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Hitoshi Omori

A NOTE ON CIUCIURA'S $\mathbf{mbC}^1$

Abstract

This note offers a non-deterministic semantics for $\mathbf{mbC}^1$, introduced by Janusz Ciuciura, and establishes soundness and (strong) completeness results with respect to the Hilbert-style proof system. Moreover, based on the new semantics, we briefly discuss an unexplored variant of $\mathbf{mbC}^1$ which has a contra-classical flavor.

Keywords: paraconsistent logic, non-deterministic semantics contra-classical logic

1. Introduction

In [10], Janusz Ciuciura introduces a system $\mathbf{mbC}^1$ of paraconsistent logic, formulated in the language of classical logic. The aim of this note is to present a non-deterministic semantics for $\mathbf{mbC}^1$, different from the semantics presented in [10], and prove its soundness and (strong) completeness. And in view of this new semantics, we will briefly discuss, in the last section, an unexplored variant of $\mathbf{mbC}^1$ which has a contra-classical flavor.

2. Proof system for $\mathbf{mbC}^1$

Let the languages $\mathcal{L}$ and $\mathcal{L}_\circ$ consist of a finite set $\{\sim, \wedge, \vee, \rightarrow\}$ and $\{\sim, \circ, \wedge, \vee, \rightarrow\}$ of propositional connectives respectively and a countable set Prop of propositional variables which we denote by p, q, etc. Furthermore, we denote by Form and $\mathsf{Form}_\circ$ the sets of formulas defined as usual in $\mathcal{L}$ and $\mathcal{L}_\circ$ respectively. We denote a formula of the languages by A, B, C, etc. and a set of formulas of the languages by Γ, Δ, Σ, etc.

First, we introduce **CLuN** which is the common core of many systems of paraconsistent logic, including $\mathbf{mbC}^1$.

DEFINITION 1. The system **CLuN** consists of the following axioms and a rule of inference.

$$A \to (B \to A) \tag{A1}$$

$$(A \to (B \to C)) \to ((A \to B) \to (A \to C)) \tag{A2}$$

$$((A \to B) \to A) \to A \tag{A3}$$

$$A \to (A \vee B) \tag{A4}$$

$$B \to (A \vee B) \tag{A5}$$

$$(A \to C) \to ((B \to C) \to ((A \vee B) \to C)) \tag{A6}$$

$$(A \wedge B) \to A \tag{A7}$$

$$(A \wedge B) \to B \tag{A8}$$

$$(C \to A) \to ((C \to B) \to (C \to (A \wedge B))) \tag{A9}$$

$$A \vee {\sim} A \tag{A10}$$

$$\frac{A \quad A \to B}{B} \tag{MP}$$

Moreover, we write $\Gamma \vdash_{\mathbf{CLuN}} A$ if there is a sequence of formulas $B_1, \ldots, B_n, A$, $n \geq 0$, such that every formula in the sequence $B_1, \ldots, B_n, A$ either (i) belongs to Γ; (ii) is an axiom of **CLuN**; (iii) is obtained by (MP) from formulas preceding it in sequence.

Second, we introduce $\mathbf{mbC}^1$ and, for the sake of comparison, **mbC**, one of the basic systems within the family of Logics of Formal Inconsistency (cf. [8, 7]).

DEFINITION 2. The system $\mathbf{mbC}^1$ is formulated in $\mathcal{L}$ and obtained by adding the following formula to **CLuN**.

$$A \to ({\sim} A \to ({\sim}{\sim} A \to B)) \tag{$*$}$$

Moreover, the system **mbC** is formulated in $\mathcal{L}_\circ$ and obtained by adding the following formula to **CLuN**.

$$\circ A \to (A \to ({\sim} A \to B))$$

We then define $\vdash_{\mathbf{mbC}^1}$ and $\vdash_{\mathbf{mbC}}$ in a similar manner.

Here are two remarks on the relation between $\mathbf{mbC}^1$ and **mbC**.

Remark 3. Ciuciura notes that $\mathbf{mbC}^1$ is "an axiomatization of **mbC** formulated directly in the language of classical propositional logic" ([10, p. 173]). This is, however, not true due to the following result:

$$\nvdash_{\mathbf{mbC}} p \to ({\sim}p \to ({\sim}{\sim}p \to q)).$$

This may be observed by the following truth table for **LFI1**, an extension of **mbC**, introduced in [9].

A	$\sim A$	$\circ A$
t	**f**	**t**
b	**b**	**f**
f	**t**	**t**

$A \wedge B$	**t**	**b**	**f**
t	**t**	**b**	**f**
b	**b**	**b**	**f**
f	**f**	**f**	**f**

$A \vee B$	**t**	**b**	**f**
t	**t**	**t**	**t**
b	**t**	**b**	**b**
f	**t**	**b**	**f**

$A \to B$	**t**	**b**	**f**
t	**t**	**b**	**f**
b	**t**	**b**	**f**
f	**t**	**t**	**t**

Note here that both **t** and **b** are designated values. Then, the axioms of **mbC** are all validated, and designated values are preserved by the above truth table. However, the concerned formula takes the non-designated value **f** when we assign **b** and **f** to p and q respectively.

Remark 4. Note that if one takes the consistency to be *defined* as $\circ A =_{\text{def.}} A{\to}{\sim}{\sim}A$, then **mbC** *with this definition of consistency* becomes equivalent to $\mathbf{mbC}^1$ since $A \wedge \circ A$ is equivalent to $A \wedge {\sim}{\sim}A$ under the above definition of $\circ$. For a system of paraconsistent logic having this kind of definition of consistency, see [22, 21].

3. Non-deterministic semantics for $\mathbf{mbC}^1$

In [10], Ciuciura already offers a semantics for $\mathbf{mbC}^1$ along the line of what is sometimes called the bivaluational semantics which has been one of the most popular semantics for a wide range of LFIs. However, there is another semantics known in the literature of LFIs, namely the non-deterministic semantics, established systematically by Arnon Avron and Iddo Lev in [4] (see [5] for a survey on non-deterministic semantics). The semantics has a nice feature being an intuitive generalization of many-valued semantics. In this section, we present non-deterministic semantics for $\mathbf{mbC}^1$.

Definition 5. A $\mathbf{mbC}^1$-*non-deterministic matrix* (*Nmatrix* for short) for $\mathcal{L}$ is a tuple $M = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where:

(a) $\mathcal{V} = \{\mathbf{t}, \mathbf{b}, \mathbf{f}\}$,

(b) $\mathcal{D} = \{\mathbf{t}, \mathbf{b}\}$,

(c) For every n-ary connective $*$ of $\mathcal{L}$, $\mathcal{O}$ includes a corresponding n-ary function $\tilde{*}$ from $\mathcal{V}^n$ to $2^{\mathcal{V}} \setminus \{\emptyset\}$ as follows (we omit the brackets for sets):

A	$\tilde{\sim}A$
t	**f**
b	**t**
f	**t, b**

$A \tilde{\wedge} B$	**t**	**b**	**f**
t	**t, b**	**t, b**	**f**
b	**t, b**	**t, b**	**f**
f	**f**	**f**	**f**

$A \tilde{\vee} B$	**t**	**b**	**f**
t	**t, b**	**t, b**	**t, b**
b	**t, b**	**t, b**	**t, b**
f	**t, b**	**t, b**	**f**

$A \tilde{\rightarrow} B$	**t**	**b**	**f**
t	**t, b**	**t, b**	**f**
b	**t, b**	**t, b**	**f**
f	**t, b**	**t, b**	**t, b**

A *legal* $\mathbf{mbC}^1$*-valuation* in an $\mathbf{mbC}^1$-Nmatrix M is a function $v : \mathsf{Form} \rightarrow \mathcal{V}$ that satisfies the following condition for every n-ary connective $*$ of $\mathcal{L}$ and $A_1, \ldots, A_n \in \mathsf{Form}$:

$$v(*(A_1, \ldots, A_n)) \in \tilde{*}(v(A_1), \ldots, v(A_n)).$$

Finally, A is a *legal* $\mathbf{mbC}^1$ *consequence* of Γ ($\Gamma \models_{\mathbf{mbC}^1} A$) iff for every legal $\mathbf{mbC}^1$-valuation v, if $v(B) \in \mathcal{D}$ for every $B \in \Gamma$ then $v(A) \in \mathcal{D}$.

Remark 6. Note that a three-valued non-deterministic semantics for **CLuN** is introduced in [2] by Avron. The only difference is that the table for negation is replaced by the following table:

A	$\tilde{\sim}A$
t	**f**
b	**t, b**
f	**t, b**

That is, there is one more non-determinacy when the negated sentence receives the value **b**. Furthermore, there is also a two-valued non-deterministic semantics for **CLuN** devised in [3]. Note also that non-deterministic semantics for **mbC** and its extensions are considered in [1] (the system **mbC** is referred to as **B** in [1]). However, the above matrix is not considered in the literature, at least to the best of author's knowledge.

Remark 7. The system $\mathbf{mbC}^1$ may be seen as a generalization of Sette's $\mathbf{P}^1$ developed in [19]. Indeed, the addition of the following formulas will eliminate the nonclassical value in the non-deterministic bits in the above matrix and give us the system $\mathbf{P}^1$.

- $\sim\sim A \to A$
- $(A * B) \to \sim\sim(A * B)$ where $* \in \{\wedge, \vee, \to\}$

For a recent discussion on discussive semantics for $\mathbf{P}^1$, see [16].

4. Soundness and completeness

We now turn to prove the soundness and completeness. The proof will be rather simple if the reader is already familiar with non-deterministic semantics, but for the purpose of making this note self-contained as much as possible, I will spell them out in some details.

The soundness is easy as usual.

PROPOSITION 1 (Soundness). *If* $\Gamma \vdash_{\mathbf{mbC}^1} A$ *then* $\Gamma \models_{\mathbf{mbC}^1} A$.

PROOF: Straightforward. □

For the completeness result, we first list some formulas that are provable in $\mathbf{mbC}^1$.

PROPOSITION 2. *The following formulas are provable in* $\mathbf{mbC}^1$*:*

$$A \vee (A \to B) \tag{4.1}$$

$$A \to (B \to (A \wedge B)) \tag{4.2}$$

PROOF: We safely leave the details to the readers. □

Second, we introduce the following standard notions.

DEFINITION 8. Let Σ be a set of formulas. Then,

- Σ is a *theory* iff it is closed under $\vdash$, i.e., if $\Sigma \vdash A$ then $A \in \Sigma$ for any formula A;
- Σ is *prime* iff $A \vee B \in \Sigma$ implies that $A \in \Sigma$ or $B \in \Sigma$ for any A and B;
- Σ is *non-trivial* iff for some formula A, $A \notin \Sigma$.

REMARK 9. Strictly speaking, we do need to specify the consequence relation in defining theories. However, in the following, we will omit that since contexts will disambiguate.

The following lemma is the well-known lemma of Lindenbaum.

LEMMA 1. *For any* $\Sigma \cup \{A\} \subseteq \mathsf{Form}$, *if* $\Sigma \nvdash A$ *then, there is a prime theory* $\Pi \supseteq \Sigma$ *such that* $\Pi \nvdash A$.

Moreover, we need the following lemma.

LEMMA 2. *Let* Σ *be a non-trivial prime theory, and define a function* v_0 *from* Form *to* $\mathcal{V}$ *as follows.*

$$v_0(B) := \begin{cases} \mathbf{t} & \textit{if } \Sigma \vdash_{\mathbf{mbC^1}} B \textit{ and } \Sigma \nvdash_{\mathbf{mbC^1}} {\sim}B \\ \mathbf{b} & \textit{if } \Sigma \vdash_{\mathbf{mbC^1}} B \textit{ and } \Sigma \vdash_{\mathbf{mbC^1}} {\sim}B \\ \mathbf{f} & \textit{if } \Sigma \nvdash_{\mathbf{mbC^1}} B \end{cases}$$

Then, v_0 *is a legal* $\mathbf{mbC}^1$*-valuation.*

PROOF: By induction on the number n of connectives.
(Base): for atomic formulas, it immediately follows that v_0 is a function.
(Induction step): We split the cases based on the connectives.
Case 1. If $B = {\sim}C$, then we have the following three cases.

Cases	$v(C)$	condition for C	$v(B)$	condition for B i.e. ${\sim}C$
(i)	$\mathbf{t}$	$\Sigma \vdash C$ and $\Sigma \nvdash {\sim}C$	$\mathbf{f}$	$\Sigma \nvdash {\sim}C$
(ii)	$\mathbf{b}$	$\Sigma \vdash C$ and $\Sigma \vdash {\sim}C$	$\mathbf{t}$	$\Sigma \vdash {\sim}C$ and $\Sigma \nvdash {\sim\sim}C$
(iii)	$\mathbf{f}$	$\Sigma \nvdash C$	$\mathbf{t}, \mathbf{b}$	$\Sigma \vdash {\sim}C$

By induction hypothesis, we have the conditions for C, and it is easy to see that the conditions for B i.e. ${\sim}C$ are provable. Indeed, (i) is obvious. For (ii), note that we have $(*)$ and that Σ is non-trivial. Finally, for (iii), note that we have (A10) and that Σ is prime.
Case 2. If $B = C \vee D$, then we have the following three cases.

Cases	$v(C)$	condition for C	$v(D)$	condition for D	$v(B)$	condition for B i.e. $C \vee D$
(i)	$\mathbf{t}, \mathbf{b}$	$\Sigma \vdash C$	any	—	$\mathbf{t}, \mathbf{b}$	$\Sigma \vdash C \vee D$
(ii)	any	—	$\mathbf{t}, \mathbf{b}$	$\Sigma \vdash D$	$\mathbf{t}, \mathbf{b}$	$\Sigma \vdash C \vee D$
(iii)	$\mathbf{f}$	$\Sigma \nvdash C$	$\mathbf{f}$	$\Sigma \nvdash D$	$\mathbf{f}$	$\Sigma \nvdash C \vee D$

By induction hypothesis, we have the conditions for C and D, and we can see that the conditions for B i.e. $C \vee D$ are provable in view of (A4), (A5) and that Σ is a prime theory for (i), (ii) and (iii) respectively.
Case 3. If $B = C \wedge D$, then we have the following three cases.

Cases	$v(C)$	condition for C	$v(D)$	condition for D	$v(B)$	condition for B i.e. $C \wedge D$
(i)	$\mathbf{f}$	$\Sigma \nvdash C$	any	—	$\mathbf{f}$	$\Sigma \nvdash C \wedge D$
(ii)	any	—	$\mathbf{f}$	$\Sigma \nvdash D$	$\mathbf{f}$	$\Sigma \nvdash C \wedge D$
(iii)	$\mathbf{t}, \mathbf{b}$	$\Sigma \vdash C$	$\mathbf{t}, \mathbf{b}$	$\Sigma \vdash D$	$\mathbf{f}$	$\Sigma \vdash C \wedge D$

By induction hypothesis, we have the conditions for C and D, and we can see that the conditions for B i.e. $C \wedge D$ are provable in view of (A7), (A8) and (4.2) for (i), (ii) and (iii) respectively.
Case 4. If $B = C \rightarrow D$, then we have the following three cases.

Cases	$v(C)$	condition for C	$v(D)$	condition for D	$v(B)$	condition for B i.e. $C \rightarrow D$
(i)	$\mathbf{f}$	$\Sigma \nvdash C$	any	—	$\mathbf{t}, \mathbf{b}$	$\Sigma \vdash C \rightarrow D$
(ii)	any	—	$\mathbf{t}, \mathbf{b}$	$\Sigma \vdash D$	$\mathbf{t}, \mathbf{b}$	$\Sigma \vdash C \rightarrow D$
(iii)	$\mathbf{t}, \mathbf{b}$	$\Sigma \vdash C$	$\mathbf{f}$	$\Sigma \nvdash D$	$\mathbf{f}$	$\Sigma \nvdash C \rightarrow D$

By induction hypothesis, we have the conditions for C and D, and we can see that the conditions for B i.e. $C \rightarrow D$ are provable in view of (4.1) and that Σ is prime, (A1) and (MP) for (i), (ii) and (iii) respectively.

This completes the proof. □

We are now ready to prove the completeness result.

Theorem 1 (Completeness). *If* $\Gamma \models_{\mathbf{mbC}^1} A$ *then* $\Gamma \vdash_{\mathbf{mbC}^1} A$.

Proof: We prove the contrapositive. Suppose that $\Gamma \nvdash_{\mathbf{mbC}^1} A$. Then by Lemma 1, we have a non-trivial prime theory Σ_0 such that $\Gamma \subseteq \Sigma_0$ and $\Sigma_0 \nvdash_{\mathbf{mbC}^1} A$. In view of Lemma 2, we can define a legal valuation v_0. Since we have $v_0(\Gamma) \in \mathcal{D}$ and $v_0(A) \notin \mathcal{D}$, we obtain $\Gamma \not\models_{\mathbf{mbC}^1} A$, as desired. □

Remark 10. Note that Ciuciura develops a hierarchy of systems $\mathbf{mbC}^n$ obtained by adding the following axiom scheme to **CLuN**:

$$A \rightarrow ({\sim}A \rightarrow ({\sim}{\sim}A \rightarrow (\cdots \rightarrow ({\sim}^{n+1}A \rightarrow B) \ldots)))$$

where ${\sim}^{n+1}A$ abbreviates the formula with $n+1$ iterated $\sim$ in front of A. The task of devising a non-deterministic semantics for $\mathbf{mbC}^n$ is left for interested readers.

5. Concluding remarks: a contra-classical variant of $\mathbf{mbC}^1$

Let us assume the three-valued non-deterministic semantics for **CLuN**, and in particular, focus on the table for negation. Then, there are two cases with non-deterministic values. Avron already observed in [2] the following.

- The refinement $\tilde{\sim}\mathbf{b} = \{\mathbf{b}\}$ corresponds to the addition of $A \to {\sim}{\sim}A$.
- The refinement $\tilde{\sim}\mathbf{f} = \{\mathbf{t}\}$ corresponds to the addition of ${\sim}{\sim}A \to A$.

Moreover, we observed in this note the following through the system $\mathbf{mbC}^1$.

- The refinement $\tilde{\sim}\mathbf{b} = \{\mathbf{t}\}$ corresponds to the addition of $A \to ({\sim}A \to ({\sim}{\sim}A \to B))$.

Then, from a purely combinatoric perspective, one may wonder what kind of formula is required in order to obtain a refinement of the three-valued non-deterministic matrix for **CLuN** with $\tilde{\sim}\mathbf{f} = \{\mathbf{b}\}$. Quick answer: $A \vee {\sim}{\sim}A$. What we need to check are the following two items.

- $A \vee {\sim}{\sim}A$ is validated in the refined matrix, and;
- (iii) of Case 1 in Lemma 2 holds for the modified case.

The first item is easy to check, and for the second item, we may confirm that if Σ is a non-trivial prime theory, then $\Sigma \nvdash C$ implies $\Sigma \vdash {\sim}{\sim}C$ thanks to the presence of $A \vee {\sim}{\sim}A$ and that Σ is prime.

Therefore, what we obtain by the unexplored refinement is a contra-classical logic obtained by adding the formula $A \vee {\sim}{\sim}A$ to **CLuN**. Note here that a logic is contra-classical "just in case not everything provable in the logic is provable in classical logic" ([12, p.438]). Moreover, the formula $A \vee {\sim}{\sim}A$ is not discussed here for the first time, but already discussed in the literature, for example, in [11, 13, 17, 18].

Finally, it is not the case that we obtain contra-classical refinements only through negation. For example, the following truth table can be seen as a refinement of the conditional of **CLuN**.

$A \to B$	**t**	**b**	**f**
t	**t**	**b**	**f**
b	**t**	**b**	**f**
f	**b**	**b**	**b**

If we combine this conditional with the negation of the *Logic of Paradox*, one of the refinements of the negation of **CLuN**, then the above conditional is *connexive* in the sense that theses of Aristotle (i.e. ${\sim}(A{\to}{\sim}A)$

and $\sim(\sim A \rightarrow A)$) and Boethius (i.e. $(A \rightarrow B) \rightarrow \sim(A \rightarrow \sim B)$ and $(A \rightarrow \sim B) \rightarrow \sim(A \rightarrow B)$) are validated. And, connexive logics are of course one of the families of contra-classical logics (see [20] for connexive logics in general, and [6, 14, 15] for systems of connexive logic with the above conditional).

A more systematic study of contra-classicality in the context of non-deterministic semantics, possibly starting with a weaker language, is yet to be seen, even for three- and four-valued logics. However, this goes well beyond the scope of this note, and I will need to leave it for another occasion.

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Young Bae Jun and Xiao Long Xin*

COMPLEX FUZZY SETS WITH APPLICATION IN BCK/BCI-ALGEBRAS

Abstract

As a generation of fuzzy set, the notion of complex fuzzy set which is an innovative concept is introduced by Ramot, Milo, Friedman and Kandel. The purpose of this article is to apply complex fuzzy set to BCK/BCI-algebras. The notions of a complex subalgebra and a complex left (right) reduced ideal in a BCK/BCI-algebra are introduced, and related properties are investigated. Characterizations of a complex subalgebra are provided, and the homomorphic image (preimage) of a complex subalgebra and a complex left (right) reduced ideal.

Keywords: complex t-norm, min-complex t-norm, complex subalgebra, complex left (right) reduced ideal, complex characteristic function.

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1. Introduction

The extension of crisp sets to fuzzy sets, in terms of membership functions, is mathematically comparable to the extension of the set of integers, $\mathbb{Z}$, to the set of real numbers, $\mathbb{R}$. That is, expanding the range of the membership function, $\mu_A(x)$, from $\{0, 1\}$ to $[0, 1]$ is mathematically analogous to the extension of $\mathbb{Z}$ to $\mathbb{R}$. Another extension of fuzzy set theory, Romat et al. [1] introduced the innovative complex fuzzy set. The complex fuzzy set, A, is characterized by a membership function, μ_A, whose range is not limited to $[0, 1]$ but extended to the unit circle in the complex plane. Hence,

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$\mu_A(x)$ is a complex valued function that assigns a grade of membership of the form $r_A \cdot e^{j\omega_A(x)}$, where $j = \sqrt{-1}$, to any element x in the universe of discourse. The value of $\mu_A(x)$ is defined by the two variables, $r_A(x)$ and $\omega_A(x)$, both real-valued, with $r_A(x) \in [0, 1]$. Tamir and Kandel [2] proposed an axiomatic framework for first order predicate complex fuzzy logic and use this framework for axiomatic definition of complex fuzzy classes. Al-Qudah and Hassan [3] introduced the concept of complex multi-fuzzy sets as a generalization of the concept of multi-fuzzy sets by adding the phase term to the definition of multi-fuzzy sets, and provided the structure of distance measure on complex multi-fuzzy sets by extending the structure of distance measure of complex fuzzy sets.

The aim of this paper is to apply the notion of complex fuzzy sets to BCK/BCI-algebras, and to generalize the fuzzy set theory in BCK/BCI-algebras. We introduce the notion of a complex subalgebra and a complex reduced left (right) ideal in a BCK/BCI-algebra, and investigate related properties. We provide characterizations of a complex subalgebra. We discuss the homomorphic image (preimage) of a complex subalgebra and a complex left (right) reduced ideal.

2. Preliminaries

By a *BCI-algebra*, we mean an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the axioms:

(I) $(\forall x, y, z \in X)\,(((x * y) * (x * z)) * (z * y) = 0)$,
(II) $(\forall x, y \in X)\,((x * (x * y)) * y = 0)$,
(III) $(\forall x \in X)\,(x * x = 0)$,
(IV) $(\forall x, y \in X)\,(x * y = y * x = 0 \Rightarrow x = y)$.

We can define a partial ordering $\leq$ by $x \leq y$ if and only if $x * y = 0$. If a BCI-algebra X satisfies $0 * x = 0$ for all $x \in X$, then we say that X is a BCK-algebra. A nonempty subset L of a BCK/BCI-algebra X is called a subalgebra of X if $x * y \in L$ for all $x, y \in L$. We refer the reader to the books [4, 5] for further information regarding BCK/BCI-algebras.

In 2011, Azam et al. [6] introduced the notion of complex valued metric space which is a generalization of the classical metric space, by defining the partial order "$\preceq$" on the set of complex numbers.

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order "$\preceq$ " on $\mathbb{C}$ as follows:

$$z_1 \preceq z_2 \text{ if and only if } \mathrm{Re}(z_1) \leq \mathrm{Re}(z_2) \text{ and } \mathrm{Im}(z_1) \leq \mathrm{Im}(z_2),$$

that is, $z_1 \preceq z_2$ if one of the following holds

(C1) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$,
(C2) $\mathrm{Re}(z_1) < \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$,
(C3) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) < \mathrm{Im}(z_2)$,
(C4) $\mathrm{Re}(z_1) < \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) < \mathrm{Im}(z_2)$.

Ramot et al. [1] introduced the notion of complex fuzzy sets.

A *complex fuzzy set* A, defined on a universe of discourse X, is characterized by a membership function $\mu_A(x)$ that assigns any element $x \in X$ a complex valued grade of membership in X, that is, the complex fuzzy set A may be represented as the set of ordered pairs

$$A = \{(x, \mu_A(x)) \mid x \in X\}, \tag{2.1}$$

where $\mu_A(x) = r_A(x) \cdot e^{j\omega_A(x)}$, $j = \sqrt{-1}$, $r_A(x)$ and $\omega_A(x)$ are both real-valued, and $r_A(x) \in [0, 1]$. Evidently, each complex grade of membership is defined by an amplitude term $r_A(x)$ and a phase term $\omega_A(x)$.

3. Complex subalgebras

Let A and B be complex fuzzy sets on X with complex valued membership functions μ_A and μ_B, respectively. We define

$$\mu_A(y) \odot \mu_B(z) = [r_A(y) \diamond r_B(z)] \cdot e^{j[\omega_A(y) \,\bar{\diamond}\, \omega_B(z)]}, \tag{3.1}$$

for all $y, z \in X$ where $\diamond$ is a t-norm and $\bar{\diamond}$ is a function

$$\bar{\diamond} : [0, \pi] \times [0, \pi] \to [0, \pi]$$

satisfying the following conditions.

1. $(a \,\bar{\diamond}\, b) \,\bar{\diamond}\, c = a \,\bar{\diamond}\, (b \,\bar{\diamond}\, c)$,
2. $a \,\bar{\diamond}\, b = b \,\bar{\diamond}\, a$,
3. $b \leq c \;\Rightarrow\; a \,\bar{\diamond}\, b \leq a \,\bar{\diamond}\, c$,
4. $a \,\bar{\diamond}\, \pi = a$,

where a, b and c are elements of $[0, \pi]$. We say that the function $\bar{\diamond}$ is an *extended t-norm*, and the operation $\odot$ is the *complex t-norm*.

In what follows, let X be a BCK/BCI-algebra and consider a complex fuzzy set A on X with complex valued membership function

$$(\forall x \in X)\left(\mu_A(x) = r_A(x) \cdot e^{j\omega_A(x)}\right)$$

where $j = \sqrt{-1}$, $r_A(x) \in [0,1]$ and $\omega_A(x) \in [0,\pi]$. It will be denoted by

$$A = \{(x, \mu_A(x)) \mid x \in X\}.$$

For any $\delta = r \cdot e^{j\omega}$ with $r \in [0,1]$ and $\omega \in [0,\pi]$, the *δ-level set* of A is denoted by $[A]_\delta$ and is defined to be the set

$$[A]_\delta := \{x \in X \mid \mu_A(x) \succeq \delta\}.$$

If, in the complex t-norm $\odot$, both the t-norm and extended t-norm are considered as "min", it is denoted by $\odot_{\min}$ and is called the *min-complex t-norm.*

Definition 3.1. A complex fuzzy set $A = \{(x, \mu_A(x)) \mid x \in X\}$ on X is called a *complex subalgebra* of X if

$$\begin{cases} \operatorname{Re}(\mu_A(x*y)) \geq \operatorname{Re}(\mu_A(x) \odot \mu_A(y)), \\ \operatorname{Im}(\mu_A(x*y)) \geq \operatorname{Im}(\mu_A(x) \odot \mu_A(y)) \end{cases} \tag{3.2}$$

or, equivalently, $\mu_A(x*y) \succeq \mu_A(x) \odot \mu_A(y)$ for all $x, y \in X$.

Note that the condition (3.2) is equivalent to the following condition:

$$\begin{cases} r_A(x*y) \cdot \cos\omega_A(x*y) \geq (r_A(x) \diamond r_A(y)) \cdot \cos(\omega_A(x) \mathbin{\bar{\diamond}} \omega_A(y)), \\ r_A(x*y) \cdot \sin\omega_A(x*y) \geq (r_A(x) \diamond r_A(y)) \cdot \sin(\omega_A(x) \mathbin{\bar{\diamond}} \omega_A(y)) \end{cases} \tag{3.3}$$

Example 3.2. Let $X = \{0, a, b, c\}$ be a BCK-algebra with the operation $*$ which is described by Table 1 (see [5]).
Let A be a complex fuzzy set on X with the complex valued membership function μ_A defined by

$$\mu_A(x) = \begin{cases} 0.7e^{j\frac{3\pi}{8}} & \text{if } x = 0, \\ 0.5e^{j\frac{3\pi}{8}} & \text{if } x = a, \\ 0.3e^{j\frac{3\pi}{8}} & \text{if } x = b, \\ 0.1e^{j\frac{3\pi}{8}} & \text{if } x = c. \end{cases}$$

Table 1. Cayley table of the operation $*$

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	b
c	c	c	c	0

It is routine to verify that $A = \{(x, \mu_A(x)) \mid x \in X\}$ is a complex subalgebra of X.

EXAMPLE 3.3. Let $X = \{0, 1, a, b, c\}$ be a BCI-algebra in which the operation $*$ is described by Table 2 (see [5]).

Table 2. Cayley table of the operation $*$

$*$	0	1	a	b	c
0	0	0	a	b	c
1	1	0	a	b	c
a	a	a	0	c	b
b	b	b	c	0	a
c	c	c	b	a	0

Let A be a complex fuzzy set on X with the complex valued membership function μ_A defined by

$$\mu_A(x) = \begin{cases} 0.7e^{j\frac{\pi}{8}} & \text{if } x = 0, \\ 0.7e^{j\frac{3\pi}{16}} & \text{if } x = 1, \\ 0.7e^{j\frac{\pi}{4}} & \text{if } x = b, \\ 0.7e^{j\frac{5\pi}{16}} & \text{if } x \in \{a, c\}. \end{cases}$$

If we use the lexicographical order, denoted by $\prec_l$, on $\mathbb{C}$, that is, for any two complex numbers $z_1 = a_1 + jb_1$ and $z_2 = a_2 + jb_2$,

$$z_1 \prec_l z_2 \text{ provided either } a_1 < a_2 \text{ or } a_1 = a_2 \text{ and } b_1 < b_2,$$

then $A = \{(x, \mu_A(x)) \mid x \in X\}$ is a complex subalgebra of X.

PROPOSITION 3.4. *Let $A = \{(x, \mu_A(x)) \mid x \in X\}$ be a complex subalgebra of a BCK-algebra X. If we use the min-complex t-norm, then $\mu_A(0) \succeq \mu_A(x)$ for all $x \in X$.*

PROOF: Let $x \in X$. Using the conditions (III) and (3.3), we have

$$\begin{aligned}\mathrm{Re}(\mu_A(0)) &= r_A(0) \cdot \cos \omega_A(0) = r_A(x * x) \cdot \cos \omega_A(x * x)\\ &\geq (r_A(x) \diamond r_A(x)) \cdot \cos(\omega_A(x) \bar{\diamond} \omega_A(x))\\ &= r_A(x) \cdot \cos \omega_A(x) = \mathrm{Re}(\mu_A(x))\end{aligned}$$

and

$$\begin{aligned}\mathrm{Im}(\mu_A(0)) &= r_A(0) \cdot \sin \omega_A(0) = r_A(x * x) \cdot \sin \omega_A(x * x)\\ &\geq (r_A(x) \diamond r_A(x)) \cdot \sin(\omega_A(x) \bar{\diamond} \omega_A(x))\\ &= r_A(x) \cdot \sin \omega_A(x) = \mathrm{Im}(\mu_A(x)).\end{aligned}$$

Therefore $\mu_A(0) \succeq \mu_A(x)$ for all $x \in X$. □

PROPOSITION 3.5. *Let $A = \{(x, \mu_A(x)) \mid x \in X\}$ be a complex subalgebra of a BCK-algebra X in which μ_A is increasing. If we use the min-complex t-norm, then μ_A is constant.*

PROOF: Straightforward. □

THEOREM 3.6. *Let $A = \{(x, \mu_A(x)) \mid x \in X\}$ be a complex subalgebra of X in which*

$$\mu_A(x) = r_A(x) \cdot e^{j\omega_A(x)}$$

with $j = \sqrt{-1}$, $r_A(x) \in [0, 1]$ and $\omega_A(x) \in [\frac{\pi}{2}, \pi]$. If we use the min-complex t-norm, then the δ-level set $[A]_\delta$ of A is a subalgebra of X for all $\delta := r \cdot e^{j\omega}$ with $r \in [0, 1]$, $\omega \in [\frac{\pi}{2}, \pi]$ and $[A]_\delta \neq \emptyset$.

PROOF: Assume that $A = \{(x, \mu_A(x)) \mid x \in X\}$ is a complex subalgebra of X. Let $x, y \in [A]_\delta$. Then $\mu_A(x) \succeq \delta$ and $\mu_A(y) \succeq \delta$. Thus

$\mathrm{Re}(\mu_A(x)) = r_A(x) \cdot \cos \omega_A(x) \geq r \cdot \cos \omega$,
$\mathrm{Im}(\mu_A(x)) = r_A(x) \cdot \sin \omega_A(x) \geq r \cdot \sin \omega$,
$\mathrm{Re}(\mu_A(y)) = r_A(y) \cdot \cos \omega_A(y) \geq r \cdot \cos \omega$,
$\mathrm{Im}(\mu_A(y)) = r_A(y) \cdot \sin \omega_A(y) \geq r \cdot \sin \omega$.

Now, we consider the following four cases.

(1) $r_A(x) \geq r_A(y)$ and $\omega_A(x) \geq \omega_A(y)$,
(2) $r_A(x) \geq r_A(y)$ and $\omega_A(x) < \omega_A(y)$,
(3) $r_A(x) < r_A(y)$ and $\omega_A(x) \geq \omega_A(y)$,
(4) $r_A(x) < r_A(y)$ and $\omega_A(x) < \omega_A(y)$.

The case (1) implies that

$$(r_A(x) \diamond r_A(y)) \cdot \cos(\omega_A(x) \,\bar{\diamond}\, \omega_A(y)) = r_A(y) \cdot \cos \omega_A(y) \geq r \cdot \cos \omega$$

and

$$(r_A(x) \diamond r_A(y)) \cdot \sin(\omega_A(x) \,\bar{\diamond}\, \omega_A(y)) = r_A(y) \cdot \sin \omega_A(y) \geq r \cdot \sin \omega.$$

For the case (2), we have

$$\begin{aligned}(r_A(x) \diamond r_A(y)) \cdot \cos(\omega_A(x) \,\bar{\diamond}\, \omega_A(y)) &= r_A(y) \cdot \cos \omega_A(x)\\ &\geq r_A(y) \cdot \cos \omega_A(y) \geq r \cdot \cos \omega\end{aligned}$$

and

$$\begin{aligned}(r_A(x) \diamond r_A(y)) \cdot \sin(\omega_A(x) \,\bar{\diamond}\, \omega_A(y)) &= r_A(y) \cdot \sin \omega_A(x)\\ &\geq r_A(y) \cdot \sin \omega_A(y) \geq r \cdot \sin \omega\end{aligned}$$

since $\cos \omega_A(y) \leq \cos \omega_A(x)$ and $\sin \omega_A(y) \leq \sin \omega_A(x)$. The case (3) induces

$$\begin{aligned}(r_A(x) \diamond r_A(y)) \cdot \cos(\omega_A(x) \,\bar{\diamond}\, \omega_A(y)) &= r_A(x) \cdot \cos \omega_A(y)\\ &\geq r_A(x) \cdot \cos \omega_A(x) \geq r \cdot \cos \omega\end{aligned}$$

and

$$\begin{aligned}(r_A(x) \diamond r_A(y)) \cdot \sin(\omega_A(x) \,\bar{\diamond}\, \omega_A(y)) &= r_A(x) \cdot \sin \omega_A(y)\\ &\geq r_A(x) \cdot \sin \omega_A(x) \geq r \cdot \sin \omega.\end{aligned}$$

From the case (4), we have

$$(r_A(x) \diamond r_A(y)) \cdot \cos(\omega_A(x) \,\bar{\diamond}\, \omega_A(y)) = r_A(x) \cdot \cos \omega_A(x) \geq r \cdot \cos \omega$$

and

$$(r_A(x) \diamond r_A(y)) \cdot \sin(\omega_A(x) \,\bar{\diamond}\, \omega_A(y)) = r_A(x) \cdot \sin \omega_A(x) \geq r \cdot \sin \omega.$$

It follows from (3.2) that
$r_A(x * y) \cdot \cos \omega_A(x * y) \geq (r_A(x) \diamond r_A(y)) \cdot \cos(\omega_A(x) \,\bar{\diamond}\, \omega_A(y)) \geq r \cdot \cos \omega$
and
$r_A(x * y) \cdot \sin \omega_A(x * y) \geq (r_A(x) \diamond r_A(y)) \cdot \sin(\omega_A(x) \,\bar{\diamond}\, \omega_A(y)) \geq r \cdot \sin \omega.$
Hence $\mu_A(x * y) \succeq \delta$, and so $x * y \in [A]_\delta$. Therefore $[A]_\delta$ is a subalgebra of X. □

Corollary 3.7. *Let $A = \{(x, \mu_A(x)) \mid x \in X\}$ be a complex subalgebra of a BCK-algebra X in which*

$$\mu_A(x) = r_A(x) \cdot e^{j\omega_A(x)}$$

with $j = \sqrt{-1}$, $r_A(x) \in [0, 1]$ and $\omega_A(x) \in [\frac{\pi}{2}, \pi]$. If we use the min-complex t-norm, then the set

$$[X] := \{x \in X \mid \mu_A(x) = \mu_A(0)\}$$

is a subalgebra of X.

Proof: Since $\mu_A(0) \succeq \mu_A(x)$ for all $x \in X$ by Proposition 3.4, we have

$$[A]_{\mu_A(0)} = \{x \in X \mid \mu_A(x) \succeq \mu_A(0)\} = \{x \in X \mid \mu_A(x) = \mu_A(0)\} = [X].$$

It follows from Theorem 3.6 that $[X]$ is a subalgebra of X. □

Theorem 3.8. *Let $A = \{(x, \mu_A(x)) \mid x \in X\}$ be a complex fuzzy set on X such that the δ-level set $[A]_\delta$ of A is a subalgebra of X for all $\delta := r \cdot e^{j\omega}$ with $r \in [0, 1]$, $\omega \in [0, \pi]$ and $[A]_\delta \neq \emptyset$. If we use the min-complex t-norm, then $A = \{(x, \mu_A(x)) \mid x \in X\}$ is a complex subalgebra of X.*

Proof: Suppose that the δ-level set $[A]_\delta$ of A is a subalgebra of X for all $\delta := r \cdot e^{j\omega}$ with $r \in [0, 1]$ and $\omega \in [0, \pi]$. Assume that there exist $a, b \in X$ such that

$$\mu_A(a * b) \prec \mu_A(a) \odot \mu_A(b).$$

We take $\delta := r \cdot e^{j\omega}$ with $r \in [0, 1]$ and $\omega \in [0, \pi]$ such that

$$\mu_A(a * b) \prec \delta \preceq \mu_A(a) \odot \mu_A(b).$$

Then $a * b \notin [A]_\delta$,

$$(r_A(a) \diamond r_A(b)) \cdot \cos(\omega_A(a) \mathbin{\bar{\diamond}} \omega_A(b)) = \mathrm{Re}(\mu_A(a) \odot \mu_A(b)) \geq \mathrm{Re}(\delta) = r \cdot \cos \omega$$

and

$$(r_A(a) \diamond r_A(b)) \cdot \sin(\omega_A(a) \mathbin{\bar{\diamond}} \omega_A(b)) = \mathrm{Im}(\mu_A(a) \odot \mu_A(b)) \geq \mathrm{Im}(\delta) = r \cdot \sin \omega.$$

It follows that

$$r_A(a) \cdot \cos \omega_A(a) \geq r \cdot \cos \omega, \;\; r_A(a) \cdot \sin \omega_A(a) \geq r \cdot \sin \omega$$

and

$$r_A(b) \cdot \cos \omega_A(b) \geq r \cdot \cos \omega, \;\; r_A(b) \cdot \sin \omega_A(b) \geq r \cdot \sin \omega.$$

This shows that $a, b \in [A]_\delta$ and this is a contradiction. Therefore $\mu_A(x * y) \succeq \mu_A(x) \odot \mu_A(y)$ for all $x, y \in X$, and $A = \{(x, \mu_A(x)) \mid x \in X\}$ is a complex subalgebra of X. □

COROLLARY 3.9. *For any subset L of X, let $A = \{(x, \mu_A(x)) \mid x \in X\}$ be a complex fuzzy set on a BCK-algebra X with the complex valued membership function μ_A defined by*

$$\mu_A(x) = \begin{cases} r_1 \cdot e^{j\omega_1} & \text{if } x \in L, \\ r_2 \cdot e^{j\omega_2} & \text{otherwise} \end{cases}$$

where $r_1 \cdot e^{j\omega_1} \succeq r_2 \cdot e^{j\omega_2}$. If L is a subalgebra of X and we use the min-complex t-norm, then $A = \{(x, \mu_A(x)) \mid x \in X\}$ is a complex subalgebra of X.

THEOREM 3.10. *Let $\delta_1, \delta_2, \cdots, \delta_n, \cdots$ be a strictly increasing sequence of complex numbers, where $\delta_k = r_k \cdot e^{j\omega_k}$ with $r_k \in [0, 1]$ and $\omega_k \in [\frac{\pi}{2}, \pi]$. For a strictly decreasing sequences $L_1 (= X), L_2, \cdots, L_n, \cdots$ of subalgebras of X, there is a complex subalgebra $A = \{(x, \mu_A(x)) \mid x \in X\}$ of X in which*

$$\mu_A(x) = r_A(x) \cdot e^{j\omega_A(x)}$$

with $j = \sqrt{-1}$, $r_A(x) \in [0, 1]$ and $\omega_A(x) \in [\frac{\pi}{2}, \pi]$ such that $[A]_{\delta_n} = L_n$ for $n \in \mathbb{N}$ if we use the min-complex t-norm.

PROOF: Define a complex fuzzy set A on X with the complex valued membership function μ_A defined by

$$\mu_A(x) = \begin{cases} \delta_n & \text{if } x \in L_n \setminus L_{n+1}, \\ \lim\limits_{n\to\infty} \delta_n & \text{if } x \in \cap_{n=1}^{\infty} L_n. \end{cases}$$

It is easy to verify that $A = \{(x, \mu_A(x)) \mid x \in X\}$ is a complex subalgebra of X and $[A]_{\delta_n} = L_n$ for $n = 1, 2, \cdots$. □

Let $f : X \to Y$ be a mapping of sets. If B is a complex fuzzy set on Y with the complex valued membership function μ_B, then the preimage of B under f, denoted by $f^{-1}(B)$, is also a complex fuzzy set on X with the complex valued membership function $\mu_{f^{-1}(B)}$ which is defined by $\mu_{f^{-1}(B)}(x) = \mu_B(f(x))$ for all $x \in X$.

THEOREM 3.11. *Let $f : X \to Y$ be a homomorphism from X to a BCK/BCI-algebra Y. If B is a complex subalgebra of Y with the complex valued membership function μ_B, then the homomorphic preimage $f^{-1}(B)$ of B under f is a complex subalgebra of X with the complex valued membership function $\mu_{f^{-1}(B)}$.*

PROOF: Assume that B is a complex subalgebra of Y with the complex valued membership function μ_B. For any $x, y \in X$, we have

$$\begin{aligned}\operatorname{Re}\left(\mu_{f^{-1}(B)}(x*y)\right) &= \operatorname{Re}\left(\mu_B(f(x*y))\right) = \operatorname{Re}\left(\mu_B(f(x)*f(y))\right)\\ &\geq \operatorname{Re}\left(\mu_B(f(x)) \odot \mu_B(f(y))\right)\\ &= \operatorname{Re}\left(\mu_{f^{-1}(B)}(x) \odot \mu_{f^{-1}(B)}(y)\right)\end{aligned}$$

and

$$\begin{aligned}\operatorname{Im}\left(\mu_{f^{-1}(B)}(x*y)\right) &= \operatorname{Im}\left(\mu_B(f(x*y))\right) = \operatorname{Im}\left(\mu_B(f(x)*f(y))\right)\\ &\geq \operatorname{Im}\left(\mu_B(f(x)) \odot \mu_B(f(y))\right)\\ &= \operatorname{Im}\left(\mu_{f^{-1}(B)}(x) \odot \mu_{f^{-1}(B)}(y)\right).\end{aligned}$$

Therefore the homomorphic preimage $f^{-1}(B)$ of B under f is a complex subalgebra of X with the complex valued membership function $\mu_{f^{-1}(B)}$.□

THEOREM 3.12. *Let f be an endomorphism of X. If A is a complex subalgebra of X with the complex valued membership function μ_A, then the complex fuzzy set $A[f]$ on X with the complex valued membership function $\mu_{A[f]}$ defined by*

$$\mu_{A[f]}(x) = \mu_A(f(x))$$

for all $x \in X$ is a complex subalgebra of X.

PROOF: Let A be a complex subalgebra of X with the complex valued membership function μ_A. For any $x, y \in X$, we get

$$\begin{aligned}\operatorname{Re}\left(\mu_{A[f]}(x*y)\right) &= \operatorname{Re}\left(\mu_A(f(x*y))\right) = \operatorname{Re}\left(\mu_A(f(x)*f(y))\right)\\ &\geq \operatorname{Re}\left(\mu_A(f(x)) \odot \mu_A(f(y))\right)\\ &= \operatorname{Re}\left(\mu_{A[f]}(x) \odot \mu_{A[f]}(y)\right)\end{aligned}$$

and

$$\begin{aligned}\operatorname{Im}\left(\mu_{A[f]}(x*y)\right) &= \operatorname{Im}\left(\mu_A(f(x*y))\right) = \operatorname{Im}\left(\mu_A(f(x)*f(y))\right)\\ &\geq \operatorname{Im}\left(\mu_A(f(x)) \odot \mu_A(f(y))\right)\\ &= \operatorname{Im}\left(\mu_{A[f]}(x) \odot \mu_{A[f]}(y)\right).\end{aligned}$$

Therefore $\mu_{A[f]}$ is a complex subalgebra of X. □

DEFINITION 3.13. A complex fuzzy set $A = \{(x, \mu_A(x)) \mid x \in X\}$ on X is called a *complex left reduced ideal* of X if

$$\operatorname{Re}(\mu_A(x*y)) \geq \operatorname{Re}(\mu_A(y)) \text{ and } \operatorname{Im}(\mu_A(x*y)) \geq \operatorname{Im}(\mu_A(y)) \tag{3.4}$$

for all $x, y \in X$. If $A = \{(x, \mu_A(x)) \mid x \in X\}$ satisfies the condition

$$\mathrm{Re}(\mu_A(x * y)) \geq \mathrm{Re}(\mu_A(x)) \text{ and } \mathrm{Im}(\mu_A(x * y)) \geq \mathrm{Im}(\mu_A(x))$$

for all $x, y \in X$, then we say $A = \{(x, \mu_A(x)) \mid x \in X\}$ is a *complex right reduced ideal* of X.

THEOREM 3.14. *Let $A = \{(x, \mu_A(x)) \mid x \in X\}$ be a complex subalgebra of a BCK-algebra X. If we use the min-complex t-norm, then A is a complex left reduced ideal of X if and only if the complex valued membership function μ_A of A is constant, that is, $\mu_A(0) = \mu_A(x)$ for all $x \in X$.*

PROOF: The sufficiency is clear. Assume that A is a complex left reduced ideal of X. For any $x \in X$, we have

$$\mathrm{Re}(\mu_A(x)) = \mathrm{Re}(\mu_A(x * 0)) \geq \mathrm{Re}(\mu_A(0))$$

and

$$\mathrm{Im}(\mu_A(x)) = \mathrm{Im}(\mu_A(x * 0)) \geq \mathrm{Im}(\mu_A(0)).$$

Since $x * x = 0$ for all $x \in X$, the condition (3.4) implies that $\mathrm{Re}(\mu_A(x)) \leq \mathrm{Re}(\mu_A(x * x)) = \mathrm{Re}(\mu_A(0))$ and $\mathrm{Im}(\mu_A(x)) \leq \mathrm{Im}(\mu_A(x * x)) = \mathrm{Im}(\mu_A(0))$. Therefore $\mu_A(0) = \mu_A(x)$ for all $x \in X$, that is, the complex valued membership function μ_A of A is constant. □

The proof of the following two theorems is the same as the proof of Theorems 3.11 and 3.12.

THEOREM 3.15. *Let $f : X \to Y$ be a homomorphism from X to a BCK/BCI-algebra Y. If B is a complex left (resp. right) reduced ideal of Y with the complex valued membership function μ_B, then the homomorphic preimage $f^{-1}(B)$ of B under f is a complex left (resp. right) reduced ideal of X with the complex valued membership function $\mu_{f^{-1}(B)}$.*

THEOREM 3.16. *Let f be an endomorphism of X. If A is a complex left (resp. right) reduced ideal of X with the complex valued membership function μ_A, then the complex fuzzy set $A[f]$ on X with the complex valued membership function $\mu_{A[f]}$ defined by*

$$\mu_{A[f]}(x) = \mu_A(f(x))$$

for all $x \in X$ is a complex left (resp. right) reduced ideal of X.

4. Conclusions

Generally, the extension of crisp sets to fuzzy sets, in terms of membership functions, is mathematically comparable to the extension of $\mathbb{Z}$ (; the set of integers) to $\mathbb{R}$ (; the set of real numbers). That is, expanding the range of the membership function from $\{0, 1\}$ to the unit interval $[0, 1]$ is mathematically analogous to the extension of $\mathbb{Z}$ to $\mathbb{R}$. The development of the number set did not end with real numbers. Historically, the introduction of real numbers was followed by their extension to the set of complex numbers, $\mathbb{C}$. Hence, it may be suggested that a further development of fuzzy set theory should be based on this extension. In the context of set theory, the result of such an extension is the complex fuzzy set, i.e., a fuzzy set characterized by a complex-valued membership function. Based on such background, Ramot et al. introduced complex fuzzy set in their paper [1]. The complex fuzzy set is characterized by a membership function μ whose range is not limited to $[0, 1]$ but extended to the unit circle in the complex plane. In this paper, we have used complex fuzzy sets to obtain the generalization of fuzzy set theory in BCK/BCI-algebras. We have introduced the notion of a complex subalgebra and a complex reduced left (right) ideal in a BCK/BCI-algebra, and have investigated related properties. We have provided characterizations of a complex subalgebra, and have discussed the homomorphic image (preimage) of a complex subalgebra and a complex left (right) reduced ideal. We will use the ideas and results of this paper to study various types of sub-structure in algebras in the future.

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TOPOLOGICAL AND MULTI-TOPOLOGICAL FRAMES IN THE CONTEXT OF INTUITIONISTIC MODAL LOGIC

Abstract

We present three examples of topological semantics for intuitionistic modal logic with one modal operator $\Box$. We show that it is possible to treat neighborhood models, introduced earlier, as topological or multi-topological. From the neighborhood point of view, our method is based on differences between properties of minimal and maximal neighborhoods. Also we propose transformation of multi-topological spaces into the neighborhood structures.

Keywords: intuitionistic modal logic, neighbourhood semantics, topological semantics, Kripke frames, soundness and completeness.

Mathematics Subject Classification (2010): 03B45, 03B20, 54A05, 54A10.

1. Introduction

Neighborhood semantics for intuitionistic propositional logic has been presented by Moniri and Maleki in [9]. Not surprisingly, it turned to be quite similar to the neighborhood semantics for classical modal logic **S4**. Moreover, the above-mentioned authors proved that their structures correspond to the well-known relational (Kripke) models for intuitionism. It seems that later they became interested rather in neighborhood semantics for subintuitionistic systems (see [7] and [8]).

Nonetheless, even in the context of relatively strong logic like intuitionism, neighborhoods still can provoke certain intuitions. For instance, Moniri and Maleki spoke about *minimal* neighborhoods (which can be identified with upper closed sets in Kripke frames). Hence, it is quite natural to ask also about *maximal* neighborhoods, i.e. to deny superset axiom. Informally speaking, in this way a place is created for modality. We can assume that necessity means satisfiability in maximal neighborhood.

This assumption led us (see [14]) to the intuitionistic logic with one modal operator $\Box$, axiomatized by the rule of necessity and two axioms (K and T). Such system has been investigated by Božic and Došen in [1] but with reference to the bi-relational frames. As we have shown, there is a strict correspondence between their setting and our neighborhood approach.

It is well-known (see [10] for more detailed survey) that neighborhood frames for S4 logic behave just like topological structures. This adequacy is true also for intuitionistic neighborhood frames, as it was proved in [9]. For this reason, it is reasonable to look for analogous results for modal logics based on intuitionism. Even if our frames can be presented as bi-relational, we still believe that neighborhoods give us better topological intuitions. In addition, they can be useful when speaking about certain generalizations of topology for weak modal logics (see [13] for details).

Topological semantics for (normal) intuitionistic modal logics has been investigated by Davoren in [4], [5] and Davoren et al. in [6]. Those authors referred to the bi-relational structures with Fischer-Servi conditions (which are not satisfied in our framework). They use specific binary relations between points of topological space. Our idea is different: we do not use any special relation. We limit ourselves to some basic notions like *topological neighborhood* or *open set*.

Another concept has been developed by Collinson et al. in [3]. It is based on the notion of *topological p-morphism*. These authors started from the relational structures and they used some methods of category theory. As for the topological p-morphism, we do not use this tool in the present work. However, we adapted it to the case of generalized topologies in [13].

In [11] we can find some considerations about neighborhood, topological and relational frames for intuitionistic systems with modality. Sotirov assumed that his topological spaces should be equipped with two operations. One of them behaves like interior and is responsible for the intuitionistic features of the logic in question. The second is used to model necessity.

In this research we present different approach. Our first intuition was that neighborhood systems assigned to the particular worlds (i.e. systems consisting of minimal and maximal neighborhood) behave like distinct topological spaces in a kind of "meta-universe". We show initial conclusions of this observation. However, in some cases it is better to assume that all these systems are in fact subspaces of one topological space. Hence, we can use the notion of *induced topology*.

We concentrate only on the basic features of structures mentioned above. In particular, we do not obtain topological completeness because our translations between neighborhood structures (for which we have completeness) and topological spaces (which are defined in three slightly different ways) are one-way. Thus, this paper can be considered as a first step in further studies.

2. Alphabet and language

Our basic system is named **IKT**$\Box$. It has rather standard syntax (i.e. alphabet and language). We use the following notations:

1. PV is a fixed denumerable set of propositional variables $p, q, r, s, \ldots$
2. Logical connectives and operators are $\wedge$, $\vee$, $\rightarrow$, $\bot$, $\Box$.
3. The only derived connective is $\neg$ (which means that $\neg\varphi$ is a shortcut for $\varphi \rightarrow \bot$).

Formulas are generated recursively in a standard manner: if φ, ψ are *wff's* then also $\varphi \vee \psi$, $\varphi \wedge \psi$, $\varphi \rightarrow \psi$ and $\Box\varphi$. Semantic interpretation of propositional variables and all the connectives introduced above will be presented in the next section. Attention: $\Leftarrow, \Rightarrow$ and $\Leftrightarrow$ are used only on the level of (classical) meta-language.

3. Neighborhood semantics

3.1. The definition of structure

Our basic structure is an intuitionistic neighborhood modal frame (**n2**-frame) defined as it follows:

Definition 3.1. **n2**-frame is an ordered pair $\langle W, \mathcal{N} \rangle$ where:

1. W is a non-empty set (of worlds, states or points)
2. $\mathcal{N}$ is a function from W into $P(P(W))$ such that:

(a) $w \in \bigcap \mathcal{N}_w$
(b) $\bigcap \mathcal{N}_w \in \mathcal{N}_w$
(c) $u \in \bigcap \mathcal{N}_w \Rightarrow \bigcap \mathcal{N}_u \subseteq \bigcap \mathcal{N}_w$ ($\to$*-condition*)
(d) $X \subseteq \bigcup \mathcal{N}_w$ and $\bigcap \mathcal{N}_w \subseteq X \Rightarrow X \in \mathcal{N}_w$ (*relativized superset axiom*)
(e) $u \in \bigcap \mathcal{N}_w \Rightarrow \bigcup \mathcal{N}_u \subseteq \bigcup \mathcal{N}_w$ ($\Box$*-condition)*
(f) $v \in \bigcup \mathcal{N}_w \Rightarrow \bigcap \mathcal{N}_v \subseteq \bigcup \mathcal{N}_w$ *(t-condition)*

The first three conditions are in fact taken from pure intuitionism and refer to the features of partial order in relational frames. For instance, $\to$-condition guarantees that forcing of implication is monotone. As for the relativized superset axiom, it creates place for modality. $\Box$-condition is necessary to assure that forcing of modal formulas is also monotone. Significance of the last restriction will be pointed out later.

3.2. Valuation and model

Definition 3.2. Neighborhood **n2**-model is a triple $F_{\mathcal{N}} = \langle W, \mathcal{N}, V_{\mathcal{N}} \rangle$, where $\langle W, \mathcal{N} \rangle$ is an **n2**-frame and $V_{\mathcal{N}}$ is a function from PV into $P(W)$ satisfying the following condition: if $w \in V_{\mathcal{N}}(q)$ then $\bigcap \mathcal{N}_w \subseteq V_{\mathcal{N}}(q)$.

Definition 3.3. For every **n2**-model $M_{\mathcal{N}} = \langle W, \mathcal{N}, V_{\mathcal{N}} \rangle$, forcing of formulas in a world $w \in W$ is defined inductively:

1. $w \nVdash \bot$
2. $w \Vdash q \Leftrightarrow w \in V_{\mathcal{N}}(q)$ for any $q \in PV$
3. $w \Vdash \varphi \vee \psi \Leftrightarrow w \Vdash \varphi$ or $w \Vdash \psi$
4. $w \Vdash \varphi \wedge \psi \Leftrightarrow w \Vdash \varphi$ and $w \Vdash \psi$
5. $w \Vdash \varphi \to \psi \Leftrightarrow \bigcap \mathcal{N}_w \subseteq \{v \in W; v \nVdash \varphi$ or $v \Vdash \psi\}$
6. $w \Vdash \Box\varphi \Leftrightarrow \bigcup \mathcal{N}_w \subseteq \{v \in W; v \Vdash \varphi\}$.

As we said, $\neg\varphi$ is a shortcut for $\varphi \to \bot$. Thus, $w \Vdash \neg\varphi \Leftrightarrow \bigcap \mathcal{N}_w \subseteq \{v \in W; v \nVdash \varphi\}$.

As usual, we say that formula φ is satisfied in a model $M_{\mathcal{N}} = \langle W, \mathcal{N}, V_{\mathcal{N}} \rangle$ when $w \Vdash \varphi$ for every $w \in W$. It is *true* (tautology) when it is satisfied in each **n2**-model.

4. Neigborhood completeness

In [14] we have shown (using slightly different symbols) that **n2**-frames are sound and complete semantics for the logic **IKT**$\Box$ defined as the following set of formulas and rules: **IPC** $\cup \{K, T, RN, MP\}$, where:

1. **IPC** is the set of all intuitionistic axiom schemes
2. K is the axiom scheme $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
3. T is the axiom scheme $\Box\varphi \to \varphi$
4. RN is the rule of necessity: $\varphi \vdash \Box\varphi$
5. MP is *modus ponens*: $\varphi, \varphi \to \psi \vdash \psi$

Completeness result has been established in two ways. First, directly – by means of prime theories and canonical model. Second, indirectly – by the transformation into certain class of bi-relational frames, introduced by Božić and Došen in [1] who proved its completeness. Basically, they used different set of axioms.

5. Multi-topological frames

5.1. The definition of structure and model

In this section we introduce the notion of *multi-topological frame (model)*. Such structure can be roughly described as a collection of topological spaces with one valuation based on open sets. Each space has its *distinguished open set* which plays crucial role in the proof of translation between neighborhood and multi-topological settings.

Definition 5.1. **mtD**-model with distinguished sets is an ordered triple $M_t = \langle W, \mathfrak{W}, V_t \rangle$ where:

1. $W \neq \emptyset$.
2. $\mathfrak{W} = \{\langle T, \tau, D^\tau \rangle : T \subseteq W,\ \tau$ is a topology on $T,\ D^\tau \in \tau, D^\tau \neq \emptyset\}$
3. $W = \bigcup \mathcal{T}$, where $\mathcal{T} = \{T : \langle T, \tau, D^\tau \rangle \in \mathfrak{W}\}$.
4. V_t is a function from PV into $P(W)$ satisfying the following condition: $V_t(q) = \bigcup \mathcal{X}$ where $\mathcal{X} \subseteq \{X \subseteq W$; there is $\langle T, \tau, D^\tau \rangle \in \mathfrak{W}$ for which $X \in \tau\}$.

The third condition can be formulated also as follows: for each $w \in W$ there is $\langle T, \tau, D^\tau \rangle \in \mathfrak{W}$ such that $w \in T$. Hence, each point of W is at least in one topological space. We can consider the whole structure as a universe with many generalized topologies[1].

For convenience, we shall often identify each $\langle T, \tau, D^\tau \rangle$ simply with τ. As for the valuation of complex formulas, it is based on the valuation of propositional variables and defined inductively:

Definition 5.2. For every **mtD**-model $M_t = \langle W, \mathfrak{W}, V_t \rangle$, valuation of formulas is defined as such:

1. $V_t(\varphi \wedge \psi) = V_t(\varphi) \cap V_t(\psi)$
2. $V_t(\varphi \vee \psi) = V_t(\varphi) \cup V_t(\psi)$
3. $V_t(\varphi \rightarrow \psi) = \bigcup_\tau Int_\tau\left(-V_t(\varphi) \cup V_t(\psi)\right)$
4. $V_t(\Box\varphi) = \bigcup \mathcal{X}$ where $\mathcal{X} = \{X \subseteq W$ such that $X = D^\tau$ for at least one τ in $\mathfrak{W}$ such that $T \subseteq V_t(\varphi)\}$.

A few words of comment should be made. We assume that $V_t(q)$ is a union of sets which are open at least in one topology. Concerning value of implication, we look for $-V_t(\varphi) \cup V_t(\psi)$ and then we sum up all τ-interiors of this set. The last important thing is modality: we check which universes are wholly contained in $V_t(\varphi)$ and then we take union of their distinguished sets. We say that formula φ is true *iff* in each **mtD**-model $M_t = \langle W, \mathfrak{W}, V_t \rangle$ we have $V_t(\varphi) = W$.

This class of models is based on the observation described above: that we have multiverse of spaces. However, our definition of forcing appears to be too weak (even if we assumed that valuation is based on unions of τ-open sets). Hence, **mtD**-structures in their most general form are not sound with respect to intuitionism. We did not develop detailed hypothesis about the logic determined by this class of frames. Certainly, some very basic axioms hold. Among them there are: $\varphi \rightarrow \varphi$, $\varphi \wedge \psi \rightarrow \psi$, $\varphi \rightarrow \varphi \vee \psi$ and $\psi \rightarrow \varphi \vee \psi$. Also $\varphi \rightarrow (\psi \rightarrow \psi)$ is true. Let us check this fact. Assume that there is a model with a world w such that $w \nVdash \varphi \rightarrow (\psi \rightarrow \psi)$. It means that for each τ, $w \notin Int_\tau(-V_t(\varphi) \cup (-V_t(\psi) \cup V_t(\varphi)))$. However, the whole expression in brackets is just $W \cup -V_t(\psi) = W$. When we take τ-interior of W, we obtain subset T. Hence, w is beyond any T. But this is contradiction.

On the other hand, it is possible that $x \nVdash (\varphi \rightarrow \psi \wedge \psi \rightarrow \gamma) \rightarrow (\varphi \rightarrow \gamma)$, i.e. for each τ, $x \notin Int_\tau(-(-V_t(\varphi) \cup V_t(\psi)) \cap (-V_t(\psi) \cup V_t(\gamma))) \cup (-V_t(\varphi) \cup$

[1]Generalized in the sense of Császár (see [2]) but with closure under finite intersections.

$V_t(\gamma))$. After some computations the whole expression can be written as $-W \cup (-V_t(\varphi) \cup V_t(\gamma)) = -V_t(\varphi) \cup V_t(\gamma)$. Now take $W = \{w, v, u, z\}$, $\tau_1 = \{\emptyset, \{w, v\}\}$, $\tau_2 = \{\emptyset, \{u, z\}\}$, $V_t(\varphi) = \{w, v\}$, distinguished sets are arbitrary, $V_t(\varphi) = \{u, z\}, V_t(\psi) = V_t(\gamma) = \emptyset$. Now v does not force the formula in question. Let us check it: $Int_{\tau_1}(-V_t(\varphi) \cup V_t(\gamma)) = Int_{\tau_1}(\{u, z\}) = \emptyset \not\ni v$. Moreover, $Int_{\tau_2}(-V_t(\varphi) \cup V_t(\gamma)) = \{u, z\} \not\ni v$.

Also we can easily build a counter-model where $V_t(\varphi \wedge \psi) \nsubseteq V_t(\top \to \varphi \wedge \psi)$[2].

As for the modal formulas: we can easily prove that axiom T (i.e. $\Box\varphi \to \varphi$) is always true. Assume that there is a model with w such that $w \nVdash T$. Hence, for any τ, $w \notin Int_\tau(-V_t(\Box\varphi) \cup V_t(\varphi)) = Int_\tau(-\bigcup \mathcal{X} \cup V_t(\varphi))$, where $\mathcal{X} = \{X \subseteq W$ such that $X = D^\tau$ for at least one τ in $\mathfrak{W}$ such that $T \subseteq V_t(\varphi)\}$. Clearly, $\bigcup \mathcal{X} \subseteq V_t(\varphi)$. Hence, $-V_t(\varphi) \subseteq -\bigcup \mathcal{X}$ which gives us that $-\bigcup \mathcal{X} \cup V_t(\varphi) = W$. Again, we obtain impossible result that $w \notin T$ for any $\langle T, \tau, D^\tau \rangle \in \mathfrak{W}$.

On the other hand, axiom *4* (i.e. $\Box\varphi \to \Box\Box\varphi$) can be falsified. Take $W = \{w, v, u\}$, $\tau_1 = \{\emptyset, \{w, v\}, \{w, v, u\}\}$, $D^{\tau_1} = \{w, v\}$, $\tau_2 = \{\emptyset, \{v\}\}$, $D^{\tau_2} = \{v\}$, $V_t(\varphi) = W$. Now $V_t(\Box\varphi) = D^{\tau_1} \cup D^{\tau_2} = \{w, v\}$, $V(\Box\Box\varphi) = D^{\tau_2} = \{v\}$, $-V_t(\Box\varphi) \cup V_t(\Box\Box\varphi) = \{u, v\}$. Hence, $Int_{\tau_1}(\{u, v\}) = \emptyset$, $Int_{\tau_2}(\{u, v\}) = \{v\}$. Clearly, u is beyond those interiors, so $u \nVdash 4$.

We see that the logic of **mtD**-frames is a kind of unknown subintuitionistic modal logic. We conjecture that it may be fruitful to study general multi-topological structures and to look for any regularities depending on mutual location of spaces or their topological properties. We signalize this possibility but it is beyond the scope of present paper. And so, overall here, we shall work only with a certain subclass of these structures, namely **i-mtD**-frames.

Definition 5.3. We say that **mtD**-frame is **i-mtD** iff there is an Alexandrov topology μ on W such that for each $\tau \in \mathfrak{W}$, τ is a subspace topology induced by μ.

If we speak about Alexandrov topology, it means that arbitrary intersections of open sets are also open. If τ on T is induced by μ, then each $U \in \tau$ can be presented as $T \cap A$ for certain $A \in \mu$. On the other hand, if $A \in \mu$, then $T \cap A \in \tau$. This subclass of models is sound with respect to

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[2] We are grateful to the anonymous reviewer for this example and some other important comments.

intuitionism what can be manually checked. It is well-known fact that subspaces of Alexandrov space also have Alexandrov property (see Theorem 7 in [12]).

6. From neighborhood frames to multi-topological structures

6.1. Basic notions

In this section we show that it is possible to treat neighborhood models as multi-topological. First, let us introduce the notion of w-open sets.

DEFINITION 6.1. We say that set $X \subseteq W$ is w-open in **n2**-frame *iff* $X \subseteq \bigcup \mathcal{N}_w$ and for every $v \in X$ we have $\bigcap \mathcal{N}_v \subseteq X$. We define $\mathcal{O}_w$ as $\{X \subseteq W : X \text{ are } w\text{-open}\}$ and call it w-topology.

Let us check that this definition is useful for our needs.

THEOREM 6.2. *Assume that we have* **n2**-*frame* $F_{\mathcal{N}} = \langle W, \mathcal{N} \rangle$. *Then* $\mathcal{O}_w$ *is a topological space for every* $w \in W$.

PROOF: Let us check standard properties of topology.

1. Take empty set. We can say that $\emptyset \in \mathcal{O}_w$ because $\emptyset \subseteq \bigcup \mathcal{N}_w$ and there are no any v in $\emptyset$.
2. Consider $\bigcup \mathcal{N}_w$. Clearly this set is contained in itself and because of T-condition we have that for every $v \in \bigcup \mathcal{N}_w$ the second condition holds: $\bigcap \mathcal{N}_v \subseteq \bigcup \mathcal{N}_w$.
3. Consider $\mathscr{X} \subseteq \mathcal{O}_w$. We show that $\bigcap \mathscr{X} \in \mathcal{O}_w$. The first condition is simple: every element of $\mathscr{X}$ belongs to $\mathcal{O}_w$ so it is contained in $\bigcup \mathcal{N}_w$. The same holds of course for intersection of all such elements. Now let $v \in \bigcap \mathscr{X}$. By the definition we have that $\bigcap \mathcal{N}_v \subseteq X$ for every $X \in \mathscr{X}$. Then $\bigcap \mathcal{N}_v \subseteq \bigcap \mathscr{X}$.
4. In the last case we deal with arbitrary unions. Suppose that $\mathscr{X} \subseteq \mathcal{O}_w$ and consider $\bigcup \mathscr{X}$. Surely this union is contained in $\bigcup \mathcal{N}_w$. Now let us take an arbitrary $v \in \bigcup \mathscr{X}$. We know that $\bigcap \mathcal{N}_v \subseteq X$ for some $X \in \mathscr{X}$ (in fact, it holds for every X which contains v). Then clearly $\bigcap \mathcal{N}_v \subseteq \bigcup \mathscr{X}$. □

One thing should be noted. Clearly, we used t-condition to assure that the whole maximal w-neighborhood is w-open. Basically, in [14], we worked with structures without t-condition (we may call them **n1**-frames). Completeness theorem holds also for them – but it would be at least problematic to treat those frames as multi-topological.

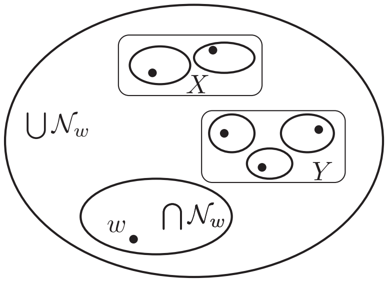

Fig. 1. Topology O_w. X, Y are w-open.

THEOREM 6.3. *Assume that we have $M_{\mathcal{N}} = \langle W, \mathcal{N}, V_{\mathcal{N}} \rangle$ and we define topology μ on W in the following way: if $A \subseteq W$, then $A \in \mu \Leftrightarrow$ for any $v \in A$, $\bigcap \mathcal{N}_v \subseteq A$. Then, for any $w \in W$, $\mathcal{O}_w$ is induced by μ (i.e. $\mathcal{O}_w$ is subspace topology).*

PROOF: Let us take $w \in W$. We shall prove that $\mathcal{O}_w$ consists strictly of intersections of $\bigcup \mathcal{N}_w$ and τ-open sets.

If $U \in \mathcal{O}_w$ then $U \in \mu$ (this is clear) and $U = U \cap \bigcup \mathcal{N}_w$. Assume now that A is μ-open and consider $Z = A \cap \bigcup \mathcal{N}_w$. Let us check that this set belongs to $\mathcal{O}_w$. Of course it is contained in $\bigcup \mathcal{N}_w$. Suppose that there is $z \in Z$ such that $\bigcap \mathcal{N}_z \nsubseteq Z$. But $\bigcap \mathcal{N}_z \subseteq A$ (because A is τ-open) and $\bigcap \mathcal{N}_z \subseteq \bigcup \mathcal{N}_w$ (because of t-condition). This is contradiction.

Additionally, one can easily check that μ is Alexandrov. □

6.2. Transformation

THEOREM 6.4. *For each* **n2**-*model $M_{\mathcal{N}} = \langle W, \mathcal{N}, V_{\mathcal{N}} \rangle$ there exists* **i-mtD**-*model $M_t = \langle W, \mathfrak{W}, V_t \rangle$ which is pointwise equivalent to $M_{\mathcal{N}}$, i.e. $w \Vdash \varphi \Leftrightarrow w \in V_t(\varphi)$.*

PROOF: Assume that we have $M_{\mathcal{N}} = \langle W, \mathcal{N}, V_{\mathcal{N}} \rangle$. Now let us consider the following structure: $M_t = \langle W, \mathfrak{W}, V_t \rangle$ where:

1. $\mathfrak{W} = \{\langle \bigcup \mathcal{N}_w, \mathcal{O}_w, \bigcap \mathcal{N}_w \rangle; w \in W\}$
2. for each $q \in PV$, $V_t(q) = V_{\mathcal{N}}(q)$

We shall identify each $\langle \bigcup \mathcal{N}_w, \mathcal{O}_w, \bigcap \mathcal{N}_w \rangle$ just with $\mathcal{O}_w$. It is easy to check that this is well-defined **i-mtD**-frame. For each $w \in W$ we can treat $\bigcup \mathcal{N}_w$ as a universe of topological subspace. Thus $\bigcap \mathcal{N}_w$ can be treated as distinguished set in this particular subspace.

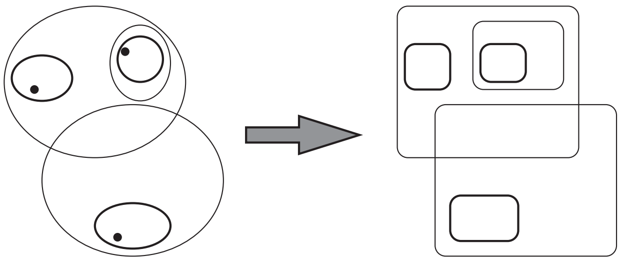

Fig. 2. From neighborhoods to multi-topological space with distinguished sets.

Now let us prove pointwise equivalency. Here we use induction by the complexity of formulas.

1. $\rightarrow$:
   ($\Rightarrow$) Suppose that $w \Vdash \varphi \rightarrow \psi$. We want to show there exists certain $\langle \bigcup \mathcal{N}_x, \mathcal{O}_x, \bigcap \mathcal{N}_x \rangle \in \mathfrak{W}$ such that $w \in Int_x((-V_t(\varphi) \cup V_t(\psi))$.
   We can say that $w \in \bigcap \mathcal{N}_w \subseteq \{x \in W; x \nVdash \varphi \text{ or } x \Vdash \psi\}$. By induction hypothesis, this set can be written as $\{x \in W; x \notin V_t(\varphi) \text{ or } x \in V_t(\psi)\} = -V_t(\varphi) \cup V_t(\psi)$. Recall the fact that $\bigcap \mathcal{N}_w \subseteq \bigcup \mathcal{N}_w$. Thus $w \in \bigcap \mathcal{N}_w \subseteq (-V_t(\varphi) \cup V_t(\psi)) \cap \bigcup \mathcal{N}_w$. But $\bigcap \mathcal{N}_w$ is w-open so it is contained in $Int_w(-V_t(\varphi) \cup V_t(\psi))$. We see that we could treat w as our x.
   ($\Leftarrow$) Now we assume that $w \in V_t(\varphi \rightarrow \psi)$. Thus we have certain $\mathcal{O}_x$ such that $w \in Int_x((-V_t(\varphi) \cap V_t(\psi))$. By induction hypothesis, $w \in Int_x(\{z \in W; z \nVdash \varphi \text{ or } z \Vdash \psi\})$. Hence, w belongs to the biggest x-open set X such that $X \subseteq \{z \in W; z \nVdash \varphi \text{ or } z \Vdash \psi\}$. But if X is x-open then $\bigcap \mathcal{N}_w \subseteq X$. Thus $w \Vdash \varphi \rightarrow \psi$.
2. $\Box$:
   ($\Rightarrow$) Assume that $w \Vdash \Box\varphi$. We want to show that $w \in V_t(\Box\varphi)$, i.e. that there is $X \subseteq W$ such that $w \in X$ and for certain $\mathcal{O}_x$ we have: $X = \bigcap \mathcal{N}_x$, $\bigcup \mathcal{N}_x \subseteq V_t(\varphi)$.

Surely, we can take $x = w$. Now, if $w \Vdash \Box\varphi$, then $\bigcup \mathcal{N}_w \subseteq V_{\mathcal{N}}(\varphi)$. By induction hypothesis, $\bigcup \mathcal{N}_w \subseteq V_t(\varphi)$.
($\Leftarrow$) Suppose that $w \in V_t(\Box\varphi)$. Thus $w \in X \subseteq W$ such that for certain $\mathcal{O}_x$ we can say that $X = \bigcap \mathcal{N}_x$ and $\bigcup \mathcal{N}_x \subseteq V_t(\varphi)$.
If $\bigcup \mathcal{N}_s \subseteq V_t(\varphi)$, then (by induction hypothesis) $\bigcup \mathcal{N}_x \subseteq V_{\mathcal{N}}(\varphi)$. Thus $x \Vdash \Box\varphi$. But $w \in \bigcap \mathcal{N}_x$. Thus, by the monotonicity of intuitionistic forcing, $w \Vdash \Box\varphi$. $\Box$

7. From multi-topological structures to neighborhood structures

In the former section we used multi-topological structures with distinguished open sets D^τ. Those sets are equivalents of minimal w-neighborhoods (while subspaces played the role of maximal w-neighborhoods). We used such unconventional approach mainly because our topology $\mathcal{O}_w$ does not "recognize" minimal neighborhoods. Thus, if we have $\bigcup \mathcal{N}_w$, then from the neighborhood point of view $\bigcap \mathcal{N}_w$ is specific – but as w-open set it is not distinguished in any way from other w-open sets. But we need such distinction to establish correspondence between $V_{\mathcal{N}}$ and V_t.

Now we are on the other side: we start from topological structures but defined in slightly different way. Here we do not have D^τ sets. We have the following definition (of frame):

Definition 7.1. **t2**-frame is an ordered pair $\langle W, \mathfrak{W}\rangle$ where:

1. $W \neq \emptyset$
2. $\mathfrak{W} = \{\langle T, \tau\rangle : T \subseteq W,\ \tau$ is an Alexandrov topology on $T\}$.
3. $W = \bigcup \mathcal{T}$, where $\mathcal{T} = \{T; \langle T, \tau\rangle \in \mathfrak{W}\}$

Each $\langle T, \tau\rangle$ is an Alexandrov space, so each $w \in T$ has its minimal τ-open neighborhood. If we denote the family of τ-open w-neighborhoods as $\mathcal{O}^w_\tau$, then we can introduce the following notation: $\bigcap \mathcal{O}^w_\tau = \min \mathcal{O}^w_\tau$.

Our definition of frame is very similar to Def. 5.1 but now we deny distinguished sets. However, in the definition of model there are bigger differences. In fact, we shall define forcing *after* introducing specific kind of neighborhoods in our topological environment.

Now let us think about intersection of all minimal τ-open w-neighborhoods. It will be denoted as $\bigcap_{\langle T,\tau\rangle \in \mathcal{T}^w}\{\min \mathcal{O}^w_\tau\}$ or shortly by $\bigcap_{\tau \in \mathcal{T}^w}\{\min \mathcal{O}^w_\tau\}$, where $\mathcal{T}^w = \{\langle T, \tau\rangle \in \mathfrak{W} : w \in T\}$. Below we define neighborhoods in the sense mentioned above.

Definition 7.2. Assume that we have **t2**-frame $\langle W, \mathfrak{W} \rangle$. Then for each $w \in W$ we define:

1. $\bigcap \mathcal{N}^t_w = \bigcap_{\tau \in \mathcal{T}^w} \{\min \mathcal{O}^w_\tau\}$
2. $\bigcup \mathcal{N}^t_w = \bigcap \mathcal{T}^w$
3. $X \in \mathcal{N}^t_w \subseteq P(P(W)) \Leftrightarrow \bigcap \mathcal{N}^t_w \subseteq X \subseteq \bigcup \mathcal{N}^t_w$

Theorem 7.3. *Assume that we have* **t2** *-frame* $\langle W, \mathfrak{W} \rangle$ *with* $\mathcal{N}^t_w$ *defined as in Def. 7.2. We state that for each* $w \in U$*,* $\mathcal{N}^t_w$ *has all the properties of neighborhood family in* **n2** *-frame.*

Proof: We must check five conditions:

1. $w \in \bigcap \mathcal{N}^t_w$. This is simple because $\bigcap \mathcal{N}^t_w$ is defined as an intersection of all τ-open w-neighborhoods (for every τ in $\mathcal{T}^w$) and certainly w is in each such neighborhood.
2. $\bigcap \mathcal{N}^t_w \in \mathcal{N}^t_w$. This is obvious by the very definition of $\mathcal{N}^t_w$.
3. $v \in \bigcap \mathcal{N}^t_w \Rightarrow \bigcap \mathcal{N}^t_v \subseteq \bigcap \mathcal{N}^t_w$. Let us note two facts. First, v is at least in all those spaces, in which w is (because it is in the intersection of all minimal w-neighborhoods). Thus, we can say that $\bigcap \mathcal{N}^t_v = \bigcap_{\tau \in \mathcal{T}^v} \{\min \mathcal{O}^v_\tau\} \subseteq \bigcap_{\tau \in \mathcal{T}^w} \{\min \mathcal{O}^v_\tau\}$.
   Second, suppose for a moment that we work with one particular Alexandrov topological space ρ. Assume that v belongs to the minimal ρ-open neighborhood of w. Of course v has its own minimal ρ-open neighborhood – but let us suppose that $\min \mathcal{O}^v_\rho \nsubseteq \min \mathcal{O}^w_\rho$. Now – from the basic properties of topology and the fact that at least v belongs to $\min \mathcal{O}^w_\rho$ – we state that $\min \mathcal{O}^v_\rho \cap \min \mathcal{O}^w_\rho$ is ρ-open. Of course, this intersection is contained in $\min \mathcal{O}^w_\rho$. Thus, we have contradiction with the assumption that minimal ρ-open v-neighborhood is not contained in $\min \mathcal{O}^w_\rho$.
   Now let us go back to the main part of the proof. The second fact allows us to say that $\bigcap_{\tau \in \mathcal{T}^w} \{\min \mathcal{O}^v_\tau\} \subseteq \bigcap_{\tau \in \mathcal{T}^w} \{\min \mathcal{O}^w_\tau\} = \bigcap \mathcal{N}^t_w$.
4. $v \in \bigcap \mathcal{N}^t_w \Rightarrow \bigcup \mathcal{N}^t_v \subseteq \bigcup \mathcal{N}^t_w$. As earlier, we say that v is at least in each space which belongs to $\mathcal{T}^w$. Thus $\bigcup \mathcal{N}^t_v = \bigcap \mathcal{T}^v = \bigcap \{\langle T, \tau \rangle \in \mathfrak{W} : v \in T\} \subseteq \bigcap \{\langle T, \tau \rangle \in \mathfrak{W} : w \in T\} = \bigcup \mathcal{N}^t_w$.
5. $v \in \bigcup \mathcal{N}^t_w \Rightarrow \bigcap \mathcal{N}^t_v \subseteq \bigcup \mathcal{N}^t_w$. Suppose that $v \in \bigcup \mathcal{N}^t_w$ defined as in Def. 7.2. Thus $v \in \bigcap \mathcal{T}^w$ which means in particular that v is in all those universes, in which w is. Now it is clear that $\bigcap \mathcal{N}^t_v$ – defined as an intersection of all τ-open minimal v-neighborhoods – must be contained at least in each element of $\mathcal{T}^w$, i.e. in $\bigcup \mathcal{N}^t_w$. □

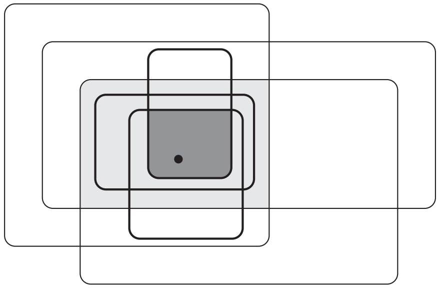

Fig. 3. Maximal and minimal neighborhoods in multi-topological space.

We have transformed our initial multi-topological structure into the neighborhood frame. Note that it is possible that for certain (and even for each) τ the set $\bigcap \mathcal{N}_w^t$ is *not* τ-open. We do not expect this. It is just intersection of all minimal w-neighborhoods. Now we shall introduce valuation and rules of forcing – thus obtaining logical model.

Definition 7.4. Assume that we have $\mathbf{t2}$-frame $\langle W, \mathfrak{W} \rangle$. Suppose that for each $w \in W$ we defined $\mathcal{N}_w^t$ as in Def. 7.2. We define valuation V_t as a function from PV into $P(W)$ satisfying the following condition: if $w \in V_t(q)$ then $\bigcap \mathcal{N}_w^t \subseteq V_t(q)$. The whole triple $\langle W, \mathfrak{W}, V_t \rangle$ is called $\mathbf{t2}$-model.

Definition 7.5. For every $\mathbf{t2}$-model $M_t = \langle W, \mathfrak{W}, V_t \rangle$, valuation of formulas is defined as such:

1. $V_t(\varphi \wedge \psi) = V_t(\varphi) \cap V_t(\psi)$
2. $V_t(\varphi \vee \psi) = V_t(\varphi) \cup V_t(\psi)$
3. $V_t(\varphi \rightarrow \psi) = \bigcup_{x \in \mathscr{I}} \{\bigcap \mathcal{N}_x^t\}$ where $\mathscr{I} = \{x \in W : \bigcap \mathcal{N}_x^t \subseteq -V_t(\varphi) \cup V_t(\psi)\}$
4. $V_t(\Box \varphi) = \bigcup_{x \in \mathscr{M}} \{\bigcap \mathcal{N}_x^t\}$ where $\mathscr{M} = \{x \in W : \bigcup \mathcal{N}_x^t \subseteq V_t(\varphi)\}$

We say that formula φ is true *iff* in each $\mathbf{t2}$-model $M_t = \langle W, \mathfrak{W}, V_t \rangle$ we have $V_t(\varphi) = W$.

The next theorem is crucial for our considerations.

Theorem 7.6. *For each $\mathbf{t2}$-model $M_t = \langle W, \mathfrak{W}, V_t \rangle$ there exists $\mathbf{n2}$-model $M_{\mathcal{N}} = \langle W, \mathcal{N}, V_{\mathcal{N}} \rangle$ which is pointwise equivalent to M_t, i.e. $w \Vdash \varphi \Leftrightarrow w \in V_t(\varphi)$.*

Proof: Let us take M_t and introduce $\mathcal{N}_w^t$ for each $w \in W$ just like in Def. 7.2. We define $V_{\mathcal{N}} : PV \rightarrow P(W)$ in the following way: $V_{\mathcal{N}} = V_t$.

Now the structure $M_{\mathcal{N}} = \langle W, \mathcal{N}^t, V_{\mathcal{N}} \rangle$ is a proper neighborhood model. In fact, we have already shown that it is **n2**-frame. By the definition of V_t we know that it is monotone in **n2**-frame. Let us check pointwise equivalency between both structures.

1. $\rightarrow$
   ($\Rightarrow$) Suppose that $w \Vdash \varphi \rightarrow \psi$. Thus $\bigcap \mathcal{N}^t_w \subseteq \{v \in W; v \nVdash \varphi$ or $v \Vdash \psi\} = -V_{\mathcal{N}}(\varphi) \cup V_{\mathcal{N}}(\psi)$. By induction this last set can be written as $-V_t(\varphi) \cup V_t(\psi)$. Thus, we can say that w belongs to $\mathscr{I}$ defined as in Def. 7.2. Of course $w \in \bigcap \mathcal{N}^t_w$. Hence, $w \in V_t(\varphi \rightarrow \psi)$.
   ($\Leftarrow$) Assume that $w \in V_t(\varphi \rightarrow \psi)$. This means that there is at least one point $x \in \mathscr{I}$ such that $w \in \bigcap \mathcal{N}^t_x$. But if $\bigcap \mathcal{N}^t_x \subseteq -V_t(\varphi) \cup V_t(\psi)$ then we can say that $\bigcap \mathcal{N}^t_x \subseteq -V_{\mathcal{N}}(\varphi) \cup V_{\mathcal{N}}(\psi)$ (by induction). Hence, $x \Vdash \varphi \rightarrow \psi$. The same can be said about w (because $w \in \bigcap \mathcal{N}^t_x$).
2. $\Box$
   ($\Rightarrow$) Suppose that $w \Vdash \Box\varphi$. Thus $\bigcup \mathcal{N}_w \subseteq V_{\mathcal{N}}(\varphi) = V_t(\varphi)$. The last equivalence is a result of induction hypothesis. Now we see that $w \in \mathscr{M}$. Of course $w \in \bigcap \mathcal{N}^t_w$. Then $w \in V_t(\Box\varphi)$.
   ($\Leftarrow$) Assume that $w \in V_t(\Box\varphi)$. Hence, there is at least one world $x \in \mathscr{M}$ such that $w \in \bigcap \mathcal{N}^t_x$. But if $\bigcup \mathcal{N}^t_x \subseteq V_t(\varphi)$, then by induction $\bigcup \mathcal{N}^t_x \subseteq V_{\mathcal{N}}(\varphi)$. This means that $x \Vdash \Box\varphi$. By monotonicity of forcing in $\bigcap \mathcal{N}^t_x$ we can say that $w \Vdash \Box\varphi$. $\Box$

8. Alternative approach

Let us go to back to the **n2**-frames. We shall define topology in a slightly different way than in Def. 6.1. Now we assume that $\bigcap \mathcal{N}_w$ is always contained in each w-open set.

Definition 8.1. Suppose that we have **n2**-frame $M_{\mathcal{N}} = \langle W, \mathcal{N} \rangle$. We say that $X \subseteq W$ is $w_{\min}$-open in **n2**-structure *iff* $X = \emptyset$ or $X \subseteq \bigcup \mathcal{N}_w$, $\bigcap \mathcal{N}_w \subseteq X$ and for every $v \in X$ we have $\bigcap \mathcal{N}_v \subseteq X$. We denote $\mathcal{Q}_w = \{X \subseteq W : X$ are $w_{\min}$-open $\} \cup \emptyset$ and call it $w_{\min}$-topology.

Theorem 8.2. *Assume that we have* **n2**-*frame* $F_{\mathcal{N}} = \langle W, \mathcal{N} \rangle$. *Then* $\langle \bigcup \mathcal{N}_w, \mathcal{Q}_w \rangle$ *is a topological space (for every* $w \in W$*).*

Proof: It is easy to check conditions of well-defined topology – just as in Th. 6.2. We leave details to the reader. $\Box$

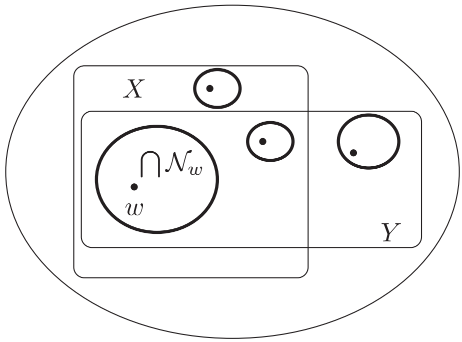

Fig. 4. Topology Q_w. X, Y are w-open.

8.1. From neighborhood frames to multi-topological structures once again

Let us introduce the new type of multi-topological structures. In fact, they are **t2**-frames but with valuation defined in a different way. Recall that $\mathcal{O}^w_\tau$ denotes the family of all τ-open w-neighborhoods and $\min \mathcal{O}^w_\tau$ is an intersection of such family.

Definition 8.3. **t3**-model is an ordered triple $M_t = \langle W, \mathfrak{W}, V\rangle$ where $\langle W, \mathfrak{W}\rangle$ is a **t2**-frame and V_t is a function from PV into $P(W)$ satisfying the following condition: $V_t(q) = \bigcup \mathcal{X}$ where $\mathcal{X} \subseteq \{X \subseteq W$; there is $\langle T, \tau\rangle \in \mathfrak{W}$ and $w \in T$ such that $X = \min \mathcal{O}^w_\tau\}$.

Definition 8.4. For every **t3**-model $M_t = \langle W, \mathfrak{W}, V_t\rangle$, valuation of formulas is defined as such:

1. $V_t(\varphi \wedge \psi) = V_t(\varphi) \cap V_t(\psi)$
2. $V_t(\varphi \vee \psi) = V_t(\varphi) \cup V_t(\psi)$
3. $V_t(\varphi \to \psi) = \bigcup \mathcal{X}$, where $\mathcal{X} = \{X \subseteq W$ such that $X \subseteq -V_t(\varphi) \cup V_t(\psi)$ and there are $\langle T, \tau\rangle \in \mathfrak{W}, x \in T$ for which $X = \min \mathcal{O}^x_\tau\}$.
4. $V_t(\Box\varphi) = \bigcup \mathcal{X}$, where $\mathcal{X} = \{X \subseteq W$ such that there are $\langle T, \tau\rangle \in \mathfrak{W}$, $x \in T$ for which $X = \min \mathcal{O}^x_\tau$ and $T \subseteq V_t(\varphi)\}$.

We say that formula φ is true *iff* in each **t3**-model $M_t = \langle W, \mathfrak{W}, V_t\rangle$ we have $V_t(\varphi) = W$.

One can see that in some sense we composed earlier definitions of multi-topological frames, valuations and models. Now our situation is similar to that from section 5. The main difference is that we can work with minimal τ-open sets, i.e. with $\min \mathcal{O}^w_\tau$.

THEOREM 8.5. *For each* $\mathbf{n2}$*-model* $M_\mathcal{N} = \langle W, \mathcal{N}, V_\mathcal{N} \rangle$ *there exists* $\mathbf{t3}$*-model* $M_t = \langle W, \mathfrak{W}, V_t \rangle$ *which is pointwise equivalent to* $M_\mathcal{N}$*, i.e.* $w \Vdash \varphi \Leftrightarrow w \in V_t(\varphi)$.

PROOF: Assume that we have $M_\mathcal{N} = \langle W, \mathcal{N}, V_\mathcal{N} \rangle$. Now let us consider the following structure: $M_t = \langle W, \mathfrak{W}, V_t \rangle$ where:

1. $\mathfrak{W} = \{\langle \bigcup \mathcal{N}_w, \mathcal{Q}_w \rangle : w \in W\}$
2. for each $q \in PV$, $V_t(q) = V_\mathcal{N}(q)$

It is easy to check that $\langle W, \mathfrak{W} \rangle$ is a well-defined $\mathbf{t2}$-frame. Let us prove pointwise equivalency by means of induction.

$\rightarrow$

($\Rightarrow$) Suppose that $w \Vdash \varphi \rightarrow \psi$. Thus $\bigcap \mathcal{N}_w \subseteq \{v \in W; v \nVdash \varphi$ or $v \Vdash \psi\}$. The last set – by induction hypothesis – is equal to $-V_t(\varphi) \cup V_t(\psi)$. Moreover, $\bigcap \mathcal{N}_w$ is an intersection of all $w_{\min}$-open sets (recall Def. 8.1) and $w \in \bigcap \mathcal{N}_w \subseteq \bigcup \mathcal{N}_w$. Thus $w \in V_t(\varphi \rightarrow \psi)$.

($\Leftarrow$) Assume that $w \in V_t(\varphi \rightarrow \psi)$. First, there is $X \subseteq W$ such that $w \in X$ and $X \subseteq -V_t(\varphi) \cup V_t(\psi)$. Second, there is $\langle \bigcup \mathcal{N}_x, \mathcal{Q}_x \rangle \in \mathfrak{W}$ such that X is minimal $\mathcal{Q}_x$-open x-neighborhood. In fact, it means that $X = \bigcap \mathcal{N}_x$. So $\bigcap \mathcal{N}_x \subseteq -V_t(\varphi) \cup V_t(\psi) =$[ind. hyp.]$-V_\mathcal{N}(\varphi) \cup V_\mathcal{N}(\psi) = \{z \in W; z \nVdash \varphi$ or $z \Vdash \psi\}$. Then, in particular, $x \Vdash \varphi \rightarrow \psi$ and also $w \Vdash \varphi \rightarrow \psi$ (because $w \in \bigcap \mathcal{N}_x$ and we have intuitionistic monotonicity of forcing).

$\Box$

($\Rightarrow$) Suppose that $w \Vdash \Box\varphi$. Thus $\bigcup \mathcal{N}_w \subseteq \{v \in W; v \Vdash \varphi\}$. The last set is – by induction hypothesis – equal to $V_t(\varphi)$. We can say that conditions from Def. 8.4 are satisfied: our X is $\bigcap \mathcal{N}_w$ and our topological space is $\langle \bigcup \mathcal{N}_w, \mathcal{Q}_w \rangle$. Thus $w \in V_t(\Box\varphi)$.

($\Leftarrow$) Assume that $w \in V_t(\Box\varphi)$. Thus, we have $X \subseteq W$ such that $w \in X$ and there are $x \in W$, $\langle \bigcup \mathcal{N}_x, \mathcal{Q}_x \rangle \in \mathfrak{W}$ such that X is $\bigcap \mathcal{N}_x$ (i.e. minimal $\mathcal{Q}_x$-open x-neighborhood) and $\bigcup \mathcal{N}_x \subseteq V_t(\varphi)$. By induction hypothesis $\bigcup \mathcal{N}_x \subseteq V_\mathcal{N}(\varphi)$. Thus, $x \Vdash \Box\varphi$. By monotonicity of forcing, $w \Vdash \varphi$. $\Box$

9. Summary

In this paper we used a lot of notions and symbols. We have introduced three different concepts of multi-topological frames (models). Moreover, we used the notion of *neighborhood* in three ways. First, we spoke about the class of all neighborhood structures (**n2**-frames). Second, we made references to neighborhoods in the standard topological sense. Third, we used those topological neighborhoods to transform multi-topological frame into certain specific **n2**-frame. Hence, we shall repeat the most important things and sum up our considerations.

In section 3 we have described neighborhood semantics for intuitionistic modal logic. It is based on the notions of *minimal* ("intuitionistic") and *maximal* ("modal") neighborhoods.

In section 5 we have introduced **mtD**-frames (models). They are collections of topological spaces. These spaces can intersect or form unions. We assumed that each space $\langle T, \tau \rangle$ has certain distinguished open set D^{τ}. Then we have shown how it is possible to treat **n2**-frames as **mtD**-frames. Shortly speaking, the main idea is to make connection between maximal (resp. minimal) neighborhoods and universes T (resp. distinguished sets).

In section 7 we spoke about **t2**-frames (models). They are similar to the class of **mtD** but each topology is Alexandrov and we do not introduce distinguished sets anymore. We have shown how to transform those structures into neighborhood models. Let us repeat main steps of this reasoning. Assume that W is the whole universe of a given **t2**-frame. Now let us take an arbitrary $w \in W$. For each topology τ we have minimal τ-open w-neighborhood (because of Alexandrov property). We take intersection of all such minimal neighborhoods and treat it as $\bigcap \mathcal{N}_w$ (as the minimal w-neighborhood in the sense of **n2**-frames). Then we take intersection of all topological spaces to which w belongs and this is our maximal neighborhood.

In section 8.1 we came back to **n2**-frames but we introduced another topology in those structures (different than in section 5). It is possible to transform **n2**-models with this topology into **t3**-multi-topological models – which are based on **t2**-frames but with different valuation than in section 7.

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A NOTE ON DISTRIBUTIVE TRIPLES

Abstract

Even if a lattice L is not distributive, it is still possible that for particular elements $x, y, z \in L$ it holds $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$. If this is the case, we say that the triple (x, y, z) is distributive. In this note we provide some sufficient conditions for the distributivity of a given triple.

Keywords: Distributive triple, dually distributive triple, covering diamond.

Standard lattice-theoretic notions can be found in [3]. Let us recall basic definitions and facts. If L is a lattice and $a, b \in L$, then the set $[a, b] = \{c \in L : a \leqslant c \leqslant b\}$ is called an *interval* (in L). Clearly, any interval is a sublattice of L. If $X \subseteq L$, then $[X]$ stands for the sublattice generated by X, i.e., the smallest sublattice of L, which contains the subset X. For any subset $X \subseteq L$ and for any interval $[a, b]$ we define

$$[\![a, b]\!]_X := [a, b] \cap [X].$$

In particular, if $X = \{x, y, z\}$, then $[\![x \wedge y \wedge z, x \vee y \vee z]\!]_X = [X]$.

A lattice L is said to be *modular* if $x \leqslant z$ implies $(x \vee y) \wedge z = x \vee (y \wedge z)$, for all $x, y, z \in L$. Moreover, L is called *distributive* if $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$, for all $x, y, z \in L$. The Dedekind–Birkhoff Theorem (cf. [3], p. 59) states that a lattice L is modular if and only if L does not contain a sublattice isomorphic to N_5 (so-called *pentagon*), and moreover, and L is distributive if and only if L does not contain a sublattice isomorphic to N_5 nor M_3 (so-called *diamond*).

Let L be an arbitrary lattice and $x, y, z \in L$. We say that (x, y, z) is a *distributive triple*, $(x, y, z)D$ in symbols, if $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$. Similarly, (x, y, z) is called a *dually distributive triple*, $(x, y, z)D^*$ in symbols, if $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$ (cf. [7], p. 76[1]). Clearly, L is distributive if and only if $(x, y, z)D$, for all x, y, z. G. Birkhoff proved the following

Theorem 1 ([1], Theorem II.12). *Let L be a modular lattice and $X = \{x, y, z\} \subseteq L$. Then:*

(*i*) $[\![x \wedge y \wedge z, x \vee y \vee z]\!]_X$ *is distributive if and only if* $(x, y, z)D$,

(*ii*) $[\![x \wedge y \wedge z, x \vee y \vee z]\!]_X$ *is distributive if and only if* $(x, y, z)D^*$.

The Dedekind–Birkhoff Theorem shows that the hypothesis of modularity is necessary as well as sufficient in Theorem 1 (cf. the lattice (a) in Figure 1).

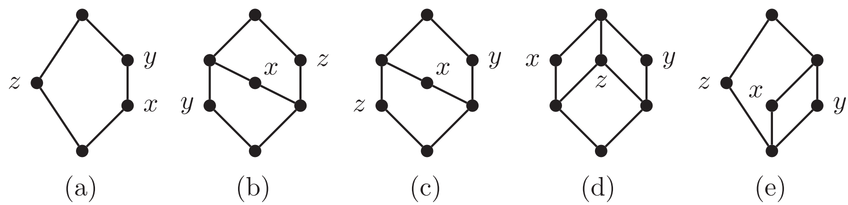

Fig. 1. Non-modular lattices satisfying $(x, y, z)D$ or $(x, y, z)D^*$.

Our result is the following

Theorem 2. *Let L be an arbitrary lattice and $X = \{x, y, z\} \subseteq L$. Then:*

(*i*) *if* $[\![x \wedge z, x \vee y \vee z]\!]_X$ *and* $[\![y \wedge z, x \vee y \vee z]\!]_X$ *are distributive, then* $(x, y, z)D$,

(*ii*) *if* $[\![x \wedge y \wedge z, x \vee y]\!]_X$ *is distributive, then* $(x, y, z)D$.

Proof: To prove (i), assume that $[\![x \wedge z, x \vee y \vee z]\!]_X$ and $[\![y \wedge z, x \vee y \vee z]\!]_X$ are distributive sublattices of L. Then

[1] Note that Birkhoff in [1], p. 37, provides a different definition: a three-element subset $\{x, y, z\}$ of a lattice L is a distributive triple if $[\{x, y, z\}]$ is a distributive sublattice of L.

$$
\begin{aligned}
z \wedge (x \vee y) &= z \wedge \Big(x \vee \big(y \vee (x \wedge z)\big)\Big) \\
&= (z \wedge x) \vee \Big(z \wedge \big(y \vee (x \wedge z)\big)\Big) \qquad \text{(by the 1st assumption)} \\
&= z \wedge \big(y \vee (x \wedge z)\big) \\
&= z \wedge \Big(y \vee \big((x \wedge z) \vee (y \wedge z)\big)\Big) \\
&= (z \wedge y) \vee \Big(z \wedge \big((x \wedge z) \vee (y \wedge z)\big)\Big) \\
&\qquad \text{(by the 2nd assumption)} \\
&= (z \wedge y) \vee \big((x \wedge z) \vee (y \wedge z)\big) \\
&= (z \wedge y) \vee (x \wedge z),
\end{aligned}
$$

which completes the proof of (i).

For (ii), we assume that $[\![x \wedge y \wedge z, x \vee y]\!]_X$ is distributive and calculate as follows:

$$
\begin{aligned}
z \wedge (x \vee y) &= \big(z \wedge (x \vee y)\big) \wedge (x \vee y) \\
&= \Big(\big(z \wedge (x \vee y)\big) \wedge x\Big) \vee \Big(\big(z \wedge (x \vee y)\big) \wedge y\Big) \\
&\qquad \text{(by the assumption)} \\
&= (z \wedge x) \vee (z \wedge y).
\end{aligned}
$$

■

By the duality principle we obtain

THEOREM 3. *Let L be an arbitrary lattice and $X = \{x, y, z\} \subseteq L$. Then:*

(i) if $[\![x \wedge y \wedge z, x \vee z]\!]_X$ and $[\![x \wedge y \wedge z, y \vee z]\!]_X$ are distributive, then $(x, y, z)D^*$,

(ii) if $[\![x \wedge y, x \vee y \vee z]\!]_X$ is distributive, then $(x, y, z)D^*$.

REMARK 1. *Lattices (b) and (c) in Figure 1 disprove the converses of Theorems 2 and 3, respectively.*

REMARK 2. *Theorem 2 allows the conclusion that $(x, y, z)D$ in lattices (d) and (e) in Figure 1. On the other hand, this fact cannot be justified on the basis of Theorem 1.*

In order to illustrate a possible use of Theorem 2 we will provide an easy inductive proof of the following

THEOREM 4. *Let L be a lattice of finite length. If L is modular but non-distributive lattice, then L contains a covering diamond, i.e., a diamond $D = \{o, a, b, c, i\}$, such that $o \prec a, b, c \prec i$.*

In the literature of lattice theory the preceding theorem is known as "folklore" (cf. [4], p. 111, or [2], p. 270). This theorem easily follows from [5] (cf. Theorem 1.4 for the case $n = 2$), or from [3] (cf. Lemma 8, p. 247). Note that [6] generalizes the theorem to the class of weakly atomic lattices.

PROOF OF THEOREM 4: Induction on $l(L)$—the length of L. If $l(L) = 1$ or $l(L) = 2$ the theorem is obvious. For the induction step, assume that for any modular, non-distributive lattice K if $l(K) < n$, then K contains a covering diamond. Moreover, fix a modular, non-distributive lattice L such that $l(L) = n \geqslant 3$. Then, by Dedekind–Birkhoff Theorem, L contains a diamond $D = \{o, a, b, c, i\}$. If $0 < o$ or $i < 1$, then $[o, i]$ satisfies premises of our induction hypothesis, thus it contains a covering diamond, so L does. If not, i.e., $D = \{0, a, b, c, 1\}$, since $l(L) \geqslant 3$ there exists some intermediate element $x \notin D$; we may assume without loss of generality that $b < x < 1$.

Let us observe that $a \wedge x > 0$, because if not, the set $\{0, a, x, b, 1\}$ would be a pentagon. For similar reasons, $c \wedge x > 0$. Now, consider intervals $[a \wedge x, 1]$ and $[c \wedge x, 1]$. If one of them is non-distributive, then by the induction hypothesis, it contains a covering diamond, so L does. On the other hand, if both intervals are distributive, then by Theorem 2, the triple (a, c, x) is distributive, thus we obtain

$$(a \wedge x) \vee (c \wedge x) = (a \vee c) \wedge x = 1 \wedge x = x.$$

Moreover, by modularity, we get $(a \wedge x) \vee b = x$ and $(c \wedge x) \vee b = x$, and obviously $(a \wedge x) \wedge (c \wedge x) = (a \wedge x) \wedge b = (c \wedge x) \wedge b = 0$, so the set $\{0, a \wedge x, b, c \wedge x, x\}$ forms a diamond. Therefore, by the induction hypothesis, the interval $[0, x]$ contains a covering diamond, and hence L does. ■

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MODAL BOOLEAN CONNEXIVE LOGICS: SEMANTICS AND TABLEAU APPROACH

Abstract

In this paper we investigate Boolean connexive logics in a language with modal operators: $\Box$, $\Diamond$. In such logics, negation, conjunction, and disjunction behave in a classical, Boolean way. Only implication is non-classical. We construct these logics by mixing relating semantics with possible worlds. This way, we obtain connexive counterparts of basic normal modal logics. However, most of their traditional axioms formulated in terms of modalities and implication do not hold anymore without additional constraints, since our implication is weaker than the material one. In the final section, we present a tableau approach to the discussed modal logics.

Keywords: Boolean connexive logics, connexive logic, modal Boolean connexive logics, modal logics, normal modal logics, possible worlds semantics, relatedness, relating logic, relating semantics, tableau methods.

1. Introduction

Aristotle's and Boethius' laws are of fundamental significance for the connexive logics. Negation and implication are the only connectives involved in them:

$$\text{(A1)} \qquad \sim (A \Rightarrow \sim A)$$

$$\text{(A2)} \qquad \sim (\sim A \Rightarrow A)$$

(B1) $$(A \Rightarrow B) \Rightarrow \sim (A \Rightarrow \sim B)$$

(B2) $$(A \Rightarrow \sim B) \Rightarrow \sim (A \Rightarrow B)$$

If we add any of these laws to the classical logic then by applying the modus ponens rule and substitution we obtain trivial, inconsistent logic – a set of all formulas. For this reason, in order to investigate connexive logics we need to interpret the negation or implication in a non-classical way. On the other hand, it is a natural idea to keep as close as possible to the classical logic while investigating the connexive logics by interpreting the Aristotelian and Boethian laws. This very idea guided us in our research published in paper [6]. There, we interpreted negation, conjunction, and disjunction in the classical Boolean way, leaving at the same time aside a broad spectrum of possible interpretations of implication by application of relating semantics to it. In the analyzed approach, the truth conditions for the implication consist not only of classical requirement that a antecedent is false or a consequent is true. There is an additional requirement that antecedent and consequent are interrelated by some binary relation R.

In [6] we constructed 32 logics by determining respective classes of relations R. In two of them Aristotelian and Boethian laws hold and at the same time the negation, conjunction, and disjunction preserve the Boolean meaning. That is why we proposed the name *Boolean connexive logics* for such logics.

It is well known that a logical system solely based on the Aristotle's and Boethius' theses allows strange interpretations either of implication or of negation, which makes it far to weak for any reasonable applications. See [9] for historical review of this subject.

In the literature one could distinguish three approaches to the connexivity. The first one consists of proposals of specific logical systems designed for a given intended aims, such as [12], where connexive systems of conditionals are investigated. In the second approach, such as [7] and [8], some additional conditions are proposed to distinguish the well behaving connexive logics. One could also find the third approach, such as [10], where a comparison of different connexive logics is conducted. In the present paper we understand the connexive logic in a very general way, just as any set of sentences closed under substitution and modus ponens containing the Aristotle's and Boethian laws. This way, we investigate the structural properties of a broad spectrum of the sentential connexive logics.

In a general way, the relating semantics was proposed in paper [4]. Its main notion – a relating relation – can be equipped with a large number of philosophical and not only philosophical motivations and interpretations. Two formulas can be related by R in many ways. For example, they could be related analytically, causally, thematically, temporally, etc. In this paper, the relating semantics is directly applied to the connexive implication. As a consequence, in this approach the connexive implication is true iff a antecedent is false or a consequent is true and simultaneously both are connected in some way. In the semantics, this connection is expressed by the relating relation.

In the present paper, we continue the investigations initiated in [6] by generalizing its results towards the area of modal logics. In a natural way, by *modal Boolean connexive logics* we mean a logic formulated in the sentential language with implication, classical negation, classical disjunction, classical conjunction, necessity and possibility operators satisfying the Aristotle's and Boethius' laws.

To express this in a short way: the modal Boolean connexive logic is a Boolean connexive logic defined in a modal language. The semantics considered here is a kind of combination of possible worlds semantics and relating semantics. As a consequence, we have two types of binary relations: a relating relation between formulas determining the meaning of implication and an accessibility relation on possible worlds defining modal operators. It appears that both kinds of relations affect each other and to some extent limit the traditional modal laws. A motivation for considering this particular combination of two semantics is natural. Since in [6] we considered the connexive logics without possible worlds, here we extend our ideas onto the possible worlds framework.

We define a number of connexive counterparts of the basic normal modal logics. However, most of their traditional axioms formulated in terms of modalities and implication do not hold anymore, since our implication is weaker than the material one. To make them valid we must impose some additional constraints on the relating semantics. In particular, we propose that modal operators have no influence on being related. We show the correspondence between both: relating as well as accessibility relations and modal axioms in all presented variants.

Additionally, as a decision procedure, in the last section we propose tableau methods for the connexive modal systems that constructed here.

2. Relating logics: syntax and semantics

Let us consider the set of formulas $\mathsf{For}_{\mathbf{CPL}}$ of Classical Propositional Logic (CPL), made up in a standard manner from: variables $\mathsf{Var} = \{p, q, r, p_1, q_1, r_1, \dots\}$, one unary connective: $\neg$, four binary connectives: $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, and brackets:), (. Let $\models_{\mathbf{CPL}}$ be a consequence relation of CPL defined on $\mathsf{For}_{\mathbf{CPL}}$ by the set of all classical valuations of formulas from $\mathsf{For}_{\mathbf{CPL}}$.

Whereas the set of formulas of Relating Logic (RL) $\mathsf{For}_{\mathbf{RL}}$ is generated with Var, negation $\neg$, four binary connectives: $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$ and four binary relating connectives that are relating counterparts of classical connectives: $\wedge^{\mathrm{w}}$, $\vee^{\mathrm{w}}$, $\rightarrow^{\mathrm{w}}$, $\leftrightarrow^{\mathrm{w}}$, and brackets:), (. Thus, $\mathsf{For}_{\mathbf{CPL}} \subset \mathsf{For}_{\mathbf{RL}}$.

A model for the relating formulas is pair $\langle v, \mathrm{R}\rangle$, where $v \colon \mathsf{Var} \mapsto \{0, 1\}$ and $\mathrm{R} \subseteq \mathsf{For}_{\mathbf{RL}} \times \mathsf{For}_{\mathbf{RL}}$. Function v to any variable assigns either truth or falsity. Relation R is called *relating relation*. We have the following, general truth conditions for the relating formulas:

$$\langle v, \mathrm{R}\rangle \models A \text{ iff } v(A) = 1, \qquad \text{if } A \in \mathsf{Var}$$

$$\langle v, \mathrm{R}\rangle \models \neg A \text{ iff } \langle v, \mathrm{R}\rangle \not\models A$$

$$\langle v, \mathrm{R}\rangle \models A \wedge^{\mathrm{w}} B \text{ iff } \langle v, \mathrm{R}\rangle \models A \;\&\; \langle v, \mathrm{R}\rangle \models B \;\&\; \mathrm{R}(A, B)$$

$$\langle v, \mathrm{R}\rangle \models A \vee^{\mathrm{w}} B \text{ iff } [\langle v, \mathrm{R}\rangle \models A \text{ or } \langle v, \mathrm{R}\rangle \models B] \;\&\; \mathrm{R}(A, B)$$

$$\langle v, \mathrm{R}\rangle \models A \rightarrow^{\mathrm{w}} B \text{ iff } [\langle v, \mathrm{R}\rangle \not\models A \text{ or } \langle v, \mathrm{R}\rangle \models B] \;\&\; \mathrm{R}(A, B)$$

$$\langle v, \mathrm{R}\rangle \models A \leftrightarrow^{\mathrm{w}} B \text{ iff } [\langle v, \mathrm{R}\rangle \models A \text{ iff } \langle v, \mathrm{R}\rangle \models B] \;\&\; \mathrm{R}(A, B).$$

As we can see, the relating connectives have intensional, and even hyper-intensional character, since the Boolean conditions are not sufficient.

The set of all models for **RL** will be denoted by $\mathbf{M}_{\mathbf{RL}}$. By taking any subset $\mathbf{M}$ of $\mathbf{M}_{\mathbf{RL}}$ in the standard way, we define relating logic $\models_{\mathbf{M}}$:

$$X \models_{\mathbf{M}} A \text{ iff for all } \mathfrak{M} \in \mathbf{M}, \text{ if } \mathfrak{M} \models X, \text{ then } \mathfrak{M} \models A.$$

The smallest relating logic is defined *modulo* all models. It is called RF in [4].

The first relating logic was probably proposed in [1], [2], [11]. However, in those studies, the authors analyzed a special kind of relating logics, called *relatedness logic*. There was considered a specific relation needed to define an extraordinary kind of content-related implication. On the other hand, the approach to the relating logics initiated in [4] is more general. We find the relatedness logic as a part of a much wider class of the relating logics with the multi-domain of applications. In principle, we could apply the relating semantics to any logic. Considering the formal conditions defining the classes of relating relations, one could determine the subclasses of $\mathbf{M_{RL}}$, and in consequence, define a multitude of specific relating logics.

However, if we only take account of the relating binary part of RL formulas, meaning the smallest subset of $\mathsf{For}_{\mathbf{RL}}$ closed under Var, $\neg$, $\wedge^{\mathrm{w}}$, $\vee^{\mathrm{w}}$, $\to^{\mathrm{w}}$, $\leftrightarrow^{\mathrm{w}}$, and brackets), (, we shall get set $\mathsf{For}^{\mathrm{w}}_{\mathbf{RL}} \subset \mathsf{For}_{\mathbf{RL}}$ that is structurally identical to $\mathsf{For}_{\mathbf{CPL}}$. Then we could get just CPL, if as models we assumed all models $\langle v, \mathrm{R}\rangle$, where R is a universal relation, so $R = \mathsf{For}^{\mathrm{w}}_{\mathbf{RL}} \times \mathsf{For}^{\mathrm{w}}_{\mathbf{RL}}$.

In order to simplify the notation, we define the language of Boolean connexive logics as identical to $\mathsf{For}_{\mathbf{CPL}}$. Although in [6] we used the language generated with: variables $\mathsf{Var} = \{p, q, r, p_1, q_1, r_1, \ldots\}$; one unary connective: $\neg$; three binary connectives: $\wedge$, $\vee$, $\to^{\mathrm{w}}$ and brackets), (. Also for the sake of simplicity, here instead of symbol $\to^{\mathrm{w}}$ we shall use symbol $\to$. The obtained set, will be denoted by $\mathsf{For}_{\mathbf{CF}}$ (*connexive formulas*), so in fact $\mathsf{For}_{\mathbf{CF}} = \mathsf{For}_{\mathbf{CPL}}$.

Now, implication $\to$ is intended to behave like a relating connective, while the other ones hold the classical, Boolean meaning.

Basic semantics for the Boolean connexive logics can be defined by the following truth conditions. Extensional for the Boolean operators:

$$\langle v, \mathrm{R}\rangle \models A \text{ iff } v(A) = 1, \qquad \text{if } A \in \mathsf{Var}$$

$$\langle v, \mathrm{R}\rangle \models \neg A \text{ iff } \langle v, \mathrm{R}\rangle \not\models A$$

$$\langle v, \mathrm{R}\rangle \models A \wedge B \text{ iff } \langle v, \mathrm{R}\rangle \models A \;\&\; \langle v, \mathrm{R}\rangle \models B$$

$$\langle v, \mathrm{R}\rangle \models A \vee B \text{ iff } \langle v, \mathrm{R}\rangle \models A \text{ or } \langle v, \mathrm{R}\rangle \models B$$

and the intensional condition for relating implication $\to$:

$$\langle v, \mathrm{R}\rangle \models A \to B \text{ iff } [\langle v, \mathrm{R}\rangle \not\models A \text{ or } \langle v, \mathrm{R}\rangle \models B] \;\&\; \mathrm{R}(A, B).$$

Similarly, as in the modal logic, we treat R as a structure of given model. So, we assume: $\mathrm{R} \models A$ iff for all valuations of letters v, $\langle v, \mathrm{R}\rangle \models A$. Obviously, in order to accomodate the specific connexive laws, we had to distinguish some class of models.

3. Quasi–connexive and connexive Boolean logics

Before we propose some extension of Boolean connexive logic to the modal language, let us recapitulate the basic facts from [6].

Let $\mathsf{For}^2_{\mathbf{CF}}$ denote $\mathsf{For}_{\mathbf{CF}} \times \mathsf{For}_{\mathbf{CF}}$. Let $\mathrm{R} \subseteq \mathsf{For}^2_{\mathbf{CF}}$. To define suitable classes of models for the Boolean connexive logics, we require the complement of relating relation R. $A\tilde{\mathrm{R}}B$ means that the relation $A\mathrm{R}B$ does not hold.

We define some classes of relations R determined by the following conditions:

(a1) R is (a1) iff for all $A \in \mathsf{For}_{\mathbf{CF}}$, $A\widetilde{\mathrm{R}}\neg A$

(a2) R is (a2) iff for all $A \in \mathsf{For}_{\mathbf{CF}}$, $\neg A\widetilde{\mathrm{R}}A$

(b1) R is (b1) iff for all $A, B \in \mathsf{For}_{\mathbf{CF}}$:
- if $A\mathrm{R}B$, then $A\widetilde{\mathrm{R}}\neg B$
- $(A \to B)\mathrm{R}\neg(A \to \neg B)$

(b2) R is (b2) iff for all $A, B \in \mathsf{For}_{\mathbf{CF}}$:
- if $A\mathrm{R}B$, then $A\widetilde{\mathrm{R}}\neg B$
- $(A \to \neg B)\mathrm{R}\neg(A \to B)$

(c1) R is (c1) iff for all $A, B \in \mathsf{For}_{\mathbf{CF}}$, if $A\mathrm{R}B$ then $\neg A\mathrm{R}\neg A$.

If R is (c1), it is often called *closed under negation*.

In paper [6] we showed that conditions (a1), (a2), (b1), (b2), (c1) were independent. Moreover, we proved the following theorem:

THEOREM 3.1 (Correspondence theorem). *Let $R \subseteq \mathsf{For}^2_{CF}$ satisfy (c1). Then:*

$$
\begin{aligned}
&R \textit{ is (a1)} \Leftrightarrow R \models \neg(A \to \neg A)\\
&R \textit{ is (a2)} \Leftrightarrow R \models \neg(\neg A \to A)\\
&R \textit{ is (b1)} \Leftrightarrow R \models (A \to B) \to \neg(A \to \neg B)\\
&R \textit{ is (b2)} \Leftrightarrow R \models (A \to \neg B) \to \neg(A \to B).
\end{aligned}
$$

Condition (c1) is needed there only for proving the inference from the left to right-hand side part of each condition, hence we have the following:

THEOREM 3.2. *Suppose that $R \subseteq \mathsf{For}^2_{CF}$:*

(1) R *is (a1)* $\Rightarrow R \models \neg(A \to \neg A)$

(2) R *is (a2)* $\Rightarrow R \models \neg(\neg A \to A)$

(3) R *is (b1)* $\Rightarrow R \models (A \to B) \to \neg(A \to \neg B)$

(4) R *is (b2)* $\Rightarrow R \models (A \to \neg B) \to \neg(A \to B)$.

The theorems above allow us to construct a dozen of logical systems by imposing some limitations on the relating models. Some of them are Boolean connexive logics, other are not. Since the conditions from set {(a1), (a2), (b1), (b2), (c1)} are independent, then any two of its subsets determine different logical systems. Since there are $2^5 = 32$ subsets, so the combinations of conditions determine 32 different logical systems. Among them, two logics are Boolean connexive logics: determined by (a1), (a2), (b1), (b2) in one case, determined by (a1), (a2), (b1), (b2), (c1) in the other. The logic determined by conditions (a1), (a2), (b1), (b2) is the least Boolean connexive logic.

Let us assume that by *Boolean quasi-connexive logic* we mean a logic determined by set of all models satisfying at least one, but not all of conditions: (a1), (a2), (b1), (b2). Then among 32 logics determined by the above models: (i) two are neither connexive, nor quasi-connexive – zero of Aristotelian or Boethian conditions are satisfied; (ii) 28 logics are quasi-connexive – at least one, but not all conditions are satisfied.

4. Emerging modal Boolean connexive logics

Now we extend the language $\mathsf{For}_{\mathbf{CF}}$ by closing it additionally under two unary modal operators $\Box$ and $\Diamond$. The language constructed this way will be denoted by $\mathsf{For}_{\mathbf{CMF}}$ (*connexive modal formulas*).

By a *model* for $\mathsf{For}_{\mathbf{CMF}}$ we mean quadruple $\langle W, Q, \{\mathrm{R}_w\}_{w \in W}, v\rangle$ where:

- W is a non empty set of "possible worlds"
- $Q \subseteq W \times W$ is an accessibility relation
- $\{\mathrm{R}_w\}_{w \in W}$ is a family of relations that contsists of $\mathrm{R}_w \subseteq \mathsf{For}_{\mathbf{CMF}} \times \mathsf{For}_{\mathbf{CMF}}$, for any $w \in W$, so a particular R_w is a relating relation, for any possible world $w \in W$
- $v \colon W \times \mathsf{Var} \longrightarrow \{0, 1\}$ v is a valuation of sentential letters in worlds.

Let us note again that a model contains two types of binary relations. Relation Q is a standard accessibility relation between possible words, while indexed R is a binary relation between formulas, one for each possible world w in W.

For any model $\mathfrak{M} = \langle W, Q, \{\mathrm{R}_w\}_{w\in W}, v\rangle$ and any $w \in W$, we define a satisfaction relation in the following way:

$$\mathfrak{M}, w \models A \text{ iff } v(w, A) = 1, \qquad \text{if } A \in \mathsf{Var}$$

$$\mathfrak{M}, w \models \neg A \text{ iff } \mathfrak{M}, w \not\models A$$

$$\mathfrak{M}, w \models A \wedge B \text{ iff } \mathfrak{M}, w \models A \;\&\; \mathfrak{M}, w \models B$$

$$\mathfrak{M}, w \models A \vee B \text{ iff } \mathfrak{M}, w \models A \text{ or } \mathfrak{M}, w \models B$$

$$\mathfrak{M}, w \models \Box A \text{ iff } \forall_{u\in W}(wQu \Rightarrow \mathfrak{M}, u \models A)$$

$$\mathfrak{M}, w \models \Diamond A \text{ iff } \exists_{u\in W}(wQu \,\&\, \mathfrak{M}, u \models A)$$

$$\mathfrak{M}, w \models A \rightarrow B \text{ iff } [\mathfrak{M}, w \not\models A \text{ or } \mathfrak{M}, w \models B] \;\&\; A\mathrm{R}_w B.$$

Let $\mathfrak{M} = \langle W, Q, \{\mathrm{R}_w\}_{w\in W}, v\rangle$. We will say that formula A is *true* in model $\mathfrak{M}$ (in symbols: $\mathfrak{M} \models A$) iff for any possible world w in W: $\mathfrak{M}, w \models A$. If $X \subseteq \mathsf{For}_{\mathbf{CMF}}$ and $w \in W$, then we say that $\mathfrak{M}, w \models X$ iff for all $A \in X$: $\mathfrak{M}, w \models A$.

Given class of models C and $X \cup \{A\} \subseteq \mathsf{For}_{\mathbf{CF}}$, we will say that X *entails* A *modulo* C (in symbols $X \models_{\mathrm{C}} A$) iff for all $\mathfrak{M} = \langle W, Q, \{\mathrm{R}_w\}_{w\in W}, v\rangle \in \mathrm{C}$ and all $w \in W$: if $\mathfrak{M}, w \models X$, then $\mathfrak{M}, w \models A$. Clearly, A is *a tautology of* C iff $\emptyset \models_{\mathrm{C}} A$ (in short: $\models_{\mathrm{C}} A$), where $\emptyset$ is an empty set. Traditionally, we say that model $\mathfrak{M} = \langle W, Q, \{\mathrm{R}_w\}_{w\in W}, v\rangle$ is based on $\langle W, Q\rangle$.

Now, we do not deal with a single relation, but with a family of relations. So, the relating structure is $\langle W, \{R_w\}_{w\in W}\rangle$, instead of a single R. But, since it is clear that set W indexes a set of relations, therefore it can be reduced to $\{R_w\}_{w\in W} = \{R_w : w \in W\}$, for some W.

Taking a modal frame $\langle W, Q\rangle$, we can mix it with relating structures, and vice versa. However, if we do not impose any constraints on accessibility relation Q, we can talk only of a family of relating relations. Consequently, we assume a definition of being true in the mixed structure and separately in the relating structure:

$\langle W, Q, \{R_w\}_{w\in W}\rangle \models A$ iff $\langle W, Q, \{\mathrm{R}_w\}_{w\in W}, v\rangle \models A$, for all valuations of letters v in W

$\{R_w\}_{w\in W} \models A$ iff $\langle W, Q, \{\mathrm{R}_w\}_{w\in W}, v\rangle \models A$, for all accessibility relations $Q \in W \times W$ of some kind (in some class of accessibility relations) and all valuations of letters v in W.

Below, we define some properties of relation R corresponding to the Aristotle's and Boethius' theses. They are natural modal counterparts of conditions (a1), (a2), (b1), (b2), (c1) from [6], extended to each family of relations indexed by worlds from some W. So, let $\{R_w\}_{w\in W}$ be a family of relating relations indexed by worlds from some W.

(Ma1) $\{\mathrm{R}_w\}_{w\in W}$ is (Ma1) iff for all $A \in \mathsf{For}_{\mathbf{CMF}}$, $\forall_{w\in W}$ $A\widetilde{\mathrm{R}_w}\neg A$

(Ma2) $\{\mathrm{R}_w\}_{w\in W}$ is (Ma2) iff for all $A \in \mathsf{For}_{\mathbf{CMF}}$, $\forall_{w\in W}$ $\neg A\widetilde{\mathrm{R}_w}A$

(Mb1) $\{\mathrm{R}_w\}_{w\in W}$ is (Mb1) iff for all $A, B \in \mathsf{For}_{\mathbf{CMF}}$, $\forall_{w\in W}$:

- if $A\mathrm{R}_w B$, then $A\widetilde{\mathrm{R}_w}\neg B$
- $(A \to B)\mathrm{R}_w\neg(A \to \neg B)$

(Mb2) $\{\mathrm{R}_w\}_{w\in W}$ is (Mb2) iff for all $A, B \in \mathsf{For}_{\mathbf{CMF}}$, $\forall_{w\in W}$:

- if $A\mathrm{R}_w B$, then $A\widetilde{\mathrm{R}_w}\neg B$
- $(A \to \neg B)\mathrm{R}_w\neg(A \to B)$

(Mc1) $\{\mathrm{R}_w\}_{w\in W}$ is (Mc1) iff for all A, $B \in \mathsf{For}_{\mathbf{CMF}}$, $\forall_{w\in W}$ $(A\mathrm{R}_w B \Rightarrow \neg A\mathrm{R}_w\neg B)$.

Clearly, if we say that a model satisfies one (or more) of the above conditions, we mean that in fact its family of relating relations does it, and the model inherits this property. For example, model $\langle W, Q, \{R_w\}_{w\in W}, v\rangle$ is (Mb2) iff $\{R_w\}_{w\in W}$ is (Mb2) etc. Let us assume that the remark applies to all properties we introduce in the paper here and further.

We have a similar theorem to the Correspondence Theorem in [6]. This time, it is extended to the modal context.

Theorem 4.1 (Modal Correspondence Theorem). *Let $\{R_w\}_{w\in W}$ be a family of relating relations for some W.*

If $\{R_w\}_{w\in W}$ is (Mc1), then:

$$\{R_w\}_{w\in W} \textit{ is (Ma1) iff } \{R_w\}_{w\in W} \models \neg(A \to \neg A)$$
$$\{R_w\}_{w\in W} \textit{ is (Ma2) iff } \{R_w\}_{w\in W} \models \neg(\neg A \to A)$$
$$\{R_w\}_{w\in W} \textit{ is (Mb1) iff } \{R_w\}_{w\in W} \models (A \to B) \to \neg(A \to \neg B)$$
$$\{R_w\}_{w\in W} \textit{ is (Mb2) iff } \{R_w\}_{w\in W} \models (A \to \neg B) \to \neg(A \to B).$$

Proof: The proof is similar to the proof of theorem 6.1 in [6], where we proved the Correspondence Theorem for conditions (a1), (a2), (b1), (b2), (c1). In fact, we show the equivalences for all relations belonging to $\{\mathrm{R}_w\}_{w \in W}$ in the identical ways as we did in [6]. □

Similarly as in the non-modal case, condition (Mc1) is necessary only to prove the inference from the left to the right part of each condition. Hence, by definitions of the conditions we have:

THEOREM 4.2. *Let $\{R_w\}_{w \in W}$ be a family of relating relations for some set W. Then:*

$$\{R_w\}_{w \in W} \textit{ is (Ma1)} \Rightarrow \{R_w\}_{w \in W} \models \neg(A \to \neg A)$$
$$\{R_w\}_{w \in W} \textit{ is (Ma2)} \Rightarrow \{R_w\}_{w \in W} \models \neg(\neg A \to A)$$
$$\{R_w\}_{w \in W} \textit{ is (Mb1)} \Rightarrow \{R_w\}_{w \in W} \models (A \to B) \to \neg(A \to \neg B)$$
$$\{R_w\}_{w \in W} \textit{ is (Mb2)} \Rightarrow \{R_w\}_{w \in W} \models (A \to \neg B) \to \neg(A \to B).$$

Here, we can repeat the maneuver we did in [6]. So, first we have some theorem on the independence of modal versions of the connexive conditions.

THEOREM 4.3. *The conditions (Ma1), (Ma2), (Mb1), (Mb2), (Mc1) are independent.*

PROOF: Since non-modal conditions (a1), (a2), (b1), (b2), (c1) are independent (theorem 6.2 in [6]), so (Ma1), (Ma2), (Mb1), (Mb2), (Mc1) are also independent, which is established by singletons $W = \{w\}$ and $\{R_w\}$. However, if a model requires more worlds (for example, because of some special requirements put on the accessibility relation Q), then since relating relations included in one model are independent from each other, so we can extend the singleton models to models with a bigger cardinality, properly. □

Secondly, similarly as in the non-modal case, since the conditions (Ma1), (Ma2), (Mb1), (Mb2), (Mc1) are independent, then each subset of set {(Ma1), (Ma2), (Mb1), (Mb2), (Mc1)} determines a different family of relations $\{\mathrm{R}_w\}_{w \in W}$, i.e. we have $2^5 = 32$ of families of such kind. Each of them determines a different logical system.

Among these 32 logics determined by these models: one is neither connexive, nor quasi-connexive; 29 logics are quasi-connexive – at least one, but not all connexive laws are valid; but two logics are really connexive – their models satisfy conditions (Ma1), (Ma2), (Mb1), (Mb2) and possibly also (Mc1). The logic determined only by conditions (Ma1), (Ma2), (Mb1), (Mb2) is the least modal Boolean connexive logic. Surely, the top of this lattice is the inconsistent logic defined on $\mathsf{For}_{\mathbf{CF}}$ by an empty set of models. Let us note that these are basic variants, since by imposing some requirements on accessibility relation Q we probably multiply the number of logics.

5. Modal aspect of the Boolean connexive logics

The very interesting question is whether our modal Boolean connexive logics are normal as modal logics. It is widely accepted that a modal logic is normal iff it is closed under necessitation rule: $\models A \Rightarrow \models \Box A$ and contains axiom K: $\Box(A \to B) \to (\Box A \to \Box B)$. However, since in the language we have classical negation $\neg$ and diamond $\Diamond$, hence also the so called dual should be satisfied: $\neg\Diamond\neg A \to \Box A$ and $\Box A \to \neg\Diamond\neg A$ (we do not dispose of the equivalence in the language, then the dual must be expressed as two opposite implications). As we know, our relating implication is much weaker than the material one. This is the reason why our modal Boolean connexive logics are not normal in the given sense. Also, if we limit the class of models by imposing specific conditions on accessibility relation Q (imposing reflexivity, transitivity or symmetry etc.), we do not get axioms that are characteristic for the extensions of the normal modal logic, expressed as the appropriate implications. The next claim states this.

Claim 5.1. *If C is a class of all models that satisfy a subset of set of conditions $\{$(Ma1), (Ma2), (Mb1), (Mb2), (Mc1)$\}$, then the following facts hold:*

(a) $\models_C A \Rightarrow \models_C \Box A$ *Necessitation rule holds*
(b) $\not\models_C \Box A \to \neg\Diamond\neg A$ *Dual does not hold*
(c) $\not\models_C \neg\Diamond\neg A \to \Box A$ *Dual does not hold*
(d) $\not\models_C \Box(A \to B) \to (\Box A \to \Box B)$ *axiom K does not hold*
(e) $\not\models_C \Box A \to \Diamond A$ *axiom D does not hold, even with serial Q*
(f) $\not\models_C \Box A \to A$ *axiom T does not hold, even with reflexive Q*
(g) $\not\models_C \Box A \to \Box\Box A$ *axiom $\mathsf{4}$ does not hold, even with transitive Q*
(h) $\not\models_C A \to \Box\Diamond A$ *axiom B does not hold, even with symmetrical Q*
(i) $\not\models_C \Diamond A \to \Box\Diamond A$ *axiom $\mathsf{5}$ does not hold, even with Euclidean Q.*

Proof: We take class of models C, consequence relation $\models_{\mathrm{C}}$, and formulas $A, B \in \mathsf{For}_{\mathbf{CMF}}$.

For (a) let us assume that $\models_{\mathrm{C}} A$. Given model $\mathfrak{M} = \langle W, Q, \{\mathrm{R}_w\}_{w \in W}, v\rangle \in \mathrm{C}$ and world $w \in W$. $\mathfrak{M}, w \models \Box A$ iff for all $u \in W$: $wQw' \Rightarrow \mathfrak{M}, u \models A$. So we take any such $u \in W$ that wQu. However, by the assumption, in all models $\mathfrak{M}' = \langle W', Q', \{\mathrm{R}_w\}_{w \in W'}, v'\rangle$ in C, all worlds w' in W': $\mathfrak{M}', w' \models A$. Therefore, $\mathfrak{M}', u \models A$, and by arbitrariness of model and world: $\models_{\mathrm{C}} \Box A$.

For (b) we take model $\mathfrak{M} = \langle W, Q, \{\mathrm{R}_w\}_{w \in W}, v\rangle$, where for some world $w \in W$: $\Box A \widetilde{\mathrm{R}_w} \neg\Diamond\neg A$. By definition, class of models C consists of all models of some kind that does not exclude such models. In consequence, $\mathfrak{M}, w \not\models \Box A \to \neg\Diamond\neg A$, so $\mathfrak{M} \not\models \Box A \to \neg\Diamond\neg A$, and finally $\not\models_{\mathrm{C}} \Box A \to \neg\Diamond\neg A$.

For (d) $\not\models_{\mathrm{C}} \Box(A \to B) \to (\Box A \to \Box B)$, it is enough to take model $\mathfrak{M} = \langle W, Q, \{\mathrm{R}_w\}_{w \in W}, v\rangle$, where for some world $w \in W$: $\Box(A \to B)\widetilde{\mathrm{R}_w}(\Box A \to \Box B)$, so axiom K does not hold.

For (i) $\not\models_{\mathrm{C}} \Diamond A \to \Box\Diamond A$, we take model $\mathfrak{M} = \langle W, Q, \{\mathrm{R}_w\}_{w \in W}, v\rangle$, where for some world $w \in W$: $\Diamond A\widetilde{\mathrm{R}_w}\Box\Diamond A$. Since relation Q is Euclidean, so $\mathfrak{M}, w \models_{\mathrm{C}} \Diamond A$ implies $\mathfrak{M}, w \models_{\mathrm{C}} \Box\Diamond A$. However, $\not\models_{\mathrm{C}} \Diamond A \to \Box\Diamond A$, because $\Diamond A\widetilde{\mathrm{R}_w}\Box\Diamond A$. Hence, axiom 5 does not hold, even with Euclidean Q.

The remaining cases we prove in a very similar way, by indicating counterexamples. They all base upon the fact that to falsify implication it is enough to find such a relating relation satisfying the conditions (Ma1), (Ma2), (Mb1), (Mb2), (Mc1) that the implication antecedent and the implication consequent do not relate. $\Box$

Now, we would like to enhance our logics a bit, by making all of the mentioned formulas true. One of the possibilities is to impose more conditions. Sometimes they turn out to correspond with the appropriate formulas, but not always is this a case. Let us start with axiom K. We can state the sufficient, but not necessary conditions for it. Beforehand, we assume some notation. We assume that if $\mathfrak{M} = \langle W, Q, \{\mathrm{R}_w\}_{w \in W}, v\rangle$ is a model for language $\mathsf{For}_{\mathbf{CMF}}$, then $W^{\mathfrak{M}} = W$, $Q^{\mathfrak{M}} = Q$.

Claim 5.2. *Let C be a class of all models such that for all $\mathfrak{M} = \langle W, Q, \{R_w\}_{w \in W}, v\rangle \in C$, for all $w \in W$, for all $A, B \in \mathsf{For}_{\boldsymbol{CMF}}$:*

(K)*:*

(1) $\Box(A \to B)R_w(\Box A \to \Box B)$

(2) $\forall_{u \in W}(wQu \Rightarrow AR_uB) \Rightarrow \Box AR_w\Box B.$

Then:

$\models_C \Box(A \to B) \to (\Box A \to \Box B)$.

Proof: Given any model $\mathfrak{M} \in C$ satisfying (1) and (2), we will show that for any possible world $w \in W$: $\mathfrak{M}, w \models \Box(A \to B) \to (\Box A \to \Box B)$. From (1) and definition of satisfaction relation it is enough to show that if $\mathfrak{M}, w \models \Box(A \to B)$ then: a) $\Box A \mathrm{R}_w \Box B$ and b) if $\mathfrak{M}, w \models \Box A$, then $\mathfrak{M}, w \models \Box B$.

a) From $\mathfrak{M}, w \models \Box(A \to B)$ we have then for all $u \in W$ such that wQu $\mathfrak{M}, u \models A \to B$, hence $A\mathrm{R}_u B$, then from (2) $\Box A \mathrm{R}_w \Box B$.

b) Suppose that $\mathfrak{M}, w \models \Box(A \to B)$ and $\mathfrak{M}, w \models \Box A$, then for any $u \in W$ such that wQu, $\mathfrak{M}, u \models A \to B$ and $\mathfrak{M}, u \models A$, hence $\mathfrak{M}, w \models \Box B$. $\Box$

Obviously condition axiom K entails (1) however it does not entail (2). Let us take the following model $\mathfrak{M} = \langle W, Q, \{\mathrm{R}_w\}_{w \in W}, v\rangle$, where $W = \{w\}$, $Q = \{\langle w, w\rangle\}$, and $R_w = \{\langle p \vee \neg p, p \wedge \neg p\rangle\} \cup \{\langle \Box(A \to B), \Box A \to \Box B\rangle : A, B \in \mathsf{For}_{\mathbf{CMF}}\} \cup \{\langle \Box A, \Box B\rangle : A, B \in \mathsf{For}_{\mathbf{CMF}}$ and $A \neq p \vee \neg p$ or $B \neq p \wedge \neg p\}$, and $v(w, x) = 1$, for all $x \in \mathsf{Var}$.

We can see that $\mathfrak{M}, w \models \Box(A \to B) \to (\Box A \to \Box B)$, for all $A, B \in \mathsf{For}_{\mathbf{CMF}}$. However, condition (2) is falsified because $\forall_{u \in W}(wQu \Rightarrow p \vee \neg p \mathrm{R}_u p \wedge \neg p)$, but $\Box(p \vee \neg p) \widetilde{\mathrm{R}}_w \Box(p \wedge \neg p)$.

For the remaining formulas we have sufficient as well as necessary conditions. The notations on the right denote conditions for the appropriate axioms on the left.

Claim 5.3. *Let C be a class of models. The following conditions are fulfilled:*

(Du1) $\models_C \Box A \to \neg\Diamond\neg A$ *iff* $\forall_{\mathfrak{M} \in C}\, \forall_{w \in W^{\mathfrak{M}}}\ \Box A R_w \neg\Diamond\neg A$

(Du2) $\models_C \neg\Diamond\neg A \to \Box A$ *iff* $\forall_{\mathfrak{M} \in C}\, \forall_{w \in W^{\mathfrak{M}}}\ \neg\Diamond\neg A R_w \Box A$

(D) *if C is based on serial frames, then:*
$\models_C \Box A \to \Diamond A$ *iff* $\forall_{\mathfrak{M} \in C}\, \forall_{w \in W^{\mathfrak{M}}}\ \Box A R_w \Diamond A$

(T) *if C is based on reflexive frames, then:*
$\models_C \Box A \to A$ *iff* $\forall_{\mathfrak{M} \in C}\, \forall_{w \in W^{\mathfrak{M}}}\ \Box A R_w A$

(4) *if C is based on transitive frames, then:*
$$\models_C \Box A \to \Box\Box A \textit{ iff } \forall_{\mathfrak{M}\in C}\, \forall_{w\in W^{\mathfrak{M}}}\ \Box A R_w \Box\Box A$$

(B) *if C is based on symmetrical frames, then:*
$$\models_C A \to \Box\Diamond A \textit{ iff } \forall_{\mathfrak{M}\in C}\, \forall_{w\in W^{\mathfrak{M}}}\ A R_w \Box\Diamond A$$

(5) *if C is based on Euclidean frames, then:*
$$\models_C \Diamond A \to \Box\Diamond A \textit{ iff } \forall_{\mathfrak{M}\in C}\, \forall_{w\in W^{\mathfrak{M}}}\ \Diamond A R_w \Box\Diamond A.$$

Proof: Please note that in each case above the right-hand side condition grants that the antecedent and the consequent of implication on the left-hand side relate with respect to the relating relation. It directly shows that each right-hand side is a necessary condition. To prove that each right-hand side condition is sufficient it is enough to note that each left-hand side condition is true if we replace $\to$ by teh material implication. $\Box$

The conditions presented in the latter fact may seem to be rather natural syntactic conditions. It is the case – we believe – because we treat modalities very literally. However there exists another option, much less obvious that is formally reduced to the demodalization.

6. Demodalization and double negation

Any modality can be treated – due to the Latin etymology of the word ”modality” – as the way a modalized proposition holds. Term *modality* comes from the Latin world *modus* which means a way; a way that something happens.

An option of non–treating modalities literally is to assume that modalities $\Box$, $\Diamond$ add nothing to the content of propositions modalized by them. For some people it may sound controversial, but modus means a way, not a content.

We begin by considering an example. Below, we have two non-modal propositions:

$p :=$ Nicolaus Copernicus was born in Toruń

$q :=$ Toruń is in Poland.

It is obvious that p and q are connected somehow (for example, by sharing the city):

$$p\mathrm{R}q.$$

The question may appear if when we add modalities to them, will they still be connected? Insisting on the option that modalities bring nothing to the content, but only change the way or the status of proposition, it does not seem strange that one can find sentences $\Diamond p$ and $\Box q$ still connected. Also the subsequent iterations should not change this situation, if $\Diamond p\mathrm{R}\Box q$, then by adding the successive modalities we get the pair of connected propositions: $\Box\Diamond p\mathrm{R}\Box\Box q$ etc. The inverse direction should be treated as intuitive, too. Why? Because if we think the modalities bring nothing to the content, then the two modalized propositions are connected through the fact their non-modal components are connected. If so, then we assume generally:

$$p\mathrm{R}q \Longleftrightarrow \circ_1, \ldots, \circ_n p\mathrm{R}\bullet_1, \ldots, \bullet_m q$$

where $1 \leq n, m$ and $\circ_1, \ldots, \circ_n, \bullet_1, \ldots, \bullet_m \in \{\Box, \Diamond\}$.

Since the initial, non-modal sentences can be more complex than only sentential letters, we introduce a special function that removes modalities from the structure of sentences. By *demodalization* we mean mapping $\mathsf{d}\colon \mathsf{For}_{\mathbf{CMF}} \longrightarrow \mathsf{For}_{\mathbf{CPL}}$, determined by conditions:

$$\begin{aligned}
&\mathsf{d}(A) = A && A \in \mathsf{Var}\\
&\mathsf{d}(\neg A) = \neg\mathsf{d}(A) &&\\
&\mathsf{d}(A * B) = \mathsf{d}(A) * \mathsf{d}(B) && * \in \{\wedge, \vee, \rightarrow\}\\
&\mathsf{d}(\circ A) = \mathsf{d}(A) && \circ \in \{\Box, \Diamond\}.^1
\end{aligned}$$

Claim 6.1. *Let C be a class of models such that for all $\mathfrak{M} \in C$ and for all $A, B \in \mathsf{For}_{\mathbf{CMF}}$:*

$$\text{(d)} \qquad \forall_{w \in W^{\mathfrak{M}}} (A R_w B \Longleftrightarrow \mathsf{d}(A) R_w \mathsf{d}(B)),$$

[1] It is worth to mention that in [5] we introduced a very similar demodalization function. It was in the context of deontic logic while our aim was to underline and preserve the deontic relationships between the sentences related to different changing possible deontic worlds. We assumed that such relationship did not depend on the modal status, but on what a content of sentences was.

Then the following conditions are fulfilled:

(K+(d)) $\models_C \Box(A \to B) \to (\Box A \to \Box B)$ *iff* $\forall_{\mathfrak{M} \in C}\ \forall_{w \in W^{\mathfrak{M}}}\ \mathsf{d}(A \to B) R_w \mathsf{d}(A \to B)$

(Du1+(d)) $\models_C \Box A \to \neg\Diamond\neg A$ *iff* $\forall_{\mathfrak{M} \in C}\ \forall_{w \in W^{\mathfrak{M}}}\ \mathsf{d}(A) R_w \mathsf{d}(\neg\neg A)$

(Du2+(d)) $\models_C \neg\Diamond\neg A \to \Box A$ *iff* $\forall_{\mathfrak{M} \in C}\ \forall_{w \in W^{\mathfrak{M}}}\ \mathsf{d}(\neg\neg A) R_w \mathsf{d}(A)$

(D+(d)) *if C is based on serial frames, then:* $\models_C \Box A \to \Diamond A$ *iff* $\forall_{\mathfrak{M} \in C}\ \forall_{w \in W^{\mathfrak{M}}}\ \mathsf{d}(A) R_w \mathsf{d}(A)$

(T+(d)) *if C is based on reflexive frames, then:* $\models_C \Box A \to A$ *iff* $\forall_{\mathfrak{M} \in C}\ \forall_{w \in W^{\mathfrak{M}}}\ \mathsf{d}(A) R_w \mathsf{d}(A)$

(4+(d)) *if C is based on transitive frames, then:* $\models_C \Box A \to \Box\Box A$ *iff* $\forall_{\mathfrak{M} \in C}\ \forall_{w \in W^{\mathfrak{M}}}\ \mathsf{d}(A) R_w \mathsf{d}(A)$

(B+(d)) *if C is based on symmetrical frames, then:* $\models_C A \to \Box\Diamond A$ *iff* $\forall_{\mathfrak{M} \in C}\ \forall_{w \in W^{\mathfrak{M}}}\ \mathsf{d}(A) R_w \mathsf{d}(A)$

(5+(d)) *if C is based on Euclidean frames, then:* $\models_C \Diamond A \to \Box\Diamond A$ *iff* $\forall_{\mathfrak{M} \in C}\ \forall_{w \in W^{\mathfrak{M}}}\ \mathsf{d}(A) R_w \mathsf{d}(A)$.

PROOF: Let us assume a class of all models C that all satisfy a subset of set of conditions {(Ma1), (Ma2), (Mb1), (Mb2), (Mc1)}, and additionally let all $\mathfrak{M} \in \mathrm{C}$ satisfy condition (d).

We will prove (Du1+(d)) and (5+(d)) as examples. The remaining cases can be shown in a similar way. Let us take $A \in \mathsf{For}_{\mathbf{CMF}}$.

(Du1+(d)) We assume $\models_{\mathrm{C}} \Box A \to \neg\Diamond\neg A$, then $\forall_{\mathfrak{M} \in \mathrm{C}}\ \forall_{w \in W^{\mathfrak{M}}} \mathfrak{M}, w \models \Box A \to \neg\Diamond\neg A$, and $\Box A \mathrm{R}_w \neg\Diamond\neg A$. Therefore, by (d), $\mathsf{d}(A)\mathrm{R}_w\mathsf{d}(\neg\neg A)$. On the other hand, suppose $\forall_{\mathfrak{M} \in \mathrm{C}}\ \forall_{w \in W^{\mathfrak{M}}}\ \mathsf{d}(A)\mathrm{R}_w\mathsf{d}(\neg\neg A)$. Given a model $\mathfrak{M} \in \mathrm{C}$ and any $w \in W^{\mathfrak{M}}$, obviously, as in the classical modal logics $\mathfrak{M}, w \not\models \Box A$ or $\mathfrak{M}, w \models \neg\Diamond\neg A$. As $\mathsf{d}(A)\mathrm{R}_w\mathsf{d}(\neg\neg A)$, from (d) we have $\Box A \mathrm{R}_w \neg\Diamond\neg A$, which shows that $\mathfrak{M}, w \models \Box A \to \neg\Diamond\neg A$. Hence $\models_{\mathrm{C}} \Box A \to \neg\Diamond\neg A$.

(5+(d)) Assume C is based on the Euclidean frames. Suppose $\models_{\mathrm{C}} \Diamond A \to \Box\Diamond A$, then $\forall_{\mathfrak{M} \in \mathrm{C}}\ \forall_{w \in W^{\mathfrak{M}}} \mathfrak{M}, w \models \Diamond A \to \Box\Diamond A$, and $\Diamond A \mathrm{R}_w \Box\Diamond A$.

Therefore, by (d), $\mathsf{d}(A)\mathrm{R}_w\mathsf{d}(A)$. On the other hand, suppose $\forall_{\mathfrak{M}\in\mathrm{C}}\,\forall_{w\in W^{\mathfrak{M}}}$ $\mathsf{d}(A)\mathrm{R}_w\mathsf{d}(A)$. Given model $\mathfrak{M}$ and any $w\in W^{\mathfrak{M}}$. As $Q^{\mathfrak{M}}$ is Euclidean then as in the classical modal logics $\mathfrak{M},w\not\models\Diamond A$ or $\mathfrak{M},w\models\Box\Diamond A$. As $A\mathrm{R}_wA$, from (d) we have $\Diamond A\mathrm{R}_w\Box\Diamond A$, which shows that $\mathfrak{M},w\models\Diamond A\rightarrow\Box\Diamond\ A$. Hence $\models_{\mathrm{C}}\Diamond A\rightarrow\Box\Diamond A$. $\Box$

The above claim introduces different conditions imposed on relation R than the former ones. They are located on the right-side of the parts of the claim. We would like to point out some interesting things. It seems that a further refinement could be imposing on models the constraint of reflexivity of relation R in all worlds, since most of the cases state reflexivity for in $\mathsf{d}(\mathsf{For}_{\mathbf{CMF}})=\mathsf{For}_{\mathbf{CF}}$ as a sufficient and necessary condition. But most is not all. In three cases we have exceptions.

First, axiom K does not imply general reflexivity $A\mathrm{R}_wA$, but its special instance $(A\rightarrow B)\mathrm{R}_w(A\rightarrow B)$. It shows its extraordinary status in the modal logic.

Second, both forms of dual Du1, Du2 are equivalent with almost reflexivity condition: $A\mathrm{R}_w\neg\neg A$, $\neg\neg A\mathrm{R}_wA$. Although formulas: A and $\neg\neg A$ are different from the syntactic point of view, let us note that we operate with the classical, Boolean negation $\neg$. So it looks reasonable to add as the next constraint:

$$(\neg\neg)\quad A\mathrm{R}_wB\iff\neg\neg A\mathrm{R}_wB\iff A\mathrm{R}_w\neg\neg B.$$

It says that double (classical!) negation has no influence on being connected. Condition $(\neg\neg)$ imposed on the models could be the next enhancement of our modal Boolean connexive logics, of course.

Finally, we may take connexive models $\mathfrak{M}=\langle W,Q,\{\mathrm{R}_w\}_{w\in W},v\rangle$ defined by (Ma1), (Ma2), (Mb1), (Mb2) (or additionally by (Mc1)) and by one of the weaker conditions:

$$(\mathsf{d1})\qquad A\mathrm{R}_wB\Rightarrow\mathsf{d}(A)\mathrm{R}_w\mathsf{d}(B)\qquad\text{for all } w\in W^{\mathfrak{M}}$$

$$(\mathsf{d2})\qquad \mathsf{d}(A)\mathrm{R}_w\mathsf{d}(B)\Rightarrow A\mathrm{R}_wB\qquad\text{for all } w\in W^{\mathfrak{M}}.$$

Both, (d1) and (d2), put together are equivalent to condition (d). Separately, they make equivalences in claim 6.1 invalid, reducing it to the suitable implications. The logics we obtain by exactly one of conditions $(\mathsf{d1})$ or $(\mathsf{d2})$ are probably stronger then the logics without the demodalization function, but weaker than the logics determined by models satisfying condition (d). The issue needs a further examination.

By CONST (*constraints*) we will denote the set of all conditions (without $(\mathsf{d1})$ and $(\mathsf{d2})$, separately) imposed on models $\langle W, Q, \{\mathrm{R}_w\}_{w\in W}, v\rangle$ we have introduced so far. Hence, CONST consists of (Ma1), (Ma2), (Mb1), (Mb2), (Mc1), (K), $(\mathsf{Du1})$, $(\mathsf{Du2})$, (D), (T), $(\mathsf{4})$, (B), $(\mathsf{5})$, (d), $(\mathsf{K}+(\mathsf{d}))$, $(\mathsf{Du1}+(\mathsf{d}))$, $(\mathsf{Du2}+(\mathsf{d}))$, $(\mathsf{D}+(\mathsf{d}))$, $(\mathsf{T}+(\mathsf{d}))$, $(\mathsf{4}+(\mathsf{d}))$, $(\mathsf{B}+(\mathsf{d}))$, $(\mathsf{5}+(\mathsf{d}))$.

7. Tableaux for the modal Boolean connexive logics

Now, we shall outline the tableau approach to our logics in a similar way we do it in [6]. Also similarly, we will be governed here by a strategy adopted in paper [3] which introduced a formalized tableau theory from some modal logics. Let us, however, disregard the formal concepts in favour of stressing the crucial points which determine the completeness of the tableau approach related to the semantically designated consequence relations.

To this end, we shall need a new language – a language of tableau proofs. We assume as a set of expressions Ex union of four sets. They are in turn:

- $\{irj : i, j \in \mathbb{N}\}$
- $\{\langle A, i\rangle : A \in \mathsf{For}_{\mathbf{CMF}}, i \in \mathbb{N}\}$
- $\{\langle ARB, i\rangle : A, B \in \mathsf{For}_{\mathbf{CMF}}, i \in \mathbb{N}\}$
- $\{\langle A\not{R}B, i\rangle : A, B \in \mathsf{For}_{\mathbf{CMF}}, i \in \mathbb{N}\}$.

We use notation R instead of R on purpose, to differentiate the tableau language notation R from the relation in model R. Intuitively, r is a tableau counterpart of the accessibility relation, so for the world denoted by i there is an accessible world denoted by j. Next, $\langle A, i\rangle$ means that formula A is true at possible world i, $\langle ARB, i\rangle$ means that the relating relation holds between A and B at world i, while $\langle A\not{R}B, i\rangle$ that it does not.

Now, all tableau proofs are carried out in language Ex. Usually, we remove external, square brackets, so instead of $\langle A, i\rangle$, $\langle ARB, i\rangle$, $\langle A\not{R}B, i\rangle$, we just write: A, i; ARB, i; $A\not{R}B, i$, respectively.

As *a tableau inconsistent set of expressions* (that closes given branch) we treat one comprising at least one of the pairs: A, i and $\neg A, i$ or ARB, i and $A\not{R}B, i$, for some $A, B \in \mathsf{For}_{\mathbf{CMF}}$ and $i \in \mathbb{N}$. Clearly, a set is *a tableau* *consistent set of expressions* iff it is not a tableau inconsistent set of expressions.

Let us go to the tableau rules. For the formulas with the main Boolean connectives: $\neg$, $\wedge$, $\vee$, we shall assume the standard tableau rules in the modal context, so with label $i \in \mathbb{N}$:

$$(\wedge)\ \frac{A \wedge B, i}{A, i;\ B, i} \qquad (\neg\wedge)\ \frac{\neg(A \wedge B), i}{\neg A, i | \neg B, i}$$

$$(\vee)\ \frac{A \vee B, i}{A, i | B, i} \qquad (\neg\vee)\ \frac{\neg(A \vee B), i}{\neg A, i;\ \neg B, i} \qquad (\neg\neg)\ \frac{\neg\neg A, i}{A, i}$$

Let us remind that the formulas do not include ones with a material implication. For the relating implication, as the main connective, we assume the tableau rules introduced in [4] also modified to the modal context. So, let $i \in \mathbb{N}$:

$$(\rightarrow)\ \frac{A \rightarrow B, i}{ARB, i;\ \neg A, i | ARB, i;\ B, i} \qquad (\neg\rightarrow)\ \frac{\neg(A \rightarrow B), i}{A, i;\ \neg B, i | A\not{R}B, i}.$$

Next, we add standard tableau rules for $\Diamond$, $\Box$ and their interactions with $\neg$.

$$(\Box)\ \frac{\Box A, i;\ irj}{A, j} \qquad (\neg\Box)\ \frac{\neg\Box A, i}{\Diamond\neg A, i} \qquad (\Diamond)\ \frac{\Diamond A, i}{irj;\ A, j} \qquad (\neg\Diamond)\ \frac{\neg\Diamond A, i}{\neg\Box A, i}$$

where $i, j \in \mathbb{N}$ and in the case of tableau rule $(\Diamond)$ index j does not appear on the branch.

The rules we have considered so far form the base for any modal Boolean connexive logic, we call the set of them BTR (*basic tableau rules*).

Now we add also well known rules for some extensions of the normal modal logics. They will be mixed with specific rules for connexive properties that are given in the further part. We assume tableau rules for the following properties of relation of accessibility Q in a model, respectively for: seriality, reflexivity, transitivity, symmetry, and Euclidean property:

$$(ser)\ \frac{}{irj} \qquad (ref)\ \frac{}{iri} \qquad (tran)\ \frac{irj;\ jrk}{irk}$$

$$(symm)\ \frac{irj}{jri} \qquad (eucl)\ \frac{irj;\ irk}{jrk}$$

for all $i, j, k \in \mathbb{N}$. Surely, in the case of (ser) i appeared on the branch, where j is new; in the case of (ref) i just appeared on the branch.

The next rules will be given for the specific conditions we introduced. First we reformulate to the modal context tableau rules for the Aristotelian and Boethian conditions given in [6]. Hence rules (Ra1), (Ra2), (Rb1), (Rb1'), (Rb2), (Rb2'), (Rc1) proposed in [6] are remade by adding indexes to the expressions. Later, in the presentation of all succeeding rules we always assume that $i \in \mathbb{N}$.

For the logics defined by conditions (Ma1), (Ma2) we have rules:

$$\text{(RMa1)}\ \frac{AR\neg A, i}{A\not R\neg A, i} \qquad \text{(RMa2)}\ \frac{\neg ARA, i}{\neg A\not RA, i}$$

For the logics defined by condition (Mb1) we have two tableau rules:

$$\text{(RMb1)}\ \frac{ARB, i}{A\not R\neg B, i}$$

$$\text{(RMb1')}\ \frac{(A \to B)\not R\neg(A \to \neg B), i}{(A \to B)R\neg(A \to \neg B), i}$$

For the logics defined by condition (Mb2) we also have two rules that work together as well:

$$\text{(RMb2)}\ \frac{AR\neg B, i}{A\not RB, i}$$

$$\text{(RMb2')}\ \frac{(A \to \neg B)\not R\neg(A \to B), i}{(A \to \neg B)R\neg(A \to B), i}$$

In fact, both (RMb1') and (RMb2') work in a similar way, since conditions (Mb1) and (Mb2) feature a common property: if $A\mathrm{R}_w B$, then

$A\widetilde{\mathrm{R}}_w\neg B$. Hence, when dealing with a logic defined in this paper by conditions (Mb1) and (Mb2) we only adopt one rule. And finally, we also have a rule for the logic defined by condition (Mc1):

$$(\text{RMc1}) \quad \frac{\neg A\not{R}\neg B, i}{A\not{R}B, i}.$$

Below, we have tableau rules for the suitable semantic conditions of the modal axioms that are related to claim 5.2 and claim 5.3, before the demodalization strategy.

For (K) we have two tableau rules:

$$(\text{RK1}) \quad \frac{\Box(A \to B)\not{R}\Box A \to \Box B, i}{\Box(A \to B)\mathrm{R}\Box A \to \Box B, i} \qquad (\text{RK2}) \quad \frac{\Box A\not{R}\Box B, i}{irj;\ A\not{R}B, j}$$

In the case of (RK2) label j must be new.

For (Du1), (Du2), we have only one tableau rule for each:

$$(\text{RDu1}) \quad \frac{\Box A\not{R}\neg\Diamond\neg A, i}{\Box AR\neg\Diamond\neg A, i} \qquad (\text{RDu2}) \quad \frac{\neg\Diamond\neg A\not{R}\Box A, i}{\neg\Diamond\neg AR\Box A, i}$$

For (D), (T), (4), (B), (5) we have two tableau rules for each: one for the relating relation, one for accessibility Q, in turn:

$$(\text{RD}) \quad \frac{\Box A\not{R}\Diamond A, i}{\Box AR\Diamond A, i} \quad (ser)$$

$$(\text{RT}) \quad \frac{\Box A\not{R}A, i}{\Box ARA, i} \quad (ref)$$

$$(\text{R4}) \quad \frac{\Box\Box A\not{R}A, i}{\Box\Box AR\Box A, i} \quad (tran)$$

$$(\text{RB}) \quad \frac{A\not{R}\Box\Diamond A, i}{AR\Box\Diamond A, i} \quad (symm)$$

$$(\text{R5}) \quad \frac{\Diamond A\not{R}\Box\Diamond A, i}{\Diamond AR\Box\Diamond A, i} \quad (eucl)$$

After assuming the demodalization property (condition (d)) (claim 5.3), we must add different tableau rules. The demodalization itself requires:

$$(\mathrm{R}\mathsf{d}{\Rightarrow})\ \ \frac{ARB, i}{\mathsf{d}(A)R\mathsf{d}(B), i} \qquad (\mathrm{R}\mathsf{d}{\Leftarrow})\ \ \frac{A\not{R}B, i}{\mathsf{d}(A)\not{R}\mathsf{d}(B), i}$$

For conditions (K+(d)), ($\mathsf{Du1}$+(d)), ($\mathsf{Du2}$+(d)), we additionally have:

$$(\mathrm{R}\mathsf{Kd})\ \ \frac{(A \to B)\not{R}(A \to B), i}{(A \to B)R(A \to B), i}$$

$$(\mathrm{R}\mathsf{Du1d})\ \ \frac{A\not{R}\neg\neg A, i}{AR\neg\neg A, i} \qquad (\mathrm{R}\mathsf{Du2d})\ \ \frac{\neg\neg A\not{R}A, i}{\neg\neg ARA, i}$$

For the remaining conditions we assume reflexivity of R_w, for all worlds in the model, so we have a specific tableau rule:

$$(\mathrm{R}\mathsf{refR})\ \ \frac{A\not{R}A, i}{ARA, i}$$

Finally, we can formulate the tableau rules for (D+(d)), (T+(d)), ($\mathsf{4}$+(d)), (B+(d)), ($\mathsf{5}$+(d)), so D, T, $\mathsf{4}$, B, $\mathsf{5}$ under demodalization (d). They are combined with (RrefR) and a suitable condition on accessibility and relating R:

(D+(d)): (ser), (RrefR)

(T+(d)): (ref), (RrefR)

($\mathsf{4}$+(d)): $(tran)$, (RrefR)

(B+(d)): $(symm)$, (RrefR)

($\mathsf{5}$+(d)): $(eucl)$, (RrefR).

For simplification, let us call the expressions in the tableau rule numerator *input*, while those in denominator *output*. Some rules, e.g. ($\to$), ($\neg\to$) and those for the Boolean connectives may have more than one output.

Now let set $\mathsf{TRCONST}$ (*tableau rules for constraints*) contain tableau rules introduced for particular conditions: (ser), (ref), $(tran)$, $(symm)$, $(eucl)$, (RMa1), (RMa2), (RMb1), (RMb1'), (RMb2), (RMb2'), (RMc1), (RK1), (RK2), (RDu1), (RDu2), (RD), (RT), (R4), (RB), (R5), (Rd$\Rightarrow$), (Rd$\Leftarrow$), (RKd), (RDu1d), (RDu2d), (RrefR).

Let us now introduce a concept which is important for the tableau issues, which in a certain sense is an extension of the concept of truthness in model from the formulas on all expressions from Ex.

Definition 7.1. [Set of indexes] By function $\mathsf{Ind}\colon \{X\colon X \subseteq \mathsf{Ex}\} \longrightarrow \mathsf{P}(\mathbb{N})$ we mean a mapping for all $i, j \in \mathbb{N}$ and for all $X \subseteq \mathsf{Ex}$ satisfying conditions:

- if $X = \{irj\}$, then $\mathsf{Ind}(X) = \{i, j\}$,
- for all $A, B \in \mathsf{For}_{\mathbf{CMF}}$:
  - $*$ if $X = \{\langle A, i\rangle\}$, then $\mathsf{Ind}(X) = \{i\}$,
  - $*$ if $X = \{\langle ARB, i\rangle\}$, then $\mathsf{Ind}(X) = \{i\}$,
  - $*$ if $X = \{\langle A\not{R}B, i\rangle\}$, then $\mathsf{Ind}(X) = \{i\}$,
- $\mathsf{Ind}(X) = \bigcup\{\mathsf{Ind}(\{y\}) : y \in X\}$.

Function Ind collects indexes contained in expressions from a given subset of Ex.

Definition 7.2 (Model suitable to the set of expressions). Let $\mathfrak{M} = \langle W, Q, \{\mathrm{R}_w\}_{w\in W}, v\rangle$ be a model and $X \subseteq \mathsf{Ex}$. Model $\mathfrak{M}$ is *suitable to* X iff there exists a function g from the set of indexes contained in expressions from X to W, i.e. $g\colon \mathsf{Ind}(X) \longrightarrow W$, such that, for any $A \in \mathsf{For}_{\mathbf{CMF}}$ and $i, j \in \mathbb{N}$:

- if $irj \in X$, then $Q(g(i), g(j))$
- if $\langle ARB, i\rangle \in X$, then $A\mathrm{R}_{g(i)}B$
- if $\langle A\not{R}B, i\rangle \in X$, then $A\widetilde{\mathrm{R}}_{g(i)}B$
- if $\langle A, i\rangle \in X$, then $\mathfrak{M}, g(i) \vDash A$.

Making use of the provided concept of suitable model and conducting the inspection of the provided tableau rules, we are able to demonstrate that if model $\langle W, Q, \{\mathrm{R}_w\}_{w\in W}, v\rangle$ of given type, fulfilling some of the conditions in CONST, is suitable to set of expressions $X \subseteq \mathsf{Ex}$, then application of a selected tableau rule relevant to the conditions extends set X to add expressions for which $\langle W, Q, \{\mathrm{R}_w\}_{w\in W}, v\rangle$ is still suitable.

For convenience with formulation of the further theorems, let us introduce certain function. Now we define: $f\colon \mathsf{CONST} \longrightarrow \{Z : Z \subseteq \mathsf{TRCONST}\}$ which to each condition assigns the corresponding tableau rules, thus:

$$f((\mathrm{Ma1})) = \{(\mathrm{RMa1})\}$$
$$f((\mathrm{Ma2})) = \{(\mathrm{RMa2})\}$$
$$f((\mathrm{Mb1})) = \{(\mathrm{RMb1}),\ (\mathrm{RMb1'})\}$$
$$f((\mathrm{Mb2})) = \{(\mathrm{RMb2}),\ (\mathrm{RMb2'})\}$$
$$f((\mathrm{Mc1})) = \{(\mathrm{RMc1})\}$$
$$f((\mathsf{K})) = \{(\mathrm{R}\mathsf{K1}),\ (\mathrm{R}\mathsf{K2})\}$$
$$f((\mathsf{Du1})) = \{(\mathrm{R}\mathsf{Du1})\}$$
$$f((\mathsf{Du2})) = \{(\mathrm{R}\mathsf{Du2})\}$$
$$f((\mathsf{D})) = \{(\mathrm{R}\mathsf{D}),\ (ser)\}$$
$$f((\mathsf{T})) = \{(\mathrm{R}\mathsf{T}),\ (ref)\}$$
$$f((\mathsf{4})) = \{(\mathrm{R}\mathsf{4}),\ (tran)\}$$
$$f((\mathsf{B})) = \{(\mathrm{R}\mathsf{B}),\ (symm)\}$$
$$f((\mathsf{5})) = \{(\mathrm{R}\mathsf{5}),\ (eucl)\}$$
$$f((\mathsf{d})) = \{(\mathrm{R}\mathsf{d}\Rightarrow),\ (\mathrm{R}\mathsf{d}\Leftarrow)\}$$
$$f((\mathsf{K}+(\mathsf{d}))) = \{(\mathrm{R}\mathsf{d}\Rightarrow),\ (\mathrm{R}\mathsf{d}\Leftarrow),\ (\mathrm{R}\mathsf{Kd})\}$$
$$f((\mathsf{Du1}+(\mathsf{d}))) = \{(\mathrm{R}\mathsf{d}\Rightarrow),\ (\mathrm{R}\mathsf{d}\Leftarrow),\ (\mathrm{R}\mathsf{Du1d})\}$$
$$f((\mathsf{Du2}+(\mathsf{d}))) = \{(\mathrm{R}\mathsf{d}\Rightarrow),\ (\mathrm{R}\mathsf{d}\Leftarrow),\ (\mathrm{R}\mathsf{Du2d})\}$$
$$f((\mathsf{D}+(\mathsf{d}))) = \{(\mathrm{R}\mathsf{d}\Rightarrow),\ (\mathrm{R}\mathsf{d}\Leftarrow),\ (\mathrm{R}\mathsf{refR}),\ (ser)\}$$
$$f((\mathsf{T}+(\mathsf{d}))) = \{(\mathrm{R}\mathsf{d}\Rightarrow),\ (\mathrm{R}\mathsf{d}\Leftarrow),\ (\mathrm{R}\mathsf{refR}),\ (ref)\}$$
$$f((\mathsf{4}+(\mathsf{d}))) = \{(\mathrm{R}\mathsf{d}\Rightarrow),\ (\mathrm{R}\mathsf{d}\Leftarrow),\ (\mathrm{R}\mathsf{refR}),\ (tran)\}$$
$$f((\mathsf{B}+(\mathsf{d}))) = \{(\mathrm{R}\mathsf{d}\Rightarrow),\ (\mathrm{R}\mathsf{d}\Leftarrow),(\mathrm{R}\mathsf{refR}),\ (symm)\}$$
$$f((\mathsf{5}+(\mathsf{d}))) = \{(\mathrm{R}\mathsf{d}\Rightarrow),\ (\mathrm{R}\mathsf{d}\Leftarrow),\ (\mathrm{R}\mathsf{refR}),\ (eucl)\}.$$

Let us now phrase a proposition.

Claim 7.1 (Rules sound to model). *Let $X \subseteq \mathsf{Ex}$ and $U \subseteq \mathsf{CONST}$. Let $\langle W, Q, \{R_w\}_{w\in W}, v\rangle$ be a model for* $\mathsf{For}_{\boldsymbol{CMF}}$ *defined by set of conditions U. Let $\langle W, Q, \{R_w\}_{w\in W}, v\rangle$ be suitable to X. If some of the tableau rules that belong to:*

1. BTR
2. $\bigcup f(U)$

were applied to set X, then $\langle W, Q, \{R_w\}_{w\in W}, v\rangle$ is suitable at least to one output obtained through the application of this rule.

PROOF: Let $X \subseteq \mathsf{Ex}$ and model $\mathfrak{M} = \langle W, Q, \{\mathrm{R}_w\}_{w \in W}, v\rangle$, for some $U \subseteq \mathsf{CONST}$, satisfy the above assumptions. If $\mathfrak{M}$ is suitable to X, then there exists function $g\colon \mathsf{Ind}(X) \longrightarrow W$ that satisfies conditions from definition 7.2.

The thesis for the rules from BTR follows from the definition of truthness for $\mathsf{For}_{\mathbf{CMF}}$, thus the proposition thesis occurs for 1, and is examined several times (see for example [3]), [4]).

The claim also holds for specific rules from $\mathsf{TRCONST}$. Most of them are negative. It means that if we use one of them we immediately obtain a tableau inconsistent set. So they are inapplicable in these instances as they would contradict the assumption. For instance, if X comprised expression $\langle AR\neg A, i\rangle$, then the model could not be suitable for X, if it meets condition (Ma1) and rule:

$$\text{(RMa1)} \quad \frac{AR\neg A, i}{A\not{R}\neg A, i}$$

would introduce the tableau inconsistency to the proof.

Only rules (RMb1), (RMb2), (RMc1), (RK2), (R$\mathsf{d}\Rightarrow$), (R$\mathsf{d}\Leftarrow$) (taking into account set $\mathsf{TRCONST}$) are positive. Thus they do not introduce the tableau inconsistency directly. The non-modal counterparts of (RMb1), (RMb2), (RMc1) we examined in [6]. Let us consider (RK2) and for example (R$\mathsf{d}\Rightarrow$) (checking of the other direction (R$\mathsf{d}\Leftarrow$) is similar).

Tableau rule:

$$\text{(R}\mathsf{K}\text{2)} \quad \frac{\Box A\not{R}\Box B, i}{irj;\ A\not{R}B, j}$$

where label j must be new, corresponds to condition (2) $\forall_{u \in W}(wQu \Rightarrow A\mathrm{R}_u B) \Rightarrow \Box A\mathrm{R}_w \Box B$, from claim 5.2.

So, let us assume that (RK2) was applied to X. Then, $\langle \Box A\not{R}\Box B, i\rangle \in X$, and $\Box A\widetilde{\mathrm{R}}_{g(i)}\Box B$ in our model $\mathfrak{M}$. The application of the rule introduced two expressions: irj and $\langle A\not{R}B, j\rangle$, where j is new in the proof. But, by the assumption that $\mathfrak{M}$ satisfies condition (2) $\forall_{u \in W}(wQu \Rightarrow A\mathrm{R}_u B) \Rightarrow \Box A\mathrm{R}_w \Box B$, there must exist such world $u \in W$ that $g(i)Qu$ and $A\widetilde{\mathrm{R}}_u B$. So, we extend function g, taking $g'\colon \mathsf{Ind}(X) \cup \{j\} \longrightarrow W$, with:

$$g'(k) = \begin{cases} g(k), & \text{if } k \in \mathsf{Ind}(X) \\ u, & \text{if } k = j. \end{cases}$$

Therefore, after application of (R$\mathsf{K2}$) to X we obtain a set to which model $\mathfrak{M}$ is still suitable, because $g'(i)Qg'(j)$ and $A\widetilde{\mathrm{R}}_{g'(j)}B$.

Now we will consider one of the rules for the demodalization:

$$(\mathrm{R}\mathsf{d}{\Rightarrow}) \quad \frac{ARB, i}{\mathsf{d}(A)R\mathsf{d}(B), i}$$

The rule corresponds to the "from-left-to-right implication" in the condition from claim 6.1:

$$(\mathsf{d}) \qquad \forall_{w \in W^{\mathfrak{M}}} (A\mathrm{R}_w B \Longleftrightarrow \mathsf{d}(A)\mathrm{R}_w\mathsf{d}(B)).$$

Let us assume that our model $\mathfrak{M}$ satisfies the condition. At the same time (R$\mathsf{d}\Rightarrow$) was applied to set X. It means that $\langle ARB, i\rangle \in X$, and after the application there appeared $\langle \mathsf{d}(A)R\mathsf{d}(B), i\rangle$. However, since $A\mathrm{R}_{g(i)}B$ in the model, so by condition (d), $\mathsf{d}(A)\mathrm{R}_{g(i)}\mathsf{d}(B)$, too. Hence the model is suitable to $X \cup \{\langle \mathsf{d}(A)R\mathsf{d}(B), i\rangle\}$. □

The proof of completeness of our tableau methods in relation to the presented semantics still requires a converse proposition in a sense. Let us introduce the concept of model produced by set of expressions.

Definition 7.3 (Model generated by branch). Let $X \subseteq \mathsf{Ex}$. Model $\langle W, Q, \{\mathrm{R}_w\}_{w\in W}, v\rangle$ is *generated by* X iff

- $W = \mathsf{Ind}(X)$
- iQj iff $irj \in X$, for all $i, j \in W$
- $A\mathrm{R}_iB$ iff $\langle ARB, i\rangle \in X$, for all $A, B \in \mathsf{For}_{\mathbf{CMF}}$, $i \in W$
- $v(i, x) = 1$ iff $\langle x, i\rangle \in X$, for all $x \in \mathsf{Var}$, $i \in W$.

Assume we have a set of tableau rules that comprises:

1. BTR
2. set of tableau rules $\bigcup f(U)$ specified by given set of constraints $U \subseteq$ CONST.

If we now take a set of expressions $X \subseteq \mathsf{Ex}$ such that:

i) it is closed under all of those rules – for all expressions from X to which one of the rules is applicable, there exists one output in X
ii) X is not a tableau inconsistent set of expressions.

then there exists a model $\langle W, Q, \{\mathrm{R}_w\}_{w \in W}, v\rangle$ generated by that set (the set will be called *minimal closure* iff it is a minimal one that satisfies i), ii); for details see [3]). It does not need to satisfy constraints $U \subseteq \mathsf{CONST}$, but it can be enhanced. In general, it is a model for language $\mathsf{For}_{\mathbf{CMF}}$. Therefore, we have one more proposition.

Claim 7.2 (Model sound to rules). *Let $U \subseteq \mathsf{CONST}$. Let X be:*

- *a tableau consistent set of expressions*
- *closed under $\mathsf{BTR} \cup \bigcup f(U)$, for some set of constraints U.*

Then there exists a model $\langle W, Q, \{R_w\}_{w \in W}, v\rangle$ such that:

1. $W = \mathsf{Ind}(X)$
2. *for all formulas $A \in \mathsf{For}_{\mathbf{CMF}}$ and index $i \in W$:*

$$\langle A, i\rangle \in X \Rightarrow \langle W, Q, \{R_w\}_{w \in W}, v\rangle, i \models A$$

3. *model $\langle W, Q, \{R_w\}_{w \in W}, v\rangle$ meets conditions U.*

Proof: Let us make all the above assumptions. We know that set X generates a model. Let $\mathfrak{M} = \langle W, Q, \{\mathrm{R}_w\}_{w \in W}, v\rangle$ be a model generated by X. Surely, $W = \mathsf{Ind}(X)$, by definition of generated model.

Now the second point:

$$\langle A, i\rangle \in X \Rightarrow \mathfrak{M}, i \models A,$$

for all $A \in \mathsf{For}_{\mathbf{CMF}}$, $i \in W$. This part of the proof is inductive.

For variables $x \in \mathsf{Var}$ and negation of variables $\neg x$, the thesis is true by definition of generated model 7.2.

For more complex expressions through examination whether the tableau rules from set $\mathsf{BTR} \cup \bigcup f(U)$ introduce expressions that are sufficient for constitution of a model. For the rules from BTR it is self-explanatory. The Boolean and modal rules were examined e.g. in [3]. (It is the same in the case of the rules for the specific properties of the accessibility relation, but it concerns point three of the thesis.) And the rules for the relating implication and its negation were examined in [4]. By virtue of the truth conditions for the combined formulas (section 4) they introduce elements that are sufficient for the construction of a verification model in the context of any possible world.

In turn, the majority of the remaining tableau rules are negative in nature or concern the character of relation R in possible worlds.

For example rules: (RMa1), (RMa2), (RMb1'), (RMb2'), (RK1), (RDu1), (RDu2), (RD), (RT), (R4), (RB), (R5), (RKd), (RDu1d), (RDu2d), (RrefR) are meant to close branches within proofs rather than to validate the verification formulas. They were not even applied to the expressions from X as X is not a tableau inconsistent set of expressions by the assumption.

So, only tableau rules (RMb1), (RMb2), (RMc1), (RK2), (Rd$\Rightarrow$) (Rd$\Leftarrow$) are worth checking here. This way, we are starting the examination of the final part of claim 7.2: if $\mathfrak{M}$ meets conditions U.

Rules (RMb1), (RMb2), (RMc1) in non-modal versions were checked in [6], but we will have a look at their modal versions here. If rule:

$$\text{(RMb1)}\quad \frac{ARB, i}{A\not{R}\neg B, i}$$

was applied then $A\mathrm{R}_iB$ and so $A\widetilde{\mathrm{R}}_i\neg B$, by definition of generated model 7.3, as condition (Mb1) states, since $\langle ARB, i\rangle, \langle A\not{R}\neg B, i\rangle \in X$. But it is similarly in the case of:

$$\text{(RMb2)}\quad \frac{AR\neg B, i}{A\not{R}B, i}$$

If it was applied then $A\mathrm{R}_i\neg B$, so $A\widetilde{\mathrm{R}}_iB$ in $\mathfrak{M}$, by definition of generated model 7.3, as condition (Mb2) states, since $\langle AR\neg B, i\rangle, \langle A\not{R}B, i\rangle \in X$. For rule:

$$\text{(RMc1)}\quad \frac{\neg A\not{R}\neg B, i}{A\not{R}B, i}$$

we proceed similarly. If (RMc1) was applied to X, then $\langle \neg A\not{R}\neg B, i\rangle$, $\langle A\not{R}B, i\rangle \in X$, by definition of generated model 7.3, $\neg A\widetilde{\mathrm{R}}_i\neg B$ and $A\widetilde{\mathrm{R}}_iB$ in $\mathfrak{M}$, as condition (Mc1) states that $(A\mathrm{R}_iB \Rightarrow \neg A\mathrm{R}\neg B)$.

The next rule corresponds to the second constraint in (K):

$$(\mathrm{R}\mathsf{K2})\quad \frac{\Box A \not R \Box B, i}{irj;\ A \not R B, j}, \qquad \text{where } j \text{ is new on the branch.}$$

Let us assume that $\langle \Box A \not R \Box B, i\rangle \in X$. By definition of generated model 7.3, $\Box A \widetilde{\mathrm{R}}_i \Box B$ in $\mathfrak{M}$. But since (R$\mathsf{K2}$) was applied to X, then $irj \in X$ and $\langle A \not R B, j\rangle \in X$, where j is a new index. Again from the definition of generated model 7.3, we get: iQj and $A\widetilde{\mathrm{R}}_j B$ in model $\mathfrak{M}$. However, after transposition this exactly states the second constraint in (K): $\Box A \widetilde{\mathrm{R}}_w \Box B \Rightarrow \exists_{u \in W}(wQu \,\&\, A\widetilde{\mathrm{R}}_u B)$.

Now, let us look at the rules for demodalization condition (d).

$$(\mathrm{R}\mathsf{d}{\Rightarrow})\quad \frac{ARB, i}{\mathsf{d}(A)R\mathsf{d}(B), i} \qquad (\mathrm{R}\mathsf{d}{\Leftarrow})\quad \frac{A \not R B, i}{\mathsf{d}(A)\not R\mathsf{d}(B), i}$$

Since X does not include all expressions Ex (it is tableau consistent), even if it is closed under the above two rules the generated model does not satisfy condition (d), since some formulas did not appear in the closure of X. So if we want to have a model that satisfies (d), we must close family $\{\mathrm{R}_w\}_{w\in W}$ in model $\mathfrak{M}$ under (d). The same we have to do for the remaining conditions: (Mb1'), (Mb2'), both points of (K), $(\mathsf{Du1})$, $(\mathsf{Du2})$,(D), (T), $(\mathsf{4})$, (B), $(\mathsf{5})$.

Next, if a model has to satisfy $(\mathsf{K}+(\mathsf{d}))$, we must close it under (d) and $(\mathsf{K}+(\mathsf{d}))$. The same applies to $(\mathsf{Du1}+(\mathsf{d}))$, $(\mathsf{Du2}+(\mathsf{d}))$.

For the rest of conditions $(\mathsf{D}+(\mathsf{d}))$, $(\mathsf{T}+(\mathsf{d}))$, $(\mathsf{4}+(\mathsf{d}))$, $(\mathsf{B}+(\mathsf{d}))$, $(\mathsf{5}+(\mathsf{d}))$ we close the model under (d) and reflexivity of $\{\mathrm{R}_w\}_{w\in W}$, so for all $w \in W$ and all $A \in \mathsf{For}_{\mathbf{CMF}}$ we put $A\mathrm{R}_w A$. Surely, each of the conditions also contains a modal component of the accessibility relation Q. But since a proper rule ((*ser*), (*ref*), (*tran*), (*symm*), or (*eucl*)) for it was used to X, so relation Q in the generated model $\mathfrak{M}$ has a suitable property as it is a standard knowledge [3], because Q can be fully defined by X.

Summing up, when we close the relations $\{\mathrm{R}_w\}_{w\in W}$ in model $\langle W, Q, \{\mathrm{R}_w\}_{w\in W}, v\rangle$ under conditions from U, we obtain model $\mathfrak{M}' = \langle W, Q, \{\mathrm{R}_w\}'_{w\in W}, v\rangle$ which meets conditions U. In this model, all formulas that are true at world i in the model based on $\{\mathrm{R}_w\}_{w\in W}$ are true at i as well, for all $i \in W$. $\Box$

Finally, we have the theorem on the completeness of tableaux and relating semantics for the discussed connexive models.

Theorem 7.3 (Completeness theorem). *Let* $U \subseteq \mathsf{CONST}$. *Let* $\models \subseteq P(\mathsf{For}_{\boldsymbol{CMF}}) \times \mathsf{For}_{\boldsymbol{CMF}}$ *be the consequence relation defined by the set of all models designated by set of conditions* U. *Then for any* $X \subseteq \mathsf{For}_{\boldsymbol{CMF}}$, $A \in \mathsf{For}_{\boldsymbol{CMF}}$ *the following facts are equivalent:*

1. $X \models A$
2. *there exists a finite subset* $Y \subseteq X$ *and some* $i \in \mathbb{N}$ *such that each minimal closure of set* $\{\langle B, i\rangle : B \in Y \cup \{\neg A\}\}$ *under set of tableau rules* $\mathsf{BTR} \cup \bigcup f(U)$ *is a tableau inconsistent set of expressions.*

Proof: Let us adopt the assumptions. In the theorem proof, we make use of the prior propositions. For implication $1 \Rightarrow 2$ claim 7.2 is sufficient. In turn, for implication $2 \Rightarrow 1$ claim 7.1 is sufficient. □

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