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A STUDY OF A SPECIAL SEMI MAXIMAL FILTER IN BL-ALGEBRAS

Abstract

In this article, a specific type of semi maximal filters is introduced, which form a lattice structure. These filters are called J- semi maximal and NJ-semi maximal, and their key properties in BL-algebras are analyzed. Additionally, these special filters are compared with other defined filters, particularly semi maximal and maximal filters. The purpose of this article is to provide a new analysis of filters in BL-algebras.

Keywords: BL-algebra, semi maximal filter, J-semi maximal filter, NJ-semi maximal filter.

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1. Introduction

Hájek introduced BL-algebras as an algebraic approach to studying many-valued logic [5]. He gave an algebraic proof of the completeness theorem for Basic Logic (BL), which is based on continuous triangular norms commonly applied in fuzzy logic. Filter theory plays a central role in the study of these algebras, since different filters correspond to different sets of provable formulas. Hájek introduced the concepts of filters and prime filters in BL-algebras [5], and employed prime filters to prove the completeness of BL. Expanding upon Hájek's foundational contributions, Turunen systematically explored the structural characteristics of filters and prime filters within BL-algebras [11, 12], thereby advancing the theoretical understanding of their underlying algebraic framework. Building on these developments, M. Bedrood et al. introduced and investigated a specific subclass of prime filters, termed *J-prime filters*, offering further insight into the structural behavior of prime filters in BL-algebras [2]. BL-algebras offer a rigorous algebraic framework for supporting logical operations within fuzzy systems. This ensures that fuzzy reasoning is not only effective for managing imprecise information but also grounded in a mathematically sound structure, guaranteeing logical consistency and reliability.

Maximal filters in BL-algebras are particularly important in various fields such as non-classical logic, lattice theory, and mathematical and practical applications. These filters are key tools for analyzing complex algebraic structures and understanding the logical relationships between elements. With the development of new concepts like semi-maximal filters and their types, researchers have gained a more precise understanding of how these filters interact with the structures of BL-algebras.

S. Motamed et al. introduced and studied radical filters based on maximal filters [8]. A. Paad et al. defined and investigated semi maximal filters [9]. Based on these studies, A. Movahed et al. presented new results and an equivalent definition for semi maximal filters, comparing them with other types of filters [6].
In this paper, we investigate a distinguished subclass of generalized semi maximal filters in BL- algebras, with an emphasis on their structural and logical characteristics. Based on the concept of semi simple BL-algebras, we concluded that every semi maximal filter is also an NJ-semi maximal filter. Furthermore, every minimal prime filter is a J-semi maximal filter in BL-algebras. Additionally, we proved that for any filter in a Hyperarchimedean BL-algebra, every proper filter of it is a J-semi maximal filter. Finally, we introduced the concept of semi factors and provided a clear framework for understanding this idea.

2. Preliminaries

We recollect some definitions and results which will be used in the sequel:

Definition 2.1 ([5]). A BL-algebra is an algebra $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ equipped with an order $\leq$ satisfying the following:

- (BL_1) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice,
- (BL_2) $(L, \odot, 1)$ is a commutative monoid,
- (BL_3) $\odot$ and $\rightarrow$ form an adjoint pair i.e., $z \leq x \rightarrow y$ if and only if $x \odot z \leq y$, for all $x, y, z \in L$,
- (BL_4) $x \wedge y = x \odot (x \rightarrow y)$, for all $x, y \in L$,
- (BL_5) $(x \rightarrow y) \vee (y \rightarrow x) = 1$, for all $x, y \in L$.

Throughout the paper, we denote L as a $BL-$algebra.

Definition 2.2 ([10]). An element, if $a \odot a = a$ and the collection of all idempotent elements is displayed with $B(L)$.

Also if every element of L is an idempotent element, then BL-algebra L is called a Boolean algebra.We say that a $BL-$algebra A is a $BL-$chain if the underlying order $\leq$ is total.

DEFINITION 2.3 ([13]). A filter is a non-empty subset F of L satisfying the following conditions:

($F1$) If $a \in F,\ b \in L$ and $a \leq b$, then $b \in F$,

($F2$) If $a, b \in F$, then $a \odot b \in F$.

We denote by $F(L)$ the set of all filters of L. Also, it has also been proven that $(F(L), \wedge, \vee, \{1\}, L)$ is a complete Brouwerian lattice.

LEMMA 2.4 ([10]).

(1) Let $X \subseteq L$. Denote by $(X]$ the filter generated by X. Then we have

$$(X] = \{a \in L : x_1 \odot x_2 \odot \ldots \odot x_n \leq a,\ n \in \mathbb{N}, x_1, x_2, \ldots, x_n \in X\}$$

In particular $(a] = \{x \in L : a^n < x, n \in \mathbb{N}\}$.

(2) For $F, G \in F(L)$ *and* $a, b \in L$

- $F \wedge G = F \cap G$;
- $F \vee G = (F \cup G] = \{x \in L | a \odot b \leq x \ for\ some\ a \in F\ and\ b \in G\}$;
- *If* $a \leq b$, *then* $(b] \subseteq (a]$;
- $(a] \vee (b] = (a \odot b] = (a \wedge b]$;
- $(a] \cap (b] = (a \vee b]$.

DEFINITION 2.5 ([13, 3]). Let F be a filter of L.

- If $F \neq L$, then F is called a proper filter of L.
- A proper filter F of L is called prime filter if for all $a, b \in L, a \vee b \in F$, satisfies $a \in F$ or $b \in F$.

We denote by $Spec(L)$ the set of all prime filters of a BL-algebra L.

- A filter P of L is called a minimal prime filter of L when:

(1) $P \in Spec(L)$;
(2) If there exists $Q \in Spec(L)$ such that $Q \subseteq P$, then $P = Q$.

We denote by $Min(L)$ the set of all minimal prime filters of L.

- A proper filter F of L is called maximal if and only if for each filter $J \neq F$, if $F \subseteq J$, implies $J = L$.
  We denote by $Max(L)$ the set of all maximal filters of L.
  The intersection of all maximal filters of L is called the radical of L and it is denoted by $Rad(L)$. The intersection of all maximal filters of L which contain the filter F is called the radical of F and it is denoted by $Rad(F)$.

Note: Prime filter P of L is called minimal prime filter over filter F, if

(1) $F \subseteq P$;
(2) If there exists $Q \in Spec(L)$ such that $F \subseteq Q \subseteq P$, then $P = Q$.

We denote by $Min(F)$ the set of all minimal prime filters over filter F.

COROLLARY 2.6 ([10]). Every prime filter of L is contained in a unique maximal filter of L.

THEOREM 2.7 ([8]). *A BL-algebra L is called semi simple if and only if $Rad(L) = \{1\}$.*

Remark 2.8 ([10]). For every $F \in F(L)$;

(1) $F = \cap \{P \in Spec(L) \mid F \subseteq P\}$.
(2) $\cap \{P \in Spec(L)\} = \{1\}$.

THEOREM 2.9 ([4]). *Let $P \in spec(L)$. Then $P \in Min(L)$ if and only if for each $a \in P$, there exists $r \in L \setminus P$ such that $r \vee a = 1$.*

DEFINITION 2.10 ([11]). Let X be a non-empty subset of L. $Co - Ann_L(X)$ is the Co- annihilator of X defined by:

$$Co - Ann_L(X) = \{a \in L | a \vee x = 1, \forall x \in X\}$$

THEOREM 2.11 ([4]). *Let $P \in Min(L)$ and F be finitely generated filter. Then $F \subseteq P$ if and only if $Co - Ann_L(F) \nsubseteq P$.*

Definition 2.12 ([10]). An element $a \in L$ is called archimedean if there is $n \in \mathbb{N}$, such that $a \vee (a^n)^* = 1$.

Theorem 2.13 ([10]). *L is Hyperarchimedean if and only if $Spec(L) = Max(L)$.*

Theorem 2.14 ([5]). *If $a \neq 1$, then there is a prime filter P of L such that $a \notin P$.*

Corollary 2.15. Let F be an filter of L and $a \in L \setminus F$. Then there exists $P \in spec(L)$ such that $F \subseteq P$ and $a \notin P$.

Theorem 2.16 ([10]). *For any L, the following statements are equivalent:*

(1) L is a BL-chain.

(2) Any proper filter of L is prime.

(3) $\{1\}$ is a prime filter.

(4) $Spec(L)$ is linearly ordered.

Proposition 2.17 ([6]). Let F be a proper filter of L and $P \in Spec(L)$ such that $F \subseteq P$. Then there exists $Q \in Min(F)$ such that $Q \subseteq P$.

Lemma 2.18 ([10]). *The proper filter P is a prime filter if and only if $F \cap G \subseteq P$, then $F \subseteq P$ or $G \subseteq P$, for all $F, G \in F(L)$.*

Definition 2.19 ([8]). Let F be a proper filter of L. If $Rad(F) = F$, then F is called a semi maximal filter of L.

Theorem 2.20 ([8]). *Every maximal filter of L is a semi maximal filter.*

Proposition 2.21 ([6]). Let F be a semi maximal filter of L and K a subset of L such that $K \nsubseteq F$. Then the set $(F : K) = \{x \in L \mid x \vee k \in F, \forall k \in K\}$ is also a semi maximal filter.

Note: Let $a \in L$ and F be a filter of L . We put

- $M_a = \cap \, \{M| \; M \in Max(L), \; a \in M\}$.
- $M(a) = \{M| \; M \in Max(L), \; a \in M\}$.
- $P_a = \cap \, \{P| \; P \in Min(L), \; a \in P\}$.

THEOREM 2.22 ([6]). *A proper filter F of L is a semi maximal filter if and only if for all $a \in F$, $M_a \subseteq F$.*

THEOREM 2.23 ([6]). *Let F be a proper filter of L. Then the following statements are equivalent:*

(1) F is a semi maximal filter in L.

(2) $M(a) \subseteq M(b)$ and $a \in F$, implies that $b \in F$.

(3) $M(a) = M(b)$ and $a \in F$ implies that $b \in F$.

THEOREM 2.24 ([6]). *If L is a semi simple $BL - algera$ and $a \in L$, then $M_a \subseteq P_a$.*

LEMMA 2.25 ([1]). *If $a \in L$, then $P_a = Co - Ann(Co - Ann(a))$.*

3. Semi maximal filters in BL-algebras

In this section, we introduce the notions of J-semi maximal and NJ-semi maximal filters, exploring their defining properties and the conditions under which various filters qualify as either J-semi maximal or NJ-semi maximal. The concept of J-semi maximal filters extends the idea of semi maximal filters, broadening their applicability. Additionally, we present the notion of semi factors and demonstrate that the set of semi factors of a filter forms a lattice.

DEFINITION 3.1. Let F and J be two filters of L. A filter F is called a J-semi maximal filter if $M_a \cap J \subseteq F$ for all $a \in F$. Also, if $J \nsubseteq F$ and F is a J-semi maximal filter, then F is called an NJ-semi maximal filter, and J is referred to as a semi factor of F.

Clearly, if $J \subseteq F$, then F is always a J-semi maximal filter. It follows that every NJ-semi maximal filter is, by definition, a J-semi maximal filter.

Example 3.2. (1) Let $L = \{0, a, b, c, d, 1\}$. where $0 < a, b < c < 1$ and $0 < b < d < 1$. Define $\odot$ and $\to$ as follows:

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	0	0	b	b
c	0	a	0	a	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	d	1
b	c	c	1	1	1	1
c	b	c	d	1	d	1
d	a	a	c	c	1	1
1	0	a	b	c	d	1

The relationships between the members are depicted in Figure 1.

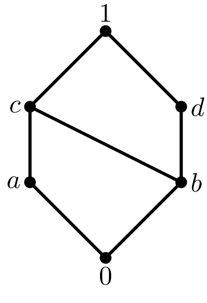


Figure 1: Relationships between the elements of $L = \{0, a, b, c, d, 1\}$.

Then $(L, \odot, \rightarrow, 0, 1)$ is a BL-algebra [10]. It has four filters: $F_0 = \{1\}, F_1 = L, F_2 = \{1, a, c\}, F_3 = \{1, d\}$. Obviously, $Max(L) = \{F_2, F_3\}$ and $M_1 = F_0$, $M_a = F_2$, $M_b = L$, $M_c = F_2$, $M_d = F_3$. Then F_0, F_2 and F_3 are semi maximal filters.
It is easy to see that F_2 is a F_3-semi maximal filter and $F_3 \nsubseteq F_2$, so F_2 is

a NF_3-semi maximal filter and F_3 is a semi factor of the filter F_2. Also, F_0 is a F_2-semi maximal filter and F_0 is a NF_2-semi maximal filter.

(2) Let $L = \overline{\mathbb{Z}} \cup \{-\infty\} \cup \{0, a, b, 1\}$, where $\overline{\mathbb{Z}}$ is the set of negative integer numbers and $-\infty < \cdots < -2 < -1 < 0 < a, b < 1$. Operations $\odot$ and $\rightarrow$ are defined as follows:

$\odot$	$-\infty$	$\cdots$	-3	-2	-1	0	a	b	1
$-\infty$	$-\infty$	$\cdots$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
-3	$-\infty$	$\cdots$	-6	-5	-4	-3	-3	-3	-3
-2	$-\infty$	$\cdots$	-5	-4	-3	-2	-2	-2	-2
-1	$-\infty$	$\cdots$	-4	-3	-2	-1	-1	-1	-1
0	$-\infty$	$\cdots$	-3	-2	-1	0	0	0	0
a	$-\infty$	$\cdots$	-3	-2	-1	0	a	0	a
b	$-\infty$	$\cdots$	-3	-2	-1	0	0	b	b
1	$-\infty$	$\cdots$	-3	-2	-1	0	a	b	1

$\rightarrow$	$-\infty$	$\cdots$	-3	-2	-1	0	a	b	1
$-\infty$	1	$\cdots$	1	1	1	1	1	1	1
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
-3	$-\infty$	$\cdots$	1	1	1	1	1	1	1
-2	$-\infty$	$\cdots$	-1	1	1	1	1	1	1
-1	$-\infty$	$\cdots$	-1	1	1	1	1	1	1
0	$-\infty$	$\cdots$	-3	-2	-1	1	1	1	1
a	$-\infty$	$\cdots$	-3	-2	-1	b	1	b	1
b	$-\infty$	$\cdots$	-3	-2	-1	a	a	1	1
1	$-\infty$	$\cdots$	-3	-2	-1	0	a	b	1

The relationships between the members are depicted in Figure 2. Then $(L, \wedge, \vee, \odot, \rightarrow, -\infty, 1)$ is a BL-algebra [7]. It has filters:
$F_0 = \{1\}$, $F_1 = L$, $F_2 = \{1, b, a, 0\}$, $F_3 = \{1, a\}$, $F_4 = \{1, b\}$ and $F_5 = \{\ldots, -3, -2, -1, 0, a, b, 1\} \setminus \{-\infty\}$ are filters. Obviously, $F_2 \in Max(L)$ and $M_a = F_2 \nsubseteq F_3$. Hence F_3 is not a semi maximal filter.

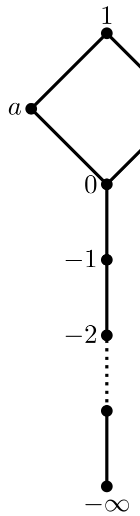


Figure 2: Relationships between the elements of $L = \overline{\mathbb{Z}} \cup \{-\infty\} \cup \{0, a, b, 1\}$.

It is easy to see that F_3 is not F_5-semi maximal filter(because $M_a \cap F_5 \nsubseteq F_3$). Also, consider $F_3 = \{1, a\}$ is not F_2-semi maximal filter.

LEMMA 3.3. *If $a, b \in L$, then the following sentences hold:*

(1) $a \in M_b$ *if and only if* $M_a \subseteq M_b$*;*

(2) $M_a \cap M_b = M_{(a \vee b)}$*;*

(3) If $a \leq b$*, then* $M(a) \subseteq M(b)$ *and* $M_b \subseteq M_a$*;*

(4) $M_{(a \odot b)} = M_a \vee M_b$*.*

PROOF: (1) It is trivial.

(2) Let $x \in M_a \cap M_b$, but $x \notin M_{(a \vee b)}$. Then there exists $M \in Max(L)$ such that $x \notin M$ and $a \vee b \in M$, since M is prime filter, Hence $a \in M$ or $b \in M$ and we get $x \notin M_a$ or $x \notin M_b$, which is a contradiction. Therefore

$M_a \cap M_b \subseteq M_{(a\vee b)}$. Now, suppose that $x \in M_{(a\vee b)}$ but $x \notin M_a \cap M_b$. Without loss of generality, suppose that $x \notin M_a$. So there exists $M \in Max(L)$ such that $x \notin M$ and $a \in M$. As $a \leq a \vee b$, hence we conclude that $a\vee b \in M$. Thus $x \notin M_{(a\vee b)}$, which is a contradiction. Hence $M_{(a\vee b)} \subseteq M_a \cap M_b$. Therefore $M_{a\vee b} = M_a \cap M_b$.

(3) Suppose that $M \in M(a)$, hence $M \in Max(L)$ and $a \in M$. As M is a filter and $a \leq b$, hence $b \in M$ and we conclude that $M \in M(b)$. It follows that

$$M(a) = \{M \in Max(L)|a \in M\} \subseteq \{M \in Max(L)|b \in M\} = M(b),$$

Also,

$$\bigcap\{M \in Max(L)|b \in M\} \subseteq \bigcap\{M \in Max(L)|a \in M\},$$

Thus $M_b \subseteq M_a$.

(4) Since $(a \odot b) \leq a, b$, by part (3) we get $M_a, M_b \subseteq M_{(a\odot b)}$. Thus we conclude that $M_a \vee M_b \subseteq M_{(a\odot b)}$. Since $a \in M_a$ and $b \in M_b$, it follows that $a, b \in M_a \vee M_b$. Thus $(a \odot b) \in M_a \vee M_b$. Consequently, we obtain $M_{(a\odot b)} \subseteq M_a \vee M_b$. Therefore $M_{(a\odot b)} = M_a \vee M_b$. □

Proposition 3.4. Let F, J and K be three filters of L.

(1) If F is a J-semi maximal filter of L and $P \in Min(F)$, then P is a J-semi maximal filter of L.
(2) Let F and J be K-semi maximal filters of L. Then $F \vee J$ is a K-semi maximal filter of L.
(3) Let $\{F_i\}_{i\in\Omega}$ be a non empty family of J-semi maximal filter. Then $\bigwedge_{i\in\Omega} F_i$ is a J-semi maximal filter.
(4) Let F be a J-semi maximal filter and $K \subseteq J$. Then F is a K-semi maximal filter.
(5) If $J = L$, then L-semi maximal filters and semi maximal filters are the same.
(6) Every maximal filter is a J-semi maximal filter.

(7) Every semi maximal filter is a J-semi maximal filter.

(8) If $F = \{1\}$ is a J-semi maximal filter, then $J \cap Rad(L) = \{1\}$.

(9) Let $J \neq \{1\}$ and $J \cap Rad(L) = \{1\}$. Then $\{1\}$ is a NJ-semi maximal filter.

Proof: (1) Suppose that $a \in P$. By Theorem 2.9 , there exists $r \in L \backslash P$ such that $r \vee a = 1 \in F$. Since F is a J-semi maximal filter, so $M_{a\vee r} \cap J \subseteq F$. It follows from Lemma 3.3 part(2) that $M_a \cap M_r \cap J \subseteq F \subseteq P$. It is clear that $M_r \nsubseteq P$ and by Lemma M we conclude that $M_a \cap J \subseteq P$. Therefore, P is a J-semi maximal filter.

(2) Let $x \in F \vee J$, hence $a \odot b \leq x$, for some $a \in F$ and $b \in J$. Since F and J are K-semi maximal filters, so we get $M_a \cap K \subseteq F$ and $M_b \cap K \subseteq J$. Hence we have $M_a \cap K \subseteq F \vee J$ and $M_b \cap K \subseteq F \vee J$. we get $K \cap (M_a \vee M_b) = (M_a \cap K) \vee (M_b \cap K) \subseteq F \vee J$. By Lemma 3.3 part (4) We get $K \cap M_{(a\odot b)} \subseteq F \vee J$. We have $a \odot b \leq x$, it follows from Lemma 3.3 part (3) that $M_x \subseteq M_{(a\odot b)}$. Therefore $M_x \cap K \subseteq M_{(a\odot b)} \cap K \subseteq F \vee J$, hence we get $M_x \cap K \subseteq F \vee J$. Thus $F \vee J$ is a K-semi maximal filter.

(3) Suppose that $a \in \bigwedge_{i\in\Omega} F_i$. Hence $a \in F_i$, for all $i \in \Omega$. Since F_i is a J-semi maximal filter, so $M_a \cap J \subseteq F_i$, for all $i \in \Omega$. Obviously, $M_a \cap J \subseteq \bigwedge_{i\in\Omega} F_i$, for $a \in \bigwedge_{i\in\Omega} F_i$. Therefore, $\bigwedge_{i\in\Omega} F_i$ is a J-semi maximal filter.

(4) Assume that F is a J-semi maximal filter. This means that for all $a \in F$, $M_a \cap J \subseteq F$. By the given hypothesis, we have $K \subseteq J$. This implies that $M_a \cap K \subseteq M_a \cap J$. Therefore, F is a K-semi maximal filter.

(5) Let F be a L-semi maximal filter. Then $M_a \cap L \subseteq F$, for all $a \in F$. Always, $M_a \cap L = M_a$, hence F is a semi maximal filter.

(6) Suppose that M is a maximal filter of L. For all $a \in M$,we have $M_a \subseteq M$. On the other hand, for all $J \in F(L)$, we always have $M_a \cap J \subseteq M_a \subseteq M$, as the intersection of M_a with any filter J is always contained within M. Therefore, we can conclude that M is a J-semi maximal filter.

(7) Let F be a semi maximal filter. Then $M_a \subseteq F$, for all $a \in F$. It is clear that $M_a \cap J \subseteq M_a$, for all $J \in F(L)$. Hence F is a J-semi maximal filter.

(8) Suppose that $F = \{1\}$ is a J-semi maximal filter. This means that $M_1 \cap J \subseteq \{1\}$. Since $M_1 = Rad(L)$ so $Rad(L) \cap J = \{1\}$.

(9) By hypothesis, $J \cap Rad(L) = \{1\}$. Hence the filter $\{1\}$ is a J-semi maximal filter. Since $J \neq \{1\}$, so the $\{1\}$ is a NJ-semi maximal filter. □

Corollary 3.5. Let J be a filter of L and $\Sigma = \{F \in F(L) |\ F$ is a J-semi maximal filter$\}$. Then

(1) $(\Sigma, \subseteq)$ is a poset.

(2) $(\Sigma, \vee, \wedge)$ is a lattice.

Proof: By Proposition 3.4, parts (2) and (3), it is clear. □

In Example 3.2 part (1) for the filter $J = F_3$, the set of all J-semi maximal filter is $\Sigma = \{F_0, F_1, F_2, F_3\}$ that forms a lattice.

Proposition 3.6. Let P be a prime filter of L and J be a proper filter of L. Then the following conditions are equivalent:

(1) P is a J-semi maximal filter.

(2) P is either a semi maximal filter or $J \subseteq P$.

Proof: (1) ⇒ (2) Suppose that $J \nsubseteq P$, we prove that P is a semi maximal filter. Since P is a J-semi maximal filter, so $M_a \cap J \subseteq P$, for all $a \in P$. By Lemma M, we deduce that $M_a \subseteq P$, for all $a \in P$. Therefore P is a J-semi maximal filter.

(2) ⇒ (1) It is clear. □

Corollary 3.7. Let F and J be two filters of L.

(1) If F is a NJ-semi maximal filter of L, then there exists $P \in Min(F)$ such that P is a semi maximal filter of L.

(2) Every prime NJ-semi maximal filter is a semi maximal filter.

(3) Let L be a BL-chain. Then every proper NJ-semi maximal filter is a semi maximal filter.

PROOF: (1) F is a NJ-semi maximal filter, hence $J \nsubseteq F$ and F is a J-semi maximal filter. Since $J \nsubseteq F$, so there exists $x \in J \setminus F$. Obviously, if we take $K := (x]$ thus F is a K-semi maximal filter. Also, $x \notin F$ hence by Corollary 2.15, we deduce that there exists $Q \in Spec(L)$ containing F such that $x \notin Q$. It follows from Theorem 2.17 that there exists $P \in Min(F)$ such that $P \subseteq Q$. Clearly, $x \notin P$, so $K \nsubseteq P$. By Proposition 3.4 part(1) and Proposition 3.6, therefore P is a semi maximal filter.

(2) By Proposition 3.6, it is clear.

(3) Suppose that F is proper J-semi maximal filter such that $J \nsubseteq F$. Hence $M_a \cap J \subseteq F$, for all $a \in F$. Since L is a BL-chain, by Theorem 2.16, F is a prime filter. Also, by part(2) of this corollary, we conclude that $M_a \subseteq F$, for all $a \in F$. Therefore F is a semi maximal filter. □

Note: Let F be a filter of L. We define:

$$F_s = \{b \in L \mid \ b \in M_a, \ for \ some \ a \in F\}.$$

LEMMA 3.8. *Let F and J be filters of L. Then the following sentences hold:*

(1) F_s is a semi maximal filter of L.

(2) $F \subseteq F_s$.

(3) $F_s = \cap\{Q \mid F \subseteq Q$ and Q is a semi maximal filter$\}$.

(4) $(F \cap J)_s = F_s \cap J_s$.

(5) If $F \subseteq J$, then $F_s \subseteq J_s$.

(6) If F is a semi maximal filter, then $F = F_s$.

PROOF: (1) First, we prove that F_s is a filter. It is clear that $1 \in F_s$. Suppose that $b, c \in F_s$, hence $b \in M_a$, for some $a \in F$ and $c \in M_t$, for some $t \in F$. By Lemma 3.3 part (1), we have $M_b \subseteq M_a$ and $M_c \subseteq M_t$. Thus $M_b \vee M_c = M_{b \odot c} \subseteq M_{a \odot t}$. Since F is a filter, so $a \odot t \in F$ and we deduce that $b \odot c \in F_s$. Also, if $b \leq c$ and $b \in F_s$, then $b \in M_a$, for some $a \in F$. It follows from Lemma 3.3 that $c \in M_a$ for some $a \in F$, that is, $c \in F_s$. Hence F_s is a filter. Now, we show that F_s is a semi maximal filter. This is proven by Theorem 2.23. Let $M(a) \subseteq M(b)$ and $a \in F_s$. Then $M_b \subseteq M_a$ and there exists $t \in F$ such that $a \in M_t$. Hence by lemma 3.3 part (1), $M_a \subseteq M_t$ and we get $b \in M_t$. Therefore $b \in F_s$.

(2) Suppose that $a \in F$. Since $a \in M_a$, so we obtain $a \in F_s$.

(3) Let $K = \cap\{\ Q \mid F \subseteq Q$ and Q is a semi maximal filter $\}$. Then by parts (1) and (2) of this lemma, it is clear that $K \subseteq F_s$. By contrary, if $F_s \nsubseteq K$. Suppose that $b \in F_s$ but $b \notin K$. Hence there exists a semi maximal filter Q such that $F \subseteq Q$ and $b \notin Q$. According to Theorem 2.23 and using the equivalent definition so, there is not $a \in Q$ such that $M_b \subseteq M_a$. We deduce that $b \notin M_a$, thus $b \notin F_s$, which is a contradiction.

(4) Assume that $b \in (F \cap J)_s$, hence there exists $a \in F \cap J$ such that $b \in M_a$. Since $a \in F$ and $b \in M_a$, so $b \in F_s$. Similarly, $b \in J_s$. Therefore $(F \cap J)_s \subseteq F_s \cap J_s$. Now, suppose that $b \in F_s \cap J_s$. Then there exist $a \in F$ and $c \in J$ such that $b \in M_a$ and $b \in M_c$. Obviously, $a \vee c \in F \cap J$ and $b \in M_a \cap M_c = M_{a \vee c}$. We conclude that $b \in (F \cap J)_s$ and $F_s \cap J_s \subseteq (F \cap J)_s$.

(5) It is clear.

(6) By part (3) of this lemma, the proof is evident. □

Example 3.9. Let $F_1 = \{1, a, c\}$ be the filter from Example 3.2 (1), it is easy to clarify $(F_1)_s = \{1, a, c\}$. (Because F_1 is a semi maximal filter.) Also, for the filter $F_2 = \{1, b, a, 0\}$ in Example 3.2 (2), $(F_2)_s = \{1, b, a, 0\}$. On the other hand, $F \subseteq F_s$.

Proposition 3.10. Suppose that F and J are filters of L. The following sentences are equivalent:

1. F is a J-semi maximal filter.
2. $F_s \cap J \subseteq F$.
3. There exists a semi maximal filter K containing F such that $K \cap J \subseteq F$.
4. For each $a \in F$ and $b \in J$ if $M_b \subseteq M_a$, then $b \in F$.

Proof: (1) $\Rightarrow$ (2) Let $b \in F_s \cap J$. Then there exists $a \in F$ such that $b \in M_a$, hence $M_b \subseteq M_a$. Since F is a J-semi maximal filter, so $M_a \cap J \subseteq F$. We have $b \in M_b \cap J \subseteq M_a \cap J \subseteq F$, thus $b \in F$.

(2) $\Rightarrow$ (3) Consider $K := F_s$.

(3) $\Rightarrow$ (4) By hypothesis $a \in F$ and $F \subseteq K$, thus $a \in K$. Since K is a semi maximal filter, so $M_a \subseteq K$. We have $b \in J$ and $b \in M_b$ hence $b \in M_b \cap J$. Therefore $b \in M_b \cap J \subseteq M_a \cap J \subseteq K \cap J \subseteq F$. As a result $b \in F$.

(4) $\Rightarrow$ (1) We show that $M_a \cap J \subseteq F$, for all $a \in F$. Assume that $b \in M_a \cap J$, then $M_b \subseteq M_a$ and $b \in J$. By part (4), we conclude that $b \in F$. Therefore F is a J-semi maximal filter. □

Proposition 3.11. Let $F, J \in F(L)$. Then F is a J-semi maximal filter (NJ-semi maximal filter) if and only if $F \cap J$ is a J-semi maximal filter (NJ-semi maximal filter).

Proof: Let $F \cap J$ be a J-semi maximal filter. By Proposition 3.10, we have $(F \cap J)_s \cap J \subseteq F \cap J$. Now, by Lemma 3.8, we get

$$(F \cap J)_s \cap J = F_s \cap J_s \cap J = F_s \cap J \text{ (since } J \subseteq J_s).$$

On the other hand, $F_s \cap J = (F \cap J)_s \cap J \subseteq F \cap J \subseteq F$. Therefore by Proposition 3.10, F is a J-semi maximal filter. Additionally, by the hypothesis, $F \cap J \subsetneqq J$. This implies that $J \nsubseteq F$ and we can conclude that F is a NJ-semi maximal filter. The other side is clear (by Prposition 3.4). Now, let F be $NJ-$semi maximal filter. For a filter $F \cap J$ there exists a filter $J \nsubseteq F \cap J$ such that $F \cap J$ is a J-semi maximal filter. Thus $F \cap J$ is a NJ-semi maximal filter. Now, suppose that $F \cap J$ is NJ-semi maximal filter. Then for a filter F there exists a filter $J \nsubseteq F$, and also we have F is J-semi maximal filter, hence we conclude that F is a NJ-semi maximal filter. □

Proposition 3.12. A filter F of L is a NJ-semi maximal filter if and only if there exists $b \in L \setminus F$ such that $(b] \cap M_a \subseteq F$, for all $a \in F$.

Proof: Let F be a NJ-semi maximal filter. Then F is a J-semi maximal filter, for some filter J of L such that $J \nsubseteq F$. It is enough to take $b \in J \setminus F$.

Conversely, by hypothesis there exists $b \in L \setminus F$, we take $J = (b]$, it is clear that F is a J-semi maximal filter and $J \nsubseteq F$. Therefore F is a NJ-semi maximal filter. □

PROPOSITION 3.13. Let F, J, K and H be filters in L. Then the following sentences hold:

(1) F is a J-semi maximal filter if and only if F is a $(F \vee J)-$semi maximal filter.

(2) Let J be a semi maximal filter and $F \subseteq J$. Then F is a J-semi maximal filter if and only if F is a semi maximal filter.

(3) If F is a semi maximal filter, then J is a $F-$semi maximal filter if and only if $F \cap J$ is a semi maximal filter.

(4) $F \cap J$ is a J-semi maximal filter and is a $F-$semi maximal filter if and only if F is a J-semi maximal filter and J is a $F-$semi maximal filter.

(5) Let M be a maximal filter of L. Then $F \cap M$ is a semi maximal filter if and only if F is a semi maximal filter.

(6) If $F \subseteq J$ and F is a J-semi maximal filter and J is a K-semi maximal filter, then F is a K-semi maximal filter.

(7) If $F \subseteq J$ and $H \subseteq K$ and F is a J-semi maximal filter and H is a K-semi maximal filter, then $F \cap H$ is a $(J \cap K)-$semi maximal filter.

(8) $F_s \cap J$ is the smallest J-semi maximal filter containing $F \cap J$.

(9) $J \cap Rad(L)$ is the smallest J-semi maximal filter.

(10) If F is a J-semi maximal filter and a K-semi maximal filter, then F is a $(J \vee K)-$semi maximal filter.

PROOF: (1) Let F be a J-semi maximal filter. By Lemma 3.8, we have $F \subseteq F_s$ and implies that $F_s \cap (F \vee J) = (F_s \cap F) \vee (F_s \cap J)$. Since F is a J-semi maximal filter, so $F_s \cap J \subseteq F$. Thus, we obtain $F_s \cap (F \vee J) = (F_s \cap F) \vee (F_s \cap J) \subseteq F \vee F = F$. It follows from Proposition 3.10, that F is a $(F \vee J)-$semi maximal filter. Conversely, it is obvious.

(2) Assume that F is a J-semi maximal filter, hence for all $a \in F$, $M_a \cap J \subseteq F$. Since J is a semi maximal filter and $a \in F \subseteq J$, so we have $M_a \subseteq J$. We deduce that $M_a \cap J = M_a \subseteq F$. So F is a semi maximal filter. As every semi maximal filter is a J-semi maximal filter, conversely is clear.

(3) First, we show that $F \cap J$ is a semi maximal filter. Since J is a $F-$semi maximal filter, so by Proposition 3.11, we deduce that $F \cap J$ is a $F-$semi maximal filter. By hypothesis, F is a semi maximal filter and $F \cap J \subseteq F$, hence by part (2) of this proposition, we conclude that $F \cap J$ is a semi maximal filter.

Conversely, it follows from Proposition 3.11 and part (2) of Proposition 3.13.

(4) By Proposition 3.11, it is clear.

(5) M is maximal filter and F is a semi maximal filter, hence M is a semi maximal filter so $F \cap M$ is a semi maximal filter. Now suppose that $F \cap M$ is a semi maximal filter. If $F \subseteq M$, then $F = F \cap M$ is a semi maximal filter. Assume that $F \nsubseteq M$, since $F \cap M$ is a semi maximal filter and M is a semi maximal filter, so by this proposition part (3), we conclude that F is a $M-$semi maximal filter. By part (1), we get F is a $(F \vee M)-$semi maximal filter. Hence F is a $M-$semi maximal filter. Therefore F is a semi maximal filter.

(6) By hypothesis and Proposition 3.10, we have $F_s \cap J \subseteq F$ and $J_s \cap K \subseteq J$. Also, $F_s \subseteq J_s$. Hence we get $F_s \cap K = F_s \cap J_s \cap K \subseteq F_s \cap J \subseteq F$. It means that F is a K-semi maximal filter.

(7) It follows from Proposition 3.10, that $F_s \cap J \subseteq F$ and $H_s \cap K \subseteq H$. So, we have

$$(F \cap H)_s \cap (J \cap K) = F_s \cap H_s \cap J \cap K = (F_s \cap J) \cap (H_s \cap K) \subseteq F \cap H.$$

(8) We know that, $F \subseteq F_s$, so $F \cap J \subseteq F_s \cap J$. It is clear that $(F_s)_s = F_s$, hence $(F_s \cap J)_s \cap J \subseteq F_s \cap J$ and we conclude that $F_s \cap J$ is a J-semi maximal filter. Now, suppose that there exists a filter K of L such that it

is a J-semi maximal filter contains $F \cap J$. Hence $F_s \cap J = F_s \cap J_s \cap J = (F \cap J)_s \cap J \subseteq K_s \cap J \subseteq K$.

(9) Suppose that F is a J-semi maximal filter, thus $J \cap Rad(L) \subseteq J \cap F_s \subseteq F$.

(10) We have $F_s \cap (J \vee K) = (F_s \cap J) \vee (F_s \cap K) \subseteq F$. Hence F is a $(J \vee K)-$semi maximal filter. □

Corollary 3.14. Let J be a semi factor of a filter F. Then $F \vee J$ is a semi factor of F containing F.

Proof: Since F is a J-semi maximal filter, so by Proposition 3.13 part(1), F is a $(F \vee J)-$semi maximal filter. Thus, $F \vee J$ is a semi factor of F. Obviously, $F \subseteq F \vee J$. □

Corollary 3.15. Let F be a filter of L. Consider,

$$\Omega = \{J \in F(L) \mid J \ is \ a \ semi \ factor \ of \ F\}.$$

(1) $(\Omega, \subseteq)$ is a poset.

(2) $(\Omega, \vee, \wedge)$ is a lattice.

Proposition 3.16. Let F, J be filters and P, Q be prime filters of L. Then

(1) If $F \cap P$ is a J-semi maximal filter (NJ-semi maximal filter), then either F is a J-semi maximal filter (NJ-semi maximal filter) or P.

(2) If $P \cap Q$ is a NJ-semi maximal filter, then either P is a semi maximal filter or Q.

Proof: (1) Obviously, if $F \subseteq P$, then F is a J-semi maximal filter (NJ-semi maximal filter.) Assume that $F \nsubseteq P$, hence there exists $a \in F \setminus P$ and for all $b \in P$, we have $a \vee b \in F \cap P$. Since $F \cap P$ is a J-semi maximal filter, so $M_{(a \vee b)} \cap J \subseteq F \cap P$. On the other hand, $M_a \cap M_b \cap J \subseteq P$ and $M_a \nsubseteq P$. By Lemma M, we conclude that $M_b \cap J \subseteq P$, for all $b \in P$. Thus, P is a J-semi maximal filter. Since $F \cap P$ is a NJ-semi maximal filter, so there is a filter J such that $J \nsubseteq F \cap P$ and $(F \cap P)_s \cap J \subseteq F \cap P$. Hence

$(F_s \cap J) \cap P_s \subseteq P$, by Lemma M, we have $F_s \cap J \subseteq P$ or $P_s \subseteq P$. We consider two cases:

Case 1: If $F_s \cap J \subseteq P$, then $J \subseteq P$. (Since $F \nsubseteq P$, so $F_s \nsubseteq P$.) On the other hand, $J \subseteq P \subseteq P_s$, hence $J \cap P_s = J$. Since $F \cap P$ is J-semi maximal filter, so $F_s \cap J = F_s \cap J \cap P_s = (F \cap P)_s \cap J \subseteq F \cap P \subseteq F$ implies that F is a J-semi maximal filter. Now, we show that $J \nsubseteq F$. We have $J \subseteq P$, if $J \subseteq F$, then $J \subseteq F \cap P$, which is a contradiction. Therefore F is a NJ-semi maximal filter.

Case 2: If $P_s \subseteq P$, then $P_s = P$ and P is a semi maximal filter. It follows from Proposition 3.6, that P is a NJ-semi maximal filter.

(2) It follows from Proposition 3.6 and part (1). □

Proposition 3.17. Let F and J be filters of L such that F is a J-semi maximal filter. Then $(F : x)$ is a J-semi maximal filter of L, for all $x \in L \backslash F$.

Proof: Following Proposition 3.10, assume that $M_b \subseteq M_a$ and $a \in (F : x)$ and $b \in J$. Since F is a J-semi maximal filter and $a \vee x \in F$, so we get $M_{(a\vee x)} \cap J \subseteq F$. Also, $b \leq b \vee x$ and J is a filter, hence $b \vee x \in J$. It is clear that by Lemma 3.3 part (2), we have $M_{(b\vee x)} \cap J = M_b \cap M_x \cap J \subseteq M_a \cap M_x \cap J = M_{(a\vee x)} \cap J \subseteq F$ and $b \vee x \in M_{(b\vee x)} \cap J$. Thus, we conclude that $b \vee x \in F$ and $b \in (F : x)$. □

Corollary 3.18. Let F, J be filters of L such that F be a J-semi maximal filter and X be a subset of L such that $X \nsubseteq F$. Then $(F : X) = \{a \in L | a \vee x \in F, for\ all\ x \in X\}$ is a J-semi maximal filter.

In the following theorem, we explain that if J is not a semi maximal filter and $Rad(L)$ does not contain J, then there is always a non-trivial J-semi maximal filter.

Theorem 3.19. *Let J be a filter of L such that $J \nsubseteq Rad(L)$ and it is not a semi maximal filter. Then there exists a filter F such that $F \subsetneqq J$ and is a J-semi maximal filter but not a semi maximal filter.*

Proof: Since $J \nsubseteq Rad(L)$, so there exists maximal filter M in L such that $J \nsubseteq M$. Consider $F := J \cap M$, it is clear that $F \subsetneqq J$. (If $F = J$, then $J \cap M = J$ and $J \subseteq M$, which is a contradiction). M is a maximal filter, so it is a semi maximal filter. Also, it is a J-semi maximal filter and by Proposition 3.11, $J \cap M$ is a J-semi maximal filter.Therefore, F is a J-semi maximal filter. We know that M is maximal and J is not a semi maximal filter, so by Proposition 3.13, part (5) we conclude that F is not a semi maximal filter. □

To summarize, if $J \nsubseteq Rad(L)$, then there are many J-semi maximal filters in L. Consider a maximal filter K such that $J \nsubseteq K$. It follows that K is a semi maximal filter. Thus, for each $k \in K$, $M_k \cap J$ is a J-semi maximal filter and, in fact, an NJ-semi maximal filter.

However, if $J \subseteq Rad(L)$, then only trivial J-semi maximal filters exist. Specifically, if K is a non-trivial J-semi maximal filter, then $J \nsubseteq K$, and for all $a \in K$, $M_a \cap J \subseteq K$. On the other hand, $J = Rad(L) \cap J \subseteq M_a \cap J \subseteq K$, which contradicts $J \nsubseteq K$.

As a result, $J \nsubseteq Rad(L)$ if and only if there exists a non-trivial J-semi maximal filter. Now, we are going to study J-semi maximal filters in Hyperarchimedean and semi-simple $BL-$algebras and get some results.
Note: In Example 3.2(2), the F_0 is not a J-semi maximal filter where $J = \{1, b\}$ (Since $M_1 \cap J = F_5 \cap J = \{1, b\} \nsubseteq F_5$). Next, we characterize $BL-$algebras in which the filter $\{1\}$ is a J-semi maximal filter.

Proposition 3.20. Let L be Hyperarchimedean and $J \in F(L)$. Then every proper filter is a J-semi maximal filter.

Proof: Suppose that F is a proper filter. for all $a \in F$;

$$\{P \in Spec(L) \mid F \subseteq P\} \subseteq \{P \in Spec(L) \mid a \in P\}$$
$$\bigcap\{P \in Spec(L) \mid a \in P\} \subseteq \bigcap\{P \in Spec(L) \mid F \subseteq P\}.$$

By Theorem 2.13 and by Corollary 2.8,

$$M_a = \bigcap\{\ P \in Spec(L) \mid a \in P\} \subseteq \bigcap\{\ P \in Spec(L) \mid F \subseteq P\} = F.$$

We always have $M_a \cap J \subseteq M_a$, for all $a \in F$. On the other hand, $M_a \cap J \subseteq M_a \subseteq F$ for all $a \in F$. Therefore, F is a J-semi maximal filter. □

Theorem 3.21. *Let L be a semi-simple $BL-$algebra, X be a subset of L and J be a filter of L.Then the following sentences hold:*

(1) The filter $\{1\}$ is a J-semi maximal filter. In particular, if $J \neq \{1\}$ is a filter, then the filter $\{1\}$ is a NJ-semi maximal filter.

(2) $Co - Ann(X)$ is a J-semi maximal filter.

(3) Every minimal prime filter of L is a J-semi maximal filter.

Proof: (1) Since L is semi-simple, so $M_1 = Rad(L) = \{1\}$. For each filter J of L, we have $M_1 \cap J \subseteq \{1\}$, hence the filter $\{1\}$ is a J-semi maximal filter.

(2) By part (1) of this proposition, filter $\{1\}$ is a J-semi maximal filter. Put $F := \{1\}$. Now by Corollary 3.18, $(\{1\} : X)$ is a J-semi maximal filter. Obviously, $(\{1\} : X) = Co - Ann(X)$. Thus $Co - Ann(X)$ is a J-semi maximal filter.

(3) In Proposition 3.4 (1), take $F = \{1\}$ and by part (1) of this proposition, we conclude that every minimal prime filter is a J-semi maximal filter. □

Theorem 3.22. *Let L be a semi-simple $BL-$algebra and F be a filter of L such that $Co - Ann(F) \neq \{1\}$. Then there is a filter J of L such that F is a NJ-semi maximal filter of L.*

Proof: Since L is semi-simple, so $Rad(L) = \{1\}$. We first prove that $M_a \cap Co - Ann(F) \subseteq Rad(L)$, for all $a \in F$. Let $b \in M_a \cap Co - Ann(F)$. Then $b \in M_a$ and $b \vee a = 1$. Hence $M_b \subseteq M_a$. We get $b \in M_b = M_b \cap M_a = M_{(a \vee b)} = M_1 = Rad(L)$. Now, take $J = Co - Ann(F)$. Moreover, $M_a \cap Co - Ann(F) = Rad(L) = \{1\} \subseteq F$, for all $a \in F$. Thus F is a J-semi maximal filter.

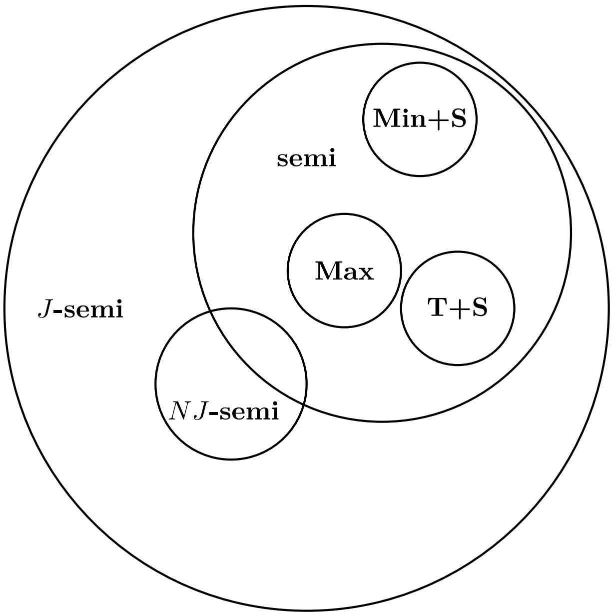
Now, we show that $J \nsubseteq F$. By contrary, suppose that $Co - Ann(F) \subseteq F$ and let $1 \neq a \in Co - Ann(F)$. Hence $a \in F$ and $a \vee t = 1$ for all $t \in F$. Hence $a \vee a = 1$, as a result $a = 1$, which is a contradiction. Therefore F is a NJ-semi maximal filter. □

Corollary 3.23. Let $F, J \in F(L)$. Then every filter F of a semi-simple $BL-$algebra L is a J-semi maximal filter or $Co - Ann(F) = \{1\}$.

Proof: First, assume that $Co - Ann(F) \neq \{1\}$. By Proposition 3.22, we conclude that there exists a filter J such that F is a NJ-semi maximal filter. Now, suppose that F is not a NJ-semi maximal filter in L. Again by Proposition 3.22, we have L is not semi-simple or $Co - Ann(F) = \{1\}$. By hypothesis L is semi-simple, hence $Co - Ann(F) = \{1\}$. □

4. Conclusions

The defined J-semi maximal filters exhibit a higher level of generality compared to semi maximal filters. We have shown that every semi maximal filter is a J-semi filter, and that every NJ-semi filter is also a J-semi filter. Furthermore, it has been proven that every prime NJ-semi filter is a semi maximal filter. We have established that the set of all J-semi filters forms a lattice. A detailed investigation of these filters has been carried out within various classes of BL-algebras. It has been proven that in any BL-chain, each proper NJ-semi filter is a semi maximal filter. Moreover, it has been demonstrated that if the intersection of two prime filters is an NJ-semi filter, then at least one of them must be a semi maximal filter. Additionally, we have shown that a prime filter is a J-semi filter if and only if it is either a semi maximal filter or contains the filter J. Finally, we concluded that there exists a minimal prime filter above every NJ-semi filter.
Next, we show a summary of the relationship between semi maximal filters in a diagram.



S:= Semi-Simple, Max:= Maximal filter, Min:= Minimal prime filter, T:=Trivial filter, J-semi:=J-semi maximal, NJ-semi:=J-semi maximal, semi:=semi maximal filter

Figure 3: The relationships between J-semi maximal filters and the other filters in BL-algebras.

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TOPOLOGY ON EQUALITY ALGEBRAS

Abstract

In this paper, by special upsets on equality algebras, we construct a topology on bounded equality algebras and investigate some of their topological properties, such as Hausdorff, T_0-space, T_1-space and disconnected. In addition, we express the relation between closed and compact sets in this topology. Moreover, by considering the binary operation $\twoheadrightarrow$ and constructing a topology on the bounded equality algebra $\mathbf{E}$, we introduce the notion of semi-topological algebra and prove that any involutive equality algebra is a right semi-topological algebra and by some conditions it can be a semi-$\curlywedge$-topological algebra. Also, we show that it is not necessarily a left semi-topological algebra. Finally, we investigate converse image, product and quotient topology on equality algebra and show that under what condition we can make finer topology.

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1. Introduction

Algebra and topology are indeed fundamental areas of mathematics that complement each other in many ways. Algebra deals with the study of mathematical operations and structures, while topology focuses on properties that are preserved under continuous transformations, such as bending and stretching. Together, these two fields provide a powerful toolkit for understanding and solving mathematical problems. They are used together in the field of mathematics known as algebraic topology. Algebraic topology is a branch of mathematics that uses algebraic tools to study topological spaces. Topological spaces are mathematical structures that capture the notion of closeness or continuity, while algebraic structures provide a way to analyze these spaces using algebraic techniques. By combining algebraic and topological methods, algebraic topology allows mathematicians to study properties of spaces that are not easily accessible through purely topological or algebraic means.

Topology on logical algebraic structures is a fascinating field that explores the relationship between the algebraic properties of structures like lattices, Boolean algebras, etc., and their topological properties. It involves studying concepts like open and closed sets, convergence, continuity, and compactness within the context of these structures.

The motivation for introducing equality algebras came from EQ-algebras which are defined by Novák in [17]. In EQ-algebras, compared to equality algebras, there is an additional operation $\otimes$, called product, which is very loosely related to the other operations. Therefore, there might not exist deep algebraic characterizations of EQ-algebras, and intention was to define a structure similar to EQ-algebras but without the product. This new logical algebra, the equality algebra, has two connectives, a meet operation and an equivalence, and a constant. Equality algebra is introduced by Jenei [8]. Since equality algebra can be a good alternative to possible algebraic semantics for fuzzy type theory, the study of equality algebra is very valuable. Recently, many mathematicians have studied this algebraic structures in various fields. For instance, the relation between equality algebra with other algebraic structure, filters, ideals, annihilators and co-annihilators are investigated that continued in [11, 12, 13, 5, 6, 20, 15, 18, 19].

Lately a lot of mathematicians have been looking into the idea of topology on logical algebraic structures specifically BL-algebras. For instance Borzooei in [4] looked at how separation axioms are connected to (semi)topological quotient BL-algebras. They explored when a (semi)topological quotient BL-algebra can be a T_1-space or Hausdorff or regular or normal. In [1], the authors introduced the concept of quasi-filter neighborhoods in (semi)topological BL-algebras and discussed some of their characteristics through statements and proofs. Furthermore by applying the idea of quasi-filter they discovered certain situations in which a BL-algebra can be turned into a metrizable structure. To learn more about this topic you can check out the articles by [9, 10, 2, 3, 7, 21, 22, 23].

In this paper, we introduce a special subset of equality algebra which is upset and by using these upsets we construct a topology on bounded equality algebras and we investigate some of their topological properties, such as some types of topological space (Hausdorff, T_0-space and T_1-space) and disconnected. In addition, we express the relation between closed and compact sets in this topology. Moreover, by considering the binary operation $\twoheadrightarrow$ and the constructed topology on the bounded equality algebra $\mathbf{E}$, we introduce the notion of a semi-topological algebra and we prove that any involutive equality algebra is a right semi-topological algebra and by some conditions it can be a semi-$\curlywedge$-topological algebra. Also, we show that it is not necessarily a left semi-topological algebra. Finally, we investigate converse image, product and quotient topology on equality algebra and show that under what condition we can make a finer topology.

2. Preliminaries

In this section, we have some basic concepts about equality algebra that will be used in later sections. We only remind some definitions and results.

Definition 2.1 ([8]). An algebraic structure $(\mathbf{E}; \curlywedge, \sim, 1)$ of type $(2,2,0)$ is called an *equality algebra* if it satisfies the following conditions, for all $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mathbf{E}$,

(E1) $(\mathbf{E}, \curlywedge, 1)$ is a commutative idempotent integral monoid,

(E2) $\mathfrak{x} \sim \mathfrak{y} = \mathfrak{y} \sim \mathfrak{x}$,

(E3) $\mathfrak{x} \sim \mathfrak{x} = 1$,

(E4) $\mathfrak{x} \sim 1 = \mathfrak{x}$,

(E5) $\mathfrak{x} \preccurlyeq \mathfrak{y} \preccurlyeq \mathfrak{z}$ implies $\mathfrak{x} \sim \mathfrak{z} \preccurlyeq \mathfrak{y} \sim \mathfrak{z}$ and $\mathfrak{x} \sim \mathfrak{z} \preccurlyeq \mathfrak{x} \sim \mathfrak{y}$,

(E6) $\mathfrak{x} \sim \mathfrak{y} \preccurlyeq (\mathfrak{x} \curlywedge \mathfrak{z}) \sim (\mathfrak{y} \curlywedge \mathfrak{z})$,

(E7) $\mathfrak{x} \sim \mathfrak{y} \preccurlyeq (\mathfrak{x} \sim \mathfrak{z}) \sim (\mathfrak{y} \sim \mathfrak{z})$.

An equality algebra $(\mathbf{E}, \preccurlyeq)$ with a binary relation defined by "$\mathfrak{x} \preccurlyeq \mathfrak{y}$ if and only if $\mathfrak{x} \curlywedge \mathfrak{y} = \mathfrak{x}$" is a poset. Also, we define the operation "$\twoheadrightarrow$" on $\mathbf{E}$ as: $\mathfrak{x} \twoheadrightarrow \mathfrak{y} = \mathfrak{x} \sim (\mathfrak{x} \curlywedge \mathfrak{y})$.

The set $\emptyset \neq \mathcal{X} \subseteq \mathbf{E}$ is called a *subequality algebra* if it is closed under all operations on $\mathbf{E}$. It means that for any $\mathfrak{x}, \mathfrak{y} \in \mathcal{X}$, $\mathfrak{x} \curlywedge \mathfrak{y}, \mathfrak{x} \sim \mathfrak{y} \in \mathcal{X}$.

An equality algebra $(\mathbf{E}, \sim, \curlywedge, 0, 1)$ is called a *bounded equality algebra* if for every $\mathfrak{x} \in \mathbf{E}$, $0 \preccurlyeq \mathfrak{x}$. In a bounded equality algebra $\mathbf{E}$, we define the negation operation "$'$" by: $\mathfrak{x}' = \mathfrak{x} \twoheadrightarrow 0 = \mathfrak{x} \sim 0$, for all $\mathfrak{x} \in \mathbf{E}$. If for any $\mathfrak{x} \in \mathbf{E}$, $\mathfrak{x}'' = \mathfrak{x}$, then $\mathbf{E}$ is called an *involutive equality algebra*.(See [8])

Note: From now on, let $(\mathbf{E}; \curlywedge, \sim, 1)$ or $\mathbf{E}$ be an equality algebra, unless otherwise stated.

Proposition 2.2 ([8, 24]). The following conditions hold, for any $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mathbf{E}$:

(i) $\mathfrak{x} \twoheadrightarrow \mathfrak{y} = 1$ if and only if $\mathfrak{x} \preccurlyeq \mathfrak{y}$,

(ii) $1 \twoheadrightarrow \mathfrak{x} = \mathfrak{x}$, $\mathfrak{x} \twoheadrightarrow 1 = 1$, and $\mathfrak{x} \twoheadrightarrow \mathfrak{x} = 1$,

(iii) $\mathfrak{x} \preccurlyeq \mathfrak{y} \twoheadrightarrow \mathfrak{x}$,

(iv) $\mathfrak{x} \preccurlyeq (\mathfrak{x} \twoheadrightarrow \mathfrak{y}) \twoheadrightarrow \mathfrak{y}$,

(v) $\mathfrak{x} \twoheadrightarrow (\mathfrak{y} \twoheadrightarrow \mathfrak{z}) = \mathfrak{y} \twoheadrightarrow (\mathfrak{x} \twoheadrightarrow \mathfrak{z})$,

(vi) $\mathfrak{x} \preccurlyeq \mathfrak{y}$ implies $\mathfrak{y} \twoheadrightarrow \mathfrak{z} \preccurlyeq \mathfrak{x} \twoheadrightarrow \mathfrak{z}$ and $\mathfrak{z} \twoheadrightarrow \mathfrak{x} \preccurlyeq \mathfrak{z} \twoheadrightarrow \mathfrak{y}$,

(vii) $\mathfrak{x} \twoheadrightarrow \mathfrak{y} = \mathfrak{x} \twoheadrightarrow (\mathfrak{x} \curlywedge \mathfrak{y})$,

(viii) If $\mathbf{E}$ is a lattice, then $(\mathfrak{x} \curlyvee \mathfrak{y}) \twoheadrightarrow \mathfrak{z} = (\mathfrak{x} \twoheadrightarrow \mathfrak{z}) \curlywedge (\mathfrak{y} \twoheadrightarrow \mathfrak{z})$.

(ix) $((\mathfrak{x} \twoheadrightarrow \mathfrak{y}) \twoheadrightarrow \mathfrak{y}) \twoheadrightarrow \mathfrak{y} = \mathfrak{x} \twoheadrightarrow \mathfrak{y}$,

(x) $\mathfrak{x} \sim \mathfrak{y} \preccurlyeq \mathfrak{x} \twoheadrightarrow \mathfrak{y}$,

(xi) $\mathfrak{x} \twoheadrightarrow \mathfrak{y} \preccurlyeq (\mathfrak{x} \curlywedge \mathfrak{z}) \twoheadrightarrow (\mathfrak{y} \curlywedge \mathfrak{z})$,

(xii) $\mathfrak{x} \twoheadrightarrow \mathfrak{y} \preccurlyeq (\mathfrak{z} \twoheadrightarrow \mathfrak{x}) \twoheadrightarrow (\mathfrak{z} \twoheadrightarrow \mathfrak{y})$ and $\mathfrak{x} \twoheadrightarrow \mathfrak{y} \preccurlyeq (\mathfrak{y} \twoheadrightarrow \mathfrak{z}) \twoheadrightarrow (\mathfrak{x} \twoheadrightarrow \mathfrak{z})$.

Definition 2.3.

(i) A lattice equality algebra is an equality algebra which is lattice.

(ii) If for any $\mathfrak{x}, \mathfrak{y} \in \mathbf{E}$, 1 is the unique upper bound of the set $\{\mathfrak{x} \twoheadrightarrow \mathfrak{y},\ \mathfrak{y} \twoheadrightarrow \mathfrak{x}\}$, then $\mathbf{E}$ is called *prelinear*. Also, any prelinear equality algebra is a distributive lattice. ([24, Theorem 3.8])

Definition 2.4 ([8]). Let $\mathbf{F}$ be a non-empty subset of $\mathbf{E}$. Then $\mathbf{F}$ is called a *filter* of $\mathbf{E}$ if for all $\mathfrak{x}, \mathfrak{y} \in \mathbf{E}$, we have

(i) $\mathfrak{x} \in \mathbf{F}$ and $\mathfrak{x} \preccurlyeq \mathfrak{y}$ imply $\mathfrak{y} \in \mathbf{F}$,

(ii) $\mathfrak{x} \in \mathbf{F}$ and $\mathfrak{x} \sim \mathfrak{y} \in \mathbf{F}$ imply $\mathfrak{y} \in \mathbf{F}$.

The set of all filters of $\mathbf{E}$ is denoted by $\mathcal{F}(\mathbf{E})$.

Proposition 2.5 ([8]). Let $\emptyset \neq \mathbf{F} \subseteq \mathbf{E}$. Then $\mathbf{F} \in \mathcal{F}(\mathbf{E})$ if and only if, for all $\mathfrak{x}, \mathfrak{y} \in \mathbf{E}$, $1 \in \mathbf{F}$, and if $\mathfrak{x} \in \mathbf{F}$ and $\mathfrak{x} \twoheadrightarrow \mathfrak{y} \in \mathbf{F}$, then $\mathfrak{y} \in \mathbf{F}$.

Clearly, for any $\mathbf{F} \in \mathcal{F}(\mathbf{E})$, $1 \in \mathbf{F}$ and $\mathbf{F}$ is called a *proper filter* of $\mathbf{E}$ if $\mathbf{F} \neq \mathbf{E}$. Clearly, if $\mathbf{E}$ is bounded, then $\mathbf{F} \in \mathcal{F}(\mathbf{E})$ is proper if and only if it is not containing 0.

Let $\mathbf{F} \in \mathcal{F}(\mathbf{E})$. Define the relation $\theta_{\mathbf{F}}$ on $\mathbf{E}$ by

$$(\mathfrak{x}, \mathfrak{y}) \in \theta_{\mathbf{F}} \quad \text{if and only if} \quad \{\mathfrak{x} \twoheadrightarrow \mathfrak{y}, \mathfrak{y} \twoheadrightarrow \mathfrak{x}\} \subseteq \mathbf{F}.$$

Let $\dfrac{\mathbf{E}}{\mathbf{F}} = \{[\mathfrak{x}] | \ \mathfrak{x} \in \mathbf{E}\}$, where $[\mathfrak{x}] = \{\mathfrak{y} \in \mathbf{E} \mid \ (\mathfrak{x}, \mathfrak{y}) \in \theta_{\mathbf{F}}\}$. Then define the binary relation $\preccurlyeq_{\mathbf{F}}$ on $\dfrac{\mathbf{E}}{\mathbf{F}}$ by:

$$[\mathfrak{x}] \preccurlyeq_{\mathbf{F}} [\mathfrak{y}] \quad \text{if and only if} \quad \mathfrak{x} \twoheadrightarrow \mathfrak{y} \in \mathbf{F},$$

which is an order relation on $\dfrac{\mathbf{E}}{\mathbf{F}}$. For any $\mathfrak{x}, \mathfrak{y} \in \mathbf{E}$, define

$$[\mathfrak{x}] \sim_{\mathbf{F}} [\mathfrak{y}] = [\mathfrak{x} \sim \mathfrak{y}] \quad \text{and} \quad [\mathfrak{x}] \curlywedge_{\mathbf{F}} [\mathfrak{y}] = [\mathfrak{x} \curlywedge \mathfrak{y}].$$

Then $\left(\dfrac{\mathbf{E}}{\mathbf{F}}, \sim_{\mathbf{F}}, \curlywedge_{\mathbf{F}}, 1_{\frac{\mathbf{E}}{\mathbf{F}}}\right)$ is called a *quotient equality algebra* and denoted by $\dfrac{\mathbf{E}}{\mathbf{F}}$, where $1_{\frac{\mathbf{E}}{\mathbf{F}}} = [1]_{\theta_{\mathbf{F}}} = \mathbf{F}$.

Definition 2.6 ([16]). Let $\emptyset \neq \mathbf{A} \subseteq \mathbf{E}$. The smallest filter of $\mathbf{E}$ containing $\mathbf{A}$ is called *the generated filter by* $\mathbf{A}$ *in* $\mathbf{E}$ which is denoted by $\langle \mathbf{A} \rangle$. Indeed, $\langle \mathbf{A} \rangle = \bigcap\limits_{\mathbf{A} \subseteq \mathbf{F} \in \mathcal{F}(\mathbf{E})} \mathbf{F}$.

Proposition 2.7 ([16]). Let $\emptyset \neq \mathbf{A} \subseteq \mathbf{E}$. Then

$$\langle \mathbf{A} \rangle = \{\mathfrak{x} \in \mathbf{E} \mid \mathfrak{a}_1 \twoheadrightarrow (\mathfrak{a}_2 \twoheadrightarrow (\cdots \twoheadrightarrow (\mathfrak{a}_n \twoheadrightarrow \mathfrak{x}) \cdots)) = 1, \\ \text{for some } n \in \mathbb{N} \text{ and } \mathfrak{a}_1, \cdots, \mathfrak{a}_n \in \mathbf{A}\}.$$

In particular, for any element $\mathfrak{a} \in \mathbf{E}$, we have $\langle \mathfrak{a} \rangle = \{\mathfrak{x} \in \mathbf{E} \mid \mathfrak{a} \twoheadrightarrow^n \mathfrak{x} = 1, \text{ for some } n \in \mathbb{N}\}$, where $\mathfrak{x} \twoheadrightarrow^0 \mathfrak{y} = \mathfrak{y}$ and $\mathfrak{x} \twoheadrightarrow^n \mathfrak{y} = \mathfrak{x} \twoheadrightarrow (\mathfrak{x} \twoheadrightarrow^{n-1} \mathfrak{y})$. If $\mathbf{F} \in \mathcal{F}(\mathbf{E})$ and $\mathfrak{a} \in \mathbf{E} \setminus \mathbf{F}$, then

$$\langle \mathbf{F} \cup \{\mathfrak{a}\} \rangle = \{\mathfrak{x} \in \mathbf{E} \mid \mathfrak{a} \twoheadrightarrow^n \mathfrak{x} \in \mathbf{F}, \text{ for some } n \in \mathbb{N}\}.$$

If $\mathbf{F}, \mathbf{G} \in \mathcal{F}(\mathbf{E})$, then

$$\begin{aligned}\langle \mathbf{F} \cup \mathbf{G} \rangle &= \{\mathfrak{x} \in \mathbf{E} \mid \ \mathfrak{g} \twoheadrightarrow \mathfrak{x} \in \mathbf{F}, \text{ for some } \mathfrak{g} \in \mathbf{G}\} \\ &= \{\mathfrak{x} \in \mathbf{E} \mid \ \mathfrak{f} \twoheadrightarrow \mathfrak{x} \in \mathbf{G}, \text{ for some } \mathfrak{f} \in \mathbf{F}\}.\end{aligned}$$

3. Topology on equality algebras

In this section, we introduce a special subset of equality algebra which is upset and by using this upsets we construct a topology on bounded equality algebras and investigate their some types of topological space properties, such as Hausdorff, T_0-space, T_1-space and disconnected. (For studying more details in topology, we refer the reader to [14].)

Note: Let $(\mathcal{X}; \preccurlyeq)$ be a poset. Then for any $\mathcal{Y} \subseteq \mathcal{X}$, we define

$$\uparrow \mathcal{Y} = \{\mathfrak{x} \in \mathcal{X} \mid \exists\, \mathfrak{y} \in \mathcal{Y} \text{ s.t } \ \mathfrak{y} \preccurlyeq \mathfrak{x}\}.$$

The set $\mathcal{Y}$ is called *upset* if $\uparrow \mathcal{Y} = \mathcal{Y}$. The set of all upsets of $\mathcal{X}$ is denoted by $\Gamma(\mathcal{X})$.

An upset $\mathcal{Y}$ is called *finitely generated* if there exists $n \in \mathbb{N}$ such that $\mathcal{Y} =\uparrow \{\mathfrak{y}_1, \mathfrak{y}_2, \ldots, \mathfrak{y}_n\}$, for some $\mathfrak{y}_1, \mathfrak{y}_2, \ldots, \mathfrak{y}_n \in \mathcal{Y}$. Obviously, $\mathcal{Y} \subseteq\uparrow \mathcal{Y}$, for any $\mathcal{Y} \subseteq \mathcal{X}$.

PROPOSITION 3.1. Suppose $(\mathcal{X}; \preccurlyeq)$ is a poset and $\{\mathcal{Y}_i\}_{i\in I} \subseteq P(\mathcal{X})$. Then

(i) If $\mathcal{Y}_i \subseteq \mathcal{Y}_j$, then $\uparrow \mathcal{Y}_i \subseteq\uparrow \mathcal{Y}_j$.

(ii) If $\{\mathcal{Y}_i\}_{i\in I} \subseteq \Gamma(\mathcal{X})$, then $\bigcup\limits_{i\in I} \mathcal{Y}_i \in \Gamma(\mathcal{X})$.

(iii) If $\{\mathcal{Y}_i\}_{i\in I} \subseteq \Gamma(\mathcal{X})$, then $\bigcap\limits_{i\in I} \mathcal{Y}_i \in \Gamma(\mathcal{X})$.

(iv) If $\mathcal{Y}_i \in \Gamma(\mathcal{X})$, then $\mathcal{Y}_i = \bigcup\limits_{\mathfrak{x}\in\mathcal{Y}_i} \uparrow \mathfrak{x}$.

PROOF: (i) The proof is clear.
(ii) Obviously, $\bigcup\limits_{i\in I} \mathcal{Y}_i \subseteq\uparrow \bigcup\limits_{i\in I} \mathcal{Y}_i$. Assume $\mathfrak{x} \in\uparrow \bigcup\limits_{i\in I} \mathcal{Y}_i$. Then there exists $\mathfrak{y} \in \bigcup\limits_{i\in I} \mathcal{Y}_i$ such that $\mathfrak{y} \preccurlyeq \mathfrak{x}$. Thus, there exists $i \in I$ such that $\mathfrak{y} \in \mathcal{Y}_i$. Since

$\mathcal{Y}_i$ is upset, we get $\mathfrak{x} \in\uparrow \mathcal{Y}_i = \mathcal{Y}_i$. Hence, $\mathfrak{x} \in \bigcup_{i\in I} \mathcal{Y}_i$. Therefore, $\bigcup_{i\in I} \mathcal{Y}_i \in \Gamma(\mathcal{X})$.
(iii) The proof is similar to the proof of (ii).
(iv) Let $\mathfrak{y} \in \mathcal{Y}_i$. Since $\mathfrak{y} \preccurlyeq \mathfrak{y}$, we have $\mathfrak{y} \in\uparrow \mathfrak{y} \subseteq \bigcup_{\mathfrak{x}\in\mathcal{Y}_i} \uparrow \mathfrak{x}$. Then $\mathcal{Y}_i \subseteq \bigcup_{\mathfrak{x}\in\mathcal{Y}_i} \uparrow \mathfrak{x}$. Furthermore, if $\mathfrak{t} \in \bigcup_{\mathfrak{x}\in\mathcal{Y}_i} \uparrow \mathfrak{x}$, then there is $\mathfrak{x} \in \mathcal{Y}_i$ such that $\mathfrak{t} \in\uparrow \mathfrak{x}$. Then $\mathfrak{x} \preccurlyeq \mathfrak{t}$, and so $\mathfrak{t} \in\uparrow \mathcal{Y}_i$. Since $\mathcal{Y}_i \in \Gamma(\mathcal{X})$, we have $\mathfrak{t} \in \mathcal{Y}_i$. Hence, $\bigcup_{\mathfrak{x}\in\mathcal{Y}_i} \uparrow \mathfrak{x} \subseteq \mathcal{Y}_i$. Therefore, $\mathcal{Y}_i = \bigcup_{\mathfrak{x}\in\mathcal{Y}_i} \uparrow \mathfrak{x}$. □

Note: From now on, $(\mathbf{E}, \curlywedge, \sim, 0, 1)$ or $\mathbf{E}$ for short is a bounded equality algebra, unless otherwise stated.

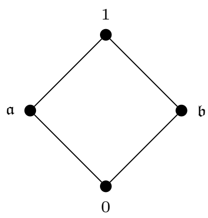
DEFINITION 3.2. Let $\mathfrak{a} \in \mathbf{E}$ and $\mathcal{X} \in \Gamma(\mathbf{E})$. Then, we define

$$\mathcal{DN}_{\mathfrak{a}}(\mathcal{X}) := \{\mathfrak{x} \in \mathbf{E} \mid \exists n \in \mathbb{N} \text{ s.t } \mathfrak{a} \overset{n}{\twoheadrightarrow} \mathfrak{x} \in \mathcal{X}\},$$

where

$$\mathfrak{a} \overset{1}{\twoheadrightarrow} \mathfrak{x} := (\mathfrak{a} \twoheadrightarrow \mathfrak{x})'',$$
$$\mathfrak{a} \overset{2}{\twoheadrightarrow} \mathfrak{x} := (\mathfrak{a} \twoheadrightarrow (\mathfrak{a} \twoheadrightarrow \mathfrak{x})'')'',$$
$$\mathfrak{a} \overset{n}{\twoheadrightarrow} \mathfrak{x} := \underbrace{(\mathfrak{a} \twoheadrightarrow \ldots (\mathfrak{a}}_{\text{n-times}} \twoheadrightarrow \mathfrak{x})'' \ldots)''.$$

Example 3.3. Let $\mathbf{E} = \{0, \mathfrak{a}, \mathfrak{b}, 1\}$ be a poset with the following Hasse diagram. Define "$\sim$" on $\mathbf{E}$ as follows:



$\sim$	0	$\mathfrak{a}$	$\mathfrak{b}$	1
0	1	$\mathfrak{b}$	$\mathfrak{a}$	0
$\mathfrak{a}$	$\mathfrak{b}$	1	0	$\mathfrak{a}$
$\mathfrak{b}$	$\mathfrak{a}$	0	1	$\mathfrak{b}$
1	0	$\mathfrak{a}$	$\mathfrak{b}$	1

$\to$	0	$\mathfrak{a}$	$\mathfrak{b}$	1
0	1	1	1	1
$\mathfrak{a}$	$\mathfrak{b}$	1	$\mathfrak{b}$	1
$\mathfrak{b}$	$\mathfrak{a}$	$\mathfrak{a}$	1	1
1	0	$\mathfrak{a}$	$\mathfrak{b}$	1

Then $(\mathbf{E}, \curlywedge, \curlyvee, \sim, 0, 1)$ is a bounded lattice equality algebra and

$$\begin{aligned}\mathcal{DN}_{\mathfrak{a}}(\{1\}) &= \mathcal{DN}_{\mathfrak{a}}(\{\mathfrak{a}, 1\}) \\ &= \{\mathfrak{a}, 1\}, \\ \mathcal{DN}_{\mathfrak{a}}(\{\mathfrak{b}, 1\}) &= \mathcal{DN}_{\mathfrak{a}}(\{\mathfrak{a}, \mathfrak{b}, 1\}) \\ &= \mathcal{DN}_{\mathfrak{a}}(\mathbf{E}) \\ &= \{0, \mathfrak{a}, \mathfrak{b}, 1\}.\end{aligned}$$

PROPOSITION 3.4. Let $\mathfrak{a}, \mathfrak{b} \in \mathbf{E}$ and $\emptyset \neq \mathcal{X}, \mathcal{Y} \in \Gamma(\mathcal{X})$. Then

(i) $\mathcal{DN}_0(\mathcal{X}) = \mathbf{E}$ and $\mathcal{DN}_1(\mathcal{X}) = \{\mathfrak{x} \in \mathbf{E} \mid \mathfrak{x}'' \in \mathcal{X}\}$. Particularly, if $\mathbf{E}$ is involutive, then $\mathcal{DN}_1(\mathcal{X}) = \mathcal{X}$.

(ii) $\mathcal{DN}_{\mathfrak{a}}(\emptyset) = \emptyset$ and $\mathcal{DN}_{\mathfrak{a}}(\mathbf{E}) = \mathbf{E}$.

(iii) $\mathfrak{a} \in \mathcal{DN}_{\mathfrak{a}}(\mathcal{X})$.

(iv) $\mathcal{DN}_{\mathfrak{a}}(\mathcal{X}) \in \Gamma(\mathbf{E})$.

(v) If $\mathcal{X} \subseteq \mathcal{Y}$, then $\mathcal{DN}_{\mathfrak{a}}(\mathcal{X}) \subseteq \mathcal{DN}_{\mathfrak{a}}(\mathcal{Y})$.

(vi) $\mathcal{X} \subseteq \mathcal{DN}_{\mathfrak{a}}(\mathcal{X})$.

(vii) If $\mathfrak{a} \preccurlyeq \mathfrak{b}$, then $\mathcal{DN}_{\mathfrak{b}}(\mathcal{X}) \subseteq \mathcal{DN}_{\mathfrak{a}}(\mathcal{X})$. Particularly, $\mathcal{DN}_{\mathfrak{b}''}(\mathcal{X}) \subseteq \mathcal{DN}_{\mathfrak{b}}(\mathcal{X})$.

(viii) If $\mathbf{E}$ is involutive, then $\mathcal{DN}_{\mathfrak{a}}(\mathcal{X}) = \langle \mathcal{X} \cup \{\mathfrak{a}\} \rangle$. Particularly, if $\mathfrak{a} \in \mathcal{X}$ and $\mathcal{X} \in \mathcal{F}(\mathbf{E})$, then $\mathcal{DN}_{\mathfrak{a}}(\mathcal{X}) = \mathcal{X}$.

PROOF: The proofs of (i) and (ii) are clear.
(iii) Let $\mathfrak{a} \in \mathbf{E}$. Then $\mathfrak{a} \overset{1}{\twoheadrightarrow} \mathfrak{a} = (\mathfrak{a} \twoheadrightarrow \mathfrak{a})'' = 1'' = 1$. Since $\mathcal{X} \in \Gamma(\mathbf{E})$, we have $1 \in \mathcal{X}$, and so $\mathfrak{a} \in \mathcal{DN}_{\mathfrak{a}}(\mathcal{X})$.
(iv) Suppose $\mathfrak{x} \in \mathcal{DN}_{\mathfrak{a}}(\mathcal{X})$ and $\mathfrak{b} \in \mathbf{E}$ such that $\mathfrak{x} \preccurlyeq \mathfrak{b}$. Then there is $n \in \mathbb{N}$ such that

$$\underbrace{(\mathfrak{a} \twoheadrightarrow (\dots (\mathfrak{a}}_{\text{n-times}} \twoheadrightarrow \mathfrak{x})'' \dots)'')'' \in \mathcal{X}.$$

Since $\mathfrak{x} \preccurlyeq \mathfrak{b}$, by Proposition 2.2(vi), $\mathfrak{a} \twoheadrightarrow \mathfrak{x} \preccurlyeq \mathfrak{a} \twoheadrightarrow \mathfrak{b}$. Thus, $(\mathfrak{a} \twoheadrightarrow \mathfrak{x})'' \preccurlyeq (\mathfrak{a} \twoheadrightarrow \mathfrak{b})''$. By repeating this method, we have

$$\mathfrak{a} \overset{n}{\twoheadrightarrow} \mathfrak{x} = \underbrace{(\mathfrak{a} \twoheadrightarrow (\ldots(\mathfrak{a}}_{\text{n-times}} \twoheadrightarrow \mathfrak{x})'' \ldots)'')'' \preccurlyeq \underbrace{(\mathfrak{a} \twoheadrightarrow (\ldots(\mathfrak{a}}_{\text{n-times}} \twoheadrightarrow \mathfrak{b})'' \ldots)'')'' = \mathfrak{a} \overset{n}{\twoheadrightarrow} \mathfrak{b}.$$

Since $\mathfrak{a} \overset{n}{\twoheadrightarrow} \mathfrak{x} \in \mathcal{X}$ and $\mathcal{X} \in \Gamma(\mathbf{E})$, we conclude $\mathfrak{a} \overset{n}{\twoheadrightarrow} \mathfrak{b} \in \mathcal{X}$. Hence, $\mathfrak{b} \in \mathcal{DN}_{\mathfrak{a}}(\mathcal{X})$. Therefore, $\mathcal{DN}_{\mathfrak{a}}(\mathcal{X}) \in \Gamma(\mathbf{E})$.

(v) If $\mathfrak{x} \in \mathcal{DN}_{\mathfrak{a}}(\mathcal{X})$, then there is $n \in \mathbb{N}$ such that $\mathfrak{a} \overset{n}{\twoheadrightarrow} \mathfrak{x} \in \mathcal{X}$. Since $\mathcal{X} \subseteq \mathcal{Y}$, we have $\mathfrak{a} \overset{n}{\twoheadrightarrow} \mathfrak{x} \in \mathcal{Y}$, and so $\mathfrak{x} \in \mathcal{DN}_{\mathfrak{a}}(\mathcal{Y})$. Hence, $\mathcal{DN}_{\mathfrak{a}}(\mathcal{X}) \subseteq \mathcal{DN}_{\mathfrak{a}}(\mathcal{Y})$.

(vi) Let $\mathfrak{x} \in \mathcal{X}$. By Proposition 2.2(iii) and (iv), we have

$$\mathfrak{x} \preccurlyeq \mathfrak{a} \twoheadrightarrow \mathfrak{x} \preccurlyeq (\mathfrak{a} \twoheadrightarrow \mathfrak{x})'' \preccurlyeq \cdots \preccurlyeq \mathfrak{a} \overset{n}{\twoheadrightarrow} \mathfrak{x}.$$

Since $\mathfrak{x} \in \mathcal{X}$ and $\mathcal{X} \in \Gamma(\mathbf{E})$, we get for any $n \in \mathbb{N}$, $\mathfrak{a} \overset{n}{\twoheadrightarrow} \mathfrak{x} \in \mathcal{X}$ and so $\mathfrak{x} \in \mathcal{DN}_{\mathfrak{a}}(\mathcal{X})$. Hence, $\mathcal{X} \subseteq \mathcal{DN}_{\mathfrak{a}}(\mathcal{X})$.

(vii) If $\mathfrak{x} \in \mathcal{DN}_{\mathfrak{b}}(\mathcal{X})$, then there is $n \in \mathbb{N}$ such that $\mathfrak{b} \overset{n}{\twoheadrightarrow} \mathfrak{x} \in \mathcal{X}$. Since $\mathfrak{a} \preccurlyeq \mathfrak{b}$, by Proposition 2.2(vi), we have $\mathfrak{b} \twoheadrightarrow \mathfrak{x} \preccurlyeq \mathfrak{a} \twoheadrightarrow \mathfrak{x}$, and so $(\mathfrak{b} \twoheadrightarrow \mathfrak{x})'' \preccurlyeq (\mathfrak{a} \twoheadrightarrow \mathfrak{x})''$. By Proposition 2.2(vi), we have $\mathfrak{b} \twoheadrightarrow (\mathfrak{b} \twoheadrightarrow \mathfrak{x})'' \preccurlyeq \mathfrak{b} \twoheadrightarrow (\mathfrak{a} \twoheadrightarrow \mathfrak{x})''$. Since $\mathfrak{a} \preccurlyeq \mathfrak{b}$, by Proposition 2.2(vi), we get $\mathfrak{b} \twoheadrightarrow (\mathfrak{a} \twoheadrightarrow \mathfrak{x})'' \preccurlyeq \mathfrak{a} \twoheadrightarrow (\mathfrak{a} \twoheadrightarrow \mathfrak{x})''$. Thus, $\mathfrak{b} \twoheadrightarrow (\mathfrak{b} \twoheadrightarrow \mathfrak{x})'' \preccurlyeq \mathfrak{a} \twoheadrightarrow (\mathfrak{a} \twoheadrightarrow \mathfrak{x})''$ and so $\mathfrak{b} \overset{2}{\twoheadrightarrow} \mathfrak{x} \preccurlyeq \mathfrak{a} \overset{2}{\twoheadrightarrow} \mathfrak{x}$. By repeating this method, we have $\mathfrak{b} \overset{n}{\twoheadrightarrow} \mathfrak{x} \preccurlyeq \mathfrak{a} \overset{n}{\twoheadrightarrow} \mathfrak{x}$. Since $\mathfrak{b} \overset{n}{\twoheadrightarrow} \mathfrak{x} \in \mathcal{X}$ and $\mathcal{X} \in \Gamma(\mathbf{E})$, we get $\mathfrak{a} \overset{n}{\twoheadrightarrow} \mathfrak{x} \in \mathcal{X}$. Hence, $\mathfrak{x} \in \mathcal{DN}_{\mathfrak{a}}(\mathcal{X})$. In addition, by Proposition 2.2(iv), $\mathfrak{y} \preccurlyeq \mathfrak{y}''$. Then the proof is clear.

(viii) Let $\mathfrak{x} \in \mathcal{DN}_{\mathfrak{a}}(\mathcal{X})$. Then

$$\mathfrak{a} \overset{n}{\twoheadrightarrow} \mathfrak{x} \in \mathcal{X} \quad \Leftrightarrow \quad \exists n \in \mathbb{N} \text{ such that } (\mathfrak{a} \twoheadrightarrow (\mathfrak{a} \twoheadrightarrow (\cdots(\mathfrak{a} \twoheadrightarrow \mathfrak{x})'' \cdots)'')'')'' \in \mathcal{X}.$$

Since $\mathbf{E}$ is involutive, we get $(\mathfrak{a} \twoheadrightarrow (\mathfrak{a} \twoheadrightarrow (\cdots(\mathfrak{a} \twoheadrightarrow \mathfrak{x})\cdots))) \in \mathcal{X}$ and by Proposition 2.7, we have $\mathfrak{x} \in \langle \mathcal{X} \cup \{\mathfrak{a}\}\rangle$. The proof of other case is clear. □

Lemma 3.5. *The sequence* $\{\mathfrak{x} \overset{n}{\twoheadrightarrow} \mathfrak{a}\}_{n=1}^{\infty}$ *is an increasing sequence in* $\mathbf{E}$*, for any* $\mathfrak{x} \in \mathbf{E}$.

PROOF: By Proposition 2.2(iii), $\mathfrak{a} \preccurlyeq \mathfrak{x} \twoheadrightarrow \mathfrak{a}$. Then by using two times Proposition 2.2(vi), $\mathfrak{a}'' \preccurlyeq (\mathfrak{x} \twoheadrightarrow \mathfrak{a})''$ and so $\mathfrak{a} \preccurlyeq (\mathfrak{x} \twoheadrightarrow \mathfrak{a})''$. Now, by Proposition 2.2(vi), we obtain $\mathfrak{x} \twoheadrightarrow \mathfrak{a} \preccurlyeq \mathfrak{x} \twoheadrightarrow (\mathfrak{x} \twoheadrightarrow \mathfrak{a})''$, and so $(\mathfrak{x} \twoheadrightarrow \mathfrak{a})'' \preccurlyeq (\mathfrak{x} \twoheadrightarrow (\mathfrak{x} \twoheadrightarrow \mathfrak{a})'')''$. Then $\mathfrak{x} \overset{1}{\twoheadrightarrow} \mathfrak{a} \preccurlyeq \mathfrak{x} \overset{2}{\twoheadrightarrow} \mathfrak{a}$. Similarly, we have $\mathfrak{x} \overset{2}{\twoheadrightarrow} \mathfrak{a} \preccurlyeq \mathfrak{x} \overset{3}{\twoheadrightarrow} \mathfrak{a}$. Therefore,

$$\mathfrak{x} \overset{1}{\twoheadrightarrow} \mathfrak{a} \preccurlyeq \mathfrak{x} \overset{2}{\twoheadrightarrow} \mathfrak{a} \preccurlyeq \mathfrak{x} \overset{3}{\twoheadrightarrow} \mathfrak{a} \preccurlyeq \cdots \preccurlyeq \mathfrak{x} \overset{n}{\twoheadrightarrow} \mathfrak{a} \preccurlyeq \cdots$$ □

PROPOSITION 3.6. If $\{\mathcal{X}_\alpha\}_{\alpha\in I} \subseteq \Gamma(\mathbf{E})$, then

(i) $\mathcal{DN}_\mathfrak{a}\left(\bigcup\limits_{\alpha\in I} \mathcal{X}_\alpha\right) = \bigcup\limits_{\alpha\in I} \mathcal{DN}_\mathfrak{a}(\mathcal{X}_\alpha)$.

(ii) $\mathcal{DN}_\mathfrak{a}\left(\bigcap\limits_{\alpha\in I} \mathcal{X}_\alpha\right) \subseteq \bigcap\limits_{\alpha\in I} \mathcal{DN}_\mathfrak{a}(\mathcal{X}_\alpha)$.

(iii) If I is finite, then $\mathcal{DN}_\mathfrak{a}\left(\bigcap\limits_{\alpha\in I} \mathcal{X}_\alpha\right) = \bigcap\limits_{\alpha\in I} \mathcal{DN}_\mathfrak{a}(\mathcal{X}_\alpha)$.

PROOF: (i) By Propositions 3.1(ii), we have $\bigcup_{\alpha\in I} \mathcal{X}_\alpha \in \Gamma(\mathbf{E})$. If $\mathfrak{x} \in \mathcal{DN}_\mathfrak{a}\left(\bigcup\limits_{\alpha\in I} \mathcal{X}_\alpha\right)$, then there is $n \in \mathbb{N}$ such that $\mathfrak{a} \overset{n}{\twoheadrightarrow} \mathfrak{x} \in \bigcup\limits_{\alpha\in I} \mathcal{X}_\alpha$. Thus, there exists $\eta \in I$ such that $\mathfrak{a} \overset{n}{\twoheadrightarrow} \mathfrak{x} \in \mathcal{X}_\eta$, and so $\mathfrak{x} \in \mathcal{DN}_\mathfrak{a}(\mathcal{X}_\eta)$. Hence, $\mathfrak{x} \in \bigcup\limits_{\alpha\in I} \mathcal{DN}_\mathfrak{a}(\mathcal{X}_\alpha)$, and so $\mathcal{DN}_\mathfrak{a}\left(\bigcup\limits_{\alpha\in I} \mathcal{X}_\alpha\right) \subseteq \bigcup\limits_{\alpha\in I} \mathcal{DN}_\mathfrak{a}(\mathcal{X}_\alpha)$. Now, since $\mathcal{X}_\alpha \subseteq \bigcup\limits_{\alpha\in I} \mathcal{X}_\alpha$ for any $\alpha \in I$, by Proposition 3.4(v) we get $\mathcal{DN}_\mathfrak{a}(\mathcal{X}_\alpha) \subseteq \mathcal{DN}_\mathfrak{a}\left(\bigcup\limits_{\alpha\in I} \mathcal{X}_\alpha\right)$. So $\bigcup\limits_{\alpha\in I} \mathcal{DN}_\mathfrak{a}(\mathcal{X}_\alpha) \subseteq \mathcal{DN}_\mathfrak{a}(\bigcup\limits_{\alpha\in I} \mathcal{X}_\alpha)$. Hence, $\mathcal{DN}_\mathfrak{a}\left(\bigcup\limits_{\alpha\in I} \mathcal{X}_\alpha\right) = \bigcup\limits_{\alpha\in I} \mathcal{DN}_\mathfrak{a}(\mathcal{X}_\alpha)$.

(ii) If $\mathfrak{x} \in \bigcap\limits_{\alpha} \mathcal{DN}_\mathfrak{a}(\mathcal{X}_\alpha)$, then $\mathfrak{x} \in \mathcal{DN}_\mathfrak{a}(\mathcal{X}_\alpha)$ for any $\alpha \in I$. Then there is

$n_\alpha \in \mathbb{N}$ for any $\alpha \in I$ such that $\mathfrak{a} \overset{n_\alpha}{\twoheadrightarrow} \mathfrak{x} \in \mathcal{X}_\alpha$. We put $\mathfrak{t} = \max\{n_\alpha\}_{\alpha \in I}$. By Lemma 3.5, $\mathfrak{a} \overset{n_\alpha}{\twoheadrightarrow} \mathfrak{x} \preccurlyeq \mathfrak{a} \overset{\mathfrak{t}}{\twoheadrightarrow} \mathfrak{x}$ for any $\alpha \in I$. Since $\mathcal{X}_\alpha \in \Gamma(\mathbf{E})$ for $\alpha \in I$, we obtain $\mathfrak{a} \overset{\mathfrak{t}}{\twoheadrightarrow} \mathfrak{x} \in \mathcal{X}_\alpha$ for any $\alpha \in I$. Hence, $\mathfrak{a} \overset{\mathfrak{t}}{\twoheadrightarrow} \mathfrak{x} \in \bigcap\limits_{\alpha \in I} \mathcal{X}_\alpha$, and so $\mathfrak{x} \in \mathcal{DN}_\mathfrak{a}\left(\bigcap\limits_{\alpha \in I} \mathcal{X}_\alpha\right)$. Hence, $\mathcal{DN}_\mathfrak{a}\left(\bigcap\limits_{\alpha \in I} \mathcal{X}_\alpha\right) \subseteq \bigcap\limits_{\alpha \in I} \mathcal{DN}_\mathfrak{a}(\mathcal{X}_\alpha)$.

(iii) If $| I | < \infty$, then by Propositions 3.1(iii), we get $\bigcap_{\alpha \in I} \mathcal{X}_\alpha \in \Gamma(\mathbf{E})$. We show $\bigcap\limits_{\alpha=1}^{k} \mathcal{DN}_\mathfrak{a}(\mathcal{X}_\alpha) = \mathcal{DN}_\mathfrak{a}\left(\bigcap\limits_{\alpha=1}^{k} \mathcal{X}_\alpha\right)$. If $\mathfrak{x} \in \mathcal{DN}_\mathfrak{a}\left(\bigcap\limits_{\alpha=1}^{k} \mathcal{X}_\alpha\right)$, then there is $n \in \mathbb{N}$ such that $\mathfrak{a} \overset{n}{\twoheadrightarrow} \mathfrak{x} \in \bigcap\limits_{\alpha=1}^{k} \mathcal{X}_\alpha$. Then $\mathfrak{a} \overset{n}{\twoheadrightarrow} \mathfrak{x} \in \mathcal{X}_\alpha$ for $1 \preccurlyeq \alpha \preccurlyeq k$. Thus, $\mathfrak{x} \in \mathcal{DN}_\mathfrak{a}(\mathcal{X}_\alpha)$ for $1 \preccurlyeq \alpha \preccurlyeq k$ and so $\mathfrak{x} \in \bigcap\limits_{\alpha=1}^{k} \mathcal{DN}_\mathfrak{a}(\mathcal{X}_\alpha)$. Conversely, holds by (ii). Therefore, $\mathcal{DN}_\mathfrak{a}\left(\bigcap\limits_{\alpha \in I} \mathcal{X}_\alpha\right) = \bigcap\limits_{\alpha \in I} \mathcal{DN}_\mathfrak{a}(\mathcal{X}_\alpha)$. □

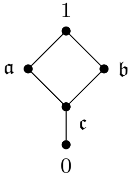
Remark 3.7. Let $\mathfrak{a} \in \mathbf{E}$. Define

$$\tau_\mathfrak{a} := \{\mathcal{DN}_\mathfrak{a}(\mathcal{X}) \mid \mathcal{X} \in \Gamma(\mathbf{E})\}.$$

THEOREM 3.8. *The structure* $(\mathbf{E}, \tau_\mathfrak{a})$ *is a topological space.*

PROOF: By Proposition 3.4(ii), $\mathcal{DN}_\mathfrak{a}(\emptyset) = \emptyset$ and $\mathcal{DN}_\mathfrak{a}(\mathbf{E}) = \mathbf{E}$. Then $\emptyset, \mathbf{E} \in \tau_\mathfrak{a}$. Furthermore, if $\{\mathcal{DN}_\mathfrak{a}(\mathcal{X}_\alpha)\}_{\alpha \in I}$ is a family of $\tau_\mathfrak{a}$, then by Propositions 3.6(i), we have $\mathcal{DN}_\mathfrak{a}(\bigcup\limits_{\alpha \in I} \mathcal{X}_\alpha) = \bigcup\limits_{\alpha \in I} \mathcal{DN}_\mathfrak{a}(\mathcal{X}_\alpha)$. Therefore, $\bigcup\limits_{\alpha \in I} \mathcal{DN}_\mathfrak{a}(\mathcal{X}_\alpha) \in \tau_\mathfrak{a}$. Now, if $\{\mathcal{DN}_\mathfrak{a}(\mathcal{X}_\alpha)\}_{\alpha \in I}$ is a finite family of $\tau_\mathfrak{a}$, then by Propositions 3.6(iii), we get $\mathcal{DN}_\mathfrak{a}(\bigcap\limits_{\alpha \in I} \mathcal{X}_\alpha) = \bigcap\limits_{\alpha \in I} \mathcal{DN}_\mathfrak{a}(\mathcal{X}_\alpha)$. Hence, $\bigcap\limits_{\alpha \in I} \mathcal{DN}_\mathfrak{a}(\mathcal{X}_\alpha) \in \tau_\mathfrak{a}$. Therefore, $\tau_\mathfrak{a}$ is a topology on $\mathbf{E}$. □

Example 3.9. Consider $\mathbf{E} = \{0, \mathfrak{a}, \mathfrak{b}, \mathfrak{c}, 1\}$ be a poset. Define "$\sim$" on $\mathbf{E}$ as the following table:



$\sim$	0	$\mathfrak{a}$	$\mathfrak{b}$	$\mathfrak{c}$	1
0	1	0	0	0	0
$\mathfrak{a}$	0	1	$\mathfrak{c}$	$\mathfrak{b}$	$\mathfrak{a}$
$\mathfrak{b}$	0	$\mathfrak{c}$	1	$\mathfrak{a}$	$\mathfrak{b}$
$\mathfrak{c}$	0	$\mathfrak{b}$	$\mathfrak{a}$	1	$\mathfrak{c}$
1	0	$\mathfrak{a}$	$\mathfrak{b}$	$\mathfrak{c}$	1

$\twoheadrightarrow$	0	$\mathfrak{a}$	$\mathfrak{b}$	$\mathfrak{c}$	1
0	1	1	1	1	1
$\mathfrak{a}$	0	1	$\mathfrak{b}$	$\mathfrak{b}$	1
$\mathfrak{b}$	0	$\mathfrak{a}$	1	$\mathfrak{a}$	1
$\mathfrak{c}$	0	1	1	1	1
1	0	$\mathfrak{a}$	$\mathfrak{b}$	$\mathfrak{c}$	1

Then $\mathbf{E} = (\mathbf{E}, \curlywedge, \sim, 1)$ is a bounded equality algebra. Clearly,

$$\Gamma(\mathbf{E}) = \{\emptyset, \{1\}, \{\mathfrak{a}, 1\}, \{\mathfrak{b}, 1\}, \{\mathfrak{a}, \mathfrak{b}, 1\}, \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, 1\}, \mathbf{E}\}\,.$$

Then since $\mathcal{DN}_{\mathfrak{b}}(\mathcal{X}) := \{\mathfrak{x} \in \mathbf{E} \mid \exists n \in \mathbb{N} \text{ s.t } \mathfrak{b} \overset{n}{\twoheadrightarrow} \mathfrak{x} \in \mathcal{X}\}$, we get that

$$\mathcal{DN}_{\mathfrak{b}}(\emptyset) = \emptyset,\ \mathcal{DN}_{\mathfrak{b}}(\mathbf{E}) = \mathbf{E},\ \mathcal{DN}_{\mathfrak{b}}(\{1\}) = \mathcal{DN}_{\mathfrak{b}}(\{\mathfrak{b}, 1\}) = \{\mathfrak{b}, 1\},$$

$$\mathcal{DN}_{\mathfrak{b}}(\{\mathfrak{a}, 1\}) = \mathcal{DN}_{\mathfrak{b}}(\{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, 1\}) = \mathcal{DN}_{\mathfrak{b}}(\{\mathfrak{a}, \mathfrak{b}, 1\}) = \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, 1\}.$$

Hence, $\tau_{\mathfrak{b}} = \{\emptyset, \{\mathfrak{b}, 1\}, \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, 1\}, \mathbf{E}\}$.

Remark 3.10. By Proposition 3.4(ii), $\mathcal{DN}_0(\emptyset) = \emptyset$. Also, for any $\emptyset \neq \mathcal{X} \in \Gamma(\mathbf{E})$, we have $\mathcal{DN}_0(\mathcal{X}) = \mathbf{E}$. Hence, $\tau_0 = \{\mathcal{DN}_0(\mathcal{X}) \mid \mathcal{X} \in \Gamma(\mathbf{E})\} = \{\emptyset, \mathbf{E}\}$.

THEOREM 3.11. *Let* $\mid \mathbf{E} \mid \geq 2$. *If* $0 \neq \mathfrak{a} \in \mathbf{E}$, *then* $\tau_{\mathfrak{a}}$ *is not a discrete topology on* $\mathbf{E}$.

PROOF: We know that if $\tau_{\mathfrak{a}}$ is a discrete topology on $\mathbf{E}$, then $\tau_{\mathfrak{a}} = \mathcal{P}(\mathbf{E})$. Also, clearly, $\tau_{\mathfrak{a}} \subseteq \mathcal{P}(\mathbf{E})$. In addition, we have some subsets of $\mathbf{E}$ that not containing 1. On the other hand, for any $\emptyset \neq \mathcal{U} \in \tau_{\mathfrak{a}}$, $\mathcal{U} \in \Gamma(\mathbf{E})$ and so $1 \in \mathcal{U}$. Hence, $\mathcal{P}(\mathbf{E}) \nsubseteq \tau_{\mathfrak{a}}$. Therefore, $\tau_{\mathfrak{a}}$ is not a discrete topology on $\mathbf{E}$. $\square$

PROPOSITION 3.12. Let $\mathfrak{a} \in \mathbf{E}$. If

$$\beta_{\mathfrak{a}} := \{\mathcal{DN}_{\mathfrak{a}}(\uparrow \mathfrak{x}) \mid \mathfrak{x} \in \mathbf{E}\} \cup \{\emptyset\},$$

then $\beta_{\mathfrak{a}}$ is a base for $\tau_{\mathfrak{a}}$.

PROOF: Let $\mathfrak{x} \in \mathbf{E}$. Since $\uparrow (\uparrow \mathfrak{x}) = \uparrow \mathfrak{x}$, we have $\mathcal{DN}_{\mathfrak{a}}(\uparrow \mathfrak{x}) \in \tau_{\mathfrak{a}}$. Then $\beta_{\mathfrak{a}}$ is a set of open subsets of $\mathbf{E}$. Furthermore, if $\emptyset \neq \mathcal{U} \in \tau_{\mathfrak{a}}$, then there is $\mathcal{X} \in \Gamma(\mathbf{E})$ such that $\mathcal{U} = \mathcal{DN}_{\mathfrak{a}}(\mathcal{X})$. By Propositions 3.1(iv) and 3.6(i) we have

$$\mathcal{U} = \mathcal{DN}_{\mathfrak{a}}(\mathcal{X}) = \mathcal{DN}_{\mathfrak{a}}(\bigcup_{\mathfrak{x}\in\mathcal{X}} \uparrow \mathfrak{x}) = \bigcup_{\mathfrak{x}\in\mathcal{X}} \mathcal{DN}_{\mathfrak{a}}(\uparrow \mathfrak{x}).$$

Therefore, β is a basis for $\tau_{\mathfrak{a}}$. □

PROPOSITION 3.13. If $\mathfrak{a} \preccurlyeq \mathfrak{b}$, then $\tau_{\mathfrak{b}}$ is finer than $\tau_{\mathfrak{a}}$, for $\mathfrak{a}, \mathfrak{b} \in \mathbf{E}$.

PROOF: Let $\mathfrak{t} \in \mathbf{E}$ and $\mathfrak{t} \in \mathbf{B} \in \beta_{\mathfrak{a}}$. Then $\mathbf{B} = \mathcal{DN}_{\mathfrak{a}}(\uparrow \mathfrak{x})$ for some $\mathfrak{x} \in \mathbf{E}$. Since $\mathfrak{t} \in \uparrow \mathfrak{t}$ and $\uparrow \mathfrak{t} \subseteq \mathcal{DN}_{\mathfrak{b}}(\uparrow \mathfrak{t})$, we have $\mathfrak{t} \in \mathcal{DN}_{\mathfrak{b}}(\uparrow \mathfrak{t})$. Also, since $\mathfrak{a} \preccurlyeq \mathfrak{b}$, by Proposition 3.4(vii) $\mathcal{DN}_{\mathfrak{b}}(\uparrow \mathfrak{t}) \subseteq \mathcal{DN}_{\mathfrak{a}}(\uparrow \mathfrak{t})$. Now, we show $\mathcal{DN}_{\mathfrak{a}}(\uparrow \mathfrak{t}) \subseteq \mathcal{DN}_{\mathfrak{a}}(\uparrow \mathfrak{x})$. Suppose $\mathfrak{r} \in \mathcal{DN}_{\mathfrak{a}}(\uparrow \mathfrak{t})$. Then there is $m \in \mathbb{N}$ such that $\mathfrak{a} \overset{m}{\twoheadrightarrow} \mathfrak{r} \in \uparrow \mathfrak{t}$. Then

$$\mathfrak{t} \preccurlyeq \underbrace{(\mathfrak{a} \twoheadrightarrow (\ldots (\mathfrak{a}}_{\text{m-times}} \twoheadrightarrow \mathfrak{r})'' \ldots)'')''.$$

Since $\mathfrak{t} \in B$, we have $\mathfrak{t} \in \mathcal{DN}_{\mathfrak{a}}(\uparrow \mathfrak{x})$. Then there is $n \in \mathbb{N}$ such that $\mathfrak{a} \overset{n}{\twoheadrightarrow} \mathfrak{t} \in \uparrow \mathfrak{x}$. Thus,

$$\mathfrak{x} \preccurlyeq \underbrace{(\mathfrak{a} \twoheadrightarrow (\ldots (\mathfrak{a}}_{\text{n-times}} \twoheadrightarrow \mathfrak{t})'' \ldots)'')''.$$

Since $\mathfrak{t} \preccurlyeq \underbrace{(\mathfrak{a} \twoheadrightarrow (\ldots (\mathfrak{a}}_{\text{m-times}} \twoheadrightarrow \mathfrak{r})'' \ldots)'')'')$, by Proposition 2.2(vi), we have

$$\mathfrak{a} \twoheadrightarrow \mathfrak{t} \preccurlyeq \mathfrak{a} \twoheadrightarrow \underbrace{(\mathfrak{a} \twoheadrightarrow (\ldots (\mathfrak{a}}_{m\ times} \twoheadrightarrow \mathfrak{r})'' \ldots)'')''.$$

By assumption, and by Proposition 2.2(vi),

$$(\mathfrak{a} \twoheadrightarrow \mathfrak{t})'' \preccurlyeq (\mathfrak{a} \twoheadrightarrow \underbrace{(\mathfrak{a} \twoheadrightarrow (\ldots (\mathfrak{a}}_{\text{m-times}} \twoheadrightarrow \mathfrak{r})'' \ldots)'')'')''.$$

So, $\mathfrak{a} \overset{1}{\twoheadrightarrow} \mathfrak{t} \preccurlyeq \mathfrak{a} \overset{m+1}{\twoheadrightarrow} \mathfrak{r}$. Similarly, we conclude $\mathfrak{a} \overset{n}{\twoheadrightarrow} \mathfrak{t} \preccurlyeq \mathfrak{a} \overset{m+n}{\twoheadrightarrow} \mathfrak{r}$, and so $\mathfrak{a} \overset{m+n}{\twoheadrightarrow} \mathfrak{r} \in \uparrow \mathfrak{x}$. Thus, $\mathfrak{r} \in \mathcal{DN}_{\mathfrak{a}}(\uparrow \mathfrak{x})$, and so $\mathcal{DN}_{\mathfrak{a}}(\uparrow \mathfrak{t}) \subseteq \mathcal{DN}_{\mathfrak{a}}(\uparrow \mathfrak{x})$. Since

$\mathcal{DN}_{\mathfrak{b}}(\uparrow \mathfrak{t}) \subseteq \mathcal{DN}_{\mathfrak{a}}(\uparrow \mathfrak{t})$, we have $\mathcal{DN}_{\mathfrak{b}}(\uparrow \mathfrak{t}) \subseteq \mathcal{DN}_{\mathfrak{a}}(\uparrow \mathfrak{x})$. Finally, it is enough to set $\mathbf{B}' = \mathcal{DN}_{\mathfrak{b}}(\uparrow \mathfrak{t})$. Then $\mathbf{B}' \in \beta_{\mathfrak{b}}$ and $\mathfrak{t} \in \mathbf{B}' \subseteq \mathbf{B}$. Thus, $\tau_{\mathfrak{b}}$ is finer than $\tau_{\mathfrak{a}}$. □

PROPOSITION 3.14. The singleton $\{\mathfrak{a}\}$ is dense in the topological space $(\mathbf{E}, \tau_{\mathfrak{a}})$, where $\mathfrak{a} \in \mathbf{E}$.

PROOF: We prove $\overline{\{\mathfrak{a}\}} = \mathbf{E}$. Let $\mathbf{B}$ be a closed subset in $\mathbf{E}$ and $\{\mathfrak{a}\} \subseteq \mathbf{B}$. Then $\mathbf{B}^c$ is open, and so there is $\mathcal{X} \in \Gamma(\mathbf{E})$ such that $\mathbf{B}^c = \mathcal{DN}_{\mathfrak{a}}(\mathcal{X})$. We know that, if $\mathcal{DN}_{\mathfrak{a}}(\mathcal{X}) \neq \emptyset$, then $\mathfrak{a} \in \mathcal{DN}_{\mathfrak{a}}(\mathcal{X})$. Thus, $\mathfrak{a} \in \mathbf{B}^c$ a contradiction. Hence, $\mathcal{DN}_{\mathfrak{a}}(\mathcal{X}) = \emptyset$, and so $\mathbf{B}^c = \emptyset$. Therefore, $\mathbf{B} = \mathbf{E}$ i.e. $\overline{\{\mathfrak{a}\}} = \bigcap_{\{\mathfrak{a}\} \subseteq \mathbf{B} = \overline{\mathbf{B}}} \mathbf{B} = \mathbf{E}$. □

PROPOSITION 3.15. The topological space $(\mathcal{X}, \tau)$ is a T_0 space if and only if $\overline{\{\mathfrak{x}\}} \neq \overline{\{\mathfrak{y}\}}$ for any distinct pair $\mathfrak{x}, \mathfrak{y} \in \mathcal{X}$.

PROOF: Let $(\mathcal{X}, \tau)$ be T_0 space and $\mathfrak{x} \neq \mathfrak{y}$. Consider $\overline{\{\mathfrak{x}\}} = \overline{\{\mathfrak{y}\}}$ and $\mathfrak{x} \in \mathcal{U} \in \tau$. Then $\mathcal{U}^c$ is closed and $\mathfrak{x} \notin \mathcal{U}^c$. If $\mathfrak{y} \in \mathcal{U}^c$, then

$$\mathfrak{x} \in \overline{\{\mathfrak{x}\}} = \overline{\{\mathfrak{y}\}} = \bigcap_{\mathfrak{y} \in \mathcal{C}} \mathcal{C} \subseteq \mathcal{U}^c \quad \text{where } \mathcal{C} \text{ is closed subset of } \mathcal{X}.$$

Then $\mathfrak{x} \in \mathcal{U}^c$, and so $\mathfrak{x} \notin \mathcal{U}$, a contradiction. Hence, $\mathfrak{y} \notin \mathcal{U}^c$ and $\mathfrak{y} \in \mathcal{U}$. Therefore, $\overline{\{\mathfrak{x}\}} \neq \overline{\{\mathfrak{y}\}}$. Conversely, consider $\overline{\{\mathfrak{x}\}} \neq \overline{\{\mathfrak{y}\}}$ for any $\mathfrak{x}, \mathfrak{y} \in \mathcal{X}$ such that $\mathfrak{x} \neq \mathfrak{y}$. Then

$$\mathfrak{y} \in \overline{\{\mathfrak{y}\}} \neq \overline{\{\mathfrak{x}\}} = \bigcap_{\mathfrak{x} \in \beta = \overline{\beta}} \beta.$$

Thus, $\mathfrak{y} \notin \bigcap_{\mathfrak{x} \in \beta = \overline{\beta}} \beta$. So, there exists a closed subset C such that $\mathfrak{x} \in C$ and $\mathfrak{y} \notin C$. Now, set $V := C^c$. Then, $V \in \tau$ such that $\mathfrak{y} \in V$ and $\mathfrak{x} \notin V$. Hence, $(\mathcal{X}, \tau)$ is T_0 space. □

THEOREM 3.16. *For any* $\mathfrak{a} \in \mathbf{E}$*,*

(i) $(\mathbf{E}, \tau_{\mathfrak{a}})$ *is not Hausdorff.*

(ii) $(\mathbf{E}, \tau_{\mathfrak{a}})$ *is not* T_0*.*

(iii) $(\mathbf{E}, \tau_{\mathfrak{a}})$ *is not* T_1*.*

(iv) *The topological space* $(\mathbf{E}, \tau_{\mathfrak{a}})$ *is connected.*

(v) *The topological space* $(\mathbf{E}, \tau_{\mathfrak{a}})$ *is irreducible.*

Proof: (i) The proof is clear.
(ii) Let $\mathfrak{a} \in \mathbf{E}$. By Proposition 3.14, $\{\mathfrak{a}\}$ is dense. Thus, $\overline{\{\mathfrak{a}\}} = \overline{\{1\}} = \mathbf{E}$ and by Proposition 3.15, $(\mathbf{E}, \tau_{\mathfrak{a}})$ is not a T_0 space.
(iii) By (ii), it is clear.
(iv) For any $\mathcal{U} \in \tau_{\mathfrak{a}}$, we have $1 \in \mathcal{U}$ and so $(\mathbf{E}, \tau_{\mathfrak{a}})$ is a connected space.
(v) Suppose $\mathbf{E} = \mathbf{B}_1 \cup \mathbf{B}_2$ such that $\mathbf{B}_1$ and $\mathbf{B}_2$ are proper closed subsets. Since $\mathbf{B}_1^c \in \tau$, there exists $\mathcal{X}_1 \in \Gamma(\mathbf{E})$ such that $\mathbf{B}_1^c = \mathcal{DN}_{\mathfrak{a}}(\mathcal{X}_1)$. Similarly, there is $\mathcal{X}_2 \in \Gamma(\mathbf{E})$ such that $\mathbf{B}_2^c = \mathcal{DN}_{\mathfrak{a}}(\mathcal{X}_2)$. In addition, from $\mathfrak{a} \in \mathcal{DN}_{\mathfrak{a}}(\mathcal{X}_1)$ and $\mathfrak{a} \in \mathcal{DN}_{\mathfrak{a}}(\mathcal{X}_2)$ we obtain, $\mathfrak{a} \notin \mathbf{B}_1$ and $\mathfrak{a} \notin \mathbf{B}_2$. Hence, $\mathfrak{a} \notin \mathbf{B}_1 \cup \mathbf{B}_2$, and so $\mathfrak{a} \notin \mathbf{E}$, a contradiction. Therefore, $\mathbf{E}$ is an irreducible topological space. □

Theorem 3.17. *If* $\emptyset \neq \mathcal{X} \subseteq \mathbf{E}$ *is finitely generated, then* $\mathcal{X}$ *is compact in* $(\mathbf{E}, \tau_{\mathfrak{a}})$*, for any* $\mathfrak{a} \in \mathbf{E}$.

Proof: Suppose $\mathcal{X} = \uparrow \{\mathfrak{x}_1, \mathfrak{x}_2, \cdots, \mathfrak{x}_n\}$, for some $\mathfrak{x}_1, \mathfrak{x}_2, \cdots, \mathfrak{x}_n \in \mathbf{E}$. In addition, let $\mathcal{C} = \{\mathcal{DN}_{\mathfrak{a}}(\mathcal{X}_i)\}_{i \in I}$ be an open cover for $\mathcal{X}$. Then $\mathcal{X} = \bigcup\limits_{i \in I} \mathcal{DN}_{\mathfrak{a}}(\mathcal{X}_i)$. For every $1 \preccurlyeq j \preccurlyeq n$, there is $i_j \in I$ such that $\mathfrak{x}_j \in \mathcal{DN}_{\mathfrak{a}}(\mathcal{X}_{i_j})$. Then $\{\mathfrak{x}_1, \mathfrak{x}_2, \cdots, \mathfrak{x}_n\} \subseteq \mathcal{DN}_{\mathfrak{a}}(\mathcal{X}_{i_1}) \cup \mathcal{DN}_{\mathfrak{a}}(\mathcal{X}_{i_2}) \cup \cdots \cup \mathcal{DN}_{\mathfrak{a}}(\mathcal{X}_{i_n})$. Thus, $\{\mathfrak{x}_1, \mathfrak{x}_2, \cdots, \mathfrak{x}_n\} \subseteq \bigcup\limits_{j=1}^{n} \mathcal{DN}_{\mathfrak{a}}(\mathcal{X}_{i_j})$. If $\mathfrak{y} \in \mathcal{X} = \uparrow \{\mathfrak{x}_1, \mathfrak{x}_2, \cdots, \mathfrak{x}_n\}$, then there exists $1 \preccurlyeq \mathfrak{t} \preccurlyeq n$ such that $\mathfrak{x}_{\mathfrak{t}} \preccurlyeq \mathfrak{y}$. Since $\mathfrak{x}_{\mathfrak{t}} \in \mathcal{DN}_{\mathfrak{a}}(\mathcal{X}_{i_{\mathfrak{t}}})$, we get $\mathfrak{y} \in \mathcal{DN}(\mathcal{X}_{i_j})$. Thus, $\mathcal{X} \subseteq \bigcup\limits_{j=1}^{n} \mathcal{DN}_{\mathfrak{a}}(\mathcal{X}_{i_j})$, and so $\mathcal{C}$ has a finite subcover for $\mathcal{X}$. Therefore, $\mathcal{X}$ is compact. □

Theorem 3.18. *If* $\mathcal{X} \subseteq \mathbf{E}$ *is closed, then* $\mathcal{X}$ *is compact in the topological space* $(\mathbf{E}, \tau_{\mathfrak{a}})$*, where* $\mathfrak{a} \in \mathbf{E}$.

Proof: Let $\mathcal{X} \subseteq \mathbf{E}$ is closed. Then $\mathcal{X}^c$ is open. In this case, either $\mathcal{X}^c = \mathbf{E}$ or $\mathcal{X}^c \neq \mathbf{E}$. If $\mathcal{X}^c = \mathbf{E}$, then $\mathcal{X} = \emptyset$, and so $\emptyset$ is compact. If $\mathcal{X}^c \neq \mathbf{E}$, since $\mathcal{X}^c$ is upset, we get $0 \notin \mathcal{X}^c$, and so $0 \in \mathcal{X}$. Let $\mathcal{C} = \{\mathcal{DN}_{\mathfrak{a}}(\mathcal{Y}_\alpha)\}_{\alpha \in I}$ be an open cover for $\mathcal{X}$. Then $0 \in \bigcup_{\alpha\in I} \mathcal{DN}_{\mathfrak{a}}(\mathcal{Y}_\alpha)$. Thus, there is $\alpha \in I$ such that $0 \in \mathcal{DN}_{\mathfrak{a}}(\mathcal{Y}_\alpha)$. Since $\mathcal{DN}_{\mathfrak{a}}(\mathcal{Y}_\alpha)$ is upset $\mathcal{DN}_{\mathfrak{a}}(\mathcal{Y}_\alpha) = \mathbf{E}$. Hence, $\{\mathbf{E}\}$ is a finite sub-cover for $\mathcal{X}$. Therefore, $\mathcal{X}$ is compact. □

The following example shows that the converse of the above theorem is not true.

Example 3.19. (i) According to Example 3.9, $\mathcal{X} = \{\mathfrak{b}, 1\}$ is finite and so it is compact but $\mathcal{X}^c = \{0, \mathfrak{a}, \mathfrak{c}\} \notin \tau_{\mathfrak{b}}$. Hence, $\mathcal{X}$ is not closed.
(ii) According to Example 3.3, $\tau_{\mathfrak{a}} = \{\emptyset, \{\mathfrak{a}, 1\}, \mathbf{E}\}$. Clearly, $\mathcal{X} = \{\mathfrak{a}, 1\}$ is finite and so it is compact but $\mathcal{X}^c = \{0, \mathfrak{b}\} \notin \tau_{\mathfrak{a}}$. Hence, $\mathcal{X}$ is not closed.

THEOREM 3.20. *If* $\mathbf{E}$ *is an involutive equality algebra, then* $(\mathbf{E}, \twoheadrightarrow, \tau_{\mathfrak{a}})$ *is a right semi-topological algebra, where* $\mathfrak{a} \in \mathbf{E}$

PROOF: We have to show that $\twoheadrightarrow_{\mathfrak{b}}\colon \mathbf{E} \to \mathbf{E}$ defined by $\mathfrak{x} \mapsto (\mathfrak{b} \twoheadrightarrow \mathfrak{x})$ is continuous. It means for any $\mathbf{B} \in \beta_{\mathfrak{a}}$, $(\twoheadrightarrow_{\mathfrak{b}})^{-1}(\mathbf{B}) \in \tau_{\mathfrak{a}}$. Let $\mathfrak{x} \in \mathbf{E}$ such that $\mathbf{B} := \mathcal{DN}_{\mathfrak{a}}(\uparrow \mathfrak{x}) \in \beta_{\mathfrak{a}}$. We prove $\twoheadrightarrow_{\mathfrak{b}}^{-1} (\mathcal{DN}_{\mathfrak{a}}(\uparrow \mathfrak{x})) \in \tau_{\mathfrak{a}}$. Thus, we have

$$\begin{aligned}(\twoheadrightarrow_b)^{-1}(\mathcal{DN}_{\mathfrak{a}}(\uparrow \mathfrak{x})) &= \{\mathfrak{y} \in \mathbf{E} \mid \twoheadrightarrow_{\mathfrak{b}} (\mathfrak{y}) \in \mathcal{DN}_{\mathfrak{a}}(\uparrow \mathfrak{x})\}\\ &= \{\mathfrak{y} \in \mathbf{E} \mid (\mathfrak{b} \twoheadrightarrow \mathfrak{y}) \in \mathcal{DN}_{\mathfrak{a}}(\uparrow \mathfrak{x})\}\\ &= \{\mathfrak{y} \in \mathbf{E} \mid \exists n \in \mathbb{N} \text{ such that } \mathfrak{a} \overset{n}{\twoheadrightarrow} (\mathfrak{b} \twoheadrightarrow \mathfrak{y}) \in \uparrow \mathfrak{x}\}\\ &= \{\mathfrak{y} \in \mathbf{E} \mid \exists n \in \mathbb{N} \text{ such that } \mathfrak{x} \preccurlyeq \mathfrak{a} \overset{n}{\twoheadrightarrow} (\mathfrak{b} \twoheadrightarrow \mathfrak{y})\}. \end{aligned} \quad (3.1)$$

First, take $\mathcal{X} := (\twoheadrightarrow_b)^{-1}(\mathcal{DN}_{\mathfrak{a}}(\uparrow \mathfrak{x}))$ and we show that $\mathcal{X} \in \Gamma(\mathbf{E})$. Let $\mathfrak{y} \preccurlyeq \mathfrak{z}$ and $\mathfrak{y} \in \mathcal{X}$. Thus by Proposition 2.2(vi) and since $\mathbf{E}$ is involutive, we have $\mathfrak{b} \twoheadrightarrow \mathfrak{y} \preccurlyeq \mathfrak{b} \twoheadrightarrow \mathfrak{z}$. Then $\mathfrak{a} \twoheadrightarrow (\mathfrak{b} \twoheadrightarrow \mathfrak{y}) \preccurlyeq \mathfrak{a} \twoheadrightarrow (\mathfrak{b} \twoheadrightarrow \mathfrak{z})$, and so

$$(\mathfrak{a} \twoheadrightarrow (\mathfrak{b} \twoheadrightarrow \mathfrak{y}))'' \preccurlyeq (\mathfrak{a} \twoheadrightarrow (\mathfrak{b} \twoheadrightarrow \mathfrak{z}))''.$$

By repeating this method, for any $n \in \mathbb{N}$ we have $\mathfrak{a} \overset{n}{\twoheadrightarrow} (\mathfrak{b} \twoheadrightarrow \mathfrak{y}) \preccurlyeq \mathfrak{a} \overset{n}{\twoheadrightarrow} (\mathfrak{b} \twoheadrightarrow \mathfrak{z})$. Since $\mathfrak{x} \preccurlyeq \mathfrak{a} \overset{n}{\twoheadrightarrow} (\mathfrak{b} \twoheadrightarrow \mathfrak{y})$ we get $\mathfrak{x} \preccurlyeq \mathfrak{a} \overset{n}{\twoheadrightarrow} (\mathfrak{b} \twoheadrightarrow \mathfrak{z})$. Thus, $\mathfrak{z} \in \mathcal{X}$, and so $\mathcal{X} \in \Gamma(\mathbf{E})$. We claim that $\mathcal{DN}_{\mathfrak{a}}(\mathcal{X}) = \mathcal{X}$. By Proposition 3.4(vi), we have $\mathcal{X} \subseteq \mathcal{DN}_{\mathfrak{a}}(\mathcal{X})$. For other side of inclusion, consider $\mathfrak{y} \in \mathcal{DN}_{\mathfrak{a}}(\mathcal{X})$. Then there exists $m \in \mathbb{N}$ such that $\mathfrak{a} \overset{m}{\twoheadrightarrow} \mathfrak{y} \in \mathcal{X} = (\twoheadrightarrow_b)^{-1}(\mathcal{DN}_{\mathfrak{a}}(\uparrow \mathfrak{x}))$. Thus there is $k \in \mathbb{N}$ such that $\mathfrak{x} \preccurlyeq \mathfrak{a} \overset{k}{\twoheadrightarrow} (\mathfrak{b} \twoheadrightarrow (\mathfrak{a} \overset{m}{\twoheadrightarrow} \mathfrak{y}))$. By Proposition 2.2(v) and since **E** is involutive we get,

$$\mathfrak{x} \preccurlyeq \mathfrak{b} \twoheadrightarrow (\mathfrak{a} \overset{k}{\twoheadrightarrow} (\mathfrak{a} \overset{m}{\twoheadrightarrow} \mathfrak{y})) = \mathfrak{b} \twoheadrightarrow (\mathfrak{a} \overset{m+k}{\twoheadrightarrow} \mathfrak{y}) = \mathfrak{a} \overset{m+k}{\twoheadrightarrow} (\mathfrak{b} \twoheadrightarrow \mathfrak{y}).$$

By (3.1), $\mathfrak{y} \in \mathcal{X}$ and so $\mathcal{DN}_{\mathfrak{a}}(\mathcal{X}) \subseteq \mathcal{X}$. Hence, $\mathcal{X} = \mathcal{DN}_{\mathfrak{a}}(\mathcal{X}) \in \tau_{\mathfrak{a}}$. Therefore, $(\mathbf{E}, \twoheadrightarrow, \tau_{\mathfrak{a}})$ is a right semi-topological algebra. □

Remark 3.21. Clearly, $\twoheadrightarrow_{\mathfrak{b}}\colon \mathbf{E} \to \mathbf{E}$ defined by $\mathfrak{x} \mapsto (\mathfrak{x} \twoheadrightarrow \mathfrak{b})$ is not continuous, since by Proposition 2.2(vi), $(\twoheadrightarrow_b)^{-1}(\mathcal{DN}_{\mathfrak{a}}(\uparrow \mathfrak{x}))$ is not upset necessarily.

Open Problem: Is there any example of a finite equality algebra such as **E** which $\tau_{\mathfrak{a}}$ is not a left semi-topology on **E**?

THEOREM 3.22. *If* **E** *satisfies the following condition*

$$\mathfrak{x} \twoheadrightarrow (\mathfrak{y} \curlywedge \mathfrak{z}) = (\mathfrak{x} \twoheadrightarrow \mathfrak{y}) \curlywedge (\mathfrak{x} \twoheadrightarrow \mathfrak{z}), \quad \textit{for any} \quad \mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mathbf{E}$$

then $(\mathbf{E}, \curlywedge, \tau_{\mathfrak{a}})$ *is a semi-topological algebra, where* $\mathfrak{a} \in \mathbf{E}$.

PROOF: We have to show that $\curlywedge_{\mathfrak{b}} : \mathbf{E} \to \mathbf{E}$ defined by $\mathfrak{x} \mapsto (\mathfrak{b} \curlywedge \mathfrak{x})$ is continuous. For this, we show that the inverse image of every element of $\beta_{\mathfrak{a}}$ is an open subset in $\tau_{\mathfrak{a}}$, i.e., for any $\mathbf{B} \in \beta_{\mathfrak{a}}$, $(\curlywedge_{\mathfrak{b}})^{-1}(\mathbf{B}) \in \tau_{\mathfrak{a}}$. Let $\mathfrak{x} \in \mathbf{E}$ such that $\mathbf{B} := \mathcal{DN}_{\mathfrak{a}}(\uparrow \mathfrak{x}) \in \beta_{\mathfrak{a}}$. We prove $\curlywedge_{\mathfrak{b}}^{-1}(\mathcal{DN}_{\mathfrak{a}}(\uparrow \mathfrak{x})) \in \tau_{\mathfrak{a}}$. Thus, we have

$$\begin{aligned}(\curlywedge_b)^{-1}(\mathcal{DN}_{\mathfrak{a}}(\uparrow \mathfrak{x})) &= \{\mathfrak{y} \in \mathbf{E} \mid \curlywedge_{\mathfrak{b}}(\mathfrak{y}) \in \mathcal{DN}_{\mathfrak{a}}(\uparrow \mathfrak{x})\} \\ &= \{\mathfrak{y} \in \mathbf{E} \mid (\mathfrak{b} \curlywedge \mathfrak{y}) \in \mathcal{DN}_{\mathfrak{a}}(\uparrow \mathfrak{x})\} \\ &= \{\mathfrak{y} \in \mathbf{E} \mid \text{there exists } n \in \mathbb{N} \text{ such that } \mathfrak{a} \overset{n}{\twoheadrightarrow} (\mathfrak{b} \curlywedge \mathfrak{y}) \in \uparrow \mathfrak{x}\} \\ &= \{\mathfrak{y} \in \mathbf{E} \mid \text{there exists } n \in \mathbb{N} \text{ such that } \mathfrak{x} \preccurlyeq \mathfrak{a} \overset{n}{\twoheadrightarrow} (\mathfrak{b} \curlywedge \mathfrak{y})\}.\end{aligned} \tag{3.2}$$

First, take $\mathcal{X} := (\curlywedge_b)^{-1}(\mathcal{DN}_{\mathfrak{a}}(\uparrow \mathfrak{x}))$ and we show that $\mathcal{X} \in \Gamma(\mathbf{E})$. Let $\mathfrak{y} \preccurlyeq \mathfrak{z}$ and $\mathfrak{y} \in \mathcal{X}$. Thus $\mathfrak{b} \curlywedge \mathfrak{y} \preccurlyeq \mathfrak{b} \curlywedge \mathfrak{z}$ and so $\mathfrak{a} \twoheadrightarrow (\mathfrak{b} \curlywedge \mathfrak{y}) \preccurlyeq \mathfrak{a} \twoheadrightarrow (\mathfrak{b} \curlywedge \mathfrak{z})$. Hence $(\mathfrak{a} \twoheadrightarrow (\mathfrak{b} \curlywedge \mathfrak{y}))'' \preccurlyeq (\mathfrak{a} \twoheadrightarrow (\mathfrak{b} \curlywedge \mathfrak{z}))''$. By repeating this method, for any $n \in \mathbb{N}$ we have $\mathfrak{a} \overset{n}{\twoheadrightarrow} (\mathfrak{b} \curlywedge \mathfrak{y}) \preccurlyeq \mathfrak{a} \overset{n}{\twoheadrightarrow} (\mathfrak{b} \curlywedge \mathfrak{z})$. Since $\mathfrak{x} \preccurlyeq \mathfrak{a} \overset{n}{\twoheadrightarrow} (\mathfrak{b} \curlywedge \mathfrak{y})$ we get $\mathfrak{x} \preccurlyeq \mathfrak{a} \overset{n}{\twoheadrightarrow} (\mathfrak{b} \curlywedge \mathfrak{z})$. Thus, $\mathfrak{z} \in \mathcal{X}$, and so $\mathcal{X} \in \Gamma(\mathbf{E})$. We claim that

$\mathcal{DN}_{\mathfrak{a}}(\mathcal{X}) = \mathcal{X}$. By Proposition 3.4(vi), we have $\mathcal{X} \subseteq \mathcal{DN}_{\mathfrak{a}}(\mathcal{X})$. For other side of inclusion, consider $\mathfrak{y} \in \mathcal{DN}_{\mathfrak{a}}(\mathcal{X})$. Then there exists $m \in \mathbb{N}$ such that $\mathfrak{a} \overset{m}{\twoheadrightarrow} \mathfrak{y} \in \mathcal{X} = (\curlywedge_b)^{-1}(\mathcal{DN}_{\mathfrak{a}}(\uparrow \mathfrak{x}))$. Thus there is $k \in \mathbb{N}$ such that

$$\mathfrak{x} \preccurlyeq \mathfrak{a} \overset{k}{\twoheadrightarrow} (\mathfrak{b} \curlywedge (\mathfrak{a} \overset{m}{\twoheadrightarrow} \mathfrak{y})).$$

By assumption we have

$$\begin{aligned}
\mathfrak{x} \preccurlyeq \mathfrak{a} \overset{k}{\twoheadrightarrow} (\mathfrak{b} \curlywedge (\mathfrak{a} \overset{m}{\twoheadrightarrow} \mathfrak{y})) &= (\mathfrak{a} \overset{k}{\twoheadrightarrow} \mathfrak{b}) \curlywedge (\mathfrak{a} \overset{k}{\twoheadrightarrow} (\mathfrak{a} \overset{m}{\twoheadrightarrow} \mathfrak{y})) \\
&\preccurlyeq (\mathfrak{a} \overset{k+m}{\twoheadrightarrow} \mathfrak{b}) \curlywedge (\mathfrak{a} \overset{k+m}{\twoheadrightarrow} \mathfrak{y})) \\
&= \mathfrak{a} \overset{k+m}{\twoheadrightarrow} (\mathfrak{b} \curlywedge \mathfrak{y})
\end{aligned}$$

By (3.2), $\mathfrak{y} \in \mathcal{X}$ and so $\mathcal{DN}_{\mathfrak{a}}(\mathcal{X}) \subseteq \mathcal{X}$. Hence, $\mathcal{X} = \mathcal{DN}_{\mathfrak{a}}(\mathcal{X}) \in \tau_{\mathfrak{a}}$. Therefore, $(\mathbf{E}, \curlywedge, \tau_{\mathfrak{a}})$ is a semi-topological algebra. □

Example 3.23. An equality algebra as in Example 3.3, satisfies in conditions of Theorem 3.22.

THEOREM 3.24. *Consider* $\mathbf{E}_1$ *and* $\mathbf{E}_2$ *be two bounded equality algebras and* $\mathbf{F} \in \mathcal{F}(\mathbf{E}_2)$. *If* $f : \mathbf{E}_1 \to \mathbf{E}_2$ *is an equality homomorphism, then*

(i) $f^{-1}(\mathcal{DN}_{f(\mathfrak{a})}(\mathbf{F})) = \mathcal{DN}_{\mathfrak{a}}(f^{-1}(\mathbf{F}))$.

(ii) $\mathcal{DN}_{\mathfrak{a}}(\ker(f)) = f^{-1}(\mathcal{DN}_{f(\mathfrak{a})}(\{1\}))$.

(iii) f *is continous map from* $(\mathbf{E}_1, \tau_{\mathfrak{a}})$ *to* $(\mathbf{E}_2, \tau_{f(\mathfrak{a})})$.

PROOF: (i) Since f is an equality homomorphism, then $f^{-1}(F) \in \mathcal{F}(\mathbf{E}_1)$ and

$$\begin{aligned} f^{-1}\left(\mathcal{DN}_{f(\mathfrak{a})}(\mathbf{F})\right) &= \{\mathfrak{x} \in \mathbf{E}_1 \mid f(\mathfrak{x}) \in \mathcal{DN}_{f(\mathfrak{a})}(\mathbf{F})\} \\ &= \{\mathfrak{x} \in \mathbf{E}_1 \mid \exists\, n \in \mathbb{N},\ s.t.\ f(\mathfrak{a}) \overset{n}{\twoheadrightarrow} f(\mathfrak{x}) \in \mathbf{F}\} \\ &= \{\mathfrak{x} \in \mathbf{E}_1 \mid \exists\, n \in \mathbb{N},\ s.t.\ f(\mathfrak{a} \overset{n}{\twoheadrightarrow} \mathfrak{x}) \in \mathbf{F}\} \\ &= \{\mathfrak{x} \in \mathbf{E}_1 \mid \exists\, n \in \mathbb{N},\ s.t.\ \mathfrak{a} \overset{n}{\twoheadrightarrow} \mathfrak{x} \in f^{-1}(\mathbf{F})\} \\ &= \mathcal{DN}_{\mathfrak{a}}(f^{-1}(\mathbf{F})). \end{aligned}$$

(ii) For $\mathfrak{x} \in \mathbf{E}_1$, we have

$$\begin{aligned} \mathfrak{x} \in \mathcal{DN}_{\mathfrak{a}}(\ker(f)) &\Leftrightarrow\ \mathfrak{a} \overset{n}{\twoheadrightarrow} \mathfrak{x} \in \ker(f),\ \text{for some } n \in \mathbb{N} \\ &\Leftrightarrow\ f\left(\mathfrak{a} \overset{n}{\twoheadrightarrow} \mathfrak{x}\right) = 1,\ \text{for some } n \in \mathbb{N} \\ &\Leftrightarrow\ f(\mathfrak{a}) \overset{n}{\twoheadrightarrow} f(\mathfrak{x}) = 1,\ \text{for some } n \in \mathbb{N} \\ &\Leftrightarrow f(\mathfrak{x}) \in \mathcal{DN}_{f(\mathfrak{a})}(\{1\}) \\ &\Leftrightarrow \mathfrak{x} \in f^{-1}\left(\mathcal{DN}_{f(\mathfrak{a})}(\{1\})\right). \end{aligned}$$

(iii) By (i), $f^{-1}\left(\mathcal{DN}_{f(\mathfrak{a})}(\mathbf{F})\right) = \mathcal{DN}_{\mathfrak{a}}(f^{-1}(\mathbf{F})) \in \tau_{\mathfrak{a}}$. Thus, f is continuous map from $(\mathbf{E}_1, \tau_{\mathfrak{a}})$ to $(\mathbf{E}_2, \tau_{f(\mathfrak{a})})$. □

Let $\mathbf{F} \in \mathcal{F}(\mathbf{E})$ and $\pi : \mathbf{E} \to \dfrac{\mathbf{E}}{\mathbf{F}}$ where $\pi(\mathfrak{x}) = [\mathfrak{x}]$. Clearly, $\left(\dfrac{\mathbf{E}}{\mathbf{F}}, \tau_{\mathfrak{a}}\right)$ is a topological space, where the set $\tau_{[\mathfrak{a}]} = \left\{\mathcal{DN}_{[\mathfrak{a}]}\left(\dfrac{\mathcal{X}}{\mathbf{F}}\right) \mid \mathbf{F} \subseteq \mathcal{X} \in \Gamma(\mathbf{E})\right\}$ is a topology on $\dfrac{\mathbf{E}}{\mathbf{F}}$ and the set $\left\{\mathcal{DN}_{[\mathfrak{a}]}\left(\uparrow [\mathfrak{x}]\right) \mid [\mathfrak{x}] \in \dfrac{\mathbf{E}}{\mathbf{F}}\right\}$ is a basis for the topology $\tau_{[\mathfrak{a}]}$ on $\dfrac{\mathbf{E}}{\mathbf{F}}$. In addition, the set $\widehat{\tau}_{\mathfrak{a}} = \{[\mathcal{DN}_{\mathfrak{a}}(\mathcal{X})] \mid \mathbf{F} \subseteq \mathcal{X} \in \Gamma(\mathbf{E})\}$ is a quotient topology on $\dfrac{\mathbf{E}}{\mathbf{F}}$ and $\left(\dfrac{\mathbf{E}}{\mathbf{F}}, \widehat{\tau}_{\mathfrak{a}}\right)$ is the quotient topology space. The set $\{[\mathcal{DN}_{\mathfrak{a}}\left(\uparrow \mathfrak{x}\right)] \mid \mathfrak{x} \in \mathbf{E}\}$ is a basis for the quotient topology $\widehat{\tau}_{\mathfrak{a}}$.

Proposition 3.25. Let $\mathbf{F}, \mathbf{G} \in \mathcal{F}(\mathbf{E})$ such that $\mathbf{F} \subseteq \mathbf{G}$. Then $\mathcal{DN}_{[\mathfrak{a}]}\left(\dfrac{\mathbf{G}}{\mathbf{F}}\right) = \dfrac{\mathcal{DN}_{\mathfrak{a}}(\mathbf{G})}{\mathbf{F}}$.

PROOF: For any $\mathfrak{x} \in \mathbf{E}$, we have

$$\begin{aligned}\mathcal{DN}_{[\mathfrak{a}]}\left(\frac{\mathbf{G}}{\mathbf{F}}\right) &= \left\{[\mathfrak{x}] \in \frac{\mathbf{E}}{\mathbf{F}} \mid \exists n \in \mathbb{N},\ s.t.\ [\mathfrak{a}] \overset{n}{\twoheadrightarrow} [\mathfrak{x}] \in \frac{\mathbf{G}}{\mathbf{F}}\right\} \\ &= \left\{[\mathfrak{x}] \in \frac{\mathbf{E}}{\mathbf{F}} \mid \exists n \in \mathbb{N},\ s.t.\ [\mathfrak{a} \overset{n}{\twoheadrightarrow} \mathfrak{x}] \in \frac{\mathbf{G}}{\mathbf{F}}\right\} \\ &= \left\{[\mathfrak{x}] \in \frac{\mathbf{E}}{\mathbf{F}} \mid \exists n \in \mathbb{N},\ s.t.\ \mathfrak{a} \overset{n}{\twoheadrightarrow} \mathfrak{x} \in \mathbf{G}\right\} \\ &= \left\{[\mathfrak{x}] \in \frac{\mathbf{E}}{\mathbf{F}} \mid \mathfrak{x} \in \mathcal{DN}_{\mathfrak{a}}(\mathbf{G})\right\} \\ &= \frac{\mathcal{DN}_{\mathfrak{a}}(\mathbf{G})}{\mathbf{F}}.\end{aligned}$$

□

THEOREM 3.26. *Let* $\mathbf{F} \in \mathcal{F}(\mathbf{E})$. *Then*

(i) $[\mathcal{DN}_{\mathfrak{a}}(\uparrow \mathfrak{x})] \subseteq \mathcal{DN}_{[\mathfrak{a}]}(\uparrow [\mathfrak{x}])$, *for any* $\mathfrak{x} \in \mathbf{E}$.

(ii) *The quotient topology* $\widehat{\tau}_{\mathfrak{a}}$ *is finer than the topology* $\tau_{[\mathfrak{a}]}$.

PROOF: (i) Consider $[\mathfrak{y}] \in [\mathcal{DN}_{\mathfrak{a}}(\uparrow \mathfrak{x})]$. Then there exists $\mathfrak{z} \in \mathcal{DN}_{\mathfrak{a}}(\uparrow \mathfrak{x})$ such that $[\mathfrak{y}] = [\mathfrak{z}]$. Thus there is $n \in \mathbb{N}$ such that $\mathfrak{x} \twoheadrightarrow (\mathfrak{a} \overset{n}{\twoheadrightarrow} \mathfrak{z}) = 1$, and so $[\mathfrak{x}] \twoheadrightarrow ([\mathfrak{a}] \overset{n}{\twoheadrightarrow} [\mathfrak{z}]) = [1]$. Since $[\mathfrak{y}] = [\mathfrak{z}]$ we get $[\mathfrak{x}] \twoheadrightarrow ([\mathfrak{a}] \overset{n}{\twoheadrightarrow} [\mathfrak{y}]) = [1]$. Hence, $[\mathfrak{y}] \in \mathcal{DN}_{[\mathfrak{a}]}(\uparrow [\mathfrak{x}])$.
(ii) By (i) the proof is clear. □

THEOREM 3.27. *Consider* $\mathbf{E}$ *be involutive and* $\mathbf{F} \in \mathcal{F}(\mathbf{E})$. *Then*

(i) $[\mathcal{DN}_{\mathfrak{a}}(\uparrow \mathfrak{x})] = \mathcal{DN}_{[\mathfrak{a}]}(\uparrow [\mathfrak{x}])$, *for any* $\mathfrak{x} \in \mathbf{E}$.

(ii) *The topology* $\widehat{\tau}_{\mathfrak{a}}$ *and the topology* $\tau_{[\mathfrak{a}]}$ *are same on* $\dfrac{\mathbf{E}}{\mathbf{F}}$.

PROOF: (i) By Theorem 3.26(i), for any $\mathfrak{x} \in \mathbf{E}$, we have $[\mathcal{DN}_{\mathfrak{a}}(\uparrow \mathfrak{x})] \subseteq \mathcal{DN}_{[\mathfrak{a}]}(\uparrow [\mathfrak{x}])$. Conversely, suppose $[\mathfrak{y}] \in \mathcal{DN}_{[\mathfrak{a}]}(\uparrow [\mathfrak{x}])$. Since

$$\mathcal{DN}_{[\mathfrak{a}]}(\uparrow[\mathfrak{x}]) = \left\{[\mathfrak{y}] \in \frac{\mathbf{E}}{\mathbf{F}} \mid \exists n \in \mathbb{N},\ s.t.\ [\mathfrak{x}] \preccurlyeq [\mathfrak{a}] \overset{n}{\twoheadrightarrow} [\mathfrak{y}]\right\}$$
$$= \left\{[\mathfrak{y}] \in \frac{\mathbf{E}}{\mathbf{F}} \mid \exists n \in \mathbb{N},\ s.t.\ \mathfrak{x} \twoheadrightarrow (\mathfrak{a} \overset{n}{\twoheadrightarrow} \mathfrak{y}) \in \mathbf{F}\right\}$$

Since $\mathfrak{x} \twoheadrightarrow (\mathfrak{a} \overset{n}{\twoheadrightarrow} \mathfrak{y}) \in \mathbf{F}$, there is $\mathfrak{t} \in \mathbf{F}$ such that $1 = \mathfrak{t} \twoheadrightarrow (\mathfrak{x} \twoheadrightarrow (\mathfrak{a} \overset{n}{\twoheadrightarrow} \mathfrak{y}))$ and by Proposition 2.2(v), we have $1 = \mathfrak{x} \twoheadrightarrow (\mathfrak{a} \overset{n}{\twoheadrightarrow} (\mathfrak{t}\twoheadrightarrow\mathfrak{y}))$. Thus, $\mathfrak{t} \twoheadrightarrow \mathfrak{y} \in \mathcal{DN}_{\mathfrak{a}}(\uparrow \mathfrak{x})$. Since $\mathfrak{t} \in \mathbf{F}$ we get $[\mathfrak{t}] = [1]$, and so $[\mathfrak{t} \twoheadrightarrow \mathfrak{y}] = [\mathfrak{t}] \twoheadrightarrow [\mathfrak{y}] = [\mathfrak{y}]$. Hence, $[\mathfrak{y}] \in [\mathcal{DN}_{\mathfrak{a}}(\uparrow \mathfrak{x})]$, and so $\mathcal{DN}_{[\mathfrak{a}]}(\uparrow [\mathfrak{x}]) \subseteq [\mathcal{DN}_{\mathfrak{a}}(\uparrow \mathfrak{x})]$. Therefore, $[\mathcal{DN}_{\mathfrak{a}}(\uparrow \mathfrak{x})] = \mathcal{DN}_{[\mathfrak{a}]}(\uparrow [\mathfrak{x}])$.
(ii) By (i) the proof is clear. □

Let $\mathbf{E}_1$ and $\mathbf{E}_2$ be two equality algebras. Then $\mathbf{G} \in \mathcal{F}(\mathbf{E}_1 \times \mathbf{E}_2)$ if and only if there exist $\mathbf{F}_1 \in \mathcal{F}(\mathbf{E}_1)$ and $\mathbf{F}_2 \in \mathcal{F}(\mathbf{E}_2)$ such that $\mathbf{G} = \mathbf{F}_1 \times \mathbf{F}_2$.

PROPOSITION 3.28. Consider $\mathbf{F}, \mathbf{G} \in \mathcal{F}(\mathbf{E})$ and $\mathfrak{a} \in \mathbf{E}$. Then

$$\mathcal{DN}_{\mathfrak{a}}(\mathbf{F}) \times \mathcal{DN}_{\mathfrak{a}}(\mathbf{G}) = \mathcal{DN}_{\mathfrak{a}}(\mathbf{F} \times \mathbf{G}).$$

PROOF: Let $\mathfrak{a} \in \mathbf{E}$. Then by Lemma 3.5, we have

$$\begin{aligned}
\mathcal{DN}_{\mathfrak{a}}(\mathbf{F})\times\mathcal{DN}_{\mathfrak{a}}(\mathbf{G}) &= \{(\mathfrak{x},\mathfrak{y})\in\mathbf{E}\times\mathbf{E} \mid \mathfrak{x} \in \mathcal{DN}_{\mathfrak{a}}(\mathbf{F}),\ \mathfrak{y} \in \mathcal{DN}_{\mathfrak{a}}(\mathbf{G})\}\\
&= \{(\mathfrak{x},\mathfrak{y}) \in \mathbf{E}\times\mathbf{E} \mid \exists n, m\in\mathbb{N}\ s.t.\ \mathfrak{a} \overset{n}{\twoheadrightarrow} \mathfrak{x} \in \mathbf{F}, \mathfrak{a} \overset{m}{\twoheadrightarrow} \mathfrak{y}\in\mathbf{G}\}\\
&= \{(\mathfrak{x},\mathfrak{y})\in\mathbf{E}\times\mathbf{E} \mid \exists \mathfrak{t} \in \mathbb{N}\ s.t.\ \mathfrak{t} \geq \max\{n,m\}\\
&\qquad s.t.\ \mathfrak{a} \overset{t}{\twoheadrightarrow} \mathfrak{x} \in \mathbf{F},\ \mathfrak{a} \overset{t}{\twoheadrightarrow} \mathfrak{y} \in \mathbf{G}\}\\
&= \{(\mathfrak{x},\mathfrak{y})\in\mathbf{E}\times\mathbf{E} \mid \exists \mathfrak{t} \in \mathbb{N}\ s.t.\ \mathfrak{t} \geq \max\{n,m\}\\
&\qquad s.t.\ (\mathfrak{a},\mathfrak{a}) \overset{t}{\twoheadrightarrow} (\mathfrak{x},\mathfrak{y})\in\mathbf{F} \times \mathbf{G}\}\\
&= \mathcal{DN}_{\mathfrak{a}}(\mathbf{F}\times\mathbf{G}).
\end{aligned}$$
□

THEOREM 3.29. *Consider* $\mathbf{F}, \mathbf{G} \in \mathcal{F}(\mathbf{E})$. *Then*

$$\psi : \frac{\mathbf{E} \times \mathbf{E}}{\mathcal{DN}_{\mathfrak{a}}(\mathbf{F} \times \mathbf{G})} \to \frac{\mathbf{E}}{\mathcal{DN}_{\mathfrak{a}}(\mathbf{F})} \times \frac{\mathbf{E}}{\mathcal{DN}_{\mathfrak{a}}(\mathbf{G})}$$

is a homeomorphism, for any $\mathfrak{a} \in \mathbf{E}$.

Proof: Define the map $\varphi : \mathbf{E} \times \mathbf{E} \to \dfrac{\mathbf{E}}{\mathcal{DN}_{\mathfrak{a}}(\mathbf{F})} \times \dfrac{\mathbf{E}}{\mathcal{DN}_{\mathfrak{a}}(\mathbf{G})}$ such that $\varphi(\mathfrak{x}, \mathfrak{y}) = \big([\mathfrak{x}]_{\mathcal{DN}_{\mathfrak{a}}(\mathbf{F})}, [\mathfrak{y}]_{\mathcal{DN}_{\mathfrak{a}}(\mathbf{G})}\big)$. Obviously, the map φ is onto and $(\mathfrak{x}, \mathfrak{y}) \in \ker\varphi$ if and only if $[\mathfrak{x}]_{\mathcal{DN}_{\mathfrak{a}}(\mathbf{F})} = [1]_{\mathcal{DN}_{\mathfrak{a}}(\mathbf{F})}$ and $[\mathfrak{y}]_{\mathcal{DN}_{\mathfrak{a}}(\mathbf{G})} = [1]_{\mathcal{DN}_{\mathfrak{a}}(\mathbf{G})}$. Hence, Proposition 3.28, $\mathcal{DN}_{\mathfrak{a}}(\mathbf{F}) \times \mathcal{DN}_{\mathfrak{a}}(\mathbf{G}) = \mathcal{DN}_{\mathfrak{a}}(\mathbf{F} \times \mathbf{G})$ and so $\ker(\varphi) = \mathcal{DN}_{\mathfrak{a}}(\mathbf{F}) \times \mathcal{DN}_{\mathfrak{a}}(\mathbf{G})$.

In addition, suppose

$$\psi : \frac{\mathbf{E} \times \mathbf{E}}{\mathcal{DN}_{\mathfrak{a}}(\mathbf{F} \times \mathbf{G})} \longrightarrow \frac{\mathbf{E}}{\mathcal{DN}_{\mathfrak{a}}(\mathbf{F})} \times \frac{\mathbf{E}}{\mathcal{DN}_{\mathfrak{a}}(\mathbf{G})}$$

defined by $\psi\big([(\mathfrak{x}, \mathfrak{y})]_{\mathcal{DN}_{\mathfrak{a}}(\mathbf{F}\times\mathbf{G})}\big) = \big([\mathfrak{x}]_{\mathcal{DN}_{\mathfrak{a}}(\mathbf{F})}, [\mathfrak{y}]_{\mathcal{DN}_{\mathfrak{a}}(\mathbf{G})}\big)$. Thus by the first isomorphism theorem ψ is an isomorphism. Suppose that $\mathcal{U}$ is an open subset of $\dfrac{\mathbf{E}}{\mathcal{DN}_{\mathfrak{a}}(\mathbf{F})} \times \dfrac{\mathbf{E}}{\mathcal{DN}_{\mathfrak{a}}(\mathbf{G})}$. Then there exist open subset $V, W \in \tau_{\mathfrak{a}}$ such that $\mathcal{U} = \dfrac{V}{\mathcal{DN}_{\mathfrak{a}}(\mathbf{F})} \times \dfrac{W}{\mathcal{DN}_{\mathfrak{a}}(\mathbf{G})}$. Clearly, $\psi^{-1}(\mathcal{U}) = \dfrac{V \times W}{\mathcal{DN}_{\mathfrak{a}}(\mathbf{F} \times \mathbf{G})}$ is an open subset of $\dfrac{\mathbf{E} \times \mathbf{E}}{\mathcal{DN}_{\mathfrak{a}}(\mathbf{F} \times \mathbf{G})}$. Hence, ψ is a continuous map. Therefore, ψ is a homeomorphism. □

4. Conclusion

In this paper, a special subset of equality algebra which is upset is introduced and by using this upsets a topology on a bounded equality algebras is constructed and some of their topological properties, such as some types of topological space (Hausdorff, T_0-space and T_1-space) and connectedness are investigated. In addition, the relation between closed and compact sets in this topology is expressed. Moreover, by considering the binary operation $\twoheadrightarrow$ and the constructed topology on the bounded equality algebra $\mathbf{E}$, the notion of a semi-topological algebra is introduced and it is proved that any involutive equality algebra is a right semi-topological algebra and a semi-$\curlywedge$-topological algebra but not a left semi-topological algebra. Finally, converse image, product and quotient topology on equality algebra are studied and showed that under what condition finer topology can be made.

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REVISITING THE ADEQUACY THEOREM FOR FRAGMENTS OF ŁUKASIEWICZ LOGIC

Abstract

A. V. Figallo introduced the 3-valued Super Łukasiewicz logic expanded with the Δ operator, denoted as $\mathcal{C}_3^{\rightarrowtail,\Delta}$, in 1990. This operator is used in the definition of 3-valued Łukasiewicz algebras, and it is not possible to recover Δ through implication and top in Super Łukasiewicz logic. On the other hand, Baaz introduced the Δ operator in Gödel logic, both in its propositional and quantified versions. Subsequently, this operator was extensively studied in the field of fuzzy logic.

In this paper, we prove a strong version of the Adequacy Theorem for $\mathcal{C}_3^{\rightarrowtail,\Delta}$. As a consequence, we demonstrate that the Deduction Theorem does not hold in this calculus. Furthermore, we introduce the first-order version of $\mathcal{C}_3^{\rightarrowtail,\Delta}$ and establish soundness and completeness results by adapting a recently developed

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algebraic technique. In this context, our presentation differs from others in the literature because we need to construct a special homomorphism, brought from the algebraic study of $\mathcal{C}_3^{\rightarrow,\Delta}$, in the syntactic setting. This homomorphism is also necessary to determine the generating algebras. While we can ascertain that the logical system is algebraizable by a (quasi-)variety of algebras, we cannot know a priori which are the subdirectly irreducible algebras.

Keywords: implicational fragment of Łukasiewicz logic, 3-valued Łukasiewicz logic, Δ operator, first-order logics.

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1. Introduction

Many-valued logics, and in particular Łukasiewicz logic, have played a central role in the development of non-classical reasoning. From its origin, Łukasiewicz's insight of assigning intermediate truth-values between truth and falsity opened the door to formalizing vagueness and uncertainty. Over the decades, this logic has influenced diverse areas such as fuzzy logic, artificial intelligence, and philosophical logic. The semantic richness of Łukasiewicz logic and its algebraic counterpart, MV-algebras, continue to serve as a testbed for deep foundational questions in logic.

However, one of the fundamental theorems of logic—the Adequacy Theorem—has a peculiar status in the context of Łukasiewicz logic and its fragments. While completeness and soundness are often provable through elegant Hilbert-style or algebraic methods, the lack of a Deduction Theorem in some fragments challenges the traditional equivalence between syntactic and semantic consequence. This invites a re-examination of what "adequacy" means in systems without full deductive strength, and how proof theory and algebra can still be aligned.

Łukasiewicz logic is perhaps the oldest and most studied many-valued logic in the literature. Its semantics was first studied by Łukasiewicz himself and formalized by Chang using the structure of MV-algebras. MV-algebras are now the standard semantics for Łukasiewicz logic and have played a central role in the development of algebraic logic. Furthermore, the connection between MV-algebras and fuzzy logic has opened new areas of application in artificial intelligence and computer science.

In this paper, we revisit the Adequacy Theorem for fragments of Łukasiewicz logic, both in the propositional and quantified case. These fragments are obtained by restricting the set of connectives to subsets that do not validate the deduction theorem. We provide new proofs of adequacy that avoid classical dependence on the deduction theorem, and we analyze how these fragments behave semantically through algebraic techniques. Our results not only clarify the status of adequacy in these fragments but also suggest possible directions for extending the concept to other non-classical logics lacking standard deductive mechanisms.

The techniques developed herein provide a new framework for establishing adequacy theorems in systems lacking the Deduction Theorem, offering a unified algebraic perspective that complements and extends existing approaches.

2. Preliminaries

In this section, we will provide the necessary background to present our paper. To this end, we discuss some algebraic properties of the class of 3-valued Łukasiewicz residuation algebras expanded by the Δ operator, which will be relevant to the Hilbert system associated with them.

First, let's recall that a 3-valued Łukasiewicz residuation algebra is an algebra $\langle A, \rightarrowtail, 1\rangle$ of type $(2,0)$ (briefly, $Ł_3^{\rightarrowtail}$-algebras) that satisfies the following identities (see, e.g., [25, 21]):

(Ł1) $x \rightarrowtail (y \rightarrowtail x) = 1$,

(Ł2) $(x \rightarrowtail y) \rightarrowtail ((y \rightarrowtail \gamma) \rightarrowtail (x \rightarrowtail \gamma)) = 1$,

(Ł3) $(x \rightarrowtail y) \rightarrowtail y = (y \rightarrowtail x) \rightarrowtail x$,

(Ł4) $((x \rightarrowtail y) \rightarrowtail (y \rightarrowtail x)) \rightarrowtail (y \rightarrowtail x) = 1$,

(Ł5) $1 \rightarrowtail x = x$,

(Ł6) $((x \rightarrowtail (x \rightarrowtail y)) \rightarrowtail x) \rightarrowtail x = 1$.

It is well known that the identities from Ł1 to Ł5 define Łukasiewicz residuation algebras. These structures were originally introduced and studied in connection with the implicative fragment of Łukasiewicz logic (see [25, 21]). Moreover, an order relation can be defined on every such algebra **A** as follows: $x \leq y$ iff $x \rightarrowtail y = 1$. We can also define a supremum for any $x, y \in A$ as $x \vee y := (x \rightarrowtail y) \rightarrowtail y$; and we also have that $z \leq 1$ for every $z \in A$. Hence, axiom Ł3 expresses the fact that for all $x, y \in A$, we have $(x \rightarrowtail y) \vee (y \rightarrowtail x) = 1$.

Now, let's move on to the class of algebras introduced and studied in [11].

Definition 2.1. An $Ł_3^{\rightarrowtail,\Delta}$-algebra is an algebra $(A, \rightarrowtail, \Delta, 1)$ of type $(2, 1, 0)$ such that $(A, \rightarrowtail, 1)$ is an $Ł_3^{\rightarrowtail}$-algebra, and the following identities are satisfied:

(ΔŁ1) $\Delta x \rightarrowtail y = x \rightarrowtail (x \rightarrowtail y)$,

(ΔŁ2) $\Delta(\Delta x \rightarrowtail y) = \Delta x \rightarrowtail \Delta y$.

In what follows, we will consider a new binary connective $\Rightarrow$ defined as follows: $x \Rightarrow y := \Delta x \rightarrowtail y$. With this definition, we can introduce the following concept:

Definition 2.2. For any $Ł_3^{\rightarrowtail,\Delta}$-algebra **A**, a subset D is considered an implicative filter of A if $1 \in D$, and if $x, x \Rightarrow y \in D$, then $y \in D$. This notion extends the classical concept of implicative filters in Łukasiewicz-type algebras (see [25]). We denote by $\mathcal{D}(A)$ the set of all implicative filters of A.

For any $Ł_3^{\rightarrowtail,\Delta}$-algebra **A**, we denote $Con(A)$ as the set of all congruences of **A**. Given an implicative filter D, the relation $R(D) = \{(x, y) \in A^2 : x \Rightarrow y, y \Rightarrow x \in D\}$ defines a congruence of **A**. Additionally, given a congruence Θ of **A**, $|1|_\Theta$ represents the class of 1 under Θ, and it is also an implicative filter. A crucial lemma in this context is:

Lemma 2.3. ([11]). *There exists a lattice isomorphism between $Con(A)$ and $D(A)$.*

Now, let's introduce a definition by A. Monteiro:

Definition 2.4. (A. Monteiro). For an $Ł_3^{\rightarrowtail,\Delta}$-algebra $\mathbf{A}$, $D \in \mathcal{D}(A)$, and $p \in A$, we say that D is an implicative filter tied to p if $p \notin D$ and for any $D' \in \mathcal{D}(A)$ such that $D \subsetneq D'$, then $p \in D'$.

Here's a proposition along with some properties:

Proposition 2.5. ([10, p. 106]). Let $\mathbf{A}$ be an $Ł_3^{\rightarrowtail,\Delta}$-algebra and for any $x, y, z \in A$, the following properties hold:

(Ł7) $1 \Rightarrow x = x$,
(Ł8) $x \Rightarrow x = 1$,
(Ł9) $x \Rightarrow (y \Rightarrow z) = (x \Rightarrow y) \Rightarrow (x \Rightarrow z)$,
(Ł10) $x \Rightarrow (y \Rightarrow x) = 1$,
(Ł11) $((x \Rightarrow y) \Rightarrow x) \Rightarrow x = 1$.

Recall that for a given $Ł_3^{\rightarrowtail,\Delta}$-algebra $\mathbf{A}$, we say that an implicative filter M is maximal if M is proper and for any $D \in \mathcal{D}(A)$, $M \subseteq D$ implies $D = A$ or $M = D$. Note that the above proposition provides fundamental properties of the implication $\Rightarrow$, which will play a key role in what follows.

Lastly, let's consider maximal implicative filters and a related lemma:

Lemma 2.6. ([17, Lemma 3.9]). *Let* $\mathbf{A}$ *be an* $Ł_3^{\rightarrowtail,\Delta}$*-algebra, and* M *is a maximal implicative filter of* A*. Then, for every* $x \in A \setminus M$*, we have that* $x \Rightarrow y \in A$ *for every* $y \in A$*.*

For an $Ł_3^{\rightarrowtail,\Delta}$-algebra $\mathbf{A}$ and according to Lemma 2.6 and (Ł11), we can conclude the following corollary:

Corollary 2.7. ([17, Section 6]). For a given $Ł_3^{\rightarrowtail,\Delta}$-algebra $\mathbf{A}$, each implicative filter tied to some element of A is maximal, and vice versa.

Finally, it is worth recalling that in [13], the authors studied n-valued Łukasiewicz residuation algebras expanded with Moisil operators. The class of $Ł_3^{\rightarrowtail,\Delta}$-algebras constitutes the particular case corresponding to $n = 3$, and was analyzed in detail within the broader context of the n-valued setting.

$\rightarrowtail$	0	$\frac{1}{2}$	1	Δx
0	1	1	1	0
$\frac{1}{2}$	$\frac{1}{2}$	1	1	0
1	0	$\frac{1}{2}$	1	1

Table 1: The operations of the $Ł_3^{\rightarrowtail,\Delta}$-algebra $\mathbb{C}_3$.

THEOREM 2.8. ([13, Theorem 3.17]). *The variety of $Ł_3^{\rightarrowtail,\Delta}$-algebras is semisimple. Furthermore, the generating algebras are $\mathbb{C}_3$ and the unique subalgebra with support $\{0,1\}$, where the support of $\mathbb{C}_3$ is the set $\{0,\frac{1}{2},1\}$, and the operations $\rightarrowtail$ and Δ are defined by Table 1.*

3. A Calculus for $Ł_3^{\rightarrowtail,\Delta}$-algebras: $CŁ_3^{\rightarrowtail,\Delta}$

In this section, we introduce a Hilbert-style calculus for $Ł_3^{\rightarrowtail,\Delta}$-algebras, which was presented in [11]. We will provide all necessary definitions and results to establish, first, a weak and, subsequently, a strong version of the Adequacy Theorem.

To this end, let us consider a denumerable set Var of propositional variables and the propositional signature $\{\rightarrowtail,\Delta\}$. The propositional language generated by this signature over Var will be denoted by For; recall that For is the absolutely free algebra of propositional formulas.

The three-valued implicative propositional calculus of Łukasiewicz, denoted $CŁ_3^{\rightarrowtail,\Delta}$, is defined by the following axiom schemes:

(Ax1) $\alpha \rightarrowtail (\beta \rightarrowtail \alpha)$,

(Ax2) $(\alpha \rightarrowtail \beta) \rightarrowtail ((\beta \rightarrowtail \gamma) \rightarrowtail (\alpha \rightarrowtail \gamma))$,

(Ax3) $((\alpha \rightarrowtail \beta) \rightarrowtail \beta) \rightarrowtail ((\beta \rightarrowtail \alpha) \rightarrowtail \alpha)$,

(Ax4) $((\alpha \rightarrowtail \beta) \rightarrowtail (\beta \rightarrowtail \alpha)) \rightarrowtail (\beta \rightarrowtail \alpha)$,

(Ax5) $((\alpha \rightarrowtail (\alpha \rightarrowtail \beta)) \rightarrowtail \alpha) \rightarrowtail \alpha$,

(Ax6) $(\Delta\alpha \rightarrowtail \Delta\beta) \rightarrowtail \Delta(\Delta\alpha \rightarrowtail \beta)$, (Ax7) $\Delta(\Delta\alpha \rightarrowtail \beta) \rightarrowtail (\alpha \rightarrowtail (\alpha \rightarrowtail \Delta\beta))$,

(Ax8) $(\alpha \rightarrowtail (\alpha \rightarrowtail \beta)) \rightarrowtail (\Delta\alpha \rightarrowtail \beta)$.

The only inference rule is Modus Ponens:

(MP) $\dfrac{\alpha, \quad \alpha \rightarrowtail \beta}{\beta}$.

Within this calculus, we define two non-primitive connectives, $\vee$ and ∇, as follows:

$$\alpha \vee \beta := (\alpha \rightarrowtail \beta) \rightarrowtail \beta,$$
$$\nabla\alpha := (\alpha \rightarrowtail \Delta\alpha) \rightarrowtail \alpha.$$

We write $\Gamma \vdash \alpha$ to denote that there exists a derivation of α in $\mathrm{CŁ}_3^{\rightarrowtail,\Delta}$ from hypotheses in the set Γ. The following well-known results, which are valid in Super-Łukasiewicz logic, also hold in our calculus:

We briefly comment on the role of axiom (Ax5) in our system. Although it is natural from the algebraic perspective of ${}_3^{\rightarrowtail,\Delta}$-algebras, we have not investigated whether it is independent from the remaining axioms. Its inclusion ensures the validity of identity (Ł6) in the associated Lindenbaum–Tarski algebra, which is used in several key arguments throughout the paper. A detailed study of its possible redundancy is left for future work.

Proposition 3.1. The following theorems and rules hold in $\mathrm{CŁ}_3^{\rightarrowtail,\Delta}$:

T1. $\vdash ((\alpha \rightarrowtail \beta) \rightarrowtail \gamma) \rightarrowtail (\beta \rightarrowtail \gamma)$,

R1. $\dfrac{\alpha \rightarrowtail \beta, \beta \rightarrowtail \gamma}{\alpha \rightarrowtail \gamma}$,

T2. $\vdash \alpha \rightarrowtail \alpha \vee \beta$,

T3. $\vdash ((\alpha \vee \gamma) \rightarrowtail \beta) \rightarrowtail (\alpha \rightarrowtail \beta)$;

T4. $\vdash ((\alpha \vee \gamma) \rightarrowtail (\beta \rightarrowtail \gamma)) \rightarrowtail (\alpha \rightarrowtail (\beta \rightarrowtail \gamma))$, T5. $\vdash (\alpha \rightarrowtail (\beta \rightarrowtail \gamma)) \rightarrowtail ((\beta \vee \gamma) \rightarrowtail (\alpha \rightarrowtail \gamma))$,

T6. $\vdash (\alpha \rightarrowtail (\beta \rightarrowtail \gamma)) \rightarrowtail (\beta \rightarrowtail (\alpha \rightarrowtail \gamma))$,

T7. $\vdash \beta \rightarrowtail (\alpha \rightarrowtail \alpha)$,

T8. $\vdash \alpha \rightarrowtail \alpha$,

R2. $\dfrac{\alpha \rightarrowtail \beta}{(\gamma \rightarrowtail \alpha) \rightarrowtail (\gamma \rightarrowtail \beta)}$,

T9. $\vdash \big(((\beta \rightarrowtail \beta) \rightarrowtail \alpha) \rightarrowtail \alpha\big)$,

R3. $\dfrac{\alpha \rightarrowtail \beta}{(\beta \rightarrowtail \gamma) \rightarrowtail (\alpha \rightarrowtail \gamma)}$.

PROOF:

T1: Follows from Ax1 and MP.

R1: Follows from Ax2 and MP.

T2: Follows from Ax1, Ax2, R1, and MP.

T3: Follows from Ax2, T2, and MP.

T4: Follows from Ax2, T2, and MP.

T5: Follows from Ax2 and the definition of $\vee$.

T6: Follows from T4, T3, and MP.

T7: Follows from T6, Ax1, and R1.

T8: Follows from T7, Ax1, and MP.

R2: Follows from T6, Ax2, and MP.

T9: Follows from T8, Ax2, Ax1, Ax3, and MP. □

In what follows, we present a lemma required for the remainder of the article. We include sketchy proofs for some theorems and rules that are derivable in $C_3^{\rightarrowtail,\Delta}$, whereas in other cases, we provide detailed proofs when the original ones are not entirely clear to us.

LEMMA 3.2. ([11]) *The following formulæ and rules hold in the logic* $C_3^{\rightarrowtail,\Delta}$*:*

(ΔT1): $\vdash \Delta(\Delta\alpha \rightarrowtail \alpha)$,

(ΔT2): $\vdash \alpha \rightarrowtail (\alpha \rightarrowtail \Delta\alpha)$,

(ΔT3): $\vdash \Delta\alpha \rightarrowtail \alpha$,

(ΔT4): $\vdash (\Delta\alpha \rightarrowtail \beta) \rightarrowtail (\alpha \rightarrowtail (\alpha \rightarrowtail \beta))$,

(ΔT5): $\vdash \Delta(\Delta\alpha \rightarrowtail \beta) \rightarrowtail (\Delta\alpha \rightarrowtail \Delta\beta)$,

(ΔR1): $\dfrac{\alpha}{\Delta\alpha}$,

(ΔR2): $\dfrac{\alpha \rightarrowtail \beta}{\Delta\alpha \rightarrowtail \Delta\beta}$,

(ΔT6): $\vdash \alpha \rightarrowtail \nabla\alpha$,

(ΔT7): $\vdash (\nabla\alpha \rightarrowtail \beta) \rightarrowtail (\alpha \rightarrowtail \beta)$.

PROOF:

(ΔT1): It follows from Ax6, T8, and MP.

(ΔT2): It follows from Ax8, Ax1, and MP.

(ΔT3): It follows from Ax8, Ax1, and MP.

(ΔT4):

1.	$\alpha \rightarrowtail (\alpha \rightarrowtail \Delta\alpha)$	
2.	$(\beta \rightarrowtail \Delta\alpha) \rightarrowtail (\alpha \rightarrowtail (\alpha \rightarrowtail \Delta\alpha))$	1. (MP) (Ax1)
3.	$\alpha \rightarrowtail ((\beta \rightarrowtail \Delta\alpha) \rightarrowtail (\alpha \rightarrowtail \Delta\alpha))$	2. and (T6)

4. $((\beta \rightarrowtail \Delta\alpha) \rightarrowtail (\alpha \rightarrowtail \Delta\alpha)) \rightarrowtail (\alpha \rightarrowtail ((\beta \rightarrowtail \Delta\alpha) \rightarrowtail \Delta\alpha))$ (T6)
5. $\alpha \rightarrowtail (\alpha \rightarrowtail ((\beta \rightarrowtail \Delta\alpha) \rightarrowtail \Delta\alpha))$ 3., 4. and (R2)
6. $((\beta \rightarrowtail \Delta\alpha) \rightarrowtail \Delta\alpha) \rightarrowtail ((\Delta\alpha \rightarrowtail \beta) \rightarrowtail \beta)$ (Ax3)
7. $(\alpha \rightarrowtail (\alpha \rightarrowtail ((\beta \rightarrowtail \Delta\alpha) \rightarrowtail \Delta\alpha))) \rightarrowtail (\alpha \rightarrowtail (\alpha \rightarrowtail ((\Delta\alpha \rightarrowtail \beta) \rightarrowtail \beta)))$ 6. and (R2)
8. $(\alpha \rightarrowtail (\alpha \rightarrowtail ((\Delta\alpha \rightarrowtail \beta) \rightarrowtail \beta)))$ 5., 7., and MP
9. $(\alpha \rightarrowtail ((\Delta\alpha \rightarrowtail \beta) \rightarrowtail \beta)) \rightarrowtail ((\Delta\alpha \rightarrowtail \beta) \rightarrowtail (\alpha \rightarrowtail \beta))$ (T6)
10. $(\alpha \rightarrowtail ((\Delta\alpha \rightarrowtail \beta) \rightarrowtail (\alpha \rightarrowtail \beta))$ (R5) and (T6)
11. $((\Delta\alpha \rightarrowtail \beta) \rightarrowtail (\alpha \rightarrowtail (\alpha \rightarrowtail \beta))$ 10., 9., and MP

(ΔT5): Follows from Ax3, Ax2, and MP.

(ΔR1):

1. $\vdash \alpha$ hyp.
2. $\vdash \Delta\alpha \rightarrowtail \Delta\alpha$ (T8)
3. $\vdash \Delta(\Delta\alpha \rightarrowtail \alpha) \rightarrowtail (\alpha \rightarrowtail (\alpha \rightarrowtail \Delta\alpha))$ (Ax7)
5. $\vdash (\Delta\alpha \rightarrowtail \Delta\alpha) \rightarrowtail (\Delta(\Delta\alpha \rightarrowtail \alpha))$ (Ax6)
5. $\vdash \Delta(\Delta\alpha \rightarrowtail \alpha)$ 2., 4., and (MP)
6. $\vdash (\alpha \rightarrowtail (\alpha \rightarrowtail \Delta\alpha))$ 5., 3., and (MP)
7. $\vdash \Delta\alpha$ 1., 6., and (MP)

(ΔR2): Follows from R2, (ΔT3), (ΔR1), (ΔT5), and MP.

(ΔT6): Follows from Ax1 and the definition of ∇.

(ΔT7): Follows from (ΔT6) and $R3$.

□

Now, we will define a relation on formulas as follows: for the formulas α and β given, we write $\alpha \equiv \beta$ if and only if $\vdash \alpha \rightarrowtail \beta$ and $\vdash \beta \rightarrowtail \alpha$. Then, we have the following Lemma 3.3 and Theorem 3.5 that was given in [11] without proofs.

Lemma 3.3 ([11]). *$\equiv$ is a congruence relation on For.*

Proof: We will start by proving that $\equiv$ is an equivalence relation:

Reflexivity. Let us see that $\alpha \equiv \alpha$, but this is immediate from T8.

Symmetry. Let us see that $\alpha \equiv \beta$ if and only if $\beta \equiv \alpha$, but this is immediate from the very definitions.

Transitivity. Let us prove that if $\alpha \equiv \beta$ and $\beta \equiv \gamma$, then $\alpha \equiv \gamma$. Indeed, we have that $\vdash \alpha \rightarrowtail \beta$ and $\vdash \beta \rightarrowtail \alpha$ as hypotheses. On the other hand, $\vdash \beta \rightarrowtail \gamma$ and $\vdash \gamma \rightarrowtail \beta$. Applying Ax2, we have that $\vdash \alpha \rightarrowtail \gamma$ and $\vdash \gamma \rightarrowtail \alpha$. Therefore, $\alpha \equiv \gamma$.

We finish by verifying that $\equiv$ is a congruential relation. Indeed:

1. If $\alpha \equiv \beta$, then $\Delta\alpha \equiv \Delta\beta$. Indeed:

   1. $\alpha \equiv \beta$ (hypothesis)
   2. $\vdash \alpha \rightarrowtail \beta$ and $\vdash \beta \rightarrowtail \alpha$
   3. $\vdash \Delta\alpha \rightarrowtail \Delta\beta$ and $\vdash \Delta\beta \rightarrowtail \Delta\alpha$ (ΔR2)
   4. $\Delta\alpha \equiv \Delta\beta$

2. If $\alpha \equiv \beta$ and $\gamma \equiv \xi$, then $\alpha \rightarrowtail \gamma \equiv \beta \rightarrowtail \xi$. Indeed:

   1. $\alpha \equiv \beta$ (hypothesis)
   2. $\vdash \alpha \rightarrowtail \beta$ and $\vdash \beta \rightarrowtail \alpha$
   3. $\gamma \equiv \xi$ (hypothesis)
   4. $\vdash \gamma \rightarrowtail \xi$ and $\vdash \xi \rightarrowtail \gamma$
   5. $\vdash (\beta \rightarrowtail \gamma) \rightarrowtail (\beta \rightarrowtail \xi)$ (Ax2), 4., (MP)
   6. $\vdash (\alpha \rightarrowtail \gamma) \rightarrowtail (\beta \rightarrowtail \gamma)$ (2. and (R2))
   7. $\vdash (\alpha \rightarrowtail \gamma) \rightarrowtail (\beta \rightarrowtail \xi)$ (5., 6., and (R1))

With a similar argument, we can see that $\vdash (\beta \rightarrowtail \xi) \rightarrowtail (\alpha \rightarrowtail \gamma)$ as desired. □

LEMMA 3.4. *The following identities hold in $For/_\equiv$:*

(ΔŁ1) $\Delta|\alpha| \rightarrowtail |\beta| = |\alpha| \rightarrowtail (|\alpha| \rightarrowtail |\beta|)$;

(ΔŁ2) $\Delta(\Delta|\alpha| \rightarrowtail |\beta|) = \Delta|\alpha| \rightarrowtail \Delta|\beta|$.

PROOF: Both identities follow from (ΔT4), (ΔT5), together with axioms (Ax6), (Ax8), and the definition of the operations on equivalence classes. □

THEOREM 3.5 ([11]). *The Lindenbaum-Tarski algebra $\langle For/_\equiv, \rightarrowtail, \Delta, 1\rangle$ is an $Ł_3^{\rightarrowtail,\Delta}$-algebra, where $|\alpha \rightarrowtail \beta| = |\alpha| \rightarrowtail |\beta|$, $|\Delta\alpha| = \Delta|\alpha|$, and $1 = |\alpha \rightarrowtail \alpha| = \{\phi \in For : \vdash \phi\}$. Moreover, the relation $|\alpha| \leq |\beta|$, defined by $\vdash \alpha \rightarrowtail \beta$, is a partial order on $For/_\equiv$.*

PROOF: • First, we prove that the relation $|\alpha| \leq |\beta|$ is a partial order on $For/_\equiv$. Indeed:

Reflexivity. From T8, we know that $\vdash \alpha \rightarrowtail \alpha$, and then $|\alpha| \leq |\alpha|$.

Antisymmetry. From the conditions $|\alpha| \leq |\beta|$ and $|\beta| \leq |\alpha|$, we have $\vdash \alpha \rightarrowtail \beta$ and $\vdash \beta \rightarrowtail \alpha$. So, $\alpha \equiv \beta$, and therefore $|\alpha| = |\beta|$.

Transitivity. From the conditions $|\alpha| \leq |\beta|$ and $|\beta| \leq |\gamma|$, we have $\vdash \alpha \rightarrowtail \beta$ and $\vdash \beta \rightarrowtail \gamma$. Then, by applying (R1), we infer that $\vdash \alpha \rightarrowtail \gamma$ and therefore $|\alpha| \leq |\gamma|$.

• Next, we show that $|\beta| \leq |\alpha \rightarrowtail \alpha| = 1$, which is an immediate consequence of (T7).

• To demonstrate that $\langle For/_\equiv, \rightarrowtail, \Delta, 1\rangle$ is an $Ł_3^{\rightarrowtail,\Delta}$-algebra, let us consider $|\alpha|, |\beta|, |\gamma| \in For/_\equiv$ and recall that $\{\phi \in For : \vdash \phi\} = 1$. Then, we have:

(Ł1): To show that $|\alpha| \rightarrowtail \big(|\beta| \rightarrowtail |\alpha|\big) = 1$ holds, note that Ax1 $\in 1$, then:

$$\begin{aligned} |\alpha \rightarrowtail (\beta \rightarrowtail \alpha)| &= 1 \\ |\alpha| \rightarrowtail |(\beta \rightarrowtail \alpha)| &= 1 \\ |\alpha| \rightarrowtail \big(|\beta| \rightarrowtail |\alpha|\big) &= 1 \end{aligned}$$

(Ł2): The identity $\big(|\alpha| \rightarrowtail |\beta|\big) \rightarrowtail \Big(\big(|\beta| \rightarrowtail |\gamma|\big) \rightarrowtail \big(|\alpha| \rightarrowtail |\gamma|\big)\Big) = 1$ is obtained from (Ax2).

(Ł3): The identity $(|\alpha| \rightarrowtail |\beta|) \rightarrowtail |\beta| = (|\beta| \rightarrowtail |\alpha|) \rightarrowtail |\alpha|$ follows from (Ax3) and the definition of $\leq$.

(Ł4): The identity $\big((|\alpha| \rightarrowtail |\beta|) \rightarrowtail (|\beta| \rightarrowtail |\alpha|)\big) \rightarrowtail (|\beta| \rightarrowtail |\alpha|) = 1$ is obtained from axiom (Ax4).

(Ł5): $1 \rightarrowtail |\alpha| = |\alpha|$.

(a) $|\alpha| \leq 1 \rightarrowtail |\alpha|$

1. $\vdash \alpha \rightarrowtail ((\beta \rightarrowtail \beta) \rightarrowtail \alpha)$ (Ax1)
2. $|\alpha| \leq |(\beta \rightarrowtail \beta) \rightarrowtail \alpha|$
3. $|\alpha| \leq |\beta \rightarrowtail \beta| \rightarrowtail |\alpha|$
4. $|\alpha| \leq 1 \rightarrowtail |\alpha|$

(b) $1 \rightarrowtail |\alpha| \leq |\alpha|$

1. $\vdash ((\beta \rightarrowtail \beta) \rightarrowtail \alpha) \rightarrowtail \alpha$ (T9)
2. $|((\beta \rightarrowtail \beta) \rightarrowtail \alpha)| \leq |\alpha|$
3. $|\beta \rightarrowtail \beta| \rightarrowtail |\alpha| \leq |\alpha|$
4. $1 \rightarrowtail |\alpha| \leq |\alpha|$

From (a) and (b), we conclude the proof.

(Ł6): To show that $\Big(\big(|\alpha| \rightarrowtail (|\alpha| \rightarrowtail |\beta|)\big) \rightarrowtail |\alpha|\Big) \rightarrowtail |\alpha| = 1$ holds, it suffices to use axiom (Ax5) and the very definitions.

Finally, the identities (ΔŁ1) and (ΔŁ2) follow from Lemma 3.4. □

We are now in a position to present the first soundness and completeness theorem in the weak sense. To that end, let us introduce the following definition:

Definition 3.6. A function $v : For \to A$ is said to be a *valuation* if it satisfies the following conditions:

(i) $v(\alpha \rightarrowtail \beta) = v(\alpha) \rightarrowtail v(\beta)$;

(ii) $v(\Delta\alpha) = \Delta v(\alpha)$;

(iii) $v(\top) = 1$.

Furthermore, a formula α is said to be *semantically valid*, denoted $\models \alpha$, if for every $_3^{\rightarrowtail,\Delta}$-algebra $\mathbf{A}$ and every valuation $v : For \to A$, it holds that $v(\alpha) = 1$.

We now establish the first (weak) soundness and completeness result, whose proof follows standard lines using Theorem 3.5.

Theorem 3.7 (Weak Adequacy Theorem). *For every formula $\alpha \in For$, we have that $\vdash \alpha$ if and only if $\models \alpha$.*

Proof: *(Soundness)*: Let $\mathbf{A}$ be a fixed $_3^{\rightarrowtail,\Delta}$-algebra, and let $v : For \to A$ be any valuation. Suppose that $\alpha \in For$ admits a formal proof $\alpha_1, \ldots, \alpha_n$ such that $\alpha_n = \alpha$. We proceed by induction on n.

If $n = 1$, then $\alpha = \alpha_1$ is an axiom. By a direct verification using Definition 3.6, we obtain that $v(\alpha_1) = 1$.

Now assume that the result holds for all proofs of length less than k, and consider a proof of length k. We distinguish two cases:

1. If α_k is an axiom, then $v(\alpha_k) = 1$ by the same reasoning as in the base case.

2. If α_k results from applying Modus Ponens to α_i and $\alpha_i \rightarrowtail \alpha_k$, with $i < k$, then by the induction hypothesis we have $v(\alpha_i) = 1$ and $v(\alpha_i \rightarrowtail \alpha_k) = 1$. Thus, $v(\alpha_i) \rightarrowtail v(\alpha_k) = 1$, and since $v(\alpha_i) = 1$, by (L7) we obtain $v(\alpha_k) = 1$.

Hence, in all cases $v(\alpha) = 1$, and thus $\models \alpha$.

(Completeness): Suppose that $\models \alpha$. Then for every $_3^{\rightarrowtail,\Delta}$-algebra $\mathbf{A}$ and every homomorphism $h : For \to \mathbf{A}$, we have $h(\alpha) = 1$. In particular, consider the canonical homomorphism $\pi : For \to For/_{\equiv}$, where $\pi(\gamma) = |\gamma|$

denotes the equivalence class of γ modulo the syntactic congruence $\equiv$. Since $\pi(\alpha) = 1$, it follows that $\alpha \in \{\beta \in For :\vdash \beta\}$. Therefore, $\vdash \alpha$. $\square$

It is worth noting that a weak version of the Adequacy Theorem is not explicitly stated in [11]; however, the following lemma is a consequence of it.

Before stating the result, let us fix the following notation:

$$\vdash \alpha \leftrightarrow \beta \text{ if and only if } \vdash \alpha \rightarrowtail \beta \text{ and } \vdash \beta \rightarrowtail \alpha.$$

LEMMA 3.8. *The following formulas and inference patterns are theorems of the logic* $\mathcal{C}_3^{\rightarrowtail,\Delta}$*:*

(Δ*T8*) $\vdash \Delta\alpha \rightarrowtail \nabla\alpha$;

(Δ*T9*) $\vdash \nabla\alpha \leftrightarrow \nabla\nabla\alpha$,

(Δ*T10*) $\vdash \nabla\Delta\alpha \leftrightarrow \Delta\alpha$,

(Δ*T11*) $\vdash \alpha \rightarrowtail \Delta\nabla\alpha$,

(Δ*T12*) $\vdash (\Delta\alpha \rightarrowtail \beta) \rightarrowtail (\nabla\beta \rightarrowtail (\alpha \rightarrowtail \beta))$,

(Δ*T13*) $\vdash (\Delta\alpha \rightarrowtail \beta) \rightarrowtail \nabla(\alpha \rightarrowtail \beta)$,

(Δ*T14*) $\vdash \Delta(\alpha \rightarrowtail \beta) \rightarrowtail (\Delta\alpha \rightarrowtail \Delta\beta)$,

(Δ*T15*) $\vdash (\nabla\alpha \rightarrowtail \nabla\beta) \rightarrowtail \nabla(\alpha \rightarrowtail \beta)$,

(Δ*T16*) $\vdash ((\alpha \rightarrowtail \beta) \rightarrowtail (\alpha \rightarrowtail \gamma)) \rightarrowtail (\alpha \rightarrowtail (\beta \rightarrowtail \gamma))$,

(Δ*T17*) $\vdash ((\alpha \rightarrowtail \beta) \rightarrowtail \beta) \rightarrowtail (\alpha \rightarrowtail \Delta((\alpha \rightarrowtail \beta) \rightarrowtail \beta))$,

(Δ*T18*) $\vdash \nabla\alpha \leftrightarrow \Delta\nabla\alpha$,

(Δ*T19*) $\vdash \alpha \rightarrowtail (\alpha \rightarrowtail (\nabla(\alpha \rightarrowtail \beta) \rightarrowtail \nabla\beta))$,

(Δ*T20*) $\vdash \Delta(\alpha \rightarrowtail \beta) \rightarrowtail (\nabla\alpha \rightarrowtail \nabla\beta)$.

PROOF: Let $v : For \to A$ be any valuation, and let $\mathbf{A}$ be an arbitrary ${}_3^{\rightarrowtail,\Delta}$-algebra. For each formula ϕ among (ΔT8) to (ΔT20), we verify that

$v(\phi) = 1$ by applying the definitions of Δ and ∇ and the properties of $\mathbf{A}$. Since these equations hold in every ${}_3^{\rightarrowtail,\Delta}$-algebra and the variety is equational, it follows that each ϕ is semantically valid. By the Weak Adequacy Theorem, we conclude that $\vdash \phi$. □

3.1. Strong Version of the Adequacy Theorem

Recall that a logic defined over a language $\mathcal{S}$ is a system $\mathcal{L} = \langle For, \vdash_{\mathbf{L}} \rangle$, where For is the set of formulas over $\mathcal{S}$, and the relation $\vdash_{\mathbf{L}} \subseteq \mathcal{P}(For) \times For$, where $\mathcal{P}(A)$ is the set of all subsets of A. This general framework follows the standard approach to abstract consequence relations (see, e.g., [3]). We adopt the standard assumption that $\vdash_{\mathbf{L}}$ is closed under uniform substitution.

The logic $\mathcal{L}$ is said to be Tarskian if it satisfies the following properties for every set $\Gamma \cup \Omega \cup \{\varphi, \beta\}$ of formulas:

(1) if $\alpha \in \Gamma$, then $\Gamma \vdash_{\mathbf{L}} \alpha$,

(2) if $\Gamma \vdash_{\mathbf{L}} \alpha$ and $\Gamma \subseteq \Omega$, then $\Omega \vdash_{\mathbf{L}} \alpha$,

(3) if $\Omega \vdash_{\mathbf{L}} \alpha$ and $\Gamma \vdash_{\mathbf{L}} \beta$ for every $\beta \in \Omega$, then $\Gamma \vdash_{\mathbf{L}} \alpha$.

A logic $\mathcal{L}$ is said to be finitary if it satisfies the following:

(4) if $\Gamma \vdash_{\mathbf{L}} \alpha$, then there exists a finite subset Γ_0 of Γ such that $\Gamma_0 \vdash_{\mathbf{L}} \alpha$.

Let $\mathcal{L}$ be a Tarskian logic, and let Γ be a set of formulas; we say that Γ is a theory. A theory Γ is said to be *consistent* if there exists a formula φ such that $\Gamma \nvdash_{\mathbf{L}} \varphi$. We also say that Γ is a maximal consistent theory if $\Gamma, \psi \vdash_{\mathbf{L}} \varphi$ for any $\psi \notin \Gamma$, and in this case, we say Γ is non-trivial maximal with respect to φ.

On the other hand, a logic is said to be standard if it is Tarskian and a finitary system. Furthermore, let $\mathscr{L}$ be a Tarskian logic. A set of formulas Γ is said to be closed in $\mathscr{L}$, or a closed theory of $\mathscr{L}$, if the following holds for every formula ψ: $\Gamma \vdash_{\mathbf{L}} \psi$ if and only if $\psi \in \Gamma$.

LEMMA 3.9. *Any non-trivial maximal set of formulas with respect to φ in $\mathcal{L}$ is closed, provided that $\mathcal{L}$ is Tarskian.*

Proof: This is a direct consequence of the definition of maximality and the Tarskian conditions, in particular the transitivity (3) and reflexivity (1) of the consequence relation. □

Lemma 3.10. (Lindenbaum-Łoś Lemma) *Let $\mathcal{L}$ be a standard logic and let $\Gamma \cup \{\varphi\}$ be a set of formulas such that $\Gamma \nvdash_{\mathbf{L}} \varphi$. Then, there exists a set of formulas Ω such that $\Gamma \subseteq \Omega$ with Ω maximal non-trivial with respect to φ in $\mathcal{L}$.*

Proof: See Theorem 2.22 of [28]. □

Proposition 3.11. The calculus $\mathcal{C}_3^{\rightarrowtail,\Delta}$ is a Tarskian and finitary logic.

Lemma 3.12. *Let $\Gamma \cup \{\varphi\}$ be a set of formulas such that Γ is non-trivial maximal with respect to φ in $\mathcal{C}_3^{\rightarrowtail,\Delta}$. Then, if $\phi \notin \Gamma$, then $\Gamma \vdash \Delta\phi \rightarrowtail \beta$ for every $\beta \in For$.*

Proof: Let us consider the set $|\Gamma| = \{|\alpha| : \alpha \in \Gamma\}$ and suppose that $\alpha \in \Gamma$ such that $\alpha \equiv \beta$. Then, $\vdash \alpha \rightarrowtail \beta$ and $\vdash \beta \rightarrowtail \alpha$. Therefore, $\beta \in \Gamma$ and then we have that $|\Gamma|$ is closed under equivalence: if $\alpha \in \Gamma$ and $|\alpha| = |\beta|$, then $\beta \in \Gamma$.

Moreover, it is not hard to see that the conditions of Definition 2.2 are verified by $|\Gamma|$. Thus, $|\Gamma|$ is an implicative filter.

Recall that $\mathbf{For}/_{\equiv}$ is an $_3^{\rightarrowtail,\Delta}$-algebra in virtue of Theorem 3.5. Now, let $D \subseteq For/_{\equiv}$ be an implicative filter that properly contains $|\Gamma|$. Then there is $|\gamma| \in D$ such that $|\gamma| \notin |\Gamma|$, so $\gamma \notin \Gamma$ and therefore $\Gamma \cup \{\gamma\} \vdash \varphi$. From the latter and taking $D' = \{\alpha : |\alpha| \in D\}$, we can infer that $D' \vdash \varphi$. Since D' is closed, we obtain that $|\varphi| \in D$. This contradicts the maximality of Γ, hence $|\Gamma|$ must be a maximal implicative filter below $|\varphi|$.

So, if $\phi \notin \Gamma$, then $|\phi| \notin |\Gamma|$. From the latter and Lemma 2.6, we have that $\Delta|\phi| \rightarrowtail |\beta| \in |\Gamma|$. By definition of $|\Gamma|$, we have that $\Delta\phi \rightarrowtail \beta \in \Gamma$ as desired. □

The last Lemma is central for the following Theorem, as it allows us to construct the special homomorphism. It is worth noting that its proof requires Lemma 2.6 and certain algebraic properties of the class of $_3^{\rightarrowtail,\Delta}$-algebras. It would be of independent interest to obtain a purely syntactic proof of Lemma 3.12, avoiding the use of maximal implicative filters and the underlying algebraic machinery. This problem remains open and is left for future research.

PROPOSITION 3.13. Let $\Gamma \cup \{\varphi\}$ be a set of formulas such that Γ is non-trivial and maximal with respect to φ in $\mathcal{C}_3^{\rightarrowtail,\Delta}$. Then, the function defined for every $\gamma \in For$ as follows:

$$v(\gamma) = \begin{cases} 1 & \text{if } \gamma \in \Gamma \\ \frac{1}{2} & \text{if } \gamma \in \Gamma_{1/2} \\ 0 & \text{if } \gamma \in \Gamma_0 \end{cases}$$

is a homomorphism from For into $\mathbb{C}_3$ such that $v^{-1}(\{1\}) = \Gamma$, where $\Gamma_{\frac{1}{2}} = \{\alpha \notin \Gamma : \Delta\alpha \notin \Gamma \text{ and } \nabla\alpha \in \Gamma\}$, $\Gamma_0 = \{\alpha \notin \Gamma : \nabla\alpha \notin \Gamma\}$, and $\mathbb{C}_3$ is the 3-element chain ${}_3^{\rightarrowtail,\Delta}$-algebra.

PROOF: We show that $v(\alpha \rightarrowtail \beta) = v(\alpha) \rightarrowtail v(\beta)$. Indeed:

- If $v(\beta) = 1$, then $\beta \in \Gamma$. By (Ax1), we have $\beta \rightarrowtail (\alpha \rightarrowtail \beta) \in \Gamma$, and by (MP), $\alpha \rightarrowtail \beta \in \Gamma$. Thus, $v(\alpha \rightarrowtail \beta) = 1$.

- If $v(\alpha) = 0$ and $v(\beta) = 1/2$, then by Lemma 3.12, $\Delta\alpha \rightarrowtail \beta \in \Gamma$. Since $\nabla\beta \in \Gamma$, using (ΔT12) and (MP), we obtain $\alpha \rightarrowtail \beta \in \Gamma$, hence $v(\alpha \rightarrowtail \beta) = 1 = v(\alpha) \rightarrowtail v(\beta)$.

- If $v(\alpha) = v(\beta) = 0$, then $\nabla\alpha \notin \Gamma$. By Lemma 3.12, $\Delta\nabla\alpha \rightarrowtail \beta \in \Gamma$. By (Ax2), $\vdash (\nabla\alpha \rightarrowtail \Delta\nabla\alpha) \rightarrowtail ((\Delta\nabla\alpha \rightarrowtail \beta) \rightarrowtail (\nabla\alpha \rightarrowtail \beta))$. From this, together with (ΔT18) and (MP), we infer that $(\Delta\nabla\alpha \rightarrowtail \beta) \rightarrowtail (\nabla\alpha \rightarrowtail \beta) \in \Gamma$, and hence $\nabla\alpha \rightarrowtail \beta \in \Gamma$. Using ($\Delta$T7) and (MP), it follows that $\alpha \rightarrowtail \beta \in \Gamma$, so $v(\alpha \rightarrowtail \beta) = 1 = v(\alpha) \rightarrowtail v(\beta)$.

- If $v(\alpha) = 1/2$ and $v(\beta) = 0$, then $\alpha, \nabla\beta \notin \Gamma$ and $\nabla\alpha \in \Gamma$. Since $\alpha \notin \Gamma$, Lemma 3.12 gives $\Delta\alpha \rightarrowtail \beta \in \Gamma$. Thus, ($\Delta$T13) and (MP) yield $\nabla(\alpha \rightarrowtail \beta) \in \Gamma$. Suppose, for contradiction, that $\alpha \rightarrowtail \beta \in \Gamma$. Then, using ($\Delta$T2) and (MP), we derive $\Delta(\alpha \rightarrowtail \beta) \in \Gamma$. From this, using ($\Delta$T20) and (MP), we obtain $\nabla\alpha \rightarrowtail \nabla\beta \in \Gamma$. Since $\nabla\alpha \in \Gamma$, it follows that $\nabla\beta \in \Gamma$, a contradiction. Therefore, $\alpha \rightarrowtail \beta \notin \Gamma$ and $v(\alpha \rightarrowtail \beta) = 1/2 = v(\alpha) \rightarrowtail v(\beta)$.

- If $v(\alpha) = v(\beta) = 1/2$, then $\alpha \notin \Gamma$ and $\nabla\beta \in \Gamma$. Lemma 3.12 ensures that $\Delta\alpha \rightarrowtail \beta \in \Gamma$. Then, ($\Delta$T12) and (MP) yield $\alpha \rightarrowtail \beta \in \Gamma$, and thus $v(\alpha \rightarrowtail \beta) = 1 = v(\alpha) \rightarrowtail v(\beta)$.

- If $v(\alpha) = 1$ and $v(\beta) = 0$, then $\alpha \in \Gamma$ and $\nabla\beta \notin \Gamma$. Suppose, for contradiction, that $\nabla(\alpha \rightarrowtail \beta) \in \Gamma$. Then, by ($\Delta$T19) and (MP), we get $\nabla\beta \in \Gamma$, a contradiction. Therefore, $\nabla(\alpha \rightarrowtail \beta) \notin \Gamma$ and $v(\alpha \rightarrowtail \beta) = 0 = v(\alpha) \rightarrowtail v(\beta)$.

- If $v(\alpha) = 1$ and $v(\beta) = 1/2$, then $\alpha, \nabla\beta \in \Gamma$ and $\beta \notin \Gamma$. By (Ax1), we have $\nabla\beta \rightarrowtail (\nabla\alpha \rightarrowtail \nabla\beta) \in \Gamma$, so by (MP), $\nabla\alpha \rightarrowtail \nabla\beta \in \Gamma$. Then, using ($\Delta$T15), we conclude that $\nabla(\alpha \rightarrowtail \beta) \in \Gamma$. Suppose $\alpha \rightarrowtail \beta \in \Gamma$. Since $\alpha \in \Gamma$, we would get $\beta \in \Gamma$, which contradicts the hypothesis. Thus, $\alpha \rightarrowtail \beta \notin \Gamma$ and $v(\alpha \rightarrowtail \beta) = 1/2 = v(\alpha) \rightarrowtail v(\beta)$.

We now show that $v(\Delta\alpha) = \Delta v(\alpha)$. Indeed:

- If $v(\alpha) = 0$, then $\alpha, \nabla\alpha \notin \Gamma$. Suppose, for contradiction, that $\nabla\Delta\alpha \in \Gamma$. Then, using ($\Delta$T10) and (MP), we derive $\Delta\alpha \in \Gamma$, and from (ΔT3) and (MP), $\alpha \in \Gamma$, a contradiction. Thus, $\nabla\Delta\alpha \notin \Gamma$ and $v(\Delta\alpha) = 0 = \Delta v(\alpha)$.

- If $v(\alpha) = 1/2$, then $\alpha \notin \Gamma$ and $\nabla\alpha \in \Gamma$. Suppose $\nabla\Delta\alpha \in \Gamma$. Then, using ($\Delta$T10) and (MP), we get $\Delta\alpha \in \Gamma$, and by (ΔT3) and (MP), $\alpha \in \Gamma$, a contradiction. Thus, $v(\Delta\alpha) = 0 = \Delta v(\alpha)$.

- If $v(\alpha) = 1$, then $\alpha \in \Gamma$. By (ΔT2) and (MP), we have $\Delta\alpha \in \Gamma$, hence $\Delta v(\alpha) = 1 = v(\Delta\alpha)$. □

Theorem 3.13 is the key ingredient in the statement of the following Completeness Theorem. It is worth noting that we were able to prove Theorem 3.13 without relying on Lemma 3.12. To conclude this section, we define the semantic entailment symbol $\Gamma \vDash \alpha$ to mean that, for every $_3^{\rightarrowtail,\Delta}$-algebra $\mathbf{A}$ and every valuation v, if $v(\gamma) = 1$ for every $\gamma \in \Gamma$, then $v(\alpha) = 1$.

Theorem 3.14. (Strong Soundness and Completeness of $C_3^{\rightarrowtail,\Delta}$ w.r.t. the class of $_3^{\rightarrowtail,\Delta}$-algebras). *Let* $\Gamma \cup \{\varphi\} \subseteq For$, $\Gamma \vdash \varphi$ *if and only if* $\Gamma \vDash \varphi$.

Proof: Soundness: It is not hard to see that every axiom is valid for every $_3^{\rightarrowtail,\Delta}$-algebra A. In addition, satisfaction is preserved by the inference rules.

Completeness: Suppose $\Gamma \vDash \varphi$ and $\Gamma \nvdash \varphi$. According to Lemma 3.10, there is a maximal consistent theory Ω such that $\Gamma \subseteq \Omega$ and $\Omega \nvdash \varphi$. From the latter and Proposition 3.13, there is a valuation $\mu : For \to \mathbb{C}_3$ such that $\mu(\Omega) = \{1\}$ but $\mu(\varphi) \neq 1$. Since $\Gamma \subseteq \Omega$, we have $\mu(\gamma) = 1$ for every $\gamma \in \Gamma$, This contradicts the assumption that $\Gamma \vDash \varphi$, and thus $\Gamma \vdash \varphi$ must hold. □

The result above establishes the strong version of completeness for our calculus, syntactically characterized and algebraically sound. It is worth mentioning that Theorem 3.14 could be obtained from [13, Theorem 4.12] for taking $n = 3$, but our contribution lies in providing a purely syntactic proof, which is independent from the general framework and therefore more elementary and self-contained. As an important consequence of this Theorem, we have that $C_3^{\rightarrowtail,\Delta}$ does not enjoy Deduction Theorem as we will see in the next Corollary.

Corollary 3.15. In the logic $C_3^{\rightarrowtail,\Delta}$, Deduction Theorem does not hold.

Proof: In virtue of Theorem 3.14 and the rule ΔR1 of Lemma 3.2, we have that $\varphi \vDash \Delta\varphi$, but it is not hard to see that $\nvDash \varphi \rightarrowtail \Delta\varphi$. Indeed, it is enough to take a valuation $v(\varphi) = \frac{1}{2}$, and so, $v(\varphi \rightarrowtail \Delta\varphi) = 0$. Hence, the implication fails in the semantics even when the entailment holds, showing that the Deduction Theorem is not valid. □

4. First-order version of $C_3^{\rightarrowtail,\Delta}$: the logic $\forall_3^{\rightarrowtail,\Delta}$

In this section we introduce the first-order extension of the logic $C_3^{\rightarrowtail,\Delta}$, denoted $\forall_3^{\rightarrowtail,\Delta}$. Our main goal is to extend the propositional framework to the first-order level, providing an appropriate semantic setting and a corresponding deductive system that preserves the essential features of the original logic. In particular, we develop the notion of valuation over first-order structures and adapt the key algebraic tools to this richer setting. However, we would like to stress that the shift from the propositional to the first-order level requires a careful reinterpretation of the semantics and the expansion of the language to accommodate variables, terms, quantifiers, and substitution mechanisms. While some notions inevitably mirror the propositional framework, they are now defined within a richer language and semantic context that substantially changes their scope and treatment.

Let us begin by fixing the propositional signature Θ of $C_3^{\rightarrowtail,\Delta}$, and extending it with two quantifier symbols $\forall$ and $\exists$, as well as the usual punctuation symbols. We consider a countable set Var of individual variables and denote by $\mathfrak{Fm}_\Sigma$ the set of formulas over a first-order signature $\Sigma = \langle \mathcal{P}, \mathcal{F}, \mathcal{C} \rangle$, where $\mathcal{P}$ is a non-empty set of predicate symbols, $\mathcal{F}$ a set of function symbols, and $\mathcal{C}$ a set of individual constants. The set Ter denotes the absolutely free term algebra over $\mathcal{F}$ and $\mathcal{C}$.

As customary, we define the notions of free and bound variables, substitution, closed terms, and sentences. Given a formula φ, we denote by $\varphi(x/t)$ the result of simultaneously replacing all free occurrences of the variable x by the term t, provided t is free for x in φ.

A Σ-structure $\mathfrak{A}$ for $\forall_3^{\rightarrowtail,\Delta}$ is a pair $\langle \mathbf{A}, \mathbf{S} \rangle$ where $\mathbf{A}$ is a complete ${}_3^{\rightarrowtail,\Delta}$-algebra and $\mathbf{S}$ provides the standard first-order interpretation over a non-empty domain S. That is:

- every constant $c \in \mathcal{C}$ is assigned an element $c^{\mathfrak{A}} \in S$,
- each n-ary function symbol $f \in \mathcal{F}$ is interpreted as a function $f^{\mathfrak{A}} : S^n \to S$,
- each n-ary predicate symbol $P \in \mathcal{P}$ is interpreted as a function $P^{\mathfrak{A}} : S^n \to A$.

Truth values of terms and formulas in a structure $\mathfrak{A}$ under a valuation $v : Var \to S$ are defined recursively in the usual way, with logical connectives interpreted via the algebraic operations of $\mathbf{A}$. Notably, quantifiers are interpreted through meet and join operations:

$$||\forall x \alpha||_v^{\mathfrak{A}} = \bigwedge_{a \in S} ||\alpha||_{v[x \to a]}^{\mathfrak{A}}, \qquad ||\exists x \alpha||_v^{\mathfrak{A}} = \bigvee_{a \in S} ||\alpha||_{v[x \to a]}^{\mathfrak{A}}.$$

Satisfaction and semantic consequence are defined analogously to the propositional case, with $\mathfrak{A} \vDash \varphi[v]$ meaning that $||\varphi||_v^{\mathfrak{A}} = 1$, and $\Gamma \vDash \varphi$ holding when every model that satisfies all formulas in Γ also satisfies φ.

The deductive system of $\forall_3^{\rightarrowtail,\Delta}$ builds on the propositional calculus $C_3^{\rightarrowtail,\Delta}$ by adding a collection of standard axiom schemas and inference rules for the quantifiers, adapted to the semantics of the operator Δ. In particular, we include two additional equivalences involving the distribution of Δ over the quantifiers, which are central to the algebraic treatment of the system and have no direct analogue in classical logic.

Axiom Schemas

- ($\forall$1) $\varphi(x/t) \rightarrowtail \exists x\varphi$, if t is free for x in φ,
- ($\forall$2) $\forall x\varphi \rightarrowtail \varphi(x/t)$, if t is free for x in φ,
- ($\forall$3) $\Delta\exists x\varphi \leftrightarrow \exists x\Delta\varphi$,
- ($\forall$4) $\Delta\forall x\varphi \leftrightarrow \forall x\Delta\varphi$,

Inference Rules

- ($\forall$R1) $\dfrac{\alpha \rightarrowtail \beta}{\exists x\alpha \rightarrowtail \beta}$, provided x does not occur free in β,
- ($\forall$R2) $\dfrac{\alpha \rightarrowtail \beta}{\alpha \rightarrowtail \forall x\beta}$, provided x does not occur free in α.

The design of $\forall_3^{\rightarrowtail,\Delta}$ is based on a conservative and modular extension of the propositional core, preserving its non-classical features while allowing for standard model-theoretic techniques in the first-order setting. Observe that, although the domain of interpretation may be infinite, the set of truth values is finite, namely $\mathbb{C}_3 = \{0, \frac{1}{2}, 1\}$. Hence, all required infima and suprema exist, and the interpretation of the quantifiers is well-defined without requiring additional completeness assumptions on the underlying algebra. In what follows, we will develop the fundamental metatheorems of this logic and establish its soundness and completeness with respect to the class of first-order ${}_3^{\rightarrowtail,\Delta}$-structures.

LEMMA 4.1. [16] *Let* $\mathbf{A}$ *be a complete* $_3^{\rightarrowtail,\Delta}$*-algebra and the set* $\{a_i\}_{i\in I}$ *of elements of* A *for any non-empty set* I*. Then, if there exists* $\bigvee_{i\in I} a_i$ $(\bigwedge_{i\in I} a_i)$*, then there exists* $\bigvee_{i\in I} \Delta a_i$ $(\bigwedge_{i\in I} \Delta a_i)$*, and also* $\bigvee_{i\in I} \Delta a_i = \Delta \bigvee_{i\in I} a_i$ *and* $\bigwedge_{i\in I} \Delta a_i = \Delta \bigwedge_{i\in I} a_i$ *hold.*

THEOREM 4.2. (Soundness Theorem). *Let* $\Gamma \cup \{\varphi\} \subseteq \mathfrak{Fm}_\Sigma$*, if* $\Gamma \vdash \varphi$ *then* $\Gamma \vDash \varphi$*.*

PROOF: Let us consider the fixed structure $\mathfrak{M} = \langle \mathbf{A}, \mathbf{S}\rangle$. Let φ be a formula such that $\Gamma \vdash \varphi$. Then, there exists $\alpha_1, \cdots, \alpha_n$ a derivation of φ from Γ. If $n = 1$ then φ is an axiom or $\varphi \in \Gamma$. If $\varphi \in \Gamma$, then it is easy to see that $\Gamma \vDash \varphi$. If φ is an axiom, then the truth of (Ax1) to (Ax8) is obtained at the propositional level.

Unlike the propositional case, we now deal with formulas containing quantifiers, and some axioms require additional semantic justification. Let us observe that on $\mathfrak{Fm}_\Sigma$ we can define an order relation $\leq$ in a similar way as was done in Theorem 3.5. So, let us suppose that φ is $\alpha(x/t) \rightarrowtail \exists x\alpha$. Then, $||\varphi||_v^{\mathfrak{M}} = ||\alpha||_{v[x\to||t||_v^{\mathfrak{M}}]}^{\mathfrak{M}} \rightarrowtail ||\exists x\alpha||_v^{\mathfrak{M}}$. It is clear that $||\alpha||_{v[x\to||t||_v^{\mathfrak{M}}]}^{\mathfrak{M}} \leq \bigvee_{a\in S} ||\alpha||_{v[x\to a]}^{\mathfrak{M}}$, then $||\alpha||_{v[x\to||t||_v^{\mathfrak{M}}]}^{\mathfrak{M}} \leq ||\exists x\alpha||_v^{\mathfrak{M}}$. Therefore, we have that $||\alpha(x/t) \rightarrowtail \exists x\alpha||_v^{\mathfrak{M}} = 1$, and so axiom (∀1) is valid on $\mathfrak{M} = \langle \mathbf{A}, \mathbf{S}\rangle$. Analogously, axiom (∀2) is also valid.

For axioms (∀3) and (∀4), which explicitly involve the operator Δ, the proof of validity requires a structural property of the algebra captured by Lemma 4.1. This is a key point where the first-order setting reveals its specific algebraic nuances, in contrast with the propositional case.

Besides, it is not difficult to see that satisfaction is preserved by the inference rules. □

To proceed with the completeness proof, we now adapt the construction of the canonical model to the first-order setting. This transition requires us to restrict the attention to closed formulas, reflecting the semantic role of sentences in classical model theory.

Let us first take the set of closed formulas denoted by $C\mathfrak{Fm}_\Sigma$ and consider the relation $\equiv$ defined by $\alpha \equiv \beta$ iff $\vdash \alpha \rightarrowtail \beta$ and $\vdash \beta \rightarrowtail \alpha$. Thus, we have that the algebra $C\mathfrak{Fm}_\Sigma/_\equiv$ is a $_3^{\rightarrowtail,\Delta}$-algebra, as in the propositional case, but now involving quantifier-free equivalence classes of closed formulas.

Let us consider the system $C\forall Ł_3^{\rightarrowtail,\Delta}$, which is obtained from $\forall Ł_3^{\rightarrowtail,\Delta}$ but is defined over sentences (closed formulas). It is clear that $C\forall Ł_3^{\rightarrowtail,\Delta}$ is a Tarskian and finitary logic, as discussed in Section 3.1. Additionally, we can introduce the notion of the set of formulas that are maximal non-trivial with respect to some closed formula φ. The concept of closed theories is defined in the same way as in the propositional case. However, unlike the purely propositional scenario, we now handle a richer language with quantifiers, and this impacts the structure of the Lindenbaum algebra. Therefore, Lindenbaum-Łoś's Theorem holds for $C\forall Ł_3^{\rightarrowtail,\Delta}$. Consequently, we have the following lemma:

Lemma 4.3. *Let $\Gamma \cup \{\varphi\}$ be a set of closed formulas, such that Γ is non-trivial and maximal with respect to φ in $C\forall Ł_3^{\rightarrowtail,\Delta}$. If $\phi \notin \Gamma$, then $\Gamma \vdash \Delta\phi \rightarrowtail \beta$ for every $\beta \in C\mathfrak{Fm}_\Sigma$.*

Proof: While the reasoning parallels that of Lemma 3.12, the key difference lies in the domain of discourse and the interpretation of closed terms. Now we consider that $C\mathfrak{Fm}_{\mathbf{\Sigma}}/_\equiv$ is an $Ł_3^{\rightarrowtail,\Delta}$-algebra. □

With this setting in place, we are ready to define a canonical first-order model based on closed terms. Unlike the earlier propositional model, the interpretation of function and predicate symbols must respect arities and term construction, thus requiring an explicit definition over a term domain.

Let us now consider the structure:

$$\mathfrak{M} = \langle \mathbb{C}_3, CTer, \cdot^{CTer} \rangle,$$

where $CTer$ is a set of closed terms. We can define the interpretation as follows:

- If $\hat{c}$ is a constant, then $||\hat{c}||_\mu^{\mathfrak{M}} := c$.
- If $f \in \mathcal{F}$, then $||f(t_1, \cdots, t_n)||_\mu^{\mathfrak{M}} = f(t_1, \cdots, t_n)$.

- If $P \in \mathcal{P}$, then $||P(t_1, \cdots, t_n)||_\mu^{\mathfrak{M}} = P^{\mathfrak{M}}(t_1, \cdots, t_n)$.

We recall that $P^{\mathfrak{M}} : (CTer)^n \to \mathbb{C}_3$ is a function that allows us to define $|| \cdot ||_\mu^{\mathfrak{M}}$ correctly. Our interpretation is defined for atomic closed formulas, but it is easy to see that $||\alpha||_\mu^{\mathfrak{M}}$ is correctly defined for every quantifier closed formula α, as we will see in the following Proposition.

PROPOSITION 4.4. Let $\Gamma \cup \{\varphi\}$ be a set of closed formulas (sentences) such that Γ is non-trivial and maximal with respect to φ in $\forall Ł_3^{\rightarrowtail,\Delta}$. Then, the function defined by

$$||\phi||_\mu^{\mathfrak{M}} = \begin{cases} 0 & \text{if } \phi \in \Gamma_0 \\ 1/2 & \text{if } \phi \in \Gamma_{1/2} \\ 1 & \text{if } \phi \in \Gamma \end{cases}$$

is a homomorphism from $C\mathfrak{Fm}$ into $\mathbb{C}_3$, where $\Gamma_{\frac{1}{2}} = \{\alpha \notin \Gamma : \Delta\alpha \notin \Gamma \text{ and } \nabla\alpha \in \Gamma\}$, $\Gamma_0 = \{\alpha \notin \Gamma : \nabla\alpha \notin \Gamma\}$, and $\mathbb{C}_3$ is the 3-element chain $Ł_3^{\rightarrowtail,\Delta}$-algebra. Moreover, $|| \cdot ||_\mu^{\mathfrak{M}}$ is a $\mathfrak{M}$-valuation, and $C\mathfrak{Fm}$ is the set of closed formulas.

PROOF: From Proposition 3.13, we can affirm that $||\varphi \rightarrowtail \phi||_\mu^{\mathfrak{M}} = ||\varphi||_\mu^{\mathfrak{M}} \rightarrowtail ||\phi||_\mu^{\mathfrak{M}}$ and $||\Delta\phi||_\mu^{\mathfrak{M}} = \Delta||\phi||_\mu^{\mathfrak{M}}$, which has the same proof as in the propositional case, but now using Lemma 4.3.

From ($\forall$1) and ($\forall$R1), we have:

$$||\forall x\alpha||_\mu^{\mathfrak{M}} = \bigwedge_{a \in T_\Theta} ||\alpha||_{\mu[x\to a]}^{\mathfrak{M}}$$

Then, by applying ($\forall$R2) (used twice), we obtain:

$$||\exists x\alpha||_\mu^{\mathfrak{M}} = \bigvee_{a \in T_\Theta} ||\alpha||_{\mu[x\to a]}^{\mathfrak{M}}$$

Hence, the proof is complete. □

We are now in a position to prove the following central theorem:

Theorem 4.5. (Completeness Theorem for Sentences). *Let* $\Gamma \cup \{\varphi\}$ *be a set of closed formulas (sentences). If* $\Gamma \vDash \varphi$*, then* $\Gamma \vdash \varphi$*.*

Proof: Suppose that $\Gamma \nvdash \varphi$. Then, there exists a theory Ω, maximal and consistent (with respect to closed formulas) in $C\forall Ł_3^{\rightarrowtail,\Delta}$ with respect to φ, such that $\Gamma \subseteq \Omega$, as proven in Lemma 3.10. According to Proposition 4.4, there exists an interpretation map $||\cdot||_\mu^{\mathfrak{M}}$ such that $||\alpha||_\mu^{\mathfrak{M}} = 1$ if and only if $\alpha \in \Omega$. Therefore, $\mathfrak{M} \vDash \gamma$ for every $\gamma \in \Gamma$, but $\mathfrak{M} \nvDash \varphi$, which contradicts our hypothesis. □

To present a completeness theorem for arbitrary formulas, we now introduce some auxiliary notions. Given a formula α, let $\{x_1, \ldots, x_n\}$ be the set of variables that occur freely in α. The *universal closure* of α is the closed formula $(\forall\alpha)$, defined as α itself if $n = 0$, and otherwise as $\forall x_1 \ldots \forall x_n \alpha$. The completeness theorem for arbitrary formulas in $\forall Ł_3^{\rightarrowtail,\Delta}$ now follows easily from the previous result:

Theorem 4.6. (Completeness of $\forall Ł_3^{\rightarrowtail,\Delta}$ with respect to the class of $Ł_3^{\rightarrowtail,\Delta}$-algebras). *Let* $\Gamma \cup \{\varphi\}$ *be a set of formulas. Then:* $\Gamma \vDash \varphi$ *implies that* $\Gamma \vdash \varphi$*.*

Proof: By ($\forall$2) and ($\forall$R2), it is easy to prove that $\alpha \vdash (\forall\alpha)$ and $(\forall\alpha) \vdash \alpha$, for every formula α. On the other hand, by the definition of $\vDash$, it is straightforward to verify that $\alpha \vDash \alpha$ and $(\forall\alpha) \vDash \alpha$, for every formula α. Then, for every $\Gamma \cup \{\varphi\}$, we have that $\Gamma \vdash \varphi$ if and only if $(\forall\Gamma) \vdash (\forall\varphi)$, and $\Gamma \vDash \varphi$ if and only if $(\forall\Gamma) \vDash (\forall\varphi)$, where $(\forall\Gamma) = \{(\forall\beta) : \beta \in \Gamma\}$. Thus, the desired result follows immediately from Theorem 4.5. □

5. Final Remarks and Conclusions

In the book [5], the authors presented Adequacy Theorems for several paraconsistent logics and Logics of Formal Inconsistency at the propositional level. In Chapter 4, they constructed a homomorphism for each of the three-valued logics studied in it; these logics are algebraizable with Blok-Pigozzi's method as established in the mentioned chapter. Clearly, this homomorphism can be constructed because of the algebraizability of the logics. This idea is present in our paper in Proposition 3.13 for the propositional case. Interestingly, we have taken this homomorphism from [13, Theorem 3.17], but our homomorphism is, in fact, the three-valued syntactic version of their presentation; in this setting, we have given a new syntactic proof for Proposition 3.13. The authors of [13] needed this homomorphism to determine the generating algebras of the variety; in our case, see Lemma 2.6, we could prove Theorem 2.8 of Section 1 using the algebraic version of our homomorphism. This homomorphism was also constructed in other classes of algebras, see, for instance, [10, 17, 13].

From a broader perspective, our work can be contrasted with several well-established approaches in the literature. In particular, while the algebraic theory of consequence developed by Blok and Pigozzi [4] and further expanded in [18] provides a general framework for algebraizable logics, our contribution focuses on a specific implicational fragment enriched with the Δ operator and emphasizes a direct syntactic treatment. Moreover, although the Δ operator has been extensively studied in fuzzy logics following Baaz [1], our setting is strictly finite-valued and algebraically simpler, which allows for a more explicit construction of the corresponding homomorphisms. In this sense, our results complement these general approaches by providing a concrete and self-contained analysis in the three-valued case.

We remark that, although the propositional homomorphism used in Proposition 4.4 is strongly inspired by the one defined in Proposition 3.13, the first-order extension involves non-trivial syntactic adjustments, particularly in the interpretation of quantifiers. These steps are not a mere repetition but rather reflect the adaptation of the method to a different logical level, and as such, constitute an original contribution.

Another issue that deserves a brief comment is the technical result given in Lemma 3.12; this is essential to construct the mentioned homomorphism, and this is a powerful syntactic property that we are only able to prove using algebraic arguments. In fact, this Lemma is a syntactic version of an algebraic one given by A. Monteiro; in the paper [17], it was established how it holds in a family of the semisimple class of algebras. In our case, see Lemma 2.6. On the other hand, this kind of homomorphisms cannot be constructed for da Costa's systems C_n $(n < \omega)$ because these logics are not algebraizable, see, for instance, [26].

On the first-order side, other kinds of proofs of Adequacy Theorems were given in [16, 17], where this technique used in them is strongly based on the study of algebraic properties of Lindenbaum-Tarski algebras for some first-order logics. This technique was recently applied to the first-order version of the logic $G'3$ ([7]) because it is not possible to apply the technique given in [5, Chapter 7]. Recall that the proofs of Adequacy Theorems for first-order logics given in [5, Chapter 7] are based on the fact that the propositional levels enjoy the Deduction Theorem, but it is not the case for the logic $G'3$ as it was proved in [7, Corollary 3.28].

In this context, the logic $\forall\mathcal{C}_3^{\rightarrowtail,\Delta}$ presents a particularly interesting case, since—despite being algebraizable—it lacks a standard Deduction Theorem, and thus the syntactic proof of adequacy had to be carefully adapted to circumvent this issue. This responds directly to the concern raised by one referee regarding the originality and necessity of our approach.

Additionally, we have proved Adequacy for the first-order version of the logic $\text{CŁ}_3^{\rightarrowtail,\Delta}$ by using the homomorphism given in Proposition 3.13, now in Proposition 4.4. The novelty here is to show that it is possible to present a more "syntactic" proof without the necessity of using algebraic properties of the corresponding first-order Lindenbaum-Tarski algebra.

This more syntactic presentation, as opposed to previous algebraic ones such as those in [17], may facilitate future extensions and proof-theoretic analyses in the study of first-order paraconsistent logics.

Another positive outcome is that our presentation can be used for the algebraizable three-valued logics studied in Carnielli and Coniglio's Chapter 4 of the book [5], where the interpretation map for quantified formulas should be given by $||\forall x\alpha||_v^{\mathfrak{S}} = \inf\{||\alpha||_{v[x\to a]}^{\mathfrak{S}} : a \in S\}$ and $||\exists x\alpha||_v^{\mathfrak{S}} = \sup\{||\alpha||_{v[x\to a]}^{\mathfrak{S}} : a \in S\}$. These kinds of interpretations are present in the celebrated Rasiowa's Book ([27]), in the first-order version of fuzzy and Δ-fuzzy logics given in [9, 19, 20], and in D'Ottaviano's work [8].

Finally, we hope that the explicit consideration of the issues raised by the reviewers—concerning the originality of our approach, the differences between propositional and first-order levels, and the role of syntactic techniques—will strengthen the clarity and value of this contribution.

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Ela Drozdowska

MATRIX SEMANTICS FOR CLASSICAL LOGIC: THE CASE OF THE LATTICE O6

Abstract

It is well established that classical propositional logic is Boolean. However, this view has recently been challenged. In their paper *Non-Orthomodular Models for Both Standard Quantum Logic and Standard Classical Logic: Repercussions for Quantum Computers*, Mladen Pavičić and Norman Megill present a non-distributive, non-orthomodular model for both classical and quantum logic based on lattice O6, and argue that classical propositional logic is non-distributive.

In this paper, we examine this claim. Pavičić and Megill's model is formulated within unital matrix semantics rather than as an algebraic model in the sense of Abstract Algebraic Logic. An analysis of the lattice O6 in the framework of matrix semantics reveals that the matrix $(\mathrm{O6}, \{1, a, b\})$ is adequate for $\mathcal{CL}$, but not reduced, and induces the same consequence relation as the two-element Boolean matrix B_2. Similarly, the unital matrix $(\mathrm{O6}, \{1\})$ is adequate for $\mathcal{CL}$ through reduction to the four-element Boolean matrix B_4. Furthermore, we present two

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lattice constructions that yield matrix models for $\mathcal{CL}$ lacking nontrivial lattice-theoretic properties.

These results show that the adequacy of O6 is not intrinsic to its algebraic structure, but is inherited from its reducibility to Boolean matrices, and more generally that classical logic admits models with highly unconstrained lattice structure. Consequently, the existence of such non-distributive models does not undermine the distributive character of classical propositional logic.

Keywords: classical propositional logic, matrix semantics, algebraic semantics, O6 lattice, distributivity.

1. Introduction

Classical propositional logic is standardly associated with Boolean algebra. However, since the 1990s several papers have challenged this view (e.g. [5, 6, 7, 4]). Mladen Pavičić and Norman Megill claim that classical propositional logic is non-distributive and quantum logic is non-orthomodular. In their paper *Non-Orthomodular Models for Both Standard Quantum Logic and Standard Classical Logic: Repercussions for Quantum Computers* they state:

> "The following theorem holds in $\mathcal{CL}$ [classical logic]:
>
> $\vdash A \vee (B \wedge C) \equiv_i (A \vee B) \wedge (A \vee C)$, where $i = 0, ..., 5$.
>
> The theorem is usually called a distributivity law. However, when its lattice mapping: $a \cup (b \cap c) \equiv_i (a \cup b) \cap (a \cup c) = 1$ is added to an ortholattice, it does not make the ortholattice even orthomodular: it does not fail in O6. We call this property a weakly distributive one and a weakly orthomodular lattice to which the property is added a weakly distributive lattice, WDL.
>
> We see that, as with the orthomodularity in quantum logic, in the syntactical structure of classical logic there is nothing distributive. The distributivity will appear as a result of the

way the relation of equivalence is usually defined in a proof of completeness of classical logic" [5].

Although it is true that $a \cup (b \cap c) \equiv (a \cup b) \cap (a \cup c) = 1$ holds in the non-orthomodular ortholattice O6 while $a \cup (b \cap c) = (a \cup b) \cap (a \cup c)$ does not, the conclusion that "in the syntactical structure of classical logic there is nothing distributive" does not follow from these observations. The purpose of this paper is to examine the grounds for this claim.

Tomasz Kowalski, Francesco Paoli, and Roberto Giuntini have already examined Pavičić and Megill's claims regarding orthomodular quantum logic and its alleged non-orthomodularity [2]. They have shown, using the tools of Abstract Algebraic Logic, that weakly orthomodular lattices provide only an algebraic semantics of quantum logic, not an equivalent one [2], so the claim of non-orthomodularity of quantum logic cannot be maintained.

In the present paper, we turn to the case of classical propositional logic, which has not been defended against the claims of non-distributivity yet. We argue that the reasoning of Pavičić and Megill relies on a conflation of algebraic semantics with unital matrix semantics. We analyse the lattice O6 in the framework of matrix semantics and show that its adequacy for $\mathcal{CL}$ is possible through reducibility to Boolean matrices. We point out that in O6, elements which are logically equivalent need not be identical. This separation between logical equivalence and lattice identity leads to the failure of the rule of replacement in O6. Furthermore, we present two lattice constructions that yield matrix models for $\mathcal{CL}$ lacking arbitrary lattice-theoretic properties. This demonstrates that the algebraic properties of matrix models are radically underdetermined by the logic.

The structure of the paper is as follows. In Section 2, we explain the difference between algebraic semantics and unital matrix semantics. In Section 3, we distinguish two possible meanings of distributivity for classical propositional logic and the main points of Pavičić and Megill's argument against it. Section 4 collects the necessary definitions. In Section 5 we examine the non-distributive lattice O6 as a matrix semantics and compare it with the Boolean matrix semantics. In Section 6, we further compare O6 to Boolean matrices and discuss certain undesirable features of the weakly distributive matrix, specifically the failure of the rule of replacement. Finally, in Section 7, we conclude the paper.

2. Algebraic semantics and matrix semantics

The notion of "algebraic semantics" is used in different senses in the literature and is not entirely uniform (cf. [1]). In earlier approaches, it was common to understand algebraic semantics in terms of logical matrices with a single designated element (cf. [10]). In this sense, a logic is interpreted in an algebra equipped with a distinguished set of values, and a formula is considered valid if it takes a designated value under every valuation. In contrast, in Abstract Algebraic Logic, the term "algebraic semantics" is used in a more specific sense, where logical systems are studied via their associated algebraic structures, typically by relating formulas to equations in algebras.

To avoid terminological ambiguity, in this paper we will distinguish between the two meanings of "algebraic semantics" by adopting the term "unital matrix semantics" for semantics based on logical matrices with a single designated value (cf. [3]).

The crucial difference between these approaches is that, unlike algebraic semantics in the sense of Abstract Algebraic Logic, matrix semantics does not require the underlying algebraic structure to reflect the laws of the logic. As a matter of fact, there are many nontrivial unital matrices which are sound and complete for Classical Logic, e.g. the three-valued matrix (cf. [9, ch. 3.5]), the non-orthomodular lattice O5 (cf. [2]). The non-distributive but weakly distributive lattice O6 is another example. Therefore, the existence of non-distributive matrix models does not by itself provide evidence that classical logic is non-distributive.

The proposed distinction is essential for our analysis. As we argue in Section 5, the constructions used by Pavičić and Megill are formulated in terms of matrix semantics, even though they are described as instances of algebraic semantics. This leads them to misleading conclusions about the basic properties of classical logic.

It is also well known that classical propositional logic is algebraizable and that the variety of Boolean algebras constitutes its equivalent algebraic semantics. Since the equivalent algebraic semantics of the classical logic is unique, it follows that no class of non-distributive lattices can serve as an equivalent algebraic semantics for $\mathcal{CL}$. Although this observation already settles the issue at the level of algebraic semantics, the matrix-theoretic analysis of O6 in the following sections reveals how O6 can nonetheless be adequate for $\mathcal{CL}$ despite its non-distributive structure.

3. What is distributivity of classical logic and the arguments against it

Before analysing Pavičić and Megill's argument, it is useful to clarify what it means for classical propositional logic $\mathcal{CL}$ to be distributive.

1. One possible meaning of distributivity of $\mathcal{CL}$ concerns the syntactic side of the logic. In this sense, a logic is distributive if the distributivity laws $a \wedge (b \vee c) \equiv (a \wedge b) \vee (a \wedge c)$ and $a \vee (b \wedge c) \equiv (a \vee b) \wedge (a \vee c)$ are its theorems.

2. Another possible meaning of distributivity concerns the semantics of $\mathcal{CL}$, in particular its models. In this case, distributivity is related to the fact that classical logic is closely connected with distributive lattices with complementation (i.e., Boolean algebras). This connection is expressed, for instance, by the following result [8]:

   Theorem 3.1. *A formula is provable in $\mathcal{CL}$ if and only if it evaluates to* 1 *under every valuation in the two-element Boolean algebra.*

   The two meanings are bridged by the theorem [8]:

   Theorem 3.2. *The Lindenbaum algebra obtained from the language of $\mathcal{CL}$ by quotienting with respect to deductive equivalence is a Boolean algebra.*

The second meaning admits a further refinement: a logic is distributive in a weaker semantic sense if it possesses a distributive model, and in a stronger sense if all of its models are distributive.

Pavičić and Megill do not challenge these results (cf. [7]). This raises the question of how their claim of non-distributivity of classical logic is to be understood.

In their work, distributivity appears in two contexts, both treated as properties of lattices. The first context (and, simultaneously, the first part of their argument) is connected with the notion of weak distributivity and the observation that adding the weak distributivity axiom to an ortholattice[1] does not yield a distributive lattice.

As they notice, in any ortholattice, if $a = b$, then $a \equiv b = 1$ ($\equiv$ is the classical equivalence, while $=$ is lattice identity). But only in Boolean algebras the reverse holds: if $a \equiv b = 1$, then $a = b$. Therefore Pavičić and Megill offer an alternative definition of Boolean algebras:

Definition 3.3 (Boolean algebra [7]). An ortholattice that satisfies the following condition:

$$\text{if } a \equiv b = 1, \text{ then } a = b$$

is called a Boolean algebra.

Let:

- $(p \vee (q \wedge r)) \equiv ((p \vee q) \wedge (p \vee r))$ be the logical distributivity law,
- $(a \cup (b \cap c)) = ((a \cup b) \cap (a \cup c))$ be the algebraic distributivity law,
- $(a \cup (b \cap c)) \equiv ((a \cup b) \cap (a \cup c)) = 1$ be its translation into the algebraic language via the notion of validity (in a unital matrix).

Pavičić and Megill notice that if one adds the algebraic version of the distributivity law to the ortholattice axioms, the resulting lattice is not distributive; i.e. $(a \cup (b \cap c)) \equiv ((a \cup b) \cap (a \cup c)) = 1$ is valid in the resulting lattice, while $(a \cup (b \cap c)) = ((a \cup b) \cap (a \cup c))$ is not. From this observation they conclude that the lattice model of $\mathcal{CL}$ need not be distributive. They

[1] For the definition of an ortholattice, see Section 4.

call the property $(a \cup (b \cap c)) \equiv ((a \cup b) \cap (a \cup c)) = 1$ *weak distributivity*. Crucially, they take this notion to correspond to the logical distributive law, since they treat the latter as the appropriate translation of logical principles into lattice-theoretic terms. This leads to models in which some logical principles are represented by identities, while others are represented only by equivalence conditions. As a result, the connection between logical equivalence and algebraic identity is weakened, and the resulting models exhibit a form of "nonstandardness", which they later interpret as evidence for the "nonstandardness" of logic. On this basis they conclude that "in the syntactical structure of classical logic there is nothing distributive" [5].

The second context (and the second part of their argument) concerns the Lindenbaum algebra of $\mathcal{CL}$ (cf. [5, 7]). Pavičić and Megill argue that the distributivity of this algebra arises from the definition of the equivalence relation rather than from the axioms or rules of inference of $\mathcal{CL}$. To support this, they define the relation $A \approx B$ by $\Gamma \vdash A \equiv B$, and argue that the property 3.3 – namely, if $a \equiv b = 1$, then $a = b$ – "has nothing to do with any axiom or rule of inference from $\mathcal{CL}$ – it is nothing but a consequence of the definition of the relation of equivalence" [7]. They then introduce a modified equivalence relation incorporating O6-valuations, and show that the resulting quotient algebra is a weakly distributive lattice, concluding that "the syntactical structure of classical logic corresponds to (maps to) the structure of the weakly distributive lattice not the one of the Boolean algebra" [7].

This line of reasoning raises several issues. First, the standard Lindenbaum equivalence is defined as $\vdash A \equiv B$, independently of any Γ. The Γ-dependent relation they use yields a different quotient and does not correspond to the usual Lindenbaum algebra of $\mathcal{CL}$. Furthermore, in the genuine Lindenbaum algebra, elements are cosets $[A]$ rather than formulas, and the condition $a \equiv b = 1$ properly reads $[A] \equiv [B] = [C \vee \neg C]$, which reduces to $[A] = [B]$, i.e. $\vdash A \equiv B$. The property they treat as characteristic of distributive lattices is therefore trivial in the standard construction: it follows directly from the meaning of coset equality, rather than from any independent algebraic constraint.

Even within their nonstandard setting, however, the conclusion does not follow in the intended sense. Since the distributivity law is a theorem of $\mathcal{CL}$, it follows from every set of premises Γ. Hence, according to their definition, for any formulas A, B such that $A \equiv B$ is a theorem of $\mathcal{CL}$, $A \approx B$, and thus $[A] = [B]$. The distributive character of this quotient algebra is therefore not an arbitrary artifact of definition, but a direct reflection of the inferential structure of the logic.

A further issue concerns the terminology: not every quotient of the formula algebra $\mathcal{F}$ deserves to be called a Lindenbaum algebra. Since $\mathcal{F}$ is an absolutely free algebra, every algebra of the same signature is its homomorphic image. Hence any such algebra arises as a quotient by some congruence, making constructions of this kind trivial, unless the congruence is determined by the consequence relation of the logic itself.

Most fundamentally, their modified construction incorporating O6-valuations (cf. [7]) is circular in the following sense: the congruence relation is defined from the outset to respect identifications induced by O6-valuations, so the resulting quotient inherits the structure of O6 by design. This does not show that the syntactic structure of $\mathcal{CL}$ naturally maps into O6. It shows only that imposing O6-based identifications on formulas produces a quotient with O6-like structure. That is a property of the chosen equivalence relation, not of classical logic.

In what follows, we will show that the lattice O6 can indeed serve as an adequate semantics for $\mathcal{CL}$, but only within the framework of matrix semantics. In particular, its comparison with Boolean semantics does not support the claim that it provides a more faithful representation of the logical structure of classical logic than Boolean semantics.

4. Preliminaries

Definition 4.1 (Ortholattice). An ortholattice $\langle L, \cap, \cup, ', 1, 0\rangle$ consists of a nonempty set L, two binary operations $\cap$ and $\cup$ called the lattice meet and join, respectively, a unary operation $'$ of orthocomplementation, and constants $1, 0 \in L$ (called the top and bottom elements), such that for every $a, b \in L$:

a) $a \cap a = a; \quad a \cup a = a,$

b) $a \cap b = b \cap a; \quad a \cup b = b \cup a,$

c) $a \cap (b \cap c) = (a \cap b) \cap c; \quad a \cup (b \cup c) = (a \cup b) \cup c,$

d) $a \cap (a \cup b) = a; \quad a \cup (a \cap b) = a,$

e) $a \cap 1 = a; \quad a \cup 1 = 1,$

f) $a \cap 0 = 0; \quad a \cup 0 = a,$

g) $a \cup a' = 1; \quad a \cap a' = 0,$

h) $a = (a')',$

i) if $a \leq b$, then $b' \leq a'$.

Definition 4.2 (Weakly orthomodular lattice WOML). A weakly orthomodular lattice is an ortholattice which satisfies the condition: $(a' \cap (a \cup b)) \cup b' \cup (a \cap b) = 1$.

Definition 4.3 (Distributive ortholattice). A distributive ortholattice is an ortholattice that satisfies the distributivity identity: $a \cup (b \cap c) = (a \cup b) \cap (a \cup c)$.

Definition 4.4 (Weakly distributive lattice WDL). A weakly distributive lattice WDL is a weakly orthomodular lattice that satisfies the condition:

$$a \cup (b \cap c) \equiv (a \cup b) \cap (a \cup c) = 1,$$

where $a \equiv b := (a' \cup b) \cap (b' \cup a)$ is the lattice-theoretic biconditional, and $=$ is the lattice identity.

Definition 4.5 (Boolean algebra). A Boolean algebra $\mathcal{B} = \langle B, \cap, \cup, ', 1, 0\rangle$ is a distributive ortholattice.

In the presence of orthocomplementation, distributivity characterizes Boolean algebras.

Since in ortholattices $\cap$ is definable by $\cup$ and $'$ via de Morgan laws, Boolean algebras may also be stated as $\mathcal{B} = \langle B, \cup, ', 1, 0\rangle$.

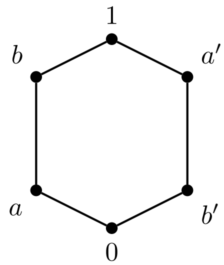

Figure 1: Lattice O6.

Definition 4.6 (O6 lattice). The lattice $O6 = \langle\{1, a, b, a', b', 0\}, \cap, \cup, ', 1, 0\rangle$ is the ortholattice shown in figure 1.

O6 is a non-orthomodular, non-distributive, weakly distributive lattice. It is called the benzene ring or the hexagon.

Definition 4.7 (Matrix). A logical matrix for a propositional language $\mathcal{F}$ is a pair $M = (\mathcal{A}, D)$, where $\mathcal{A}$ is an algebra of the same signature as $\mathcal{F}$ and $D \subseteq A$ is a set of designated elements.

Definition 4.8 (Matrix homomorphism). Let $M = (\mathcal{A}, D_M)$ and $N = (\mathcal{B}, D_N)$ be matrices, where $\mathcal{A}$ and $\mathcal{B}$ are algebras of the same signature. A mapping $h : \mathcal{A} \to \mathcal{B}$ is a homomorphism if:

1. $h(a \vee b) = h(a) \vee h(b)$,
2. $h(\neg a) = \neg h(a)$,
3. $h(D_M) \subseteq D_N$.

Definition 4.9 (Valuation). A valuation in a matrix $M = (\mathcal{A}, D)$ is a homomorphism $v : \mathcal{F} \to \mathcal{A}$.

Definition 4.10 (Satisfaction). A formula $A \in \mathcal{F}$ is satisfied in a matrix M under a valuation v if $v(A) \in D$.

Definition 4.11 (Validity in a matrix). Let $M = (\mathcal{A}, D)$ be a matrix. A formula $A \in \mathcal{F}$ is valid in M, written $\models_M A$, if for every valuation v, $v(A) \in D$.

A formula A is a consequence of $\Gamma \subseteq \mathcal{F}$ relative to M, written $\Gamma \models_M A$, if for every valuation v, whenever $v(X) \in D$ for all $X \in \Gamma$, then $v(A) \in D$.

Definition 4.12 (Rule validity in a matrix). An inference rule Γ/A is valid in a matrix $M = (\mathcal{A}, D)$ if for every valuation v, whenever $v(X) \in D$ for all $X \in \Gamma$, then $v(A) \in D$.

Definition 4.13 (Weakly adequate matrix). A matrix M is weakly adequate for a logic $\mathcal{L}$ if a formula A is a theorem of $\mathcal{L}$ if and only if it is valid in M.

Definition 4.14 (Adequate matrix). A matrix M is adequate for a logic $\mathcal{L}$ if it is weakly adequate and every rule of inference of $\mathcal{L}$ is valid in M.

Definition 4.15 (Matrix B_2). The matrix B_2 is the two-element Boolean algebra with $\{1\}$ as the set of designated elements: $\mathrm{B}_2 = ((\{1, 0\}, \vee, \neg), \{1\})$.

Matrix B_2 is an adequate matrix for $\mathcal{CL}$.

Throughout the paper, we use the logical symbols $\vee, \wedge$ and $\neg$ for matrix operations, while $\cup, \cap$, and $'$ denote their lattice-theoretic counterparts.

5. Lattice O6 as matrix semantics for classical logic

Pavičić and Megill showed that weakly distributive models, in particular the lattice O6, are sound and complete for $\mathcal{CL}$. Although they describe their approach in terms of algebraic semantics, their definitions of model and validity have the form typical of matrix semantics. Let us examine their definitions.

Definition 5.1 (Model [7]). We call $\mathcal{M} = \langle \mathcal{A}, h \rangle$ a model if $\mathcal{A}$ is an algebra and $h : \mathcal{F} \to \mathcal{A}$, called a valuation, is a morphism of formulas $\mathcal{F}$ into $\mathcal{A}$, preserving the operations $\neg, \vee$ while turning them into $', \cup$.

Definition 5.2 (Validity in model [7]). We call a formula $A \in \mathcal{F}$ valid in the model $\mathcal{M}$, and write $\models_{\mathcal{M}} A$, if $h(A) = 1$ for all valuations h on the model, i.e. for all h associated with the base set $\mathcal{A}$ of the model. We call a formula $A \in \mathcal{F}$ a consequence of $\Gamma \subseteq \mathcal{F}$ in the model $\mathcal{M}$ and write $\Gamma \models_{\mathcal{M}} A$ if $h(X) = 1$ for all $X \in \Gamma$ implies $h(A) = 1$, for all valuations h.

Note that these definitions correspond, in substance, to the standard definitions of validity in a (unital) matrix (Definitions 4.11, 4.12).

These definitions blur the distinction between satisfaction under a single valuation and validity across all valuations. Pavičić and Megill partially address this by allowing the term “model” to refer either to a specific pair $\langle \mathcal{A}, h \rangle$ or to the class of all such pairs based on a fixed algebra $\mathcal{A}$ (cf. [7, sec. 3]).

However, once this ambiguity is resolved, the underlying structure becomes clear: a model is determined by an algebra together with homomorphisms from the algebra of formulas, and validity is defined by requiring that formulas take the value 1 under all such homomorphisms. In other words, their notion of validity coincides with validity in a unital matrix $(\mathcal{A}, \{1\})$.

This differs fundamentally from algebraic semantics in the sense of Abstract Algebraic Logic, where formulas are related to equations in algebras.

Thus, despite the terminology used by Pavičić and Megill, their framework is most naturally understood as a unital matrix semantics. In what follows, we adopt this matrix-theoretic perspective and analyse the lattice O6 accordingly.

5.1. O6 as a matrix semantics of classical logic

We first introduce the matrix O6 and its operations. We defined the lattice $\text{O6} = (\{1, a, b, a', b', 0\}, \cap, \cup, ', 1, 0)$ in Section 4. Interpreting the logical connective $\vee$ by the lattice operation $\cup$ and the logical negation $\neg$ by orthocomplementation $'$, we obtain the matrix O6.

Definition 5.3 (Matrix O6). A matrix O6 for $\mathcal{CL}$ is the matrix $\text{O6} = ((\{1, a, b, a', b', 0\}, \vee, \neg), D)$ with the operations $\vee, \neg$ defined in Table 1 and with D being the set of designated elements.

We will consider O6 matrices with various choices of subset D. We will

Table 1: Operations $\vee$ and $\neg$ in O6.

$\vee$	1	a	b	a’	b’	0		$\neg$
1	1	1	1	1	1	1		0
a	1	a	b	1	1	a		a’
b	1	b	b	1	1	b		b’
a’	1	1	1	a’	a’	a’		a
b’	1	1	1	a’	b’	b’		b
0	1	a	b	a’	b’	0		1

also use O6 to denote the algebra $(\{1, a, b, a', b', 0\}, \vee, \neg)$, e.g. in $(\text{O6}, \{1\})$ to denote the unital matrix O6 or in $(\text{O6}, \{1, a, b\})$ to denote an O6 matrix with $D = \{1, a, b\}$.

Conjunction is defined in accordance with the lattice infimum (meet) and presented in Table 2. Implication is defined in Table 3 in accordance with the definition $a \to b := \neg a \vee b$. Equivalence is defined in Table 4 in accordance with the standard definition $a \equiv b := (a \to b) \wedge (b \to a)$.

From Table 4 we observe that elements a and b are equivalent ($a \equiv b = 1$), although non-identical. The same holds for a' and b'. This fact has a direct consequence for the distributivity law $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$. Consider a, b, and b'. Then $a \vee (b \wedge b') = a \vee 0 = a$, while $(a \vee b) \wedge (a \vee b') = b \wedge 1 = b$. Thus, the equality fails, while the equivalence of both sides is still 1. So the matrix is weakly distributive, but not distributive.

5.2. O6 and the two-element Boolean matrix B_2

Recall the following theorem [9]:

THEOREM 5.4. *If there is a homomorphism f from matrix M to N that maps undesignated elements of M into undesignated elements on N, then all the propositions that are valid in N are valid in M.*

PROPOSITION 5.5. O6 is an adequate matrix for classical propositional logic $\mathcal{CL}$.

Table 2: Operation $\wedge$ in O6.

$\wedge$	1	a	b	a'	b'	0
1	1	a	b	a'	b'	0
a	a	a	a	0	0	0
b	b	a	b	0	0	0
a'	a'	0	0	a'	b'	0
b'	b	0	0	b'	b'	0
0	0	0	0	0	0	0

Table 3: Operation $\rightarrow$ in O6.

$\rightarrow$	1	a	b	a'	b'	0
1	1	a	b	a'	b'	0
a	1	a	1	a'	a'	a'
b	1	1	1	a'	b'	0
a'	1	a	b	1	1	a
b'	1	b	b	1	1	b
0	1	1	1	1	1	1

Table 4: Operation $\equiv$ in O6.

$\equiv$	1	a	b	a'	b'	0
1	1	a	b	a'	b'	0
a	a	1	1	0	0	a'
b	b	1	1	0	0	b'
a'	a'	0	0	1	1	a
b'	b'	0	0	1	1	b
0	0	a'	b'	a	b	1

PROOF: Define a mapping $f : \mathrm{B}_2 \to \mathrm{O6}$ by:

$$f(1) = 1$$
$$f(0) = 0.$$

Then f is a homomorphism and maps undesignated elements of B_2 into undesignated elements of O6. Therefore, by Theorem 5.4, all propositions valid in O6 are valid in B_2. Since B_2 is an adequate matrix for $\mathcal{CL}$, it follows that all propositions valid in B_2 are derivable in $\mathcal{CL}$. Thus, O6 is complete for $\mathcal{CL}$.

Conversely, define a mapping $g : \mathrm{O6} \to \mathrm{B}_2$ as:

$$g(1) = 1$$
$$g(a) = 1$$
$$g(b) = 1$$
$$g(a') = 0$$
$$g(b') = 0$$
$$g(0) = 0.$$

It is straightforward to verify in the truth tables for O6 that g is a homomorphism. We take the set of designated elements to be $D = \{1, a, b\}$. Then, by Theorem 5.4, all propositions valid in B_2 are valid in O6. Since B_2 is an adequate matrix for $\mathcal{CL}$, all propositions derivable in $\mathcal{CL}$ are valid in B_2. It follows that every theorem of $\mathcal{CL}$ is valid in O6. Thus, every theorem of $\mathcal{CL}$ is valid in O6, i.e. O6 is sound for $\mathcal{CL}$.

Finally, for rules of inference, if we consider Modus Ponens, it is straightforward to verify (see Table 3) that whenever A and $(A \to B)$ are designated, B is also designated. Hence, the rules of inference of $\mathcal{CL}$ are valid in O6.

Therefore, O6 is an adequate matrix for $\mathcal{CL}$. □

Remark 5.6. The choice of designated elements in O6 is essential in this proof of adequacy. The only possible choices of D for which there exists a matrix homomorphism $g : \mathrm{O6} \to \mathrm{B}_2$ are $D = \{1, a, b\}$ and $D = \{1, a', b'\}$ (and these sets are the maximal filters in the lattice O6). Note that the map $f : \mathrm{O6} \to \mathrm{O6}$ such that $f(1) = 1, f(a) = a', f(b) = b', f(a') = a, f(b') =$

b, $f(0) = 0$, is an automorphism, and the two choices of designated elements differ only by a symmetry of the lattice O6.

If $D = \{1\}$, then any mapping $g_1 : \mathrm{O6} \to \mathrm{B}_2$ must satisfy $g_1(1) = 1$ and $g_1(x) = 0$ for all other elements. However, such a mapping fails to preserve negation, since

$$g_1(\neg b) = g_1(b') = 0 \neq 1 = \neg 0 = \neg g_1(b).$$

Hence, no such homomorphism exists.

A similar argument shows that for any other choice of D containing 1 (e.g. $D = \{1, a\}$, $D = \{1, a, a'\}$, $D = \{1, a, b, a', b'\}$, etc.), either $\vee$ or $\neg$ is not preserved. Therefore, no homomorphism with B_2 exists in these cases.

Consequently, this proof can only establish the adequacy of matrix $(\mathrm{O6}, \{1, a, b\})$ for $\mathcal{CL}$, and not of the unital matrix $(\mathrm{O6}, \{1\})$.

For completeness, we briefly recall the standard notions of congruence and quotient constructions, which will be used in the proof below.

Definition 5.7 (Congruence). An equivalence relation θ on an algebra $\mathcal{A}$ is a congruence if it is preserved by all operations of $\mathcal{A}$.

Every homomorphism determines a congruence relation given by $a\,\theta\,b$ iff $h(a) = h(b)$.

Definition 5.8 (Quotient algebra). Let $\mathcal{A}$ be an algebra with operations $\{O_i\}$, let θ be a congruence relation on $\mathcal{A}$, and A/θ the collection of equivalence classes determined by θ (where $[a]_\theta = \{b \in A : a\,\theta\,b\}$ is the equivalence class of a). The quotient algebra determined by θ is the algebra $\mathcal{A}/\theta = \langle A/\theta, \{Q_i\}\rangle$, where operations Q_i on equivalence classes are defined as:

$Q_i([a_1], ..., [a_n]) = [O_i(a_1, ..., a_n)]$.

Definition 5.9 (Matrix congruence). Let $M = (\mathcal{A}, D)$ be a logical matrix. A congruence θ in $\mathcal{A}$ is a matrix congruence if $a \in D$ implies $[a]_\theta \subseteq D$.

Definition 5.10 (Quotient matrix). The matrix $(\mathcal{A}/\theta, D/\theta)$, where $\mathcal{A}/\theta$ is the quotient of $\mathcal{A}$ and $D/\theta = \{[a]_\theta : a \in D\}$ is called the matrix quotient of M by θ, M/θ.

THEOREM 5.11 (Homomorphism Theorem). *Every homomorphic image of an algebra $\mathcal{A}$ is isomorphic to a quotient of $\mathcal{A}$, and vice versa.*

THEOREM 5.12 ([10]). *Let $M = (\mathcal{A}, D)$ be a logical matrix, and let θ be a matrix congruence in M. Then for every language interpreted in M:*
$M^{\models} = (M/\theta)^{\models}$.

Where $M^{\models}$ is the matrix consequence of matrix M.

PROPOSITION 5.13. Let O6 be the matrix $(\mathrm{O6}, \{1, a, b\})$, and B_2 be the two-valued Boolean matrix. Then:
$\mathrm{O6}^{\models} = \mathrm{B}_2^{\models}$.

PROOF: By Theorem 5.11, B_2 is the homomorphic image of O6 under the homomorphism g, and hence is isomorphic to the quotient $\mathrm{O6}/G$, where G is the congruence defined by: aGb iff $g(a) = g(b)$.

$D = \{1, a, b\}$. Then $[1]_G = [a]_G = [b]_G = \{1, a, b\} \subseteq D$, so G is a matrix congruence. Therefore, $\mathrm{O6}/G$ is a quotient matrix.

By Theorem 5.12, $\mathrm{O6}^{\models} = (\mathrm{O6}/G)^{\models}$, and since $\mathrm{O6}/G$ is isomorphic to B_2, it follows that $\mathrm{O6}^{\models} = \mathrm{B}_2^{\models}$. □

The weakly distributive matrix $(\mathrm{O6}, \{1, a, b\})$ and the Boolean matrix B_2 have the same matrix consequence.

DEFINITION 5.14 (Reduced matrix). A matrix M is called reduced if it admits no nontrivial matrix congruences.

As the proof shows, under congruence G, the matrix $(\mathrm{O6}, \{1, a, b\})$ reduces to B_2.

5.3. O6 and the unital four-element Boolean matrix B_4

Let $B_4 = ((\{1, a, b, 0\}, \vee, \neg), \{1\})$ be the unital four-element Boolean matrix, and let $(\mathrm{O6}, \{1\})$ be the unital matrix O6.

PROPOSITION 5.15. $(\mathrm{O6}, \{1\})$ is an adequate (unital) matrix for classical propositional logic $\mathcal{CL}$.

PROOF: Define a mapping $f : B_4 \to \mathrm{O6}$ as:

$$
\begin{aligned}
f(1) &= 1 \\
f(a) &= a \\
f(b) &= a' \\
f(0) &= 0.
\end{aligned}
$$

Then f is a homomorphism and maps undesignated elements of B_4 to undesignated elements of O6.

Conversely, define a mapping $g : \mathrm{O6} \to \mathrm{B}_4$ as:

$$
\begin{aligned}
g(1) &= 1 \\
g(a) &= a \\
g(b) &= a \\
g(a') &= b \\
g(b') &= b \\
g(0) &= 0.
\end{aligned}
$$

Then g is a homomorphism and maps undesignated elements of O6 to undesignated elements of B_4.

Since there exist matrix homomorphisms in both directions between $(\mathrm{O6}, \{1\})$ and B_4, it follows by Theorem 5.4 that both matrices validate the same formulas. Since B_4 is sound and complete for $\mathcal{CL}$, the same holds for $(\mathrm{O6}, \{1\})$. □

Observe that B_4 is isomorphic to the quotient matrix $\mathrm{O6}/\theta$, where θ is the congruence with equivalence classes $\{1\}$, $\{a, b\}$, $\{a', b'\}$, and $\{0\}$. The mapping g corresponds to the canonical quotient map. Since $g^{-1}(1) = \{1\}$, the matrix $(\mathrm{O6}, \{1\})$ reduces to B_4 via a nontrivial matrix congruence. Thus, the adequacy of $(\mathrm{O6}, \{1\})$ is not grounded in its internal algebraic structure, but is inherited via its reduction to a Boolean matrix.

As we have shown, the matrix O6 is an adequate matrix for classical logic, with D being either $\{1\}$ or $\{1, a, b\}$. The unital matrix O6 provides an adequate semantics for $\mathcal{CL}$ not because of its internal algebraic structure, but because it admits a reduction to the Boolean matrix B_4.

Furthermore, it is also a special case of the following construction, which allows us to obtain from B_4 various unital matrices that are sound and complete for classical logic but which do not have any specific nontrivial lattice properties (such as, e.g., distributivity)[2].

The construction goes as follows:

Let $B4 = (\{1, a, b, 0\}, \vee, \neg)$ be the 4-element Boolean algebra and $B_4 = (B4, \{1\})$ be the unital Boolean matrix. For any two disjoint bounded lattices L_0 and L_1, we can construct a new unital matrix M by "replacing" the element a with L_0 and b with L_1. The join in the extended lattice is defined in an obvious way, while for the complement $'$ we require:

$$1' = 0, 0' = 1,$$
$$a' \in L_1 \text{ for all } a \in L_0,$$
$$b' \in L_0 \text{ for all } b \in L_1.$$

Then we can define a map $h : M \to B_4$ as:

$$h(1) = 1,$$
$$h(0) = 0,$$
$$h(x) = a, \text{ for all } x \text{ in } L_0,$$
$$h(y) = b, \text{ for all } y \text{ in } L_1.$$

The map h is a matrix homomorphism and maps undesignated elements of M into undesignated elements of B_4. A mapping $h^{-1} : B_4 \to M$ can be defined as follows:

$$h^{-1}(1) = 1,$$
$$h^{-1}(0) = 0,$$
$$h^{-1}(a) = x, \text{ where } x \text{ is some element of } L_0,$$
$$h^{-1}(b) = y, \text{ where } y \text{ is some element of } L_1,$$
$$x' = y \text{ and } y' = x.$$

The map h^{-1} is a matrix homomorphism and maps undesignated elements of B_4 into undesignated elements of M. Therefore, M and B_4 validate the same formulas, and M is a sound and complete semantics for classical logic. Consequently, the existence of such matrices does not reflect any intrinsic logical properties of their underlying lattices, but rather

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[2] I would like to thank the anonymous referee for suggesting this construction.

the fact that matrix semantics allows arbitrary algebraic structure to be combined with a fixed Boolean core.

In particular, the lattice O6 appears as just one instance of a much broader class of constructions.

Another construction can be formulated in a more explicitly lattice-theoretic manner[3]. Let $\mathrm{L} = (L, \cap, \cup)$ be a bounded lattice and L^* be its dual lattice, obtained by reversing the partial order on L. Assume that the set L is disjoint from the universe of the algebra $B4$. For each $x \in L$, let x^* denote its corresponding element in L^*, and for each $y \in L^*$, let y^* denote its counterpart in L.

Define a new algebra $\mathcal{A} = (A, \cap, \cup, ')$ such that $A = \{0\} \cup L \cup L^* \cup \{1\}$. The lattice operations $\cap$ and $\cup$ are determined by the natural partial order extending those of L and L^*, while the unary operation $'$ is defined by:

$$1' = 0, 0' = 1,$$
$$x' = x^* \text{ for all } x \in L \cup L^*.$$

The resulting lattice $\mathcal{A}$ forms an ortholattice. As in the previous construction, one can define matrix homomorphisms $h : \mathcal{A} \to B_4$ and $k : B_4 \to \mathcal{A}$ such that designated elements are preserved, which shows that $\mathcal{A}$ is sound and complete for classical propositional logic.

Moreover, the partition of A into the sets $\{0\}$, L, L^*, and $\{1\}$ determines a matrix congruence, and the corresponding quotient algebra is isomorphic to $B4$.

Since L was an arbitrary bounded lattice, it follows that models of classical logic may contain sublattices with completely unrestricted lattice-theoretic properties, and in particular need not satisfy any nontrivial lattice identities.

These two constructions share a common feature: in each case, the resulting matrix admits a congruence whose quotient is isomorphic to B_4. Thus, their adequacy for $\mathcal{CL}$ does not stem from their internal lattice structure, which may lack arbitrary properties, but from their reducibility to Boolean matrices.

[3] I would like to thank the anonymous referee for suggesting this construction.

6. Lattice O6 and Boolean algebras

The homomorphism $g : \mathrm{O6} \to \mathrm{B}_4$ from the matrix $(\mathrm{O6}, \{1, a, b\})$ into the unital Boolean matrix $(\mathrm{B4}, \{1\})$ suggests an interesting possible relationship between weakly distributive lattices and distributive lattices. First, notice that O6 is similar to B_4, but with the elements a and b split into distinct nodes. In distributive lattices, complementation is unique (meaning that for each element a there is only one element a' such that $a \cap a' = 0$ and $a \cup a' = 1$), while in the weakly distributive lattice O6 it is not. For example, for the element a both a' and b' serve as complements, for a' both a and b serve as complements, etc. In the Boolean case, if a and b were complements to a', a would be identified with b, and a' with b'. At the same time, Table 4 shows that a and b are equivalent in O6, and so are a' and b' (but e.g. a and 1 are not). From the point of view of the logical operation of equivalence, a and b "act" like one element, while from the point of view of lattice identity, they are not identical. This suggests that O6 can be viewed as a "broken" Boolean lattice in which identity has been separated from equivalence.

This feature reveals a fundamental limitation of O6 as a model of $\mathcal{CL}$.

First of all, for B_2 we have the feature [8]:

THEOREM 6.1. *If the formula $\Phi \equiv \Psi$ is a theorem of $\mathcal{CL}$, then for every valuation $v : \mathcal{F} \to B_2$ we have $v(\Phi) = v(\Psi)$.*

Formulas Φ and Ψ, such that for every $v : \mathcal{F} \to \mathrm{B}_2$ we have $v(\Phi) = v(\Psi)$, are called semantically equivalent in matrix B_2. An equivalence relation that has the property described in Theorem 6.1. is said to have a normal interpretation in matrix B_2.

In the case of O6, as we have seen, this feature does not hold. Furthermore, formulas that are logically equivalent in $\mathcal{CL}$ are not semantically equivalent in matrix O6, and equivalence does not have a normal interpretation in matrix O6.

Secondly, another problem is that in the lattice O6 the rule of replacement of equivalents does not hold.

The rule of replacement of logical equivalents is a derivable rule of $\mathcal{CL}$:

$$\frac{\Phi \equiv \Psi}{\Omega \equiv \Omega(\Phi/\Psi)}$$

which states that if a formula Φ is logically equivalent to Ψ, then Ω is logically equivalent to a formula obtained from Ω through replacement of one or more occurrences of Φ in Ω by Ψ [8].

If we take Pavičić and Megill's interpretation of the language of $\mathcal{CL}$ in ortholattices, the rule will take the form:

$$\frac{\Phi \equiv \Psi = 1}{\Omega \equiv \Omega(\Phi/\Psi) = 1}.$$

Let Φ be the formula $a \vee (b \wedge c)$, Ψ the formula $((a \vee b) \wedge (a \vee c))$, and Ω the formula $(a \vee (b \wedge c)) \rightarrow (a \vee (b \wedge c))$. If we interpret these formulas in O6 and replace the first occurrence of Φ (the antecedent of Ω) with Ψ, we obtain:

$$\frac{(a \cup (b \cap c)) \equiv ((a \cup b) \cap (a \cup c)) = 1}{((a \cup (b \cap c)) \rightarrow (a \cup (b \cap c))) \equiv (((a \cup b) \cap (a \cup c)) \rightarrow (a \cup (b \cap c))) = 1}.$$

Consider the valuation assigning a to a, b to b, and b' to c. Then the weak distributivity as the premise holds. In the conclusion, however, the left side has the value: $a \cup (b \cap b') \rightarrow (a \cup (b \cap b')) = (a \cup 0) \rightarrow (a \cup 0) = a \rightarrow a = a$, while the right side has value: $((a \cup b) \cap (a \cup b')) \rightarrow (a \cup (b \cap b')) = (b \cap 1) \rightarrow (a \cup 0) = b \rightarrow a = 1$. Hence $a \equiv 1 = a \neq 1$. Therefore, the rule fails in O6.

This failure shows that O6 does not preserve one of the fundamental structural properties of classical logic, namely the substitutivity of logically equivalent formulas. Consequently, although O6 may reproduce the set of valid formulas, it does not fully preserve the inferential structure of $\mathcal{CL}$.

7. Conclusion

The aim of this paper was to analyse the lattice O6 as a matrix semantics for classical propositional logic $\mathcal{CL}$, and to examine what the existence of such non-distributive matrix models reveals about the distributive character of classical logic.

The matrix-theoretic analysis yields the following results:

- The matrix $(\mathrm{O6}, \{1, a, b\})$ is adequate for $\mathcal{CL}$ but not reduced, and it reduces via a matrix homomorphism to the Boolean matrix B_2.
- The unital matrix $(\mathrm{O6}, \{1\})$ is also adequate for $\mathcal{CL}$, and reduces via a matrix homomorphism to the Boolean matrix B_4.
- More generally, the class of adequate matrix semantics for $\mathcal{CL}$ is very broad: as shown in Sections 5 and 6, it includes structures lacking any nontrivial lattice-theoretic properties, including distributivity, double negation, and the law of excluded middle.

These results can be situated within the framework introduced in Section 3. Syntactic distributivity of $\mathcal{CL}$ is not affected: distributivity remains a theorem. Weak semantic distributivity trivially holds, as Boolean matrices are distributive models of $\mathcal{CL}$. Strong semantic distributivity fails, as witnessed by the constructions of Sections 5 and 6 – but this failure was never in doubt, and does not bear on the logical character of $\mathcal{CL}$. Pavičić and Megill's result, charitably interpreted, bears only on this last sense.

The analysis reveals not only that O6 is adequate for $\mathcal{CL}$, but also why: its adequacy is mediated by matrix homomorphisms to Boolean matrices, and is therefore inherited rather than intrinsic. This illustrates a general phenomenon: the flexibility of matrix semantics allows many algebraically diverse structures to validate the same set of formulas without preserving the underlying logical structure. In particular, O6 fails to preserve essential structural features of $\mathcal{CL}$, such as the substitutivity of logically equivalent formulas, further undermining its status as a faithful model.

The existence of non-distributive matrix models therefore does not constitute evidence that classical logic itself lacks distributivity. Distributivity is a theorem of $\mathcal{CL}$, and its presence in the equivalent algebraic semantics follows from the uniqueness guaranteed by Abstract Algebraic Logic. The claim that weakly distributive lattices better correspond to the syntactic structure of $\mathcal{CL}$ than Boolean algebras rests on a conflation of matrix semantics with algebraic semantics in the sense of Abstract Algebraic Logic. The definitions of model and validity employed by Pavičić and Megill are matrix-theoretic in character, and their construction yields a weakly distributive quotient only by incorporating O6-valuations into the equivalence relation from the outset. Consequently, the distributive character of classical propositional logic remains intact.

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MODAL LOGIC OF LATTICES

Abstract

We prove that the modal logic of lattices with the accessibility relation of being isomorphic to a sublattice is S4.2. The same is proven for modular and distributive lattices.

Keywords: modal logic of classes of structures, lattices, order, distributivity.

1. Introduction

Let $\mathcal{L}$ be a first-order language and T a theory in said language. We consider the class $Mod(T)$ of all models of the theory T, along with the relation $\subsetneqq$ interpreted as *embeddability*; we write $M \subsetneqq N$ if there is an embedding $f \colon M \to N$. This gives rise to a Kripke frame $(Mod(T), \subsetneqq)$, whose modal logic we shall investigate.

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The paper is concerned with the modal logic of the lattice theory (by which we mean the modal logic of the frame $(Mod(T), \subsetneq)$, for $T =$ lattice theory), as well as some stronger theories: theory of modular lattices and theory of distributive lattices.

The study of modal logics of classes of structures began with the study of modal logics of set theory and arithmetic. In these and the following work the modal operators were interpreted a little bit differently than in our case — $\Box$ was interpreted as "in all forcing extensions" [5], "in all ground models" [7] etc. In these works it was demanded from the language $\mathcal{L}$ to be strong enough to be able to express the interpretation of the operator $\Box$ [9]. This has been generalised and more recent papers on the topic consider the case where $\mathcal{L}$ is a first-order language, which is not strong enough [6, 1]. We follow these authors in taking $\mathcal{L}$ to be first-order, as well as investigating the relation $\subsetneq$ on the class $Mod(T)$. Intuitively, as Saveliev and Shapirovsky describe "robust" theories to be the "true" modal logics of a given relation, we think of the modal logic of $(Mod(T), \subsetneq)$ to be the "true" modal logic of a theory (especially since the said frame is bisimilar to a frame where the relation $\subsetneq$ is replaced with the direct extension relation $\subseteq$ [9]).

The cases where $T =$ graph theory and $T =$ theory of abelian groups are known thanks to the authors of [1] and [6]. Lattice theory seems to be a natural theory to be investigated in this manner, in order to see if the same is true for graphs, abelian groups and lattices. In this paper we prove that this is in fact true.

2. Preliminaries

Let $\mathcal{L}_\Box$ be a standard propositional modal language, that is a countable set of propositional variables Var together with the set of logical symbols $\{\wedge, \neg, \Box\}$ — the symbols $\to, \leftrightarrow, \vee$ and $\Diamond$ are defined as usual. The set $Fm_\Box$ of modal formulas is defined in a standard way.

The set of modal formulas Λ is called a *modal logic* if it is closed under modus ponens and substitution. If it is furthermore closed under necessitation ($\frac{\alpha}{\Box\alpha}$) and contains the K axiom (the formula $\Box(p \to q) \to (\Box p \to \Box q)$) then it is called a *normal modal logic*.

The modal logic relevant to our research is well known logic S4.2, which is the smallest normal modal logic containing the following formulas:

$$\begin{aligned} \mathrm{T}: &\quad \Box p \to p, \\ 4: &\quad \Box p \to \Box\Box p, \\ .2: &\quad \Diamond\Box p \to \Box\Diamond p. \end{aligned}$$

Validity in a Kripke model $\mathbb{M} = (\mathcal{W}, R, v)$ — where $\mathcal{W}$ is a nonempty collection of *possible worlds*, R is an accessibility relation on $\mathcal{W}$, and $v\colon Var \to \mathcal{P}(\mathcal{W})$ is a valuation — is defined in a standard way. If $M \in \mathcal{W}$, then:

$$\begin{aligned} M \Vdash p &\iff M \in v(p), \\ M \Vdash \alpha \wedge \beta &\iff M \Vdash \alpha \text{ and } M \Vdash \beta, \\ M \Vdash \neg\alpha &\iff M \nVdash \alpha, \\ M \Vdash \Box\alpha &\iff R(M, N) \Rightarrow N \Vdash \alpha, \text{ for all } N \in \mathcal{W}. \end{aligned}$$

A formula α is valid in a Kripke model $\mathbb{M}$ if for all $M \in \mathcal{W}$, $M \Vdash \alpha$. Similarly, a formula is valid in a Kripke frame $\mathbb{F} = (\mathcal{W}, R)$ if it is valid in every model built on that frame.

Our research concerns the modal logic of the frame $(Mod(T), \subsetneq)$. Since members of $Mod(T)$ are first-order structures, it is important to define valuations accordingly. First, any function $t\colon Var \to Fm_{\mathcal{L}}$, where $Fm_{\mathcal{L}}$ is the set of well formed formulas of the language $\mathcal{L}$, is called an $\mathcal{L}$*-translation*. Every $\mathcal{L}$-translation t gives rise to a valuation v_t:

$$v_t(p) = \{M \in Mod(T) : M \models t(p)\}.$$

It is important to stress out, that our collection of possible worlds is a proper class, and so the ranges of valuations v_t are power sets of proper classes. The functions are definable, since the relation of satisfiablity is definable in the language of set theory, so we use only valuations v_t (for all $\mathcal{L}$-translations t) in the frame in order to avoid metamathematical issues.

In order to validate the soundness of a given modal logic in the frame one needs to check certain properties of the accessibility relation; if the relation satisfies properties that are characteristic for a certain logic, then this logic is valid in a frame.

THEOREM 2.1. *If $\mathbb{F}$ is a frame such that its accessibility relation is transitive, reflexive and directed, then S4.2 is valid in $\mathbb{F}$.*

PROOF: This is a simple proof one can find as an exercise in many handbooks concerning modal logic. See for example [2]. $\square$

The main tool for proving completeness results is the method of *control statements*, developed by Hamkins and Löwe [5]. Said technique is based on the Jankov-Fine formulas. It works in an environment where a given logic conjectured to be logic complete with respect to that frame (its upper bound) is a normal modal logic with the finite frame property. In this technique various kinds of control statements are used. In our case only two of those kinds will be useful: buttons and dials.

A sentence $\varphi \in Fm_{\mathcal{L}}$ is called a *button* in a first-order model M iff for every $\mathcal{L}$-translation t and every propositional variable p such that $t(p) = \varphi$ we have $(Mod(T), \subsetneq, v_t), M \Vdash \Diamond\Box p$. A set of sentences $\{\varphi_0, \dots, \varphi_n\} \subseteq Fm_{\mathcal{L}}$ is called a *dial* in a model M iff for every $\mathcal{L}$-translation t and every $p \in Var$ such that $t(p_i) = \varphi_i$ we have that $(Mod(T), \subsetneq, v_t), M \Vdash \Diamond p_i$, for every $i \in \{0, \dots, n\}$, $(Mod(T), \subsetneq, v_t), M \Vdash \Box \bigvee_{j\in\{0,\dots,n\}} p_j$ and $(Mod(T), \subsetneq, v_t), M \Vdash p_k$, for exactly one $k \in \{0, \dots, n\}$. To put it in simpler words, the idea behind buttons, is that they are statements that are possibly necessary. They are true in some extension, and from that point onward they are true in all further extensions. A button β in a model M is said to be *pushed* if $M \Vdash \Box p$, for all propositional variables p, and all translations t such that $t(p) = \beta$. Otherwise, the button is said to be *unpushed*. So one can push any buttons that are yet unpushed in a model by going to an extension in which they are already pushed — but once it is done, they will never become unpushed again. This idea works in this way only in the directed environment, so when the logic S4.2 is valid – in weaker logics, like S4 a stronger notion is needed for the same result (Hamkins, Leibman and Löwe introduce a notion of a weak button [4]). Dials on the other hand, can be set as desired always. For a given dial $\{\psi_0, \dots, \psi_n\}$ model M is said to have a dial value j (for $j \leqslant n$) if $M \models \psi_j$. Dial is a set of statements in a model: exactly one of them is true, but one can switch to any other statement, changing the value of the dial — because all of the statements are possible. The value can go back and forth indefinitely.

A finite set of buttons is said to be *independent* of a finitely long dial in a model M iff one can push only the desired buttons and set the dial as desired as well without interfering with one another. To put it more formally, if $\{\varphi_0, \ldots, \varphi_n\}$ is a set of buttons and $\{\psi_0, \ldots, \psi_k\}$ is a dial, then for every $I_0 \subseteq I_1 \subseteq \{0, \ldots, n\}$ and for every $m \in \{0, \ldots, k\} = J$ and every $\mathcal{L}$-translation t such that $t(p_i) = \varphi_i$ and $t(q_j) = \psi_j$ we have that a formula:

$$(\bigwedge_{i\in I_0} \Box p_i \wedge \bigwedge_{i\notin I_0} \neg\Box p_i \wedge q_j \wedge \bigwedge_{l\in J\setminus\{j\}} \neg q_l) \to \Diamond(\bigwedge_{i\in I_1} \Box p_i \wedge \bigwedge_{i\notin I_1} \neg\Box p_i \wedge q_m \wedge \bigwedge_{l\in J\setminus\{m\}} \neg q_l)$$

is valid in $(Mod(T), \subsetneq, v_t), M$, for some $j \in J$.

This terminology is used by Hamkins and Löwe in the proof of the following theorem:

THEOREM 2.2 ([5, 4]). *If a world M in a frame validates arbitrarily large finite sets of buttons independent of an arbitrarily large finite dial, then its modal logic is contained within S4.2, as long as the buttons are unpushed — if they all are pushed, the validities are contained within S5.*

3. The modal logic of lattice theory

In this section we shall consider the case, where T = lattice theory, that is the first-order theory in a first-order language augmented by two binary functional symbols $\wedge, \vee$ and characterised by the universal closure of the following axioms:

$$x \vee (y \vee z) = (x \vee y) \vee z, \qquad x \wedge (y \wedge z) = (x \wedge y) \wedge z;$$
$$x \vee y = y \vee x, \qquad x \wedge y = y \wedge x;$$
$$x \vee (x \wedge y) = x, \qquad x \wedge (x \vee y) = x.$$

The theorem we aim to prove is the following:

THEOREM 3.1. *The modal logic of the lattice theory (the modal logic of the frame $(Mod(T), \subsetneq)$, for T = lattice theory) is exactly S4.2.*

Proof: To establish S4.2 as a lower bound we simply need to check that the relation $\subsetneq$ is reflexive, transitive and directed. The first two are trivial, and lattices exhibit even a stronger quality with respect to embeddings than directedness — any two lattices L_1, L_2 have a common extension (take for example a direct product of lattices L_1 and L_2).

In order to establish an upper bound, we are going to use the beforementioned technique of control statements. For an arbitrary lattice L, we are going to find arbitrarily long sets of buttons, as well as an arbitrarily long dial.

Dials: For dials of any finite length (larger than 1) we take for each $n \in \mathbb{N}$ the sequences $\{\varphi_0, \ldots, \varphi_{n-1}, \varphi_n\}$, where φ_0 states that there are no atoms or no least element, φ_i, for $1 \leqslant i < n$ states that "there are exactly i atoms", and φ_n states that "there are at least n atoms" (atoms are understood in the standard way, as elements immediately above the least element of the lattice, a statement "there are exactly i atoms" implies existence of the least element). In any lattice L one of the sentences of any of such dials is always true, as the lattice always has some amount of atoms: let i be the cardinality of the set of atoms of the lattice L (i may be infinite). If $i < n$, then $L \models \varphi_i$, otherwise $L \models \varphi_n$. So any lattice satisfies exactly one of the sentences φ_i from a given dial.

Any of the sentences φ_i is possible, necessarily so: one can always take any lattice L, no matter the dial volume it satisfies, and add a descending infinite chain below all the elements of L to bring the value down to φ_0 in such a lattice L'. From there one can add any lattice L_k, with k atoms below all of L''s elements. Let L_k be a lattice M_k (a k-wide diamond). This new extension satisfies any desired dial value.

Buttons: The idea used for buttons is that of a cycle from graph theory. To mimic a notion of a cycle of a length n (for $n \geqslant 3$) in a lattice we construct a lattice W_n in a following way: take sets $\{1\}, \ldots, \{n\}$. Then take each pair of numbers of the form $\{m, m+2\}$ (for $1 \leqslant m \leqslant n-2$), as well as the pairs $\{1,2\}$ and $\{n-1, n\}$ additionally. Those sets together with the sets $\{1, \ldots, n\}, \varnothing$ form a lattice W_n together with two operations: $W_n = (L_n, \cup^*, \cap)$, where $L_n = \{\varnothing, \{1, \ldots, n\}, \bigcup\{\{m\} : m \in \{1, \ldots, n\}\}, \bigcup\{\{k, k+2\} : k \in \{1, \ldots, n-2\}\}, \{1,2\}, \{n-1, n\}\}$ and $a \cup^* b =$

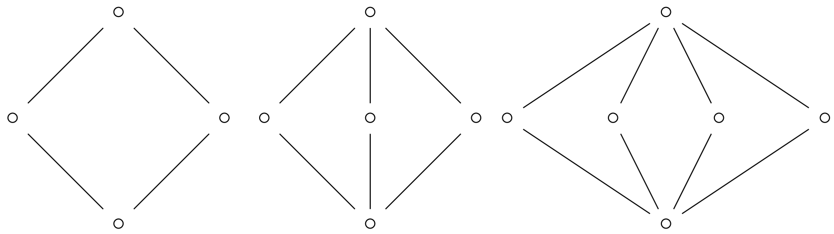

Figure 1: Lattices M_2, M_3 and M_4 respectively.

$min\{x \in L : a \cup b \subseteq x\}$. Each of these lattices is finite, so any lattice W_n can be characterised as a sublattice by a first-order sentence ψ_n. For buttons we take those sentences ψ_n, stating that a lattice W_n is a sublattice of a lattice under consideration.

We need to prove that they are independent of dials: this is easy, since the dial is concerned about what happens at the lowest parts of the lattice, and one can easily extend the lattice downwards, so that other parts of the lattice are not interfered with. The fact that none of the lattices M_n contain any of W_m as sublattice is trivial. What remains to be proven is that none of the lattices W_m contain any lattice W_l as a sublattice, for $m \neq l$.

Let m, l be any two natural numbers such that $l < m$. We will show W_m does not contain W_l as a subblattice.

Suppose that $f \colon W_l \to W_m$ is a injection. We will show that f is not a homomorphism. First, observe that if there is $i \leqslant l$ such that $f(\{i\}) \neq \{n\}$, for some $n \leqslant m$, then f is not a homomorphism. Of course, if $f(\{i\}) = \varnothing$ or $\{1, \ldots, m\}$ then f is not a homomorphism. Let us assume, that $f(\{i\}) = \{n, k\}$, for some $n, k \leqslant m$. By the definition of W_l, there exists $j \leqslant l$ such that $\{i\} \vee_{W_l} \{j\} = \{i, j\}$. In W_l we have $\{i\} \leqslant_{W_l} \{i, j\} \leqslant_{W_l} \{1, \ldots, l\}$, but since f is an injection and $f(\{i\}) = \{n, k\}$ this chain cannot be replicated by $f(\{i\}), f(\{i, j\}), f(\{1, \ldots, l\})$.

Suppose then, that for all $i \leqslant l$ there exists $n \leqslant m$ such that $f(\{i\}) = \{n\}$. Let us call $\{a\}, \{b\} \in W_s$ neighbours in W_s iff $\{a, b\} \in W_s$. Because $l < m$, there exists $n \leqslant m$ such that there is no $i \leqslant j$ such that $f(\{i\}) = \{n\}$

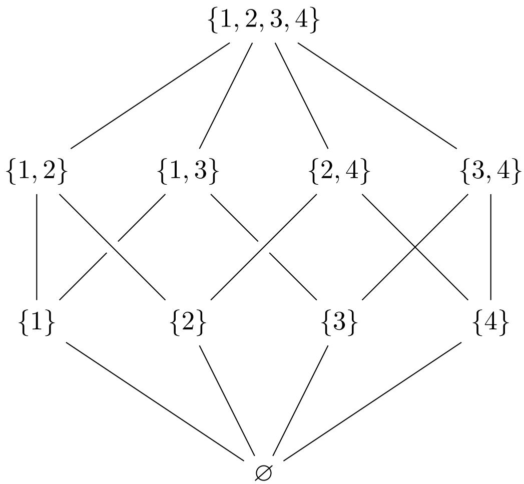

Figure 2: Lattice W_4.

and $\{n\}$ has a neighbour $\{k\}$ in W_m such that there is some $a \leqslant l$ such that $f(\{a\}) = \{k\}$. It follows that $\{a\}$ has a neighbour $\{b\}$ in W_l, and $f(\{b\})$ is not a neighbour of $f(\{a\})$ in W_m, and hence $f(\{a\}) \vee_{W_m} f(\{b\}) \neq f(\{a, b\})$ or $f(\{a, b\}) > f(\{1, \ldots, l\})$. Hence, f is not a homomorphism. Obviously, it is also not the other way around, so the sets of buttons are independent. It is also fairly obvious that not all lattices have these buttons pushed.

Using Theorem 2.2 we establish a completeness result and conclude the proof of the theorem. □

4. Modal logic of modular and distributive lattices

Since the buttons (of index $\geqslant 4$) used to prove the above result are implying that there is a N_5 sublattice in any lattice satisfying them (so those lattices are not modular), one may wonder what the modal logic of stronger theories is, mainly the cases where T = theory of modular lattices and T = theory of distributive lattices, so the lattice theory augmented with the universal closures of the following axioms, respectively:

$$x \leqslant y \Rightarrow x \vee (z \wedge y) = (x \vee z) \wedge y, \tag{M}$$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z). \tag{D}$$

Let us start with investigating modular lattices. It turns out that the modal logic of their theory is the same:

THEOREM 4.1. *The modal logic of modular lattice theory is exactly S4.2.*

PROOF: Lower bound is as easy as before: $\subsetneq$ on modular lattices is directed, as every product of modular lattices is a modular lattice. Obviously it is also transitive and reflexive.

For dials we do not need to change anything. We can use the same idea as before.

Buttons: We shall formulate different buttons, as the ones used before do not work in a modular environment. Instead of working as before and defining an infinite sequence of statements, we will define different sets of buttons for every natural number n:

In the paper [10] Wroński introduces the operation $\oplus$ on lattices. The finite lattice L is said to be a *sum* of A and B, $L = A \oplus B$ in symbols, if A and B are proper sublattices of L, such that $A \cup B = L$, and moreover, $A \cap B$ is a filter in A, and $A \cap B$ is an ideal in B (see [8], and [3, Chapter 4] for details). We will write $L_1 \oplus_i L_2$ for Wroński's sum where $L_1 \cap L_2 = S_i$, for some $i \in I$.

Consider a lattice C_n that is a chain of length n (for $n \geqslant 3$). Let $D_n = C_{n+2} \times C_2$ be a product lattice. Consider two copies of D_n (D_n and D_n^*) and all Wroński's sums $L_{n_{i-2}} = D_n \oplus_i D_n^*$ where $D_n \cap D_n^*$ is a chain of cardinality $i \geqslant 3$. For each $n \in \mathbb{N}$ there are n such sums.

For any $n \in \mathbb{N}$ all L_{n_k} (for $1 \leqslant k \leqslant n$) are finite, hence characterizable by a first order formula φ_k. For a set of buttons of length n we take sentences $\varphi_1, \ldots, \varphi_n$ charaterising respectively $L_{n_1}, \ldots, L_{n_n}$ as sublattices.

The cardinality of the lattice L_{n_k} is bigger than the cardinality of the lattice L_{n_j}, for $j > k$ and all $n \in \mathbb{N}$, so in order to prove independence of the buttons, we need to check that $L_{n_k} \not\subseteq L_{n_j}$. This is also the case, because in each L_{n_i} there are exactly i different triples of elements such that they are incomparable and together with their suprema and infima they form a covering sublattice L_{1_1} — one cannot embed k such triples into a lattice where only j of them exist without identifying some of them.

The dial is independent of the buttons as well, since it concerns what happens at the lowest parts of the lattice, and one can always extend a lattice adding more elements at the bottom to accommodate a given dial value, without interfering in the upper parts. Furthermore, lattices M_n do not contain any of the lattices L_{n_k}, and Theorem 2.2 concludes the proof. □

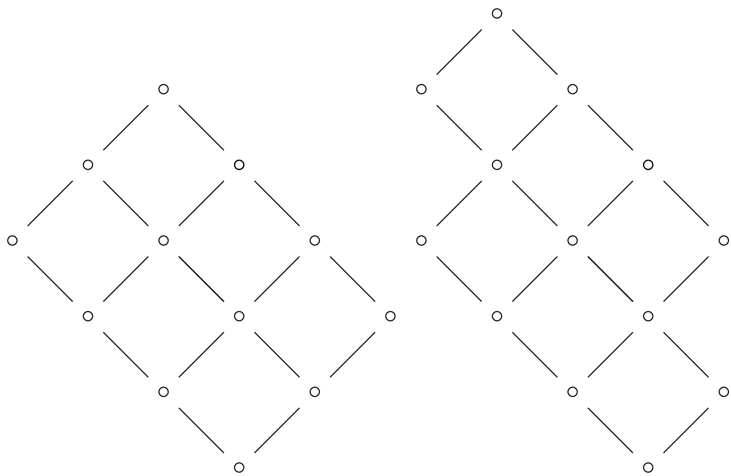

Figure 3: Lattices L_{2_2} and L_{2_1}.

Corollary 4.2. The modal logic of the theory of distributive lattices is exactly S4.2.

Proof: Just as before, the product of distributive lattices is a distributive lattice. Thus, $S4.2 \subseteq$ modal logic of distributive lattices.

The lattices implied by the buttons used to prove theorem 4.1 are distributive, so the same idea works here as well.
The dials need to be changed, as lattices M_n are not distributive. We cannot use the idea of atoms, because we might by mistake embed a lattice that satisfies one of our buttons. The idea here is that $\{\psi_0, \ldots, \psi_{n-1}, \psi_n\}$ is a dial, where $\psi_0 =$ "there is no least element", $\psi_m =$ "there are exactly $m-1$ $\wedge$-irreducible elements between the least element and the least $\wedge$-reducible element" and $\psi_n =$ "there are at least $n-1$ $\wedge$-irreducible elements between the least element and the least $\wedge$-reducible element". This works similarly as before. The theorem 2.2 concludes the proof. □

5. Summary

Our results in fact are a little bit more general and state that the robust modal logic of the above frames is S4.2 (see [9] section 5). Furthermore, our results can be easily extended to some stronger theories i.e. Stone algebras, or more narrow classes of lattices (they not need to be first-order theories, so in this case the theorem concerns the modal logic of the frame $(\mathcal{S}, \subsetneq)$, where $\mathcal{S}$ = class of all such lattices) using a similar idea. It cannot be done indefinitely, since complete theories have Triv as their modal logic [6]. This leads us to ask a following question:

Question 5.1. Is there a theory extending lattice theory that its modal logic is S4.2, and all strictly stronger theories have a different modal logic?

We can ask as well not only about theories, but classes of lattices:

Question 5.2. Is there a class $\mathcal{C}$ of lattices such whose modal logic (the modal logic of the frame $(\mathcal{C}, \subsetneq)$) is still S4.2, yet all proper subclasses of $\mathcal{C}$ have a different modal logic?

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PROOF TRANSLATIONS BETWEEN LABEL-FREE AND LABELED SEQUENT CALCULI IN ISCI

Abstract

In this paper we consider the Intuitionistic Sentential Calculus with Identity (ISCI). We study two main families of sequent calculi. The first one, called $\mathsf{G3}_{\mathsf{ISCI}}$, is based on a label-free multi-succedent sequent calculus that is sound and complete w.r.t. Kripke models and the second, called $\mathsf{L3}_{\mathsf{ISCI}}$, is based on a multi-succedent labeled sequent calculus that is sound and complete w.r.t. Beth models. Our goal is to investigate how the calculi, that capture distinct semantics of the logic, relate to each other through proof translations. Proof translations from $\mathsf{G3}_{\mathsf{ISCI}}$ to $\mathsf{L3}_{\mathsf{ISCI}}$ provide new results about the soundness and (cut-free) completeness of $\mathsf{G3}_{\mathsf{ISCI}}$ w.r.t. Beth models. Proof translations from

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$\mathsf{L3}_{\mathsf{ISCI}}$ to $\mathsf{G3}_{\mathsf{ISCI}}$ are more difficult and require the definition of new calculi for ISCI that provide intermediate steps in the translation process.

Keywords: non-Fregean sentential logic, sequent calculi, labeled calculi, nested sequents, proof translations.

1. Introduction

In this work we consider an extension of intuitionistic logic (IL) that arises from adding a non Fregean operator ($\approx$) called Suszko's identity. The resulting logic is known as the Intuitionistic Sentential Calculus (or Logic) with Identity (ISCI). Suszko's identity has first been investigated as an extension of classical logic called SCI [1, 10].

The motivation behind SCI is related to the ontology of situations. In classical logic, only two situations can exist, truth and falsity, that are witnessed by any true or false proposition. According to [1], this is unfortunate and could be improved with a new non Fregean operator, written $\approx$, that witnesses two identical situations. In SCI, one acknowledges the fact that there could possibly be more than two situations. Under the usual Fregean interpretation, two formulas are equivalent if they share the same logical value. Under Suszko's identity, two formulas with the same logical value might be considered non-identical if they do not describe the same situations, for instance, two formulas might be valid (and thus logically equivalent) while not having the same sets of proofs. Deduction in SCI has been thoroughly studied, resulting in a Hilbert style proof system [1], various Gentzen sequent calculi [12, 19, 20] and dual tableaux [8, 14]. One drawback of those systems is their lack of analyticity. In particular, they do not enjoy the subformula property and could therefore not provide any kind of decision procedures although SCI is known to be decidable [10]. An alternative decision procedure for SCI based on labeled tableaux has been recently proposed [9].

In the case of ISCI, we consider two main works. The first one is [4], where a Kripke semantics for ISCI is introduced along with a related (Kripke) sound and complete label-free single-succedent sequent calculus

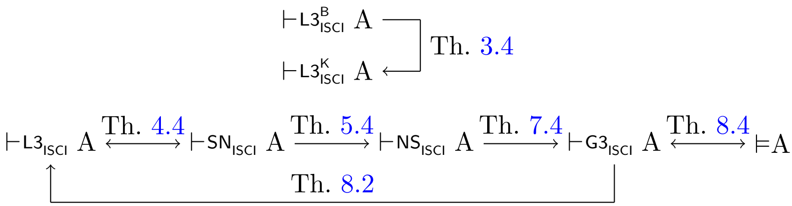

Figure 1: Proof Translation Cycle.

called $\mathsf{sG3}_{\mathsf{ISCI}}$. The second one is [6], where a new TB semantics is introduced along with two related (TB) sound and complete labeled sequent calculi called $\mathsf{L}^1_{\mathsf{ISCI}}$ and $\mathsf{L}^2_{\mathsf{ISCI}}$. Unlike the Kripke semantics, the TB semantics does not rely on partially ordered sets, but on bounded distributive lattices. Both semantics are proven sound and complete w.r.t. the Hilbert system of axioms $\mathsf{H}_{\mathsf{ISCI}}$ [2, 6]. The completeness of the Kripke semantics is achieved by the construction of a canonical model built from (Lindenbaum's idea of) equivalence classes of formulas. On the contrary, the completeness of the TB semantics relies on a canonical model built from (Beth's idea of) theories of formulas (TB standing as either "Topological Beth", or "Theory Based"). The decidability results of ISCI have been settled, from the labeled calculus $\mathsf{L}^2_{\mathsf{ISCI}}$, for a fragment involving decreasing sentential substitutions [6] and from $\mathsf{sG3}_{\mathsf{ISCI}}$ by counter-model construction, for a fragment including only implication and sentential identity [18]. Let us note that the BHK-interpretation (in terms of sets of proofs) for ISCI has been recently described in [3] by introducing a new semantics that captures the notion of identity within a constructive framework.

The main goal of the paper is to study proof translations between families of labeled and label-free sequent calculi for ISCI. We focus on two main families of sequent calculi: $\mathsf{G3}_{\mathsf{ISCI}}$, arising from the $\mathsf{sG3}_{\mathsf{ISCI}}$ calculus that is sound and complete w.r.t. Kripke models, and $\mathsf{L3}_{\mathsf{ISCI}}$, arising from the $\mathsf{L}^1_{\mathsf{ISCI}}$ and $\mathsf{L}^2_{\mathsf{ISCI}}$ calculi proven sound and complete w.r.t. Beth models. Translating proofs from a labeled to a label-free calculus is usually a more difficult problem than in the other direction. Therefore, our approach to translate $\mathsf{L3}_{\mathsf{ISCI}}$ into $\mathsf{G3}_{\mathsf{ISCI}}$ consists in defining two new families of calculi, namely $\mathsf{SN}_{\mathsf{ISCI}}$ and $\mathsf{NS}_{\mathsf{ISCI}}$, as intermediate steps in the translation process.

All of our families of calculi come with two disjoint sets of rules for sentential identities (denoted 1 and 2) and two disjoint sets of rules for disjunction and falsity (denoted K for Kripke and B for Beth). For simplicity and technical consistency, all of our calculi deal with sequents that are sets (and not multisets) of formulas, thus making the contraction rule implicit. We also consider weakening as implicit in our calculi (as it can easily be proven admissible).

The main results are depicted in Figure 1, with two translation cycles: a first one for the Beth variants and a second one for the Kripke variants, with a connection between both by showing that Beth proofs can always be turned into Kripke proofs. We deduce that the K and B proof systems are sound and cut-free complete w.r.t. the Kripke or TB semantics.

In Section 2 we recall the syntax of ISCI and the basics of its TB and Kripke semantics.

In Section 3 we introduce the family $\mathsf{L3}_{\mathsf{ISCI}}$ of labeled sequent calculi, that subsumes the $\mathsf{L}^1_{\mathsf{ISCI}}$ and $\mathsf{L}^2_{\mathsf{ISCI}}$ calculi given in [6]. $\mathsf{L3}_{\mathsf{ISCI}}$ uses sets of integers as labels and implicitly captures the labeling algebra via set union. Then we show that any Beth proof in $\mathsf{L3}^{\mathsf{B}}_{\mathsf{ISCI}}$ can be translated into a Kripke proof in $\mathsf{L3}^{\mathsf{K}}_{\mathsf{ISCI}}$.

In Section 4 we define $\mathsf{SN}_{\mathsf{ISCI}}$ that is a family of labeled sequent calculi, where labels are single letters and where the labeling algebra is captured via explicit relational atoms. In addition to the family with Beth rules for disjunction and falsity, we provide a stepwise and height-preserving translation of $\mathsf{L3}_{\mathsf{ISCI}}$-proofs into $\mathsf{SN}_{\mathsf{ISCI}}$-proofs. A complementary result is a translation in the reverse direction. In Section 5 we define $\mathsf{NS}_{\mathsf{ISCI}}$ that is a family of nested sequent calculi with both Beth and Kripke variants and we show that $\mathsf{SN}_{\mathsf{ISCI}}$-proofs can be translated into $\mathsf{NS}_{\mathsf{ISCI}}$-proofs.

In Section 6, we extend the single-succedent calculus $\mathsf{sG3}_{\mathsf{ISCI}}$ [4] to a multi-succedent calculus and also define a new Beth variant of the calculus that is sound and complete w.r.t. the TB semantics of ISCI. This gives rise to the family $\mathsf{G3}_{\mathsf{ISCI}}$ of multi-succedent label-free sequent calculi.

In Section 7 we show that $\mathsf{NS}_{\mathsf{ISCI}}$-proofs can be translated into $\mathsf{G3}_{\mathsf{ISCI}}$-proofs and then we deduce, from previous translations, that $\mathsf{L3}_{\mathsf{ISCI}}$-proofs can be translated into $\mathsf{G3}_{\mathsf{ISCI}}$-proofs.

In Section 8 we show how $\mathsf{G3}_{\mathsf{ISCI}}$-proofs can be translated into $\mathsf{L3}_{\mathsf{ISCI}}$-proofs. From this translation we deduce new results: $\mathsf{G3}^{\mathsf{K}}_{\mathsf{ISCI}}$ ($\mathsf{G3}^{\mathsf{B}}_{\mathsf{ISCI}}$) is sound and complete w.r.t. the Kripke (TB) semantics and all of the $\mathsf{L3}_{\mathsf{ISCI}}$ calculi are cut-free complete.

In Section 9, we emphasize how proof translations can help us gain a better understanding of how semantics reveals itself in a calculus and how they allow us to transpose properties from one calculus to the other.

2. The Logic ISCI: Syntax and Semantics

Definition 2.1. Let $\mathbf{P} = \{\mathrm{p}, \mathrm{q}, \dots\}$ be a countable set of propositional letters. The formulas of ISCI, the set of which is denoted by $\mathbf{F}$, are given by the grammar:

$$\mathrm{A} ::= \mathrm{p} \in \mathbf{P} \mid \bot \mid \mathrm{A} \wedge \mathrm{A} \mid \mathrm{A} \vee \mathrm{A} \mid \mathrm{A} \supset \mathrm{A} \mid \mathrm{A} \approx \mathrm{A}.$$

As usual, negation $\neg\mathrm{A}$ is defined as a shorthand $\mathrm{A} \supset \bot$ and $\top$ is then defined as $\bot \supset \bot$. We write $\mathbf{F}_{\approx}$ for the restriction of $\mathbf{F}$ to sentential identities.

ISCI admits various semantics. We first recall the TB semantics introduced in [6] as it is less widely known and more elaborated than the known Kripke semantics [4].

Definition 2.2. Let $\mathbf{M}$ be a set of elements, called *worlds*, such that $\omega, \pi \in \mathbf{M}$ and $\omega \neq \pi$. A TB *frame* is a bounded distributive lattice $\mathcal{F} = (\mathbf{M}, \leqslant, \sqcup, \omega, \sqcap, \pi)$ with ω and π as least and greatest elements respectively.

Definition 2.3. A TB *pre-model* is a triple $\mathcal{M} = (\mathcal{F}, [\cdot], \Vdash)$, where $\mathcal{F}$ is a TB frame, and $[\cdot]$ is a *valuation function* from $\mathbf{M}$ to $\wp(\mathbf{P} \cup \mathbf{F}_{\approx})$, such that for all worlds m and n:

($\mathcal{M}_\pi$) $[\pi] = \mathbf{P} \cup \mathbf{F}_{\approx}$,

($\mathcal{M}_\mathrm{K}$) if $\mathrm{m} \leqslant \mathrm{n}$, then $[\mathrm{m}] \subseteq [\mathrm{n}]$,

$(\mathcal{M}_{\approx_1})$ $A \approx A \in [m]$,

$(\mathcal{M}_{\approx_4})$ for all $\otimes \in \{\wedge, \vee, \supset, \approx\}$, if $A \approx B \in [m]$ and $C \approx D \in [m]$, then $A \otimes C \approx B \otimes D \in [m]$[1]

The *forcing relation* $\Vdash$ is inductively defined as the smallest relation on $\mathbf{M} \times \mathbf{F}$ such that:

- $m \Vdash p$ iff $p \in [m]$,
- $m \Vdash A \approx B$ iff $A \approx B \in [m]$,
- $m \Vdash \bot$ iff $\pi \leqslant m$,
- $m \Vdash A \wedge B$ iff $m \Vdash A$ and $m \Vdash B$,
- $m \Vdash A \supset B$ iff for all $n \in \mathbf{M}$, if $n \Vdash A$, then $m \sqcup n \Vdash B$,
- $m \Vdash A \vee B$ iff for some $n_1, n_2 \in \mathbf{M}$ such that $n_1 \sqcap n_2 \leqslant m$, $n_1 \Vdash A$ and $n_2 \Vdash B$.

Definition 2.4. A TB *model* is a TB pre-model satisfying the admissibility and regularity conditions:

$(\mathcal{M}_{\approx_3})$ if $m \Vdash A \approx B$, then $m \Vdash B \supset A$,

$(\mathcal{M}_{\mathrm{R}})$ for all $A \in \mathbf{F}$, there exists an A-*minimal* world, i.e., there exists $m_A \in \mathbf{M}$ such that $m_A \Vdash A$ and for all $n \in \mathbf{M}$, if $n \Vdash A$ then $m_A \leqslant n$.

As usual, a formula A is *true* (or *satisfied*) in a TB model $\mathcal{M}$, written $\mathcal{M} \vDash A$, iff $m \Vdash A$ for all worlds m in $\mathcal{M}$ and *valid*, written $\vDash A$, iff it is true in all models.

The Kripke semantics of ISCI is built on the simple notion of *Kripke frame (more shortly,* K *frame)* which is a partially ordered set of worlds $\mathcal{F} = (\mathbf{M}, \leqslant)$. Kripke models are obtained from Definition 2.3 by discarding conditions $(\mathcal{M}_\pi)$ and $(\mathcal{M}_{\mathrm{R}})$ and replacing the forcing clause for the intuitionistic connectives with their standard interpretation. We write $\vDash_{\mathsf{K}}$ or $\vDash_{\mathsf{B}}$ instead of $\vDash$ whenever confusion may arise.

[1] Let us note that $(\mathcal{M}_{\approx_2})$ if $A \approx B \in [m]$ then $\neg A \approx \neg B \in [m]$ can be derived from $(\mathcal{M}_{\approx_4})$.

3. The Labeled Sequent Calculi $\mathsf{L3}_{\mathsf{ISCI}}$

Let us introduce the $\mathsf{L3}_{\mathsf{ISCI}}$ family of labeled sequent calculi that subsumes the ones in [6]. A label is either a (possibly empty) finite subset of $\mathbb{N}$, or $\mathbb{N}$ itself. We write $\mathbf{L}$ for the set of labels and $\mathbf{L}^n$ for the restriction of $\mathbf{L}$ to labels of size n (sets of cardinal n). $\emptyset$ and $\mathbb{N}$ are called *label units*. We use the (possibly subscripted or primed) letters $\mathrm{a}, \mathrm{b}, \mathrm{c}$ to denote singletons and save the letters $\mathrm{x}, \mathrm{y}, \mathrm{z}$ to denote arbitrary labels. Since all of the examples in this paper use labels built from singletons $\{\, i \mid 1 \leqslant i \leqslant 9 \,\}$, we use the more concise notation 13 to unambiguously refer to the label $\{\, 1, 3 \,\}$ and not to the singleton $\{\, 13 \,\}$.

A label x is a *sublabel* of a label y if $\mathrm{x} \subseteq \mathrm{y}$. Labels are interpreted w.r.t. a labeling algebra $\mathcal{L}$ defined as the bounded lattice $(\mathbf{L}, \subseteq, \cup, \emptyset, \cap, \mathbb{N})$, where join $\cup$ and meet $\cap$ are standard set union and intersection. We consider that $\cup$ binds stronger than $\cap$ and we shall frequently write xy instead of $\mathrm{x} \cup \mathrm{y}$ ($\mathrm{xx}' \cap \mathrm{yy}'$ should therefore be read as $(\mathrm{x} \cup \mathrm{x}') \cap (\mathrm{y} \cup \mathrm{y}')$).

A *labeled formula* is a pair (C, z), written $\mathrm{C} : \mathrm{z}$, where C is a formula and z is a label. A *labeled sequent* is a pair (Γ, Δ), written $\Gamma \vdash \Delta$, of sets of formulas. Γ and Δ are respectively called the *antecedent* and the *succedent* of the sequent.

The proof rules of $\mathsf{L3}_{\mathsf{ISCI}}$ are given in Figure 2. $\mathsf{L3}^1_{\mathsf{ISCI}}$ and $\mathsf{L3}^2_{\mathsf{ISCI}}$ are respectively defined with the sets $\{\, \mathrm{L}^1_{\approx}, \mathrm{L}^2_{\approx}, \mathrm{L}^3_{\approx}, \mathrm{L}^{3*}_{\approx} \,\}$ and $\{\, \mathrm{L}^1_{\approx}, \mathrm{aL}^2_{\approx}, \mathrm{L}^{\mathrm{r}}_{\approx} \,\}$ for sentential identity rules. For both sets of identity rules we have a K (Kripke) and a B (Beth) version of the calculus depending on the rules for disjunction and falsity. In the rule $\mathrm{L}^{\mathrm{r}}_{\approx}$ (left replacement), $\mathrm{D}^{\mathrm{A}}_{\mathrm{B}}$ denotes the result of replacing some (possibly all) occurrences of A with B in D. Given a set or multiset S of labeled formulas and a label x, the notation $\mathrm{x} \in \mathrm{S}$ is a shorthand for $(\exists (\mathrm{A} : \mathrm{xy}) \in \mathrm{S})$. Therefore, the side conditions of the rules $\mathrm{L}_{\supset}$, $\mathrm{L}^2_{\approx}$, $\mathrm{L}^3_{\approx}$ and $\mathrm{L}^{\mathrm{r}}_{\approx}$ mean that the labels introduced in their premises must already occur in the succedent of their conclusion. A sequent $\Gamma \vdash \Delta$ is *right connected* if $(\forall \mathrm{A} : \mathrm{x} \in \Gamma)(\mathrm{x} \in \Delta)$. A close inspection of the rules shows that they preserve right connectedness upwards.

DEFINITION 3.1. Let C be a formula. An $\mathsf{L3}_{\mathsf{ISCI}}$-*proof of* C is a proof of the sequent $\vdash \mathrm{C} : \emptyset$ with the $\mathsf{L3}_{\mathsf{ISCI}}$ rules.

$$\frac{}{\Gamma, p : x \vdash \Delta, p : xy}\,\mathrm{id_p} \qquad \frac{}{\Gamma, A \approx B : x \vdash \Delta, A \approx B : xy}\,\mathrm{id}_{\approx}$$

$$\frac{}{\Gamma, \bot : x \vdash \Delta, C : xy}\,L^{B}_{\bot} \qquad \frac{}{\Gamma, \bot : x \vdash \Delta}\,L^{K}_{\bot}$$

$$\frac{\Gamma, A_1 : xa_1 \vdash \Delta, C : xya_1 \quad \Gamma, A_2 : xa_2 \vdash \Delta, C : xya_2}{\Gamma, A_1 \vee A_2 : x \vdash \Delta, C : xy}\,L^{B}_{\vee}$$

$$\frac{\Gamma, A_1 : x \vdash \Delta \quad \Gamma, A_2 : x \vdash \Delta}{\Gamma, A_1 \vee A_2 : x \vdash \Delta}\,L^{K}_{\vee} \qquad \frac{\Gamma \vdash \Delta, A_1 : y, A_2 : y}{\Gamma \vdash \Delta, A_1 \vee A_2 : y}\,R_{\vee}$$

$$\frac{\Gamma, A \supset B : x \vdash \Delta, A : xy \quad \Gamma, B : xy \vdash \Delta}{\Gamma, A \supset B : x \vdash \Delta}\,L_{\supset}(xy \in \Delta) \qquad \frac{\Gamma, A : a \vdash \Delta, B : xa}{\Gamma \vdash \Delta, A \supset B : x}\,R_{\supset}$$

$$\frac{\Gamma, A : x, B : x \vdash \Delta}{\Gamma, A \wedge B : x \vdash \Delta}\,L_{\wedge} \qquad \frac{\Gamma \vdash \Delta, A : y \quad \Gamma \vdash \Delta, B : y}{\Gamma \vdash \Delta, A \wedge B : y}\,R_{\wedge}$$

$$\frac{\Gamma, A \approx A : \emptyset \vdash \Delta}{\Gamma \vdash \Delta}\,L^{1}_{\approx} \qquad \frac{\Gamma, A \approx B : x, B \supset A : x \vdash \Delta}{\Gamma, A \approx B : x \vdash \Delta}\,aL^{2}_{\approx}$$

$$\frac{\Gamma, A \approx B : x \vdash \Delta, B : xy \quad \Gamma, A \approx B : x, A : xy \vdash \Delta}{\Gamma, A \approx B : x \vdash \Delta}\,L^{2}_{\approx}(xy \in \Delta)$$

$$\frac{\Gamma, A \approx B : x, C \approx D : y, A \otimes C \approx B \otimes D : xy \vdash \Delta}{\Gamma, A \approx B : x, C \approx D : y \vdash \Delta}\,L^{3}_{\approx}(xy \in \Delta)$$

$$\frac{\Gamma, A \otimes A \approx B \otimes B : x \vdash \Delta}{\Gamma, A \approx B : x \vdash \Delta}\,L^{3*}_{\approx} \qquad \frac{\Gamma, A \approx B : x, D : y, D^{A}_{B} : xy \vdash \Delta}{\Gamma, A \approx B : x, D : y \vdash \Delta}\,L^{r}_{\approx}(xy \in \Delta)$$

Eigenvariable conditions: In $R_{\supset}$ and $L^{B}_{\vee}$, a, a_1, a_2 are fresh singletons and $a_1 \neq a_2$.

Figure 2: Rules for the $\mathsf{L3}_{\mathsf{ISCI}}$ Family of Labeled Calculi.

A *label substitution* is a total function $\sigma : \mathbf{L} \to \mathbf{L}$ whose restriction $\sigma^* : (\mathbf{L}^0 \cup \mathbf{L}^1) \to \mathbf{L}$ differs from the identity only for a finite number of elements in the domain and such that for all labels $\mathrm{z} \notin (\mathbf{L}^0 \cup \mathbf{L}^1)$, $\mathrm{z}\sigma = \bigcup\{\,\mathrm{x}\sigma^* \mid \mathrm{x} \in (\mathbf{L}^0 \cup \mathbf{L}^1) \text{ and } \mathrm{x} \subseteq \mathrm{z}\,\}$. Label substitutions extend to labeled formulas, multisets of labeled formulas and sequents as follows: $(\mathrm{A}:\mathrm{x})\sigma = \mathrm{A}:\mathrm{x}\sigma$, $\mathrm{S}\sigma = \{\,(\mathrm{A}:\mathrm{x})\sigma \mid \mathrm{A}:\mathrm{x} \in \mathrm{S}\,\}$ and $(\Gamma \vdash \Delta)\sigma = \Gamma\sigma \vdash \Delta\sigma$. We write $[\mathrm{x}_1/\mathrm{y}_1; \ldots; \mathrm{x}_n/\mathrm{y}_n]$, where $\mathrm{x}_i \in \mathbf{L}$ and $\mathrm{y}_i \in (\mathbf{L}^0 \cup \mathbf{L}^1)$ for all $1 \leqslant i \leqslant n$, to denote the label substitution σ such that $\mathrm{z}\sigma = \mathrm{x}_i$ if $\mathrm{z} = \mathrm{y}_i$ for some i and $\mathrm{z}\sigma = \mathrm{z}$ otherwise. Hence, x/y means that x replaces y. For instance, let $\sigma = [17/\emptyset; 2/7]$, then since $347 = \bigcup\{\,\emptyset, \{\,3\,\}, \{\,4\,\}, \{\,7\,\}\,\}$, we have $(347)\sigma = \bigcup\{\,\{\,1,7\,\}, \{\,3\,\}, \{\,4\,\}, \{\,2\,\}\,\} = 12347$. It is easy to check that $\mathrm{z}[\mathrm{x}/\mathrm{y}] = (\mathrm{z} - \mathrm{y}) \cup \mathrm{x}$ if $\mathrm{y} \subseteq \mathrm{z}$, and $\mathrm{z}[\mathrm{x}/\mathrm{y}] = \mathrm{z}$ otherwise. Thus, $(347)[\emptyset/7] = \{\,3,4,7\,\} - \{\,7\,\} \cup \emptyset = \{\,3,4\,\} = 34$.

Lemma 3.2. *Let* $\mathrm{s} = \Gamma \vdash \Delta$. *If* $\vdash_{\mathsf{L3_{ISCI}}} \mathrm{s}$ *then* $\vdash_{\mathsf{L3_{ISCI}}} \mathrm{s}\sigma$.

Proof: By induction on the height of the proof (see Appendix C.2). □

The fact that only labels in $\mathbf{L}^0 \cup \mathbf{L}^1$ can be replaced is essential for the soundness of Lemma 3.2. Indeed, consider the instance of id_p whose conclusion is $\mathrm{s} = \mathrm{p}:12 \vdash \mathrm{p}:123$, we have $\vdash_{\mathsf{L3_{ISCI}}} \mathrm{s}$ but not $\vdash_{\mathsf{L3_{ISCI}}} \mathrm{s}[4/23] = \mathrm{p}:12 \vdash \mathrm{p}:14$.

Let us now show that any $\mathsf{L3^B_{ISCI}}$-proof translates to a $\mathsf{L3^K_{ISCI}}$-proof by erasing all of the singletons introduced by instances of $\mathrm{L}^{\mathrm{B}}_{\vee}$ and globally renaming $\mathrm{L}^{\mathrm{B}}_{\perp}$ and $\mathrm{L}^{\mathrm{B}}_{\vee}$ as $\mathrm{L}^{\mathrm{K}}_{\perp}$ and $\mathrm{L}^{\mathrm{K}}_{\vee}$.

Definition 3.3. Let Π be a proof of a sequent s in $\mathsf{L3^B_{ISCI}}$. B_Π is defined as the set $\{\,\mathrm{c}_1, \ldots, \mathrm{c}_n\,\}$ of all the fresh singletons introduced by an instance of $\mathrm{L}^{\mathrm{B}}_{\vee}$ in Π. Moreover, σ_Π is defined as the erasing substitution $[\emptyset/\mathrm{c}_1; \ldots; \emptyset/\mathrm{c}_n]$ that replaces all occurrences of c_i in B_Π with $\emptyset$.

Theorem 3.4. *If* Π *is a proof of a sequent* s *in* $\mathsf{L3^B_{ISCI}}$, *then* $\Pi\sigma_\Pi$ *is a proof of* $\mathrm{s}\sigma_\Pi$ *in* $\mathsf{L3^K_{ISCI}}$.

Proof: By induction on the height of Π. We only consider the cases $\mathrm{L}^{\mathrm{B}}_{\perp}$ and $\mathrm{L}^{\mathrm{B}}_{\vee}$ since they are the ones that differ in $\mathsf{L3^K_{ISCI}}$.

Base case $L^{B}_{\bot}$: Since Π is of height 0, we have that $B_\Pi = \emptyset$ and $\sigma_\Pi = \emptyset$. Hence,

$$\frac{}{\Gamma, \bot : x \vdash \Delta, C : xy} L^{B}_{\bot} \quad \rightsquigarrow \quad \frac{}{\Gamma, \bot : x \vdash \Delta, C : xy} L^{K}_{\bot}$$

Case $L^{B}_{\vee}$: We start with a proof Π:

$$\frac{\dfrac{\Pi_1}{\Gamma, A_1 : xa_1 \vdash \Delta, C : xya_1} \qquad \dfrac{\Pi_2}{\Gamma, A_2 : xa_2 \vdash \Delta, C : xya_2}}{\Gamma, A_1 \vee A_2 : x \vdash \Delta, C : xy} L^{B}_{\vee}$$

By induction hypothesis on $\Pi_i, i \in \{1, 2\}$ we get:

$$\frac{\Pi_i \sigma_{\Pi_i}}{\Gamma\sigma_{\Pi_i}, A_i : xa_i\sigma_{\Pi_i} \vdash \Delta\sigma_{\Pi_i}, C : xya_i\sigma_{\Pi_i}}$$

Let σ be the label substitution $[\emptyset/a_1; \emptyset/a_2]$. After applying σ on $\Pi_i\sigma_{\Pi_i}$ using Lemma 3.2 we get new $\mathsf{L3}^{K}_{\mathsf{ISCI}}$-proofs and since a_1 and a_2 are fresh in $\Gamma \vdash \Delta$, we have $\sigma_{\Pi_i}\sigma = \sigma_\Pi$ and $xa_i\sigma_{\Pi_i}\sigma = x\sigma_\Pi = x$, which allows us to apply an instance of $L^{K}_{\vee}$ as follows:

$$\frac{\dfrac{\Pi_1\sigma_\Pi}{\Gamma\sigma_\Pi, A_1 : xa_1\sigma_\Pi \vdash \Delta\sigma_\Pi, C : xya_1\sigma_\Pi} \qquad \dfrac{\Pi_2\sigma_\Pi}{\Gamma\sigma_\Pi, A_2 : xa_2\sigma_\Pi \vdash \Delta\sigma_\Pi, C : xya_2\sigma_\Pi}}{\Gamma\sigma_\Pi, A_1 \vee A_2 : x\sigma_\Pi \vdash \Delta\sigma_\Pi, C : xy\sigma_\Pi} L^{K}_{\vee}$$

The other cases are similar. □

4. From $\mathsf{L3}_{\mathsf{ISCI}}$ to $\mathsf{SN}_{\mathsf{ISCI}}$

We introduce the $\mathsf{SN}_{\mathsf{ISCI}}$ family of labeled calculi that extends the labeled sequent calculus $\mathsf{SN}_{\mathsf{IL}}$ for intuitionistic logic [13]. Labels in $\mathsf{SN}_{\mathsf{ISCI}}$ are not sets or multisets but single atomic symbols over a predefined alphabet which we take as the set of natural numbers in this paper. When we write 5 and 24 we actually mean the singleton $\{5\}$ and the set $\{2, 4\}$ in $\mathsf{L3}_{\mathsf{ISCI}}$. In $\mathsf{SN}_{\mathsf{ISCI}}$, 5 and 24 are the actual natural numbers five and twenty four. To avoid confusion, we use the letters u, v, w to denote labels in $\mathsf{SN}_{\mathsf{ISCI}}$ and keep x, y, z for labels in $\mathsf{L3}_{\mathsf{ISCI}}$.

$$\frac{}{\mathcal{R}, \Gamma, p : u \vdash \Delta, p : v}\ \mathrm{id_p}(u \overset{\mathcal{R}}{\rightsquigarrow} v) \qquad \frac{}{\mathcal{R}, \Gamma, A \approx B : u \vdash \Delta, A \approx B : v}\ \mathrm{id}_{\approx}(u \overset{\mathcal{R}}{\rightsquigarrow} v)$$

$$\frac{}{\mathcal{R}, \Gamma, \bot : u \vdash \Delta, C : v}\ L^{B}_{\bot}(u \overset{\mathcal{R}}{\rightsquigarrow} v) \qquad \frac{}{\mathcal{R}, \Gamma, \bot : u \vdash \Delta}\ L^{K}_{\bot}$$

$$\frac{\mathcal{R}, v \sqsubset u_1, \Gamma, A_1 : u_1 \vdash \Delta, C : u_1 \quad \mathcal{R}, v \sqsubset u_2, \Gamma, A_2 : u_2 \vdash \Delta, C : u_2}{\mathcal{R}, \Gamma, A_1 \vee A_2 : u \vdash \Delta, C : v}\ L^{B}_{\vee}(u \overset{\mathcal{R}}{\rightsquigarrow} v)$$

$$\frac{\mathcal{R}, \Gamma, A_1 : u \vdash \Delta \quad \mathcal{R}, \Gamma, A_2 : u \vdash \Delta}{\mathcal{R}, \Gamma, A_1 \vee A_2 : u \vdash \Delta}\ L^{K}_{\vee} \qquad \frac{\mathcal{R}, \Gamma \vdash \Delta, A_1 : u, A_2 : u}{\mathcal{R}, \Gamma \vdash \Delta, A_1 \vee A_2 : u}\ R_{\vee}$$

$$\frac{\mathcal{R}, \Gamma, A \supset B : u \vdash \Delta, A : v \quad \mathcal{R}, \Gamma, B : v \vdash \Delta}{\mathcal{R}, \Gamma, A \supset B : u \vdash \Delta}\ L_{\supset}(u \overset{\mathcal{R}}{\rightsquigarrow} v) \qquad \frac{\mathcal{R}, u \sqsubset v, \Gamma, A : v \vdash \Delta, B : v}{\mathcal{R}, \Gamma \vdash \Delta, A \supset B : u}\ R_{\supset}$$

$$\frac{\mathcal{R}, \Gamma, A : u, B : u \vdash \Delta}{\mathcal{R}, \Gamma, A \wedge B : u \vdash \Delta}\ L_{\wedge} \qquad \frac{\mathcal{R}, \Gamma \vdash \Delta, A : u \quad \mathcal{R}, \Gamma \vdash \Delta, B : u}{\mathcal{R}, \Gamma \vdash \Delta, A \wedge B : u}\ R_{\wedge}$$

$$\frac{\mathcal{R}, \Gamma, A \approx A : 0 \vdash \Delta}{\mathcal{R}, \Gamma \vdash \Delta}\ L^{1}_{\approx} \qquad \frac{\mathcal{R}, \Gamma, A \approx B : u, B \supset A : u \vdash \Delta}{\mathcal{R}, \Gamma, A \approx B : u \vdash \Delta}\ aL^{2}_{\approx}$$

$$\frac{\mathcal{R}, \Gamma, A \approx B : u \vdash \Delta, B : v \quad \mathcal{R}, \Gamma, A \approx B : u, A : v \vdash \Delta}{\mathcal{R}, \Gamma, A \approx B : u \vdash \Delta}\ L^{2}_{\approx}(u \overset{\mathcal{R}}{\rightsquigarrow} v)$$

$$\frac{\mathcal{R}, \Gamma, A \approx B : u, C \approx D : v, A \otimes C \approx B \otimes D : v \vdash \Delta}{\mathcal{R}, \Gamma, A \approx B : u, C \approx D : v \vdash \Delta}\ L^{3}_{\approx}(u \overset{\mathcal{R}}{\rightsquigarrow} v)$$

$$\frac{\mathcal{R}, \Gamma, A \approx B : u, A \otimes A \approx B \otimes B : u \vdash \Delta}{\mathcal{R}, \Gamma, A \approx B : u \vdash \Delta}\ L^{3*}_{\approx}$$

$$\frac{\mathcal{R}, \Gamma, A \approx B : u, D : v, D^{A}_{B} : v \vdash \Delta}{\mathcal{R}, \Gamma, A \approx B : u, D : v \vdash \Delta}\ L^{r}_{\approx}(u \overset{\mathcal{R}}{\rightsquigarrow} v)$$

Eigenvariable Conditions:

In $R_{\supset}$, v is fresh in $\Gamma \cup \Delta$.

In $L^{B}_{\vee}$, u_1 and u_2 are fresh in $\Gamma \cup \Delta$.

Figure 3: Rules for the $\mathsf{SN}_{\mathsf{ISCI}}$ Family of Labeled Calculi.

Labeled sequents in $\mathsf{SN}_{\mathsf{ISCI}}$ have the form $\mathcal{R}, \Gamma \vdash \Delta$, where Γ, Δ are sets of labeled formulas and $\mathcal{R}$ is the set of *relational atoms*, which are expressions of the form $\mathrm{u} \sqsubset \mathrm{v}$, where u, v are labels. The proof rules of $\mathsf{SN}_{\mathsf{ISCI}}$ are given in Figure 3. $\mathsf{SN}^1_{\mathsf{ISCI}}$ and $\mathsf{SN}^2_{\mathsf{ISCI}}$ are respectively defined with the sets $\{ \mathrm{L}^1_{\approx}, \mathrm{L}^2_{\approx}, \mathrm{L}^3_{\approx}, \mathrm{L}^{3*}_{\approx} \}$ and $\{ \mathrm{L}^1_{\approx}, \mathrm{aL}^2_{\approx}, \mathrm{L}^{\mathrm{r}}_{\approx} \}$ for sentential identity rules. For both sets of identity rules we a K (Kripke) and B (Beth) version of the calculus depending on the rules for disjunction and falsity. An $\mathsf{SN}_{\mathsf{ISCI}}$-proof of a formula C is a proof of the sequent $\vdash \mathrm{C} : 0$.

$\mathsf{SN}_{\mathsf{IL}}$ is usually formulated with rules for the reflexivity and the transitivity of $\sqsubset$ [13]. Such rules can be eliminated by introducing a *reachability predicate* $\mathrm{u} \overset{\mathcal{R}}{\rightsquigarrow} \mathrm{v}$ that can be either defined as the reflexive and transitive closure of $\sqsubset$ (denoted $\sqsubset^*$) in [7], i.e. $\mathrm{u} \overset{\mathcal{R}}{\rightsquigarrow} \mathrm{v}$ iff $(\mathrm{u} \sqsubset \mathrm{v}) \in \sqsubset^*$, or equivalently via the notion of *directed path* as in [11], defined as a chain $\mathrm{w}_1 \sqsubset \ldots \sqsubset \mathrm{w}_\mathrm{n}$ in $\mathcal{R}$ such that $\mathrm{w}_1 = \mathrm{u}$, $\mathrm{w}_\mathrm{n} = \mathrm{v}$, with the special case $\mathrm{u} \overset{\mathcal{R}}{\rightsquigarrow} \mathrm{v}$ if $\mathrm{u} = \mathrm{v}$.

Let us now explain how to translate $\mathsf{L3}_{\mathsf{ISCI}}$-proofs into $\mathsf{SN}_{\mathsf{ISCI}}$-proofs. Any non-empty (finite) label x in $\mathsf{L3}_{\mathsf{ISCI}}$ can be written as an ordered set $\{ k_1 < k_2 < \ldots < k_n \}$ of natural numbers. Let us write $\mu(\mathrm{x})$ for the singleton $\{ k_n \}$ containing the greatest element in x, with the special cases $\mu(\emptyset) = 0$ and $\mu(\mathbb{N}) = \infty$. The set $\mathcal{R}(\mathrm{x})$ of relational atoms associated with a label x is defined as the set $\{ k_i \sqsubset k_{i+1} \mid 0 \leqslant i < n \}$ where $k_0 = 0$. For example, $\mathcal{R}(\{ 1, 2, 5, 8 \}) = \{ 0 \sqsubset 1, 1 \sqsubset 2, 2 \sqsubset 5, 5 \sqsubset 8 \}$. In order to save space, let us write chains $k_1 \sqsubset k_2, k_2 \sqsubset k_3, \ldots, k_{n-1} \sqsubset k_n$ more concisely as $1 \sqsubset 2 \sqsubset \ldots \sqsubset k-1 \sqsubset k$.

Let S be set of labeled formulas (in $\mathsf{L3}_{\mathsf{ISCI}}$ or $\mathsf{SN}_{\mathsf{ISCI}}$). The set $\lceil \mathrm{S} \rceil$ is defined as the restriction of S to the formulas whose labels are maximal w.r.t. $\subseteq$ ($\sqsubset$) in $\mathsf{L3}_{\mathsf{ISCI}}$ ($\mathsf{SN}_{\mathsf{ISCI}}$). A label x is *maximal* in S if $\mathrm{x} \in \lceil \mathrm{S} \rceil$. Given a sequent $\mathrm{s} = \Gamma \vdash \Delta$, a label x (or labeled formula $\mathrm{A} : \mathrm{x}$) is *right maximal* in s if x (or $\mathrm{A} : \mathrm{x}$) $\in \lceil \Delta \rceil$. Left maximality is defined similarly w.r.t. $\lceil \Gamma \rceil$. An instance of a rule r is *right (left) maximal* in a proof if all of its principal formulas occurring in the succedent (antecedent) are right (left) maximal. A proof is *right (left) maximal* if all of its rules are right (left) maximal.

Definition 4.1. Let $\Gamma \vdash \Delta$ be a labeled sequent in $\mathsf{L3}_{\mathsf{ISCI}}$. Let $\mathrm{A}_1 : \mathrm{x}_1, \ldots,$ $\mathrm{A}_\mathrm{m} : \mathrm{x}_\mathrm{m}$ and $\mathrm{B}_1 : \mathrm{y}_1$, ..., $\mathrm{B}_\mathrm{n} : \mathrm{y}_\mathrm{n}$ be enumerations of Γ and Δ respectively.

The translation $\mathsf{LS}(\Gamma \vdash \Delta)$ is the $\mathsf{SN}_{\mathsf{ISCI}}$ sequent $\mathcal{R}', \Gamma' \vdash \Delta'$ where $\Gamma' = \{ \mathrm{A_i} : \mu(\mathrm{x_i}) \mid 1 \leqslant \mathrm{i} \leqslant \mathrm{m} \}$, $\Delta' = \{ \mathrm{B_j} : \mu(\mathrm{y_j}) \mid 1 \leqslant \mathrm{j} \leqslant \mathrm{n} \}$ and $\mathcal{R}' = \bigcup \{ \mathcal{R}(\mathrm{z}) \mid \mathrm{z}$ is maximal in $\Gamma \cup \Delta \}$.

For instance, the translation of the sequent $\mathrm{A} : 1, \mathrm{B} : 34 \vdash \mathrm{C} : 2345, \mathrm{D} : 18$ in $\mathsf{L3}_{\mathsf{ISCI}}$ is the sequent $0 \sqsubset 2 \sqsubset 3 \sqsubset 4 \sqsubset 5, 0 \sqsubset 1 \sqsubset 8, \mathrm{A} : 1, \mathrm{B} : 4 \vdash \mathrm{C} : 5, \mathrm{D} : 8$ in $\mathsf{SN}_{\mathsf{ISCI}}$, where the set of relational atoms is given by the two maximal labels 2345 and 18.

Definition 4.2. A proof Π in $\mathsf{L3}_{\mathsf{ISCI}}$ is *standard* if it does not contain any occurrence of $\mathbb{N}$ and all instances of $\mathrm{R}_{\supset}$ and $\mathrm{L}^{\mathrm{B}}_{\vee}$ introduce fresh labels that are maximal in their premises (for instance, by setting $\mathrm{a} = \{ 3k + 1 \}$ and $\mathrm{a_i} = \{ 3k + \mathrm{i} + 1 \}$ for the smallest suitable k).

Theorem 4.3. *Any* $\mathsf{L3}_{\mathsf{ISCI}}$*-proof can be transformed into a right maximal standard proof.*

Proof: Any $\mathsf{L3}_{\mathsf{ISCI}}$-proof of formula C can be turned into a standard proof by successive applications of Lemma 3.2 since no rule in $\mathsf{L3}_{\mathsf{ISCI}}$ can introduce $\mathbb{N}$ (which can therefore only be present in arbitrarily defined sequents). Showing that a standard proof Π can be turned into a right maximal proof follows from a routine induction on the height of Π. □

Theorem 4.4. *Any right maximal standard* $\mathsf{L3}_{\mathsf{ISCI}}$*-proof can be translated into a (right maximal standard)* $\mathsf{SN}_{\mathsf{ISCI}}$*-proof.*

Proof: By induction on the height of the proof in $\mathsf{L3}_{\mathsf{ISCI}}$ (see Appendix B). □

Example 4.5. As an illustration of the translation, let us consider an $\mathsf{L3}^{\mathsf{B}}_{\mathsf{ISCI}}$-proof of the formula $((\mathrm{p} \supset \mathrm{r}) \wedge (\mathrm{q} \supset \mathrm{r})) \supset ((\mathrm{p} \vee \mathrm{q}) \supset \mathrm{r})$.

$$\Pi_1 \left\{ \dfrac{\dfrac{}{\mathrm{p} \supset \mathrm{r} : 1, \mathrm{q} \supset \mathrm{r} : 1, \mathrm{p} : 45 \vdash \mathrm{r} : 145, \mathrm{p} : 145}\,\mathrm{id_p} \qquad \dfrac{}{\mathrm{q} \supset \mathrm{r} : 1, \mathrm{p} : 45, \mathrm{r} : 145 \vdash \mathrm{r} : 145}\,\mathrm{id_p}}{\mathrm{p} \supset \mathrm{r} : 1, \mathrm{q} \supset \mathrm{r} : 1, \mathrm{p} : 45 \vdash \mathrm{r} : 145}\,\mathrm{L}_{\supset} \right.$$

$$\Pi_2 \left\{ \dfrac{\dfrac{}{\mathrm{p} \supset \mathrm{r} : 1, \mathrm{q} \supset \mathrm{r} : 1, \mathrm{q} : 46 \vdash \mathrm{r} : 146, \mathrm{q} : 146}\,\mathrm{id_p} \qquad \dfrac{}{\mathrm{p} \supset \mathrm{r} : 1, \mathrm{q} : 46, \mathrm{r} : 146 \vdash \mathrm{r} : 146}\,\mathrm{id_p}}{\mathrm{p} \supset \mathrm{r} : 1, \mathrm{q} \supset \mathrm{r} : 1, \mathrm{q} : 46 \vdash \mathrm{r} : 146}\,\mathrm{L}_{\supset} \right.$$

$$\Pi \left\{ \dfrac{\dfrac{\dfrac{\dfrac{\Pi_1 \qquad \Pi_2}{p \supset r:1, q\supset r:1, p\vee q:4 \vdash r:14}\, L^{B}_{\vee}}{p\supset r:1, q\supset r:1 \vdash (p\vee q)\supset r:1}\, R_{\supset}}{(p\supset r)\wedge(q\supset r):1 \vdash (p\vee q)\supset r:1}\, L_{\wedge}}{\vdash ((p\supset r)\wedge(q\supset r))\supset((p\vee q)\supset r):\emptyset}\, R_{\supset} \right.$$

The translation of Π in $\mathsf{SN}^{\mathsf{B}}_{\mathsf{ISCI}}$ is given below.

$$\mathsf{LS}(\Pi_1) \left\{ \dfrac{\dfrac{}{\begin{array}{l}0\sqsubset 1\sqsubset 4\sqsubset 5,\\ p\supset r:1, q\supset r:1, p:5\end{array} \vdash \begin{array}{l} r:5,\\ p:5\end{array}}\, id_p \qquad \dfrac{}{\begin{array}{l}0\sqsubset 1\sqsubset 4\sqsubset 5,\\ q\supset r:1, p:5, r:5\end{array} \vdash r:5}\, id_p}{0\sqsubset 1\sqsubset 4\sqsubset 5, p\supset r:1, q\supset r:1, p:5 \vdash r:5}\, L_{\supset} \right.$$

$$\mathsf{LS}(\Pi_2) \left\{ \dfrac{\dfrac{}{\begin{array}{l}0\sqsubset 1\sqsubset 4\sqsubset 6,\\ p\supset r:1, q\supset r:1, q:6\end{array} \vdash \begin{array}{l} r:6,\\ q:6\end{array}}\, id_p \qquad \dfrac{}{\begin{array}{l}0\sqsubset 1\sqsubset 4\sqsubset 6,\\ p\supset r:1, q:6, r:6\end{array} \vdash r:6}\, id_p}{0\sqsubset 1\sqsubset 4\sqsubset 6, p\supset r:1, q\supset r:1, q:6 \vdash r:6}\, L_{\supset} \right.$$

$$\mathsf{LS}(\Pi) \left\{ \dfrac{\dfrac{\dfrac{\dfrac{\mathsf{LS}(\Pi_1) \qquad \mathsf{LS}(\Pi_2)}{0\sqsubset 1\sqsubset 4, p \supset r:1, q\supset r:1, p\vee q:4 \vdash r:4}\, L^{B}_{\vee}}{0\sqsubset 1, p\supset r:1, q\supset r:1, \vdash (p\vee q)\supset r:1}\, R_{\supset}}{0\sqsubset 1, (p\supset r)\wedge(q\supset r):1 \vdash (p\vee q)\supset r:1}\, L_{\wedge}}{\vdash ((p\supset r)\wedge(q\supset r))\supset((p\vee q)\supset r):0}\, R_{\supset} \right.$$

Since $\mathsf{L3}_{\mathsf{ISCI}}$ does not contain any transitivity or reflexivity rules, the completeness of $\mathsf{SN}_{\mathsf{ISCI}}$ without such rules is a corollary of Theorem 4.4. An important consequence of this result is that for all sequents s in a structural free $\mathsf{SN}_{\mathsf{ISCI}}$-proof Π of a formula C, the set $\mathcal{R}$ of relational atoms describes a tree structure such that if $u \sqsubset v \in \mathcal{R}$, then the node corresponding to v is an immediate successor of the node corresponding to u.

Definition 4.6. A *labeled tree sequent* is a labeled sequent $\tau = \mathcal{R}, \Gamma \vdash \Delta$ such that $\mathcal{R}$ forms a (minimal) tree and all labels in $\Gamma\cup\Delta$ occur in $\mathcal{R}$ (unless $\mathcal{R}$ is empty, in which case every labeled formula in $\Gamma\cup\Delta$ must share the same label). A *labeled tree proof* is a proof containing only labeled tree sequents. A labeled tree proof has the *fixed root property* iff every labeled sequent in the proof has the same root, in which case it is called *standard proof*.

Let us remark that Definition 4.1 maps $\mathsf{L3}_{\mathsf{ISCI}}$-proofs to labeled tree proofs in $\mathsf{SN}_{\mathsf{ISCI}}$ with the fixed root property (the root being 0) since $\emptyset \subseteq x$ for any label x in $\mathsf{L3}_{\mathsf{ISCI}}$.

DEFINITION 4.7. Let $\mathcal{R}, \Gamma \vdash \Delta$ be a labeled sequent in $\mathsf{SN}_{\mathsf{ISCI}}$. Let $A_1 : u_1$, ..., $A_m : u_m$ and $B_1 : v_1$, ..., $B_n : v_n$ be enumerations of Γ and Δ respectively. Let $\mathcal{R}$ be set of relational atoms and u be a label occurring in $\mathcal{R}$, $\mathcal{R}(u) = \{ v \mid v \neq 0 \text{ and } v \overset{\mathcal{R}}{\leadsto} u \}$. The translation $\mathsf{SL}(\Gamma \vdash \Delta)$ is the $\mathsf{L3}_{\mathsf{ISCI}}$ sequent $\Gamma' \vdash \Delta'$ where $\Gamma' = \{ A_i : \mathcal{R}(x_i) \mid 1 \leqslant i \leqslant m \}$, $\Delta' = \{ B_j : \mathcal{R}(y_j) \mid 1 \leqslant j \leqslant n \}$.

For instance, the sequent $0 \sqsubset 2 \sqsubset 3 \sqsubset 4 \sqsubset 5, 0 \sqsubset 1 \sqsubset 8, A : 1, B : 4 \vdash C : 5, D : 8$ in $\mathsf{SN}_{\mathsf{ISCI}}$ translates into the sequent $A : 1, B : 234 \vdash C : 2345, D : 18$ in $\mathsf{L3}_{\mathsf{ISCI}}$.

THEOREM 4.8. *Any standard* $\mathsf{SN}_{\mathsf{ISCI}}$*-proof can be translated into an* $\mathsf{L3}_{\mathsf{ISCI}}$*-proof.*

PROOF: A direct consequence of the translation cycle depicted in Figure 1 or more directly proven by induction on the height of the $\mathsf{SN}_{\mathsf{ISCI}}$-proof. □

5. From $\mathsf{SN}_{\mathsf{ISCI}}$ to $\mathsf{NS}_{\mathsf{ISCI}}$

We introduce the new (family of) nested sequent calculi $\mathsf{NS}_{\mathsf{ISCI}}$ as extensions of $\mathsf{NS}_{\mathsf{mLJ}}$, the nested sequent calculus given for IL in [15].

DEFINITION 5.1. A *nested sequent* is inductively defined as follows:

1. if $s = \Gamma \vdash \Delta$ is a sequent, where Γ, Δ sets of formulas, then s is a nested sequent;
2. if s is a sequent and $\nu_1, \ldots, \nu_n$ are nested sequents then $s, [\nu_1], \ldots, [\nu_n]$ is a nested sequent.

We use the letters ν and Λ (possibly primed or subscripted) to denote nested sequents and sets of nestings respectively. A nested sequent can more conveniently be written as an expression $\Gamma \vdash \Delta, \Lambda$, where all members of Λ are expressions $[\Gamma' \vdash \Delta', \Lambda']$. As usual, we introduce the standard notion of nested-holed contexts [15].

$$\frac{}{\mathcal{S}\{\Gamma, \mathrm{p} \vdash \Delta, \mathrm{p}, \Lambda\}}\,\mathrm{Nid_p} \qquad \frac{}{\mathcal{S}\{\Gamma, \mathrm{A} \approx \mathrm{B} \vdash \Delta, \mathrm{A} \approx \mathrm{B}, \Lambda\}}\,\mathrm{Nid}_{\approx}$$

$$\frac{}{\mathcal{S}\{\Gamma, \bot \vdash \Delta, \mathrm{C}, \Lambda\}}\,\mathrm{NL}^{\mathrm{B}}_{\bot} \qquad \frac{}{\mathcal{S}\{\Gamma, \bot \vdash \Delta, \Lambda\}}\,\mathrm{NL}^{\mathrm{K}}_{\bot}$$

$$\frac{\mathcal{S}\{\Gamma \vdash \Delta, \Lambda, [\mathrm{A} \vdash \mathrm{C}]\} \quad \mathcal{S}\{\Gamma \vdash \Delta, \Lambda, [\mathrm{B} \vdash \mathrm{C}]\}}{\mathcal{S}\{\Gamma, \mathrm{A} \vee \mathrm{B} \vdash \Delta, \mathrm{C}, \Lambda\}}\,\mathrm{NL}^{\mathrm{B}}_{\vee}$$

$$\frac{\mathcal{S}\{\Gamma, \mathrm{A} \vdash \Delta, \Lambda\} \quad \mathcal{S}\{\Gamma, \mathrm{B} \vdash \Delta, \Lambda\}}{\mathcal{S}\{\Gamma, \mathrm{A} \vee \mathrm{B} \vdash \Delta, \Lambda\}}\,\mathrm{NL}^{\mathrm{K}}_{\vee} \qquad \frac{\mathcal{S}\{\Gamma \vdash \Delta, \mathrm{A}, \mathrm{B}, \Lambda\}}{\mathcal{S}\{\Gamma \vdash \Delta, \mathrm{A} \vee \mathrm{B}, \Lambda\}}\,\mathrm{NR}_{\vee}$$

$$\frac{\mathcal{S}\{\Gamma, \mathrm{A} \supset \mathrm{B} \vdash \Delta, \mathrm{A}, \Lambda\} \quad \mathcal{S}\{\Gamma, \mathrm{B} \vdash \Delta, \Lambda\}}{\mathcal{S}\{\Gamma, \mathrm{A} \supset \mathrm{B} \vdash \Delta, \Lambda\}}\,\mathrm{NL}_{\supset} \qquad \frac{\mathcal{S}\{\Gamma \vdash \Delta, \Lambda, [\mathrm{A} \vdash \mathrm{B}]\}}{\mathcal{S}\{\Gamma \vdash \Delta, \mathrm{A} \supset \mathrm{B}, \Lambda\}}\,\mathrm{NR}_{\supset}$$

$$\frac{\mathcal{S}\{\Gamma, \mathrm{A}, \mathrm{B} \vdash \Delta, \Lambda\}}{\mathcal{S}\{\Gamma, \mathrm{A} \wedge \mathrm{B} \vdash \Delta, \Lambda\}}\,\mathrm{NL}_{\wedge} \qquad \frac{\mathcal{S}\{\Gamma \vdash \Delta, \mathrm{A}, \Lambda\} \quad \mathcal{S}\{\Gamma \vdash \Delta, \mathrm{B}, \Lambda\}}{\mathcal{S}\{\Gamma \vdash \Delta, \mathrm{A} \wedge \mathrm{B}, \Lambda\}}\,\mathrm{NR}_{\wedge}$$

$$\frac{\mathcal{S}\{\Gamma \vdash \Delta, \Lambda, [\Gamma', \mathrm{A} \vdash \Delta', \Lambda']\}}{\mathcal{S}\{\Gamma, \mathrm{A} \vdash \Delta, \Lambda, [\Gamma' \vdash \Delta', \Lambda']\}}\,\mathrm{lift}$$

$$\frac{\mathcal{S}\{\Gamma, \mathrm{A} \approx \mathrm{A} \vdash \Delta, \Lambda\}}{\mathcal{S}\{\Gamma \vdash \Delta, \Lambda\}}\,\mathrm{NL}^{1}_{\approx} \qquad \frac{\mathcal{S}\{\Gamma, \mathrm{B} \supset \mathrm{A} \vdash \Delta, \Lambda\}}{\mathcal{S}\{\Gamma, \mathrm{A} \approx \mathrm{B} \vdash \Delta, \Lambda\}}\,\mathrm{NaL}^{2}_{\approx}$$

$$\frac{\mathcal{S}\{\Gamma \vdash \Delta, \mathrm{B}, \Lambda\} \quad \mathcal{S}\{\Gamma, \mathrm{A} \vdash \Delta, \Lambda\}}{\mathcal{S}\{\Gamma, \mathrm{A} \approx \mathrm{B} \vdash \Delta, \Lambda\}}\,\mathrm{NL}^{2}_{\approx} \qquad \frac{\mathcal{S}\{\Gamma, \mathrm{A} \otimes \mathrm{C} \approx \mathrm{B} \otimes \mathrm{D} \vdash \Delta, \Lambda\}}{\mathcal{S}\{\Gamma, \mathrm{A} \approx \mathrm{B}, \mathrm{C} \approx \mathrm{D} \vdash \Delta, \Lambda\}}\,\mathrm{NL}^{3}_{\approx}$$

$$\frac{\mathcal{S}\{\Gamma, \mathrm{A} \otimes \mathrm{A} \approx \mathrm{B} \otimes \mathrm{B} \vdash \Delta, \Lambda\}}{\mathcal{S}\{\Gamma, \mathrm{A} \approx \mathrm{B} \vdash \Delta, \Lambda\}}\,\mathrm{NL}^{3*}_{\approx} \qquad \frac{\mathcal{S}\{\Gamma, \mathrm{D}^{\mathrm{A}}_{\mathrm{B}} \vdash \Delta, \Lambda\}}{\mathcal{S}\{\Gamma, \mathrm{A} \approx \mathrm{B}, \mathrm{D} \vdash \Delta, \Lambda\}}\,\mathrm{NL}^{\mathrm{r}}_{\approx}$$

Figure 4: Rules for the $\mathsf{NS}_{\mathsf{ISCI}}$ Family of Nested Sequent Calculi.

DEFINITION 5.2. A *nested-holed context* is a nested sequent that contains a hole of the form $\{\}$ in place of nestings. Such a context is denoted by $\mathcal{S}\{\}$. Given a nested-holed context and a nested sequent ν, $\mathcal{S}\{\nu\}$ denotes the nested sequent where the hole $\{\}$ has been replaced with $[\nu]$, assuming that it is removed if ν is empty and if $\mathcal{S}$ is empty then $\mathcal{S}\{\nu\} = \nu$.

The translation is similar in principle to the one given for IL in [5] for proofs with prefixes and more thoroughly studied in [11] in the context of labeled systems with relational atoms. Given a fixed root labeled tree sequent $\mathcal{R}, \Gamma \vdash \Delta$ in $\mathsf{SN}_{\mathsf{ISCI}}$, the main idea is to use the tree structure described by the relational atoms in $\mathcal{R}$ to determine the depth of the nested sequents: if $\mathrm{u} \sqsubset \mathrm{v} \in \mathcal{R}$ then all of the formulas labeled with u should be nested one level deeper than the ones labeled with v.

DEFINITION 5.3. Let $\tau = \mathcal{R}, \Gamma \vdash \Delta$ be a labeled tree sequent with root u. Let $\mathrm{w}_1, \ldots, \mathrm{w}_\mathrm{n}$ be all of the labels such that $\mathrm{u} \sqsubset \mathrm{w}_\mathrm{i} \in \mathcal{R}$. S_u is the restriction of a set S of labeled formulas to the formulas labeled with u, i.e. $\mathrm{S}_\mathrm{u} = \{\, A \;/\; \mathrm{A} : \mathrm{u} \in \mathrm{S} \,\}$. $\mathsf{N}(\tau) = \mathsf{N}_\mathrm{u}(\tau)$ is recursively defined on the tree structure of $\mathcal{R}$ as follows: $\mathsf{N}_\mathrm{v}(\tau) = \Gamma_\mathrm{v} \vdash \Delta_\mathrm{v}, \Lambda_\mathrm{v}$ with $\Lambda_\mathrm{v} = [\mathsf{N}_{\mathrm{w}_1}(\tau)], \ldots, [\mathsf{N}_{\mathrm{w}_\mathrm{n}}(\tau)]$.

The rules of the nested sequent calculi $\mathsf{NS}_{\mathsf{ISCI}}$ are given in Figure 4. $\mathsf{NS}^1_{\mathsf{ISCI}}$ and $\mathsf{NS}^2_{\mathsf{ISCI}}$ are defined as having the sets $\{\, \mathrm{NL}^1_{\approx}, \mathrm{NL}^2_{\approx}, \mathrm{NL}^3_{\approx}, \mathrm{NL}^{3*}_{\approx} \,\}$ and $\{\, \mathrm{NL}^1_{\approx}, \mathrm{NaL}^2_{\approx}, \mathrm{NL}^\mathrm{r}_{\approx} \,\}$ for sentential identity rules respectively. For both sets we have two K (Kripke) and B (Beth) variants depending on the rules for disjunction and falsity. Following the standard terminology for nested systems, we distinguish *creation rules* that introduce new nestings in their premises from *upgrade rules* that only move information between nestings without creating new ones. For example, $\mathrm{NR}_{\supset}$ is a creation rule, while lift is an upgrade rule. One noticeable difference between the Beth and the Kripke variants of the rules is that the former gives rise to a nested-like creation rule for left disjunction, while the latter only gives rise to a sequent-like rule. We now state the following translation result.

THEOREM 5.4. *Any standard* $\mathsf{SN}_{\mathsf{ISCI}}$*-proof can be translated into an* $\mathsf{NS}_{\mathsf{ISCI}}$*-proof.*

PROOF: By induction on the height of the $\mathsf{SN}_{\mathsf{ISCI}}$-proof Π (see Appendix A). □

Example 5.5.

The translation of the proof given in Example 4.5 after the erasure of the Beth labels 5 and 6 (to get an $\mathsf{SN}^{\mathsf{K}}_{\mathsf{ISCI}}$-proof) is given below:

$$\Pi^1_3 \left\{ \dfrac{\dfrac{}{\vdash [\vdash [p \supset r, q \supset r, p \vdash r, p]]}\,\mathrm{Nid_p} \qquad \dfrac{}{\vdash [\vdash [p \supset r, q \supset r, p, r \vdash r]]}\,\mathrm{Nid_p}}{\vdash [\vdash [p \supset r, q \supset r, p \vdash r]]}\,\mathrm{NL}_{\supset} \right.$$

$$\Pi^2_3 \left\{ \dfrac{\dfrac{}{\vdash [\vdash [p \supset r, q \supset r, q \vdash r, q]]}\,\mathrm{Nid_p} \qquad \dfrac{}{\vdash [\vdash [p \supset r, q \supset r, q, r \vdash r]]}\,\mathrm{Nid_p}}{\vdash [\vdash [p \supset r, q \supset r, q \vdash r]]}\,\mathrm{NL}_{\supset} \right.$$

$$\Pi_3 \left\{ \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\Pi^1_3 \\ \vdots \\ \vdash [\vdash [p \supset r, q \supset r, p \vdash r]]\end{matrix} \qquad \begin{matrix}\Pi^2_3 \\ \vdots \\ \vdash [\vdash [p \supset r, q \supset r, q \vdash r]]\end{matrix}}{\vdash [\vdash [p \supset r, q \supset r, p \vee q \vdash r]]}\,\mathrm{NL}^{\mathrm{K}}_{\vee}}{\vdash [p \supset r \vdash [q \supset r, p \vee q \vdash r]]}\,\mathrm{lift}}{\vdash [p \supset r, q \supset r \vdash [p \vee q \vdash r]]}\,\mathrm{lift}}{\vdash [p \supset r, q \supset r \vdash (p \vee q) \supset r]}\,\mathrm{NR}_{\supset}}{\vdash [(p \supset r) \wedge (q \supset r) \vdash (p \vee q) \supset r]}\,\mathrm{NL}_{\wedge}}{\vdash ((p \supset r) \wedge (q \supset r)) \supset ((p \vee q) \supset r)}\,\mathrm{NR}_{\supset} \right.$$

6. The Label-Free Sequent Calculi $\mathsf{G3}_{\mathsf{ISCI}}$

In this section we start from the single-succedent label-free sequent calculus $\mathsf{sG3}_{\mathsf{ISCI}}$ that deals with multisets and for which the weakening, contraction and cut rules are proved admissible [4]. We first extend it to a multi-succedent calculus. Then we devise new label-free disjunction and falsity rules that are sound and complete w.r.t. the TB semantics of ISCI to achieve Beth variants of the calculi. The $\mathsf{G3}_{\mathsf{ISCI}}$ proof rules are given in Figure 5, with sequents $\Gamma \vdash \Delta$, where Γ and Δ are sets of formulas. $\mathsf{G3}^1_{\mathsf{ISCI}}$ and $\mathsf{G3}^2_{\mathsf{ISCI}}$ are respectively defined with the sets $\{\, \mathrm{L}^1_{\approx}, \mathrm{L}^2_{\approx}, \mathrm{L}^3_{\approx}, \mathrm{L}^{3*}_{\approx} \,\}$ and $\{\, \mathrm{L}^1_{\approx}, \mathrm{aL}^2_{\approx}, \mathrm{L}^{\mathrm{r}}_{\approx} \,\}$) for sentential identity rules. For both sets of identity rules, we have a K version of the calculus if we consider the rules $L_\bot^K$ and $L_\vee^K$ respectively for falsity and disjunction. We also have a B (Beth) version of the calculus if we consider the rules $L_\bot^B$ and $L_\vee^B$ respectively for falsity and disjunction.

$$\frac{}{\Gamma, p \vdash \Delta, p}\,id_p \qquad \frac{}{\Gamma, A \approx B \vdash \Delta, A \approx B}\,id_\approx$$

$$\frac{}{\Gamma, \bot \vdash \Delta, C}\,L_\bot^B \qquad \frac{}{\Gamma, \bot \vdash \Delta}\,L_\bot^K$$

$$\frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash \Delta, C}\,L_\vee^B \qquad \frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\,L_\vee^K \qquad \frac{\Gamma \vdash \Delta, A_1, A_2}{\Gamma \vdash \Delta, A_1 \vee A_2}\,R_\vee$$

$$\frac{\Gamma, A \supset B \vdash \Delta, A \quad \Gamma, B \vdash \Delta}{\Gamma, A \supset B \vdash \Delta}\,L_\supset \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash \Delta, A \supset B}\,R_\supset$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\,L_\wedge \qquad \frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}\,R_\wedge$$

$$\frac{\Gamma, A \approx A \vdash \Delta}{\Gamma \vdash \Delta}\,L_\approx^1 \qquad \frac{\Gamma, A \approx B, B \supset A \vdash \Delta}{\Gamma, A \approx B \vdash \Delta}\,aL_\approx^2$$

$$\frac{\Gamma, A \approx B \vdash \Delta, B \quad \Gamma, A \approx B, A \vdash \Delta}{\Gamma, A \approx B \vdash \Delta}\,L_\approx^2 \qquad \frac{\Gamma, A \approx B, C \approx D, (A \otimes C) \approx (B \otimes D) \vdash \Delta}{\Gamma, A \approx B, C \approx D \vdash \Delta}\,L_\approx^3$$

$$\frac{\Gamma, A \approx B, (A \otimes A) \approx (B \otimes B) \vdash \Delta}{\Gamma, A \approx B \vdash \Delta}\,L_\approx^{3*} \qquad \frac{\Gamma, A \approx B, D, D_B^A \vdash \Delta}{\Gamma, A \approx B, D \vdash \Delta}\,L_\approx^r$$

Figure 5: Rules for the $\mathsf{G3}_{\mathsf{ISCI}}$ Family of Sequent Calculi.

We note that in the rule $L_\bot^B$, the presence of C on the right-hand side prevents the succedent from being empty. In the rule $L_\vee^B$, the context Δ gets discarded, as in $R_\supset$, from the conclusion to the premises. Moreover one can observe also the rule $L_\approx^r$ is not explicit in [4] but can be introduced in some variants of the calculus. Here we propose to consider a similar rule extended to multisets.

Example 6.1. A $\mathsf{G3}^1_{\mathsf{ISCI}}$-proof that sentential identity is commutative is given below:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\begin{array}{l}p\approx q, q\approx q,\\(q\approx p)\approx(q\approx q)\end{array}\vdash q\approx p, q\approx q}\,\mathrm{id}_{\approx} \qquad \dfrac{}{\begin{array}{l}p\approx q,q\approx q,\\(q\approx p)\approx(q\approx q), q\approx p\end{array}\vdash q\approx p}\,\mathrm{id}_{\approx}}{p\approx q, q\approx q,(q\approx p)\approx(q\approx q)\vdash q\approx p}\,\mathrm{L}^2_\approx}{p\approx q, q\approx q\vdash q\approx p}\,\mathrm{L}^3_\approx}{p\approx q\vdash q\approx p}\,\mathrm{L}^1_\approx$$

THEOREM 6.2. $\mathsf{G3}^1_{\mathsf{ISCI}}$*-proofs translate into* $\mathsf{G3}^2_{\mathsf{ISCI}}$*-proofs.*

PROOF: We show that the sentential identity rules of $\mathsf{G3}^2_{\mathsf{ISCI}}$ can simulate those of $\mathsf{G3}^1_{\mathsf{ISCI}}$.

Case $\mathrm{L}^2_\approx$**:**

$$\dfrac{\dfrac{\Gamma, A\approx B, B\supset A\vdash\Delta, B \qquad \Gamma, A\approx B, B\supset A, A\vdash\Delta}{\Gamma, A\approx B, B\supset A\vdash\Delta}\,\mathrm{L}_\supset}{\Gamma, A\approx B\vdash\Delta}\,\mathrm{aL}^2_\approx$$

Case $\mathrm{L}^3_\approx$**:**

$$\dfrac{\dfrac{\dfrac{\begin{array}{l}\Gamma, A\approx B, C\approx D,\\(A\otimes C)\approx(A\otimes C),(A\otimes C)\approx(B\otimes C),(A\otimes C)\approx(B\otimes D)\end{array}\vdash\Delta}{\Gamma, A\approx B, C\approx D,(A\otimes C)\approx(A\otimes C),(A\otimes C)\approx(B\otimes C)\vdash\Delta}\,\mathrm{L}^{\mathrm{r}}_\approx}{\Gamma, A\approx B, C\approx D,(A\otimes C)\approx(A\otimes C)\vdash\Delta}\,\mathrm{L}^{\mathrm{r}}_\approx}{\Gamma, A\approx B, C\approx D\vdash\Delta}\,\mathrm{L}^1_\approx$$

Case $\mathrm{L}^{3*}_\approx$**:**

$$\dfrac{\dfrac{\Gamma, A\approx B,(A\otimes A)\approx(A\otimes A),(A\otimes A)\approx(B\otimes B)\vdash\Delta}{\Gamma, A\approx B,(A\otimes A)\approx(A\otimes A)\vdash\Delta}\,\mathrm{L}^{\mathrm{r}}_\approx}{\Gamma, A\approx B\vdash\Delta}\,\mathrm{L}^1_\approx$$

□

THEOREM 6.3. $\mathsf{G3}^1_{\mathsf{ISCI}}$ *and* $\mathsf{G3}^2_{\mathsf{ISCI}}$ *are cut-free complete.*

PROOF: The single-succedent $\mathsf{sG3}^1_{\mathsf{ISCI}}$ is proven cut-free complete in [4], which implies the result for (multi-succedent) $\mathsf{G3}^1_{\mathsf{ISCI}}$. As the translation of Theorem 6.2 does not require the cut rule we conclude that $\mathsf{G3}^2_{\mathsf{ISCI}}$ is cut-free complete. □

7. From $\mathsf{NS}_{\mathsf{ISCI}}$ to $\mathsf{G3}_{\mathsf{ISCI}}$

An important result proven in [15] is that any basic nested calculus can be sequentialized: if NS is a basic nested system which sequentialises to a sequent system SC, then the sequent $\Gamma \vdash \Delta$ is provable in NS iff it is provable in SC. The result is too technical to be described here in full details but it relies on the fact that any proof in a basic nested system can be turned into what we will call here a *standard proof*.

Definition 7.1. The *depth of an application of a rule* in a derivation is the depth of its principal formula. A *sequential* $\mathcal{B}^s$ (nested block $\mathcal{B}^n$) in a proof Π is a maximal bottom-up sequence of applications of sequent-like (nested-like) rules in a branch of Π having the same depth d. The depth of such a sequential (nested) block is defined by $\mathsf{dp}(\mathcal{B}^s) = d$ ($\mathsf{dp}(\mathcal{B}^n) = d$).

Definition 7.2. A proof Π in a basic nested system NS is *standard* iff:

1. axioms are applied eagerly;
2. Π is *end-active*, i.e. all rules are applied only in the deepest nestings of a sequent;
3. if a sequential block $\mathcal{B}^s$ immediately follows a nested block $\mathcal{B}^n$ then $\mathsf{dp}(\mathcal{B}^s) = \mathsf{dp}(\mathcal{B}^n) + 1$;
4. if a nested block $\mathcal{B}^n$ immediately follows a sequential block $\mathcal{B}^s$ then $\mathsf{dp}(\mathcal{B}^n) = \mathsf{dp}(\mathcal{B}^s)$;
5. nested blocks have exactly one occurrence of a creation rule.

In order to be able to translate $\mathsf{NS}_{\mathsf{ISCI}}$-proofs to $\mathsf{G3}_{\mathsf{ISCI}}$-proofs, all we have to do is to show that $\mathsf{NS}_{\mathsf{ISCI}}$ calculi are basic nested systems in the sense of [15].

Definition 7.3. A nested system NS is *basic* if it satisfies the following conditions:

1. nested-like rules must have exactly one nesting in the premises or conclusion;

2. nested-like rules always move information deeper inside nestings;
3. upgrade rules must have exactly one principal and auxiliary formula;
4. upgrade rules move only one piece of information at a time.

Theorem 7.4. *Any standard* $\mathsf{NS}_{\mathsf{ISCI}}$*-proof can be translated into a* $\mathsf{G3}_{\mathsf{ISCI}}$*-proof.*

Proof: Since the Kripke nested calculi $\mathsf{NS}^{\mathsf{K}}_{\mathsf{ISCI}}$ only add sequent-like rules to the basic nested system $\mathsf{NS}_{\mathsf{mLJ}}$ [15], they all qualify as basic nested systems. For the Beth nested calculi $\mathsf{NS}^{\mathsf{B}}_{\mathsf{ISCI}}$, we observe that the new creation rule $\mathrm{NL}^{\mathrm{B}}_{\vee}$ satisfies the conditions of Definition 7.3. Therefore, the result follows as an immediate consequence of Theorem 29 in [15]. □

From [15] we know that nested blocks in a standard proof Π of a formula C should be thought of as macros turning nested sequents into sequents, which actually means that we only need to replay in $\mathsf{G3}_{\mathsf{ISCI}}$, bottom-up from the end sequent ⊢ C, the same sequent-like rule application order encoded in Π, to obtain the corresponding $\mathsf{G3}_{\mathsf{ISCI}}$-proof of C.

Example 7.5. Since the $\mathsf{NS}_{\mathsf{ISCI}}$-proof given in Example 5.5 is standard, one can translate it into a $\mathsf{G3}_{\mathsf{ISCI}}$-proof using the same rule application order as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p \supset r, q \supset r, p \vdash r, p}\,\mathrm{id_p} \quad \dfrac{}{q \supset r, p, r \vdash r}\,\mathrm{id_p}}{p \supset r, q \supset r, p \vdash r}\,\mathrm{L}_{\supset} \quad \dfrac{\dfrac{}{p \supset r, q \supset r, q \vdash r, q}\,\mathrm{id_p} \quad \dfrac{}{p \supset r, q, r \vdash r}\,\mathrm{id_p}}{p \supset r, q \supset r, q \vdash r}\,\mathrm{L}_{\supset}}{p \supset r, q \supset r, p \vee q \vdash r}\,\mathrm{L}^{\mathrm{B}}_{\vee}}{p \supset r, q \supset r \vdash (p \vee q) \supset r}\,\mathrm{R}_{\supset}}{(p \supset r) \wedge (q \supset r) \vdash (p \vee q) \supset r}\,\mathrm{L}_{\wedge}}{\vdash ((p \supset r) \wedge (q \supset r)) \supset ((p \vee q) \supset r)}\,\mathrm{R}_{\supset}$$

Theorem 7.6. $\mathsf{L3}_{\mathsf{ISCI}}$*-proofs can be translated into* $\mathsf{G3}_{\mathsf{ISCI}}$*-proofs*

Proof: The result follows from Theorems 4.4, 5.4 and 7.4. □

8. From $\mathsf{G3}_{\mathsf{ISCI}}$ to $\mathsf{L3}_{\mathsf{ISCI}}$

In this section we consider proof translations from the label-free $\mathsf{G3}_{\mathsf{ISCI}}$ sequent calculi to the labeled sequent calculi $\mathsf{L3}_{\mathsf{ISCI}}$. The main problem is to turn label-free sequents into labeled sequents. Let $\Gamma = C_1, \ldots, C_m$ be a sequence of m formulas. A label vector $\vec{c}$ is a non-empty sequence of labels $c_1, \ldots, c_n$ such that for any i such that $1 \leqslant i \leqslant n$ we have $c_i \in \mathbf{L}^1$ and for any i, j such that $1 \leqslant i, j \leqslant n$, if $i \neq j$ then $c_i \neq c_j$.

In other words, a label vector is a non-empty finite sequence of pairwise distinct singleton labels. We define $\check{c}$ as $c_1 \cup \ldots \cup c_n$ In particular, given a strictly positive integer n, $\vec{n}$ is defined as the sequence of singletons $\{1\}, \{2\}, \ldots, \{n\}$ and $\check{n}$ is the label $\{1, 2, \ldots, n\}$. As a special case we set $\vec{0} = \emptyset$. Finally, we define $\Gamma : \vec{c}$ as $C_1 : d_1, \ldots, C_n : d_n$, where $d_i = c_i$ if $i \leqslant n$ and $d_i = c_n$ otherwise.

Definition 8.1. Let $\Gamma \vdash \Delta$ be a $\mathsf{G3}_{\mathsf{ISCI}}$ label-free sequent. Given a label vector $\vec{c}$ for Γ, the translation L of $\Gamma \vdash \Delta$ under $\vec{c}$, written $\mathsf{L}(\Gamma \vdash \Delta, \vec{c})$ is defined as the $\mathsf{L3}_{\mathsf{ISCI}}$ labeled sequent $\Gamma : \vec{c} \vdash \Delta : \check{c}$. In particular, $\mathsf{L}(\Gamma \vdash \Delta) = \mathsf{L}(\Gamma \vdash \Delta, \vec{n})$, where $n = |\Gamma|$.

Theorem 8.2. *$\mathsf{G3}_{\mathsf{ISCI}}$-proofs translate into $\mathsf{L3}_{\mathsf{ISCI}}$-proofs.*

Proof: By induction on the height of $\mathsf{G3}_{\mathsf{ISCI}}$-proofs (see Appendices C and C.1). □

Example 8.3. Let us translate the $\mathsf{G3}^1_{\mathsf{ISCI}}$-proof given in Example 6.1 into a right maximal $\mathsf{L3}^1_{\mathsf{ISCI}}$-proof. The $\mathsf{G3}^1_{\mathsf{ISCI}}$-proof starts with two axioms $\mathrm{id}_{\approx}$. Applying Definition 8.1 and the translation pattern for $\mathrm{id}_{\approx}$, we get the following two $\mathsf{L3}^1_{\mathsf{ISCI}}$-proofs:

$$\Pi^0_1 \left\{ \frac{}{p \approx q : 1, q \approx q : 2, (q \approx p) \approx (q \approx q) : 3 \vdash q \approx q : 123} \mathrm{id}_{\approx}(2 \subseteq 123) \right.$$

$$\Pi^0_2 \left\{ \frac{}{p \approx q : 1, q \approx q : 2, (q \approx p) \approx (q \approx q) : 3, q \approx p : 4 \vdash q \approx p : 1234} \mathrm{id}_{\approx}(4 \subseteq 1234) \right.$$

The next rule is $\mathrm{L}^2_{\approx}$ and the translation of its conclusion yields:

$$p \approx q : 1, q \approx q : 2, (q \approx p) \approx (q \approx q) : 3 \vdash q \approx p : 123$$

The restriction $xy \in \lceil\Delta\rceil$ then leads to the replacement of 4 with 123 in Π^0_2 since the active formula of $L^2_\approx$ is $q \approx p$, which is labeled with 4 in Π^0_2, while the succedent formula in the conclusion of $L^2_\approx$ is labeled with 123:

$$\Pi^1_2 \left\{ \frac{}{p \approx q : 1, q \approx q : 2, (q \approx p) \approx (q \approx q) : 3, q \approx p : 123 \vdash q \approx p : 123} \, id_\approx (123 \subseteq 123) \right.$$

We combine Π^0_1 and Π^1_2 using $L^2_\approx$:

$$\Pi^1 \left\{ \frac{\Pi^0_1 \qquad \Pi^1_2}{p \approx q : 1, q \approx q : 2, (q \approx p) \approx (q \approx q) : 3 \vdash q \approx p : 123} \, L^2_\approx \right.$$

The next rule is $L^3_\approx$. The conclusion translation is $p \approx q : 1, q \approx q : 2 \vdash q \approx p : 12$ Before translating $L^3_\approx$ in $\mathsf{G3}^1_{\mathsf{ISCI}}$ to $L^3_\approx$ in $\mathsf{L3}^1_{\mathsf{ISCI}}$, we need to replace 3 with 12 in Π^1 since the active formula of $L^3_\approx$ is $(q \approx p) \approx (q \approx q) : 3$.

$$\Pi^2 \left\{ \frac{\dfrac{\Pi^1[12/3]}{p \approx q : 1, q \approx q : 2, (q \approx p) \approx (q \approx q) : 12 \vdash q \approx p : 12}}{p \approx q : 1, q \approx q : 2 \vdash q \approx p : 12} \, L^3_\approx \right.$$

The next rule is $L^1_\approx$. The conclusion translation yields: $p \approx q : 1 \vdash q \approx p : 1$ The restriction $x \in \lceil\Delta\rceil$ for $L^1_\approx$ leads to the replacement of 2 with 1 in Π^2:

$$\Pi^3 \left\{ \frac{\dfrac{\Pi^2[1/2]}{p \approx q : 1, q \approx q : 1 \vdash q \approx p : 1}}{p \approx q : 1 \vdash q \approx p : 1} \, L^1_\approx \right.$$

The final result is given below:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p \approx q : 1, q \approx q : 1, (q \approx p) \approx (q \approx q) : 1 \vdash q \approx q : 1} \, id_\approx \qquad \dfrac{}{p \approx q : 1, q \approx q : 1, q \approx p : 1, (q \approx p) \approx (q \approx q) : 1 \vdash q \approx p : 1} \, id_\approx}{p \approx q : 1, q \approx q : 1, (q \approx p) \approx (q \approx q) : 1 \vdash q \approx p : 1} \, L^2_\approx}{p \approx q : 1, q \approx q : 1 \vdash q \approx p : 1} \, L^3_\approx}{p \approx q : 1 \vdash q \approx p : 1} \, L^1_\approx$$

THEOREM 8.4. $\mathsf{G3}^{\mathsf{K}}_{\mathsf{ISCI}}$ *(*$\mathsf{G3}^{\mathsf{B}}_{\mathsf{ISCI}}$*) is sound and complete w.r.t. Kripke (*TB*) semantics.*

PROOF: For the soundness, we have $\vdash_{\mathsf{G3}^{\mathsf{K}}_{\mathsf{ISCI}}} A \Rightarrow \models_{\mathsf{K}} A$, that is proven in Appendix D and $\vdash_{\mathsf{G3}^{\mathsf{B}}_{\mathsf{ISCI}}} A \Rightarrow \models_{\mathsf{B}} A$ proven as follows: $\vdash_{\mathsf{G3}^{\mathsf{B}}_{\mathsf{ISCI}}} A$ implies $\vdash_{\mathsf{L3}^{\mathsf{B}}_{\mathsf{ISCI}}}$

A by Theorem 8.2 and the implication $\vdash_{\mathsf{L3}^{\mathsf{B}}_{\mathsf{ISCI}}} A \Rightarrow \models_{\mathsf{B}} A$ is proven in [6]. The cut-free completeness of the single-succedent calculus $\mathsf{sG3}^{\mathsf{K}}_{\mathsf{ISCI}}$ w.r.t. the Kripke semantics is proven in [4]. Since $\vdash_{\mathsf{sG3}^{\mathsf{K}}_{\mathsf{ISCI}}} A$ implies $\vdash_{\mathsf{G3}^{\mathsf{K}}_{\mathsf{ISCI}}} A$, we get $\models_{\mathsf{K}} A \Rightarrow \vdash_{\mathsf{G3}^{\mathsf{K}}_{\mathsf{ISCI}}} A$. Now since $\models_{\mathsf{B}} A \Rightarrow \vdash_{\mathsf{L3}^{\mathsf{B}}_{\mathsf{ISCI}}} A$ is proven in [6], it follows from the translation cycle in Figure 1 that $\vdash_{\mathsf{L3}^{\mathsf{B}}_{\mathsf{ISCI}}} A$ implies $\vdash_{\mathsf{G3}^{\mathsf{B}}_{\mathsf{ISCI}}} A$, hence $\models_{\mathsf{B}} A \Rightarrow \vdash_{\mathsf{G3}^{\mathsf{B}}_{\mathsf{ISCI}}} A$. □

COROLLARY 8.5. All of the $\mathsf{L3}_{\mathsf{ISCI}}$ calculi are cut-free complete.

PROOF: Since the translation in Definition 8.1 does not require the cut rule, the cut-free completeness of $\mathsf{G3}^{1}_{\mathsf{ISCI}}$ and $\mathsf{G3}^{2}_{\mathsf{ISCI}}$ (see Theorem 6.3) entails the cut-free completeness of both $\mathsf{L3}^{1}_{\mathsf{ISCI}}$ and $\mathsf{L3}^{2}_{\mathsf{ISCI}}$ by Theorem 8.2. □

9. Conclusion and Perspectives

In this paper we studied how families of labeled and label-free sequent calculi, that capture distinct semantics of the logic ISCI, relate to each other. We considered a syntactical approach based on proof translations between calculi rather than working directly within the semantics. Although the long translation cycle depicted in Figure 1 might at first sight appear as an unnecessary hassle, we now point out some of the merits of this approach.

Firstly, while most labeled calculi indeed reflect the main properties of a given semantics in their labeling algebras, they do not necessarily fully and faithfully capture all of such properties. For example, the family $\mathsf{L3}_{\mathsf{ISCI}}$ of labeled calculi was carefully crafted so that the Beth variants would differ as little as possible from the Kripke variants while still enabling one of the most important and interesting feature of the TB semantics: remaining sound when the eigenvariable conditions are dropped to allow the reuse of the singletons introduced by a previous instance of a $L^{\mathsf{B}}_{\vee}$ or $R_{\supset}$ rule with the same principal formula. The soundness of the $\mathsf{L3}^{\mathsf{B}}_{\mathsf{ISCI}}$ family without the eigenvariable conditions comes from the regularity property of the TB semantics (condition of Definition 2.3) although the minimality of (the world realizing) a reused singleton (in a liberalized soundness proof as the one given in [6]) cannot be syntactically witnessed inside the calculus (as comparing labels via set inclusion alone is too weak). Let us recall that the

Kripke variants $\mathsf{L3}^{\mathsf{K}}_{\mathsf{ISCI}}$ are unsound without the eigenvariable conditions. The TB semantics of ISCI and its associated Beth labeled calculi therefore play a central role in the decidability arguments given in [6].

Secondly, we argue that proof translations can help us gain a better understanding of how the semantics reveals itself in a calculus. Indeed, the proof translation approach allows us to depart from the traditional way of devising labeled proof systems, which consists in turning the forcing clauses of the underlying semantics into logical proof rules while capturing its properties inside a labeling algebra. For instance, to devise a Beth $\mathsf{SN}_{\mathsf{ISCI}}$ labeled system, we would have introduced a syntactic operator "|" to reflect the lattice meet of the TB semantics, which would have led to extended relational atoms of the form $(\mathrm{v} \mid \mathrm{w})\,\mathrm{R}\,\mathrm{u}$. The rule for left and right disjunction would have respectively taken the following forms:

$$\frac{\mathcal{R}, (\mathrm{v} \mid \mathrm{w})\ \mathrm{R}\ \mathrm{u}, \Gamma, \mathrm{A} : \mathrm{v}, \mathrm{B} : \mathrm{w} \vdash \Delta}{\mathcal{R}, \Gamma, \mathrm{A} \vee \mathrm{B} : \mathrm{u} \vdash \Delta}\ \mathrm{v}, \mathrm{w}\ \text{fresh}$$

$$\frac{\mathcal{R}, (\mathrm{v} \mid \mathrm{w})\ \mathrm{R}\ \mathrm{u}, \Gamma \vdash \Delta, \mathrm{A} : \mathrm{v} \qquad \mathcal{R}, (\mathrm{v} \mid \mathrm{w})\ \mathrm{R}\ \mathrm{u}, \Gamma \vdash \Delta, \mathrm{B} : \mathrm{w}}{\mathcal{R}, (\mathrm{v} \mid \mathrm{w})\ \mathrm{R}\ \mathrm{u}, \Gamma \vdash \Delta, \mathrm{A} \vee \mathrm{B} : \mathrm{u}}$$

Proceeding by translation from $\mathsf{L3}^{\mathsf{B}}_{\mathsf{ISCI}}$ enabled a simpler and more concise account of the TB semantics in the $\mathsf{SN}^{\mathsf{B}}_{\mathsf{ISCI}}$ calculi since they do not require any of the extra machinery described previously (no special meet operator "|", no extended relational atoms). Guessing such calculi directly from the semantics would not have been that obvious. Guessing the rules for Beth disjunction and falsity in $\mathsf{NS}^{\mathsf{B}}_{\mathsf{ISCI}}$ and $\mathsf{G3}^{\mathsf{B}}_{\mathsf{ISCI}}$ would have been even more difficult. The shape of the rule $\mathrm{NL}^{\mathrm{B}}_{\vee}$ as depicted in Figure 4 and the fact that it should behave as a creation rule in $\mathsf{NS}_{\mathsf{ISCI}}$ just like $\mathrm{NR}_{\supset}$ actually came from the translation described in Section 5 after noticing the similarity of $\mathrm{L}^{\mathrm{B}}_{\vee}$ with $\mathrm{R}_{\supset}$ in $\mathsf{SN}^{\mathsf{B}}_{\mathsf{ISCI}}$. In turn, the similarity of $\mathrm{NL}^{\mathrm{B}}_{\vee}$ with $\mathrm{NR}_{\supset}$ gave rise to the idea of a discarding context Δ from the conclusion to the premises for the Beth disjunction rule $\mathrm{L}^{\mathrm{B}}_{\vee}$ in $\mathsf{G3}^{\mathsf{B}}_{\mathsf{ISCI}}$.

Thirdly, the translation approach makes it easier to transpose results from one proof system to another one. For example, the decidability result proven in [18] relies on the fact that the proof search space for a formula

A in (single-succedent) $\mathsf{sG3}_{\mathsf{ISCI}}$ can be restricted to a bounded set of formulas generated from A. Such a result can directly be exported to $\mathsf{L3}_{\mathsf{ISCI}}$ via the translation of Theorem 8.2. We argue that it is not the technical complexity (of the soundness proof) of a translation that matters, but what we can learn from it and what we can do with it. Our translations are all proven sound by standard inductive proofs that can convincingly be checked or rebuilt by a human reader. For us, this is a feature. Moreover our translations provide cut-free completeness for all the calculi in the paper, knowing that cut-free completeness via cut-elimination involves much harder proofs, with a significantly higher number of cases, thus making them more error-prone.

Lastly, another benefit of proof translations is to give people proofs in the formalism they understand better. For example, Theorem 3.4 is a key contribution for labeled systems (but can be transposed to all of our calculi). Since Beth proofs remain sound without the eigenvariable, Beth calculi are better suited for giving decidability arguments as explained previously. With Kripke proofs, one would have to devise additional mechanisms to mitigate the introduction of fresh labels. On the other hand, Kripke proofs are more easily understood because Kripke disjunction seem more "natural" to grasp for most people. Well, use a Beth-like calculus under the hood, then use our results to provide a Kripke proof.

In future works we expect to find direct translations from $\mathsf{L3}_{\mathsf{ISCI}}$ and $\mathsf{SN}_{\mathsf{ISCI}}$ to $\mathsf{G3}_{\mathsf{ISCI}}$, without the intermediate step via proofs in $\mathsf{NS}_{\mathsf{ISCI}}$. We conjecture that the heavy machinery of end-active nested proofs described in Section 6 can be mimicked in $\mathsf{L3}_{\mathsf{ISCI}}$ and $\mathsf{SN}_{\mathsf{ISCI}}$ by following a rule application strategy that always expands formulas with labels that are maximal. Moreover we want to tackle the challenging problem of translating multi-succedent $\mathsf{G3}_{\mathsf{ISCI}}$-proofs into single-succedent $\mathsf{sG3}_{\mathsf{ISCI}}$-proofs. Achieving direct translations from multi-succedent to single-succedent calculi is a notoriously difficult task. For propositional IL, although there are indirect translations requiring intermediate steps (one in [16] involving nested sequents and a detour through bi-intuitionistic logic), the only actual direct translation we are aware of is the one by proof reconstruction from the connection method [17]. Unfortunately, such a method fails in our case as it very strongly depends on the subformula property to calculate atomic paths, a property that current calculi for ISCI fail to enjoy.

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A. Appendix: From $\mathsf{SN}_{\mathsf{ISCI}}$ to $\mathsf{NS}_{\mathsf{ISCI}}$

THEOREM A.1. *Any standard $\mathsf{SN}_{\mathsf{ISCI}}$-proof can be translated into an $\mathsf{NS}_{\mathsf{ISCI}}$-proof.*

PROOF: By induction on the height of the $\mathsf{SN}_{\mathsf{ISCI}}$-proof Π mapping each rule in Π to its corresponding rule in $\mathsf{NS}_{\mathsf{ISCI}}$ (with possible additional steps involving the rule lift) and stepwise translating each sequent in Π into an $\mathsf{NS}_{\mathsf{ISCI}}$ nested sequent using Definition 5.3. To avoid confusion, we add a superscript i to the objects Γ, Δ, Λ described in Definition 5.3 when translating the i-th premiss of a rule and we keep the original non-superscripted notation for the translation of its conclusion.

Base case id**:** This case subsumes the base cases $\mathrm{id_p}$ and $\mathrm{id}_{\approx}$. We start with an axiom

$$\frac{}{\mathcal{R}, \Gamma, \mathrm{A} : \mathrm{u} \vdash \Delta, \mathrm{A} : \mathrm{v}}\ \mathrm{id}(\mathrm{u} \overset{\mathcal{R}}{\rightsquigarrow} \mathrm{v})$$

If $\mathrm{u} = \mathrm{v}$ then formulas have the same depth and we have an axiom in $\mathsf{NS}_{\mathsf{ISCI}}$ directly. Otherwise, $\mathrm{u} \overset{\mathcal{R}}{\rightsquigarrow} \mathrm{v}$ derives from the transitive closure of a chain $\mathrm{w_1} \sqsubset \ldots \sqsubset \mathrm{w_n}$ with $\mathrm{w_1} = \mathrm{u}$ and $\mathrm{w_n} = \mathrm{v}$. We then get an axiom in $\mathsf{NS}_{\mathsf{ISCI}}$ after n applications of the rule lift.

Base cases $\mathrm{L}_{\bot}^{\mathrm{B}}$ and $\mathrm{L}_{\bot}^{\mathrm{K}}$: Similar to Base case id.

Case $\mathrm{R}_{\supset}$: We start with

$$\frac{\begin{array}{c}\Pi \\ \hdashline \mathcal{R}, \mathrm{u} \sqsubset \mathrm{v}, \Gamma, \mathrm{A} : \mathrm{v} \vdash \Delta, \mathrm{B} : \mathrm{v}\end{array}}{\mathcal{R}, \Gamma \vdash \Delta, \mathrm{A} \supset \mathrm{B} : \mathrm{u}}\ \mathrm{R}_{\supset}$$

The translation of the conclusion is a nested sequent of the form $\mathcal{S}\{\Gamma_{\mathrm{u}} \vdash \Delta_{\mathrm{u}}, \Lambda_{\mathrm{u}}\}$. The translation of the premiss is a nested sequent of the form $\mathcal{S}\{\Gamma_{\mathrm{u}}^1 \vdash \Delta_{\mathrm{u}}^1, \Lambda_{\mathrm{u}}^1\}$ for which, by induction hypothesis, we have a proof Π'. Since v is fresh and $\mathrm{u} \sqsubset \mathrm{v} \in \mathcal{R}$, v represents a nesting level one deep w.r.t. u. Therefore, by Definition 5.3, we have:

$$\Gamma_{\mathrm{u}} = \Gamma_{\mathrm{u}}^1 \qquad \Delta_{\mathrm{u}} = \Delta_{\mathrm{u}}^1, \mathrm{A} \supset \mathrm{B} \qquad \Lambda_{\mathrm{u}}^1 = \Lambda_{\mathrm{u}}, [\mathrm{A} \vdash \mathrm{B}].$$

We can then conclude with an instance of $\mathrm{NR}_\supset$ as follows:

$$\dfrac{\dfrac{\dfrac{\Pi'}{\mathcal{S}\{\Gamma^1_u \vdash \Delta^1_u, \Lambda^1_u\}}}{\mathcal{S}\{\Gamma^1_u \vdash \Delta^1_u, \Lambda_u, [A \vdash B]\}}\ (\Lambda^1_u = \Lambda_u, [A \vdash B])}{\dfrac{\mathcal{S}\{\Gamma^1_u \vdash \Delta^1_u, A \supset B, \Lambda_u\}}{\mathcal{S}\{\Gamma_u \vdash \Delta_u \Lambda_u\}}\ (\Gamma_u = \Gamma^1_u,\ \Delta_u = \Delta^1_u, A \supset B)}\ \mathrm{NR}_\supset$$

Case $\mathrm{L}_\supset$: We start with

$$\dfrac{\dfrac{\Pi_1}{\mathcal{R}, \Gamma, A \supset B : u \vdash \Delta, A : v} \quad \dfrac{\Pi_2}{\mathcal{R}u, \Gamma, B : v \vdash \Delta}}{\mathcal{R}, \Gamma, A \supset B : u \vdash \Delta}\ \mathrm{L}_\supset (u \overset{\mathcal{R}}{\rightsquigarrow} v)$$

The translation of the conclusion is a nested sequent $\mathcal{S}\{\Gamma_u \vdash \Delta_u, \Lambda_u\}$. The translations of the premises are nested sequents $\mathcal{S}\{\Gamma^1_u \vdash \Delta^1_u, \Lambda^1_u\}$ and $\mathcal{S}\{\Gamma^2_u \vdash \Delta^2_u, \Lambda^2_u\}$ for which we have, by induction hypothesis, proofs ${\Pi_1}'$ and ${\Pi_2}'$.

Since $u \overset{\mathcal{R}}{\rightsquigarrow} v$ we know that v represents a deeper nesting level than u, by Definition 5.3:

$$\begin{gathered}
\Gamma_u = \Gamma^1_u, A \supset B \qquad \Delta_u = \Delta^1_u = \Delta^2_u \\
\Lambda^1_u = \Lambda_u(\Gamma^1_v \vdash \Delta^1_v, A, \Lambda^1_v)) \\
\Gamma^2_u = \Gamma^1_u \qquad \Lambda^2_u = \Lambda_u(\Gamma^2_v, B \vdash \Delta^2_v, \Lambda^2_v)) \\
\Delta^2_v = \Delta^1_v = \Delta_v \qquad \Lambda^2_v = \Lambda^1_v = \Lambda_v \qquad \Gamma^2_v = \Gamma^1_v = \Gamma_v.
\end{gathered}$$

After an application of $\mathrm{NL}_\supset$ and some applications of lift rule we can conclude as follows:

$$\dfrac{\dfrac{\dfrac{{\Pi_1}'}{\mathcal{S}\{\Gamma^1_u \vdash \Delta^1_u, \Lambda^1_u\}}}{\mathcal{S}\{\Gamma_u, A \supset B \vdash \Delta_u, \Lambda_u(\Gamma_v \vdash \Delta_v, A, \Lambda_v)\}} \quad \dfrac{\dfrac{{\Pi_2}'}{\mathcal{S}\{\Gamma^2_u \vdash \Delta^2_u, \Lambda^2_u\}}}{\mathcal{S}\{\Gamma_u \vdash \Delta_u, \Lambda_u(\Gamma_v, B \vdash \Delta_v, \Lambda_v)\}}}{\dfrac{\mathcal{S}\{\Gamma_u, A \supset B \vdash \Delta_u, \Lambda_u(\Gamma_v, A \supset B \vdash \Delta_v, \Lambda_v)\}}{\mathcal{S}\{\Gamma_u, A \supset B \vdash \Delta_u, \Lambda_u(\Gamma_v \vdash \Delta_v, \Lambda_v)\}}\ \mathrm{lift}*}\ \mathrm{NL}_\supset$$

Case $L_\wedge$ Since there is no label modification in the sequents we have

$$\dfrac{\begin{array}{c}\Pi\\ \hdashline \mathcal{R},\Gamma,A:u,B:u\vdash\Delta\end{array}}{\mathcal{R},\Gamma,A\wedge B:u\vdash\Delta}\,L_\wedge$$

$$\downarrow$$

$$\dfrac{\begin{array}{c}\Pi'\\ \hdashline \mathcal{S}\{\Gamma_u,A,B\vdash\Delta_u,\Lambda_u\}\end{array}}{\mathcal{S}\{\Gamma_u,A\wedge B\vdash\Delta_u,\Lambda_u\}}\,NL_\wedge$$

Case $R_\wedge$ Since there is no label modification in the sequents we have

$$\dfrac{\begin{array}{c}\Pi_1\\ \hdashline \mathcal{R},\Gamma\vdash\Delta,A:u\end{array}\quad\begin{array}{c}\Pi_2\\ \hdashline \mathcal{R},\Gamma\vdash\Delta,B:u\end{array}}{\mathcal{R},\Gamma\vdash\Delta,A\wedge B:u}\,R_\wedge$$

$$\downarrow$$

$$\dfrac{\begin{array}{c}\Pi_1{}'\\ \hdashline \mathcal{S}\{\Gamma_u\vdash\Delta_u,A,\Lambda_u\}\end{array}\quad\begin{array}{c}\Pi_2{}'\\ \hdashline \mathcal{S}\{\Gamma_u\vdash\Delta_u,B,\Lambda_u\}\end{array}}{\mathcal{S}\{\Gamma_u\vdash\Delta_u,A\wedge B,\Lambda_u\}}\,NR_\wedge$$

Case $R_\vee$ Since there is no label modification in the sequents we have

$$\dfrac{\begin{array}{c}\Pi\\ \hdashline \mathcal{R},\Gamma\vdash\Delta,A:u,B:u\end{array}}{\mathcal{R},\Gamma\vdash\Delta,A\vee B:u}\,R_\vee$$

$$\downarrow$$

$$\dfrac{\begin{array}{c}\Pi'\\ \hdashline \mathcal{S}\{\Gamma_u\vdash\Delta_u,A,B,\Lambda_u\}\end{array}}{\mathcal{S}\{\Gamma_u\vdash\Delta_u,A\vee B,\Lambda_u\}}\,NR_\vee$$

Case $L_\vee^\kappa$ Since there is no label modification we have

$$\dfrac{\begin{array}{c}\Pi_1\\ \hdashline \mathcal{R},\Gamma,A:u\vdash\Delta\end{array}\quad\begin{array}{c}\Pi_2\\ \hdashline \mathcal{R},\Gamma,B:u\vdash\Delta\end{array}}{\mathcal{R},\Gamma,A\vee B:u\vdash\Delta}\,R_\wedge$$

$$\downarrow$$

$$\dfrac{\dfrac{\Pi_1{}'}{\mathcal{S}\{\Gamma_u, A \vdash \Delta_u, \Lambda_u\}} \quad \dfrac{\Pi_2{}'}{\mathcal{S}\{\Gamma_u, B \vdash \Delta_u, \Lambda_u\}}}{\mathcal{S}\{\Gamma_u, A \vee B \vdash \Delta_u, \Lambda_u\}}\ NL^K_\vee$$

Case $L^B_\vee$ This case is similar to case $R_\supset$, but with two premises and two fresh labels. We start with :

$$\dfrac{\dfrac{\Pi_1}{\mathcal{R}, v \sqsubset u_1 \quad \Gamma, A_1 \vee A_2 : u, A_1 : u_1 \vdash \Delta, C : u_1} \quad \dfrac{\Pi_2}{\mathcal{R}, v \sqsubset u_2 \quad \Gamma, A_1 \vee A_2 : u, A_2 : u_2 \vdash \Delta, C : u_2}}{\mathcal{R}, \Gamma, A_1 \vee A_2 : u \vdash \Delta, C : v}\ R_\wedge$$

The translation of the conclusion is the following nested sequent $\mathcal{S}\{\Gamma_u \vdash \Delta_u, \Lambda_u(\Gamma_v \vdash \Delta_v, \Lambda_v)\}$. The translation of the premises are nested sequents $\mathcal{S}\{\Gamma^1_u \vdash \Delta^1_u, \Lambda^1_u\}$ and $\mathcal{S}\{\Gamma^2_u \vdash \Delta^2_u, \Lambda^2_u\}$ for which, by induction hypothesis, we have proofs Π'_1 and Π'_1. Since u_1 and u_2 are fresh and $v \sqsubset u_1 \in \mathcal{R}$ and $v \sqsubset u_2 \in \mathcal{R}$, u_1 and u_2 both represent a nesting level one deep w.r.t. v. Also, since $u \overset{\mathcal{R}}{\rightsquigarrow} v$ for the application of the rule, v represents a nesting level deeper than u (they could be equal).

Therefore, by Definition 5.3, we have:

$$\Gamma_u = \Gamma^1_u, A_1 \vee A_2 \quad \Delta_u = \Delta^1_u = \Delta^2_u \quad \Lambda^1_u = \Lambda_u(\Gamma^1_v \vdash \Delta^1_v, \Lambda^1_v([A_1 \vdash C])$$
$$\Gamma^2_u = \Gamma^1_u$$
$$\Lambda^2_u = \Lambda_u(\Gamma^2_v \vdash \Delta^2_v, \Lambda^2_v([A_2 \vdash C]) \quad \Lambda_v = \Lambda^1_v = \Lambda^2_v \quad \Delta_v = \Delta^1_v, C = \Delta^2_v, C.$$

We can then conclude with an instance of $NL^B_\vee$ and possible multiple lift rule application as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Pi'_1}{\mathcal{S}\{\Gamma^1_u \vdash \Delta^1_u, \Lambda^1_u\}}}{\mathcal{S}\{\Gamma^1_u \vdash \Delta^1_u, \Lambda_u(\Gamma^1_v \vdash \Delta^1_v, \Lambda^1_v([A_1 \vdash C])\}} \quad \dfrac{\dfrac{\Pi'_2}{\mathcal{S}\{\Gamma^2_u \vdash \Delta^2_u, \Lambda^2_u\}}}{\mathcal{S}\{\Gamma^2_u \vdash \Delta^2_u, \Lambda_u(\Gamma^2_v \vdash \Delta^2_v, \Lambda^2_v([A \vdash C])\}}}{\mathcal{S}\{\Gamma^1_u \vdash \Delta^2_u, \Lambda_u(\Gamma^2_v, A_1 \vee A_2 \vdash \Delta^2_v, C, \Lambda^2_v)\}}\ NL^B_\vee}{\dfrac{\mathcal{S}\{\Gamma^1_u, A_1 \vee A_2 \vdash \Delta^2_u, \Lambda_u(\Gamma^2_v \vdash \Delta^2_v, C, \Lambda^2_v)\}}{\mathcal{S}\{\Gamma_u \vdash \Delta_u, \Lambda_u(\Gamma_v \vdash \Delta_v, \Lambda_v)\}}}\ lift*$$

If u = v then there is no lift applied.

Case $L^1_{\approx}$ Since there is no label modification we have

$$\dfrac{\begin{array}{c}\Pi_1\\ \hdashline \mathcal{R}, \Gamma, A \approx A : 0 \vdash \Delta\end{array}}{\mathcal{R}, \Gamma \vdash \Delta}\, L^1_{\approx}$$

$$\downarrow$$

$$\dfrac{\begin{array}{c}\Pi_1{}'\\ \hdashline \mathcal{S}\{\Gamma_0, A \approx A \vdash \Delta_0, \Lambda_0\}\end{array}}{\mathcal{S}\{\Gamma_0 \vdash \Delta_0, \Lambda_0\}}\, NL^1_{\approx}$$

Case $aL^2_{\approx}$ Since there is no label modification we have

$$\dfrac{\begin{array}{c}\Pi_1\\ \hdashline \mathcal{R}, \Gamma, A \approx B : u, B \approx A : u \vdash \Delta\end{array}}{\mathcal{R}, \Gamma, A \approx B : u \vdash \Delta}\, L^1_{\approx}$$

$$\downarrow$$

$$\dfrac{\begin{array}{c}\Pi_1{}'\\ \hdashline \mathcal{S}\{\Gamma_u, A \approx B, B \approx A \vdash \Delta_u, \Lambda_u\}\end{array}}{\mathcal{S}\{\Gamma_u, A \approx B \vdash \Delta_u, \Lambda_u\}}\, NL^1_{\approx}$$

Case $L^2_{\approx}$ This case is very similar to case $L_{\supset}$.

$$\dfrac{\begin{array}{c}\Pi_1\\ \hdashline \mathcal{R}, \Gamma \vdash \Delta, B : v\end{array} \quad \begin{array}{c}\Pi_2\\ \hdashline \mathcal{R}u, \Gamma, A : v \vdash \Delta\end{array}}{\mathcal{R}, \Gamma, A \approx B : u \vdash \Delta}\, L^2_{\approx}(u \overset{\mathcal{R}}{\rightsquigarrow} v)$$

$$\downarrow$$

$$\dfrac{\dfrac{\Pi_1'}{\mathcal{S}\{\Gamma_u \vdash \Delta_u, \Lambda_u(\Gamma_v \vdash \Delta_v, \Lambda_v, B)\}} \quad \dfrac{\Pi_2'}{\mathcal{S}\{\Gamma_u \vdash \Delta_u, \Lambda_u(\Gamma_v, A \vdash \Delta_v, \Lambda_v)\}}}{\dfrac{\mathcal{S}\{\Gamma_u \vdash \Delta_u, \Lambda_u(\Gamma_v, A \approx B \vdash \Delta_v, \Lambda_v)\}}{\mathcal{S}\{\Gamma_u, A \approx B \vdash \Delta_u, \Lambda_u(\Gamma_v \vdash \Delta_v, \Lambda_v)\}}\,\text{lift}}\,\text{NL}^2_{\approx}$$

Case $L^3_{\approx}$ This case is very similar to case $L_{\supset}$.

$$\dfrac{\dfrac{\Pi}{\mathcal{R}u, \Gamma, A \approx B : u, C \approx D : v, A \otimes C \approx B \otimes D : v \vdash \Delta}}{\mathcal{R}, \Gamma, A \approx B : u, C \approx D : v \vdash \Delta}\,L^3_{\approx}(u \overset{\mathcal{R}}{\rightsquigarrow} v)$$

$$\downarrow$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\Pi_1'}{\mathcal{S}\{\Gamma_u, A \approx B \vdash \Delta_u, \Lambda_u(\Gamma_v, C \approx D, A \otimes C \approx B \otimes D \vdash \Delta_v, \Lambda_v)\}}}{\mathcal{S}\{\Gamma_u, A \approx B, C \approx D \vdash \Delta_u, \Lambda_u(\Gamma_v, C \approx D, A \otimes C \approx B \otimes D \vdash \Delta_v, \Lambda_v)\}}\,\text{lift}}{\mathcal{S}\{\Gamma_u, A \approx B, C \approx D, A \otimes C \approx B \otimes D \vdash \Delta_u, \Lambda_u(\Gamma_v, C \approx D \vdash \Delta_v, \Lambda_v)\}}\,\text{lift}}{\mathcal{S}\{\Gamma_u, A \approx B, C \approx D \vdash \Delta_u, \Lambda_u(\Gamma_v, C \approx D \vdash \Delta_v, \Lambda_v)\}}\,\text{NL}^3_{\approx}$$

Since we work with sets of formulas, the duplication of formulas is implicit in the rules.

Case $L^3_{\approx}$ Similar to the previous case, and we do not need to manage the labels.

$$\dfrac{\dfrac{\Pi}{\mathcal{R}u, \Gamma, A \approx B : u, A \otimes A \approx B \otimes B : u \vdash \Delta}}{\mathcal{R}, \Gamma, A \approx B : u \vdash \Delta}\,L^3_{\approx}$$

$$\downarrow$$

$$\dfrac{\dfrac{\Pi_1'}{\mathcal{S}\{\Gamma_u, A \approx B, A \otimes A \approx B \otimes B \vdash \Delta_u, \Lambda_u\}}}{\mathcal{S}\{\Gamma_u, A \approx B \vdash \Delta_u, \Lambda_u\}}\,\text{NL}^{3*}_{\approx}$$

Case $L^r_{\approx}$ This case is very similar to the case of $L_{\supset}$.

$$\dfrac{\dfrac{\Pi}{\mathcal{R}u, \Gamma, A \approx B : u, D : v, D^A_B : v \vdash \Delta}}{\mathcal{R}, \Gamma, A \approx B : u, D : v \vdash \Delta} \, L^r_\approx(u \overset{\mathcal{R}}{\rightsquigarrow} v)$$

$$\downarrow$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\Pi_1{}'}{\mathcal{S}\{\Gamma_u, A \approx B \vdash \Delta_u, \Lambda_u(\Gamma_v, D, D^A_B \vdash \Delta_v, \Lambda_v)\}}}{\mathcal{S}\{\Gamma_u, A \approx B, D \vdash \Delta_u, \Lambda_u(\Gamma_v, D, D^A_B \vdash \Delta_v, \Lambda_v)\}} \text{lift}}{\mathcal{S}\{\Gamma_u, A \approx B, C \approx D, D^A_B \vdash \Delta_u, \Lambda_u(\Gamma_v, C \approx D \vdash \Delta_v, \Lambda_v)\}} \text{lift}}{\mathcal{S}\{\Gamma_u, A \approx B, D \vdash \Delta_u, \Lambda_u(\Gamma_v, D \vdash \Delta_v, \Lambda_v)\}} \, NL^r_\approx$$

Since we work with sets of formulas, the duplication of formulas is implicit in the rules. □

B. Appendix: From $\mathsf{L3}_{\mathsf{ISCI}}$ to $\mathsf{SN}_{\mathsf{ISCI}}$

LEMMA B.1. *Let* s *be a sequent in* $\mathsf{L3}_{\mathsf{ISCI}}$ *and let* x *and* y *be two labels in* s *such that* $x \subseteq y$*, then* $\mu(x) \overset{\mathcal{R}}{\rightsquigarrow} \mu(y)$ *in* $\mathsf{LS}(s)$*.*

PROOF: If y is maximal in s, then by Definition 4.1 we have a chain $0 \sqsubset \ldots \sqsubset \mu(x) \sqsubset \ldots \sqsubset \mu(y)$. Hence, $\mu(x) \overset{\mathcal{R}}{\rightsquigarrow} \mu(y)$. Otherwise, there is some maximal $z \in s$ such that $x \subseteq y \subseteq z$ and by Definition 4.1 we have a chain $0 \sqsubset \ldots \sqsubset \mu(x) \sqsubset \ldots \sqsubset \mu(y) \sqsubset \ldots \sqsubset \mu(z)$. Hence, $\mu(x) \overset{\mathcal{R}}{\rightsquigarrow} \mu(y)$. □

THEOREM B.2. *Any standard* $\mathsf{L3}_{\mathsf{ISCI}}$*-proof can be translated into an* $\mathsf{SN}_{\mathsf{ISCI}}$*-proof.*

PROOF: The proof is by induction on the height of the proof in $\mathsf{L3}_{\mathsf{ISCI}}$, with a case analysis on the last rule r applied in the $\mathsf{L3}_{\mathsf{ISCI}}$ proof. We write $\mathcal{R}$ and $\mathcal{R}_i$ for the sets of relational atoms obtained from the translation of the conclusion and of the i^{th} premiss of r respectively.

Base case id**:**

This case subsumes both id_p and $id_\approx$. By Lemma B.1, since $x \subseteq xy$, we have $\mu(x) \overset{\mathcal{R}}{\rightsquigarrow} \mu(y)$. Hence,

$$\dfrac{}{\Gamma, \mathrm{A} : \mathrm{x} \vdash \Delta, \mathrm{A} : \mathrm{xy}}\,\mathrm{id} \quad \leadsto \quad \dfrac{}{\mathcal{R}, \mathsf{LS}(\Gamma), \mathrm{A} : \mu(\mathrm{x}) \vdash \mathsf{LS}(\Delta), \mathrm{A} : \mu(\mathrm{xy})}\,\mathrm{id}$$

Base case $L_{\bot}^{B}$:

Similar to the base case id.

$$\dfrac{}{\Gamma, \bot : \mathrm{x} \vdash \Delta, \mathrm{A} : \mathrm{xy}}\,\mathrm{L}_{\bot}^{\mathrm{B}} \quad \leadsto \quad \dfrac{}{\mathcal{R}, \mathsf{LS}(\Gamma), \bot : \mu(\mathrm{x}) \vdash \mathsf{LS}(\Delta), \mathrm{A} : \mu(\mathrm{xy})}\,\mathrm{id}$$

Base case $L_{\bot}^{K}$:

$$\dfrac{}{\Gamma, \bot : \mathrm{x} \vdash \Delta}\,\mathrm{L}_{\bot}^{\mathrm{K}} \quad \leadsto \quad \dfrac{}{\mathcal{R}, \mathsf{LS}(\Gamma), \mathrm{A} : \mu(\mathrm{x}) \vdash \mathsf{LS}(\Delta)}\,\mathrm{L}_{\bot}^{\mathrm{K}}$$

Case $R_{\supset}$:

Since the $\mathsf{L3}_{\mathsf{ISCI}}$ proof is standard (Definition 4.2) , a must be a fresh singleton $\{ i \}$ (with $i = 3k$ for some k) such that i is the greatest natural number in the premiss. Therefore, $\mu(\mathrm{a}) = \mu(\mathrm{xa}) = i$. We now show that $\mathcal{R}_1 = \mathcal{R} \cup \{ \mu(\mathrm{x}) \sqsubset \mu(\mathrm{xa}) \}$. Since x is right maximal, by right connectedness we have that x is maximal in the whole conclusion, which by the freshness of a implies that xa is maximal in the whole premiss. Therefore, $\mathcal{R}_1 = \mathcal{R} - \mathcal{R}(\mathrm{x}) \cup \mathcal{R}(\mathrm{xa})$. Let us write a as the ordered set $\{ i_1 < i_2 < \ldots < i_n \}$. Then, $\mathrm{xa} = \{ i_1 < i_2 < \ldots < i_n < i \}$. Thus, we get $\mathcal{R}(\mathrm{x}) = \{ 0 \sqsubset i_1, i_1 \sqsubset i_2, \ldots, i_{n-1} \sqsubset i_n \}$ and $\mathcal{R}(\mathrm{xa}) = \{ 0 \sqsubset i_1, i_1 \sqsubset i_2, \ldots, i_{n-1} \sqsubset i_n, i_n \sqsubset i \}$, where $i_n = \mu(\mathrm{x})$ and $i = \mu(\mathrm{xa})$. Hence, $\mathcal{R}_1 = \mathcal{R} \cup \{ \mu(\mathrm{x}) \sqsubset \mu(\mathrm{xa}) \}$ and we can conclude the translated proof $\mathsf{LS}(\Pi_1)$ obtained by induction hypothesis with an instance of $R_{\supset}$ in $\mathsf{SN}_{\mathsf{ISCI}}$ as follows:

$$\dfrac{\dfrac{\Pi_1}{\Gamma, \mathrm{A} : \mathrm{a} \vdash \Delta, \mathrm{A} \supset \mathrm{B} : \mathrm{xa}}}{\Gamma \vdash \Delta, \mathrm{A} \supset \mathrm{B} : \mathrm{x}}\,\mathrm{R}_{\supset} \quad \leadsto \quad \dfrac{\dfrac{\dfrac{\mathsf{LS}(\Pi_1)}{\mathcal{R}, \mu(\mathrm{x}) \sqsubset \mu(\mathrm{xa}), \mathsf{LS}(\Gamma), \mathrm{A} : \mu(\mathrm{a}) \vdash \mathsf{LS}(\Delta), \mathrm{B} : \mu(\mathrm{xa})}\,(\mathcal{R}_1)}{\mathcal{R}_1, \mathsf{LS}(\Gamma), \mathrm{A} : \mu(\mathrm{a}) \vdash \mathsf{LS}(\Delta), \mathrm{B} : \mu(\mathrm{xa})}}{\mathcal{R}, \mathsf{LS}(\Gamma) \vdash \mathsf{LS}(\Delta), \mathrm{A} \supset \mathrm{B} : \mu(\mathrm{x})}\,\mathrm{R}_{\supset}$$

Case $L_{\supset}$:

Since no new label is created then both premises share the same label relations ($\mathcal{R}_1 = \mathcal{R}_2 = \mathcal{R}$). As $\mathrm{x} \subseteq \mathrm{xy}$, by application of Lemma B.1, we know that $\mu(x) \sqsubset \mu(xy)$. Therefore the following transformation is valid.

$$\dfrac{\dfrac{\Pi_1}{\Gamma, A \supset B : x \vdash \Delta, A : xy} \qquad \dfrac{\Pi_2}{\Gamma, B : xy \vdash \Delta}}{\Gamma, A \supset B : x \vdash \Delta} L_\supset$$

$$\downarrow$$

$$\dfrac{\dfrac{\mathsf{LS}(\Pi_1)}{\mathcal{R}_1, \mathsf{LS}(\Gamma), A \supset B : \mu(x) \vdash \mathsf{LS}(\Delta), A : \mu(xy)} \qquad \dfrac{\mathsf{LS}(\Pi_2)}{\mathcal{R}_2, \mathsf{LS}(\Gamma), B : \mu(xy) \vdash \mathsf{LS}(\Delta)}}{\mathcal{R}, \mathsf{LS}(\Gamma), A \supset B : \mu(x) \vdash \mathsf{LS}(\Delta)} L_\supset$$

Case $L_\wedge$**:**

Since no new label is created the translation is the following:

$$\dfrac{\dfrac{\Pi_1}{\Gamma, A : x, B : x \vdash \Delta}}{\Gamma, A \wedge B : x \vdash \Delta} L_\wedge \quad \rightsquigarrow \quad \dfrac{\dfrac{\mathsf{LS}(\Pi_1)}{\mathcal{R}_1, \mathsf{LS}(\Gamma), A : \mu(x), B : \mu(x) \vdash \mathsf{LS}(\Delta)}}{\mathcal{R}, \mathsf{LS}(\Gamma), A \wedge B : \mu(x) \vdash \mathsf{LS}(\Delta)} L_\wedge(\mathcal{R}_1 = \mathcal{R})$$

Case $R_\wedge$**:**

The translation is straightforward since we do not worry about the labels.

$$\dfrac{\dfrac{\Pi_1}{\Gamma \vdash \Delta, A : x} \qquad \dfrac{\Pi_2}{\Gamma \vdash \Delta, B : x}}{\Gamma \vdash \Delta, A \wedge B : x} R_\wedge$$

$$\downarrow$$

$$\dfrac{\dfrac{\mathsf{LS}(\Pi_1)}{\mathcal{R}_1, \mathsf{LS}(\Gamma) \vdash \mathsf{LS}(\Delta), A : \mu(x)} \qquad \dfrac{\mathsf{LS}(\Pi_2)}{\mathcal{R}_2, \mathsf{LS}(\Gamma) \vdash \mathsf{LS}(\Delta), B : \mu(x)}}{\mathcal{R}, \mathsf{LS}(\Gamma) \vdash \mathsf{LS}(\Delta), A \wedge B : \mu(x)} R_\wedge(\mathcal{R}_1 = \mathcal{R}_2 = \mathcal{R})$$

Case $R_\vee$**:**

Similar to case $L_\wedge$.

$$\dfrac{\dfrac{\Pi_1}{\Gamma \vdash \Delta, \mathrm{A} : \mathrm{x}, \mathrm{B} : \mathrm{x}}}{\Gamma \vdash \Delta, \mathrm{A} \vee \mathrm{B} : \mathrm{x}}\,\mathrm{R}_\vee \quad \rightsquigarrow \quad \dfrac{\dfrac{\mathsf{LS}(\Pi_1)}{\mathcal{R}_1, \mathsf{LS}(\Gamma) \vdash \mathsf{LS}(\Delta), \mathrm{A} : \mu(\mathrm{x}), \mathrm{B} : \mu(\mathrm{x})}}{\mathcal{R}, \mathsf{LS}(\Gamma) \vdash \mathsf{LS}(\Delta), \mathrm{A} \vee \mathrm{B} : \mu(\mathrm{x})}\,\mathrm{R}_\vee(\mathcal{R}_1 = \mathcal{R})$$

Case $\mathrm{L}_\vee^{\mathrm{K}}$:

Similar to case $\mathrm{R}_\wedge$.

$$\dfrac{\dfrac{\Pi_1}{\Gamma, \mathrm{A} : \mathrm{x} \vdash \Delta} \quad \dfrac{\Pi_2}{\Gamma, \mathrm{B} : \mathrm{x} \vdash \Delta}}{\Gamma \mathrm{A} \vee \mathrm{B} : \mathrm{x} \vdash \Delta}\,\mathrm{L}_\vee^{\mathrm{K}}$$

$$\downarrow$$

$$\dfrac{\dfrac{\mathsf{LS}(\Pi_1)}{\mathcal{R}_1, \mathsf{LS}(\Gamma), \mathrm{A} : \mu(\mathrm{x}) \vdash \mathsf{LS}(\Delta)} \quad \dfrac{\mathsf{LS}(\Pi_2)}{\mathcal{R}_2, \mathsf{LS}(\Gamma), \mathrm{B} : \mu(\mathrm{x}) \vdash \mathsf{LS}(\Delta)}}{\mathcal{R}, \mathsf{LS}(\Gamma), \mathrm{A} \vee \mathrm{B} : \mu(\mathrm{x}) \vdash \mathsf{LS}(\Delta)}\,\mathrm{L}_\vee^{\mathrm{K}}(\mathcal{R}_1 = \mathcal{R}_2 = \mathcal{R})$$

Case $\mathrm{L}_\vee^{\mathrm{B}}$:

Similar to case $\mathrm{R}_\supset$, except with two premises.
Since the $\mathsf{L3}_{\mathsf{ISCI}}$ proof is standard (Definition 4.2), a_1 and a_2 must be fresh singletons, respectively $\{ i \}$ and $\{ j \}$ (with $i = 3k + 1$ and $j = 3k + 2$ for some k), such that i and j are the greatest natural numbers in their respective premises. Therefore, $\mu(\mathrm{a}_1) = \mu(\mathrm{xa}_1) = \mu(\mathrm{xya}_1) = i$ and $\mu(\mathrm{a}_2) = \mu(\mathrm{xa}_2) = \mu(\mathrm{xa}_2) = i$. We now show that $\mathcal{R}_1 = \mathcal{R} \cup \{ \mu(\mathrm{xy}) \sqsubset \mu(\mathrm{xya}_1) \}$ and $\mathcal{R}_2 = \mathcal{R} \cup \{ \mu(\mathrm{xy}) \sqsubset \mu(\mathrm{xya}_2) \}$. Since xy is right maximal, by right connectedness we have that xy is maximal in the whole conclusion, which by the freshness of a_1 and a_2 implies that xya_1 and xya_1 are maximal in their respective premises. Therefore, $\mathcal{R}_1 = \mathcal{R} - \mathcal{R}(\mathrm{xy}) \cup \mathcal{R}(\mathrm{xya})$ and $\mathcal{R}_2 = \mathcal{R} - \mathcal{R}(\mathrm{xy}) \cup \mathcal{R}(\mathrm{xya}_2)$.

Let us write a_1 and a_2 as the ordered set $\{ i_1 < i_2 < \ldots < i_n \}$. Then, $\mathrm{xya}_1 = \{ i_1 < i_2 < \ldots < i_n < i \}$ and $\mathrm{xya}_2 = \{ i_1 < i_2 < \ldots < i_n < j \}$. Thus, we get $\mathcal{R}(\mathrm{xy}) = \{ 0 \sqsubset i_1, i_1 \sqsubset i_2, \ldots, i_{n-1} \sqsubset i_n \}$ and $\mathcal{R}(\mathrm{xya}_1) = \{ 0 \sqsubset i_1, i_1 \sqsubset i_2, \ldots, i_{n-1} \sqsubset i_n, i_n \sqsubset i \}$, where $i_n = \mu(\mathrm{xy})$ and $i = \mu(\mathrm{xa})$ and $\mathcal{R}(\mathrm{xya}_2) = \{ 0 \sqsubset i_1, i_1 \sqsubset i_2, \ldots, i_{n-1} \sqsubset i_n, i_n \sqsubset j \}$, where $i_n = \mu(\mathrm{xy})$ and $j = \mu(\mathrm{xa})$. Hence, $\mathcal{R}_1 = \mathcal{R} \cup \{ \mu(\mathrm{xy}) \sqsubset \mu(\mathrm{xya}_1) \}$ and $\mathcal{R}_2 = \mathcal{R} \cup \{ \mu(\mathrm{xy}) \sqsubset \mu(\mathrm{xya}_2) \}$. Now since by applying

Lemma B.1 we have $\mu(x) \sqsubset \mu(xy)$ we can consider the translated proofs $\mathsf{LS}(\Pi_1)$ and $\mathsf{LS}(\Pi_1)$ obtained by induction hypothesis with an instance of $L^B_\vee$. in $\mathsf{SN}_{\mathsf{ISCI}}$ as follows:

$$\dfrac{\dfrac{\Pi_1}{\Gamma, A_1 : xa_1 \vdash \Delta, C : xya_1} \quad \dfrac{\Pi_2}{\Gamma, A_2 : xa_2 \vdash \Delta, C : xya_2}}{\Gamma A \vee B : x \vdash \Delta, C : xy} L^B_\vee$$

$$\downarrow$$

$$\Pi_3 \left\{ \dfrac{\dfrac{\mathsf{LS}(\Pi_1)}{\mathcal{R}, \mu(xy) \sqsubset \mu(xya_1), \mathsf{LS}(\Gamma), A_1 : \mu(xa_1) \vdash \mathsf{LS}(\Delta), C : \mu(xya_1)}}{\mathcal{R}_1, \mathsf{LS}(\Gamma), A_1 : \mu(xa_1) \vdash \mathsf{LS}(\Delta), C : \mu(xya_1)} \right.$$

$$\Pi_4 \left\{ \dfrac{\dfrac{\mathsf{LS}(\Pi_2)}{\mathcal{R}, \mu(xy) \sqsubset \mu(xya_2), \mathsf{LS}(\Gamma), A_2 : \mu(xa_2) \vdash \mathsf{LS}(\Delta), C : \mu(xya_2)}}{\mathcal{R}_2, \mathsf{LS}(\Gamma), A_2 : \mu(xa_2) \vdash \mathsf{LS}(\Delta), C : \mu(xya_2)} \right.$$

$$\dfrac{\Pi_3 \qquad \Pi_4}{\mathcal{R}, \mathsf{LS}(\Gamma), A \vee B : \mu(x) \vdash \mathsf{LS}(\Delta), C : \mu(xya_1)} L^B_\vee$$

Case $L^1_\approx$:

Since $\mu(\emptyset) = 0$ we have the following translation

$$\dfrac{\dfrac{\Pi_1}{\Gamma, A \approx A : \emptyset \vdash \Delta}}{\Gamma \vdash \Delta} L^1_\approx \quad \rightsquigarrow \quad \dfrac{\dfrac{\mathsf{LS}(\Pi_1)}{\mathcal{R}_1, \mathsf{LS}(\Gamma), A \approx A : 0 \vdash \mathsf{LS}(\Delta)}}{\mathcal{R}, \mathsf{LS}(\Gamma) \vdash \mathsf{LS}(\Delta)} L^1_\approx(\mathcal{R}_1 = \mathcal{R})$$

Case $aL^2_\approx$:

Since we do not change any label we have

$$\dfrac{\dfrac{\Pi_1}{\Gamma, A \approx B : x, B \approx A : x \vdash \Delta}}{\Gamma, A \approx B : x \vdash \Delta} aL^2_\approx$$

$$\downarrow$$

$$\rightsquigarrow \quad \dfrac{\dfrac{\Pi_1}{\mathcal{R}_1, \mathsf{LS}(\Gamma), A \approx B : \mu(x), B \approx A : \mu(x) \vdash \mathsf{LS}(\Delta)}}{\mathcal{R}, \mathsf{LS}(\Gamma), A \approx B : \mu(x) \vdash \mathsf{LS}(\Delta)} aL^2_\approx(\mathcal{R}_1 = \mathcal{R})$$

Case $\mathrm{L}_{\approx}^{2}$**:**

Similar to case $\mathrm{L}_{\supset}$.

$$\dfrac{\dfrac{\Pi_1}{\Gamma, A \approx B : x \vdash \Delta, B : xy} \quad \dfrac{\Pi_2}{\Gamma, A \approx B : x, A : xy \vdash \Delta}}{\Gamma, A \approx B : x \vdash \Delta}\ \mathrm{L}_{\approx}^{2}$$

$$\downarrow$$

$$\dfrac{\dfrac{\mathsf{LS}(\Pi_1)}{\mathcal{R}_1, \mathsf{LS}(\Gamma), A \approx B : \mu(x) \vdash \mathsf{LS}(\Delta), B : \mu(xy)} \quad \dfrac{\mathsf{LS}(\Pi_2)}{\mathcal{R}_2, \mathsf{LS}(\Gamma), A \approx B : \mu(x), A : \mu(xy) \vdash \mathsf{LS}(\Delta) :}}{\mathcal{R}, \mathsf{LS}(\Gamma), A \approx B : \mu(x) \vdash \mathsf{LS}(\Delta)}\ \mathrm{L}_{\approx}^{2}$$

with $\mathcal{R}_1 = \mathcal{R}_2 = \mathcal{R}$.

Case $\mathrm{L}_{\approx}^{3}$**:**

Since $x \subseteq xy$ and $y \subseteq xy$, by application of Lemma B.1 we have $\mu(x) \sqsubset \mu(xy)$ and $\mu(y) \sqsubset \mu(xy)$, and we have the following translation

$$\dfrac{\dfrac{\Pi_1}{\Gamma, A \approx B : x, C \approx D : y, A \otimes C \approx B \otimes D : xy \vdash \Delta}}{\Gamma, A \approx B : x, C \approx D : y \vdash \Delta}\ \mathrm{L}_{\approx}^{3}$$

$$\downarrow$$

$$\dfrac{\dfrac{\mathsf{LS}(\Pi_1)}{\mathcal{R}_1, \mathsf{LS}(\Gamma), A \approx B : \mu(x), C \approx D : \mu(x), A \otimes C \approx B \otimes D : \mu(xy) \vdash \mathsf{LS}(\Delta)}}{\mathcal{R}, \mathsf{LS}(\Gamma), A \approx B : \mu(x), C \approx D : \mu(x) \vdash \mathsf{LS}(\Delta)}\ \mathrm{L}_{\approx}^{3}(\mathcal{R}_1 = \mathcal{R})$$

Case $\mathrm{L}_{\approx}^{3*}$**:**

Similar to the previous case.

$$\dfrac{\dfrac{\Pi_1}{\Gamma, A \approx B : x, A \otimes A \approx B \otimes B : x \vdash \Delta}}{\Gamma, A \approx B : x \vdash \Delta}\ \mathrm{L}_{\approx}^{3*}$$

$$\downarrow$$

$$\dfrac{\dfrac{\mathsf{LS}(\Pi_1)}{\mathcal{R}_1, \mathsf{LS}(\Gamma), A \approx B : \mu(x), A \otimes A \approx B \otimes B : \mu(x) \vdash \mathsf{LS}(\Delta)}}{\mathcal{R}, \mathsf{LS}(\Gamma), A \approx B : \mu(x) \vdash \mathsf{LS}(\Delta)} \; L^{3*}_{\approx} (\mathcal{R}_1 = \mathcal{R})$$

Case $L^{r}_{\approx}$:

Since $x \subseteq xy$ and $y \subseteq xy$, by application of Lemma B.1 we have $\mu(x) \sqsubset \mu(xy)$ and $\mu(y) \sqsubset \mu(xy)$, and we have the following translation

$$\dfrac{\dfrac{\Pi_1}{\Gamma, A \approx B : x, D : y, D^{A}_{B} : xy \vdash \Delta}}{\Gamma, A \approx B : x, D : y \vdash \Delta} \; L^{3*}_{\approx}$$

$$\downarrow$$

$$\dfrac{\dfrac{\mathsf{LS}(\Pi_1)}{\mathcal{R}_1, \mathsf{LS}(\Gamma), A \approx B : \mu(x), D : \mu(y), D^{A}_{B} : \mu(xy) \vdash \mathsf{LS}(\Delta)}}{\mathcal{R}, \mathsf{LS}(\Gamma), A \approx B : \mu(x), D : \mu(y) \vdash \mathsf{LS}(\Delta)} \; L^{3*}_{\approx} (\mathcal{R}_1 = \mathcal{R})$$

□

C. Appendix: From G3$_{\mathsf{ISCI}}$ to L3$_{\mathsf{ISCI}}$

In this appendix we give the full proof of Theorem 8.2 for the translation of G3$_{\mathsf{ISCI}}$-proofs into L3$_{\mathsf{ISCI}}$-proofs. In fact we show a more general result that also includes the translation cases for the maximal variants of the rules given in Figure 6. Such rules only introduce (in their premises) active formulas whose label is right maximal and therefore help reduce the number of choices for the labels to introduce in a bottom-up application of the rules. An immediate corollary of such translations is that restricting the proof-search process to the class of right maximality preserving proofs does not change the set of provable formulas, i.e. replacing the original rules with the right maximal ones still yields a calculus that is complete w.r.t. the Kripke and TB semantics of ISCI.

THEOREM C.1. G3$_{\mathsf{ISCI}}$*-proofs translate into* L3$_{\mathsf{ISCI}}$*-proofs.*

PROOF: By induction on the height of G3$_{\mathsf{ISCI}}$-proofs. We start with a G3$_{\mathsf{ISCI}}$-proof Π of a sequent $\Gamma \vdash \Delta$. We transform Π into a L3$_{\mathsf{ISCI}}$-proof of

$$\frac{\Gamma, A_1 : xa_1 \vdash \Delta, C : xya_1 \quad \Gamma, A_2 : xa_2 \vdash \Delta, C : xya_2}{\Gamma, A_1 \vee A_2 : x \vdash \Delta, C : xy}\ \nu L^{B}_{\vee}(xy \in \lceil \Delta, C : xy \rceil)$$

$$\frac{\Gamma \vdash \Delta, A : xy \quad \Gamma, B : xy \vdash \Delta}{\Gamma, A \supset B : x \vdash \Delta}\ \nu L_{\supset}(xy \in \lceil \Delta \rceil) \qquad \frac{\Gamma, A \approx A : x \vdash \Delta}{\Gamma \vdash \Delta}\ \nu L^{1}_{\approx}(x \in \lceil \Delta \rceil)$$

$$\frac{\Gamma, A \approx B : x \vdash \Delta, B : xy \quad \Gamma, A \approx B : x, A : xy \vdash \Delta}{\Gamma, A \approx B : x \vdash \Delta}\ \nu L^{2}_{\approx}(xy \in \lceil \Delta \rceil)$$

$$\frac{\Gamma, A \approx B : x, C \approx D : y, A \otimes C \approx B \otimes D : xyz \vdash \Delta}{\Gamma, A \approx B : x, C \approx D : y \vdash \Delta}\ \nu L^{3}_{\approx}(xyz \in \lceil \Delta \rceil)$$

$$\frac{\Gamma, A \approx B : x, D : y, D^{A}_{B} : xyz \vdash \Delta}{\Gamma, A \approx B : x, D : y \vdash \Delta}\ \nu L^{r}_{\approx}(xyz \in \lceil \Delta \rceil)$$

Figure 6: Maximal Rules for $\mathsf{L3}_{\mathsf{ISCI}}$.

$\mathsf{L}(\Gamma \vdash \Delta, \vec{n})$, where $n = |\Gamma|$. For convenience, we write a_i as a shorthand for the singleton $\{n + i\}$. In the base case, we give a direct translation of the axioms of $\mathsf{G3}_{\mathsf{ISCI}}$. In the inductive case, we suppose that Π has height $h + 1$ and that it ends with a rule r of arity k. We first apply the induction hypothesis to all of the $\mathsf{G3}_{\mathsf{ISCI}}$-subproofs Π_j $(1 \leqslant j \leqslant k)$ to get the corresponding $\mathsf{L3}_{\mathsf{ISCI}}$-proofs Π'_j. If necessary, we perform some label substitutions σ_j in the subproofs Π_j using Lemma 3.2 to get the new subproofs $\Pi'_j\sigma_j$ and further extend them to new conclusions s_j with weakening steps r_j when required. Finally, the resulting subproofs are combined into a $\mathsf{L3}_{\mathsf{ISCI}}$-proof of $\mathsf{L}(\Gamma \vdash \Delta, \vec{n})$ that ends with the rule r.

The proof principle is depicted below:

$$\frac{\Pi_1 \quad \cdots \quad \Pi_k}{\Gamma \vdash \Delta}\ r \quad \rightsquigarrow \quad \frac{\dfrac{\Pi'_1\sigma_1}{s_1}\ r_1 \quad \cdots \quad \dfrac{\Pi'_k\sigma_k}{s_k}\ r_k}{\Gamma : \vec{n} \vdash \Delta : \check{n}}\ r$$

The soundness of label substitutions in $\mathsf{L3}_{\mathsf{ISCI}}$ is proven in Lemma C.13. The weakening admissibility property is proven in Lemma C.14. Given a proof Π and a labeled sequent $\Gamma \vdash \Delta$, we write $\Pi + \Gamma \vdash \Delta$ for the proof obtained from Π by appending Γ and Δ respectively to left-hand and right-hand side of all of the sequents occurring in Π.

Base case $\mathrm{id_p}$**:**

$$\dfrac{}{\Gamma, p \vdash \Delta, p}\,\mathrm{id_p} \quad \rightsquigarrow \quad \dfrac{}{\Gamma : \vec{n}, p : a_1 \vdash \Delta : \check{n}a_1, p : \check{n}a_1}\,\mathrm{id_p}$$

Base case $\mathrm{id}_{\approx}$**:**

$$\dfrac{}{\Gamma, A \approx B \vdash \Delta, A \approx B}\,\mathrm{id}_{\approx} \quad \rightsquigarrow \quad \dfrac{}{\Gamma : \vec{n}, A \approx B : a_1 \vdash \Delta : \check{n}a_1, A \approx B : \check{n}a_1}\,\mathrm{id}_{\approx}$$

Base case $\mathrm{L}^{\mathrm{K}}_{\bot}$**:**

$$\dfrac{}{\Gamma, \bot \vdash \Delta}\,\mathrm{L}^{\mathrm{K}}_{\bot} \quad \rightsquigarrow \quad \dfrac{}{\Gamma : \vec{n}, \bot : a_1 \vdash \Delta : \check{n}a_1}\,\mathrm{L}^{\mathrm{K}}_{\bot}$$

Base case $\mathrm{L}^{\mathrm{B}}_{\bot}$**:**

$$\dfrac{}{\Gamma, \bot \vdash \Delta, C}\,\mathrm{L}^{\mathrm{B}}_{\bot} \quad \rightsquigarrow \quad \dfrac{}{\Gamma : \vec{n}, \bot : a_1 \vdash \Delta : \check{n}a_1, C : \check{n}a_1}\,\mathrm{L}^{\mathrm{B}}_{\bot}$$

Case $\mathrm{L}_{\wedge}$**:** We start with a proof Π whose height is $h + 1$ and apply the induction hypothesis on the subproof Π_1 which has height h to get Π'_1. We then use Lemma 3.2 to replace a_2 with a_1.

$$\dfrac{\begin{array}{c}\Pi_1 \\ \hdashline \Gamma, A, B \vdash \Delta\end{array}}{\Gamma, A \wedge B \vdash \Delta}\,\mathrm{L}_{\wedge} \quad \rightsquigarrow \quad \begin{array}{c}\Pi'_1 \\ \hdashline \Gamma : \vec{n}, A : a_1, B : a_2 \vdash \Delta : \check{n}a_1a_2\end{array}$$

$$\rightsquigarrow \quad \begin{array}{c}\Pi'_1[a_1/a_2] \\ \hdashline \Gamma : \vec{n}, A : a_1, B : a_1 \vdash \Delta : \check{n}a_1\end{array}$$

We finally conclude with an application of $\mathrm{L}_{\wedge}$ as follows:

$$\dfrac{\begin{array}{c}\Pi'_1[a_1/a_2] \\ \hdashline \Gamma : \vec{n}, A : a_1, B : a_1 \vdash \Delta : \check{n}a_1\end{array}}{\Gamma : \vec{n}, A \wedge B : a_1 \vdash \Delta : \check{n}a_1}\,\mathrm{L}_{\wedge}$$

Case $\mathrm{R}_{\wedge}$**:** We apply the induction hypothesis on the subproofs Π_1 and Π_2 to get Π'_1 and Π'_2. Then we conclude with $\mathrm{R}_{\wedge}$ as depicted below:

$$\dfrac{\dfrac{\Pi_1}{\Gamma \vdash \Delta, A} \quad \dfrac{\Pi_2}{\Gamma \vdash \Delta, B}}{\Gamma \vdash \Delta, A \wedge B}\, R_\wedge \quad \rightsquigarrow \quad \dfrac{\dfrac{\Pi'_1}{\Gamma : \vec{n} \vdash \Delta : \check{n}, A : \check{n}} \quad \dfrac{\Pi'_2}{\Gamma : \vec{n} \vdash \Delta : \check{n}, B : \check{n}}}{\Gamma : \vec{n} \vdash \Delta : \check{n}, A \wedge B : \check{n}}\, R_\wedge$$

Case $L_\vee^K$: We start with the following proof and apply the induction hypothesis to the subproofs Π_1 and Π_2 to get Π'_1 and Π'_2:

$$\dfrac{\dfrac{\Pi_1}{\Gamma, A \vdash \Delta} \quad \dfrac{\Pi_2}{\Gamma, B \vdash \Delta}}{\Gamma, A \vee B \vdash \Delta}\, L_\vee^K \quad \rightsquigarrow \quad \dfrac{\Pi'_1}{\Gamma : \vec{n}, A : a_1 \vdash \Delta : \check{n}a_1} \quad \dfrac{\Pi'_2}{\Gamma : \vec{n}, B : a_1 \vdash \Delta : \check{n}a_1}$$

We use Lemma 3.2 to replace a_1 with $\check{n}$ in both Π'_1 and Π'_2. Finally, we conclude with $L_\vee^K$:

$$\dfrac{\dfrac{\Pi'_1[\check{n}/a_1]}{\Gamma : \vec{n}, A : \check{n} \vdash \Delta : \check{n}} \quad \dfrac{\Pi'_2[\check{n}/a_1]}{\Gamma : \vec{n}, B : \check{n} \vdash \Delta : \check{n}}}{\Gamma : \vec{n}, A \vee B : \check{n} \vdash \Delta : \check{n}}\, L_\vee^K$$

Case $L_\vee^B$: We start with the following proof and apply the induction hypothesis to the subproofs Π_1 and Π_2 to get Π'_1 and Π'_2:

$$\dfrac{\dfrac{\Pi_1}{\Gamma, A \vdash C} \quad \dfrac{\Pi_2}{\Gamma, B \vdash C}}{\Gamma, A \vee B \vdash \Delta, C}\, L_\vee^B \quad \rightsquigarrow \quad \dfrac{\Pi'_1}{\Gamma : \vec{n}, A : a_1 \vdash C : \check{n}a_1} \quad \dfrac{\Pi'_2}{\Gamma : \vec{n}, B : a_1 \vdash C : \check{n}a_1}$$

We use Lemma 3.2 to replace a_1 with a_1a_2 in Π_1 and with a_1a_3 in Π'_2 to obtain the new proofs $\Pi'_1[a_1a_2/a_1]$ and $\Pi'_2[a_1a_3/a_1]$. We then use weakening admissibility (Lemma C.14) on $\Pi'_1[a_1a_2/a_1]$ and $\Pi'_2[a_1a_3/a_1]$ to add $\Delta : \check{n}a_1$ on the right-hand side of all the sequents occurring in $\Pi'_1[a_1a_2/a_1]$ and $\Pi'_2[a_1a_3/a_1]$. Finally, since a_2 and a_3 are fresh in the conclusion, we can conclude with $L_\vee^B$:

$$\dfrac{\dfrac{\Pi'_1[a_1a_2/a_1] + \vdash \Delta : \check{n}a_1}{\Gamma : \vec{n}, A : a_1a_2 \vdash \Delta : \check{n}a_1, C : \check{n}a_1a_2} \quad \dfrac{\Pi'_2[a_1a_3/a_1] + \vdash \Delta : \check{n}a_1}{\Gamma : \vec{n}, B : a_1a_3 \vdash \Delta : \check{n}a_1, C : \check{n}a_1a_3}}{\Gamma : \vec{n}, A \vee B : a_1 \vdash \Delta : \check{n}a_1, C : \check{n}a_1}\, L_\vee^B$$

with $\check{n}a_1 \in \lceil \Delta : \check{n}a_1 C : \check{n}a_1 \rceil$.

Remark C.2. The instance of $L^{B}_{\vee}$ preserves right connectedness and $\check{n}a_1 \in \lceil \Delta : \check{n}a_1, C : \check{n}a_1 \rceil$ implies $\check{n}a_1 \in \Delta : \check{n}a_1, C : \check{n}a_1$.

Case $R_\vee$**:** This case is similar to Case $R_\wedge$.

$$\dfrac{\begin{array}{c}\Pi_1\\ \hdashline \Gamma \vdash \Delta, A_1, A_2\end{array}}{\Gamma \vdash \Delta, A_1 \vee A_2} R_\vee \quad \rightsquigarrow \quad \dfrac{\begin{array}{c}\Pi'_1\\ \hdashline \Gamma : \vec{n} \vdash \Delta : \check{n}, A_1 : \check{n}, A_2 : \check{n}\end{array}}{\Gamma : \vec{n} \vdash \Delta : \check{n}, A_1 \vee A_2 : \check{n}} R_\vee$$

Case $L_\supset$**:** We apply the induction hypothesis on the subproofs Π_1 and Π_2 to obtain the proofs Π'_1 and Π'_2.

$$\dfrac{\begin{array}{c}\Pi_1\\ \hdashline \Gamma, A \supset B \vdash \Delta, A\end{array} \quad \begin{array}{c}\Pi_2\\ \hdashline \Gamma, B \vdash \Delta\end{array}}{\Gamma, A \supset B \vdash \Delta} L_\supset$$

$$\rightsquigarrow \quad \begin{array}{c}\Pi'_1\\ \hdashline \Gamma : \vec{n}, A \supset B : a_1 \vdash \Delta : \check{n}a_1, A : \check{n}a_1\end{array} \quad \begin{array}{c}\Pi'_2\\ \hdashline \Gamma : \vec{n}, B : a_1 \vdash \Delta : \check{n}a_1\end{array}$$

We use Lemma 3.2 to replace a_1 with $\check{n}a_1$ in Π'_2. Finally, since $\check{n}a_1$ is right maximal in the conclusion, we can apply $L_\supset$.

$$\dfrac{\begin{array}{c}\Pi'_1\\ \hdashline \Gamma : \vec{n}, A \supset B : a_1 \vdash \Delta : \check{n}a_1, A : \check{n}a_1\end{array} \quad \begin{array}{c}\Pi'_2[\check{n}a_1/a_1]\\ \hdashline \Gamma : \vec{n}, B : \check{n}a_1 \vdash \Delta : \check{n}a_1\end{array}}{\Gamma : \vec{n}, A \supset B : a_1 \vdash \Delta : \check{n}a_1} L_\supset(\check{n}a_1 \subseteq \Delta)$$

Remark C.3. The instance of $L_\supset$ preserves right connectedness in $\Pi'_2[\check{n}a_1/a_1]$. It also preserves right connectedness in Π'_1, even in the single-succedent restriction.

Case $R_\supset$**:** We apply the induction hypothesis on the subproof Π_1 to obtain the proof Π'_1 and we use weakening admissibility (Lemma C.14) to add $\Delta : \check{n}$ to the right-hand side of all of the sequents occurring in Π'_1. Since a_1 is fresh in the conclusion, we can apply $R_\supset$.

$$\dfrac{\genfrac{}{}{0pt}{}{\Pi_1}{\overline{\Gamma, A \vdash B}}}{\Gamma \vdash \Delta, A \supset B}\,R_{\supset} \quad \rightsquigarrow \quad \dfrac{\genfrac{}{}{0pt}{}{\Pi_1' + \vdash \Delta : \check{n}}{\overline{\Gamma : \vec{n}, A : a_1 \vdash \Delta : \check{n}, B : \check{n}a_1}}}{\Gamma : \vec{n} \vdash \Delta : \check{n}, A \supset B : \check{n}}\,R_{\supset}$$

Case $L^1_{\approx}$: We get Π_1' from Π_1 by induction hypothesis and replace a_1 with $\emptyset$ ((Lemma 3.2).

$$\dfrac{\genfrac{}{}{0pt}{}{\Pi_1}{\overline{\Gamma, A \approx A \vdash \Delta}}}{\Gamma \vdash \Delta}\,L^1_{\approx} \quad \rightsquigarrow \quad \genfrac{}{}{0pt}{}{\Pi_1'}{\overline{\Gamma : \vec{n}, A \approx A : a_1 \vdash \Delta : \check{n}a_1}} \quad \rightsquigarrow \quad \dfrac{\genfrac{}{}{0pt}{}{\Pi_1'[\emptyset/a_1]}{\overline{\Gamma : \vec{n}, A \approx A : \emptyset \vdash \Delta : \check{n}}}}{\Gamma : \vec{n} \vdash \Delta : \check{n}}\,L^1_{\approx}$$

Remark C.4. The instance of $L^1_{\approx}$ preserves right connectedness. The active formula is left minimal.

Case $\nu L^1_{\approx}$: We get Π_1' from Π_1 by induction hypothesis and replace a_1 with $\check{n}$ ((Lemma 3.2).

$$\dfrac{\genfrac{}{}{0pt}{}{\Pi_1}{\overline{\Gamma, A \approx A \vdash \Delta}}}{\Gamma \vdash \Delta}\,L^1_{\approx} \quad \rightsquigarrow \quad \genfrac{}{}{0pt}{}{\Pi_1'}{\overline{\Gamma : \vec{n}, A \approx A : a_1 \vdash \Delta : \check{n}a_1}}$$

$$\rightsquigarrow \quad \dfrac{\genfrac{}{}{0pt}{}{\Pi_1'[\check{n}/a_1]}{\overline{\Gamma : \vec{n}, A \approx A : \check{n} \vdash \Delta : \check{n}}}}{\Gamma : \vec{n} \vdash \Delta : \check{n}}\,\nu L^1_{\approx}(\check{n} \in \lceil \Delta \rceil \subseteq \Delta)$$

Remark C.5. The active formula is right maximal by definition. The instance of $\nu L^1_{\approx}$ preserves right connectedness. Thus, the active formula is also left maximal.

Case $L^2_{\approx}$: We start with proof Π and apply the induction hypothesis to the subproofs Π_1 and Π_2 to get Π_1' and Π_2'.

$$\dfrac{\genfrac{}{}{0pt}{}{\Pi_1}{\overline{\Gamma, A \approx B \vdash \Delta, B}} \quad \genfrac{}{}{0pt}{}{\Pi_2}{\overline{\Gamma, A \approx B, A \vdash \Delta}}}{\Gamma, A \approx B \vdash \Delta}\,L^2_{\approx}$$

$$\rightsquigarrow \quad \dfrac{\Pi'_1}{\Gamma : \vec{n}, A \approx B : a_1 \vdash \Delta : \check{n}a_1, B : \check{n}a_1} \qquad \dfrac{\Pi'_2}{\Gamma : \vec{n}, A \approx B : a_1, A : a_2 \vdash \Delta : \check{n}a_1 a_2}$$

Then we use Lemma 3.2 to replace a_2 with $\check{n}a_1$ in Π'_2 and conclude with $L^2_\approx$ as follows:

$$\dfrac{\dfrac{\Pi'_1}{\Gamma : \vec{n}, A \approx B : a_1 \vdash \Delta : \check{n}a_1, B : \check{n}a_1} \quad \dfrac{\Pi'_2[\check{n}a_1/a_2]}{\Gamma : \vec{n}, A \approx B : a_1, A : \check{n}a_1 \vdash \Delta : \check{n}a_1}}{\Gamma : \vec{n}, A \approx B : a_1 \vdash \Delta : \check{n}a_1} L^2_\approx (\check{n}a_1 \in \Delta)$$

Remark C.6. The instance of $L^2_\approx$ preserves right connectedness in $\Pi'_2[\check{n}a_1/a_2]$. It also preserves right connectedness in Π'_1, even in the single-succedent restriction.

Case $L^3_\approx$: We start with the following proof:

$$\dfrac{\dfrac{\Pi_1}{\Gamma, A \approx B, C \approx D, (A \otimes C) \approx (B \otimes D) \vdash \Delta}}{\Gamma, A \approx B, C \approx D \vdash \Delta} L^3_\approx$$

We apply the induction hypothesis to the subproof Π_1 to get Π'_1.

$$\dfrac{\Pi'_1}{\Gamma : \vec{n}, A \approx B : a_1, C \approx D : a_2, (A \otimes C) \approx (B \otimes D) : a_3 \vdash \Delta : \check{n}a_1 a_2 a_3}$$

Then we use Lemma 3.2 to replace a_3 with $a_1 a_2$.

$$\dfrac{\dfrac{\Pi'_1[a_1 a_2/a_3]}{\Gamma : \vec{n}, A \approx B : a_1, C \approx D : a_2, (A \otimes C) \approx (B \otimes D) : a_1 a_2 \vdash \Delta : \check{n}a_1 a_2}}{\Gamma : \vec{n}, A \approx B : a_1, C \approx D : a_2 \vdash \Delta : \check{n}a_1 a_2} L^3_\approx (a_1 a_2 \in \Delta)$$

Remark C.7. The instance of $L^3_\approx$ preserves right connectedness.

Case $\nu L^3_\approx$: We start with the following proof:

$$\dfrac{\dfrac{\Pi_1}{\Gamma, A \approx B, C \approx D, (A \otimes C) \approx (B \otimes D) \vdash \Delta}}{\Gamma, A \approx B, C \approx D \vdash \Delta} L^3_\approx$$

We apply the induction hypothesis to the subproof Π_1 to get Π_1'.

$$\dfrac{\Pi_1'}{\Gamma : \vec{n}, A \approx B : a_1, C \approx D : a_2, (A \otimes C) \approx (B \otimes D) : a_3 \vdash \Delta : \check{n}a_1a_2a_3}$$

Then we use Lemma 3.2 to replace a_3 with $\check{n}a_1a_2$.

$$\dfrac{\dfrac{\Pi_1'[\check{n}a_1a_2/a_3]}{\Gamma : \vec{n}, A \approx B : a_1, C \approx D : a_2, (A \otimes C) \approx (B \otimes D) : \check{n}a_1a_2 \vdash \Delta : \check{n}a_1a_2}}{\Gamma : \vec{n}, A \approx B : a_1, C \approx D : a_2 \vdash \Delta : \check{n}a_1a_2}\ \nu L^3_{\approx}(\check{n}a_1a_2 \in \lceil \Delta \rceil)$$

Remark C.8. The instance of $\nu L^3_{\approx}$ preserves right connectedness.

Case $L^{3*}_{\approx}$: The proof is similar to the one for $L^3_{\approx}$. We start with the proof:

$$\dfrac{\dfrac{\Pi_1}{\Gamma, A \approx B, (A \otimes A) \approx (B \otimes B) \vdash \Delta}}{\Gamma, A \approx B \vdash \Delta}\ L^{3*}_{\approx}$$

We apply the induction hypothesis to the subproof Π_1 to get Π_1'.

$$\dfrac{\Pi_1'}{\Gamma : \vec{n}, A \approx B : a_1, (A \otimes B) \approx (A \otimes B) : a_2 \vdash \Delta : \check{n}a_1a_2}$$

Then we use Lemma 3.2 to replace a_2 with a_1 and conclude as follows:

$$\dfrac{\dfrac{\Pi_1'[a_1/a_2]}{\Gamma : \vec{n}, A \approx B : a_1, (A \otimes B) \approx (A \otimes B) : a_1 \vdash \Delta : \check{n}a_1}}{\Gamma : \vec{n}, A \approx B : a_1 \vdash \Delta : \check{n}a_1}\ L^{3*}_{\approx}$$

Case $aL^2_{\approx}$: We first apply the induction hypothesis to the subproof Π_1 to get Π_1':

$$\dfrac{\dfrac{\Pi_1}{\Gamma, A \approx B, B \supset A \vdash \Delta}}{\Gamma, A \approx B \vdash \Delta}\ aL^2_{\approx} \quad \rightsquigarrow \quad \dfrac{\Pi_1'}{\Gamma : \vec{n}, A \approx B : a_1, B \supset A : a_2 \vdash \Delta : \check{n}a_1a_2}$$

Then we use Lemma 3.2 to replace a_2 with a_1 and conclude as follows:

$$\dfrac{\dfrac{\Pi_1'[a_1/a_2]}{\Gamma : \vec{n}, A \approx B : a_1, B \supset A : a_1 \vdash \Delta : \check{n}a_1}}{\Gamma : \vec{n}, A \approx B : a_1 \vdash \Delta : \check{n}a_1}\ aL^2_{\approx}$$

Case $L^r_{\approx}$: We first apply the induction hypothesis to the subproof Π_1 to get Π_1':

$$\dfrac{\dfrac{\Pi_1}{\Gamma, A \approx B, D, D^A_B \vdash \Delta}}{\Gamma, A \approx B, D \vdash \Delta}\ L^r_{\approx} \quad \rightsquigarrow \quad \dfrac{\Pi_1'}{\Gamma : \vec{n}, A \approx B : a_1, D : a_2, D^A_B : a_3 \vdash C : \check{n}a_1a_2a_3}$$

Then we use Lemma 3.2 to replace a_3 with a_1a_2.

$$\dfrac{\dfrac{\Pi_1'[a_1a_2/a_3]}{\Gamma : \vec{n}, A \approx B : a_1, D : a_2, D^A_B : a_1a_2 \vdash \Delta : \check{n}a_1a_2}}{\Gamma : \vec{n}, A \approx B : a_1, D : a_2 \vdash \Delta : \check{n}a_1a_2}\ L^r_{\approx}(a_1a_2 \in \Delta)$$

Remark C.9. The instance of $L^r_{\approx}$ preserves right connectedness.

Case $\nu L^r_{\approx}$: We first apply the induction hypothesis to the subproof Π_1 to get Π_1':

$$\dfrac{\dfrac{\Pi_1}{\Gamma, A \approx B, D, D^A_B \vdash \Delta}}{\Gamma, A \approx B, D \vdash \Delta}\ L^r_{\approx} \quad \rightsquigarrow \quad \dfrac{\Pi_1'}{\Gamma : \vec{n}, A \approx B : a_1, D : a_2, D^A_B : a_3 \vdash C : \check{n}a_1a_2a_3}$$

Then we use Lemma 3.2 to replace a_3 with $\check{n}a_1a_2$.

$$\dfrac{\dfrac{\Pi_1'[\check{n}a_1a_2/a_3]}{\Gamma : \vec{n}, A \approx B : a_1, D : a_2, D^A_B : \check{n}a_1a_2 \vdash \Delta : \check{n}a_1a_2}}{\Gamma : \vec{n}, A \approx B : a_1, D : a_2 \vdash \Delta : \check{n}a_1a_2}\ \nu L^r_{\approx}(\check{n}a_1a_2 \in \lceil \Delta \rceil)$$

Remark C.10. The instance of $\nu L^r_{\approx}$ preserves right connectedness. □

C.1. Rule Variants that Fail and How to Fix Them

Let us consider the following increasing (Kripke-monotonic) and synchronizing (same-label) variants of $L^3_{\approx}$ and $L^r_{\approx}$:

$$\frac{\Gamma, A \approx B : x, C \approx D : y, A \otimes C \approx B \otimes D : y \vdash \Delta}{\Gamma, A \approx B : x, C \approx D : y \vdash \Delta}\, kL^3_{\approx}(x \subseteq y)$$

$$\frac{\Gamma, A \approx B : x, C \approx D : x, A \otimes C \approx B \otimes D : x \vdash \Delta}{\Gamma, A \approx B : x, C \approx D : x \vdash \Delta}\, sL^3_{\approx}$$

$$\frac{\Gamma, A \approx B : x, D : y, D^A_B : y \vdash \Delta}{\Gamma, A \approx B : x, D : y \vdash \Delta}\, kL^r_{\approx}(x \subseteq y) \qquad \frac{\Gamma, A \approx B : x, D : x, D^A_B : x \vdash \Delta}{\Gamma, A \approx B : x, D : x \vdash \Delta}\, sL^r_{\approx}$$

We show that Theorem 8.2 extends to the variants discussed above only in the presence of the following explicit left Kripke monotonicity rule:

$$\frac{\Gamma, A : x, A : y \vdash \Delta}{\Gamma, A : x \vdash \Delta}\, L_k(x \subseteq y)$$

An immediate consequence is that using the variants in place of the original rules does not allow fully structural-free complete calculi. Let us also note that L_k subsumes contraction, which corresponds to the special case when $x = y$.

Case $kL^3_{\approx}$ **(fails):** We start with the following proof:

$$\frac{\dfrac{\Pi_1}{\Gamma, A \approx B, C \approx D, (A \otimes C) \approx (B \otimes D) \vdash \Delta}}{\Gamma, A \approx B, C \approx D \vdash \Delta}\, L^3_{\approx}$$

We apply the induction hypothesis to the subproof Π_1 to get Π_1'.

$$\frac{\Pi_1'}{\Gamma : \vec{n}, A \approx B : a_1, C \approx D : a_2, (A \otimes C) \approx (B \otimes D) : a_3 \vdash \Delta : \check{n} a_1 a_2 a_3}$$

Then we use Lemma 3.2 to replace a_2 and a_3 with $a_1 a_2$ and since $a_1 \subseteq a_1 a_2$ we can apply an instance of $kL^3_{\approx}$.

$$\dfrac{\dfrac{\Pi_1'[a_1a_2/a_2; a_1a_2/a_3]}{\Gamma : \vec{n}, A \approx B : a_1, C \approx D : a_1a_2, (A \otimes C) \approx (B \otimes D) : a_1a_2 \vdash \Delta : \check{n}a_1a_2}}{\Gamma : \vec{n}, A \approx B : a_1, C \approx D : a_1a_2 \vdash \Delta : \check{n}a_1a_2}\ kL^3_{\approx}(a_1 \subseteq a_1a_2)$$

However, the conclusion in the labeled proof is not a translation of the conclusion in the label-free proof because $C \approx D$ should be labeled with a_2.

With an explicit rule for left Kripke monotonicity, we can fix the problem as follows:

$$\dfrac{\dfrac{\dfrac{\Pi_1'[a_1a_2/a_2; a_1a_2/a_3]}{\Gamma : \vec{n}, A \approx B : a_1, C \approx D : a_1a_2, (A \otimes C) \approx (B \otimes D) : a_1a_2 \vdash \Delta : \check{n}a_1a_2}}{\Gamma : \vec{n}, A \approx B : a_1, C \approx D : a_1a_2 \vdash \Delta : \check{n}a_1a_2}\ kL^3_{\approx}(a_1 \subseteq a_1a_2)}{\Gamma : \vec{n}, A \approx B : a_1, C \approx D : a_2 \vdash \Delta : \check{n}a_1a_2}\ L_k(a_2 \subseteq a_1a_2)$$

Remark C.11. The instances of L_k and $kL^3_{\approx}$ preserve right connectedness.

Case $sL^3_{\approx}$ **(fails):** We start with the following proof:

$$\dfrac{\dfrac{\Pi_1}{\Gamma, A \approx B, C \approx D, (A \otimes C) \approx (B \otimes D) \vdash \Delta}}{\Gamma, A \approx B, C \approx D \vdash \Delta}\ L^3_{\approx}$$

We apply the induction hypothesis to the subproof Π_1 to get Π_1'.

$$\dfrac{\Pi_1'}{\Gamma : \vec{n}, A \approx B : a_1, C \approx D : a_2, (A \otimes C) \approx (B \otimes D) : a_3 \vdash \Delta : \check{n}a_1a_2a_3}$$

Then we use Lemma 3.2 to replace a_2 and a_3 with a_1.

$$\dfrac{\dfrac{\Pi_1'[a_1/a_2; a_1/a_3]}{\Gamma : \vec{n}, A \approx B : a_1, C \approx D : a_1, (A \otimes C) \approx (B \otimes D) : a_1 \vdash \Delta : \check{n}a_1a_2}}{\Gamma : \vec{n}, A \approx B : a_1, C \approx D : a_1 \vdash \Delta : \check{n}a_1}\ sL^3_{\approx}$$

However, the conclusion in the labeled proof is not a translation of the conclusion in the label-free proof because $\mathrm{C}\approx\mathrm{D}$ should be labeled with a_2.

With an explicit rule for left Kripke monotonicity, we can fix the problem as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Pi'_1[\mathrm{a}_1\mathrm{a}_2/\mathrm{a}_2;\mathrm{a}_1\mathrm{a}_2/\mathrm{a}_3]}{\Gamma:\vec{\mathrm{n}},\mathrm{A}\approx\mathrm{B}:\mathrm{a}_1\mathrm{a}_2,\mathrm{C}\approx\mathrm{D}:\mathrm{a}_1\mathrm{a}_2,(\mathrm{A}\otimes\mathrm{C})\approx(\mathrm{B}\otimes\mathrm{D}):\mathrm{a}_1\mathrm{a}_2\vdash\Delta:\check{\mathrm{n}}\mathrm{a}_1\mathrm{a}_2}}{\Gamma:\vec{\mathrm{n}},\mathrm{A}\approx\mathrm{B}:\mathrm{a}_1\mathrm{a}_2,\mathrm{C}\approx\mathrm{D}:\mathrm{a}_1\mathrm{a}_2\vdash\Delta:\check{\mathrm{n}}\mathrm{a}_1\mathrm{a}_2}\,\mathrm{sL}^3_\approx}{\Gamma:\vec{\mathrm{n}},\mathrm{A}\approx\mathrm{B}:\mathrm{a}_1\mathrm{a}_2,\mathrm{C}\approx\mathrm{D}:\mathrm{a}_2\vdash\Delta:\check{\mathrm{n}}\mathrm{a}_1\mathrm{a}_2}\,\mathrm{L_k}(\mathrm{a}_2\subseteq\mathrm{a}_1\mathrm{a}_2)}{\Gamma:\vec{\mathrm{n}},\mathrm{A}\approx\mathrm{B}:\mathrm{a}_1,\mathrm{C}\approx\mathrm{D}:\mathrm{a}_2\vdash\Delta:\check{\mathrm{n}}\mathrm{a}_1\mathrm{a}_2}\,\mathrm{L_k}(\mathrm{a}_1\subseteq\mathrm{a}_1\mathrm{a}_2)$$

Remark C.12. The instances of $\mathrm{L_k}$ and $\mathrm{sL}^3_\approx$ preserve right connectedness.

Case $\mathrm{kL}^{\mathrm{r}}_\approx$ **(fails):** Similar to $\mathrm{kL}^3_\approx$.

Case $\mathrm{sL}^{\mathrm{r}}_\approx$ **(fails):** Similar to $\mathrm{sL}^3_\approx$.

C.2. Technical Height-Preserving Lemmas

Let us write $\overset{\mathrm{n}}{\vdash}_{\mathsf{S}}$ instead of $\vdash_{\mathsf{S}}$ to indicate provability in a proof-system S but only for proofs with height less than some natural number n.

LEMMA C.13. *Let* s *be a labeled sequent* $\Gamma\vdash\Delta$, $[\mathrm{u}/\mathrm{c}]$ *be a label substitution such that* $\mathrm{c}\in\mathbf{L}^1$ *or* $\mathrm{c}=\emptyset$, *and* $\mathrm{s}[\mathrm{u}/\mathrm{c}]$ *be the sequent obtained from* s *by simultaneously applying* $[\mathrm{u}/\mathrm{c}]$ *to all of the labeled formulas occurring in* s. *If* $\overset{\mathrm{n}}{\vdash}_{\mathsf{L3_{ISCI}}}\mathrm{s}$ *then* $\overset{\mathrm{n}}{\vdash}_{\mathsf{L3_{ISCI}}}\mathrm{s}[\mathrm{u}/\mathrm{c}]$.

PROOF: By induction on the height h of the proof of $\Gamma\vdash\Delta$. The base case $h=0$ is when s is the conclusion of an axiom.

Case id**:** This case subsumes both $\mathrm{id_p}$ and $\mathrm{id}_\approx$. Suppose that s is of the form $\Gamma,\mathrm{A}:\mathrm{x}\vdash\Delta,\mathrm{A}:\mathrm{y}$ with $\mathrm{x}\subseteq\mathrm{y}$. If $\mathrm{c}\not\subseteq\mathrm{y}$ then $\mathrm{s}[\mathrm{u}/\mathrm{c}]=\mathrm{s}$ and the result is immediate. Otherwise, $\mathrm{c}\subseteq\mathrm{y}$ and $\mathrm{y}=(\mathrm{y}-\mathrm{c})\cup\mathrm{c}$. Since $\mathrm{x}\subseteq\mathrm{y}$, $\mathrm{y}=(\mathrm{y}-\mathrm{x})\cup\mathrm{x}$ implies $\mathrm{y}=(\mathrm{y}-(\mathrm{x}\cup\mathrm{c}))\cup(\mathrm{x}-\mathrm{c})\cup\mathrm{c}$. Hence, $\mathrm{y}[\mathrm{u}/\mathrm{c}]=(\mathrm{y}-(\mathrm{x}\cup\mathrm{c}))\cup(\mathrm{x}-\mathrm{c})\cup\mathrm{u}$. We then show that $\mathrm{s}[\mathrm{u}/\mathrm{c}]$ remains an axiom for

A by showing that $x[u/c] \subseteq y[u/c]$. If $c \not\subseteq x$ then $x[u/c] = x = x - c$ and $x - c \subseteq y[u/c]$. If $c \subseteq x$ then $x[u/c] = ((x-c) \cup c)[u/c] = (x-c) \cup u$ and $(x - c) \cup u \subseteq y[u/c]$.

Cases $L_{\bot}^{K}$ and $L_{\bot}^{B}$: Similar to Case id.

For the inductive case $h = n + 1$, let r be the last rule applied (which has s as a conclusion). If r requires the introduction of eigenvariables we proceed as follows.

Case $L_{\vee}^{B}$: Suppose that s is of the form $\Gamma, A \vee B : x \vdash \Delta, C : y$ and is obtained by the rule $L_{\vee}^{B}$ from the premises $s_1 = \Gamma, A : xa \vdash \Delta, C : ya$ and $s_2 = \Gamma, B : xb \vdash \Delta, C : yb$, where $a, b \not\subseteq \Gamma \cup \Delta$, which have proofs Π_1, Π_2 such that $h(\Pi_1), h(\Pi_2) \leqslant n$. We choose two labels $a' \neq b'$ such that $a', b' \not\subseteq \Gamma \cup \Delta$ and $a', b' \not\subseteq xyuabc$. By induction hypothesis on Π_1 and Π_2 with substitutions $[a'/a]$ and $[b'/b]$ we get proofs Π_1' and Π_2' of $\Gamma, A : xa' \vdash \Delta, C : ya'$ and $\Gamma, B : xb' \vdash \Delta, C : yb'$. Then, by induction hypothesis on Π_1' and Π_2' with substitution $[u/c]$, we get proofs Π_1'' and Π_2'' of $\Gamma[u/c], A \vee B : x[u/c], A : x[u/c]a' \vdash \Delta[u/c], C : y[u/c]a'$ and $\Gamma[u/c], A \vee B : x[u/c], B : x[u/c]b' \vdash \Delta[u/c], C : y[u/c]b'$ from which we infer the conclusion $\Gamma[u/c], A \vee B : x[u/c] \vdash \Delta[u/c]$ by the rule $L_{\vee}^{B}$.

Case $R_{\supset}$: Similar to Case $L_{\vee}^{B}$.

If r does not require eigenvariables, we apply the induction hypothesis on the premises of r since they have proofs of height strictly less than $n + 1$ and we conclude $s[u/c]$ by reapplying r. □

Lemma C.14 shows that weakening is height-preserving admissible for all calculi in the $\mathsf{L3_{ISCI}}$ family.

LEMMA C.14. *Weakening is height-preserving eliminable in $\mathsf{L3_{ISCI}}$, that is, if $\models^{n}_{\mathsf{L3_{ISCI}}} \Gamma \vdash \Delta$, then $\models^{n}_{\mathsf{L3_{ISCI}}} \Gamma, \Gamma' \vdash \Delta$ and $\models^{n}_{\mathsf{L3_{ISCI}}} \Gamma \vdash \Delta, \Delta'$.*

PROOF: The proof is by induction on the height h of a proof Π of $\Gamma \vdash \Delta$. For $h = 0$, it is clear that if $\Gamma \vdash \Delta$ is an axiom, then so are $\Gamma, \Gamma' \vdash \Delta$ and $\Gamma \vdash \Delta, \Delta'$.

For $h = n + 1$, let r be the last rule applied in Π. If r is not $R_{\supset}$ or $L^{B}_{\vee}$, we apply the induction hypothesis on the premises of r and conclude by reapplying r. Otherwise, we first use Lemma C.13 to replace the eigenvariables in all of the premises of r with variables not occurring in $\Gamma \cup \Gamma' \cup \Delta \cup \Delta'$ and then apply the induction hypothesis to the modified premises before concluding with a new instance of r. □

COROLLARY C.15. The proof translation given in Theorem 8.2 is height-preserving. Hence, if $\models^{n}_{\mathsf{G3}_{\mathsf{ISCI}}} A$, then $\models^{n}_{\mathsf{L3}_{\mathsf{ISCI}}} A$.

PROOF: Let us first observe that label substitution and weakening admissibility are height-preserving. It then follows that the translation described in Theorem 8.2 is also height-preserving since it is a one-to-one mapping of each rule in $\mathsf{G3}_{\mathsf{ISCI}}$ to the corresponding rule in $\mathsf{L3}_{\mathsf{ISCI}}$. □

D. Appendix: Kripke Soundness of $\mathsf{G3}^{\mathsf{K}}_{\mathsf{ISCI}}$

THEOREM D.1. $\mathsf{G3}^{\mathsf{K}}_{\mathsf{ISCI}}$ *is sound w.r.t. the Kripke semantics of* ISCI*: if* $\vdash_{\mathsf{G3}^{\mathsf{K}}_{\mathsf{ISCI}}} A$ *then* $\models_{\mathsf{K}} A$*.*

PROOF: Let S be a finite set of formulas $\{ F_1, \ldots, F_n \}$. We define $\bigwedge S$ and $\bigvee S$ as the formulas $F_1 \wedge F_2 \wedge \ldots \wedge F_n$ and $F_1 \vee F_2 \ldots \vee F_n$ with the special cases $\bigwedge \emptyset = \top$ and $\bigvee \emptyset = \bot$.

We define the realizability of a sequent $\Gamma \vdash \Delta$ as the following property: forall Kripke models $\mathcal{M}$ and all words m in $\mathcal{M}$, if $m \Vdash \bigwedge \Gamma$ then $m \Vdash \bigvee \Delta$. For all rules in $\mathsf{G3}^{\mathsf{K}}_{\mathsf{ISCI}}$, we show that if all premises are realizable, then so is the conclusion.

Case id: This case subsumes both id_p and $id_{\approx}$.

Suppose we have $\overline{\Gamma, A \vdash \Delta, A}^{\,id}$ and let m be a world in a Kripke model. If $m \Vdash \bigwedge \Gamma \wedge A$, then $m \Vdash A$. Hence $m \Vdash \bigvee \Delta \vee A$.

Case $L_\bot$: Suppose we have $\overline{\Gamma, \bot \vdash \Delta}^{\,L_\bot}$ and let m be a world in a Kripke model. Since $m \nVdash \bot$, we immediately have that $m \Vdash \bigwedge \Gamma \wedge \bot$ implies $m \Vdash \bigvee \Delta$.

Case $R_\vee$**:** We consider the rule $\dfrac{\Gamma \vdash \Delta, A_1, A_2}{\Gamma \vdash \Delta, A_1 \vee A_2} R_\vee$ and m a world in a Kripke model such that $m \Vdash \bigwedge \Gamma$.
By assumption from the premiss we get $m \Vdash \bigvee \Delta \vee A_1 \vee A_2$.

Case $L^K_\vee$**:** We consider the rule $\dfrac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta} L^K_\vee$ and m a world in a Kripke model such that $m \Vdash \bigwedge \Gamma \wedge (A \vee B)$.
If $m \Vdash A$ then $m \Vdash \bigwedge \Gamma \wedge A$ and by assumption from the first premiss $m \Vdash \bigwedge \Gamma \wedge A$ implies $m \Vdash \bigwedge \Delta$. Otherwise, since $m \Vdash A \vee B$, we necessarily have $m \Vdash B$ and by assumption from the second premiss $m \Vdash \bigwedge \Gamma \wedge B$ implies $m \Vdash \bigvee \Delta$. Hence, $m \Vdash \bigvee \Delta$.

Case $R_\supset$**:** We consider the rule $\dfrac{\Gamma, A \vdash B}{\Gamma \vdash \Delta, A \supset B} R_\supset$ and m a world in a Kripke model such that $m \Vdash \bigwedge \Gamma$.
If $m \Vdash \bigvee \Delta$ then $m \Vdash \bigvee \Delta \vee (A \supset B)$. Otherwise, we need to show that $m \Vdash A \supset B$. Suppose some arbitrary n such that $m \leqslant n$. If $n \Vdash A$ then, since by Kripke monotonicity we also have $n \Vdash \bigwedge \Gamma$, it follows that $n \Vdash \bigwedge \Gamma \wedge A$. Hence, by assumption from the premiss, we have $n \Vdash B$. Hence, $m \Vdash A \supset B$, which implies $m \Vdash \bigvee \Delta \vee (A \supset B)$.

Case $L^1_\approx$**:** We consider the rule $\dfrac{\Gamma, A \approx A \vdash \Delta}{\Gamma \vdash \Delta} L^1_\approx$ and m a world in a Kripke model such that $m \Vdash \bigwedge \Gamma$.
By condition $\mathcal{M}_{\approx_1}$ of Kripke models, we have $m \Vdash A \approx A$. Hence, by assumption from the premiss, we get $m \Vdash \bigvee \Delta$.

Case $L^2_\approx$**:** Similar to Case $L^3_\approx$.

Case $aL^2_\approx$**:** Similar to Case $L^3_\approx$.

Case $L^3_\approx$**:** We consider the rule $\dfrac{\Gamma, A \approx B, C \approx D, A \otimes C \approx B \otimes D \vdash \Delta}{\Gamma, A \approx B, C \approx D \vdash \Delta} L^3_\approx$ and m a world in a Kripke model such that $m \Vdash \bigwedge \Gamma \wedge A \approx B \wedge C \approx D$.
Since $m \Vdash A \approx B$ and $m \Vdash C \approx D$, we have $m \Vdash A \otimes C \approx B \otimes D$ by

condition $\mathcal{M}_{\approx_4}$ of Kripke models. Therefore, by assumption from the premiss, we get $m \Vdash \bigvee \Delta$.

Case $L^{3*}_{\approx}$: Similar to Case $L^{3}_{\approx}$.

Case $L^{r}_{\approx}$: We consider the rule $\dfrac{\Gamma, A \approx B, D, D^{A}_{B} \vdash \Delta}{\Gamma, A \approx B, D \vdash \Delta}\, L^{r}_{\approx}$ and m a world in a Kripke model such that $m \Vdash \bigwedge \Gamma \wedge A \approx B \wedge D$.
By the replacement law, we have $m \Vdash D^{A}_{B}$. Hence, by assumption from the premiss, we get $m \Vdash \Delta$.

The other cases are similar. □

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