UNIVERSITY OF ŁÓDŹ
DEPARTMENT OF LOGIC

BULLETIN
OF THE SECTION OF LOGIC

VOLUME 48, NUMBER 4

ŁÓDŹ, DECEMBER 2019

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UNIVERSITY OF ŁÓDŹ
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BULLETIN

OF THE SECTION OF LOGIC

VOLUME 48, NUMBER 4

ŁÓDŹ, DECEMBER 2019

Layout
Beata Promińska

Initiating Editor
Katarzyna Smyczek

Printed directly from camera-ready materials provided
to the Łódź University Press

Published by Łódź University Press
First edition. W.09721.19.0.C

Printing sheets 5.75

Łódź University Press
90-131 Łódź, 8 Lindleya St.
www.wydawnictwo.uni.lodz.pl
e-mail: ksiegarnia@uni.lodz.pl
tel. +48 42 665 58 63

The **Bulletin of the Section of Logic** (*BSL*) is a quarterly peer-reviewed journal published with the support from the University of Łódź. Its aim is to act as a forum for a wide and timely dissemination of new and significant results in logic through rapid publication of short papers. *BSL* publishes contributions on topics dealing directly with logical calculi, their methodology, and algebraic interpretation.

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TABLE OF CONTENTS

Bulletin of the Section of Logic
Volume 48/4 (2019), pp. 245–257

http://dx.doi.org/10.18778/0138-0680.48.4.01

Daishi Yazaki

SEMANTICAL PROOF OF SUBFORMULA PROPERTY FOR THE MODAL LOGICS K4.3, KD4.3, AND S4.3

Abstract

The main purpose of this paper is to give alternative proofs of syntactical and semantical properties, i.e. the subformula property and the finite model property, of the sequent calculi for the modal logics **K4.3**, **KD4.3**, and **S4.3**. The application of the inference rules is said to be acceptable, if all the formulas in the upper sequents are subformula of the formulas in lower sequent. For some modal logics, Takano analyzed the relationships between the acceptable inference rules and semantical properties by constructing models. By using these relationships, he showed Kripke completeness and subformula property. However, his method is difficult to apply to inference rules for the sequent calculi for **K4.3**, **KD4.3**, and **S4.3**. Looking closely at Takano's proof, we find that his method can be modified to construct finite models based on the sequent calculus for **K4.3**, if the calculus has (cut) and all the applications of the inference rules are acceptable. Similarly, we can apply our results to the calculi for **KD4.3** and **S4.3**. This leads not only to Kripke completeness and subformula property, but also to finite model property of these logics simultaneously.

Keywords: modal logic, analytic cut, subformula property, finite model property.

1. Introduction

The sequent calculi for some modal logics possess subformula property and finite model property. Takano [2] proved that the sequent calculi for **K5** and **K5D** enjoy these properties through semantical method. Then, he generalized the method by introducing special unprovable sequent, *analytically saturated sequent*, in Takano [3].

In [3], Takano analyzed the relationships between acceptable inference rules and semantical properties by constructing Kripke models using the set of all analytically saturated sequents. (The application of the inference rules is said to be acceptable, if all formulas in the upper sequents are subformulas of the formulas in the lower sequent.) We call here this method as Takano's method. Then, he showed that the sequent calculi for modal logics which are obtained from **K** by adding axioms from **T**, **4**, **5**, **D**, and **B** enjoy subformula property and finite model property.

The main purpose of this paper is to give alternative proofs of subformula property and finite model property of the sequent calculi for the modal logics **K4.3**, **KD4.3**, and **S4.3**. For this purpose, we consider the relationships between the semantical properties and the inference rules ($\Box$4.3) and (S4.3) (introduced by Shimura [1]) based on Takano's method. However, the straightforward application of Takano's method does not work well for ($\Box$4.3) and (S4.3). Taking a close look at his proof, we find that Takano's method can be modified to construct finite models based on the sequent calculus for **K4.3**, if the calculus has ($\Box$4.3) and (*cut*), and all the applications of inference rules are acceptable. Similarly, we can apply this result to the inference rule (S4.3). This implies Kripke completeness of the sequent calculi for **K4.3** and **S4.3**, and these calculi enjoy not only subformula property, but also finite model property.

In Section 2, we introduce the definition and property of an analytically saturated sequent based on Takano [3]. In Section 3 and 4, we consider ($\Box$4.3) and (S4.3), respectively, and give the procedure for constructing finite models.

2. Preliminaries

In this paper, we use only $\neg$ (negation), $\supset$ (implication), and $\Box$ (necessity) as logical symbols, and other are considered as abbreviations. Propositional letters and formulas are denoted by $p,\ q,\ r,\cdots$ and $A,\ B,\ C,\cdots$, respectively. Finite sequences of formulas are denoted by Γ, Δ, Θ, $\Lambda,\cdots$, and a sequent is an expression of the form $\Gamma \to \Theta$. A $\Box$-formula is a formula whose outermost logical symbol is $\Box$. We mean by $\mathrm{Sf}(\Gamma)$ the set of all the subformulas of some formulas in Γ, and by $\Box\Gamma$ the set $\{\Box A \mid A \in \Gamma\}$.

Let us consider the following structural rules:

$$\frac{\Gamma \to \Theta}{A,\ \Gamma \to \Theta}\,(w \to) \qquad \frac{\Delta,\ B,\ A,\ \Gamma \to \Theta}{\Delta,\ A,\ B,\ \Gamma \to \Theta}\,(e \to) \qquad \frac{A,\ A,\ \Gamma \to \Theta}{A,\ \Gamma \to \Theta}\,(c \to)$$

$$\frac{\Gamma \to \Theta}{\Gamma \to \Theta,\ A}\,(\to w) \qquad \frac{\Gamma \to \Theta,\ B,\ A,\ \Lambda}{\Gamma \to \Theta,\ A,\ B,\ \Lambda}\,(\to e) \qquad \frac{\Gamma \to \Theta,\ A,\ A}{\Gamma \to \Theta,\ A}\,(\to c)$$

Every sequent calculus which we treat in this paper enjoys the following stipulation.

STIPULATION 1. *The sequent calculus has $A \to A$ as an initial sequent for every A, and contains the structural rules $(\to w)$, $(w \to)$, $(\to e)$, $(e \to)$, $(\to c)$, and $(c \to)$.*

Due to this in the rest of this paper, we recognize Γ, Δ, Θ, $\Lambda, \cdots$ as finite sets.

DEFINITION 2.1. (Takano [3, Definition 1.1]) Let GL be a sequent calculus with Stipulation 1. The sequent $\Gamma \to \Theta$ is *analytically saturated* in GL, iff the following properties hold.

(a) $\Gamma \to \Theta$ is unprovable in GL.
(b) Suppose $A \in \mathrm{Sf}(\Gamma \cup \Theta)$. If $A,\ \Gamma \to \Theta$ is unprovable in GL, then $A \in \Gamma$; while if $\Gamma \to \Theta,\ A$ is unprovable in GL, then $A \in \Theta$.

The set of all analytically saturated sequents is denoted by W_{GL}.

We denote the analytically saturated sequents by u, v, $w, \cdots$, and denote the antecedent and succedent of u by $a(u)$ and $s(u)$, respectively.

LEMMA 2.2. (Takano [3, Lemma 1.3]) *For a sequent calculus GL with Stipulation 1, if the sequent $\Gamma \to \Theta$ is unprovable in GL, then there is an analytically saturated sequent u with the following properties;*

(*i*) $\Gamma \subseteq a(u)$ *and* $\Theta \subseteq s(u)$
(*ii*) $a(u) \cup s(u) \subseteq \mathrm{Sf}(\Gamma \cup \Theta)$

DEFINITION 2.3. (Takano [3, Definition 1.5]) An inference is *admissible* in a sequent calculus GL, iff either some of the upper sequents of the inference is unprovable in GL, or the lower one in provable in GL.

For a sequent calculus GL with Stipulation 1, there are relationships between properties of analytically saturated sequents and inferences which are admissible in GL. For example, we consider the following inferences.

$$\frac{\Gamma \to \Theta,\ A}{\neg A,\ \Gamma \to \Theta}\,(\neg \to) \qquad \frac{A,\ \Gamma \to \Theta}{\Gamma \to \Theta,\ \neg A}\,(\to \neg)$$

$$\frac{\Gamma \to \Theta,\ A \qquad B,\ \Gamma \to \Theta}{A \supset B,\ \Gamma \to \Theta}\,(\supset\to) \quad \frac{A,\ \Gamma \to \Theta,\ B}{\Gamma \to \Theta,\ A \supset B}\,(\to\supset)$$

$$\frac{\Gamma \to \Theta,\ C \qquad C,\ \Delta \to \Lambda}{\Gamma, \Delta \to \Theta, \Lambda}\,(cut)^a \quad \textit{where } C \in \mathrm{Sf}(\Gamma \cup \Theta \cup \Delta \cup \Lambda)$$

Note that inference rule $(cut)^a$ is obtained from (cut) by applying appropriate restriction.

PROPOSITION 2.4. (Takano [3, Proposition 1.6]) *For a sequent calculus GL with Stipulation 1, the following equivalences hold for every A and B.*

(1) *The inference $(\neg \to)$ is admissible in GL for every Γ and Θ, iff $\neg A \in a(u)$ implies $A \in s(u)$ for every u.*
(2) *The inference $(\to \neg)$ is admissible in GL for every Γ and Θ, iff $\neg A \in s(u)$ implies $A \in a(u)$ for every u.*
(3) *The inference $(\supset\to)$ is admissible in GL for every Γ and Θ, iff $A \supset B \in a(u)$ implies $A \in s(u)$ or $B \in a(u)$ for every u.*
(4) *The inference $(\to\supset)$ is admissible in GL for every Γ and Θ, iff $A \supset B \in s(u)$ implies $A \in a(u)$ and $B \in s(u)$ for every u.*

PROPOSITION 2.5. (Takano [3, Proposition 3.1]) *For a sequent calculus GL with Stipulation 1, the inference $(cut)^a$ is admissible for every $\Gamma,\ \Theta,\ \Delta,\ \Lambda$, and C with the restriction that $C \in \mathrm{Sf}(\Gamma \cup \Theta \cup \Delta \cup \Lambda)$, iff $\mathrm{Sf}(a(u) \cup s(u)) \subseteq a(u) \cup s(u)$.*

We introduce Stipulation 2 as well.

STIPULATION 2. *The sequent calculus contains $(\neg \to)$, $(\to \neg)$, $(\supset\to)$, and $(\to\supset)$ as inference rules.*

The aim of introducing analytically saturated sequents is obtaining the proof of Kripke completeness.

LEMMA 2.6. (Takano [3, Proposition 1.4]) *Let GL be a sequent calculus with Stipulation 1. Suppose that (W, R) is a Kripke frame such that $W \subseteq W_{GL}$, and the following properties hold for every $A,\ B$ and every $u \in W$;*

$(\neg - a)$ $\neg A \in a(u)$ *implies* $A \in s(u)$.
$(\neg - s)$ $\neg A \in s(u)$ *implies* $A \in a(u)$.
$(\supset -a)$ $A \supset B \in a(u)$ *implies* $A \in s(u)$ *or* $B \in a(u)$.

$(\supset -s)$ $A \supset B \in s(u)$ *implies* $A \in a(u)$ *and* $B \in s(u)$.

$(\Box - a)$ $\Box A \in a(u)$ *implies* $A \in a(v)$ *for every* $v \in W$ *such that* uRv.

$(\Box - s)$ $\Box A \in s(u)$ *implies* $A \in s(v)$ *for some* $v \in W$ *such that* uRv.

Let $\models$ *be the satisfaction relation on* (W, R) *such that* $u \models p$ *iff* $p \in a(u)$ *for every* $u \in W$ *and every* p. *Then for every* C *and every* $u \in W$, *if* $C \in a(u)$ *then* $u \models C$; *while if* $C \in s(u)$ *then* $u \not\models C$.

The proof of this lemma is given by induction on the construction of C.

For a sequent calculus GL with Stipulation 1, assume that any $u \in W_{GL}$ has a Kripke frame (W, R) which satisfies following properties.

- $u \in W \subseteq W_{GL}$.
- (W, R) meets conditions of Lemma 2.6.
- the accessibility relation R meets the condition of Kripke frame for L.

Then, if $\Gamma \to \Theta$ is unprovable in GL, there is an analytically saturated sequent u such that $\Gamma \subseteq a(u)$ and $\Theta \subseteq s(u)$ by Lemma 2.2. And u has a Kripke frame (W, R) which satisfies the above properties. Adding satisfaction relation $\models$ introduced in Lemma 2.6, we obtain Kripke model $(W, R, \models)$ in which $C \in \Gamma$ implies $u \models C$ and $C \in \Theta$ implies $u \not\models C$. This leads to Kripke completeness of GL.

The key point is whether every $u \in W_{GL}$ has such Kripke frame or not. It depends on admissibility of inferences in GL. From Proposition 2.4, for any Kripke frame of sequent calculus GL with Stipulation 1 and 2 holds $(\neg - a)$, $(\neg - s)$, $(\supset -a)$, and $(\supset -s)$. The remaining conditions $(\Box - a)$ and $(\Box - s)$ depend not only on admissibility of inferences, but also on properties of accessibility relation. We will discuss them in the remaining sections.

3. The logics K4.3 and KD4.3

Modal logic **K4.3** is obtained from the least normal logic **K** by adding axioms $\Box p \supset \Box\Box p$ and $\Box((p \wedge \Box p) \supset q) \vee \Box((q \wedge \Box q) \supset p)$. Kripke frame (W, R) meets condition of **K4.3** iff the frame is transitive and *weakly connected*; the Kripke frame is said to be weakly connected, if a binary relation R enjoys the following condition.

$$\forall u, v, w(uRv \textit{ and } uRw \ \Rightarrow \ vRw \textit{ or } u = w \textit{ or } wRv)$$

Modal logic **KD4.3** is obtained from **K4.3** by adding axiom $\Box p \supset \neg\Box\neg p$. In this section, we consider the inference rule for **K4.3** introduced by Shimura [1].

Definition 3.1. Suppose that $\Delta \neq \emptyset$. $P(\Delta)$ is defined as the set of all pairs (Σ, Λ) with following properties:

(1) $\Sigma \cup \Lambda = \Delta$ and $\Sigma \cap \Lambda = \emptyset$,

(2) $\Lambda \neq \emptyset$.

For example, if $\Delta = \{A,\ B\}$, then

$$P(\Delta) = \{(\{A\}, \{B\}),\ (\{B\}, \{A\}),\ (\emptyset, \{A, B\})\}$$

The inference rule ($\Box$4.3) is defined as follows:

$$\frac{\{\Gamma,\ \Box\Gamma \to \Box\Sigma,\ \Lambda |\ (\Sigma,\ \Lambda) \in P(\Delta)\}}{\Box\Gamma \to \Box\Delta}\ (\Box 4.3)$$

If $\Delta = \{A, B\}$, ($\Box$4.3) is of the form:

$$\frac{\Gamma,\ \Box\Gamma \to \Box A,\ B \qquad \Gamma,\ \Box\Gamma \to \Box B,\ A \qquad \Gamma,\ \Box\Gamma \to A,\ B}{\Box\Gamma \to \Box A,\ \Box B}\ (\Box 4.3)$$

Sequent calculus G(**K4.3**) is obtained from Gentzen's original LK by adding inference rule ($\Box$4.3). Shimura proved that G(**K4.3**) satisfies cut elimination using the syntactic method.

Restricted sequent calculi $G(\mathbf{K4.3})^-$ and $G(\mathbf{KD4.3})^-$ are obtained from LK by replacing (cut) with $(cut)^a$ and adding following rules.

	Rules	Condition on relations
$G(\mathbf{K4.3})^-$	LK with $(cut)^a$, ($\Box$4.3)	transitive and weakly connected
$G(\mathbf{KD4.3})^-$	LK with $(cut)^a$, ($\Box$4.3), $(4D)$	transitive, serial, and weakly connected

Where $(4D)$ is

$$\frac{\Gamma, \Box\Gamma \to}{\Box\Gamma \to}\ (4D).$$

We can prove their Kripke completeness by modifying Takano's method as follows. Note that the condition on Kripke frame is not equivalent to admissibility of ($\Box$4.3). (See Section 5.)

Definition 3.2. For a sequent calculus GL with Stipulation 1, the binary relation $R_{\mathbf{K4}}$ on W_{GL} is defined by: $uR_{\mathbf{K4}}v$, iff $\Box B \in a(u)$ implies B, $\Box B \in a(v)$ for every B.

From this definition, it follows that for every nonempty set $W \subseteq W_{GL}$, Kripke frame $(W, R_{\mathbf{K4}})$ is transitive and meets $(\Box - a)$.

Proposition 3.3. *Let GL be a sequent calculus with Stipulation 1 and the inference rule $(cut)^a$. If $(\Box 4.3)$ is admissible in GL for every Γ and Δ $(\Delta \neq \emptyset)$, then for every $u \in W_{GL}$, there is a finite set $W \subseteq W_{GL}$ with the following properties.*

- *(i)* $u \in W$
- *(ii)* *Kripke frame $(W,\ R_{\mathbf{K4}})$ enjoys the property $(\Box - a)$ and $(\Box - s)$, and meets condition for* **K4.3***.*

Proof: Suppose $u \in W_{GL}$. We construct analytically saturated sequents $v_1, ..., v_n$ as follows.

- $v_1 := u$
- Suppose that $v_1, \cdots, v_k$ are constructed. Put Γ_k, Θ_k, L_k and Δ_k as follows:

$$\Gamma_k = \{B \mid \Box B \in a(v_k)\},\ \Theta_k = \{B \mid \Box B \in s(v_k)\},$$

$$L_k = \{B \in \Theta_k \mid \exists w \in \{v_1, ..., v_k\}\ s.t.\ v_k R_{\mathbf{K4}} w \ and\ B \in s(w)\},$$

$$\Delta_k = \Theta_k \setminus L_k.$$

We have two cases: $\Delta_k \neq \emptyset$ and $\Delta_k = \emptyset$.
Case (1): $\Delta_k \neq \emptyset$. By following procedure, we construct the analytically saturated v_{k+1} which satisfies $v_k R_{\mathbf{K4}} v_{k+1}$ and $B \in s(v_{k+1})$ for some $B \in \Delta_k$. Since $\Box\Gamma_k \to \Box\Delta_k$ is unprovable, $\Gamma_k, \Box\Gamma_k \to \Box\Sigma, \Lambda$ is unprovable for some $(\Sigma, \Lambda) \in P(\Delta_k)$. So, $\Gamma_k \cup \Box\Gamma_k \subseteq a(v)$, $\Box\Sigma \cup \Lambda \subseteq s(v)$ and $a(v) \cup s(v) \subseteq \mathrm{Sf}(\Gamma_k \cup \Box\Gamma_k \cup \Box\Sigma \cup \Lambda)$ for some v by Lemma 2.2. Put $v_{k+1} := v$, it is clear that $v_k R_{\mathbf{K4}} v_{k+1}$ and $B \in s(v_{k+1})$ for some $B \in \Delta_k$. Furthermore, $v_{k+1} \notin \{v_1, ..., v_k\}$. (Suppose $v_{k+1} \in \{v_1, ..., v_k\}$. Since $v_k R_{\mathbf{K4}} v_{k+1}$ and $\Lambda \subseteq s(v_{k+1})$, $\Lambda \subseteq L_k$ would follow, which is a contradiction.) Note that for all $\Box B \in s(v_k)$, $B \in s(v_{k+1})$, $\Box B \in s(v_{k+1})$, or $B \in L_k$ is satisfied.
Case (2): $\Delta_k = \emptyset$. Stop the construction.

To prove that this construction stops with finite steps, we will show that $\Delta_{k+1} \subsetneq \Delta_k$ or $\Theta_{k+1} \subsetneq \Theta_k$ holds. Since $a(v) \cup s(v) \subseteq \mathrm{Sf}(\Gamma_k \cup \Box\Gamma_k \cup \Box\Sigma \cup \Lambda) \subseteq \mathrm{Sf}(a(v_k) \cup s(v_k))$, it follows $a(v_{k+1}) \cup s(v_{k+1}) \subseteq a(v_k) \cup s(v_k)$ by $(cut)^a$. It is clear that $\Gamma_k \subseteq \Gamma_{k+1}$, so $\Theta_{k+1} \subseteq \Theta_k$. Suppose that $\Theta_{k+1} = \Theta_k$. We will derive $\Delta_{k+1} \subsetneq \Delta_k$ in this case. Since $a(v_{k+1}) \cup s(v_{k+1}) \subseteq a(v_k) \cup s(v_k)$ and $\Gamma_k \subseteq \Gamma_{k+1}$, it follows $\Gamma_k = \Gamma_{k+1}$. This implies that if $v_k R_{\mathbf{K4}} w$, then $v_{k+1} R_{\mathbf{K4}} w$ for any $w \in W_{GL}$, so $L_k \subseteq L_{k+1}$. Moreover, since $v_k R_{\mathbf{K4}} v_{k+1}$, it follows $v_{k+1} R_{\mathbf{K4}} v_{k+1}$ and $\Lambda \subseteq L_{k+1}$, although $\Lambda \not\subseteq L_k$. This implies $L_k \subsetneq L_{k+1}$; hence $\Delta_{k+1} \subsetneq \Delta_k$.

There is an analytically saturated sequent v_n with $\Delta_n = \emptyset$ by repeating this procedure. Put $W = \{v_1, \cdots, v_n\}$, it is clear that Kripke frame $(W, R_{\mathbf{K4}})$ is transitive and weakly connected frame, and enjoys $(\Box - a)$. If $\Box B \in s(v_n)$, then $B \in L_n$ since $\Delta_n = \emptyset$. So $(W, R_{\mathbf{K4}})$ enjoys $(\Box - s)$. $\Box$

From the above proposition, we can show Kripke completeness for $G(\mathbf{K4.3})^-$, and this leads to subformula property for $G(\mathbf{K4.3})$. Furthermore, this leads to finite model property simultaneously because the constructed model is finite.

Similarly, $G(\mathbf{KD4.3})$ has subformula property and finite model property.

Lemma 3.4. *Let GL be a sequent calculus with Stipulation 1 and the inference rule $(cut)^a$. If $(\Box 4.3)$ and $(4D)$ are admissible in GL for every Γ and Δ $(\Delta \neq \emptyset)$, then for every $u \in W_{GL}$, there is a finite set $W \subseteq W_{GL}$ with the following properties.*

- *(i)* $u \in W$
- *(ii)* *Kripke frame $(W,\ R_{\mathbf{K4}})$ enjoys the property $(\Box - a)$ and $(\Box - s)$, and meets condition for* **KD4.3**.

Proof: Suppose $u \in W_{GL}$. From Proposition 3.3, there is a finite set $\{v_1, \cdots, v_n\}$ which meets the condition of Proposition 3.3 with $v_1 = u$. If v_n has $R_{\mathbf{K4}}$ successor in $\{v_1, \cdots, v_n\}$, then the set is the desired one. If not so, we construct the analytically saturated v_{n+1} by following procedure.

Put Γ_n and Θ_n same as Proposition 3.3. It is clear that $\Theta_n = \emptyset$ and $\Gamma_n \neq \emptyset$. (Suppose otherwise $\Gamma_n = \emptyset$. Then, all analytically saturated sequents of W_{GL} are $R_{\mathbf{K4}}$ successors of v_n. This is a contradiction.) Since $\Box\Gamma_n \rightarrow$ is unprovable, $\Gamma_n, \Box\Gamma_n \rightarrow$ is unprovable. Then, by Lemma 2.2, $\Gamma_n,\ \Box\Gamma_n \subseteq a(v)$ and $a(v) \cup s(v) \subseteq \mathrm{Sf}(\Box\Gamma_n)$ for some v. Put $v = v_{n+1}$, it is clear $v_n R_{\mathbf{K4}} v_{n+1}$. Furthermore, since GL has inference rule $(cut)^a$, $a(v_{n+1}) \cup s(v_{n+1}) \subseteq a(v_n) \cup s(v_n)$. So, $s(v_{n+1})$ has no $\Box$-formulas. (Suppose

$\Box B \in s(v_{n+1})$, this implies $\Box B \in a(v_n)$ or $\Box B \in s(v_n)$. Since $\Theta_n = \emptyset$, $\Box B \in a(v_n)$. It follows $\Box B \in \Box\Gamma_n \subseteq a(v_{n+1})$, which is a contradiction.) Similarly, suppose $\Box B \in a(v_{n+1})$, it follows $B \in a(v_{n+1})$, and this implies $v_{n+1} R_{\mathbf{K4}} v_{n+1}$. Thus, $W = \{v_1, \cdots, v_n, v_{n+1}\}$ meets the conditions. $\Box$

4. The logic S4.3

Modal logic **S4.3** is obtained form **K4.3** by adding axiom $\Box p \supset p$. Kripke frame (W, R) meets condition of **S4.3** iff the frame is transitive, weakly connected, and reflexive. Shimura [1] also introduced inference rule for **S4.3**.

$$\frac{\{\Box\Gamma \to \Box(\Delta \setminus \{A\}),\ A |\ A \in \Delta\}}{\Box\Gamma \to \Box\Delta}\ (S4.3)$$

If $\Delta = \{A, B\}$, ($\Box$4.3) is of the form:

$$\frac{\Box\Gamma \to \Box A,\ B \qquad \Box\Gamma \to \Box B,\ A}{\Box\Gamma \to \Box A,\ \Box B}\ (S4.3)$$

Sequent calculus $G(\mathbf{S4.3})$ is obtained from LK by adding inference rule $(S4.3)$ and (T).

$$\frac{A,\ \Gamma \to \Theta}{\Box A, \Gamma \to \Theta}\ (T)$$

Shimura proved that $G(\mathbf{S4.3})$ satisfies cut elimination using the syntactic method.

Restricted sequent calculus $G(\mathbf{S4.3})^-$ is obtained from LK by replacing (cut) with $(cut)^a$ and adding the following rules.

	Rules	Condition on relations
$G(\mathbf{S4.3})^-$	LK with $(cut)^a$, $(S4.3)$, (T)	transitive, reflexive and weakly connected

We can prove its Kripke completeness by modifying Takano's method as follows.

Definition 4.1. For a sequent calculus GL with Stipulation 1, the binary relation $R_{\mathbf{S4}}$ on W_{GL} is defined by: $uR_{\mathbf{S4}}v$, iff $\Box B \in a(u)$ implies $\Box B \in a(v)$ for every B.

By this definition, for every nonempty set $W \subseteq W_{GL}$, Kripke frame $(W, R_{\mathbf{S4}})$ is transitive and reflexive.

Proposition 4.2. *Let GL be a sequent calculus with Stipulation 1 and the inference rule $(cut)^a$. If $(S4.3)$ is admissible in GL for every Γ and Δ $(\Delta \neq \emptyset)$, then for every $u \in W_{GL}$, there is a finite set $W \subseteq W_{GL}$ with the following properties.*

(i) $u \in W$

(ii) *Kripke frame* $(W,\ R_{\mathbf{S4}})$ *enjoys the property* $(\Box - s)$, *and meets condition for* **S4.3**.

The proof is similar to Proposition 3.3. Note that the Kripke frame constructed by the above proposition does not enjoy $(\Box - a)$. If GL has (T) as inference rule, then the constructed model enjoys $(\Box - a)$ by following lemma.

Lemma 4.3. *Let GL be a sequent calculus with Stipulation 1. If the inference (T) is admissible in GL for every A, Γ, and Θ, then Kripke frame $(W, R_{\mathbf{S4}})$ holds the property $(\Box - a)$ for every $W \subseteq W_{GL}$.*

Proof: Suppose that $u \in W$. If $\Box B \in a(u)$, then $uR_{\mathbf{K4}}v$ implies $\Box B \in a(v)$ for every $v \in W$. Since $\Box B, a(v) \to s(v)$ is unprovable, we have that $B, a(v) \to s(v)$ is unprovable by applying rule (T). Hence, $B \in a(v)$. □

We can show Kripke completeness of $G(\mathbf{S4.3})^-$ by Propositions 4.2 and 4.3. This implies not only subformula property, but also finite model property of $G(\mathbf{S4.3})$.

5. Concluding remark

In this paper, we gave alternative proofs of Kripke completeness, subformula property and finite model property for **K4.3**, **KD4.3** and **S4.3** by modifying Takano's method in [3].

Takano's method in [3] was developed originally to analyze relationships between admissibility of acceptable inference rules and semantical properties. Then, by using these relationships, he showed Kripke completeness of some modal logics as well. But, the straightforward application of Takano's method does not work well for $(\Box 4.3)$ and $(S4.3)$. Takano's method is useful to prove Kripke completeness, but has limitations. Let us explain this with examples. We consider the following inference.

$$\frac{\Gamma,\ \Box\Gamma \to A}{\Box\Gamma \to \Box A}\ (4) \qquad \frac{\Box\Gamma \to A}{\Box\Gamma \to \Box A}\ (S4)$$

PROPOSITION 5.1. (Takano [3, Proposition 2.2]) *For a sequent calculus GL with Stipulation 1, the following equivalences hold for every A.*

(*i*) *The inference* (4) *is admissible in GL for every* Γ*, iff the Kripke frame* $(W_{GL},\ R_{\mathbf{K4}})$ *enjoys the property* $(\Box - s)$.

(*ii*) *The inference* $(S4)$ *is admissible in GL for every* Γ*, iff the Kripke frame* $(W_{GL},\ R_{\mathbf{S4}})$ *enjoys the property* $(\Box - s)$.

Sequent calculi $G(\mathbf{K4})$ and $G(\mathbf{S4})$ are obtained on the basis sequent calculus with Stipulation 1 and 2 by adding the following inference rules, respectively.

	Additional rules	Condition on relations
$G(\mathbf{K4})$	(4)	transitive
$G(\mathbf{S4})$	$(S4)$, (T)	transitive and reflexive

From Proposition 5.1 and Lemma 4.3, it follows that $(W_{G(\mathbf{K4})}, R_{\mathbf{K4}})$ and $(W_{G(\mathbf{S4})}, R_{\mathbf{S4}})$ meet conditions in Lemma 2.6. Hence, we have Kripke completeness for $G(\mathbf{K4})$ and $G(\mathbf{S4})$.

In this way, we can show Kripke completeness of sequent calculi for some modal logics by using the conditions of Kripke frame which are equivalent to admissibility of their inferences.

On the other hands, we cannot deal with $(\Box 4.3)$ and $(S4.3)$ in a similar way, although the following Propositions 5.2 and 5.3 give conditions of Kripke frames which are equivalent to admissibility of $(\Box 4.3)$ and $(S4.3)$.

PROPOSITION 5.2. *For a sequent calculus GL with Stipulation 1, the following conditions are equivalent for every nonempty set* Δ.

(*i*) *The inference* $(\Box 4.3)$ *is admissible in GL for every* Γ.

(*ii*) *For every* $u \in W_{GL}$*, if* $\Box\Delta \subseteq s(u)$*, then there is an analytically saturated sequent v with the following properties.*

- $*$ $uR_{\mathbf{K4}}v$
- $*$ $\forall B \in \Delta,\ B \in s(v)$ *or* $\Box B \in s(v)$
- $*$ $\exists B \in \Delta$ *s.t.* $B \in s(v)$

PROOF: $(\Rightarrow)$ Suppose that $\Box\Delta \subseteq s(u)$. Put $\Gamma = \{B \mid \Box B \in a(u)\}$. $\Box\Gamma \to \Box\Delta$ is unprovable in GL. Since $(\Box 4.3)$ is admissible in GL, $\Gamma, \Box\Gamma \to \Box\Sigma, \Lambda$ is unprovable for some $(\Sigma, \Lambda) \in P(\Delta)$. By Lemma 2.2, we have $\Gamma, \Box\Gamma \subseteq a(v)$ and $\Box\Sigma, \Lambda \subseteq s(v)$ for some v. It is clear that $uR_{\mathbf{K4}}v$. Since $\Lambda \neq \emptyset$, v satisfies remaining properties.

($\Leftarrow$) Take a finite set Γ such that $\Box\Gamma \to \Box\Delta$ is unprovable. By Lemma 2.2, $\Box\Gamma \subseteq a(u)$, $\Box\Delta \subseteq s(u)$ and $a(u) \cup s(u) \subseteq \mathrm{Sf}(\Box\Gamma \cup \Box\Delta)$ for some u. Since $\Box\Delta \subseteq s(u)$, there is an analytically saturated v which satisfies properties. Note that $uR_{\mathbf{K4}}v$ leads to $\Gamma, \Box\Gamma \subseteq a(v)$. Put Λ and Σ as follows:

$$\Lambda = \{B \in \Delta \mid B \in s(v)\},$$

$$\Sigma = \Delta \setminus \Lambda.$$

It is clear that $(\Sigma, \Lambda) \in P(\Delta)$ since $\Lambda \neq \emptyset$ by the third condition. Note that $\Gamma,\ \Box\Gamma \to \Box\Sigma,\ \Lambda$ is one of upper sequents of $\Box\Gamma \to \Box\Delta$. Therefore ($\Box$4.3) is admissible for this Γ. $\Box$

Proposition 5.3. *For a sequent calculus GL with Stipulation 1, the following conditions are equivalent for every nonempty set* Δ.

(*i*) *The inference* (S4.3) *is admissible in GL for every* Γ.

(*ii*) *For every* $u \in W_{GL}$, *if* $\Box\Delta \subseteq s(u)$, *then there is an analytically saturated sequent* v *with the following properties.*

- $uR_{\mathbf{S4}}v$
- $\exists B \in \Delta$ *s.t.* $B \in s(v)$ *and* $\Box(\Delta \setminus \{B\}) \subseteq s(v)$

By the above propositions, we can show that if GL with Stipulation 1 has ($\Box$4.3) or (S4.3), then $(W_{GL},\ R_{\mathbf{K4}})$ or $(W_{GL},\ R_{\mathbf{S4}})$ enjoys $(\Box - s)$ respectively. But these Kripke frames are not weakly connected. Thus, we cannot use these conditions to the proof of Kripke completeness of the calculi with ($\Box$4.3) and (S4.3). So, we extended Takano's method and established our results in this paper.

As of now, we do not have the Kripke frame condition suitable for the proof of Kripke completeness and for the weak connectedness. In order to obtain the condition, author expects that it is necessary to improve the definitions of analytically saturated or binary relation.

Acknowledgement. The author thanks Professors N.-Y. Suzuki and M. Takano for their comments. He also thanks an anonymous referee for helpful suggestions.

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Bulletin of the **S**ection of **L**ogic
Volume 48/4 (2019), pp. 259–283

http://dx.doi.org/10.18778/0138-0680.48.4.02

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AN INVESTIGATION INTO INTUITIONISTIC LOGIC WITH IDENTITY

Abstract

We define Kripke semantics for propositional intuitionistic logic with Suszko's identity (ISCI). We propose sequent calculus for ISCI along with cut-elimination theorem. We sketch a constructive interpretation of Suszko's propositional identity connective.

Keywords: non-Fregean logics, intuitionistic logic, admissibility of cut, propositional identity, congruence.

Introduction

In this paper we propose a constructive interpretation of Suszko's propositional identity operator [10, 1] along with a sequent calculus for the logic ISCI. The name 'ISCI' was introduced in [5]; however, already in [1] the authors, Bloom and Suszko, noted that SCI can be modified by taking intuitionistic logic as a base. ISCI is an extension of the propositional intuitionistic logic by a set of axioms which characterizes propositional identity operator '$\approx$'.

The strongest connective of classical propositional logic that may be used to express *sameness of situations* is the equivalence connective. But

*Corresponding author. During the work on this paper the first author benefited from the financial support of National Science Centre, Poland, grant no. 2015/17/B/HS1/02232.

**The work of the second author was supported financially by National Science Centre, Poland, grant no. 2017/26/E/HS1/00127.

the question of equivalence of two formulas reduces to the question whether the formulas have the same logical value. This is not the case with the propositional identity – two formulas may be equivalent yet not identical in Suszko's sense. The philosophical motivation behind SCI was related to the ontology of situations – in classical logic, there are only two situations: *Truth* and *Falsity*, and the Truth (Falsity) is described by any true (false) proposition. According to Suszko, this is unfortunate, and could be remedied by allowing new *identity* connective, which describes the fact that two propositions describe the same situation. From this point of view, SCI can be considered as a generalization of classical logic in which we assume that there are *at least* two different situations.

The language of intuitionistic propositional logic also has the equivalence connective, and we can ask, again, whether the connective is suitable to express sameness of situations. In intuitionistic terms, we are not interested in propositions being true or false but in *constructions* which prove them. Equivalence of two formulas, A and B, means that whenever A is provable, B is provable as well, and *vice versa*. But we can still think of a stronger notion which says that the classes of constructions proving A and B are *exactly* the same. As we shall see, this is the intended interpretation of the identity connective on the grounds of intuitionistic logic. Thus also in the intuitionistic setting, the identity connective gains an interpretation stronger than that of equivalence.

1. Intuitionistic logic and Suszko's identity

1.1. BHK-interpretation and propositional identity

Here is a version of the BHK-interpretation of logical constants. The last row depicts the first author's original interpretation of the propositional identity connective in constructive environment.

there is no proof of $\perp$	
a is a proof of $A \wedge B$	$a = (a_1, a_2)$; a_1 is a proof of A and a_2 is a proof of B
a is a proof of $A \vee B$	$a = (a_1, a_2)$; $a_1 = 0$ and a_2 is a proof of A or $a_1 = 1$ and a_2 is a proof of B

a is a proof of $A \supset B$	a is a construction that converts each proof a_1 of A into a proof $a_2(a_1)$ of B
a is a proof of $A \approx B$	a is a construction which shows that the classes of proofs of A and B are the same

In the present context a formula A may be thought of as an expression which represents the set of its own proofs. If A is identical to B, then A and B represent the same set. A construction, which can be used in establishing that the classes of proofs of two formulas are the same is the identity function $\lambda x.x$. We do not claim, however, that the identity function is the only object belonging to the identity type, but only that it is a natural choice to use this construction to intuitively validate identity axioms.

1.2. Hilbert-style system for ISCI

The language $\mathcal{L}_{\mathsf{ISCI}}$ of the logic ISCI is defined by the following grammar:

$$A \ ::= \ V \mid \bot \mid A \wedge A \mid A \vee A \mid A \supset A \mid A \approx A$$

where V is a denumerable set of propositional variables. Intuitionistic negation '$\sim A$' is defined as '$A \supset \bot$'. Sometimes we will call formulas of the form '$A \approx B$' *equations*. The axiom system for ISCI is obtained from any such system for INT, for example that from Table 1 (quoted after [4]) by the addition of $\approx$-specific axioms following under the four schemes ($\approx_1$)-($\approx_4$), where $\otimes \in \{\wedge, \vee, \supset, \approx\}$.

The presented axiom system for ISCI is called '$\mathsf{H_{ISCI}}$'. By '$S \vdash_{\mathsf{H_{ISCI}}} A$' we mean that A is *derivable in* $\mathsf{H_{ISCI}}$ by means of axioms and formulas in S, where *derivability* is understood in a standard manner. If $\emptyset \vdash_{\mathsf{H_{ISCI}}} A$, then we will say that A is a *thesis of* $\mathsf{H_{ISCI}}$. (Here is an example of a thesis of $\mathsf{H_{ISCI}}$ other than an axiom: $\bot \supset \bot$. We shall use it in Lemma 3.)

If $A \approx B$ holds, we will say that A and B are identical. By the symbol $|A|$ we denote the class of proofs of A. Let us note that axioms of Suszko's identity are valid under the interpretation proposed in Subsection 1.1.

($\approx_1$) Naturally, $|A| = |A|$.

($\approx_2$) Assume that $|A| = |B|$. In this case, a construction that converts any element of $|A|$ into a proof of $\bot$ is a construction that converts any element of $|B|$ into a proof of $\bot$. Therefore there exists a function which transforms each proof of the identity of A and B into a proof of the identity of $A \supset \bot$ and $B \supset \bot$.

Table 1. Axioms of intuitionistic logic INT

$\mathbf{H}_1$	$A \supset (B \supset A)$
$\mathbf{H}_2$	$(A \supset B) \supset ((A \supset (B \supset C)) \supset (A \supset C))$
$\mathbf{H}_3$	$A \supset (B \supset (A \wedge B))$
$\mathbf{H}_4$	$(A \wedge B) \supset A$
$\mathbf{H}_5$	$(A \wedge B) \supset B$
$\mathbf{H}_6$	$(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$
$\mathbf{H}_7$	$A \supset (A \vee B)$
$\mathbf{H}_8$	$B \supset (A \vee B)$
$\mathbf{H}_9$	$(A \supset B) \supset ((A \supset \sim B) \supset \sim A)$
$\mathbf{H}_{10}$	$\sim A \supset (A \supset B)$
MP	from A and $A \supset B$ conclude B
$(\approx_1)$	$A \approx A$
$(\approx_2)$	$(A \approx B) \supset ((A \supset \bot) \approx (B \supset \bot))$
$(\approx_3)$	$(A \approx B) \supset (B \supset A)$
$(\approx_4)$	$((A \approx B) \wedge (C \approx D)) \supset ((A \otimes C) \approx (B \otimes D))$

$(\approx_3)$ If A is identical to B, then each proof of B can be transformed (by the identity function $\lambda x.x$) into a proof of A. Therefore there is a function which transforms each proof of the identity of A and B into a proof of $B \supset A$ (and $A \supset B$, but this is implied by the other conditions).

$(\approx_4)$ We shall argue that each pair (a_1, a_2), where a_1 is a proof of the identity of A and B and a_2 is a proof of the identity of C and D, can be transformed into a proof of the identity of $A \otimes C$ and $B \otimes D$. Assume $|A| = |B|$ and $|C| = |D|$ and:

(a) $\otimes = \wedge$. If a_1 is a proof of A and a_2 is a proof of C, then, by assumption, pair (a_1, a_2) constitutes a proof of B and D; and *vice versa*: if a pair proves B and D, then it proves A and C, respectively. It follows that $|A| \times |C| = |B| \times |D|$.

(b) $\otimes = \vee$. Let $(0, a_2)$ be a proof of $A \vee C$ (thus a_2 is a proof of A). Since $|A| = |B|$, $(0, a_2)$ is also a proof of $B \vee D$. For a similar reason, if $(1, a_2)$ is a proof of $A \vee C$, then it is also a proof of $B \vee D$. And *vice versa*: from $B \vee D$ to $A \vee C$.

(c) $\otimes = \supset$. Assume $\lambda x.y$ is a proof of $A \supset C$. Since $|A| = |B|$ and $|C| = |D|$ this function is also a proof of $B \supset D$ (and *vice versa*).

(d) $\otimes = \approx$. Assume $\lambda x.x$ is a proof of $A \approx C$. Since $|A| = |B|$ and $|C| = |D|$ this function also proves $B \approx D$.

The identity of formulas A and B amounts to the existence of a function showing the identity of sets $|A|$ and $|B|$. Let us note that according to the proposed interpretation, the identity connective is stronger than intuitionistic equivalence. If A and B are identical, then they are intuitionisticaly equivalent (that is, every proof of A can be transformed into a proof of B and *vice versa*). But the converse does not hold. From the fact that A and B are intuitionisticaly equivalent one cannot derive the conclusion that the function which converts proofs of A into proofs of B is the identity function $\lambda x.x$ between $|A|$ and $|B|$.

In type-theoretical terms [11, 3, 2], a formula $A \supset B$ corresponds to the type of functions which take arguments of the type A and return values of type B

$$(\lambda x^A.t^B)^{A \supset B},$$

whereas a type $A \approx B$ is certainly inhabited by identity functions

$$(\lambda x^A.x^B)^{A \approx B}.$$

Note that the set of all functions of the type $A \approx B$ is a subset of the set of all functions of the type $A \supset B$; each function of the type $A \approx B$ is also of the type $A \supset B$. Let us stress once again that identity is stronger than intuitionistic equivalence. This point becomes clear if we realise that an equation $A \approx B$ is a thesis of ISCI if and only if it represents a function: $(\lambda x^A.x^A)^{A \approx A}$, *i.e.*, when '$A$' and '$B$' is the same formula.

1.3. Semantics

An algebraic semantic for ISCI is given in [5] along with a sketch of completeness proof. Here we propose a simple semantic approach based on Kripke frames.

Definition 1 (ISCI frame). *By an* ISCI *frame we mean an ordered pair* $\boldsymbol{F} = \langle W, \leq \rangle$*, where* W *is a non-empty set and* $\leq$ *is a reflexive and transitive binary relation on* W*.*

By 'For_0' we shall mean the sum of V (the set of all variables) and the set of all equations. If $\boldsymbol{F} = \langle W, \leq \rangle$ is an ISCI frame, then by *assignment in* $\boldsymbol{F}$ we mean a function:

$$v : For_0 \times W \longrightarrow \{0, 1\}.$$

An assignment is called ISCI-*admissible*, provided that for each $w \in W$, and for arbitrary formulas A, B, C, D: (1) $v(A \approx A, w) = 1$, (2) if $v(A \approx B, w) = 1$, then $v((A \supset \bot) \approx (B \supset \bot), w) = 1$, and (3) if $v(A \approx B, w) = 1$ and $v(C \approx D, w) = 1$, then $v((A \otimes C) \approx (B \otimes D), w) = 1$. The three conditions capture $\approx$-specific axioms falling under $(\approx_1)$, $(\approx_2)$, $(\approx_4)$, respectively. The third scheme will be captured in the notion of *forcing*.

Definition 2 (Forcing). *Let v be an* ISCI*-admissible assignment in a given frame $\boldsymbol{F}$. A forcing relation $\Vdash$ determined by v in $\boldsymbol{F}$ is a relation between elements of W and elements of $\mathcal{L}_{\mathsf{ISCI}}$ which satisfies, for arbitrary $w \in W$, the following conditions:*

(1) $w \Vdash p_i$ *iff* $v(p_i, w) = 1$;

(2) $w \nVdash \bot$;

(3) $w \Vdash A \wedge B$ *iff* $w \Vdash A$ *and* $w \Vdash B$;

(4) $w \Vdash A \vee B$ *iff* $w \Vdash A$ *or* $w \Vdash B$;

(5) $w \Vdash A \supset B$ *iff for each* w' *such that* $w \leq w'$, *if* $w' \Vdash A$ *then* $w' \Vdash B$;

(mon) *if* $w \Vdash p_j$ *and* $w \leq w'$, *then* $w' \Vdash p_j$;

($mon_{\approx}$) *if* $w \Vdash A \approx B$ *and* $w \leq w'$, *then* $w' \Vdash A \approx B$;

($\approx$) *if* $w \Vdash A \approx B$, *then* $w \Vdash B \supset A$.

Note that the condition (mon) can be strengthened to:

(mon') where A is not an equation, if $w \Vdash A$ and $w \leq w'$, then $w' \Vdash A$,

and combined with ($mon_{\approx}$), the conditions yield monotonicity for all formulas of $\mathcal{L}_{\mathsf{ISCI}}$.

Definition 3. *An* ISCI *model is a triple $\boldsymbol{M} = \langle W, \leq, \Vdash \rangle$, where $\boldsymbol{F} = \langle W, \leq \rangle$ is an* ISCI *frame and $\Vdash$ is a forcing relation determined by some* ISCI-*admissible assignment in $\boldsymbol{F}$.*

A formula A which is forced by every world of an ISCI *model, that is, such that $w \Vdash A$ for each $w \in W$, is called true in the model.*

A formula true in every ISCI *model is called* ISCI-*valid.*

THEOREM 1 (soundness). *Let A be a formula of $\mathcal{L}_{\mathsf{ISCI}}$. If $\emptyset \vdash_{\mathsf{H}_{\mathsf{ISCI}}} A$, then A is* ISCI*-valid.*

PROOF: Since the semantics for ISCI is based on Kripke semantics for INT, we omit most of the proof. One fact worth noting is that showing that axioms $\mathbf{H}_1$ and $\mathbf{H}_3$ are ISCI-valid requires not only (*mon*), but ($mon_{\approx}$) as well. The argument for $\mathbf{H}_1$ goes as follows. Assume that $\mathbf{H}_1 : A \supset (B \supset A)$ is not true in some ISCI model $\boldsymbol{M} = \langle W, \leq, \Vdash \rangle$. Then there is $w \in W$ such that: $w \nVdash A \supset (B \supset A)$. Hence, there is a world w^1 available from w, such that $w^1 \Vdash A$, but $w^1 \nVdash B \supset A$. Again, there is a world w^2 visible from w^1 such that $w^2 \Vdash B$ and $w^2 \nVdash A$. But since we have proved A at w^1 and the next step is w^2, A is also proved at w^2 (monotonicity), which results in a contradiction. However, in the last sentence we cannot rely on (*mon*) only, since A can be an equation.

Further, the three conditions in the definition of ISCI-admissible assignment and condition ($\approx$) in the definition of forcing warrant that the $\approx$-specific axioms are ISCI-valid. Needless to say, **MP** preserves ISCI-validity, therefore each thesis of $\mathsf{H}_{\mathsf{ISCI}}$ is ISCI-valid. □

1.4. Discussion

Independently of the solutions which we have adapted in the previous subsection, it seems worth to consider the idea of monotonicity of identity, *i.e.*, the condition:

($mon_{\approx}$) if $w \Vdash A \approx B$ and $w \leq w'$, then $w' \Vdash A \approx B$.

First of all, ($mon_{\approx}$) does not follow by induction from (*mon*). The reason is that the components of identity may be both true at a given world, but a formula which states their identity may be false. Hence there are two possibilities which are worth considering:

1. We accept ($mon_{\approx}$), as we did above. There is a good reason for that from the intuitionistic viewpoint. A proof of *each* formula should be remembered, *i.e.*, if it has been proved at a given point, then it should also be provable at a later point. According to this interpretation also a proved equation $A \approx B$ remains proved, irrespective of the proofs of A and/or B available at further points (further in the sense of $\leq$).
2. We reject ($mon_{\approx}$). One can find good reasons for rejecting ($mon_{\approx}$); under the BHK-interpretation the truth of $A \approx B$ yields the existence

of a construction which shows that the classes of proofs of A and B are the same. But when we move ahead along with $\leq$, some new proofs of A and/or of B can be found, and then the classes of the appropriate proofs $|A|$ and $|B|$ may become distinct. Hence the connective of constructive identity without ($mon_{\approx}$) seems more adequate to account for *provability* as acquired by a human being.

However, rejecting ($mon_{\approx}$) yields some serious consequences. As shown in the proof of Theorem 1, in the presented setting ($mon_{\approx}$) is necessary to prove that axioms $\mathbf{H}_1$ and $\mathbf{H}_3$ are ISCI-valid. Hence if ($mon_{\approx}$) is rejected, one needs to warrant the validity of the axioms in some other way.

One the other hand, let us observe that ($mon_{\approx}$) is neither necessary nor sufficient in proving that the $\approx$-specific axioms are ISCI-valid. In the presented setting this is warranted in the notion of ISCI-admissibility and in the additional condition ($\approx$) that *forcing* must preserve.

1.5. Completeness of Hilbert-style system for ISCI

Here we give a Henkin-style completeness proof of $\mathsf{H}_{\mathsf{ISCI}}$ with respect to the presented semantics.

Let S and F stand for a set of formulas and a single formula of $\mathcal{L}_{\mathsf{ISCI}}$, respectively. We will say that S is F-$\mathsf{H}_{\mathsf{ISCI}}$-*consistent* iff $S \nvdash_{\mathsf{H}_{\mathsf{ISCI}}} F$; otherwise S is called F-$\mathsf{H}_{\mathsf{ISCI}}$-*inconsistent*. S is called *maximally* F-$\mathsf{H}_{\mathsf{ISCI}}$-*consistent* iff it is F-$\mathsf{H}_{\mathsf{ISCI}}$-consistent and no proper superset of S is F-$\mathsf{H}_{\mathsf{ISCI}}$-consistent.

For simplicity, we will write 'F-(in)consistent' instead of 'F-$\mathsf{H}_{\mathsf{ISCI}}$-(in)consistent'.

Lemma 1 (Lindenbaum's lemma). *Let F stand for a formula of $\mathcal{L}_{\mathsf{ISCI}}$. For every F-consistent set S there is a maximally F-consistent set $\mathbf{S} \supseteq S$.*

Proof: Let us recall the well-known construction. We enumerate all formulas of $\mathcal{L}_{\mathsf{ISCI}}$:

$$B_1, B_2, \ldots, B_n, \ldots$$

Suppose that S is an F-consistent set, that is, $S \nvdash_{\mathsf{H}_{\mathsf{ISCI}}} F$. We construct an infinite sequence of sets by means of the following rules:

$$S_0 = S$$

$$S_{n+1} = \begin{cases} S_n & \text{if } S_n \cup \{B_{n+1}\} \vdash_{\mathsf{H_{ISCI}}} F \\ S_n \cup \{B_{n+1}\} & \text{otherwise.} \end{cases}$$

It follows from the construction that each member of this sequence is F-consistent and the set $\mathbf{S} = \bigcup_{n=0}^{\infty} S_n$ is maximally F-consistent. □

If for some formula F, a set S is (maximally) F-consistent, and the formula is irrelevant in a given context, then we will say simply that S is (maximally) *consistent*. Let us now prove:

LEMMA 2. *A formula of language $\mathcal{L}_{\mathsf{ISCI}}$ is a thesis of $\mathsf{H_{ISCI}}$ if and only if it is an element of each maximally consistent set.*

PROOF: Assume that: (a) it is the case that $\emptyset \vdash_{\mathsf{H_{ISCI}}} A$, but there is a maximally consistent set S such that (b) $A \notin S$. By definition, there is some formula F such that $S \nvdash_{\mathsf{H_{ISCI}}} F$ (that is, for some F, S is F-consistent), and by the construction of S and by (b), for some $S_n \subseteq S$, $S_n \cup \{A\} \vdash_{\mathsf{H_{ISCI}}} F$. Hence, and by deduction theorem, $S_n \vdash_{\mathsf{H_{ISCI}}} A \supset F$, and since (a) yields $S_n \vdash_{\mathsf{H_{ISCI}}} A$ (weakening), also $S_n \vdash_{\mathsf{H_{ISCI}}} F$ (by **MP**). But then $S \supseteq S_n$ is not F-consistent. A contradiction.

For the *only-if* part assume that A is not a thesis of $\mathsf{H_{ISCI}}$, that is, $\emptyset \nvdash_{\mathsf{H_{ISCI}}} A$. It follows that the empty set is A-consistent. Thus, by Lemma 1, there is a maximally A-consistent set $\mathbf{S} \supseteq \emptyset$, and hence $A \notin \mathbf{S}$. □

LEMMA 3. *Let $\mathbf{S}$ be a maximally consistent set. The following conditions are satisfied:*

1. $\perp \notin \mathbf{S}$;
2. $A \in \mathbf{S}$ *iff* $\mathbf{S} \vdash_{\mathsf{H_{ISCI}}} A$;
3. $A \wedge B \in \mathbf{S}$ *iff* $A \in \mathbf{S}$ *and* $B \in \mathbf{S}$;
4. $A \vee B \in \mathbf{S}$ *iff* $A \in \mathbf{S}$ *or* $B \in \mathbf{S}$;
5. *if* $A \supset B \in \mathbf{S}$ *and* $A \in \mathbf{S}$, *then* $B \in \mathbf{S}$;
6. *if* $A \supset B \notin \mathbf{S}$, *then* $\mathbf{S} \cup \{A\}$ *is* B*-consistent.*

PROOF: Suppose that $\mathbf{S}$ is a maximally consistent set. Then for some formula F, (a) $\mathbf{S} \nvdash_{\mathsf{H_{ISCI}}} F$.

(ad1) If $\perp \in \mathbf{S}$, then for each formula, in particular for F, we have $\mathbf{S} \vdash_{\mathsf{H_{ISCI}}} F$,[1] which contradicts (a). Hence $\perp \notin \mathbf{S}$.

[1] A justification is such that: $(\perp \supset \perp) \supset (\perp \supset F)$ is an instance of $\mathbf{H}_{10}$. Moreover, $\mathbf{S} \vdash_{\mathsf{H_{ISCI}}} \perp \supset \perp$, thus $\mathbf{S} \vdash_{\mathsf{H_{ISCI}}} \perp \supset F$ and $\mathbf{S} \vdash_{\mathsf{H_{ISCI}}} F$ by **MP**.

(ad2) The *if-then* direction holds by reflexivity of $\vdash_{\mathsf{H_{ISCI}}}$. For the *only-if* part assume that $A \notin \mathbf{S}$. Then $\mathbf{S} \cup \{A\} \vdash_{\mathsf{H_{ISCI}}} F$ by the construction of a maximally consistent set, and hence also $\mathbf{S} \vdash_{\mathsf{H_{ISCI}}} A \supset F$ (deduction theorem). If, in addition, we assumed that $\mathbf{S} \vdash_{\mathsf{H_{ISCI}}} A$, then we would obtain $\mathbf{S} \vdash_{\mathsf{H_{ISCI}}} F$ by **MP**, therefore $\mathbf{S} \nvdash_{\mathsf{H_{ISCI}}} A$.

(ad3) For the *if-then* part assume that $A \wedge B \in \mathbf{S}$. Then $\mathbf{S} \vdash_{\mathsf{H_{ISCI}}} A \wedge B$; also $\mathbf{S} \vdash_{\mathsf{H_{ISCI}}} (A \wedge B) \supset A$, since the formula is an axiom of $\mathsf{H_{ISCI}}$. Thus $\mathbf{S} \vdash_{\mathsf{H_{ISCI}}} A$ by **MP**, and, by clause 2 of this lemma, $A \in \mathbf{S}$. The reasoning is similar for $B \in \mathbf{S}$.
For the *only-if* direction assume that $A \in \mathbf{S}$ and $B \in \mathbf{S}$. Then $\mathbf{S} \vdash_{\mathsf{H_{ISCI}}} A$ and $\mathbf{S} \vdash_{\mathsf{H_{ISCI}}} B$. By using axiom $\mathbf{H}_3$ and **MP** we get $\mathbf{S} \vdash_{\mathsf{H_{ISCI}}} A \wedge B$, and finally $A \wedge B \in \mathbf{S}$ by clause 2.

(ad4) *If-then* direction: if $A \notin \mathbf{S}$ and $B \notin \mathbf{S}$, then $\mathbf{S} \vdash_{\mathsf{H_{ISCI}}} A \supset F$ and $\mathbf{S} \vdash_{\mathsf{H_{ISCI}}} B \supset F$. Then by $\mathbf{H}_6$ and **MP** we have $\mathbf{S} \vdash_{\mathsf{H_{ISCI}}} A \vee B \supset F$. By (a) and clause 2, $A \vee B \notin \mathbf{S}$.
For the *only-if* side assume that $A \in \mathbf{S}$, but $A \vee B \notin \mathbf{S}$. Using clause 2, deduction theorem, axiom $\mathbf{H}_7$ and **MP**, one arrives at $\mathbf{S} \vdash_{\mathsf{H_{ISCI}}} F$. Thus $A \vee B \in \mathbf{S}$. The reasoning is similar if $B \in \mathbf{S}$.

(ad5) By clause 2 and **MP**.

(ad6) If $A \supset B \notin \mathbf{S}$, then, by clause 2, $\mathbf{S} \nvdash_{\mathsf{H_{ISCI}}} A \supset B$. Hence also $\mathbf{S} \cup \{A\} \nvdash_{\mathsf{H_{ISCI}}} B$ (deduction theorem). In other words, $\mathbf{S} \cup \{A\}$ is B-consistent. □

Definition 4 (Canonical ISCI model). *Canonical* ISCI *model is a triple* $\boldsymbol{M} = \langle W, \subseteq_W, \in_W \rangle$, *where* W *is the set of all maximally consistent sets of formulas of* $\mathcal{L}_{\mathsf{ISCI}}$, $\subseteq_W$ *is the set inclusion in* W, *and* $\in_W$ *is the membership relation between formulas of* $\mathcal{L}_{\mathsf{ISCI}}$ *and elements of* W.

Frame $\langle W, \subseteq_W \rangle$ is an ISCI-frame, because, first, W is non-empty (the empty set is a consistent set, and by Lemma 1, it has a maximally consistent superset, hence at least one maximally consistent set exists), and the set inclusion $\subseteq_W$ is reflexive and transitive. We still need to show that the canonical ISCI model is in fact an ISCI model.

Lemma 4. *The canonical* ISCI *model satisfies Definition 3, that is, is an* ISCI *model.*

Proof: Let $\boldsymbol{M} = \langle W, \subseteq_W, \in_W \rangle$ be the canonical ISCI model. We already know that the structure $\langle W, \subseteq_W \rangle$ is an ISCI-frame. Hence what is left to show is that the membership relation $\in_W$ satisfies Definition 2 of forcing.

As the required assignment in $\langle W, \subseteq_W \rangle$ we take function $v : For_0 \times W \longrightarrow \{0,1\}$ defined: $v(A,w) = 1$ iff $A \in w$. The assignment is ISCI-admissible, since each $\approx$-specific axiom belongs to each maximally consistent set by the definition of consistent sets (see clause 2 of Lemma 3). (We obtain conditions (2) and (3) defining ISCI-admissibility by **MP**.) Clearly, $\in_W$ extends v, and thus it satisfies clause (1) of Definition 2. Clauses (2)-(4) hold by Lemma 3.

Clause (5), *if-then* direction: assume that $A \supset B \in_W w$, and that $w \subseteq_W w^*$; hence also $A \supset B \in_W w^*$. By clause 5 of Lemma 3, if $A \in_W w^*$, then also $B \in_W w^*$. Clause (5), *only-if* direction: suppose that $A \supset B \notin_W w$. By clause 6 of Lemma 3, $w \cup \{A\}$ is B-consistent. By Lemma 1, there is a maximally B-consistent set $w^* \supseteq w \cup \{A\}$. It follows that $w \subseteq_W w^*$, $A \in_W w^*$ and $B \notin_W w^*$.

Monotonicity conditions (*mon*) and ($mon_\approx$) hold trivially by the fact that the relation between worlds is set inclusion.

Finally, since each $\approx$-specific axiom, $(\approx_3) : (A \approx B) \supset (B \supset A)$ in particular, belongs to each maximally consistent set, condition ($\approx$) holds as well. $\square$

Let us note that if A is a thesis of $\mathsf{H}_{\mathsf{ISCI}}$, then A is an element of each maximally s-consistent set (Lemma 2), and thus A is true in the canonical ISCI-model.

Theorem 2 (completeness). *If a formula is* ISCI*-valid, then it is a thesis of* $\mathsf{H}_{\mathsf{ISCI}}$.

Proof: The proof is by contraposition. Assume that a formula is not a thesis of $\mathsf{H}_{\mathsf{ISCI}}$. Thus, by Lemma 2, there exists maximally consistent set w such that $A \notin w$. Thus there is a world w in the canonical ISCI model which does not contain A. Hence A is not ISCI-valid. $\square$

2. Sequent calculi for ISCI

2.1. Axioms and rules

There is a number of strategies of building sequent calculi or natural deduction systems for axiomatic theories based on a certain logic (see for example [12, 7, 9, 8]). The strategy we are interested in enables one to turn each axiom of a given axiomatic system into a rule of a corresponding sequent calculus in such a way that all structural rules – the cut rule in particular – are admissible in the generated calculus. The strategy requires that the initial axioms, from which the rules will be generated, are of the form:

$$P_1 \wedge \ldots \wedge P_m \rightarrow Q_1 \vee \ldots \vee Q_n \tag{2.1}$$

where P_i and Q_j are propositional variables. Naturally, the specific $\approx$-axioms do not fit into this form. Thus we will generalize this strategy to axioms of the form:

$$A_1 \wedge \ldots \wedge A_m \rightarrow B_1 \vee \ldots \vee B_n \tag{2.2}$$

where A_i, B_j are arbitrary formulas. The sequent rules corresponding to (2.2) should present as follows (Γ and Δ stand for *multisets* of formulas):

$$\frac{B_1, A_1, \ldots, A_m, \Gamma \Rightarrow \Delta \quad \ldots \quad B_n, A_1, \ldots, A_m, \Gamma \Rightarrow \Delta}{A_1, \ldots, A_m, \Gamma \Rightarrow \Delta}\ L$$

$$\frac{\Gamma \Rightarrow \Delta, B_1, \ldots, B_n, A_1 \quad \ldots \quad \Gamma \Rightarrow \Delta, B_1, \ldots, B_n, A_n}{\Gamma \Rightarrow \Delta, B_1, \ldots, B_n}\ R$$

If each axiom is transformed into a left (right) rule, then we obtain a *left* (*right*) *system*. Let us observe, however, that the right rule is problematic in the constructive setting, due to the usual restriction on the consequent of a sequent in intuitionistic logic. Sequents used in constructing the sequent calculus for ISCI will be of the form:

$$\Gamma \Rightarrow A$$

where Γ is a finite, possibly empty, multiset of formulas of $\mathcal{L}_{\mathsf{ISCI}}$ and A is a *single* formula of $\mathcal{L}_{\mathsf{ISCI}}$.

The restriction on the consequent of a sequent forces us to define only left system for ISCI. However, according to the presented strategy of rules construction, each left system constructed by means of this method needs to satisfy the following additional condition:

Definition 5 (Closure Condition, [6]). *If a system with nonlogical rules has a rule, where a substitution instance in the atoms produces a rule of the form:*

$$\frac{B_1, A_1, \ldots, A_{m-2}, A, A, \Gamma \Rightarrow \Delta \quad \ldots \quad B_n, A_1, \ldots, A_{m-2}, A, A, \Gamma \Rightarrow \Delta}{A_1, \ldots, A_{m-2}, A, A, \Gamma \Rightarrow \Delta}\ R$$

then it also has to contain the rule:

$$\frac{B_1, A_1, \ldots, A_{m-2}, A, \Gamma \Rightarrow \Delta \quad \ldots \quad B_n, A_1, \ldots, A_{m-2}, A, \Gamma \Rightarrow \Delta}{A_1, \ldots, A_{m-2}, A, \Gamma \Rightarrow \Delta}\ R^*$$

The closure condition ensures the existence of rules in a given system which are essential for the admissibility of contraction in that system.

Table 2. Structural rules

$$\frac{\Gamma \Rightarrow C}{A, \Gamma \Rightarrow C}\ L_w \qquad \frac{A, A, \Gamma \Rightarrow C}{A, \Gamma \Rightarrow C}\ L_{ctr}$$

$$\frac{\Gamma' \Rightarrow D \quad D, \Gamma'' \Rightarrow C}{\Gamma', \Gamma'' \Rightarrow C}\ cut$$

Sequent calculus $\mathbf{G3}_{\mathsf{ISCI}}$ is composed of the structural rules displayed in Table 2, and the logical rules presented in Table 3.

Following the presented strategy, we obtain the $\approx$-specific (left) rules listed in Table 3. It is worth noticing that none of the rules is directly obtained from axiom $(\approx)_2$. In the absence of negation as a primitive connective, axiom $(\approx)_2$ is provable with the use of rules $L^3_\approx$ and $L^1_\approx$ (in a sense, it is a particular case of congruence). The rule $L^3_\approx *$ belongs to the system due to the closure condition.

Let $\mathbf{M} = \langle W, \leq, \Vdash \rangle$ be an ISCI model and $w \in W$. We will say that a sequent $\Gamma \Rightarrow C$ is *satisfied at w in $\boldsymbol{M}$* iff the fact that w forces each member of Γ implies that it also forces C. A sequent is said to be *true in a model* iff it is satisfied at each world in this model.

LEMMA 5. *Each rule of* $\mathbf{G3}_{\mathsf{ISCI}}$ *preserves truth in an* ISCI *model.*

PROOF: Let us consider $L^1_\approx$ only. Let $\boldsymbol{M} = \langle W, \leq, \Vdash \rangle$ be an arbitrary model. Assume that sequent $A \approx A, \Gamma \Rightarrow C$ is true in $\boldsymbol{M}$ and that $\Gamma \Rightarrow C$ is not. Thus there exists a world w such that w forces each member of Γ, but $w \nVdash C$. Naturally $w \Vdash A \approx A$. Thus $A \approx A, \Gamma \Rightarrow C$ is not true in $\boldsymbol{M}$, contrary to our assumption. $\square$

Table 3. Logical rules of $\mathbf{G3}_{\mathsf{ISCI}}$

$$p_i, \Gamma \Rightarrow p_i \qquad\qquad A \approx B, \Gamma \Rightarrow A \approx B$$

$$\frac{}{\bot, \Gamma \Rightarrow C}\ L_\bot$$

$$\frac{A, B, \Gamma \Rightarrow C}{A \wedge B, \Gamma \Rightarrow C}\ L_\wedge \qquad\qquad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\ R_\wedge$$

$$\frac{A, \Gamma \Rightarrow C \quad B, \Gamma \Rightarrow C}{A \vee B, \Gamma \Rightarrow C}\ L_\vee$$

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B}\ R_\vee \qquad\qquad \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B}\ R_\vee$$

$$\frac{A \supset B, \Gamma \Rightarrow A \quad B, \Gamma \Rightarrow C}{A \supset B, \Gamma \Rightarrow C}\ L_\supset \qquad\qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \supset B}\ R_\supset$$

$$\frac{A \approx A, \Gamma \Rightarrow C}{\Gamma \Rightarrow C}\ L^1_\approx \qquad\qquad \frac{A \approx B, \Gamma \Rightarrow B \quad A, A \approx B, \Gamma \Rightarrow C}{A \approx B, \Gamma \Rightarrow C}\ L^2_\approx$$

$$\frac{(A \otimes C) \approx (B \otimes D), A \approx B, C \approx D, \Gamma \Rightarrow E}{A \approx B, C \approx D, \Gamma \Rightarrow E}\ L^3_\approx$$

$$\frac{(A \otimes A) \approx (B \otimes B), A \approx B, \Gamma \Rightarrow E}{A \approx B, \Gamma \Rightarrow E}\ L^{3*}_\approx$$

2.2. Completeness

The easiest way to show completeness of $\mathbf{G3}_{\mathsf{ISCI}}$ is to prove that it can simulate axiomatic system.

First, the following is easy to show by induction on the structure of formula A:

Corollary 1. *For each formula A of $\mathcal{L}_{\mathsf{ISCI}}$, sequent $A, \Gamma \Rightarrow A$ is provable in $\mathbf{G3}_{\mathsf{ISCI}}$.*

Theorem 3. $\mathbf{G3}_{\mathsf{ISCI}} + [cut]$ *is complete with respect to* ISCI*-semantics.*

Proof: The Hilbert-style system $\mathsf{H}_{\mathsf{ISCI}}$ is complete with respect to the presented ISCI-semantics, as we have shown in the previous section. One can simulate $\mathsf{H}_{\mathsf{ISCI}}$ in $\mathbf{G3}_{\mathsf{ISCI}} + [cut]$. Each axiom is derivable – see for example $\approx_3$:

$$\dfrac{\dfrac{\dfrac{A \approx B, B \Rightarrow B \quad A, A \approx B, B \Rightarrow A}{A \approx B, B \Rightarrow A}\, L^2_{\approx}}{A \approx B \Rightarrow B \supset A}\, R_{\supset}}{\Rightarrow (A \approx B) \supset (B \supset A)}\, R_{\supset}$$

and, by Corollary 1, the following derivation proves that **MP** can be reconstructed in $\mathbf{G3}_{\mathsf{ISCI}} + [cut]$.

$$\dfrac{\Rightarrow A \qquad \dfrac{\Rightarrow A \supset B \qquad \dfrac{A \supset B, A \Rightarrow A \quad B, A \Rightarrow B}{A \supset B, A \Rightarrow B}\, L_{\supset}}{A \Rightarrow B}\, cut}{\Rightarrow B}\, cut$$

$\square$

Thus $\mathbf{G3}_{\mathsf{ISCI}} + [cut]$ enable us to prove formulas in a similar manner to the one used in $\mathsf{H}_{\mathsf{ISCI}}$, but in a tree-like form, and with a significant use of *cut*.

2.3. Admissibility results

Due to the fact that we have relaxed the form of axioms accepted ((2.2) instead of (2.1), see page 270), there is no guarantee that the resulting system will be cut-free. This result will be proved below.

We refer to formulas specified in the premisses of a rule schema as *active* and to those specified in the conclusion as *principal*. Following [7], by *height of a derivation* we mean the maximal number of successive applications of the logical rules of $\mathbf{G3}_{\mathsf{ISCI}}$. Moreover, by:

$$\vdash_n \phi$$

we shall mean that the sequent ϕ is derivable in $\mathbf{G3}_{\mathsf{ISCI}}$ with height no greater than n.

The terms of cut-height and formula weight defined below are used in proving admissibility of structural rules and follow the definitions from [7].

Definition 6 (Cut-height). *The cut-height of an application of the cut rule in a derivation $\mathcal{D}$ is the sum of heights of derivations of two premisses of cut.*

Definition 7 (Formula weight). *The weight is a function from the set of all formulas of $\mathcal{L}_{\mathsf{ISCI}}$ to the set of natural numbers, which fulfils the following conditions:*

1. $w(\bot) = 0$,
2. $w(p_i) = 1$, *for each* $p_i \in V$,
3. $w(A \otimes B) = w(A) + w(B) + 1$, *where* $\otimes \in \{\approx, \vee, \wedge, \rightarrow\}$.

Theorem 4 (Admissibility of weakening). *If* $\vdash_n \Gamma \Rightarrow C$, *then* $\vdash_n A, \Gamma \Rightarrow C$.

Proof: A very straightforward proof relies on the observation that one can always transform a given derivation of $\Gamma \Rightarrow C$ into a derivation of $A, \Gamma \Rightarrow C$ by adding a formula A to the antecedent of each sequent in the original derivation. □

Lemma 6 (Height-preserving invertibility).

1. *If* $\vdash_n A \wedge B, \Gamma \Rightarrow C$, *then* $\vdash_n A, B, \Gamma \Rightarrow C$.
2. *If* $\vdash_n A \vee B, \Gamma \Rightarrow C$, *then* $\vdash_n A, \Gamma \Rightarrow C$ *and* $\vdash_n B, \Gamma \Rightarrow C$.
3. *If* $\vdash_n A \supset B, \Gamma \Rightarrow C$, *then* $\vdash_n B, \Gamma \Rightarrow C$.
4. *If* $\vdash_n \Gamma \Rightarrow C$, *then* $\vdash_n A \approx A, \Gamma \Rightarrow C$.
5. *If* $\vdash_n A \approx B, \Gamma \Rightarrow C$, *then* $\vdash_n A, A \approx B, \Gamma \Rightarrow C$.
6. *If* $\vdash_n A \approx B, C \approx D, \Gamma \Rightarrow E$, *then* $\vdash_n (A \otimes C) \approx (B \otimes D), A \approx B, C \approx D, \Gamma \Rightarrow E$.
7. *If* $\vdash_n A \approx B, \Gamma \Rightarrow C$, *then* $\vdash_n (A \otimes A) \approx (B \otimes B), A \approx B, \Gamma \Rightarrow C$.

Proof: The argument for clauses 1.-3. is essentially the same as in the classical case and we skip it (see [7], if necessary). In the case of clauses 4.-7., each clause holds due to the admissibility of weakening. The second identity rule, $L^2_{\approx}$, is invertible only with respect to the right premiss. □

Theorem 5 (Height-preserving admissibility of contraction). *If* $\vdash_n A, A, \Gamma \Rightarrow C$, *then* $\vdash_n A, \Gamma \Rightarrow C$.

Proof: By induction on the height of derivation. Assume $n = 0$. Then the sequent $A, A, \Gamma \Rightarrow C$ is (i) an axiom or (ii) a conclusion of $L_\bot$. Naturally, in these cases sequent $A, \Gamma \Rightarrow C$ is an axiom or a conclusion of $L_\bot$.

Assume that the theorem holds up to n, and let $\vdash_{n+1} A, A, \Gamma \Rightarrow C$. If the contraction formula A is not principal in the last applied rule R of a given derivation, then we have to consider two cases: either R is a one-premiss rule or a two-premisses rule. When the former is the case we have to consider the following situation:

$$\frac{\vdash_n A, A, \Gamma' \Rightarrow \Delta'}{\vdash_{n+1} A, A, \Gamma \Rightarrow \Delta}\ R$$

By the inductive hypothesis we have that $\vdash_n A, \Gamma' \Rightarrow \Delta'$. By applying R to this sequent we obtain $\vdash_{n+1} A, \Gamma \Rightarrow \Delta$.

Similarly, if R is a two-premisses rule:

$$\frac{\vdash_n A, A, \Gamma' \Rightarrow \Delta' \quad \vdash_n A, A, \Gamma'' \Rightarrow \Delta''}{\vdash_{n+1} A, A, \Gamma \Rightarrow \Delta}\ R$$

then we apply the inductive hypothesis to the two premisses of R in order to obtain (by means of R) the sequent $A, \Gamma \Rightarrow \Delta$ provable with height at most $n+1$.

If one of the contraction formulas is principal, then we have three cases where the formula is not an identity. Let us consider only implication.

$$\frac{\vdash_n A \supset B, A \supset B, \Gamma \Rightarrow A \quad \vdash_n A \supset B, B, \Gamma \Rightarrow C}{\vdash_{n+1} A \supset B, A \supset B, \Gamma \Rightarrow C}\ L_{\supset}$$

By the inductive hypothesis applied to the left premiss we know that

$$\vdash_n A \supset B, \Gamma \Rightarrow A. \tag{2.3}$$

For the right premiss we apply clause 3. of Theorem 6 which yields that $\vdash_n B, B, \Gamma \Rightarrow C$. Now the inductive hypothesis can be applied, which results in $B, \Gamma \Rightarrow C$ being provable with height at most n. Application of $L_{\supset}$ to (2.3) and $B, \Gamma \Rightarrow C$ gives us

$$\vdash_{n+1} A \supset B, \Gamma \Rightarrow C$$

The only non-standard cases are when the contracted formula is an equation, and the last rule used is one of $L^2_{\approx}$, $L^3_{\approx}$ or $L^{3*}_{\approx}$. Let us consider the case when the last rule applied is $L^2_{\approx}$:

$$\frac{\vdash_n A \approx B, A \approx B, \Gamma \Rightarrow B \quad \vdash_n A, A \approx B, A \approx B, \Gamma \Rightarrow C}{\vdash_{n+1} A \approx B, A \approx B, \Gamma \Rightarrow C}\ L^2_{\approx}$$

By inductive hypothesis, $\vdash_n A \approx B, \Gamma \Rightarrow B$ and $\vdash_n A, A \approx B, \Gamma \Rightarrow C$. We apply the rule $L^2_{\approx}$, to conclude $A \approx B, \Gamma \Rightarrow C$ in at most $n+1$ steps:

$$\frac{\vdash_n A \approx B, \Gamma \Rightarrow B \quad A, A \approx B, \Gamma \Rightarrow C}{\vdash_{n+1} A \approx B, \Gamma \Rightarrow C}\ L_2^{\approx}$$

Due to the fact that in one of the rules of the system two formulas are principal ($L^3_{\approx}$), we have to consider a situation, where both contraction formulas are principal.

$$\frac{\vdash_n (B \otimes B) \approx (C \otimes C), B \approx C, B \approx C, \Gamma \Rightarrow \Delta}{\vdash_{n+1} B \approx C, B \approx C, \Gamma \Rightarrow \Delta}\ L^3_{\approx}$$

By inductive hypothesis applied to the premiss we get $\vdash_n (B \otimes B) \approx (C \otimes C), B \approx C, \Gamma \Rightarrow \Delta$. Now an application of the rule $L^{3*}_{\approx}$, results in $B \approx C, \Gamma \Rightarrow \Delta$ with at most $n+1$ steps:

$$\frac{\vdash_n (B \otimes B) \approx (C \otimes C), B \approx C, \Gamma \Rightarrow \Delta}{\vdash_{n+1} B \approx C, \Gamma \Rightarrow \Delta}\ L^{3*}_{\approx}$$

This case clearly shows how the rule obtained by the closure condidtion is necessary for proving admissibility of contraction. □

Theorem 6. *The cut rule*

$$\frac{\Gamma' \Rightarrow D \quad D, \Gamma'' \Rightarrow C}{\Gamma', \Gamma'' \Rightarrow C}\ cut$$

is admissible in $\mathbf{G3}_{\mathsf{ISCI}}$.

Proof: The proof is organized as in [6]. The idea is to divide all the cases to consider into some classes. The first class enhances the cases where at least one of the premisses of the cut-rule is an axiom or a conclusion of $L_\perp$. Assume it is the left premiss. Then (the case of an axiom) D is a propositional variable or an equation and it belongs to Γ'. In this case *cut* can be completely eliminated by (possibly multiple) application(s) of weakening:

$$\frac{D, \Gamma'' \Rightarrow C}{\Gamma', \Gamma'' \Rightarrow C}\ L_w$$

If $\perp$ occurs in Γ' (the case of $L_\perp$), then the conclusion of *cut*, that is, $\Gamma', \Gamma'' \Rightarrow C$, also follows by $L_\perp$, so the application of the cut rule can be eliminated.

Similar simplifications can be applied, if we assume that the right premiss was one of the axioms or a conclusion of $\bot$. There is one subtlety here. Assume $D = \bot$. Thus we arrive at:

$$\dfrac{\Gamma' \Rightarrow \bot \quad \bot, \Gamma'' \Rightarrow C}{\Gamma', \Gamma'' \Rightarrow C}\ cut$$

If $\bot$ occurs in Γ' or Γ'', then *cut* can be eliminated. But if it does not occur in Γ', we have to consider the rule which was applied in order to obtain the left premiss. In fact, this case follows under class **I** of cases considered below, so there is no need to consider it separately.

All the other cases (i.e. those where a premiss is neither an axiom nor a conclusion of $L_\bot$) can be divided into the following three classes.

I The cut formula D is not principal in the left premiss. We consider only the cases where the left premiss is itself a conclusion of a $\approx$-specific rules.

(1) The last rule applied was $L^1_\approx$. Cut-height equals $(m + 1) + m'$:

$$\dfrac{\dfrac{\overset{m}{A \approx A, \Gamma' \Rightarrow D}}{\Gamma' \Rightarrow D}\ L^1_\approx \qquad \overset{m'}{D, \Gamma'' \Rightarrow C}}{\Gamma', \Gamma'' \Rightarrow C}\ cut$$

This derivation is transformed into a derivation of smaller cut-height (equal to $m + m'$):

$$\dfrac{\dfrac{A \approx A, \Gamma' \Rightarrow D \quad D, \Gamma'' \Rightarrow C}{A \approx A, \Gamma', \Gamma'' \Rightarrow C}\ cut}{\Gamma', \Gamma'' \Rightarrow C}\ L^1_\approx$$

(2) The last rule applied was $L^2_\approx$. The cut-height equals $(max(m, n) + 1)$ $+ m'$:

$$\dfrac{\dfrac{\overset{m}{A \approx B, \Gamma' \Rightarrow B} \quad \overset{n}{A, A \approx B, \Gamma' \Rightarrow D}}{A \approx B, \Gamma' \Rightarrow D}\ L^2_\approx \qquad \overset{m'}{D, \Gamma'' \Rightarrow C}}{A \approx B, \Gamma', \Gamma'' \Rightarrow C}\ cut$$

This derivation is transformed into derivation with *cut* of cut-height $n + m'$:

$$\dfrac{\dfrac{A \approx B, \Gamma' \Rightarrow B}{A \approx B, \Gamma', \Gamma'' \Rightarrow B}\,L_w \quad \dfrac{A \approx B, A, \Gamma' \Rightarrow D \quad D, \Gamma'' \Rightarrow C}{A, A \approx B, \Gamma', \Gamma'' \Rightarrow C}\,cut}{A \approx B, \Gamma', \Gamma'' \Rightarrow C}\,L^2_{\approx}$$

(3) The last rule applied was $L^3_{\approx}$. Cut-height equals $(m + 1) + m'$:

$$\dfrac{\dfrac{\overset{m}{(A \otimes C^*) \approx (B \otimes D^*), A \approx B, C^* \approx D^*, \Gamma' \Rightarrow D}}{A \approx B, C^* \approx D^*, \Gamma' \Rightarrow D}\,L^3_{\approx} \quad \overset{m'}{D, \Gamma'' \Rightarrow C}}{A \approx B, C^* \approx D^*, \Gamma', \Gamma'' \Rightarrow C}\,cut$$

This derivation is transformed into a derivation with a lesser cut-height $(m + m')$:

$$\dfrac{\dfrac{(A \otimes C^*) \approx (B \otimes D^*), A \approx B, C^* \approx D^*, \Gamma' \Rightarrow D \quad D, \Gamma'' \Rightarrow C}{(A \otimes C^*) \approx (B \otimes D^*), A \approx B, C^* \approx D^*, \Gamma', \Gamma'' \Rightarrow C}\,cut}{A \approx B, C^* \approx D^*, \Gamma', \Gamma'' \Rightarrow C}\,L^3_{\approx}$$

(4) The last rule applied was $L^{3*}_{\approx}$. Cut-height equals $(m + 1) + m'$:

$$\dfrac{\dfrac{\overset{m}{(A \otimes A) \approx (B \otimes B), A \approx B, \Gamma' \Rightarrow D}}{A \approx B, \Gamma' \Rightarrow D}\,L^{3*}_{\approx} \quad \overset{m'}{D, \Gamma'' \Rightarrow C}}{A \approx B, \Gamma', \Gamma'' \Rightarrow C}\,cut$$

This derivation is transformed into a derivation with a lesser cut-height $(m + m')$:

$$\dfrac{\dfrac{(A \otimes A) \approx (B \otimes B), A \approx B, \Gamma' \Rightarrow D \quad D, \Gamma'' \Rightarrow C}{(A \otimes A) \approx (B \otimes B), A \approx B, \Gamma', \Gamma'' \Rightarrow C}\,cut}{A \approx B, \Gamma', \Gamma'' \Rightarrow C}\,L^{3*}_{\approx}$$

II When the cut-formula is principal in the left premiss only, we consider the last rule applied to the right premiss of cut. Note that in this case the

cut formula cannot be an equation, due to the fact that there are no right identity rules. The transformations are analogous to the ones in [7].
III If the cut formula D is principal in both premisses, only classical rules can be applied to D. □

3. Discussion

3.1. Subformula property

Note that in the described system subformula property is not entailed by cut elimination due to the non-analytic character of the identity rules. Nevertheless, the notion of a subformula can be modified as follows.

Definition 8. *Let F be a formula of $\mathcal{L}_{\mathsf{ISCI}}$. The set of subformulas of F, in symbols, $\mathsf{sub}(F)$, is the smallest set satisfying the following conditions:*

- $F \in \mathsf{sub}(F)$*;*
- *if F is of the form $B \otimes C$, then $\mathsf{sub}(B) \subseteq \mathsf{sub}(B \otimes C)$ and $\mathsf{sub}(C) \subseteq \mathsf{sub}(B \otimes C)$.*

Moreover, the set $\mathsf{sub}(F)$ is closed under the following rules:

$$\frac{A \approx C \in \mathsf{sub}(F) \quad B \approx D \in \mathsf{sub}(F)}{\mathsf{sub}((A \otimes B) \approx (C \otimes D)) \subseteq \mathsf{sub}(F)}$$

$$\frac{A \in \mathsf{sub}(F)}{A \approx A \in \mathsf{sub}(F)}$$

Let d be a derivation in $\mathbf{G3}_{\mathsf{ISCI}}$. By $\mathsf{labels}(d)$ we denote the set of all formulas occurring in sequents labelling nodes of d. Now we can state:

Theorem 7. *If a sequent $\Rightarrow A$ is provable, then there exists a proof d of $\Rightarrow A$, such that*

$$\mathsf{labels}(d) \subseteq \mathsf{sub}(A)$$

Note that this result cannot be extended to derivations which are not proofs, due to the lack of restrictions imposed on the first identity rule.

3.2. Variants of the calculus

Note that the second SCI axiom can be turned into a rule in such a way that intuitionistic implication occurs explicitly in the premiss:

$$\frac{A \approx B, B \supset A, \Gamma \Rightarrow C}{A \approx B, \Gamma \Rightarrow C}\; aL^2_{\approx}$$

In a system in which we exchange $L^2_{\approx}$ with $aL^2_{\approx}$ the first rule becomes derivable:

$$\cfrac{\cfrac{\cfrac{A \approx B, \Gamma \Rightarrow B}{A \approx B, B \supset A, \Gamma \Rightarrow B}\; L_w \qquad A, A \approx B, \Gamma \Rightarrow C}{A \approx B, B \supset A, \Gamma \Rightarrow C}\; L_{\supset}}{A \approx B, \Gamma \Rightarrow C}\; aL^2_{\approx}$$

On the other hand, $aL^2_{\approx}$ cannot be derived in a system with $L^2_{\approx}$, since the latter is analytic, while the former introduces new formula $B \supset A$. In both this systems the rule of cut is admissible.

In our strategy of building a sequent calculus for ISCI we kept close to the syntactic structure of the identity axioms, as they are expressed in the Hilbert-style system. Another strategy can be applied in order to obtain a different system. This time we make use of the analogy between propositional and term identity.

We have to assume that:

- identity is reflexive, and
- identical propositions can be exchanged in arbitrary contexts, and
- identity is stronger than intuitionistic equivalence.

These assumptions can be transformed into rules in the following way:

$$\frac{A \approx A, \Gamma \Rightarrow C}{\Gamma \Rightarrow C}\; ref \qquad\qquad \frac{D^A_B, \Gamma \Rightarrow C}{A \approx B, D, \Gamma \Rightarrow C}\; rep$$

$$\frac{A \approx B, B \supset A, \Gamma \Rightarrow C}{A \approx B, \Gamma \Rightarrow C}\; aL^2_{\approx}$$

where D^A_B is the result of replacing A with B in some (possibly all) contexts. Let us note that each identity axiom can be derived in the system. Let us prove the fourth axiom:

$$\dfrac{\dfrac{\dfrac{\dfrac{(A \otimes B) \approx (C \otimes D) \Rightarrow (A \otimes B) \approx (C \otimes D)}{(A \otimes B) \approx (C \otimes B), B \approx D \Rightarrow (A \otimes B) \approx (C \otimes D)}\ rep}{(A \otimes B) \approx (A \otimes B), A \approx C, B \approx D \Rightarrow (A \otimes B) \approx (C \otimes D)}\ rep}{A \approx C, B \approx D \Rightarrow (A \otimes B) \approx (C \otimes D)}\ ref}{\Rightarrow ((A \approx C) \wedge (B \approx D)) \supset ((A \otimes B) \approx (C \otimes D))}\ L_\wedge, R_\supset$$

Let us also note that the rule of contraction is not height-preserving admissible in this system due to the shape of the replacement rule. This however can be fixed up by replacing *rep* by the following, more general version:

$$\dfrac{D^A_B, A \approx B, D, \Gamma \Rightarrow C}{A \approx B, D, \Gamma \Rightarrow C}\ rep^*$$

The rule of cut is admissible in the system and the argument is very similar to the one presented in Section 2.3.

Let us also note that each rule of our initial system $\mathbf{G3}_{\mathsf{ISCI}}$ can be derived in the system we have just defined – the new one is thus more general. Here is a derivation of $L^3_\approx$ (Z stands for $(A \otimes B) \approx (C \otimes D)$):

$$\dfrac{\dfrac{\dfrac{\dfrac{Z, A \approx C, B \approx D, \Gamma \Rightarrow E}{Z, (A \otimes B) \approx (C \otimes B), (A \otimes B) \approx (A \otimes B), A \approx C, B \approx D, \Gamma \Rightarrow E}\ L_w}{Z, (A \otimes B) \approx (A \otimes B), A \approx C, B \approx D, \Gamma \Rightarrow E}\ rep^*}{Z, A \approx C, B \approx D, \Gamma \Rightarrow E}\ rep^*}{A \approx C, B \approx D, \Gamma \Rightarrow E}\ ref$$

The rule $L^{3*}_\approx$ can be derived by the same mechanism (with the use of weakening). $L^2_\approx$ is also derivable, in exactly the same manner as is shown at the beginning of this section.

The problem is that the approach now is semantical – we *know* that $\approx$ denotes identity, thus we can construct the rules. Therefore the described approach is not mechanical and strongly depends on our ability to interpret the corresponding axioms.

4. Conclusions

We showed that Suszko's propositional identity connective has a natural constructive interpretation. Therefore, the logic ISCI can be considered as a legitimate (in the sense of the underlying philosophical intuitions) extension of intuitionistic logic. We defined possible world semantics for ISCI along with two cut-free sequent calculi for ISCI.

The future work will cover the construction and the analysis of natural deduction system for ISCI along with the typed lambda calculus corresponding to it, which will put more light on the constructive interpretation of Suszko's propositional identity connective.

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Bulletin of the **S**ection of **L**ogic
Volume 48/4 (2019), pp. 285–298

http://dx.doi.org/10.18778/0138-0680.48.4.03

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MANY FACES OF LATTICE TOLERANCES

Abstract

Our aim is to overview and discuss some of the most popular approaches to the notion of a tolerance relation in algebraic structures with the special emphasis on lattices.

Keywords: lattice, tolerance, congruence, covering system, gluing.

1. Introduction

The idea of tolerance relations seen as a formalization of the intuitive notion of resemblance was present in the late works of Poincaré who introduced the sets of impressions to describe sensations concerning objects hardly indiscernible [2].

In 1962 Zeeman formally introduced the notion of a tolerance as a relation that is reflexive and symmetric, but not necessarily transitive [29]. Studying models of visual perceptions, Zeeman found it useful to axiomatize the notion of similarity and formalized the notion of tolerance spaces. The idea of "being within tolerance" or of "closeness" or "resemblance" is universal enough to appear, quite naturally, in almost any setting. It is particularly natural in practical applications: real-life problems, more often than not, deal with approximate input data and require only viable results with a tolerable level of exactness. Therefore, the topic became popular among researchers from different areas such as linguistics, information theory, humanities, social sciences, but also logic and mathematics. Studies on tolerance spaces were conducted by Szrejder [28], Arbib [1], Pogonowski [27] and others.Since then, numerous books and papers concerning this topic have appeared.

There are different approaches to tolerance relations, so we decided to present here a short survey of them. This paper is meant as an inspiration to wider and more complete studies of results obtained in this area, particularly in algebraic structures.

As a natural generalization of congruences, tolerances appeared to be a very useful tool in universal algebra. In an algebraic structure $\mathcal{A} = (A, F)$ by *tolerances* we mean only those reflexive and symmetric relations which are compatible with the operations from F. Some authors (see, *e.g.*, [23]) call them *admissible relations*. Formally speaking, a reflexive and symmetric relation R on an algebra $\mathcal{A} = (A, F)$ is a tolerance iff for every n-ary operation $f \in F$ and for all $a_1, \dots, a_n, b_1, \dots, b_n \in A$, such that $(a_i, b_i) \in T$ for $i = 1, \dots, n$ we have $(f(a_1, \dots, a_n), f(b_1, \dots, b_n)) \in T$.

Therefore, $R \subseteq A^2$ is a tolerance on an algebra $\mathcal{A} = (A, F)$ iff R is reflexive, symmetric and closed under the operations from F (according to coordinates). In other words, it means that R is a reflexive and symmetric subalgebra of $\mathcal{A}^2$.

It is clear that every congruence of an algebra $\mathcal{A}$ is a tolerance on it and then the notion of a tolerance in universal algebra can be regarded as a generalization of the notion of a congruence.

Tolerances play an important role in the theory of Maltsev conditions (see for example [25] or [12]). Moreover, as we show in the next sections, they are particularly useful in lattice theory.

There are many ways to deal with tolerances, the most common ones consist in describing tolerances as some types of covering systems (*e.g.*, [4, 19]) or characterizing them as homomorphic images of congruences (see [11]). In the case of finite lattices we can also use polarities [23] or Galois connections [17].

2. Covering systems

The most natural way of studying a tolerance is by considering the system of its blocks.

Let T be a tolerance on a given algebra $\mathcal{A} = (A, F)$ (or even more general, on a given set A). Since tolerances are a generalization of congruences (equivalence relations, respectively) and every congruence determines uniquely a quotient algebra (a partition of a given set), we should look for a generalization of the concept of a congruence class (or an equivalence class, in general).

In fact, there are two possible ways of performing this task. First, we can consider a *class* $[a]_T \subseteq A$ containing a given element $a \in A$ defined as the set of all elements from A tolerant with a:

$$[a]_T = \{b \in A \colon (a, b) \in T\}.$$

On the other hand, we can consider subsets $B \subseteq A$ such that $B^2 \subseteq T$. Such subsets are called *preblocks* of T. *Blocks* of T are maximal (under inclusion) preblocks of T.

It is easy to check that in the case of equivalence relations both the above definitions coincide and lead to the same notion of equivalence classes. However, in general, it is not true for tolerance relations.

Example 1. *Let us consider the set of cookies of two sizes: big and small, each of them being either red or yellow. We say that two cookies are similar if they are both of the same size or colour. It is obvious that this relation of similarity among cookies is a tolerance. Let us observe that the tolerance class containing a small red cookie consists of all cookies that are small or red. At the same time there are two different blocks containing this cookie: the block of red cookies and the block of small cookies. Of course, they are not disjoint and their common part consists of small red cookies.*

Example 2. *Let us consider the lattice $\mathcal{L}$ shown in Figure 1 and the tolerance T generated on it by all pairs $(x, y) \in L^2$ such that $x \prec y$.*

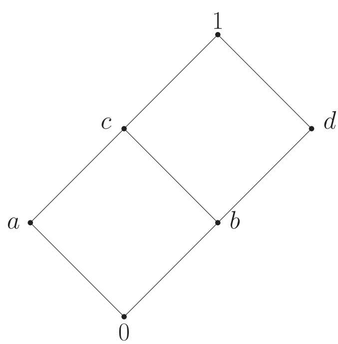

Fig. 1. Lattice $\mathcal{L}$ from Example 2

Since $(0, a), (0, b) \in T$, by compatibility of T with lattice operations, we get $(0, c) = (0 \vee 0, a \vee b) \in T$. Similarly, we obtain $(b, 1) \in T$. However, it is not true that $(0,d) \in T$. Therefore, there are two different blocks of T containing b: $\{0,a,b,c\}$ and $\{b,c,d,1\}$ but $[b]_T = L$.

In general, neither classes nor blocks of a tolerance are disjunctive. The fact that they can overlap constitutes an essential difference between congruences and tolerances, which makes the second ones useful in many practical applications, for example in the rough set theory (see, *e.g.,* [26] or [24]) or in the conceptual analysis (see [17]).

In fact, as it follows from the result below, only the notion of blocks is significant.

Lemma 3. *Let T be a tolerance relation on a set A and let A/T denote the set of all blocks of T on A. Then, for every $a \in A$, we have*

(i) $[a]_T = \bigcup\{\alpha \in A/T\colon a \in \alpha\}$;

(ii) $\alpha \in A/T$ *iff* $\alpha = \bigcap\{[b]_T\colon b \in \alpha\}$.

A proof is identical as in the case of algebraic tolerances, which can be found in [4]. If we deal with tolerances on an algebraic structure, blocks can be characterized by means of an algebraic function over the given structure ([4]).

There are many papers investigating blocks of tolerances of different algebras. In particular, Chajda and Duda proved in [7] what follows.

Theorem 4. *Let T be a tolerance on an algebra $\mathcal{A}$. Every block of T is a subalgebra of $\mathcal{A}$ iff $\mathcal{A}$ is idempotent.*

Let us recall that an algebra $\mathcal{A} = (A,F)$ is idempotent iff its every operation is idempotent, *i.e.,* $f(a,\dots,a) = a$ for every $f \in F$ and $a \in A$.

Since every lattice is an idempotent algebra, we conclude immediately that blocks of tolerances of a lattice are its sublattices. What is more, they are always convex sublattices (see [9] or [3]).

It is well-known that there is a one-to-one correspondence between congruences of a given algebra $\mathcal{A} = (A,F)$ and these partitions of the underlying set A which preserve the substitution property for all operations from F. As it was proved independently by Grätzer and Wenzel in [18] and by Chajda, Niederle and Zelinka in [8], there is a similar one-to-one correspondence between tolerance relations of an algebra $\mathcal{A} = (A,F)$ and a normal covering system of subsets of the set A.

The covering system $\{\alpha_i\}_{i\in I}$ of the set A is called *normal* if it fulfills the following conditions:

(i) it forms an antichain;

(ii) it fulfills the substitution property for the set F of operations of the algebra $\mathcal{A}$, *i.e.,* for every n-ary operation $f \in F$ and all $i_1, \dots, i_n \in I$ there exists $j \in I$ such that $f(\alpha_{i_1}, \dots, \alpha_{i_n}) \subseteq \alpha_j$;

(iii) for every $B \subseteq A$, if $B \nsubseteq \alpha_i$ for every $i \in I$, then there exists $C \subseteq B$ such that C contains exactly two elements and $C \nsubseteq \alpha_i$ for every $i \in I$.

All classes of a given congruence of an algebra $\mathcal{A} = (A, F)$ form an algebra (called the quotient algebra) which belongs to the variety generated by $\mathcal{A}$. In the case of tolerances the situation is different. It is not difficult (see, *e.g.,* [4]) to find examples of an algebra $\mathcal{A}$ and a tolerance T on it such that the basic operations from $\mathcal{A}$ cannot be defined uniquely on the blocks of T.

Example 5. *Let us consider an algebra $\mathcal{A} = (\{a, b, c\}, f)$ with a binary operation defined by $f(x, y) = a$ for all $x, y \in \{a, b, c\}$. Let T be a tolerance on $\mathcal{A}$ generated by the set $\{(a, b), (a, c)\}$. Since $(b, c) \notin T$, there are two blocks of T: $\alpha = \{a, b\}$ and $\beta = \{a, c\}$. Notice that*

$$f(\alpha, \beta) = \{f(x, y)\colon x \in \alpha,\ y \in \beta\} = \{b\}.$$

Thus, we cannot define uniquely the result of operation f on blocks α and β, as $\{b\}$ is a subset both α and β.

Even if it is possible to define the operations on blocks of a tolerance T of an algebra $\mathcal{A}$ uniquely, the corresponding quotient algebra need not belong to the variety generated by $\mathcal{A}$, as we can see later in Example 4.

Formally speaking, an algebra $\mathcal{A} = (A, F)$ is called *tolerance factorable* if for every tolerance T on $\mathcal{A}$, every $f \in F$ and every blocks $\alpha_1, \dots, \alpha_n$ of T there is a unique block β of T such that

$$\{f(a_1, \dots, a_n)\colon a_1 \in \alpha_1, \dots, a_n \in \alpha_n\} \subseteq \beta.$$

If $\mathcal{A}$ is tolerance factorable, then for every tolerance T on $\mathcal{A}$ it is possible to form the quotient algebra $\mathcal{A}/T$ by defining $\beta := f(\alpha_1, \dots, \alpha_n)$ for every n-ary operation $f \in F$ and every system of blocks $\alpha_1, \dots, \alpha_n$ of T. The quotient algebra in the case of tolerances not being congruences is called the *factor algebra.*

We say that a variety $\mathcal{V}$ is tolerance factorable if all its algebras are tolerance factorable. A variety $\mathcal{V}$ is called *strongly tolerance factorable* if it is tolerance factorable and for every $\mathcal{A} \in \mathcal{V}$ and every tolerance T on $\mathcal{A}$ it holds $\mathcal{A}/T \in \mathcal{V}$.

It is clear that every tolerance-trivial algebra, *i.e.,* algebra without tolerances not being congruences, is tolerance factorable. Therefore, for example the variety of Boolean algebras is not only tolerance factorable but also strongly tolerance factorable. The essential question remains, which algebras with proper tolerances enjoy that property.

In 1982 Czédli [10] proved that the variety of lattices is strongly tolerance factorable, although it is not tolerance-trivial. Up to 2012 no other example of such a variety was known. In [5] Chajda, Czédli and Halaš proved that this property is preserved by forming independent joins of varieties, providing infinitely many strongly tolerance factorable varieties with proper tolerances.

Anyway, it is the variety of lattices which behaves surprisingly well with respect to tolerance relations. The fact was discovered and investigated by Czédli, who proved in [10] that if $\mathcal{L}$ is a lattice and T is a tolerance on $\mathcal{L}$, then for any blocks α, β of T there is exactly one block containing $\alpha \wedge \beta = \{a \wedge b \colon a \in \alpha,\ b \in \beta\}$ and exactly one block containing $\alpha \vee \beta = \{a \vee b \colon a \in \alpha, b \in \beta\}$, which means that we can form a quotient structure $\mathcal{L}/T$. Moreover, the structure is a lattice and the partial order of blocks in it coincides with the order of ideals generated by them in the lattice of ideals of $\mathcal{L}$ and – by the duality principle – it is dual to the order of filters generated by the blocks in the lattice of filters of $\mathcal{L}$. Formally, if we denote by $(A]$ and $[A)$, respectively, an ideal and a filter generated by the subset A of a lattice $\mathcal{L}$, then for any $\alpha, \beta \in L/T$,

$$\alpha \leq \beta \text{ iff } (\alpha] \subseteq (\beta] \text{ iff } [\beta) \subseteq [\alpha).$$

However, even in the case of lattices, many properties typical for quotient structures are not, in general, valid for factor structures. Some of them, like the homomorphism theorem and the second isomorphism theorem for lattice congruences, can be imitated in the set of all tolerances partially ordered by a particular restriction of a regular (*i.e.,* inclusion) order of tolerances (see [20]). The resulted poset is not always a lattice, but it can be converted into a specific commutative join-directoid.

In [10], Czédli formulated also a set of necessary and sufficient conditions for a covering of the underlying set L of a lattice $\mathcal{L}$ to be its normal covering.

THEOREM 6. *Let $\mathcal{L} = (L, \wedge, \vee)$ be a lattice. The covering $\{\alpha_i\}_{i \in I}$ of L is the family of blocks of a certain tolerance T on $\mathcal{L}$ iff the following conditions hold:*

1. *Every* α_i *for* $i \in I$ *is a sublattice of* $\mathcal{L}$.
2. *For every* $i, j \in I$
$$[\alpha_i) = [\alpha_j) \text{ iff } (\alpha_i] = (\alpha_j].$$
3. *For every* $i, j \in I$ *there exist* $m, n \in I$ *such that*
$$\{a \vee b \colon a \in [\alpha_i),\ b \in [\alpha_j)\} = [\alpha_m),$$
$$\{a \vee b \colon a \in (\alpha_i],\ b \in (\alpha_j]\} \subseteq (\alpha_m],$$
$$\{a \wedge b \colon a \in [\alpha_i),\ b \in [\alpha_j)\} \supseteq [\alpha_n),$$
$$\{a \wedge b \colon a \in (\alpha_i],\ b \in (\alpha_j]\} = (\alpha_n].$$
4. *Let* $x \in L$, $b \in \alpha_i$ *for some* $i \in I$. *If for any* $b \in \alpha_i \cap (a]$ *there is* $j \in J$ *such that* $\{b, x\} \subseteq \alpha_j$, *then* $x \in (\alpha_j]$. *Dually, if for any* $b \in \alpha_i \cap [a)$ *there is* $j \in J$ *such that* $\{b, x\} \subseteq \alpha_j$, *then* $x \in [\alpha_j)$.
5. *If for any convex sublattice* $\mathcal{K}$ *of* $\mathcal{L}$ *and any* $a, b \in K$ *there is* $i \in I$ *such that* $a, b \in \alpha_i$, *then* $K \subseteq \alpha_j$ *for some* $j \in I$.

Moreover, if $\mathcal{L}$ *is a finite lattice, then the last two conditions follow from the conditions* 1–3.

In the case of lattices of finite height, blocks of any tolerance relation – as convex sublattices – are intervals and hence as the corollary of the above theorem we obtain the following theorem (see [18]).

Theorem 7. *Let* $\mathcal{L} = (L, \wedge, \vee)$ *be a lattice of a finite height. The covering* $\{\alpha_i\}_{i \in I}$ *of* L *is the family of blocks of a certain tolerance* T *on* $\mathcal{L}$ *iff the following conditions hold:*

1. *Every* α_i *for* $i \in I$ *is of the form* $[0_i, 1_i]$, *where* $0_i, 1_i \in L$ *and* $0_i < 1_i$.
2. *For distinct* $i, j \in I$, $0_i \neq 0_j$ *and* $1_i \neq 1_j$.
3. *For any* $i, j \in I$ *there are* $m, n \in I$ *such that*
$$1_i \vee 1_j \leq 1_m, \ \ 1_i \wedge 1_j = 1_n,$$
$$0_i \vee 0_j = 0_m, \ \ 0_i \wedge 0_j \geq 0_n.$$

3. Gluings and polarities

As we see from the previous section, any tolerance on a finite lattice (or more general, on a lattice of a finite height) decomposes the lattice into intervals, which themselves form a lattice (the quotient lattice). The intervals can be seen as the partial maps in an atlas and the factor lattice can be regarded as a general map indicating the relations between the partial maps. This attitude was adopted particularly by Wille in the theory of concept lattices, which forms a part of the theoretical basis for investigations on artificial intelligence ([17]).

The situation simplifies further when we focus on *glued* tolerances of a lattice $\mathcal{L}$. A tolerance T on $\mathcal{L}$ is called glued if its transitive closure is the total relation on the lattice.

In [13], Herrmann and Day proved that every finite lattice can be seen as an $\mathcal{L}/T$-gluing of a family of blocks of any glued tolerance T on $\mathcal{L}$. The notion of $\mathcal{K}$-gluing, where $\mathcal{K}$ is a finite lattice, was introduced by Herrmann in [21] as a natural generalization of the original Hall–Dilworth construction of gluing a filter and an ideal ([15]).

Wille describes the idea of Herrmann's construction as analogous to that used in drawing maps. If the area supposed to be set out in a map is too big to make the map legible, we can split it up into an atlas – a collection of maps covering the area together with an additional map which provides the information how the partial maps are related. It is common that the partial maps overlap to make data more accessible. Of course, the partial maps can be "glued" together to obtain the map of the whole area.

Let $(\mathcal{L}_i)_{i\in K}$ be a family of finite lattices and let an index set K be the underlying set of a finite lattice $\mathcal{K} = (K, \leq)$. The family $(\mathcal{L}_i)_{i\in K}$ is called a *$\mathcal{K}$-atlas with overlapping neighbours* if the following conditions hold for all $i, j \in K$:

(i) if $L_i \subseteq L_j$, then $i = j$;

(ii) if $i < j$ and there is no $k \in K$ such that $i < k < j$, then $L_i \cap L_j \neq \emptyset$;

(iii) if $i < j$ and $L_i \cap L_j \neq \emptyset$, then the orders of lattices $\mathcal{L}_i$ and $\mathcal{L}_j$ coincide on $L_i \cap L_j$ and the interval $L_i \cap L_j$ is at the same time a filter of $\mathcal{L}_i$ and an ideal of $\mathcal{L}_j$;

(iv) $L_i \cap L_j = L_{i\wedge j} \cap L_{i\vee j}$.

The structure $\mathcal{L} = (\bigcup L_{i\in K}, \leq)$, where $\leq$ is the transitive closure of the union of orders of all lattices $\mathcal{L}_i$ for $i \in K$, is called the *sum of the $\mathcal{K}$-atlas with overlapping neighbours* or simply the *$\mathcal{K}$-gluing* of the family $(\mathcal{L}_i)_{i\in K}$.

One can prove the following (see [13] or [17]).

THEOREM 8. *Let $(\mathcal{L}_i)_{i\in K}$ be a $\mathcal{K}$-atlas with overlapping neighbours. The sum of the $\mathcal{K}$-atlas is a lattice $\mathcal{L}$ for which the lattices $\mathcal{L}_i$, where $i \in K$, are blocks of some glued tolerance T on $\mathcal{L}$ and the mapping $i \mapsto L_i$ is an isomorphism of $\mathcal{K}$ onto the factor lattice $\mathcal{L}/T$.*

Conversely, if T is a glued tolerance on a lattice $\mathcal{L}$, then the family of blocks of T forms an $\mathcal{L}/T$-atlas with overlapping neighbours, whose $\mathcal{L}/T$-gluing is the lattice $\mathcal{L}$.

Since, as it is easy to notice, the intersection of a family of glued tolerances on a lattice $\mathcal{L}$ is again a glued tolerance, then there is the smallest (under inclusion) glued tolerance, which is called the *skeleton tolerance* of $\mathcal{L}$. The factor lattice of $\mathcal{L}$ by the skeleton tolerance is said to be the *skeleton* $S(\mathcal{L})$ of $\mathcal{L}$. Therefore, any finite lattice $\mathcal{L}$ can be seen as the $S(\mathcal{L})$-gluing of the blocks of its skeleton. It is particularly useful in the case of finite modular or distributive lattices since the blocks of their skeleton tolerance are regular and easy to describe (see for example [22, 14, 19]).

Example 9. *Figure 3 depicts a $\mathcal{K}$-atlas with overlapping neighbours along with the lattice $\mathcal{K}$. The $\mathcal{K}$-gluing of this atlas gives the three generated free distributive lattice $\mathcal{F}_D(3)$ presented in Figure 2. On the other hand, $\mathcal{B}_i$, where $i \in K$, are the blocks of the skeleton tolerance of $\mathcal{F}_D(3)$ and $\mathcal{K} = S(\mathcal{F}_D(3))$.*

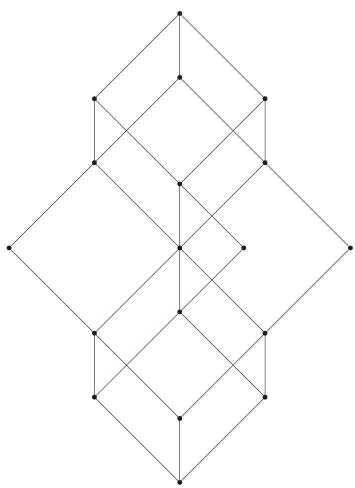

Fig. 2. The lattice $\mathcal{F}_D(3)$

We can notice now that the factor lattice of a distributive lattice need not be distributive, so it need not belong to the variety generated by it.

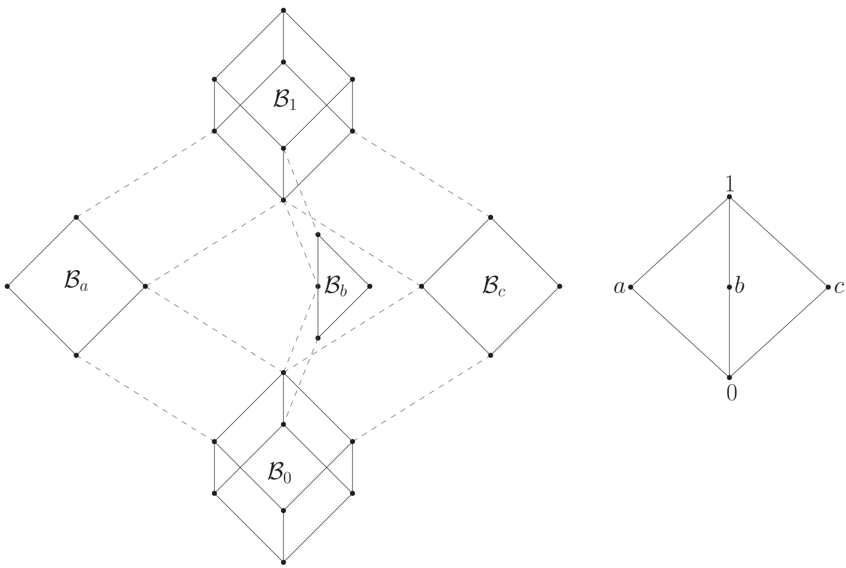

Fig. 3. The $\mathcal{K}$-atlas with overlapping neighbours (left) and the lattice $\mathcal{K}$ (right)

There is also another way of describing tolerances on finite lattices, namely by means of polarities (see [23]).

By a *polarity* in a lattice $\mathcal{L}$ we mean a pair (f, g) of mappings $L \to L$ such that f is a decreasing $\vee$-endomorphism, g is an increasing $\wedge$-endomorphism and $f(g(x)) \leq x \leq g(f(x))$ for every $x \in L$.

Hobby and McKenzie observed that there is a one-to-one connection between polarities and tolerance relations in finite lattices.

Theorem 10. *Let $\mathcal{L}$ be a finite lattice. If T is a tolerance on $\mathcal{L}$, then*

$$f(x) := \bigwedge\{y\colon (x, y) \in T\},$$

$$g(x) := \bigvee\{y\colon (x, y) \in T\}$$

define the polarity (f, g) such that

$$T = \{(x, y)\colon f(x \vee y) \leq x \wedge y\}.$$

On the other hand, if (f, g) is a polarity in a lattice $\mathcal{L}$, then there is exactly one tolerance on $\mathcal{L}$ such that f, g, T fulfill the above conditions.

Additionally, the tolerance T is glued iff the $\wedge$-endomorphism g defined above is strictly decreasing (or, what is equivalent, the $\vee$-endomorphism f is strictly increasing).

4. Homomorphic images of congruences

Let $\mathcal{A} = (A, F)$ and $\mathcal{B} = (B, F')$ be algebras with the same signature. If $\phi : \mathcal{A} \to \mathcal{B}$ is a surjective homomorphism and T is a tolerance on $\mathcal{A}$, then $\phi(T) = \{(\phi(x), \phi(y)) \colon (x, y) \in T\}$ is a tolerance on $\mathcal{B}$ (see [16]). In particular, if T is congruence, then $\phi(T)$ is a tolerance, but not necessarily a congruence. In other words, the homomorphic image of any congruence is a tolerance.

Czédli and Grätzer proved in [11] that in the case of lattices the inverse holds, *i.e.*, every lattice tolerance is a homomorphic image of some lattice congruence.

Theorem 11. *Let T be a tolerance of a lattice $\mathcal{L}$. Then there are a lattice $\mathcal{K}$, a congruence θ on $\mathcal{K}$ and a surjective homomorphism $\phi : \mathcal{K} \to \mathcal{L}$ such that $T = \phi(\theta)$.*

The construction of the lattice $\mathcal{K}$, the congruence θ and the homomorphism ϕ is very natural and based on facts proved by Czédli in [10]. Namely,

$$K := \{(\alpha, a) \colon \alpha \in L/T,\ a \in \alpha\},$$

the lattice operations on K are defined coordinate-wise, *i.e.*,

$$(\alpha, a) \wedge (\beta, b) := (\alpha \wedge \beta, a \wedge b),$$

and dually for $\vee$. The congruence θ is defined by

$$((\alpha, a), (\beta, b)) \in \theta \ \text{ iff } \alpha = \beta$$

and the homomorphism is given by $(\alpha, a) \mapsto a$.

Czédli and Kiss in [12] characterized by means of a Maltsev-like condition those varieties in which tolerances are homomorphic images of congruences of some algebras within the variety. They observed that among them there are all varieties of lattices, all varieties of unary algebras and the variety of semilattices. Chajda, Czédli, Halaŝ and Lipparini proved in [6] that also all varieties defined by a set of linear equations enjoy that property. By a linear identity they mean an identity $s = t$ such that each variable occurs at most once in each of the terms s and t.

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Bulletin of the **S**ection of **L**ogic
Volume 48/4 (2019), pp. 299–317

http://dx.doi.org/10.18778/0138-0680.48.4.04

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TWO TREATMENTS OF DEFINITE DESCRIPTIONS IN INTUITIONIST NEGATIVE FREE LOGIC

Abstract

Sentences containing definite descriptions, expressions of the form 'The F', can be formalised using a binary quantifier ι that forms a formula out of two predicates, where $\iota x[F, G]$ is read as 'The F is G'. This is an innovation over the usual formalisation of definite descriptions with a term forming operator. The present paper compares the two approaches. After a brief overview of the system $\mathbf{INF}^\iota$ of intuitionist negative free logic extended by such a quantifier, which was presented in [4], $\mathbf{INF}^\iota$ is first compared to a system of Tennant's and an axiomatic treatment of a term forming ι operator within intuitionist negative free logic. Both systems are shown to be equivalent to the subsystem of $\mathbf{INF}^\iota$ in which the G of $\iota x[F, G]$ is restricted to identity. $\mathbf{INF}^\iota$ is then compared to an intuitionist version of a system of Lambert's which in addition to the term forming operator has an operator for predicate abstraction for indicating scope distinctions. The two systems will be shown to be equivalent through a translation between their respective languages. Advantages of the present approach over the alternatives are indicated in the discussion.

Keywords: definite descriptions, binary quantifier, term forming operator, Lambert's Law, intuitionist negative free logic, natural deduction.

1. Introduction

Sentences of the form 'The F is G' can be formalised by using a binary quantifier ι that forms a formula out of two predicates as $\iota x[F, G]$. This provides an alternative to the usual way of formalising definite descriptions by means of an operator ι that forms a term out of a predicate, where ιxF is read as 'The F'. This paper is a comparison of the two approaches. The use of the same symbol ι for the binary quantifier and the term-forming operator should not lead to confusion, as context will make clear which one is meant. In [4], I presented the system $\mathbf{INF}^\iota$ of natural deduction for intuitionist negative free logic extended by the binary quantifier ι and proved a normalisation theorem for it.[1] The present paper begins with a brief overview of $\mathbf{INF}^\iota$, so that it can be read independently of the previous one. I will then compare $\mathbf{INF}^\iota$ to a system of Tennant's sketched in [9] and [8]. Tennant provides rules of natural deduction for a term-forming ι operator within the version of intuitionist negative free logic used here. After some clarification related to scope distinctions, it will be shown that Tennant's system is equivalent to the subsystem of $\mathbf{INF}^\iota$ in which the G of $\iota x[F, G]$ is restricted to identity. Both systems are also shown to be equivalent to an axiomatic treatment of a term forming ι operator within intuitionist negative free logic. I then compare $\mathbf{INF}^\iota$ to an intuitionist version of a system proposed by Lambert in [6], which in addition to the term forming operator has an operator for predicate abstraction for indicating scope distinctions. Both systems are shown to be equivalent by means of a translation between their respective languages. As we go along proving these equivalences, the present paper will also illustrate the workings of the rules for the binary quantifier ι with numerous examples of deductions in $\mathbf{INF}^\iota$, and advantages of the present approach over the common one will become apparent. In particular, in the formalisation of definite descriptions it is desirable to have a device for scope distinctions. The sole purpose of the abstraction operator in Lambert's system is as an indicator of scope. The formalism of the present system, by contrast, incorporates scope distinctions directly. Thus the formal treatment of definite descriptions with a binary quantifier is in this sense more economical than the approach using a term forming operator.[2]

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[1] For the proof-theory of term forming ι operators in the context of sequent calculi for classical logic, see [3] and [2].

[2] I would like to thank a referee for the *Bulletin* for the careful and helpful comments.

2. $\mathbf{INF}^{\iota}$

Let's begin with a review of intuitionist negative free logic **INF**. The rules for the propositional connectives are just those of intuitionist logic:

$$\wedge I: \quad \frac{A \qquad B}{A \wedge B} \qquad\qquad \wedge E: \quad \frac{A \wedge B}{A} \qquad \frac{A \wedge B}{B}$$

$$\to I: \quad \frac{\begin{array}{c}\overline{A}^{\,i}\\ \Pi \\ B\end{array}}{A \to B}\,i \qquad\qquad \to E: \quad \frac{A \to B \qquad A}{B}$$

$$\vee I: \quad \frac{A}{A \vee B} \qquad \frac{B}{A \vee B} \qquad\qquad \vee E: \quad \frac{A \vee B \qquad \begin{array}{c}\overline{A}^{\,i}\\ \Pi \\ C\end{array} \qquad \begin{array}{c}\overline{B}^{\,i}\\ \Sigma \\ C\end{array}}{C}\,i$$

$$\bot E: \quad \frac{\bot}{B}$$

where the conclusion of $\bot E$ is restricted to atomic formulas.

The rules for the quantifiers are relativised to an existence predicate:

$$\forall I: \quad \frac{\begin{array}{c}\overline{\exists! y}^{\,i}\\ \Pi \\ A^x_y\end{array}}{\forall x A}\,i \qquad\qquad \forall E: \quad \frac{\forall x A \qquad \exists! t}{A^x_t}$$

where in $\forall I$, y is not free in any undischarged assumption of Π except $\exists! y$, and either y is the same as x or y is not free in A; and in $\forall E$, t is free for x in A.

$$\exists I: \quad \frac{A^x_t \qquad \exists! t}{\exists x A} \qquad\qquad \exists E: \quad \frac{\exists x A \qquad \begin{array}{c}\underbrace{\overline{A^x_y}^{\,i}, \overline{\exists! y}^{\,i}}\\ \Pi \\ C\end{array}}{C}\,i$$

where in $\exists I$, t is free for x in A; and in $\exists E$, y is not free in C nor in any undischarged assumption of Π except A^x_y and $\exists! y$, and either y is the same as x or y is not free in A.

The existence predicate also appears in the premise of the introduction rule for identity; the elimination rule for $=$ is Leibniz' Law:

$$=I^n: \quad \frac{\exists! t}{t = t} \qquad\qquad =E: \quad \frac{t_1 = t_2 \qquad A^x_{t_1}}{A^x_{t_2}}$$

where A is an atomic formula and to exclude vacuous applications of $=E$, we can require that x occurs in A and that t_1 and t_2 are different.

Finally, there is the rule of atomic denotation:

$$AD: \quad \frac{At_1 \ldots t_n}{\exists! t_i}$$

where A is an n-place predicate letter (including identity) and $1 \leq i \leq n$. AD captures the semantic intuition that an atomic sentence can only be true if the terms that occur in it refer.

INF$^\iota$ has in addition the binary quantifier ι with the following rules:

$$\iota I: \quad \cfrac{F^x_t \qquad G^x_t \qquad \exists! t \qquad \begin{matrix} \underbrace{\overline{F^x_z}^{\,i}, \ \overline{\exists! z}^{\,i}} \\ \Pi \\ z = t \end{matrix}}{\iota x[F, G]}\ i$$

where t is free for x in F and in G, and z is different from x, not free in t and does not occur free in any undischarged assumption in Π except F^x_z and $\exists! z$.

$$\iota E^1: \quad \cfrac{\iota x[F, G] \qquad \begin{matrix} \underbrace{\overline{F^x_z}^{\,i}, \ \overline{G^x_z}^{\,i}, \ \overline{\exists! z}^{\,i}} \\ \Pi \\ C \end{matrix}}{C}\ i$$

where z is not free in C nor in any undischarged assumption of Π except F^x_z, G^x_z and $\exists! z$, and either z is the same as x or it is not free in F nor in G.

$$\iota E^2: \quad \frac{\iota x[F, G] \qquad \exists! t_1 \qquad \exists! t_2 \qquad F^x_{t_1} \qquad F^x_{t_2}}{t_1 = t_2}$$

where t_1 and t_2 are free for x in F.

INF$^\iota$ formalises a Russellian theory of definite descriptions, as $\iota x[F, G]$ and $\exists x(F \wedge \forall y(F^x_y \rightarrow x = y) \wedge G)$ are interderivable.

3. Comparison of $\mathbf{INF}^\iota$ with Tennant's system

To formalise definite descriptions using a term forming ι operator within intuitionist negative free logic, Tennant adds introduction and elimination rules for formulas of the form $\iota xF = t$ to **INF**:

$$\iota I^T: \quad \dfrac{\exists! t \qquad \begin{array}{c} \overline{z = t}^{\,i} \\ \Xi \\ F^x_z \end{array} \qquad \begin{array}{c} \underbrace{\overline{F^x_z}^{\,i}, \ \overline{\exists! z}^{\,i}} \\ \Pi \\ z = t \end{array}}{\iota xF = t}\, i$$

where in Ξ, z does not occur in any undischarged assumption except $z = t$, and either z is the same as x or it is not free in F; and in Π, z does not occur in any undischarged assumption except F^x_z and $\exists! z$.

$$\iota E^{1T}: \quad \dfrac{\iota xF = t \qquad u = t}{F^x_u} \qquad\qquad \iota E^{2T}: \quad \dfrac{\iota xF = t \qquad F^x_u \qquad \exists! u}{u = t}$$

$$\iota E^{3T}: \quad \dfrac{\iota xF = t}{\exists! t}$$

where u is free for x in F.

It is fairly evident that there are reduction procedures for removing maximal formulas of the form $\iota xF = t$ from deductions. ιE^{3T} is a special case of the rule of atomic denotation AD. Notice however that it is more properly regarded as an elimination rule for ι, as there is a reduction procedure for maximal formulas of the form $\iota xF = t$ that have been concluded by ιI^T and are premise of ιE^{3T}.

When negation is applied to $G(\iota xF)$, an ambiguity arises: is $\neg$ an internal negation, so that $\neg G(\iota xF)$ means 'The F is not G', or is it an external negation, so that the formula means 'It is not the case that the F is G'? Conventions or a syntactic device are needed to disambiguate. The language of Tennant's system makes no provision for distinguishing different scopes of negation. For this reason, in this section I shall restrict consideration to cases in which terms of the form ιxF occur to the left or right of $=$. I will consider a more complete system after the comparison of a restricted version of $\mathbf{INF}^\iota$ with Tennant's system.

It might be worth noting that there is a sense in which it suffices to consider occurrences of ι terms to the left or right of identity. Whenever we are tempted to use a formula $G(\iota xF)$, we can introduce a new individual constant c and use $G(c)$ and $\iota xF = c$ instead. Furthermore, in negative free logic, if G is a predicate letter, then $G(\iota xF)$ can be interpreted as $\exists y(G(y) \wedge \iota xF = y)$, and instead of the former, we can use the latter.[3] There is also no need to apply the existence predicate to ι terms, as instead of $\exists!\iota xA$ we can use $\exists y\ \iota xA = y$.

It is generally agreed that the minimal condition on a formalisation of a term forming ι operator is that it should obey *Lambert's Law*:

$$(LL) \qquad \forall y(\iota xF = y \leftrightarrow \forall x(F \leftrightarrow x = y))$$

Tennant's rules of ι are Lambert's Law cast in the form of natural deduction.

Call **INF** with its language modified to contain a term forming ι operator restricted to occurrences to the left or right of = and augmented by Tennant's rules $\mathbf{INF}^T$. Call the same modified system augmented by Lambert's Law as an axiom $\mathbf{INF}^{LL}$.

Under the current proposal of treating ι as a binary quantifier, where 'The F is G' is formalised as $\iota x[F, G]$, formulas of the form $\iota xF = t$ employing the term forming ι operator, which intuitively mean 'The F is identical to t', can be rendered as $\iota x[F, x = t]$. Treating $\iota x[F, x = t]$ and $\iota xF = t$ as notational variants, it is not difficult to show that $\mathbf{INF}^T$ is equivalent to the fragment of $\mathbf{INF}^\iota$ where the G of $\iota x[F, G]$ is restricted to identity. Call the latter system $\mathbf{INF}^{\iota R}$. For clarity, I will refer to the rules for the binary quantifier ι restricted to suit $\mathbf{INF}^{\iota R}$ by ιI^R, ιE^{1R} and ιE^{2R}.

It is now convenient to have rules for the biconditional $\leftrightarrow$:

$$\leftrightarrow I: \ \frac{\begin{array}{cc} \overline{A}^{\,i} & \overline{B}^{\,i} \\ \Pi & \Pi \\ B & A \end{array}}{A \leftrightarrow B}\,i$$

$$\leftrightarrow E^1: \ \frac{A \leftrightarrow B \qquad A}{B} \qquad \leftrightarrow E^2: \ \frac{A \leftrightarrow B \qquad B}{A}$$

For perspicuity, we will mark applications of the rules for the biconditional, of Tennant's rules for ι, and of ιI^R, ιE^{1R} and ιE^{2R} in the deductions to

[3] In positive free logic, only half of the insinuated equivalence holds, if predicates are allowed to form sentences from ι terms: then $\exists y(G(y) \wedge \iota xF = y)$ implies $G(\iota xF)$, but not conversely.

follow in the next paragraphs; unmarked inferences are by the more familiar rules of **INF**.

To show that $\mathbf{INF}^T$ is a subsystem of $\mathbf{INF}^{LL}$, we observe that, treating formulas of the form $\iota xF = t$ as atomic, ιE^{3T} is a special case of AD, and that ιE^{1T} and ιE^{2T} are derivable from (LL) by $\leftrightarrow E^1$. The following construction shows that ιI^T is also a derived rule of $\mathbf{INF}^{LL}$:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\overline{z=t}^{\;1}\\ \Xi\\ F^x_z\end{matrix}\qquad\qquad \begin{matrix}\underbrace{\overline{F^x_z}^{\;1},\ \overline{\exists! z}^{\;2}}\\ \Pi\\ z=t\end{matrix}}{F^x_z \leftrightarrow z=t}\,{\scriptstyle 1\ \leftrightarrow I}}{\forall x(F\leftrightarrow x=t)}\,{\scriptstyle 2}\qquad \dfrac{(LL)\qquad \exists! t}{\iota xF=t\leftrightarrow \forall x(F\leftrightarrow x=t)}}{\iota xF=t}\,{\scriptstyle \leftrightarrow E^2}$$

Hence $\mathbf{INF}^T$ is a subsystem of $\mathbf{INF}^{LL}$.

The next three paragraphs show that, if we write $\iota xF = t$ for $\iota x[F, x = t]$, the rules ιI^R, ιE^{1R} and ιE^{2R} of $\mathbf{INF}^{\iota R}$ are derived rules of $\mathbf{INF}^T$.

1. Due to the restriction on $\mathbf{INF}^{\iota R}$, applications of ιI^R are those cases of ιI in which G^x_t is an identity. So it can be any identity in which x is replaced by t and the other term is arbitrary, i.e. any identity $(x = u)^x_t$ or $t = u$ for short:

$$\dfrac{F^x_t\qquad t=u\qquad \exists! t\qquad \begin{matrix}\underbrace{\overline{F^x_z}^{\;i},\ \overline{\exists! z}^{\;i}}\\ \Pi\\ z=t\end{matrix}}{\iota x[F, x=t]}\,{\scriptstyle i\ \iota I^R}$$

To derive the rule it suffices to change notation and write $\iota xF = t$ instead of $\iota x[F, x = t]$, and to observe that $F^x_t, z = t \vdash F^x_z$ by Leibniz' Law and apply ιI^T:

$$\dfrac{\exists! t\qquad \dfrac{F^x_t\qquad \overline{z=t}^{\;i}}{F^x_z}\qquad \begin{matrix}\underbrace{\overline{F^x_z}^{\;i},\ \overline{\exists! z}^{\;i}}\\ \Pi\\ z=t\end{matrix}}{\iota xF=t}\,{\scriptstyle i\ \iota I^T}$$

The premise $t = u$ of ιI^R is redundant: a suitable identity can always be provided by deriving $(x = t)^x_t$, i.e. $t = t$, from the first premise $\exists!t$ by $= I^n$.

2. ιE^{1R} is derivable by changing notation and applying $\exists E$ with the major premise $\exists x(Fx \wedge x = t)$ derived from $\iota xF = t$ by ιE^{1T}, multiple applications of $= I^n$ and ιE^{T3}, and $\exists I$:

$$\dfrac{\dfrac{\dfrac{\iota xF = t \qquad \dfrac{\dfrac{\iota xF = t}{\exists!t}}{t = t}}{Ft}\,{\scriptstyle \iota E^{1T}} \qquad \dfrac{\dfrac{\iota xF = t}{\exists!t}}{t = t}}{Ft \wedge t = t} \qquad \dfrac{\iota xF = t}{\exists!t}}{\exists x(Fx \wedge x = t)}$$

For a more elegant deduction that does not make the detour through introducing and eliminating $\exists x(Fx \wedge x = t)$, given a deduction Π of C from F^z_x, $z = t$ and $\exists!z$, replace z with t throughout Π, and add deductions of $\iota xF = t \vdash Ft$, $\iota xF = t \vdash t = t$ and $\iota xF = t \vdash \exists!t$ to derive the three open premises.

3. Change of notation and two applications of ιE^{2T} and one of Leibniz' Law derive ιE^{2R}:

$$\dfrac{\dfrac{\iota xF = t \qquad \exists!t_1 \qquad F^x_{t_1}}{t = t_1}\,{\scriptstyle \iota E^{2T}} \qquad \dfrac{\iota xF = t \qquad \exists!t_2 \qquad F^x_{t_2}}{t = t_2}\,{\scriptstyle \iota E^{2T}}}{t_1 = t_2}$$

Thus $\mathbf{INF}^{\iota R}$ is a subsystem of $\mathbf{INF}^T$.

Finally, we derive (LL) in the version appropriate to $\mathbf{INF}^{\iota R}$, i.e. with $\iota xA = y$ replaced by $\iota x[A, x = y]$:

(LL') Lambert's Law: $\forall y(\iota x[A, x = y] \leftrightarrow \forall x(A \leftrightarrow x = y))$

1. $\iota x[A, x = y] \vdash \forall x(A \leftrightarrow x = y)$

$$\dfrac{\iota x[A, x = y] \qquad \dfrac{\dfrac{\xi \qquad \eta}{A \leftrightarrow x = y}\,{\scriptstyle 2\ \leftrightarrow I}}{\forall x(A \leftrightarrow x = y)}\,{\scriptstyle 3}}{\forall x(A \leftrightarrow x = y)}\,{\scriptstyle 4\ \iota E^{1R}}$$

where $\xi =$

$$\dfrac{\iota x[A, x = y] \qquad \overline{\exists! x}^{\,3} \qquad \dfrac{\overline{\exists! z}^{\,4} \qquad \overline{z = y}^{\,4}}{\exists! y} \qquad \dfrac{\overline{A^x_z}^{\,4} \qquad \overline{z = y}^{\,4}}{A^x_y} \qquad \overline{A}^{\,2}}{x = y}\,\iota E^{2R}$$

and $\eta =$

$$\dfrac{\iota x[A, x = y] \qquad \dfrac{\dfrac{\overline{A^x_z}^{\,1} \qquad \overline{z = y}^{\,1}}{A^x_y} \qquad \overline{x = y}^{\,2}}{A}}{A}\,1\ \iota E^{1R}$$

2. $\forall x(A \leftrightarrow x = y), \exists! y \vdash \iota x[A, x = y]$

$$\dfrac{\xi' \qquad \dfrac{\exists! y}{y = y} \qquad \exists! y \qquad \eta'}{\iota x[A, x = y]}\,1\ \iota I^{R}$$

where $\xi' =$

$$\dfrac{\dfrac{\forall x(A \leftrightarrow x = y) \qquad \exists! y}{A^x_y \leftrightarrow y = y} \qquad \dfrac{\exists! y}{y = y}}{A^x_y}$$

and $\eta' =$

$$\dfrac{\dfrac{\forall x(A \leftrightarrow x = y) \qquad \overline{\exists! z}^{\,1}}{A^x_z \leftrightarrow z = y} \qquad \overline{A^x_z}^{\,1}}{z = y}$$

Now from 1 and 2 by $\leftrightarrow I$, we have $\exists! y \vdash \iota x[A, x = y] \leftrightarrow \forall x(A \leftrightarrow x = y)$, and so by $\forall I$, $\vdash \forall y(\iota x[A, x = y] \leftrightarrow \forall x(A \leftrightarrow x = y))$.

Hence $\mathbf{INF}^{LL}$ is a subsystem of $\mathbf{INF}^{\iota R}$. This completes the circle, and we have shown:

THEOREM 1. *$\mathbf{INF}^{T}$, $\mathbf{INF}^{LL}$ and $\mathbf{INF}^{\iota R}$ are equivalent.*

4. Comparison of $\mathbf{INF}^\iota$ with an intuitionist version of a system of Lambert's

As noted towards the beginning of the previous section, in the absence of a formal device or a convention for distinguishing two ways of applying negation to $G(\iota xF)$, $\neg G(\iota xF)$ is ambiguous: $\neg$ can either be internal or external negation. To eliminate ambiguity, Lambert introduces an abstraction operator Δ that forms complex predicate terms ΔxB from open formulas B, and with the formation rule that if ΔxB is a predicate term and t an individual term, then $\Delta xB, t$ is a formula. Semantically, $\Delta xB, t$ is interpreted as true just in case t exists and Bt is true.[4] In this section I will compare $\mathbf{INF}^\iota$ to an intuitionist version of Lambert's system. Like Lambert, I will only consider unary predicates and keep the discussion fairly informal.[5]

In Lambert's system, Δ is governed by a principle regarded either as an axiom or as a contextual definition:

$$(\Delta t) \qquad \Delta xB, t \leftrightarrow (\exists! t \wedge B^x_t) \qquad (t \text{ free for } x \text{ in } B \text{ and } x \text{ not free in } t)$$

To formalise a free Russellian theory of definite descriptions, Lambert adds Lambert's Law and the following principle to negative free logic, also regarded either as an axiom or as a contextual definition:

$$(\Delta\iota) \qquad \Delta xB, \iota xA \leftrightarrow \exists z(\iota xA = z \wedge B^x_z)$$

Lambert uses a classical negative free logic, but in this section I will consider adding (LL), (Δt) and $(\Delta\iota)$ to **INF**. Call the resulting system $\mathbf{INF}^{LL\Delta}$. In this system, what we may call the *primary occurrences* of ι terms are those to the left or right of identity and which are governed by Lambert's Law. What we may call the *secondary occurrences* of ι terms are those introduced on the basis of the primary ones by the contextual definition $(\Delta\iota)$.

Lambert notes three characteristically Russellian theorems that are consequences of (LL), (Δt) and $(\Delta\iota)$:

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[4]For this and the following, see [6, 39ff].

[5]Lambert provides a more general treatment of an abstraction operator in classical positive free logic, but without a description operator, in [5]. A more complete and precise comparison of my treatment of definite description with Lambert's is reserved for sequels to this paper on the binary quantifier ι in intuitionist positive free logic and in negative and positive classical free logic. Fitting and Mendelsohn also employ predicate abstraction as a device for distinguishing scope within modal logic [1, Ch 12].

$(R1)$ $\quad \exists!\iota xA \leftrightarrow \exists y\forall x(A \leftrightarrow x = y)$

$(R2)$ $\quad \Delta xB, \iota yA \leftrightarrow \exists z(\forall y(A \leftrightarrow y = z) \wedge B^x_z)$

$(R3)$ $\quad \iota xA = t \rightarrow A^x_t \qquad$ (t free for x in A and x not free in t)

A further characteristically Russellian thesis mentioned by Morscher and Simons [7, 19] is worth listing:

$(R4)$ $\quad \exists!\iota xA \rightarrow A(\iota xA)$

We will show that $\mathbf{INF}^{LL\Delta}$ and $\mathbf{INF}^{\iota}$ are equivalent, and then, to take a convenient opportunity to illustrate the workings of the latter system, derive formulas corresponding to $(R1)$ to $(R4)$ in $\mathbf{INF}^{\iota}$.

In the present formalisation of ι as a binary quantifier, no conventions or syntactic devices are needed for the disambiguation of complex formulas involving ι. Ambiguity is avoided by the notation for the operator itself, which incorporates the relevant scope distinction. In this sense, the current formalisation of definite descriptions is more versatile than a formalisation using a term forming operator: it does the work of both, the term forming ι operator and the abstraction operator.

There is a certain redundancy in Lambert's axioms. $\exists!t \wedge B^x_t$ is equivalent to $\exists z(t = z \wedge B^x_z)$:[6]

$$\cfrac{\cfrac{\cfrac{\cfrac{\exists!t \wedge B^x_t}{\exists!t}}{t = t} \qquad \cfrac{\exists!t \wedge B^x_t}{B^x_t}}{t = t \wedge B^x_t} \qquad \cfrac{\exists!t \wedge B^x_t}{\exists!t}}{\exists z(t = z \wedge B^x_z)}$$

$$\cfrac{\exists z(t = z \wedge B^x_z) \qquad \cfrac{\cfrac{\cfrac{\overline{t = z \wedge B^x_z}^{\,1}}{z = t} \qquad \overline{\exists!z}^{\,1}}{\exists!t} \qquad \cfrac{\cfrac{\overline{t = z \wedge B^x_z}^{\,1}}{z = t} \qquad \cfrac{\overline{t = z \wedge B^x_z}^{\,1}}{B^x_z}}{B^x_t}}{\exists!t \wedge B^x_t}}{\exists!t \wedge B^x_t}\,1$$

[6]The second deduction is constructed so as not to appeal to any rules of **INF** that are not also rules of the system **IPF** of [4, Sec 3]. The first deduction can be adjusted to **IPF** by deducing $t = t$ from no premises by $= I$.

This means that there is a uniform treatment of the Δ operator, irrespective of whether the term a predicate abstract is applied to is an ι term or not, and one axiom suffices to replace (Δt) and $(\Delta\iota)$:

$$(\Delta t') \quad \Delta xB, t \leftrightarrow \exists z(t = z \wedge B^x_z) \quad (t \text{ free for } z \text{ in } B \text{ and } z \text{ not free in } t)$$

This works only for a Russellian theory of definite descriptions, however: an alternative theory of definite descriptions within positive free logic may be intended to provide room for the option that $\Delta xB, \iota xA$ is true even though there is no unique A: such a theory may contain (Δt) but not $(\Delta\iota)$.

Furthermore, $\Delta xB, t$ is equivalent to $\Delta xB, \iota x(x = t)$, both being equivalent to $\exists z(t = z \wedge B^x_z)$. Thus there is a sense in which nothing is lost from Lambert's system if the formation rules for the abstraction operator were reformulated so as to require a predicate and an ι term to form a formula out of them. The ι symbol, being embedded within the Δ operator, could then just as well be omitted, so that Δ forms a formula out of two predicates, which is exactly how the ι operator works in $\mathbf{INF}^\iota$. Of course what is crucial for Lambert's system is Lambert's Law, and in his formulation of it Δ does not occur. The present system is thus in a sense more economical than Lambert's.

We can emulate Lambert's use of both, the abstraction operator and the term forming ι operator, in the present system: $\Delta xG, \iota xF$ is translated as $\iota x[F, G]$, and where t is not an ι term, $\Delta xA, t$ is translated as $\iota x[t = x, A]$: instead of naming an object and applying a predicate to it, we pick out the object by a predicate that is true at most of it. Then what is expressed by $\iota xA = y$ in Lambert's system is expressed in $\mathbf{INF}^\iota$ by $\iota x[A, x = y]$, and what is expressed by $\exists!\iota xA$ is expressed by $\iota x[A, \exists!x]$.

A little more precisely, to show that $\mathbf{INF}^{LL\Delta}$ and $\mathbf{INF}^\iota$ are equivalent, observe that their languages differ only in that the former has Δ and the term forming ι, which the latter lacks, and in that the latter has the binary quantifier ι, which the former lacks. We construct a translation τ from the language of $\mathbf{INF}^{LL\Delta}$ to the language of $\mathbf{INF}^\iota$. Atomic sentences and those containing operators other than Δ and ι are translated homophonically:

(a) if A is atomic formula not containing any ι terms, then $\tau(A) = A$,
(b) if the main operator of A is a unary operator $*$ (i.e. $*$ is $\neg$, $\exists$ or $\forall$), then $\tau(*B) = *\tau(B)$,
(c) if $*$ is a binary sentential operator, then $\tau(A * B) = \tau(A) * \tau(B)$.

Next, the primary occurrences of ι terms:

(d.i) $\tau(\iota xA = t) = \iota x[\tau(A), x = t]$; similarly for $t = \iota xA$ (i.e. $\tau(\iota xA = \iota yB) = \iota x[\tau(A), \iota y[\tau(B), x = y]]$).

For formulas containing Δ and the secondary occurrences of ι terms, we need a distinction:

(e.i) if t is not an ι term, then $\tau(\Delta xB, t) = \iota x[t = x, \tau(B)]$,
(e.ii) if t is an ι term ιxA, then $\tau(\Delta xB, t) = \iota x[\tau(A), \tau(B)]$.

To construct a translation υ from the language of $\mathbf{INF}^\iota$ to the language of $\mathbf{INF}^{LL\Delta}$, we recycle clauses (a) to (c) of τ and add only $\upsilon(\iota x[A, B]) = \Delta x\upsilon(B), \iota x\upsilon(A)$, letting the contextual definitions (Δt) and $(\Delta\iota)$ do the rest.

Let $\tau(\Gamma)$, $\upsilon(\Gamma)$ be the set of formulas in Γ translated by τ, υ. We have:

THEOREM 2. $\mathbf{INF}^\iota$ *is equivalent to* $\mathbf{INF}^{LL\Delta}$*: (a) if* $\Gamma \vdash A$ *in* $\mathbf{INF}^\iota$*, then* $\upsilon(\Gamma) \vdash \upsilon(A)$ *in* $\mathbf{INF}^{LL\Delta}$*; (b) if* $\Gamma \vdash A$ *in* $\mathbf{INF}^{LL\Delta}$*, then* $\tau(\Gamma) \vdash \tau(A)$ *in* $\mathbf{INF}^\iota$*.*

Proof. (a) It suffices to observe that the introduction and elimination rules for ι of $\mathbf{INF}^\iota$ remain valid under the translation υ, due to the equivalence of $\iota x[F, G]$ with $\exists x(F \wedge \forall y(F^x_y \rightarrow y = x) \wedge G)$ and $(R2)$. (b) It suffices to prove the translations of (LL), (Δt) and $(\Delta\iota)$ under τ in $\mathbf{INF}^\iota$:

(LL^τ) $\forall y(\iota x[\tau(A), x = y] \leftrightarrow \forall x(\tau(A) \leftrightarrow x = y))$

(Δt^τ) $\iota x[x = t, \tau(A)] \leftrightarrow (\exists! t \wedge \tau(A)^x_t)$ (t free for x in $\tau(A)$ and x not free in t)

$(\Delta\iota^\tau)$ $\iota x[\tau(A), \tau(B)] \leftrightarrow \exists z(\iota x[\tau(A), x = z] \wedge \tau(B)^x_z)$

For readability I will prove these equivalences 'schematically', it being understood that the formulas A and B in the deductions to follow are translations under τ.[7] Then (LL^τ) is (LL'), which we proved earlier. The other two we prove next.

(Δt^τ) $\iota x[x = t, A] \leftrightarrow (\exists! t \wedge A^x_t)$ (t free for x in A and x not free in t)

[7]From an alternative perspective, the provability of these equivalences shows that adding (LL), (Δt) and $(\Delta\iota)$ to $\mathbf{INF}^\iota$ does not increase its expressive power, as for each formula containing the term forming ι operator and Δ, there is a provably equivalent one containing only the binary quantifier ι.

1. $\iota x[x=t,A] \vdash \exists! t \wedge A^x_t$

$$\dfrac{\iota x[t=x,A] \qquad \dfrac{\dfrac{\overline{\exists! x}^{\,1} \quad \overline{t=x}^{\,1}}{\exists! t} \qquad \dfrac{\overline{A}^{\,1} \quad \overline{t=x}^{\,1}}{A^x_t}}{\exists! t\wedge A^x_t}}{\exists! t\wedge A^x_t}\;{\scriptstyle 1\ \iota E^1}$$

2. $\exists! t \wedge A^x_t \vdash \iota x[x=t,A]$

$$\dfrac{\dfrac{\dfrac{\exists! t\wedge A^x_t}{\exists! t}}{t=t} \qquad \dfrac{\exists! t\wedge A^x_t}{A^x_t} \qquad \dfrac{\exists! t\wedge A^x_t}{\exists! t} \qquad \overline{z=t}^{\,1}}{\iota x[x=t,A]}\;{\scriptstyle 1\ \iota I}$$

This is a correct application of ιI: F^x_t is $(x=t)^x_t$, i.e. $t=t$, and F^x_z is $(x=t)^x_z$, i.e. $z=t$. $\exists! z$ is discharged vacuously.

$(\Delta\iota^\tau)$ $\iota x[A,B] \leftrightarrow \exists z(\iota x[A,x=z] \wedge B^x_z)$

1. $\iota x[A,B] \vdash \exists z(\iota x[A,x=z] \wedge B^x_z)$

$$\dfrac{\iota x[A,B] \qquad \dfrac{\dfrac{\xi \qquad \overline{B^x_z}^{\,2}}{\iota x[A,x=z]\wedge B^x_z} \qquad \overline{\exists! z}^{\,2}}{\exists z(\iota x[A,x=z]\wedge B^x_z)}}{\exists z(\iota x[A,x=z]\wedge B^x_z)}\;{\scriptstyle 2\ \iota E^1}$$

where $\xi =$

$$\dfrac{\overline{A^x_z}^{\,2} \qquad \dfrac{\overline{\exists! z}^{\,2}}{z=z} \qquad \overline{\exists! z}^{\,2} \qquad \dfrac{\iota x[A,B] \quad \overline{\exists! x}^{\,1} \quad \overline{\exists! z}^{\,2} \quad \overline{A}^{\,1} \quad \overline{A^x_z}^{\,2}}{x=z}\;{\scriptstyle \iota E^2}}{\iota x[A,x=z]}\;{\scriptstyle 1\ \iota I}$$

2. $\exists z(\iota x[A,x=z] \wedge B^x_z) \vdash \iota x[A,B]$

First, $\iota x[A,x=z], B^x_z \vdash \iota x[A,B]$:

$$\dfrac{\iota x[A,x=z] \qquad \dfrac{\dfrac{\overline{A}^{\,2} \quad \overline{x=z}^{\,2}}{A^x_z} \qquad B^x_z \qquad \overline{\exists! z}^{\,2} \qquad \eta}{\iota x[A,B]}\;{\scriptstyle 1\ \iota I}}{\iota x[A,B]}\;{\scriptstyle 2\ \iota E^1}$$

where $\eta =$

$$\dfrac{\iota x[A, x = z] \qquad \overline{\exists ! y}^{\,1} \qquad \overline{\exists ! z}^{\,2} \qquad \overline{A^x_y}^{\,1} \qquad \dfrac{\overline{A}^{\,2} \qquad \overline{x = z}^{\,2}}{A^x_z}}{y = z}\,\iota E^2$$

Thus $\iota x[A, x = z] \wedge B^x_z \vdash \iota x[A, B]$, and so $\exists z(\iota x[A, x = z] \wedge B^x_z) \vdash \iota x[A, B]$. In this last application of $\exists E$, $\exists ! z$ is discharged vacuously. Notice that it would have been possible to discharge only one (or indeed none) of the $\exists ! z$ by ιE^2, and the discharge the other (or both) by the application of $\exists E$.

This completes the proof of Theorem 2.

Under translation τ, $(R1)$, $(R2)$, $(R3)$ and $(R4)$ become:

$(R1^\tau)$ $\iota x[\tau(A), \exists ! x] \leftrightarrow \exists y \forall x(\tau(A) \leftrightarrow x = y)$

$(R2^\tau)$ $\iota x[\tau(A), \tau(B)] \leftrightarrow \exists z(\forall y(\tau(A) \leftrightarrow y = z) \wedge \tau(B)^x_z)$

$(R3^\tau)$ $\iota x[\tau(A), x = t] \rightarrow \tau(A)^x_t$ (t free for x in $\tau(A)$ and x not free in t)

$(R4^\tau)$ $\iota x[\tau(A), \exists ! x] \rightarrow \iota x[\tau(A), \tau(A)]$

$(R2^\tau)$ follows from the interderivability of $\exists x(A \wedge \forall y(A^x_y \rightarrow x = y) \wedge B)$ with $\iota x[A, B]$ (see [4, 90f]). The rest are proved on the following pages, once more 'schematically' and with τ suppressed for readability. The proofs presuppose a judicious choice of variables.

$(R1^\tau)$ $\iota x[A, \exists ! x] \leftrightarrow \exists y \forall x(A \leftrightarrow x = y)$

1. $\iota x[A, \exists ! x] \vdash \exists y \forall x(A \leftrightarrow x = y)$

$$\dfrac{\iota x[A, \exists ! x] \qquad \dfrac{\dfrac{\dfrac{\xi \qquad \dfrac{\overline{A^x_y}^{\,3} \qquad \overline{x = y}^{\,1}}{A}}{A \leftrightarrow x = y}\,1 \leftrightarrow I}{\forall x(A \leftrightarrow x = y)}\,2 \qquad \overline{\exists ! y}^{\,3}}{\exists y \forall x(A \leftrightarrow x = y)}}{\exists y \forall x(A \leftrightarrow x = y)}\,3\ \iota E^1$$

where $\xi =$

$$\dfrac{\iota x[A, \exists ! x] \qquad \overline{\exists ! x}^{\,2} \qquad \overline{\exists ! y}^{\,3} \qquad \overline{A}^{\,1} \qquad \overline{A^x_y}^{\,3}}{x = y}\,\iota E^2$$

2. $\exists y \forall x(A \leftrightarrow x = y) \vdash \iota x[A, \exists! x]$

$$\cfrac{\exists y \forall x(A \leftrightarrow x = y) \qquad \cfrac{\eta_1 \qquad \cfrac{}{\exists!^x_y}\,{}^{2} \qquad \cfrac{}{\exists! y}\,{}^{2} \qquad \eta_2}{\iota x[A, \exists! x]}\ {\scriptstyle 1\ \iota I}}{\iota x[A, \exists! x]}\ {\scriptstyle 2}$$

where $\eta_1 =$

$$\cfrac{\cfrac{\cfrac{}{\forall x(A \leftrightarrow x = y)}\,{}^{2} \qquad \cfrac{}{\exists! y}\,{}^{2}}{A^x_y \leftrightarrow y = y} \qquad \cfrac{\cfrac{}{\exists! y}\,{}^{2}}{y = y}}{A^x_y}\ {\scriptstyle \leftrightarrow E^2}$$

and $\eta_2 =$

$$\cfrac{\cfrac{\cfrac{}{\forall x(A \leftrightarrow x = y)}\,{}^{2} \qquad \cfrac{}{\exists! v}\,{}^{1}}{A^x_v \leftrightarrow v = y} \qquad \cfrac{}{A^x_v}\,{}^{1}}{v = y}\ {\scriptstyle \leftrightarrow E^1}$$

$(R3^\tau)$ $\iota x[A, x = t] \rightarrow A^x_t$ (t free for x in A and x not free in t)

$$\cfrac{\iota x[A, x = t] \qquad \cfrac{\cfrac{}{A}\,{}^{1} \qquad \cfrac{}{x = t}\,{}^{1}}{A^x_t}}{A^x_t}\ {\scriptstyle 1\ \iota E^1}$$

We also have $\iota x[A, x = t] \rightarrow \exists! t$ (x not free in t):

$$\cfrac{\iota x[A, x = t] \qquad \cfrac{\cfrac{}{\exists! x}\,{}^{1} \qquad \cfrac{}{x = t}\,{}^{1}}{\exists! t}}{\exists! t}\ {\scriptstyle 1\ \iota E^1}$$

Hence $\iota x[A, x = t] \rightarrow (\exists! t \wedge A^x_t)$, and so by (Δt^τ), $\iota x[A, x = t] \rightarrow \iota x[x = t, A]$. We do not, however, have the converse. $(\exists! t \wedge A^x_t) \rightarrow \iota x[A, x = t]$ is not true. $\iota x[A, x = t]$ means 'The A is identical to t', and this does not follow from the existence of a t which is A, i.e. $\exists! t \wedge A^x_t$.

$(R4^\tau)$ $\iota x[A, \exists! x] \vdash \iota x[A, A]$

$$\cfrac{\iota x[A, \exists! x] \qquad \cfrac{\cfrac{}{A^x_z}\,{}^{2} \qquad \cfrac{}{A^x_z}\,{}^{2} \qquad \cfrac{}{\exists! z}\,{}^{2} \qquad \xi}{\iota x[A, A]}\ {\scriptstyle 1\ \iota I}}{\iota x[A, A]}\ {\scriptstyle 2\ \iota E^1}$$

where $\xi =$

$$\dfrac{\iota x[A,\exists!x] \qquad \overline{\exists!v}^{\,1} \qquad \overline{\exists!z}^{\,2} \qquad \overline{A^x_v}^{\,1} \qquad \overline{A^x_z}^{\,2}}{v=z}\;\iota E^2$$

To close this section, a few words about $\forall E$, $\exists I$ and $=E$. In systems where ι is a term forming operator, ι terms can be used as terms instantiating universal generalisations, as terms over which to generalise existentially and as terms to the left or right of identity in Leibniz's Law. To establish that the current system is as versatile as a system in which this is possible, it remains to be shown that these uses of ι terms can be reconstructed in the present formalism. In other words, we need to show:

$(\forall\iota)$ $\quad \forall xB, \iota x[A,\exists!x] \vdash \iota x[A,B]$

$(\exists\iota)$ $\quad \iota x[A,B], \iota x[A,\exists!x] \vdash \exists xB$

$(=\iota)$ $\quad B^x_t, \iota x[A,x=t] \vdash \iota x[A,B]$

An inference concluding the existence of an ι term by AD is a special case of $\iota x[F,G] \vdash \iota x[F,\exists!x]$, which holds by $(R2^\tau)$, $(R1^\tau)$ and general logic. I will only show that $(\forall\iota)$ and $(=\iota)$ hold, the proof of $(\exists\iota)$ being similar.

$(\forall\iota)$ $\;\forall xB, \iota x[A,\exists!x] \vdash \iota x[A,B]$

$$\dfrac{\iota x[A,\exists!x] \qquad \dfrac{\overline{A^x_z}^{\,2} \qquad \dfrac{\forall xB \qquad \overline{\exists!z}^{\,2}}{B^x_z} \qquad \overline{\exists!z}^{\,2} \qquad \xi}{\iota x[A,B]}\;1\;\iota I}{\iota x[A,B]}\;2\;\iota E^1$$

where $\xi =$

$$\dfrac{\iota x[A,\exists!x] \qquad \overline{\exists!y}^{\,1} \qquad \overline{\exists!z}^{\,2} \qquad \overline{A^x_y}^{\,1} \qquad \overline{A^x_z}^{\,2}}{y=z}\;\iota E^2$$

$(=\iota)$ $\;B^x_t, \iota x[A,x=t] \vdash \iota x[A,B]$

$$\dfrac{\iota x[A,x=t] \qquad \dfrac{\dfrac{\overline{x=t}^{\,2} \qquad \overline{A}^{\,2}}{A^x_t} \qquad B^x_t \qquad \dfrac{\overline{\exists!x}^{\,2} \qquad \overline{x=t}^{\,2}}{\exists!t} \qquad \eta}{\iota x[A,B]}\;1\;\iota I}{\iota x[A,B]}\;2\;\iota E^1$$

where $\eta =$

$$\dfrac{\iota x[A, x=t] \qquad \overline{\exists! v}^{\,1} \qquad \dfrac{\overline{x=t}^{\,2} \qquad \overline{\exists! x}^{\,2}}{\exists! t} \qquad \overline{A^x_v}^{\,1} \qquad \dfrac{\overline{x=t}^{\,2} \qquad \overline{A}^{\,2}}{A^x_t}}{v=t}\,\iota E^2$$

5. Conclusion and further work

The present formalism has certain advantages over the use of ι as a term forming operator. It incorporates scope distinctions within the notation, without the need for an abstraction operator or other syntactic devices or conventions. It provides a natural formalisation of a theory of definite descriptions, here developed within intuitionist negative free logic. The resulting system has desirable proof-theoretic properties, as deductions in it normalise, and it is equivalent to well known axiomatic theories of definite descriptions.

Scope distinctions are of particular interest to the development of a theory of definite descriptions within modal logic. Fitting and Mendelsohn, for instance, provide a detailed account of definite descriptions within quantified modal logic [1, Ch 12], which uses an abstraction operator for scope distinction. They observe that scope distinctions are already needed for formulas containing individual constants, if they are not interpreted rigidly, and so they introduce predicate abstraction well before definite descriptions. However, in their system, as in Lambert's, predicate abstraction does not appear to play any further role than marking scope distinctions. The present notation provides a perspicuous way of distinguishing the scope of modal operators that is independent of abstraction operators:

It is possible that the F is G: $\Diamond \iota x[F, G]$
The F is possibly G: $\iota x[F, \Diamond G]$.
The possible F is G: $\iota x[\Diamond F, G]$

For scope distinctions with regard to non-rigidly interpreted individual constants, we can use the technique of simulating the use of a constant t by a predicate $x = t$ introduced earlier. It would be worth comparing the approach proposed here with Fitting's and Mendelsohn's, but this must wait for another occasion.

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Bulletin of the **S**ection of **L**ogic
Volume 48/4 (2019), pp. 319–328

http://dx.doi.org/10.18778/0138-0680.48.4.05

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JM13714

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DISJUNCTIVE MULTIPLE-CONCLUSION CONSEQUENCE RELATIONS

Abstract

The concept of multiple-conclusion consequence relation from [8] and [7] is considered. The closure operation C assigning to any binary relation r (defined on the power set of a set of all formulas of a given language) the least multiple-conclusion consequence relation containing r, is defined on the grounds of a natural Galois connection. It is shown that the very closure C is an isomorphism from the power set algebra of a simple binary relation to the Boolean algebra of all multiple-conclusion consequence relations.

Keywords: multiple-conclusion consequence relation, closure operation, Galois connection.

1. Preliminaries

Given a set A, any mapping $C : \wp(A) \longrightarrow \wp(A)$ such that for each $X, Y \subseteq A$, $X \subseteq C(X)$, $C(C(X)) \subseteq C(X)$ and C is monotone: $X \subseteq Y \Rightarrow C(X) \subseteq C(Y)$, is called a *closure operation* defined on the power set $\wp(A)$ of A. Any subset $\mathcal{B} \subseteq \wp(A)$ is said to be a *closure system over* A (or *of the complete lattice* $(\wp(A), \subseteq)$), if for each $\mathcal{X} \subseteq \mathcal{B}$, $\bigcap \mathcal{X} \in \mathcal{B}$. Given a closure operation C on $\wp(A)$, the set of all its fixed points called *closed elements*: $Cl(C) = \{X \subseteq A : X = C(X)\}$, is a closure system over A. Conversely, given a closure system $\mathcal{B}$ over A, the mapping $C : \wp(A) \longrightarrow \wp(A)$ defined by $C(X) = \bigcap\{Y \in \mathcal{B} : X \subseteq Y\}$, is a closure operation on $\wp(A)$. The closure system $\mathcal{B}$ is just the set of all its closed elements. On the other hand, the closure system $Cl(C)$ of all closed elements of a given closure operation C defines, in that way, just the operation C. Thus, there is a one to one correspondence between the class of all closure operations defined on $\wp(A)$ and of all closure systems of $(\wp(A), \subseteq)$, in fact, it is a dual isomorphism between the respective complete lattices of all closure operations and closure systems (the poset $(\mathcal{C}(A), \leq)$ of all closure operations defined on $\wp(A)$, where $C_1 \leq C_2$ iff $C_1(X) \subseteq C_2(X)$ for each $X \subseteq A$, forms a complete lattice such that for any class $\mathcal{E} \subseteq \mathcal{C}(A)$ its infimum, $\inf \mathcal{E}$, is a closure operation defined on $\wp(A)$ by $(\inf \mathcal{E})(X) = \bigcap\{C(X) : C \in \mathcal{E}\}$). Any closure system $\mathcal{B}$ of $(\wp(A), \subseteq)$ forms a complete lattice with respect to the order $\subseteq$ such that $\inf \mathcal{X} = \bigcap \mathcal{X}$ and $\sup \mathcal{X} = C(\bigcup \mathcal{X})$, for each $\mathcal{X} \subseteq \mathcal{B}$, where C is the closure operation corresponding to closure system $\mathcal{B}$. Given a family $\mathcal{X} \subseteq \wp(A)$, there exists the least closure system $\mathcal{B}$ of $(\wp(A), \subseteq)$ such that $\mathcal{X} \subseteq \mathcal{B}$. It is called a *closure system generated by* $\mathcal{X}$ and shall be denoted by $[\mathcal{X}]$. It is simply the intersection of all closure systems of $(\wp(A), \subseteq)$ containing $\mathcal{X}$ and is expressed by $[\mathcal{X}] = \{\bigcap \mathcal{Y} : \mathcal{Y} \subseteq \mathcal{X}\}$. The closure operation C corresponding to closure system $[\mathcal{X}]$ is defined by $C(X) = \bigcap\{Y \in \mathcal{X} : X \subseteq Y\}$, any $X \subseteq A$.

When A is a set of all formulae of a given formal language, a closure operation C defined on $\wp(A)$ is called a *consequence operation (in the sense of Tarski)*.

We shall apply here the standard (called sometimes archetypal) anti-monotone Galois connection (f, g) defined on the complete lattices $(\wp(A), \subseteq)$, $(\wp(B), \subseteq)$ of all subsets of given sets A, B by a binary relation $R \subseteq A \times B$ (cf. [3], a general theory is to be found for example in [1, 2, 4]). That is, $f : \wp(A) \longrightarrow \wp(B)$ and $g : \wp(B) \longrightarrow \wp(A)$ are the mappings defined for any $X \subseteq A,\ a \in A,\ Y \subseteq B,\ b \in B$ by

$$b \in f(X) \ \text{ iff } \ \text{for all } x \in X,\ (x, b) \in R,$$

$$a \in g(Y) \ \text{ iff } \ \text{for all } y \in Y,\ (a, y) \in R.$$

The following three facts are useful for our goals.

The compositions $f \circ g,\ g \circ f$ are closure operations on $\wp(A),\ \wp(B)$, respectively.

The set $Cl(f \circ g)$ of all closed sets with respect to closure operation $f \circ g$ is the counterdomain of map g: $\{X \subseteq A : g(f(X)) = X\} = \{g(Y) : Y \subseteq B\}$ and similarly, $Cl(g \circ f) = \{Y \subseteq B : f(g(Y)) = Y\} = \{f(X) : X \subseteq A\}$.

The mapping f restricted to $Cl(f \circ g)$ is a dual isomorphism of the complete lattices $(Cl(f \circ g), \subseteq)$, $(Cl(g \circ f), \subseteq)$ as well as the map g restricted to $Cl(g \circ f)$ is the inverse dual isomorphism.

2. The concept of disjunctive multiple-conclusion consequence relation

This what will be called here a *disjunctive* consequence relation recalls the concept of multiple-conclusion entailment or multiple-conclusion consequence relation [7, 8]. In [8, p. 28] the following definition of multiple-conclusion consequence relation was introduced. Let V be a set of all formulae of a given language. For any $\mathcal{T} \subseteq \wp(V)$ a binary relation $\vdash_{\mathcal{T}}$ is defined on $\wp(V)$ by

(mc) $X \vdash_{\mathcal{T}} Y$ iff $\forall\, T \in \mathcal{T}\ (X \subseteq T \ \Rightarrow\ Y \cap T \neq \emptyset)$.

We say that $\vdash \subseteq \wp(V) \times \wp(V)$ is a multiple-conclusion consequence relation iff $\vdash = \vdash_{\mathcal{T}}$ for some $\mathcal{T} \subseteq \wp(V)$. Next the authors of [8] prove the theorem (2.1, p. 30):

A relation $\vdash$ is a multiple-conclusion consequence relation iff it satisfies the following conditions for any $X, Y \subseteq V$:

$(overlap)$ $X \cap Y \neq \emptyset \ \Rightarrow\ X \vdash Y$,

$(dilution)$ $X \vdash Y,\ X \subseteq X',\ Y \subseteq Y' \ \Rightarrow\ X' \vdash Y'$,

$(cut for sets)$ $\forall S \subseteq V\ ((\forall Z \subseteq S,\ X \cup Z \vdash Y \cup (S - Z)) \ \Rightarrow\ X \vdash Y)$.

Given $S \subseteq V$, the part $(\forall Z \subseteq S,\ X \cup Z \vdash Y \cup (S - Z)) \ \Rightarrow\ X \vdash Y$ of the condition $(cut for sets)$ is called $(cut for S)$. In turn, $(cut for formulae)$ denotes the family of all the conditions $(cut for \{\alpha\})$, $\alpha \in V$:

$(cut for \{\alpha\})$ $X \vdash Y \cup \{\alpha\}\ \&\ X \cup \{\alpha\} \vdash Y \ \Rightarrow\ X \vdash Y$,

that is, stands to the cut rule of [5] from 1934. In general, granted $(dilution)$, the conditions $(cut for sets)$ and $(cut for V)$ are equivalent (Theorem 2.2 in [8], p. 31). Moreover, when a binary relation $\vdash \subseteq \wp(V) \times \wp(V)$ satisfies not only $(dilution)$ but also is compact, i.e fulfils the condition

$(compactness)$ $X \vdash Y \ \Rightarrow\$ there exist finite subsets $X' \subseteq X,\ Y' \subseteq Y$ such that $X' \vdash Y'$, both conditions $(cut for sets)$, $(cut for formulae)$ are equivalent (Theorem 2.9 in [8], p. 37).

The conditions $(overlap), (dilution), (cut for formulae)$, under different names, were used to define on finite sets of formulas, the relation of multiple-conclusion entailment by D. Scott [7].

In [11] it was proved that when a family $\mathcal{T} \subseteq \wp(V)$ is a closure system over V, the consequence relation $\vdash_{\mathcal{T}}$ defined by (mc), may be expressed by

(dis) $X \vdash_{\mathcal{T}} Y$ iff $Y \cap C_{\mathcal{T}}(X) \neq \emptyset$,

where $C_{\mathcal{T}}$ is the closure operation determined by closure system $\mathcal{T}$. As it is seen, given a set of premises X some of conclusions of the consequence relation $\vdash_{\mathcal{T}}$ are conclusions of ordinary consequence operation $C_{\mathcal{T}}$ associated with the relation. So, one may say that the relation $\vdash_{\mathcal{T}}$ has a *disjunctive* character. It is worth to notice that in general, for arbitrary family $\mathcal{T} \subseteq \wp(V)$ only the implication $(\Leftarrow)$ from right to left holds true, where in case, $C_{\mathcal{T}}$ is the closure operation (consequence operation) determined by the family $\mathcal{T}$ (that is, by $[\mathcal{T}]$ – the least closure system over V containing $\mathcal{T}$): for a formula $\alpha \in V$, $\alpha \in C_{\mathcal{T}}(X)$ iff for any $T \in \mathcal{T}$, $X \subseteq T \Rightarrow \alpha \in T$.

Hereafter the consequence relations $\vdash_{\mathcal{T}}$, $\mathcal{T} \subseteq \wp(V)$ will be called *disjunctive*. Let $DR = \{\vdash_{\mathcal{T}} : \mathcal{T} \subseteq \wp(V)\}$.

3. Galois connection for disjunctive consequence relation

Taking into account the very definition of disjunctive consequence relation from the previous section (cf. (mc)), the following Galois connection (f, g) should be considered. Put $R \subseteq \wp(V)^2 \times \wp(V)$ of the form $((X, Y), T) \in R$ iff $X \subseteq T \Rightarrow Y \cap T \neq \emptyset$. So $f : (\wp(\wp(V) \times \wp(V)), \subseteq) \longrightarrow (\wp(\wp(V)), \subseteq)$, $g : (\wp(\wp(V)), \subseteq) \longrightarrow (\wp(\wp(V) \times \wp(V)), \subseteq)$ are defined for any relation $r \subseteq \wp(V) \times \wp(V)$ and any family $\mathcal{T} \subseteq \wp(V)$ by

$T \in f(r)$ iff for all $X, Y \subseteq V$ such that $(X, Y) \in r$, $X \subseteq T$ implies that $Y \cap T \neq \emptyset$, any $T \subseteq V$,

$(X, Y) \in g(\mathcal{T})$ iff for all $T \in \mathcal{T}$, $X \subseteq T$ implies that $Y \cap T \neq \emptyset$, any $X, Y \subseteq V$.

In more handy formulation,

(1) $T \in f(r)$ iff $\forall X, Y \subseteq V\ (X \subseteq T \subseteq -Y \;\Rightarrow\; (X, Y) \notin r)$,

(2) $(X, Y) \in g(\mathcal{T})$ iff $\forall T \subseteq V\ (X \subseteq T \subseteq -Y \;\Rightarrow\; T \notin \mathcal{T})$,

where "$-$" is the operation of complementation in the Boolean algebra of all subsets of V.

Let us put $C = f \circ g$ and $C' = g \circ f$, that is, C is a closure operation defined on $\wp(\wp(V) \times \wp(V))$ assigning to each binary relation r defined on $\wp(V)$ the least relation from DR containing r (the operation C is the counterpart of closure introduced in [6, p. 1006, definition 3.1] for Scott's multiple-conclusion relations from [7]); in turn C' is a closure operation whose closed sets correspond via dual isomorphism f restricted to DR to disjunctive consequence relations. Using (1) and (2) we obtain that for any binary relation $r \subseteq \wp(V) \times \wp(V)$, $(X, Y) \in C(r)$ iff $(X, Y) \in g(f(r))$ iff $\forall T \subseteq V\ (X \subseteq T \subseteq -Y \;\Rightarrow\; T \notin f(r))$ iff $\forall T \subseteq V\ (X \subseteq T \subseteq -Y \;\Rightarrow\; \exists U, Z \subseteq V\ (U \subseteq T \subseteq -Z\ \&\ (U, Z) \in r))$. Finally,

(3) $(X, Y) \in C(r)$ iff $[X, -Y] \subseteq \bigcup\{[U, -Z] : (U, Z) \in r\}$,

where for any $X, Y \subseteq V$, $[X, Y] = \{U \subseteq V : X \subseteq U \subseteq Y\}$. However, the equivalence:

(4) $(X, Y) \in C(r)$ iff $\forall T \subseteq V\ (X \subseteq T \subseteq -Y \;\Rightarrow\; T \notin f(r))$,

is also interesting since from it one may derive that for any set $T \subseteq V$ and any binary relation $r \subseteq \wp(V) \times \wp(V)$,

(5) $T \in f(r)$ iff $(T, -T) \notin C(r)$.

Similarly, for any family $\mathcal{T} \subseteq \wp(V)$: $T \in C'(\mathcal{T})$ iff $T \in f(g(\mathcal{T}))$ iff $\forall X, Y \subseteq V\ (X \subseteq T \subseteq -Y \;\Rightarrow \exists T' \subseteq V (X \subseteq T' \subseteq -Y\ \&\ T' \in \mathcal{T}))$ iff $T \in \mathcal{T}$. In this way, C' is the identity mapping on $\wp(\wp(V))$ so $Cl(C') = Cl(g \circ f) = \wp(\wp(V))$. On the other hand, $Cl(C) = Cl(f \circ g) = \{g(\mathcal{T}) : \mathcal{T} \subseteq \wp(V)\} = \{\vdash_{\mathcal{T}} : \mathcal{T} \subseteq \wp(V)\} = DR$. Thus we have the following corollary.

Corollary. *The mapping f restricted to DR (that is f defined for each $r \in DR$ by $f(r) = \{T \subseteq V : (T, -T) \notin r\}$ due to (5)) is a dual isomorphism of the complete lattices $(DR, \subseteq)$, $(\wp(\wp(V)), \subseteq)$ and the mapping g is the inverse dual isomorphism.*

This result, obtained first in [11] without application of Galois connection, can be strengthened (cf. also [11]) to a dual isomorphism of complete and atomic Boolean algebras $(DR, \cap, \vee, -, \vdash_0, \wp(V)^2)$, $(\wp(\wp(V)), \cap, \cup, -, \emptyset, \wp(V))$, by equipping the family DR of disjunctive relations with the operation of Boolean complementation – in such a way that the dual isomorphism of complete lattices preserves it : $-r = -g(f(r)) = g(\wp(V) - f(r)) = g(\{T \subseteq V : (T, -T) \in r\})$. Here for any $r_1, r_2 \in DR$, $r_1 \vee r_2 = C(r_1 \cup r_2)$ and $\vdash_0 = g(\wp(V)) = \{(X, Y) : X \cap Y \neq \emptyset\}$ is the least disjunctive relation.

4. Isomorphism theorem for disjunctive consequence relations

Let us put $\mathcal{R}_0 = \{(T, -T) : T \subseteq V\}$. Consider the mapping $p : \wp(\mathcal{R}_0) \longrightarrow \wp(\wp(V))$ defined by $p(\rho) = \{T \subseteq V : (T, -T) \in \rho\}$. It is obvious that p is a Boolean and complete isomorphism of Boolean algebras $(\wp(\mathcal{R}_0), \cap, \cup, -, \emptyset, \mathcal{R}_0)$, $(\wp(\wp(V)), \cap, \cup, -, \emptyset, \wp(V))$. Consider the following composition of mappings:

$$\wp(\mathcal{R}_0) \ni \rho \longmapsto p(\rho) \longmapsto \wp(V) - p(\rho) \longmapsto g(\wp(V) - p(\rho)) \in DR.$$

The correspondence $\wp(\wp(V)) \ni \mathcal{T} \longmapsto \wp(V) - \mathcal{T}$ is obviously a dual Boolean complete isomorphism from $(\wp(\wp(V)), \cap, \cup, -, \emptyset, \wp(V))$ onto itself. So the composition $\wp(\mathcal{R}_0) \ni \rho \longmapsto g(\wp(V) - p(\rho)) \in DR$ (one isomorphism and two dual isomorphisms are here composed) is a complete Boolean isomorphism from $(\wp(\mathcal{R}_0), \cap, \cup, -, \emptyset, \mathcal{R}_0)$ onto $(DR, \cap, \vee, -, \vdash_0, \wp(V)^2)$.

Using (2) one may calculate the value of that isomorphism on a $\rho \subseteq \mathcal{R}_0$: for any $X, Y \subseteq V$, $(X, Y) \in g(\wp(V) - p(\rho))$ iff $[X, -Y] \subseteq p(\rho)$. Moreover, from (3) we have

(6) $(X, Y) \in C(\rho)$ iff $[X, -Y] \subseteq \bigcup\{[T, T] : (T, -T) \in \rho\}$ iff $[X, -Y] \subseteq p(\rho)$.

Therefore, for any $\rho \subseteq \mathcal{R}_0$, $C(\rho) = g(\wp(V) - p(\rho))$. Furthermore, one may consider the inverse isomorphism as the following composition:

$DR \ni r \longmapsto f(r) \longmapsto \wp(V) - f(r) = \{T \subseteq V : (T, -T) \in r\}$ (by (5)) $\longmapsto r \cap \mathcal{R}_0$.

In this way the following result is proved.

PROPOSITION. *The closure operation C (assigning to each binary relation r defined on $\wp(V)$ the least disjunctive relation containing r) restricted to the power set of $\mathcal{R}_0 = \{(T, -T) : T \subseteq V\}$ is a Boolean and complete isomorphism from the power set algebra $(\wp(\mathcal{R}_0), \cap, \cup, -, \emptyset, \mathcal{R}_0)$ onto atomic and complete Boolean algebra $(DR, \cap, \vee, -, \vdash_0, \wp(V)^2)$ of all disjunctive relations defined on the language V. The inverse isomorphism, say $h : DR \longrightarrow \wp(\mathcal{R}_0)$ is defined by $h(r) = r \cap \mathcal{R}_0$. In this way, for any $r \in DR$ and $\rho \subseteq \mathcal{R}_0$, $r = C(r \cap \mathcal{R}_0)$ and $\rho = C(\rho) \cap \mathcal{R}_0$.*

5. Some applications

Applying (6) one may show that for any $T_1, T_2 \subseteq V$ such that $T_1 \subseteq T_2$ and for any $X, Y \subseteq V$,

(7) $(X, Y) \in C(\{(T, -T) : T \in [T_1, T_2]\})$ iff either $X \vdash_0 Y$ or $T_1 \subseteq X \subseteq -Y \subseteq T_2$.

In particular, using (7) and Proposition, one may find a form of atoms in the Boolean algebra $(DR, \cap, \vee, -, \vdash_0, \wp(V)^2)$ of all disjunctive relations. Let us take any atom $\{(T, -T)\}$, $T \subseteq V$, of $(\wp(\mathcal{R}_0), \cap, \cup, -, \emptyset, \mathcal{R}_0)$. Then the corresponding atom in the Boolean algebra of all disjunctive relations is of the form:

(8) $C(\{(T, -T)\}) = \vdash_0 \cup \{(T, -T)\}$.

The coatoms of $(DR, \cap, \vee, -, \vdash_0, \wp(V)^2)$ are much more interesting. Take any $T \subseteq V$. Then the corresponding coatom in this Boolean algebra to the coatom $\mathcal{R}_0 - \{(T, -T)\}$ of $(\wp(\mathcal{R}_0), \cap, \cup, -, \emptyset, \mathcal{R}_0)$ is, due to (6) and (mc), of the form

(9) $(X, Y) \in C(\mathcal{R}_0 - \{(T, -T)\})$ iff $[X, -Y] \subseteq \wp(V) - \{T\}$ iff either $X \not\subseteq T$ or $Y \cap T \neq \emptyset$ iff $X \vdash_{\{T\}} Y$.

More figuratively,

(10) $C(\mathcal{R}_0 - \{(T, -T)\}) = \vdash_{\{T\}} = \bigcup\{[(\{\alpha\}, \emptyset)) : \alpha \notin T\} \cup \bigcup\{[(\emptyset, \{\alpha\})) : \alpha \in T\}$,

where for any $X, Y \subseteq V$, $[(X, Y)) = \{(X', Y') \in \wp(V)^2 : X \subseteq X'$ & $Y \subseteq Y'\}$.

The following lemma provides a useful characteristics of coatoms.

LEMMA. *For any* $\vdash \in DR$ *and* $T \subseteq V$, $\vdash = \vdash_{\{T\}}$ *iff for each* $\alpha \in V$, $(\emptyset \vdash \{\alpha\}$ *iff* $\alpha \in T)$ *and* $(\{\alpha\} \vdash \emptyset$ *iff* $\alpha \notin T)$.

PROOF. Consider any disjunctive relation $\vdash$ and $T \subseteq V$.

($\Rightarrow$): By (10).

($\Leftarrow$): Assume that for each $\alpha \in V$, $(\emptyset \vdash \{\alpha\}$ iff $\alpha \in T)$ and $(\{\alpha\} \vdash \emptyset$ iff $\alpha \notin T)$. First we show that $\vdash_{\{T\}} \subseteq \vdash$. So suppose that $X \vdash_{\{T\}} Y$, that is, either $X \nsubseteq T$ or $Y \cap T \neq \emptyset$. In the first case, from the assumption it follows that $\{\alpha\} \vdash \emptyset$ for some $\alpha \in X$ so $X \vdash Y$ by (*dilution*). In the second case, analogously, $\emptyset \vdash \{\alpha\}$ for some $\alpha \in Y$ so $X \vdash Y$. Now notice that $\vdash_{\{T\}}$ is a coatom in the Boolean algebra of all disjunctive relations, therefore the inclusion $\vdash_{\{T\}} \subseteq \vdash$ implies that $\vdash_{\{T\}} = \vdash$ or $\vdash = \wp(V)^2$. Since the relation $\wp(V)^2$ does not satisfy the assumption we obtain $\vdash_{\{T\}} = \vdash$. □

The coatoms in the Boolean algebra of all disjunctive consequence relations are easily expressible in terms of [7]. In order to show this let us apply the definition from [7, p. 416], for any disjunctive relation. A relation $\vdash \in DR$ is said to be *consistent* (*complete*) iff for any $\alpha \in V$, either $\emptyset \nvdash \{\alpha\}$ or $\{\alpha\} \nvdash \emptyset$ (for any $\alpha \in V$, either $\emptyset \vdash \{\alpha\}$ or $\{\alpha\} \vdash \emptyset$). In this way, for any $\vdash \in DR$,

(11) $\vdash$ is consistent and complete iff for any $\alpha \in V$, $\emptyset \vdash \{\alpha\}$ iff $\{\alpha\} \nvdash \emptyset$.

FACT. *For any* $\vdash \in DR$, $\vdash$ *is consistent and complete iff for some* $T \subseteq V$, $\vdash = \vdash_{\{T\}}$.

PROOF. Consider any disjunctive relation $\vdash$.

($\Rightarrow$): Assume that $\vdash$ is consistent and complete. Put $T = \{\alpha \in V : \emptyset \vdash \{\alpha\}\}$. Then from the assumption and (11) it follows that $-T = \{\alpha \in V : \{\alpha\} \vdash \emptyset\}$. In this way, $\vdash = \vdash_{\{T\}}$ due to Lemma.

($\Leftarrow$): Immediately from Lemma and (11). □

In the light of this fact, the result of [7] that any multiple-conclusion consequence relation is an intersection of all consistent and complete relations containing it, becomes absolutely clear. Since for every $\vdash \in DR$, the identity $\vdash = \bigcap\{\vdash_{\{T\}} : \vdash \subseteq \vdash_{\{T\}}\}$ holds. In turn, the latter connection is an obvious consequence of the following one: $\rho = \bigcap\{\mathcal{R}_0 - \{(T, -T)\} : (T, -T) \notin \rho\}$, any $\rho \subseteq \mathcal{R}_0$ (implying together with Proposition and (9) that $C(\rho) = \bigcap\{C(\mathcal{R}_0 - \{(T, -T)\}) : \rho \subseteq \mathcal{R}_0 - \{(T, -T)\}\} = \bigcap\{\vdash_{\{T\}} : C(\rho) \subseteq \vdash_{\{T\}}\})$.

Notice that the power set $\wp(\mathcal{R}_0)$ is closed on the operation $^\sim$ of taking the converse relation. Applying (6) for a given $\rho \subseteq \mathcal{R}_0$ we have $(X,Y) \in C(\rho^\sim)$ iff $[X,-Y] \subseteq p(\rho^\sim)$ iff $[X,-Y] \subseteq \{-T : T \in p(\rho)\}$ iff $[Y,-X] \subseteq p(\rho)$ iff $(Y,X) \in C(\rho)$ iff $(X,Y) \in C(\rho)^\sim$. Hence, $C(\rho^\sim) = C(\rho)^\sim$ so the operation $^\sim$ is preserved under the isomorphism C and the set DR is closed on this operation. Denoting for a given family $\mathcal{T} \subseteq \wp(V)$, $\mathcal{T}^\sim = \{-T : T \in \mathcal{T}\}$ we have $g(\mathcal{T}^\sim) = g(\mathcal{T})^\sim$ due to (2), that is, in terms of (mc):

(12) $\vdash_{\mathcal{T}^\sim} = \vdash_{\mathcal{T}}^\sim$.

Given $\vdash \in DR$ the relation $\vdash^\sim$ could be called *dual with respect to* $\vdash$. For example, assume that V is the set of all formulas of propositional language equipped with the standard connectives $\neg, \wedge, \vee, \rightarrow$ and let Val be the set of all Boolean valuations of the language into $\{0,1\}$. Consider the disjunctive relation $\vdash_{\mathcal{T}_{Max}}$ determined (according to (mc)) by the family of all maximal theories of classical propositional logics $\mathcal{T}_{Max} = \{T_v : v \in Val\}$, where for each $v \in Val$, $T_v = \{\alpha \in V : v(\alpha) = 1\}$ (cf. also [9, p. 242, definition 1]):

$X \vdash_{\mathcal{T}_{Max}} Y$ iff $\forall v \in Val(X \subseteq T_v \Rightarrow Y \cap T_v \neq \emptyset)$ iff $\forall v \in Val(v[X] \subseteq \{1\} \Rightarrow \exists \alpha \in Y,\ v(\alpha) = 1)$.

The dual relation with respect to $\vdash_{\mathcal{T}_{Max}}$ is, according to (12), determined by the family $\mathcal{T}_{Max}^\sim = \{\{\alpha \in V : v(\alpha) = 0\} : v \in Val\}$ (notice that the consequence operation corresponding to the closure system $[\mathcal{T}_{Max}^\sim]$ over V is dual in the sense of Wójcicki [10] with respect to the consequence operation of classical propositional logic, that is, corresponding to the closure system $[\mathcal{T}_{Max}]$). One may consider the dual disjunctive relation with respect to a coatom $\vdash_{\{T\}}$, $T \subseteq V$ which is the coatom $\vdash_{\{-T\}}$ (cf. also (10)). In particular $\vdash_{\{-T_v\}}$, $v \in Val$ is considered in [9, p. 245, definition 3].

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