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Editorial Office: Department of Logic, University of Łódź
ul. Lindleya 3/5, 90–131 Łódź, Poland
e-mail: *bulletin@uni.lodz.pl*

Homepage: *http://czasopisma.uni.lodz.pl/bulletin*

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C O P E
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Formal Methods and Science in Philosophy: Introduction to the Special Issue

This special issue of the *Bulletin of the Section of Logic* contains five papers, which were originally presented at the *Formal Methods and Science in Philosophy III* conference, which was held at the Inter-University Centre, Dubrovnik, Croatia, 11–13 April 2019. The two previous editions of this meetings were held at the same location in 2015 and 2017, and as the third call for papers made clear, the Dubrovnik meeting again emphasized:

> *Problems of philosophical ontology, epistemology, philosophy of science, and philosophy of mind that are formulated or solved using formal methods (as defined in logic, mathematics, formal linguistics, theoretical computer science, information science, AI) and/or with references to the results of natural and social sciences.*

The 2019 edition drew participants from Europe and further afield, and during the three-day event, a total of 46 talks were presented. The keynote talks were given by Christoph Benzmüller (Freie Universität Berlin), María Manzano (Universidad de Salamanca), and Edward Zalta (Stanford University), and plenary session talks were given by Patrick Blackburn (Roskilde University), Elena Dragalina-Chernaya (National Research University, Moscow), Robert Piłat (Cardinal S. Wyszyński University, Warsaw), and Georg Schiemer (University of Vienna). Twenty four other submitted talks were presented across (sometimes two, sometimes three) parallel sessions, and in addition there were fifteen talks spread over three special parallel sessions for PhD students. The conference committee members were Gianfranco Basti (Pontifical Lateran University, Vatican City), Grzegorz Bugajak (Cardinal S. Wyszyński University, Warsaw), Filip Grgić (Institute of Philosophy, Zagreb), Srećko Kovač (Institute of Philosophy, Zagreb), and Kordula Świętorzecka (Cardinal S. Wyszyński University, Warsaw). The institutions coordinating the event were the Institute of Philosophy (Zagreb) and the Cardinal Stefan Wyszyński University (Warsaw).

The event was intense and lively, marked by spirited discussion: it has clearly found its niche and its voice. On the last day of the meeting, participants were offered the chance to submit a new version of their work for a further round of refereeing. We hoped, in this way, to attract submissions for a special issue that would convey something of the variety and flavour of the Dubrovnik meeting, and we believe that we have succeeded. Here you will find five papers drawing on mathematics, computer science, philosophy, and linguistics, with approaches ranging technical, historical, conceptual, or computational explanation. But as well as variety, there is coherence: the coherence provided by the core of logic. Let us briefly note what the five papers in this special issue discuss.

Víctor Aranda (Universidad Autónoma de Madrid): *Completeness, categoricity and imaginary numbers: the debate on Husserl.*

This paper explores Husserl's two notions of "definiteness", notions which had enabled him to clarify the extension of the number concept through the realm of the imaginary. However the exact meaning of these notions remains controversial. A "definite" axiom system has been interpreted as a syntactically complete theory, but also as a categorical one. Do either of these readings successfully capture Husserl's goal of elucidating the status of imaginary numbers? The author raises objections to both approaches, and then suggests an interpretation of "absolute definiteness" as semantic completeness – an approach, he argues, that does not suffice to explain Husserl's solution.

Christoph Benzmüller and David Fuenmayor (Freie Universität Berlin): *Computer-supported analysis of positive properties, ultrafilters and modal collapse in variants of Gödel's ontological argument.*

This paper reports the result of using the Isabelle/HOL proof-assistant, coupled with shallow semantic embeddings of various logical embeddings, to rigorously assess three versions of Gödel's ontological argument. Two of these versions prove the existence of a Godlike being, and avoid modal collapse, but superficially they appear very different. This computational experiments discussed in this paper, however, reveal an intriguing correspondence between the two: both link the positive properties of Gödel's argument to the mathematical notion of a principal modal ultrafilter on intensional properties.

Piotr Błaszczyk and Marlena Fila (Pedagogical University of Cracow): *Cantor on infinitesmals. Historical and modern perspective.*

This paper discusses in detail Cantor's attempt to prove that infinitesmal numbers are inconsistent. Much of the paper is historical, reaching back to Book V of Euclid's *Elements*, covering the theory of magnitudes in the late 19th century, and drawing attention to Cantor and Dedekind's mutual uncertainty as to whether their accounts of continuity for the real numbers were equivalent. The paper concludes with a counterexample to Cantor's hypothesis about products of ordinal and infinitesmal numbers that makes use of Conway numbers.

Zvonimir Šikić (University of Zagreb): *Compounding objects.*

Forming complex structures by building objects component-wise from elements of simple structures (for example, to define $\mathbb{R}^3$ from $\mathbb{R}$) is an important technique. But this compounding process may destroy desirable first-order properties (for example, when component-wise combined, the total order on $\mathbb{R}$ yields a partial order on $\mathbb{R}^3$). In this short paper, the author proves "a kind of converse" to the Łos Theorem, that characterizes the properties of component-wise defined equality in terms of filters, proper filters and ultrafilters.

Urszula Wybraniec-Skardowska (Cardinal Stefan Wyszyński University, Warsaw): *What is the sense in logic and philosophy of language?*

This paper characterizes and formalizes various notions of logical and philosophical sense. The author distinguishes between syntactic, intensional, and *extensional* sense. The approach is categorial, with functor-argument syntactic structure linked to intensional and extensional meanings of appropriate semantic categories. Three principles of compositionality are derived and, together with generalized version of Ajdukiewicz-style cancellation rules, are applied to the problem of determining the categories of first-order quantifiers.

Acknowledgements. The special issue editors would like the thank the participants and referees of the Dubrovnik 2019 meeting for providing inspiration, the authors and referees of this special issue for all their hard work, and Andrzej Indrzejczak for saying "yes" to this project in the first place.

Patrick Blackburn, Srećko Kovač, and Kordula Świętorzecka
Editors of the Special Issue

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Víctor Aranda

COMPLETENESS, CATEGORICITY AND IMAGINARY NUMBERS: THE DEBATE ON HUSSERL[1]

Abstract

Husserl's two notions of "definiteness" enabled him to clarify the problem of imaginary numbers. The exact meaning of these notions is a topic of much controversy. A "definite" axiom system has been interpreted as a *syntactically complete* theory, and also as a *categorical* one. I discuss whether and how far these readings manage to capture Husserl's goal of elucidating the problem of imaginary numbers, raising objections to both positions. Then, I suggest an interpretation of "absolute definiteness" as *semantic completeness* and argue that this notion does not suffice to explain Husserl's solution to the problem of imaginary numbers.

Keywords: Husserl, completeness, categoricity, relative and absolute definiteness, imaginary numbers.

1. Introduction

Since the publication of Hill [10] and Majer [17], much attention has been devoted to Husserl's two notions of "definiteness" (*relative* and *absolute definiteness*), which were introduced in a Double Lecture (henceforth, *Doppelvortrag*) for the Göttingen Mathematical Society in 1901. These notions enabled him "to clarify the logical sense of the computational transition through the 'imaginary"' and, in connection with that, to bring

[1]This work was supported by the Spanish Ministry of Education under the Grant FPU15/00830. I thank the anonymous referee for his/her constructive comments and suggestions.

out the sound core of Hermann Hankel's[2] renowned, but logically unsubstantiated and unclear, "principle of the permanence of formal laws" (Husserl [13], p. 97).

What Husserl describes as the "computational transition through the 'imaginary'" is the extension of the number-concept[3]. The Principle of Permanence says that the progressive extension of the number-concept should preserve (to the greatest extent possible) the arithmetical laws of the positive whole numbers. Strictly speaking, it asserts that the laws governing the newly introduced numbers have to be *consistent* with the laws constraining the old ones.

Husserl's *Doppelvortrag* was an attempt to find a justification for this Principle. A consensus has emerged that, according to Husserl, if every level of the hierarchy of numbers has a *definite* axiom system, then the extension of the number-concept can never lead to contradictions. There is, however, disagreement in the literature as to the exact meaning of the word "definite". A passionate debate has opposed those like da Silva [4] and [5], who read "definiteness" as *syntactic completeness*, and those like Hartimo [8] and [9], who favor reading it in terms of *categoricity*. Centrone [3] pointed out that Husserl himself seems to oscillate between both characterizations.

In the present paper, I discuss the plausibility of the different interpretations of "definiteness" in the literature. Is a syntactically complete axiom system compatible with the extension of the number-concept? And a categorical one? I will provide a new interpretation of "absolute definiteness"[4] (as *semantic completeness*) which is, I think, conceptually stronger. I will also maintain that "definiteness" does not suffice to explain Husserl's justification of the transition through the imaginary: the hierarchy of numbers *must contain a copy* of the previous levels.

[2] Although the Principle of Permanence is discussed in Hankel [7], it was formulated by Peacock.

[3] "Here I of course take the term 'imaginary' in the widest possible sense, according to which also the negative, indeed even the fraction, the irrational number, and so forth, can be regarded as imaginary" (Husserl [14], p. 412).

[4] In my opinion, a *relatively definite theory* is not semantically complete. However, this would require a separate paper.

2. State of the art

In the *Doppelvortrag*, Husserl's notion of *definiteness* was introduced in a twofold manner and served a double purpose. First of all, it shows the perfect delimitation of a "domain" (or "sphere of existence")[5] by its axiom system:

> The further question: Would such a system be definite? It would be definite if, for the demarcated sphere of existence, for the given individuals, and for the individuals not given, no further new axiom were possible (Husserl [14], p. 424).

Secondly, it guarantees that every meaningful proposition of the language of the system *is decided* from the axioms:

> A formal axiom system which contains no extra-essential closure axiom is said to be a definite one if each proposition that has a sense at all through the axiom system *eo ipso* falls under the axiom system, be it as consequence or be it as contradiction (Husserl [14], p. 431).

Almost thirty years after the *Doppelvortrag*, this duality between the "full description of the domain" and "maximality of the axiom system" remains invariable in Husserl's definition of his central notion. He [13] explicitly asserted that, if a domain is wholly captured by an axiom system (in modern terms, *if a theory axiomatizes a structure*), then every proposition constructed in the system has to be either a consequence of the axioms or an "analytic contradiction" (see p. 96).

Husserl also split the notion of definiteness in *relative* (the axiom system for "the whole and the fractional numbers") and *absolute definiteness*[6] (the axiom system for the "continuous number sequence" i.e., for the reals) in the context of the transition through the imaginary. The exact meaning of these notions, as well as their role in the extension of the number-concept, are matters of much controversy.

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[5] Husserl speaks of "domain" (or "sphere of existence") of a group of axioms in the sense that a system of objects satisfies certain general laws. I will use the term "domain" to refer to such a system of objects, because using the term "structure" seems quite anachronistic (see Hodges [11] and Husserl [14], pp. 437–38).

[6] "Therefore, absolutely definite = complete, in *Hilbert*'s sense" (Husserl [14], p. 127).

2.1. Centrone

Centrone [3] maintained an interpretation of "relative definiteness" as syntactic completeness and "absolute definiteness" as categoricity. Regarding the extension of the number-concept, she makes the following claim:

> The thesis that Husserl proposes in the *Doppelvortrag* is a conditional claim: if T is consistent and syntactically complete (definite) then every consistent extension of T is conservative, so that the transition through the imaginary is justified (Centrone [3], p. 178).

Syntactic completeness is a very unusual property of a set of sentences[7], because the set is so strong that, for every sentence φ of its language, either φ or $\neg\varphi$ has to be provable from the set. It follows that, if a sentence ψ formulated in the language of a complete set T is not provable from T, then $T \cup \{\psi\}$ will be inconsistent. This property is often known as the *maximality* of a consistent set.

But Centrone's solution does not function. Suppose for the sake of argument that T is a complete axiom system for the naturals. The sentence $\varphi :=$ "there exists an x such that when added to 1 gives 0" is not provable from T. Since T is complete, $\neg\varphi$ has to be provable from T. Let T' be an axiom system for the integers which is an extension of T. It is easy to see that φ is a theorem of T', which means that the extension T' of T is inconsistent. Thus, "definite" cannot be syntactically complete.

Furthermore, an extension T' of a theory T is *conservative* if T' is just a theory containing T. More precisely, every sentence of the language of T which is provable from T' is also a theorem of T. Is the extension of the number-concept a conservative extension?

Let T and T' be the axiom systems for the fields of real and complex numbers, respectively. While the reals can be ordered, there is no total ordering of the complexes that is compatible with the field operations. The sentence $\psi :=$ "there exists an x such that $x < 0$ and $-x < 0$" is provable from T' if we suppose that the complexes can be totally ordered. If Centrone were right, then ψ would be also provable from T, which contradicts the axioms of a total order. Consequently, the extension T' of T is not conservative.

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[7] A set of sentences is a *theory* (see Hodges [12], p. 33).

Contrary to Centrone's interpretation, Husserl did not believe that the extension of the number-concept had to be conservative. In the double lecture, he argued that the "expansion of the numbers series" leads to a new domain in which new relations and elements may be defined:

> The series of the positive whole numbers is a part of the series of numbers that is infinite at both ends. This in turn is part of the two-fold manifold of the complex numbers. The system of the positive whole numbers is defined by certain elementary relations. In these latter nothing is modified through expansion of the number series [...] In the new domain new relations as well as new elements may be defined. In the new domain there then will be such conceivable relations as include the old elements and old relations (Husserl [14], p. 457).

Husserl explicitly stated that a domain of numbers cannot be extended in a way that the same axiom system describes the broader domain (see [14], p. 427). If the same axiom system holds for both domains, then the narrower domain will not be extended at all. New propositions must be true in the broader domain (and hence the extension from T to T' cannot be conservative).

2.2. Da Silva

Da Silva [4] and [5] read "relative definiteness" as syntactic completeness *relative to a particular set of expressions* and "absolute definiteness" as syntactic completeness. The former is the central notion for understanding Husserl's solution to the problem of imaginary numbers:

> Husserl's solution for the problem of imaginary elements has, I believe, the following form: given systems A and B such that A and B are consistent and B extends A, let $\mathcal{D}$ be the formal manifold determined by A [...] and suppose that A is complete *relative* to the assertions of $L_{\mathcal{D}}(A)$, i.e., the assertions of $L(A)$ with all variables restricted to $\mathcal{D}$. Now, if any of these assertions (i.e., assertions of $L_{\mathcal{D}}(A)$) is proved by B, it can also be accepted from the perspective of A (Da Silva [4], p. 423).

A theory is (syntactically) complete relative to a particular set of expressions Δ of its language if, for every sentence $\varphi \in \Delta$, either φ or $\neg\varphi$ has to be a theorem of the theory. Therefore, the set of expressions Δ is the collection of all statements that the theory can either prove or disprove, and it is called its *apophantic domain*[8]. This domain is obtained by restricting quantification to the domain of $\mathcal{D}$, so the sentences of Δ refer exclusively to the narrower domain (i.e., they do not contain terms denoting imaginary numbers).

The restriction of syntactic completeness to a particular set of sentences intends to avoid the difficulties of Centrone's approach. The sentence $\varphi :=$ "there exists an x such that when added to 1 gives 0" is now undecidable starting from the axioms of the natural numbers, because it refers to a number which belongs to the integers. Theorems of T are preserved in theories that extend T *provided that* they are about the narrower domain. For this reason, the provability of φ by means of the axioms of the integers does not imply a contradiction anymore. Does da Silva's restriction explain the transition through the imaginary?

Let T and T' be the axiom systems for the rationals and the reals, respectively. The sentence $\theta :=$ "$\sqrt{2}$ is an irrational number" does not belong to the apophantic domain of T, as it refers to a number which is imaginary from the point of view of T. If da Silva were right, then θ would be undecidable starting from the axioms of the rationals. However, the proof that shows the irrationality of $\sqrt{2}$ can be achieved by means of T and the rational root theorem. Hence, da Silva's restriction of syntactic completeness to a particular set of sentences does not account for the extension of the number-concept.

In the *Doppelvortrag*, Husserl claimed that the truth-value of an expression[9] that alludes to a broader domain is decided *on the basis* of the axioms for the narrower, for the reason that it is *false* in the old domain.

> Let us consider, for example, the axiom system of the whole numbers, positive and negative. Then $x^2 = -a$, $x = \pm\sqrt{-a}$ certainly has a sense. For square is defined, and $-a$, and $=$ also. But "in the domain" there exists no $\sqrt{-a}$. The equation

[8] "If an assertion belongs to the apophantic domain of a system, then it is either true on the basis of the axioms of the system, if they can prove it, or it is false on the basis of these axioms, if they can prove its negation" (Da Silva [4], p. 427).

[9] Since quantifiers had not been introduced in 1901, Husserl's "expressions" are probably just equations or operations among numbers. However, in the scholarly debate on Husserl these "expressions" are understood as "sentences" (in the modern sense). See, for instance, da Silva [4] and da Silva [5].

> is false in the domain, inasmuch as such an equation cannot hold at all in the domain (Husserl [14], pp. 438–39).

He also defended that an axiom system is definite if "it leaves open or undecided no question related to the domain and meaningful in terms of this system of axioms" ([14], pp. 438), which implies that no proposition will be undecidable from a definite set of axioms.

2.3. Hartimo

Hartimo [8] and [9] interpreted "relative" and "absolute definiteness" as categoricity. The usage of imaginary numbers in calculations is justified if both the narrower and the broader domain are fully described by a categorical set of axioms.

> Our suggestion is that Husserl's remarks in the *Doppelvortrag* are best understood if by the formal domain Husserl means something like a domain of a categorical theory [...] Each axiom system defines a unique formal domain that is included in the unique formal domain of the more extended axiom system (Hartimo [8], pp. 302–03).

A theory is *categorical* if for every pair $\mathcal{M}$ and $\mathcal{N}$ of its models there is an isomorphism between $\mathcal{M}$ and $\mathcal{N}$. In other words, a categorical theory has exactly only one model. It still remains to be explained how categoricity relates to justifying the extension of the number-concept. Hartimo [8] suggested that, according to Husserl, categoricity implies some kind of "maximality" which guarantees that the transition through the imaginary can never lead to contradictions. She also [9] argued that this maximality corresponds to syntactic completeness.

In favor of Hartimo's reading, it has to be said that the axiomatically constructed second-order arithmetic of natural numbers is categorical. But it is also incomplete by Gödel's theorems. Hartimo alleged that Husserl's view of "definiteness" combines *expressive* power (categoricity) and *deductive* power (syntactic completeness). Both ideals combined, which are not simultaneously attainable in the interesting cases, were called "monomathematics" by Tennant [23].

From these ideals, we can draw some important conclusions regarding the problem of imaginary numbers. If a definite axiom system is categorical *and* complete, then Hartimo's proposal is open to the same objections as

Centrone's. If it is categorical *and* complete relative to a particular set of expressions, then Hartimo is forced to address the objections against da Silva.

In short, it seems that the interpretation of "definiteness" as syntactic completeness (or implying syntactic completeness) does not make plausible Husserl's idea of how the number-concept should be extended.

3. Semantic completeness

In a lecture probably delivered in 1939, which Tarski never published and entitled "On the Completeness and Categoricity of Deductive Systems"[10], he introduced the notion of "semantic completeness". After remarking that every theory affected by Gödel's first incompleteness theorem is essentially incomplete (i.e., it always contains undecidable propositions), Tarski aimed to present semantic analogues of syntactic completeness (he called "absolute completeness" to syntactic completeness):

> On the basis of the foregoing we see that absolute completeness occurs rather as an exception in the domain of the deductive sciences, and by no means can it be treated as a universal methodological demand. In this connection, I want to call your attention to certain concepts very closely related to the concept of absolute completeness, which are the result of a weakening of this concept and whose occurrence is not such an exceptional phenomenon. (Tarski, [22], p. 488).

Tarski believed that the notion of *provability* developed in modern logic was not the formal counterpart of the intuitive concept of consequence[11]. For this reason, the notion of semantic completeness is obtained by replacing "provability" with "logical consequence" in the definition of syntactic completeness. A (consistent) theory is *semantically complete* if, for every sentence φ of its language, φ or $\neg\varphi$ is a logical consequence of the axioms. Awodey and Reck [1] stated the following four equivalent conditions for semantic completeness:

1. For all sentences φ and all models $\mathcal{M}$ and $\mathcal{N}$ of T, if $\models_{\mathcal{M}} \varphi$ then $\models_{\mathcal{N}} \varphi$.

[10] It is published in Mancosu [18].

[11] See Tarski [21], p. 409 and Tarski [22], p. 489.

2. For all sentences φ, either $T \models \varphi$ or $T \models \neg\varphi$.
3. For all sentences φ, either $T \models \varphi$ or $T \cup \{\varphi\}$ is not satisfiable.
4. There is no sentence φ such that both $T \cup \{\varphi\}$ and $T \cup \{\neg\varphi\}$ are satisfiable (p. 3).

As Centrone [3] rightly noticed, one cannot seriously defend that Husserl already distinguished between the syntactic notion of provability and the semantic concept of logical consequence. However, the interpretation of "definiteness" as semantic completeness, instead of syntactic completeness, certainly makes more plausible Husserl's attempts to link the full description of a domain with the maximality of its axiom system.

To begin with, it is clear that a semantically complete theory is maximal in some general sense. Consider, for instance, the fourth condition above. It corresponds to Carnap's notion of "non-forkability"[12], which was identified by Fraenkel [6] and states that there is no sentence φ (of the language of T) such that $T \cup \{\varphi\}$ and $T \cup \{\neg\varphi\}$ have a model. In other words, a theory is *non-forkable* if it does not branch out to other sets of sentences containing both φ and $\neg\varphi$. A proof of the implication from semantic completeness to relative completeness, which Tarski considered equivalent to non-forkability, is given in Mancosu [18] (see pp. 457–58).

The implication from categoricity to semantic completeness also holds[13]. We saw that reading (absolute and relative) definiteness as categoricity implying syntactic completeness weakened Husserl's position, because categoricity and syntactic completeness are not both simultaneously attainable for the interesting cases. In contrast, Tarski [22] showed that every categorical theory – semantically categorical, in Tarski's terminology – is semantically complete.

Finally, a few words about the transition through the imaginary. If according to Husserl the axiom systems for the naturals, integers, rationals, reals, and complexes must be *definite*, then these systems must be semantically complete. Otherwise, my interpretation will be flawed. Fortunately:

> We know many systems of sentences that are categorical; we know, for instance, categorical systems of axioms for the arithmetic of natural, integral, rational, real, and complex numbers,

[12] See Carnap [2], pp. 130–33.

[13] See Carnap [2], p. 138, and Lindenbaum and Tarski [15], pp. 390–92.

> for the metric, affine, projective geometry of any number of dimensions etc [...] From theorems I and II, we see that all mentioned systems are at the same time semantically or relatively complete. Thus, in opposition to absolute completeness, relative or semantical completeness occurs as a common phenomenon (Tarski [22], p. 492).

For instance, from the categoricity of second-order Peano arithmetic we conclude that this theory is semantically complete (see Manzano [19], p. 128). One could be tempted to infer that the extension of the number-concept will be justified if every logical consequence of T (the axiom system for the narrower domain) is likewise a logical consequence of T' (the axiom system for the broader one), but this is clearly not true. Instead, I will argue that such an extension is permitted iff every sentence which is true in the narrower domain $\mathcal{M}$ is also true in the copy of $\mathcal{M}$ contained in the broader domain $\mathcal{N}$. In model theory, we say that there is an *embedding* of $\mathcal{M}$ in $\mathcal{N}$.

4. Not a sufficient condition

The debate on Husserl's notions of definiteness presupposes that a *definite* axiom system is a sufficient condition for the transition through the imaginary. But there is another necessary condition that has not been emphasized as deserved in the literature. Let me quote the entire relevant passage.

> According to this the following general law seems to result: A transition through the imaginary is permitted 1) if the imaginary can be formally defined in a consistent and comprehensive system of deduction, and 2) if the original domain of deduction when formalized has the property that every proposition falling within that domain is either true on the basis of the axioms of that domain or else is false on the same basis (i.e., is contradictory to the axioms).
> However, it is easily seen that this formulation does not suffice, although it already brings to expression the most essential part of the truth [...]
> But there is still the question whether the derived propositions of the broader domain fall in this sense within the narrower

> domain. If that is not determined in advance, we can say absolutely nothing about it (Husserl [14], pp. 428–29).

There are two points that are important here. First, Husserl highlights the role of consistency and definiteness in the extension of the number-concept. The axiom system for the original domain has to be consistent and definite. Second, he claims that both requirements do not suffice. Propositions about the narrower domain *but obtained from the axioms of the broader* are permitted if they are true propositions in the narrower domain[14]. The question is: How can such a result be established?

In the passages following the above, Husserl argues that this result can be proved if the extension of the number-concept does not induce new determinations on the old domain. For instance, the sentence $\chi :=$ "there exists an x whose square is -1", which extends the number-concept when added to the axiom system for the reals, does not define any arithmetical law of the real numbers. Husserl believed that, "if I expand an $\mathcal{M}_0$ to $\mathcal{M}$, then the $\mathcal{M}_0$ remains in $\mathcal{M}$ thus as structure still an $\mathcal{M}_0$. It is not thereby modified in species" (Husserl [14], p. 456).

Notice that for Husserl, the broader domain must contain a copy of the narrower one. The textual evidence for this is given in the first appendix of his *Doppelvortrag*:

> $\mathcal{M}_E$ is to be an expansion of $\mathcal{M}_0$. Thus $\mathcal{M}_E$ consists of the elements of $\mathcal{M}_0$ plus other elements. But that does not suffice. The $\mathcal{M}_0$ must be a part of $\mathcal{M}_E$. $\mathcal{M}_E$ has a part that falls under the concept $\mathcal{M}_0$. But that too is not sufficient. The expansion to $\mathcal{M}_E$ must not disturb $\mathcal{M}_0$ as that which it is, and above all must not specialize it (Husserl [14], p. 454).
>
> If a manifold is given to me as an $\mathcal{M}_0$, then $\mathcal{M}$ is an expansion of $\mathcal{M}_0$ if $\mathcal{M}_0$ undergoes no further "specialization" within $\mathcal{M}$ (Husserl [14], p. 456).

Furthermore, in the *Doppelvortrag* he stated that every domain of numbers of a lower level is completely contained in the higher levels. When a domain is contained in another one, Husserl explicitly speaks about "expansion" of the narrower domain or "contraction" of the broader one (see

[14] "The inference from the imaginary is permitted in the singular case or for a class, if we can know in advance and can see that for this case or for this class the inference is decided by the narrower system" (Husserl [14], p. 437).

[14], p. 421). If every object a of a domain $\mathcal{M}$ must occur in $\mathcal{N}$, and if every operation f defined on $\mathcal{M}$ must be defined on $\mathcal{N}$, then, Husserl says, $\mathcal{M}$ is contained in $\mathcal{N}$. The inclusion of the narrower domain *as a part* of the broader one "is the presupposition for the possibility of the transition through the Imaginary" (Husserl [14], p. 451).

This "presupposition" is also coherent with the construction of numbers. As it is well-known, the hierarchy of numbers is formally expressed as $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$. However, these inclusions are an abuse of notation because the set of integers is a quotient set of $\mathbb{N} \times \mathbb{N}$; the set of rationals is a quotient set of $\mathbb{Z} \times \mathbb{Z}^*$; the set of real numbers is the set of *all* the Dedekind cuts; and the set of complex numbers is the set $\mathbb{R} \times \mathbb{R}$. Let me briefly explain why the construction of numbers *speaks in favor of Husserl's presupposition.*

The inclusion $\mathbb{N} \subset \mathbb{Z}$ really expresses the identification of $\mathbb{N}$ with $\mathbb{Z}_+$, which means that there is an isomorphism between $\mathbb{N}$ and the subset $\mathbb{Z}_+$ of $\mathbb{Z}$. Therefore, we can put into one-one correspondence every number n of the naturals with every number $[(n, 0)]$ of $\mathbb{Z}_+$. Likewise, $\mathbb{Q}$ contains an ordered ring *isomorphic* to the ordered ring of the integers, and so on. Every level of the hierarchy of numbers contains a copy of the previous levels, which is mathematically indistinguishable from them. Hence, the extension of the number-concept does not introduce new determinations on the narrower domains, just as Husserl required.

5. Isomorphism and elementary equivalence

At the beginning of the *Doppelvortrag*, Husserl faces the problem of calculating with those numbers which are "absurd" or "imaginary" from the point of view of the original domain[15]. The main challenge, related to the Principle of Permanence, was introduced next.

> How is it to be explained that one can operate with the absurd according to rules, and that, if the absurd is then eliminated from the propositions, the propositions obtained are correct? (Husserl [14], p. 433).

[15] "Imaginary objects = objects which do not occur in A, are not defined there, are not established by means of the axioms and existential definitions of A, so that, therefore, if we regard A as the axiom system of a domain which has no other axioms – and thus also no other objects – those objects are in fact 'impossible"' (Husserl [14], p. 433).

Before looking at this passage in detail, I want to call your attention to Husserl's concept of "proposition'." He argued that the equation "$7 + 5 = 12$" *is a proposition*, which is correct iff its truth necessarily follows from the definitions of the numbers "7'," '5," and "12," and from the definition of addition (see Husserl [14], p. 194). If we extend the number-concept to solve the equation "$7 + 5 + x = 0$'," then our domain of numbers must include the number "-12,", which is "absurd" from the point of view of the naturals. But we still can single out propositions about the old domain (i.e., equations without imaginary numbers). The question is: Why correct propositions about natural numbers, such as "$7+5 = 12$", are still correct if we *restrict* a broader domain of numbers to the copy of the naturals contained in such a domain?

Consider, for instance, the truth of the proposition "$7+5 = 12$" in the domain of the positive integers. This proposition is true in both the natural numbers and the positive integers because the result of adding "$7_{\mathbb{Z}}$" (the equivalence class representing the number "7") to "$5_{\mathbb{Z}}$" (the equivalence class representing the number "5") is "$12_{\mathbb{Z}}$" (the equivalence class representing the number "12"). Let h be the isomorphism between $\mathbb{N}$ and $\mathbb{Z}_{+}$. More formally, we would say that "$7 + 5 = 12$" is true in the positive integers, for the reason that $h(7) + h(5) = h(7 + 5)$.

We only need to generalize these reflections on the preservation of truth to arrive at the solution of the problem quoted above. True propositions of a certain domain must also be true propositions of every isomorphic domain. In contemporary model theory, the isomorphism theorem[16] states that, if there is an isomorphism between $\mathcal{M}$ and $\mathcal{N}$, then every formula φ satisfied by $\mathcal{M}$ will be satisfied by $\mathcal{N}$. Thus, every n-tuple $a_1, ..., a_n$ of $\mathcal{M}$ satisfies φ if $h(a_1, ..., a_n)$ satisfies φ. It also establishes that, if a term t denotes an individual a in $\mathcal{M}$, then its denotation in $\mathcal{N}$ will be $h(a)$, where h is the isomorphism from $\mathcal{M}$ to $\mathcal{N}$.

Although many commentators have read Husserl's *Doppelvortrag* through the glasses of modern logic (see, for instance, da Silva [5], p. 1928), he never proved an isomorphism theorem. Since the oldest theorem of model theory is probably due to Löwenheim [16], it would be anachronistic to look for such a proof in the *Doppelvortrag*. However, Husserl felt that this kind of result could be actually achieved.

[16] See Manzano [20], pp. 68–69.

> The utilization of a broader system in order to bring forth propositions of the narrower one can only be permitted *if we possess some characterizing mark* by which we recognize that every proposition that has a sense in the narrower domain also is decided in the broader one, thus must be its consequence or its contradictory (Husserl [14], p. 437; my emphasis).

I claim that this "characterizing mark" is the fact that the isomorphism between $\mathcal{M}$ and $\mathcal{N}$ implies elementary equivalence. For instance, the ordered ring $\mathbb{Q}$ contains an ordered ring isomorphic (and hence elementarily equivalent) to $\mathbb{Z}$. It follows that every sentence that is true in the narrower ring *is also true in its copy* contained in the broader ring. "The laws of the expanded domain include those of the narrower one, but in such a way, however, that for the old domain no new laws are established" (Husserl [14], p. 457).

Let me conclude by pointing out the main difference between the other readings of Husserl's *Doppelvortrag* and my own approach. Whereas the justification of the transition through the imaginary has usually been associated with the preservation of the theorems of a (syntactically) complete theory, I have argued that it is better understood as the preservation of the true sentences of certain isomorphic domains ($\mathbb{N}$ and $\mathbb{Z}_+$, and so on).

6. Conclusions

This paper began with a discussion of the recent contributions to the debate on Husserl's two notions of "definiteness". We saw that the interpretation of (relative) definiteness as syntactic completeness seems unsatisfactory, because it presupposes that every extension from T (the axiom system for the narrower domain) to T' (the axiom system for the broader) must be *conservative*. Furthermore, if T is complete, then *proper extensions* of T will be inconsistent. On the other hand, the interpretation of (relative) definiteness as syntactic completeness is *relative to a set of sentences* flaws, for the reason that certain provable sentences (from the axioms of the old domain) are considered to be *undecidable*. Finally, the reading of definiteness as categoricity implying syntactic completeness (due to a pre-gödelian predicament which is called "monomathematics") is open to the same conceptual difficulties.

I claimed that the interpretation of absolute definiteness as *semantic completeness* makes Husserl's position more plausible. There are categorical axiom systems for the natural numbers, the integers, and so on, which are also semantically complete, as categoricity implies semantic completeness. Semantic completeness is not such an uncommon phenomenon. However, this implication does not suffice to explain Husserl's justification of the "transition through the imaginary". He remarks that the extension of the number-concept must not induce any new determinations on the narrower domains. This necessary condition has not been fairly emphasized in the literature.

I offered textual evidence in favor of understanding this requirement as the fact that the highest domains of the hierarchy of numbers contain a copy of the previous levels. For instance, the set of the integers includes a subset that is mathematically indistinguishable from the natural numbers ($\mathbb{N} \cong \mathbb{Z}_+$). There is also an isomorphism from the integers to a certain subset of the rationals, and so on. Every true sentence of $\mathbb{N}$ is a true sentence of $\mathbb{Z}_+$ by the isomorphism theorem, which explains that true formulas about the naturals are preserved *if we restrict the integers* to the positive ones. Husserl never proved such a result, but the fact that isomorphism implies elementary equivalence enabled me to explain his solution (definiteness + a hierarchy of numbers containing the lowest levels) to the problem of imaginary numbers.

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Universidad Autónoma de Madrid
Deptartamento de Lingüística, Lenguas Modernas,
Lógica y Filosofía de la Ciencia
Madrid, Spain
e-mail: victor.aranda@uam.es

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Christoph Benzmüller, David Fuenmayor

COMPUTER-SUPPORTED ANALYSIS OF POSITIVE PROPERTIES, ULTRAFILTERS AND MODAL COLLAPSE IN VARIANTS OF GÖDEL's ONTOLOGICAL ARGUMENT

Abstract

Three variants of Kurt Gödel's ontological argument, proposed by Dana Scott, C. Anthony Anderson and Melvin Fitting, are encoded and rigorously assessed on the computer. In contrast to Scott's version of Gödel's argument the two variants contributed by Anderson and Fitting avoid modal collapse. Although they appear quite different on a cursory reading they are in fact closely related. This has been revealed in the computer-supported formal analysis presented in this article. Key to our formal analysis is the utilization of suitably adapted notions of (modal) ultrafilters, and a careful distinction between extensions and intensions of positive properties.

Keywords: Computational metaphysics, ontological argument, higher-order modal logic, higher-order logic, automated reasoning, modal ultrafilters.

1. Introduction

The premises of the variant of the modal ontological argument [20] which was found in Kurt Gödel's "Nachlass" are inconsistent; this holds already in base modal logic **K** [11, 9]. The premises of Scott's [28] variant of Gödel's work, in contrast, are consistent [9, 11], but they imply the modal collapse, $\varphi \to \Box\varphi$, which has by many philosophers been considered an undesirable side effect; cf. Sobel [30] and the references therein.[1]

[1] The modal collapse was already noted by Sobel [29, 30]. One might conclude from it, that the premises of Gödel's argument imply that everything is determined, or alternatively, that there is no free will. Srećko Kovač [25] argues that modal collapse was eventually intended by Gödel.

In this article we formally encode and analyze, starting with Scott's variant, two prominent further emendations of Gödel's work both of which successfully avoid modal collapse. These two variants have been contributed by C. Anthony Anderson [1, 2] and Melvin Fitting [16], and on a cursory reading they appear quite different. Our formal analysis, however, shows that from a certain mathematical perspective they are in fact closely related.

Two notions are particularly important in our analysis. From set theory, resp. topology, we borrow and suitably adapt, for use in our modal logic context, the notion of ultrafilter and apply it in two different versions to the set of positive properties. From the philosophy of language we adopt the distinction between intensions and extensions of (positive) properties. Such a distinction has been suggested already by Fitting in his book *"Types, Tableaus and Gödel's God"* [16], which we take as a starting point in our formalization work.

Utilizing these notions, and extending Fitting's analysis, the modifications as introduced by Anderson and Fitting to Gödel's concept of positive properties are formally studied and compared. Our computer-supported analysis, which is carried out in the proof assistant system Isabelle/HOL [27], is technically enabled by the universal logical reasoning approach [4], which exploits shallow semantical embeddings (SSEs) of various logics of interest – such as intensional higher-order modal logics (IHOML) in the present article – in Church's simple type theory [5], aka. classical higher-order logic (HOL). This approach enables the reuse of existing, interactive and automated, theorem proving technology for HOL to mechanize also non-classical higher-order reasoning.

Some of the findings reported in this article have, at an abstract level, already been summarized in the literature before [24, 6, 17], but they have not been published in full detail yet (for example, the notions of "modal" ultrafilters, as employed in our analysis, have not been made precise in these papers). This is the contribution of this article.

In fact, we present and explain in detail the SSE of intensional higher-order modal logic (IHOML) in HOL (§3.1), the encoding of different types of modal filters and modal ultrafilters in HOL (§3.2), and finally the encoding and analysis of the three mentioned variants of Gödel's ontological argument in HOL utilizing the SSE approach (§4, §5 and §6). We start out (§2) by pointing to related prior work and by outlining the SSE approach.

2. Prior Work and the SSE Approach

The key ideas of the shallow semantical embedding (SSE) approach, as relevant for the remainder of this article, are briefly outlined. This section is intended to make the article sufficiently self-contained and to give references to related prior work. The presentation in this section is taken and adapted from a recently published related article [24, §1.1]; readers already familiar with the SSE approach may simply skip it, and those who need further details may consult further related documents [7, 4].

Earlier papers, cf. [4] and the references therein, focused on the development of SSEs. These papers show that the standard translation from propositional modal logic to first-order logic can be concisely modeled (i.e., embedded) within higher-order theorem provers, so that the modal operator $\Box$, for example, can be explicitly defined by the λ-term $\lambda\varphi.\lambda w.\forall v.(Rwv \rightarrow \varphi v)$, where R denotes the accessibility relation associated with $\Box$. Then one can construct first-order formulas involving $\Box\varphi$ and use them to represent and proof theorems. Thus, in an SSE, the target logic is internally represented using higher-order constructs in a proof assistant system such as Isabelle/HOL. The first author, in collaboration with Paulson [7], developed an SSE that captures quantified extensions of modal logic (and other non-classical logics). For example, if $\forall x.\phi x$ is shorthand in higher-order logic (HOL) for $\Pi(\lambda x.\phi x)$, then $\Box\forall x P x$ would be represented as $\Box\Pi'(\lambda x.\lambda w.Pxw)$, where Π' stands for the λ-term $\lambda\Phi.\lambda w.\Pi(\lambda x.\Phi xw)$, and the $\Box$ gets resolved as described above.

To see how these expressions can be resolved to produce the right representation, consider the following series of reductions:

$$
\begin{aligned}
\Box\forall x P x \quad &\equiv \quad \Box\Pi'(\lambda x.\lambda w.Pxw)\\
&\equiv \quad \Box((\lambda\Phi.\lambda w.\Pi(\lambda x.\Phi xw))(\lambda x.\lambda w.Pxw))\\
&\equiv \quad \Box(\lambda w.\Pi(\lambda x.(\lambda x.\lambda w.Pxw)xw))\\
&\equiv \quad \Box(\lambda w.\Pi(\lambda x.Pxw))\\
&\equiv \quad (\lambda\varphi.\lambda w.\forall v.(Rwv \rightarrow \varphi v))(\lambda w.\Pi(\lambda x.Pxw))\\
&\equiv \quad (\lambda\varphi.\lambda w.\Pi(\lambda v.Rwv \rightarrow \varphi v))(\lambda w.\Pi(\lambda x.Pxw))\\
&\equiv \quad (\lambda w.\Pi(\lambda v.Rwv \rightarrow (\lambda w.\Pi(\lambda x.Pxw))v))\\
&\equiv \quad (\lambda w.\Pi(\lambda v.Rwv \rightarrow \Pi(\lambda x.Pxv)))\\
&\equiv \quad (\lambda w.\forall v.Rwv \rightarrow \forall x.Pxv)\\
&\equiv \quad (\lambda w.\forall vx.Rwv \rightarrow Pxv)
\end{aligned}
$$

Thus, we end up with a representation of $\Box\forall x P x$ in HOL. Of course, types are assigned to each term of the HOL language. More precisely, in the

SSE presented in Fig. 1, we will assign individual terms (such as variable x above) the type `e`, and terms denoting worlds (such as variable w above) the type `i`. From such base choices, all other types in the above presentation can be inferred. While types have been omitted above, they will often be given in the remainder of this article.

The SSE technique provided a fruitful starting point for a natural encoding of Gödel's ontological argument in second-order modal logics **S5** and **KB** [9]. Initial studies investigated Gödel's and Scott's variants of the argument within the higher-order automated theorem prover (henceforth ATP) LEO-II [8]. Subsequent work deepened these assessment studies [11, 12]. Instead of using LEO-II, these studies utilized the higher-order proof assistant Isabelle/HOL, which is interactive and which also supports strong proof automation. Some experiments were also conducted with the proof assistant Coq [10]. Further work (see the references in [24, 4]) contributed a range of similar studies on variants of the modal ontological argument that have been proposed by Anderson [1], Anderson and Gettings [2], Hájek [21, 22, 23], Fitting [16], and Lowe [26]. Particularly relevant for this article is some prior formalization work by the authors that has been presented in [18, 17]. The use of ultrafilters to study the distinction between extensional and intensional positive properties in the variants of Scott, Anderson and Fitting has first been mentioned in invited keynotes presented at the AISSQ-2018 [6] and the FMSPh-2019 [3] conferences.

3. Further Preliminaries

The formal analysis in this article takes Fitting's book [16] as a starting point; see also [18, 17]. Fitting suggests to carefully distinguish between intensions and extensions of positive properties in the context of Gödel's argument, and, in order to do so within a single framework, he introduces a sufficiently expressive higher-order modal logic enhanced with means for the explicit representation of intensional terms and their extensions, which we have termed intensional higher-order modal logic (IHOML) in previous work [17]. The SSE of IHOML in HOL, that we utilize in the remainder of this article, is presented in §3.1. Notions of ultrafilters on sets of intensions, resp. extensions, of (positive) properties are then introduced in §3.2. Since we develop, explain and discuss our formal encodings directly in Isabelle/HOL [27], some familiarity with this proof assistant and its background logic HOL [5] is assumed.

3.1. Intensional Higher-Order Modal Logic in HOL

An encoding of IHOML in Isabelle/HOL utilizing the SSE approach, is presented in Fig. 1. It starts in line 3 with the declaration of two base types in HOL as mentioned before: type `i` stands for possible worlds and type `e` for entities/individuals. To keep the encoding concise some type synonyms are introduced in lines 4–7, which we explain next.

δ and σ abbreviate the types of predicates `e⇒bool` and `i⇒bool`, respectively. Terms of type δ represent (extensional) properties of individuals. Terms of type σ can be seen to represent world-lifted propositions, i.e., truth-sets in Kripke's modal relational semantics [19]. Note that the explicit transition from modal propositions to terms (truth-sets) of type σ is a key aspect in SSE approach; see the literature [4] for further details. In the remainder of this article we make use of phrases such as "world-lifted" or "σ-type" terms to emphasize this conversion in the SSE approach.

τ, which abbreviates the type `i⇒i⇒bool`, stands for the type of accessibility relations in modal relational semantics, and γ, which stands for `e`$\Rightarrow\sigma$, is the type of world-lifted, intensional properties.

In lines 8–32 in Fig. 1 the modal logic connectives are introduced. For example, in line 15 we find the definition of the world-lifted $\boldsymbol{\vee}$-connective (which is of type $\sigma\Rightarrow\sigma\Rightarrow\sigma$; type information is given here explicitly after the `::`-token for '`mor`', which is the ASCII-denominator for the infix-operator $\boldsymbol{\vee}$ as introduced in parenthesis shortly after). $\varphi_\sigma \boldsymbol{\vee} \psi_\sigma$ is defined as abbreviation for the truth-set $\lambda w_i.\varphi_\sigma w_i \vee \psi_\sigma w_i$ (i.e., $\boldsymbol{\vee}$ is associated with the lambda-term $\lambda\varphi_\sigma.\lambda\psi_\sigma.\lambda w_i.\varphi_\sigma w_i \vee \psi_\sigma w_i$). In the remainder we generally use bold-face symbols for world-lifted connectives (such as $\boldsymbol{\vee}$) in order to rigorously distinguish them from their ordinary counterparts (such as $\vee$) in the meta-logic HOL.

The world-lifted $\boldsymbol{\neg}$-connective is introduced in line 11, $\boldsymbol{\bot}$ and $\boldsymbol{\top}$ in lines 9–10, and respective further abbreviations for conjunction, implication and equivalence are given in lines 14, 16 and 17, respectively. The operators $\rightharpoondown$ and $\rightharpoondown$, introduced in lines 12 and 13, negate properties of types δ and γ, respectively; these operations occur in the premises in the works of Scott, Anderson and Fitting which govern the definition of positive properties.

As we see in Fig. 1, types can often be omitted in Isabelle/HOL due to the system's internal type inference mechanism. This feature is exploited in our formalization to some extend to improve readability. However, for all *new* abbreviations and definitions, we always explicitly declare the types

Isabelle2019/HOL - IHOML.thy

```
theory IHOML imports Main
begin
  typedecl i (*Possible worlds*)  typedecl e (*Individuals*)
  type_synonym δ = "e⇒bool" (*Type of ordinary predicates*)
  type_synonym σ = "i⇒bool" (*Type of world-lifted propositions, truth-sets*)
  type_synonym τ = "i⇒i⇒bool" (*Type of accessibility relations*)
  type_synonym γ = "e⇒σ" (*Type of lifted predicates*)
(**Logical operators lifted to truth-sets**)
  abbreviation mbot::"σ" ("⊥") where "⊥ ≡ λw. False"
  abbreviation mtop::"σ" ("⊤") where "⊤ ≡ λw. True"
  abbreviation mnot::"σ⇒σ" ("¬_"[52]53) where "¬φ ≡ λw. ¬(φ w)"
  abbreviation negpred::"δ⇒δ" ("⇁_"[52]53) where "⇁Φ ≡ λx. ¬(Φ x)"
  abbreviation mnegpred::"γ⇒γ" ("⇁_"[52]53) where "⇁Φ ≡ λx.λw. ¬(Φ x w)"
  abbreviation mand::"σ⇒σ⇒σ" (infixr"∧"51) where "φ∧ψ ≡ λw. (φ w)∧(ψ w)"
  abbreviation mor::"σ⇒σ⇒σ" (infixr"∨"50) where "φ∨ψ ≡ λw. (φ w)∨(ψ w)"
  abbreviation mimp::"σ⇒σ⇒σ" (infixr"→"49) where "φ→ψ ≡ λw. (φ w)⟶(ψ w)"
  abbreviation mequ::"σ⇒σ⇒σ" (infixr"↔"48) where "φ↔ψ ≡ λw. (φ w)⟷(ψ w)"
(**Polymorphic possibilist quantification**)
  abbreviation mforall::"('a⇒σ)⇒σ" ("∀") where "∀Φ ≡ λw.∀x. (Φ x w)"
  abbreviation mforallB (binder"∀"[8]9) where "∀x. φ(x) ≡ ∀φ"
  abbreviation mexists::"('a⇒σ)⇒σ" ("∃") where "∃Φ ≡ λw.∃x. (Φ x w)"
  abbreviation mexistsB (binder"∃"[8]9) where "∃x. φ(x) ≡ ∃φ"
(**Actualist quantification for individuals**)
  consts Exists::γ ("existsAt")
  abbreviation mforallAct::"γ⇒σ" ("∀ᴱ") where "∀ᴱΦ ≡ λw.∀x. (existsAt x w)⟶(Φ x w)"
  abbreviation mforallActB (binder"∀ᴱ"[8]9) where "∀ᴱx. φ(x) ≡ ∀ᴱφ"
  abbreviation mexistsAct::"γ⇒σ" ("∃ᴱ") where "∃ᴱΦ ≡ λw.∃x. (existsAt x w) ∧ (Φ x w)"
  abbreviation mexistsActB (binder"∃ᴱ"[8]9) where "∃ᴱx. φ(x) ≡ ∃ᴱφ"
(**Modal operators**)
  consts aRel::τ (infixr "r" 70)  (*Accessibility Relation r*)
  abbreviation mbox::"σ⇒σ" ("□_"[52]53) where "□φ ≡ λw.∀v. (w r v)⟶(φ v)"
  abbreviation mdia::"σ⇒σ" ("◇_"[52]53) where "◇φ ≡ λw.∃v. (w r v)∧(φ v)"
(**Meta-logical predicates**)
  abbreviation globalvalid::"σ⇒bool" ("⌊_⌋"[8]) where "⌊ψ⌋ ≡  ∀w.(ψ w)"
(**Definition of rigidly intensionalised predicates**)
  (*⦇φ⦈ converts an extensional object φ into `rigid' intensional one*)
  abbreviation trivialConversion::"bool⇒σ" ("⦇_⦈") where "⦇φ⦈ ≡ (λw. φ)"
  (*Q↓φ: the extension of a (possibly) non-rigid predicate φ is turned into a rigid intensional
   one and Q is applied to the latter; ↓φ is read as "the rigidly intensionalised predicate φ"*)
  abbreviation mextPredArg::"(γ⇒σ)⇒γ⇒σ" (infix"↓"60) where "φ↓P ≡ λw. φ (λx. ⦇P x w⦈) w"
  lemma "∀φ P. φ P = φ↓P" nitpick[user_axioms] oops (*Countermodel: notions are not the same*)
(**Some further definitions required for Fitting's variant**)
  (*φ's argument is a relativized term of extensional type derived from an intensional predicate*)
  abbreviation extPredArg::"(δ⇒σ)=>γ=>σ" (infix"↓"60) where "φ↓P ≡ λw. φ (λx. P x w) w"
  (*Another variant where φ has two arguments (the first one being relativized)*)
  abbreviation extPredArg1::"(δ⇒γ)=>γ=>γ" (infix"↓₁"60) where "φ↓₁P ≡ λz. λw. φ (λx. P x w) z w"
(**Consistency**)
  lemma True nitpick[satisfy] oops  (*Model found by Nitpick*)
end
```

Fig. 1. Shallow semantical embedding of IHOML in HOL.

of the freshly introduced symbols; this not only supports a better intuitive understanding of these notions but also reduces the number of polymorphic terms in the formalization (since polymorphism may generally cause decreased proof automation performance).

The world-lifted modal $\Box$-operator and the polymorphic, world-lifted universal quantifier $\forall$, as already discussed in §2, are introduced in lines 31 and 19, respectively (the `'a` in the type declaration for $\forall$ represents a type variable). In line 20, user-friendly binder-notation for $\forall$ is additionally defined. In addition to the (polymorphic) possibilist quantifiers, $\forall$ and $\exists$, defined this way in lines 19–22, further actualist quantifiers, $\forall^E$ and $\exists^E$, are introduced in lines 24–28; their definition is guarded by an explicit, possibly empty, `existsAt` predicate, which encodes whether an individual object actually "exists" at a particular given world, or not. These additional actualist quantifiers are declared non-polymorphic, so that they support quantification over individuals only. In the subsequent analysis of the variants of Gödel's argument, as contributed by Scott, Anderson and Fitting, we will indeed use $\forall$ and $\exists$ for different types in the type hierarchy of HOL, while keeping $\forall^E$ and $\exists^E$ for quantification over individuals only.

The notion of global validity of a world-lifted formula ψ_σ, denoted as $\lfloor\psi\rfloor$, is introduced in line 34 as an abbreviation for $\forall w_{\mathtt{i}}.\psi w$.

Note that an (intensional) base modal logic **K** is introduced in the theory `IHOML` (Fig. 1). In later sections we will switch to logics **KB** and **S5** by postulating respective conditions (symmetry, and additionally reflexivity and transitivity) on the accessibility relation `r`.

In lines 35–46 some further abbreviations are declared, which address the mediation between intensions and extensions of properties. World-lifted propositions and intensional properties are modeled as terms of types σ and γ respectively, i.e., they are technically handled in HOL as functions over worlds whose extensions are obtained by applying them to a given world w in context. The operation $(\!|\varphi|\!)$ in line 37 is trivially converting a world-independent proposition of Boolean type into a *rigid* world-lifted proposition of type σ; the rigid world-lifted propositions obtained from this trivial conversion have identical evaluations in all worlds.

The $\downarrow$-operator in line 40, which is of type $(\gamma{\Rightarrow}\sigma){\Rightarrow}\gamma{\Rightarrow}\sigma$, is slightly more involved. It evaluates its second argument, which is a property P of type γ, for a given world w, and it then *rigidly intensionalizes* the obtained extension of P in w. For technical reasons, however, $\downarrow$ is introduced as a binary operator, with its first argument being a world-lifted predicate $\varphi_{\gamma\Rightarrow\sigma}$ that is being applied to the rigidly intensionalized $\downarrow P_\gamma$; in fact, all occurrences of the $\downarrow$-operator in our subsequent sections will have this binary pattern.

The lemma statement in line 41 confirms that intensional properties P_γ are generally different from their rigidly intensionalized counterparts $\downarrow P_\gamma$: Isabelle/HOL's model finder Nitpick [14] generates a countermodel to the claim that they are (Leibniz-)equal.

A related (non-bold) binary operator $\downarrow$, of type $(\delta{\Rightarrow}\sigma){\Rightarrow}\gamma{\Rightarrow}\sigma$, is introduced in line 44. Its first argument is a predicate $\varphi_{\delta\Rightarrow\sigma}$ applicable to extensions of properties, and its second argument is an intensional property. The $\downarrow$-operator evaluates its second argument P_γ in a given world w, thereby obtaining an extension $\downarrow P_\gamma$ of type δ, and then it applies its first argument $\varphi_{\delta\Rightarrow\sigma}$ to this extension. The $\downarrow_1$-operator is analogous, but its first argument φ is now of type $\delta{\Rightarrow}\gamma$, which can be understood as world-lifted binary predicate whose first argument is of type δ and its second argument of type `e`. The $\downarrow_1$-operator evaluates the intensional argument P_γ, given to it in second position, in a given world w, and it then applies $\varphi_{\delta\Rightarrow(e\Rightarrow\sigma)}$ to the result of this operation and subsequently to its (unmodified) second argument z_e.

In line 48, consistency of the introduced concepts is confirmed by the model finder Nitpick [14]. Since only abbreviations and no axioms have been introduced so far, the consistency of the Isabelle/HOL theory `IHOML` in Fig. 3.1 is actually evident.

3.2. Filters and Ultrafilters

Two related world-lifted notions of modal filters and modal ultrafilters are defined in Fig. 2; for a general introduction to filters and ultrafilters we refer to the corresponding mathematical literature (e.g. [15]).

δ-Ultrafilters are introduced in line 26 as world-lifted characteristic functions of type $(\delta{\Rightarrow}\sigma){\Rightarrow}\sigma$. They thus denote σ-sets of σ-sets of objects of type δ. In other words, a δ-Ultrafilter is a σ-subset of the σ-powerset of δ-type property extensions.

A δ-Ultrafilter ϕ is defined as a δ-Filter satisfying an additional maximality condition: $\forall\varphi.\varphi \in^\delta \phi \vee (^{-1\delta}\varphi) \in^\delta \phi$, where $\in^\delta$ is elementhood of δ-type objects in σ-sets of δ-type objects (see line 4), and where $^{-1\delta}$ is the relative set complement operation on sets of entities (see line 14).

The notion of δ-Filter is introduced in lines 17 and 18. A δ-Filter ϕ is required to

- be large: $\mathbf{U}^\delta \in^\delta \phi$, where $\mathbf{U}^\delta$ denotes the full set of δ-type objects we start with (see line 8),

```
Isabelle2019/HOL - ModalUltrafilter.thy
1 theory ModalUltrafilter imports IHOML
2 begin
3 (**Some abbreviations for operations on δ/γ-sets in modal context**)
4   abbreviation elem_delta::"δ⇒(δ⇒σ)⇒σ" (infixr"∈^δ"99) where "x∈^δS ≡ S x"
5   abbreviation elem_gamma::"γ⇒(γ⇒σ)⇒σ" (infixr"∈^γ"99) where "x∈^γS ≡ S x"
6   abbreviation emptySet_delta::δ ("∅^δ") where "∅^δ ≡ λx. False"
7   abbreviation emptySet_gamma::γ ("∅^γ") where "∅^γ ≡ λx. ⊥"
8   abbreviation fullSet_delta::δ ("U^δ") where "U^δ ≡ λx. True"
9   abbreviation fullSet_gamma::γ ("U^γ") where "U^γ ≡ λx. ⊤"
10  abbreviation entails_delta::"δ⇒δ⇒σ" (infixr"⊆^δ"51) where "φ⊆^δψ ≡ λw. ∀x. φ x ⟶ ψ x"
11  abbreviation entails_gamma::"γ⇒γ⇒σ" (infixr"⊆^γ"51) where "φ⊆^γψ ≡ ∀x. φ x → ψ x"
12  abbreviation andPred_delta::"δ⇒δ⇒δ" (infixr"⊓^δ"51) where "φ⊓^δψ ≡ λx. φ x ∧ ψ x"
13  abbreviation andPred_gamma::"γ⇒γ⇒γ" (infixr"⊓^γ"51) where "φ⊓^γψ ≡ λx. φ x ∧ ψ x"
14  abbreviation negpred_delta::"δ⇒δ" ("^-1δ_"[52]53) where "^-1δψ ≡ λx. ¬(ψ x)"
15  abbreviation negpred_gamma::"γ⇒γ" ("^-1γ_"[52]53) where "^-1γψ ≡ λx. ¬(ψ x)"
16 (**Definition of δ/γ-Filter in modal context**)
17  abbreviation filter_delta::"(δ⇒σ)⇒σ" ("δ-Filter") where "δ-Filter Φ ≡
18     U^δ∈^δΦ ∧ ¬∅^δ∈^δΦ ∧ (∀φ ψ.(φ∈^δΦ ∧ φ⊆^δψ) → ψ∈^δΦ) ∧ (∀φ ψ.(φ∈^δΦ ∧ ψ∈^δΦ) → ((φ⊓^δψ)∈^δΦ))"
19  abbreviation filter_gamma::"(γ⇒σ)⇒σ" ("γ-Filter") where "γ-Filter Φ ≡
20     U^γ∈^γΦ ∧ ¬∅^γ∈^γΦ ∧ (∀φ ψ.(φ∈^γΦ ∧ φ⊆^γψ) → ψ∈^γΦ) ∧ (∀φ ψ.(φ∈^γΦ ∧ ψ∈^γΦ) → ((φ⊓^γψ)∈^γΦ))"
21 (**δ/γ-Filter are consistent**)
22  lemma cons_delta: "⌊∀Φ φ. δ-Filter Φ → ¬(φ∈^δΦ ∧ (^-1δφ)∈^δΦ)⌋" by fastforce
23  lemma cons_gamma: "⌊∀Φ φ. γ-Filter Φ → ¬(φ∈^γΦ ∧ (^-1γφ)∈^γΦ)⌋" by fastforce
24 (**Definition of δ/γ-Ultrafilter in modal context**)
25  abbreviation ultrafilter_delta::"(δ⇒σ)⇒σ" ("δ-Ultrafilter") where
26    "δ-Ultrafilter Φ ≡ δ-Filter Φ ∧ (∀φ. φ∈^δΦ ∨ (^-1δφ)∈^δΦ)"
27  abbreviation ultrafilter_gamma::"(γ⇒σ)⇒σ" ("γ-Ultrafilter") where
28    "γ-Ultrafilter Φ ≡ γ-Filter Φ ∧ (∀φ. φ∈^γΦ ∨ (^-1γφ)∈^γΦ)"
29 end
```

Fig. 2. Definition of δ/γ-Filters and δ/γ-Ultrafilters.

- exclude the empty set: $\emptyset^\delta \notin^\delta \phi$, where $\emptyset^\delta$ is the world-lifted empty set of δ-type objects (see line 6),
- be closed under supersets: $\forall\varphi\,\psi.(\varphi \in^\delta \phi \wedge \varphi \subseteq^\delta \psi) \rightarrow \psi \in^\delta \phi$ (the world-lifted subset relation $\subseteq^\delta$ is defined in line 10), and
- be closed under intersections: $\forall\varphi\,\psi.(\varphi \in^\delta \phi \wedge \psi \in^\delta \phi) \rightarrow (\varphi \sqcap^\delta \psi) \in^\delta \phi$ (the intersection operation $\sqcap^\delta$ is defined in line 12).

γ-Ultrafilters, which are of type $(\gamma{\Rightarrow}\sigma){\Rightarrow}\sigma$, are analogously defined as a σ-subset of the σ-powerset of γ-type property extensions.

The distinction of both notions of ultrafilters is needed in our subsequent investigation. This is because we will rigorously distinguish between positive property intensions (as used by Scott and Anderson) and positive property extensions (as utilized by Fitting).

By using polymorphic definitions, several "duplications" of abbreviations in the theory `ModalUltrafilter` (Fig. 2) could be avoided. To support a more precise understanding of δ- and γ-Ultrafilters, and their differences, however, we have decided to be very transparent and explicit regarding type information in the provided definitions.

4. Scott's Variant of Gödel's Argument

Scott's variant of Gödel's argument has been reproduced by Fitting in his book [16]. It is Fitting's formalization of Scott's variant that we have encoded and verified first in our computer-supported analysis of positive properties, ultrafilters and modal collapse. This encoding of Scott's variant is presented in Fig. 3 and its presentation is continued in Fig. 4.

Part I of the argument is reconstructed in lines 4-11 of Fig. 3 and verified with automated reasoning tools.[2] In this part we conclude from the premises and definitions (lines 5–8) that a Godlike being possibly exists (theorem T3 in line 11): $\lfloor \Diamond \exists^E \mathtt{G} \rfloor$; this follows from theorems T1 and T2 that are proved in lines 9 and 10. Note that, using binder notation, $\lfloor \Diamond \exists^E \mathtt{G} \rfloor$ can be more intuitively presented as $\lfloor \Diamond \exists^E x. \mathtt{G}x \rfloor$. The most essential definition, the definition of property G, which is of type γ and which defines a Godlike being $x_{\mathtt{e}}$ to possess all (intensional!) positive properties $\mathcal{P}$, is given in line 5. Premises that govern the notion of (intensional) positive properties $\mathcal{P}$ are A1 (which is split into A1a and A1b), A2 and A3; see lines 6–8. Scott [28] actually avoids axiom A3 and instead directly postulates T2 (the sole purpose of A3 is to support T2). Although we here explicitly include the inference from A3 to T2, it could also be left out without any implications for the rest of the proof.

Part II of the argument is presented in lines 12–20. In line 13 we switch from base modal logic **K** to logic **KB** by postulating symmetry of the accessibility relation `r`. Utilizing the same tools as before, and by exploiting theorems T3, T4 and T5, we finally prove, in line 20, the main theorem T6, which states that a Godlike being necessarily exists: $\lfloor \Box \exists^E \mathtt{G} \rfloor$, resp. $\lfloor \Box \exists^E x. \mathtt{G}x \rfloor$ using binder notation.

Consistency of the Isabelle/HOL theory `ScottVariant`, as introduced up to here, is confirmed by the model finder Nitpick [14] in line 21 (which constructs a model with one world and one Godlike entity).

---

[2] The automated reasoning tools that are integrated with Isabelle/HOL, and which we utilize in this article, include `metis`, `smt`, `simp`, `blast`, `force`, and `auto`. In fact, in each case where those occur in the presented Isabelle/HOL formalizations, we have actually first used a generic hammer-tool, called sledgehammer [13], which calls state-of-the-art ATPs to prove the statements in question fully automatically and without the need for specifying the particularly required premises; sledgehammer, in case of success, subsequently attempts to reconstruct the external proofs reported by the ATPs in Isabelle/HOL's trusted kernel by applying the mentioned automated reasoning tools.

Scott's Axioms and Definitions

(df.G)	$\mathtt{G}\,x \equiv \forall Y_{\gamma}.\mathcal{P}\,Y \rightarrow Y x$	
(A1a)	$\forall X.\mathcal{P}(\rightharpoondown X) \rightarrow \neg(\mathcal{P}\,X)$	where $\rightharpoondown$ is set/predicate negation
(A1b)	$\forall X.\neg(\mathcal{P}\,X) \rightarrow \mathcal{P}(\rightharpoondown X)$	
(A2)	$\forall XY.(\mathcal{P}\,X \wedge \Box(\forall^{E} z.Xz \rightarrow Y z)) \rightarrow \mathcal{P}\,Y$	
(A3)	$\forall ZX.((\forall Y.Z\,Y \rightarrow \mathcal{P}\,Y) \wedge \Box(\forall x.Xx \leftrightarrow (\forall Y.Z\,Y \rightarrow Y x))) \rightarrow \mathcal{P}X$	
(A4)	$\forall X.\mathcal{P}\,X \rightarrow \Box(\mathcal{P}\,X)$	
(df.$\mathcal{E}$)	$\mathcal{E}\,Y x \equiv Y x \wedge (\forall Z.Zx \rightarrow \Box(\forall^{E} z.Y z \rightarrow Zz))$	
(df.NE)	$\mathrm{NE}\;x \equiv \forall Y.\mathcal{E}\,Y x \rightarrow \Box\exists^{E}\,Y$	
(A5)	$\mathcal{P}\,\mathrm{NE}$	

```
ScottVariant1
Isabelle2019/HOL - ScottVariant.thy
1  theory ScottVariant imports IHOML ModalUltrafilter
2  begin
3  (**Positiveness**) consts positiveProperty::"γ⇒σ" ("𝒫")
4  (**Part I**)
5   (*D1*) definition G::γ ("G") where "G x ≡ ∀Y. 𝒫 Y → Y x"
6   (*A1*) axiomatization where A1a: "⌊∀X. 𝒫(⇁X) → ¬(𝒫 X)⌋" and A1b:"⌊∀X. ¬(𝒫 X) → 𝒫 (⇁X)⌋" and
7   (*A2*) A2: "⌊∀X Y. (𝒫 X ∧ □(∀ᴱz. X z → Y z)) → 𝒫 Y⌋" and
8   (*A3*) A3: "⌊∀Z X. ((∀Y. Z Y → 𝒫 Y) ∧ □(∀x. X x ↔ (∀Y. Z Y → Y x))) → 𝒫 X⌋"
9   (*T1*) theorem T1: "⌊∀X. 𝒫 X → ◇∃ᴱ X⌋" by (metis A1a A2)
10  (*T2*) theorem T2: "⌊𝒫 G⌋" by (smt A3 G_def)
11  (*T3*) theorem T3: "⌊◇∃ᴱ G⌋" using T1 T2 by simp
12 (**Part II**)
13  (*Logic KB*)  axiomatization where symm: "∀x y. x r y ⟶ y r x"
14  (*A4*) axiomatization where A4: "⌊∀X. 𝒫 X → □(𝒫 X)⌋"
15  (*D2*) definition ess::"γ⇒γ" ("ℰ") where  "ℰ Y x ≡ Y x ∧ (∀Z. Z x → □(∀ᴱz. Y z →  Z z))"
16  (*T4*) theorem T4: "⌊∀x. G x → (ℰ G x)⌋" by (metis A1b A4 G_def ess_def)
17  (*D3*) definition NE::γ ("NE") where "NE x ≡ ∀Y. ℰ Y x → □∃ᴱ Y"
18  (*A5*) axiomatization where A5: "⌊𝒫 NE⌋"
19  (*T5*) theorem T5: "⌊(◇∃ᴱ G) → □∃ᴱ G⌋" by (metis A5 G_def NE_def T4 symm)
20  (*T6*) theorem T6: "⌊□∃ᴱ G⌋" using T3 T5 by blast
21 (**Consistency**) lemma True nitpick[satisfy,user_axioms] oops  (*Model found by Nitpick*)
22 (**Modal collapse**)
23   lemma MC: "⌊∀Φ.(Φ → (□ Φ))⌋"
24     proof - {fix w fix Q
25       have "∀x. G x w ⟶ (∀Z. Z x → □(∀ᴱz. G z → Z z)) w" by (metis A1b A4 G_def)
26       hence 1: "(∃x. G x w) ⟶ ((Q → □(∀ᴱz. G z → Q)) w)" by force
27       have "∃x. G x w" using T3 T6 symm by blast
28       hence "(Q → □Q) w" using 1 T6 by blast}
29     thus ?thesis by auto qed
30 (**Analysis of positive properties using ultrafilters**)
```

Fig. 3. Scott's variant of Gödel's argument, following Fitting [16].

In lines 23–29 modal collapse is proved. This is one of the rare cases in our experiments where direct proof automation with Isabelle/HOL's integrated automated reasoning tools (incl. sledgehammer [13]) still fails. A little interactive help is needed here to show that modal collapse indeed follows from the premises in Scott's variant of Gödel's argument.

Isabelle2019/HOL - ScottVariant.thy

```
30 (**Analysis of positive properties using ultrafilters**)
31  (*unimportant*) declare [[smt_solver = cvc4, smt_oracle]]
32  (*U1*) theorem U1: "⌊γ-Ultrafilter 𝒫⌋" (*Sledgehammer succeeds, reconstruction fails*)
33          proof - have 1: "⌊U⁾∈⁾𝒫 ∧ ¬∅⁾∈⁾𝒫⌋" using A1b T1 by auto
34                  have 2: "⌊∀φ ψ.(φ∈⁾𝒫 ∧ φ⊆⁾ψ) → ψ∈⁾𝒫⌋" by (metis A1b G_def T1 T6 symm)
35                  have 3: "⌊∀φ ψ.(φ∈⁾𝒫 ∧ ψ∈⁾𝒫) → ((φ⊓⁾ψ)∈⁾𝒫)⌋" by (metis A1b G_def T1 T6 symm)
36                  have 4: "⌊∀φ. φ∈⁾𝒫 ∨ (⁻¹⁾φ)∈⁾𝒫⌋" using A1b by blast
37                  thus ?thesis by (simp add: 1 2 3 4) qed
38  (*U2*) abbreviation "𝒫' ≡ λφ.(𝒫↓φ)" (*Set of φ's whose rigidly intens. extensions are positive*)
39 theorem U2: "⌊γ-Ultrafilter 𝒫'⌋"  (*Sledgehammer succeeds, reconstruction fails*)
40          proof - have 1: "⌊U⁾∈⁾𝒫 ∧ ¬∅⁾∈⁾𝒫⌋" using A1b T1 by auto
41                  have 2: "⌊∀φ ψ.(φ∈⁾𝒫 ∧ φ⊆⁾ψ) → ψ∈⁾𝒫⌋" by (metis A1b G_def T1 T6 symm)
42                  have 3: "⌊∀φ ψ.(φ∈⁾𝒫 ∧ ψ∈⁾𝒫) → ((φ⊓⁾ψ)∈⁾𝒫)⌋" by (metis A1b G_def T1 T6 symm)
43                  have 4: "⌊∀φ. φ∈⁾𝒫 ∨ (⁻¹⁾φ)∈⁾𝒫⌋" using A1b by blast
44                  thus ?thesis by (simp add: 1 2 3 4) qed
45  (*U3*) theorem U3: "⌊∀φ. φ∈⁾𝒫' ↔ φ∈⁾𝒫⌋" by (metis A1b G_def T1 T6 symm) (*𝒫' and 𝒫 are equal*)
46 (**Modal logic S5**)
47   axiomatization where refl: "∀x. x r x" and trans: "∀x y z. x r y ∧ y r z ⟶ x r z"
48   lemma True nitpick[satisfy,user_axioms] oops (*Model found by Nitpick*)
49 (**Barcan and converse Barcan formula for individuals (type e) and properties (type e⇒i⇒bool)**)
50   lemma Bind1: "⌊(∀ᴱx::e.□(φ x)) → □(∀ᴱx::e. φ x)⌋" using MC symm by blast
51   lemma CBind1: "⌊□(∀ᴱx::e. φ x) → (∀ᴱx::e.□(φ x))⌋" using MC by blast
52   lemma Bpred1: "⌊(∀x::e⇒i⇒bool. □(φ x)) → □(∀x::e⇒i⇒bool. φ x)⌋" by simp
53   lemma CBpred1: "⌊□(∀x::e⇒i⇒bool. φ x) → (∀x::e⇒i⇒bool. □(φ x))⌋" by simp
54 end
```

Fig. 4. Ultrafilter-analysis of Scott's variant (continued from Fig. 3).

For more background information and details on the formalization of Scott's argument, and also on the arguments by Anderson and Fitting as presented in the following sections, we refer to Fitting's book [16, §11] and our previous work [17].

4.1. Positive Properties and Ultrafilters: Scott

Interesting findings regarding positive properties and ultrafilters in Scott's variant are revealed in Fig. 4.

Theorem U1, which is proved in lines 32–37, states that the set of positive properties $\mathcal{P}$ in Scott's variant constitutes a γ-Ultrafilter.

In line 38, a modified notion of positive properties $\mathcal{P}'$ is defined as the set of properties φ whose rigidly intensionalized extensions $\downarrow\varphi$ are in $\mathcal{P}$. It is then shown in theorem U2 (lines 39–44), that also $\mathcal{P}'$ constitutes a γ-Ultrafilter. And theorem U3 in line 45 shows that these two sets, $\mathcal{P}$ and $\mathcal{P}'$, are in fact equal.

In line 47 we switch from logic **KB** to logic **S5** by postulating reflexivity and transitivity of the accessibility relation `r` in addition to symmetry (line 13 in Fig. 3); and we show consistency again (line 48). In the remaining lines 49–53 in Fig. 4 we show that the Barcan and the converse Barcan formulas are valid for types `e` and γ; we use for the former type actualist quantifiers (as in the argument) and for the latter type possibilist quantifiers.

5. Anderson's Variant of Gödel's Argument

Anderson's variant of Gödel's argument is presented in Fig. 5.

A central change in comparison to Scott's variant concerns Scott's premises A1a and A1b. Anderson drops A1b and only keeps A1a: "If a property is positive, then its negation is not positive". This modification, however, has the effect that the necessary existence of a Godlike being would no longer follow (and the reasoning tools in Isabelle/HOL can confirm this; not shown here). Anderson's variant therefore introduces further emendations: it strengthens the notions of Godlikeness (in line 5) and essence (in line 14). The emended notions, referred to by $\mathtt{G}^A$ and $\mathcal{E}^A$, are as follows:

$\mathtt{G}^A$ An individual x is Godlike $\mathtt{G}^A$ if and only if all and only the necessary/essential properties of x are positive, i.e., $\mathtt{G}^A x \equiv \forall Y(\mathcal{P}\,Y \leftrightarrow \Box(Yx))$.

$\mathcal{E}^A$ A property Y is an essence $\mathcal{E}^A$ of an individual x if and only if all of x's necessary/essential properties are entailed by Y and (conversely) all properties entailed by Y are necessary/essential properties of x.

As is shown in lines 3–19, no further modifications are required to ensure that the intended theorem T6, the necessary existence of a $\mathtt{G}^A$-like being, can (again) be proved.[3]

In line 20, the model finder Nitpick confirms that modal collapse is indeed countersatisfiable in Anderson's variant of Gödel's argument. As expected, the reported countermodel consists of two worlds and one entity.

Consistency of theory `AndersonVariant` is confirmed by Nitpick in line 21, by finding a model with only one world and one entity (not shown).

[3]In a very stringent interpretation this statement is not entirely true: Theorem T2 in Scott's argument, which was derived in Fig. 3 from axiom A3 and the definition of `G`, is now directly postulated here (for simplicity reasons) and axiom A3, which had no other purpose besides supporting T2, is dropped. This simplification, however, is obviously independent from the aspects as discussed.

Anderson's Axioms and Definitions

(df.$\mathtt{G}^A$)	$\mathtt{G}^A\, x \equiv \forall Y_\gamma . \mathcal{P}\, Y \leftrightarrow \Box(Y x)$	
(A1a)	$\forall X . \mathcal{P}(\rightharpoondown X) \rightarrow \neg(\mathcal{P}\, X)$	where $\rightharpoondown$ is set/predicate negation
(A2)	$\forall X Y . (\mathcal{P}\, X \wedge \Box(\forall^E z . X z \rightarrow Y z)) \rightarrow \mathcal{P}\, Y$	
(T2)	$\mathcal{P}\, \mathtt{G}^A$	
(A4)	$\forall X . \mathcal{P}\, X \rightarrow \Box(\mathcal{P}\, X)$	
(df.$\mathcal{E}^A$)	$\mathcal{E}^A\, Y x \equiv \forall Z . \Box(Z x) \leftrightarrow \Box(\forall^E z . Y z \rightarrow Z z)$	
(df.NE^A)	$\mathrm{NE}^A\, x \equiv \forall Y . \mathcal{E}^A\, Y x \rightarrow \Box\exists^E\, Y$	
(A5)	$\mathcal{P}\, \mathrm{NE}^A$	

Isabelle2019/HOL - AndersonVariant.thy

```
theory AndersonVariant imports IHOML ModalUltrafilter
begin
(**Positiveness**) consts positiveProperty::"γ⇒σ" ("𝒫")
(**Part I**)
 (*D1'*) definition GA::γ ("G^A") where "G^A x ≡ ∀Y::γ. 𝒫 Y ↔ □(Y x)"
 (*A1a*) axiomatization where A1a:"⌊∀X. 𝒫(⇁X) → ¬(𝒫 X)⌋"
 (*A2*)  axiomatization where A2: "⌊∀X Y. (𝒫 X ∧ □(∀^Ez. X z → Y z)) → 𝒫 Y⌋"
 (*T1*)  theorem T1: "⌊∀X. 𝒫 X → ◇∃^E X⌋" using A1a A2 by metis
 (*T2*)  axiomatization where T2: "⌊𝒫 G^A⌋"  (*here we postulate T2 instead of proving it*)
 (*T3*)  theorem T3: "⌊◇∃^E G^A⌋" by (metis A1a A2 T2)
(**Part II**)
 (*Logic KB*) axiomatization where symm: "∀x y. x r y ⟶ y r x"
 (*A4*)  axiomatization where A4: "⌊∀X. 𝒫 X → □(𝒫 X)⌋"
 (*D2'*) abbreviation essA::"γ⇒γ" ("ℰ^A") where "ℰ^A Y x ≡ ∀Z. □(Z x) ↔ □(∀^Ez. Y z → Z z)"
 (*T4*)  theorem T4: "⌊∀x. G^A x → (ℰ^A G^A x)⌋" by (metis (mono_tags) A2 GA_def T2 symm)
 (*D3*)  abbreviation NEA::γ ("NE^A") where "NE^A x  ≡ ∀Y. ℰ^A Y x → □∃^E Y"
 (*A5*)  axiomatization where A5: "⌊𝒫 NE^A⌋"
 (*T5*)  theorem T5: "⌊◇∃^E G^A⌋ ⟶ ⌊□∃^E G^A⌋" by (metis A2 GA_def T2 symm)
 (*T6*)  theorem T6: "⌊□∃^E G^A⌋"  using T3 T5 by blast
(**Modal collapse**) lemma "⌊∀Φ.(Φ → (□ Φ))⌋" nitpick[user_axioms,show_all] oops(*Countermodel*)
(**Consistency**) lemma True nitpick[satisfy,user_axioms] oops  (*Model found by Nitpick*)
(**Analysis of positive properties using ultrafilters**)
 (*U1*) theorem U1: "⌊γ-Ultrafilter 𝒫⌋" nitpick[user_axioms,show_all] oops (*Countermodel*)
 (*U2*) abbreviation "𝒫' ≡ λφ.(𝒫↓φ)" (*Set of φ's whose rigidly intens. extensions are positive*)
        theorem U2: "⌊γ-Ultrafilter 𝒫'⌋"  (*Sledgehammer succeeds, reconstruction fails*)
          proof - have 1: "⌊U^γ∈^γ𝒫' ∧ ¬∅^γ∈^γ𝒫'⌋" using A2 T1 T2 by fastforce
                  have 2: "⌊∀φ ψ.(φ∈^γ𝒫' ∧ φ⊆^γψ) → ψ∈^γ𝒫'⌋" by (metis A2)
                  have 3: "⌊∀φ ψ.(φ∈^γ𝒫' ∧ ψ∈^γ𝒫') → ((φ⊓^γψ)∈^γ𝒫')⌋" by (smt GA_def T3 T5 symm)
                  have 4: "⌊∀φ. φ∈^γ𝒫' ∨ (^{-1γ}φ)∈^γ𝒫'⌋"by (smt GA_def T3 T5 symm)
                  thus ?thesis by (simp add: 1 2 3 4) qed
 (*U3*) theorem U3: "⌊∀φ. φ∈^γ𝒫' ↔ φ∈^γ𝒫⌋" nitpick[user_axioms] oops(*Counterm.: 𝒫',𝒫 not equal*)
(**Modal logic S5: consistency and modal collapse**)
  axiomatization where refl: "∀x. x r x" and trans: "∀x y z. x r y ∧ y r z ⟶ x r z"
  lemma True nitpick[satisfy,user_axioms] oops  (*Model found by Nitpick*)
  lemma MC: "⌊∀Φ.(Φ → (□ Φ))⌋" nitpick[user_axioms] oops (*Countermodel*)
(**Barcan and converse Barcan formula for individuals (type e) and properties (type e⇒i⇒bool)**)
  lemma Bind1: "⌊(∀^Ex::e.□(φ x)) → □(∀^Ex::e. φ x)⌋" nitpick[user_axioms] oops (*Countermodel*)
  lemma CBind1: "⌊□(∀^Ex::e. φ x) → (∀^Ex::e.□(φ x))⌋" nitpick[user_axioms] oops (*Countermodel*)
  lemma Bpred1: "⌊(∀x::e⇒i⇒bool. □(φ x)) → □(∀x::e⇒i⇒bool. φ x)⌋" by simp
  lemma CBpred1: "⌊□(∀x::e⇒i⇒bool. φ x) → (∀x::e⇒i⇒bool. □(φ x))⌋" by simp
end
```

Fig. 5. Anderson's variant of Gödel's argument, following Fitting [16].

5.1. Positive Properties and Ultrafilters: Anderson

Regarding positive properties and ultrafilters an interesting difference to our prior observations for Scott's version is revealed by the automated reasoning tools: the set of positive properties $\mathcal{P}$ in Anderson's variant does *not* constitute a γ-Ultrafilter; Nitpick finds a countermodel to statement U1 in line 23 that consists of two worlds and one entity. However, the modified notion $\mathcal{P}'$, i.e., the set of all properties φ, whose rigidly intensionalized extensions are in $\mathcal{P}$ (line 24), still is a γ-Ultrafilter; see theorem U2, which is proved in lines 25–30. Consequently, the sets $\mathcal{P}$ and $\mathcal{P}'$ are not generally equal anymore and Nitpick reports a countermodel for statement U3 in line 31.

In lines 32–40, we once again switch from logic **KB** to logic **S5**, we again show consistency, and we again analyze the Barcan and the converse Barcan formulas for types `e` and γ. In contrast to before, the Barcan and converse Barcan formulas for type `e`, when formulated with actualist quantifiers, are not valid anymore; Nitpick presents countermodels with two worlds and two entities.

6. Fitting's Variant of Gödel's Argument

In Fitting's variant of Gödel's Argument, see Fig. 6, the notion of positive properties $\mathcal{P}$ in the definition of Godlikeness `G` ranges over extensions of properties, i.e., over terms of type δ, and not over γ-type intensional properties as in Scott's and Anderson's variants. In Fitting's understanding, positive properties are thus fixed from world to world, while they are world-dependent in Scott's and Anderson's. In technical terms, Scott (resp. Gödel) defines $\mathtt{G}\,x$ as $\forall Y_\gamma.\mathcal{P}Y \rightarrow Yx$ (line 5 in Fig. 3), whereas Fitting modifies this into $\forall Y_\delta.\mathcal{P}Y \rightarrow (\!|Yx|\!)$ (line 5 in Fig. 6). In an analogous way, the notion of essence is emended by Fitting: in Scott's variant, see line 15 in Fig. 3, $\mathcal{E}\,Yx$ is defined as $Yx \wedge (\forall Z.Zx \rightarrow \Box(\forall^E z.Yz \rightarrow Zz))$, while it becomes $(\!|Yx|\!) \wedge (\forall Z.(\!|Zx|\!) \rightarrow \Box(\forall^E z.(\!|Yz|\!) \rightarrow (\!|Zz|\!))$ in Fitting's variant (see line 15 in Fig. 6).

The definition of necessary existence NE in line 17 is adapted accordingly, and in several other places of Fitting's variant respective emendations are required to suitably address his alternative interpretation of Gödel's notion of positive properties (see, e.g., theorem T2 in line 9 or axiom A5 in

Fitting's Axioms and Definitions

(df.G)	$\mathtt{G}\, x \equiv \forall Y_\delta . \mathcal{P}\, Y \rightarrow (\!| Y x |\!)$	
(A1a)	$\forall X . \mathcal{P}(\rightharpoondown X) \rightarrow \neg(\mathcal{P}\, X)$	where $\rightharpoondown$ is set/predicate negation
(A1b)	$\forall X . \neg(\mathcal{P}\, X) \rightarrow \mathcal{P}(\rightharpoondown X)$	
(A2)	$\forall X Y . (\mathcal{P}\, X \wedge \Box(\forall^E z . (\!| X z |\!) \rightarrow (\!| Y z |\!))) \rightarrow \mathcal{P}\, Y$	
(T2)	$\mathcal{P} {\downarrow} \mathtt{G}$	
(df.$\mathcal{E}$)	$\mathcal{E}\, Y x \equiv (\!| Y x |\!) \wedge (\forall Z . (\!| Z x |\!) \rightarrow \Box(\forall^E z . (\!| Y z |\!) \rightarrow (\!| Z z |\!))$	
(df.NE)	$\mathrm{NE}\ x \equiv \forall Y . \mathcal{E}\, Y x \rightarrow \Box(\exists^E\, z . (\!| Y z |\!))$	
(A5)	$\mathcal{P} {\downarrow} \mathrm{NE}$	

Isabelle2019/HOL - FittingVariant.thy

```
theory FittingVariant imports IHOML ModalUltrafilter
begin
(**Positiveness**) consts Positiveness::"δ⇒σ" ("𝒫")
(**Part I**)
 (*D1*) abbreviation God::γ ("G") where "G x ≡ ∀Y::δ. 𝒫 Y → ⦇Y x⦈"
 (*A1*) axiomatization where A1a:"⌊∀X. 𝒫(⇁X) → ¬(𝒫 X)⌋" and A1b:"⌊∀X. ¬(𝒫 X) → 𝒫 (⇁X)⌋"
 (*A2*) axiomatization where A2: "⌊∀X Y. (𝒫 X ∧ (□(∀ᴱz. ⦇X z⦈→⦇Y z⦈))) → 𝒫 Y⌋"
 (*T1*) theorem T1: "⌊∀X. 𝒫 X → ◇(∃ᴱz. ⦇X z⦈)⌋" using A1a A2 by blast
 (*T2*) axiomatization where T2: "⌊𝒫↓G⌋"
 (*T3*) theorem T3deRe: "⌊(λX. ◇∃ᴱ X) ↓G⌋" using T1 T2 by simp
        theorem T3deDicto: "⌊◇∃ᴱ ↓G⌋" nitpick[user_axioms] oops (*Countermodel*)
(**Part II*)
 (*Logic KB*)  axiomatization where symm: "∀x y. x r y ⟶ y r x"
 (*A4*) axiomatization where A4: "⌊∀X. 𝒫 X → □(𝒫 X)⌋"
 (*D2*) abbreviation ess::"δ⇒γ" ("ℰ") where "ℰ Y x ≡ ⦇Y x⦈ ∧ (∀Z.⦇Z x⦈→(□(∀ᴱz. ⦇Y z⦈→⦇Z z⦈)))"
 (*T4*) theorem T4: "⌊∀x. G x → ((ℰ↓₁G) x)⌋" using A1b by metis
 (*D3*) definition NE::γ ("NE") where "NE x  ≡ ∀Y.  ℰ Y x → □(∃ᴱz. ⦇Y z⦈)"
 (*A5*) axiomatization where A5: "⌊𝒫↓NE⌋"
    lemma help1: "⌊∃ ↓G → □∃ᴱ ↓G⌋"  sorry (*...longer interactive proof, omitted here...*)
    lemma help2: "⌊∃ ↓G → ((λX. □∃ᴱ X)↓G)⌋" by (metis A4 help1)
 (*T5*) theorem T5deDicto:"⌊◇∃ᴱ ↓G⌋⟶⌊□∃ᴱ ↓G⌋" using T3deRe help1 by blast
        theorem T5deRe:"⌊(λX.◇∃ᴱ X)↓G⌋ ⟶ ⌊(λX. □∃ᴱ X)↓G⌋" by (metis help2)
 (*T6*) theorem T6deDicto: "⌊□∃ᴱ ↓G⌋" using T3deRe help1 by blast
        theorem T6deRe: "⌊(λX. □∃ᴱ X)↓G⌋" using T3deRe help2 by blast
(**Modal collapse**) lemma MC: "⌊∀Φ.(Φ → (□ Φ))⌋" nitpick[user_axioms] oops (*Countermodel*)
(**Consistency**) lemma True nitpick[satisfy,user_axioms] oops  (*Model found by Nitpick*)
(**Analysis of positive properties using ultrafilters**)
 (*U1*) theorem U1: "⌊δ-Ultrafilter 𝒫⌋"  (*Sledgehammer succeeds, reconstruction fails*)
          proof - have 1: "⌊Uᵟ∈ᵟ𝒫 ∧ ¬∅ᵟ∈ᵟ𝒫⌋" using A1b T1 by auto
                  have 2: "⌊∀φ ψ.(φ∈ᵟ𝒫 ∧ φ⊆ᵟψ) → ψ∈ᵟ𝒫⌋" by (metis A1b T3deRe)
                  have 3: "⌊∀φ ψ.(φ∈ᵟ𝒫 ∧ ψ∈ᵟ𝒫) → ((φ⊓ᵟψ)∈ᵟ𝒫)⌋" by (metis A1b T3deRe)
                  have 4: "⌊∀φ. φ∈ᵟ𝒫 ∨ (⁻¹ᵟφ)∈ᵟ𝒫⌋" using A1b by blast
                  thus ?thesis by (simp add: 1 2 3 4) qed
(**Modal logic S5: consistency and modal collapse**)
  axiomatization where refl: "∀x. x r x" and trans: "∀x y z. x r y ∧ y r z ⟶ x r z"
  lemma True nitpick[satisfy,user_axioms] oops  (*Model found by Nitpick*)
  lemma MC: "⌊∀Φ.(Φ → (□ Φ))⌋" nitpick[user_axioms] oops (*Countermodel*)
(**Barcan and converse Barcan formula for individuals (type e) and properties (type e⇒i⇒bool)**)
  lemma Bind1: "⌊(∀ᴱx::e.□(φ x)) → □(∀ᴱx::e. φ x)⌋" nitpick[user_axioms] oops (*Countermodel*)
  lemma CBind1: "⌊□(∀ᴱx::e. φ x) → (∀ᴱx::e.□(φ x))⌋" nitpick[user_axioms] oops (*Countermodel*)
  lemma Bpred1: "⌊(∀x::e⇒i⇒bool. □(φ x)) → □(∀x::e⇒i⇒bool. φ x)⌋" by simp
  lemma CBpred1: "⌊□(∀x::e⇒i⇒bool. φ x) → (∀x::e⇒i⇒bool. □(φ x))⌋" by simp
end
```

Fig. 6. Fitting's variant of Gödel's argument.

line 18). Fitting's expressive logical system (IHOML) also allows us to distinguish between *de dicto* and *de re* readings of theorems T3, T5, and T6. Except for the *de dicto* reading of T3, which has a countermodel with two worlds and two entities, all of these statements are proved automatically by the reasoning tools integrated with Isabelle/HOL.

As intended by Fitting, modal collapse is not provable anymore, which can be seen in line 25, where Nitpick reports a countermodel with two worlds and one entity.

Consistency of the Isabelle/HOL theory `FittingVariant`, as introduced up to here, is confirmed by Nitpick in line 26 (one world, one entity).

6.1. Positive Properties and Ultrafilters: Fitting

The type of $\mathcal{P}$ has changed in Fitting's variant from the prior $\gamma{\Rightarrow}\sigma$ to $\delta{\Rightarrow}\sigma$. Hence, in our ultrafilter analysis, the notion of a γ-Ultrafilter no longer applies and we must consult the corresponding notion of a δ-Ultrafilter. Theorem U1, which is proved in lines 28–33 of Fig. 6, confirms that Fitting's emended notion of $\mathcal{P}$ indeed constitutes a δ-Ultrafilter.

In line 35 we again switch from modal logic **KB** to logic **S5**. Consistency of the Isabelle/HOL theory `FittingVariant` in S5 is confirmed in line 36, and countersatisfiability of modal collapse is reconfirmed in line 37.

Moreover, like for Anderson's variant before, we get a countermodel for the Barcan formula and the converse Barcan formula on type `e`, when formulated with actualist quantifiers. The Barcan formula and its converse are proved valid for type γ.

7. Conclusion

Anderson and Fitting both succeed in altering Gödel's modal ontological argument in such a way that the intended result, the necessary existence of a Godlike being, is maintained while modal collapse is avoided. And both solutions, from a cursory reading, are quite different.

We conclude by rephrasing in more precise, technical terms what has been mentioned at abstract level already in the mentioned related article [24, §2.3]:

In order to compare the argument variants by Scott, Anderson, and Fitting, two notions of ultrafilters were formalized in Isabelle/HOL: A δ-Ultrafilter, of type $(\delta{\Rightarrow}\sigma){\Rightarrow}\sigma$, is defined on the powerset of individuals, i.e., on the set of rigid properties, and a γ-Ultrafilter, which is of type $(\delta{\Rightarrow}\sigma){\Rightarrow}\sigma$, is defined on the powerset of concepts, i.e., on the set of non-rigid, world-dependent properties. In our formalizations of the variants, a careful distinction was made between the original notion of a positive property $\mathcal{P}$ that applies to (intensional) properties and a restricted notion $\mathcal{P}'$ that applies to properties whose rigidified extensions are $\mathcal{P}$-positive. Using these definitions the following results were proved computationally:

- In Scott's variant both $\mathcal{P}$ and $\mathcal{P}'$ coincide, and both are γ-Ultrafilters.
- In Anderson's variant $\mathcal{P}$ and $\mathcal{P}'$ do not coincide, and only $\mathcal{P}'$, but not $\mathcal{P}$, is a γ-Ultrafilter.
- In Fitting's variant, the $\mathcal{P}$ in the sense of Scott and Anderson is not considered an appropriate notion. However, Fitting's emended notion of a positive property $\mathcal{P}$, which applies to extensions of properties, corresponds to our definition of $\mathcal{P}'$ in Scott's and Anderson's variants; and, as was to be expected, Fitting's emended notion of $\mathcal{P}$ constitutes a δ-Ultrafilter.

The presented computational experiments thus reveal an intriguing correspondence between the variants of the ontological argument by Anderson and Fitting, which otherwise seem quite different. The variants of Anderson and Fitting require that only the restricted notion of a positive property is an ultrafilter.

The notion of positive properties in Gödel's ontological argument is thus aligned with the mathematical notion of a (principal) *modal* ultrafilter on intensional properties, and to avoid modal collapse it is sufficient to restrict the modal ultrafilter-criterion to property extensions. In a sense, the notion of Godlike being "$\mathtt{G}x$" of Gödel is thus in close correspondence to the x-object in a principal modal ultrafilter "F_x" of positive properties. This appears interesting and relevant, since metaphysical existence of a Godlike being is now linked to existence of an abstract object in a mathematical theory.

Further research could look into a formal analysis of monotheism and polytheism for the studied variants of Gödel's ontological argument. We conjecture that different notions of equality will eventually support both views, and a respective formal exploration study could take Kordula Świętorzecka's related work [31] as a starting point.

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Freie Universität Berlin
Institute of Computer Science
Berlin, Germany

University of Luxembourg
Faculty of Science, Technology, and Communication
Luxembourg
e-mail: c.benzmueller@gmail.com

Freie Universität Berlin
Institute of Computer Science
Berlin, Germany
e-mail: david.fuenmayor@fu-berlin.de

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Piotr Błaszczyk*, Marlena Fila

CANTOR ON INFINITESIMALS HISTORICAL AND MODERN PERSPECTIVE

Abstract

In his 1887's *Mitteilungen zur Lehre von Transfiniten*, Cantor seeks to prove inconsistency of infinitesimals. We provide a detailed analysis of his argument from both historical and mathematical perspective. We show that while his historical analysis are questionable, the mathematical part of the argument is false.

Keywords: Infinitesimals, infinite numbers, real numbers, hyperreals, ordinal numbers, Conway numbers.

1. Introduction

It is well-known that Cantor praised Bolzano for developing the arithmetic of *proper-infinite numbers*. The famous quotation reads:

"Bolzano is perhaps the only one for whom the proper-infinite numbers are legitimate (at any rate, he speaks about them a great deal); but I absolutely do *not* agree with the manner in which he handles them without being able to give a correct definition, and I regard, for example, §§ 29–33 of that book [*Paradoxien des Unendlichen*] as unsupported and erroneous. The author lacks two things necessary for a genuine grasp of the concept of determinate-infinite number: both the general *concept of power* and the precise *concept of Anzahl*".[1]

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[1] [**?**, p. 181] translated by W. Ewald [**?**, p. 895, note in square brackets added].

Interestingly, the specified paragraphs of *Paradoxien* develop calculus in a way that appeals to Euler's 1748 *Introductio in Analysin Infinitorum*, i.e., calculus that employs infinitely small and infinitely large numbers along with the relation *is infinitely close.* Euler's masterpiece was written in Latin, then it was translated into French by J. B. Labey in 1796, and into German by J. A. C. Michelsen in 1788–1791, another German translation by H. Maser was released in 1885. It was arguably one of the most important 18th century mathematical treaties. But there are only two significant references to [?] in Cantor's *Gesammelte Abhandlungen*: [?] and [?]. They deal with a specific issue raised by Euler in the section § 328 called *De partitione numerorum.* Any other references to Euler are also of minor importance. Furthermore, in Cantor's *Briefe* [?] the name Euler occurs twice, and, again, in very short remarks. Quite strange.

Cantor has never addressed Euler's technique of infinite numbers as employed in determining infinite sums and products. Whether knowingly or unknowingly, or just by correcting supposed errors, in [?] and [?] he interprets Euler's infinite operations within the framework of standard analysis, albeit limits do not occur in [?]. In [?] he discusses infinitesimals through distinction actual vs. potential infinity.[2] Due to the theory of limits, developed, among others, by Cauchy – Cantor argues – the mistake of ascribing them actual infinity had been fixed. The letter to Mittag-Leffler dated March 3, 1883 contains a hint that Euler's and Bolzano's infinite sums are inconsistent.[3] Notes such as these as well as concerning infinitesimals are scattered all throughout Cantor's papers, however he has never developed them into a thorough criticism. Instead, they attest Cantor's aversion to infinitesimals – an aversion based on prejudices rather than concrete arguments.

As to Euler, in [?], the number line is explicitly revealed as consisting of infinitesimals, infinitely large numbers, and *assignable quantities*, i.e., numbers representing line segments, while infinite numbers are viewed as inverses of infinitesimals. In [?], infinitesimals and infinitely large numbers are employed to expand $\sin x$ and $\cos x$ functions into series, and then to derive the famous formula $e^{ix} = \cos x + i \sin x$, to mention the most spectacular achievements. Thus, Euler's infinite numbers provide a ma-

[2] See [?, p. 410].

[3] See [?, p. 117–118]. Another reference to Euler occurs in the letter to Lipschitz dated October 18, 1885 [?, p. 247]; it regards an arithmetic problem.

chinery which enable crucial mathematical results. And yet, there are no references to this technique neither in Cantor's papers, nor in his letters. Why then, instead of referring to the mathematical treatise that explicitly develops the analysis of infinity, does Cantor prefer to discuss ancient Greek and medieval philosophers? Moreover, whereas his theory of infinity is formal, his philosophical considerations explore distinctions that he could never formalize, namely actual infinity vs. potential infinity. Instead, Euler's approach to infinity builds on the easily formalized opposition of finite vs. non-finite that turned out to be equivalent to the Archimedean vs. non-Archimedean opposition.

We believe that the following quotation is crucial when it comes to understanding Cantor's perspective:

"The fact of actual infinite numbers is thus so little ground for the existence of actual infinitely small magnitudes that, on the contrary, the impossibility of the latter can be proven with the former".[4]

In fact, there are many similar declarations scattered throughout Cantor's papers, showing that he considered infinitesimals as the most serious rival of his theory of infinite numbers.

Cantor's position is more understandable when we realize that he was absolutely certain about his characterization of infinity being the only possible characterization. Similarly, he was absolutely certain there was only one possible domain to develop the calculus.[5] This no-alternative philosophy, whether applied to infinity or to a domain of calculus, motivated Cantor's struggle with infinitesimals.

In his 1887 paper *Mitteilungen zur Lehre von Transfiniten*, Cantor goes beyond declarations and seeks to prove that infinitesimals are inconsistent. The general idea of his argument is this: Let ζ be a positive infinitesimal, which, for him, means it fulfills the condition $(\forall n \in \mathbb{N})(\zeta < \frac{1}{n})$, then for any infinite ordinal number ν the product $\zeta \cdot \nu$ is smaller than any *finite magnitude*, in symbols

$$(\forall n \in \mathbb{N})(\zeta < \tfrac{1}{n}) \Rightarrow (\forall \nu \in Ord)(\forall n \in \mathbb{N})(\zeta \cdot \nu < \tfrac{1}{n}), \qquad (1.1)$$

where Ord stands for the class of ordinal numbers. In other words, provided ζ is infinitesimal, every product $\zeta \cdot \nu$ is also infinitesimal. Hereinafter, we will also refer to the more suggestive and equivalent version of (1), namely

[4] [**?**, p. 408], translated by P. Enrlich [**?**, p. 42].

[5] See [**?**, pp. 233–236].

$$(\forall n \in \mathbb{N})(\zeta < \tfrac{1}{n}) \Rightarrow (\forall \nu \in Ord)(\zeta \cdot \nu < 1). \quad (1.2)$$

Cantor's argument looks like a *reductio ad absurdum* proof, while the supposed contradiction is to consist of the following statements: ζ is a *linear magnitude*, ζ *cannot be made finite through any actual infinite multiplication*. Arguably, to get a real contradiction, Cantor's argument requires interpretation. One has to decide (a) what is the meaning of the term *linear magnitude*, (b) what does the product $\zeta \cdot \nu$ mean.

In this paper, we will provide a detailed analysis of Cantor's argument and will argue that *linear magnitude* means real numbers. We will also present some modern interpretations of the product $\zeta \cdot \nu$ that do not entail the conclusion $(\forall \nu \in Ord)(\zeta \cdot \nu < 1)$.

2. *Linear magnitude* and Archimedean property

Provided that Cantor's *linear magnitude* means positive real numbers, his proof of the inconsistency of infinitesimals aims to show that the concept of *linear magnitude* implies the Archimedean property.

Indeed, Cantor's definition of infinitesimals is the same as the one provided in Bolzano's *Paradoxien*, §§ 10, 16; it is, in fact, the same definition as the modern one. Cantor, thus, sought to show that no infinitesimal is a real number. From the modern, axiomatic perspective, it is an obvious observation, as the completeness of the field of real numbers implies the Archimedean property, and the Archimedean property excludes infinitesimals. In fact, Cantor's reasoning in its full version involves the Archimedean property and can be paraphrased as follows: *Linear magnitude* has the Archimedean property, while infinitesimals and the Archimedean property are mutually exclusive.

However, the argument is not that simple, as Cantor adopts a specific interpretation of what we nowadays consider to be the Archimedean axiom, AA in short. Namely, he allows multiplications by any ordinal rather than any natural number. More importantly, his characteristic of real numbers differs from our modern one. In the 1872's *Über die Ausdehnung eines Satzes der Theorie der trigonometrischen Reihen*, he identified the completeness (continuity) of real numbers with a condition currently called Cauchy completeness, CC in short. While nowadays we know that AA does not follow from CC, this was not the case at the turn of the 19th and 20th century.

In 1887, neither Cantor nor Dedekind were quite sure whether their versions of continuity of real numbers were equivalent. It was partly because there was no obvious framework that would enable to establish or dismiss the equivalence of Cantor's and Dedekind's versions. Adopting a modern perspective, we can say that in [**?**] Cantor sought to characterize continuum as an ordered field $(\mathbb{R}, +, \cdot, 0, 1, <)$, in [**?**] as a subset of the metric space $\mathbb{R}^n$, and in [**?**] as a totally ordered set $(\mathbb{R}, <)$. The Dedekind cut principle does not apply to the subsets of metric space; in the context of totally ordered sets, it does not provide unique characteristics (i.e. up to isomorphism); in the framework of an ordered field, the cut principle implies CC, yet not vice versa.[6] As all these facts were not clear at the time, it is no wonder Cantor objected whether the Dedekind cut principle really reveals the "essence of continuity".[7] Basically – in our interpretation – it can-not be applied in every context he considered continuum.

In modern mathematics, the concept of an ordered field provides such a framework, yet the concept itself was introduced only in Hilbert's 1899 *Grundlagen der Geometrie*. The first widely-known proof that completeness of real numbers implies the Archimedean axiom was given in 1901 by Otto Hölder; to this end, he applied the Dedekind cut version of completeness, and the result was established for an ordered group. In 1900's *Über den Zahlbegriff*, Hilbert presents the continuity of real numbers in the form of a conjunction: AA plus *Axiom of Completeness*; the second condition could be paraphrased as follows: Real numbers are the biggest Archimedean field. In 1932's *Anschauliche Geometrie*, Hilbert characterized the continuity of real numbers as a conjunction of AA plus the condition, which he named *Cantor's axiom*, namely: If (A_n) is a descending sequence of closed line segments, then $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$.

In fact, we can not determine who was the first to show that CC does not imply AA. [**?**] proves that the field of Laurent series over real numbers is a non-Archimedean, Cauchy-complete field. [**?**] shows that Levi-Civita fields are non-Archimedean and Cauchy-complete. We can show that the field of hyperreals is yet another example of a non-Archimedean, Cauchy-complete field. Still, these are relatively recent results.

In what follows, next to Cantor's construction of real numbers and its accompanied characteristics of continuity, we will also discuss Cantor's

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[6] We develop these claims in section 7.

[7] The very phrase occurs in [**?**].

topological characterization of *linear magnitude* given in [**?**], as well as his characterization of real numbers in terms of total order given in [**?**]. None of these characteristics implies the Archimedean property. Nevertheless, suggestions that the line of real numbers $(\mathbb{R}, <)$ has the Archimedean property permeated Cantor's milieu.

Indeed, there were some earlier attempts to prove the Archimedean property in the 19th century. The first one we know of was made by Bolzano in his *Reine Zahlenlehre*; yet the paper was published posthumously in 1962.[8] In 1817's *Rein analytischer Beweis*, he applied the seemingly self-evident condition $\lim_{n\to\infty} \frac{1}{n} = 0$; nowadays we know that it is but another version of the Archimedean axiom. In 1885's *Vorlesungen über allgemeine Arithmetik*, Stolz sought to show that the Archimedean property follows from the Dedekind cut principle; his proof, however, proceeded in a geometrical framework. In the 1890's *Teoria delle grandezze*, Bottazzi proved that in the abelian group the Dedekind cut principle implies the Archimedean property.[9]

Whether correct or not, they were attempts to derive the Archimedean property from the Dedekind cut principle.[10] Since Cantor could not decide whether his account of continuum differs from that proposed by Dedekind, it is no wonder he was seeking his own proof of the Archimedean property. Moreover, he knew that in some contexts, e.g. in a theory of totally ordered sets, the Dedekind cut principle was insufficient to characterize real numbers up to isomorphism. This could be the reason behind his search for a genuine proof.

The rest of paper is as follows: in section 3 we present basic mathematical facts concerning infinitesimals, the Archimedean axiom, continuity and related issues. Then, in subsequent sections, we provide a detailed analysis of chapter VI of [**?**] which includes Cantor's inconsistency proof of infinitesimals.

[8] See [**?**, § 69] and [**?**, § 74]. There is, however, an alternative interpretation of these paragraphs to the effect that instead of proving the Archimedean property of real numbers, Bolzano sought to show that his *measurable numbers* had the Archimedean property, and to this end he assumed that real numbers had the Archimedean property. Then, Bolzano's *measurable numbers* are viewed as assignable hyperreal numbers.

[9] See [**?**, p. 80].

[10] [**?**] identifies flaws in Stolz's 1885 proof. In fact, [**?**] provides a corrected version of his 1885 proof. Still, the 1902 proof is incomplete.

3. Basic Facts

A commutative field $(\mathbb{F}, +, \cdot, 0, 1)$ together with a total order $<$ is an ordered field when the sums and products are compatible with the order, that is

$$x < y \Rightarrow x + z < y + z, \quad x < y,\, 0 < z \Rightarrow xz < yz.$$

In any ordered field we define in a usual way an absolute value, $|x|$, and a limit of sequence, $\lim_{n\to\infty} a_n$. Note, however, that while in real analysis the formula $\forall \varepsilon > 0$ stands for $\forall \varepsilon \in \mathbb{R}_+$, in an ordered field $(\mathbb{F}, +, \cdot, 0, 1, <)$ it means $\forall \varepsilon \in \mathbb{F}_+$.

The term n is defined by

$$n =_{df} \underbrace{1 + 1 + ... + 1}_{n-times},$$

while $\frac{n}{m} =_{df} n \cdot m^{-1}$. On this basis we assume that any ordered field includes natural numbers $\mathbb{N}$ and rational numbers $\mathbb{Q}$.

We define the following subsets of $\mathbb{F}$:

$$\begin{aligned}
\mathbb{L} &= \{x \in \mathbb{F} : (\exists n \in \mathbb{N})(|x| < n)\},\\
\mathbb{A} &= \{x \in \mathbb{F} : (\exists n \in \mathbb{N})(\tfrac{1}{n} < |x| < n)\},\\
\Psi &= \{x \in \mathbb{F} : (\forall n \in \mathbb{N})(|x| > n)\},\\
\Omega &= \{x \in \mathbb{F} : (\forall n \in \mathbb{N})(|x| < \tfrac{1}{n})\}.
\end{aligned}$$

The elements of these sets are called limited, assignable, infinitely large, and infinitely small numbers respectively. Here are some obvious relationships between these kinds of elements, we will call them $\Omega\Psi$ rules,

$(\forall x, y \in \Omega)(x + y \in \Omega, xy \in \Omega)$,
$(\forall x \in \Omega)(\forall y \in \mathbb{L})(xy \in \Omega)$,
$(\forall x)(x \in \mathbb{A} \Rightarrow x^{-1} \in \mathbb{A})$,
$(\forall x \neq 0)(x \in \Omega \Leftrightarrow\ x^{-1} \in \Psi)$.

Referring to the set Ω, an equivalence relation is defined by

$$x \approx y \Leftrightarrow x - y \in \Omega.$$

We say that x *is infinitely close to* y, when the relation $x \approx y$ holds.

Although we present the above relations within the modern framework, all of them were explicitly discussed in [**?**, ch. 3].

3.1. Archimedean axiom

Here are some equivalent forms of the Archimedean axiom:

(A1) $(\forall x, y \in \mathbb{F})(\exists n \in \mathbb{N})(0 < x < y \Rightarrow nx > y)$,
(A2) $(\forall x \in \mathbb{F})(\exists n \in \mathbb{N})(n > x)$,
(A3) $\lim\limits_{n\to\infty} \frac{1}{n} = 0$,
(A4) $(\forall x, y \in \mathbb{F})(\exists q \in \mathbb{Q})(x < y \Rightarrow x < q < y)$,
(A5) For any Dedekind cut (A, B) of $(\mathbb{F}, <)$ obtains[11]

$$(\forall n \in \mathbb{N})(\exists a \in A)(\exists b \in B)(b - a < \tfrac{1}{n}),$$

(A6) $\Omega = \{0\}$.

Versions A1 and A2 are well-known, both in the mathematical as well as the historical context. In calculus courses, A3 is usually presented as a theorem rather than an axiom, however the Archmimedean axiom follows from some versions of the continuity of real numbers, or is explicitly included in other versions (see section 3.2. below). A6 reveals that in a non-Archimedean field the set of infinitesimals Ω contains at least one positive element, say ε. Then, by $\Omega\Psi$ rules, $\frac{\varepsilon}{n}$, as well as, $n \cdot \varepsilon$ are also infinitesimals.

The versions A1 to A6 above are equivalent within the framework of an ordered field while some of them, for instance A1, apply to an ordered group $(\mathbb{G}, +, <)$. Then, the term nx is defined by

$$nx =_{df} \underbrace{x + x + ... + x}_{n-times}.$$

We can also apply versions A4 and A5, provided the concept of fraction is interpretable in a group. Versions A3 and A6 involve the concept of an absolute value. While the very definition makes sense in any ordered group, some properties of the absolute value, such as $|x \cdot y| = |x| \cdot |y|$, require the order to be compatible both with sums and products. Hence, these versions need to be applied carefully.

At the end of the 19th century, a few non-Archimedean structures were introduced, however they contained rather exotic mathematical entities that provoked distrust.[12] We present a non-Archimedean group made up

[11] For the remainder, a pair (A, B) of non-empty sets is a Dedekind cut of a totally ordered set $(X, <)$ iff: (1) $A \cup B = X$, (2) $(\forall x \in A)(\forall y \in B)(x < y)$.

[12] [**?**] provides a thorough overview of these structures.

of then well-known objects, namely complex numbers; the simplicity of the model makes us wonder why it was not involved in the dispute concerning infinitesimals.

Let $(\mathbb{C}, +, 0, \prec)$ be the additive group of complex numbers with the lexicographical order, i.e.,

$$a + bi \prec c + di \Leftrightarrow a < c \vee (a = c,\, b < d).$$

The order is compatible with sums, although not with products. One can easily show that $0 \prec i \prec 1$, moreover, for every natural number n the inequality $ni \prec 1$ holds. The set $\{ri : r \in \mathbb{R}\}$ includes infinitesimals of the group $(\mathbb{C}, +, 0, \prec)$.[13]

3.2. Real numbers

The field of real numbers is a commutative ordered field $(\mathbb{F}, +, \cdot, 0, 1, <)$ in which every Dedekind cut (L, U) of $(\mathbb{F}, <)$ satisfies the following condition:

$$(\exists x \in \mathbb{F})(\forall y \in L)(\forall z \in U)(y \leq x \leq z). \tag{C1}$$

Throughout the paper, we consider the condition C1 the Dedekind cut principle. Here are some other equivalent forms of C1:

(C2) If $A \subset \mathbb{F}$ is a nonempty set which is bounded above, then there exists $a \in \mathbb{F}$ such that $a = \sup A$.

(C3) The field is Archimedean and every Cauchy (fundamental) sequence $(a_n) \subset \mathbb{F}$ has a limit in $\mathbb{F}$.

(C4) The field is Archimedean and if $\{A_n |\ n \in \mathbb{N}\} \subset \mathbb{F}$ is a family of descending, closed line segments, then $\bigcap_{n \in \mathbb{N}} A_n \neq \emptyset$.

Any equivalent form of C1 usually gets the name of continuity or completeness, and then the real numbers system is called the continuous ordered field or the complete ordered field. The version C2 is also known as Dedekind completeness or the least upper bound (LUB) principle, whereas the second part of C3 is called Cauchy completeness. Since Dedekind and Cauchy completeness are not equivalent, we prefer to use more specific names like Dedekind cut or LUB principle.

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[13] [**?**] provides historical account of the Archimedean axiom, from Euclid and Archimedes, through Heiberg's edition of Greek text, to the 19th century theories of magnitudes developed by Stolz, Weber, Hölder and others.

The above definition is based on the so called categoricity theorem which states that every two ordered fields that satisfy C1 are isomorphic. In that sense, the field of real numbers is the unique complete ordered field.

When dealing with non-Archimedean fields, the following theorem is of crucial importance: The field of real numbers is *the biggest* Archimedean field, that is, for any Archimedean field $(A, +, \cdot, 0, 1, <)$, there is a subfield of the field of real numbers that is isomorphic to $(A, +, \cdot, 0, 1, <)$. As a result, any field extension of the system of real numbers is a non-Archimedean field and includes infinitely many infinitesimal numbers. Below we present such an extension, namely the field of hyperreals $(\mathbb{R}^*, +, \cdot, 0, 1, <^*)$.

One way to obtain hyperreals is by the ultrapower construction. Here is a sketch of that approach.[14] Let $(\mathbb{R}, +, \cdot, 0, 1, <)$ be the field of real numbers, let $\mathcal{U}$ be a nonprincipal ultrafilter on $\mathbb{N}$. The set $\mathbb{R}^*$ is defined as the quotient class of $\mathbb{R}^\mathbb{N}$ with respect to the following relation

$$(r_n) \equiv (s_n) \Leftrightarrow \{n \in \mathbb{N} : r_n = s_n\} \in \mathcal{U},$$

thus, $\mathbb{R}^* = \mathbb{R}^\mathbb{N}/_\mathcal{U}$. New sums and products are defined pointwise, while the total order is defined by

$$[(r_n)] <^* [(s_n)] \Leftrightarrow \{n \in \mathbb{N} : r_n < s_n\} \in \mathcal{U}.$$

Hence, the product of hyperreal $[(r_1, r_2, ...)]$ and $[(s_1, s_2, ...)]$ gives $[(r_1 \cdot s_1, r_2 \cdot s_2, ...)]$, and the relation $[(r_1, r_2, ...)] <^* [(s_1, s_2, ...)]$ holds when, for example, the set $\{n \in \mathbb{N} : r_n < s_n\}$ equals $\mathbb{N}$ minus some finite set (though the definition of order $<^*$ also includes other cases).

Standard real number, $r \in \mathbb{R}$, is represented by the class $[(r, r, r, ...)]$, i.e., the class of a constant sequence $(r, r, r, ...)$. Owing to these definitions, we employ the same symbols for real numbers in the standard and non-standard context; we will also employ the same symbols for sums, products and order relation in the standard and non-standard context.

The equivalence class of the sequence $(\frac{1}{n})$, i.e., the hyperreal number $\varepsilon = [(\frac{1}{n})]$, is a model example of infinitesimal. As another infinitesimal, let us consider a hyperreal δ represented by the sequence $(\frac{1}{n^2})$, that is $\delta = [(\frac{1}{n^2})]$. It is easy to check that $0 < \delta < \varepsilon$.

To study products of infinitely small and infinitely large numbers, let us define two infinite numbers,

[14] For details, see [**?**, **?**].

$$K = [(n)] = [(1, 2, 3, ...)], \quad L = [(n^2)] = [(1, 4, 9, ...)].$$

Since products are defined pointwise, we easily obtain the following equalities

$$K \cdot \varepsilon = 1, \quad L \cdot \varepsilon = K, \quad K \cdot \delta = \varepsilon.$$

Although K and L are not Cantor's ordinal numbers, these results undermine the seemingly obvious supposition that the product of an infinitely large and infinitely small number has to be infinitesimal regardless of the framework. In fact, within the field of hypereals we can realize all three options: the product can be an appreciable, infinitely large, or infinitely small number.

As already mentioned, in his argument Cantor employs the specific interpretation of the Archimedean property. Namely, instead of sums, he allows for multiplications.[15] Adopting that perspective, axiom A1 will take the following form

$(A1^*)$ $(\forall x, y \in \mathbb{R}^*)(\exists n \in \mathbb{N}^*)(0 < x < y \Rightarrow n \cdot x > y)$,

where the set of hypernatural numbers $\mathbb{N}^*$ is defined by

$$\mathbb{N}^* = \{[(n_j)] \in \mathbb{R}^* : \ (n_j) \in \mathbb{N}^{\mathbb{N}}\}.$$

The field of hyperreals is non-Archimedean in the sense of A1, yet it is Archimedean in the sense of $A1^*$. Indeed, for any positive hyperreal numbers $x = [(r_1, r_2, r_3, ...)]$, $y = [(s_1, s_2, s_3, ...)]$, due to the Archimedean property of real numbers, there is a sequence of natural numbers (n_j) such that $n_j \cdot r_j > s_j$. Thus, the hypernatural number K defined by $K = [(n_j)]$ is such that $K \cdot x > y$.

Finally, let us note that the set of assignable hyperreals $(\mathbb{A}, +, <)$ is Archimedean in the sense A1, A2, A5, and non-Archimedean in the sense A4, while versions A3 and A6 do not apply to this structure; $(\mathbb{A}, +, <)$ is not an ordered group, not to mention an ordered field. Therefore to deal with the Archimedean property we need a broader algebraic context.

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[15]Under some interpretations, Cantor's term $\zeta\nu$ stands for specific infinite sum; see [**?**]. Whether it is a product or infinite sum, Cantor's interpretation differs from our modern understanding of A1, as well as from the version of the Archimedean property introduced by [**?**]; see section 6 below.

4. Cantor's proof

Cantor's proof was presented in his letter to Goldscheider (Cantor, 13 May, 1887). Its crucial part was also included in the letter to Weierstrass (Cantor, 16 May, 1887). Then the former letter was made into section VI of the paper [**?**]. Below, we present the Philip Ehrlich translation of [**?**, ch. VI, pp. 406–409]; numerals 1–11 as well as some Greek and German words in square brackets are added; to enhance the numbered sentences, we also changed the setting of original sections.

"You mention in your letter the question of actual infinitely small magnitudes [Grössen]. At several places of my works you will find expressed the opinion that this is impossible, i.e., they are self-contradictory in thought, and I already implied in my work "Foundations of a General Theory of Manifolds", p. 8, §4, even though still with a certain reserve, that a rigorous proof of this position could be derived from the theory of transfinite numbers. During this winter, the time was first found to express my ideas on this subject in the form of a formal proof. It concerns the theorem:

[1] Non-zero linear numbers ζ (i.e., numbers which may be regarded as bounded, continuous lengths of straight lines) which would be smaller than each arbitrarily small finite number do not exist, i.e., they contradict the concept of linear numbers.

The thought process of my proof is simply as follows:

[2] I proceed from the assumption of a linear magnitude [linearen Grössen] ζ which is so small that its n-fold product $\zeta \cdot n$ is less than unity for each whole number, and prove from the concept of linear magnitude with the help of certain propositions of transfinite number theory,

[3] that even when ν is an arbitrarily large transfinite ordinal (i.e., the order type of a well-ordered set) $\zeta \cdot \nu$ is smaller than any finite magnitude that is as small as you please.

[4] This means that ζ cannot be made finite through any actual infinite multiplication [Vervielfachung], and is therefore certainly not an element of finite magnitude.

[5] Thus, the assumption we made contradicts the concept of a linear magnitude, which is of the sort that, according to it each linear magnitude must be thought of as an integral part of another, in particular of finite linear magnitude. So nothing is left but to let go of the assumption that there is a magnitude ζ which for any finite whole number n would be smaller than $\frac{1}{n}$, and with this our proposition has been proven.

It seems to me that this is an important application of the theory of transfinite numbers, which is capable of pushing aside widespread prejudices.

[6] The fact of actual infinite numbers is thus so little ground for the existence of actual infinitely small magnitudes that, on the contrary, the impossibility of the latter can be proven with the former.

[7] I also don't believe that this result can be reached fully and strictly in any other way.

[8] The need of our theorem is especially clear for the purpose of opposing the newer attempts of O. Stolz and P. Dubois-Reymond to derive the legitimacy of actual infinitely small magnitudes from the so-called "Archimedean axiom" (cf. O. Stolz, "Zur Geometrie der Alten, insbesondere über ein Axiom des Archimedes" 1881–1882, 1883; "Die unendlich kleinen Grössen" 1884; "Vorlesungen über allgemeine Arithmetik", Part 1, Leipzig 1885, p. 205).

[9] Archimedes appears to be the first to remark that, the assertion used in Euclid's *Elements*, where upon from any arbitrarily small line segment can be produced through sufficiently large multiplication [Vervielfachung] an arbitrarily large line segment, requires proof, and for that reason he believed that this assertion should be called an "Assumption".

[10] (Cf. Euclid's *Elements*, Book V, Definition 4: Magnitudes are said to have a ratio to one another which are capable, when multiplied [πολλαπλασιαζόμενα], of exceeding one another; also, especially *Elements*, Book X, Proposition 1, Archimedes' *The Sphere and Cylinder* I, Postulate 5 and the *Introduction* to his work: *The Quadrature of the Parabola*).

[11] Now it is the reasoning of those authors (O. Stolz loc. cit.), that if one deletes this supposed "axiom", the permissibility of actual infinitely small magnitudes, which are there called "moments", would emerge.

[12] But if the above theorem of mine is applied to the continuous straight line, the necessity of the Euclidean assumption immediately follows".

5. Magnitudes and Archimedean property in Greek mathematics

Cantor's sentences [9] and [10] as well as the bracketed apposition in sentence [1] explicitly refer to Greek mathematics, therefore we dedicate this section to ancient versions of the Archimedean property.

We start with a brief description of the interest in ancient mathematics prevalent in the second half of the 19th century. In the 1880s, Johan L. Heiberg published *Archimedis opera Omnia* [**?**] and then Euclid's *Elements* [**?**]. They are both arranged in the same format: Greek text and Latin translation authored by Heiberg are provided alternately page by page. For the mathematicians of that time, these were standard source books for Greek understanding of magnitude. Cantor, Peano, and Hilbert cite them when discussing Euclid and Archimedes. However, the reading of ancient texts hinges upon a philosophical disposition. While, for instance, Hilbert's *Grundlagen der Geometrie* provides an interpretation of Euclid's *Elements*, Cantor's comments on the Archimedean property and the concept of magnitude reveal his Platonic propensity and belief that with new definitions, he complements the Greek idea of continuum rather than introducing a new concept. The 19th-century renaissance of Greek mathematics, specifically mathematicians' interest in the concept of magnitude and the theory of proportion, was initiated by German mathematician and historian Hermann Hankel. In 1874's *Zur Geschichte der Mathematik in Altertum und Mittelalter*, he developed modern formalizations of books V and VI of the *Elements*, and his symbolic representations of Euclid schematic phrases were then adopted in Heiberg's Latin translation.[16]

We must also note Hermann Grassmann's *Lehrbuch der Arithmetik*, published 1861. It was the first monograph dedicated to totally ordered groups. The idea of an order compatible with sums was employed in every axiomatic characteristic of a magnitude of that time, whereas Dedekind employed the idea of a total order compatible with sums and products in his definition of rational and real numbers, as developed in [**?**].

5.1. Book V of Euclid's *Elements*

The term *linear number* as it occurs in the sentence [1] refers to a closed line segment. The bracketed original German phrase reads: *d.h. kurz gesagt, solche Zahlgrößen, welche sich unter dem Bilde begrenzter geradliniger stetiger Strecken vorstellen lassen.* Thus, *linear numbers* are to represent – unter dem Bilde [...] vorstellen lassen – closed line segments, the model example of ancient Greek magnitudes.

[16] See [**?**, pp. 389–404].

To elaborate, the general term μέγεθος covers line segments, triangles, convex polygons, circles, solids, angles, and arcs of circles. We formalize (write down in symbols) Euclid's magnitudes of the same kind (line segments being of one kind, triangles being of another, etc.) as an additive semigroup with a total order, $(M, +, <)$, characterized by the following five axioms:

E1 $(\forall x, y)(\exists n \in \mathbb{N})(nx > y)$,
E2 $(\forall x, y)(\exists z)(x < y \Rightarrow x + z = y)$,
E3 $(\forall x, y, z)(x < y \Rightarrow x + z < y + z)$,
E4 $(\forall x)(\forall n \in \mathbb{N})(\exists y)(x = ny)$,
E5 $(\forall x, y, z)(\exists v)(x : y :: z : v)$.

The total order, both in book V as well as throughout the *Elements*, is a primitive notion characterized by transitivity and the trichotomy law. Unlike modern mathematics, in Greek mathematics it is applied not only to line segments, but to figures at all, e.g. in proposition I.6, triangles are compared in terms of lesser-greater.

E1 is Euclid definition 4 of book V. The sign :: represents proportion as provided in definition 5 of book V. The fact that Greek line segments are closed segments is explicated in definition 3 of book I, which reads: "And the extremities of a line are points". Moreover, all throughout the *Elements*, line segments are represented by their end-points, such as A, B, whether in the text as AB, or on diagrams, when A, B stand next to intersections of lines, or next to short vertical lines depicting the ends of segments; for instance, all throughout book V magnitudes are represented by line segments with short, bounding vertical lines.

To be clear, we do not suggest that Cantor based his argument on such exegesis. Our point is that over the course of history, from ancient to modern times, line segments were considered what we now call closed line segments, i.e., segments with their ends. In the second half of the 19th century, when the idea of totally ordered sets was introduced, it made distinguishing between closed and open line segments easy. These new concepts launched the mathematical career of open segments.

Throughout the ages, essentially owing to Descartes and Euler, the structure of Greek magnitudes $(M, +, <)$ was transformed into an ordered field $(M, +, \cdot, 0, 1, <)$, and then, a number line. In the late 19th century, the number line was turned into the continuous line of real numbers. In this way, Cantor's sentence [12] refers to the 19th century characteristics of real numbers, rather than the ancient Greek structure of line segments.

In translations, the Greek word μέγεϑος has been rendered in Latin as *quantitas*, in English as *quantity* or *magnitude*, in French – as *quantité* or *grandeur*, in German – as *Quantität*, or *Grösse*. In the 20th century mathematics, the term *magnitude* was slowly replaced with *real numbers*; still, in the late 40s, Nicolas Bourbaki used the term *grandeur* meaning *real numbers*.[17] In [**?**], the term *Zahlengrössen* stands for what we consider to be real numbers, while already in [**?**] the term *reellen Zahlen* occurs.[18] [**?**] defines real numbers as all rational and irrational numbers: *aller reellen, d.h. aller rationalen und irrationalen Zahlen.*

In sentence [10], Cantor cites the Greek text of definition V.4, however, in sentences [4] and [9], he interprets the word πολλαπλασιασμός as *Vervielfachung* (multiplication). Euclid's πολλαπλασιασμός means *multiplicity* rather than *multiplication*. Whereas multiplicity of a magnitude x means the reiterated addition of that magnitude, that is $x + ... + x$, there was no multiplication of any kind in Greek geometry. Although the reading $nx = x + ... + x$ was standard at that time, Cantor adopted a specific interpretation of the Archimdean property, namely, at first, instead of nx he assumed $n \cdot x$, then in the place of $(\forall n \in \mathbb{N})$, he allowed $(\forall \nu \in Ord)$.

In section 3.2 above, we have shown that by the same kind of interpretation, when instead of nx we take $n \cdot x$, and then change the range of the variable n from $\mathbb{N}$ to $\mathbb{N}^*$, we reach the conclusion that the field of hyperreal numbers is Archimedean in the modified sense $A1^*$, although it is non-Archimedean in the standard sense A1.

5.2. Archimedean property

There are two versions of the Archimedean property in the *Opera omnia* [**?**]. In the treaty *On spiral lines*, Archimedes applies Euclid's version E1, although he calls it *Lemma*. In *On the sphere and cylinder*, the Lemma reads:

"Further, of unequal lines, unequal surfaces, and unequal solids, the greater exceeds the lesser by such a magnitude, as when added to itself, can be

[17] See. [**?**].

[18] See Section 7 below.

made to exceed any assigned magnitude among those which are comparable with and one another".[19]

We formalize it with the following formula:

LA $(\forall x, y, z)(\exists n \in \mathbb{N})(x < y \rightarrow n(y - x) > z)$.

This version is placed among four lemmas, such as e.g. the following one:

"Of all lines which have the same extremities the straight line is the least".[20]

Clearly, Archimedes could not prove his lemmas. In fact, there is no mention throughout the treaty that any proof was needed at all. Yet, when, in the *Quadrature of the parabola* he reiterates the LA version, he also adds this comment:

"The earlier geometers have also used this lemma; for it is by the use of this same lemma that they have shown that circles are to one another in the duplicate ratio of their diameters, and that spheres are to one another in the triplicate ratio of their diameters, and further that every pyramid is one third part of the prism which has the same base with the pyramid and equal height; also, that every cone is one third part of the cylinder having the same base as the cone and equal height they proved by assuming a certain lemma similar to that aforesaid. And, in the result, each of the aforesaid theorems has been accepted no less than those proved without the lemma".[21]

The results mentioned in this passage are Euclid's propositions XII.2, XII.18, XII.7, and XII.10 respectively. Archimedes, thus, evokes the authority of Euclid to justify his reference to LA. Cantor could have considered Archimedes' restraint revealed in these lines when he wrote: "Archimedes appears to be the first to remark that, the assertion used in Euclid's *Elements* [...] requires proof".

Viewed from the mathematical perspective, there were no other axioms, lemmas or definitions in Greek mathematics allowing to deduce the Archimedean property. Book V of the *Elements* encapsulates all Greek science of magnitudes, while in our axiomatic account of the theory, E1 is an independent axiom.

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[19] [**?**, p. 4].

[20] [**?**, p. 3].

[21] [**?**, p. 234].

Archimedes was the unsurpassed champion of the exhaustion method, he also contributed to the foundations of mathematics by expanding the scope of the concept of magnitude to curve lines and surfaces. Nevertheless, he certainly did not seek to prove the LA lemma.

To sum up, Cantor's note in sentence [9] on Archimedes' will to prove LA reflects his attitude towards the Archimedean property rather than historical facts.

Finally, let us note that it was Stolz who coined the name *Archimedean axiom*. He referred to the lemma as presented in *On the sphere and cylinder* and *Quadratura of the parabola* as a model version of the axiom.[22] The name prevailed in mathematics due to his often-cited books *Vorlesungen über Allgemeine Arithmetik* and *Theoretische Arithmetik*. On the other hand, Heiberg, in his comment on the Archimedean lemma, cites Euclid's definition V.4 and notes that these two are the same axiom.[23] This comment of Heiberg, it seems, confirmed the name *Archimedean axiom* for Euclid's definition V.4.

6. Theory of magnitudes in the late 19th century

In sentences [8] and [11] Cantor refers to Stolz's idea of introducing non-Archimedean numbers via axioms for magnitudes, while sentences [9] and [10] contain comments on Archimedes and Euclid. In fact, it was Stolz who confirmed mathematical studies of the concept of magnitude, as opposed to historical studies. Although many names were involved in this process, we present the mathematical extracts of the concept of magnitude as developed in [**?**, **?**, **?**, **?**, **?**]. This is how the movement has been characterized by Hölder's 1901 [**?**]:

"The theory of measurable magnitudes was developed to a high level by Euclid. Recently, it has been treated in depth from a number of different points of view. Nevertheless, it seems that the theory has not been treated exhaustively; further, errors and obscurities have appeared in some of the more recent treatments. This is why I think that a reformulation of this important and fundamental theory will be profitable".[24]

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[22]See [**?**, pp. 70, 332].

[23]See [**?**, p. 11].

[24][**?**, p. 238]. It is an English translation of [**?**] by J. Mitchell.

In a way, [?] finished and crowned these studies. In 1899, in the first edition of his *Grundlagen der Geometrie*, Hilbert provided axioms for an ordered field. Then, in [?], he provided the first ever axioms for real numbers. From that moment on, the mathematical studies of the concept of magnitude were redirected to axioms for real numbers, the continuity axiom specifically.

In what follows, we present symbolic accounts of the concept of magnitude. To be clear, no one in the 19th century applied such symbols. We decided on that form of presentation to clarify the mathematical background of Cantor's considerations. The common feature of all of these accounts was that the structure of magnitudes is a totally ordered, semigroup $\mathfrak{M} = (M, +, <)$ equipped with different axioms depending on the author.[25]

6.1. Du Bois-Reymond, 1882/1887

B1 $(\forall x)(\exists y, z)(y < x \wedge z > x)$,
B2 $(\forall x, y)(x + y > x)$,
B3 $(\forall x)(\forall n \in \mathbb{N})(\exists y)x = ny)$,
B4 $(\forall x, y)(\exists z)(x < y \Rightarrow x + z = y)$,
B5 $(\forall x, y, z)(x < y \Rightarrow x + z < y + z)$,
B6 $(\forall x, y)(\exists n \in \mathbb{N})(x \leq y \Rightarrow nx \geq y)$,
B7 $(\forall x, y)(\forall n \in \mathbb{N})(\exists z)(z < y \wedge x = nz)$.

6.2. Otto Stolz, 1885

S1 $(\forall x, y)(\exists n \in \mathbb{N})(x < y \Rightarrow nx > y)$,
S2 $(\forall x, y)(\exists! z)(x < y \Rightarrow x + z = y)$,
S3 $(\forall x, y, z)(x < y \Rightarrow x + z < y + z)$,
S4 $(\forall x, y)(x + y > x)$,
S5 $(\forall x)(\forall n \in \mathbb{N})(\exists y)(x = ny)$.

6.3. Heinrich Weber, 1895

W1 $(\forall x, y)(\exists n \in \mathbb{N})(nx > y)$,

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[25] For details, see [?].

W2 $(\forall x, y)(\exists z)(x < y \Rightarrow x + z = y)$,
W3 $(\forall x, y)(x + y > x)$,
W4 The order $<$ is dense and for every Dedekind cut (A, B) of $(M, <)$, obtains
$(\exists z)(\forall x \in A)(\forall y \in B)(x \leq z \leq y)$.

6.4. Otto Hölder, 1901

Hölder clarifies the concept of addition by following axioms:

$M \times M \ni (x, y) \mapsto x + y \in M$.
$(\forall x, y, z)[(x + y) + z = x + (y + z)]$.

Specifically, he does not assume the commutativity of addition. As regards the order $<$, his only assumption is the so called trichotomy law, namely

> For every two elements x, y, one and only one of the three possibilities obtains: $x < y \vee x = y \vee x > y$.

Here are his axioms.

H1 $(\forall x)(\exists y)(y < x)$,
H2 $(\forall x)(\forall y)(x + y > x \wedge x + y > y)$,
H3 $(\forall x, y)(\exists z, w)(x < y \Rightarrow (x + z = y \wedge w + x = y))$,
H4 For every Dedekind cut (A, B) of the set $(M, <)$, obtains
$(\exists z)(\forall x \in A)(\forall y \in B)(x \leq z \leq y)$.

Hölder managed to show that axioms H1-H4 entail the transitivity of the order $<$, the commutativity of addition, and above all, the Archimedean property.

6.5. Non-Archimedean group

As we can see, Du Bois-Reymond, Stolz, and Weber explicitly assume the Archimedean property. These are axioms B6, S1, and W1 respectively. When adopting their perspective, rejection of the Archimedean property does not imply inconsistency; we could say it would lead to a concept of a non-Archimedean ordered group.

Yet, in sentence [12], Cantor claims: "if the above theorem of mine is applied to the continuous straight line, the necessity of the Euclidean assumption immediately follows". These are Hölder's results that could support this belief, rather than Cantor's alleged theorem given in sentence [3]. Instead of infinitesimals, "continuous straight line" proved to be the key concept in these considerations. However, Cantor never adopted Dedekind's version of continuity.

7. Cantor's continuum

Neither Cantor, nor Dedekind was quite sure whether their versions of continuity of real numbers were equivalent. It was partly because there were no obvious framework that would enable to establish or dismiss the equivalence of Cantor's and Dedekind's versions of continuity. While Dedekind Cut Principle applies to totally ordered sets, Cantor sought for universal formula which could be applied in any framework. In fact, he considered an ordered field, a metric space and a totally ordered set. In each context, he tried to apply his newly discovered idea of derived set, P'.

Nevertheless, both Cantor and Dedekind developments had a clear reference object: Euclid's geometrical line.

7.1. Cantor on the field of real numbers

In [**?**], real numbers are made up of fundamental sequences (Cauchy sequences). Cantor managed to define field operations as well as the total order of real numbers. Dealing with numbers, he applies the concept of sequence limit. From the perspective of the continuity of real numbers, the following sentence is crucial:

"While domains B and A are so related, that although each **a** is assigned to a certain **b**, but not each **b** can be assigned to **a**, it turns out that both **b** can be assigned to a certain **c**, and each **c** can be assigned to a certain **b**".[26]

The phrase "each **a** is assigned to a certain **b**" means that each rational number can be represented as a Cauchy sequence of rational numbers. The phrase "not every **b** can be assigned to any **a**" means that the space of rational numbers is not Cauchy-complete. The phrase "each **c** can be assigned to a certain **b**" means that the space of real numbers is Cauchy-complete. It could be rendered as follows: $A' = B$, $B' = B$, where prime

[26]([**?**, p. 95]. Letters **a**, **b**, **c** stand for elements of the sets A, B, C respectively.

represents Cantor's derivative set. Cantor does not prove any of these claims and – as far as we know – never returned to this issue.

At the end of section § 1, Cantor declares that he develops the results of Book X of Euclid *Elements.* Interestingly, while integers and rational numbers are called *Zahlen*, for real numbers Cantor adopted the term *Zahlengrössen*, which could mean numbers assigned to magnitudes.

In the next section, Cantor introduces his famous axiom relating real numbers and geometric line. First, he shows how to assign for a point on the straight line a real number. To this end, he determines a unit segment and assumes that any point on the line is in a rational (*rationales Verhältnis*) or irrational (*im andern Falle*) ratio to the unit. Due to this very assumption he is really in the heir to Euclid, specifically in his understanding of commensurable and in-commensurable line segments as presented in Book X of the *Elements.*[27]

Then Cantor writes:

"To make complete the relationship of the domain of number magnitudes [*Zahhlengrössen*] defined in §1 with straight line geometry outlined in this §, I should only add an axiom, which is simply the converse, to every number magnitude there corresponds a definite point of the line whose coordinate is equal to that number magnitude, and equal in the sense as explained herein §. I call this statement an axiom, because it is in its nature that it is not generally provable".[28]

When points on the straight line and real numbers are identified, Cantor continues to study subsets of the line. Within the geometrical context, he prefers to apply the derivative set P' of a set P, rather than the concept of sequence limit, as defined in an ordered field. In fact, to define P' one only needs a structure of open line segments, thus the idea of P' can be transferred to a totally ordered set $(X, <)$.

7.2. Correspondence with Dedekind

In [**?**, § 3], Dedekind coined his cut principle as the "essence of continuity". The *Preface* mentions [**?**] and reads:

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[27]In terms of the unit segment, Cantor belongs to the tradition which goes back to [**?**]. There was, of course, no universal unit segment in Greek mathematics.

[28][**?**, p. 97]. As for the last sentence, both Cantor's axioms turned out to be theorems within the framework of axiomatic account of Euclid's geometry; see [**?**, § 20] or [**?**, § 21].

"After a hasty reading, it seems to me that the axiom given in Section II of that paper (except for the form of presentation) agrees with what I designate in Section III as the essence of continuity".[29]

Dedekind seems never had a time to study Cantor's paper in depth and decide whether their axioms really agree.

In the letter to Dedekind dated May 17, 1887, Cantor raised the objection that the cut principle applies both to integers and to real numbers: "this property also holds of the system of all integers".[30] In the *Post Scriptum*, he reiterated his objection by writing:

"you lay special emphasis on IV [i.e. the cut principle] because this property distinguishes the complete domain of numbers from the domain of all rational numbers; however it seems to me for the above reasons that one cannot give property IV the name *essence of continuity*".[31]

On May 17, 1887, Dedekind replied: "you worry that my exclusive stressing of IV as the property in which the essence of continuity is expressed could lead to misunderstanding. I do not share this concern".[32] Then he adds that the cut principle is the *essence of continuity* when applied to a dense total order.

Cantor refers to Dedekind's reply in the first sentences of the letter dated June 20, 1887. It reads:

"Thank you for your letter of 18 May. I completely agree with its contents; and I acknowledge that the difference in our opinion of view was merely external".[33]

In the rest of the letter Cantor presents his proof to the effect subsets of $\mathbb{R}^2$ can be in one-to-one relation with "continuous line".[34]

The standard reading of that exchange is that Cantor simply misinterpreted Dedekind by applying the cut principle to a totally ordered set, rather than to a densely ordered set. Nevertheless in [**?**, § 9], Dedekind's

[29] [**?**, p. 767].

[30] [**?**, p. 852].

[31] [**?**, p. 852]. [**?**, § 2] provides a characteristic of the total order of rational numbers, R in his notation, which we paraphrase as follows: I transitivity, II density, III every rational number determines Dedekind cut of the set $(R, <)$. The cut principle, the condition IV, occurs in section § 3 called *Continuity of the Straight Line*.

[32] [**?**, p. 852].

[33] [**?**, p. 853].

[34] [**?**, p. 853].

construction of real numbers was perfectly summarized. Thus, there was no misinterpretation. The point is a philosophical question: the "essence of continuity". Doubts whether Dedekind's cut principle provides a universal characteristic of the continuum have been voiced already in [**?**, § 10]. In fact, they have never been dispelled.

7.3. Cantor on continuum in metric space

[**?**, § 9] summaries some 19th century theories of real numbers, namely: Weierstrass', as developed in 1872' Kossak *Die Elemente der Arithmetik*, Dedekind's, as developed in [**?**], and Cantor's, as developed in [**?**].

[**?**, § 10] is dedicated to the continuum. Tracing back the history of this concept, Cantor discusses ancient Greek and Medieval philosophers. He believes that "the underlying idea has taken on different meanings". Therefore, his definition could be compared with, for instance, Aristotle's *ex partibus sine fine divisibilibus*.[35] This very section initiated a branch of point-set topology, namely continuum theory, that is the study of compact and connected spaces. Yet Cantor's own definition is a bit different.

According to Cantor, a subset T of the space $\mathbb{R}^n$ with Euclidean metric is *connected* if for any of its points t_0 and t^0 and any positive real number ε there are finitely many points $t_1, t_2, ..., t_n$ of T such that the distances $d(t_0, t_1), d(t_1, t_2), ..., d(t_n, t^0)$ are all less than ε.

T is perfect if $T = T'$, where

$$x \in T' \Leftrightarrow \lim_{n\to\infty} x_n = x,$$

for some $(x_n) \subset \mathbb{R}^n \setminus \{x\}$; convergence of a sequence is defined in the metric space $\mathbb{R}^n$. The latter condition is equivalent to the following assertion: Every convergent series $(t_n) \subset$ T has a limit in T, and for every $t \in T$, there exists a sequence $(t_n) \subset T$ such that $\lim_{n\to\infty} t_n = t$.

Then comes the famous definition:

"I therefore define a point-continuum inside G_n [$\mathbb{R}^n$ in our notation] as *a perfect-connected*".[36]

[35] It is a scholastic version of Aristotles characterization of magnitude (μέγεθος) as provided in *Physics*, VI: divisible into divisibles that are infinitely divisible. It can be show that Aristole's definition is compatible with Euclid's characteristic of a line segment; see [**?**].

[36] See [**?**, pp. 903–906].

Based on this definition, Cantor rebukes definition of the continuum as given in [**?**, § 38] and Dedekind's cut principle. He claims that non-connected sets exemplify Bolzano's definition.[37] As for Dedekind, Cantor writes:

"Likewise, it seems to me that in the article (*Continuity and irrational numbers*) only another property of the continuum has been one-sidedly emphasized, namely, that property which it has in common with all 'perfect' sets".[38]

7.4. Order type of linear continuum (*Linearkontinuum*)

[**?**, § 11] provides the order type characteristic of the segment $[0,1]$ of numbers (*reellen Zahlen*) with their natural order (ihrer natürlichen Rangordung).[39] Cantor proves that any linearly ordered set $(X,<)$ with the first and the last element, that is (1) perfect, and (2) contains a subset $A \subset X$ which is dense in $(X,<)$ and of cardinality $\aleph_0$ is isomorphic to the set $([0,1],<)$.

Since in 1883 Cantor interprets Dedekind Cut Principle as a property of perfect set, in 1895 he could be certain that Dedekind Cut Principle did not provide the "essence of continuity". Nowadays we can support his belief by a simple example. Namely, let $X = [0,1] \times [0,1]$, be the Cartesian product of real numbers segments with lexicographical order. The set $(X,<)$ is continuous in terms of the cut principle, however, it is not a separable space.

8. Modern account of the product $\zeta \cdot \nu$

In this section, we firstly provide an alternative arithmetic for Cantor's sums and products of ordinal numbers. Then, we introduce a non-Archimedean field *ONAG* which includes the class of ordinal numbers, *Ord*. As the field *ONAG* includes both ordinal and infinitesimal numbers, we can show that Cantor hypothesis concerning products of ordinal and infinitesimal numbers, as presented in sentence [3], fails.

[37] Indeed, Bolzano definition of the continuum boils to the fact that the continuum has no isolated points.

[38] [**?**, p. 906].

[39] Cantor had never explained what *natural order* means in mathematical terms.

8.1. Normal sums and products of ordinal numbers

Let us start with a remainder of the normal form theorem [**?**]: For every ordinal number $\alpha \in Ord$, there are ordinal numbers $\eta_1, \ldots, \eta_h$, and natural numbers $h, p_i \in \mathbb{N}$ such that

$$\alpha = \omega^{\eta_1} \cdot p_1 + \ldots + \omega^{\eta_h} \cdot p_h,$$

where $\eta_1 > \ldots > \eta_h$.

This representation of α is unique. Moreover, it is finite, due to the assumption concerning the index h.

Based on this theorem, [**?**] introduced the so-called normal sums and products of ordinal numbers. Namely, for

$$\alpha = \omega^{\eta_1} \cdot p_1 + \ldots + \omega^{\eta_h} \cdot p_h, \quad \beta = \omega^{\eta_1} \cdot q_1 + \ldots + \omega^{\eta_h} \cdot q_h$$

their normal sum $+_n$ and normal product $\cdot_n$ is defined by[40]

$$\begin{aligned} \alpha +_n \beta &=_{df} \omega^{\eta_1} \cdot (p_1 + q_1) + \ldots + \omega^{\eta_h} \cdot (p_h + q_h) \\ \alpha \cdot_n \beta &=_{df} \sum_{1 \leq i,j \leq h} \omega^{\eta_i +_n \eta_j} \cdot p_i q_j \end{aligned}$$

Contrary to Cantor's sums and products of ordinal numbers, normal sums and products are commutative and compatible with the standard order of ordinal numbers, that is

$$\alpha +_n \beta = \beta +_n \alpha, \quad \alpha \cdot_n \beta = \beta \cdot_n \alpha,$$

$$\alpha < \beta \Rightarrow \alpha +_n \gamma < \beta +_n \gamma, \quad \alpha < \beta \Rightarrow \alpha \cdot_n \gamma < \beta \cdot_n \gamma.$$

Thus, the structure $(Ord, +_n, \cdot_n, 0, 1, <)$ is an abelian semigroup.

Hence, e.g. since $\omega = \omega \cdot 1 + 0$, and $1 = \omega \cdot 0 + 1$, we calculate the normal sums of $\omega +_n 1$ and $1 +_n \omega$ as follows,

$$1 +_n \omega = (\omega \cdot 0 + 1) +_n (\omega \cdot 1 + 0) = \omega \cdot (0 + 1) + 1 = \omega + 1,$$

$$\omega +_n 1 = (\omega \cdot 1 + 0) +_n (\omega \cdot 0 + 1) = \omega \cdot (1 + 0) + 1 = \omega + 1.$$

Similarly, we calculate

$$2 \cdot_n \omega = (\omega \cdot 0 + 2) \cdot_n (\omega \cdot 1 + 0) = \omega^2 \cdot 0 + \omega \cdot 2 + 0 = \omega \cdot 2,$$

$$\omega \cdot_n 2 = (\omega \cdot 1 + 0) \cdot_n (\omega \cdot 0 + 2) = \omega^2 \cdot 0 + \omega \cdot 2 + 0 = \omega \cdot 2.$$

As is well known, in Cantor's arithmetic the inequalities hold $1 + \omega < \omega + 1$, and $2 \cdot \omega < \omega \cdot 2$.

[40] We assume for the use of the definition, that some p_i or q_i could equal 0.

8.2. Conway numbers

[?, ?] introduces a very special non-Archimedean ordered field; it is usually called the field of surreal numbers or in short ONAG (the acronym for on numbers and games). In fact, [?] proves that $(ONAG, +.\cdot, 0, 1, <)$ is the biggest non-Archimedean field.

While Conway develops his theory beyond the framework of the set theory, [?] manages to rediscover surreal numbers in the set theory, and provides a suggestive representation. Namely, a surreal number is a function $\mathbf{a}$ from an ordinal α into the set $\{+, -\}$, that is

$$\mathbf{a} : \alpha \mapsto \{+, -\}.$$

Hence, every ordinal number α is represented by the α-length string of pluses

$$\alpha \ \sim \ (\underbrace{+ + ...}_{\alpha}).$$

To compare surreal numbers $\mathbf{a}, \mathbf{b}$ in terms of lesser-greater, when $\alpha < \beta$, where α and β are domains of $\mathbf{a}, \mathbf{b}$ respectively, we make up the sequence $\mathbf{a}$ by 0s, to the sequence of β-length. Then, the total order $\mathbf{a} < \mathbf{b}$ is defined by lexicographical order, given

$$- < 0 < + \ .$$

For example,

$$(--) < (-) < (-+) < (+) < (++-) < (++).$$

We can show that the field $ONAG$ includes the structure $(Ord, +_n, \cdot_n, 0, 1, <)$. Therefore, within the framework of surreal numbers, Cantor's ordinal numbers are subject to field operations. Next to the ordinal number ω, in the field $ONAG$, there are also elements such as

$$-\omega, \quad \omega - 1, \quad \frac{\omega}{2}, \quad \frac{1}{\omega}.$$

Due to Gonshor's development, we can represent these numbers as follows

$$-\omega = (\underbrace{- - ....}_{\omega}),$$

$$\omega - 1 = (\underbrace{+ + +...}_{\omega} -),$$

$\frac{\omega}{2} = (\underbrace{+++...}_{\omega}\underbrace{---...}_{\omega}),$

$\frac{1}{\omega} = (\underbrace{+---...}_{\omega}).$

Since every infinite ordinal number α is an infinite element of the field *ONAG*, i.e. it is an element of the class Ψ, as defined in section 3 above, the element α^{-1} is infinitesimal. In this way, the field of surreal numbers provides a framework to test Cantor's hypothesis concerning the products of infinitesimal and ordinal numbers.

8.3. Falsifying Cantor's hypothesis

Sentence [3] includes the key mathematical part of Cantor's argument, we call it the Infinitesimals Hypotheses (IH): when ζ is infinitesimal and ν "is an arbitrarily large transfinite ordinal [...] $\zeta \cdot \nu$ is smaller than any finite magnitude", in symbols

$$(\forall \nu \in Ord)(\zeta \cdot \nu < 1). \tag{IH}$$

Cantor had never defined the product of infinitesimal and ordinal numbers, especially he had never proved the claim IH. The framework of surreal numbers enables, both make sense of the product $\zeta \cdot \nu$, and falsify the claim IH.

For the falsification part, let ζ be a positive infinitesimal. Then $\zeta^{-1} \in \Psi$, i.e. ζ^{-1} is infinitely large number in the field *ONAG*. Due to Gonshor's representation of surreal numbers, we can find and ordinal number α greater than ζ^{-1}. By the standard rules of an ordered field, we have

$$\zeta^{-1} < \alpha \Rightarrow \alpha^{-1} < \zeta.$$

Similarly, by the standard rules of an ordered field

$$\alpha^{-1} < \zeta \Rightarrow \alpha^{-1} \cdot \alpha < \zeta \cdot \alpha.$$

Hence, the product $\zeta \cdot \alpha$ is greater than 1. In the same manner, we can show that the product $\zeta \cdot \alpha^2$ is an infinite surreal number, as it is greater than α. □

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Pedagogical University of Cracow
Institute of Mathematics
Podchorążych 2
30-084 Cracow, Poland
e-mail: `{piotr.blaszczyk,marlena.fila}@up.krakow.pl`

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Zvonimir Šikić

COMPOUNDING OBJECTS

Abstract

We prove a characterization theorem for filters, proper filters and ultrafilters which is a kind of converse of Łoś's theorem. It is more natural than the usual intuition of these terms as large sets of coordinates, which is actually unconvincing in the case of ultrafilters. As a bonus, we get a very simple proof of Łoś's theorem.

Keywords: Łoś's theorem, converse of Łoś's theorem, filter, proper filter, ultrafilter.

One of the useful methods in formal sciences is the construction of complex structures by compounding objects of simpler structures. For example, by compounding real numbers in triples we construct $(\mathbb{R}^3, +^3, <^3)$ from $(\mathbb{R}, +, <)$. The operation $+^3$ and the relation $<^3$ are defined coordinate-wise e.g. $(2,3,1) +^3 (1,-1,0) = (3,2,1)$ and $(2,3,1) <^3 (3,4,2)$, but we have to be aware that the total order $<$ turns into the partial order $<^3$ (e.g. neither $(2,3,1) <^3 (3,2,1)$ nor $(3,2,1) <^3 (2,3,1)$). The interesting question is whether it is possible to construct a compound system with the same 1-order properties as the systems it is compound of. In this way we could construct nonstandard models of standard (intended) structures. For example, by compounding standard PA structures of natural numbers we could get a nonstandard (non-isomorphic) model of standard PA. For the systems $\mathbf{S}_i = (S_i, \ldots, \circ_i, \ldots, R_i, \ldots)$, $i \in J$, we may always construct the compound system $\Pi\mathbf{S}_i = (\Pi S_i, \ldots, \Pi\circ_i, \ldots, \Pi R_i, \ldots) = (S, \ldots, \circ, \ldots, R, \ldots)$, with sequences $a = (a_1, a_2, a_3, \ldots)$, $a_i \in S_i$, as elements of S and operations o and relations R defined coordinate-wise:

$$a \circ b = (a_1, a_2, a_3, \ldots) \circ (b_1, b_2, b_3, \ldots) = (a_1 \circ_1 b_1, a_2 \circ_2 b_2, a_3 \circ_3 b_3, \ldots)$$

$$aRb \equiv \forall i (a_i R_i b_i) \text{ i.e. } aRb \equiv \{i : a_i R_i b_i\} = J$$

But, as we have already pointed out, the compound system will not share the properties of the components (compare the totality of $<$ and the partiality of $<^3$). It could share them if instead of

$$aRb \equiv (\forall i)(a_i R_i b_i) \equiv \{i : a_i R_i b_i\} = J$$

we define

$$aRb \equiv (\underline{\forall} i)(a_i R_i b_i) \equiv \{i : a_i R_i b_i\} \in B$$

with some appropriate B. We may think of B as a family of "big" subsets of J and of $\underline{\forall}$ as meaning "for almost all". It means that something is true $\underline{\forall} i \in J$ if and only if it is true on a big subset of J. It was proved by Łoś (in the famous Łoś 's Theorem) that the appropriate "big" families are ultrafilters. Here we want to prove a kind of converse which is the following characterization theorem for filters, proper filters and ultrafilters:

THEOREM 1 (Characterization theorem).

(i) *The equality in the compound system, defined by $a = b \equiv \{i : a_i = b_i\} \in B$, is an equivalence relation if and only if B is a filter. Moreover, the equivalence relation is then a congruence i.e. if $a = a^*$ and $b = b^*$ then $a \circ b = a^* \circ b^*$.*

(ii) *The equality $a = b \equiv \{i : a_i = b_i\} \in B$ is an equivalence relation and obeys the principle of contradiction i.e. $\neg((a = b) \wedge (a \neq b))$ if and only if B is a proper filter, where $a \neq b$ if $\{i : a_i \neq b_i\} \in B$. Furthermore, compound relations defined by $aRb \equiv \{i : a_i R_i b_i\} \in B$ then obey the principle of contradiction too i.e. $\neg((aRb) \wedge (a\hat{R}b))$, where $a\hat{R}b$ if $\{i : a_i \hat{R}_i b_i\} \in B$.*

(iii) *The equality $a = b \equiv \{i : a_i = b_i\} \in B$ is an equivalence relation, satisfies the principle of contradiction and obeys the principle of excluded middle i.e. $(a = b) \vee (a \neq b)$ if and only if B is an ultrafilter. Furthermore, compound relations defined by $aRb \equiv \{i : a_i R_i b_i\} \in B$ then obey the principle of excluded middle too i.e. $(aRb) \vee (a\hat{R}b)$.*

From the characterization theorem it easily follows that $\underline{\forall}$ distributes through every truth-functional connective. Namely, if X_i and Y_i are formulae evaluated in the component S_i, we have the following:

COROLLARY 1.

1. $(\underline{\forall} i)(X_i \wedge Y_i) \equiv (\underline{\forall} i)X_i \wedge (\underline{\forall} i)Y_i$
2. $(\underline{\forall} i)(\neg X_i) \equiv \neg(\underline{\forall} i)X_i$

Note that $\forall$ satisfies (1) but does not satisfy (2). Using this corollary and the process of Skolemization, it is easy to prove Łoś's Theorem.

THEOREM 2 (Łoś's Theorem). *For every* 1*-order formula* $F, S \models F$ *if and only if* $(\underline{\forall} i)S_i \models F_i$*, where every operation symbol* $\circ$ *and every relation symbol* R *in* F *is replaced by the corresponding operation symbol* $\circ_i$ *and the corresponding relation symbol* R_i *in* F_i*.*

PROOF OF THE CHARACTERIZATION THEOREM: In what follows $X = \{i : a_i = b_i\}$, $Y = \{i : b_i = c_i\}$ and $Z = \{i : a_i = c_i\}$.

Proof of (i):

$a = a$ if and only if $\{i : a_i = a_i\} = J \in B$

$a = b \wedge b = c \to a = c$ if and only if $X \in B \wedge Y \in B \to X \cap Y \subset Z \in B$ if and only if $(X \in B \wedge Y \in B \to X \cap Y \in B) \wedge (Z \in B \wedge Z \subset U \to U \in B)$.

But $J \in B$, $(X \in B \wedge Y \in B \to X \cap Y \in B)$ and $(Z \in B \wedge Z \subset U \to U \in B)$ define a filter. Furthermore, if $a = a^* \wedge b = b^*$ then $\{i : a_i = a_i^*\} \in B$ and $\{i : b_i = b_i^*\} \in B$ and it follows that $\{i : a_i \circ b_i = a_i^* \circ b_i^*\} \in B$ because $\{i : a_i = a_i^*\} \cap \{i : b_i = b_i^*\} \subset \{i : a_i \circ b_i = a_i^* \circ b_i^*\}$. □

Proof of (ii):
$\neg((a = b) \wedge (a \neq b))$ if and only if $\neg(X \in B \wedge X^c \in B)$ i.e. $X^c \in B \to \neg(X \in B)$ i.e. the filter is proper. Furthermore, then $\neg((aRb) \wedge \neg(aRb))$ for every R because $\neg(X \in B \wedge X^c \in B)$ for every X. □

Proof of (iii):
$(a = b) \vee (a \neq b)$ if and only if $X \in B \vee X^c \in B$ i.e. $\neg(X \in B) \to X^c \in B$ i.e. the filter is ultrafilter. Furthermore, then $(aRb) \vee \neg(aRb)$ for every R because $\neg X \in B \to X^c \in B$ for every X. □

PROOF OF THE COROLLARY: (1) is evidently true and (2) follows from $\neg(X \in B) \leftrightarrow X^c \in B$. □

PROOF OF THE ŁOŚ'S THEOREM: For atomic formulae F, "$S \models F$ if and only if $(\underline{\forall} i)S_i \models F_i$" is the definition of $\models$. For truth functional F we have to prove that $\forall$ distributes through truth functional connectives and this follows from the corollary. For quantified $F = \exists x G$: $S \models_v \exists x G$ means $(\exists a) S \models_{v(a/x)} G$. By induction $S \models_{v(a/x)} G \leftrightarrow (\underline{\forall} i) S_i \models_{v_i(a_i/x)} G_i$. By skolemization $(\exists a)(\underline{\forall} i) S_i \models_{v_i(a_i/x)} G_i \leftrightarrow (\underline{\forall} i)(\exists a) S_i \models_{v_i(a_i/x)} G_i$. By definition of $\models$ this is equivalent to $(\underline{\forall} i) S_i \models_{v_i} \exists x G_i$. □

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University of Zagreb
Zagreb, Croatia
email: zsikic@math.hr

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Urszula Wybraniec-Skardowska

WHAT IS THE SENSE IN LOGIC AND PHILOSOPHY OF LANGUAGE?[1]

Abstract

In the paper, various notions of the logical semiotic sense of linguistic expressions – namely, syntactic and semantic, intensional and extensional – are considered and formalised on the basis of a formal-logical conception of any language L characterised categorially in the spirit of certain Husserl's ideas of pure grammar, Leśniewski-Ajdukiewicz's theory of syntactic/semantic categories and, in accordance with Frege's ontological canons, Bocheński's and some of Suszko's ideas of language adequacy of expressions of L. The adequacy ensures their unambiguous syntactic and semantic senses and mutual, syntactic and semantic correspondence guaranteed by the acceptance of a postulate of categorial compatibility of syntactic and semantic (extensional and intensional) categories of expressions of L. This postulate defines the unification of these three logical senses. There are three principles of compositionality which follow from this postulate: one syntactic and two semantic ones already known to Frege. They are treated as conditions of homomorphism of partial algebra of L into algebraic models of L: syntactic, intensional and extensional. In the paper, they are applied to some expressions with quantifiers. Language adequacy connected with the logical senses described in the logical conception of language L is, obviously, an idealisation. The syntactic and semantic unambiguity of its expressions is not, of course, a feature of natural languages, but every syntactically and semantically ambiguous expression of such languages may be treated as a schema representing all of its interpretations that are unambiguous expressions.

[1] The paper was delivered at the conference ,,Formal Methods and Science in Philosophy III" at Inter-University Centre, Dubrovnik, Croatia, April 11–13, 2019. It is based on results presented in the author's papers [65, 66].

Keywords: Logic and philosophy of language, categorial language, syntactic and semantic senses, intensional semantics, meaning, extensional semantics, denotation, categorisation, syntactic and semantic compatibility, algebraic models, truth, structural compatibility, compositionality, language communication.

1. Introduction

The word 'sense' has many meanings, and it appeals to us in many ways. On the basis of philosophy (and/or theology), it is for centuries that we have been trying to grasp and understand what the sense of our life is; likewise the sense of existence, the sense of our action and endeavour, and what the sense of the world is in general. From the point of view of philosophy, there are various visions and many theories regarding the sense of the world, the sense of life, our actions, etc. To discover their rational justifications, logical knowledge is needed, but, obviously, it is not enough. This philosophical meaning of the word 'sense' must clearly be distinguished from the logical, semiotic one. In the philosophical meaning, the word 'sense' is used as a certain property of extra-linguistic objects when it is said that something has or does not have sense, while referring to such objects. It derives from the basic, logical and semiotic meaning of this word, the meaning referring to linguistic objects, verbal signs. It should be noted, however, that it is not only the non-semiotic, but also the semiotic usage of the word 'sense' that is homogenous. Thus, one can speak of many notions of sense.

In this paper, we would like to characterise and formalise various notions of the *logical and philosophical sense* of linguistic expressions; from the viewpoint of logic, only these notions of sense can be of interest to us. The contemporary logic, logic of language (logical semiotics) can define the semiotic sense, *logical sense* strictly with regards to some general aspects of developing the cognition of the world and, at the same time, contributing to an explication of one of the most important traditional philosophical problems: *Language adequacy of our knowledge in relation to cognition of reality*, or, briefly: *language adequacy.* It is connected with the mutual relations between the three elements of the triad, reality–knowledge–language, and an adequate reflection of fragments of reality via expressions of language and inter-subjective knowledge of these fragments [67].

The above-mentioned adequacy requires, first of all, syntactic and semantic characterisation of language expressions as generalised by a grammar [57, 66]. Languages structured by grammar and logic are important tools of thinking, cognition of reality and knowledge acquisition, which stand for the foundations of our sense of existence [43]. In modern logic and philosophy of language, an approach based on Frege functions. It is implemented by the trend of formal and logical reflection on language and Fregean senses.

Logical sense, in its different variants, is considered and formalised on the basis of the conception of formalisation of language L, which is sketched below. The *syntactic sense* of these expressions is defined on the basis of language syntax and *semantic senses* – on the basis of bi-level language semantic: *intensional* and *extensional.*

From the logical point of view, the three notions of the sense of expressions of language L are understood as follows [62, 68] (see Fig. 1):

- *syntactic sense* is found in expressions of L which are well-formed; it is their *essence*; it is defined in the syntax of L, and-in accordance with Carnap's distinction, *intension-extension* [21], or Frege's differentiation, *Sinn-Bedeutung* [23]–two kinds of *semantic sense*:
  - *intensional sense* is proper to the expressions of L which have a meaning, *intension*; it is defined in intensional semantics of L,
  - *extensional sense* is proper to the expressions of L which have a denotation, *extension*; it is defined in extensional semantics of L.

The syntactic and semantic notions of sense must be differentiated and explicated. This is possible through a conceptualisation of these notions that will lead to a formal-logical theory of syntax and semantics of language L, which specifies and describes these notions.

There are different points of view on the grammar of language, its syntax and semantics. In the paper, any language L, its syntax and bi-level semantic: intensional and extensional, is characterised and formalised *categorially* in the spirit of some ideas of Husserl (see [30]) and Leśniewski-Ajdukiewicz's theory of syntactic/semantic categories [3, 4, 36, 37], in accordance with Frege's ontological canons [21], Bocheński's motto, *syntax mirrors ontology* [13], and some ideas of Suszko: *language should be a linguistic scheme of ontological reality and simultaneously a tool of its cognition* [50, 51, 52, 53].

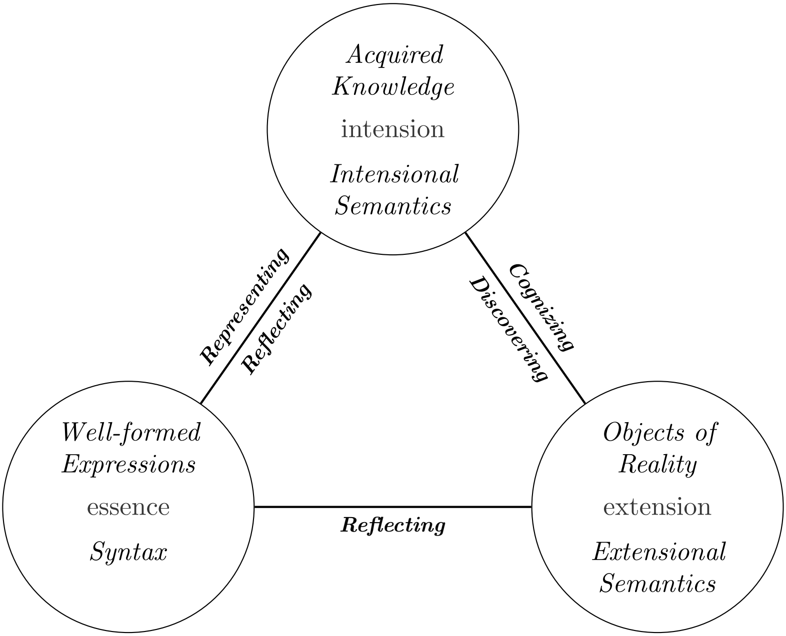

Fig. 1. Three notions of linguistic sense: essence, intension, extension.

2. Main ideas of the formalisation of categorial language L

Categorial language L is defined if the set S of all *well-formed expressions* (*wfes*) of L is determined. These expressions must satisfy the requirements of categorial syntax and categorial semantics. The categorial syntax is connected with generating the set S by the classical categorial grammar, the idea of which originated from Ajdukiewicz [3, 4] under the influence of Leśniewski's theory of semantic (syntactic) categories in his systems of protothetics and ontology [36, 37], under Husserl's ideas of pure grammar (see [30]), and under the influence of Russel's theory of logical types. The notion of categorial grammar was shaped by Bar-Hillel (see [6, 7, 8]) and developed by Lambek, Montague, Cresswell, Buszkowski, Marciszewski, Simons, Tałasiewicz and others [14, 15, 16, 17, 18, 19, 20, 21, 34, 35, 38, 39, 40, 45, 46, 52].The first formalisation of languages generated by the aforementioned classical categorial grammar, the notion introduced and explicated by Buszkowski was presented in the author's book in Polish [55] and its English translation, as well as some extension [56] (see also [63]).

In the categorial approach to language L, *wfes* of S should belong to appropriate syntactic categories. A characteristic feature of categorial syntax is that each composed *wfe* of the set S has a *functor-argument* structure, so that it is possible to distinguish in it the main part (the so-called *main functor*) and the other parts (called *arguments* of this functor), yet each constituent of the *wfe* has a determined syntactic category. Categorial intensional and extensional semantics is connected with *meaning* and *denotation* of *wfes* of S and with their membership in appropriate semantic categories: intensional and extensional, respectively (see [25, 61, 63, 64, 65, 66]). Each constituent of the composed *wfe* has a determined semantic (intensional and extensional) category, can have a *meaning* (*intension*) assigned to it, and thus also a *category of knowledge* (the category of constituents of knowledge) and also *denotation* (*extension*), and thus – an *ontological category* (the *category of ontological objects*).

The meanings (intensions) of *wfes* of L are treated as certain constituents of inter-subjective knowledge: logical concepts, logical judgments, operations on such notions or judgments, on the former and the latter, on other operations.

Object references (references) of *wfes* of L, and also constituents of knowledge, are objects of the cognised reality: individuals (concretes or abstract), states of things, operation on the indicated objects, and the like. *Denotations (extensions)* of *wfes* of L and constituents of knowledge are sets of such objects. The compatibility of these denotations is called *semantic compatibility* of L.

3. General assumption concerning the logical sense of expressions of language L

In the logical conception of language L and the semiotic senses outlined in the paper, expressions of L have syntactic, intensional and extensional senses and satisfy some general conditions of the logical sense of these expressions. Baseline conditions apply to syntactic and semantic unambiguity expressions of language L and the subsequent – relate to categorial compatibility and structural compatibility.

3.1. Syntactic and semantic unambiguity

The starting point is the syntactic and semantic unambiguity of the language expressions of language L. They should be:

- syntactically coherent and *wfes* of the set S (its essences),
- structurally unambiguous: have one syntactic sense (essence), i.e. do not contain amphiboly and have the one *mentioned functor-argument structure*,
- semantically unambiguous: have one *intension* and one *extension*, thus, one meaning and one denotation.

Remark 1. Syntactic and semantic unambiguity is not, of course, a feature of natural languages and not often of languages of non-exact sciences, but every syntactically and semantically ambiguous expression of these languages may be treated as a schema representing all of its interpretations that are unambiguous expressions (with exactly one syntactic and/or semantic sense) and which serve for an adequate description of specified fragments of reality.

For example, the sentence:

Teachers are tired because they teach students in various schools and they have a lot of them.

is structurally ambiguous (contains amphiboly), but it can be treated as a schema of two unambiguous sentences:

Teachers are tired because they teach students in various schools and they have a lot of students.

and

Teachers are tired because they teach students in various schools and they have a lot of schools.

On the other hand, the structurally unambiguous sentences

I came back tomorrow on foot on the colourful black-and-white train of 25:66.

She laughed with sweet tears which fell weightlessly onto the ceiling.

have no meaning or intensional and extensional sense; they are semantic nonsense.

In the categorial approach to language L, generated by the classical categorial grammar, a categorial index (type) *i(e)* of a certain set T of types is unambiguously assigned to every *wfe* e of the set S, and every composed *wfe* of S has the functor-argument structure. Categorial indices (types) were introduced into logical semiotics by Ajdukiewicz [3] with the goal of determining the syntactic role of expressions and to examine their *syntactic connection*, in compliance with the *principle of syntactic connection* (Sc) which, in a free formulation, says that:
(Sc) *The categorial type of the main functor of each functor-argument expression of language L is formed out of the categorial type of the expression which the functor forms together with its arguments, as well as out of the subsequent types' arguments of this functor.*

Every functor-argument expression e of L can be written in a functional-argument form as follows:

$$e = f(e_1, e_2, \ldots, e_n), \tag{e}$$

where f is the main functor of e and $e_1, e_2, \ldots, e_n$ are its subsequent arguments. Then, assuming that t is the type of e and $t_1, t_2, \ldots, t_n$ are successive types of its arguments, the type of the functor f satisfying the principle (Sc) can be written in the following quasi-fractional form:

$$i(f) = i(e)/i(e_1)i(e_2)\ldots i(e_n) = t/t_1t_2\ldots t_n. \tag{$i(f)$}$$

Then, the set S of all *wfes* of L is defined as the smallest set including the vocabulary V of L and closed under the principle (Sc):

Definition 1.

$$S = \bigcap \big\{X : V \subset X \wedge \forall e = f(e_1, e_2, \ldots, e_n)(Sc(e)) \to e \in X\big\} \text{ where,}$$
$$Sc(e) = (i(e) = t \wedge \forall j = 1, 2, \ldots n \; i(e_j) = t_j) \to i(f) = t/t_1t_2\ldots t_n.$$

In the formal definition of set S, it is required that each functor-argument constituent of the given expression should satisfy the principle (Sc).

Every *wfe* e of S is a *meaningful expression* of L possessing one *intension*, i.e. one meaning $\mu(e)$, where μ is the *operation of indicating the meaning* defined on the set S:

$$\mu : S \to \mu(S) = K.$$

The **meaning** $\mu(e)$ of the *wfe* e of the set S may be intuitively understood, in accordance with the understanding of meaning of expressions by Ajdukiewicz [1, 2] and, independently, by Wittgenstein [55] as a common property of all the *wfes* of S which possess the same manner of using as does e by competent users of language L (cf. [41]). Formalisation of thus conceived notion of meaning (and related notions) is given by Wybraniec-Skardowska in [62]. In [62], its different philosophical conceptions, in particular those originating from Richard Montague, Donald Davison or Michael Dummett, are sketched. In my approach to the meaning of an expression of L, it is treated as a constituent of knowledge $K = \mu(S)$.

Every *wfe* e of S is a *meaningful expression* of L possessing one denotation, *extension* $\delta(e)$, where δ is the operation of denoting defined on set S:

$$\delta : S \to \delta(S) = O.$$

The notion of *denoting* can, however, be introduced also as the *operation of denoting* δ_K, defined on the set of constituents of knowledge K:

$$\delta_K : K \to \delta_K(K) \subseteq O.$$

The **denotation** $\delta(e)$ of the meaningful expression e is defined as the set of all ontological objects (or the ontological object) of the set $O = \delta(S)$, whose occurrences the expression e refers to. The **denotation** $\delta_K(k)$ of the constituent k of knowledge K is defined as the set of all extra linguistic, ontological objects to which k refers. Semantic compatibility takes place iff $\delta(S) = \delta_K(K) = O$ (see Fig. 2).

3.2. Categorial compatibility

In the logical conception of language L, the three distinguished kinds of logical sense of expressions of L must be compatible: any *wfe* of L having the syntactic sense, *essence* (belonging to a syntactic category of the defined kind), has a semantic, intensional sense (*intension*) and an extensional sense (*extension*) and is, simultaneously, a meaningful expression of L belonging to a defined intensional and, respectively, to a defined extensional semantic category. The logical sense of *wfes* of L is connected with the compatibility of their syntactic and semantic, intensional and extensional categories. In the categorial approach to language, the aforementioned categories of *wfes* of L are determined by attributing to them, as to their

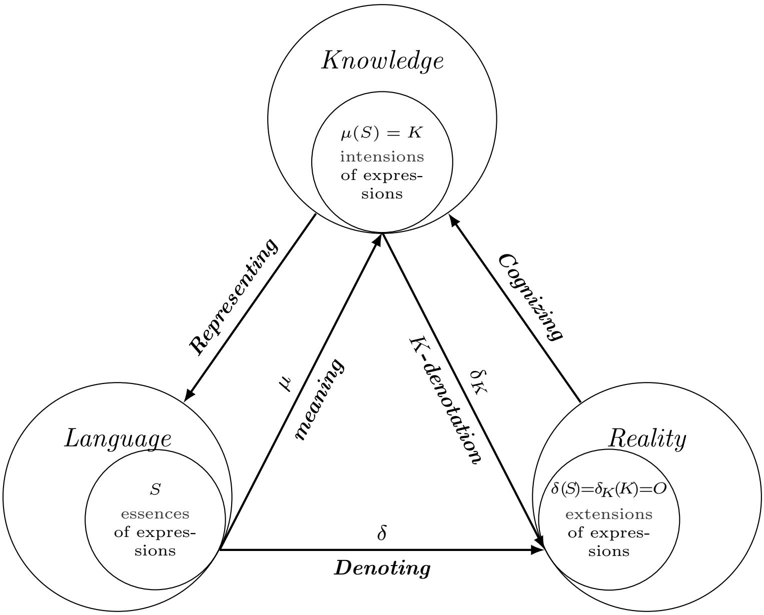

Fig. 2. Semantic compatibility and language adequacy.

expressions, categorial indices (types) of the set T. Compatible categories have the same categorial type that unifies these three notions of sense (see Fig. 3).

Categorial types play here the role of a tool coordinating meaningful expressions and extralinguistic objects: *intensions* and *extensions* [4, 47, 50, 51, 52].

3.2.1. Postulate of categorial compatibility

The postulate of categorial compatibility of syntactic and semantic categories is one of the most important conditions of the logical sense of *wfes* of language L. Here is a more formal description of this postulate. Let

1. S be the set of all *wfes* of L,
2. K – the set of all *intensions* of expressions of the set $S; K = \mu(S)$,
3. O – the set of all *extensions* of expressions of the set S; $O = \delta(S) = \delta_K(K)$.

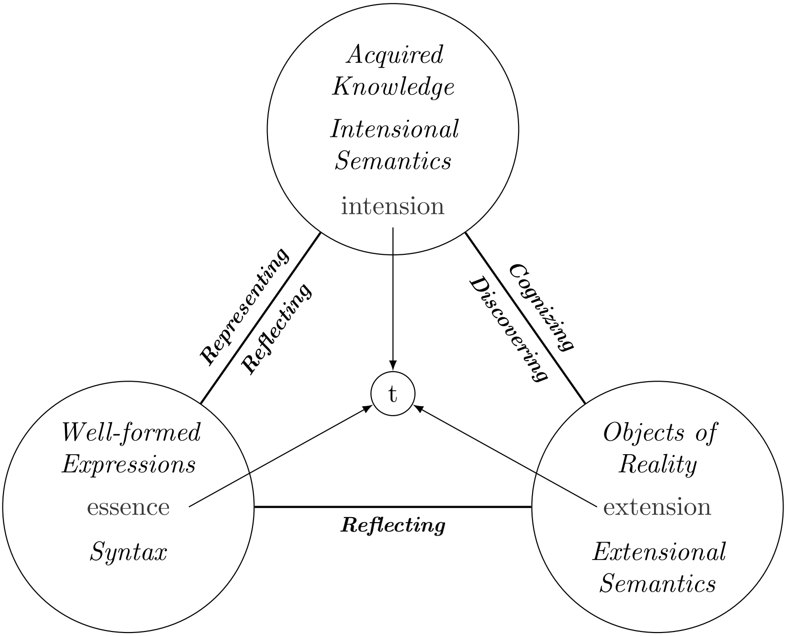

Fig. 3. Type-unifying three notions of logical sense: essence, intension, extension.

The above-discussed syntactic and semantic categories of meaningful *wfes* of L are the following subsets of the set S:

Definition 2. $Syn_{\boldsymbol{t}} = \{\boldsymbol{e} \in S : \boldsymbol{i(e)} = t\}$, where $\boldsymbol{i} : S \to T$,

Definition 3. $Int_{\boldsymbol{t}} = \{\boldsymbol{e} \in S : \boldsymbol{i_K}(\mu(\boldsymbol{e})\} = t\}$, where $\boldsymbol{i_K} : K \to T$,

Definition 4. $Eks_{\boldsymbol{t}} = \{\boldsymbol{e} \in S : \boldsymbol{i_O}(\delta(\boldsymbol{e})\} = t\}$, where $\boldsymbol{i_O} : O \to T$,

The syntactic (resp. **intensional**, resp. **extensional**) **category with the index** t is the set of all *wfes* of S that have the categorial index t (resp. *intensions* of which, resp. *extensions* of which have the index t).

The postulate of categorial compatibility defining an aspect of the logical sense of *wfes* of L has the following form [65, 66, 67]:

$$Syn_{\boldsymbol{t}} = Int_{\boldsymbol{t}} = Eks_{\boldsymbol{t}} \qquad \text{for any } t \in T. \qquad \text{AXIOM(P)}.$$

3.2.2. Type-unifying logical senses

The formal postulate (P) does not grasp the problem of the logical sense of language expressions of L adequately, because it does not show the relationships of the distinguished categories of *wfes* (*essences*) with the corresponding extra-linguistic categories of *intensions* and ontological categories of *extensions* in such a way that the mutual correspondence of elements of the triad: reality–knowledge-language, and the language adequacy of syntax with bi-level semantics, intensional and extensional, have been preserved (see Fig. 2).

As it was mentioned, unambiguous determined meanings (*intensions*) and denotations (*extensions*) should be assigned to *wfes* of L. They belong, respectively, to suitable extra linguistic categories of objects: *categories of meanings, intensions* (e.g. logical notions, logical judgments, operations on them) and *ontological categories of denotations, extensions* (e.g. individuals, set of individuals, states of affairs, or operations on them).

The categories of meanings, *intensions*, are subsets of the set K of constituents of knowledge, and ontological categories – subsets of the set O ontological objects. They are determined by categorial indices (types). And so, for any type $t \in T$:

Definition 5. $K_t = \{m \in K : i_K(m) = t\}$,

Definition 6. $O_t = \{o \in O : i_O(o) = t\}$.

Sematic categories (see Definitions 3 and 4) can by defined by formulas:

Corollary 1. $Int_t = \{e \in S : \mu(e) \in K_t\}$,

Corollary 2. $Ext_t = \{e \in S : \delta(e) \in O_t\}$,

stating that the semantic intensional (resp. extensional) category with the index t is the set of all *wfes* of L, the meanings, *intensions* (resp. denotations, *extensions*) of which belong to the category of constituents of knowledge (resp. to the ontological category) with the type t.

It is easy to prove that for any $e \in S$ and $t \in T$, by Corollaries 1 and 2 we can state that the Axiom (P) of categorial compatibility can be replaced by the following equivalent conditions:

Theorem 1. $e \in Syn_t$ *iff* $\mu(e) \in K_t$ *iff* $\delta(e) \in O_t$,

Theorem 2. $i(e) = i_K(\mu(e)) = i_O(\delta(e))$.

So, we see that categorial types serve also as a tool coordinating *wfes* of L and corresponding extra-linguistic objects, and that they unify the three notions of logical sense (see Fig. 4).

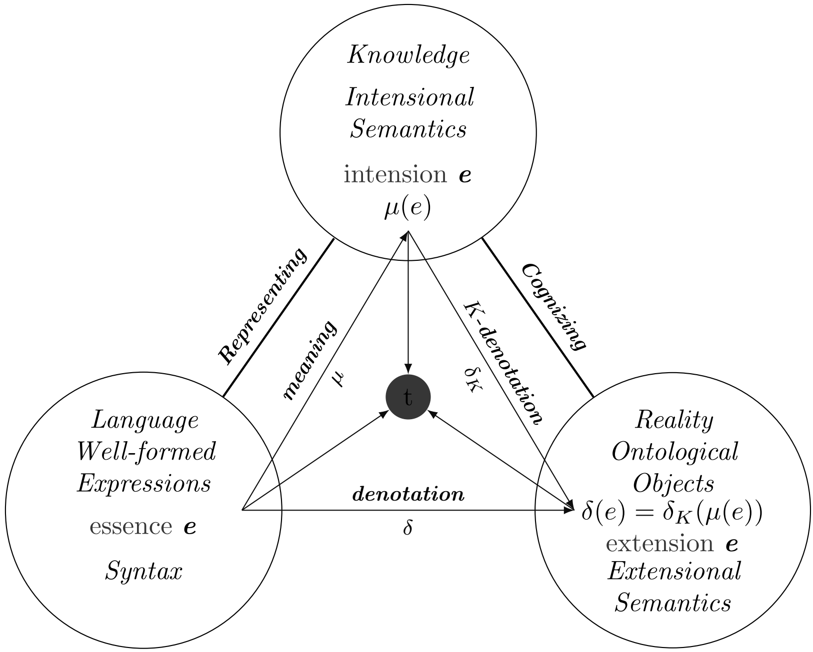

Fig. 4. Semantic compatibility and type – unifying the three notions of sense.

The idea of unification of the type of the logical term of the natural language, its intension and extension, is also one of the features of the type-theoretic object theory of E. Zalta [70, 71].

3.2.3. Semantic compatibility

From Diagrams 2 and 4, we conclude that ontological objects of the set O are not only denotations of *wfes* of the set S (its *essences*), but also denotations of *intensions* of knowledge K corresponding to them.

Semantic compatibility for language L is defined by the following formula:

Definition 7. $\delta(e) = \delta_K(\mu(e) \in O$,
where δ_K is the operation on intensions, meanings of knowledge K.

From Definition 7, it immediately follows that if two expressions of L have the same meaning, then they have the same denotation:

Corollary 3. $\mu(e) = \mu(e') \to \delta(e) = \delta(e')$.

It is well known that the reverse implication does not hold. For example, the extensions of the terms 'equilateral triangle' and 'equiangular triangle' are the same, but their intensions are not.

3.3. Structural compatibility

3.3.1. On the structure of expressions and their semantic counterparts

The form of language expressions, their connectivity, well-formedness and logical sense, are connected with the structure of our knowledge and the structure of the cognising part of reality. Its language description is composed of parts that can be separated. Some of them are independent or relatively independent and are counted as basic language categories. In categorial languages, these are *names* and *sentences*. Others are auxiliary, dependent constituents of language expressions, which allow for the construction of more composed expressions from simpler ones. They are *functors*.

The categorial approach to L allows us to define the structural compatibility of its composed expressions and their corresponding meanings and denotations. Every *wfe* of L has one functor-argument structure. Functors of such expressions may be treated as partial functions defined on a proper subset of the set S and with the values in this set. Language L can then be characterised as the following partial algebra:

$$\boldsymbol{L} = \langle S, F \rangle, \text{ where } F \subset S, \text{ and } F \text{ is the set of all functors of } L.$$

As we mentioned in Sec. 3.1, (e), every composed expression e of the set S can be written in the functional-argument form:

$$e = f(e_1, e_2, \ldots, e_n),$$

where f is the main functor and $e_1, e_2, \ldots, e_n$ its subsequent arguments.

If the expression e is *wfe* of the set S, then – in accordance with the principle of syntactic connection (Sc) – the index of its main functor f, formed from the type t of e and successive types $t_1, t_2, \ldots, t_n$ of successive arguments $e_1, e_2, \ldots, e_n$ of the functor f, can be written in the following quasi-fractional form $(i(f))$:

$$i(f) = t/t_1 t_2 \ldots t_n. \qquad (i(f))$$

The functor-function f corresponds to the function defined on meanings (*intensions*), respectively denotations (*extensions*) of arguments of this functor with subsequent types $t_1, t_2, \ldots, t_n$, the value of which is the intension, respectively the extension, of the expression e, which the functor f forms, with the type t.

If, in language L, we have two basic syntactic categories, names and sentences with respective types n and s, then meanings, *intensions* – logical notions with the type n – are assigned to names, and meanings, *intensions* – logical judgments with the type s – are assigned to sentences. Denotations of names are usually individuals or their sets, and denotations of sentences (in situational semantics) are states of affairs, situations. They also have, respectively, indices n and s.

Example. Let us consider the following sentence of a natural language:

Robert practices football. (i)

with the index s, the main functor of which is the word 'practices' of two name arguments, 'Robert' and 'football', with the index n. The expression (i) can be written in the following function-argument form:

practices(Robert, football). (ii)

The index of the functor 'practices' is s/nn. The meaning of the functor (with the same index) is the function which, being defined on the notions of 'Robert' and 'football' with the index n, as meanings (*intensions*) of these names in the sentence (ii), has, as the value, its meaning, i.e. the logical judgment with the index s stating that Robert practices football. Denotation of the functor is the mapping which, being defined on denotations (*denotates*) of names in (ii) with index n, so on person Robert and the sport discipline football, has, as its value, the state of affairs: the fact that Robert practices football, being the denotation of the sentence (ii); it has, like the sentence, the index s.

If somebody accepts, in accordance with Chomsky's phrase-structural grammar, that in (i) the main functor is 'practices football' (the predicate) of one argument 'Robert', then the function-argument form of (i) is as follows:

$$\text{practices football (Robert)} = \text{(practices(football))(Robert)} \qquad \text{(iii)}$$

The index of the composed functor 'practices football' is s/n, and the index of the functor 'practices' in it is $(s/n)/n$. Then the meaning and the denotation of the latter functor differ essentially from those used in (ii).

Remark 2. As we can see in a natural language, sentences may have a different functor-argument structure, thus different semantic senses: intensions and extensions. Therefore, they can be treated as skeletons, schemas which represent unambiguous expressions with one functor-argument structure, one meaning and one denotation.

3.3.2. Principles of compositionality

From the axiom (P) of categorial compatibility three principles of compositionality follow [63, 64, 65, 66]: one syntactic (compositionality of essences, syntactic forms) and two semantic: compositionality of meaning (intension) and compositionality of denotation (extension). For every composed expression of L, the form $e = f(e_1, e_2, \ldots, e_n)$ and functions $h = i, \mu, \delta$, their common schema has the form:

$$h(e) = h((f(e_1, e_2, \ldots, e_n)) = h(f)(h(e_1), h(e_2), \ldots, h(e_n)). \qquad (COMP_h)$$

For $h = i$, we have the syntactic principle, for $h = \mu, \delta$ we obtain the semantic principles corresponding to the ones already known to Frege [23] (cf. also [25, 31, 32, 39, 40, 42, 43, 26, 27, 28, 33]).

Speaking freely, these principles state that:
The categorial type (the syntactic form), resp. *the meaning*, resp. *the denotation of a well-formed functor-argument expression of language L is the value of the function of the type, resp. the function of the meaning,* resp. *the function of the denotation, of its main functor defined on types*, resp. *on meanings*, resp. *on denotations subsequent arguments of this functor.*

3.3.3. Main properties of functions $h(f)$

The formulation of the principle $(COMP_h)$ defines $h(f)$ as functions. Indeed, index $i(f)$ of functor f is the function:

$$i(f) : \{(i(e_1)\} \times \{i(e_2)\} \times \ldots \times \{i(e_n))\} \rightarrow \{i(e)\},$$

which, defined on n-tuple indices $(i(e_1), i(e_2), \ldots, i(e_n))$, has the value $i(e)$; hence there follows the syntactic principle of compositionality $(COMP_i)$.

Similarly, the meaning and the denotation of the functor f, defined on meanings and, respectively, on denotations of its arguments, are functions whose values are, respectively, meanings and denotations of the expression e. However, let us remember that the same n argument functor $(n \geq 1)$, e.g. 'practices' in (ii) of Example 3.3.1, may have different arguments, though its meaning, respectively denotation, is uniquely determined.

Thus, for any *wfe* $e = f(e_1, e_2, \ldots, e_n)$ such that for types $i(e) = t, i(e_k) = t_k$, where $k = 1, \ldots, n$, $\mu(f)$, is the function:

$$\mu(f) : K_{t1} \times K_{t2} \times \ldots \times K_{tn} \rightarrow K_t,$$

which for intensions of arguments of functor f has the value $\mu(e)$ compatible with the principle $(COMP_\mu)$, and $\delta(f)$ is the function:

$$\delta(f) : O_{t1} \times O_{t2} \times \ldots \times O_{tn} \rightarrow O_t,$$

which for denotations of arguments of functor f has the value $\delta(e)$ compatible with the principle $(COMP_\delta)$.

Remark 3. Note that the logical sense of language expressions, including functors, assumes that they have both intensions and extensions. Thus, any functor f forming the complex expression e has the meaning $\mu(f)$ and at the same time denotation (reference) $\delta(f)$, and its meaning and denotation are functions that meet the conditions listed above in accordance with the semantic principles of compatibility.
In semiotic literature, however, we encounter some controversy regarding the sense of functors, which are predicates of name arguments in natural language sentences. Debate on Geach-Dummet controversy about the sense of a predicate is reconstructed by M. Tałasiewicz [53]. For Peter T. Geach sense of a predicate is its meaning and a function satisfying the principle of compositionality of meaning, while for Michael Dummet the sense is rather something that determines its denotation (reference). Tałasiewicz in [53]

proposed a solution giving predicates both semantic senses as functions fulfilling the relevant conditions of semantic compositionality principles. This solution is interesting because it allows us to maintain semantic compatibility (see Def. 7).

3.3.4. Generalisation of Ajdukiewicz's cancellation principles

Just like the index of functor f in expression $e = f(e_1, e_2, \dots, e_n)$ (see $(i(f))$ in Sec. 3.1), we write its meaning and denotation in a quasi-fractional form. The general quasi-fractional form of the functions $h(f)$, for $h = i, \mu, \delta$ is given as the schema:

$$h(f) = h(e)/h(e_1)h(e_2)\dots h(e_n). \qquad (h(f))$$

At the established quasi-fractional records $(\boldsymbol{i}(f))$: the type of the functor $f, (\mu(f))$ of its meaning (intension) and $(\delta(f))$ of its denotation (extension), some counterparts of Ajdukiewicz's rules of cancellation of fractional indices (types) that serve to check the syntactic connection of complex expressions, correspond to the principles of compositionality $(COMP_h)$. They follow from them. To justify these rules, it is sufficient to use the equality $(COMP_h)$ from the left to the right and $(h(f))$. They allow us to calculate types, meanings (intensions) and denotations (extensions) of functor-argument expressions of L. Their schema, for $h = i, \mu, \delta$, can be written in the following way:

$$h(e)/h(e_1)h(e_2)\dots h(e_n)(h(e_1), h(e_2), \dots, h(e_n)) = h(e). \qquad \text{(CANCh)}$$

Example. For the functor 'practices' in the functor-argument sentence

$$\text{practices(Robert, football)} \qquad \text{(ii)}$$

the cancellation principles for $h = i, \mu, \delta$ have the forms:

$$s/nn(n, n) = s,$$
$$\mu((ii))/\mu(\text{Robert}\mu(\text{football})(\mu(\text{Robert}), \mu(\text{football})) = \mu(\text{(ii)}),$$
$$\delta((ii))/\delta(\text{Robert})\delta(\text{football})(\delta(\text{Robert}, \delta(\text{football})) = \delta(\text{(ii)}),$$

while for the functor 'practices football' in the sentence

$$\text{practices football(Robert)} = \text{(practices(football))(Robert)} \qquad \text{(iii)}$$

the cancellation principles for $h = i, \mu$ are the following:

$((s/n)//n(n))(n) = s/n(n) = s,$
$\mu(\text{practices football(Robert)}) = ((\mu(\text{practices(football)})(\mu(\text{Robert})) =$
$= (\mu(\text{practices})(\mu(\text{football}))(\mu(\text{Robert})) =$
$= (((\mu(iii)/\mu(\text{Robert}))//\mu(\text{football})(\mu(\text{football}))(\mu(\text{Robert})) =$
$= ((\mu(iii)/\mu(\text{Robert}))(\mu(\text{Robert})) = \mu(\text{iii}).$

Similarly, for $h = \delta$.

Let us observe that sentences (ii) and (iii) have the same categorial type s, and, according to Theorem 2, their intensions and extensions also have the type s. However, the appropriate constituents of these sentences and their intensions and extensions have different categorial types.

3.3.5. Models of L and the notion of truth

The principles of compositionality can be considered as some conditions of homomorphisms $h = i, \mu, \delta$ of the syntactic algebra of language $\boldsymbol{L}$ into algebras of its images $h(\boldsymbol{L})$, i.e.

$$\boldsymbol{L} = \langle S, F\rangle \underset{h}{\longrightarrow} h(\boldsymbol{L})\langle h(S), h(F)\rangle,$$

where F is the set of all simple functor-partial functions mapping subsets of set S into set S, and $h(F)$, for $h = i, \mu, \delta$, is the set of functions corresponding to the functor-functions of set F.

Let us notice that the algebraisation of language can already be found in Leibnitz's papers. We can also find the algebraic approach to issues connected with syntax, semantics and compositionality in Montague's *Universal Grammar* [39] and in papers of Dutch logicians, especially in those by J. van Benthem [9, 10, 11, 12] and T.M.V. Jansen [31, 32]. The difference between their approaches and the approach which is presented here lies in fact that carriers of the syntactic and semantic algebras include functors, or, respectively, their suitable correlates, i.e. their i- or semantic-function μ- and δ- images; simple functors and their suitable i-, μ-, δ- images are simultaneously partial operations of this algebras. They are set-theoretical functions, determining those operations.

The algebra $i(L) = \langle i(S), i(F)\rangle$ is called the *syntactic model of language* L, while the algebras

$$\mu(L) = \langle \mu(S),\ \mu(F)\rangle = \langle K, \mu(F)\rangle \quad \text{and} \quad \delta(\boldsymbol{L}) = \langle \delta(S), \delta(F)\rangle = \langle O, \delta(F)\rangle$$

are the *semantic models* for L; the first is called the *intensional model* for L, the other one, the *extensional model* for L.

In the process of cognition of reality, we want the sentences of the language L, representing the knowledge acquired about it, to be the carriers of true information about cognised portion of reality; they should be true in the above-mentioned models of L. Language as a tool for describing reality must distinguish the category of *sentences* among its syntactic categories. True sentences have informative content and allow us to enrich our knowledge. If for $h = i, \mu, \delta$, it is the case that the sentence e of language L is true in models $h(\boldsymbol{L})$, we may say that our cognition by means of the sentence e is *true*.

The notions of truthfulness in appropriate models are introduced theoretically by means of three new primitive notions Th, satisfying for $h = i, \mu, \delta$ the schema of axioms:

$$\emptyset \neq Th \subseteq h(S) \qquad \text{AXIOM}(Th)$$

and are understood intuitively, respectively, as the singleton consisting of the index of true sentences, the set of all true logical judgments, the set composed of the states of affairs that take place (in situational semantics) or the singleton composed of the value of truth (in Frege's semantics).

For $h = i, \mu, \delta$, we assume that:

DEFINITION 8. The sentence e of language L *is true in the model* $h(\boldsymbol{L})$ iff $h(e) \in Th$.

In particular, if $h = \delta$, then we may state that the sentence e of L *is true in the extensional model* iff its extension is the state of affairs that takes place (in situational semantics), or it is the value of truth (in Fregean semantics).

3.3.6. Some remarks concerning the problem of categories of first-order quantifiers

There is a well-known problem with determining syntactic and semantic categories, and therefore a problem with categorial types of quantifiers, and, in particular, of quantifiers of the first order language $L1$ and types of their intensions and extensions. To solve this problem, we can apply the principles of compositionality and the cancellation rules. Some general findings relating to the solution to the problem of syntactic categories of quantifiers, their denotation or/and meaning are presented in the following papers: [58, 59, 69, 70]. In this work, I will limit myself to dealing with this problem for the quantifier in the simple formulas of $L1$.

Example. Let us consider the quantifier expressions:

$$(1) \quad \forall_x P(x) \qquad \text{and} \qquad (2) \quad \exists_x P(x),$$

in which P is an established one-argument predicate treated as a one-argument functor-function, and the quantifiers $\forall$ and $\exists$ are treated as two-argument functors-functions defined on a variable standing next to them and a sentential function with a free variable bound by the given quantifier. The categorial type for x is n_1, i.e. $i(x) = n_1$, the type for P is s_1/n_1, i.e. $i(P) = s_1/n_1$, because we assume that the type for the sentential function $P(x)$ is s_1, since $i(P(x)) = i(P)(i(x)) = s_1/n_1(n_1) = s_1$. The type of quantifiers $\forall$ and $\exists$ is then: s/n_1s_1, i.e. $i(\forall) = i(\exists) = s/n_1s_1$. Using the principles of compositionality and cancellation, we can 'compute' the type of the expression (1) in its functor-argument form:

$$\begin{aligned} i(\forall(x, P(x)) &= i(\forall)(i(x), i(P(x))) = i(\forall)(i(x), i(P)(i(x))) \\ &= s/n_1s_1(n_1, s_1/n_1(n_1)) = s/n_1s_1(n_1, s_1) = s. \end{aligned}$$

In a similar way, we 'calculate' the index of the expression (2) $= \exists(x, P(x))$. Thus, expressions (1) and (2) are sentences.

We will now define the denotation of the discussed quantifiers in Fregean semantics. We assume that $\delta(x) = U$, where U is the universe of individuals in an established model $\boldsymbol{M}_{L1}$; $\delta(P) : U \to \delta(P(x))$, where $\delta(P(x)) = \delta(P)(\delta(x)) = \{u \in U : \delta(P(x/u)) = 1\}$ and $P(x/u)$ is a sentence which we get for replacing in the sentential function $P(x)$ its free variable x by the name of the individual u, and 1 is the value truth. Then,

$$\delta(\forall_x P(x)) = \delta(\forall)(\delta(x), \delta(P(x)) = \begin{cases} 1 & \text{if} \quad \delta(x) = U = \delta(P(x)) \\ 0 & \text{if} \quad \delta(x) = U \neq \delta(P(x)) \end{cases}$$

$$\delta(\exists_x P(x)) = \delta(\exists)(\delta(x), \delta(P(x)) = \begin{cases} 1 & \text{if} \quad \delta(x) \cap \delta(P(x)) \neq \emptyset \\ 0 & \text{if} \quad \delta(x) \cap \delta(P(x)) = \emptyset. \end{cases}$$

So, *the denotation* $\delta(\forall)$ (resp. $\delta(\exists)$) of the quantifier $\forall$(resp. $\exists$) *is the function which, for the universe U and the denotation of the scope of the quantifier, has the truth value iff the denotation of its scope is the universe (resp. the denotation of this scope has at least one individual of the universe).*

In a similar way, we define the meanings of the quantifiers $\forall$ and $\exists$ in (1) and (2).

EXAMPLE. It is obvious that the quantifiers $\forall$ and $\exists$ are typically ambiguous in logic, depending on a type. In other contexts, e.g., in the expressions

(3) $\forall_{x,y}R(x,y)$ and (4) $\exists_{x,y}R(x,y)$ or
(5) $\forall_{x}R(x,y)$ and (6) $\exists_{y}R(x,y)$

they have other categorial types, intensions and extensions. Their categorial type in expressions (3) and (4) is $s/n_1n_1s_2$, where s_2 is the index of the sentential function of two individual variables, while in expressions (5) and (6) they have the type s_1/n_1s_2. The predicate-functor's R categorial type is, of course, s_2/n_1n_1.

It is easy to check and 'compute' that exemplary expressions are syntactically connective, therefore *wfes*. The first of them, (3) and (4), are sentences, because they have the index s, while the others, (5) and (6), are sentential functions with one free variable, because they have the index s_1.

4. Final remarks

The logical sense of language expressions is, of course, a kind of idealisation. In the logical and categorial conception of language, the sense of its expressions, both syntactic and semantic, intensional and extensional, ensures their structural and semantic unambiguity and mutual syntactic and semantic compatibility.

A natural language, and often also the scientific variation, is a living creature, still developing. The degree of syntactic and semantic senses of its expressions changes, it can be narrower or higher, depending on its skilful precision. However, structural or semantically ambiguous expressions can always be split into expressions having unambiguous syntactic and semantic senses and be categorially analysed. Also, expressions that are imprecise or vague can be replaced by sets of sentences with precise meanings and denotations. Moreover, they can be considered separately with respect to their categorial structure, because only expressions with a high degree of logical sense, syntactical and semantical (intensional and extensional), get closer to the sense and may, after a proper justification, become theorems of a given discipline of knowledge and be a base for satisfactory interpersonal communication about our world.

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Cardinal Stefan Wyszyński University in Warsaw
Department of Philosophy
Warsaw, Poland
email: `skardowska@gmail.com`

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