UNIVERSITY OF ŁÓDŹ
DEPARTMENT OF LOGIC

BULLETIN
OF THE SECTION OF LOGIC

VOLUME 49, NUMBER 1

ŁÓDŹ, MARCH 2020

The **Bulletin of the Section of Logic** (*BSL*) is a quarterly peer-reviewed journal published with the support from the University of Łódź. Its aim is to act as a forum for a wide and timely dissemination of new and significant results in logic through rapid publication of short papers. *BSL* publishes contributions on topics dealing directly with logical calculi, their methodology, and algebraic interpretation.

Papers may be submitted to the Editor-in-Chief or to any of the Collecting Editors. While preparing the munuscripts for publication please consult Submission Information.

* * *

Editorial Office: Department of Logic, University of Łódź
ul. Lindleya 3/5, 90–131 Łódź, Poland
e-mail: *bulletin@uni.lodz.pl*

Homepage: *http://czasopisma.uni.lodz.pl/bulletin*

TABLE OF CONTENTS

Bulletin of the **S**ection of **L**ogic
Volume 49/1 (2020), pp. 1–11

http://dx.doi.org/10.18778/0138-0680.2020.01

Eunsuk Yang

NILPOTENT MINIMUM LOGIC NM AND PRETABULARITY

Abstract

This paper deals with pretabularity of fuzzy logics. For this, we first introduce two systems NM^{nfp} and $\mathrm{NM}^{\overline{\frac{1}{2}}}$, which are expansions of the fuzzy system NM (Nilpotent minimum logic), and examine the relationships between NM^{nfp} and the another known extended system NM^{-}. Next, we show that NM^{nfp} and $\mathrm{NM}^{\overline{\frac{1}{2}}}$ are pretabular, whereas NM is not. We also discuss their algebraic completeness.

Keywords: Pretabularity, nilpotent minimum logic, algebraic semantics, fuzzy logic, finite model property.

1. Fuzzy logic and pretabularity

This paper is a contribution to the study of pretabularity of fuzzy logics. In general, a logic L is said to be pretabular if it does not itself have a finite characteristic matrix (algebra, or frame), but every normal extension of it does (see [4, 7, 8, 11, 13]). Note that Dunn (and Meyer) [3, 5] investigated the pretabularity of the semi-relevance logic RM (R with mingle) and the Dummett-Gödel logic G. One interesting fact is that these systems can be also regarded as *fuzzy* logics.[1] Then, a natural question is now raised as follows.

[1] According to Cintula (and Běhounek) [1, 2], a (weakly implicative) logic L is said to be *fuzzy* if it is complete with respect to (w.r.t.) linearly ordered matrices (or algebras) and *core fuzzy* if it is complete w.r.t. *standard* algebras (i.e., algebras on the real unit interval $[0, 1]$).

Which fuzzy logics are pretabular?

This question, on the one hand, is not interesting in the sense that most basic fuzzy logics such as UL (Uninorm logic), MTL (Monoidal t-norm logic), and BL (Basic fuzzy logic) are not pretabular because such logics have some axiomatic extensions (henceforth, extensions for short) without finite characteristic matrices. On the other hand, it is interesting in that while, since then, no further pretabular fuzzy logics have been introduced, we can still introduce other concrete fuzzy logic systems.

We introduce two new pretabular systems as fuzzy logics, which we shall call the fixed-pointed nilpotent minimum logic $\mathrm{NM}^{\overline{\frac{1}{2}}}$ and the non-fixed-pointed nilpotent minimum logic NM^{nfp}. These two are the systems expanding and extending, respectively, the well-known fuzzy system NM (Nilpotent minimum logic) [6].[2] In particular, the system NM^{nfp} can be regarded as a Hilbert-style presentation of NM^{-} (the NM with (BP) below), which is one of the extensions of NM introduced in [9, 10]. For this purpose, we first introduce these two systems and examine the relationship between NM^{nfp} and NM^{-}. We then show that NM^{nfp} and $\mathrm{NM}^{\overline{\frac{1}{2}}}$ are pretabular while NM is not. We also discuss their algebraic completeness.

2. Nilpotent minimum logics

The nilpotent minimum logic NM can be based on a countable propositional language with formulas Fm built inductively as usual from a set of propositional variables VAR, binary connectives $\rightarrow, \&, \wedge$, and constant $\mathbf{F}$, with defined connectives: (df1) $\neg A := A \rightarrow \mathbf{F}$; (df2) $A \vee B := ((A \rightarrow B) \rightarrow B) \wedge ((B \rightarrow A) \rightarrow A)$; (df3) $A \leftrightarrow B := (A \rightarrow B) \wedge (B \rightarrow A)$.

The constant $\mathbf{T}$ is defined as $\mathbf{F} \rightarrow \mathbf{F}$. For the rest of this paper, we use the customary notations and terminology, and the axiom systems to provide a consequence relation.

We start with the following axiomatizations of NM and its two expansions.

Definition 1.

(i) ([6]) NM consists of the following axiom schemes and rules:
A1. $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$;

[2] For the definitions of expansion and extension, see Definition 9 in [2].

A2. $(A\&B) \to A$;
A3. $(A\&B) \to (B\&A)$;
A4. $(A \wedge B) \to A$;
A5. $(A \wedge B) \to (B \wedge A)$;
A6. $(A\&(A \to B)) \to (A \wedge B)$;
A7. $(A \to (B \to C)) \leftrightarrow ((A\&B) \to C)$;
A8. $((A \to B) \to C) \to (((B \to A) \to C) \to C)$;
A9. $\mathbf{F} \to A$;
A10. $\neg\neg A \to A$;
A11. $((A\&B) \to \mathbf{F}) \vee ((A \wedge B) \to (A\&B))$;
$A \to B, A \vdash B$ (modus ponens, mp);
$A, B \vdash A \wedge B$ (adjunction, adj).

(ii)
- Non-fixed-pointed nilpotent minimum logic NM^{nfp} is NM plus $(A \vee \neg A) \to ((A\&A) \vee (\neg A\&\neg A))$ (Non-fixed-point, Nfp).
- Fixed-pointed nilpotent minimum logic NM$^{\overline{\frac{1}{2}}}$ is NM plus $\overline{\frac{1}{2}}$ and $\overline{\frac{1}{2}} \leftrightarrow \neg\overline{\frac{1}{2}}$ (Fixed-point, Fp).[3]

For convenience, '$\neg$,' '$\wedge$,' '$\vee$,' and '$\to$' are used ambiguously as propositional connectives and as algebraic operators, but context should clarify their meaning.

The algebraic counterpart of L $\in \{NM, NM^{nfp}, NM^{\overline{\frac{1}{2}}}\}$ is defined as follows.

DEFINITION 2.

(i) An *NM-algebra* is a structure $\mathcal{A} = (A, \top, \bot, \wedge, \vee, *, \to, \neg)$, where $\neg x := x \to \bot$ for all $x \in A$ and $x \vee y := ((x \to y) \to y) \wedge ((y \to x) \to x)$ for all $x, y \in A$, such that:
  - $(A, \top, \bot, \wedge, \vee)$ is a bounded lattice with top element $\top$ and bottom element $\bot$;
  - $(A, *, \top)$ is an integral commutative monoid;
  - $y \leq x \to z$ iff $x * y \leq z$ (residuation);
  - $\top = (x \to y) \vee (y \to x)$ (prelinearity);

[3] The constant $\overline{\frac{1}{2}}$ does not necessarily correspond to the actual fraction $\frac{1}{2}$. Since the standard negation $\neg x$ is defined as $1 - x$ in $[0, 1]$ and $\frac{1}{2}$ has the role of fixed-point in that $\frac{1}{2} = \neg\frac{1}{2}$ in $[0, 1]$, $\frac{1}{2}$ is used as a representative of fixed-point. Therefore, here we use $\overline{\frac{1}{2}}$ as the constant for denoting a fixed-point element of any algebra.

– $\neg\neg x = x$ (involution);
– $\top = ((x*y) \to \bot) \vee ((x \wedge y) \to (x*y))$ (weak nilpotent minimum).

(ii) • An *NM^{nfp}-algebra* is an NM-algebra satisfying $x \vee \neg x \leq (x * x) \vee (\neg x * \neg x)$ (non-fixed-point).
• An *$NM^{\overline{\frac{1}{2}}}$-algebra* is an NM-algebra with $\frac{1}{2}$ satisfying $\frac{1}{2} = \neg\frac{1}{2}$ (fixed-point).

Consider the system NM^{-}, which is NM plus (BP) $\neg(\neg(A \,\&\, A) \,\&\, \neg(A \,\&\, A)) \leftrightarrow (\neg(\neg A \,\&\, \neg A) \,\&\, \neg(\neg A \,\&\, \neg A))$. This system was introduced as the logic with semantics on $[0,1]$ minus the fixed-point in [9]. Let linearly ordered algebras be chains. We finally consider the relationships between NM^{nfp} and NM^{-}.

THEOREM 1.

(1) ([9]) *A nontrivial NM-chain satisfies* $(BP^{\mathcal{A}})$ $\neg(\neg(x*x)* \neg(x*x)) = \neg(\neg x * \neg x) * \neg(\neg x * \neg x)$ *iff it does not contain a fixed-point.*

(2) A nontrivial NM-chain satisfies (non-fixed-point) iff it does not contain a fixed-point.

PROOF: For the left-to-right direction of (2), we assume that there is an element $x > \bot$ such that $x = \neg x$ and show that $x \vee \neg x > (x*x) \vee (\neg x * \neg x)$. Let $x = \neg x$. Then, since $x * x = \neg x * \neg x = \bot$, we have that $x \vee \neg x > (x * x) \vee (\neg x * \neg x) = \bot$. For the right-to-left direction of (2), assume that $x \neq \neg x$ for all $x \in A$. First, consider the case $x < \neg x$. Using (weak nilpotent minimum), we can obtain that $x * x = \bot$ and $\neg x * \neg x = \neg x$ and thus $x \vee \neg x = \neg x = (x*x) \vee (\neg x * \neg x)$; therefore, $x \vee \neg x \leq (x*x) \vee (\neg x * \neg x)$. Consider the case $\neg x < x$. Its proof is analogous to that of the case $x < \neg x$. □

COROLLARY 1. *A nontrivial NM-chain satisfies* $(BP^{\mathcal{A}})$ *iff it satisfies (non-fixed-point).*

Now consider the systems NM^{nfp} and NM^{-} synthetically. We can show the following.

THEOREM 2. *The system NM^{nfp} proves:*
(BP) $\neg(\neg(A\&A)\&\neg(A\&A)) \leftrightarrow (\neg(\neg A\&\neg A)\&\neg(\neg A\&\neg A))$.

PROOF: First, note that the following are theorems of NM: (a) $(A \to B) \vee (B \to A)$; (b) $A \to (\neg A \to B)$; (c) $A \to (B \to A)$; (d) $\neg\neg A \leftrightarrow A$; (e) $((A\&A) \vee (\neg A\&\neg A)) \to (A \vee \neg A)$.

($\Rightarrow$) 1. $(A \to \neg A) \vee (\neg A \to A)$ (a);
2. $((A \to \neg A)\&(A \to \neg A)) \vee ((\neg A \to A)\&(\neg A \to A))$ (1, $\mathbf{T}\&\mathbf{T} \leftrightarrow \mathbf{T}$);
3. $((A \to \neg A)\&(A \to \neg A)) \to (\neg((A \to \neg A)\&(A \to \neg A)) \to ((\neg A \to A)\&(\neg A \to A)))$ (b);
4. $((\neg A \to A)\&(\neg A \to A)) \to (\neg((A \to \neg A)\&(A \to \neg A)) \to ((\neg A \to A)\&(\neg A \to A)))$ (c);
5. $\neg((A \to \neg A)\&(A \to \neg A)) \to ((\neg A \to A)\&(\neg A \to A))$ (2, 3, 4, adj, mp);
6. $\neg(\neg(A\&A)\&\neg(A\&A)) \to (\neg(\neg A\&\neg A)\&\neg(\neg A\&\neg A))$ (5, d, (df4) $A\&B := \neg(A \to \neg B)$).

($\Leftarrow$) 1. $(A \vee \neg A) \leftrightarrow ((A\&A) \vee (\neg A\&\neg A))$ (e, Nfp, adj, df3);
2. $\neg(A \vee \neg A) \leftrightarrow \neg((A\&A) \vee (\neg A\&\neg A)) \leftrightarrow \mathbf{F}$ (1, df1, A9, adj);
3. $(A \wedge \neg A) \leftrightarrow (\neg(A\&A) \wedge \neg(\neg A\&\neg A)) \leftrightarrow \mathbf{F}$ (2, d, De Morgan);
4. $(\neg(A\&A)\&\neg(A\&A)) \wedge (\neg(\neg A\&\neg A)\&\neg(\neg A\&\neg A)) \leftrightarrow \mathbf{F}$ (3, $\mathbf{F}\&\mathbf{F} \leftrightarrow \mathbf{F}$);
5. $\neg(\neg(A\&A)\&\neg(A\&A)) \vee \neg(\neg(\neg A\&\neg A)\&\neg(\neg A\&\neg A)) \leftrightarrow \mathbf{T}$ (4, $\neg\mathbf{F} \leftrightarrow \mathbf{T}$, De Morgan);
6. $(\neg(\neg A\&\neg A)\&\neg(\neg A\&\neg A)) \to \neg(\neg(A\&A)\&\neg(A\&A))$ (4, 5, Boolean property). $\square$

Then, from Theorem 2, the following question arises when we just think of the systems synthetically.

- *Open Problem*: Does the system NM$^-$ prove (Nfp) synthetically?

According to Corollary 1, it seems possible to show this since the conditions $(BP^{\mathcal{A}})$ and (non-fixed-point) both correspond to the condition 'no fixed-point.' However, we have not yet proved this. To the author, it seems that the correct axiomatization of the extension of NM with the semantics on $[0,1]^-$, i.e., $[0,1] \setminus \{\frac{1}{2}\}$, is not the axiomatization of NM$^-$, but that of NM^{nfp}.

3. Pretabularity

For L $\in \{NM, NM^{nfp}, NM^{\overline{\frac{1}{2}}}\}$, by an *L-algebra*, we henceforth denote any of NM-, NM^{nfp}-, and NM^{nfp}-algebras. By *1* and *0*, we express $\top$ and $\bot$, respectively, on the real unit interval $[0,1]$ or on a subset of it with top and bottom elements $1, 0$. We refer to L-algebras on such a carrier set as S^L*-algebras*. S^L-algebras are defined as follows:

DEFINITION 3. The operations for an S^L-algebra are defined as follows.

(1) ([6]) Let the carrier set S be $[0, 1]$. An S^{NM}*-algebra* is an algebra satisfying: T1. $x \wedge y = min(x, y)$; T2. $x \vee y = max(x, y)$; T3. $x \rightarrow y = 1$ if $x \leq y$, and otherwise $x \rightarrow y = max(1 - x, y)$; T4. $\neg x = 1 - x$.[4]

(2) Let the carrier set S be a subset of $[0, 1]$ with top and bottom elements $1, 0$.
- An $S^{NM^{nfp}}$*-algebra* is an S^{NM}-algebra whose carrier set S has no fixed-point.
- *An* $S^{NM^{\frac{1}{2}}}$*-algebra* is an S^{NM}-algebra whose carrier set S has $\frac{1}{2}$, a fixed-point.

By $S^L_{[0,1]}$*-algebra*, we henceforth denote the S^L-algebra on $[0, 1]$; by $S^L_{[0,1]^-}$*-algebra*, the S^L-algebra on $[0, 1]\backslash\{\frac{1}{2}\}$; by S^L_n*-algebra*, the S^L-algebra whose elements are in $\{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}$. Generalizing, S*-algebra* refers to any algebra whose elements form a chain with the greatest and least elements, and whose operations are defined in an analogous way.

Note that S-algebras having $\frac{1}{2}$ as an element x such that $x = \neg x$ are said to be *fixed-pointed*, and otherwise *non-fixed-pointed.* A logic L is said to be *fixed-pointed* if L is characterized by an S-algebra having a fixed-point, and otherwise is *non-fixed-pointed.* An extension of L is said to be *proper* if it does not have exactly the same theorems as L.

DEFINITION 4.

(i) (Tabularity) A logic L is *tabular* if L has some finite characteristic algebra.

(ii) (Pretabularity) A logic L is *pretabular* if (a) L is not tabular and (b) every proper extension of L has some finite characteristic algebra.

Now, we show that $\mathrm{L} \in \{NM^{nfp}, NM^{\overline{\frac{1}{2}}}\}$ is pretabular, but the systems NM is not. We first introduce some known pretabular logics.

FACT 1. ([3, 5]) Each of RM and G is pretabular.

[4] In general, the *involutive* negation is defined as the negation n satisfying $n(n(x)) = x$ for all $x \in [0, 1]$. Since any involutive negation $[0, 1]$ can be isomorphic to $1 - x$, for convenience, we take this definition.

We then divide the work into a number of propositions following the line in [3, 5].

PROPOSITION 1. *Let $\mathcal{X}$ be an extension of $L \in \{NM^{nfp}, NM^{\overline{\frac{1}{2}}}\}$, $\mathcal{A}$ be an $\mathcal{X}$-algebra, and $a \in \mathcal{A}$ be such that $a < \top$. Then, there is a homomorphism h of $\mathcal{A}$ onto an S-algebra which is an $\mathcal{X}$-algebra, such that $h(a) < 1$.*

PROOF: The proof is analogous to Theorem 3 in [3] and Theorem 11.10.4 in [4]. □

PROPOSITION 2.

(i) Let L be the system NM^{nfp}. Let $S_1^L, S_2^L, S_4^L, S_6^L, \ldots$, i.e., S_1^L and $S_{2n}^L, 1 \leq n \in N$, be the sequence of S^L-algebras relabeled in order as $M_1^L, M_2^L, M_3^L, \ldots$. If a sentence A is valid in M_i^L, then A is valid in M_j^L, for all j, $j \leq i$.

(ii) Let L be the system NM$^{\overline{\frac{1}{2}}}$. Let $S_1^L, S_3^L, S_5^L, S_7^L, \ldots$, i.e., $S_{2n-1}^L, 1 \leq n \in N$, be the sequence of S^L-algebras relabeled in order as $M_1^L, M_2^L, M_3^L, \ldots$. If a sentence A is valid in M_i^L, then A is valid in M_j^L, for all j, $j \leq i$.

PROOF: Since each S_j^L is (isomorphic to) a subalgebra or a homomorphic image of S_i^L, (i) and (ii) are immediate. □

PROPOSITION 3. *In S^{NM}-algebras, when i is even (≥ 4), S_i^{NM} validates a sentence A that is not valid in any odd-valued $S_j^{NM}, 3 \leq j \leq i$.*

PROOF: The claim can be verified by considering the sentence (Nfp), which is valid in every even-valued S_i^{NM}, but not in S_3^{NM} (and thus not in any odd-valued $S_j^{NM}, j \geq 3$). □

Remark 1. Proposition 3 implies that every valid sentence in $S_{[0,1]}^{NM}$ must be valid in $S_{[0,1]^-}^{NM}$, but there is a valid sentence in $S_{[0,1]^-}^{NM}$ that is not in $S_{[0,1]}^{NM}$.

Now, we recall the concept of a Lindenbaum-Tarski algebra. Let L $\in$ $\{NM^{nfp}, NM^{\overline{\frac{1}{2}}}\}$ and T be a theory in L. We define $[A] = \{B : T \vdash_L A \leftrightarrow B\}$ and L $= \{[A] : A \in Fm\}$. The *Lindenbaum-Tarski algebra* $\mathbf{Lind}_T$ w.r.t. L and T is L-algebra having the domain L, operations $\#^{\mathbf{Lind}}([A_1], \ldots, [A_n]) = [\#(A_1, \ldots, A_n)]$, where $\# \in \{\wedge, \&, \rightarrow\}$, and the top and bottom elements are [**T**] and [**F**], respectively. We call this algebra the Lindenbaum-Tarski algebra $\mathcal{A}(L)$.

Where $\mathcal{X}$ is a propositional system and $\mathbf{V}$ is a set of atomic sentences, let $\mathcal{X}/\mathbf{V}$ be that propositional system like $\mathcal{X}$ except that its sentences contain no atomic sentences other than those in $\mathbf{V}$ and thus $\mathcal{A}(\mathcal{X}/\mathbf{V})$ be its corresponding Lindenbaum-Tarski algebra. The following is obvious.

Proposition 4. *Let $\mathcal{X}$ be an extension of $L \in \{NM^{nfp}, NM^{\overline{\frac{1}{2}}}\}$. Then, $\mathcal{A}(\mathcal{X}/\boldsymbol{V})$ is an $\mathcal{X}$-algebra and is characteristic for $\mathcal{X}/\boldsymbol{V}$, since any non-theorem may be falsified under the canonical evaluation v_c, which sends every sentence A to $[A]$, where $[A]$ is the set of all sentences B such that $B \leftrightarrow A$.*

Also, it follows from Propositions 1 and 4 that:

Proposition 5. *Let $\mathcal{X}$ be an extension of $L \in \{NM^{nfp}, NM^{\overline{\frac{1}{2}}}\}$. Then, if a sentence A is not a theorem of $\mathcal{X}$, there is some S^L-algebra S^L_n such that S^L_n is an $\mathcal{X}$-algebra and A is not valid in S^L_n.*

Proof: If A is not a theorem of $\mathcal{X}$, then, by Proposition 4, A is falsifiable in the $\mathcal{X}$-algebra $\mathcal{A}(\mathcal{X}/\mathbf{V})$, where $\mathbf{V}$ is the set of sentential variables occurring in A, by the canonical evaluation v_c. However, since $[A]$ is undesignated in $\mathcal{A}(\mathcal{X}/\mathbf{V})$, then, by Proposition 1, there is a homomorphism h of $\mathcal{A}(\mathcal{X}/\mathbf{V})$ onto an S^L-algebra S^L such that S^L is an $\mathcal{X}$-algebra and $h([A]) < 1$ in S^L. However, the composition of h and v_c, $h \circ v_c(B) = h([B])$, is an evaluation that falsifies A in S^L. Note that an S^L-subalgebra, the image $h(\mathcal{A}(\mathcal{X}/\mathbf{V})$, is finitely generated since it is the homomorphic image of $\mathcal{A}(\mathcal{X}/\mathbf{V})$, which is finitely generated by the elements $[p]$ such that $p \in \mathbf{V}$. Thus, this algebra is finitely generated by the elements $[p]$ such that $p \in \mathbf{V}$. It is obvious that every finitely generated S^L-subalgebra is finite and isomorphic to some S^L_n. Thus, this algebra is isomorphic to some S^L_n, which completes the proposition. □

If $\mathcal{X}$ is L itself, we have the following completeness theorem as a corollary.

Corollary 2. (*Completeness*) *For $L \in \{NM^{nfp}, NM^{\overline{\frac{1}{2}}}\}$ and the set of S^L-algebras $\mathcal{S}^{\mathcal{L}}$, if a sentence A is valid in $\mathcal{S}^{\mathcal{L}}$, then A is a theorem of L.*

Proof: By proposition 5, we have that if a sentence A is not a theorem of L, there is some S^L-algebra S^L_n such that A is not valid in S^L_n. Thus, by contraposition, we obtain the claim. □

Finally, we turn to a proof of our principal results.

THEOREM 3.

(i) $L \in \{NM^{nfp}, NM^{\overline{\frac{1}{2}}}\}$ *is pretabular.*

(ii) *NM is not pretabular.*

PROOF: For (i), we show that every proper extension of L has a finite characteristic algebra. Let $M_1^L, M_2^L, M_3^L, \ldots$ be the sequence of S^L-algebras defined in Proposition 2. Let I be the set of indices of those S^L-algebras that are $\mathcal{X}$-algebras, where $\mathcal{X}$ is the given proper extension of L.

First, if I contains an infinite number of indices, then I contains every index because of Proposition 2. However, since every S^L-algebra M_i^L is an L-algebra, it follows from Proposition 5 and Corollary 2 that $\mathcal{X}$ is identical with L, which contradicts the hypothesis that $\mathcal{X}$ is a proper extension of L.

Second, if I contains only a finite number of indices, then, by Proposition 2, there must be some index i such that I contains exactly those indices less then or equal to i. By construction, S_i^L is an $\mathcal{X}$-algebra. Let a sentence A not be a theorem of $\mathcal{X}$. Then, by Proposition 5, A is not valid in some $\mathcal{X}$-algebra M_h^L, and, by our choice of i, $h \leq i$. However, by Proposition 2, A is not valid in M_i^L. Therefore, M_i^L is the desired finite characteristic algebra.

L itself has no finite characteristic algebra, which can easily be shown by a proof similar to that of Sugihara in [12]. Therefore, it can be ensured that L is pretabular.

(ii) directly follows from (i), Proposition 3, and Remark 1. (Note that the system NM^{nfp} is a pretabular extension of NM.) □

We finally remark some relationships between the results in Theorem 3 and algebraic results introduced in [9, 10].

Remark 2.

(1) The fact that NM^{nfp} is pretabular but NM is not can be *algebraically* obtained as a consequence of the full description of the lattice of subvarieties of the variety $\mathcal{NM}$ (see Theorems 2 and 3 and Figure 2 in [9] and Figure 1 in [10]).

(2) Pretabularity is a property related to logics whose associated varieties of algebras are locally finite. A variety of algebras is said to be *locally finite* if each of its finitely generated members is a finite algebras. We first note that the variety $\mathcal{NM}$ is locally finite (see [9, 10]).

Thus, since the varieties $\mathcal{NM}^{nfp}$ (the variety of non-fixed-pointed NM-algebras) and $\mathcal{NM}^{\overline{\frac{1}{2}}}$ (the variety of fixed-pointed NM-algebras) are subvarieties of $\mathcal{NM}$, $\mathcal{NM}^{nfp}$ and $\mathcal{NM}^{\overline{\frac{1}{2}}}$ are locally finite. These results show that every pretabular variety is locally finite, but not conversely.

4. Concluding remarks

We showed that the two fuzzy systems NM^{nfp}, $\mathrm{NM}^{\overline{\frac{1}{2}}}$ are pretabular while NM is not. We also showed that NM^{nfp} and NM^{-} are semantically equivalent. However, we have not yet shown this syntactically. This problem should be addressed in future research. We also have another interesting question as follows: Let L_1 and L_2 be two pretabular logics complete w.r.t. characteristic algebras S^{L_1} and S^{L_2}, and consider the logic L induced by the ordinal sum $S^{L_1} \oplus S^{L_2}$. Then, we can ask: Under which condition L is pretabular?

Acknowledgment. The author must thank the anonymous referees for their helpful comments. This work was supported by the Ministry of Education of the Republic of Korea and the National Research Foundation of Korea (NRF-2016S1A5A8018255).

References

[1] L. Běhounek and P. Cintula, *Fuzzy logics as the logics of chains*, **Fuzzy Sets and Systems**, Vol. 157 (2006), pp. 604–610.

[2] P. Cintula, *Weakly Implicative (Fuzzy) Logics I: Basic properties*, **Archive for Mathematical Logic**, Vol. 45 (2006), pp. 673–704.

[3] J. M. Dunn, *Algebraic completeness for R-mingle and its extensions*, **The Journal of Symbolic Logic**, Vol. 35 (1970), pp. 1–13.

[4] J. M. Dunn and G. Hardegree, **Algebraic Methods in Philosophical Logic**, Oxford University Press, Oxford, 2001.

[5] J. M. Dunn and R. K. Meyer, *Algebraic completeness results for Dummett's LC and its extensions*, **Mathematical Logic Quarterly**, Vol. 17 (1971), pp. 225–230.

[6] F. Esteva and L. Godo, *Monoidal t-norm based logic: towards a logic for left-continuous t-norms*, **Fuzzy Sets and Systems**, Vol. 124 (2001), pp. 271–288.

[7] D. Gabbay and V. B. Shetman, *Undecidability of modal and intermediate first-order logics with two individual variables*, **The Journal of Symbolic Logic**, Vol. 58 (1993), pp. 800–823.

[8] L. Galminas and J. G. Mersch, *A pretabular classical relevance logic*, **Studia Logica**, Vol. 100 (2012), pp. 1211–1221.

[9] J. Gispert, *Axiomatic extensions of the nilpotent minimum logic*, **Reports on Mathematical Logic**, Vol. 37 (2003), pp. 113–123.

[10] C. Noguera, F. Esteva, and J. Gispert, *On triangular norm based axiomatic extensions of the weak nilpotent minimum logic*, **Mathematical Logic Quarterly**, Vol. 54 (2008), pp. 387–409.

[11] V. Rybakov, V. Kiyatkin, and M. Terziler, *Independent bases for rules admissible in pretabular logics*, **Logic Journal of the Interest Group in Pure and Applied Logics**, Vol. 7 (1999), pp. 253–266.

[12] T. Sugihara, *Strict implication free from implicational paradoxes*, **Memoirs of the Faculty of Liberal Arts**, Fukui University, Series 1, 1955, pp. 55–59.

[13] K. Świrydowicz, *There exists an uncountable set of pretabular extensions of the relevant logic R and each logic of this set is generated by a variety of finite height*, **The Journal of Symbolic Logic**, Vol. 73 (2008), pp. 1249–1270.

Jeonbuk National University
Department of Philosophy & Institute of Critical Thinking and Writing
Rm 417 Center for Humanities & Social Sciences
Jeonju 54896, Korea
e-mail: eunsyang@jbnu.ac.kr

Bulletin of the **S**ection of **L**ogic
Volume 49/1 (2020), pp. 13–51

http://dx.doi.org/10.18778/0138-0680.2020.02

Aldo V. Figallo, Inés Pascual, Gustavo Pelaitay

A TOPOLOGICAL APPROACH TO TENSE $\mathrm{LM}_{n\times m}$-ALGEBRAS

Abstract

In 2015, tense $n \times m$-valued Łukasiewicz–Moisil algebras (or tense $LM_{n\times m}$-algebras) were introduced by A. V. Figallo and G. Pelaitay as an generalization of tense n-valued Łukasiewicz–Moisil algebras. In this paper we continue the study of tense $LM_{n\times m}$-algebras. More precisely, we determine a Priestley-style duality for these algebras. This duality enables us not only to describe the tense $LM_{n\times m}$-congruences on a tense $LM_{n\times m}$-algebra, but also to characterize the simple and subdirectly irreducible tense $LM_{n\times m}$-algebras.

Keywords: Tense $LM_{n\times m}$-algebras, Priestley-style topological duality, Priestley spaces, tense De Morgan algebras.

1. Introduction

In 1975, Suchoń ([36]) defined matrix Łukasiewicz algebras so generalizing n-valued Łukasiewicz algebras without negation ([29]). In 2000, A. V. Figallo and C. Sanza ([23]) introduced $n \times m$-valued Łukasiewicz algebras with negation which are both a particular case of matrix Łukasiewicz algebras and a generalization of n-valued Łukasiewicz–Moisil algebras ([1]). It is worth noting that unlike what happens in n-valued Łukasiewicz–Moisil algebras, generally the De Morgan reducts of $n \times m$-valued Łukasiewicz algebras with negation are not Kleene algebras. Furthermore, in [34] an important example which legitimated the study of this new class of algebras is provided. Following the terminology established in [1], these algebras were called $n \times m$-valued Łukasiewicz–Moisil algebras (or $\mathrm{LM}_{n\times m}$-algebras for short). $LM_{n\times m}$-algebras were studied in [24, 25, 15, 34] and [35].

Propositional logics usually do not incorporate the dimension of time; consequently, in order to obtain a tense logic, a propositional logic is enriched by the addition of new unary operators (or connectives) which are usually denoted by G, H, F and P. We can define F and P by means of G and H as follows: $F(x) = \neg G(\neg x)$ and $P(x) = \neg H(\neg x)$, where $\neg x$ denotes negation of the proposition x. Tense algebras (or tense Boolean algebras) are algebraic structures corresponding to the propositional tense logic (see [4, 19]). An algebra $\langle A, \vee, \wedge, \neg, G, H, 0, 1\rangle$ is a tense algebra if $\langle A, \vee, \wedge, \neg, 0, 1\rangle$ is a Boolean algebra and G, H are unary operators on A which satisfy the following axioms for all x, $y \in A$:

$$G(1) = 1,\ H(1) = 1,$$

$$G(x \wedge y) = G(x) \wedge G(y),\ H(x \wedge y) = H(x) \wedge H(y),$$

$$x \leq GP(x),\ x \leq HF(x),$$

where $P(x) = \neg H(\neg x)$ and $F(x) = \neg G(\neg x)$.

Taking into account that tense algebras constitute the algebraic basis for the bivalent tense logic, D. Diaconescu and G. Georgescu introduced in [12] the tense MV-algebras and the tense Łukasiewicz–Moisil algebras (or tense n-valued Łukasiewicz–Moisil algebras) as algebraic structures for some many-valued tense logics. In recent years, these two classes of algebras have become very interesting for several authors (see [2, 6, 8, 9, 15, 7, 17, 18]). In particular, in [8, 9], Chiriţă, introduced tense θ-valued Łukasiewicz–Moisil algebras and proved an important representation theorem which made it possible to show the completeness of the tense θ-valued Moisil logic (see [8]). In [12], the authors formulated an open problem about representation of tense MV-algebras, this problem was solved in [21, 3] for semisimple tense MV-algebras. Also, in [2], tense basic algebras which are an interesting generalization of tense MV-algebras, were studied.

The main purpose of this paper is to give a topological duality for tense $n \times m$-valued Łukasiewicz–Moisil algebras. In order to achieve this we will extend the topological duality given in [27], for $n \times m$-valued Łukasiewicz–Moisil algebras. In [35] another duality for $n\times m$-valued Łukasiewicz–Moisil algebras was developed, starting from De Morgan spaces and adding a family of continuous functions.

The paper is organized as follows: In Section 2, we briefly summarize the main definitions and results needed throughout this article. In Section 3, we developed a topological duality for tense $n \times m$-valued Łukasiewicz–Moisil algebras, extending the one obtained in [27] for $n \times m$-valued Łukasiewicz–Moisil algebras. In Section 4, the results of Section 3 are applied. Firstly, we characterize congruences on tense $n \times m$-valued Łukasiewicz–Moisil algebras by certain closed and increasing subsets of the space associated with them. This enables us to describe the subdirectly irreducible tense $n \times m$-valued Łukasiewicz–Moisil algebras and the simple tense $n \times m$-valued Łukasiewicz–Moisil algebras.

2. Preliminaries

2.1. Tense De Morgan algebras

In [16] A. V. Figallo and G. Pelaitay introduced the variety of algebras, which they call tense De Morgan algebras, and they also developed a representation theory for this class of algebras.

First, recall that an algebra $\langle A, \vee, \wedge, \sim, 0, 1\rangle$ is a De Morgan algebra if $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice and $\sim$ is a unary operation on A satisfying the following identities for all x, $y \in A$:

1. $\sim (x \vee y) = \sim x \wedge \sim y$,
2. $\sim\sim x = x$,
3. $\sim 0 = 1$.

In what follows a De Morgan algebra $\langle A, \vee, \wedge, \sim, 0, 1\rangle$ will be denoted briefly by $(A, \sim)$.

Definition 1. An algebra $(A, \sim, G, H)$ is a tense De Morgan algebra if $(A, \sim)$ is a De Morgan algebra and G and H are two unary operations on A such that for any $x, y \in A$:

1. $G(1) = 1$ and $H(1) = 1$,
2. $G(x \wedge y) = G(x) \wedge G(y)$ and $H(x \wedge y) = H(x) \wedge H(y)$,
3. $x \leq GP(x)$ and $x \leq HF(x)$, where $F(x) = \sim G(\sim x)$ and $P(x) = \sim H(\sim x)$,
4. $G(x \vee y) \leq G(x) \vee F(y)$ and $H(x \vee y) \leq H(x) \vee P(y)$.

In [16] a duality for tense De Morgan algebras is described taking into account the results established by W. Cornish and P. Fowler in [11]. To this purpose, the topological category **tmPS** of tmP-spaces and tmP-functions was considered, which we indicate below:

Definition 2. A tense De Morgan space (or tmP-space) is a system (X, g, R, R^{-1}), where

(i) (X, g) is an mP-space ([11]). More precisely,

(mP1) X is a Priestley space (or P-space),

(mP2) $g : X \longrightarrow X$ is an involutive homeomorphism and an anti-isomorphism,

(ii) R is a binary relation on X and R^{-1} is the converse of R such that:

(tS1) For each $U \in D(X)$ it holds that $G_R(U)$, $H_{R^{-1}}(U) \in D(X)$, where G_R and $H_{R^{-1}}$ are two operators on $\mathcal{P}(X)$ defined for any $U \subseteq X$ as follows:

$$G_R(U) = \{x \in X \mid R(x) \subseteq U\}, \tag{2.1}$$

$$H_{R^{-1}}(U) = \{x \in X \mid R^{-1}(x) \subseteq U\}, \tag{2.2}$$

and $D(X)$ is the set of all increasing and clopen subsets of X,

(tS2) $(x, y) \in R$ implies $(g(x), g(y)) \in R$ for any x, $y \in X$,

(tS3) for each $x \in X$, $R(x)$ is a closed set in X,

(tS4) for each $x \in X$, $R(x) =\downarrow R(x) \cap \uparrow R(x)$, where $\downarrow Y$ $(\uparrow Y)$ denotes the set of all $x \in X$ such that $x \leq y$ $(y \leq x)$ for some $y \in Y \subseteq X$.

Definition 3. A tmP-function from a tmP-space $(X_1, g_1, R_1, R_1^{-1})$ into another one, $(X_2, g_2, R_2, R_2^{-1})$, is a continuous and increasing function (P-function) $f : X_1 \longrightarrow X_2$, which satisfies the following conditions:

(mf) $f \circ g_1 = g_2 \circ f$ (mP-function [11]),

(tf1) $(x, y) \in R_1$ implies $(f(x), f(y)) \in R_2$ for any $x, y \in X_1$,

(tf2) if $(f(x), y) \in R_2$, then there is an element $z \in X_1$ such that $(x, z) \in R_1$ and $f(z) \leq y$,

(tf3) if $(y, f(x)) \in R_2$, then there is an element $z \in X_1$ such that $(z, x) \in R_1$ and $f(z) \leq y$.

Next, A. V. Figallo and G. Pelaitay (see [16, Section 5]) showed that the category **tmPS** is dually equivalent to the category **TDMA** of tense De Morgan algebras and tense De Morgan homomorphisms. The following results are used to show the dual equivalence:

- Let (X, g, R, R^{-1}) be a tmP-space. Then, $(D(X), \sim_g, G_R, H_{R^{-1}})$ is a tense De Morgan algebra, where for all $U \in D(X)$, $\sim_g U$ is defined by

$$\sim_g U = X \setminus g(U), \tag{2.3}$$

  and $G_R(U)$ and $H_{R^{-1}}(U)$ are defined as in (2.1) and (2.2), respectively.
- Let $(A, \sim, G, H)$ be a tense De Morgan algebra and $X(A)$ be the Priestley space associated with A, i.e. $X(A)$ is the set of all prime filters of A, ordered by inclusion and with the topology having as a sub-basis the following subsets of $X(A)$:

$$\sigma_A(a) = \{S \in X(A) : a \in S\} \text{ for each } a \in A, \tag{2.4}$$

  and

$$X(A) \setminus \sigma_A(a) \text{ for each } a \in A.$$

  Then, $(X(A), g_A, R_G^A, R_H^A)$ is a tmP-space, where $g_A(S)$ is defined by

$$g_A(S) = \{x \in A : \sim x \notin S\}, \text{ for all } S \in X(A), \tag{2.5}$$

  and the relations R_G^A and R_H^A are defined for all $S, T \in X(A)$ as follows:

$$(S, T) \in R_G^A \Longleftrightarrow G^{-1}(S) \subseteq T \subseteq F^{-1}(S), \tag{2.6}$$

$$(S, T) \in R_H^A \Longleftrightarrow H^{-1}(S) \subseteq T \subseteq P^{-1}(S). \tag{2.7}$$

- Let $(A, \sim, G, H)$ be a tense De Morgan algebra; then, the function $\sigma_A : A \longrightarrow D(X(A))$ is a tense De Morgan isomorphism, where σ_A is defined as in (2.4).
- Let (X, g, R, R^{-1}) be a tmP-space; then, $\varepsilon_X : X \longrightarrow X(D(X))$ is an isomorphism of tmP-spaces, where ε_X is defined by

$$\varepsilon_X(x) = \{U \in D(X) : x \in U\}, \text{ for all } x \in X. \tag{2.8}$$

- Let $h : (A_1, \sim_1, G_1, H_1) \longrightarrow (A_2, \sim_2, G_2, H_2)$ be a tense De Morgan morphism. Then, the map $\Phi(h) : X(A_2) \longrightarrow X(A_1)$ is a morphism of tmP-spaces, where

$$\Phi(h)(S) = h^{-1}(S), \text{ for all } S \in X(A_2). \tag{2.9}$$

- Let $f : (X_1, g_1, R_1, R_1^{-1}) \longrightarrow (X_2, g_2, R_2, R_2^{-1})$ be a morphism of tmP-spaces. Then, $\Psi(f) : D(X_2) \longrightarrow D(X_1)$ is a tense De Morgan morphism, where

$$\Psi(f)(U) = f^{-1}(U), \text{ for all } U \in D(X_2). \tag{2.10}$$

In [16], the duality described above was used to characterize the congruence lattice $Con_{tM}(A)$ of a tense De Morgan algebra $(A, \sim, G, H)$. First the following notion was introduced:

Definition 4. Let $(X, \leq, g, R, R^{-1})$ be a tmP-space. An involutive (i.e. $Y = g(Y)$ [11]) closed subset Y of X is a tmP-subset if it satisfies the following conditions for $u, v \in X$:

(ts1) if $(v, u) \in R$ and $u \in Y$, then there exists, $w \in Y$ such that $(w, u) \in R$ and $w \leq v$.

(ts2) if $(u, v) \in R$ and $u \in Y$, then there exists, $z \in Y$ such that $(u, z) \in R$ and $z \leq v$.

The lattice of all tmP-subsets of the tmP-space associated with a tense De Morgan algebra was taken into account to characterize the congruence lattice of this algebra as it is indicated in the following theorem:

Theorem 1. ([16, Theorem 6.4]) *Let $(A, \sim, G, H)$ be a tense De Morgan algebra and $(X(A), \subseteq, g_A, R_G^A, R_H^A)$ be the tmP-space associated with A. Then, the lattice $\mathcal{C}_T(X(A))$ of all tmP-subsets of $X(A)$ is anti-isomorphic to the lattice $Con_{tM}(A)$ of the tense De Morgan congruences on A, and the anti-isomorphism is the function Θ_T defined by the prescription:*

$$\Theta_T(Y) = \{(a, b) \in A \times A : \sigma_A(a) \cap Y = \sigma_A(b) \cap Y\}, \text{ for all } Y \in \mathcal{C}_T(X(A)). \tag{2.11}$$

2.2. $n \times m$-valued Łukasiewicz–Moisil algebras

In the sequel n and m are positive integer numbers and we use the notation $[n] := \{1, \ldots, n-1\}$ and so the cartesian product $\{1, \ldots, n-1\} \times \{1, \ldots, m-1\}$ is denoted by $[n] \times [m]$.

Definition 5. ([34, Definition 3.1.]) Let $n \geq 2$ and $m \geq 2$. An $n \times m$-valued Łukasiewicz–Moisil algebra (or LM$_{n\times m}$-algebra) is an algebra $\langle A, \wedge, \vee, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]}, 0, 1\rangle$, such that:

(a) the reduct $\langle A, \wedge, \vee, \sim, 0, 1\rangle$ is a De Morgan algebra,

(b) $\{\sigma_{ij}\}_{(i,j)\in[n]\times[m]}$ is a family of unary operations on A which fulfills the following conditions for any x, $y \in A$ and any $(i,j), (r,s) \in [n]\times[m]$:

(C1) $\sigma_{ij}(x \vee y) = \sigma_{ij}x \vee \sigma_{ij}y$,
(C2) $\sigma_{ij}x \leq \sigma_{(i+1)j}x$,
(C3) $\sigma_{ij}x \leq \sigma_{i(j+1)}x$,
(C4) $\sigma_{ij}\sigma_{rs}x = \sigma_{rs}x$,
(C5) $\sigma_{ij}x = \sigma_{ij}y$ for all $(i,j) \in [n]\times[m]$ imply $x = y$,
(C6) $\sigma_{ij}x \vee \sim \sigma_{ij}x = 1$,
(C7) $\sigma_{ij}(\sim x) = \sim \sigma_{(n-i)(m-j)}x$.

In what follows and where no confusion might arise, we denote these algebras by A or $\left(A, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]}\right)$, in the case we need to specify unary operators.

In Lemma 1 we summarize the most important properties of these algebras necessary in what follows.

Lemma 1. ([34, Lemma 3.1.]) *Let $\left(A, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]}\right)$ be an $LM_{n\times m}$-algebra. Then, the following properties are satisfied for all x, $y \in A$ and for all $(i,j) \in [n]\times[m]$:*

(C8) $\sigma_{ij}(x \wedge y) = \sigma_{ij}x \wedge \sigma_{ij}y$,

(C9) $\sigma_{ij}x \wedge \sim \sigma_{ij}x = 0$,

(C10) $x \leq y$ *if and only if* $\sigma_{ij}x \leq \sigma_{ij}y$ *for all* $(i,j) \in [n]\times[m]$,

(C11) $x \leq \sigma_{(n-1)(m-1)}x$,

(C12) $\sigma_{ij}0 = 0,\ \sigma_{ij}1 = 1$,

(C13) $\sigma_{11}x \leq x$,

(C14) $\sim x \vee \sigma_{(n-1)(m-1)}x = 1$,

(C15) $x \vee \sim \sigma_{11}x = 1$.

Definition 6. ([28, Definition 2.1.]) Let $\left(A, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]}\right)$ and $(A', \sim', \{\sigma'_{ij}\}_{(i,j)\in[n]\times[m]})$ be two $LM_{n\times m}$-algebras. A function $h : A \longrightarrow A'$ is an $LM_{n\times m}$-homomorphism if it satisfies the following conditions for all x, $y \in A$ and for all $(i,j) \in [n]\times[m]$:

(a) h is a lattice homomorphism,
(b) $h(\sim x) = \sim' h(x)$,
(c) $h(\sigma_{ij}x) = \sigma'_{ij}h(x)$.

LEMMA 2. ([28, Remark 2.2.]) *Let* $(A, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]})$ *and* $(A', \sim',$ $\{\sigma'_{ij}\}_{(i,j)\in[n]\times[m]})$ *be two* $LM_{n\times m}$*-algebras and* $h : A \longrightarrow A'$ *be a lattice homomorphism. Then the following conditions are equivalent:*

(a) *h is an $LM_{n\times m}$-homomorphism,*
(b) *$h(\sigma_{ij}x) = \sigma'_{ij}h(x)$ for all $x \in A$ and for all $(i,j) \in [n] \times [m]$.*

The results announced here for $LM_{n\times m}$-algebras are used throughout the paper.

(LM1) $\sigma_{ij}(A) = B(A)$ for all $(i,j) \in [n] \times [m]$, where $B(A)$ is the set of all complemented elements of A ([33, Proposition 2.5]).

(LM2) Every $LM_{n\times 2}$-algebra is isomorphic to an n-valued Łukasiewicz–Moisil algebra. It is worth noting that $LM_{n\times m}$-algebras constitute a non-trivial generalization of the latter (see [34, Remark 2.1]).

(LM3) The class of $LM_{n\times m}$-algebras is a variety and two equational bases for it can be found in [33, Theorem 2.7] and [34, Theorem 4.6].

(LM4) Let X be a non-empty set and let A^X be the set of all functions from X into A. Then A^X is an $LM_{n\times m}$-algebra, where the operations are defined componentwise.

(LM5) Let $B(A) \uparrow^{[n]\times[m]} = \{f : [n] \times [m] \longrightarrow B(A)$ such that for arbitrary i, j, if $r \leq s$, then $f(r,j) \leq f(s,j)$ and $f(i,r) \leq f(i,s)\}$. Then $\langle B(A) \uparrow^{[n]\times[m]}, \wedge, \vee, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]}, 0, 1\rangle$ is an $LM_{n\times m}$-algebra, where for all $f \in B(A) \uparrow^{[n]\times[m]}$ and $(i,j) \in [n] \times [m]$ the operations $\sim$ and σ_{ij} are defined as follows:

$$(\sim f)(i,j) = \neg f(n-i, m-j), \tag{2.12}$$

where $\neg x$ is the Boolean complement of x,

$$(\sigma_{ij}f)(r,s) = f(i,j) \text{ for all } (r,s) \in [n] \times [m], \tag{2.13}$$

and the remaining operations are defined componentwise ([34, Proposition 3.2]). It is worth noting that this result can be generalized by replacing $B(A)$ by any Boolean algebra B. Furthermore, if B is a complete Boolean algebra, it is simple to check that $B \uparrow^{[n]\times[m]}$ is also a complete $LM_{n\times m}$-algebra.

(LM6) Every $LM_{n\times m}$-algebra $(A, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]})$ can be embedded into the $LM_{n\times m}$-algebra $B(A)\uparrow^{[n]\times[m]}$ ([34, Theorem 3.1]). Besides, A is isomorphic to $B(A)\uparrow^{[n]\times[m]}$ if and only if A is *centred* ([34, Corollary 3.1]), where A is centred if for each $(i,j)\in[n]\times[m]$ there exists $c_{ij}\in A$ such that

$$\sigma_{rs}c_{ij} = \begin{cases} 0 & \text{if } i > r \quad \text{or} \quad j > s, \\ \\ 1 & \text{if } i \leq r \quad \text{and} \quad j \leq s. \end{cases}$$

(LM7) Let $\mathbf{2}\uparrow^{[\mathbf{n}]\times[\mathbf{m}]}$ be the set of all increasing functions from $[n]\times[m]$ to the Boolean algebra $\mathbf{2}$ with two elements. Then every simple $LM_{n\times m}$-algebra is a subalgebra of $\langle \mathbf{2}\uparrow^{[\mathbf{n}]\times[\mathbf{m}]}, \wedge, \vee, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]}, \mathbf{0}, \mathbf{1}\rangle$, where the operations of this $LM_{n\times m}$-algebra are defined as in statement (LM5) and $\mathbf{0}$, $\mathbf{1}\in\mathbf{2}\uparrow^{[\mathbf{n}]\times[\mathbf{m}]}$ are the functions $\mathbf{0}, \mathbf{1} : [n]\times[m]\longrightarrow\mathbf{2}$, defined for all $(i,j)\in[n]\times[m]$ by $\mathbf{0}((i,j)) = 0$ and $\mathbf{1}((i,j)) = 1$, respectively (see [34, Theorem 5.5]).

(LM8) Let A be an $LM_{n\times m}$-algebra. Then, the following conditions are equivalent:

(a) A is a subdirectly irreducible $LM_{n\times m}$-algebra,

(b) $B(A) = \{0, 1\}$, where $B(A) = \{\sigma_{ij}a : a\in A,\ (i,j)\in[n]\times[m]\}$.

In [27], A. V. Figallo, I. Pascual and G. Pelaitay determined a topological duality for $LM_{n\times m}$-algebras. To this aim, these authors considered the topological category $\mathbf{LM_{n\times m}P}$ of $LM_{n\times m}$-spaces and $LM_{n\times m}$-functions. Specifically:

Definition 7. A system $\left(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]}\right)$ is an $n\times m$-valued Łukasiewicz–Moisil space (or shortly $LM_{n\times m}$-space) if the following properties are fulfilled for all x, $y\in X$ and $(i,j), (r,s)\in[n]\times[m]$:

(LP1) (X, g) is an m-space,

(LP2) $f_{ij} : X\longrightarrow X$ is a continuous function,

(LP3) $f_{ij}(x)\leq f_{(i+1)j}(x)$,

(LP4) $f_{ij}(x)\leq f_{i(j+1)}(x)$,

(LP5) $x\leq y$ implies $f_{ij}(x) = f_{ij}(y)$ for all $(i,j)\in[n]\times[m]$,

(LP6) $f_{ij}\circ f_{rs} = f_{ij}$,

(LP7) $f_{ij}\circ g = f_{ij}$,

(LP8) $g \circ f_{ij} = f_{(n-i)(m-j)}$,
(LP9) $\bigcup\limits_{(i,j)\in[n]\times[m]} f_{ij}(X) = X$.

Remark 1. The axiom (LP5) is omitted in the Sanza's definition of $LM_{n\times m}$-space (see [35, Definition 2.1]). This axiom plays a fundamental role in the characterization of $LM_{n\times m}$-spaces and consequently in the characterization of congruences on $LM_{n\times m}$-algebras as we prove next.

Definition 8. If $\left(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]}\right)$ and $\left(X', g', \{f'_{ij}\}_{(i,j)\in[n]\times[m]}\right)$ are two $LM_{n\times m}$-spaces, then an $LM_{n\times m}$-function f from X to X' is a continuous and increasing function (P-function), which satisfies the following conditions:

(mPf) $f \circ g = g' \circ f$, (i.e., f is an m-function as in Defintion 6),
(LPf) $f'_{ij} \circ f = f \circ f_{ij}$ for all $(i,j) \in [n] \times [m]$.

Remark 2. The condition (mPf) in Definition 8 can be omitted.

Proposition 1. ([27]) Let $\left(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]}\right)$ be a system which satisfies the properties (LP1) to (LP8), and let $D(X)$ be the lattice of all increasing clopen (closed and open) of X. Then, the following conditions are equivalent:

(LP9) $\bigcup\limits_{(i,j)\in[n]\times[m]} f_{ij}(X) = X$,
(LP10) $\overline{\bigcup\limits_{(i,j)\in[n]\times[m]} f_{ij}(X)} = X$, where $\overline{Z}$ denotes the closure of $Z \subseteq X$,
(LP11) if $U, V \in D(X)$ and $f_{ij}^{-1}(U) = f_{ij}^{-1}(V)$ for all $(i,j) \in [n] \times [m]$, then $U = V$,
(LP12) for each $x \in X$, there is $(i_0, j_0) \in [n] \times [m]$ such that $f_{i_0 j_0}(x) = x$,
(LP13) if Y, $Z \subseteq X$ and $f_{ij}^{-1}(Y) = f_{ij}^{-1}(Z)$ for all $(i,j) \in [n] \times [m]$, then $Y = Z$.

Definition 9. Let $(X, \leq)$ be a partial ordered set. For all $x, y \in X$ such that $x \leq y$, the subset $[x; y] := \{z \in X : x \leq z \leq y\}$ is said to be a segment or a closed interval in X.

It is worth mentioning the following properties of $LM_{n\times m}$-spaces because they are useful to describe these spaces:

Lemma 3. *Let* $\left(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]}\right)$ *be an* $LM_{n\times m}$*-space. Then, for any* $x \in X$,

(a) $[f_{11}(x); f_{(n-1),(m-1)}(x)] = \{f_{ij}(x) : (i,j) \in [n] \times [m]\}$,
(b) $x \in [f_{11}(x); f_{(n-1),(m-1)}(x)]$.

Proposition 2. Let $\Big(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]}\Big)$ be an $LM_{n\times m}$-space. Then X is the cardinal sum of the sets $[f_{11}(x); f_{(n-1)(m-1)}(x)]$, $x \in X$.

Corollary 1. Let $\Big(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]}\Big)$ be an $LM_{n\times m}$-space. Then it holds that

(LP14) $\min X = \{f_{11}(x) : x \in X\}$,
(LP15) $\max X = \{f_{(n-1)(m-1)}(x) : x \in X\}$.

Corollary 2. Let $\Big(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]}\Big)$ be an $LM_{n\times m}$-space. Then for any $x \in X$ it holds that

(LP16) $f_{11}(x) \leq x$ and $f_{11}(x)$ is the unique minimal element in X that precedes x,
(LP17) $x \leq f_{(n-1)(m-1)}(x)$ and $f_{(n-1)(m-1)}(x)$ is the unique maximal element in X that follows x.

Corollary 3. Let $\Big(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]}\Big)$ be an $LM_{n\times m}$-space. Then, for all interval $I \subseteq X$, the following conditions are equivalent:

(a) $I = [f_{11}(x); f_{(n-1)(m-1)}(x)]$ for some $x \in X$,
(b) I is a maximal interval in X.

In addition, in [27], the following results were established:

- If $(X, g, \{f_{ij}\}_{i\in[n]\times[m]})$ is an $LM_{n\times m}$-space. Then,

$$(D(X), \sim_g, \{\sigma^X_{ij}\}_{i\in[n]\times[m]})$$

is an $LM_{n\times m}$-algebra, where for every $U \in D(X)$, $\sim_g U$ is defined as in (2.3) and

$$\sigma^X_{ij}(U) = f^{-1}_{ij}(U) \text{ for all } (i,j) \in [n] \times [m]. \tag{2.14}$$

- If $(A, \sim, \{\sigma_{ij}\}_{i\in[n]\times[m]})$ is an $LM_{n\times m}$-algebra and $X(A)$ is the Priestley space associated with A, then $(X(A), g_A, \{f^A_{ij}\}_{i\in[n]\times[m]})$ is an $LM_{n\times m}$-space, where for every $S \in X(A)$, $g_A(S)$ is defined as (2.5) and

$$f^A_{ij}(S) = \sigma^{-1}_{ij}(S) \text{ for all } (i,j) \in [n] \times [m]. \tag{2.15}$$

- $(A, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]}) \cong (D(X(A)), \sim, \{\sigma_{ij}^{X(A)}\}_{(i,j)\in[n]\times[m]})$ and
- $(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]}) \cong (X(D(X)), g_{D(X)}, \{f_{ij}^{D(X)}\}_{(i,j)\in[n]\times[m]})$, via the natural isomorphisms denoted by σ_A and ε_X respectively, which are defined as in (2.4) and (2.8), respectively.
- The correspondences between the morphisms of both categories are defined in the usual way as in (2.9) and (2.10).

Then, from these results it was concluded that the category $\mathbf{LM_{n\times m}P}$ is dually equivalent to the category $\mathbf{LM_{n\times m}A}$ of $LM_{n\times m}$-algebras and $LM_{n\times m}$-homomorphisms. Moreover, this duality was taken into account to characterize the congruence lattice on an $LM_{n\times m}$-algebra as is indicated in Theorem 2. In order to obtain this characterization the modal subsets of the $LM_{n\times m}$-spaces were taken into account, which we mention below:

Definition 10. Let $(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]})$ be an LM_n-space. A subset Y of X is modal if $Y = f_i^{-1}(Y)$ for all $(i, j) \in [n] \times [m]$.

Theorem 2. ([27]) *Let $(A, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]})$ be an $LM_{n\times m}$-algebra and $(X(A), g_A, \{f_{ij}^A\}_{(i,j)\in[n]\times[m]})$ be the $LM_{n\times m}$-space associated with A. Then, the lattice $C_M(X(A))$ of all modal and closed subsets of $X(A)$ is anti-isomorphic to the lattice $Con_{LM_{n\times m}}(A)$ of $LM_{n\times m}$-congruences on A, and the anti-isomorphism is the function $\Theta_M : C_M(X(A)) \longrightarrow Con_{LM_{n\times m}}(A)$ defined by the same prescription in* (2.11).

The previous results allow us to prove the following theorem.

Theorem 3. ([27]) *Let $\left(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]}\right)$ be an $LM_{n\times m}$-space and let $\left(D(X), \sim_g, \{\sigma_{ij}^X\}_{(i,j)\in[n]\times[m]}\right)$ be the $LM_{n\times m}$-algebra associated with X. Then the following conditions are equivalent:*

(a) $X = [f_{11}(x), f_{(n-1)(m-1)}(x)]$ *for all* $x \in X$,

(b) $\left(D(X), \sim_g, \{\sigma_{ij}^X\}_{(i,j)\in[n]\times[m]}\right)$ *is a simple $LM_{n\times m}$-algebra,*

(c) $\left(D(X), \sim_g, \{\sigma_{ij}^X\}_{(i,j)\in[n]\times[m]}\right)$ *is a subdirectly irreducible $LM_{n\times m}$-algebra,*

(d) $D(X)$ *is finite and* $D(X) \setminus \{\emptyset, X\}$ *has least and greatest element.*

2.3. Tense $n \times m$-valued Łukasiewicz–Moisil algebras

In [17], A. V. Figallo and G. Pelaitay introduce the following notion:

Definition 11. An algebra $\langle A, \vee, \wedge, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]}, G, H, 0, 1\rangle$ is a tense $n \times m$-valued Łukasiewicz–Moisil algebra (or tense $LM_{n\times m}$-algebra) if $\langle A, \vee, \wedge, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]}, 0, 1\rangle$, is an $LM_{n\times m}$-algebra and G, H are two unary operators on A which satisfy the following properties:

(T1) $G(1) = 1$ and $H(1) = 1$,
(T2) $G(x \wedge y) = G(x) \wedge G(y)$ and $H(x \wedge y) = H(x) \wedge H(y)$,
(T3) $G\sigma_{ij}(x) = \sigma_{ij}G(x)$ and $H\sigma_{ij}(x) = \sigma_{ij}H(x)$,
(T4) $x \leq GP(x)$ and $x \leq HF(x)$, where $P(x) =\sim H(\sim x)$ and $F(x) =\sim G(\sim x)$, for any x, $y \in X$ and $(i, j) \in [n] \times [m]$.

A tense $LM_{n\times m}$-algebra $\langle A, \vee, \wedge, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]}, G, H, 0, 1\rangle$ will be denoted in the rest of this paper by (A, G, H) or by $(A, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]}, G, H)$.

The following lemma contains properties of tense LM_n-algebras that are useful in what follows.

Lemma 4. ([27]) *The following properties hold in every tense $LM_{n\times m}$-algebra (A, G, H):*

(T5) $x \leq y$ *implies* $G(x) \leq G(y)$ *and* $H(x) \leq H(y)$,
(T6) $x \leq y$ *implies* $F(x) \leq F(y)$ *and* $P(x) \leq P(y)$,
(T7) $F(0) = 0$ *and* $P(0) = 0$,
(T8) $F(x \vee y) = F(x) \vee F(y)$ *and* $P(x \vee y) = P(x) \vee P(y)$,
(T9) $PG(x) \leq x$ *and* $FH(x) \leq x$,
(T10) $GP(x) \wedge F(y) \leq F(P(x) \wedge y)$ *and* $HF(x) \wedge P(y) \leq P(F(x) \wedge y)$,
(T11) $G(x) \wedge F(y) \leq F(x \wedge y)$ *and* $H(x) \wedge P(y) \leq P(x \wedge y)$,
(T12) $G(x \vee y) \leq G(x) \vee F(y)$ *and* $H(x \vee y) \leq H(x) \vee P(y)$, *for any* $x, y \in X$.

Definition 12. ([27]) If (A, G, H) and (A', G', H') are two tense $LM_{n\times m}$-algebras, then a morphism of tense $LM_{n\times m}$-algebras $f : (A, G, H) \longrightarrow (A', G', H')$ is a morphism of $LM_{n\times m}$-algebras such that

(tf) $f(G(a)) = G'(f(a))$ and $f(H(a)) = H'(f(a))$, for any $a \in A$.

LEMMA 5. ([27]) *Let (A, G, H) be a tense $LM_{n\times m}$-algebra and let $C(A) := \{a \in A : d(a) = a\}$. Then, $\langle C(A), \vee, \wedge, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]}, 0, 1\rangle$ is an $LM_{n\times m}$-algebra.*

3. Topological duality for tense $LM_{n\times m}$-algebras

In this section, we will develop a topological duality for tense $n \times m$-valued Łukasiewicz–Moisil algebras, taking into account the results established by A. V. Figallo, I. Pascual and G. Pelaitay in [27] and the results obtained by A. V. Figallo and G. Pelaitay in [16]. In order to determine this duality, we introduce a topological category whose objects and their corresponding morphisms are described below.

DEFINITION 13. A system $(X, g, \{f_{ij}\}_{i\in[n]\times[m]}, R)$ is a tense $LM_{n\times m}$-space if the following conditions are satisfied:

(i) $(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]})$ is an $LM_{n\times m}$-space (Definition 7),

(ii) R is a binary relation on X and R^{-1} is the converse of R such that:

(tS1) $(x, y) \in R$ implies $(g(x), g(y)) \in R$,

(tS2) for each $x \in X$, $R(x)$ and $R^{-1}(x)$ are closed subsets of X,

(tS3) for each $x \in X$, $R(x) =\downarrow R(x) \cap \uparrow R(x)$,

(tS4) $(x, y) \in R$ implies $(f_{ij}(x), f_{ij}(y)) \in R$ for any $(i, j) \in [n] \times [m]$,

(tS5) $(f_{ij}(x), y) \in R$, $(i, j) \in [n] \times [m]$, implies that there exists $z \in X$ such that $(x, z) \in R$ and $f_{ij}(z) \leq y$,

(tS6) $(y, f_{ij}(x)) \in R$, $(i, j) \in [n] \times [m]$, implies that there exists $z \in X$ such that $(z, x) \in R$ and $f_{ij}(z) \leq y$,

(tS7) for each $U \in D(X)$, $G_R(U), H_{R^{-1}}(U) \in D(X)$, where G_R and $H_{R^{-1}}$ are operators on $P(X)$ defined as in (2.1) and (2.2), respectively.

DEFINITION 14. A tense $LM_{n\times m}$-function f from a tense $LM_{n\times m}$-space $(X_1, g_1, \{f^1_{ij}\}_{(i,j)\in[n]\times[m]}, R_1)$ into another one, $(X_2, g_2, \{f^2_{ij}\}_{(i,j)\in[n]\times[m]}, R_2)$ is a function $f : X_1 \longrightarrow X_2$ such that:

(i) $f : X_1 \longrightarrow X_2$ is an $LM_{n\times m}$-function (Definition 8),

(ii) $f : X_1 \longrightarrow X_2$ satisfies the following conditions, for all $x \in X_1$:

(tf1) $f(R_1(x)) \subseteq R_2(f(x))$ and $f(R_1^{-1}(x)) \subseteq R_2^{-1}(f(x))$,

(tf2) $R_2(f(x)) \subseteq\uparrow f(R_1(x))$,

(tf3) ${R_2}^{-1}(f(x)) \subseteq\uparrow f({R_1}^{-1}(x))$.

The category that has tense $LM_{n\times m}$-spaces as objects and tense $LM_{n\times m}$-functions as morphisms will be denoted by $\mathbf{tLM_{n\times m}S}$, and $\mathbf{tLM_{n\times m}A}$ will denote the category of tense $LM_{n\times m}$-algebras and tense $LM_{n\times m}$-homomorphisms. Our next task will be to determine that the category $\mathbf{tLM_{n\times m}S}$ is naturally equivalent to the dual category of $\mathbf{tLM_{n\times m}A}$.

Now we will show a characterization of tense $LM_{n\times m}$-functions which will be useful later.

Lemma 6. *Let $(X_1, g_1, \{f^1_{ij}\}_{i\in[n]\times[m]}, R_1)$ and $(X_2, g_2, \{f^2_{ij}\}_{i\in[n]\times[m]}, R_2)$ be two tense $LM_{n\times m}$-spaces and*
$f : X_1 \longrightarrow X_2$ be a tense $LM_{n\times m}$-function. Then, f satisfies the following conditions:

(tf4) $\uparrow f(R_1(x)) =\uparrow R_2(f(x))$,
(tf5) $\uparrow f({R_1}^{-1}(x)) =\uparrow {R_2}^{-1}(f(x))$, *for any $x \in X$.*

Proof: It can be proved using a similar technique to that used in the proof of Lemma 3.4 in [14]. □

Lemma 7. *Let $(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]}, R)$ be a tense $LM_{n\times m}$-space. Then for all x, $y \in X$ such that $(x, y) \notin R$, the following conditions are satisfied:*

(i) *There is $U \in D(X)$ such that $y \notin U$ and $x \in G_R(U)$ or $y \in U$ and $x \notin F_R(U)$, where $F_R(U) := \{x \in X : R(x) \cap U \neq \emptyset\}$.*
(ii) *There is $V \in D(X)$ such that $y \notin V$ and $x \in H_{R^{-1}}(V)$ or $y \in V$ and $x \notin P_{R^{-1}}(V)$, where $P_{R^{-1}}(V) := \{x \in X : R^{-1}(x) \cap V \neq \emptyset\}$.*

Proof: It can be proved in a similar way to Lemma 3.5 of [14]. □

Lemma 8. *Let $(X_1, g_1, \{f^1_{ij}\}_{(i,j)\in[n]\times[m]}, R_1)$ and $(X_2, g_2, \{f^2_{ij}\}_{(i,j)\in[n]\times[m]}, R_2)$ be two tense $LM_{n\times m}$-spaces. Then, the following conditions are equivalent:*

(i) *$f : X_1 \longrightarrow X_2$ is a tense $LM_{n\times m}$-function,*
(ii) *$f : X_1 \longrightarrow X_2$ is an $LM_{n\times m}$-function such that, for any $U \in D(X_2)$:*
  (tf6) $f^{-1}(G_{R_2}(U)) = G_{R_1}(f^{-1}(U))$,
  (tf7) $f^{-1}(H_{R_2^{-1}}(U)) = H_{R_1^{-1}}(f^{-1}(U))$.

Proof: The proof is similar in spirit to Lemma 3.6 of [14]. □

Lemma 9 and Corollary 4 can be proved in a similar way to Lemma 3.8 and Corollary 3.9, respectively of [14].

LEMMA 9. *Let* $(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]}, R)$ *be a tense* $LM_{n\times m}$*-space. Then, the following conditions are satisfied for any* $x, y, \in X$ *and* $(i,j) \in [n]\times[m]$*:*

(tS11) $R(g(x)) = g(R(x))$, $R^{-1}(g(x)) = g(R^{-1}(x))$,

(tS12) $R(f_{ij}(x)) \subseteq \bigcup\limits_{y\in R(f_{ij}(x))} \uparrow f_{ij}(y)$,

(tS13) $R^{-1}(f_{ij}(x)) \subseteq \bigcup\limits_{y\in R^{-1}(f_{ij}(x))} \uparrow f_{ij}(y)$,

(tS14) $\uparrow f_{ij}(R_1(x)) = \uparrow R_2(f(x))$,

(tS15) $\uparrow f_{ij}({R_1}^{-1}(x)) = \uparrow {R_2}^{-1}(f(x))$,

(tS16) $f_{ij}^{-1}(G_R(U)) = G_R(f_{ij}^{-1}(U))$,

(tS17) $f_{ij}^{-1}(H_{R^{-1}}(U)) = H_{R^{-1}}(f_{ij}^{-1}(U))$,

(tS18) $f_{ij}^{-1}(\sim_g U) = \sim_g (f_{(n-i)(m-j)}^{-1}(U))$,

(tS19) $f_{ij}^{-1}(F_R(U)) = F_R(f_{ij}^{-1}(U))$,

(tS20) $f_{ij}^{-1}(P_{R^{-1}}(U)) = P_{R^{-1}}(f_{ij}^{-1}(U))$.

COROLLARY 4. Let $(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]})$ be a tense $LM_{n\times m}$-space. Then, the conditions (tS4), (tS5) and (tS6) can be replaced by the following conditions:

(tS16) $f_{ij}^{-1}(G_R(U)) = G_R(f_{ij}^{-1}(U))$ for any $U \in D(X)$

(tS17)$f_{ij}^{-1}(H_{R^{-1}}(U)) = H_{R^{-1}}(f_{ij}^{-1}(U))$ for any $U \in D(X)$.

Next, we will define a contravariant functor from $\mathbf{tLM_{n\times m}S}$ to $\mathbf{tLM_{n\times m}A}$.

LEMMA 10. *Let* $(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]}, R)$ *be a tense* $LM_{n\times m}$*-space. Then,*

$$\Psi(X) = \langle D(X), \sim_g, \{\sigma_{ij}^X\}_{(i,j)\in[n]\times[m]}, G_R, H_{R^{-1}}, \emptyset, X\rangle$$

is a tense $LM_{n\times m}$*-algebra, where for all* $U \in D(X)$, $\sim_g U$, $\sigma_{ij}^X(U)$, $(i,j) \in [n]\times[m]$, $G_R(U)$ *and* $H_{R^{-1}}(U)$ *are defined as in (2.3), (2.14), (2.1) and (2.2), respectively.*

PROOF: From [27] and [16, Lemma 4.3] it follows that the reduct $\langle D(X), \sim_g, \{\sigma_{ij}^X\}_{(i,j)\in[n]\times[m]}, \emptyset, X\rangle$ is an $LM_{n\times m}$-algebra and the structure $\langle D(X), \sim_g, G_R, H_{R^{-1}}, \emptyset, X\rangle$ is a tense De Morgan algebra, respectively. Therefore, the properties (T1), (T2) and (T4) of tense $LM_{n\times m}$-algebras (Definition 11) hold. In addition, since any $U \in D(X)$ satisfies properties (tS16) and (tS17) in Lemma 9, then we can assert that property (T3) holds too, and so the proof is complete. □

Lemma 11. *Let $f : (X_1, g_1, \{f^1_{ij}\}_{(i,j)\in[n]\times[m]}) \longrightarrow (X_2, g_2, \{f^2_i\}_{(i,j)\in[n]\times[m]})$ be a morphism of tense $LM_{n\times m}$-spaces. Then, the map $\Psi(f) : D(X_2) \longrightarrow D(X_1)$ defined by $\Psi(f)(U) = f^{-1}(U)$ for all $U \in D(X_2)$, is a tense $LM_{n\times m}$-homomorphism.*

Proof: It follows from the results established in [27] and Lemma 8. □

The previous two lemmas show that Ψ is a contravariant functor from $\mathbf{tLM_nS}$ to $\mathbf{tLM_nA}$. To achieve our goal we need to define a contravariant functor from $\mathbf{tLM_nA}$ to $\mathbf{tLM_nS}$.

Lemma 12. *Let (A, G, H) be a tense $LM_{n\times m}$-algebra and let $S, T \in X(A)$. Then the following conditions are equivalent:*

(i) $G^{-1}(S) \subseteq T \subseteq F^{-1}(S)$,

(ii) $H^{-1}(T) \subseteq S \subseteq P^{-1}(T)$.

Proof: In a similar way to [18, Lemma 3.8]. □

Definition 15. Let (A, G, H) be a tense $LM_{n\times m}$-algebra and let R^A be the relation defined on $X(A)$ by the prescription:

$$(S, T) \in R^A \Longleftrightarrow G^{-1}(S) \subseteq T \subseteq F^{-1}(S). \tag{3.1}$$

Remark 3. Lemma 12 means that we have two ways to define the relation R^A, either by using G and F, or by using H and P.

The following lemma, whose proof can be obtained as in [18, Lemma 3.11], will be essential for the proof of Lemma 14.

Lemma 13. *Let (A, G, H) be a tense $LM_{n\times m}$-algebra and let $S \in X(A)$ and $a \in A$. Then,*

(i) *$G(a) \notin S$ if and only if there exists $T \in X(A)$ such that $(S, T) \in R^A$ and $a \notin T$,*

(ii) *$H(a) \notin S$ if and only if there exists $T \in X(A)$ such that $(S, T) \in {R^A}^{-1}$ and $a \notin T$.*

Lemma 14. *Let (A, G, H) be an $LM_{n\times m}$-algebra and $X(A)$ be the Priestley space associated with A. Then, $\Phi(A) = (X(A), g_A, \{f^A_{ij}\}_{(i,j)\in[n]\times[m]}, R^A)$ is a tense $LM_{n\times m}$-space, where for every $S \in X(A)$, $g_A(S)$ and $f^A_{ij}(S)$ are defined as in (2.5) and (2.15), respectively and R^A is the relation defined on $X(A)$ as in (3.1). Besides, $\sigma_A : A \longrightarrow D(X(A))$, defined by the prescription (2.4), is a tense $LM_{n\times m}$-isomorphism.*

Proof: From [27] and [16, Lemma 5.6] it follows that the system $(X(A), g_A, \{f_{ij}^A\}_{(i,j)\in[n]\times[m]})$ is an $LM_{n\times m}$-space and $(X(A), g_A, R^A, {R^A}^{-1})$ is a tense mP-space, and so properties (tS1), (tS2) (tS3) and (tS7) of tense $LM_{n\times m}$-spaces hold (Definition 13). Also, from Corollary 4 we have that the conditions (tS4), (tS5) and (tS6) are satisfied. Therefore, we have that $(X(A), g_A, \{f_{ij}^A\}_{(i,j)\in[n]\times[m]}, R^A)$ is a tense $LM_{n\times m}$-space. In addition, from [27] we have that σ_A is an $LM_{n\times m}$-isomorphism. Also for all $a \in A$, $G_{R^A}(\sigma_A(a)) = \sigma_A(G(a))$ and $H_{{R^A}^{-1}}(\sigma_A(a)) = \sigma_A(H(a))$. Indeed, let us take a prime filter S such that $G(a) \notin S$. By Lemma 13, there exists $T \in X(A)$ such that $(S,T) \in R^A$ and $a \notin T$. Then, $R^A(S) \nsubseteq \sigma_A(a)$. So, $S \notin G_{R^A}(\sigma_A(a))$ and, therefore, $G_{R^A}(\sigma(a)) \subseteq \sigma_A(G(a))$. Moreover, it is immediate that $\sigma_A(G(a)) \subseteq G_{R_A}(\sigma_A(a))$. Similarly we obtain that $H_{{R^A}^{-1}}(\sigma_A(a)) = \sigma_A(H(a))$ and so σ_A is a tense $LM_{n\times m}$-isomorphism. □

Lemma 15. *Let (A_1, G_1, H_1) and (A_2, G_2, H_2) be two $LM_{n\times m}$-algebras and $h : A_1 \longrightarrow A_2$ be a tense $LM_{n\times m}$-homomorphism. Then, the map $\Phi(h) : X(A_2) \longrightarrow X(A_1)$, defined by $\Phi(h)(S) = h^{-1}(S)$ for all $S \in X(A_2)$, is a tense $LM_{n\times m}$-function.*

Proof: It follows from the results established in [27] and [16, Lemma 5.7]. □

Lemmas 14 and 15 show that Φ is a contravariant functor from $\mathbf{tLM_{n\times m}A}$ to $\mathbf{tLM_{n\times m}S}$.

The following characterization of isomorphisms in the category $\mathbf{tLM_{n\times m}S}$ will be used to determine the duality that we set out to prove.

Proposition 3. Let $(X_1, g_1, \{f_{ij}^1\}_{(i,j)\in[n]\times[m]}, R_1)$ and $(X_2, g_2, \{f_{ij}^2\}_{(i,j)\in[n]\times[m]}, R_2)$ be two tense $LM_{n\times m}$-spaces. Then, the following conditions are equivalent, for every function $f : X_1 \longrightarrow X_2$:

(i) f is an isomorphism in the category $\mathbf{tLM_nS}$,

(ii) f is a bijective $LM_{n\times m}$-function such that for all $x, y \in X_1$:

(itf) $(x, y) \in R_1 \iff (f(x), f(y)) \in R_2$.

Proof: It is routine. □

The map $\varepsilon_X : X \longrightarrow X(D(X))$, defined as in (2.8), leads to another characterization of tense $LM_{n\times m}$-spaces, which also allow us to assert that this map is an isomorphism in the category $\mathbf{tLM_{n\times m}S}$, as we will describe below:

LEMMA 16. *Let $(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]}, R)$ be a tense $LM_{n\times m}$-space, $\varepsilon_X : X \longrightarrow X(D(X))$ be the map defined by the prescription (2.8) and let $R^{D(X)}$ be the relation defined on $X(D(X))$ by means of the operators G_R and F_R as follows:*

$$(\varepsilon_X(x), \varepsilon_X(y)) \in R^{D(X)} \Longleftrightarrow G_R^{-1}(\varepsilon_X(x)) \subseteq \varepsilon_X(y) \subseteq F_R^{-1}(\varepsilon_X(x)). \quad (3.2)$$

Then, the following property holds:

(tS5) $(x, y) \in R$ *implies* $(\varepsilon_X(x), \varepsilon_X(y)) \in R^{D(X)}$.

PROOF: It is routine. □

PROPOSITION 4. Let $(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]}, R)$ be a tense $LM_{n\times m}$-space, $\varepsilon_X : X \longrightarrow X(D(X))$ be the function defined by the prescription (2.8) and let $R^{D(X)}$ be the relation defined on $X(D(X))$ by the prescription (3.2). Then, the condition (tS3) can be replaced by the following one:

(tS18) $(\varepsilon_X(x), \varepsilon_X(y)) \in R^{D(X)} \Longleftrightarrow (x, y) \in R$.

PROOF: It can be proved in a similar way to [16, Proposition 5.5]. □

COROLLARY 5. Let $(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]}, R)$ be a tense $LM_{n\times m}$-space. Then, the map $\varepsilon_X : X \longrightarrow X(D(X))$ is an isomorphism in the category $\mathbf{tLM_{n\times m}S}$.

PROOF: It follows from the results established in [27], Lemma 16, Propositions 3 and 4. □

Then, from the above results and using the usual procedures we can prove that the functors $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are naturally equivalent to the identity functors on $\mathbf{tLM_{n\times m}S}$ and $\mathbf{tLM_{n\times m}A}$, respectively, from which we conclude:

THEOREM 4. *The category $\mathbf{tLM_{n\times m}S}$ is naturally equivalent to the dual of the category $\mathbf{tLM_nA}$.*

4. Subdirectly irreducible tense $\mathbf{LM_{n\times m}}$-algebras

In this section, our first objective is the characterization of the congruence lattice on a tense $LM_{n\times m}$-algebra by means of certain closed and modal subsets of its associated tense $LM_{n\times m}$-space. Later, this result will be taken into account to characterize simple and subdirectly irreducible tense

$LM_{n\times m}$-algebras. With this purpose, we will start by introducing the following notion.

DEFINITION 16. Let $(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]}, R)$ be a tense $LM_{n\times m}$-space. A subset Y of X is a tense subset if it satisfies the following conditions for all $y, z \in X$:

(ts1) if $y \in Y$ and $z \in R(y)$, then there is $w \in Y$ such that $w \in R(y) \cap \downarrow z$,

(ts2) if $y \in Y$ and $z \in R^{-1}(y)$, then there is $v \in Y$ such that $v \in R^{-1}(y) \cap \downarrow z$.

In [27] the following characterizations of a modal subset of an $LM_{n\times m}$-space were obtained.

PROPOSITION 5. ([27]) Let $(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]})$ be an $LM_{n\times m}$-space and Y be a nonempty subset of X. Then, the following conditions are equivalent:

(a) Y is modal,

(b) Y is involutive and increasing,

(c) $Y = \bigcup_{y\in Y} [f_{11}(y), f_{(n-1)(m-1)}(y)]$ (i.e. Y is the cardinal sum of certain maximal intervals of X).

COROLLARY 6. Let $(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]})$ be an $LM_{n\times m}$-space. If $\{Y_i\}_{i\in I}$ is a family of modal subsets of X, then $\bigcap_{i\in I} Y_i$ is a modal subset of X.

PROOF: It is a direct consequence of Proposition 5. $\square$

The notion of a modal and tense subset of a tense $LM_{n\times m}$-space has several equivalent formulations, which will be useful later:

PROPOSITION 6. Let $(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]}, R)$ be a tense $LM_{n\times m}$-space. If Y is a modal subset of X, then the following conditions are equivalent:

(i) Y is a tense subset,

(ii) for all $y \in Y$, the following conditions are satisfied:

(ts3) $R(y) \subseteq Y$,

(ts4) $R^{-1}(y) \subseteq Y$,

(iii) $Y = G_R(Y) \cap Y \cap H_{R^{-1}}(Y)$, where $G_R(Y) := \{x \in X : R(x) \subseteq Y\}$ and $H_{R^{-1}}(Y) := \{x \in X : R^{-1}(x) \subseteq Y\}$.

PROOF: (i) $\Rightarrow$ (ii): Let $y \in Y$ and $z \in R(y)$, then by (i) and (ts1), there is $w \in Y$ such that $w \in R(y)$ and $w \leq z$. Since Y is modal, from Proposition 5 it follows that $z \in Y$ and therefore $R(y) \subseteq Y$. Using an analogous reasoning we get that $R^{-1}(y) \subseteq Y$.

(ii) $\Rightarrow$ (i): It is immediate.

(ii) $\Leftrightarrow$ (iii): It is immediate. □

The closed, modal and tense subsets of the tense $LM_{n\times m}$-space associated with a tense $LM_{n\times m}$-algebra perform a fundamental roll in the characterization of the tense $LM_{n\times m}$-congruences on these algebras as we will show next.

THEOREM 5. *Let (A, G, H) be a tense $LM_{n\times m}$-algebra, and $(X(A), g_A, \{f_{ij}^A\}_{(i,j)\in[n]\times[m]}, R^A)$ be the tense $LM_{n\times m}$-space associated with A. Then, the lattice $C_{MT}(X(A))$ of all closed, modal and tense subsets of $X(A)$ is anti-isomorphic to the lattice $Con_{tLM_{n\times m}}(A)$ of tense $LM_{n\times m}$-congruences on A, and the isomorphism is the function Θ_{MT} defined by the same prescription as in* (2.11).

PROOF: It immediately follows from Theorems 1 and 2 and the fact that $C_{MT}(X(A)) = C_M(X(A)) \cap C_T(X(A))$ and for all $\varphi \subseteq A \times A$, $\varphi \in Con_{tLM_{n\times m}}(A)$ iff φ is both an $LM_{n\times m}$-congruence on A and a tense De Morgan congruence on A. □

Next, we will use the results already obtained in order to determine the simple and subdirectly irreducible tense $LM_{n\times m}$-algebras.

COROLLARY 7. Let (A, G, H) be a tense $LM_{n\times m}$-algebra, and $(X(A), g_A, \{f_i^A\}_{(i,j)\in[n]\times[m]}, R^A)$ be the tense $LM_{n\times m}$-space associated with A. Then, the following conditions are equivalent:

(i) (A, G, H) is a simple tense $LM_{n\times m}$-algebra,
(ii) $C_{MT}(X(A)) = \{\emptyset, X(A)\}$.

PROOF: It is a direct consequence of Theorem 5. □

COROLLARY 8. Let (A, G, H) be a tense $LM_{n\times m}$-algebra, and $(X(A), g_A, \{f_{ij}^A\}_{(i,j)\in[n]\times[m]}, R^A)$ be the tense $LM_{n\times m}$-space associated with A. Then, the following conditions are equivalent:

(i) (A, G, H) is a subdirectly irreducible tene $LM_{n\times m}$-algebra,
(ii) there is $Y \in C_{MT}(X(A)) \setminus \{X(A)\}$ such that $Z \subseteq Y$ for all $Z \in C_{MT}(X(A)) \setminus \{X(A)\}$.

PROOF: It is a direct consequence of Theorem 5. □

PROPOSITION 7. Let $(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]}, R)$ be a tense $LM_{n\times m}$-space. If Y is a modal subset of X, then $G_R(Y)$ and $H_{R^{-1}}(Y)$ are also modal.

PROOF: Let Y be a modal subset of X. From Proposition 2 it follows immediately that (1) $G_R(Y) \subseteq \bigcup\limits_{z\in G_R(Y)} [f_{11}(z), f_{(n-1)(m-1)}(z)]$. Let (2) $z \in G_R(Y)$ and let (3) $w \in [f_{11}(z), f_{(n-1)(m-1)}(z)]$, then from (3) and properties (LP5) and (LP6), we obtain that (4) $f_{ij}(w) = f_{rs}(z)$ for all $(i,j),(r,s) \in [n]\times[m]$. Let (5) $t \in R(w)$, then by (4), (5) and property (tS3), we infer that $f_{11}(t) \in R(f_{11}(z))$ and therefore, from properties (tS4), (LP5) and (LP6), we can assert that there exists $y \in X$ such that (5) $y \in R(z)$ and (6) $f_{ij}(y) = f_{rs}(t)$ for all $(i,j),(r,s) \in [n]\times[m]$. From (2) and (5) we get that $y \in Y$. Since Y is modal, then from this last assertion and (6) it results that $f_{ij}(t) \in Y$ for all $(i,j) \in [n]\times[m]$. Then, since Y is modal, we have that $t \in Y$, from which we deduce by (5) that $R(w) \subseteq Y$, which allows to assert that $w \in G_R(Y)$. Therefore, from (3) we can set that $\bigcup\limits_{z\in G_R(Y)} [f_{11}(z), f_{(n-1)(m-1)}(z)] \subseteq G_R(Y)$. Then, from (1) it follows that $G_R(Y) = \bigcup\limits_{z\in G_R(Y)} [f_{11}(z), f_{(n-1)(m-1)}(z)]$, and so from Proposition 5, we conclude that $G_R(Y)$ is modal. The proof that $H_{R^{-1}}(Y)$ is modal is similar. $\square$

The characterization of modal and tense subsets of a tense $LM_{n\times m}$-space, given in Proposition 6, prompts us to introduce the following definition:

DEFINITION 17. Let $(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]}, R)$ be a tense $LM_{n\times m}$-space and let $d_X : P(X) \longrightarrow P(X)$ defined by:

$$d_X(Z) = G_R(Z) \cap Z \cap H_{R^{-1}}(Z), \text{ for all } Z \in P(X). \tag{4.1}$$

For each $n \in \omega$, let $d_X^n : P(X) \longrightarrow P(X)$, defined by:

$$d_X^0(Z) = Z, \; d_X^{n+1}(Z) = d_X(d_X^n(Z)), \text{ for all } Z \in P(X). \tag{4.2}$$

By using the above functions d_X, d_X^n, $n \in \omega$, we obtain another equivalent formulation of the notion of modal and tense subset of a tense $LM_{n\times m}$-space.

LEMMA 17. *Let $(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]}, R)$ be a tense $LM_{n\times m}$-space. If Y is modal subset of X, then the following conditions are equivalent:*

(i) *Y is a tense subset,*

(ii) $Y = d^n_X(Y)$, *for all* $n \in \omega$,
(iii) $Y = \bigcap\limits_{n\in\omega} d^n_X(Y)$.

PROOF: It is an immediate consequence of Proposition 6 and Definition 17. □

PROPOSITION 8. Let $(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]}, R)$ be a tense $LM_{n\times m}$-space and $(D(X), G_R, H_{R^{-1}})$ be the tense $LM_{n\times m}$-algebra associated with X. Then, for all $n \in \omega$, for all $U, V \in D(X)$ and for all $(i,j) \in [n]\times[m]$, the following conditions are satisfied:

(d0) $d^n_X(U) \in D(X)$,
(d1) $d^n_X(X) = X$ and $d^n_X(\emptyset) = \emptyset$,
(d2) $d^{n+1}_X(U) \subseteq d^n_X(U)$,
(d3) $d^n_X(U \cap V) = d^n_X(U) \cap d^n_X(V)$,
(d4) $U \subseteq V$ implies $d^n_X(U) \subseteq d^n_X(V)$,
(d5) $d^n_X(U) \subseteq U$,
(d6) $d^{n+1}_X(U) \subseteq G_R(d^n_X(U))$ and $d^{n+1}_X(U) \subseteq H_{R^{-1}}(d^n_X(U))$,
(d7) $d^n_X(f^{-1}_{ij}(U)) = f^{-1}_{ij}(d^n_X(U))$ for any $n \in \omega$ and $(i,j) \in [n]\times[m]$,
(d8) if U is modal, then $d^n_X(U)$ is modal,
(d9) $\bigcap\limits_{n\in\omega} d^n_X(f^{-1}_{ij}(U))$ is a closed, modal and tense subset of X and therefore $d_X(\bigcap\limits_{n\in\omega} d^n_X(f^{-1}_{ij}(U))) = \bigcap\limits_{n\in\omega} d^n_X(f^{-1}_{ij}(U))$.

PROOF: From Definition 17, Lemma 14 and the fact that G_R, $H_{R^{-1}}$ and d^n_X, $n \in \omega$, are monotonic operations it immediately follows that properties (d0), (d1), (d2), (d3), (d4), (d5) and (d6) hold.

(d7): Let $U \in D(X)$ and $(i,j) \in [n]\times[m]$, then $d_X(f^{-1}_{ij}(U)) = f^{-1}_{ij}(U) \cap G_R(f^{-1}_{ij}(U)) \cap H_{R^{-1}}(f^{-1}_{ij}(U))$. From the last assertion and properties (tS17) and (tS18) in Lemma 9, we infer that (1) $d_X(f^{-1}_{ij}(U)) = f^{-1}_{ij}(U \cap G_R(U) \cap H_{R^{-1}}(U)) = f^{-1}_{ij}(d_X(U))$ for any $U \in D(X)$ and $(i,j) \in [n]\times[m]$. Suppose that $d^{n-1}_X(f^{-1}_{ij}(U)) = f^{-1}_{ij}(d^{n-1}_X(U))$, for any $n \in \omega$ and $(i,j) \in [n]\times[m]$, then (2) $d^n_X(f^{-1}_{ij}(U)) = d_X(d^{n-1}_X(f^{-1}_{ij}(U))) = d_X(f^{-1}_{ij}(d^{n-1}_X(U)))$. Taking into account that $d^{n-1}_X(U) \in D(X)$ and (1), we get that $d_X(f^{-1}_{ij}(d^{n-1}_X(U))) = f^{-1}_{ij}(d_X(d^{n-1}_X(U))) = f^{-1}_{ij}(d^n_X(U))$, and so from (2) the proof is complete.

(d8): It is a direct consequence of Corollary 6 and Proposition 7.

(d9): Let $U \in D(X)$. Then, from Lemma 14 and the prescription (2.14), we have that $f_{ij}^{-1}(U) \in D(X)$. Also, from (LP5), $f_{ij}^{-1}(U)$ is a modal subset of X for all $(i,j) \in [n] \times [m]$, from which it follows by (d7) that for $n \in \omega$ and $(i,j) \in [n] \times [m]$, $d_X^n(f_{ij}^{-1}(U))$ is a modal and closed subset of X, and so by Corollary 6 and the fact that the arbitrary intersection of closed subsets of X is closed, we get that $\bigcap\limits_{n\in\omega} d_X^n(f_{ij}^{-1}(U))$ is a modal and closed subset of X. If $\bigcap\limits_{n\in\omega} d_X^n(f_{ij}^{-1}(U)) = \emptyset$, then it is verified that $\bigcap\limits_{n\in\omega} d_X^n(f_{ij}^{-1}(U))$ is a closed, modal and tense subset of X. Suppose now that there exists $y \in \bigcap\limits_{n\in\omega} d_X^n(f_{ij}^{-1}(U))$. Since, $f_{ij}^{-1}(U) \in D(X)$ for any $(i,j) \in [n] \times [m]$, then from (d6) it follows that $y \in G_R(d_X^{n-1}(f_{ij}^{-1}(U)))$ and $y \in H_{R^{-1}}(d_X^{n-1}(f_{ij}^{-1}(U)))$ for all $n \in \omega$. Therefore, $R(y) \subseteq d_X^{n-1}(f_{ij}^{-1}(U))$ and $R^{-1}(y) \subseteq d_X^{n-1}(f_{ij}^{-1}(U))$ for all $n \in \omega$ and consequently $R(y) \subseteq \bigcap\limits_{n\in\omega} d_X^n(f_{ij}^{-1}(U))$ and $R^{-1}(y) \subseteq \bigcap\limits_{n\in\omega} d_X^n(f_{ij}^{-1}(U))$ for all $(i,j) \in [n] \times [m]$. From these last assertions, the fact that $\bigcap\limits_{n\in\omega} d_X^n(f_{ij}^{-1}(U))$ is a modal and closed subset of X and Proposition 8, we have that $\bigcap\limits_{n\in\omega} d_X^n(f_{ij}^{-1}(U))$ is a tense subset, from which we conclude, by Lemma 17, that $d_X(\bigcap\limits_{n\in\omega} d_X^n(f_{ij}^{-1}(U))) = \bigcap\limits_{n\in\omega} d_X^n(f_{ij}^{-1}(U))$. □

As consequences of Proposition 8 and the above duality for tense $LM_{n\times m}$-algebras (Lemma 14) we obtain the following corollaries.

Corollary 9. Let $(A, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]}, G, H)$ be a tense $LM_{n\times m}$-algebra and consider the function $d : A \longrightarrow A$, defined by $d(a) = G(a) \wedge a \wedge H(a)$, for all $a \in A$. For all $n \in \omega$, let $d^n : A \longrightarrow A$ be a function, defined by $d^0(a) = a$ and $d^{n+1}(a) = d(d^n(a))$, for all $a \in A$. Then, for all $n \in \omega$ and a, $b \in A$, the following conditions are satisfied:

(d1) $d^n(1) = 1$ and $d^n(0) = 0$,

(d2) $d^{n+1}(a) \leq d^n(a)$,

(d3) $d^n(a \wedge b) = d^n(a) \wedge d^n(b)$,

(d4) $a \leq b$ implies $d^n(a) \leq d^n(b)$,

(d5) $d^n(a) \leq a$,

(d6) $d^{n+1}(a) \leq G(d^n(a))$ and $d^{n+1}(a) \leq H(d^n(a))$, (d7) for all $(i,j) \in [n] \times [m]$ and $n \in \omega$, $d^n(\sigma_{ij}(a)) = \sigma_{ij}(d^n(a))$.

Corollary 10. Let (A, G, H) be a tense $LM_{n\times m}$-algebra, $(X(A), g_A, \{f_{ij}^A\}_{(i,j)\in[n]\times[m]}, R^A)$ be the tense $LM_{n\times m}$-space associated with A and let $\sigma_A : A \longrightarrow D(X(A))$ be the map defined by the prescription (2.4). Then, $\sigma_A(d^n(a)) = d^n_{X(A)}(\sigma_A(a))$ for all $a \in A$ and $n \in \omega$.

Proof: It is a direct consequence of Lemma 14. □

It seems worth mentioning that the operator d defined in Corollary 9 was previously defined in [19] for tense algebras, in [12] for tense MV-algebras, and in [8, 9] for tense θ-valued Łukasiewicz–Moisil algebras, respectively.

Lemma 18. *Let (A, G, H) be a tense $LM_{n\times m}$-algebra. If $\bigwedge_{i\in I} a_i$ exists, then the following conditions hold:*

(i) $\bigwedge_{i\in I} G(a_i)$ *exists and* $\bigwedge_{i\in I} G(a_i) = G(\bigwedge_{i\in I} a_i)$,

(ii) $\bigwedge_{i\in I} H(a_i)$ *exists and* $\bigwedge_{i\in I} H(a_i) = H(\bigwedge_{i\in I} a_i)$,

(iii) $\bigwedge_{i\in I} d(a_i)$ *exists and* $\bigwedge_{i\in I} d^n(a_i) = d^n(\bigwedge_{i\in I} a_i)$ *for all* $n \in \omega$.

Proof:

(i): Assume that $a_i \in A$ for all $i \in I$ and $\bigwedge_{i\in I} a_i$ exists. Since $\bigwedge_{i\in I} a_i \leq a_i$, we have by (T2) that $G(\bigwedge_{i\in I} a_i) \leq G(a_i)$ for each $i \in I$. Thus, $G(\bigwedge_{i\in I} a_i)$ is a lower bound of the set $\{G(a_i) : i \in I\}$. Assume now that b is a lower bound of the set $\{G(a_i) : i \in I\}$. From (T5) and (T6) we have that $P(b) \leq PG(a_i) \leq a_i$ for each $i \in I$. So, $P(b) \leq \bigwedge_{i\in I} a_i$. Besides, the pair (G, P) is a Galois connection, this means that $x \leq G(y) \iff P(x) \leq y$, for all $x, y \in A$. So, we can infer that $b \leq G(\bigwedge_{i\in I} a_i)$. This proves that $\bigwedge_{i\in I} G(a_i)$ exists and $\bigwedge_{i\in I} G(a_i) = G(\bigwedge_{i\in I} a_i)$.

(ii): The proof for the operator H is analogous to the proof for G.

(iii): It is a direct consequence of (i) and (ii). □

For invariance properties we have:

Lemma 19. *Let $(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]}, R)$ be a tense $LM_{n\times m}$-space and $(D(X), G_R, H_{R^{-1}})$ be the tense $LM_{n\times m}$-algebra associated with X. Then,* *for all* U, $V, W \in D(X)$ *such that* $U = d_X(U)$, $V = d_X(V)$ *and for some* $(i_0, j_0) \in [n] \times [m]$, $d_X(f^{-1}_{i_0 j_0}(W)) = f^{-1}_{i_0 j_0}(W)$, *the following properties are satisfied:*

(i) $U \cap V = d_X(U \cap V)$,
(ii) $U \cup V = d_X(U \cup V)$,
(iii) $\sim_g U = d_X(\sim_g U)$,
(iv) $d_X(f^{-1}_{ij}(W)) = f^{-1}_{ij}(W)$ *for all* $(i, j) \in [n] \times [m]$.

PROOF:
(i): It immediately follows from the definition of the function d_X and property (T2) of tense $LM_{n\times m}$-algebras.

(ii): Taking into account that $U = d_X(U)$ and $V = d_X(V)$ and the fact that the operations G_R and $H_{R^{-1}}$ are increasing, we infer that $U \cup V \subseteq G_R(U \cup V)$ and $U \cup V \subseteq H_{R^{-1}}(U \cup V)$, which imply that $U \cup V = d_X(U \cup V)$.

(iii): Let $U \in D(X)$ such that (1) $U = d_X(U)$. Then, it is verified that $\sim_g U \subseteq G_R(\sim_g U)$. Indeed, let $x \in \sim_g U$ and (2) $y \in R(x)$. Then, $x \in X \setminus g(U)$ and hence (3) $x \notin g(U)$. Suppose that $y \in g(U)$, then there is $z \in U$ such that $y = g(z)$, and by (tS11) in Lemma 9, we get that $R^{-1}(y) = R^{-1}(g(z)) = g(R^{-1}(z))$. Since $z \in U$, from (1) it follows that $R^{-1}(z) \subseteq U$ and so $g(R^{-1}(z))) \subseteq g(U)$. Thus, $R^{-1}(y) \subseteq g(U)$. From the last statement and (2), we infer that $x \in g(U)$, which contradicts (3). Consequently, $y \in \sim_g U$, which allows us to assert that $R(x) \subseteq \sim_g U$ and therefore $\sim_g U \subseteq G_R(\sim U)$. In a similar way, we can prove that $\sim_g U \subseteq H_{R^{-1}}(\sim_g U)$. From the two last assertions we conclude that $\sim_g U = d_X(\sim_g U)$.

(iv): If $W \in D(X)$ and $d_X(f^{-1}_{i_0 j_0}(W)) = f^{-1}_{i_0 j_0}(W)$ for some $(i_0, j_0) \in [n] \times [m]$, then from (d7) it follows that $f^{-1}_{i_0 j_0}(d_X(W)) = f^{-1}_{i_0 j_0}(W)$. From the last assertion and (LP5) we infer that $f^{-1}_{ij}(d_X(W)) = f^{-1}_{ij}(W)$ for all $(i, j) \in [n] \times [m]$, and so from (d7), we get that $d_X(f^{-1}_{ij}(W)) = f^{-1}_{ij}(W)$ for all $(i, j) \in [n] \times [m]$. □

COROLLARY 11. Let $(A, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]}, G, H)$ be a tense $LM_{n\times m}$-algebra. Then, for all $a, b, c \in A$, such that $a = d(a)$, $b = d(b)$ and $\varphi_{i_0 j_0}(c) = d(\varphi_{i_0 j_0}(c))$ for some $(i_0, j_0) \in [n] \times [m]$, the following properties are satisfied:

(i) $d(a \wedge b) = a \wedge b$,
(ii) $d(a \vee b) = a \vee b$,
(iii) $d(\sim a) =\sim a$,
(iv) $\sigma_{ij}(c) = d(\sigma_{ij}(c))$ for all all $(i,j) \in [n] \times [m]$.

Proof: It is a direct consequence of Lemmas 14 and 19. □

Lemma 20. *Let (A, G, H) be a tense $LM_{n\times m}$-algebra. Then, for all $a \in A$, the following conditions are equivalent:*
(i) $a = d(a)$,
(ii) $a = d^n(a)$ *for all* $n \in \omega$.

Proof: It immediately follows from Corollary 9. □

Lemma 21. *Let $(A, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]}, G, H)$ be a tense $LM_{n\times m}$-algebra and $C(A) := \{a \in A : d(a) = a\}$. Then, $\langle C(A), \vee, \wedge, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]}, 0, 1\rangle$ is an $LM_{n\times m}$-algebra.*

Proof: From Corollary 11 and property (d1) in Corollary 9, we have that $\langle C(A), \vee, \wedge, \sim, 0, 1\rangle$ is a De Morgan algebra. Taking into account that $a = d(a)$ for all $a \in C(A)$, and the property (iv) in Corollary 11 it follows that $\sigma_{ij}(a) = \sigma_{ij}(d(a)) = d(\sigma_{ij}(a))$ for all $a \in C(A)$ and $(i,j) \in [n] \times [m]$. Therefore, $\sigma_{ij}(a) \in C(A)$ for all $a \in C(A)$ and $(i,j) \in [n] \times [m]$, from which we conclude that $\langle C(A), \vee, \wedge, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]}, 0, 1\rangle$ is an $LM_{n\times m}$-algebra. □

Corollary 12. Let (A, G, H) be a tense $LM_{n\times m}$-algebra. Then, the structure $(B(C(A)), G, H)$ is a tense Boolean algebra, where $B(C(A))$ is the Boolean algebra of all complemented elements of $C(A)$.

Proof: It is a direct consequence of Lemmas 5 and 21 and property (iv) in Corollary 11. □

Remark 4. Let us recall that under the Priestley duality, the lattice of all filters of a bounded distributive lattice is dually isomorphic to the lattice of all increasing closed subsets of the dual space. Under that isomorphism, any filter T of a bounded distributive lattice A corresponds to the increasing closed set

$$Y_T = \{S \in X(A) : T \subseteq S\} = \bigcap\{\sigma_A(a) : a \in T\} \tag{4.3}$$

and $\Theta_C(Y_T) = \Theta(T)$, where Θ_C is defined as in (2.11) and $\Theta(T)$ is the lattice congruence associated with T.

Conversely any increasing closed subset Y of $X(A)$ corresponds to the filter

$$T_Y = \{a \in A : Y \subseteq \sigma_A(a)\}, \tag{4.4}$$

and $\Theta(T_Y) = \Theta_C(Y)$, where Θ_C is defined as in (2.11), and $\Theta(T_Y)$ is the lattice congruence associated with T_Y.

Taking into account these last remarks on Priestley duality, Theorem 5 and Proposition 5, we can say that the congruences on a tense $LM_{n\times m}$-algebra are the lattice congruences associated with certain filters of this algebra. So our next goal is to determine the conditions that a filter of a tense $LM_{n\times m}$-algebra must fulfill for the associated lattice congruence to be a tense $LM_{n\times m}$-congruence.

Theorem 6. *Let $(A, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]}, G, H)$ be a tense $LM_{n\times m}$-algebra. If S is a filter of A, then, the following conditions are equivalent:*

(i) $\Theta(S) \in Con_{tLM_{n\times m}}(A)$,

(ii) $d(\sigma_{ij}(a)) \in S$ *for any* $a \in S$ *and* $(i,j) \in [n]\times[m]$,

(iii) $d^n(\sigma_{ij}(a)) \in S$ *for any* $a \in S$, $n \in \omega$ *and* $(i,j) \in [n]\times[m]$.

Proof: (i) $\Rightarrow$ (ii): Let S be a filter of A such that $\Theta(S) \in Con_{tLM_{n\times m}}(A)$. Then, from Priestley duality and Theorem 5 it follows that $\Theta(S) = \Theta_{MT}(Y_S)$, where $\Theta(S)$ is the lattice congruence associated with S, and $Y_S = \{x \in X(A) : S \subseteq x\} = \bigcap\limits_{a\in S} \sigma_A(a)$ is a closed, modal and tense subset of the tense $LM_{n\times m}$-space $X(A)$, associated with A. Since Y_S is modal and σ_A is an $LM_{n\times m}$-isomorphism, then $Y_S = f_{ij}^{A^{-1}}(Y_S) = f_{ij}^{A^{-1}}\Big(\bigcap\limits_{a\in S}\sigma_A(a)\Big) = \bigcap\limits_{a\in S}\sigma_A(\sigma_{ij}(a))$ for any $(i,j)\in[n]\times[m]$. From the last assertion, and taking into account that Y is a tense subset, Lemmas 17 and 9, Corollary 10 and the fact that the function $d_{X(A)} : X(A) \longrightarrow X(A)$ is monotone, we infer that $Y_S = d_{X(A)}\Big(\bigcap\limits_{a\in S}\sigma_A(\sigma_{ij}(a))\Big) \subseteq$

$\bigcap_{a\in S} d_{X(A)}\left(\sigma_A(\sigma_{ij}(a))\right) = \bigcap_{a\in S} \sigma_A(d(\sigma_{ij}(a))) \subseteq \bigcap_{a\in S} \sigma_A(\sigma_{ij}(a)) = Y_S$, for any $(i,j) \in [n]\times[m]$. Hence $Y_S = \bigcap_{a\in S} \sigma_A(d(\sigma_{ij}(a))$ for any $(i,j) \in [n]\times[m]$, from which we conclude that $d(\sigma_{ij}(a)) \in S$ for any $a \in S$ and $(i,j) \in [n]\times[m]$. Indeed, assume that $a \in S$, then $a \in x$ for all $x \in Y_S$, from which it follows that $x \in \bigcap_{a\in S} \sigma_A(d(\sigma_{ij}(a))$ for any $(i,j) \in [n]\times[m]$, and thus $d(\sigma_{ij}(a)) \in x$ for all $x \in Y_S$ and $(i,j) \in [n]\times[m]$. Therefore, $d(\sigma_{ij}(a)) \in \bigcap_{x\in Y_S} x$ for any $(i,j) \in [n]\times[m]$, and taking into account that $S = \bigcap_{x\in Y_S} x$, we obtain that $d(\sigma_{ij}(a)) \in S$ for any $(i,j) \in [n]\times[m]$.

(ii) $\Rightarrow$ (i): From Priestley duality and (4.3), we have that $\bigcap_{a\in S} \sigma_A(a) = Y_S = \{x \in X(A) : S \subseteq x\}$ is an increasing and closed subset of $X(A)$ and $\Theta(S) = \Theta(Y_S)$. By Theorem 5, it remains to show that Y_S is a modal and tense subset of $X(A)$. From the hypothesis (ii), it follows that for all $a \in S$, $(i,j) \in [n]\times[m]$ and $x \in Y_S$, $d(\sigma_{ij}(a)) \in x$. Therefore, from this last fact and Corollary 11, it results that $\sigma_{ij}(d(a)) \in x$ for all $(i,j) \in [n]\times[m]$ and all $x \in Y_S$, and hence (1) $Y_S \subseteq \bigcap_{a\in S} \sigma_A(\sigma_{ij}(d(a)))$ for all $(i,j) \in [n]\times[m]$. Consequently, by Corollary 9, $Y_S \subseteq \bigcap_{a\in S} \sigma_A(\sigma_{ij}(a))$ for all $(i,j) \in [n]\times[m]$, and from this assertion it follows that $Y_S \subseteq \bigcap_{a\in S} \sigma_A\left(\varphi_1(a)\right) \subseteq \bigcap_{a\in S} \sigma_A(a) = Y_s$. Since σ_A is an $LM_{n\times m}$-isomorphism, then we get that (2) $Y_s = \bigcap_{a\in S} \sigma_A\Big(\sigma_{11}(a)\Big) = \bigcap_{a\in S} f_{11}^{A^{-1}}(\sigma_A(a)) = f_{11}^{A^{-1}}\left(\bigcap_{a\in S} \sigma_A(a)\right) = f_{11}^{A^{-1}}(Y_S)$. Therefore from the last statement and (LP6) we conclude that $Y_S = f_{ij}^{A}(Y_S)$ for all $(i,j) \in [n]\times[m]$ and so Y_S is modal. In addition, from (1), (2) and Corollary 9 we infer that $Y_S \subseteq \bigcap_{a\in S} \sigma_A(d(\sigma_{11}(a)) \subseteq \bigcap_{a\in S} \sigma_A(\sigma_{11}(a)) = Y_S$ and hence, $Y_S = \bigcap_{a\in S} \sigma_A(d(\sigma_{11}(a))$. Then, taking into account Corollary 10 and that $\bigcap_{a\in S} d_{X(A)}(\sigma_A(\sigma_{11}(a))) = d_{X(A)}\left(\bigcap_{a\in S} \sigma_A(\sigma_{11}(a))\right)$, we obtain that $Y_S = d_{X(A)}(Y_S)$, and thus, from Lemma 17 and the fact that Y_S is modal, we infer that Y_S is a tense subset of $X(A)$. Finally, since Y_S is a closed, modal and tense subset of $X(A)$ and $\Theta(S) = \Theta_{MT}(Y_S)$, we conclude, from Theorem 5, that $\Theta(S) \in Con_{tLM_{n\times m}}(A)$.

(ii) $\Leftrightarrow$ (iii): It is trivial. □

Theorem 6 leads us to introduce the following definition:

DEFINITION 18. Let (A, G, H) be a tense $LM_{n\times m}$-algebra. A filter S of A is a tense filter iff

(tf) $d(a) \in S$ for all $a \in S$ or equivalently $d^n(a) \in S$ for all $a \in S$ and $n \in \omega$.

Now, we remember the notion of Stone filter of an $LM_{n\times m}$-algebra.

DEFINITION 19. Let $(A, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]})$ be an $LM_{n\times m}$-algebra. A filter S of A is a Stone filter iff

(sf) $\sigma_{ij}(a) \in S$ for all $a \in S$ and $(i, j) \in [n] \times [m]$, or equivalently $\sigma_{11}(a) \in S$ for all $a \in S$.

LEMMA 22. *Let $(A, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]}, G, H)$ be a tense $LM_{n\times m}$-algebra. If S is a Stone filter of A, then the following conditions are equivalent:*

(i) *S is a tense filter of A,*

(ii) *$d^n(\sigma_{ij}(a)) \in S$ for all $a \in S$, $n \in \omega$ and $(i, j) \in [n] \times [m]$.*

PROOF:

(i) $\Rightarrow$ (ii): Let S be a Stone filter of A, $a \in S$, $n \in \omega$ and $(i, j) \in [n] \times [m]$. Since S is an Stone filter of A, we have that $\sigma_{ij}(a) \in S$. From this last assertion and the fact that S is a tense filter we conclude that $d^n(\sigma_{ij}(a)) \in S$.

(ii) $\Rightarrow$ (i): Let $a \in S$. Then, from the hypothesis (ii) we obtain that $d^n(\sigma_{11}(a)) \in S$. From the last assertion, properties (C13) and (d5) and the fact that S is a filter of A we infer that $d^n(a) \in S$ for all $n \in \omega$, and therefore S is a tense filter of A. □

We will denote by $F_{TS}(A)$ the set of all tense Stone filters of a tense $LM_{n\times m}$-algebra (A, G, H).

PROPOSITION 9. Let $(A, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]}, G, H)$ be a tense $LM_{n\times m}$-algebra. Then, the following conditions are equivalent for all $\theta \subseteq A \times A$:

(i) $\theta \in Con_{tLM_{n\times m}}(A)$,

(ii) there is $S \in F_{TS}(A)$ such that $\theta = \Theta(S)$, where $\Theta(S)$ is the lattice congruence associated with the filter S.

PROOF:

(i) $\Rightarrow$ (ii): From (i) and Theorem 5, it follows that there exists $Y \in C_{MT}(X(A))$ such that (1) $\Theta_{MT}(Y) = \theta$. Then, from Remark 4, we infer that $T_Y = \{a \in A : Y \subseteq \sigma_A(a)\}$ is a filter on A and (2) $\Theta(T_Y) =$

$\Theta(Y) = \Theta_{MT}(Y)$. Therefore $\Theta(T_Y) \in Con_{tLM_{n\times m}}(A)$, and so from Theorem 6, we obtain that $Y \in F_{TS}(A)$. This last assertion, (1) and (2) enable us to conclude the proof.

(ii) $\Rightarrow$ (i): It immediatly follows from Theorem 6. $\square$

Corollary 13. Let (A, G, H) be a tense $LM_{n\times m}$-algebra. Then,

(i) (A, G, H) is a simple tense $LM_{n\times m}$-algebra if and only if $F_{TS}(A) = \{A, \{1\}\}$.

(ii) (A, G, H) is a subdirectly irreducible tense $LM_{n\times m}$-algebra if and only if there is $T \in F_{TS}(A)$, $T \neq \{1\}$ such that $T \subseteq S$ for all $S \in F_{TS}(A)$, $S \neq \{1\}$.

Proof: It is a direct consequence of Corollaries 7 and 8, Remark 4 and Proposition 9. $\square$

Finally, we will describe the simple and subdirectly irreducible tense $LM_{n\times m}$-algebras.

In the proof of the following proposition we will use the finite intersection property of compact spaces, which establishes that if X is a compact topological space, then for each family $\{M_i\}_{i\in I}$ of closed subsets of X satisfying $\bigcap\limits_{i\in I} M_i = \emptyset$, there is a finite subfamily $\{M_{i_1}, \ldots, M_{i_n}\}$ such that $\bigcap\limits_{j=1}^{n} M_{i_j} = \emptyset$.

Proposition 10. Let (A, G, H) be a tense $LM_{n\times m}$-algebra and $(X(A), g_A, \{f_{ij}^A\}_{(i,j)\in[n]\times[m]}, R^A)$ be the tense $LM_{n\times m}$-space associated with A. Then, the following conditions are equivalent:

(i) (A, G, H) is a simple tense $LM_{n\times m}$-algebra,

(ii) for every $U \in D(X(A)) \setminus \{X(A)\}$ and for every $(i, j) \in [n] \times [m]$ such that ${f_{ij}^A}^{-1}(U) \neq X(A)$, $\bigcap\limits_{n\in\omega} d^n_{X(A)}({f_{ij}^A}^{-1}(U)) = \emptyset$,

(iii) for every $U \in D(X(A)) \setminus \{X(A)\}$ and for every $(i, j) \in [n] \times [m]$ such that ${f_{ij}^A}^{-1}(U) \neq X(A)$, $d^{n_{ij}^U}_{X(A)}({f_{ij}^A}^{-1}(U)) = \emptyset$ for some $n_{ij}^U \in \omega$,

(iv) for every $U \in B(D(X(A))) \setminus \{X(A)\}$, there is $n_U \in \omega$ such that $d^{n_U}_{X(A)}(U) = \emptyset$,

(v) $F_{TS}(D(X(A))) = \{D(X(A)), \{X(A)\}\}$.

Proof:
(i) $\Rightarrow$ (ii): Let $U \in D(X(A)) \setminus \{X(A)\}$. Now, let $(i,j) \in [n] \times [m]$ such that $f_{ij}^{A^{-1}}(U) \neq X(A)$, then from (d5) in Proposition 8 we have that $d^n_{X(A)}(f_{ij}^{A^{-1}}(U)) \neq X(A)$. From this last assertion and (d9) in Proposition 8, we obtain that $\bigcap\limits_{n\in\omega} d^n_{X(A)}(f_{ij}^{A^{-1}}(U)) \in C_{MT}(X(A)) \setminus \{X(A)\}$. From this last assertion, the hypothesis (i) and Corollary 7, we conclude that $\bigcap\limits_{n\in\omega} d^n_{X(A)}(f_{ij}^{A^{-1}}(U)) = \emptyset$.
(ii) $\Rightarrow$ (iii): Let $U \in D(X(A)) \setminus \{X(A)\}$ and $(i,j) \in [n] \times [m]$ such that $f_{ij}^{A^{-1}}(U) \neq X(A)$. Then, from the hypothesis (ii), we have that

$$(1) \quad \bigcap_{n\in\omega} d^n_{X(A)}(f_{ij}^{A^{-1}}(U)) = \emptyset.$$

Besides, for all $n \in \omega$, $d^n_{X(A)}(f_{ij}^{A^{-1}}(U))$ is a closed subset of $X(A)$ and $d^n_{X(A)}(f_{ij}^{A^{-1}}(U)) = \bigcap\limits_{k=1}^{n} d^k_{X(A)}(f_{ij}^{A^{-1}}(U))$. Then, from (1), the last statement, the fact that $X(A)$ is compact and the finite intersection property of compact spaces, we conclude that there is $n^U_{ij} \in \omega$ such that $d^{n^U_{ij}}_{X(A)}(f_{ij}^{A^{-1}}(U)) = \emptyset$.
(iii) $\Rightarrow$ (iv): From Lemma 5, we have that $U \in B(D(X(A)))$ if and only if $U = f_{ij}^{A^{-1}}(U)$ for all $(i,j) \in [n] \times [m]$, and so from property (LP10) of $LM_{n\times m}$-spaces, we infer that $U \in B(D(X(A))) \setminus \{X(A)\}$ iff $f_{ij}^{A^{-1}}(U) \neq X(A)$ for all $(i,j) \in [n] \times [m]$. Therefore, from the previous assertion and the hypothesis (iii), we obtain that for each $U \in B(D(X(A)))$ and each $(i,j) \in [n] \times [m]$, there is $n^U_{ij} \in \omega$ such that $d^{n^U_{ij}}_{X(A)}(U) = \emptyset$. Since, from (1) it follows that for all $(i,j),(r,s) \in [n] \times [m]$, $n^U_{ij} = n^U_{rs} = n^U$, then the proof is complete.
(iv) $\Rightarrow$ (v): Assume that $S \in F_{TS}(D(X(A)))$, $S \neq \{X(A)\}$. Then there is (1) $U \in S$, $U \neq X(A)$ and so from property (LP10) of $LM_{n\times m}$-spaces, we infer that there is $(i,j) \in [n]\times[m]$ such that $f_{ij}^{A^{-1}}(U) \neq X(A)$. Considering (2) $V = f_{ij}^{A^{-1}}(U)$, then from Lemma 5, we obtain that $V \in B(D(X(A)))$, $V \neq X(A)$. Hence, from the hypothesis (iv), we can assert that there is $n_V \in \omega$ such that $d^{n_V}_{X(A)}(V) = \emptyset$. From (1), (2), the preceding assertion and Definitions 18 and 19, we deduce that $\emptyset \in S$, which implies that $S = D(X(A))$.

(v) $\Rightarrow$ (i): It immediately follows from Corollary 13 and the fact that (A, G, H) is isomorphic to the tense $LM_{n\times m}$-algebra $(D(X(A)), G_{R^A}, H_{R^{A^{-1}}})$. □

COROLLARY 14. Let $(A, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]}, G, H)$ be a tense $LM_{n\times m}$-algebra. Then, the following conditions are equivalent:

(i) $(A, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]}, G, H)$ is a simple tense $LM_{n\times m}$-algebra,

(ii) for every $a \in A \setminus \{1\}$ and for every $(i, j) \in [n] \times [m]$ such that $\sigma_{ij}(a) \neq 1$, $d^{n^a_{ij}}(\sigma_{ij}(a)) = 0$ for some $n^a_{ij} \in \omega$,

(iii) for each $a \in B(A) \setminus \{1\}$, there is $n_a \in \omega$ such that $d^{n_a}(a) = 0$,

(iv) $F_{TS}(A) = \{A, \{1\}\}$.

PROOF: It is a direct consequence of Proposition 10 and the fact that $\sigma_A : A \longrightarrow D(X(A))$ is a tense $LM_{n\times m}$-isomorphism. □

COROLLARY 15. If $(A, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]}, G, H)$ is a simple tense $LM_{n\times m}$-algebra, then $B(C(A)) = \{0, 1\}$ and therefore $(C(A), \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]})$ is a simple $LM_{n\times m}$-algebra.

PROOF: From Lemmas 5 and 20, property (iv) in Corollary 11 and property (ii) in Corollary 14 it follows that $B(C(A)) = \{0, 1\}$. From this last assertion, (LM8) and Lemma 21 the proof is complete. □

Next, we will recall two concepts which will play a fundamental role in this paper. Let Y be a topological space and $y_0 \in Y$. A net in a space Y is a map $\varphi : D \longrightarrow Y$ of some directed set $(D, \prec)$ (i.e. $D \neq \emptyset$ and $\prec$ is a preorder on D and for all d_1, $d_2 \in D$ there is $d_3 \in D$ such that $d_1 \prec d_3$ and $d_2 \prec d_3$). Besides, we say that φ converges to y_0 (written $\varphi \to y_0$) if for all neighborhoods $U(y_0)$ of y_0 there is $d_0 \in D$ such that for all $d \in D$, $d_0 \prec d$, $\varphi(d) \in U(y_0)$. We also say that φ accumulates at y_0 (written $\varphi \succ y_0$) if for all neighborhoods $U(y_0)$ of y_0 and for all $d \in D$, there is $d_c \in D$ such that $d \prec d_c$ and $\varphi(d_c) \in U(y_0)$. If $\varphi : D \longrightarrow Y$ is a net and $y_d = \varphi(d)$ for all $d \in D$, then the net φ it will be denoted by $(y_d)_{d\in D}$. If $\varphi \to y_0$, it will be denoted by $(y_d) \underset{d\in D}{\longrightarrow} y_0$. If $\varphi \succ y_0$, it will be denoted $(y_d)_{d\in D} \succ y_0$.

PROPOSITION 11. Let (A, G, H) be a tense $LM_{n\times m}$-algebra and $(X(A), g_A, \{f^A_{ij}\}_{(i,j)\in[n]\times[m]}, R^A)$ be the tense $LM_{n\times m}$-space associated with A. Then, the following conditions are equivalent:

(i) (A,G,H) is a subdirectly irreducible tense $LM_{n\times m}$-algebra,

(ii) there is $V \in B(D(X(A)))$, $V \neq X(A)$, such that for each $U \in D(X(A))$, $U \neq X(A)$ and for each $(i,j) \in [n]\times[m]$ such that $f_{ij}^{A^{-1}}(U) \neq X(A)$, $\bigcap\limits_{n\in\omega} d^n_{X(A)}(f_{ij}^{A^{-1}}(U)) \subseteq V$,

(iii) there is $V \in B(D(X(A)))$, $V \neq X(A)$, such that for each $U \in D(X(A))$, $U \neq X(A)$ and for each $(i,j) \in [n]\times[m]$ such that $f_{ij}^{A^{-1}}(U) \neq X(A)$, $d^{n_i^U}_{X(A)}(f_{ij}^{A^{-1}}(U)) \subseteq V$ for some $n_{ij}^U \in \omega$,

(iv) there is $V \in B(D(X(A)))$, $V \neq X(A)$, such that for all $U \in B(D(X(A)))$, $U \neq X(A)$, $d^{n^U}_{X(A)}(U) \subseteq V$, for some $n^U \in \omega$,

(v) there is $T \in F_{TS}(D(X(A))$, $T \neq \{X(A)\}$, such that $T \subseteq S$ for all $S \in F_{TS}(D(X(A)))$, $S \neq \{X(A)\}$.

Proof:
(i) $\Rightarrow$ (ii): From (i) and Corollary 8 we infer that there exists $Y \in C_{MT}(X(A)) \setminus \{X(A)\}$ such that (1) $Z \subseteq Y$ for all $Z \in C_{MT}(X(A)) \setminus \{X(A)\}$. Since Y is modal, then by Proposition 5, there is (2) $x \in \max X(A) \setminus Y$. Taking into account that Y is a closed subset of $X(A)$ and hence it is compact, we can assert that there is $W \in D(X(A))$, such that (3) $Y \subseteq W$ and (4) $x \notin W$. In addition from (2) and (LP15) in Corollary 1, we have that $x = f^A_{(n-1)(m-1)}(x)$ and so by (4) we infer that $x \notin f^{A^{-1}}_{(n-1)(m-1)}(W)$. If $V = f^{A^{-1}}_{(n-1)(m-1)}(W)$, then $V \in B(D(X(A)))\setminus\{X(A)\}$. Besides, from (3) and the fact that $Y = f^{A^{-1}}_{(n-1)(m-1)}(Y)$, we get that (5) $Y \subseteq f^{A^{-1}}_{(n-1)(m-1)}(W) = V$. On the other hand, if $U \in D(X(A)) \setminus \{X(A)\}$, then from Lemma 14 and property (LP10) of $LM_{n\times m}$-spaces, we infer that there is at least $(i_0,j_0) \in [n]\times[m]$ such that $f^{A^{-1}}_{i_0j_0}(U) \neq X(A)$. Now, let $(i,j) \in [n]\times[m]$ such that $f_{ij}^{A^{-1}}(U) \neq X(A)$, then from Proposition 8 we obtain that $\bigcap\limits_{n\in\omega} d^n_{X(A)}(f_{ij}^{A^{-1}}(U)) \in C_{MT}(X(A)) \setminus \{X(A)\}$, from which we conclude, by the assertions (1) and (5), that $\bigcap\limits_{n\in\omega} d^n_{X(A)}(f_{ij}^{A^{-1}}(U)) \subseteq V$.

(ii) $\Rightarrow$ (iii): From the hypothesis (ii), we have that there is $V \in B(D(X(A))) \setminus \{X(A)\}$, such that (1) $\bigcap\limits_{n\in\omega} d^n_{X(A)}(f_{ij}^{A^{-1}}(U)) \subseteq V$ for each $U \in D(X(A)) \setminus \{X(A)\}$ and each $i \in [n]\times[m]$ such that $f_{ij}^{A^{-1}}(U) \neq X(A)$. Suppose that there is $U \in D(X(A)) \setminus \{X(A)\}$ and there is $i_0 \in [n]\times[m]$, which satisfy (1) and $d^{n}_{X(A)}({f^{A}_{i_0j_0}}^{-1}(U)) \not\subseteq V$ for all $n \in \omega$. Then for each $n \in \omega$, there exists (2) $x_n \in d^{n}_{X(A)}({f^{A}_{i_0j_0}}^{-1}(U))$ and $x_n \notin V$. Hence $(x_n)_{n\in\omega}$ is a sequence in $X(A) \setminus V$ and since $X(A) \setminus V$ is compact, we can assert that there exists (3) $x \in X(A) \setminus V$ such that $(x_n)_{n\in\omega}$ accumulates at x. In addition, by (1) and (3), we have that $x \notin \bigcap\limits_{n\in\omega} d^{n}_{X(A)}({f^{A}_{i_0j_0}}^{-1}(U))$, and thus $x \in X(A) \setminus d^{n_0}_{X(A)}({f^{A}_{i_0j_0}}^{-1}(U))$ for some $n_0 \in \omega$. Since x is an accumulation point of $(x_n)_{n\in\omega}$, then the preceding assertion and the fact that $X(A) \setminus d^{n_0}_{X(A)}({f^{A}_{i_0j_0}}^{-1}(U))$ is an open subset of $X(A)$ allows us to infer that for all $n \in \omega$ there is $m_n \in \omega$ such that $n \leq m_n$ and $x_{m_n} \in X(A) \setminus d^{n_0}_{X(A)}({f^{A}_{i_0j_0}}^{-1}(U))$. Thus $x_{m_{n_0}} \in X(A) \setminus d^{n_0}_{X(A)}({f^{A}_{i_0j_0}}^{-1}(U))$ and $n_0 \leq m_{n_0}$. As a consequence of Proposition 8 we have that $X(A) \setminus d^{n_0}_{X(A)}({f^{A}_{i_0j_0}}^{-1}(U)) \subseteq X(A) \setminus d^{m_{n_0}}_{X(A)}({f^{A}_{i_0j_0}}^{-1}(U))$ and so $x_{m_{n_0}} \notin d^{m_{n_0}}_{X(A)}({f^{A}_{i_0j_0}}^{-1}(U)))$, which contradicts (2). Therefore, for every $U \in D(X(A)) \setminus \{X(A)\}$ and $(i,j) \in [n] \times [m]$ such that ${f^{A}_{ij}}^{-1}(U) \neq X(A)$, $d^{n^{U}_{i}}_{X(A)}({f^{A}_{ij}}^{-1}(U)) \subseteq V$ for some $n^{U}_{i} \in \omega$.

(iii) $\Rightarrow$ (iv): From Lemma 5 and the property (LP10) of $LM_{n\times m}$-spaces, we infer that for all $U \in B(D(X(A)))$, $U \neq X(A)$ if and only if ${f^{A}_{ij}}^{-1}(U) \neq X(A)$ for all $(i,j) \in [n] \times [m]$. Therefore, from the last statement and the hypothesis (iii), we obtain that for each $U \in B(D(X(A)))$, $U \neq X(A)$ and each $(i,j) \in [n] \times [m]$, there is $n^{U}_{ij} \in \omega$ such that $d^{n^{U}_{ij}}_{X(A)}(U) \subseteq V$. Then, considering $n_U = \max\{n^{U}_{ij} : (i,j) \in [n] \times [m]\}$, from (d2) in Proposition 8 we conclude that $d^{n_U}_{X(A)}(U) \subseteq V$.

(iv) $\Rightarrow$ (v): Let $S \in F_{TS}(D(X(A)))$, $S \neq \{X(A)\}$. Then there exists (1) $U \in S \setminus \{X(A)\}$ and so from property (LP10) we infer that there is $(i,j) \in [n] \times [m]$ such that ${f^{A}_{ij}}^{-1}(U) \neq X(A)$. Let (2) $W = {f^{A}_{ij}}^{-1}(U)$. Then, from Lemma 5 we have that $W \in B(D(X(A)))$, $W \neq X(A)$ and thus by the hypothesis (iv), we can assert that there is $n_W \in \omega$ such that (3) $d^{n_W}_{X(A)}(W) \subseteq V$. Besides, from the assertions (1) and (2) and Lemma 22, we obtain that $d^{n_W}_{X(A)}(W) \in S$. From the last statement, (3) and the fact that S is a filter of $D(X(A))$, we get that $V \in S$, and so $V \in \bigcap\limits_{S\in\Omega} S$, where $\Omega = \{S \in F_{TS}(D(X(A))) : S \neq \{X(A)\}\}$. Therefore, considering $T = \bigcap\limits_{S\in\Omega} S$ and taking into account that $V \neq X(A)$, we conclude that $T \in \Omega$ and $T \subseteq S$, for all $S \in \Omega$.

(v) $\Rightarrow$ (i): It follows from the fact that (A, G, H) and $(D(X(A)), G_{R^A}, H_{R^{A^{-1}}})$ are isomorphic tense $LM_{n\times m}$-algebras. $\square$

COROLLARY 16. Let $(A, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]}, G, H)$ be a tense $LM_{n\times m}$-algebra. Then, the following conditions are equivalent:

(i) $(A, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]}, G, H)$ is a subdirectly irreducible tense $LM_{n\times m}$-algebra,

(ii) there is $b \in B(A) \setminus \{1\}$ such that for every $a \in A \setminus \{1\}$ and for every $(i,j) \in [n] \times [m]$ such that $\sigma_{ij}(a) \neq 1$, $d^{n^a_{ij}}(\sigma_{ij}(a)) \leq b$ for some $n^a_{ij} \in \omega$,

(iii) there is $b \in B(A) \setminus \{1\}$ such that for every $a \in B(A) \setminus \{1\}$, there is $n_a \in \omega$ such that $d^{n_a}(a) \leq b$,

(iv) there is $T \in F_{TS}(A)$, $T \neq \{1\}$ such that $T \subseteq S$ for all $S \in F_{TS}(A)$, $S \neq \{1\}$.

PROOF: It is a direct consequence of Proposition 11 and the fact that $\sigma_A : A \longrightarrow D(X(A))$ is a tense $LM_{n\times m}$-isomorphism. $\square$

COROLLARY 17. Let $(A, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]}, G, H)$ be a subdirectly irreducible tense $LM_{n\times m}$-algebra such that for every $a \in B(A) \setminus \{1\}$, $d^n(a) = d^{n_a}(a)$ for some $n_a \in \omega$ and for all $n \in \omega$, $n_a \leq n$. Then, $(C(A) \sim\ , \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]})$ is a simple $LM_{n\times m}$-algebra.

PROOF: From Corollary 16, we can assert that there exists $b \in B(A) \setminus \{1\}$ such that (1) for every $a \in B(A) \setminus \{1\}$, $d^{n_a}(a) \leq b$ for some $n_a \in \omega$. Also, from hypothesis we have that there is $n_b \in \omega$ such that $d^n(b) = d^{n_b}(b)$ for all $n \in \omega$, $n_b \leq n$. Considering $u = d^{n_b}(b)$, then from the last assertion, properties (d5) and (d7) in Corollary 9 and the fact that $b \in B(A) \setminus \{1\}$, we obtain that $u \in B(C(A))$, $u \neq 1$. In addition, let $c \in B(C(A))$, $c \neq 1$, then by Lemma 20, $c = d^n(c)$ for all $n \in \omega$, and thus from (1) we get that $c = d^{n_c}(c) \leq b$. Then from property (d4) in Corollary 9, we infer that $c = d^{n_b}(c) \leq d^{n_b}(b) = u$. Consequently, from Corollary 12, $B(C(A))$ is a totally ordered Boolean algebra and so $B(C(A)) = \{0, 1\}$. Therefore, from (LM8) and Lemma 21, we conclude that $(C(A) \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]})$ is a simple $LM_{n\times m}$-algebra. $\square$

5. Conclusion and future research

Priestley spaces arise more naturally in relation with logics, as Priestley spaces incorporate the now widely used Kripke semantics in them. As a result, Priestley's duality became rather popular among logicians, and most dualities for distributive lattices with operators have been performed in terms of Priestley spaces. In particular, in this paper we have determined a topological duality for tense $n \times m$-valued Łukasiewicz–Moisil algebras, extending the one obtained for $n \times m$-valued Łukasiewicz–Moisil algebras in [27]. By means of the above duality we have characterized simple and subdirectly irreducible tense $n \times m$-valued Łukasiewicz–Moisil algebras. We expect that our method can be easily applied to modal operators or monadic operators on $n \times m$-valued Łukasiewicz–Moisil algebras (see, [25], [27]).

References

[1] V. Boicescu, A. Filipoiu, G. Georgescu and S. Rudeanu, *Łukasiewicz–Moisil Algebras*, **Annals of Discrete Mathematics**, Vol. 49 (1991), North-Holland.

[2] M. Botur, I. Chajda, R. Halaš and M. Kolařík, *Tense operators on Basic Algebras*, **International Journal of Theoretical Physics**, Vol. 50, No. 12 (2011), pp. 3737–3749.

[3] M. Botur, J. Paseka, *On tense MV-algebras*, **Fuzzy Sets and Systems**, Vol. 259 (2015), pp. 111–125.

[4] J. Burges, *Basic tense logic*, [in:] D. M. Gabbay, F. Günter (eds.), **Handbook of Philosophical Logic**, Vol. II, Reidel, Dordrecht (1984), pp. 89–139.

[5] I. Chajda, *Algebraic axiomatization of tense intuitionistic logic*, **Central European Journal of Mathematics**, Vol. 9, No. 5 (2011), pp. 1185–1191.

[6] I. Chajda and J. Paseka, *Dynamic effect algebras and their representations*, **Soft Computing**, Vol. 16, No. 10 (2012), pp. 1733–1741.

[7] I. Chajda and M. Kolařík, *Dynamic Effect Algebras*, **Mathematica Slovaca**, Vol. 62, No. 3 (2012), pp. 379–388.

[8] C. Chiriţă, *Tense θ-valued Moisil propositional logic*, **International Journal of Computers Communications and Control**, Vol. 5 (2010), pp. 642–653.

[9] C. Chiriţă, *Tense θ-valued Łukasiewicz–Moisil algebras*, **Journal of Multiple-Valued Logic and Soft Computing**, Vol. 17, No. 1 (2011), pp. 1–24.

[10] R. Cignoli, *Moisil Algebras*, **Notas de Lógica Matemática**, Vol. 27 (1970), Instituto de Matemática, Universidad Nacional del Sur, Bahía Blanca.

[11] W. Cornish and P. Fowler, *Coproducts of De Morgan algebras*, **Bulletin of the Australian Mathematical Society**, Vol. 16 (1977), pp. 1–13.

[12] D. Diaconescu and G. Georgescu, *Tense operators on MV-algebras and Łukasiewicz–Moisil algebras*, **Fundamenta Informaticae**, Vol. 81, No. 4 (2007), pp. 379–408.

[13] A. V. Figallo, G. Pelaitay, *$n \times m$-valued Łukasiewicz–Moisil algebras with two modal operators*, **South American Journal of Logic**, Vol. 1, No. 1 (2015), pp. 267–281.

[14] A. V. Figallo, I. Pascual, G. Pelaitay, *A topological duality for tense LM_n-algebras and applications*, **Logic Journal of the IGPL**, Vol. 26, No. 4 (2018), pp. 339–380.

[15] A. V. Figallo and G. Pelaitay, *Note on tense SHn-algebras*, **Analele Universitatii din Craiova. Seria Matematica-Informatica**, Vol. 38, No. 4 (2011), pp. 24–32.

[16] A. V. Figallo and G. Pelaitay, *Tense operators on De Morgan algebras*, **Logic Journal of the IGPL**, Vol. 22, No. 2, (2014), pp. 255–267.

[17] A. V. Figallo and G. Pelaitay, *A representation theorem for tense $n \times m$-valued Łukasiewicz–Moisil algebras*, **Mathematica Bohemica**, Vol. 140, No. 3 (2015), pp. 345–360.

[18] A. V. Figallo and G. Pelaitay, *Discrete duality for tense Łukasiewicz–Moisil algebras*, **Fundamenta Informaticae**, Vol. 136, No. 4 (2015), pp. 317–329.

[19] T. Kowalski, *Varieties of tense algebras*, **Reports on Mathematical Logic**, Vol. 32 (1998), pp. 53–95.

[20] Gr. C. Moisil, *Recherches sur les logiques non-chrysippiennes*, **Annales Scientifiques de l'Université de Jassy**, Vol. 26 (1940), pp. 431–466.

[21] J. Paseka, *Operators on MV-algebras and their representations*, **Fuzzy Sets and Systems**, Vol. 232 (2013), pp. 62–73.

[22] H. A. Priestley, *Representation of distributive lattices by means of ordered Stone spaces*, **Bulletin of the London Mathematical Society**, Vol. 2 (1970), pp. 186–190.

[23] A. V. Figallo and C. Sanza, *Álgebras de Łukasiewicz $n \times m$-valuadas con negación*, **Noticiero de la Unión Matemática Argentina**, Vol. 93 (2000).

[24] A. V. Figallo and C. Sanza, *The $NS_{n\times m}$-propositional calculus*, **Bulletin of the Section of Logic**, Vol. 35, No. 2 (2008), pp. 67–79.

[25] A. V. Figallo and C. Sanza, *Monadic $n \times m$-valued Łukasiewicz–Mosil algebras*, **Mathematica Bohemica**, Vol. 137, No. 4 (2012), pp. 425–447.

[26] A. V. Figallo and G. Pelaitay, *A representation theorem for tense $n \times m$-valued Łukasiewicz–Moisil algebras*, **Mathematica Bohemica**, Vol. 140, No. 3 (2015), pp. 345–360.

[27] A. V. Figallo, I. Pascual, G. Pelaitay, *A new topological duality for $n \times m$-valued Łukasiewicz–Moisil algebras*, **Asian–European Journal of Mathematics** (2019).

[28] C. Gallardo, C. Sanza and A. Ziliani, *F-multipliers and the localization of $LM_{n\times m}$-algebras*, **Analele Stiintifice ale Universitatii Ovidius Constanta**, Vol. 21, No. 1 (2013), pp. 285–304.

[29] Gr. C. Moisil, **Essais sur les logiques non Chrysippiennes**, Ed. Academiei, Bucarest, 1972.

[30] H. Priestley, *Representation of distributive lattices by means of ordered Stone spaces*, **Bulletin of the London Mathematical Society**, Vol. 2 (1970), pp. 186–190.

[31] H. Priestley, *Ordered topological spaces and the representation of distributive lattices*, **Proceedings of the London Mathematical Society**, Vol. 3 (1972), pp. 507–530.

[32] H. Priestley, *Ordered sets duality for distributive lattices*, **Annals of Discrete Mathematics**, Vol. 23 (1984), pp. 39–60.

[33] C. Sanza, *Notes on $n \times m$-valued Łukasiewicz algebras with negation*, **Logic Journal of the IGPL**, Vol. 6, No. 12 (2004), pp. 499–507.

[34] C. Sanza, *$n \times m$-valued Łukasiewicz algebras with negation*, **Reports on Mathematical Logic**, Vol. 40 (2006), pp. 83–106.

[35] C. Sanza, *On $n\times m$-valued Łukasiewicz–Moisil algebras*, **Central European Journal of Mathematics**, Vol. 6, No. 3 (2008), pp. 372–383.

[36] W. Suchoń, *Matrix Łukasiewicz Algebras*, **Reports on Mathematical Logic**, Vol. 4 (1975), pp. 91–104.

Universidad Nacional de San Juan
Instituto de Ciencias Básicas
5400 San Juan, Argentina
e-mail: `avfigallo@gmail.com`
`ipascual756@hotmail.com`
`gpelaitay@gmail.com`

Bulletin of the **S**ection of **L**ogic
Volume 49/1 (2020), pp. 53–78

http://dx.doi.org/10.18778/0138-0680.2020.03

Young Bae Jun, Seok-Zun Song*

INF-HESITANT FUZZY IDEALS IN BCK/BCI-ALGEBRAS[1]

Abstract

Based on the hesitant fuzzy set theory which is introduced by Torra in the paper [12], the notions of Inf-hesitant fuzzy subalgebras, Inf-hesitant fuzzy ideals and Inf-hesitant fuzzy p-ideals in BCK/BCI-algebras are introduced, and their relations and properties are investigated. Characterizations of an Inf-hesitant fuzzy subalgebras, an Inf-hesitant fuzzy ideals and an Inf-hesitant fuzzy p-ideal are considered. Using the notion of BCK-parts, an Inf-hesitant fuzzy ideal is constructed. Conditions for an Inf-hesitant fuzzy ideal to be an Inf-hesitant fuzzy p-ideal are discussed. Using the notion of Inf-hesitant fuzzy (p-) ideals, a characterization of a p-semisimple BCI-algebra is provided. Extension properties for an Inf-hesitant fuzzy p-ideal is established.

Keywords: p-semisimple BCI-algebra, Inf-hesitant fuzzy subalgebra, Inf-hesitant fuzzy ideal, Inf-hesitant fuzzy p-ideal.

Mathematics Subject Classification (2010): 06F35, 03G25, 08A72.

---

*Corresponding author.

[1]This research was supported by Basic Science Research Program to Research Institute for Basic Sciences (RIBS) of Jeju National University through the National Research Foundation of Korea (NRF) funded by the Ministry of Education. (2019R1A6A1A10072987).

1. Introduction

Several generalizations and extensions of Zadeh's fuzzy sets have been introduced in the literature, for example, intuitionistic fuzzy sets, interval-valued fuzzy sets, type 2 fuzzy sets and fuzzy multisets etc. As another generalization of fuzzy sets, Torra [12] introduced the notion of hesitant fuzzy sets which are a very useful to express peoples hesitancy in in daily life. The hesitant fuzzy set is a very useful tool to deal with uncertainty, which can be accurately and perfectly described in terms of the opinions of decision makers. Xu and Xia [17] proposed a variety of distance measures for hesitant fuzzy sets, based on which the corresponding similarity measures can be obtained. They investigated the connections of the aforementioned distance measures and further develop a number of hesitant ordered weighted distance measures and hesitant ordered weighted similarity measures. Also, hesitant fuzzy set theory is used in decision making problem etc. (see [10, 14, 15, 16, 18]). In the algebraic structures, Jun et al. [6, 8] applied the hesitant fuzzy sets to BCK/BCI-algebras and MTL-algebras. They introduced the notions of hesitant fuzzy subalgebras and hesitant fuzzy ideals of BCK/BCI-algebras, and the notions of a (Boolean, prime, ultra, good) hesitant fuzzy filter and a hesitant fuzzy MV-filter of MTL-algebras. They investigated related relations and properties, and considered characterizations of hesitant fuzzy subalgebras, hesitant fuzzy ideals, (Boolean, ultra) hesitant fuzzy filters in BCK/BCI-algebras and MTL-algebras. Recently BCK/BCI-algebras have been widely applied to soft set theory, cubic structure, bipolar and m-polar fuzzy set theory etc. (see [1], [2], [3], [4], [7], [11]).

In this paper, based on the hesitant fuzzy set theory which is introduced by Torra [12], we introduce the notions of Inf-hesitant fuzzy subalgebras, Inf-hesitant fuzzy ideals and Inf-hesitant fuzzy p-ideals in BCK/BCI-algebras. We investigate their relations and properties, and find conditions for an Inf-hesitant fuzzy ideal to be an Inf-hesitant fuzzy p-ideal. We discuss caracterizations of an Inf-hesitant fuzzy subalgebras, an Inf-hesitant fuzzy ideals and an Inf-hesitant fuzzy p-ideal. We construct an Inf-hesitant fuzzy ideal by using the notion of BCK-parts. Using the notion of Inf-hesitant fuzzy (p-) ideals, we provide a characterization of a p-semisimple BCI-algebra. Finally, we establish the extension properties for an Inf-hesitant fuzzy p-ideal.

2. Preliminaries

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers.

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a BCI-algebra if it satisfies the following conditions:

(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,
(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,
(III) $(\forall x \in X)\ (x * x = 0)$,
(IV) $(\forall x, y \in X)\ (x * y = 0, y * x = 0 \Rightarrow x = y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\ (0 * x = 0)$,

then X is called a BCK-algebra. A BCK-algebra X is said to be positive implicative if it satisfies:

$$(\forall x, y, z \in X)\,((x * y) * z = (x * z) * (y * z))\,. \tag{2.1}$$

A BCK-algebra X is said to be implicative if it satisfies:

$$(\forall x, y \in X)\,(x = x * (y * x))\,. \tag{2.2}$$

Any BCK/BCI-algebra X satisfies the following conditions:

$$(\forall x \in X)\,(x * 0 = x)\,, \tag{2.3}$$
$$(\forall x, y, z \in X)\,(x \leq y \ \Rightarrow\ x * z \leq y * z,\ z * y \leq z * x)\,, \tag{2.4}$$
$$(\forall x, y, z \in X)\,((x * y) * z = (x * z) * y)\,, \tag{2.5}$$
$$(\forall x, y, z \in X)\,((x * z) * (y * z) \leq x * y) \tag{2.6}$$

where $x \leq y$ if and only if $x * y = 0$.

Any BCI-algebra X satisfies the following conditions:

$$(\forall x, y, z \in X)\,(0 * (0 * ((x * z) * (y * z))) = (0 * y) * (0 * x))\,, \tag{2.7}$$
$$(\forall x, y \in X)\,(0 * (0 * (x * y)) = (0 * y) * (0 * x))\,, \tag{2.8}$$
$$(\forall x \in X)\,(0 * (0 * (0 * x)) = 0 * x)\,. \tag{2.9}$$

A BCI-algebra X is said to be p-semisimple (see [5]) if $0 * (0 * x) = x$ for all $x \in X$.

Every p-semisimple BCI-algebra X satisfies:

$$(\forall x, y, z \in X)\,((x * z) * (y * z) = x * y)\,. \tag{2.10}$$

A nonempty subset S of a BCK/BCI-algebra X is called a subalgebra of X if $x * y \in S$ for all $x, y \in S$. A subset A of a BCK/BCI-algebra X is called an ideal of X if it satisfies:

$$0 \in A, \tag{2.11}$$

$$(\forall x \in X)\,(x * y \in A,\ y \in A \;\Rightarrow\; x \in A)\,. \tag{2.12}$$

A subset A of a BCI-algebra X is called a p-ideal of X (see [19]) if it satisfies (2.11) and

$$(\forall x, y, z \in X)\,((x * z) * (y * z) \in A,\ y \in A \;\Rightarrow\; x \in A)\,. \tag{2.13}$$

Note that every p-ideal is an ideal, but the converse is not true in general (see [19]). Note that an ideal A of a BCI-algebra X is a p-ideal of X if and only if the following assertion is valid:

$$(\forall x, y, z \in X)\,((x * z) * (y * z) \in A \;\Rightarrow\; x * y \in A)\,. \tag{2.14}$$

We refer the reader to the books [5, 9] for further information regarding BCK/BCI-algebras.

3. Inf-hesitant fuzzy subalgebras and ideals

Torra [12] introduced a new extension for fuzzy sets to manage those situations in which several values are possible for the definition of a membership function of a fuzzy set.

Definition 3.1 ([12, 13]). Let X be a reference set. A hesitant fuzzy set on X is defined in terms of a function that when applied to X returns a subset of $[0, 1]$, which can be viewed as the following mathematical representation:

$$H := \{(x, h(x)) \mid x \in X\}$$

where $h : X \to \mathscr{P}([0, 1])$.

In what follows, the power set of $[0, 1]$ is denoted by $\mathscr{P}([0, 1])$ and

$$\mathscr{P}^*([0, 1]) = \mathscr{P}([0, 1]) \setminus \{\emptyset\}.$$

For any element $D \in \mathscr{P}^*([0,1])$, the infimum of D is denoted by $\inf D$. For any hesitant fuzzy set $H := \{(x, h(x)) \mid x \in X\}$ and $D \in \mathscr{P}^*([0,1])$, consider the set

$$\mathrm{Inf}[H; D] := \{x \in X \mid \inf h(x) \geq \inf D\}.$$

Definition 3.2. Let X be a BCK/BCI-algebra. Given an element $D \in \mathscr{P}^*([0,1])$, a hesitant fuzzy set $H := \{(x, h(x)) \mid x \in X\}$ is called an Inf-hesitant fuzzy subalgebra of X related to D (briefly, D-Inf-hesitant fuzzy subalgebra of X) if the set $\mathrm{Inf}[H; D]$ is a subalgebra of X whenever it is non-empty. If $H := \{(x, h(x)) \mid x \in X\}$ is a D-Inf-hesitant fuzzy subalgebra of X for all $D \in \mathscr{P}^*([0,1])$ with $\mathrm{Inf}[H; D] \neq \emptyset$, then we say that $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy subalgebra of X.

Example 3.3.

(1) Let $X = \{0, a, b, c\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	a	0

Let $H := \{(x, h(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by

$$H = \{(0, (0.8, 1]), (a, (0.3, 0.5) \cup \{0.9\}), (b, [0.5, 0.7]), (c, (0.3, 0.5) \cup \{0.7\})\}.$$

Since $\inf h(0) = 0.8$, $\inf h(a) = 0.3 = \inf h(c)$ and $\inf h(b) = 0.5$, it is routine to verify that $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy subalgebra of X.

(2) Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	0
b	b	a	0	0	0
c	c	c	c	0	0
d	d	c	c	a	0

Let $H := \{(x, h(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by

$$H = \{(0, \{0.8, 0.9\}), (a, [0.2, 0.9)), (b, (0.7, 0.8]), \\ (c, \{0.5\} \cup (0.7, 0.9)), (d, [0.1, 0.5])\}.$$

Note that $\inf h(0) = 0.8$, $\inf h(a) = 0.2$, $\inf h(b) = 0.7$, $\inf h(c) = 0.5$ and $\inf h(d) = 0.1$. It is easy to check that $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy subalgebra of X.

(3) Consider a BCI-algebra $X = \{0, 1, a, b, c\}$ with the following Cayley table.

$*$	0	1	a	b	c
0	0	0	c	c	a
1	1	0	c	c	a
a	a	a	0	0	c
b	b	a	1	0	c
c	c	c	a	a	0

Let $H := \{(x, h(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by

$$H = \{(0, [0.8, 0.9]), (1, (0.6, 0.7]), (a, [0.5, 0.6]), (b, [0.5, 0.6]), (c, [0.3, 0.7])\}.$$

Then $H := \{(x, h(x)) \mid x \in X\}$ is a D_1-Inf-hesitant fuzzy subalgebra of X with $D_1 := [0.55, 0.65]$. But it is not a D_2-Inf-hesitant fuzzy subalgebra of X with $D_2 := [0.4, 0.6]$ since $\mathrm{Inf}[H; D_2] = \{0, 1, a, b\}$ is not a subalgebra of X.

(4) Consider a BCK-algebra $X = \{0, a, b, c, d\}$ with the following Cayley table.

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	a
b	b	a	0	0	b
c	c	b	a	0	c
d	d	d	d	d	0

Let $H := \{(x, h(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by

$$H = \{(0, [0.7, 0.8]), (a, (0.6, 0.7]), (b, [0.3, 0.6]), (c, [0.5, 0.7]), (d, [0.2, 0.4])\}.$$

Then $H := \{(x, h(x)) \mid x \in X\}$ is a D_1-Inf-hesitant fuzzy subalgebra of X with $D_1 := [0.2, 0.4]$. If we take $D_2 := (0.4, 0.6]$, then $\mathrm{Inf}[H; D_2] = \{0, a, c\}$ which is not a subalgebra of X. Hence $H := \{(x, h(x)) \mid x \in X\}$ is not a D_2-Inf-hesitant fuzzy subalgebra of X.

THEOREM 3.4. *A hesitant fuzzy set $H := \{(x, h(x)) \mid x \in X\}$ on a BCK/BCI-algebra X is an Inf-hesitant fuzzy subalgebra of X if and only if the following assertion is valid:*

$$(\forall x, y \in X)\left(\inf h(x * y) \geq \min\{\inf h(x), \inf h(y)\}\right). \tag{3.1}$$

PROOF: Assume that $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy subalgebra of X. Assume that there exists $Q \in \mathscr{P}^*([0, 1])$ such that

$$\inf h(x * y) < \inf Q \leq \min\{\inf h(x), \inf h(y)\}.$$

Then $x, y \in \mathrm{Inf}[H; D]$ and $x * y \notin \mathrm{Inf}[H; D]$. This is a contradiction, and so

$$\inf h(x * y) \geq \min\{\inf h(x), \inf h(y)\}$$

for all $x, y \in X$.

Conversely, suppose that (3.1) is valid. Let $D \in \mathscr{P}^*([0, 1])$ and $x, y \in \mathrm{Inf}[H; D]$. Then $\inf h(x) \geq \inf D$ and $\inf h(y) \geq \inf D$. It follows from (3.1) that

$$\inf h(x * y) \geq \min\{\inf h(x), \inf h(y)\} \geq \inf D$$

and that $x * y \in \mathrm{Inf}[H; D]$. Hence the set $\mathrm{Inf}[H; D]$ is a subalgebra of X, and so $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy subalgebra of X. □

LEMMA 3.5. *If $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy subalgebra of a BCK/BCI-algebra X, then*

$$(\forall x \in X)\left(\inf h(0) \geq \inf h(x)\right). \tag{3.2}$$

PROOF: Using (III) and (3.1), we have

$$\inf h(0) = \inf h(x * x) \geq \min\left\{\inf h(x), \inf h(x)\right\} = \inf h(x)$$

for all $x \in X$. □

PROPOSITION 3.6. *Let $H := \{(x, h(x)) \mid x \in X\}$ be an Inf-hesitant fuzzy subalgebra of a BCK-algebra X. For any elements $a_1, a_2, \cdots, a_n \in X$, if there exists $a_k \in \{a_1, a_2, \cdots, a_n\}$ such that $a_1 = a_k$, then*

$$(\forall x \in X)\,(\inf h((\cdots((a_1 * a_2) * a_3) * \cdots) * a_n) \geq \inf h(x))\,.$$

PROOF: Using (2.5), (III) and (IV), we have $(\cdots((a_1{*}a_2){*}a_3){*}\cdots){*}a_n = 0$. Thus the desired result follows from Lemma 3.5. □

DEFINITION 3.7. Let X be a BCK/BCI-algebra. Given an element $D \in \mathscr{P}^*([0,1])$, a hesitant fuzzy set $H := \{(x, h(x)) \mid x \in X\}$ is called an Inf-hesitant fuzzy ideal of X related to D (briefly, D-Inf-hesitant fuzzy ideal of X) if the set $\mathrm{Inf}[H; D]$ is an ideal of X whenever it is non-empty. If $H := \{(x, h(x)) \mid x \in X\}$ is a D-Inf-hesitant fuzzy ideal of X for all $D \in \mathscr{P}^*([0,1])$ with $\mathrm{Inf}[H; D] \neq \emptyset$, then we say that $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy ideal of X.

EXAMPLE 3.8.

(1) The hesitant fuzzy set $H := \{(x, h(x)) \mid x \in X\}$ in Example 3.3(1) is an Inf-hesitant fuzzy ideal of X.

(2) Let $(Y, *, 0)$ be a BCI-algebra and $(\mathbb{Z}, +, 0)$ an additive group of integers. Let $(\mathbb{Z}, -, 0)$ be the adjoint BCI-algebra of $(\mathbb{Z}, +, 0)$ and let $X := Y \times \mathbb{Z}$. Then $(X, \otimes, (0,0))$ is a BCI-algebra where the operation $\otimes$ is given by

$$(\forall (x, m), (y, n) \in X)\,((x, m) \otimes (y, n) = (x * y, m - n))\,.$$

For a subset $A := Y \times \mathbb{N}_0$ of X where $\mathbb{N}_0$ is the set of nonnegative integers, let $H := \{(x, h(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by

$$H = \{(x, (0.5, 1]), (y, [0.4, 0.9]) \mid x \in A,\ y \in X \setminus A\}\,.$$

Then $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy ideal of X.

(3) Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	0	0
c	c	b	a	0	0
d	d	d	d	d	0

Let $H := \{(x, h(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by

$$H = \{(0, [0.8, 1)), (a, [0.4, 0.7]), (b, \{0.3\} \cup (0.4, 0.6]), \\ (c, [0.6, 0.9]), (d, [0.1, 0.5])\}.$$

If $D_1 := [0.5, 0.8)$, then $\mathrm{Inf}[H; D_1] = \{0, c\}$ which is not an ideal of X since $b * c = 0 \in \mathrm{Inf}[H; D_1]$ but $b \notin \mathrm{Inf}[H; D_1]$. Thus $H := \{(x, h(x)) \mid x \in X\}$ is not a D_1-Inf-hesitant fuzzy ideal of X. We can easily verify that $H := \{(x, h(x)) \mid x \in X\}$ is a D_2-Inf-hesitant fuzzy ideal of X with $D_2 = [0.25, 0.5]$.

THEOREM 3.9. *A hesitant fuzzy set $H := \{(x, h(x)) \mid x \in X\}$ on a BCK/BCI-algebra X is an Inf-hesitant fuzzy ideal of X if and only if it satisfies* (3.2) *and*

$$(\forall x, y \in X) \left(\inf h(x) \geq \min\{\inf h(x * y), \inf h(y)\}\right). \tag{3.3}$$

PROOF: Let $H := \{(x, h(x)) \mid x \in X\}$ be an Inf-hesitant fuzzy ideal of X. If (3.2) is not valid, then there exists $D \in \mathscr{P}^*([0, 1])$ and $a \in X$ such that $\inf h(0) < \inf D \leq \inf h(a)$. It follows that $a \in \mathrm{Inf}[H; D]$ and $0 \notin \mathrm{Inf}[H; D]$. This is a contradiction, and so (3.2) is valid. Now assume that there exist $a, b \in X$ such that $\inf h(a) < \min\{\inf h(a * b), \inf h(b)\}$. Then there exists $K \in \mathscr{P}^*([0, 1])$ such that

$$\inf h(a) < \inf K \leq \min\{\inf h(a * b), \inf h(b)\},$$

which implies that $a * b \in \mathrm{Inf}[H; K]$, $b \in \mathrm{Inf}[H; K]$ but $a \notin \mathrm{Inf}[H; K]$. This is a contradiction, and thus (3.3) holds.

Conversely, suppose that $H := \{(x, h(x)) \mid x \in X\}$ satisfies two conditions (3.2) and (3.3). Let $K \in \mathscr{P}^*([0, 1])$ be such that $\mathrm{Inf}[H; K] \neq \emptyset$. Obviously, $0 \in \mathrm{Inf}[H; K]$. Let $x, y \in X$ be such that $x * y \in \mathrm{Inf}[H; K]$ and $y \in \mathrm{Inf}[H; K]$. Then $\inf h(x * y) \geq \inf K$ and $\inf h(y) \geq \inf K$. It follows from (3.3) that

$$\inf h(x) \geq \min\{\inf h(x * y), \inf h(y)\} \geq \inf K$$

and that $x \in \mathrm{Inf}[H; K]$. Hence $\mathrm{Inf}[H; K]$ is an ideal of X for all $K \in \mathscr{P}^*([0, 1])$, and therefore $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy ideal of X. □

THEOREM 3.10. *Let $H := \{(x, h(x)) \mid x \in X\}$ be a hesitant fuzzy set on a BCI-algebra X defined by*

$$H = \{(x, D), (y, E) \mid x \in B,\ y \in X \setminus B,\ \inf D \geq \inf E\}$$

where $D, E \in \mathscr{P}^*([0,1])$ and B is the BCK-part of X. Then $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy ideal of X.

PROOF: Since $0 \in B$, we have $\inf h(0) = \inf D \geq \inf h(x)$ for all $x \in X$. Let $x, y \in X$. If $x \in B$, then it is clear that

$$\inf h(x) \geq \min\{\inf h(x * y), \inf h(y)\}.$$

Assume that $x \in X \setminus B$. Since B is an ideal of X, it follows that $x * y \in X \setminus B$ or $y \in X \setminus B$ and that

$$\inf h(x) = \min\{\inf h(x * y), \inf h(y)\}.$$

Therefore $H := \{(x, h(x)) \mid x \in X\}$ is an Int-hesitant fuzzy ideal of X by Theorem 3.9. □

PROPOSITION 3.11. *Every Inf-hesitant fuzzy ideal $H := \{(x, h(x)) \mid x \in X\}$ of a BCK/BCI-algebra X satisfies:*

$$(\forall x, y \in X)\,(x \leq y \ \Rightarrow\ \inf h(x) \geq \inf h(y))\,. \tag{3.4}$$

PROOF: Let $x, y \in X$ be such that $x \leq y$. Then $x * y = 0$, and so

$$\inf h(x) \geq \min\{\inf h(x * y), \inf h(y)\} = \min\{\inf h(0), \inf h(y)\} = \inf h(y) \tag{3.5}$$

by (3.3) and (3.2). □

THEOREM 3.12. *Let $H := \{(x, h(x)) \mid x \in X\}$ be a hesitant fuzzy set on a BCK/BCI-algebra X which satisfies the condition (3.2). Then $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy ideal of X if and only if the following assertion is valid.*

$$(\forall x, y, z \in X)\,(x * y \leq z \ \Rightarrow\ \inf h(x) \geq \min\{\inf h(y), \inf h(z)\})\,. \tag{3.6}$$

PROOF: Assume that $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy ideal of X and let $x, y, z \in X$ be such that $x * y \leq z$. Then $(x * y) * z = 0$, and thus

$$\begin{aligned}\inf h(x*y) &\geq \min\{\inf h((x*y)*z), \inf h(z)\} \\ &= \min\{\inf h(0), \inf h(z)\} \\ &= \inf h(z).\end{aligned} \tag{3.7}$$

It follows that $\inf h(x) \geq \min\{\inf h(x*y), \inf h(y)\} \geq \min\{\inf h(y), \inf h(z)\}$.

Conversely, suppose that the condition (3.6) is valid. Since $x*(x*y) \leq y$ for all $x, y \in X$, it follows from (3.6) that $\inf h(x) \geq \min\{\inf h(x*y), \inf h(y)\}$ for all $x, y \in X$. Therefore $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy ideal of X. □

PROPOSITION 3.13. *For any Inf-hesitant fuzzy ideal $H := \{(x, h(x)) \mid x \in X\}$ of a BCK/BCI-algebra X, the following assertions are equivalent.*

(1) $\inf h((x*y)*y) \leq \inf h(x*y)$,
(2) $\inf h((x*y)*z) \leq \inf h((x*z)*(y*z))$

for all $x, y, z \in X$.

PROOF: Assume that (1) holds. Note that

$$((x*(y*z))*z)*z = ((x*z)*(y*z))*z \leq (x*y)*z$$

for all $x, y, z \in X$. It follows from Proposition 3.11, (1) and (2.5) that

$$\begin{aligned}\inf h((x*y)*z) &\leq \inf h(((x*(y*z))*z)*z) \\ &\leq \inf h((x*(y*z))*z) \\ &= \inf h((x*z)*(y*z))\end{aligned} \tag{3.8}$$

for all $x, y, z \in X$.

Conversely, suppose that (2) is valid and if we put $z := y$ in (2), then

$$\begin{aligned}\inf h((x*y)*y) &\leq \inf h((x*y)*(y*y)) \\ &= \inf h((x*y)*0) \\ &= \inf h(x*y)\end{aligned} \tag{3.9}$$

for all $x, y \in X$. □

THEOREM 3.14. *In a BCK-algebra X, every Inf-hesitant fuzzy ideal is an Inf-hesitant fuzzy subalgebra.*

PROOF: Let $H := \{(x, h(x)) \mid x \in X\}$ be an Inf-hesitant fuzzy ideal of a BCK-algebra X. Using (3.3), (2.5), (III), (V) and (3.2), we have

$$\begin{aligned}\inf h(x*y) &\geq \min\{\inf h((x*y)*x), \inf h(x)\}\\ &\geq \min\{\inf h((x*x)*y), \inf h(x)\}\\ &= \min\{\inf h(0*y), \inf h(x)\}\\ &= \min\{\inf h(0), \inf h(x)\}\\ &\geq \min\{\inf h(x), \inf h(y)\}\end{aligned}$$

for all $x, y \in X$. Therefore $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy subalgebra of X. □

The converse of Theorem 3.14 is not true in general as seen in the following example.

EXAMPLE 3.15. The Inf-hesitant fuzzy subalgebra $H := \{(x, h(x)) \mid x \in X\}$ in Example 3.3(2) is not an Inf-hesitant fuzzy ideal of X since

$$\inf h(d) = 0.1 < 0.5 = \min\{\inf h(d*b), \inf h(b)\}.$$

In a BCI-algebra X, Theorem 3.14 is not true. In fact, the Inf-hesitant fuzzy ideal $H := \{(x, h(x)) \mid x \in X\}$ in Example 3.8 is not an Inf-hesitant fuzzy subalgebra of X since

$$\begin{aligned}\inf h((0,0) \otimes (0,1)) &= \inf h(0,-1) = 0.4\\ &< 0.5 = \min\{\inf h(0,0), \inf h(0,1)\}.\end{aligned}$$

Let $H := \{(x, h(x)) \mid x \in X\}$ be a hesitant fuzzy set on a BCK-algebra X. For any $a, b \in X$ and $n \in \mathbb{N}$, let

$$\operatorname{Inf}[b; a^n] := \{x \in X \mid \inf h((x*b)*a^n) = \inf h(0)\}$$

where $(x*b)*a^n = ((\cdots((x*b)*a)*a)*\cdots)*a$ in which a appears n-times. Obviously, $a, b, 0 \in \operatorname{Inf}[b; a^n]$.

PROPOSITION 3.16. *Let $H := \{(x, h(x)) \mid x \in X\}$ be a hesitant fuzzy set on a BCK-algebra X in which the condition* (3.2) *is valid and*

$$(\forall x, y \in X)\,(\inf h(x*y) \geq \max\{\inf h(x), \inf h(y)\})\,. \tag{3.10}$$

For any $a, b \in X$ and $n \in \mathbb{N}$, if $x \in \operatorname{Inf}[b; a^n]$ then $x*y \in \operatorname{Inf}[b; a^n]$ for all $y \in X$.

PROOF: Let $x \in \inf h[b; a^n]$. Then $\inf h((x*b)*a^n) = \inf h(0)$, and thus

$$\begin{aligned}\inf h(((x*y)*b)*a^n) &= \inf h(((x*b)*y)*a^n)\\ &= \inf h(((x*b)*a^n)*y)\\ &\geq \max\{\inf h((x*b)*a^n), \inf h(y)\}\\ &= \max\{\inf h(0), \inf h(y)\} = \inf h(0)\end{aligned}$$

for all $y \in X$. Hence $\inf h(((x*y)*b)*a^n) = \inf h(0)$, that is, $x*y \in \inf h[b;a^n]$ for all $y \in X$. $\square$

PROPOSITION 3.17. *Let $H := \{(x, h(x)) \mid x \in X\}$ be a hesitant fuzzy set on a BCK-algebra X. If an element $a \in X$ satisfies:*

$$(\forall x \in X)\,(x \leq a)\,, \tag{3.11}$$

then $\mathrm{Inf}[b;a^n] = X = \mathrm{Inf}[a;b^n]$ for all $b \in X$ and $n \in \mathbb{N}$.

PROOF: Let $b, x \in X$ and $n \in \mathbb{N}$. Then

$$\begin{aligned} \inf h((x*b)*a^n) &= \inf h(((x*b)*a)*a^{n-1}) \\ &= \inf h(((x*a)*b)*a^{n-1}) \\ &= \inf h((0*b)*a^{n-1}) \\ &= \inf h(0) \end{aligned}$$

by (2.5), (3.11) and (V), and so $x \in \mathrm{Inf}[b;a^n]$, which shows that $\mathrm{Inf}[b;a^n] = X$. Similarly $\mathrm{Inf}[a;b^n] = X$. $\square$

COROLLARY 3.18. *If $H := \{(x, h(x)) \mid x \in X\}$ is a hesitant fuzzy set on a bounded BCK-algebra X, then $\mathrm{Inf}[b;u^n] = X = \mathrm{Inf}[u;b^n]$ for all $b \in X$ and $n \in \mathbb{N}$ where u is the unit of X.*

PROPOSITION 3.19. *Let $H := \{(x, h(x)) \mid x \in X\}$ be a hesitant fuzzy subalgebra of a BCK-algebra X satisfying the condition (3.4). Then the following assertion is valid.*

$$(\forall a, b, c \in X)\,(\forall n \in \mathbb{N})\,(b \leq c \;\Rightarrow\; \mathrm{Inf}[b;a^n] \subseteq \mathrm{Inf}[c;a^n])\,. \tag{3.12}$$

PROOF: Let $b, c \in X$ be such that $b \leq c$. For any $a \in X$ and $n \in \mathbb{N}$, if $x \in \mathrm{Inf}[b;a^n]$ then

$$\begin{aligned} \inf h(0) = \inf h((x*b)*a^n) &= \inf h((x*a^n)*b) \\ \leq \inf h((x*a^n)*c) &= \inf h((x*c)*a^n) \end{aligned}$$

by (2.4) and (3.4), and so $\inf h((x*c)*a^n) = \inf h(0)$. Thus $x \in \mathrm{Inf}[c;a^n]$, and therefore $\mathrm{Inf}[b;a^n] \subseteq \mathrm{Inf}[c;a^n]$ for all $a \in X$ and $n \in \mathbb{N}$. $\square$

Corollary 3.20. *Every Inf-hesitant fuzzy ideal* $H := \{(x, h(x)) \mid x \in X\}$ *of a* BCK*-algebra* X *satisfies the condition* (3.12).

The following example shows that there exists a hesitant fuzzy set $H := \{(x, h(x)) \mid x \in X\}$ on a BCK-algebra X such that

(1) $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy ideal of X,
(2) There exist $a, b \in X$ and $n \in \mathbb{N}$ such that the set $\mathrm{Inf}[b; a^n]$ is not an ideal of X.

Example 3.21. Let $X = \{0, a, b, c\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Let $H := \{(x, h(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by

$$H = \{(0, (0.8, 0.9]), (a, [0.6, 0.8]), (b, [0.6, 0.8]), (c, \{0.3\} \cup [0.4, 0.6))\}.$$

Then $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy ideal of of X and

$$\mathrm{Inf}[a; c^n] = \{x \in X \mid \inf h((x * a) * c^n) = \inf h(0)\} = \{0, a, c\}$$

which is not an ideal of X for any $n \in \mathbb{N}$ since $b * a = a \in \mathrm{Inf}[a; c^n]$ but $b \notin \mathrm{Inf}[a; c^n]$.

We now consider conditions for a set $\mathrm{Inf}[b; a^n]$ to be an ideal of X.

Theorem 3.22. *Let* $H := \{(x, h(x)) \mid x \in X\}$ *be a hesitant fuzzy set on a* BCK*-algebra* X *such that*

$$(\forall x, y \in X)\, (\inf h(x) = \inf h(y) \;\Rightarrow\; x = y)\,. \tag{3.13}$$

If X *is positive implicative, then* $\mathrm{Inf}[b; a^n]$ *is an ideal of* X *for all* $a, b \in X$ *and* $n \in \mathbb{N}$.

Proof: Let $a, b, x, y \in X$ and $n \in \mathbb{N}$ be such that $x * y \in \mathrm{Inf}[b; a^n]$ and $y \in \mathrm{Inf}[b; a^n]$. Then $\inf h((y * b) * a^n) = \inf h(0)$, which implies from (3.13) that $(y * b) * a^n = 0$. Hence

$$
\begin{aligned}
\inf h(0) &= \inf h(((x*y)*b)*a^n) \\
&= \inf h((((x*y)*b)*a)*a^{n-1}) \\
&= \inf h((((x*b)*(y*b))*a)*a^{n-1}) \\
&= \inf h(((((x*b)*a)*((y*b)*a))*a)*a^{n-2}) \\
&= \cdots \\
&= \inf h(((x*b)*a^n)*((y*b)*a^n)) \\
&= \inf h(((x*b)*a^n)*0) \\
&= \inf h((x*b)*a^n),
\end{aligned}
$$

which shows that $x \in \mathrm{Inf}[b; a^n]$. Therefore $\mathrm{Inf}[b; a^n]$ is an ideal of X for all $a, b \in X$ and $n \in \mathbb{N}$. □

Since every implicative BCK-algebra is a positive implicative BCK-algebra, we have the following corollary.

COROLLARY 3.23. *Let $H := \{(x, h(x)) \mid x \in X\}$ be a hesitant fuzzy set on a BCK-algebra X satisfying* (3.13)*. If X is implicative, then* $\mathrm{Inf}[b; a^n]$ *is an ideal of X for all $a, b \in X$ and $n \in \mathbb{N}$.*

Theorem 3.22 is illustrated by the following example.

EXAMPLE 3.24. Let $X = \{0, a, b, c\}$ be a set with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	b
c	c	c	c	0

Then X is a positive implicative BCK-algebra. Let $H := \{(x, h(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by

$$H = \{(0, (0.6, 0.9]), (a, [0.7, 0.8)), (b, \{0.4, 0.5, 0.6\}), (c, (0.2, 0.4])\}.$$

Then $\inf h(0) = 0.6$, $\inf h(a) = 0.7$, $\inf h(b) = 0.4$ and $\inf h(c) = 0.2$. Thus $H := \{(x, h(x)) \mid x \in X\}$ satisfies the condition (3.13), but it does not satisfy the condition (3.2). Hence $H := \{(x, h(x)) \mid x \in X\}$ is not an Inf-hesitant fuzzy ideal of X. Note that
$\mathrm{Inf}[0 : 0^n] = \{0\}$, $\mathrm{Inf}[0; a^n] = \{0, a\}$, $\mathrm{Inf}[0; b^n] = \{0, a, b\}$, $\mathrm{Inf}[0; c^n] = \{0, c\}$,

$\mathrm{Inf}[a; 0^n] = \{0, a\}$, $\mathrm{Inf}[a; a^n] = \{0, a\}$, $\mathrm{Inf}[a; b^n] = \{0, a, b\}$, $\mathrm{Inf}[a; c^n] = \{0, a, c\}$,
$\mathrm{Inf}[b; 0^n] = \{0, a, b\}$, $\mathrm{Inf}[b; a^n] = \{0, a, b\}$, $\mathrm{Inf}[b; b^n] = \{0, a, b\}$, $\mathrm{Inf}[b; c^n] = X$,
$\mathrm{Inf}[c; 0^n] = \{0, c\}$, $\mathrm{Inf}[c; a^n] = \{0, a, c\}$, $\mathrm{Inf}[c; b^n] = X$, $\mathrm{Inf}[c; c^n] = \{0, c\}$, and they are ideals of X.

Proposition 3.25. *Let $H := \{(x, h(x)) \mid x \in X\}$ be a hesitant fuzzy set on a BCK-algebra X in which the condition* (3.13) *is valid. If J is an ideal of X, then the following assertion holds.*

$$(\forall a, b \in J)\, (\forall n \in \mathbb{N})\, (\mathrm{Inf}[b; a^n] \subseteq J)\,. \tag{3.14}$$

Proof: For any $a, b \in J$ and $n \in \mathbb{N}$, let $x \in \mathrm{Inf}[b; a^n]$. Then

$$\inf h(((x * b) * a^{n-1}) * a) = \inf h((x * b) * a^n) = \inf h(0)$$

and so $((x * b) * a^{n-1}) * a = 0 \in J$ by (3.13). Since J is an ideal of X, it follows from (2.12) that $(x * b) * a^{n-1} \in J$. Continuing this process, we have $x * b \in J$ and thus $x \in J$. Therefore $\mathrm{Inf}[b; a^n] \subseteq J$ for all $a, b \in J$ and $n \in \mathbb{N}$. □

Theorem 3.26. *Let $H := \{(x, h(x)) \mid x \in X\}$ be a hesitant fuzzy set on a BCK-algebra X. For any subset J of X, if the condition* (3.14) *holds, then J is an ideal of X.*

Proof: Suppose that the condition (3.14) is valid. Not that $0 \in \mathrm{Inf}[b; a^n] \subseteq J$. Let $x, y \in X$ be such that $x * y \in J$ and $y \in J$. Taking $b := x * y$ implies that

$$\begin{aligned} \inf h((x * b) * y^n) &= \inf h((x * (x * y)) * y^n) \\ &= \inf h(((x * (x * y)) * y) * y^{n-1}) \\ &= \inf h(((x * y) * (x * y)) * y^{n-1}) \\ &= \inf h(0 * y^{n-1}) = \inf h(0), \end{aligned}$$

and so $x \in \mathrm{Inf}[b; y^n] \subseteq J$ with $b = x * y$. Therefore J is an ideal of X. □

Theorem 3.27. *If $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy ideal of a BCK/BCI-algebra X, then the set*

$$H_a := \{x \in X \mid \inf h(a) \leq \inf h(x)\}$$

is an ideal of X for all $a \in X$.

PROOF: Let $x, y \in X$ be such that $x * y \in H_a$ and $y \in H_a$. Then $\inf h(a) \leq \inf h(x * y)$ and $\inf h(a) \leq \inf h(y)$. It follows from (3.3) and (3.2) that

$$\inf h(a) \leq \min\{\inf h(x * y), \inf h(y)\} \leq \inf h(x) \leq \inf h(0)$$

and that $0 \in H_a$ and $x \in H_a$. Therefore H_a is an ideal of X for all $a \in X$. □

COROLLARY 3.28. *If $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy ideal of a BCK/BCI-algebra X, then the set*

$$H_0 := \{x \in X \mid \inf h(0) = \inf h(x)\}$$

is an ideal of X for all $a \in X$.

THEOREM 3.29. *Let $a \in X$ and let $H := \{(x, h(x)) \mid x \in X\}$ be a hesitant fuzzy set on a BCK/BCI-algebra X. Then*

(1) *If H_a is an ideal of X, then $H := \{(x, h(x)) \mid x \in X\}$ satisfies:*

$$(\forall x, y \in X)(\inf h(a) \leq \min\{\inf h(x * y), \inf h(y)\} \Rightarrow \inf h(a) \leq \inf h(x)). \tag{3.15}$$

(2) *If $H := \{(x, h(x)) \mid x \in X\}$ satisfies two condition* (3.2) *and* (3.15), *then H_a is an ideal of X.*

PROOF:

(1) Assume that H_a is an ideal of X and let $x, y \in X$ be such that $\inf h(a) \leq \min\{\inf h(x * y), \inf h(y)\}$. Then $x * y \in H_a$ and $y \in H_a$, which imply that $x \in H_a$, that is, $\inf h(a) \leq \inf h(x)$.

(2) Let $H := \{(x, h(x)) \mid x \in X\}$ be a hesitant fuzzy set on X in which two conditions (3.2) and (3.15) are valid. Then $0 \in H_a$. Let $x, y \in X$ be such that $x * y \in H_a$ and $y \in H_a$. Then $\inf h(a) \leq \inf h(x * y)$ and $\inf h(a) \leq \inf h(y)$, and so $\inf h(a) \leq \min\{\inf h(x * y), \inf h(y)\}$. It follows from (3.15) that $\inf h(a) \leq \inf h(x)$, that is, $x \in H_a$. Therefore H_a is an ideal of X. □

4. Inf-hesitant fuzzy p-ideals

In what follows, we take a BCI-algebra X as a reference set unless otherwise specified.

DEFINITION 4.1. Given an element $D \in \mathscr{P}^*([0,1])$, a hesitant fuzzy set $H := \{(x, h(x)) \mid x \in X\}$ on X is called an Inf-hesitant fuzzy p-ideal of X related to D (briefly, D-Inf-hesitant fuzzy p-ideal of X) if the set $\mathrm{Inf}[H; D]$ is a p-ideal of X whenever it is non-empty. If $H := \{(x, h(x)) \mid x \in X\}$ is a D-Inf-hesitant fuzzy p-ideal of X for all $D \in \mathscr{P}^*([0,1])$ with $\mathrm{Inf}[H; D] \neq \emptyset$, then we say that $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy p-ideal of X.

EXAMPLE 4.2.

(1) Let $X = \{0, a, b, c\}$ be a BCI-algebra with the following Cayley table.

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Let $H := \{(x, h(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by

$$H_X = \{(0, (0.7, 0.9]), (a, (\{0.5\} \cup (0.6, 0.7)), (b, [0.3, 0.6]), (c, [0.3, 0.6])\}.$$

It is easy to verify that $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy p-ideal of X.

(2) Let $X = \{0, 1, a, b\}$ be a BCI-algebra with the following Cayley table.

$*$	0	1	a	b
0	0	0	a	a
1	1	0	b	a
a	a	a	0	0
b	b	a	1	0

Let $H := \{(x, h(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by

$$H_X = \{(0, [0.7, 0.9]), (1, ([0.3, 0.6)), (a, [0.5, 0.8]), (b, [0.3, 0.6))\}.$$

Then $H := \{(x, h(x)) \mid x \in X\}$ is a D_1-Inf-hesitant fuzzy p-ideal of X with $D_1 = [0.3, 0.6)$. But if $D_2 = (0.4, 0.7]$, then $\mathrm{Int}[H; D_2] = \{0, a\}$ is not a

p-ideal of X since $(1*b)*(a*b) = a \in \mathrm{Int}[H; D_2]$ and $b \notin \mathrm{Int}[H; D_2]$. Hence $H := \{(x, h(x)) \mid x \in X\}$ is not a D_2-Inf-hesitant fuzzy p-ideal of X.

THEOREM 4.3. *A hesitant fuzzy set $H := \{(x, h(x)) \mid x \in X\}$ on X is an Inf-hesitant fuzzy p-ideal of X if and only if if it satisfies* (3.2)

$$(\forall x, y, z \in X)\,(\min\{\inf h((x*z)*(y*z)), \inf h(y)\} \leq \inf h(x))\,. \quad (4.1)$$

PROOF: Let $H := \{(x, h(x)) \mid x \in X\}$ be an Inf-hesitant fuzzy p-ideal of X. If (3.2) is not valid, then there exists $D \in \mathscr{P}^*([0,1])$ and $a \in X$ such that $\inf h(0) < \inf D \leq \inf h(a)$. It follows that $a \in \mathrm{Inf}[H; D]$ and $0 \notin \mathrm{Inf}[H; D]$. This is a contradiction, and so (3.2) is valid. Now assume that (4.1) is not valid. Then

$$\min\{\inf h((a*c)*(b*c)), \inf h(b)\} > \inf h(a)$$

for some $a, b, c \in X$. Thus there exists $B \in \mathscr{P}^*([0,1])$ such that

$$\min\{\inf h((a*c)*(b*c)), \inf h(b)\} \geq \inf B > \inf h(a).$$

which implies that $(a*c)*(b*c) \in \mathrm{Inf}[H; B]$, $b \in \mathrm{Inf}[H; B]$ but $a \notin \mathrm{Inf}[H; B]$. This is a contradiction, and thus (4.1) holds.

Conversely, suppose that $H := \{(x, h(x)) \mid x \in X\}$ satisfies two conditions (3.2) and (4.1). Let $D \in \mathscr{P}^*([0,1])$ be such that $\mathrm{Inf}[H; D] \neq \emptyset$. Obviously, $0 \in \mathrm{Inf}[H; D]$. Let $x, y, z \in X$ be such that $(x*z)*(y*z) \in \mathrm{Inf}[H; D]$ and $y \in \mathrm{Inf}[H; D]$. Then $\inf h((x*z)*(y*z)) \geq \inf D$ and $\inf h(y) \geq \inf D$. It follows from (4.1) that

$$\inf h(x) \geq \min\{\inf h((x*z)*(y*z)), \inf h(y)\} \geq \inf D$$

and that $x \in \mathrm{Inf}[H; D]$. Hence $\mathrm{Inf}[H; D]$ is a p-ideal of X for all $D \in \mathscr{P}^*([0,1])$ with $\mathrm{Inf}[H; D] \neq \emptyset$, and therefore $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy p-ideal of X. □

THEOREM 4.4. *The hesitant fuzzy set $H := \{(x, h(x)) \mid x \in X\}$ on X which is described in Theorem 3.10 is an Inf-hesitant fuzzy p-ideal of X.*

PROOF: In the proof of Theorem 3.10, we know that the condition (3.2) is valid. Let $x, y, z \in X$. If $(x*z)*(y*z) \in X \setminus B$ or $y \in X \setminus B$, then we have

$$\min\{\inf h((x*z)*(y*z)), \inf h(y)\} \leq \inf h(x).$$

Assume that $(x*z)*(y*z) \in B$ and $y \in B$. Since $(x*z)*(y*z) \leq x*y$ and B is the BCK-part of X, it follows from (2.4) and (III) that

$(x * y) * ((x * z) * (y * z)) \in B$ and from (2.12) that $x \in B$ since B is an ideal of X. Hence

$$\min\{\inf h((x * z) * (y * z)), \inf h(y)\} = \inf h(x).$$

Therefore $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy p-ideal of X by Theorem 4.3. □

Proposition 4.5. *Every Inf-hesitant fuzzy p-ideal $H := \{(x, h(x)) \mid x \in X\}$ of X satisfies:*

$$(\forall x \in X)\,(\inf h(0 * (0 * x)) \leq \inf h(x))\,. \tag{4.2}$$

Proof: If we put $z := x$ and $y := 0$ in (4.1), then

$$\begin{aligned}\inf h(x) &\geq \min\{\inf h((x * x) * (0 * x)), \inf h(0)\}\\ &= \min\{\inf h(0 * (0 * x)), \inf h(0)\}\\ &= \inf h(0 * (0 * x))\end{aligned}$$

for all $x \in A$ by (III) and (3.2). □

Theorem 4.6. *Every Inf-hesitant fuzzy p-ideal of X is an Inf-hesitant fuzzy ideal of X.*

Proof: Let $H := \{(x, h(x)) \mid x \in X\}$ be an Inf-hesitant fuzzy p-ideal of X. Since $x * 0 = x$ for all $x \in X$, it follows from (4.1) that

$$\begin{aligned}\inf h(x) &\geq \min\{\inf h((x * 0) * (y * 0)), \inf h(y)\}\\ &= \min\{\inf h(x * y), \inf h(y)\}\end{aligned}$$

for all $x, y \in X$. Therefore $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy ideal of X. □

The converse of Theorem 4.6 is not true in general as seen in the following example.

Example 4.7. Consider a BCI-algebra $X = \{0, 1, a, b, c\}$ with the following Cayley table.

$*$	0	1	a	b	c
0	0	0	c	b	a
1	1	0	c	b	a
a	a	a	0	c	b
b	b	b	a	0	c
c	c	c	b	a	0

Let $H := \{(x, h(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by

$$\begin{aligned} H = \{&(0, [0.6, 0.7] \cup \{0.8, 0.9\}), (1, (\{0.5, 0.6, 0.7, 0.8\}), \\ &(a, [0.4, 0.9]), (b, [0.4, 0.9]), (c, [0.4, 0.9])\}. \end{aligned}$$

Then $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy ideal of X, but it is not an Inf-hesitant fuzzy p-ideal of X since

$$\inf h(1) = 0.5 < 0.6 = \min\{\inf h((1 * a) * (0 * a)), \inf h(0)\}.$$

Proposition 4.8. *Every Inf-hesitant fuzzy p-ideal $H := \{(x, h(x)) \mid x \in X\}$ of X satisfies:*

$$(\forall x, y, z \in X)\,(\inf h(x * y) \le \inf h((x * z) * (y * z)))\,. \tag{4.3}$$

Proof: Let $H := \{(x, h(x)) \mid x \in X\}$ be an Inf-hesitant fuzzy p-ideal of X. Then it is an Inf-hesitant fuzzy ideal of X by Theorem 4.6. Hence

$$\begin{aligned} \inf h((x * z) * (y * z)) &\ge \min\{\inf h(((x * z) * (y * z)) * (x * y)), \inf h(x * y)\} \\ &= \min\{\inf h(0), \inf h(x * y)\} = \inf h(x * y) \end{aligned}$$

for all $x, y, z \in X$. □

We provide conditions for an Inf-hesitant fuzzy ideal to be an Inf-hesitant fuzzy p-ideal.

Theorem 4.9. *Let $H := \{(x, h(x)) \mid x \in X\}$ be an Inf-hesitant fuzzy ideal of X such that*

$$(\forall x, y, z \in X)\,(\inf h(x * y) \ge \inf h((x * z) * (y * z)))\,. \tag{4.4}$$

Then $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy p-ideal of X.

Proof: If the condition (4.4) is valid, then

$$\begin{aligned}\inf h(x) &\geq \min\{\inf h(x*y), \inf h(y)\}\\ &\geq \min\{\inf h((x*z)*(y*z)), \inf h(y)\}\end{aligned}$$

for all $x, y, z \in X$. Therefore $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy p-ideal of X. □

Lemma 4.10. *Every Inf-hesitant fuzzy ideal $H := \{(x, h(x)) \mid x \in X\}$ of X satisfies the following condition:*

$$(\forall x \in X)\,(\inf h(x) \leq \inf h(0*(0*x)))\,.$$

Proof: For every $x \in X$, we have

$$\begin{aligned}\inf h(x) &= \min\{\inf h(0), \inf h(x)\}\\ &= \min\{\inf h((0*(0*x))*x), \inf h(x)\}\\ &\leq \inf h(0*(0*x))\end{aligned}$$

which is the desired result. □

Theorem 4.11. *If an Inf-hesitant fuzzy ideal $H := \{(x, h(x)) \mid x \in X\}$ of X satisfies the condition* (4.2)*, then it is an Inf-hesitant fuzzy p-ideal of X.*

Proof: Let $x, y, z \in A$. Using Lemma 4.10, (2.7), (2.8) and (4.2), we have

$$\begin{aligned}\inf h((x*z)*(y*z)) &\leq \inf h(0*(0*((x*z)*(y*z))))\\ &= \inf h((0*y)*(0*x))\\ &= \inf h(0*(0*(x*y)))\\ &\leq \inf h(x*y).\end{aligned}$$

It follows from Theorem 4.9 that $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy p-ideal of X. □

Theorem 4.12. *If $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy p-ideal of X, then the set*

$$I := \{x \in X \mid \inf h(x) = \inf h(0)\}$$

is a p-ideal of X.

Proof: Obviously $0 \in I$. Let $x, y, z \in X$ be such that $(x*z)*(y*z) \in I$ and $y \in I$. Then

$$\inf h(x) \geq \min\{\inf h((x*z)*(y*z)), \inf h(y)\} = \inf h(0),$$

and so $\inf h(x) = \inf h(0)$, that is, $x \in I$. Therefore I is a p-ideal of X. □

For any subset I of X, let $H_X^I = \{(x, \inf h^I(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by

$$\inf h^I(x) = \begin{cases} \{1\} & \text{if } x \in I, \\ [0,1] & \text{otherwise.} \end{cases}$$

LEMMA 4.13. *For any subset I of X, the following are equivalent:*

(1) *I is an ideal (resp. p-ideal) of X.*
(2) *The hesitant fuzzy set $H_X^I = \{(x, \inf h^I(x)) \mid x \in X\}$ on X is an Inf-hesitant fuzzy ideal (resp. Inf-hesitant fuzzy p-ideal) of X.*

PROOF: The proof is straightforward. □

THEOREM 4.14. *A BCI-algebra X is p-semisimple if and only if every Inf-hesitant fuzzy ideal of X is an Inf-hesitant fuzzy p-ideal of X.*

PROOF: Assume that X is a p-semisimple BCI-algebra and let $H := \{(x, h(x)) \mid x \in X\}$ be an Inf-hesitant fuzzy ideal of X. Then

$$\inf h(x) \geq \min\{\inf h(x * y), \inf h(y)\} = \min\{\inf h((x * z) * (y * z)), \inf h(y)\}$$

by using (3.3) and (2.10). Hence $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy p-ideal of X.

Conversely, suppose that every Inf-hesitant fuzzy ideal of X is an Inf-hesitant fuzzy p-ideal of X. Since the hesitant fuzzy set $H_X^{\{0\}} = \{(x, \inf h^{\{0\}}(x)) \mid x \in X\}$ on X is an Inf-hesitant fuzzy ideal of X, it is also an Inf-hesitant fuzzy p-ideal of X. It follows from Lemma 4.13 that $\{0\}$ is a p-ideal of X. For any $x \in X$, we have

$$\begin{aligned} ((x * (0 * (0 * x))) * x) * (0 * x) &= ((x * x) * (0 * (0 * x))) * (0 * x) \\ &= (0 * (0 * (0 * x))) * (0 * x) \\ &= (0 * (0 * x)) * (0 * (0 * x)) = 0 \in \{0\} \end{aligned}$$

by using (2.5) and (III), which implies from (2.13) that $x*(0*(0*x)) \in \{0\}$. Hence $x * (0 * (0 * x)) = 0$, that is, $x \leq 0 * (0 * x)$. Since $0 * (0 * x) \leq x$, we get $0 * (0 * x) = x$. Therefore X is a p-semisimple BCI-algebra. □

THEOREM 4.15. (Extension property for Inf-hesitant fuzzy p-ideals) *Let*

$$H := \{(x, h(x)) \mid x \in X\} \text{ and } G := \{(x, g(x)) \mid x \in X\}$$

be Inf-hesitant fuzzy ideals of X such that $\inf h(0) = \inf g(0)$ *and* $\inf h(x) \subseteq \inf g(x)$ *for all $x \in X$. If $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy p-ideal of X, then so is $G := \{(x, g(x)) \mid x \in X\}$.*

Proof: Assume that $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy p-ideal of X. Using (2.8), (2.9) and (III), we have $0*(0*(x*(0*(0*x)))) = 0$ for all $x \in X$. It follows from hypothesis and (4.2) that

$$\begin{aligned}\inf g(x*(0*(0*x))) &\geq \inf h(x*(0*(0*x)))\\ &\geq \inf h(0*(0*(x*(0*(0*x)))))\\ &= \inf h(0) = \inf g(0).\end{aligned}$$

Hence

$$\begin{aligned}\inf g(x) &\geq \min\{\inf g(x*(0*(0*x))), \inf g(0*(0*x))\}\\ &\geq \min\{\inf g(0), \inf g(0*(0*x))\}\\ &= \inf g(0*(0*x)),\end{aligned}$$

and thus $G := \{(x, g(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy p-ideal of X by Theorem 4.11. □

5. Conclusions

Since hesitant fuzzy set theory was introduced by Torra in 2010, this concept has been applied to many areas including algebraic structures. The aim of this paper is introduce the notion of Inf-hesitant fuzzy set, and applied it to BCK/BCI-algebras. We have introduced the notions of Inf-hesitant fuzzy subalgebras, Inf-hesitant fuzzy ideals and Inf-hesitant fuzzy p-ideals in BCK/BCI-algebras, and have investigated their relations and properties. We have discussed caracterizations of an Inf-hesitant fuzzy subalgebras, an Inf-hesitant fuzzy ideals and an Inf-hesitant fuzzy p-ideal, and have constructed an Inf-hesitant fuzzy ideal by using the notion of BCK-parts. We have provided conditions for an Inf-hesitant fuzzy ideal to be an Inf-hesitant fuzzy p-ideal, and have provided a characterization of a p-semisimple BCI-algebra. We have considered characterizations of Inf-hesitant fuzzy p-ideals. We finally have established extension property for an Inf-hesitant fuzzy p-ideal. Future research will focus on applying the notions/contents to other types of ideals in BCK/BCI-algebras and related algebraic structures.

Acknowledgements. The authors wish to thank the anonymous reviewer(s) for their valuable suggestions.

References

[1] A. Al-Masarwah and A. G. Ahmad, *m-polar fuzzy ideals of BCK/BCI-algebras*, **Journal of King Saud University-Science**, Vol. 31 (2019), pp. 1220–1226. http://dx.doi.org/10.1016/j.jksus.2018.10.002

[2] A. Al-Masarwah and A. G. Ahmad, *m-Polar* (α, β)*-fuzzy ideals in BCK/BCI-algebras*, **Symmetry**, Vol. 11 (2019), p. 44. http://dx.doi.org/10.3390/sym11010044

[3] A. Al-Masarwah and A. G. Ahmad, *Novel concepts of doubt bipolar fuzzy H-ideals of BCK/BCI-algebras*, **International Journal of Innovative Computing, Information and Control**, Vol. 14, No. 6 (2018), pp. 2025–2041.

[4] A. Al-Masarwah and A. G. Ahmad, *On some properties of doubt bipolar fuzzy H-ideals in BCK/BCI-algebras*, **European Journal of Pure and Applied Mathematics**, Vol. 11, No. 3 (2018), pp. 652–670.

[5] Y. Huang, ***BCI*-algebra**, Science Press, Beijing, 2006.

[6] Y. B. Jun and S. S. Ahn, *Hesitant fuzzy set theory applied to BCK/BCI-algerbas*, **Journal of Computational Analysis and Applications**, Vol. 20, No. 4 (2016), pp. 635–646.

[7] Y. B. Jun, G. Muhiuddin, M. A. Ozturk and E. H. Roh, *Cubic soft ideals in BCK/BCI-algebras*, **Journal of Computational Analysis and Applications**, Vol. 22, No. 5 (2017), pp. 929–940.

[8] Y. B. Jun and S. Z. Song, *Hesitant fuzzy set theory applied to filters in MTL-algebras*, **Honam Mathematical Journal**, Vol. 36, No. 4 (2014), pp. 813–830.

[9] J. Meng and Y. B. Jun, ***BCK*-algebras**, Kyungmoon Sa Co. Seoul, 1994.

[10] R. M. Rodriguez, L. Martinez and F. Herrera, *Hesitant fuzzy linguistic term sets for decision making*, **IEEE Transactions on Fuzzy Systems**, Vol. 20, No. 1 (2012), pp. 109–119.

[11] T. Senapati, Y. B. Jun, G. Muhiuddin and K. P. Shum, *Cubic intuitionistic structures applied to ideals of BCI-algebras*, **Analele Universitatii Ovidius Constanta. Seria Matematic**, Vol. 27, No. 2 (2019), pp. 213–232.

[12] V. Torra, *Hesitant fuzzy sets,* **International Journal of Intelligent Systems**, Vol. 25 (2010), pp. 529–539.

[13] V. Torra and Y. Narukawa, *On hesitant fuzzy sets and decision*, [in:] **The 18th IEEE International Conference on Fuzzy Systems**, Jeju Island, Korea, 2009, pp. 1378–1382.

[14] F. Q. Wang, X. Li and X. H. Chen, *Hesitant fuzzy soft set and its applications in multicriteria decision making*, **Journal of Applied Mathematics**, Vol. 2014, Article ID 643785, 10 pages.

[15] G. Wei, *Hesitant fuzzy prioritized operators and their application to multiple attribute decision making*, **Knowledge-Based Systems**, Vol. 31 (2012), pp. 176–182.

[16] M. Xia and Z. S. Xu, *Hesitant fuzzy information aggregation in decision making*, **International Journal of Approximate Reasoning**, Vol. 52, No. 3 (2011), pp. 395–407.

[17] Z. S. Xu and M. Xia, *Distance and similarity measures for hesitant fuzzy sets*, **Information Sciences**, Vol. 181, No. 11 (2011), pp. 2128–2138.

[18] Z. S. Xu and M. Xia, *On distance and correlation measures of hesitant fuzzy information*, **International Journal of Intelligent Systems**, Vol. 26, No. 5 (2011), pp. 410–425.

[19] X. H. Zhang, H. Jiang and S. A. Bhatti, *On p-ideals of a BCI-algebra*, **Punjab University Journal of Mathematics (Lahore)**, Vol. 27 (1994), pp. 121–128.

Gyeongsang National University
Department of Mathematics Education
Jinju 52828, Korea
e-mail: `skywine@gmail.com`

Jeju National University
Department of Mathematics
Jeju 63243, Korea
e-mail: `szsong@jejunu.ac.kr`

Bulletin of the **S**ection of **L**ogic
Volume 49/1 (2020), pp. 79–96

http://dx.doi.org/10.18778/0138-0680.2020.04

Arkadiusz Wójcik

THE DYNAMIC EPISTEMIC LOGIC FOR ACTUAL KNOWLEDGE

Abstract

The dynamic epistemic logic for actual knowledge models the phenomenon of actual knowledge change when new information is received. In contrast to the systems of dynamic epistemic logic which have been discussed in the past literature, our system is not burdened with the problem of logical omniscience, that is, an idealized assumption that the agent explicitly knows all classical tautologies and all logical consequences of his or her knowledge. We provide a sound and complete axiomatization for this logic.

Keywords: Dynamic epistemic logic, logic of public announcements, knowledge representation, problem of logical omniscience, actual knowledge, epistemic change, multi-agent systems.

1. Introduction

During the mid-twentieth century, the attention of many logicians and philosophers focused on epistemic modalities, and the first systems of epistemic logic were developed, such as those of Jerzy Łoś [10] and Arthur Pap [11]. An interest in these systems was then heightened when Jaakko Hintikka [5] applied the concept of possible worlds semantics to epistemic operators. Originally, possible worlds semantics was formulated by Saul Kripke [6, 7] for the logic of necessity and possibility. The semantics developed by Kripke has since been adapted to epistemic logic, prompting the development of modal epistemic logics. Modal epistemic logics have become not only helpful tools for the formalization of certain intuitions connected with the concepts of knowledge, belief and information; but have also become a subject of interest to scientists in fields such as game theory, computer science, cognitive science, decision theory, artificial intelligence (AI) and cryptology.

The increase of interest in epistemic logic among representatives of other scientific disciplines has lead to new goals for the application of the logic of knowledge and beliefs. Epistemic logics started to be perceived as formal systems whose aim is to capture the phenomenon of epistemic change as a result of the flow of information between various agents. But standard modal epistemic logics have a clearly static character: they model the agent's information state at a given time, but do not enable us to express how this state can change when new information is received. Thus, the need to develop formal systems that allow for capturing the phenomenon of epistemic change led to a dynamic turnover in epistemic logic.

Dynamic epistemic logics are not free of the shortcomings typical of standard epistemic logics. Both make use of possible worlds semantics, and inherit its drawbacks. This concerns logical omniscience, i.e. the controversial assumption underlying epistemic logics built on the basis of possible worlds semantics, according to which the agent knows all classical (propositional) tautologies, and all logical consequences of his or her knowledge.

So far, at least a dozen different proposals have been made to solve the problem of logical omniscience[1]. Many authors have recognized that this problem illustrates the fact that epistemic logics do not model the actual knowledge of agents (explicit knowledge), but only potential knowledge (implicit knowledge), for which the discussed assumptions are not problematic (Levesque [8], Fagin, Halpern [2], van Benthem [1]). If standard epistemic logic models potential knowledge, it is still an open question how to develop a system that enables us to model actual knowledge. Ronald Fagin and Joseph Y. Halpern [2] have constructed such a logic for actual knowledge of non-omniscient agents. The main aim of this article is to develop a dynamic epistemic logic that is built on the ideas formulated by Fagin and Halpern. We present the axiomatization for such a system, we propose a semantics for it and finally we prove the soundness and completeness theorem.

[1] The most important of these proposals are discussed in detail by Fagin et al. [3] and Sim [15].

2. Modal epistemic logics

The formal language of modal epistemic logics $\mathcal{L}_{MEL}$, is the common language for a wide class of logics.

Definition 1. Let Var denote the set of sentential variables and let Ag denote the set of agents. The language of modal epistemic logics $\mathcal{L}_{MEL}$ is defined inductively as follows:

$$\varphi \ ::= \ p \mid \neg\varphi \mid \varphi \to \varphi \mid K_i\varphi,$$

where $p \in Var$ and $i \in Ag$. The set of all $\mathcal{L}_{MEL}$ formulas is denoted by $\Gamma_{\mathcal{L}_{MEL}}$.

$\mathcal{L}_{MEL}$ is an extension of the language of propositional logic with epistemic operators K_i for every agent $i \in Ag$. The intended interpretation of $K_i\varphi$ is "agent i knows that φ". In the case where we are only dealing with one agent, we can omit the index. Other classical logical constants can be defined in the standard way.

The semantics of modal epistemic logics are constructed on the basis of the semantics that Kripke [6, 7] proposed for the logic of possibility and necessity.

Definition 2. An epistemic model is a structure $\mathcal{M} = (W, \{R_i : i \in Ag\}, v)$, where

- $W \neq \emptyset$ is a set of epistemic states,
- $Ag \neq \emptyset$ is a set of agents,
- $R_i \subseteq W \times W$ is an epistemic accessibility relation for any $i \in Ag$,
- $v : Var \mapsto \mathcal{P}(W)$ is a valuation function which to every $p \in Var$ assigns the set of epistemic states in which p is true.

Definition 3. Let $\mathcal{M} = (W, \{R_i : i \in Ag\}, v)$ be an epistemic model. The satisfiaction relation $\models$ is defined inductively in the following way:

$$\begin{array}{lll}
\mathcal{M}, s \models p & \text{iff} & s \in v(p), \\
\mathcal{M}, s \models \neg\varphi & \text{iff} & \mathcal{M}, s \not\models \varphi, \\
\mathcal{M}, s \models \varphi \to \psi & \text{iff} & \text{if } \mathcal{M}, s \models \varphi, \text{ then } \mathcal{M}, s \models \psi, \\
\mathcal{M}, s \models K_i\varphi & \text{iff} & \text{for all } t \in W\text{: if } (s,t) \in R_i, \text{ then } \mathcal{M}, t \models \varphi,
\end{array}$$

where $p \in Var$, $\varphi, \psi \in \Gamma_{\mathcal{L}_{MEL}}$, and $i \in Ag$.

Definition 4. A formula $\varphi \in \Gamma_{\mathcal{L}_{MEL}}$ is true in an epistemic model $\mathcal{M} = (W, \{R_i : i \in Ag\}, v)$ whenever for any $s \in W$, $\mathcal{M}, s \models \varphi$. We denote this by $\mathcal{M} \models \varphi$. A formula $\varphi \in \Gamma_{\mathcal{L}_{MEL}}$ is valid whenever for any epistemic model $\mathcal{M}$, $\mathcal{M} \models \varphi$. We denote this by $\models \varphi$.

Let us start from the minimal modal epistemic logic which is denoted as **K**.

DEFINITION 5. A proof system for the logic **K** is given by the following axiom schemes and inference rules:

all instantiations of propositional tautologies PC

$K_i(\varphi \to \psi) \to (K_i\varphi \to K_i\psi)$ K

The inference rules:

from $\varphi \to \psi$ and φ infer ψ modus ponens

from $\vdash \varphi$ infer $\vdash K_i\varphi$ Gödel's rule for K_i

A formula is a **K**-theorem if it belongs to the least set of formulas that contain all the axioms, and is closed under the inference rules. If φ is a **K**-theorem, we write $\vdash_{\mathrm{K}} \varphi$.

The axiom K states that the knowledge operator is closed under implication. This axiom is accepted in every system of modal epistemic logic. A list of other familiar epistemic axioms is provided in Table 1.

Table 1. Axioms for knowledge

Name	**Axiom**
D	$K_i\varphi \to \neg K_i \neg\varphi$
T	$K_i\varphi \to \varphi$
4	$K_i\varphi \to K_iK_i\varphi$
5	$\neg K_i\varphi \to K_i\neg K_i\varphi$

Proof systems for the epistemic logics stronger than **K** are obtained by adding other axioms to the system. Table 2 lists the most important of these logics. It should be emphasized that the **S5** logic is generally considered to be the standard epistemic logic. We get a proof system for **S5** by expanding the proof system for **K** with the axioms T, 4, and 5.

Table 2. Some basic epistemic systems

Name	Axioms	Properties of R_i
KD	$\mathbf{K} \cup \{\mathsf{D}\}$	serial
T	$\mathbf{K} \cup \{\mathsf{T}\}$	reflexive
S4	$\mathbf{T} \cup \{\mathsf{4}\}$	reflexive, transitive
S5	$\mathbf{S4} \cup \{\mathsf{5}\}$	reflexive, symmetric, transitive
KD4	$\mathbf{KD} \cup \{\mathsf{4}\}$	serial, transitive
KD45	$\mathbf{KD4} \cup \{\mathsf{5}\}$	serial, transitive, euclidean

The standard modal epistemic logics are sound and complete with respect to classes of models whose accessibility relations have the properties expressed in the first-order language, and specified in Table 2.

THEOREM 1. *Logic* **S5** *is sound and complete with respect to the class of all equivalence epistemic models, i.e. any formula* $\varphi \in \Gamma_{\mathcal{L}_{MEL}}$ *is an* **S5**-*theorem iff* φ *is true in all epistemic models, where the accessibility relations are equivalence relations.*

The corresponding theorems for other epistemic logics can be formulated in the same way based on the content of Table 2. The logic **K** is sound and complete with respect to the class of epistemic models with arbitrary accessibility relations.

Epistemic logics based on possible worlds semantics suffer from the problem of logical omniscience. The problem of logical omniscience is connected to two rules of inference. This is the Gödel rule according to which the agent knows all theorems of a given epistemic logic, including all classical propositional tautologies, and the monotonicity rule which may be formulated in the following way:

if $\vdash \varphi \to \psi$, then $\vdash K_i\varphi \to K_i\psi$.

This rule is a consequence of the application of the Gödel rule and the rule of modus ponens to the axiom K. It implies that the agent knows all logical consequences of his or her knowledge. In some applications of epistemic logics, e.g. in epistemology and game theory, representing the knowledge of agents with unlimited deductive abilities may be accepted as a justified idealization. But in the case of representing knowledge of real cognitive agents, these unrealistic assumptions are undesirable.

3. The Fagin and Halpern logic

In order to solve the problem of logical omniscience, Fagin and Halpern [2] propose to add a new epistemic operator to the standard, modal epistemic logic: an awareness operator. The set of formulas of such an extended language is characterized in accordance with the following definition.

Definition 6. Let Var be the set of sentential variables, and let Ag be the set of agents. The language of the modal epistemic logics with awareness operator $\mathcal{L}_{MEL\text{-}A}$ is defined inductively as follows:

$$\varphi \ ::= \ p \mid \neg\varphi \mid \varphi \to \varphi \mid K_i\varphi \mid A_i\varphi,$$

where $p \in Var$ and $i \in Ag$. The set of all $\mathcal{L}_{MEL\text{-}A}$ formulas is denoted by $\Gamma_{\mathcal{L}_{MEL\text{-}A}}$.

The intended interpretation of $A_i\varphi$ is "the agent i is aware that φ" or "the agent i is informed that φ"[2]. The A_i operator can be applied to a formula independently of the K_i operator.

Fagin and Halpern [2] note that if an agent knows something, then the agent cannot be completely unaware of it. For this reason, the authors introduce a distinction between potential and actual knowledge. The potential knowledge is modeled by the operator K_i for any $i \in Ag$, while we are concerned with the actual knowledge that φ, when φ is a subject of potential knowledge and the agent is aware that φ. Thus, we have a new epistemic operator – the actual knowledge operator defined in the language $\mathcal{L}_{MEL\text{-}A}$ for any formula φ and any agent $i \in Ag$ as follows:

$$E_i\varphi \overset{\text{def}}{=} A_i\varphi \wedge K_i\varphi.$$

The semantics for $\mathcal{L}_{MEL\text{-}A}$ is a modified version of the semantics for $\mathcal{L}_{MEL}$.

Definition 7. A model of $\mathcal{L}_{MEL\text{-}A}$ is the epistemic model of Definition 2 with the awareness function: $\mathcal{M} = (W, \{R_i : i \in Ag\}, \{\mathcal{A}_i : i \in Ag\}, v)$, where $\mathcal{A}_i : W \mapsto \mathcal{P}(\Gamma_{\mathcal{L}_{MEL\text{-}A}})$ is the awareness function for any $i \in Ag$.

[2] The philosophical aspects of the awareness approach are discussed in detail by Sillari [14].

DEFINITION 8. Let $\mathcal{M} = (W, \{R_i : i \in Ag\}, \{\mathcal{A}_i : i \in Ag\}, v)$ be an epistemic model with the awareness function and let $i \in Ag$. Then Definition 3 is extended with the following condition:

$\mathcal{M}, s \models A_i\varphi$ iff $\varphi \in \mathcal{A}_i(s)$.

Referring to this definition and the way the E_i operator was defined for any $i \in Ag$, we get the following condition for the actual knowledge operator:

$\mathcal{M}, s \models E_i\varphi$ iff $\varphi \in \mathcal{A}_i(s)$ and for all $t \in W$: if $(s, t) \in R_i$, then $\mathcal{M}, t \models \varphi$.

Let **L** denote the modal epistemic logic, while **L-A** denotes its extension obtained by adding to the proof system of **L** all formulas which are instantiations of the following schema:

$E_i\varphi \leftrightarrow (A_i\varphi \wedge K_i\varphi)$.

Fagin and Halpern [2, p. 67] proved the soundness and completeness of **KD45-A** logic with respect to the above-mentioned semantics with the awareness function[3]. The soundness and completeness of **S5-A** can be proved in a completely analogous fashion.

THEOREM 2. *Logic* **S5-A** *is sound and complete with respect to the class of all equivalence epistemic models with the awareness function, i.e. any formula $\varphi \in \Gamma_{\mathcal{L}_{MEL\text{-}A}}$ is an* **S5-A***-theorem iff φ is true in all epistemic models with the awareness function, where the accessibility relations are equivalence relations.*

It should be noted that the E operator does not behave like a normal modal operator. In particular, the formula $E_i(p \vee \neg p)$ is not valid in epistemic logics with awareness, because agents may not realize that $p \vee \neg p$. There is also no equivalence between the formulas $E_i(p \wedge q)$ and $E_i(q \wedge p)$, because we may want to model the knowledge of an agent who does not have to see the equivalence relation between the formulas $p \wedge q$ and $q \wedge p$. The equivalence of Axiom 4 and 5 do not have to apply to the actual knowledge – although, of course, it is possible to add such axioms to the proof systems. Neither $(E_i\varphi \wedge A_iE_i\varphi) \rightarrow E_iE_i\varphi$, nor $(\neg E_i\varphi \wedge A_i\neg E_i\varphi) \rightarrow E_i\neg E_i\varphi$ has to be valid in Fagin and Halpern's logic. It should be emphasized, however,

[3]The authors use the concepts of knowledge and belief interchangeably. They work primarily on the logic **KD45-A**, because the acceptance of the T axiom for beliefs is not justified.

that some properties of explicit knowledge are equivalent to properties of implicit knowledge. The schema for actual knowledge equivalent to the K axiom is $(E_i\varphi \wedge E_i(\varphi \rightarrow \psi) \wedge A_i\psi) \rightarrow E_i\psi$. The rule corresponding to the Gödel rule has now the following form: if $\vdash \varphi$, then $\vdash A_i\varphi \rightarrow E_i\varphi$.

Obviously, actual knowledge will have additional properties once we put some further restrictions on the awareness function. For example, the fact that the order of presentation of the conjuncts does not matter can be captured by the axiom $A_i(\varphi \wedge \psi) \leftrightarrow A_i(\psi \wedge \varphi)$, and in systems satisfying this restriction $E_i(\varphi \wedge \psi) \leftrightarrow E_i(\psi \wedge \varphi)$ is a valid formula. The fact that an agent is aware of a formula if and only if he is aware of its negation can be captured by the axiom $A_i\varphi \leftrightarrow A_i\neg\varphi$, and in systems satisfying this restriction $E_i\varphi \leftrightarrow E_i\neg\neg\varphi$ is valid.

4. Dynamic epistemic logic for non-omniscient agents

Our goal will be to construct a dynamic epistemic logic for actual knowledge of non-omniscient agents. Although similar motivations have been formulated by Rasmussen [13], the author developed his system only from an axiomatic point of view and has not provided a model theory for his logic.

In the case of the epistemic logics discussed so far, we have considered only static semantics, and logics of this kind do not allow for modeling the phenomenon of the agent's knowledge change when new information is received. Dynamic logics, starting from the logic of public announcements presented by Plaza [12] and Gerbrandy and Groeneveld [4], enable us to model the phenomenon of knowledge change.

The main idea associated with modeling updated knowledge is that whenever an agent receives new information, all epistemic states that are contradictory with this information are removed from an epistemic model that represents the agent's knowledge before the new information is received. The new information causes a transition from a certain initial epistemic model to its sub-model, that is, a model bounded by this new information. This is best illustrated by the so-called Muddy Children Puzzle, which comes from the book written by Littlewood (1953).

Example 1. Three children come back home after playing in the garden. During the game, children could get mud on their foreheads. Each child sees the foreheads of the other two, however no child can see his or her own forehead. Father arranges the children in a row, and then says:

Father: At least one of you has a dirty forehead.

After this announcement he asks:

Father: If you know whether your forehead is dirty, then step forward

None of the children step forward. Father repeats himself a second time. Again nothing happens. Yet, when the father repeats himself for a third time, all of the remaining children step forward.
Let us assume that we have the following trio of children: a, b, c. We will use propositional variables p_a, p_b, p_c to denote that relevant children have dirty foreheads. The graphical representation of the initial model of this situation is shown in Figure 1. The relation of accessibility in this model is reflexive, but the arrows symbolising this fact are left out from the figures in order to achieve better graphical clarity.
The sequence of epistemic interactions begins from the father's first announcement: "At least one of you has a dirty forehead", which means that $p_a \vee p_b \vee p_c$ holds. Let us note that $\mathcal{M}, 7 \not\models p_a \vee p_b \vee p_c$. Therefore world 7 is contradictory to the introduced information and should be eliminated. We leave it to the reader to analyze consequences of the remaining announcements and to solve the puzzle.

Definition 9. Let $\mathcal{M} = (W, \{R_i : i \in Ag\}, \{\mathcal{A}_i : i \in Ag\}, v)$ be an epistemic model with the awareness function. The updated model by the new information φ is defined as a tuple $\mathcal{M}|\varphi = (W', \{R'_i : i \in Ag\}, \{\mathcal{A}'_i : i \in Ag\}, v')$, where:

$$W' = \{s \in W : \mathcal{M}, s \models \varphi\},$$
$$R'_i = R_i \cap (W' \times W'), \text{ for any } i \in Ag,$$
$$\mathcal{A}'_i = \mathcal{A}|_{W'}, \text{ for any } i \in Ag,$$
$$v'(p) = v(p) \cap W', \text{ for any } p \in Var.$$

In other words, the model $\mathcal{M}|\varphi$ is the model $\mathcal{M}$ restricted to all those epistemic states where φ holds.

To express that a sentence is true as a result of an announcement, we expand the language $\mathcal{L}_{MEL-A}$ by the dynamic operator $[\varphi]\psi$ for any formulas $\varphi, \psi \in \Gamma_{\mathcal{L}_{MEL\text{-}A}}$.

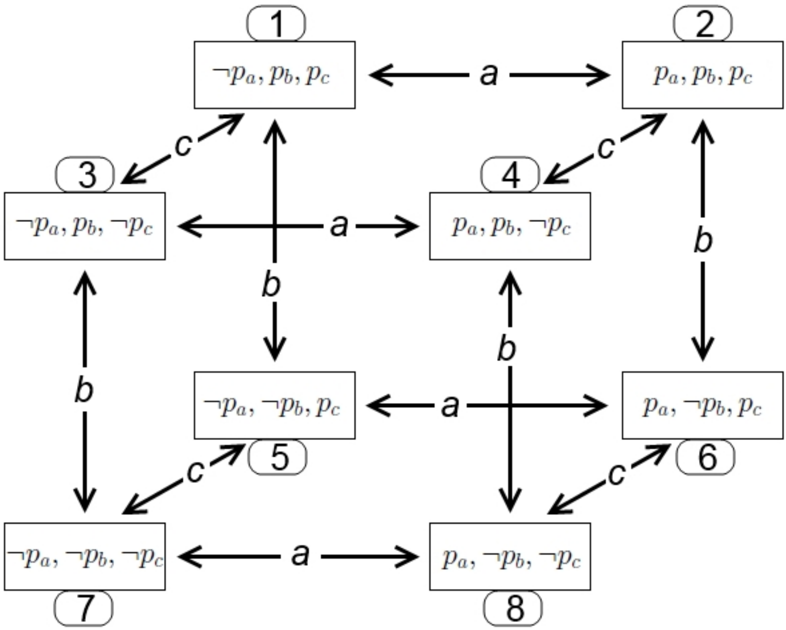

Fig. 1. Model $\mathcal{M}$ (initial model)

Definition 10. Let Var be the set of sentential variables and let Ag be the set of agents. The language of dynamic epistemic logic with the awareness operator $\mathcal{L}_{DEL\text{-}A}$ is defined inductively as follows:

$$\varphi \;\; ::= \;\; p \mid \neg\varphi \mid \varphi \rightarrow \varphi \mid K_i\varphi \mid A_i\varphi \mid [\varphi]\varphi,$$

where $p \in Var$ and $i \in Ag$. The set of all $\mathcal{L}_{DEL\text{-}A}$ formulas is denoted by $\Gamma_{\mathcal{L}_{DEL\text{-}A}}$.

The narrow interpretation of the formula $[\varphi]\psi$ is "after a public announcement φ, it holds that ψ". However, the default interpretation of the dynamic operator is wider, and $[\varphi]\psi$ should be read as "after an epistemic update with φ, it holds that ψ" or simply "after obtaining information φ, it holds that ψ". Thus, one may get the updated epistemic model not only as a result of public announcements but also as a result of other acts, such as observation or positive verification. Depending on the context, a formula $[\varphi]\psi$ can be interpreted, for example, as "after the observation that φ, it holds that ψ" or "after publicly verifying the truth of φ, it holds that ψ".

DEFINITION 11. Let $\mathcal{M} = (W, \{R_i : i \in Ag\}, \{\mathcal{A}_i : i \in Ag\}, v)$ be an epistemic model with the awareness function and let $s \in W$. Then Definition 8 is extended with the following condition:

$$\mathcal{M}, s \models [\varphi]\psi \quad \text{iff} \quad \text{if } \mathcal{M}, s \models \varphi, \text{ then } \mathcal{M}|\varphi, s \models \psi,$$

where $\mathcal{M}|\varphi$ is a model $\mathcal{M}$ restricted to all those epistemic states where φ holds.

We propose the following proof system for the logic **DEL-A**, i.e. the dynamic epistemic logic for actual knowledge of non-omniscient agents.

DEFINITION 12. A proof system for the logic **DEL-A** with the operators K_i and A_i for any $i \in Ag$, and the dynamic operator $[\ ]$ is given by the following axioms and inference rules:

all instantiations of propositional tautologies

$K_i(\varphi \to \psi) \to (K_i\varphi \to K_i\psi)$

$K_i\varphi \to \varphi$

$K_i\varphi \to K_iK_i\varphi$

$\neg K_i\varphi \to K_i\neg K_i\varphi$

$E_i\varphi \leftrightarrow (A_i\varphi \land K_i\varphi)$

$[\varphi]p \leftrightarrow (\varphi \to p)$

$[\varphi]\neg\psi \leftrightarrow (\varphi \to \neg[\varphi]\psi)$

$[\varphi](\psi \land \chi) \leftrightarrow ([\varphi]\psi \land [\varphi]\chi)$

$[\varphi]K_i\psi \leftrightarrow (\varphi \to K_i[\varphi]\psi)$

$[\varphi]A_i\psi \leftrightarrow (\varphi \to A_i\psi)$

$[\varphi][\psi]\chi \leftrightarrow [\varphi \land [\varphi]\psi]\chi$

The inference rules:

from $\varphi \to \psi$ and φ infer ψ	modus ponens
if $\vdash \varphi$, then $\vdash K_i\varphi$	Gödel's rule for K_i
if $\vdash \varphi$, then $\vdash [\psi]\varphi$, for any $\psi \in \Gamma_{\mathcal{L}_{DEL-A}}$	Gödel's rule for $[\ ]$

Let us note that the axioms for the dynamic operator extending the axioms of the **S5-A** logic to the axioms of the **DEL-A** logic enable us to eliminate announcements, one by one, from a formula of the language $\mathcal{L}_{DEL\text{-}A}$, by giving a logically equivalent formula without announcements.

To prove the soundness and completeness theorems for the logic **DEL-A**, we shall define a translation function $t : \Gamma_{\mathcal{L}_{DEL\text{-}A}} \mapsto \Gamma_{\mathcal{L}_{MEL\text{-}A}}$ which will enable us to translate any formula of the language of dynamic epistemic logic with the awareness operator into a formula of static epistemic logic with such an operator.

Definition 13. The translation function $t : \Gamma_{\mathcal{L}_{DEL\text{-}A}} \mapsto \Gamma_{\mathcal{L}_{MEL\text{-}A}}$ is defined as follows:

$t(p) = p$,
$t(\neg\varphi) = \neg t(\varphi)$,
$t(\varphi \wedge \psi) = t(\varphi) \wedge t(\psi)$,
$t(K_i\varphi) = K_i t(\varphi)$,
$t(A_i\varphi) = A_i t(\varphi)$,
$t([\varphi]p) = t(\varphi \to p)$,
$t([\varphi]\neg\psi) = t(\varphi \to \neg[\varphi]\psi)$,
$t([\varphi](\psi \wedge \chi)) = t([\varphi]\psi \wedge [\varphi]\chi)$,
$t([\varphi]K_i\psi) = t(\varphi \to K_i[\varphi]\psi)$,
$t([\varphi]A_i\psi) = t(\varphi \to A_i\psi)$,
$t([\varphi][\psi]\chi) = t([\varphi \wedge [\varphi]\psi]\chi)$.

In the next step, we shall define a measure of complexity of formulas of the language $\mathcal{L}_{DEL\text{-}A}$, that is, the function that assigns a natural number to each formula of that language.

Definition 14. The complexity measure $m : \Gamma_{\mathcal{L}_{DEL\text{-}A}} \mapsto \mathbb{N}$ is defined in the following way:

$m(p) = 1$,
$m(\neg\varphi) = 1 + m(\varphi)$,
$m(\varphi \wedge \psi) = 1 + max(m(\varphi), m(\psi))$,
$m(K_i\varphi) = 1 + m(\varphi)$,
$m(A_i\varphi) = 1 + m(\varphi)$,
$m([\varphi]\psi) = (4 + m(\varphi)) \cdot m(\psi)$.

The choice of such values for the complexity measure of formulas of the language $\mathcal{L}_{DEL\text{-}A}$ seems arbitrary, but it allows us to prove the following lemma:

Lemma 1. *For all* $\varphi, \psi, \chi \in \Gamma_{\mathcal{L}_{DEL\text{-}A}}$ *:*

(i) $m(\psi) \geq m(\varphi)$, *if* φ *is a sub-formula of* ψ,

(ii) $m([\varphi]p) > m(\varphi \to p)$,

(iii) $m([\varphi]\neg\psi) > m(\varphi \to \neg[\varphi]\psi)$,

(iv) $m([\varphi](\psi \wedge \chi)) > m([\varphi]\psi \wedge [\varphi]\chi)$,

(v) $m([\varphi]K_i\psi) > m(\varphi \to K_i[\varphi]\psi)$,

(vi) $m([\varphi]A_i\psi) > m(\varphi \to A_i\psi)$,

(vii) $m([\varphi][\psi]\chi) > m([\varphi \wedge [\varphi]\psi]\chi)$.

Proof: By induction on the complexity of a formula.

(i) Proof by induction on ψ. In the base step, it is enough to note that if ψ is a sentential variable, then $m(\psi) = 1$. In this case $m(\psi) \geq m(\psi)$ holds. Let us assume the following inductive hypothesis: $m(\psi) \geq m(\varphi)$ if φ is a sub-formula of ψ, and $m(\chi) \geq m(\varphi)$ if φ is a sub-formula of χ.

(Case of negation) Suppose that φ is a sub-formula of $\neg\psi$. Then $\varphi = \psi$ or φ is a sub-formula of ψ. In the first case, the theorem holds because $m(\varphi) = m(\psi) < m(\psi) + 1 = m(\neg\psi)$. In the second case, we have $m(\psi) \geq m(\varphi)$ due to the inductive hypothesis. Therefore, $m(\neg\psi) = m(\psi) + 1 \geq m(\varphi)$.

(Case of conjunction) Let us assume that φ is a sub-formula of $\psi \wedge \chi$. Then $\varphi = \psi \wedge \chi$ or φ is a sub-formula of ψ or χ. In the first case, trivially $m(\psi \wedge \chi) \geq m(\varphi)$. Let us consider the second case and assume that φ is a sub-formula of ψ. Then, from the inductive hypothesis, $m(\psi) \geq m(\varphi)$ and consequently $m(\psi \wedge \chi) = 1 + \max(m(\psi), m(\chi)) > m(\psi) \geq m(\varphi)$. If φ is a sub-formula of χ, then by analogy we get $m(\psi \wedge \chi) \geq m(\varphi)$.

(Case of the operators K_i and A_i) By analogy to the case of negation.

(Case of the dynamic operator) Let us assume that φ is a sub-formula of $\psi[\chi]$. Then $\varphi = [\psi]\chi$ or φ is a sub-formula of ψ or χ. In the first case, $m([\psi]\chi) \geq m(\varphi)$. Let us consider the second case, and assume that φ is a sub-formula of ψ. Since we know that $m([\psi]\chi) = (4 + m(\psi)) \cdot m(\chi)$ and $m(\chi) \geq 1$, from the inductive hypothesis it follows that $m(\psi) \geq m(\varphi)$, therefore $m([\psi]\chi) \geq m(\varphi)$. By analogy, we get $m([\psi]\chi) \geq m(\varphi)$, when φ is a sub-formula of χ.

(ii) Since $m([\varphi]p) = (4 + m(\varphi)) \cdot m(p) = 4 + m(\varphi)$, and $m(\varphi \to p) = m(\neg(\varphi \wedge \neg p)) = 1 + m(\varphi \wedge \neg p) = 1 + max(m(\varphi), m(\neg p)) = 1 + max(m(\varphi), 2)$, therefore $m(\varphi \to p) = 1 + m(\varphi)$ or $m(\varphi \to p) = 3$. In both cases $m([\varphi]p) > m(\varphi \to p)$.

(iii) Since $m([\varphi]\neg\psi) = (4 + m(\varphi)) \cdot m(\neg\psi) = (4 + m(\varphi)) \cdot (1 + m(\psi)) = 4 + m(\varphi) + 4 \cdot m(\psi) + m(\varphi) \cdot m(\psi)$, and $m(\varphi \to \neg[\varphi]\psi) = m(\neg(\varphi \wedge \neg\neg[\varphi]\psi)) = 1 + m(\varphi \wedge \neg\neg[\varphi]\psi) = 2 + max(m(\varphi), m(\neg\neg[\varphi]\psi)) = 2 + max(m(\varphi), 2 + ((4 + m(\varphi)) \cdot m(\psi))) = 2 + max(m(\varphi), 2 + 4 \cdot m(\psi) + m(\varphi) \cdot m(\psi))$, therefore $m(\varphi \to \neg[\varphi]\psi) = 2 + m(\varphi)$ or $m(\varphi \to \neg[\varphi]\psi) = 4 + 4 \cdot m(\psi) + m(\varphi) \cdot m(\psi)$. In both cases $m([\varphi]\neg\psi) > m(\varphi \to \neg[\varphi]\psi)$.

(iv) Let us assume that $m(\psi) \geq m(\chi)$. Since $m([\varphi](\psi \wedge \chi)) = (4 + m(\varphi)) \cdot m(\psi \wedge \chi) = (4 + m(\varphi)) \cdot (1 + max(m(\psi), m(\chi))) = (4 + m(\varphi)) \cdot (1 + m(\psi)) = 4 + m(\varphi) + 4 \cdot m(\psi) + m(\varphi) \cdot m(\psi)$, and $m([\varphi]\psi \wedge [\varphi]\chi) = 1 + max(m([\varphi]\psi), m([\varphi]\chi)) = 1 + max((4 + m(\varphi)) \cdot m(\psi), (4 + m(\varphi)) \cdot m(\chi)) = 1 + ((4 + m(\varphi)) \cdot m(\psi)) = 1 + 4 \cdot m(\psi) + m(\varphi) \cdot m(\psi)$, therefore $m([\varphi](\psi \wedge \chi)) > m([\varphi]\psi \wedge [\varphi]\chi)$. The case where $m(\chi) \geq m(\psi)$ is analogous.

(v) and (vi) are proved in an analogous way to the proof of (iii).

(vii) Since $m([\varphi][\psi]\chi) = (4 + m(\varphi)) \cdot m([\psi]\chi) = (4 + m(\varphi)) \cdot ((4 + m(\psi)) \cdot m(\chi)) = ((4 + m(\varphi)) \cdot (4 + m(\psi))) \cdot m(\chi) = (16 + 4 \cdot m(\varphi) + 4 \cdot m(\psi) + m(\varphi) \cdot m(\psi)) \cdot m(\chi)$, and $m([\varphi \wedge [\varphi]\psi]\chi) = (4 + m(\varphi \wedge [\varphi]\psi)) \cdot m(\chi) = (4 + (1 + max(m(\varphi), (4 + m(\varphi)) \cdot m(\psi)))) \cdot m(\chi) = (5 + ((4 + m(\varphi)) \cdot m(\psi))) \cdot m(\chi) = (5 + 4 \cdot m(\psi) + m(\varphi) \cdot m(\psi)) \cdot m(\chi)$, therefore $m([\varphi][\psi]\chi) > m([\varphi \wedge [\varphi]\psi]\chi)$.
$\square$

Making use of Lemma 1, we can prove that each formula of $\mathcal{L}_{DEL\text{-}A}$ is equivalent to its translation in the logic **DEL-A**.

LEMMA 2. *For any formula* $\varphi \in \Gamma_{\mathcal{L}_{DEL\text{-}A}}$ *it holds that* $\vdash_{\text{DEL-A}} \varphi \leftrightarrow t(\varphi)$.

PROOF: We conduct a proof by induction on $m(\varphi)$. In the base case, i.e. when φ is a sentential variable, according to Definition 13, $t(p) = p$ and since by Definition 12 all instances of propositional tautologies are theorems of **DEL-A**, we obtain $\vdash_{\text{DEL-A}} p \leftrightarrow p$. Let us assume the following inductive hypothesis: for any formula ψ such that $m(\psi) \leq n$, it holds that $\vdash_{\text{DEL-A}} \psi \leftrightarrow t(\psi)$.

(Case of negation) Let us assume that φ is a formula of the form $\neg\psi$ such that $m(\neg\psi) = n + 1$. Then $m(\psi) = n$. Therefore, by the inductive hypothesis, $\vdash_{\text{DEL-A}} \psi \leftrightarrow t(\psi)$. Hence, $\vdash_{\text{DEL-A}} \neg\psi \leftrightarrow \neg t(\psi)$. Finally, according to Definition 13, $\neg t(\psi) = t(\neg\psi)$, so $\vdash_{\text{DEL-A}} \neg\psi \leftrightarrow t(\neg\psi)$.

(Case of conjunction) Let us assume that φ is a formula of the form $\psi \wedge \chi$, such that $m(\psi \wedge \chi) = n + 1$. Referring to Lemma 1(i) and Definition 14, we get $m(\psi) \leq n$ and $m(\chi) \leq n$. Therefore, by virtue of the inductive hypothesis, $\vdash_{\text{DEL-A}} \psi \leftrightarrow t(\psi)$ and $\vdash_{\text{DEL-A}} \chi \leftrightarrow t(\chi)$. Therefore, $\vdash_{\text{DEL-A}} (\psi \wedge \chi) \leftrightarrow (t(\psi) \wedge t(\chi))$. Finally, by Definition 13, $\vdash_{\text{DEL-A}} (\psi \wedge \chi) \leftrightarrow t(\psi \wedge \chi)$.

(Case of the operators K_i and A_i) By analogy to the case of negation.

(Case of the dynamic operator) Let us assume that φ is a formula of the form $[\psi]p$, such that $m([\psi]p) = n + 1$. By virtue of Lemma 1(ii), $m([\psi]p) > m(\psi \to p)$. Hence, $m(\psi \to p) \leq n$, and by virtue of the inductive hypothesis, $\vdash_{\text{DEL-A}} (\psi \to p) \leftrightarrow t(\psi \to p)$. It follows from the DEL-A axioms that $\vdash_{\text{DEL-A}} [\psi]p \leftrightarrow (\psi \to p)$. Therefore, $\vdash_{\text{DEL-A}} [\psi]p \leftrightarrow t(\psi \to p)$, and according to Definition 13, $\vdash_{\text{DEL-A}} [\psi]p \leftrightarrow t([\psi]p)$. The cases where φ takes the form of $[\psi]\neg\chi$, $[\psi]\chi \wedge \xi$, $[\psi]K_i\chi$, $[\psi]A_i\chi$ and $[\psi][\chi]\xi$, are proved in an analogous way, referring to the corresponding points of Lemma 1, the inductive hypothesis, the axioms of the **DEL-A** logic and Definition 13. □

Our Lemma 2 is crucial for proving the completeness theorem for the **DEL-A** logic.

THEOREM 3. *Logic* **DEL-A** *is sound and complete with respect to the class of all equivalence epistemic models with the awareness function, i.e. any formula* $\varphi \in \Gamma_{\mathcal{L}_{DEL\text{-}A}}$ *is a* **DEL-A** *theorem iff* φ *is true in all epistemic models with the awareness function, where the accessibility relations are equivalence relations.*

PROOF: ($\rightarrow$) The soundness follows from the axioms which are shown to be valid and the rules of inference which are validity-preserving. The validity of the axioms without the dynamic operator is guaranteed by the soundness of the logic **S5-A**. As an illustration of the validity of the remaining axioms we shall prove that $\models_{\text{DEL-A}} [\varphi]A_i\psi \leftrightarrow (\varphi \to A_i\psi)$.

Let $\mathcal{M} = (W, \{R_i : i \in Ag\}, \{\mathcal{A}_i : i \in Ag\}, v)$ be a model of the logic **DEL-A** and let $s \in W$ be such that $\mathcal{M}, s \models [\varphi]A_i\psi$. We shall show that $\mathcal{M}, s \models \varphi \to A_i\psi$. Let us assume that $\mathcal{M}, s \models \varphi$. Then, according to Definition 11, we have $\mathcal{M}|\varphi, s \models A_i\psi$, which implies $\mathcal{M}, s \models A_i\psi$.

Let $\mathcal{M} = (W, \{R_i : i \in Ag\}, \{\mathcal{A}_i : i \in Ag\}, v)$ be a model of the logic **DEL-A** and let $s \in W$ be such that $\mathcal{M}, s \models \varphi \to A_i\psi$. We shall show that $\mathcal{M}, s \models [\varphi]A_i\psi$. Let us assume that $\mathcal{M}, s \models \varphi$, which implies that $\mathcal{M}, s \models A_i\psi$. Since $\mathcal{M}, s \models \varphi$, then $s \in \mathcal{M}|\varphi$. Since $s \in \mathcal{M}|\varphi$ and $\mathcal{M}, s \models A_i\psi$, then finally $\mathcal{M}|\varphi, s \models A_i\psi$. This proves that $\models_{\text{DEL-A}} [\varphi]A_i\psi \leftrightarrow (\varphi \to A_i\psi)$.

($\leftarrow$) In order to prove completeness, let us assume that φ is **DEL-A**-valid, i.e. $\models_{\text{DEL-A}} \varphi$. According to Lemma 2, $\vdash_{\text{DEL-A}} \varphi \leftrightarrow t(\varphi)$. Hence, from the soundness, $\models_{\text{DEL-A}} \varphi \leftrightarrow t(\varphi)$. Therefore, if $\models_{\text{DEL-A}} \varphi$, then $\models_{\text{DEL-A}} t(\varphi)$. Since $t(\varphi)$ does not contain the dynamic operator, we have $\models_{\text{S5-A}} t(\varphi)$, and according to Theorem 2, $\vdash_{\text{S5-A}} t(\varphi)$. This implies that $\vdash_{\text{DEL-A}} t(\varphi)$, because the axioms of the logic **DEL-A** contain the axioms of **S5-A**. Hence, if $\vdash_{\text{DEL-A}} t(\varphi)$ and $\vdash_{\text{DEL-A}} \varphi \leftrightarrow t(\varphi)$, then $\vdash_{\text{DEL-A}} \varphi$. $\square$

5. Summary

The problem of logical omniscience is a drawback not only of static epistemic logics, but also of dynamic epistemic logics. If those systems of logic are considered to model the concept of potential knowledge represented by the operator K_i for any agent i, then modeling actual knowledge remains an open question. This is an interesting problem, since the possible worlds semantics account has proven to be a highly successful framework for modeling not only epistemic notions such as knowledge, belief and information, but also the act of epistemic change. Since the possible worlds semantics framework has been widely adopted not only by philosophers, but also by computer and cognitive scientists, linguists, and artificial intelligence researchers, it is desirable to establish a dynamic model theory of knowledge that solves the problem of logical omniscience. In this article we showed that it is possible to construct a system of dynamic epistemic logic for actual knowledge of non-omniscient agents. We presented a proof system and a natural semantics for such a logic, and finally we proved the soundness and completeness theorem. In our system, actual knowledge will have additional properties once we apply some further restrictions on the awareness function, so it may be seen as a general framework of representing the actual knowledge change for different, more or less logically competent, agents. An open problem is extending the logic **DEL-A** in such a way that enables us to model the notion of common knowledge.

References

[1] J. van Benthem, *Epistemic Logic and Epistemology: The State of Their Affairs*, **Philosophical Studies**, Vol. 128 (2006), pp. 49–76.

[2] R. Fagin, J. Y. Halpern, *Belief, Awareness, and Limited Reasoning*, **Artificial Intelligence**, Vol. 34 (1988), pp. 39–76.

[3] R. Fagin, J. Y. Halpern, Y. Moses, M. Vardi, **Reasoning about Knowledge**, MIT Press, Cambridge (1995).

[4] J. Gerbrandy, W. Groeneveld, *Reasoning About Information Change*, **Journal of Logic, Language, and Information**, Vol. 6 (1997), pp. 147–169.

[5] J. Hintikka, **Knowledge and Belief**, Cornell University Press, Ithaca (1962).

[6] S. Kripke, *Semantical Analysis of Modal Logic (abstract)*, **Journal of Symbolic Logic**, Vol. 24 (1959), pp. 323–324.

[7] S. Kripke, *Semantical Analysis of Modal Logic I. Normal Modal Propositional Calculi*, **Zeitschrift für Mathematische Logik und Grundlagen der Mathematik**, Vol. 9 (1963), pp. 67–96.

[8] H. J. Levesque, *A Logic of Implicit and Explicit Belief*, **Proceedings of AAAI-84**, Austin (1984), pp. 198–202.

[9] J. E. Littlewood, **A Mathematician's Miscellany**, Meuthen, London (1953).

[10] J. Łoś, *Logiki wielowartościowe a formalizacja funkcji intensjonalnych*, **Kwartalnik Filozoficzny**, Vol. 17 (1948), pp. 57–78.

[11] A. Pap, *Belief and Propositions*, **Philosophy of Science**, Vol. 24 (1957), pp. 123–136.

[12] J. Plaza, *Logics of Public Communications*, [in:] M. Emrich, M. Pfeifer, M. Hadzikadic, Z. Ras (eds.), **Proceedings of the 4th International Symposium on Methodologies for Intelligent Systems: Poster Session Program**, Oak Ridge National Laboratory, Oak Ridge (1989), pp. 201–216.

[13] M. S. Rasmussen, *Dynamic Epistemic Logic and Logical Omniscience*, **Logic and Logical Philosophy**, Vol. 24 (2015), pp. 377–399.

[14] G. Sillari, *Models of awareness*, [in:] G. Bonanno, W. van der Hoek, M. Wooldridge (eds.), **Logic and the Foundations of Game and Decision Theory. Vol. 2: Texts in Logic and Games**, Amsterdam University Press, Amsterdam (2008), pp. 209–240.

[15] K. M. Sim, *Epistemic Logic and Logical Omniscience: A Survey*, **International Journal of Intelligent Systems**, Vol. 12 (1997), pp. 57–81.

University of Warsaw
Institute of Philosophy
Department of Logic
ul. Krakowskie Przedmieście 3
00–927 Warszawa, Poland
e-mail: arkadiuszwojcik@uw.edu.pl

Bulletin of the **S**ection of **L**ogic
Volume 49/1 (2020), pp. 97–102

http://dx.doi.org/10.18778/0138-0680.2020.05

Bożena Pięta

MAY THE SEA-BATTLE TOMORROW NOT HAPPEN?

Abstract

This note provides a review of the book 'On the Sea-Battle Tomorrow That May Not Happen' by Tomasz Jarmużek.

Keywords: Aristotle, future contingents, structure of time.

The headline question is a pivotal problem of a book entitled 'On the Sea-Battle Tomorrow That May Not Happen' [5].[1] The aim of the monograph is to reconstruct and analyze the reasoning of the Megarian philosopher and logician, Diodorus Cronus. Unfortunately, the knowledge about his considerations is highly uncertain and fragmentary, nevertheless, it can be inferred from some, not only historical sources that Diodorus Cronus have plunged into a polemic with Aristotle. Stagyrite announced the problem of the logical value of sentences about the future by raising a question that became the inspiration for the book's title: Will be a sea-battle tomorrow? In response, Diodorus Cronus proposed a reasoning that went down in history as the Master Argument. The author reconstructs its form by means of modern logic in five different ways. The polemics between philosophers resulted from their approaches – Aristotle as an indeterminist argued that sentences about future are not necessary and moreover he adopted the principle of bivalence. In contrast to him, Diodorus Cronus approach has been regarded as deterministic. One of the fundamental issues in the monograph

[1] It is an English and improved version of the Polish book 'Jutrzejsza bitwa morska. Rozumowanie Diodora Kronosa' that was published six years ago [4].

is to analyze whether Master Argument factually forces the linear structure of time, which is a differentiator of determinism. Although Diodorus Cronus reasoning is a major problem of philosophy that has been studied multiple times, e.g. by such philosopher as Jakko Hintikka [3], the book under review provides a new fresh approach to the problem. Even though already existing reconstructions of the problem appear in the book, the author analyzes them in an inventive way.

The work consists of three parts. The first part is the introduction, where the first chapter concerns the ancient dispute about definitions of modality and the status of sentences about the future. The issue itself has been formulated by Aristotle in the Chapter 9 of 'On Interpretation' [1], where he considers the problem of the necessity or impossibility of tomorrow's sea battle. According to Aristotle, we cannot make assertion that something will happen until it actually happens. As opposed to this view, Diodorus Cronus claims that the expressions referring to the future events could bear a logical value. What is significant, chapter one outlines three essential terms that shape the frame of the problem: determinism, time and truth, discussed in following chapters.

The second chapter is an attempt to present the notion of language in the context of the notion of time. Nevertheless it is not a comprehensive discussion, but it is justified by further analysis, which does not require such an extensive exploration. In this chapter, the role of the meaning of the sentences is emphasized. The author makes a historical review of the issue, referring to philosophers such as Willard Van Orman Quine or Gottlob Frege. The relationship between the sentence and the proposition is underlined, inclining towards an objectivist approach, represented by Frege, where the proposition is the meaning expressed by the sentence. Attention is drawn to pragmatic component of the statement and sentences that are temporarily determined, making them context invariant.

Then, the most important concepts of truth employed in monograph are discussed. They are divided into the epistemic concepts and the non-epistemic concepts of truth. What is important, the study mainly takes into account non-epistemic concepts of truth, based on a belief that logical value does not depend on the cognising subjects. This limitation is justified by the composition of the study and the narrative conducted in it. Subsequently, the Tarski's concept of truth is distinguished, which is further modified by limiting domain of sentences to sentences that are temporarily determined. The problem of bivalence of sentences and dynamics of logical values in the context of determinism and indeterminism is discussed.

The third chapter presents the problem of determinism. The ontological, physical and metaphysical determinism is discussed and the consequences of determinism in the form of logical, epistemological, temporal and anthropological determinism are emphasized. In the context of Diodorus Kronos reasoning, the temporal determinism is accented, a concept according to which, if the world is determined, there exists only one correct description of the future. At the end of the chapter branching and linear structures of time are presented. They are discussed in the following part of monograph.

The next chapter discusses the subject of time. Any use of term 'time' equips this concept with a different meaning. Therefore, the reasoning about time is briefly discussed from many perspectives – cultural, psychological, phenomenological and physical. Next, the author moves on to philosophy of time and its problems, limiting considerations in an intentional way. The purpose of reviewing chosen view is not a comprehensive analysis, aimed at finding solutions, but only emphasizing the problematic issues in this area. Problems related to ontological autonomy from physical world, or those related to the passage of time [2] are accentuated. The monograph excludes the possibility of the passage of time with an undefined direction, which is justified by the necessity to take into account the past and the present time. The nomological and idiographical character of time is therefore not considered. Then, the problem of McTaggart is presented, in which he argues in favor of the thesis that time is not something real. The question of time is also associated with Diodorus Cronus resoning. Since there are not many testimonies about his views or philosophy, in the monograph the attention is mainly focused on the formal aspects of the approach to time. The aim is to reach a compromise, which was named 'formal-ontological approach to the time' and in which the set-theoretical tools are used. The considerations are limited to the so-called 'pointwise concept of time' [5, p. 125].

In the second part entitled 'The issues', there is a discussion about the origin of the problem of tomorrow's sea-battle. Problems related to modalities in Aristotle's view are highlighted and then his reflections are placed as polemical with regard to Diodorus Cronus considerations. The reasoning of Diodorus Cronus is presented as a trilemma with the following form [5, p. 147]:

1. that everything past must of necessity be true;
2. that an impossibility does not follow a possibility;
3. and that a thing is possible which neither is nor will be true.

Owing to the fact that contradiction occurs between these three propositions, Diodorus Cronus postulated to reject the third premise, which would allow to achieve some version of logical determinism and it was related to his philosophical orientation. Nonetheless, the approach of other philosophers differed from the one adopted by Diodorus Cronus and the problem remained open. Then, the issue of futura contingentia and interesting approach to time, called the logics of branching time is raised. In the theory RDC (Reasoning of Diodorus Cronus) allowing branching, determinism would not be sine qua non condition – this problem is studied later in the monograph.

The sixth chapter deals with dates, since the understanding of the dates and their inclusion in a certain metrological system is necessary to establish the logical value of sentences. Denotations of dates are called intervals, which in turn refers to states of affairs. Some of these states can be broken down into simpler states of affairs (more detailed) and if we define logical values for sentences concerning points, we are able to determine a logical value for sentences which refer to the larger interval that these sentences refer to.

In the next section, formal issues are presented. After presenting the most important facts regarding the adopted notation, various types of temporal logics are described. It is noted that the classical logic is a theory of the classical functors and the temporal logics are theories of the temporal functors [5, p. 173]. Next, attention is dedicated to the most important concepts, useful in further research on the problem of determinism in the light of RDC. Finally, the tense logic is described, which is the logic of the sentences undetermined in time. Also some ideas of positional logics are work out in the book. The application of positional logic (a logic of realization operator) are based on former papers [6] and [7]. It results in a new kind of positional logic $\mathcal{R}_n^+$ with calculations on a metric time.

The last part entitled 'Solutions' deals with possible factors of the problem of determinism (in RDC). Different reconstructions of the Master's Argument are discussed. The first of the presented reconstructions is the reconstruction elaborated by F. S. Michael, where among others, calculation of moments are presented. Then, the author works out Rescher's reconstruction which include interpretation of the method proposed by Zeller (the second interpretation of the premise number two). All these reconstructions are described in the chapter 'Reconstructions with operator R'. In the next chapter entitled 'Other reconstructions' reconstruction of A. N. Prior and P. Øhstrøm are presented. Within most of these reconstructions (apart from Reserch's reconstruction), it turns out that Diodorus Cronus reasoning does not necessarily leads to determinism.

The monograph has many advantages – it has a coherent structure, it is written in an accessible language and has the extensive bibliography, however, its greatest merit is originality. The author not only presents the original solution to the problem of determinism, but also makes it possible to look at the problem from a wider, also historical, perspective. The main result in the monograph is that Master's Argument, with certain interpretations, does not have to be deterministic, which means it can be reconstructed without assumption of linear time structures. Thus, despite the universal agreement on the deterministic orientation of Diodorus Cronus, its reasoning can be reconstructed in an indeterministic way. Indirectly, this monograph also presents the power of temporal logic as a tool for formal analysis of philosophical problems. The book is worth recommending to both logicians who want to deepen their philosophical knowledge and philosophers who want to get to know the power of formal methods for analyzing philosophical problems.

References

[1] Aristotle, *On Interpretation*, E. M. Edghill (trans.), [in:] W. D. Ross and J. A. Smith (eds.), **The Works of Aristotle**, Vol. 1, Oxford University Press, London (1996), pp. 46–79.

[2] M. Bunge, **Philosophy of Science**, Vols. 1 and 2, Transaction Publisher, New Brunswick/London (1998).

[3] J. Hintikka, *Aristotle and the 'Master Argument' of Diodorus*, **American Philosophical Quarterly**, Vol. 1(2) (1964), pp. 101–114.

[4] T. Jarmużek, **Jutrzejsza bitwa morska. Rozumowanie Diodora Kronosa**, Wydawnictwo Naukowe Uniwersytetu Mikołaja Kopernika, Toruń (2013).

[5] T. Jarmużek, **On the Sea-Battle Tommorow That May Not Happen. A Logical and Philosophical Analysis of the Master Argument**, Peter Lang, Berlin (2018).

[6] T. Jarmużek, *Minimal Logical Systems with R-operator: Their Metalogical Properties and Ways of Extensions*, [in:] J. Béziau and A. Costa-Leite (eds.), **Perspectives on Universal Logic**, Polimetrica Publisher, Italy (2007), pp. 319–333.

[7] T. Jarmużek, A. Pietruszczak, *Completenes of minimal positional calculus*, **Logic and Logical Philosophy**, Vol. 13 (2004), pp. 147–162.

Nicolaus Copernicus University
Faculty of Humanities
Toruń, Poland
e-mail: `bozena.pieta95@gmail.com`

Submission Information

Manuscripts can be submitted to the Editor-in-Chief or any member of the Editorial Board. The *BSL* prefers submissions in standard LaTex, Tex, or AMS-LaTex. For the purpose of refereeing, papers with abstracts only may be submitted either in hard copy (a diskette with the .tex version of the paper should also be included) or via e-mail. Prospective authors should follow the layout of the published *BSL* papers in the preparation of their manuscripts. Authors who are unable to comply with these requirements should contact the Editorial Office in advance.

Paper Length should not exceed 14 pages when typesetted in LaTex with the following parameters: 12pt bookstyle, \textheight 552.4 pt, \textwidth 5in. In exceptional cases an Editor can accept and subsequently recommend a longer paper for publication if the significance and/or the presentation of the paper warrants an additional space.

Footnotes should be avoided as much as possible. When essential, they should be brief and consecutively numbered throughout, with superscript Arabic numbers.

References should be listed in alphabetic order, numbered consecutively, and typed in the same way as the following examples:

[1] L. Henkin, *Some remarks on infinitely long formulas*, **Proceedings of the Symposium on Foundations of Mathematics: Infinitistic Methods, Warsaw, 1959**, Pergamon Press, New-York, and PWN, Warsaw (1961), pp. 167–183.

[2] S. C. Kleene, **Mathematical Logic**, John Wiley & Sons, Inc., New York (1967).

[3] J. Łoś and R. Suszko, *Remarks on sentential logic*, **Indagationes Mathematicae**, 20 (1958), pp. 177–183.

Affiliation and mailing addresses of all the authors should be placed at the end of the paper.

It is the author's responsibility to obtain the necessary copyright permission from the copyright owner(s) to publish the submitted material in *BSL*.