E)$. It is routine to see that $W(N) \cup W(M) = W(N \cap M)$, for every $N, M \in Sb_P(E)$. We show that $\bigcap_{i \in I} W(N_i) = W(JE)$, where $J = \bigvee_{i \in I}(N_i : E)$. The first, we prove that JE is a submodule of E. Let $a, b \in JE$ and $a + b$ is defined in E. Then

$$a \leq a_1x_1 + \cdots + a_nx_n \textit{ and } b \leq b_1y_1 + \cdots + b_my_m$$

where $a_1x_1 + \cdots + a_nx_n$ and $b_1y_1 + \cdots + b_my_m$ are defined in E, for some $a_1, \cdots, a_n, b_1, \cdots, b_m \in J$ and $x_1, \cdots, x_n, y_1, \cdots, y_m \in E$. So

$$a + b \leq a_1x_1 + \cdots + a_nx_n + b_1y_1 + \cdots + b_my_m$$

If $a_1x_1 + \cdots + a_nx_n + b_1y_1 + \cdots + b_my_m$ is not defined in E, then we can rewrite it by new a_i's and b_i's such that is defined in E (since $a+b$ is defined in E, it is possible). Thus, I_1 is true. Note that (I_2) is clear. Now, let $e \in P$ and $a \in JE$. Then $a \leq a_1x_1+\cdots+a_nx_n$, where $a_1x_1+\cdots+a_nx_n$ is defined, for any $a_1, \cdots, a_n \in J$ *and* $x_1, \cdots, x_n \in E$. Since $a \leq a_1x_1 + \cdots + a_nx_n$, there is $c \in E$ such that $a + c = a_1x_1 + \cdots + a_nx_n$ and so by $(PE2)$ and $(PE3)$,

$$ea + ec = (e.a_1)x_1 + \cdots + (e.a_n)x_n$$

It means that $ea \leq (e.a_1)x_1 + \cdots + (e.a_n)x_n$, where by Lemma 4.22, $(e.a_1), \cdots, (e.a_n) \in J$ and so $ea \in JE$. Hence JE is a submodule of E. Now, it is routine to see that $\bigcap_{i \in I} W(N_i) = W(JE)$. Therefore, Υ_E satisfies the axioms of topology defined by open sets. □

Example 4.24.

(i) In Example 3.2 (iii), we have $Spec_P(E) = \{I, J\}$. Then $W(I) = \{I\}$, $W(J) = \{J\}$, $W(\emptyset) = Spec_P(E)$ and $W(E) = \{\emptyset\}$. Then $\Upsilon_E = \{\emptyset, \{I\}, \{J\}, \{I, J\}\}$ is topology on $Spec_P(E)$.

(ii) In Example 4.15 (iii), we have $Spec_P(E) = \{I,\ J,\ K\}$. Then $W(E) = \emptyset$,

$$W(I) = W(J) = W(K) = Spec_P(E)\ ,\ \Upsilon_E = \{\emptyset, Spec_P(E)\}.$$

Therefore, Υ_E is topology on $Spec_P(E)$.

5. Conclusion

Effect algebras generalize many examples of quantum structures, like Boolean algebras, orthomodular lattices or posets, orthoalgebras, MV-algebras, etc. Recently, module structures have been defined over some algebraic structures, and some researches have been interested in this topic. We presented definition of effect modules. Next researchers can study free effect modules, projective (injective) modules and many of the other concepts of modules. In the study of effect algebras (or more general, quantum structures) as carriers of states and probability measures, an important tool is the study of topologies on them. Also, the studying of certain topological properties of algebraic structures characterize also their certain algebraic properties. We studied and introduced some topologies on effect modules. We wish that the obtained results can encourage us to continue this long way. In fact, we hope that we could open new fields to anyone that is interested to studying and development of modules.

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EQUALITY LOGIC

Abstract

In this paper, we introduce and study a corresponding logic to equality-algebras and obtain some basic properties of this logic. We prove the soundness and completeness of this logic based on equality-algebras and local deduction theorem. We show that this logic is regularly algebraizable with respect to the variety of equality-algebras but it is not Fregean. Then we introduce the concept of (prelinear) equality$_\triangle$-algebras and investigate some related properties. Also, we study $\triangle$-deductive systems of equality$_\triangle$-algebras. In particular, we prove that every prelinear equality$_\triangle$-algebra is a subdirect product of linearly ordered equality$_\triangle$-algebras. Finally, we construct prelinear equality$_\triangle$ logic and prove the soundness and strong completeness of this logic respect to prelinear equality$_\triangle$-algebras.

Keywords: Many-valued logic, equality logic, completness, prelinear equality$_\triangle$-algebra, prelinear equality$_\triangle$ logic.

1. Introduction

Novák introduced the concept of EQ-algebras in [17] as candidates for a possible algebraic semantics of fuzzy-type theory (see [16]). These algebras are meet semilattices endowed with two additional binary operations: fuzzy equality and multiplication. Implication is derived from the fuzzy equality and it is not a residuation with respect to multiplication. Consequently, EQ-algebras is a generalization of residuated lattices in the sense that each residuated lattice is an EQ-algebra but not vice-versa.

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Dyba and Novák introduced EQ-logic in [9] as a specific formal logic in which the basic connective is fuzzy equality and the implication is derived from the fuzzy equality. They formulated the basic EQ-logic and proved the completeness of this logic. Also, see [19, 10, 11].

Recently, Dyba and et all in [8], introduced and studied the prelinear $\mathrm{EQ}_{\triangle}$-algebras and the corresponding propositional $\mathrm{EQ}_{\triangle}$-logic.

As Jenei mentioned in [13], if the product operation in EQ-algebras is replaced by another binary operation smaller or equal than the original (viewed as a two-place function) we still obtain an EQ-algebra. This fact might make it difficult to obtain certain algebraic results. For this reason, Jenei introduced a new structure in [13], called equality-algebra, to find something similar to EQ-algebras but without a product. The equality-algebras have two binary operations meet and equivalence, and a constant 1. Jenei proved the term equivalence of the closed algebras to BCK-meet-semilattices. In [23], F. Zebardast and et all studied and proved that there are relations among equality algebras and some of other logical algebras such as residuated lattice, MTL-algebra, BL-algebra, MV-algebra, Hertz-algebra, Heyting-algebra, Boolean-algebra, EQ-algebra and hoop-algebra. Some types of filters of equality algebras are introduced in [3]. Since then many researchers have worked on this area (see [4, 6, 14, 12]).

In this paper, we will show that equality-algebras are semantics of fuzzy-type theory. In the next section, we review some notions which are needed in the sequel. In section 3, the corresponding equality logic is constructed and some related properties are proved. Also, the soundness and completeness of this logic are proved. We prove that this logic is regularly algebraizable with respect to the variety of the equality-algebras. In section 4, we investigate (prelinear) equality$_{\triangle}$-algebras and $\triangle$-deductive systems on equality$_{\triangle}$-algebras. We obtain some related results. Finally in section 5, we introduce prelinear equality$_{\triangle}$ logic and prove strong completeness.

2. Preliminaries

In this section, we recall the basic definitions and some known results about equality-algebras that we need in the rest of the paper.

Definition 2.1 ([13]). An equality-algebra is an algebra $\mathcal{A} = (A, \wedge, \sim, 1)$ of the type (2, 2, 0) such that satisfies the following axioms for all $x, y, z \in A$:

(E1) $(A, \wedge, 1)$ is a meet-semilattice with top element 1,

(E2) $x \sim y = y \sim x$,

(E3) $x \sim x = 1$,

(E4) $x \sim 1 = x$,

(E5) $x \leq y \leq z$ implies $x \sim z \leq y \sim z$ and $x \sim z \leq x \sim y$,

(E6) $x \sim y \leq (x \wedge z) \sim (y \wedge z)$,

(E7) $x \sim y \leq (x \sim z) \sim (y \sim z)$.

The operation $\wedge$ is called meet (infimum) and $\sim$ is an equality operation. We write $x \leq y$ (and $y \geq x$) iff $x \wedge y = x$. Define the following two derived operations, the implication and the equivalence operation of the equality-algebra $\mathcal{A}$ by

(I) $x \rightarrow y = x \sim (x \wedge y)$,

(II) $x \leftrightarrow y = (x \rightarrow y) \wedge (y \rightarrow x)$.

An equality-algebra $\mathcal{A} = (A, \wedge, \sim, 1)$ is bounded if there exists an element $0 \in A$ such that $0 \leq x$, for all $x \in A$.

Proposition 2.2 ([13]). Let $\mathcal{A} = (A, \wedge, \sim, 1)$ be an equality-algebra and consider

(E5a) $x \sim (x \wedge y \wedge z) \leq x \sim (x \wedge y)$,

(E5a') $x \rightarrow (y \wedge z) \leq x \rightarrow y$,

Then (E5) is equivalent to (E5a), which in turn is equivalent to (E5a').

Definition 2.3 ([23]). Let $\mathcal{A} = (A, \wedge, \sim, 1)$ be an equality-algebra.

(1) Then $\mathcal{A}$ is called prelinear, if 1 is the unique upper bound of the set $\{x \rightarrow y, y \rightarrow x\}$ for all $x, y \in A$.

(2) A lattice equality-algebra is an equality-algebra which is a lattice.

Theorem 2.4 ([8]). *Any prelinear equality -algebra is a distributive lattice.*

Proposition 2.5 ([13, 23]). Let $\mathcal{A} = (A, \wedge, \sim, 1)$ be an equality-algebra. Then the following hold for all $x, y, z \in A$:

(1) $x \sim y \leq x \leftrightarrow y \leq x \rightarrow y$,

(2) $x \leq (x \sim y) \sim y$,

(3) $x \sim y = 1$ iff $x = y$,

(4) $x \rightarrow y = 1$ iff $x \leq y$,

(5) $x \rightarrow y = 1$ and $y \rightarrow x = 1$ implies $x = y$,

(6) $1 \rightarrow x = x$, $x \rightarrow 1 = 1$and $x \rightarrow x = 1$,

(7) $x \leq y \rightarrow x$,

(8) $x \leq (x \rightarrow y) \rightarrow y$,

(9) $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$,

(10) $x \leq y \rightarrow z$ iff $y \leq x \rightarrow z$,

(11) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,

(12) $x \leftrightarrow x = 1$, $1 \leftrightarrow x = x$,

(13) $x \leq y$ implies $x \leftrightarrow y = y \rightarrow x = y \sim x$,

(14) $x \leq y$ implies $x \leq x \sim y$,

(15) $x \leq y$ implies that $z \rightarrow x \leq z \rightarrow y$ and $y \rightarrow z \leq x \rightarrow z$,

(16) if $\mathcal{A}$ is a lattice equality-algebra, then $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$,

(17) if $\mathcal{A}$ is a prelinear equality-algebra, then $x \sim y = (x \rightarrow y) \wedge (y \rightarrow x)$.

Definition 2.6 ([13]). Let $\mathcal{A} = (A, \wedge, \sim, 1)$ be an equality-algebra and F be a subset of A. Then F is called a deductive system of $\mathcal{A}$ if for all $x, y \in A$,

(i) $1 \in F$,

(ii) if $x \in F$ and $x \leq y$, then $y \in F$,

(iii) if $x \in F$ and $x \sim y \in F$, then $y \in F$.

PROPOSITION 2.7 ([13]). Let $\mathcal{A} = (A, \wedge, \sim, 1)$ be an equality-algebra and F be a subset of A. Then F is a deductive system of $\mathcal{A}$ if and only if

(i) $1 \in F$,

(ii) if $x \in F$ and $x \to y \in F$, then $y \in F$.

A deductive system F of an equality-algebra $\mathcal{A} = (A, \wedge, \sim, 1)$ is called a proper deductive system if $F \neq A$. If $\mathcal{A}$ is a bounded equality-algebra, then a deductive system is proper if and only if it does not contain 0 (see [3]).

DEFINITION 2.8 ([13]). An equivalence relation θ on an equality-algebra $\mathcal{A} = (A, \wedge, \sim, 1)$ is called congruence, if $(x, z), (y, w) \in \theta$, then $(x \sim y, z \sim w), (x \wedge y, z \wedge w) \in \theta$.

PROPOSITION 2.9 ([6, 13]). Let F be a deductive system of an equality-algebra $\mathcal{A} = (A, \wedge, \sim, 1)$. Define the relation $\theta_{\overrightarrow{F}}$ and θ_F as follows:

$$(x, y) \in \theta_{\overrightarrow{F}} \text{ iff } \{x \to y, y \to x\} \subseteq F$$

and

$$(x, y) \in \theta_F \text{ iff } x \sim y \in F,$$

then $\theta_{\overrightarrow{F}}$ and θ_F are congruence and $\theta_{\overrightarrow{F}} = \theta_F$.

PROPOSITION 2.10 ([6, 13]). Let F be a deductive system of an equality-algebra $\mathcal{A} = (A, \wedge, \sim, 1)$ and $A/\theta_F = \{[x] : x \in A\}$, where $[x] = \{y \in A : (x, y) \in \theta_F\}$. Then $\mathcal{A}/\theta_{\mathcal{F}} = (A/\theta_F, \wedge, \sim, 1)$ is an equality-algebra, where for every $x, y \in A$, $1 := [1]$, $[x] \wedge [y] := [x \wedge y]$ and $[x] \sim [y] := [x \sim y]$.

DEFINITION 2.11 ([3]). A proper deductive system F of an equality-algebra $\mathcal{A} = (A, \wedge, \sim, 1)$ is called a prime deductive system if $x \to y \in F$ or $y \to x \in F$ for all $x, y \in A$.

THEOREM 2.12 ([3]). *Let F be a proper deductive system of prelinear equality-algebra $\mathcal{A} = (A, \wedge, \sim, 1)$. Then the following statements are equivalent:*

(i) *F is a prime deductive system,*

(ii) *for each $x, y \in A$, if $x \vee y \in F$, then $x \in F$ or $y \in F$,*

(iii) *$\mathcal{A}/\theta_F$ is a chain.*

Definition 2.13 ([21]). Let $\mathcal{A} = (A, \wedge, \sim, 1)$ be an equality-algebra. The mapping $\tau : A \to A$ is called a very true operator, if it satisfies the following conditions:

($V1$) $\tau(1) = 1$,

($V2$) $\tau(x) \leq x$,

($V3$) $\tau(x) \leq \tau(\tau(x))$,

($V4$) $\tau(x \sim y) \leq \tau(x) \sim \tau(y)$,

($V5$) $\tau(x \wedge y) = \tau(x) \wedge \tau(y)$.

3. Equality logic

In this section, we introduce and study a propositional equality logic and we obtain some its properties.

Definition 3.1.

(i) The language of propositional equality logic $\mathcal{EL}$ has propositional variables p, q, r,... binary connectives $\sqcap$, $\equiv$ and a truth (logical) constant $\top$.

(ii) Formulas of $\mathcal{EL}$ are defined in the following way: each propositional variable is a formula, $\top$ is a formula and if φ, ψ are formulas, then $\varphi \sqcap \psi$ (conjunction) and $\varphi \equiv \psi$ are formulas. Implication and equivalence connectives are defined as the following short:

$$\varphi \Rightarrow \psi := (\varphi \sqcap \psi) \equiv \varphi, \quad \varphi \Leftrightarrow \psi := (\varphi \Rightarrow \psi) \sqcap (\psi \Rightarrow \varphi).$$

The set of all formulas of $\mathcal{EL}$ is denoted by $\mathcal{F}$.

(iii) The following formulas are axioms of $\mathcal{EL}$:

(EL1) $\varphi \sqcap \varphi \equiv \varphi$,
(EL2) $\varphi \sqcap \psi \equiv \psi \sqcap \varphi$,
(EL3) $(\varphi \sqcap \psi) \sqcap \chi \equiv \varphi \sqcap (\psi \sqcap \chi)$,
(EL4) $\varphi \sqcap \top \equiv \varphi$,
(EL5) $(\varphi \equiv \top) \equiv \varphi$,
(EL6) $(\varphi \equiv \psi) \equiv (\psi \equiv \varphi)$,

(EL7) $((\varphi \sqcap \psi \sqcap \chi) \equiv \varphi) \Rightarrow ((\varphi \sqcap \psi) \equiv \varphi)$,
(EL8) $(\varphi \equiv \psi) \Rightarrow (\varphi \sqcap \chi \equiv \psi \sqcap \chi)$,
(EL9) $(\varphi \equiv \psi) \Rightarrow ((\varphi \equiv \chi) \equiv (\psi \equiv \chi))$.

(iv) The inference rules are :

(EA) from ψ and $\varphi \equiv \psi$, we infer φ,
(MP) from φ and $\varphi \Rightarrow \psi$, we infer ψ.

The rule (EA) is the equanimity rule (cf. [18]) and (MP) is the modus ponens rule for formulas.

Definition 3.2. Let $\mathcal{A} = (A, \wedge, \sim, 1)$ be an equality-algebra. An $\mathcal{A}$-evaluation of formulas is a mapping $e : \mathcal{F} \to A$, defined as follows:

(1) $e(\top) = 1$,

(2) $e(\varphi \sqcap \psi) = e(\varphi) \wedge e(\psi)$,

(3) $e(\varphi \equiv \psi) = e(\varphi) \sim e(\psi)$,

for all formulas $\varphi, \psi \in \mathcal{F}$. A formula φ is a $\mathcal{A}$-tautology if $e(\varphi) = 1$ for each $\mathcal{A}$- evaluation $e : \mathcal{F} \to A$.

Lemma 3.3. *All axioms of $\mathcal{EL}$ are $\mathcal{A}$-tautologies for all equality-algebra $\mathcal{A} = (A, \wedge, \sim, 1)$.*

Proof: Suppose that $\mathcal{A} = (A, \wedge, \sim, 1)$ is an arbitrary equality-algebra and $e : \mathcal{F} \to A$ is an arbitrary $\mathcal{A}$-evaluation.

(EL1) By (E1) and (E2), we have $e(\varphi \sqcap \varphi \equiv \varphi) = e(\varphi \sqcap \varphi) \sim e(\varphi) = (e(\varphi) \wedge e(\varphi)) \sim e(\varphi) = e(\varphi) \sim e(\varphi) = 1$.

(EL2) Using (E1) and (E3), $e(\varphi \sqcap \top \equiv \varphi) = e(\varphi \sqcap \top) \sim e(\varphi) = (e(\varphi) \wedge e(\top)) \sim e(\varphi) = (e(\varphi) \wedge 1) \sim e(\varphi) = e(\varphi) \sim e(\varphi) = 1$ by (A2).

Similarly, we can prove the (EL3)–(EL9). □

Lemma 3.4. *The inference rules of propositional equality logic $\mathcal{EL}$ sound in the following sense: Let $e : \mathcal{F} \to A$ be an $\mathcal{A}$-evaluation where $\mathcal{A}$ is an equality-algebra:*

(1) *if φ and $\varphi \equiv \psi$ are $\mathcal{A}$-tautology, then ψ is also $\mathcal{A}$-tautology,*

(2) *if φ and $\varphi \Rightarrow \psi$ are $\mathcal{A}$-tautology, then ψ is also $\mathcal{A}$-tautology.*

Proof:

(1) Suppose that $e(\varphi) = 1$ and $e(\varphi \equiv \psi) = 1$. Then $1 = e(\psi) \sim 1 = e(\psi)$ by (E4).

Similarly, we can prove (2). □

Theorem 3.5 (Soundness). *The propositional equality logic $\mathcal{EL}$ is sound.*

Proof: It follows from Lemma 3.3 and Lemma 3.4. □

A proof in propositional equality logic $\mathcal{EL}$ is a sequence $\varphi_1, \ldots, \varphi_n$ of formulas such that each φ_i either is an axiom of $\mathcal{EL}$ or follows from some preceding $\varphi_j, \varphi_k (j, k < i)$ by inference rules. A formula is provable (notation $\vdash \varphi$) if it is the least member a proof in propositional equality logic $\mathcal{EL}$. By Theorem 3.5, each provable formula in propositional equality logic $\mathcal{EL}$ is A-tautology for all equality-algebra $\mathcal{A}$.

A theory over propositional equality logic $\mathcal{EL}$ is any subset $\Gamma \subseteq \mathcal{F}$. A proof in a theory Γ is a sequence $\varphi_1, \ldots, \varphi_n$ of formulas whose each member is either an axiom $\mathcal{EL}$ or a member of Γ (spacial axiom) or follows from some preceding members of the sequence using the inference rules. $\Gamma \vdash \varphi$ means that φ is provable in Γ, that is the last member of a proof in Γ. An $\mathcal{A}$-evaluation e is a model of Γ, if $e(\varphi) = 1$ for each $\varphi \in \Gamma$. If $\Gamma = \{\varphi\}$, then we write $\varphi \vdash \psi$ instead of $\{\varphi\} \vdash \psi$.

In the following, we will verify provability of several formulas in propositional equality logic $\mathcal{EL}$.

Proposition 3.6. Let $\varphi, \psi, \chi \in \mathcal{F}$ be formulas. $\mathcal{EL}$ proves the following properties of equality:

(1) $\vdash \varphi \equiv \varphi$,

(2) $\varphi \vdash \varphi \equiv \top$,

(3) $\varphi \equiv \top \vdash \varphi$,

(4) $\varphi \equiv \psi \vdash \psi \equiv \varphi$,

(5) $\varphi \sqcap \psi \equiv \chi \vdash \psi \sqcap \varphi \equiv \chi$,

(6) $\varphi \equiv \psi \vdash (\varphi \equiv \chi) \equiv (\psi \equiv \chi)$,

(7) $\{\varphi \equiv \psi, \psi \equiv \chi\} \vdash \varphi \equiv \chi$,

(8) $\varphi \equiv \psi \vdash (\chi \equiv \varphi) \equiv (\chi \equiv \psi)$,

(9) $\varphi \equiv \psi \vdash (\chi \sqcap \varphi) \equiv (\chi \sqcap \psi)$,

(10) $\{\psi, \psi \equiv \varphi\} \vdash \varphi$.

PROOF:

(1) We have $\vdash (\varphi \sqcap \varphi \equiv \varphi) \Rightarrow ((\varphi \sqcap \varphi \equiv \varphi) \equiv (\varphi \equiv \varphi))$ by (EL9). Applying (EL1) and (MP), we get $\vdash (\varphi \sqcap \varphi \equiv \varphi) \equiv (\varphi \equiv \varphi)$. Using (EL1) and (EA), we obtain $\vdash \varphi \equiv \varphi$.

(2) By (EL6), we have $\vdash ((\varphi \equiv \top) \equiv \varphi) \equiv (\varphi \equiv (\varphi \equiv \top))$. Applying (EL5) and then (EA), we obtain $\vdash \varphi \equiv (\varphi \equiv \top)$. Using assumption and (EA), we get the result.

(3) It follows from (EL5), assumption and (EA).

(4) By assumption, (EL6) and (EA), we obtain the result.

(5) Using (EL9), we have $\vdash (\varphi \sqcap \psi \equiv \psi \sqcap \varphi) \Rightarrow ((\varphi \sqcap \psi \equiv \chi) \equiv (\psi \sqcap \varphi \equiv \chi))$. Applying (EL2) and (MP), we get $\vdash (\varphi \sqcap \psi \equiv \chi) \equiv (\psi \sqcap \varphi \equiv \chi)$. Using assumption and (EA), we have $\varphi \sqcap \psi \equiv \chi \vdash \psi \sqcap \varphi \equiv \chi$.

(6) It is immediate consequence of (EL9).

(7) We have $\varphi \equiv \psi \vdash (\varphi \equiv \chi) \equiv (\psi \equiv \chi)$ by assumption and part (6). Using part (4), we get $\varphi \equiv \psi \vdash (\psi \equiv \chi) \equiv (\varphi \equiv \chi)$. By assumption and (EA), we get the result.

(8) By assumption, part (4) and then part (6), we have $\varphi \equiv \psi \vdash (\psi \equiv \chi) \equiv (\varphi \equiv \chi)$. Since we have $\vdash ((\psi \equiv \chi) \equiv (\varphi \equiv \chi)) \Rightarrow (((\psi \equiv \chi) \equiv (\chi \equiv \psi)) \equiv ((\varphi \equiv \chi) \equiv (\chi \equiv \psi)))$ by (EL9), then $\vdash ((\psi \equiv \chi) \equiv (\chi \equiv \psi)) \equiv ((\varphi \equiv \chi) \equiv (\chi \equiv \psi))$ by (MP). Using (EL6), we have $\vdash (\psi \equiv \chi) \equiv (\chi \equiv \psi)$. So by (EA), we get $\vdash (\varphi \equiv \chi) \equiv (\chi \equiv \psi)$. Applying (EL9) $\vdash ((\varphi \equiv \chi) \equiv (\chi \equiv \varphi)) \Rightarrow (((\varphi \equiv \chi) \equiv (\chi \equiv \psi)) \equiv ((\chi \equiv \varphi) \equiv (\chi \equiv \psi)))$. By (EL6) $\vdash (\varphi \equiv \chi) \equiv (\chi \equiv \varphi)$ and (MP), we get $\vdash ((\varphi \equiv \chi) \equiv (\chi \equiv \psi)) \equiv ((\chi \equiv \varphi) \equiv (\chi \equiv \psi))$. Hence by (EA), we obtain $\varphi \equiv \psi \vdash (\chi \equiv \varphi) \equiv (\chi \equiv \psi)$.

(9) It follows from (EL8), (EL2) and part (8).

(10) Using assumptions, part (4) and (EA), we get result. $\square$

Proposition 3.7. Let $\varphi, \varphi_1, \psi, \psi_1, \chi, \chi_1 \in \mathcal{F}$ be formulas. $\mathcal{EL}$ proves the following:

(1) $\{\varphi \sqcap \psi \equiv \chi, \varphi \equiv \varphi_1\} \vdash \varphi_1 \sqcap \psi \equiv \chi$,

(2) $\{(\varphi \equiv \psi) \equiv \chi, \varphi \equiv \varphi_1\} \vdash (\varphi_1 \equiv \psi) \equiv \chi$,

(3) $\{\varphi \Rightarrow (\psi \equiv \chi), \psi \equiv \psi_1\} \vdash \varphi \Rightarrow (\psi_1 \equiv \chi)$,

(4) $\{\varphi \Rightarrow (\psi \equiv \chi), \chi \equiv \chi_1\} \vdash \varphi \Rightarrow (\psi \equiv \chi_1)$,

(5) $\{\varphi \Rightarrow \psi, \varphi \equiv \varphi_1\} \vdash \varphi_1 \Rightarrow \psi$,

(6) $\{\varphi \Rightarrow (\psi \equiv \chi), \psi\} \vdash \varphi \Rightarrow \chi$,

(7) $\{\varphi, \psi\} \vdash \varphi \sqcap \psi$,

(8) $\varphi \equiv \psi \vdash \varphi \Rightarrow \psi$,

(9) $\{(\varphi \equiv \psi) \Rightarrow \chi, \varphi \equiv \varphi_1\} \vdash (\varphi_1 \equiv \psi) \Rightarrow \chi$.

Proof:

(1) Suppose that $\Gamma = \{\varphi \sqcap \psi, \varphi \equiv \varphi_1\}$. By assumption $\Gamma \vdash \varphi \equiv \varphi_1$, (EL8) and (MP), we obtain $\Gamma \vdash (\varphi \sqcap \psi) \equiv (\varphi_1 \sqcap \psi)$. Using (EL9), $\vdash ((\varphi \sqcap \psi) \equiv (\varphi_1 \sqcap \psi)) \Rightarrow ((\varphi \sqcap \psi \equiv \chi) \equiv (\varphi_1 \sqcap \psi \equiv \chi))$ and (MP), we have $\Gamma \vdash (\varphi \sqcap \psi \equiv \chi) \equiv (\varphi_1 \sqcap \psi \equiv \chi)$. Applying assumption $\Gamma \vdash \varphi \sqcap \psi \equiv \chi$ and (EA), we get the result.

(2) Let $\Gamma = \{(\varphi \equiv \psi) \equiv \chi, \varphi \equiv \varphi_1\}$. Using assumption $\Gamma \vdash \varphi \equiv \varphi_1$, (EL9) and (MP), we have $\Gamma \vdash (\varphi \equiv \psi) \equiv (\varphi_1 \equiv \psi)$. Since $\vdash ((\varphi \equiv \psi) \equiv (\varphi_1 \equiv \psi)) \equiv ((\varphi \equiv \psi) \equiv \chi) \equiv ((\varphi_1 \equiv \psi) \equiv \chi))$ by (EL9), then $\Gamma \vdash (\varphi \equiv \psi) \equiv \chi) \equiv ((\varphi_1 \equiv \psi) \equiv \chi)$. Applying assumption $\Gamma \vdash (\varphi \equiv \psi) \equiv \chi$ and (EA), we have $\Gamma \vdash ((\varphi_1 \equiv \psi) \equiv \chi)$.

(3) Suppose that $\Gamma = \{\varphi \Rightarrow (\psi \equiv \chi), \psi \equiv \psi_1\}$. Since $\Gamma \vdash \psi \equiv \psi_1$, then $\Gamma \vdash (\psi \equiv \chi) \equiv (\psi_1 \equiv \chi)$ by Proposition 3.6 part (6). We have $\Gamma \vdash (\varphi \sqcap (\psi \equiv \chi)) \equiv \varphi$ by assumption. Hence $\Gamma \vdash (\varphi \sqcap (\psi_1 \equiv \chi)) \equiv \varphi$ by part (1), that is $\Gamma \vdash \varphi \Rightarrow (\psi_1 \equiv \chi)$.

(4) Let $\Gamma = \{\varphi \Rightarrow (\psi \equiv \chi), \chi \equiv \chi_1\}$. By assumption $\Gamma \vdash \chi \equiv \chi_1$ and Proposition 3.6 part (8) we get $\Gamma \vdash (\psi \equiv \chi) \equiv (\psi \equiv \chi_1)$.

Applying part (1) and assumption $\Gamma \vdash (\varphi \sqcap (\psi \equiv \chi)) \equiv \varphi$, we obtain $\Gamma \vdash (\varphi \sqcap (\psi \equiv \chi_1)) \equiv \varphi$. Hence $\Gamma \vdash \varphi \Rightarrow (\psi \equiv \chi_1)$.

(5) It follows from assumptions and part (1).

(6) Suppose that $\Gamma = \{\varphi \Rightarrow (\psi \equiv \chi), \psi\}$. By assumption $\Gamma \vdash \psi$ and Proposition 3.6 part (2), we obtain $\Gamma \vdash \psi \equiv \top$. Using Proposition 3.6 part (6), we get $\Gamma \vdash (\psi \equiv \chi) \equiv (\top \equiv \chi)$. By Proposition 3.6 part (9) and (MP), we have $\Gamma \vdash \varphi \sqcap (\psi \equiv \chi) \equiv \varphi \sqcap (\top \equiv \chi)$. Applying (EL5) and part (1), we obtain $\Gamma \vdash \varphi \sqcap (\psi \equiv \chi) \equiv \varphi \sqcap \chi$. Thus $\vdash \varphi \sqcap \chi \equiv \varphi \sqcap (\psi \equiv \chi)$ by Proposition 3.6 part (4). Using assumption $\Gamma \vdash \varphi \sqcap (\psi \equiv \chi) \equiv \varphi$ and Proposition 3.6 part (7), we have $\Gamma \vdash \varphi \sqcap \chi \equiv \varphi$, that is $\Gamma \vdash \varphi \Rightarrow \chi$.

(7) Let $\Gamma = \{\varphi, \psi\}$. By assumption $\Gamma \vdash \varphi$, Proposition 3.6 part (2), (EL8) and (EA), we have $\Gamma \vdash (\varphi \sqcap \psi) \equiv (\top \sqcap \psi)$. By assumption $\Gamma \vdash \psi$, (EL4) and (EA), we get $\Gamma \vdash \psi \sqcap \top$. Thus $\Gamma \vdash \top \sqcap \psi$ by Proposition 3.6 part (4). By Proposition 3.6 part (10), we get result.

(8) By assumption and Proposition 3.6 part (9), we have $\varphi \equiv \psi \vdash (\varphi \sqcap \varphi) \equiv (\varphi \sqcap \psi)$. Since $\Gamma \vdash ((\varphi \sqcap \varphi) \equiv (\varphi \sqcap \psi)) \Rightarrow (((\varphi \sqcap \varphi) \equiv \varphi) \equiv ((\varphi \sqcap \psi) \equiv \varphi))$ by (EL9), then $\Gamma \vdash ((\varphi \sqcap \varphi) \equiv \varphi) \equiv ((\varphi \sqcap \psi) \equiv \varphi)$. Applying (EL1) and (EA), we have $\Gamma \vdash (\varphi \sqcap \psi) \equiv \varphi$, that is (6) $\varphi \equiv \psi \vdash \varphi \Rightarrow \psi$.

(9) Suppose that $\Gamma = \{(\varphi \equiv \psi) \Rightarrow \chi, \varphi \equiv \varphi_1\}$. By assumption $\Gamma \vdash \varphi \equiv \varphi_1$ and Proposition 3.6 part (6), we have $\vdash (\varphi \equiv \psi) \equiv (\varphi_1 \equiv \psi)$. Applying (EL8), we obtain $\Gamma \vdash ((\varphi \equiv \psi) \sqcap \chi) \equiv ((\varphi_1 \equiv \psi) \sqcap \chi)$. Thus $\Gamma \vdash ((\varphi_1 \equiv \psi) \sqcap \chi) \equiv ((\varphi \equiv \psi) \sqcap \chi)$ by Proposition 3.6 part (4). Also, using assumption $\Gamma \vdash ((\varphi \equiv \psi) \sqcap \chi) \equiv (\varphi \equiv \psi)$ and twice Proposition 3.6 part (7), we get $\Gamma \vdash ((\varphi_1 \equiv \psi) \sqcap \chi) \equiv (\varphi_1 \equiv \psi)$, that is $\Gamma \vdash (\varphi_1 \equiv \psi) \Rightarrow \chi$. □

Proposition 3.8. Let $\varphi, \psi, \chi \in \mathcal{F}$ be formulas. $\mathcal{EL}$ proves the following properties of implication:

(1) $\vdash \varphi \Rightarrow \varphi$,

(2) $\vdash (\top \Rightarrow \varphi) \equiv \varphi$,

(3) $\vdash (\varphi \Rightarrow \psi) \Rightarrow ((\varphi \sqcap \chi) \Rightarrow \psi)$,

(4) $\vdash \varphi \Rightarrow (\psi \Rightarrow \varphi)$,

(5) $\vdash \varphi \Rightarrow (\varphi \equiv \top)$,

(6) $\vdash (\varphi \sqcap \psi) \Rightarrow \varphi$, $\vdash (\varphi \sqcap \psi) \Rightarrow \psi$,

(7) $\{\varphi \Rightarrow \psi, \psi \Rightarrow \chi\} \vdash \varphi \Rightarrow \chi$,

(8) $\vdash (\varphi \Rightarrow \psi) \Rightarrow ((\varphi \sqcap \chi) \Rightarrow \psi)$.

PROOF:

(1) It follows from (EL1).

(2) The proof is straightforward by (EL5), (EL4) and Proposition 3.7 part (1).

(3) We have $\vdash ((\varphi \sqcap \psi) \equiv \varphi) \Rightarrow (((\varphi \sqcap \psi) \sqcap \chi) \equiv (\varphi \sqcap \chi))$ by (EL8). Using (EL2), (EL3) and Proposition 3.7 part (2), we obtain $\vdash ((\varphi \sqcap \psi) \equiv \varphi) \Rightarrow (((\varphi \sqcap \chi) \sqcap \psi) \equiv (\varphi \sqcap \chi))$. Hence $\vdash (\varphi \Rightarrow \psi) \Rightarrow ((\varphi \sqcap \chi) \Rightarrow \psi)$ by definition $\Rightarrow$.

(4) We have $\vdash (\top \Rightarrow \varphi) \Rightarrow ((\top \sqcap \psi) \Rightarrow \varphi)$ by part (3). Since $\vdash (\top \Rightarrow \varphi) \equiv \varphi$ by part (2), then $\vdash \varphi \Rightarrow ((\top \sqcap \psi) \Rightarrow \varphi)$ by Proposition 3.7 part (5). By definition implication, $\vdash \varphi \Rightarrow (((\top \sqcap \psi) \sqcap \varphi) \equiv (\top \sqcap \psi))$. Using (EL4) and Proposition 3.7 part (4), we obtain $\vdash \varphi \Rightarrow (((\top \sqcap \psi) \sqcap \varphi) \equiv \psi)$. Applying (EL4), (EL8) and (MP), we have $\vdash ((\top \sqcap \psi) \sqcap \varphi) \equiv (\psi \sqcap \varphi)$. Hence $\vdash \varphi \Rightarrow (((\psi \sqcap \varphi) \equiv \psi)$ by Proposition 3.7 part (3), that is $\vdash \varphi \Rightarrow (\psi \Rightarrow \varphi)$.

(5) Applying (EL5), (EL8), we have $\vdash (\varphi \sqcap (\varphi \equiv \top)) \equiv (\varphi \sqcap \varphi)$. By (EL1) and Proposition 3.6 part (7), we get $\vdash (\varphi \sqcap (\varphi \equiv \top)) \equiv \varphi$. Hence $\vdash \varphi \Rightarrow (\varphi \equiv \top)$.

(6) By (EL9), We have $\vdash (((\varphi \sqcap \varphi) \sqcap \psi) \equiv (\varphi \sqcap (\varphi \sqcap \psi))) \Rightarrow (((\varphi \sqcap \varphi) \sqcap \psi \equiv (\varphi \sqcap \psi)) \equiv (\varphi \sqcap (\varphi \sqcap \psi) \equiv (\varphi \sqcap \psi)))$. Using (EL3) and (MP), we get $\vdash ((\varphi \sqcap \varphi) \sqcap \psi \equiv (\varphi \sqcap \psi)) \equiv (\varphi \sqcap (\varphi \sqcap \psi) \equiv (\varphi \sqcap \psi))$. We have $\vdash (\varphi \sqcap \varphi) \sqcap \psi \equiv (\varphi \sqcap \psi)$ by (EL1), (EL8) and (MP). Thus $\vdash \varphi \sqcap (\varphi \sqcap \psi) \equiv (\varphi \sqcap \psi)$ by (EA). Hence $\vdash (\varphi \sqcap \psi) \Rightarrow \varphi$ by definition of implication.

(7) By assumptions, definition of implication and Proposition 3.6 part (4), we have $\vdash (\psi \sqcap \varphi) \equiv \varphi$ and $\vdash \psi \equiv (\psi \sqcap \chi)$. Using Proposition 3.7 part (1), we obtain $\vdash ((\psi \sqcap \chi) \sqcap \varphi) \equiv \varphi$. By Proposition 3.6 part (5), $\vdash \varphi \sqcap (\psi \sqcap \chi) \equiv \varphi$. Hence we get the result by (EL7) and (MP).

(8) It follows from (EL8) and definition $\Rightarrow$. □

In the following, we will use the standard Lindenbaum Tarski technique to show that propositional equality logic $\mathcal{EL}$.

LEMMA 3.9. *Let $\varphi_1, \varphi_2, \psi_1, \psi_2 \in \mathcal{F}$ be formulas. $\mathcal{EL}$ proves the following properties:*

(1) $\{\varphi_1 \equiv \psi_1, \varphi_2 \equiv \psi_2\} \vdash (\varphi_1 \sqcap \varphi_2) \equiv (\psi_1 \sqcap \psi_2)$,

(2) $\{\varphi_1 \equiv \psi_1, \varphi_2 \equiv \psi_2\} \vdash (\varphi_1 \equiv \varphi_2) \equiv (\psi_1 \equiv \psi_2)$.

PROOF: Suppose that $\Gamma = \{\varphi_1 \equiv \psi_1, \varphi_2 \equiv \psi_2\}$.

(1) By assumption $\Gamma \vdash \varphi_1 \equiv \psi_1$, (EL8) and (MP), we have $\Gamma \vdash (\varphi_1 \sqcap \varphi_2) \equiv (\psi_1 \sqcap \varphi_2)$. By Proposition 3.6 part (9) we obtan $\Gamma \vdash (\psi_1 \sqcap \varphi_2) \equiv (\psi_1 \sqcap \psi_2)$. By Proposition 3.6 part (7), we get $\Gamma \vdash (\varphi_1 \sqcap \varphi_2) \equiv (\psi_1 \sqcap \psi_2)$.

(2) Using assumption $\Gamma \vdash \varphi_1 \equiv \psi_1$, (EL9) and (MP), we have $\Gamma \vdash (\varphi_1 \equiv \varphi_2) \equiv (\psi_1 \equiv \varphi_2)$. Applying assumption $\Gamma \vdash \varphi_2 \equiv \psi_2$, Proposition 3.6 part (8) and (MP), we obtain $\Gamma \vdash (\psi_1 \equiv \varphi_2) \equiv (\psi_1 \equiv \psi_2)$. Therefore $\Gamma \vdash (\varphi_1 \equiv \varphi_2) \equiv (\psi_1 \equiv \psi_2)$ by Proposition 3.6 part (7). □

PROPOSITION 3.10. Let Γ be a theory over the propositional equality logic $\mathcal{EL}$. Put $\varphi \approx_\Gamma \psi$ iff $\Gamma \vdash \varphi \equiv \psi$. Then $\approx_\Gamma$ is an equivalence relation on $\mathcal{F}$.

PROOF: It follows from Proposition 3.6 part (1), part (4) and part (7) that $\approx_\Gamma$ is an equivalence on relation on $\mathcal{F}$. □

Let Γ be a theory over the propositional equality logic $\mathcal{EL}$. Denote $M_\Gamma = \{[\varphi]_\Gamma : \varphi \in \mathcal{F}\}$ where $[\varphi]_\Gamma = \{\psi \in \mathcal{F} | \varphi \approx_\Gamma \psi\}$. Finally, we define

$[\varphi]_\Gamma \wedge [\psi]_\Gamma = [\varphi \sqcap \psi]_\Gamma$,

$[\varphi]_\Gamma \sim [\psi]_\Gamma = [\varphi \equiv \psi]_\Gamma$,

$1 = [\top]_\Gamma$.

Proposition 3.11. The algebra $\mathcal{M}_\Gamma = (M_\Gamma, \wedge, \sim, 1)$ is an equality-algebra.

Proof: By Lemma 3.9, we know that the operations $\wedge$ and $\sim$ are well defined. By (EL1)–(EL4), we can see that $(M_\Gamma, \wedge, \sim, 1)$ i a meet-semilattice with top element 1. Now, we will show that $[\varphi]_\Gamma \leq [\psi]_\Gamma$ iff $T \vdash \varphi \Rightarrow \psi$. Suppose that $\Gamma \vdash \varphi \Rightarrow \psi$. Then $\Gamma \vdash (\varphi \sqcap \psi) \equiv \varphi$. So $[\varphi]_\Gamma \wedge [\psi]_\Gamma = [\varphi]_\Gamma$. Hence $[\varphi]_\Gamma \leq [\psi]_\Gamma$. Similarly, we can prove if $[\varphi]_\Gamma \leq [\psi]_\Gamma$, then $\Gamma \vdash \varphi \Rightarrow \psi$. The proof of (EL5)–(EL9) is straightforward. □

Theorem 3.12 (Completeness). *The propositional equality logic $\mathcal{EL}$ is complete, i.e. the following are equivalent:*

(i) $\vdash \varphi$,

(ii) *for every equality-algebra $\mathcal{A} = (A, \wedge, \sim, 1)$, φ is an $\mathcal{A}$-tautology.*

Proof: (i)$\Rightarrow$ (ii) follows from Theorem 3.5. Conversely, for every theory Γ of the propositional equality logic $\mathcal{EL}$, $\mathcal{M}_\Gamma = (M_T, \wedge, \sim, 1)$ is an equality-algebra. Let Γ to be the set of all axioms of $\mathcal{EL}$. Thus φ is an $\mathcal{M}_\Gamma$-tautology by assumption. Consider the mapping e defined by $e(p) = [p]_\Gamma$ for all propositional variables p. Then e is a $\mathcal{M}$-evaluation from $\mathcal{F}$ to the equality-algebra $\mathcal{M}$. By Definition 3.2, $e(\varphi) = [1]_\Gamma$. Then $[\varphi]_\Gamma = [1]_\Gamma$, that is $\Gamma \vdash \varphi \equiv \top$. Hence $\vdash \varphi$. □

Now, we will show the locally deduction theorem for the propositional equality logic $\mathcal{EL}$. For this, we need the following proposition.

For convenience, we shall abbreviate the formulas $\varphi \equiv (\cdots \equiv (\varphi \equiv \psi)\cdots)$ and $\varphi \Rightarrow (\cdots \Rightarrow (\varphi \Rightarrow \psi)\cdots)$ by $\varphi \equiv^n \psi$ and $\varphi \Rightarrow^n \psi$, $n \in \mathbb{N}_0$ indicating the number of occurrences of φ.

Proposition 3.13. Let $\varphi, \psi, \chi \in \mathcal{F}$ be formulas. $\mathcal{EL}$ proves the following:

(1) $\varphi \Rightarrow \psi \vdash (\chi \Rightarrow \varphi) \Rightarrow (\chi \Rightarrow \psi)$,

(2) $\varphi \Rightarrow \psi \vdash (\psi \Rightarrow \chi) \Rightarrow (\varphi \Rightarrow \chi)$,

(3) $\vdash \varphi \Rightarrow [(\psi \equiv \varphi) \equiv \psi]$,

(4) $\vdash (\varphi \equiv \psi) \Rightarrow (\varphi \Rightarrow \psi)$,

(5) $\vdash \varphi \Rightarrow ((\varphi \Rightarrow \psi) \Rightarrow \psi)$,

(6) $\vdash (\varphi \Rightarrow \psi) \Rightarrow [(\psi \Rightarrow \chi) \Rightarrow (\varphi \Rightarrow \chi)]$,

(7) $\vdash (\chi \Rightarrow (\varphi \Rightarrow \psi)) \Rightarrow (\varphi \Rightarrow (\chi \Rightarrow \psi))$,

(8) $\vdash (\varphi \Rightarrow^n (\psi \Rightarrow \chi)) \Rightarrow (\psi \Rightarrow (\varphi \Rightarrow^n \chi))$,

(9) $\vdash (\varphi \Rightarrow \psi) \Rightarrow ((\psi \Rightarrow \varphi) \Rightarrow (\varphi \equiv \psi))$,

(10) $\{\varphi \Rightarrow \chi, \varphi \Rightarrow \psi\} \vdash \varphi \Rightarrow (\psi \sqcap \chi)$,

(11) $\vdash (\varphi \equiv \psi) \Rightarrow ((\varphi \Rightarrow \psi) \sqcap (\psi \Rightarrow \varphi))$.

PROOF:

(1) Let $\Gamma = \{\varphi \Rightarrow \psi\}$. Applying (EL7) and definition $\Rightarrow$, we obtain $\vdash ((\varphi \sqcap \psi \sqcap \chi) \equiv \chi) \Rightarrow (\chi \Rightarrow \psi)$. By assumption, (EL8) and (MP), we get $\Gamma \vdash (\varphi \sqcap \psi \sqcap \chi) \equiv \varphi \sqcap \chi$. By Proposition 3.7 part (9), we have $\Gamma \vdash ((\varphi \sqcap \chi) \equiv \chi) \Rightarrow (\chi \Rightarrow \psi)$, that is $\Gamma \vdash (\chi \Rightarrow \varphi) \Rightarrow (\chi \Rightarrow \psi)$.

(2) Let $\Gamma = \{\varphi \Rightarrow \psi\}$. We have $\vdash ((\psi \sqcap \chi) \equiv \psi) \Rightarrow ((\varphi \sqcap (\psi \sqcap \chi)) \equiv (\varphi \sqcap \psi))$ by (EL8)and $\Gamma \vdash (\varphi \sqcap \psi) \equiv \varphi$ by assumption. Using Proposition 3.7 part (4), we get $\Gamma \vdash ((\psi \sqcap \chi) \equiv \psi) \Rightarrow ((\varphi \sqcap \psi \sqcap \chi) \equiv \varphi)$. By (EL7) and Proposition 3.8 part (7), we obtain $\Gamma \vdash ((\psi \sqcap \chi) \equiv \psi) \Rightarrow ((\varphi \sqcap \chi) \equiv \varphi)$. Hence $\Gamma \vdash (\psi \Rightarrow \chi) \Rightarrow (\varphi \Rightarrow \chi)$.

(3) We have $\vdash (\varphi \equiv \top) \Rightarrow ((\varphi \equiv \psi) \equiv (\top \equiv \psi))$ and $\vdash (\psi \equiv \top) \equiv \psi)$ by (EL9) and (EL5) respectively. Therefore $\vdash (\varphi \equiv \top) \Rightarrow ((\psi \equiv \varphi) \equiv \psi)$ by Proposition 3.7 part (4). Again by (EL5) and Proposition 3.7 part (5), we get $\vdash \varphi \Rightarrow [(\psi \equiv \varphi) \equiv \psi]$.

(4) Applying (EL8), (EL1) and Proposition 3.7 part (4), we have $\vdash (\varphi \equiv \psi) \Rightarrow ((\varphi \sqcap \psi) \equiv \psi)$, that is $\vdash (\varphi \equiv \psi) \Rightarrow (\varphi \Rightarrow \psi)$.

(5) By part (3) and definition $\Rightarrow$, we have $\vdash \varphi \Rightarrow ((\varphi \Rightarrow \psi) \equiv (\varphi \sqcap \psi))$. By part (4) and Proposition 3.8 part (7), we get $\vdash \varphi \Rightarrow ((\varphi \Rightarrow \psi) \Rightarrow (\varphi \sqcap \psi))$. By Proposition 3.8 part (6) and then part (1), we obtain $\vdash ((\varphi \Rightarrow \psi) \Rightarrow (\varphi \sqcap \psi)) \Rightarrow ((\varphi \Rightarrow \psi) \Rightarrow \psi)$. Using Proposition 3.8 part (7), we obtain the result.

(6) By (EL8) and part (2), we have $\vdash [((\varphi \sqcap \psi) \equiv (\varphi \sqcap \psi \sqcap \chi)) \Rightarrow (\varphi \equiv (\varphi \sqcap \psi \sqcap \chi))] \Rightarrow [(\psi \sqcap \chi) \equiv \psi) \Rightarrow (\varphi \equiv (\varphi \sqcap \psi \sqcap \chi))]$. By (EL7) and part (1), we have $\vdash [((\psi \sqcap \chi) \equiv \psi) \Rightarrow (\varphi \equiv (\varphi \sqcap \psi \sqcap \chi))] \Rightarrow [((\psi \sqcap \chi) \equiv \psi) \Rightarrow ((\varphi \sqcap \chi) \equiv \chi)]$. By (EL9), we have $\vdash ((\varphi \sqcap \psi) \equiv \varphi) \Rightarrow [((\varphi \sqcap \psi) \equiv (\varphi \sqcap \psi \sqcap \chi)) \Rightarrow ((\varphi \sqcap \psi \sqcap \chi)) \equiv \varphi]$. Using Proposition 3.8 part (7) twice, we obtain $\vdash ((\varphi \sqcap \psi) \equiv \varphi) \Rightarrow [((\psi \sqcap \chi) \equiv \psi) \Rightarrow ((\varphi \sqcap \chi) \equiv \chi)]$. Hence $\vdash (\varphi \Rightarrow \psi) \Rightarrow [(\psi \Rightarrow \chi) \Rightarrow (\varphi \Rightarrow \chi)]$.

(7) By part (7), we have $\vdash (\chi \Rightarrow (\varphi \Rightarrow \psi)) \Rightarrow [((\varphi \Rightarrow \psi) \Rightarrow \psi) \Rightarrow (\chi \Rightarrow \psi)]$. By part (5) and then part (2), we have $\vdash [((\varphi \Rightarrow \psi) \Rightarrow \psi) \Rightarrow (\chi \Rightarrow \psi)] \Rightarrow [\varphi \Rightarrow (\chi \Rightarrow \psi)]$. Using Proposition 3.8 part (6), we obtain the result.

(8) It can be proved by part (7) and induction.

(9) It follows from (EL9) and part (4).

(10) Let $\Gamma = \{\varphi \Rightarrow \chi, \varphi \Rightarrow \psi\}$. By assumption $\Gamma \vdash \varphi \Rightarrow \chi$, (EL8) and (MP), we get $\Gamma \vdash ((\varphi \sqcap \chi) \sqcap \psi) \equiv (\varphi \sqcap \psi)$. Using assumption $\Gamma \vdash \varphi \Rightarrow \psi$ and (EL9), we obtain the result.

(11) It follows from part (4) and part (10). □

Theorem 3.14 (Local Deduction Theorem). *Let Γ be a theory over the propositional equality logic $\mathcal{EL}$ and φ, ψ be formulas. Then $\Gamma \cup \{\varphi\} \vdash \psi$ if and only if $\Gamma \vdash \varphi \Rightarrow^n \psi$ where $n \in \mathbb{N}_0$.*

Proof: Suppose that $\Gamma \cup \{\varphi\} \vdash \psi$. We will prove it by induction on the number of formulas on the sequence of deduction of ψ from $\Gamma \cup \{\varphi\}$. Let $\chi_1, \chi_2, \ldots, \chi_k$ be a corresponding $\Gamma \cup \{\varphi\}$-proof of ψ. We should consider four cases:

Case 1: ψ is an axiom of $\mathcal{EL}$ or $\psi \in \Gamma$. By Proposition 3.8 part (4) and (MP), we obtain $\Gamma \vdash \varphi \Rightarrow \psi$.

Case 2: ψ is φ. By Proposition 3.8 part (1), we have $\Gamma \vdash \varphi \Rightarrow \varphi$.

Case 3: ψ is obtained from two pervious formulas on the corresponding $\Gamma \cup \{\varphi\}$-proof of ψ by an application (MP). These two formulas must have the form χ_i and $\chi_i \Rightarrow \psi$ where $1 < i < k$. By the induction hypothesis, there exist $n, m \in \mathbb{N}_0$ such that $T \vdash \varphi \Rightarrow^n \chi_i$ and $\Gamma \vdash \varphi \Rightarrow^m (\chi_i \Rightarrow \psi)$.
By Proposition 3.13 part (8) and (MP), we get $\Gamma \vdash \chi_i \Rightarrow (\varphi \Rightarrow^m \chi)$. Using Proposition 3.13 part (1), we have $\Gamma \vdash (\varphi \Rightarrow^n \chi_i) \Rightarrow (\varphi \Rightarrow^n (\varphi \Rightarrow^m \psi))$. Applying (MP), we obtain $\Gamma \vdash \varphi \Rightarrow^n (\varphi \Rightarrow^m \psi)$. Hence $\Gamma \vdash \varphi \Rightarrow^{n+m} \psi$.

Case 4: ψ results by (EA) from pervious member χ_i and $\chi_i \equiv \psi$ $(1 < i < k)$ of the corresponding $\Gamma \cup \{\varphi\}$-proof of ψ. Thus $\Gamma \cup \{\varphi\} \vdash \chi_i$ and $T \cup \{\varphi\} \vdash$

$\chi_i \equiv \psi$. By Proposition 3.7 part (8), we have $\Gamma \cup \{\varphi\} \vdash \chi_i \Rightarrow \psi$. As Case 3 above, we can show that $\Gamma \vdash \varphi \Rightarrow^{n+m} \psi$.

Conversely, suppose that $\Gamma \vdash \varphi \Rightarrow^n \psi$ for $n > 1$. Then $\Gamma \vdash \varphi \Rightarrow (\varphi \Rightarrow^{n-1} \psi)$. Thus $\Gamma \cup \{\varphi\} \vdash \varphi \Rightarrow^{n-1} \psi$. Replacing this, we obtain $\Gamma \cup \{\varphi\} \vdash \varphi \Rightarrow \psi$. Hence $\Gamma \cup \{\varphi\} \vdash \psi$. $\square$

Remark. The deduction theorem in the form of $\Gamma \cup \{\varphi\} \vdash \psi$ if and only if $\Gamma \vdash \varphi \Rightarrow \psi$ does not hold in the propositional equality logic $\mathcal{EL}$. Suppose that it holds and $\varphi \in \mathcal{F}$ be arbitrary formula. Then $\{\varphi, \varphi \Rightarrow (\varphi \Rightarrow \psi)\} \vdash \varphi$. Hence $\vdash (\varphi \Rightarrow (\varphi \Rightarrow \psi)) \Rightarrow (\varphi \Rightarrow \psi)$. Therefore $(\varphi \Rightarrow (\varphi \Rightarrow \psi)) \Rightarrow (\varphi \Rightarrow \psi)$ is an $\mathcal{A}$-tautology for every equality-algebra $\mathcal{A} = (A, \exists, \forall)$ by Theorem 3.12. Now, consider equality-algebra in Example 4.7 and define $e(\varphi) = a$ and $e(\psi) = b$. Then $(e(\varphi) \to (e(\varphi) \to e(\psi))) \to (e(\varphi) \to e(\psi)) = 1 \to d = d$ which is a contradiction.

In the following, we will show that the propositional equality logic $\mathcal{EL}$ algebraizable with respect to the variety of equality-algebras in the sense of [1] (Also see [2]).

THEOREM 3.15. *The propositional equality logic $\mathcal{EL}$ is algebraizable with the defining equation $\varphi = \top$ and the equivalence formulas $\{\varphi \equiv \psi\}$.*

PROOF: Suppose that $\varphi \Delta \psi = \{\varphi \equiv \psi\}$, $\delta(\varphi) = \varphi$ and $\epsilon(\varphi) = \top$. By the intrinsic characterization given by Blok and Pigozzi [1, Theorem 4.7], it is sufficient to check that the following conditions hold for all formulas:

(1) $\vdash \varphi \Delta \varphi$,

(2) $\varphi \Delta \psi \vdash \psi \Delta \varphi$,

(3) $\varphi \Delta \psi, \psi \Delta \chi \vdash \varphi \Delta \chi$,

(4) $\varphi_1 \Delta \psi_1, \varphi_2 \Delta \psi_2 \vdash (\varphi_1 \sqcap \varphi_2) \Delta (\psi_1 \sqcap \psi_2)$,

(5) $\varphi_1 \Delta \psi_1, \varphi_2 \Delta \psi_2 \vdash (\varphi_1 \equiv \varphi_2) \Delta (\psi_1 \equiv \psi_2)$,

(6) $\varphi \dashv\vdash \delta(\varphi) \Delta \epsilon(\varphi)$.

Now, we will prove them as follows:

(1) Since $\varphi \Delta \varphi = \{\varphi \equiv \varphi\}$, then $\vdash \varphi \Delta \varphi$ by Proposition 3.6 part (1).

(2) By Proposition 3.6 part (4), $\varphi \equiv \psi \vdash \psi \equiv \varphi$. Hence $\varphi \Delta \psi \vdash \psi \Delta \varphi$.

(3) It follows from Proposition 3.6 part (7).

(4) and (5) We obtain them by Lemma 3.9.

(6) Applying Proposition 3.6 part (2) and part (3), we have $\varphi \dashv\vdash \varphi \equiv \top$. Hence $\varphi \dashv\vdash \delta(\varphi)\Delta\epsilon(\varphi)$. □

Theorem 3.16. *The propositional equality logic $\mathcal{EL}$ is algebraizable with respect to the variety of equality-algebras, with equivalence formulas $\{\varphi \equiv \psi\}$ and defining equation $\varphi = \top$.*

Proof: Let $Alg^*\mathcal{EL}$ be the algebraic semantics of the propositional equality logic $\mathcal{EL}$. By Theorem 3.15, it exists and we can take $\{\varphi \equiv \psi\}$ for the equivalence formulas, and $\delta(p) = p$, $\epsilon(p) = \top$ for the defining equation. By [1, Theorem 2.17], the variety $Alg^*\mathcal{EL}$ is axiomatized as follows:

(1) $(x \wedge x) \sim x = 1$.

(2) $(x \wedge y) \sim (y \wedge x) = 1$,

(3) $((x \wedge y) \wedge z) \sim (x \wedge (y \wedge z)) = 1$.

(4) $((x \wedge 1) \sim x) = 1$,

(5) $(x \sim 1) \sim x = 1$,

(6) $(x \sim y) \sim (y \sim x) = 1$,

(7) $((x \wedge y \wedge z) \sim x) \to ((x \wedge y) \sim x) = 1$,

(8) $(x \sim y) \to ((x \wedge z) \sim (y \wedge z)) = 1$,

(9) $(x \sim y) \to ((x \sim z) \sim (y \sim z)) = 1$,

(10) $x = 1$ and $x \sim y = 1$ imply $y = 1$,

(11) $x = 1$ and $x \to y = 1$ imply $y = 1$,

(12) $x \sim y = 1$ imply $x = y$.

It is obvious that every equality-algebra satisfies (1)–(12). Hence the variety of equality-algebras is included in $Alg^*\mathcal{EL}$. Conversely, let

$\mathcal{A} = (A, \wedge, \sim, 1)$ be an algebra belonging to $Alg^*\mathcal{EL}$. Then $(A, \wedge, 1)$ is a meet-semilattice with top element 1 by part (1)–(5) and part (12). (E5) follows from part (7) and Proposition 2.2. It is clear that $\mathcal{A}$ satisfies the other conditions of Definition 2.1.

Therefore We conclude that $Alg^*\mathcal{EL}$ is precisely the variety of all equality-algebras. □

In 1990, Skolem semilattices were defined by Büchi and Owens (see [5]). Let x, y be arbitrary elements of a meet-semilattice $(S, \wedge, 1)$ with the greatest element 1. If the largest element of the set $\{z \in S : a \wedge x = b \wedge x\}$ exists, then it is called the symmetric relative pseudo-complement (the symmetric RPC) of x and y, and is denoted by $x \leftrightarrow_s y$. If the the symmetric RPC exists for every pair of elements x, y, then the enriched structure $(S, \wedge, \leftrightarrow_s, 1)$ is called a Skolem semilattice.

The class of Skolem semilattices is a strongly point-regular and forms a Hilbertian variety and hence Fregean. Skolem semilattices form the algebraic semantics of the conjunctive-equivalential fragment of intuitionistic logic ([7]).

Proposition 3.17. The Skolem semilattices form a proper subvariety of the variety of the equality algebras.

Proof: Let x, y be arbitrary elements of the Skolem semilattice $(S, \wedge, \leftrightarrow_s, 1)$. We define $x \sim y := x \leftrightarrow_s y$.Then (E2)–(E4) hold by parts (1)–(3) of Theorem 6.5.2 in [7]. The proof of (E5) and (E6) is easy. Let $t \in S$ such that $t \wedge x = t \wedge x$. By part of (4) of Theorem 6.5.2 in [7], we have $t \wedge (x \leftrightarrow_s z) = t \wedge ((x \wedge t) \leftrightarrow_s z)$. Thus $t \wedge (x \sim z) = t \wedge ((x \wedge t) \sim z) = t \wedge ((y \wedge t) \sim z) = t \wedge (y \sim z)$. We obtain $t \leq \sup\{w \in S : w \wedge (x \sim z) = w \wedge (x \sim z)\} = (x \sim z) \sim (y \sim z)$. Then (E7) hold. Hence $(S, \wedge, \sim, 1)$ is an equality algebra. It follows that Skolem semilattices form a subvariety of the variety of the equality algebras. This inclusion is proper, because the logic determined by Skolem semilattice admits the standard deduction theorem while the logic determined by equality algebras admits merely a local deduction theorem by Theorem 3.14. □

Corollary 3.18. The propositional equality logic $\mathcal{EL}$ with respect to the variety of equality-algebras is regularly algebraizable but it is not Fregean.

Proof: Let $E(\varphi, \psi) := \{\varphi \equiv \psi\}$. Then $E(\varphi, \psi)$ is a (finite) system of equivalence sentences for $\mathcal{EL}$ and the G-rule determined by E is valid in

$\mathcal{EL}$. Thus $\mathcal{EL}$ is finitely regularly algebraizable. By Corollary 6.5.11 in [7] and Theorem 3.15, we conclude that $\mathcal{EL}$ is not Fregean. $\square$

In 1966, famous Polish logician Roman Suszko create a new logical calculus called by him Non-Fregean Logic (see [20]). He introduced the identity connective to metalogic and, relying on Wittgenstein's writings, he has initiated systematic investigations of deductive systems endowed with identity. By the above corollary, the equality algebras are the algebraic counterparts of a strengthening of the pure Suszko logic with identity and additionally equipped with the connective that possesses the properties of conjunction.

4. Equality$_\triangle$-algebras

In this section, the concept of (prelinear) equality$_\triangle$-algebra is introduced and some related properties are investigated.

Definition 4.1. An equality$_\triangle$-algebra is an algebra $(A, \wedge, \sim, \triangle, 0, 1)$ of type $(2, 2, 1, 0, 0)$ where $(A, \wedge, \sim, 0, 1)$ is a bounded equality-algebra expanded by a unary operation $\triangle : A \to A$ satisfying the following:

- ($\triangle$1) $\triangle 1 = 1$,
- ($\triangle$2) $\triangle x \leq x$,
- ($\triangle$3) $\triangle x \leq \triangle\triangle x$,
- ($\triangle$4) $\triangle(x \sim y) \leq \triangle x \sim \triangle y$,
- ($\triangle$5) $\triangle(x \wedge y) = \triangle x \wedge \triangle y$,
- ($\triangle$6) if $x \vee y$ and $\triangle x \vee \triangle y$ exist, then $\triangle(x \vee y) \leq \triangle x \vee \triangle y$,
- ($\triangle$7) $\triangle x \vee \neg\triangle x = 1$, that is 1 is unique upper bound of the set $\{\triangle x, \neg\triangle x\}$ in A.

Example 4.2.

1. Let $(A, \wedge, \sim, 0, 1)$ be a bounded equality-algebra. Define $\triangle : A \to A$ by $\triangle 1 = 1$ and $\triangle x = 0$ for any $x < 0$. Then $(A, \wedge, \sim, \triangle, 0, 1)$ is an equality$_\triangle$-algebra.
2. Let $A = \{0, a, b, c, d, 1\}$ be a lattice in Fig. 1. Consider the operations $\sim$ and $\to$ given by the following tables:

$\sim$	0	a	b	c	d	1
0	1	d	c	b	a	0
a	d	1	a	d	c	a
b	c	a	1	0	d	b
c	b	d	0	1	a	c
d	a	c	d	a	1	d
1	1	1	1	1	1	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	a	c	c	1
b	c	1	1	c	c	1
c	b	a	b	1	a	1
d	a	1	a	1	1	1
1	0	a	b	c	d	1

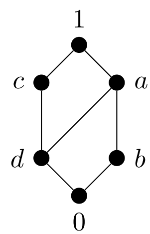

Figure 1

Then $\mathcal{A} = (A, \wedge, \sim, 0, 1)$ is a bounded equality-algebra ([22]). Define the unary operation $\triangle$ on A as $\triangle 0 = \triangle d = 0$, $\triangle a = \triangle b = b$, $\triangle c = c$ and $\triangle 1 = 1$. Then $(A, \wedge, \sim, \triangle, 0, 1)$ is an equality$_\triangle$-algebra.

Remark 4.3. It is obvious that every equality$_\triangle$-algebra is a true equality algebra. But the converse may not be true in general. Consider the following example:

Example 4.4. Let $A = \{0, a, b, 1\}$ be a chain such that $0 < a < b < 1$. Consider the operations $\sim$ and $\to$ given by the following tables:

$\sim$	0	a	b	1
0	1	a	a	0
a	1	1	a	a
b	1	1	1	b
1	1	1	1	1

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

Then $\mathcal{A} = (A, \wedge, \sim, 0, 1)$ is a bounded equality-algebra ([8]). Define the unary operation τ on A as follows:

$$\tau(0) = 0,\ \tau(a) = \tau(b) = a \text{ and } \tau(1) = 1.$$

Then τ is a very true operation on A ([21]). Since $\tau(a) \vee \neg\tau(a) = a \neq 1$, then τ is not $(A, \wedge, \sim, \tau, 0, 1)$ is not an equality$_\triangle$-algebra.

PROPOSITION 4.5. Let $(A, \wedge, \sim, \triangle, 0, 1)$ be an equality$_\triangle$-algebra. Then the following properties hold, for all $x, y, z \in A$:

(1) $\triangle x = 1$ if and only if $x = 1$,

(2) $x \leq y$ implies $\triangle x \leq \triangle y$,

(3) $\triangle\triangle x = \triangle x$,

(4) $\triangle x \leq y$ if and only if $\triangle x \leq \triangle y$,

(5) $Im(\triangle) = Fix(\triangle)$ where $Fix(\triangle) = \{x \in A : \triangle x = x\}$,

(6) if $\triangle$ is surjective, then $\triangle = Id_A$,

(7) $Ker(\triangle) = \{1\}$, where $Ker(\triangle) = \{x \in A : \triangle x = 1\}$,

(8) $Ker(\triangle)$ is a deductive system of A,

(9) $\triangle(x \to y) \leq \triangle x \to \triangle y$,

(10) if $x \vee y$ and $\triangle x \vee \triangle y$ exist, then $\triangle(x \vee y) = \triangle x \vee \triangle y$.

PROOF: Since every equality$_\triangle$-algebra is a very true equality-algebra, then part (1)–(9) follow from Proposition in [21]. (10) follows from ($\triangle$6) and part (2). $\square$

DEFINITION 4.6. A prelinear equality$_\triangle$-algebra is an equality$_\triangle$-algebra $(A, \wedge, \sim, \triangle, 0, 1)$ satisfies the following: for all $x, y, z \in A$

($\triangle$8) $\triangle(x \to y) \to z \leq (\triangle(y \to x) \to z) \to z$.

Example 4.7.

(1) An equality$_\triangle$-algebra in Example 4.2 part (2) is a prelinear equality$_\triangle$-algebra.

(2) Let $A = \{0, a, b, c, d, 1\}$ be a lattice in Fig. 2. Consider the operations $\sim$ and $\to$ given by the following tables:

$\sim$	0	a	b	c	d	1
0	1	d	d	d	c	0
a	d	1	c	d	c	a
b	d	c	1	d	c	b
c	d	d	d	1	d	c
d	c	c	c	d	1	d
1	0	a	b	c	d	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	1	1
b	d	d	1	1	1	1
c	d	d	d	1	1	1
d	c	c	c	d	1	1
1	0	a	b	c	d	1

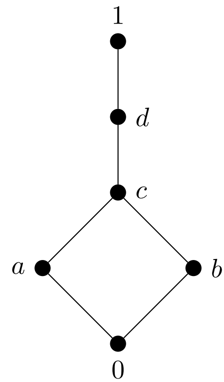

Figure 2

Then $\mathcal{A} = (A, \wedge, \sim, 0, 1)$ is a bounded equality-algebra ([22]). Define the unary operation $\triangle$ on A as $\triangle 0 = \triangle a = \triangle b = \triangle c = \triangle d = 0$ and $\triangle 1 = 1$. Then $(A, \wedge, \sim, \triangle, 0, 1)$ is an equality$_\triangle$-algebra but it is not a prelinear equality$_\triangle$-algebra because $\triangle(a \to b) \to c = 1 \not\leq c = (\triangle(b \to y) \to c) \to c$.

(3) Let $(A, \wedge, \sim, 0, 1)$ be a prelinear bounded equality-algebra. Define $\triangle : A \to A$ by $\triangle 1 = 1$ and $\triangle x = 0$ for any $x < 0$. Then $(A, \wedge, \sim, \triangle, 0, 1)$ is a prelinear equality$_\triangle$-algebra.

The proof of the following proposition is similar to Lemma 8 in [8].

Proposition 4.8. Let $\triangle$ be a unary operation on a bounded equality-algebra $\mathcal{A} = (A, \wedge, \sim, 0, 1)$ such that satisfies $(\triangle 1)$, $(\triangle 2)$, $(\triangle 8)$ and

($\triangle$9) $\triangle(x \to y) \leq \triangle x \to \triangle y$.

Then, we have

(1) $\triangle(x \to y) \vee \triangle(y \to x) = 1$,

(2) if $x \leq y$, then $\triangle x \leq \triangle y$,

(3) $(x \to y) \vee (y \to x) = 1$.

(4) $\triangle(x \wedge y) = \triangle x \wedge \triangle y$,

(5) if $x \vee y$ and $\triangle x \vee \triangle y$ exist, then $\triangle(x \vee y) = \triangle x \vee \triangle y$,

(6) $\triangle(x \sim y) \leq \triangle x \sim \triangle y$.

Proof:

(1) Suppose that u is an upper bound of the set $\{\triangle(x \to y), \triangle(y \to x)\}$. By ($\triangle$8), Proposition 2.5 part (15) and part (6), we get $1 = (x \to y) \to u \leq ((x \to y) \to u) \to u = 1 \to u = u$. Hence $u = 1$.

(2) It follows from Proposition 2.5 part (4), ($\triangle$9) and ($\triangle$1).

(3) Suppose that u is an upper bound of the set $\{x \to y, y \to x\}$. Then $\triangle(x \to y) \leq \triangle u$ and $\triangle(y \to x) \leq \triangle u$ by part (2). By part (1), we obtain $\triangle u = 1$. Hence $u = 1$ by ($\triangle$2).

(4) By part (2), we have $\triangle(x \wedge y) \leq \triangle x \wedge \triangle y$. On the other hand, by Proposition 2.5 part (17), ($\triangle$9) and Proposition 2.5 part (16) $1 = \triangle(x \to y) \vee \triangle(y \to x) = \triangle(x \to (x \wedge y)) \vee \triangle(y \to (x \wedge y)) \leq (\triangle x \to \triangle(x \wedge y)) \vee (\triangle y \to \triangle(x \wedge y)) = (\triangle x \wedge \triangle y) \to \triangle(x \wedge y)$. Thus $(\triangle x \wedge \triangle y) \leq \triangle(x \wedge y)$.

(5) By part (2), we have $\triangle x \vee \triangle y \leq \triangle(x \vee y)$. On the other hand, by part (4), ($\triangle$9), Proposition 2.5 part (15) and part (16), we obtain

$$\begin{aligned}
\triangle(x \vee y) &= \triangle(((x \to y) \to y) \wedge ((y \to x) \to x)) \\
&\leq ((\triangle(x \to y) \to \triangle y) \wedge (\triangle(y \to x) \to \triangle x)) \\
&\leq ((\triangle(x \to y) \to (\triangle x \vee \triangle y)) \wedge (\triangle(y \to x) \to (\triangle x \vee \triangle y)) \\
&\leq (\triangle(x \to y) \vee \triangle(y \to x)) \to (\triangle x \vee \triangle y) = (\triangle x \vee \triangle y).
\end{aligned}$$

(6) It follows from Proposition 2.5 part (7),($\triangle$9), part (4) and part (2). $\square$

Corollary 4.9. Let $\triangle$ be a unary operation on bounded equality-algebra $\mathcal{A} = (A, \wedge, \sim, 0, 1)$. Then $(A, \wedge, \sim, \triangle, 0, 1)$ is a prelinear equality$_\triangle$- algebra if and only if it satisfies, for all $x, y, z \in A$.

($\triangle$1) $\triangle 1 = 1$,

($\triangle$2) $\triangle x \leq x$,

($\triangle$3) $\triangle x \leq \triangle\triangle x$,

($\triangle$8) $\triangle(x \to y) \to z \leq (\triangle(y \to x) \to z) \to z$,

($\triangle$9) $\triangle(x \to y) \leq \triangle x \to \triangle y$,

($\triangle$7) $\triangle x \vee \neg\triangle x = 1$, that is 1 is unique upper bound of the set $\{\triangle x, \neg\triangle x\}$ in A.

Proof: It follows from Proposition 4.5 and Proposition 4.8. □

Corollary 4.10. A prelinear equality$_\triangle$-algebra is an equality$_\triangle$-algebra satisfying the prelinearity. Moreover, it is a distributive lattice.

Proof: It follows from Proposition 4.8 part (3) an Theorem 2.4. □

Definition 4.11. A $\triangle$-deductive system of an equality$_\triangle$-algebra $(A, \wedge, \sim, \triangle, 0, 1)$ is a deductive system F of $(A, \wedge, \sim, 0, 1)$ that satisfies for all $x \in F$, $\triangle x \in F$.

Example 4.12. Consider the prelinear equality$_\triangle$-algebra $(A, \wedge, \sim, \triangle, 0, 1)$ in Example 4.2 part (2). It is easy to see that $F_1 = \{c, 1\}$ is a $\triangle$-deductive system of A. Also, $F_2 = \{c, 1\}$ is a deductive system of A but it is not a $\triangle$-deductive system.

Let $(A, \wedge, \sim, \triangle, 0, 1)$ be an equality$_\triangle$-algebra and X be a nonempty subset of A. We denote by $\langle X\rangle_\triangle$ the $\triangle$-deductive system of A generated by X, that is, $\langle X\rangle_\triangle$ is the smallest $\triangle$-deductive system of A containing X. If F is a $\triangle$-deductive system of A and $x \notin F$, then $\langle F, x\rangle_\triangle := \langle F \cup \{x\}\rangle_\triangle$.

Theorem 4.13. *Let X be a nonempty subset of an equality$_\triangle$-algebra $(A, \wedge, \sim, \triangle, 0, 1)$. Then*

(i) $\langle X\rangle_\triangle = \{x \in A | \exists n \in \mathbb{N}, y_1, ..., y_n \in A \ni \triangle y_1 \to (\triangle y_2 \to ...(\triangle y_n \to x)...) = 1\}$,

(ii) *If F is a deductive system of A and $S \subseteq A$, then* $\langle F \cup S \rangle_\triangle = \{x \in A | \exists n \in \mathbb{N}, s_1, ..., s_n \in S \ni \triangle s_1 \to (\triangle s_2 \to ...(\triangle s_n \to x)...) \in F\}$,

(iii) $\langle a \rangle_\triangle = \{x \in A | \exists n \in \mathbb{N}, \triangle x \to^n a = 1\}$.

Proof: The proof is straightforward. □

Definition 4.14. Let $\mathcal{A}_\triangle = (A, \wedge, \sim, \triangle, 0, 1)$ be an equality$_\triangle$-algebra and θ be a congruence on an equality-algebra $(A, \wedge, \sim, 0, 1)$. Then θ is called a $\triangle$-congruence on $\mathcal{A}_\triangle$, if $(x, y) \in \theta$, then $(\triangle x, \triangle y) \in \theta$, for any $x, y \in A$.

Proposition 4.15. Let $\mathcal{A}_\triangle = (A, \wedge, \sim, \triangle, 0, 1)$ be an equality$_\triangle$-algebra and let F be a $\triangle$-deductive system. Put $(x, y) \in \theta_F$ iff $x \sim y \in F$. Then

(i) θ_F is a $\triangle$-congruence and the corresponding quotient algebra $(\mathcal{A}/\theta_\mathcal{F})_\triangle = (A/\theta_F, \wedge, \sim, \triangle, 1)$ is an equality$_\triangle$-algebra, where for every $x, y \in A$, $[x] \wedge [y] := [x \wedge y]$, $[x] \sim [y] := [x \sim y]$, $\triangle[x] := [\triangle x]$ and $1 := [1]$.

(ii) $(\mathcal{A}/\theta_\mathcal{F})_\triangle$ is linearly ordered iff F is a prime $\triangle$-deductive system of $\mathcal{A}$.

(iii) if $\mathcal{A}_\triangle$ is a prelinear equality$_\triangle$-algebra, then $(\mathcal{A}/\theta_\mathcal{F})_\triangle$ is a prelinear equality$_\triangle$-algebra.

Proof: The proof is straightforward. □

Let $\mathcal{A} = (A, \wedge, \sim, 0, 1)$ be an equality-algebra. For $x, y \in A$ and $n \in \mathbb{N}_0$, we define $x \to^n y$ inductively as follows:

$$x \to^0 y = y,$$
$$x \to^n y = x \to (x \to^{n-1} y) \text{ for } n \geq 1.$$

The proof of the following lemma is similar to the proof of lemma 3.3 in [15].

Lemma 4.16. *Let* $\mathcal{A}_\triangle = (A, \wedge, \sim, \triangle, 0, 1)$ *be an equality$_\triangle$-algebra satisfying prelinearity and P be a prime $\triangle$-deductive system of $\mathcal{A}$. If $x \to^n z \in P$ and $y \to^m z \in P$ for $m, n \in \mathbb{N}$, then $(x \vee y) \to^r z \in P$ for some $r \in \mathbb{N}$.*

Proof: Suppose that $l = \max\{n, m\}$. Then $x \to^l z, y \to^l z \in P$. We will prove by induction on l. For $l = 1$, we have $(x \vee y) \to z = (x \to z) \wedge (y \to z) \in P$ by Proposition 2.5 part (16). Thus $r = 1$.

Now, suppose that the statement holds for all $k \in N$ with $k \leq l$ and $x \rightarrow^{l+1} z, y \rightarrow^{l+1} z \in P$. Since

$$\begin{aligned} y \rightarrow^{l+1} z \leq x \rightarrow^{l} (y \rightarrow^{l+1} z) &= x \rightarrow^{l} (y \rightarrow (y \rightarrow^{l} z)) \\ &= y \rightarrow (x \rightarrow^{l} (y \rightarrow^{l+1} z)), \\ x \rightarrow^{l+1} z \leq x \rightarrow^{l+1} (y \rightarrow^{l} z) &= x \rightarrow (x \rightarrow^{l} (y \rightarrow^{l} z), \end{aligned}$$

then $y \rightarrow (x \rightarrow^{l} (y \rightarrow^{l+1} z)), x \rightarrow (x \rightarrow^{l} (y \rightarrow^{l} z) \in P$. Using Proposition 2.5 part (11) and part (16), we get

$$\begin{aligned} x \rightarrow (x \rightarrow^{l-1} (y \rightarrow^{l} ((x \vee y) \rightarrow z))) &= (x \vee y) \rightarrow ((x \rightarrow^{l} (y \rightarrow^{l} z) \\ &= [y \rightarrow (x \rightarrow^{l} (y \rightarrow^{l+1} z))] \wedge [x \rightarrow (x \rightarrow^{l} (y \rightarrow^{l} z)] \in P. \end{aligned} \tag{2.1}$$

By Proposition 2.5 part (15)

$$\begin{aligned} y \rightarrow^{l+1} z &\leq y \rightarrow^{l+1} ((x \vee y) \rightarrow z) \\ &\leq x \rightarrow^{l-1} (y \rightarrow^{l+1} ((x \vee y) \rightarrow z)) \\ &= y \rightarrow (x \rightarrow^{l-1} (y \rightarrow^{l} ((x \vee y) \rightarrow z)). \end{aligned}$$

Thus

$$y \rightarrow (x \rightarrow^{l-1} (y \rightarrow^{l} ((x \vee y) \rightarrow z)) \in P. \tag{2.2}$$

By Proposition 2.5 part (11) and part (16), (2.1) and (2.2), we get $x \rightarrow (x \rightarrow^{l-2} (y \rightarrow^{l} ((x \vee y) \rightarrow^{2} z))) = (x \vee y) \rightarrow ((x \rightarrow^{l-1} (y \rightarrow^{l} (x \vee y) \rightarrow z)) = [y \rightarrow (x \rightarrow^{l-1} (y \rightarrow^{l} ((x \vee y) \rightarrow z))) \wedge x] \rightarrow [(x \rightarrow^{l-1} (y \rightarrow^{l-1} ((x \vee y) \rightarrow z))] \in P$. By repeating this, we get

$$y \rightarrow^{l} ((x \vee y) \rightarrow^{l+1} z)) \in P. \tag{2.3}$$

by interchanging x, y, we obtain

$$x \rightarrow^{l} ((x \vee y) \rightarrow^{l+1} z)) \in P. \tag{2.4}$$

Using induction hypothesi to (2.3) and (2.4), there exists $s \in N$ such that $(x \vee y) \rightarrow^{s+l+1} z = (x \vee y) \rightarrow^{s} ((x \vee y) \rightarrow^{l+1} z) \in P$. Hence $r = s + l + 1$. $\square$

Proposition 4.17. Let $(A, \wedge, \sim, \triangle, 0, 1)$ be a prelinear equality$_{\triangle}$-algebra and let $a \in A$, $a \neq 1$. Then there is a prime $\triangle$-deductive system F on A not containing a.

Proof: Suppose that $\mathcal{P} = \{F : F$ is a proper $\triangle$-deductive system and $a \notin F\}$. Then $\mathcal{P}$ is a partially set under inclusion relation. Since $\{1\} \in \mathcal{P}$, then $\mathcal{P}$ is a nonempty set. It is easy to see that every chain in $\mathcal{P}$ has an upper bound in $\mathcal{P}$. By Zorn's Lemma, there exists a maximal element P in $\mathcal{P}$. Since $P \in \mathcal{P}$, then P is a $\triangle$-deductive system of A not containing a. We will prove that P is prime. If P is not prime, then there exist $x, y \in A$ such that $x \to y, y \to x \notin P$. Since P is strictly contained in $\langle P, x \to y\rangle_\triangle$ and $\langle P, y \to x\rangle_\triangle$, then $\langle P, x \to y\rangle_\triangle \notin \mathcal{P}$ and $\langle P, y \to x\rangle_\triangle \notin \mathcal{P}$ by the maximality of P. Thus $a \in \langle P, y \to x\rangle_\triangle$ and $a \in \langle P, x \to y\rangle_\triangle$. Then there exist $n, m \in N$ such that $\triangle(x \to y) \to^n a \in P$ and $\triangle(y \to x) \to^m a \in P$ by Theorem 4.13 part (iii). By Lemma 4.16, there exists $r \in \mathbb{N}$ such that $(\triangle(x \to y) \vee \triangle(y \to x)) \to^r a \in P$. By Proposition 4.8 part (1), we obtain $a \in P$ which is a contradiction. □

Proposition 4.18. Each prelinear equality$_\triangle$-algebra is a subalgebra of the direct product of a system of linearly ordered equality$_\triangle$-algebra.

Proof: Suppose that $\mathcal{P}$ is the class of all prime $\triangle$-deductive systems of a prelinear equality$_\triangle$-algebra $(A, \wedge, \sim, \triangle, 0, 1)$. Then $B = \prod_{\theta \in \mathcal{P}} A/\theta_F$ is a direct product of linearly ordered equality$_\triangle$- algebra by Proposition 4.15 part (iii). Define $f : A \to B$ by $f(x) = \{x/\theta_F : F \in \mathcal{P}\}$. It is easy to prove that f preserves operations. We will prove that f is one to one. Suppose that $x, y \in A$ such that $x \neq y$. Then $x \not\leq y$ or $y \not\leq x$. Suppose that $x \not\leq y$. Then $x \to y \neq 1$. By Proposition 4.17, there exists a prime $\triangle$-deductive system F such that $x \to y \notin F$. Thus $x/\theta_F \not\leq y/\theta_F$ in A/θ_F. So $x/\theta_F \neq y/\theta_F$ in A/θ_F. Hence $f(x) \neq f(y)$. □

5. Prelinear equality$_\triangle$ logic

In this section, we introduce the logic corresponding to prelinear equality$_\triangle$-algebras and prove that the resulting logic, i.e. propositional prelinear equality$_\triangle$ logic $\mathcal{EL}_\triangle$ is sound and complete with respect to the variety of prelinear equality$_\triangle$-algebras.

Definition 5.1.

(i) The language of propositional prelinear equality$_\triangle$ logic $\mathcal{EL}_\triangle$ is the language of propositional equality logic $\mathcal{EL}$ expanded by the unary connective Δ and the truth constant $\bot$.

(ii) Formulas of $\mathcal{EL}_\triangle$ are defined in the following way:
each formula of $\mathcal{EL}$ is a formula of $\mathcal{EL}_\triangle$, $\bot$ is a formula and if φ is a formula, then $\Delta(\varphi)$ is a formula. Disjunction and negation connectives are defined as the following short:

$$\varphi \sqcup \psi := ((\varphi \Rightarrow \psi) \Rightarrow \psi) \sqcap ((\psi \Rightarrow \varphi) \Rightarrow \varphi), \quad \neg\varphi =: \varphi \Rightarrow \bot.$$

The set of all formulas of $\mathcal{EL}_\triangle$ is denoted by $\mathcal{F}_\triangle$.

(iii) The logical axioms of $\mathcal{EL}_\triangle$ consist of the logical axioms of $\mathcal{EL}$ plus the following axioms :

(E10) $(\varphi \sqcap \bot) \equiv \bot$,
(EΔ1) $\Delta\top$,
(EΔ2) $\Delta\varphi \Rightarrow \varphi$,
(EΔ3) $\Delta\varphi \Rightarrow \Delta\Delta\varphi$,
(EΔ4) $(\Delta(\varphi \Rightarrow \psi) \Rightarrow \chi) \Rightarrow ((\Delta(\psi \Rightarrow \varphi) \Rightarrow \chi) \Rightarrow \chi)$,
(EΔ5) $\Delta(\varphi \Rightarrow \psi) \Rightarrow (\Delta\varphi \Rightarrow \Delta\psi)$,
(EΔ6) $(\Delta\varphi \Rightarrow \neg\Delta\varphi) \Rightarrow \neg\Delta\varphi$,
(EΔ7) $(\neg\Delta\varphi \Rightarrow \Delta\varphi) \Rightarrow \Delta\varphi$.

(iv) The inference rules of $\mathcal{EL}_\triangle$ are (EA), (MP) and generalization (Gen): from φ derive $\Delta\varphi$.

Definition 5.2. Let $\mathcal{A}_\triangle = (A, \wedge, \sim, \triangle, 0, 1)$ be a prelinear equality$_\triangle$-algebra. An $\mathcal{A}_\triangle$-evaluation of formulas is a mapping $e : \mathcal{F}_\triangle \to A$, defined as follows:

(1) $e(\bot) = 0$,

(2) $e(\top) = 1$,

(3) $e(\Delta\varphi) = \triangle e(\varphi)$,

(4) $e(\varphi \sqcap \psi) = e(\varphi) \wedge e(\psi)$,

(5) $e(\varphi \equiv \psi) = e(\varphi) \sim e(\psi)$,

for all formulas $\varphi, \psi \in \mathcal{F}_\triangle$. A formula φ is a $\mathcal{A}_\triangle$-tautology if $e(\varphi) = 1$ for each $\mathcal{A}_\triangle$- evaluation $e : \mathcal{F}_\triangle \to A$. If an $\mathcal{A}_\triangle$-evaluation e satisfies $e(\varphi) = 1$ for every φ in theory Γ, then it is called an $\mathcal{A}_\triangle$-model of Γ.

The propositional prelinear equality$_\triangle$ logic $\mathcal{EL}_\triangle$ is an extension of the propositional equality logic $\mathcal{EL}$. Thus every the theorems and inferences of

$\mathcal{EL}$ is valid in $\mathcal{EL}_\triangle$. In the following Lemma, we prove properties that we will use in the strong completeness of $\mathcal{EL}_\triangle$.

Lemma 5.3. *Let $\varphi, \psi, \chi \in \mathcal{F}$ be formulas. $\mathcal{EL}_\triangle$ proves the following properties:*

(1) $\{\Delta(\varphi \Rightarrow \psi) \Rightarrow \chi, \Delta(\psi \Rightarrow \varphi) \Rightarrow \chi\} \vdash \chi$,

(2) $\{(\varphi \Rightarrow \psi) \Rightarrow \chi, (\psi \Rightarrow \varphi) \Rightarrow \chi\} \vdash \chi$,

(3) $\vdash ((\varphi \Rightarrow \psi) \sqcap (\psi \Rightarrow \varphi)) \Rightarrow (\varphi \equiv \psi)$,

(4) $\vdash \varphi \equiv \psi \vdash \Delta\varphi \equiv \Delta\psi$,

(5) $\vdash \Delta\top \equiv \top$.

Proof:

(i) The results follows from assumptions, (EΔ4) and (MP).

(ii) Suppose that $\Gamma = \{(\varphi \Rightarrow \psi) \Rightarrow \chi, (\psi \Rightarrow \varphi) \Rightarrow \chi\}$. By assumption, (Gen), (EΔ5) and (MP), we have $\Gamma \vdash \Delta(\varphi \Rightarrow \psi) \Rightarrow \Delta\chi$ and $\Gamma \vdash \Delta(\psi \Rightarrow \varphi) \Rightarrow \Delta\chi$. Using part (1), we obtain $\Gamma \vdash \chi$.

(iii) Using Proposition 3.13 part (9), Proposition 3.8 part (8) and part (7), we get $\vdash (\varphi \Rightarrow \psi) \Rightarrow [((\varphi \Rightarrow \psi) \sqcap (\psi \Rightarrow \varphi)) \Rightarrow (\varphi \equiv \psi)]$, $\vdash (\psi \Rightarrow \varphi) \Rightarrow [((\varphi \Rightarrow \psi) \sqcap (\psi \Rightarrow \varphi)) \Rightarrow (\varphi \equiv \psi)]$. Applying part (2), the result is obtained.

(iv) It is easy to prove by assumption, Proposition 3.13 part (11) , (Gen), (EΔ5), (MP), Proposition 3.7 part (7) and then part (3) and (MP).

(v) Using (EL5), we have $\vdash (\Delta\top \equiv \top) \equiv \Delta\top$. By (EL6) and (EA), we obtain $\vdash \Delta\top \equiv (\Delta\top \equiv \top)$. Applying (E$\Delta$1) and (EA), we have $\vdash \Delta\top \equiv \top$. □

Proposition 5.4. Let Γ be a theory over the propositional equality logic $\mathcal{EL}_\triangle$. Then algebra $\mathcal{M}_\Gamma = (M_\Gamma, \wedge, \sim, \triangle, 0, 1)$ is a prelinear equality$_\triangle$-algebra where $1 = [\top]_\Gamma$, $0 = [\bot]_\Gamma$, $\triangle[\psi]_\Gamma := [\Delta(\varphi)]_\Gamma$, $[\varphi]_\Gamma \wedge [\psi]_\Gamma := [\varphi \sqcap \psi]_\Gamma$ and $[\varphi]_\Gamma \sim [\psi]_\Gamma := [\varphi \equiv \psi]_\Gamma$.

Proof: Let Γ be a theory over the propositional equality logic $\mathcal{EL}_\triangle$. Since Γ be a theory over the propositional equality logic $\mathcal{EL}$, then $(M_\Gamma, \wedge, \sim, 1)$ is an equality algebra by Proposition 3.11. By (E10),

$(M_\Gamma, \wedge, \sim, 0, 1)$ is bounded. By Lemma 5.3, $\triangle$ is well defined. Using Lemma 5.3 and Corollary 4.9, it is easy to prove $\mathcal{M}_\Gamma = (M_\Gamma, \wedge, \sim, \triangle, 0, 1)$ is a prelinear equality$_\triangle$- algebra. □

DEFINITION 5.5. Let Γ be a theory over the propositional equality logic $\mathcal{EL}_\triangle$.

(1) A theory Γ is contradictory if for some φ, Γ proves φ and Γ proves $\neg\varphi$. Γ is consistent if it is not contradictory.

(2) Γ is complete if for every pair φ and ψ of formulas, then $\Gamma \vdash \varphi \Rightarrow \psi$ or $\Gamma \vdash \psi \Rightarrow \varphi$.

LEMMA 5.6. *Let Γ be a theory over the propositional equality logic $\mathcal{EL}_\triangle$.*

(1) *Γ is complete iff the prelinear equality$_\triangle$-algebra $\mathcal{M}_\Gamma$ is linearly ordered.*

(2) *If $\Gamma \nvdash \varphi$, then there exists a consistent complete supertheory $T \subseteq T'$ such that $T' \nvdash \varphi$.*

PROOF:

(1) It is obvious.

(2) It follows similarly with the proof of Proposition 4.17. □

THEOREM 5.7 (Strong completeness). *Let Γ be a theory over $\mathcal{EL}_\triangle$ and φ be a formula. Then the following are equivalent:*

(i) $\vdash_{\mathcal{EL}_\triangle} \varphi$,

(ii) *For each linearly ordered equality$_\triangle$-algebra $\mathcal{A}$ and each $\mathcal{A}$-model e of Γ, $e(\varphi) = 1$,*

(iii) *For each prelinear equality$_\triangle$-algebra $\mathcal{A}$ and each $\mathcal{A}$-model e of Γ, $e(\varphi) = 1$.*

PROOF:

(i) $\Rightarrow$ (ii) This is because all axioms of $\mathcal{EL}_\triangle$ are true in all $\mathcal{A}$-models of Γ, axioms of Γ are true in all models of Γ by the definition of a model and the inference rules of $\mathcal{EL}_\triangle$ are sound in the following sense:

(1) If for all prelinear equality$_\triangle$-algebra $\mathcal{A}$ and for all $\mathcal{A}$-model e of Γ, $e(\varphi) = 1$ and $e(\varphi \equiv \psi) = 1$, then for all prelinear equality$_\triangle$-algebra $\mathcal{A}$ and for all $\mathcal{A}$-model e of Γ, $e(\psi) = 1$.

(2) If for all prelinear equality$_\triangle$-algebra $\mathcal{A}$ and for all $\mathcal{A}$-model e of Γ, $e(\varphi) = 1$ and $e(\varphi \Rightarrow \psi) = 1$, then for all prelinear equality$_\triangle$-algebra $\mathcal{A}$ and for all $\mathcal{A}$-model e of Γ, $e(\psi) = 1$.

(3) If for all prelinear equality$_\triangle$-algebra $\mathcal{A}$ and for all $\mathcal{A}$-model e of Γ, $e(\varphi) = 1$, then for all prelinear equality$_\triangle$-algebra $\mathcal{A}$ and for all $\mathcal{A}$-model e of Γ, $e(\Delta(\varphi)) = 1$.

(ii) $\Rightarrow$ (i) Suppose that $\Gamma \not\vdash \varphi$. Then then there exists a consistent complete supertheory $\Gamma \subseteq \Gamma'$ such that $\Gamma' \not\vdash \varphi$ by Lemma 5.6 part (2). Since Γ' is complete, then the prelinear equality$_\triangle$- algebra $\mathcal{M}_\Gamma$ is linearly ordered. For each propositional variable p, define $e(\psi) = [\psi]_{\Gamma'}$. Then we have an $\mathcal{M}_\Gamma$-model of Γ such that $e(\varphi) < 1$, which is a contradiction.

(ii) $\Rightarrow$ (iii) follows from Proposition 4.17.

(iii) $\Rightarrow$ (ii) is obvious. $\square$

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