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Antonio Ledda*
Hanamantagouda P. Sankappanavar
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THE AMALGAMATION PROPERTY IN THE VARIETY OF REGULAR DOUBLE STONE ALGEBRAS: A CONSTRUCTIVE VIEW

Abstract

In this paper we give a constructive proof that the variety of Boolean algebras has the strong amalgamation property by describing constructively the strong amalgams in the variety. Then, capitalizing on this construction, we investigate several forms of amalgamation, such as the strong amalgamation property and Maksimova super-amalgamation for the varieties of regular double Stone algebras and centered regular double Stone algebras. In fact, we prove that the amalgamation property holds for the variety **RDS**. Then, we introduce the variety $\mathbf{RDS}^k$ of centered regular double Stone algebras and prove that $\mathbf{RDS}^k$ enjoys

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the strong amalgamation property. It is also proved that the varieties of Boolean algebras and centered regular double Stone algebras have the super-amalgamation property. We close the paper by providing a number of concrete examples and applications to illustrate the theory developed in the paper.

Keywords: Boolean algebras, regular double Stone algebras, Kleene algebras, amalgamation, strong amalgamation, super-amalgamation.

2020 Mathematical Subject Classification: 06A06, 06B05, 06B15, 06B20, 06B23, 06C15, 06D05, 06D15, 06D30, 06D35, 06E05, 06E75.

1. Introduction

In model theory and algebraic logic, there are properties of classes of structures that are "reflected" in the logics that are associated to those classes of structures, thus providing a useful interplay between those classes of algebras and their logics. One such property of paramount importance is the amalgamation property. Under standard algebraizability assumptions, the amalgamation property of a variety is reflected in the Craig's interpolation property of the associated logic; in other words, a variety $\mathbf{V}$ has an amalgamation property if and only if its corresponding logic has the Craig's interpolation property.

In this paper we mostly focus on amalgamation property, strong amalgamation property, and super-amalgamation property for the varieties of Boolean algebras, regular double Stone algebras and centered regular double Stone algebras.

Double Stone algebras are a natural generalization of Boolean algebras. Recall that, in a Boolean algebra, the complement of an element a is characterized both as the greatest element x such that $a \wedge x = 0$ and as the least element y such that $a \vee y = 1$. Dropping one of these two requirements leads to the notions of *pseudocomplement* $^{\sim}$ and *dual pseudocomplement* $^{+}$, which give rise, respectively, to the classes of p-algebras and dual p-algebras. From a logical perspective, this amounts to a splitting of classical negation into two unary operations: $^{\sim}$, which captures a form of negation enforcing non-contradiction, and $^{+}$, which captures a form of negation enforcing the law of excluded middle.

Following an important result of Ribenboim in 1949 [48], it was shown that the class of pseudocomplemented distributive lattices forms in fact an equational class of algebras of type $(2, 2, 1, 0)$ in the language $(\vee, \wedge, ^{\sim}, 0)$. Early on, it was also observed that the identity $x^{\sim} \vee x^{\sim\sim} = 1$, corresponding to the so-called *weak law of excluded middle*, holds in some p-algebras. In fact, already in the mid-1930s, Stone posed the problem of investigating the (sub)class of p-algebras satisfying this identity. In response to Stone's proposal, Grätzer and Schmidt initiated a systematic study of this subclass, which they termed the class of Stone algebras, with their dual counterparts naturally called dual Stone algebras.

As a consequence, the class of Stone algebras, being itself a variety, became the object of intensive investigation. It was then natural to ask what would arise from expanding a Stone algebra by equipping it with the additional structure of a dual Stone algebra. This line of inquiry led to the introduction and subsequent systematic study of the class of DS double Stone algebras. In this paper, we are actually interested in an important subvariety of **DS**, called the variety **RDS** of regular double Stone algebras (see Section 2 for the definition). The variety **RDS** is a subvariety of the variety of regular double p-algebras, the latter was first introduced as a quasi-variety, in 1972, by Varlet [52] in connection with the problem of characterizing the congruence-regular double p-algebras. A little later, Katriňák [25] proved, in 1973, that the regular double p-algebras, indeed, form a variety (see also [14]). Since then, there is a considerable amount of literature on the variety of regular double p-algebras and, in particular, on the variety **RDS**; see e.g. [1, 14] (and the references therein). It is shown in [1] that there are $2^{\aleph_0}$ subvarieties of the variety of regular double p-algebras. Our present paper is a further addition to the already existing rich literature on **RDS**.

It may be worth noticing that the study of regular double Stone algebras provides algebraic tools for modeling uncertainty and partial information. From a logical perspective, regular double Stone algebras generalize classical logic to a substructural logic [17] where the principles of non-contradiction and excluded middle are not valid.

Interestingly, there is a connection between the theory of rough sets due to Pawlak ([43] and [44]) and regular double Stone algebras; for example,[47] and [13] have shown that every regular double Stone algebra is isomorphic to the algebra arising from an approximation space. Thus, regular double Stone algebras provide the interplay between lower and upper approximations in rough set theory. This approach is central to artificial intelligence and cognitive science, with applications in machine learning, knowledge discovery, data mining, expert systems, approximate reasoning, and pattern recognition [43, 44]. Moreover, these structures can be regarded as the algebraic counterpart of three-valued Łukasiewicz logic[7], a paradigmatic system for reasoning under indeterminacy, which itself may be seen as a special case of fuzzy logic [20] with a three-element chain of truth values.

Regular double Stone algebras play a structural role as distributive "sharp" contexts within broader non-classical frameworks, just as Boolean algebras serve as classical blocks in orthomodular lattices. In this sense, they provide natural building blocks for unsharp quantum logics [18, 31, 32].

As mentioned earlier, our first goal in this paper is to provide a novel proof of the strong amalgamation property (AP) for the variety of Boolean algebras. As a second objective, we use it to provide a constructive proof for the amalgamation property for regular double Stone algebras, even though this result is already known (see [15]). Our constructive proof relies on a construction due to Johnstone [24].

The facts that the variety **BA** has the strong amalgamation property (SAP) and the variety **RDS** fails to have (SAP) led us to consider the variety of centered regular double Stone algebras, an expansion of **RDS** by a center. The notion of a center is not new; for example, already in 1940, Moisil [38] introduced it in the context of 3-valued Łukasiewicz algebras. Later, in 1972, Cignoli [11] used it to show that the variety of centered n-valued Łukasiewicz algebras is term equivalent to the variety of Post algebras of order n and Cignoli [12] used it to characterize injective 3-valued Łukasiewicz algebras.

Let $\mathbf{A} \in \mathbf{RDS}$ and let $k \in \mathbf{A}$. We say that k is called a *center* of $\mathbf{A}$ if it satisfies:

$$k^{\sim} = 0 \text{ and } k^{+} = 1. \tag{K}$$

We expand the language $L = (\wedge, \vee, ^{\sim}, ^{+}, 0, 1)$ of regular double Stone algebras, by adding a new constant k to obtain the language:

$$L^k = (\wedge, \vee, ^{\sim}, ^{+}, 0, k, 1),$$

of type $(2, 2, 1, 1, 0, 0, 0)$.

Our third objective is to introduce a new variety of algebras called "centered regular double Stone algebras" in the language L^k, and show that it has the (SAP). In fact, we prove a stronger result: the variety of centered regular double Stone algebras satisfies Maksimova super-amalgamation property.

The introduction of the constant k can be thought of as the explicit algebraic counterpart of an intermediate designated value, that separates the two extremal truth values 0 and 1.[1] This expansion provides a way to capture contexts of uncertainty or indeterminacy that are central in rough set theory, where k corresponds to the boundary region between lower and upper approximations, as well as in three-valued Łukasiewicz logic, where it represents the "undetermined" truth value. From a computational perspective, k can also be interpreted as a marker of error or inconsistency, thus offering a formal tool for distinguishing reliable from unreliable information states.

The paper is structured as follows. In Section 2 we provide all the specific notions from algebra and category theory, that may be expedient for a comprehensive reading of our discourse. In Section 3, we give a constructive proof of the strong amalgamation property for the variety of Boolean algebras by providing a constructive description of the strong amalgams. In Section 4, we use the construction of the strong amalgams of V-formations of Boolean algebras from Section 3 to prove the amalgamation property for the variety **RDS**. In Section 5 we introduce the variety $\mathbf{RDS}^k$ of centered regular double Stone algebras and prove that $\mathbf{RDS}^k$ enjoys the strong amalgamation property. In Section 6 it is shown that the varieties

[1] A paramount example of this approach traces back to the early works of Pavelka on multiple-valued logics [40, 41, 42].

of Boolean algebras and centered regular double Stone algebras have the super-amalgamation property. We conclude the paper by discussing several examples and applications to illustrate the theory we have developed in this paper.

2. Preliminaries

For standard facts about double Stone algebras we refer the reader to Grätzer [19] or Balbes and Dwinger [2].

2.1. Regular double Stone algebras

Definition 2.1 ([19, 2]). An algebra $\mathbf{L} = (L, \wedge, \vee, {}^{\sim}, 0, 1)$ is a *p-algebra* if $(L, \wedge, \vee, 0, 1)$ is a bounded distributive lattice and ${}^{\sim}$ satisfies:

$$x \wedge y = 0 \text{ if and only if } x \leq y^{\sim}.$$

Dually, an algebra $\mathbf{L} = (L, \wedge, \vee, {}^{+}, 0, 1)$ is a *dual p-algebra* if $(L, \wedge, \vee, 0, 1)$ is a bounded distributive lattice and ${}^{+}$ satisfies:

$$x \vee y = 1 \text{ if and only if } x \geq y^{+}.$$

An algebra $\mathbf{L} = (L, \wedge, \vee, {}^{\sim}, {}^{+}, 0, 1)$ is a *double p-algebra* if the following conditions are satisfied:

1. $\mathbf{L} = (L, \vee, \wedge, {}^{\sim}, 0, 1)$ is a p-algebra.
2. $\mathbf{L} = (L, \vee, \wedge, {}^{+}, 0, 1)$ is a dual p-algebra.

As mentioned in the introduction, the class of double p-algebras is a variety [2]. Let us now introduce the notions of a Stone algebra and a double Stone algebra.

Definition 2.2 ([19, 2]).

1. A p-algebra $\mathbf{L} = (L, \wedge, \vee, {}^{\sim}, 0, 1)$ is a *Stone algebra* if it satisfies:

$$x^{\sim} \vee x^{\sim\sim} = 1. \qquad \text{(Stone Condition)}$$

2. A dual p-algebra $\mathbf{L} = (L, \wedge, \vee, ^+, 0, 1)$ is a *dual Stone algebra* if it satisfies:
$$x^+ \wedge x^{++} = 0. \qquad \text{(Dual Stone Condition)}$$

3. An algebra $\mathbf{L} = (L, \wedge, \vee, ^\sim, ^+, 0, 1)$ is a double Stone algebra if $(L, \wedge, \vee, ^\sim, 0, 1)$ is a Stone algebra and $(L, \wedge, \vee, ^+, 0, 1)$ is a dual Stone algebra.

As mentioned earlier, the class of double Stone algebras form a variety. Varlet [52] investigated the following important condition on double Stone algebras:

$$\text{if } \ x^\sim = y^\sim \quad \text{and} \quad x^+ = y^+ \quad \text{then} \quad x = y. \qquad \text{(Regularity)}$$

Following Varlet [52], we call a double Stone algebra *regular* if it satisfies the Condition (Regularity). In fact, "regularity" is an appropriate name. Actually, in [52], Varlet proved that Condition (Regularity) is equivalent to congruence regularity: if two congruences coincide on a congruence class, then they are in fact the same congruence. In 1973, Katriňák [25] proved that Condition (Regularity) is equivalent to the following identity:

$$x \wedge x^+ \leq y \vee y^\sim. \qquad \text{(M1)}$$

Thus, the class of regular double p-algebras is a variety.

Theorem 2.3. *[6] Let* $\mathbf{L}$ *be a p-algebra, then the following statements are equivalent:*

1. $\mathbf{L}$ *satisfies* (Stone Condition)*;*
2. $(x \vee y)^{\sim\sim} = x^{\sim\sim} \vee y^{\sim\sim}$*;*
3. $(x \wedge y)^\sim = x^\sim \vee y^\sim$*;*
4. $S_K(\mathbf{L}) = \{x^{\sim\sim} : x \in L\}$ *is a Boolean subalgebra.*

Given a regular double Stone algebra $\mathbf{L}$, three of its subsets, namely $S_K(\mathbf{L})$, $D^\sim(\mathbf{L})$ and $D^+(\mathbf{L})$, will play a significant role in what follows.

$S_K(\mathbf{L})$ is defined above in Theorem 2.3 and its elements are called *sharp elements of* $\mathbf{L}$. It is easy to see that

$$S_K(\mathbf{L}) = \{x \in L : x = x^{\sim\sim}\} = \{x^{\sim} : x \in L\} = \{x \in L : x^{+} = x^{\sim}\}.$$

Let us also notice that if $\mathbf{L}$ is a regular double Stone algebra, then $S_K(\mathbf{L})$ is the largest Boolean subalgebra of $\mathbf{L}$.

As observed in [39], it is possible to define two unary operations that behave as the modal operators *necessarily* "$\Box$" and *possibly* "$\Diamond$" as follows:

$$\Box x = x^{++} \quad \text{and} \quad \Diamond x = x^{\sim\sim}. \tag{2.1}$$

Moreover, *all elements in* $S_K(\mathbf{L})$ are *stable* under both $\Diamond$, and $\Box$:

$$\Diamond x = \Box x = x.$$

It is straightforward to verify that in $S_K(\mathbf{L})$ (Regularity) is nothing but a triviality. Indeed, as we mentioned earlier $S_K(\mathbf{L})$ is in fact a Boolean algebra.

Next, we define $D^{\sim}(\mathbf{L})$ by

$$D^{\sim}(\mathbf{L}) = \{x \in L : x^{\sim} = 0\}.$$

The elements of $D^{\sim}(\mathbf{L})$ are called *dense elements of* $\mathbf{L}$.

Note that $S_K(\mathbf{L}) \cap D^{\sim}(\mathbf{L}) = \{1\}$. Furthermore, for all $x \in L$,

$$x \vee x^{\sim} \in D^{\sim}(\mathbf{L}).$$

Lastly, we define the set $D^{+}(\mathbf{L})$ of *dually dense elements of* $\mathbf{L}$ by:

$$D^{+}(\mathbf{L}) = \{x \in L : x^{+} = 1\}.$$

Also, we have that $S_K(\mathbf{L}) \cap D^{+}(\mathbf{L}) = \{0\}$. Moreover, for all $x \in L$,

$$x \wedge x^{+} \in D^{+}(\mathbf{L}).$$

2.2. A representation theorem for RDS

We need the representation theorem proved in [26], see also [10]. So we recall it below.

Given a Boolean algebra $\mathbf{B} = (B, \wedge, \vee, ', 0, 1)$ and a filter F on $\mathbf{B}$, let us consider the following set:

$$[B, F] = \{(x, y) \in B^2 : (x \leq y) \text{ and } (x \vee y' \in F)\}. \tag{A}$$

We will turn the set $[B, F]$ into an algebra $[\mathbf{B}, F]$. We will define the algebra $[\mathbf{B}, F]$, in the language $(\wedge, \vee, ^\sim, ^+, 0, 1)$ as follows:

Let $[\mathbf{B}, F] = ([B, F], \wedge, \vee, ^\sim, ^+, 0, 1)$, where $\wedge, \vee$ are defined componentwise, and the unary operations $^\sim$ and $^+$ are defined as follows:

$$(x, y)^\sim = (y', y'); \quad (x, y)^+ = (x', x'), \text{ where } (x, y) \in [B, F]. \tag{2.2}$$

It turns out that the algebra $[\mathbf{B}, F]$ is a regular double Stone algebra. Even more importantly, we have the following representation theorem proved in [26, 27] (see also [32]).

Theorem 2.4. *Every regular double Stone algebra $\mathbf{A}$ is isomorphic to $[S_K(\mathbf{A}), g(D^\sim(\mathbf{A}))]$, where the isomorphism f is given, for each a, by*

$$f(a) = (\Box a, \Diamond a), \tag{2.3}$$

and $g : D^\sim(\mathbf{A}) \to S_K(\mathbf{A})$ is such that for any $a \in D^\sim(\mathbf{A})$:

$$g(a) = a^{++}.$$

Furthermore, any homomorphism h between regular double Stone algebras $\mathbf{A}_1$ and $\mathbf{A}_2$ factorizes through a Boolean homomorphism acting componentwise:

$$h(a) = f((\Box a, \Diamond a)) = (h \restriction_{S_K(\mathbf{A})} (\Box a), h \restriction_{S_K(\mathbf{A})} (\Diamond a)).$$

Additionally, in any regular double Stone algebra $\mathbf{A}$, a unary operation $'$ can be defined as given in Equation (2.4):

$$x' = x^{\sim} \vee (x \wedge x^{+}). \tag{2.4}$$

The operation $'$ turns out to be the Kleene negation, i.e., an antitone and involutive negation satisfying

$$x \wedge x' \leq y \vee y'. \tag{Kleene}$$

On the algebra $[\mathbf{B}, F]$, the Kleene operation $'$ is defined by:

$$(x, y)' = (y', x')$$

Let us now close the present section discussing an important subset of a regular double Stone algebra.

Definition 2.5. Let $\mathbf{A}$ be a regular double Stone algebra. The *core of* $\mathbf{A}$ is defined as the intersection of the set of dense elements and the set of dually dense elements; that is,

$$D^{\sim}(\mathbf{A}) \cap D^{+}(\mathbf{A}).$$

Let us notice that if a regular double Stone algebra $\mathbf{A}$ has a non-empty core, then it possesses rather remarkable properties. In fact, it can be seen that algebras of this sort are all of the form $[\mathbf{B}, B]$, for a certain Boolean algebra $\mathbf{B}$. Actually, the element $(0, 1)$ will be in the core of $[\mathbf{B}, B]$. Moreover, $(0, 1)' = (0, 1)$, see Equation (2.4). Lemma 2.6 summarizes these facts.

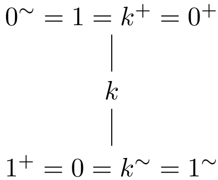
Lemma 2.6. *[32] Let* $\mathbf{A}$ *be a regular double Stone algebra and* $x \in A$. *We have that:*

1. *if* $x \in D^{\sim}(\mathbf{A}) \cap D^{+}(\mathbf{A})$, *then* $x = (0, 1)$.
2. *the cardinality of the core of* $\mathbf{A}$ *is at most* 1 *[51].*
3. x *belongs to the core of* $\mathbf{A}$ *if and only if* $x = x'$, *i.e.* x *is a fixpoint.*

In other words, Lemma 2.6 expresses the fact that a regular double Stone algebra $\mathbf{A}$ admits at most one fixpoint $k = k'$ which would be dense and dually dense, and

$$D^{\sim}(\mathbf{A}) \cap D^{+}(\mathbf{A}) = \{k\}.$$

Moreover, for a regular double Stone algebra, the conditions of having a non-empty core, satisfying Condition (K), and possessing a fixed point of $'$ are all equivalent. A prominent example of a regular double Stone algebra having a non-empty core is the three-element algebra **3** whose Hasse diagram is

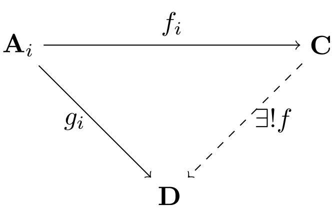


(**3**)

2.3. Basics of category theory

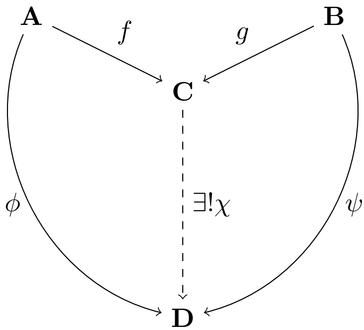
In order to fix the notation needed from category theory, we begin the present subsection by recalling a few categorial notions that will be needed in this paper. For our purposes we can restrict our attention to algebraic categories. For a comprehensive account we refer the reader to the classical textbooks [33, 21, 46].

Let $\mathcal{A}$ be a category. The *coproduct* of a family of objects $(\mathbf{A}_i : i \in I)$ in $\mathcal{A}$ is an object $\mathbf{C}$ in $\mathcal{A}$ equipped with the morphisms $f_i : \mathbf{A}_i \to \mathbf{C}$, $i \in I$, such that for any $\mathbf{D}$ in $\mathcal{A}$ and any collection of morphisms $g_i : \mathbf{A}_i \to \mathbf{D}$ there exists a unique $f : \mathbf{C} \to \mathbf{D}$ such that Diagram (2.5) commutes:

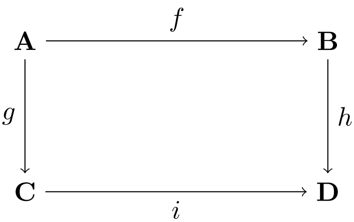


(2.5)

In other words there is a unique morphism f such that $f \circ f_i = g_i$. In case the family contains only two objects and the morphisms, there is a unique morphism χ such that the following diagram commutes:

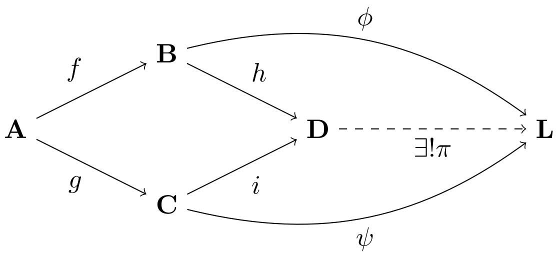


Given a pair $f : \mathbf{A} \to \mathbf{B}$, $g : \mathbf{A} \to \mathbf{C}$ of maps in $\mathcal{A}$ with a common domain $\mathbf{A}$, the *pushout* of (f, g) is a commutative square, such as given in the commutative Diagram (2.6)



(2.6)

and, furthermore, for any morphisms $\phi : \mathbf{B} \to \mathbf{L}$, and $\psi : \mathbf{C} \to \mathbf{L}$ there is a unique $\pi : \mathbf{D} \to \mathbf{L}$ such that Diagram (2.7) commutes:



(2.7)

that is,

$$\phi = \pi \circ f \text{ and } \psi = \pi \circ g.$$

We will conclude this section by defining the amalgamation property and the strong amalgamation property (for a brief historical account, the reader may refer to [34, 15]).

The category $\mathcal{A}$ possesses the *amalgamation property* (AP) if given $\mathbf{A}, \mathbf{B}$, $\mathbf{C}$ in $\mathcal{A}$ and the embeddings $f : \mathbf{A} \to \mathbf{B}$, $g : \mathbf{A} \to \mathbf{C}$ (referred to as *V-formation* or *amalgam*), then there is some $\mathbf{D}$ in $\mathcal{A}$ and embeddings $h : \mathbf{B} \to \mathbf{D}$, $i : \mathbf{C} \to \mathbf{D}$ with $h \circ f = i \circ g$.

A category $\mathcal{A}$ enjoys the *strong amalgamation property* (SAP), if for every V-formation with $\mathbf{A}, \mathbf{B}, \mathbf{C}$ in $\mathcal{A}$, and the embeddings $f : \mathbf{A} \to \mathbf{B}$, $g : \mathbf{A} \to \mathbf{C}$, there exist both an object $\mathbf{D}$ in $\mathcal{A}$ and embeddings $h : \mathbf{B} \to \mathbf{D}$, $i : \mathbf{C} \to \mathbf{D}$ such that $h \circ f = i \circ g$ and

$$h(\mathbf{B}) \cap i(\mathbf{C}) = h \circ f(\mathbf{A}) = i \circ g(\mathbf{A}).$$

3. Revisiting the strong amalgamation property for Boolean algebras: a constructive approach

In this section we present a novel proof of the well-known result that the variety $\mathbf{BA}$ has the strong amalgamation property. Our proof is constructive and uses a simplified version of an outstanding construction proposed by Banaschewski in [3], which partly relies on a previous work by Lagrange [30].

First, we introduce the notation $a * b$. Let $\mathbf{A} \coprod \mathbf{B}$ denote the coproduct of $\mathbf{A}, \mathbf{B} \in \mathbf{BA}$, with the canonical injections

$$f : \mathbf{A} \to \mathbf{A} \coprod \mathbf{B} \text{ and } g : \mathbf{B} \to \mathbf{A} \coprod \mathbf{B}.$$

For $a \in \mathbf{A}$ and $b \in \mathbf{B}$, we define $a * b$ by

$$a * b = f(a) \wedge g(b) \tag{3.1}$$

in $\mathbf{A} \coprod \mathbf{B}$.

We will now provide an explicit description of the coproduct. We follow below the construction by Johnstone from the context of frames [23, 24].

Let **A** and **B** be Boolean algebras. We consider the set

$$A \times B = \{(a, b) : a \in A,\ b \in B\}$$

and let **D** be the free bounded distributive lattice generated by the set $A \times B$. We think of the ordered pairs (a, b), with $a \in A$ and $b \in B$, as formal symbols. Observe that the following terms are (some of the) elements of **D**:

$$(a, 0), (0, b), (a, b) \wedge (c, d), (a \wedge c, b \wedge d), (a, b) \vee (c, b), (a \vee c, b),$$
$$(a, b) \vee (a, d), (a, b \vee d),$$

where $0, a, c \in A$ and $0, b, d \in B$. Let R be a binary relation on **D** consisting, precisely, of the following ordered pairs:

$$\begin{aligned} &((0, 0), (a, 0)), ((a, 0), (0, b));\\ &((a, b) \wedge (c, d), (a \wedge c, b \wedge d));\\ &((a, b) \vee (a, d), (a, b \vee d));\\ &((a, b) \vee (c, b), (a \vee c, b)). \end{aligned} \tag{3.2}$$

Let $\theta(R)$ (or simply, θ) be the congruence on **D** generated by R.

We denote by $\mathbf{C} = \mathbf{D}/\theta$. We will indicate the elements of **C** by x/θ, where $x \in \mathbf{D}$. Then the following lemma trivially holds.

Lemma 3.1. **C** *is a bounded distributive lattice satisfying the following conditions:*

$$\begin{aligned} &(1)\ 0^{\mathbf{C}} = (a, 0^{\mathbf{B}})/\theta = (0^{\mathbf{A}}, b)/\theta;\\ &(2)\ (a, b)/\theta\ \wedge\ (c, d)/\theta = (a\ \wedge^{\mathbf{A}} c,\ b\ \wedge^{\mathbf{B}} d)/\theta;\\ &(3)\ (a, b)/\theta\ \vee\ (a, d)/\theta = (a,\ b \vee^{\mathbf{B}} d)/\theta;\\ &(4)\ (a, b)/\theta\ \vee\ (c, b)/\theta = (a \vee^{\mathbf{A}} c,\ b)/\theta;\\ &(5)\ 1^{\mathbf{C}} = (1^{\mathbf{A}}, 1^{\mathbf{B}})/\theta. \end{aligned} \tag{Relations}$$

Let us recall that any element x in free distributive lattice $\mathbf{L}$ can be represented as

$$x = \bigvee_i \bigwedge_j y_{i,j}$$

where all $y_{i,j}$ are in the set of generators of $\mathbf{L}$ (see e.g. [5, Lemma III.3]). As a consequence,

LEMMA 3.2. *Any element x of $\mathbf{D}$ is of the form*

$$x = \bigvee_i \bigwedge_j (y, z)_{i,j},$$

where all $(y, z)_{i,j}$ are in the set of generators of $\mathbf{D}$.

We say that two generators (a, b) and (c, d) of $\mathbf{D}$ are extremely distinct if $a \neq c$ and $b \neq d$.

By virtue of the fact that $\mathbf{C} = \mathbf{D}/\theta$, Lemma 3.2 applies to obtain a normal form lemma also for $\mathbf{C}$ in terms of congruence classes of extremely distinct generators of $\mathbf{D}$.

LEMMA 3.3 (Normal Form). *Each element of $\mathbf{C}$ admits a canonical representation as a finite join of congruence classes of pairwise extremely distinct generators of $\mathbf{D}$. Moreover, any $x \in C$ is of the form*

$$\bigvee_i \left((\bigwedge_j a_{i,j}, 1)/\theta \wedge (1, \bigwedge_j b_{i,j})/\theta \right).$$

PROOF: Let $x \in \mathbf{C}$. By Lemma 3.2,

$$\begin{aligned} x =& y/\theta \\ =& (\bigvee_i \bigwedge_j (a, b)_{i,j})/\theta \\ =& \bigvee_i (\bigwedge_j a_{i,j}, \bigwedge_j b_{i,j})/\theta \end{aligned}$$

where the third equality is by virtue of the definition of θ, (Relations)-(2).

Next, we wish to show that the generators can be chosen to be pairwise extremely distinct in $\mathbf{D}$. Without loss of generality we can assume that x is a join of the congruence classes of two generators of $\mathbf{D}$, say:

$$x = (a, b)/\theta \ \vee^{\mathbf{C}} (c, d)/\theta.$$

If $a = c$, then from (Relations)-(3), we get $x = (a, b \vee^{\mathbf{B}} d)/\theta$, which is a congruence class of a generator of $\mathbf{D}$.
If $b = d$, then from (Relations)-(4), we get $x = (a \vee^{\mathbf{A}} c, b)/\theta$, which is also a congruence class of a generator of $\mathbf{D}$. Thus we can conclude that x can be written as a finite join of congruence classes of pairwise extremely distinct generators of $\mathbf{D}$. The moreover part is straightforward. □

This normal form will allow us to extend definitions given on congruence classes of generators, such as complementation, to arbitrary elements of $\mathbf{C}$ by recursion.

When there is no danger of confusion, to ease the notation we will omit unnecessary superscripts and subscripts.

Informally speaking, as we will see later, (Relations)-(3) and (Relations)-(4) are meant to "mimic" in $\mathbf{C}$ the operations of the factors $\mathbf{A}, \mathbf{B}$ which are to be preserved in the embeddings. Next, we show that it is in fact possible to define a complement operation $'$ on $\mathbf{C}$ which renders the distributive lattice $\mathbf{C}$ into a Boolean algebra.

Remark 3.4. From now on, we will refer to an element of $\mathbf{C}$ by one of its representatives. In particular, if x is the congruence class of a generator of $\mathbf{D}$, say (a, b), we will (by a mild abuse of language) denote x simply by (a, b) (instead of $(a, b)/\theta$) and call it a *generator* of $\mathbf{C}$. By Lemma 3.3, every element of $\mathbf{C}$ can be expressed as a finite join of congruence classes of generators of $\mathbf{D}$, that is, of elements of the form (a, b). Furthermore, the *moreover* part of Lemma 3.3 shows that each such generator admits a decomposition

$$(a, b) = (a, 1) \wedge (1, b).$$

As a consequence, the algebra $\mathbf{C}$ is generated by the set

$$\{(a, 1) : a \in A\} \;\cup\; \{(1, b) : b \in B\},$$

which is a generating set strictly contained in the set of all generators (a, b).

We will freely switch between generators of the form (a, b) and the more specific generators $(a, 1)$ and $(1, b)$, depending on the context. The former provide a uniform and symmetric description of elements of $\mathbf{C}$ (e.g. in the Definition 3.5 of complementation), while the latter form a smaller generating set, which is technically convenient in arguments involving universal properties and homomorphism extensions (see, e.g., the application of Sikorski's Criterion in Theorem 3.10).

Definition 3.5. Let $x \in C \setminus \{0, 1\}$. The unary operation $'$ on $\mathbf{C}$ is defined by recursion as follows:

Step 1. Let x be a generator, say $x = (a, b)$. Then x' is given by

$$x' = (a, b)' = (a', 1) \vee (1, b').$$

Step 2. Let x be a non-generator. Then by normal form Lemma 3.3, we have

$$x = \bigvee_{i=1}^{n} t_i, \text{ where } t_i \text{ is a generator.}$$

In this case, we define complementation as follows:

$$x' = \Big(\bigvee_{i=1}^{n} t_i\Big)' = \bigwedge_{i=1}^{n} t_i'.$$

Lemma 3.6. *The algebra* $\mathbf{C}$ *with the operation* $'$ *is a Boolean algebra.*

Proof: We already know that the algebra $\mathbf{C}$ is a bounded distributive lattice. So we only need to show that $'$ is an involution and satisfies the complementation laws. Let $t \in \mathbf{C}$.

(1) Involution. We rely on Lemma 3.3.

For a generator (a,b) we have

$$((a,b)')' = ((a',1) \vee (1,b'))' = (a',1)' \wedge (1,b')'$$

where the latter equation is from Definition 3.5-Step 2. By definition of complement on generators,

$$(a',1)' = (a'',1) \vee (1,1') = (a,1) \vee (1,0) = (a,1) \vee 0 = (a,1),$$
$$(1,b')' = (1',1) \vee (1,(b')') = (0,1) \vee (1,b) = 0 \vee (1,b) = (1,b).$$

Hence

$$\big((a,b)'\big)' = (a',1)' \wedge (1,b')' = (a,1) \wedge (1,b) = (a,b).$$

For compound terms, the claim follows by induction using De Morgan laws in Definition 3.5.

(2) Complementation law. Again we rely on Lemma 3.3.

(i) We first prove that $t \wedge t' = 0$.

Step 1. Let t be a generator, say $t = (a,b)$. We note that

$$\begin{aligned} t \wedge t' =& (a,b) \wedge (a,b)' \\ =& (a,b) \wedge ((a',1) \vee (1,b')) \\ =& ((a,b) \wedge (a',1)) \vee ((a,b) \wedge (1,b')) \\ =_{(2)}& (a \wedge a', b) \vee (a, b \wedge b') \\ =& (0,b) \vee (a,0) \\ =_{(1)}& 0 \vee 0 = 0. \end{aligned}$$

Step 2. Let t be a non-generator. Then, $t = \bigvee_{i=1}^{n}(a_i,b_i)$. We have that

$$t' = (\bigvee_{i=1}^{n}(a_i,b_i))' = \bigwedge_{i=1}^{n}(a_i,b_i)' = \bigwedge_{i=1}^{n}((a_i',1) \vee (1,b_i')).$$

Making use of distributivity,

$$t \wedge t' = \bigvee_{i=1}^{n} \bigwedge_{i=1}^{n} ((a_i, b_i) \wedge ((a_i', 1) \vee (1, b_i'))) = 0 \vee 0 = 0.$$

Next we prove that $t \vee t' = 1$. We confine our argument only to the generators, since the argument to the case of a non-generator is dual to the previous case. Let again $t = (a, b)$. Then

$$\begin{aligned} t \vee t' =& (a, b) \vee (a, b)' \\ =& (a, b) \vee (a', 1) \vee (1, b') \\ \geq_{(4)}& (a, b) \vee (a', b) \vee (1, b') \\ =_{(3)}& (a \vee a', b) \vee (1, b') \\ =& (1, b) \vee (1, b') \\ =_{(3)}& (1, b \vee b') \\ =& (1, 1) =_{(5)} 1. \end{aligned}$$

In conclusion, **C** is a Boolean algebra. □

Next we wish to show that **C** is the coproduct of **A** and **B**. We will achieve this through a few lemmas.

LEMMA 3.7. *Define the mappings* $f : \mathbf{A} \to \mathbf{C}$ *and* $g : \mathbf{B} \to \mathbf{C}$ *as in Display* (3.3)*:*

$$f(a) = (a, 1) \quad \textit{and} \quad g(b) = (1, b). \tag{3.3}$$

Then, the maps f, g *are injective homomorphisms.*

PROOF: The fact that f, g are injective is due to the definition of the mappings and the fact that $\ker(f) = \ker(g) = \{0\}$, since

$$0/\theta = \{(x, y) : x = 0 \text{ or } y = 0\}$$

by construction of θ. To show that f, g are homomorphisms, we have

$$\begin{aligned} f(a_1 \wedge a_2) =& (a_1 \wedge a_2, 1) \\ =& (a_1, 1) \wedge (a_2, 1) \\ =& f(a_1) \wedge f(a_2). \end{aligned}$$

Also,

$$\begin{aligned} f(a_1 \vee a_2) =& (a_1 \vee a_2, 1) \\ =& (a_1, 1) \vee (a_2, 1) \\ =& f(a_1) \vee f(a_2). \end{aligned}$$

Finally,

$$f(a') = (a', 1) = (a', 1) \vee 0 = (a', 1) \vee (1, 0) = (a, 1)' = f(a)'$$

by Lemma 3.6. □

Corollary 3.8 (Independent Algebras). The algebras $f(\mathbf{A})$, $g(\mathbf{B})$ are independent in $\mathbf{C}$, i.e., for all $x \in f(A)$, $y \in g(B)$

$$x \wedge^{\mathbf{C}} y = 0 \text{ if and only if } x = 0 \text{ or } y = 0.$$

In other words, Corollary (3.8) entails that, for any finite set $F \subseteq f(A) \cup g(B)$ such that $0 \notin F$, whenever

$$\bigwedge_{x \in F} x = 0,$$

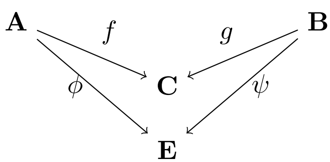
then

$$0 = \bigwedge_{x \in F} x = \bigwedge_{a \in f(A) \cap F} a \wedge \bigwedge_{b \in f(B) \cap F} b$$

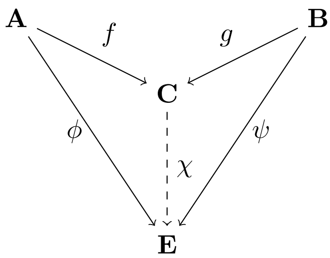
if and only if either $\bigwedge_{a \in f(A) \cap F} a = 0$ or $\bigwedge_{b \in f(B) \cap F} b = 0$, or both $\bigwedge_{a \in f(A) \cap F} a = \bigwedge_{b \in f(B) \cap F} b = 0$.

We are now ready to prove that $\mathbf{C}$ is the coproduct $\mathbf{A} \coprod \mathbf{B}$.

In fact, suppose that there are mappings ϕ, ψ so that



We wish to find a unique morphism χ such that Diagram (3.4) commutes:



(3.4)

To this aim, Sikorski Criterion will be expedient. Consider the mapping defined for any element x of a Boolean algebra $\mathbf{B}$, and a homomorphism $\varphi : \mathbf{B} \to \mathbf{2}$:

$$p(x, \varphi(x)) = \begin{cases} x, & \text{if } \varphi(x) = 1; \\ x', & \text{if } \varphi(x) = 0. \end{cases}$$

THEOREM 3.9 (Sikorski Criterion [50, 29]). *A mapping h from a generating set K of a Boolean algebra $\mathbf{B}$ into a Boolean algebra $\mathbf{A}$ can be extended to a homomorphism from $\mathbf{B}$ into $\mathbf{A}$ just in case for every 2-valued function φ on a finite subset F of K,*

$$\bigwedge_{a \in F} p(a, \varphi(a)) = 0 \textit{ implies } \bigwedge_{a \in F} p(h(a), \varphi(a)) = 0.$$

THEOREM 3.10. *$\mathbf{C}$ is the coproduct $\mathbf{A} \coprod \mathbf{B}$.*

PROOF: Recall that for $a \in A, b \in B$ we have by Display (3.3) in Lemma 3.7

$$f(a) = (a, 1) \text{ and } g(b) = (1, b).$$

Now consider a Boolean algebra $\mathbf{E}$ and homomorphisms $\phi : \mathbf{A} \to \mathbf{E}$, $\psi : \mathbf{B} \to \mathbf{E}$ (see Diagram (3.4)). By Lemma 3.3, it follows that the set

$$K = \{(x, 1) : x \in A\} \cup \{(1, y) : y \in B\}$$

generates $\mathbf{C}$. We define a map $\chi : K \to \mathbf{E}$ as follows:

$$\chi((x, y)) = \phi(x) \wedge^{\mathbf{E}} \psi(y).$$

We shall see that this assignment extends by Theorem 3.9 to a Boolean homomorphism. Moreover, we will also prove that this extension will be unique in making Diagram (3.4) commutative.

Suppose that $\bigwedge_{a \in F} p(a, \varphi(a)) = 0$ for any 2-valued function, and $F \subseteq_{fin} K$. By Corollary 3.8, without loss of generality we can assume all $a \in F$ of the form $(x, 1)$. This means,

$$\begin{aligned}
\bigwedge_{a \in F} p(a, \varphi(a)) = & \bigwedge_{a \in F} p((x, 1), \varphi(a)) \\
= & \bigwedge_{a \in F} p(f(x), \varphi(a)) \\
= & f(\bigwedge_{a \in F} p(x, \varphi(a))) \\
= & f(0) = 0.
\end{aligned}$$

Then,

$$\bigwedge_{a \in F} p(\chi((x, 1)), \varphi(a)) = \bigwedge_{a \in F} p(\phi(x), \varphi(a)) = \phi(\bigwedge_{a \in F} p(x, \varphi(a))) = \phi(0) = 0.$$

It is also easy to observe that for $a \in A$:

$$\begin{aligned}
\chi \circ f(a) = & \chi((a, 1)) \\
= & \phi(a) \wedge \psi(1) \\
= & \phi(a) \wedge 1 \\
= & \phi(a).
\end{aligned}$$

An analogous argument applies to any $b \in B$. Therefore, Diagram (3.4) commutes.

Finally, we prove that χ is unique in making Diagram (3.4) commutative. Suppose that there exists a $\delta : \mathbf{C} \to \mathbf{E}$ such that the diagram commutes. Hence, $\delta \circ f(a) = \phi(a)$ and $\delta \circ g(b) = \psi(b)$, for $a \in A, b \in B$. As a consequence of this observation, it readily follows that

$$\begin{aligned} \delta \circ f(a) =& \delta((a,1)) \\ =& \phi(a) \\ =& \phi(a) \wedge \psi(1) \\ =& \chi((a,1)) \end{aligned}$$

and dually for g. Therefore, the proof of the theorem is complete. □

Using our explicit description of $\mathbf{A} \coprod \mathbf{B}$, we can prove a technical lemma that will turn useful for our arguments that follow.

Lemma 3.11 (Comparison Lemma). *Let $\mathbf{A}, \mathbf{B} \in \mathbf{BA}$. For any $a_1, a_2 \in A$, $b_1, b_2 \in B$, if $a_1 * b_1 \le a_2 * b_2$, then $a_1 \le a_2$ and $b_1 \le b_2$.*

Proof: Let $a_1, a_2 \in A$, $b_1, b_2 \in B$ be such that $a_1 * b_1 \le a_2 * b_2$, where $*$ is defined as in Display (3.1): $x * y = f(x) \wedge g(y)$. Now, by hypothesis, we have

$$f(a_1) \wedge g(b_1) \le f(a_2) \wedge g(b_2),$$

which implies that

$$(a_1, 1) \wedge (1, b_1) \le (a_2, 1) \wedge (1, b_2),$$

whence

$$(a_1, b_1) \le (a_2, b_2).$$

Therefore,

$$(a_1, b_1) = (a_1, b_1) \wedge (a_2, b_2) = (a_1 \wedge a_2, b_1 \wedge b_2).$$

by (Relations)–(2), and by the injectivity of the mappings f, g (see Lemma 3.7) we have that $a_1 = a_1 \wedge a_2$ and $b_1 = b_1 \wedge b_2$. Thus $a_1 \le a_2$ and $b_1 \le b_2$. □

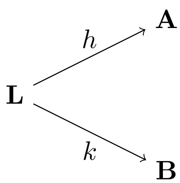
For a different perspective, we would refer the interested reader also to Banaschewski [3]. However, in [3] the result is mentioned without an explicit proof, and the author refers to it as the "Comparison Principle".

For a subset X of a Boolean algebra $\mathbf{B}$, we denote by $\langle X\rangle$ the ideal generated by X. Recall that for $x \in \mathbf{B}$, we have that

$$x \in \langle X\rangle \text{ if and only if } x \leq \bigvee_{i=1}^{n} x_i \qquad \text{for some } x_1, \ldots, x_n \in X. \tag{3.5}$$

Let us now use the coproduct $\mathbf{A} \coprod \mathbf{B}$ to construct a strong amalgam of any V-formation in $\mathbf{BA}$.

Given the V-formation in Display (3.6):



(3.6)

let us consider the coproduct $\mathbf{A} \coprod \mathbf{B}$ of $\mathbf{A}$ and $\mathbf{B}$. Define the subset H of the coproduct by

$$H = \langle h(a) * k(a)' : a \in L\rangle, \tag{3.7}$$

where "$*$" is defined in Display (3.1). Clearly, H is an ideal of the Boolean algebra $\mathbf{A} \coprod \mathbf{B}$ (this idea modifies an intuition due to Banaschewski [3]).

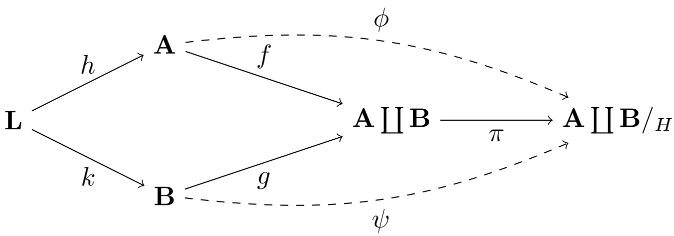
Lemma 3.12 will be fundamental to our discourse:

Lemma 3.12 (Lagrange [30], Banaschewski[3]). *For any $a \in A$, $b \in B$, if $a * b \in H$, then there already exists $l \in L$ such that $a * b \leq h(l) * k(l)'$.*

Let us now take into account the *pushout* diagram of the formation in Display (3.6), which will be the coproduct $\mathbf{A} \coprod \mathbf{B}$ modulo the ideal H in Display (3.7).[2]

[2] In fact, the interested reader may verify that H assumes the role of the coequalizer.

The corresponding diagram is the following:



(3.8)

where the maps ϕ and ψ are defined by:

$$\phi = \pi \circ f \text{ and } \psi = \pi \circ g.$$

THEOREM 3.13. *The maps ϕ and ψ are embeddings. Moreover, Diagram (3.8) commutes.*

PROOF: We show that $\ker(\phi) = \{0\}$, the case for ψ being similar. Suppose by contradiction that $0 \neq a \in A$ and $\phi(a) = 0$. Then $f(a) \in H$. By the definition of H in Display (3.7), this means that

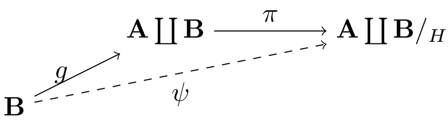
$$f(a) \leq \bigvee_{i=1}^{n} \big(h(l_i) * k(l_i)'\big) \qquad \text{for some } l_1, \ldots, l_n \in L. \tag{3.9}$$

By Lemma 3.12, there exists $l \in L$ such that

$$f(a) = (a, 1) \leq h(l) * k(l)' = f \circ h(l) \wedge g \circ k(l)' = (h(l), k(l)').$$

This entails both $a \wedge h(l) \neq 0$ and $k(l)' = 1$, i.e. $k(l) = 0$. Since k is an embedding, $k(l) = 0$ implies $l = 0$; but then $h(l) = 0$, and thus $a \wedge h(l) = a \wedge 0 = 0$, contradicting $a \wedge h(l) \neq 0$.

Thus no nonzero $a \in A$ can belong to $\ker(\phi)$, and ϕ is injective. The same argument applies dually to the other cone of the diagram:



Finally, commutativity of Diagram (3.8) follows directly from the definition $\phi = \pi \circ f$, $\psi = \pi \circ g$. □

It is well known (see e.g. [28, Lemma 5.22]) that, in Boolean algebras, for any ideal I,

$$x/I = y/I \text{ if and only if } x\Delta y = (x \wedge y') \vee (x' \wedge y) \in I.$$ [3]

Now note that, by definition of H, for any $a \in L$ we have

$$f \circ h(a) \wedge (g \circ k(a))' = (h(a), k(a)') \in H,$$

and moreover

$$(f \circ h(a))' \wedge g \circ k(a) = (h(a)', k(a)) \in H.$$

By closure of H under joins, it follows that

$$(f \circ h(a) \wedge (g \circ k(a))') \vee ((f \circ h(a))' \wedge g \circ k(a)) \in H.$$

Hence,

$$(f \circ h(a))\Delta(g \circ k(a)) \in H,$$

and therefore

$$(f \circ h(a))/_H = (g \circ k(a))/_H.$$

By virtue of Theorem 3.13, this is equivalent to

$$\phi \circ h(a) = \pi \circ f \circ h(a) = \pi \circ g \circ k(a) = \psi \circ k(a).$$

In other words, the algebra $\mathbf{A} \coprod \mathbf{B}/_H$ provides an amalgam of the V-formation $(\mathbf{L}, \mathbf{A}, \mathbf{B})$ in Diagram (3.6).

[3] In other words, the *symmetric difference* Δ of x, y is in I.

Our next step is to show that this merging is in fact *strong*. Namely, if $\phi(a) = \psi(b)$ in $\mathbf{A}\coprod\mathbf{B}/_H$, for any $a \in A$, $b \in B$, then $a \in L$, i.e. $h^{-1}(a) = k^{-1}(b) = \{l\}$ for a unique $l \in L$. In other words, $\mathbf{L}$ is indistinguishably mapped into $\mathbf{A}\coprod\mathbf{B}/_H$ by any cone of Diagram (3.8).

This is exactly the content of Theorem 3.14:

THEOREM 3.14. *The Boolean V-formation* $(\mathbf{L}, \mathbf{A}, \mathbf{B})$ *is strongly amalgamated into* $\mathbf{A}\coprod\mathbf{B}/_H$.

PROOF: Suppose $\phi(a) = \psi(b)$ in $\mathbf{A}\coprod\mathbf{B}/_H$. This means $f(a)/_H = g(b)/_H$, i.e.

$$f(a) \wedge g(b)' \in H.$$

By Lagrange Lemma 3.12, there is a $l \in L$ such that

$$f(a) \wedge g(b)' = (a, b') \leq (h(l), k(l)').$$

By the Comparison Lemma 3.11 this implies

$$a \leq h(l) \quad \text{and} \quad b' \leq k(l)'.$$

Equivalently $b \geq k(l)$.

Now, since $\mathbf{A}\coprod\mathbf{B}/_H$ is an amalgam, we have $\phi(h(l)) = \psi(k(l))$. Thus

$$\phi(a) \leq \phi(h(l)) = \psi(k(l)) \leq \psi(b).$$

But $\phi(a) = \psi(b)$ by assumption, so we must have equalities throughout:

$$\phi(a) = \phi(h(l)), \qquad \psi(b) = \psi(k(l)).$$

As ϕ, ψ are embeddings, it follows that $a = h(l)$ and $b = k(l)$. Therefore a, b both come from the same $l \in L$, proving that the amalgam is strong. □

4. The amalgamation property for regular double Stone algebras: a constructive view

The first studies on the amalgamation property are to be traced back to the early work of Fraïssé, and consequences of this property have found fruitful applications in contemporary model theory, logic, and algebra.

Capitalizing on the results from Section 3, we investigate in this section the amalgamation property for the variety of regular double Stone algebras. We provide, in Theorem 4.1, a constructive proof of the amalgamation property for regular double Stone algebras. While this result had already been proved in [15], our proof proceeds along different lines and is constructive.

Theorem 4.1. *The variety of regular double Stone algebras enjoys the amalgamation property.*

Proof: Consider the V-formation

$$\begin{array}{ccc} & & \mathbf{C}_1 \\ & \overset{f_1}{\nearrow} & \\ \mathbf{A} & & \\ & \underset{f_2}{\searrow} & \\ & & \mathbf{C}_2 \end{array} \tag{4.1}$$

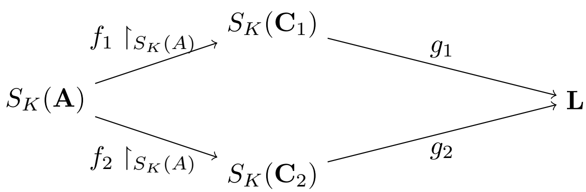
in the variety of regular double Stone algebras. By virtue of Theorem 2.4, we can describe any regular double Stone algebra $\mathbf{D}$ in the form $[\mathbf{B}, F]$, where $\mathbf{B}$ is a Boolean algebra isomorphic to $S_K(\mathbf{D})$, and F is a filter on $\mathbf{B}$ isomorphic to $D^{\sim}(\mathbf{D})$(see Theorem 2.4). Therefore, the formation in Display (4.1) can be rewritten as in Diagram (4.2).

$$\begin{array}{ccc} & & [S_K(\mathbf{C}_1), G_1] \\ & \overset{f_1}{\nearrow} & \\ [S_K(\mathbf{A}), F] & & \\ & \underset{f_2}{\searrow} & \\ & & [S_K(\mathbf{C}_2), G_2] \end{array} \tag{4.2}$$

Now by Theorem 2.4 (see Display (2.1) for the notation), any element a in A can be factored out as a pair $(\Box a, \Diamond a)$. As a consequence, by virtue of Theorem 2.4 we can factor the morphisms f_i, $i \in \{1, 2\}$, as

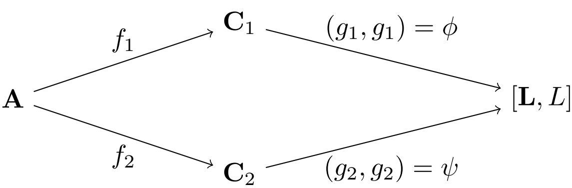
$$f_i(a) = f_i(\Box a, \Diamond a) = (f_i \restriction_{S_K(A)} (\Box a), f_i \restriction_{S_K(A)} (\Diamond a)). \tag{4.3}$$

Due to Theorem 3.14, which proves the strong amalgamation property for the case of Boolean algebras, we can construct a Boolean algebra $\mathbf{L}$ so that the V-formation finds a strong amalgam $\mathbf{L}$:



(4.4)

Then, we can close Diagram (4.2):



(4.5)

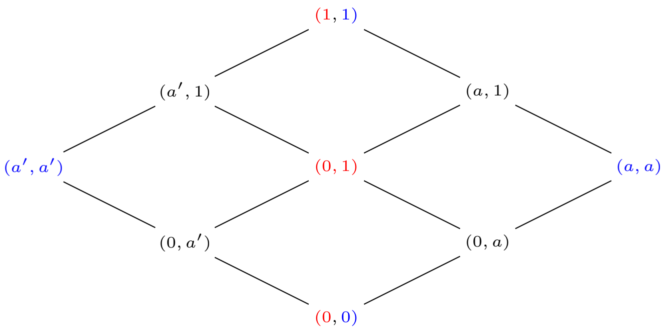
Furthermore, we have

$$\begin{aligned} (g_1, g_1) \circ f_1(x) &= (g_1, g_1)(f_1 \restriction_{S_K(A)} (\Box x), f_1 \restriction_{S_K(A)} (\Diamond x)) \\ &= (g_1(f_1 \restriction_{S_K(A)} (\Box x)), g_1(f_1 \restriction_{S_K(A)} (\Diamond x))) \\ &= (g_2(f_2 \restriction_{S_K(A)} (\Box x)), g_2(f_2 \restriction_{S_K(A)} (\Diamond x))) \\ &= (g_2, g_2) \circ f_2(x) \end{aligned} \tag{4.6}$$

which implies that the proof is complete. □

Example 4.2 below provides a concrete illustration of the construction of an amalgam for a V-formation in the variety of regular double Stone algebras.

Example 4.2. Consider the V-formation $(\mathbf{2}, \mathbf{3}, \mathbf{4})$, where $\mathbf{2}, \mathbf{4}$ are the two and four-element Boolean algebras respectively, and $\mathbf{3}$ is the three-element regular double Stone algebra. Then, the algebra $\mathbf{9} = [\mathbf{4}, 4]$:



is an *amalgam* for the V-formation $(\mathbf{2}, \mathbf{3}, \mathbf{4})$, where the images of $\mathbf{3}$ and $\mathbf{4}$ are coloured in red and blue, respectively. Indeed, $\mathbf{9}$ is a *strong amalgam*. In fact, the intersection of the images of $\mathbf{3}$, $\mathbf{4}$ is exactly the image of $\mathbf{2}$ in $\mathbf{9}$.

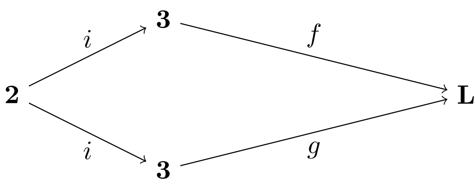
Let us remark that Example 4.2 is a clear case of an amalgam that closes a V-formation that presents Boolean and non-Boolean components, namely $\mathbf{2}, \mathbf{4}$ are Boolean algebras, whilst $\mathbf{3}$ is non-Boolean. It is worth noting that such an amalgam must necessarily have a non-empty core (or, equivalently, it must possess a fixpoint for Kleene negation). We shall later see that the presence or absence of this fixpoint plays a crucial role in the development of our arguments.

We have seen that Theorem 4.1 smoothly extends the construction of the amalgam that we have discussed in Section 3. However, we shall see that the theory of regular double Stone algebras diverges from that of Boolean algebras if we aim at stronger forms of amalgamation.

4.1. Failure of the strong amalgamation property for RDS

Example 4.3 below describes a V-formation of regular double Stone algebras which is not strongly amalgamable in **RDS**, thus showing that the variety **RDS** does not possess the strong amalgamation property.

Example 4.3. Consider the V-formation $(\mathbf{2}, \mathbf{3}, \mathbf{3})$, where $\mathbf{2}, \mathbf{3}$ are as in Example 4.2. Note that if the strong amalgamation property were true in the variety of regular double Stone algebras, i.e.



(4.7)

for some $\mathbf{L}$, then $f(\mathbf{3}) \cap g(\mathbf{3}) = \mathbf{2}$. However, this is impossible because $\mathbf{L}$ must possess a non-empty core, in particular $f(\mathbf{3}) \cap g(\mathbf{3}) = \mathbf{3}$.

In fact, Theorem 4.4 was first observed by Fussner.[4]

THEOREM 4.4. *The variety of regular double Stone algebras fails to have the strong amalgamation property.*

We will see in Section 5 that Fussner Theorem does not hold any longer if we expand the language by a special element k. The motivation of this failure will become evident in the proof of Theorem 5.6.

5. The variety of centered regular double Stone algebras

In this section we define and investigate a new variety, closely related to the variety **RDS**, called centered regular double Stone algebras.

Let $\mathbf{A} \in \mathbf{RDS}$. An element k in $\mathbf{A}$ is called a *center* of $\mathbf{A}$ if k satisfies:

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[4] Personal communication.

$$k^{\sim} = 0 \text{ and } k^{+} = 1. \tag{K}$$

According to Lemma 2.6, Condition (K) is equivalent to $k = k'$, where the operation $'$ is defined as in Equation (2.4). In other words, k is a fixed point of the operation $'$.

We expand the language $L = (\wedge, \vee, ^{\sim}, ^{+}, 0, 1)$ of regular double Stone algebras by a new constant k to the language

$$L^k = (\wedge, \vee, ^{\sim}, ^{+}, 0, k, 1),$$

of type $(2, 2, 1, 1, 0, 0, 0)$.

We will now introduce the variety of centered regular double Stone algebras.

Definition 5.1. An algebra $\mathbf{A} = (A, \wedge, \vee, ^{\sim}, ^{+}, 0, k, 1)$ in the language L^k is called a *centered regular double Stone algebra* if and only if:

1. $(A, \wedge, \vee, ^{\sim}, ^{+}, 0, 1)$ is a regular double Stone algebra;
2. the constant k is a center of $\mathbf{A}$.

It is evident that the class of all centered regular double Stone algebras in the language L^k forms a variety. Let $\mathbf{RDS}^k$ denote the variety of all centered regular double Stone algebras.

It is proved in [49] that the variety **RDS** is a discriminator variety (see [8] for an extensive discussion on discriminator varieties). It immediately follows that the variety $\mathbf{RDS}^k$ is a discriminator variety.

Lemma 5.2. *[9, Corollary 6.9] Let* $\mathbf{V}$ *be a discriminator variety and let* S *be the class of simple algebras in* $\mathbf{V}$. *Then the following are equivalent:*

1. *Every* $\mathbf{V}$*-epimorphism is a surjection.*[5]
2. *Every* S*-epimorphism is a surjection.*

Lemma 5.3. *[22, Corollary 2.5.23, page 52] Let* $\mathbf{V}$ *be a variety. Then* $\mathbf{V}$ *has the strong amalgamation property if and only if* $\mathbf{V}$ *has the amalgamation property and every* $\mathbf{V}$*-epimorphism is a surjection.*

[5] A morphism f in a class of algebras is an *epimorphism* in case $k \circ f = h \circ f$ implies that $k = h$.

Theorem 5.4 will be useful to prove Theorem 5.6:

THEOREM 5.4 (Beazer [4]). *The unique non-Boolean subdirectly irreducible regular double Stone algebra is* $\mathbf{3}$.

Let $\mathbf{3}^k$ denote the 3-element centered regular double Stone algebra. Then the following corollary is immediate from 5.4.

COROLLARY 5.5. The only non-trivial subdirectly irreducible (simple) centered regular double Stone algebra is $\mathbf{3}^k$.

THEOREM 5.6. *The variety* $\mathbf{RDS}^k$ *(in the language* L^k*) enjoys the strong amalgamation property.*

PROOF: By virtue of Theorem 4.1 any V-formation in the variety of regular double Stone algebras has an amalgam, see Diagram (4.5). We are left with the task of showing that this amalgam is strong. We have already noted that the variety $\mathbf{RDS}^k$ is a discriminator variety. Moreover, since, by Corollary 5.5, $\mathbf{RDS}^k$ has $\mathbf{3}^k$ as the only simple algebra in which every epimorphism is trivially surjective, it follows from Lemma 5.2 that every $\mathbf{V}$-epimorphism is a surjection. Therefore, by Lemma 5.3 $\mathbf{RDS}^k$ enjoys the strong amalgamation property. □

The proof of Theorem 5.6 is, in our view, interesting also because it explains, from a different perspective the motivation for which the strong amalgamation property fails if we consider the variety of regular double Stone algebras in the language without the constant k.[6] In fact, in such a case the simple members of the variety are $\mathbf{2}, \mathbf{3}$, and $i : \mathbf{2} \to \mathbf{3}$ is the obvious inclusion, and the sole possible morphism. Now, for $f : \mathbf{A} \to \mathbf{2}$ and $g : \mathbf{A} \to \mathbf{2}$, if $f \circ i = g \circ i$, clearly $f = g$. However, i is not a surjection, i.e. in the variety of regular double Stone algebras without fixpoint epimorphisms are not surjective. So by Lemma 5.3 the strong amalgamation property need not hold in general, as stated in Theorem 4.4.

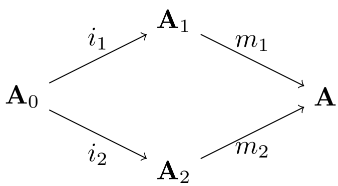
[6]This was also observed by Düntsch [16, Corollary 3] in a different context as a direct corollary of Katriňák's Theorem [27]. Actually, the variety of regular double Stone algebras contains epimorphisms that are not surjective. We sketch here a new constructive and direct argument that proves this fact.

6. The super-amalgamation property

The super-amalgamation property is a rather strong algebraic property that entails quite important logical properties, such as the Craig interpolation property and the Maehara property [17], if the class of algebras under consideration is the equivalent algebraic semantics of a certain logic. The investigation of these connections traces back to the seminal works of Maximova [35, 36, 37], and Pitt [45] from the mid seventies.

Let us begin with the definition of the super-amalgamation property for a class of partially ordered algebras.

Definition 6.1 (Maksimova). Let K be a class of partially ordered algebras. We say that K has the *super-amalgamation property* (*SUPAP* for short) if for any V-formation $(\mathbf{A}_0, \mathbf{A}_1, \mathbf{A}_2)$ in K, there is an algebra $\mathbf{A}$ in K and embeddings m_1, m_2 such that Diagram (6.1) commutes:



(6.1)

where, for embeddings $i_1 : \mathbf{A}_0 \to \mathbf{A}_1$ and $i_2 : \mathbf{A}_0 \to \mathbf{A}_2$, there exist an $\mathbf{A} \in K$ and embeddings $m_1 : \mathbf{A}_1 \to \mathbf{A}$ and $m_2 : \mathbf{A}_2 \to \mathbf{A}$ such that $m_1 \circ i_1 = m_2 \circ i_2$, and the following *Maksimova property* holds:

$$(\forall x \in \mathbf{A}_j)(\forall y \in \mathbf{A}_k)\,(m_j(x) \leq m_k(y) \Rightarrow (\exists z \in \mathbf{A}_0)(x \leq i_j(z) \,\&\, i_k(z) \leq y))\,, \tag{MP}$$

where $\{j, k\} = \{1, 2\}$.

Capitalizing on the results in Section 3, we firstly provide a novel proof of the fact that the variety $\mathbf{BA}$ of Boolean algebras enjoy the SUPAP.

Theorem 6.2. *The variety* $\mathbf{BA}$ *enjoys the super-amalgamation property.*

PROOF: Let $(\mathbf{L}, \mathbf{A}, \mathbf{B})$ be a V-formation of Boolean algebras with embeddings $h : L \to A$, $k : L \to B$. Let $\mathbf{A} \coprod \mathbf{B}/_H$ be the amalgam constructed in Theorem 3.14. Suppose $\phi(a) \le \psi(b)$ in $\mathbf{A} \coprod \mathbf{B}/_H$, with $a \in A$ and $b \in B$. Then

$$\pi(f(a)) \le \pi(g(b)).$$

By definition of the order in the quotient, this means

$$\pi(f(a) \wedge g(b)') = 0,$$

hence

$$f(a) \wedge g(b)' \in H.$$

By Lagrange Lemma 3.12, for some $l \in L$,

$$(a, b') \le (h(l), k(l)').$$

By the Comparison Lemma 3.11, this inequality implies

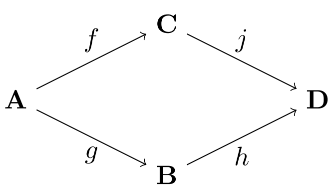
$$a \le h(l) \quad \text{and} \quad b' \le k(l)',$$

that is, $a \le h(l)$ and $k(l) \le b$.

Therefore, whenever $\phi(a) \le \psi(b)$ in the amalgam, there exists $l \in L$ with $a \le h(l)$ and $k(l) \le b$. This is precisely the order-reflecting property required for super-amalgamation. □

Let us recall from Pitt [45] the notion of interpolation (we refer the reader also to the extensive discussion by Maksimova [35, 36, 37]).

DEFINITION 6.3. Consider the commutative square

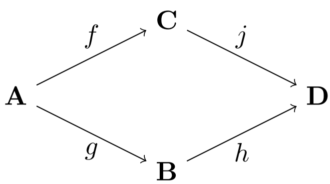


(6.2)

where $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ are partially ordered algebras with order preserving maps.

We say that Diagram (6.3) enjoys the *interpolation property* if whenever for $b \in B, c \in C$ such that $h(b) \leq j(c)$, there is $a \in A$ so that $b \leq g(a)$ and $f(a) \leq c$.

THEOREM 6.4. *If a regular double Stone algebra* $\mathbf{A}$ *possesses a non-empty core, then for every* V*-formation* $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ *the commutative diagram:*



(6.3)

enjoys the interpolation property.

PROOF: Observe first that if $\mathbf{A}$ has a non-empty core, then every algebra appearing in the diagram must also have a non-empty core; otherwise, no homomorphisms between the algebras could exist, since the central element k can only be mapped to another central element.

Suppose that, for some $b \in B$ and $c \in C$, we have $h(b) \leq j(c)$ in $\mathbf{D}$, where $\mathbf{D}$ can be regarded as a strong amalgam of $(\mathbf{A}, \mathbf{B}, \mathbf{C})$, due to Theorem 5.6 and the fact that $\mathbf{A}$ possesses a non-empty core. By virtue of Theorem 2.4, b and c can be identified with the pairs $(\Box b, \Diamond b)$ and $(\Box c, \Diamond c)$, respectively. We may observe that in the V-formation

$$[S_K(\mathbf{A}), S_K(\mathbf{B}), S_K(\mathbf{C})]$$

the elements $\Box b, \Diamond b, \Box c, \Diamond c$ have Boolean interpolant elements a, d in the Boolean algebra $S_K(\mathbf{A})$. Specifically, we can assume that a, d satisfy

$$\Box b \leq g(a) \quad f(a) \leq \Box c$$

and

$$\Diamond b \leq g(d) \quad f(d) \leq \Diamond c.$$

Fix such elements $a, d \in S_K(\mathbf{A})$. In accordance with Theorem 2.4, since a, d are Boolean elements, they may be regarded as pairs of the form

$$a = (\Box a, \Diamond a) = (a, a), \quad d = (\Box d, \Diamond d) = (d, d).$$

This observation will play a pivotal role in the construction of the interpolant.

Since $\mathbf{A}$ has a non-empty core, by the construction in Display (A), $\mathbf{A}$ is of the form $[S_K(\mathbf{A}), S_K(A)]$.

Our goal is to identify an element $x = (\Box x, \Diamond x) \in A$ such that

$$b = (\Box b, \Diamond b) \leq g((\Box x, \Diamond x)) = g(x)$$

and

$$f(x) = f((\Box x, \Diamond x)) \leq (\Box c, \Diamond c) = c.$$

To this end, consider the element

$$x = (a \wedge d, a \vee d).$$

Clearly, $a \wedge d \leq a \vee d$, and since $\mathbf{A}$ has a non-empty core, this suffices to ensure that $(a \wedge d, a \vee d) \in A$, because both $a \wedge d, a \vee d \in S_K(A)$, and the universe of A is

$$\{(x, y) \in S_K(A)^2 : x \leq y\}.$$

We now verify that the element $x = (a \wedge d, a \vee d)$ is indeed the desired interpolant. To check that $b \leq g((a \wedge d, a \vee d))$, we focus on the maps g and f restricted to the sharp elements (see Theorem 2.4). From the fact that $\Box b \leq g(a)$ and $\Box b \leq \Diamond b \leq g(d)$, we obtain that

$$\Box b \leq g(a \wedge d).$$

Moreover, because $\Diamond b \leq g(d)$, then

$$\Diamond b \leq g(a \vee d).$$

Therefore, $b = (\Box b, \Diamond b) \leq g((a \wedge d, a \vee d))$. To verify that

$$f((a \wedge d, a \vee d)) \leq c,$$

the argument is dual. This establishes our claim. □

Therefore, we immediately obtain Theorem 6.5:

THEOREM 6.5. *The variety* $\mathbf{RDS}^k$ *enjoys the super-amalgamation property.*

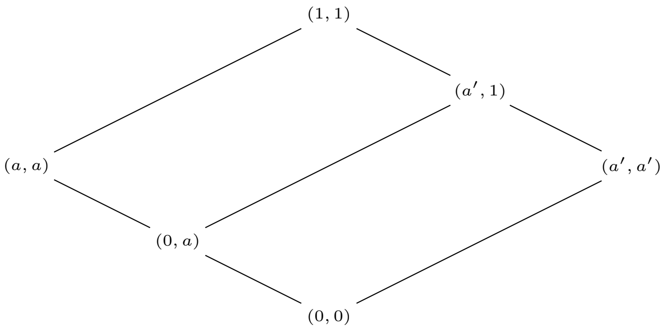
PROOF: By Theorem 4.1, every V-formation admits an amalgam that renders commutative the corresponding diagram. Moreover, by Theorem 6.4 any diagram satisfies the interpolation property. Therefore, the super-amalgamation property (Definition 6.1) follows. □

7. Examples and applications

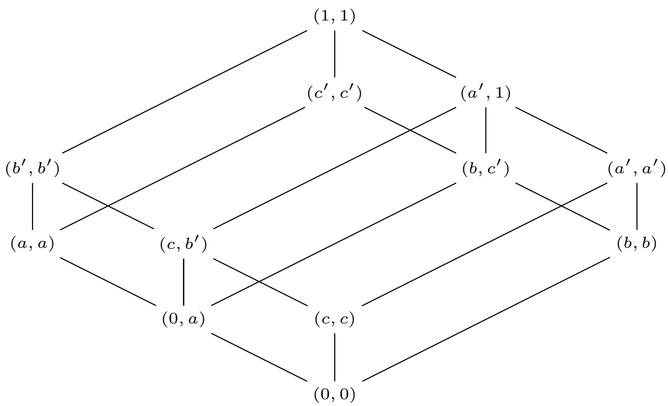
In this section we provide a collection of examples of regular double Stone algebras that will be expedient to describe concretely the construction of some interesting amalgams in the variety of regular double Stone algebras.

Furthermore, we elaborate on the significance of Theorem 4.4 by providing examples that show the existence of V-formations that fail to have a strong amalgam in the variety of regular double Stone algebras.

Example 7.1. Consider the 4-element Boolean algebra $\mathbf{4}$, whose carrier is $\{0, a, a', 1\}$ together with the filter $\{a', 1\}$. Then, the regular double Stone algebra $[\mathbf{4}, \{a', 1\}]$ will be the following:



Example 7.2. Consider the 8-element Boolean algebra **8**, whose carrier is $\{0, a, b, c, a', b', c', 1\}$ together with the filter $\{a', 1\}$. Then, the regular double Stone algebra $[\mathbf{8}, \{a', 1\}]$ will be the following:



As mentioned in Section 4, the variety of regular double Stone algebras does not fulfill the strong amalgamation property in general. However, in case a V-formation $(\mathbf{A}_0, \mathbf{A}_1, \mathbf{A}_2)$ is *homogeneous*, in the sense that all algebras in the formation possess a non-empty core, then the formation admits a strong amalgam. Actually, it is straightforward to observe that in case $\mathbf{A}_0$ possesses a non-empty core the whole V-formation must be homogeneous.

Example 7.1 will be relevant to this discourse.

In fact, it can be seen that $([\mathbf{4}, \{a', 1\}], [\mathbf{4}, 4], [\mathbf{8}, \{a', 1\}])$ is a V-formation, where the mappings f_1, f_2 are self-evident. To ease the notation, let us rename the V-formation as $(\mathbf{A}, \mathbf{C}_1, \mathbf{C}_2)$. We can observe that

$$f_1((a', 1)) = (f_1 \restriction_{S_K(A)} (a'), f_1 \restriction_{S_K(A)} (1)) = (a', 1) \text{ in } C_1.$$

Clearly, $(\mathbf{4}, \mathbf{4}, \mathbf{8})$ is a Boolean V-formation of finite algebras. By Theorem 3.14, the algebra $\mathbf{4} \coprod \mathbf{8}/_H$ is an amalgam. As we have seen in Theorem 4.1 and Theorem 5.6 we construct

$$[\mathbf{4} \coprod \mathbf{8}/_H, \mathbf{4} \coprod \mathbf{8}/_H], \tag{7.1}$$

and we obtain a regular double Stone algebra with a non-empty core which amalgamates the formation

$$([\mathbf{4}, \{a', 1\}], [\mathbf{4}, 4], [\mathbf{8}, \{a', 1\}]).$$

Let us recall that the algebra in Display (7.1) is not a strong amalgam. Indeed, the same reasoning applies in Example 7.3.

Theorem 4.4 follows from general algebraic facts. However, it may be interesting to find a concrete case in which a V-formation of regular double Stone algebras does not have a strong amalgam within the same variety.

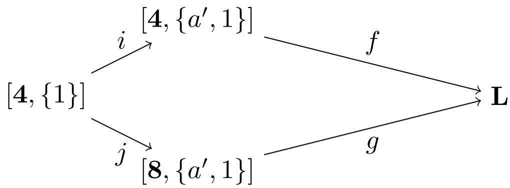
In Example 7.3, we propose a direct application of Theorem 4.4. In fact, making a straightforward use of the subdirectly irreducible members in the variety of regular double Stone algebras it can be shown that in general this variety does not fulfill the strong amalgamation property.

A more elaborated counterexample is Example 7.3. In fact, Example 7.3 presents the case of a V-formation homogeneous for not possessing a fixpoint which can not be strongly amalgamated in the variety of regular double Stone algebras.

Example 7.3. Consider the V-formation

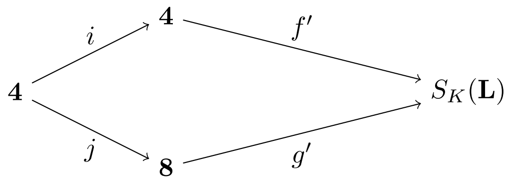
$$([\mathbf{4}, \{1\}], [\mathbf{4}, \{a', 1\}], [\mathbf{8}, \{a', 1\}]),$$

where $[\mathbf{4}, \{a', 1\}]$ and $[\mathbf{8}, \{a', 1\}]$ are as in Example 7.1 and Example 7.2, respectively. Suppose that there exists a strong amalgam $\mathbf{L}$ as in Diagram (7.2):



(7.2)

Note that Diagram (7.2) is determined by Diagram (7.3):



(7.3)

Moreover, $S_K(\mathbf{L})$ is a strong Boolean amalgam. Indeed, for any element $x = (\Box x, \Diamond x)$, and any homomorphism in Diagram (7.2):

$$f((\Box x, \Diamond x)) = (f'(\Box x), f'(\Diamond x)).$$

By construction, the element $(a', 1)$ is in both $[\mathbf{4}, \{a', 1\}]$ and $[\mathbf{8}, \{a', 1\}]$. Moreover, by Condition (Regularity), $(a', 1)$ is the unique unsharp element in the interval $[(a', a'), (1, 1)]$.

Therefore, we have that

$$f((a', 1)) = (f'(a)', f'(1)) = (g'(a)', g'(1)) = g((a', 1)).$$

However, $(a', 1)$ is not the image of any element in $[\mathbf{4}, \{1\}]$, since $(a', 1)$ is unsharp, and $[\mathbf{4}, \{1\}]$ is a Boolean algebra.

Thus, the V-formation $([\mathbf{4}, \{1\}], [\mathbf{4}, \{a', 1\}], [\mathbf{8}, \{a', 1\}])$ does not admit any strong amalgam in the variety of regular double Stone algebras.

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A PROOF-THEORETIC INTERPOLATION THEOREM FOR INQUISITIVE PROPOSITIONAL LOGIC

Abstract

This paper presents a sequent calculus for Inquisitive Propositional Logic obtained by expanding the sequent calculus g3ip for intuitionistic propositional logic with suitable rules for double negation elimination for atoms and the Split Property. A suitable rule for the Split Property is obtained by taking advantage of the connection between the truth-conditional fragment in Inquisitive Logic and Harrop formulas. The paper proves admissibility of cut for the sequent calculus and uses the sequent calculus to prove interpolation for Inquisitive Propositional Logic. Interpolation is obtained using Maehara's lemma.

Keywords: Inquisitive Logic, Harrop formulas, interpolation, Maehara's lemma, intermediate logics.

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1. Introduction

Inquisitive propositional logic, referred to as basic inquisitive logic (InqB) by [3], is a propositional logic obtained by expanding intuitionistic propositional logic (IPL) with double negation elimination for atoms (DNEp) and the Kreisel-Putnam axiom (KP). The result is a logic for which uniform substitution of arbitrary formulas for atoms does not hold because some formulas will not satisfy double negation elimination. The logic was first presented by [6] and the research programme in its contemporary shape can be found in [4, 3]. The logic is developed to model reasoning involving both statements and questions by interpreting the intuitionistic disjunction $A \vee B$ as "*whether* A *or* B?".

Natural deduction systems for InqB can be found in [3, 9] and labelled sequent calculi based on the support semantics for InqB can found in [2, 9]. However, there is no simple and standard two-sided sequent calculus for InqB, that is, a calculus obtained by extending a standard sequent calculus for IPL with appropriate introduction rules equivalent to DNEp and KP and for which cut is admissible. In addition, there is no syntactic proof of an interpolation theorem for InqB. In fact, I am not aware of any interpolation theorem for InqB. While there is an interpolation theorem for the variant of inquisitive propositional logic explored in [5], and the "folklore" is that their result transfers to InqB, that remains unpublished. The aim of this paper is to rectify this situation by developing such a sequent calculus for InqB and use it to provide a syntactic proof of interpolation specifically for InqB.

The next section presents InqB and highlights the connection with Harrop formulas. The third section presents the sequent calculus. The fourth section establishes that cut is admissible for the sequent calculus and the fifth section presents the interpolation theorem. The final section concludes by briefly discussing decidability and future work.

2. Towards a sequent calculus for InqB

I will use a standard propositional language with countably many atoms, $\to$, $\wedge$, $\vee$ and $\bot$. Lower case Latin letters starting from p represent atoms, and upper case Latin letters starting from A represent arbitrary formulas. $\neg A$ is defined as $A \to \bot$. As mentioned above, InqB can be obtained by expanding an axiomatisation of IPL with the following axioms:

$$\neg\neg p \to p \tag{DNEp}$$

$$(\neg C \to (A \vee B)) \to ((\neg C \to A) \vee (\neg C \to B)) \tag{KP}$$

The last axiom is the Kreisel-Putnam axiom. Going forward, I will for readability drop the innermost parentheses in both the antecedent and the consequent by writing

$$(\neg C \to A \vee B) \to (\neg C \to A \vee \neg C \to B)$$

This also goes for variations of this axiom and rules based on them. Thus, unless otherwise specified, a formula of the form $C \to A \vee B$ should be read as $C \to (A \vee B)$ while a formula of the form $C \to A \vee C \to B$ should be read as $(C \to A) \vee (C \to B)$.

KP can be replaced with the following axiom where α is a $\vee$-free formula:

$$(\alpha \to A \vee B) \to (\alpha \to A \vee \alpha \to B) \tag{Split$\vee$}$$

They are intersubstitutable because every formula for which DNE holds is equivalent to a $\vee$-free formula and every $\vee$-free formula is equivalent to some negated formula within InqB [3, p. 65].

In IPL, DNE only holds for a formula A if A or $A \to \bot$ is a theorem of IPL. InqB strictly extends the set of formulas for which DNE holds. This fragment is referred to as the truth-conditional fragment [4, 3].

As demonstrated by [12], there is an intimate connection between Harrop formulas and the truth-conditional fragment of the language. A (propositional) Harrop formula as introduced in [7] is defined inductively as follows: atoms and $\bot$ are Harrop formulas; if A and B are Harrop formulas, then $A \wedge B$ is a Harrop formula; if A is a Harrop formula, then for any formula

C, $C \to A$ is a Harrop formula. Harrop formulas were introduced to extend the disjunctive property for IPL from just theorems as follows:

Proposition 2.1 (Harrop [7]). For every Harrop formula $\mathcal{H}$, If $\mathcal{H} \to A \vee B$ is a theorem of IPL, then either $\mathcal{H} \to A$ or $\mathcal{H} \to B$ is a theorem of IPL.

With InqB, this becomes an *object-theoretic* statement in the form of the Split property which then holds for every truth-conditional formula.

Trivially, every negated formula and every $\vee$-free formula is a Harrop formula, and it is shown in [9] that every Harrop formula is truth-conditional. Of course, there are truth-conditional formulas that are not Harrop formulas. This includes not only theorems such as the instances of $A \to (A \vee B)$ but also other formulas such as $(p \to (p \vee q)) \wedge p$. DNE holds for this formula in InqB. That being said, the truth-conditionality of the latter kind of formulas can be accounted for through the notion of a Harrop expansion. In particular, Harrop expansions of truth-conditional formulas are truth-conditional formulas:

- If α and β are truth-conditional, then $\alpha \wedge \beta$ is truth-conditional.
- If α is truth-conditional and C is any formula, then $C \to \alpha$ is truth-conditional.

Establishing whether every truth-conditional formula is either a Harrop formula, a theorem or a Harrop expansion is beyond the scope of this paper. Instead, it suffices for our purposes to observe that Split for Harrop formulas implies Split for $\vee$-free formulas and KP. One can thus define InqB by expanding IPL with DNEp and the following Split axiom for Harrop formulas where $\mathcal{H}$ is a Harrop formula:

$$(\mathcal{H} \to A \vee B) \to (\mathcal{H} \to A \vee \mathcal{H} \to B) \qquad (\text{Split}\mathcal{H})$$

This will turn out to be extremely practical in the following.

3. A sequent calculus for InqB

I present in this section the sequent calculus for InqB together with some basic observations about it. Roughly, the sequent calculus is obtained by expanding the single-succedent sequent calculus g3ip for IPL from [10] with suitable rules to capture DNEp and the Split axiom for Harrop formulas.

Definition 3.1. Let g3InqB be the sequent calculus based on sequents of the form $\Gamma \Rightarrow A$ where Γ is a multiset of formulas with the following initial sequents and rules where A, B and C are arbitrary formulas, p is an atom and $\mathcal{H}$ is a Harrop formula:

$$p, \Gamma \Rightarrow p \qquad \bot, \Gamma \Rightarrow C$$

$$\frac{A, \Gamma \Rightarrow C \qquad B, \Gamma \Rightarrow C}{A \vee B, \Gamma \Rightarrow C}\ \vee\text{L} \qquad \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B}\ \vee\text{R}_1 \qquad \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B}\ \vee\text{R}_2$$

$$\frac{A, B, \Gamma \Rightarrow C}{A \wedge B, \Gamma \Rightarrow C}\ \wedge\text{L} \qquad \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\ \wedge\text{R}$$

$$\frac{A \to B, \Gamma \Rightarrow A \qquad B, \Gamma \Rightarrow C}{A \to B, \Gamma \Rightarrow C}\ \to\text{L} \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \to B}\ \to\text{R}$$

$$\frac{\neg p, \Gamma \Rightarrow \bot}{\Gamma \Rightarrow p}\ \text{Raa-at}$$

$$\frac{\mathcal{H} \to A, \Gamma \Rightarrow C \qquad \mathcal{H} \to B, \Gamma \Rightarrow C}{\mathcal{H} \to A \vee B, \Gamma \Rightarrow C}\ \mathcal{H}\mathit{Split}\text{L}$$

The sequent calculus g3ip is thus g3InqB without *Raa-at* and $\mathcal{H}$*Split*L. For each application of a rule, I will refer to Γ and C as the context, while the other formula displayed in the conclusion-sequent is the principal formula, and the other formulas displayed in the premise-sequents are the active formulas. The formulas in Γ and C are parametric formulas.

As in the case of g3ip, the following *structural* rules are admissible in g3InqB:

$$\frac{\Gamma \Rightarrow C}{\Gamma', \Gamma \Rightarrow C}\ \text{Weakening} \qquad \frac{A, A, \Gamma \Rightarrow C}{A, \Gamma \Rightarrow C}\ \text{Contraction}$$

$$\frac{\Gamma \Rightarrow A \qquad A, \Gamma' \Rightarrow C}{\Gamma, \Gamma' \Rightarrow C}\text{ cut}$$

This will be demonstrated below and in the next section.

The rule *Raa-at* is from [10, p. 158]. I am not aware of any previous presentation of the rule $\mathcal{H}$*Split*L (or the corresponding one for KP or Split∨). Instead, one typically finds the following rule or the corresponding one for the ∨-free fragment:

$$\frac{\Gamma, \neg C \Rightarrow (A \vee B)}{\Gamma \Rightarrow \neg C \rightarrow A \vee \neg C \rightarrow B}\text{ KP-R}$$

See for example [11] for a natural deduction variant of this rule.

However, this rule is not particularly practical from the perspective of proof analysis and admissibility of cut, a property which is desirable for our purposes since proving interpolation through *Maehara's lemma* requires that atoms occurring in the premise-sequents also occur in the conclusion-sequents. If cut (or a similar rule) must be included as primitive rule in the sequent calculus, then that will not be the case.

Consider for example the following sequent as an instance of Split∨:

$$p \rightarrow A \vee B \Rightarrow p \rightarrow A \vee p \rightarrow B$$

There is no cut-free derivation of that sequent with KP-R. KP-R is thus not suitable for our purposes. Replacing KP-R with the corresponding rule for ∨-free formulas will not solve the issue, since there is in that case no cut-free derivation of for example

$$((p \vee q) \rightarrow r) \rightarrow (A \vee B) \Rightarrow (((p \vee q) \rightarrow r) \rightarrow A) \vee (((p \vee q) \rightarrow r) \rightarrow B)$$

With $\mathcal{H}$*Split*L and the corresponding rule obtained by formulating KP-R for Harrop formulas, this is not an issue.

I suspect that the strategy for establishing admissibility of cut employed below in section 4 will also work with KP-R for Harrop formulas. However, I prefer $\mathcal{H}$*Split*L over the corresponding right introduction rule for various aesthetic reasons. Most importantly, I understand the rule as a structural rule concerning the structure of the antecedent of a sequent: it imitates the use of set-theoretic union of two states in a proof of the validity of (Split$\mathcal{H}$) in the support semantics for InqB. Presenting the support semantics and the proof in question goes beyond the scope of this paper, but the reader may refer to [3] for a presentation of the support semantics. As I see it, a nice hypersequent calculus for InqB would include a structural rule with the appropriate effect.

In this regard, it is also worth comparing $\mathcal{H}$*Split*L with the "higher-level" natural deduction elimination rule $\vee E_+$ for $\vee$ presented by [9, p. 18]. That rule is employed to obtain a normalization theorem for a natural deduction system for InqB. While I will not present the rule itself, it corresponds directly to the following sequent calculus rule:

$$\dfrac{\mathcal{H},\Gamma \Rightarrow A \vee B \qquad \dfrac{\mathcal{D}[A/\mathcal{H}]}{\Gamma \Rightarrow C} \qquad \dfrac{\mathcal{D}[B/\mathcal{H}]}{\Gamma \Rightarrow C}}{\Gamma \Rightarrow C}\ \mathrm{S}_E$$

The expression $\mathcal{D}[A/\mathcal{H}]$ means that the derivation of the premise-sequent includes one or more applications of a rule of the following form:

$$\dfrac{A,\Gamma \Rightarrow C}{\mathcal{H},\Gamma \Rightarrow C}$$

It is assumed that the calculus includes one rule of this form for each pair of a Harrop formula and any formula. The formulation of a precise notion of a derivation is beyond the scope of this paper, but the rule S_E must be understood as "discharging" applications of such rules, and the root only counts as derivable if each application of such rules are "discharged" through an appropriate application of S_E.

The rule S_E is currently an elimination rule, but it corresponds to the following introduction rule:

$$\dfrac{\dfrac{}{\Gamma \Rightarrow C}\,[A/\mathcal{H}] \qquad \dfrac{}{\Gamma \Rightarrow C}\,[B/\mathcal{H}]}{\mathcal{H} \to A \vee B, \Gamma \Rightarrow C}\ \mathrm{S}_I$$

The rule $\mathcal{H}$*Split*L is now obtained by replacing the requirement on the derivations with the conditionals $\mathcal{H} \to A$ and $\mathcal{H} \to B$ as active formulas in the premise-sequent. This paper thus demonstrates that the more complicated notion of a derivation which is required to accommodate the "discharging" of applications of $A/\mathcal{H}$-rules is not needed to obtain a sequent calculus for InqB for which cut is admissible and *Maehara's lemma* is applicable.

The rule $\mathcal{H}$*Split*L can also be understood as a simplification of the approach by [1] to a sequent calculus for a related logic which can be described as propositional team logic expanded with inquisitive disjunction. The language in [1] includes two disjunctions, a tensor disjunction and an inquisitive disjunction, where $\vee$ is used for tensor disjunction. I will in my brief discussion continue to use $\vee$ for the inquisitive disjunction. The following *deep inference* rules are presented by [1] where $\Psi(A \vee B)$ is *roughly* a formula in which the subformula $A \vee B$ does not occur within the scope of a negation:

$$\frac{\Psi(A), \Gamma \Rightarrow C \qquad \Psi(B), \Gamma \Rightarrow C}{\Psi(A \vee B), \Gamma \Rightarrow C}\ \text{Deep-}\vee\text{L} \qquad \frac{\Gamma \Rightarrow \Delta, \Psi(A)}{\Gamma \Rightarrow \Delta, \Psi(A \vee B)}\ \text{Deep-}\vee\text{R}_1$$

Like the rule $\mathcal{H}$*Split*L, the deep inference rules decompose a formula from within. With deep inference rules, one requires only two rules for $\vee$ rather than $\vee$L, $\vee$R and $\mathcal{H}$*Split*L. However, $\mathcal{H}$*Split*L is considerably simpler by only considering a context of a specific form, which in turn simplifies the strategy for establishing admissibility of cut. Specifically, whereas the proof presented below is relatively direct, the proof that cut is admissible in the sequent calculus presented by [1] ends up being considerably more complex because the deep inference rules must be combined with certain syntactic restrictions on the context in other rules.

Finally, I note that the subformula property fails for g3InqB because $\mathcal{H} \to A$ is not a subformula of $\mathcal{H} \to (A \vee B)$ in the rule $\mathcal{H}$*Split*L, and $\neg p$ is not a subformula of p in the rule Raa-at. However, g3InqB still has a *subatom property* in the sense that every atom occurring as a subformula in a leaf of a derivation is a subformula of a formula in the root of that derivation. This holds for g3InqB because there is no rule that removes an atom from the premise-sequents. This property is key for the application of *Maehara's lemma* to establish the interpolation property.

In any case, the admissibility of cut implies the admissibility of modus ponens:

$$\dfrac{\Gamma \Rightarrow A \to B \qquad \Gamma' \Rightarrow A}{\Gamma, \Gamma' \Rightarrow B}\,\text{MP}$$

With this rule admissible, one can easily show the equivalence of g3InqB and InqB.

Proposition 3.2. If A is a theorem of InqB, then $\Rightarrow A$ is derivable in g3InqB.

Proof: InqB is IPL expanded with DNEp and KP. KP is subsumed by the Split axiom for Harrop formulas. Since every theorem of IPL is already derivable in g3InqB because the calculus extends g3ip (which is g3InqB without *Raa-at* and $\mathcal{H}$*Split*L), it suffices to show that MP remains admissible, and that the axioms DNEp and Split for Harrop formulas are derivable in g3InqB. Both axioms are obviously derivable, so the desired result follows from theorem 4.2 which implies that MP is admissible. □

Proposition 3.3. If $\Rightarrow A$ is derivable in g3InqB, then A is a theorem of InqB.

Proof: Since every theorem of IPL is a theorem of InqB, it follows that if $\Rightarrow A$ is derivable in g3ip, then A is a theorem of InqB. It is also the case that the sequent calculus obtained by expanding g3ip with MP, the Split axiom for Harrop formulas and DNEp defines InqB since InqB is IPL expanded with those axioms. It is thus left to show that the rules *Raa-at* and $\mathcal{H}$*Split*L are admissible in that sequent calculus. I illustrate the case of $\mathcal{H}$*Split*L where $\mathcal{H}(D)$ abbreviates $\mathcal{H} \to D$:

$$\dfrac{\Rightarrow \mathcal{H}(A\vee B)\to \mathcal{H}(A)\vee\mathcal{H}(B) \qquad \dfrac{\dfrac{\Gamma, \mathcal{H}(A\vee B) \Rightarrow \mathcal{H}(A\vee B) \qquad \dfrac{\mathcal{H}(A), \Gamma \Rightarrow C \quad \mathcal{H}(B), \Gamma \Rightarrow C}{\mathcal{H}(A)\vee\mathcal{H}(B), \Gamma \Rightarrow C}\,\vee\text{L}}{\mathcal{H}(A\vee B), \mathcal{H}(A\vee B)\to\mathcal{H}(A)\vee\mathcal{H}(B), \Gamma \Rightarrow C}\,\to\text{L}}{\mathcal{H}(A\vee B), \Gamma \Rightarrow (\mathcal{H}(A\vee B)\to\mathcal{H}(A)\vee\mathcal{H}(B))\to C}\,\to\text{R}}{\mathcal{H}(A \vee B), \Gamma \Rightarrow C}\,\text{MP}$$

□

I now state some basic lemmas that are required for admissibility of cut. As is usual, the height of a derivation is defined as the longest branch in a derivation. I write $n \vdash \Gamma \Rightarrow C$ for the claim that there is a derivation with height at most n of $\Gamma \Rightarrow C$.

Lemma 3.4. *The following properties hold in* g3InqB*:*

(a) Height-preserving weakening:

- *if* $n \vdash \Gamma \Rightarrow C$ *then* $n \vdash \Gamma', \Gamma \Rightarrow C$

(b) Height-preserving inversion:

- *For every rule* $\dfrac{\sigma_0 \quad \dots \quad \sigma_m}{\sigma}$ *except* $\to$*L and* $\vee$*R, if* $n \vdash \sigma$ *then* $n \vdash \sigma_i$ *for* $i \leq m$.
- *For* $\to$ *L, if* $n \vdash A \to B, \Gamma \Rightarrow C$ *then* $n \vdash B, \Gamma \Rightarrow C$.

(c) Height-preserving admissibility of contraction:

- *if* $n \vdash A, A, \Gamma \Rightarrow C$ *then* $n \vdash A, \Gamma \Rightarrow C$

Proof: The proofs are standard, and the reader is referred to [10] for the details with only two exceptions. Inversion for $\mathcal{H}$*Split*L differs slightly from that of $\vee$L, and relies on the inversion of $\vee$L. In addition, one must also consider $\mathcal{H}$*Split*L in the case of inversion for the right premise of $\to$L.

The proofs proceed by induction on the height of a derivation. In the case of the inductive step for the inversion of $\mathcal{H}$*Split*L I reason as follows. Assume that $n{+}1 \vdash \mathcal{H} \to A \vee B, \Gamma \Rightarrow C$. If $\mathcal{H} \to A \vee B$ is not principal, then the inductive hypothesis is applied on the premise-sequent(s) before the rule in question is re-applied. If it is principal, is it obtained with either $\mathcal{H}$*Split*L or $\to$L. In the former case, the premise-sequents are the desired conclusions themselves. The latter case involves the premise-sequents $\mathcal{H} \to A \vee B, \Gamma \Rightarrow \mathcal{H}$ and $A \vee B, \Gamma \Rightarrow C$ derivable with height at most n. The inductive hypothesis is applied on the first, and $\vee$L inversion on the second. Applying $\to$L yields $n{+}1 \vdash \mathcal{H} \to A, \Gamma \Rightarrow C$ and $n{+}1 \vdash \mathcal{H} \to B, \Gamma \Rightarrow C$.

Regarding inversion of the right premise of $\to$L, I reason as follows if $\mathcal{H} \to A \vee B, \Gamma \Rightarrow C$ is obtained with $\mathcal{H}$*Split*L and $\mathcal{H} \to A \vee B$ is principal. Then it is obtained from $\mathcal{H} \to A, \Gamma \Rightarrow C$ and $\mathcal{H} \to B, \Gamma \Rightarrow C$. I apply the inductive hypothesis and then $\vee$L to obtain the desired result. □

This concludes the preliminary discussion of the sequent calculus. I now proceed to show that cut is admissible in g3InqB by proof analysis.

4. Admissibility of cut

A proof that cut is admissible by proof analysis proceeds typically by induction on the weight of the cut-formula with a subinduction on the cut-height defined as the sum of the heights of the derivations of the premise-sequents. The weight of a complex formula is defined as the sum of the weights of its direct subformulas plus 1. Atoms are assigned 1 and $\bot$ is assigned 0. The cut-formula is the formula displayed in the premise-sequents in the rule. The proof consists in providing transformations that remove the cut (if at least one premise is an initial sequent), or reduce either the weight of the cut-formula or the cut-height.

Consider for example the case where the cut-formula is of the form $A \wedge B$ and it is the principal formula of the rules applied to obtain the premise-sequents:

$$\dfrac{\dfrac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\wedge\text{R} \qquad \dfrac{A, B, \Gamma' \Rightarrow C}{A \wedge B, \Gamma' \Rightarrow C}\wedge\text{L}}{\Gamma, \Gamma' \Rightarrow C}\text{cut}$$

That application of cut is permuted into two applications on lower weight as follows:

$$\dfrac{\dfrac{\Gamma \Rightarrow B \qquad \dfrac{\Gamma \Rightarrow A \qquad A, B, \Gamma' \Rightarrow C}{B, \Gamma, \Gamma' \Rightarrow C}\text{cut}}{\Gamma, \Gamma, \Gamma' \Rightarrow C}\text{cut}}{\Gamma, \Gamma' \Rightarrow C}\text{Contr.}$$

To obtain the desired result, contraction is applied. More details can be found in for example [10].

Whereas the rule *Raa-at* is dealt with in a straight-forward manner, the rule $\mathcal{H}$*Split*L complicates things considerably. Consider the following case:

$$\dfrac{\dfrac{\mathcal{H}, \Gamma \Rightarrow A \vee B}{\Gamma \Rightarrow \mathcal{H} \to A \vee B}\to\text{R} \qquad \dfrac{\mathcal{H} \to A, \Gamma' \Rightarrow C \qquad \mathcal{H} \to B, \Gamma' \Rightarrow C}{\mathcal{H} \to A \vee B, \Gamma' \Rightarrow C}\mathcal{H}\textit{Split}\text{L}}{\Gamma, \Gamma' \Rightarrow C}\text{cut}$$

As things stand, one cannot apply the inductive hypothesis (i.e. cut) directly on the premise-sequents of →R and $\mathcal{H}$*Split*L since the active formulas $\mathcal{H} \rightarrow A$ and $\mathcal{H} \rightarrow B$ in the antecedents of premise-sequents of $\mathcal{H}$*Split*L are not matched by the active formula $A \vee B$ in the succedent of the premise-sequent of →R. Moreover, applying the inversion lemma on the conclusion-sequent of $\mathcal{H}$*Split*L will only yield $A \vee B, \Gamma' \Rightarrow C$ which leaves us with $\mathcal{H}, \Gamma, \Gamma' \Rightarrow C$ after cut. Finally, there is obviously no transformation of the derivation of the premise-sequent of →R into $\Gamma \Rightarrow \mathcal{H} \rightarrow A, \mathcal{H} \rightarrow B$ since the sequents are single-succedents.

However, with $\mathcal{H}$ being a Harrop formula, there is a solution inspired by the strategy employed by [14] to establish admissibility of cut for Gentzen's original sequent calculus without multicut, namely by tracing formulas up through the derivation and applying cut there. In particular, the derivation of $\mathcal{H}, \Gamma \Rightarrow A \vee B$ will have one or more branches with the property that $A \vee B$ is introduced into the succedent position through a leaf or an application of ∨R and remains parametric in that position until the root. In each such case where $\mathcal{H}$ isn't already in the context, the introduction of $\mathcal{H}$ can be permuted upwards in the derivation until that is the case. One can then apply cut there before the derivation below is reconstructed. To that purpose, the following lemma is required.

Lemma 4.1. *The rules ∧L and →L permute up with respect to every rule in* g3InqB*.*

Proof: In the case of ∧L, I illustrate the permutation with respect to →L. The others proceed in the same manner.

$$\dfrac{\dfrac{A, B, D \rightarrow E, \Gamma \Rightarrow D \qquad A, B, E, \Gamma \Rightarrow C}{A, B, D \rightarrow E, \Gamma \Rightarrow C}\rightarrow\mathrm{L}}{A \wedge B, D \rightarrow E, \Gamma \Rightarrow C}\wedge\mathrm{L}$$

It is permuted into the following:

$$\dfrac{\dfrac{A, B, D \rightarrow E, \Gamma \Rightarrow D}{A \wedge B, D \rightarrow E, \Gamma \Rightarrow D}\wedge\mathrm{L} \qquad \dfrac{A, B, E, \Gamma \Rightarrow C}{A \wedge B, E, \Gamma \Rightarrow C}\wedge\mathrm{L}}{A \wedge B, D \rightarrow E, \Gamma \Rightarrow C}\rightarrow\mathrm{L}$$

One simply applies $\wedge$L prior to applying the other rule since the context contains A, B.

In the case of $\rightarrow$L, the subcases differ slightly between each other, and inversion is crucial for the permutation up with respect to left introduction rules. Let's consider some subcases.
The case $\rightarrow$L below $\vee$L:

$$\dfrac{A \rightarrow B, D \vee E, \Gamma \Rightarrow A \qquad \dfrac{B, D, \Gamma \Rightarrow C \qquad B, E, \Gamma \Rightarrow C}{B, D \vee E, \Gamma \Rightarrow C}\,\vee\text{L}}{A \rightarrow B, D \vee E, \Gamma \Rightarrow C}\,\rightarrow\text{L}$$

It is permuted as follows:

$$\dfrac{\dfrac{\dfrac{A \rightarrow B, D \vee E, \Gamma \Rightarrow A}{A \rightarrow B, D, \Gamma \Rightarrow A}\,\text{Inv.} \qquad B, D, \Gamma \Rightarrow C}{A \rightarrow B, D, \Gamma \Rightarrow C}\,\rightarrow\text{L}}{}$$

$$\vdots$$

$$\dfrac{\dfrac{\vdots}{A \rightarrow B, D, \Gamma \Rightarrow C} \qquad \dfrac{\dfrac{A \rightarrow B, D \vee E, \Gamma \Rightarrow A}{A \rightarrow B, E, \Gamma \Rightarrow A}\,\text{Inv.} \qquad B, E, \Gamma \Rightarrow C}{A \rightarrow B, E, \Gamma \Rightarrow C}\,\rightarrow\text{L}}{A \rightarrow B, D \vee E, \Gamma \Rightarrow C}\,\vee\text{L}$$

The case $\rightarrow$L below $\vee$R:

$$\dfrac{A \rightarrow B, \Gamma \Rightarrow A \qquad \dfrac{B, \Gamma \Rightarrow D}{B, \Gamma \Rightarrow D \vee E}\,\vee\text{R}}{A \rightarrow B, \Gamma \Rightarrow D \vee E}\,\rightarrow\text{L}$$

This can be permuted into the following derivation:

$$\dfrac{\dfrac{A \rightarrow B, \Gamma \Rightarrow A \qquad B, \Gamma \Rightarrow D}{A \rightarrow B, \Gamma \Rightarrow D}\,\rightarrow\text{L}}{A \rightarrow B, \Gamma \Rightarrow D \vee E}\,\vee\text{R}$$

The case of $\rightarrow$L below *Raa-at* proceeds in the same way as $\rightarrow$L below $\vee$R.
The case $\rightarrow$L $(A \rightarrow B)$ below $\rightarrow$L $(E \rightarrow D)$:

$$\dfrac{A \to B, D \to E, \Gamma \Rightarrow A \qquad \dfrac{B, D \to E, \Gamma \Rightarrow D \qquad B, E, \Gamma \Rightarrow C}{B, D \to E, \Gamma \Rightarrow C}\,{\to}\mathrm{L}}{A \to B, D \to E, \Gamma \Rightarrow C}\,{\to}\mathrm{L}$$

This can be permuted into the following derivation:

$$\dfrac{\dfrac{A{\to}B, D{\to}E, \Gamma \Rightarrow A \quad B, D{\to}E, \Gamma \Rightarrow D}{A{\to}B, D \to E, \Gamma \Rightarrow D}\,{\to}\mathrm{L} \qquad \dfrac{\dfrac{A{\to}B, D \to E, \Gamma \Rightarrow A}{A{\to}B, E, \Gamma \Rightarrow A}\,\mathrm{Inv.} \quad B, E, \Gamma \Rightarrow C}{A{\to}B, E, \Gamma \Rightarrow C}\,{\to}\mathrm{L}}{A \to B, D \to E, \Gamma \Rightarrow C}\,{\to}\mathrm{L}$$

The remaining subcases proceed in an analogous manner. □

To illustrate the issue with $\vee$L (and $\mathcal{H}Split$L), and thus why the upward permutation only works for Harrop formulas, consider the following application of $\vee$L below an application of $\vee$R:

$$\dfrac{A, \Gamma \Rightarrow D \vee E \qquad \dfrac{B, \Gamma \Rightarrow D}{B, \Gamma \Rightarrow D \vee E}\,\vee\mathrm{R}}{A \vee B, \Gamma \Rightarrow D \vee E}\,\vee\mathrm{L}$$

The problem is that $A, \Gamma \Rightarrow D \vee E$ must be transformed into $A, \Gamma \Rightarrow D$, but $\vee$R is not invertible. Thus, one cannot permute the application of $\vee$L through the application of $\vee$R.

Theorem 4.2 (Admissibility of cut). *If $\Gamma \Rightarrow A$ and $A, \Gamma' \Rightarrow C$ are derivable, then $\Gamma, \Gamma' \Rightarrow C$ is also derivable.*

Proof: The proof proceeds by induction on the weight of A with a subinduction on the sum of the heights of the derivations of $\Gamma \Rightarrow A$ and $A, \Gamma' \Rightarrow C$. The proof may be organised like the proof for g3ip in [10]. If neither premise-sequent is an initial sequent, then there are three cases to consider: whether the cut-formula is not principal in the left premise-sequent, not principal only in the right premise-sequent, or principal in both premise-sequents. The cases involving the rule *Raa-at* are already described in [10, p.159], and I only need to consider the cases for the rule $\mathcal{H}Split$L. In the first two cases, one proceeds as in the case of $\vee$L. If however the cut-formula is principal in both premise-sequents and the right premise-sequent is obtained with $\mathcal{H}Split$L, then the derivation ends as follows:

$$\dfrac{\dfrac{\mathcal{H}, \Gamma \Rightarrow A \vee B}{\Gamma \Rightarrow \mathcal{H} \to A \vee B}\,{\to}\mathrm{R} \qquad \dfrac{\mathcal{H} \to A, \Gamma' \Rightarrow C \qquad \mathcal{H} \to B, \Gamma' \Rightarrow C}{\mathcal{H} \to A \vee B, \Gamma' \Rightarrow C}\,\mathcal{H}\mathit{Split}\mathrm{L}}{\Gamma, \Gamma' \Rightarrow C}\,\mathrm{cut}$$

Consider now the derivation of the premise-sequent $\mathcal{H}, \Gamma \Rightarrow A \vee B$. The occurrence of $A \vee B$ in the succedent can be traced upwards in the derivation. For each branch, there is a top-most sequent after which $A \vee B$ remains parametric until the root. For this sequent, one of the following holds:

- The sequent is the conclusion-sequent of an application of $\vee$R with $A \vee B$ as the principal formula. I will refer to this as a *final* application of $\vee$R.
- The sequent is a leaf of the form $\bot, \Gamma' \Rightarrow A \vee B$. Let this be a *initial occurrence* of $A \vee B$.
- The sequent is the conclusion-sequent of an application of $\to$L where $A \vee B$ is parametric (and the branch thus continues upwards through the left premise-sequent of the application of $\to$L).

As an illustration, consider the following snippet of a derivation with three (displayed) branches:

$$\dfrac{\mathcal{H} \to \bot \vee A, \mathcal{H} \Rightarrow \mathcal{H} \qquad \dfrac{\bot \Rightarrow A \vee B \qquad \dfrac{A \Rightarrow A}{A \Rightarrow A \vee B}\,\vee\mathrm{R}}{\bot \vee A \Rightarrow A \vee B}\,\vee\mathrm{L}}{\mathcal{H}, \mathcal{H} \to \bot \vee A \Rightarrow A \vee B}\,{\to}\mathrm{L}$$

The right-most branch has a final application of $\vee$R, the middle branch has an initial occurrence of $A \vee B$ and the occurrence of $A \vee B$ in the left-most branch is introduced through $\to$L where $A \vee B$ is parametric.

In fact, every derivation with the root $\mathcal{H}, \Gamma \Rightarrow A \vee B$ has at least one branch with either an initial occurrence of $A \vee B$ or a final application of $\vee$R. This follows by induction on the height of a derivation. If the root is a leaf, then it is itself a branch with an initial occurrence of $A \vee B$. Assume instead for the inductive step that it is obtained with another rule. If that rule is $\vee$R, then every branch of the derivation has a final application of $\vee$R. Otherwise, the sequent is obtained with $\mathcal{H}$*Split*L, $\vee$L, $\wedge$L or $\to$L. The desired result follows by applying the induction hypothesis on the premise-sequent in the case of $\wedge$L, on each premise-sequent in the case of the rules $\mathcal{H}$*Split*L and $\vee$L, and finally on the right premise-sequent in the case of $\to$L.

To simplify the rest of the proof, it is useful to transform every initial occurrence of $A \vee B$ into a final application of $\vee$R by replacing the leaf with a derivation of $\bot, \Gamma' \Rightarrow A \vee B$ from the new leaf $\bot, \Gamma' \Rightarrow A$. The result is a final application of $\vee$R because $A \vee B$ still remains parametric until the root from the inserted application of $\vee$R.

Using A/B as a metalinguistic device to represent that the formula is either A or B, it follows that each branch $\mathcal{B}$ with a final application of $\vee$R in the resulting derivation of $\mathcal{H}, \Gamma \Rightarrow A \vee B$ is of the form

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma_{\mathcal{B}} \Rightarrow A/B}}{\Gamma_{\mathcal{B}} \Rightarrow A \vee B}\,\vee\mathrm{R}}{\dfrac{\vdots}{\Gamma, \mathcal{H} \Rightarrow A \vee B}}$$

where $A \vee B$ remains parametric after the displayed application of $\vee$R. If $\mathcal{H}$ is not in $\Gamma_{\mathcal{B}}$, then $\mathcal{H}$ is neither an atom nor $\bot$, and there are two subcases to consider, depending on whether $\mathcal{H}$ is $D \wedge E$ or $D \to E$. Depending on the main connective, each such branch is of the following shape where the displayed application of $\vee$R is final:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\Gamma_{\mathcal{B}} \Rightarrow A/B}}{\Gamma_{\mathcal{B}} \Rightarrow A \vee B}\,\vee\mathrm{R}}{\vdots}}{\Gamma^*, D, E \Rightarrow A \vee B}}{\Gamma^*, D \wedge E \Rightarrow A \vee B}\,\wedge\mathrm{L}}{\dfrac{\vdots}{\Gamma, D \wedge E \Rightarrow A \vee B}} \qquad \dfrac{\dfrac{D \to E, \Gamma^* \Rightarrow D \qquad \dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\Gamma_{\mathcal{B}} \Rightarrow A/B}}{\Gamma_{\mathcal{B}} \Rightarrow A \vee B}\,\vee\mathrm{R}}{\vdots}}{\Gamma^*, E \Rightarrow A \vee B}}{\Gamma^*, D \to E \Rightarrow A \vee B}\,\to\mathrm{L}}{\dfrac{\vdots}{\Gamma, D \to E \Rightarrow A \vee B}}$$

$\Gamma_{\mathcal{B}}$ is the context of that final application of $\vee$R at that node in the branch $\mathcal{B}$ of the derivation, and Γ^* is the context at the application of $\wedge$L or $\to$L. It is possible, but not necessary, that $\Gamma_{\mathcal{B}}$ just is Γ^*, D, E or Γ^*, E respectively. For example, if the displayed application of $\vee$R is directly above the displayed application of $\wedge$L, then $\Gamma_{\mathcal{B}}$ is Γ^*, D, E.

The above permutation lemma is now applied *iteratively* to transform the derivation into a derivation in which $\mathcal{H}$ occurs in the context of each final application of $\vee$R. For example, in the simplest case where the displayed application of $\vee$R is directly above the displayed application of $\wedge$L, one obtains the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\Gamma^*, D, E \Rightarrow A/B}}{\Gamma^*, D \wedge E \Rightarrow A/B}\,\wedge\mathrm{L}}{\Gamma^*, D \wedge E \Rightarrow A \vee B}\,\vee\mathrm{R}}{\vdots}}{\Gamma, D \wedge E \Rightarrow A \vee B}$$

One can now proceed as follows for every branch $\mathcal{B}$ with a final application of $\vee$R where $\Gamma_{\mathcal{B}}*, \mathcal{H} \Rightarrow A/B$ is the premise-sequent of that final application of $\vee$R, and $\Gamma_{\mathcal{B}}* \subset \Gamma_{\mathcal{B}}$ (and $\Gamma_{\mathcal{B}}*$ just is Γ^* in the above example):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\Gamma_{\mathcal{B}}*, \mathcal{H} \Rightarrow A/B}}{\Gamma_{\mathcal{B}}* \Rightarrow \mathcal{H} \rightarrow A/B} \qquad \mathcal{H} \rightarrow A/B, \Gamma' \Rightarrow C}{\Gamma_{\mathcal{B}}*, \Gamma' \Rightarrow C}\,\text{cut}}{\vdots}}{\Gamma, \Gamma' \Rightarrow C}$$

Each cut is on a formula of lesser weight. □

The next section shows how to prove interpolation for InqB using Maehara's lemma which is available because the subatom property holds for g3InqB.

5. Interpolation for InqB

A propositional logic has the interpolation property, or also, that Craig's interpolation theorem holds for it, just in case, if $A \rightarrow B$ is a theorem, then there is a formula F such that $A \rightarrow F$ and $F \rightarrow B$ are theorems and every atom in F is contained in both A and B. F is said to be an interpolant of A and B.

The standard approach to prove interpolation using sequent calculus is through Maehara's lemma. I follow here the presentation of [13]. The strategy consists in proving a more general statement by considering every partition of a sequent.

Let $\Gamma_0; \Gamma_1 \Rightarrow C$ be a partition of a sequent $\Gamma \Rightarrow C$ where Γ is Γ_0, Γ_1. The expression $\Gamma_0; \Gamma_1 \overset{F}{\Longrightarrow} C$ means that F is a split-interpolant of $\Gamma_0 \Rightarrow$ and $\Gamma_1 \Rightarrow C$, that is, if $\Gamma_0, \Gamma_1 \Rightarrow C$ is derivable then $\Gamma_0 \Rightarrow F$ and $F, \Gamma_1 \Rightarrow C$ are derivable and $\mathit{Atoms}(F) \subseteq \mathit{Atoms}(\Gamma_0) \cap \mathit{Atoms}(\Gamma_1, C)$ where *Atoms* returns the set of atoms contained in the input multiset. The aim is now to show that for every partition of a derivable sequent $\Gamma \Rightarrow C$, there is a formula F such that F is its split-interpolant. Craig's interpolation theorem is then the special case where Γ_1 is empty, Γ_0 is A and C is B.

THEOREM 5.1. *Every partition of a derivable sequent of* g3InqB *has a split-interpolant.*

PROOF: The proof proceeds by induction on the height of a derivation. I present one subcase for the axioms and then the subcases involving the new rules, referring to [13] for the other details. One subcase of the axiom $\Gamma, p \Rightarrow p$ is the following partition and its split-interpolant:

$$\Gamma_0, p; \Gamma_1 \stackrel{p}{\Longrightarrow} p$$

After all, $\Gamma_0, p \Rightarrow p$ and $p, \Gamma_1 \Rightarrow p$ are derivable and the atom p is in the intersection of $Atoms(\Gamma_0, p)$ and $Atoms(\Gamma_1, p)$. For the inductive step with the rule *Raa-at*, there is only one subcase to consider. Moreover, it is trivial to see that whatever is a split-interpolant for the premise is also a split-interpolant for the conclusion:

$$\dfrac{\Gamma_0; \neg p, \Gamma_1 \stackrel{F}{\Longrightarrow} \bot}{\Gamma_0; \Gamma_1 \stackrel{F}{\Longrightarrow} p}\ \text{Raa-at}$$

Assume that $\Gamma_0 \Rightarrow F$ and $F, \neg p, \Gamma_1 \Rightarrow \bot$ is derivable, then $F, \Gamma_1 \Rightarrow p$ is also derivable. There is no change in atoms from premise to conclusion.

In the case of $\mathcal{H}Split$L, there are two subcases depending on the location of the principal formula. The first subcase goes as follows:

$$\dfrac{\Gamma_0; \mathcal{H} \to A, \Gamma_1 \stackrel{F}{\Longrightarrow} C \qquad \Gamma_0; \mathcal{H} \to B, \Gamma_1 \stackrel{G}{\Longrightarrow} C}{\Gamma_0; \mathcal{H} \to A \vee B, \Gamma_1 \stackrel{F \wedge G}{\Longrightarrow} C}\ \mathcal{H}Split\text{L}$$

This holds because one can proceed as follows after applying the inductive hypothesis on the premise-sequents:

$$\dfrac{\Gamma_0 \Rightarrow F \qquad \Gamma_0 \Rightarrow G}{\Gamma_0 \Rightarrow F \wedge G} \qquad \dfrac{\dfrac{F, \mathcal{H} \to A, \Gamma_1 \Rightarrow C \qquad G, \mathcal{H} \to B, \Gamma_1 \Rightarrow C}{F, G, \Gamma_1, \mathcal{H} \to A \vee B, \Gamma_1 \Rightarrow C}}{F \wedge G, \Gamma_1, \mathcal{H} \to A \vee B, \Gamma_1 \Rightarrow C}$$

The atoms of $F \wedge G$ are in the intersection of Γ_0 and $\Gamma_1, \mathcal{H} \to A \vee B, C$ since those of F are in the intersection of Γ_0 and $\Gamma_1, \mathcal{H} \to A, C$ while those of G in the intersection of Γ_0 and $\Gamma_1, \mathcal{H} \to B, C$. For the other subcase, the split-interpolant is $F \vee G$. □

This establishes that every partition of every derivable sequent of g3InqB has a split-interpolant. It follows that InqB has the interpolation property.

Basic Inquisitive Logic is an intermediate logic, that is, a logic inclusively between intuitionistic and classical logic, and it was established in [8] that there are only seven intermediate logics that satisfy Craig's interpolation theorem, including intuitionistic and classical logic. The theorem presented in this section does not contradict that result. Instead, the result in [8] concerns logics that satisfy uniform substitution, which Basic Inquisitive Logic does not.

6. Conclusions

I have in this paper presented a sequent calculus for InqB obtained by expanding g3ip for IPL with suitable rules for the Split property and double negation elimination for atoms, established that cut is admissible for the sequent calculus, and demonstrated that it can be used to show that InqB has the interpolation property.

The proof-theoretic approach presented in this paper can also be employed to provide a syntactic decidability proof for InqB. This will proceed more or less in the same way as described in [10, p. 45] for the corresponding sequent calculus g3ip for IPL. It will use the same halting condition regarding the reduction of conditionals in the antecedent which are not of the form $\mathcal{H} \to (A \vee B)$. Formulas of that form will instead be reduced with $\mathcal{H}$*Split*L. They should, like other invertible rules, be reduced prior to succedent disjunctions to avoid unnecessary back-tracking. The additional complexity for the decidability of InqB beyond that of IPL consists thus in determining whether the antecedent of a conditional with a disjunctive consequent is a Harrop formula or not. But this is linear on the length of the antecedent.

I leave two interesting questions about the sequent calculus g3InqB presented in this paper for future research:

- Can a countermodel in the support semantics for InqB be extracted from a failed proof search?

- Can it be expanded to a sequent calculus for first-order Inquisitive Logic, InqBQ?

There is currently no deductive system which is known to be sound and complete with regard to InqBQ, and it is also not known whether entailment in InqBQ is compact [3, p. 157]. A positive answer to the first question would encourage investigations into the properties of the sequent calculus obtained by expanding g3InqB with appropriate sequent calculus rules based on the sound (but possibly incomplete) natural deduction system for InqBQ presented by [3].

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İskender Taşdelen

AN ELEMENTARY PROOF OF THE CHARACTERIZATION THEOREM FOR CONJUNCTIVE MULTIPLE-CONCLUSION CONSEQUENCE RELATIONS

Abstract

We give a characterization theorem for multiple-conclusion consequence relations with the conjunctive reading of conclusions. As in the case of disjunctive multiple-conclusion consequence relations, we define consequence relations in terms of sets of two-set partitions of formulae. We see that a binary relation between sets of formulae is a conjunctive multiple-conclusion consequence relation if it is closed under the properties of inclusion, transitivity and reducibility. To prove this result we use only the definition and some basic properties of conjunctive multiple-conclusion consequence relations.

Keywords: multiple-conclusion consequence relation, conjunctive set of conclusions, characterization theorem.

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1. Introduction

Smiley [3, p. 30] proves a characterization theorem for multiple-conclusion consequence relations. There a set of conclusions Y is said to follow from a set X of premises if at least one formula in Y is true whenever all formulae in X are true. In other words, the set of conclusions is read disjunctively. Later, Šikić [4] improved Smiley's result by characterizing disjunctive consequence relations in terms of properties that explicitly refer to the relevant formal language.

Here we deal with the characterization problem for multiple-conclusion consequence relations with a conjunctive set of conclusions. On this reading, a set of conclusions Y is said to follow from a set X of premises if all formulae in Y are true whenever all formulae in X are true. The characterization theorem (See Theorem 3.2 in Section 3) provides a list of necessary and sufficient conditions for a binary relation between sets of formulas to be a conjunctive multiple-conclusion consequence relation.

Nowak [2, p. 1141] also gives a characterization theorem as a corollary in his study on disjunctive and conjunctive multiple-conclusion consequence relations. He obtains that result within the theory of Galois connections and closure systems. Here we present an elementary proof of a characterization theorem using solely the definition of conjunctive multiple-conclusion consequence relations and some of their basic properties (Compare Theorem 2.1 and its proof in [3, p. 30]). We also give a proof of the equivalence of Nowak's characterization theorem with our result.

2. Terminology, Notation and Preliminaries

In this section we review some notions and previous results from the theory of multiple-conclusion logic. Those who are already familiar with this material may skip this section and go directly to Section 3.

Every formal language $\mathcal{L}$ will be considered as identical with its set of formulae. We write $T, U, X, Y, Z, \ldots, T', U', X', Y', Z', \ldots$ for sets of formulae and $A, B, \ldots$ for formulae. For every language $\mathcal{L}$, there are mainly two natural multiple-conclusion consequence relations between sets of $\mathcal{L}$-formulae: Given a language $\mathcal{L}$, for all sets of $\mathcal{L}$-formulae X and Y:

- Y is a *disjunctive set of conclusions of* X if and only if at least one formula in Y is true whenever all formulae in X are true.
- Y is a *conjunctive set of conclusions of* X if and only if all formulae in Y are true whenever all formulae in X are true.

Here we will be dealing mainly with conjunctive consequence relations. The classic reference on disjunctive consequence relations is [3]. See also [1] for an order-theoretic study of consequence relations that connects them with closure operations on complete lattices. Following the works cited in this paper, to give a formal definition of a multiple-conclusion consequence relation for a language $\mathcal{L}$, we make use of sets of two-set partitions of $\mathcal{L}$.[1] Let T represent the set of all true $\mathcal{L}$-formulae and U the set of all untrue $\mathcal{L}$-formulae according to a possible states of affairs. Since every formula is either true or untrue, and no formula is both true and untrue, $T \cup U = \mathcal{L}$ and $T \cap U = \emptyset$. In other words, the sets T and U form a two-set partition (T, U) of $\mathcal{L}$. Considered as the collection of all possible states of affairs, every set of two-set partitions of $\mathcal{L}$ enables us to give formal definitions of the two types of multiple-conclusion consequence relations mentioned above:

Definition 2.1. Let $\mathcal{L}$ be a formal language and $\mathcal{I}$ be a set of two-set partitions of $\mathcal{L}$. Let $\models^{d}_{\mathcal{I}}$ and $\models^{c}_{\mathcal{I}}$ symbolize, respectively, the disjunctive and conjunctive multiple-conclusion consequence relations with regard to $\mathcal{I}$:

- $X \models^{d}_{\mathcal{I}} Y$ if and only if there is no partition (T, U) in $\mathcal{I}$ such that all formulae in X are true and all formulae in Y are untrue with regard to that partition:

[1] As remarked by an anonymous reviewer, in the setting of classical logic it would be more convenient to use families of subsets of the set of all formulae: once we are given a set T consisting of the true formulae of $\mathcal{L}$ in any particular states of affairs, the corresponding two-set partition would be $(T, (\mathcal{L} - T))$. We have opted for following the notation settled in this area starting from Smiley's monograph [3] and using pairs (T, U) to represent the set of all true and untrue formulae in a possible states of affairs.

$$X \models^{d}_{\mathcal{I}} Y \Leftrightarrow \neg\exists((T,U) \in \mathcal{I})\,(X \subseteq T \;\&\; Y \subseteq U) \tag{2.1}$$

- $X \models^{c}_{\mathcal{I}} Y$ if and only if whenever all formulae in X are true with regard to a partition (T,U) in $\mathcal{I}$, all formulae in Y are also true with regard to that partition:

$$X \models^{c}_{\mathcal{I}} Y \Leftrightarrow \forall((T,U) \in \mathcal{I})(X \subseteq T \Rightarrow Y \subseteq T) \tag{2.2}$$

For every set X and every formula A, we may abbreviate $X \models^{c}_{\mathcal{I}} \{A\}$ as $X \models^{c}_{\mathcal{I}} A$. We may use the comma as the symbol of set-theoretic union. Thus, for example, we may write $X, Y \models^{c}_{\mathcal{I}} Z, A$ for $X \cup Y \models^{c}_{\mathcal{I}} Z \cup \{A\}$. Moreover, for every set $\mathcal{I}$ of two-set partitions of $\mathcal{L}$, we let:

$$\begin{aligned} \mathcal{T} &= \{T \subseteq \mathcal{L} : \exists(U \subseteq \mathcal{L})(T,U) \in \mathcal{I}\} \\ \mathcal{U} &= \{U \subseteq \mathcal{L} : \exists(T \subseteq \mathcal{L})(T,U) \in \mathcal{I}\} \end{aligned} \tag{2.3}$$

Note that in Definition 2.1 we impose nothing either on the structure of formulae, or on the semantic relations among formulae based on their structure. Therefore, *every* set $\mathcal{I}$ of two-set partitions of $\mathcal{L}$ gives us a disjunctive and a conjunctive multiple-conclusion consequence relation:

Definition 2.2. A relation $\vdash \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{P}(\mathcal{L})$ is a *disjunctive multiple-conclusion consequence relation* if $\vdash = \models^{d}_{\mathcal{I}}$ for some set $\mathcal{I}$ of two-set partitions of $\mathcal{L}$. Similarly, a relation $\vdash \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{P}(\mathcal{L})$ is a *conjunctive multiple-conclusion consequence relation* if $\vdash = \models^{c}_{\mathcal{I}}$ for some set $\mathcal{I}$ of two-set partitions of $\mathcal{L}$.

To establish our results we will need the following basic properties of conjunctive consequence relations:

Fact 2.3. For every language $\mathcal{L}$ and every set $\mathcal{I}$ of partitions of $\mathcal{L}$,

(a) If $Y \subseteq X$, then $X \models^{c}_{\mathcal{I}} Y$.

(b) If $X' \models^{c}_{\mathcal{I}} Y$ and $X' \subseteq X$, then $X \models^{c}_{\mathcal{I}} Y$.

(c) If $X \models^{c}_{\mathcal{I}} Y'$ and $Y \subseteq Y'$, then $X \models^{c}_{\mathcal{I}} Y$.

(d) If $X \models^{c}_{\mathcal{I}} Y$ and $Y \models^{c}_{\mathcal{I}} Z$, then $X \models^{c}_{\mathcal{I}} Z$.

(e) Let $T \in \mathcal{T}$. Then $T \models^{c}_{\mathcal{I}} X$ if and only if $X \subseteq T$.

(f) $X \models^{c}_{\mathcal{I}} Y$ if and only if $\forall A(A \in Y \Rightarrow X \models^{c}_{\mathcal{I}} A)$

Proof: Proofs of the first four clauses are immediate from the definitions. We only prove the less trivial 2.3 and 2.3 below:

2.3 Assume that $T \in \mathcal{T}$ and $T \models^{c}_{\mathcal{I}} X$. Thus, for all $T' \in \mathcal{T}$, if $T \subseteq T'$, then $X \subseteq T'$. From the assumption that $T \in \mathcal{T}$ and the fact that $T \subseteq T$, it follows that $X \subseteq T$. The converse is an immediate result of the clause 2.3 of this Fact.

2.3 The left-to-right conditional is due to the clauses 2.3 and 2.3 of this Fact. To prove the converse, assume that $\forall A(A \in Y \Rightarrow X \models^{c}_{\mathcal{I}} A)$ and let $T \in \mathcal{T}$. If $X \subseteq T$, then $\{A\} \subseteq T$ for every $A \in Y$. Therefore, $Y \subseteq T$. □

It is remarked in [4] that for every set $\mathcal{I}$ of two-set partitions of $\mathcal{L}$ and for every two-set partition (T, U):

$$(T, U) \in \mathcal{I} \text{ if and only if } \neg(T \models^{d}_{\mathcal{I}} U)$$

If we consider conjunctive consequence relations and look for a similar result, we first deduce from Fact 2.3.2.3 that $T \models^{c}_{\mathcal{I}} U$ if and only if $(T, U) = (\mathcal{L}, \emptyset)$. Moreover, we may deduce the following proposition from this observation:

Fact 2.4. Let $\mathcal{L}$ be a language and $\mathcal{I} = \{(T, U) : T \cap U = \emptyset \ \&\ T \cup U = \mathcal{L} \ \&\ \exists X(T \subseteq X \ \&\ X \neq \mathcal{L})\}$. For every (T, U):

$$T \models^{c}_{\mathcal{I}} U \Leftrightarrow (T, U) \notin \mathcal{I}$$

In the next section we give a characterization theorem for conjunctive multiple-conclusion consequence relations. We finish this section by recalling the characterization theorem for disjunctive consequence relations:

Theorem 2.5 ([3, p. 30]). *A binary relation* $\mid\!\!-$ *on* $\mathcal{P}(\mathcal{L})$ *is a disjunctive multiple-conclusion consequence relation if and only if it is closed under the following properties:*

(Overlap) *If* $X \cap Y \neq \emptyset$*, then* $X \mid\!\!- Y$*.*

(Dilution) *If* $X' \mid\!\!- Y'$ *where* $X' \subseteq X$ *and* $Y' \subseteq Y$*, then* $X \mid\!\!- Y$*.*

(Cut for sets) *If* $X, Z_1 \mid\!\!- Z_2, Y$ *for every partition* (Z_1, Z_2) *of* Z*, then* $X \mid\!\!- Y$*.*

3. The Characterization Theorem

In this section we prove a characterization theorem for conjunctive multiple-conclusion consequence relations. We replace the overlap property of Theorem 2.5 with inclusion. We also replace dilution with weakening and split it into two statements, namely the left and right weakening properties. For the proof of our main theorem we only need the right weakening. As is shown below, each of these weakening properties results from inclusion and transitivity. Finally, we introduce a property that we call reducibility.

Lemma 3.1. *Let inclusion and transitivity be the following properties:*

(Inclusion) *If* $Y \subseteq X$*, then* $X \mid\!\!- Y$*.*

(Transitivity) *If* $X \mid\!\!- Z$ *and* $Z \mid\!\!- Y$*, then* $X \mid\!\!- Y$*.*

Then, any binary relation that satisfies inclusion and transitivity also satisfies the following properties of weakening:

(Left weakening) *If* $X' \mid\!\!- Y$ *and* $X' \subseteq X$*, then* $X \mid\!\!- Y$*.*

(Right weakening) *If* $X \mid\!\!- Y'$ *and* $Y \subseteq Y'$*, then* $X \mid\!\!- Y$*.*

Proof: We first prove the left weakening. Assume that $X' \subseteq X$ and $X' \mid\!\!- Y$. By inclusion, $X \mid\!\!- X'$. It follows by transitivity that $X \mid\!\!- Y$. The proof of the right weakening from inclusion and transitivity is similar: Let

$X \mid\!\!- Y'$ and $Y \subseteq Y'$. Bu inclusion, $Y' \mid\!\!- Y$. It then follows by transitivity that $X \mid\!\!- Y$. □

Theorem 3.2. *A binary relation* $\mid\!\!-$ *on* $\mathcal{P}(\mathcal{L})$ *is a conjunctive multiple-conclusion consequence relation if and only if it is closed under inclusion, transitivity and the following property of reducibility:*

(Reducibility) $X \mid\!\!- Y$ *if and only if* $\forall A(A \in Y \Rightarrow X \mid\!\!- A)$.

Proof: Let $\mid\!\!-$ be a conjunctive multiple-conclusion consequence relation, that is, let $\mid\!\!- = \models^{c}_{\mathcal{I}}$ for some set of partitions $\mathcal{I}$. We already have seen that every conjunctive multiple-conclusion consequence relation is closed under inclusion, transitivity and reducibility (See the clauses 2.3, 2.3 and 2.3 of Fact 2.3).

To prove the converse, let $\mid\!\!-$ be closed under inclusion, transitivity and reducibility. Let:

$$\mathcal{I} = \{(T, U) : \forall Z(T \mid\!\!- Z \Leftrightarrow Z \subseteq T)\} \tag{3.1}$$

We claim that $\mid\!\!- = \models^{c}_{\mathcal{I}}$, where $\mathcal{I}$ is as in (3.1). To demonstrate this, we first assume that $X \mid\!\!- Y$. Let $(T, U) \in \mathcal{I}$ and $X \subseteq T$. Then, by inclusion $T \mid\!\!- X$. From this result and the assumption that $X \mid\!\!- Y$, it follows by transitivity that $T \mid\!\!- Y$. From the definition of $\mathcal{I}$ in (3.1), we conclude that $Y \subseteq T$. Therefore, $X \models^{c}_{\mathcal{I}} Y$.

Now let us assume that $X \not\mid\!\!- Y$. Let $T = \{A : X \mid\!\!- A\}$ and $U = \{A : X \not\mid\!\!- A\}$. One can easily see that (T, U) is a partition. To see that $(T, U) \in \mathcal{I}$, we must only prove that for all Z, if $T \mid\!\!- Z$, then $Z \subseteq T$. (The converse holds by inclusion. Note that we do not yet know that $T \in \mathcal{T}$. Thus, we could not use the clause 2.3 of Fact 2.3 to prove that $X \subseteq T$ from the assumption that $T \models^{c}_{\mathcal{I}} X$.) Let $T \mid\!\!- Z$ and $A \in Z$. Since $X \mid\!\!- A$ for all $A \in T$ (by the definition of T), by reducibility it follows that $X \mid\!\!- T$. By the assumption that $T \mid\!\!- Z$, it follows by transitivity that $X \mid\!\!- Z$. Since $A \in Z$, we conclude by right weakening that $X \mid\!\!- A$. Thus, $A \in T$, by the definition of T, and we conclude that $Z \subseteq T$. Therefore, $(T, U) \in \mathcal{I}$. We now prove that, $Y \nsubseteq T$, although $X \subseteq T$, thus proving that $\neg(X \mathrel{|\!\frac{c}{\mathcal{I}}} Y)$. Since $X \vdash A$ for every $A \in X$, we conclude that $X \subseteq T$, by the definition of T. If $Y \subseteq T$ were also true, by inclusion it would follow that $T \vdash Y$. Since we have also seen above that $X \vdash T$, it would follow by transitivity that $X \vdash Y$, contrary to our assumption. Therefore $Y \nsubseteq T$. Since $X \subseteq T$, the partition $(T, U) \in \mathcal{I}$ demonstrates that $\neg(X \mathrel{|\!\frac{c}{\mathcal{I}}} Y)$. □

As a corollary of Theorem 3.2, we now prove Nowak's proposition that also gives a characterization of conjunctive multiple-conclusion consequence relations:

Corollary 3.3 ([2, p. 1141]). A binary relation $\vdash \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{P}(\mathcal{L})$ is a conjunctive multiple-conclusion consequence relation if and only if it is closed under the following properties:

(Transitivity) If $X \vdash Y$ and $Y \vdash Z$, then $X \vdash Z$.

(Ext. of converse order) If $Y \subseteq X$, then $X \vdash Y$.

(Closure on sup) $X \vdash \bigcup\{Y \subseteq \mathcal{L} : X \vdash Y\}$.

Proof: Note that the property here called ext. of converse order is what we have named as inclusion. We only need to show that reducibility and closure on sup are deducible from each other in the presence of inclusion and transitivity.

We first assume that $\vdash$ is closed under reducibility together with inclusion and transitivity. If $A \in \bigcup\{Y \subseteq \mathcal{L} : X \vdash Y\}$, then $A \in Y$ for some Y such that $X \vdash Y$. Since $A \in Y$, it follows by inclusion $Y \vdash A$. By transitivity, $X \vdash A$. Therefore, $\forall A(A \in \bigcup\{Y \subseteq \mathcal{L} : X \vdash Y\} \Rightarrow X \vdash A)$. We can now conclude by reducibility that $X \vdash \bigcup\{Y \subseteq \mathcal{L} : X \vdash Y\}$.

We now assume closure on sup, together with inclusion and transitivity. If $X \vdash Y$, and $A \in Y$, then by means of inclusion and transitivity it follows that $X \vdash A$. Therefore, $\forall A(A \in Y \Rightarrow X \vdash A)$. To prove the converse of reducibility, assume that $X \vdash A$ for every formula A in Y. Then $\{A\} \in \{Z \subseteq \mathcal{L} : X \vdash Z\}$, for all $A \in Y$. Therefore, by basic set theory,

$Y = \bigcup_{A \in Y}\{A\} \subseteq \bigcup\{Z \subseteq \mathcal{L} : X \vdash Z\}$. From the property of closure on sup, $X \vdash \bigcup\{Z \subseteq \mathcal{L} : X \vdash Z\}$. Since $\vdash$ satisfies right weakening, we conclude that $X \vdash Y$. □

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CLASSICAL LOGIC, UNIFORMITY, AND WEAK EXCLUDED MIDDLE IN NON-MONOTONIC PROOF-THEORETIC SEMANTICS

Abstract

Non-monotonic base-extension semantics (nB-eS), a kind of *non-monotonic proof-theoretic semantics* (nPTS), is known to validate classical logic when its meta-logic is classical. Schroeder-Heister has remarked that classical meta-logic is as problematic for the project of modelling intuitionistic logic, as an intuitionistic proof of incompleteness would be. It may be unclear, though, whether Schroeder-Heister's remark holds for *non-monotonic proof-theoretic validity* (nP-tV) as well, i.e., for Prawitz's original version of nPTS. We only know that, with classical meta-logic again, classical logic is sound over a variant of nP-tV, which I shall call *liberal non-monotonic proof-theoretic validity* (LnP-tV). The latter, in turn, differs from nP-tV in that reductions for the rewriting of proof-structures are not required to be *uniform*. After drawing attention to a number of divergences between nB-eS, nP-tV and LnP-tV, I show that Schroeder-Heister's remark might

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after all apply to nP-tV too. In particular, Weak Excluded Middle (WEM) is logically valid via uniform reductions (with a meta-logic which is non-intuitionistic, but non-classical either).

Keywords: classical logic, uniformity, weak excluded middle, non-monotonic proof-theoretic semantics.

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1. Introduction

By *non-monotonic proof-theoretic semantics* (nPTS) I shall understand in what follows the kind of constructive semantics introduced by Prawitz in [23].

Prawitz's original approach, called today (non-monotonic) *proof-theoretic validity* (nP-tV), is based on the notion of *valid argument structure*. An argument structure is a Natural Deduction derivation with arbitrary inferences, and it is said to be valid when it reduces, modulo a set of proof-rewriting functions called *reductions* (and possibly modulo substitution of unbound assumptions with closed valid arguments for these assumptions), to an argument structure which ends by a Natural Deduction introduction, and whose immediate sub-structures are also valid—the prior role of introductions stems from Gentzen's claim in [6], that introductions define the meaning of the logical constants, while eliminations (or better, inferences which are not in introduction-form, given that nP-tV appeals to arbitrary inferences) are consequences of this definition. Validity of arguments is first relativised to atomic proof-systems, called *atomic bases*. An argument is *logically* valid when it is valid over all atomic bases. Consequence over an atomic base means existence of an argument which is valid on the given base, whereas logical consequence means existence of a logically valid argument. The approach is non-monotonic in the sense that both validity and consequence may hold on a given atomic base, while failing on extensions of this base—a monotonic variant can be developed too, and is in fact more investigated nowadays, but I will leave it aside here.

In a more recent approach, called (non-monotonic) *base-extension semantics* (nB-eS), argument structures and reductions are left out. After introducing atomic bases, one defines a notion of consequence on an atomic base by direct induction on the complexity of the formulas of the underlying language. Logical consequence is just consequence over all atomic bases. Once again, the framework is non-monotonic since consequence might hold on a given atomic base, but fail on some of its extensions—and, also in this case, a more investigated monotonic picture is available, but I will not discuss it here.

Classical logic (`CL`) has been proved by Schroeder-Heister to be sound and complete on nB-eS, provided the meta-logic is classical [30]. Schroeder-Heister remarks that his result is as fatal as an intuitionistic proof of incompleteness would be for Prawitz's project of a semantics which intuitionistic logic (`IL`) be complete over—so-called *Prawitz's conjecture*, see also [17, 19]. Schroeder-Heister's remark appeals to the joint facts that an intuitionistic proof of completeness would imply a classical contradiction, that the proofs in question can be coded in first-order arithmetic, and finally that classical arithmetic and Heyting arithmetic are equi-consistent.

As said, however, nB-eS differs from Prawitz's original nP-tV, since it does without argument structures and reductions. This means, in particular, that the nP-tV and the nB-eS notions of consequence may not coincide, so the answer to the question whether Schroeder-Heister's remark applies to nP-tV too is not straightforward.

In this paper, I aim to shed a bit of light on this issue. I shall start by recalling the proof in [12] that `CL` is sound with classical meta-logic on a variant of nP-tV, called *liberal* (non-monotonic) *proof-theoretic validity* (LnP-tV). I note that from this one cannot infer that Schroeder-Heister's remark applies to nP-tV, since the proof at issue, besides classical meta-logic, forces reductions and the reduction sequences they induce to be *non-uniform*—namely, the rewriting of the argument structures depends on non-invariant features of the input-values, specifically, the atomic bases which these values are valid over. Prawitz's original nP-tV seems to take reductions to be uniform instead.

Next, I refine and improve some recent results established in [14]. One of these is that nB-eS and LnP-tV are in fact equivalent. This is not so relevant for the question whether `IL` is complete over LnP-tV which, as said, had been already settled negatively in [12] by proving CL to be sound over it—although it permits one to infer that CL is also complete over LnP-tV, via Schroeder-Heister's result for nB-eS mentioned above. Rather, the equivalence shows that nB-eS and LnP-tV share some structural principles, in particular, that the derivability of a rule on an atomic base is tantamount to the admissibility of that rule on that atomic base, and that logical consequence is tantamount to consequence on every atomic base. I highlight the connection between the potential failure of these principles in nP-tV, and the requirement that reductions and reduction sequences be uniform. On the other hand, I provide a result of completeness of the implication-free fragment of IL over nP-tV. The interest of this proof of completeness stems from the fact that it is obtained via the *iuxta propria principia* of nP-tV, namely, without using the disputed structural principles.

I also prove the equivalence between the nB-eS and the nP-tV notions of consequence on an atomic base when the meta-logic is classical. This, in turn, shows two things. First, that the non-uniform reading of reductions and reduction sequences is morally equivalent to a "local" usage of classical meta-logic—as I shall specify below, a similar phenomenon is encountered by Barroso Nascimento, Pereira and Pimentel in [2] in the somewhat different context of a proof-theoretic semantics for (a variant of) Prawitz's ecumenical logic [25]. Second, that the "local" equivalence is not enough for incompleteness of IL, due to the fact that, when reductions and reduction sequences are uniform, logical consequence is not reducible to a collection of "local" consequence relations. One potential conclusion one may draw from this is that, unless some stronger result will be proved in the future, the requirement of uniformity on reductions and reduction sequences is in a sense stronger than classical meta-logic.

From what said, one can positively conclude that whether one can apply to nP-tV Schroeder-Heister's argument that a classical proof of incompleteness is enough for refuting Prawitz's conjecture, boils down to whether one can give a non-intuitionistic proof of incompleteness of IL over nP-tV, while insisting at the same time on the uniform character of reductions or reduction sequences. This is indeed the case, as shown by the final result of the paper, which establishes the logical validity of Weak

Excluded Middle (WEM) with a meta-logic where WEM itself holds, and in a framework which might be said to cope with most of the requirements of nP-tV. Note that this is not in contrast with the claim that the requirement of uniformity is stronger than classical meta-logic, since the proof of the logical validity of WEM employs as said a less-than-classical meta-logic—nor is it in contrast with [9] and [16], two sources I shall also touch upon below whose counter-examples to completeness of `IL` over nP-tV are not closed under replacements of atoms with arbitrary formulas.

The reason why I say that Schroeder-Heister's remark *might be*—rather than just *is*—applicable to nP-tV is that, while improving the soundness result for `CL` from [12], my proof of the logical validity of WEM over nP-tV highlights that there might be further features besides uniformity, especially concerning how *falsum* (noted $\bot$) is semantically dealt with, which are compatible with the nP-tV approach, and which the reductions that I shall put forward for the proof itself *do not* enjoy. Concerning the semantic treatment of $\bot$, I shall work in what follows under the assumption that atomic bases are always consistent, i.e., never prove $\bot$—a convention whose importance has been recently unfolded by Barroso Nascimento, Pereira and Pimentel in [2], as I remark below. These issues are dealt with in the concluding remarks.

Before starting, it may be useful to locate this work within the broader field of researches into completeness of `IL` over the kind of semantics which nPTS belongs to. Some crucial results proved by Sandqvist [26] and, in a more general framework, by de Campos Sanz, Piecha and Schroeder-Heister [5, 18, 19], showed that `IL` is incomplete over a monotonic variant of nB-eS. This was later on extended to a monotonic variant of LnP-tV by [13], via an equivalence theorem similar to the one proved in [14] for nB-eS and LnP-tV[1]. Incompleteness of `IL` over a monotonic variant of nP-tV was finally established in [15]. All these achievements, however, pertain as said to the monotonic picture which, again, I shall not be interested in what

[1]The equivalence theorem between the monotonic variants of nB-eS and LnP-tV is expressly used in [13] to establish incompleteness of `IL` over LnP-tV, but it had been already achieved by Stafford in [31]. A more general proof of this result has been recently provided also by [16].

follows[2]. As for the non-monotonic picture, besides the results I have already mentioned [12, 14, 30], an early proof of incompleteness of IL over nB-eS was provided by Piecha and Schroeder-Heister in [19]. The latter, however, may not carry over to nP-tV, precisely in the same way as was observed above for the other related findings. This was remarked in [16]—and, as anticipated, it will be discussed also below—following similar analyses carried out in [10, 11, 12, 14]. Further insights into the non-monotonic behaviour induced by classical connectives in the framework of a proof-theoretic semantics for (a variant of) Prawitz's ecumenical logic [25] have been recently provided by Barroso Nascimento, Pereira and Pimentel in [2]. As I shall hint at below, these insights are related to the topics at issue here mostly relative to the semantic treatment of $\bot$. Finally, [15, 16] provide proofs of incompleteness of IL over nP-tV which differ from the one presented in this paper in that, while sticking more strictly to Prawitz's "pure" version of nP-tV, they rely upon counterexamples to completeness which, as already pointed out, are not closed under replacements of atoms with formulas.

In what follows, I will limit myself to a propositional language $\mathscr{L}$ with connectives $\wedge, \vee$ and $\rightarrow$. The set of the formulas of $\mathscr{L}$ is written $\mathtt{FORM}_{\mathscr{L}}$, while the set of the atoms of $\mathscr{L}$, written $\mathtt{ATOM}_{\mathscr{L}}$, is $\{p_i \mid i \in \mathbb{N}\} \cup \{\bot\}$. Negation is not primitive, i.e., $\neg A$ is interpreted as $A \rightarrow \bot$.

Although I am concerned with the relation between nP-tV and nB-eS, I shall not work with the latter directly, but with its variant LnP-tV. This depends on what I said above, i.e., that a proof of soundness of CL with classical meta-logic is available for LnP-tV, that nP-tV obtains from LnP-tV by restricting the notion of reduction, and that the issue about reductions boils down to structural differences between nB-eS and nP-tV.

[2] IL has been proved by Sandqvist to be complete over a monotonic variant of nB-eS with an elimination-like clause for disjunction and atomic bases of level ≥ 2 [27]. This was later on used by Gheorghiu and Pym to prove that IL is complete over a monotonic variant of nP-tV with elimination rules as primitive and atomic bases of level ≥ 2 [7], and further extended by Barroso Nascimento, Pereira and Pimentel in the broader context of Prawitz's ecumenical logic [2, 25]. By modifying the notion of atomic base, moreover, IL was proved to be complete on a monotonic variant of nB-eS by Schroeder-Heister [30] and, with a different strategy, by Stafford and Barroso Nascimento [1, 32].

2. Atomic bases, nPTS and classical logic

Let us first of all deal with LnP-tV and nB-eS, and their relation with $\mathtt{CL}$.

2.1. Atomic bases

The starting notion is that of atomic base. The latter, as well as the notion of derivability in atomic bases, requires however the preliminary definitions of the concepts of atomic rule and atomic derivation.

Definition 2.1. *Atomic rules of level n* are defined by induction as follows:

- $A \in \mathtt{ATOM}_{\mathscr{L}} \Longrightarrow$

$$\frac{}{A}$$

  is an atomic rule of level 0

- $A_1, ..., A_n, A \in \mathtt{ATOM}_{\mathscr{L}} \Longrightarrow$

$$\frac{A_1 \qquad \dots \qquad A_n}{A}$$

  is an atomic rule of level 1

- $A_1, ..., A_n, A \in \mathtt{ATOM}_{\mathscr{L}}$ and $\mathfrak{C}_i$ are atomic rules of level at most k ($i \leq n$) $\Longrightarrow$

$$\frac{\begin{matrix}[\mathfrak{C}_1] & & [\mathfrak{C}_n] \\ A_1 & \dots & A_n\end{matrix}}{A}$$

  is an atomic rule of level $k + 2$.

Brackets indicate dischargement of assumptions or of assumed rules—via Schroeder-Heister's *higher-level rules* [28].

Definition 2.2. *Atomic derivations* are defined by standard induction on the length of applications of atomic rules, starting from the basic case of a single-node derivation consisting of an application of a rule of level 0.

Definition 2.3. An *atomic base of level n* is a set of atomic rules whose maximal level is n.

Definition 2.4. A is *derivable* from $\mathfrak{C}$ in the atomic base $\mathfrak{B}$, written $\mathfrak{C} \vdash_{\mathfrak{B}} A$, iff there is an atomic derivation $\mathscr{D}$ such that, for every atomic rule $\mathfrak{c}$ applied and not discharged in $\mathscr{D}$, it holds that $\mathfrak{c} \notin \mathfrak{C} \Longrightarrow \mathfrak{c} \in \mathfrak{B}$.

I assume the following convention of consistency of atomic bases.

Convention 2.5. For every $\mathfrak{B}$, $\nvdash_{\mathfrak{B}} \bot$.

As we shall see in Section 5, Convention 2.5 will play a crucial role in the proof of the logical validity of WEM in nP-tV. The importance of this convention has been recently unfolded by Barroso Nascimento, Pereira and Pimentel in [2], although in a somewhat different framework from the one at stake in this paper. They provide some proof-theoretic semantics for (a variant of) Prawitz's ecumenical logic [25], the latter being, roughly, a logic where classical and intuitionistic connectives coexist. Also, they work in a context which is more akin to—albeit eventually richer than—that of Sandqvist in [27], i.e., a monotonic variant of nB-eS with an elimination-like clause for disjunction. In spite of this, they provide a number of new results which seem to be of interest for the kind of non-monotonic approach that I am interested in here. In particular, they show that Convention 2.5, when combined with a "Hilbertian" understanding of the semantics of classical proofs as consistency of formulas over an atomic base, might have as an effect that formulas with classical connectives may behave non-monotonically—whereas purely intuitionistic formulas are always monotonic. They end up with a motto to the effect that, in the specific kind of semantics where such a phenomenon occurs, monotonic classicality equals intuitionistic double negation[3]. Now, as we shall prove below, nPTS entertains strict connections

[3]This specific kind of semantics is the one that Barroso Nascimento, Pereira and Pimentel qualify as *weak*. Here, a distinction between *local* and *global validity* over an atomic base in the sense of Cobreros [4] is at play. Let me additionally remark that

with classical or quasi-classical logics, whence the interest for nPTS of Barroso Nascimento, Pereira and Pimentel's results is crystal-clear. In fact, it is an interesting open question what the precise interactions are between the "standard" picture provided by nB-eS, LnP-tV and nP-tV, on the one hand, and the one provided by Barroso Nascimento, Pereira and Pimentel—possibly enriched with argument structures and reductions—on the other. This question can be addressed in future works, but the general impression is that Barroso Nascimento, Pereira and Pimentel have singled out a sufficiently broad frame where "standard" variants of proof-theoretic semantics in Prawitz's style (both monotonic and non-monotonic) can be studied with the same peaceful coexistence as the one showed by the classical and the intuitionistic connectives in Prawitz's ecumenical logic.

2.2. LnP-tV

Let us now introduce LnP-tV. As said, the main notion here is that of valid argument structure.

DEFINITION 2.6. An *argument structure* is a pair $\langle T, f \rangle$ where:

- T is a finite rooted tree with order relation ω, nodes labelled by formulas of $\mathscr{L}$, and top-nodes partitioned into two groups, i.e., axiomatic and non-axiomatic;
- f is a function mapping onto lower nodes elements of: (a) a sub-set of the non-axiomatic top-nodes of T; (b) a sub-set of the axiomatic top-nodes of T labelled by atoms; (c) a sub-set of $\wp(\omega)$, all the elements of which contain only the pairs linking a node labelled by an atom to all its children, labelled by atoms too, with no non-axiomatic top-node

Barroso Nascimento, Pereira and Pimentel also show, very convincingly, that an approach with consistent atomic bases is able to work out a number of conceptual problems related to other existing semantic treatments of $\bot$ in the monotonic variant of nB-eS, see [5, 17, 18, 19, 27, 33], but see also the concluding remarks below. The idea of using consistent atomic bases is moreover historically very faithful to the "spirit" of Prawitz's original semantic project, since Prawitz himself has always understood in such terms a proof-theoretic treatment of the semantics of intuitionistic logic, e.g., [22, 24], but in fact already [21].

of T mapped by f onto its argument, or onto the image of the latter through f.

The non-axiomatic top-nodes and the root of T are called, respectively, the *assumptions* and the *conclusion* of $\langle T, f\rangle$.

f is to be understood as a discharge function which operates, not only on assumptions, but also on atomic rules of any level.

DEFINITION 2.7. With $\mathscr{D} = \langle T, f\rangle$, $\mathscr{D}$ is *closed* iff the domain of f contains all the non-axiomatic top-nodes of T, otherwise $\mathscr{D}$ is *open*. When $\mathscr{D}$ has undischarged assumptions Γ and conclusion A, it is called an argument structure *from* Γ *to* or *for* A.

An argument structure $\mathscr{D}$ from Γ to or for A is indicated by the figure

$$\begin{matrix}\Gamma \\ \mathscr{D} \\ A\end{matrix}$$

The notion of (*immediate*) *sub-structure* of an argument structure is defined in a standard way. The same happens for the notion of *replacement* of a sub-structure $\mathscr{D}^*$ of an argument structure $\mathscr{D}$ with an argument structure $\mathscr{D}^{**}$—written $\mathscr{D}[\mathscr{D}^{**}/\mathscr{D}^*]$—although in this case one must take care of re-indexings of the discharge functions of $\mathscr{D}$ and $\mathscr{D}^{**}$. I will abstract from these details here.

DEFINITION 2.8. Given $\mathscr{D}$ from $\Gamma = \{A_1, ..., A_n\}$ $(n \geq 0)$ to A, and a function σ from $\texttt{FORM}_{\mathscr{L}}$ to argument structures such that $\sigma(A_i)$ is an argument structure for A_i $(i \leq n)$, $\mathscr{D}^\sigma = \mathscr{D}[\sigma(A_1), ..., \sigma(A_n)/A_1, ..., A_n]$ is an *instance* of $\mathscr{D}$.

The notions of inference and inference rule can be thoroughly defined via the previous definitions—see [14, 23]. For what concerns us here, though, we can content ourselves with conceiving of an inference as an argument structure looked at, so to say, from below, i.e., as an argument structure $\mathscr{D}$ obtained by conjoining certain argument structures $\mathscr{D}_1, ..., \mathscr{D}_n$ $(n \geq 0)$ for some elements $A_1, ..., A_n \in \texttt{FORM}_{\mathscr{L}}$ (the premises of the inference) through a new root node labelled by $A \in \texttt{FORM}_{\mathscr{L}}$ (the conclusion of the inference), possibly plus dischargements δ that extend the discharge functions involved in $\mathscr{D}_1, ..., \mathscr{D}_n$. We also say that $\mathscr{D}$ ends by the inference identified by premises $A_1, ..., A_n$, conclusion A, and possibly dischargements δ. An inference rule is a set of inferences. It is understood that inference rules are recursive, in the sense that they can be identified by meta-linguistic descriptions in familiar Natural Deduction style. The elements of the rule are also called its instances, or applications. We shall also say that these instances or applications end by instantiating or applying the corresponding rule.

DEFINITION 2.9. $\mathscr{D}$ is *canonical* iff it ends by applying a Natural Deduction introduction rule.

It should not be difficult to see that an argument structure can be morally understood as a derivation-tree in the Natural Deduction style, where arbitrary inferences occur, or where arbitrary inference rules are instantiated or applied—with potential dischargements. Observe also that atomic derivations are argument structures of a special kind—but I defined them separately because of the special role they play, also in the nB-eS approach, relative to the notions of (logical) validity and (logical) consequence.

We can now define the notion of reduction and the reducibility relation. For doing this, I shall follow Schroeder-Heister's approach in [29].

DEFINITION 2.10. A *reduction* is a pair $\langle \mathscr{D}, \mathscr{D}^* \rangle$ with $\mathscr{D}$ from Γ to A and $\mathscr{D}^*$ from $\Delta \subseteq \Gamma$ to A. A *reduction system* is a set $\mathfrak{S}$ of reductions such that, for every $\langle \mathscr{D}, \mathscr{D}^* \rangle \in \mathfrak{S}$ and every σ, $\langle \mathscr{D}^\sigma, (\mathscr{D}^*)^\sigma \rangle \in \mathfrak{S}$. A *reduction sequence* is a sequence $\langle \mathscr{D}_1^1, \mathscr{D}_2^1 \rangle, ..., \langle \mathscr{D}_1^n, \mathscr{D}_2^n \rangle$ $(n \geq 0)$ of reductions such that $\mathscr{D}_1^{i+1} = \mathscr{D}_2^i$ $(i \leq n)$. The sequence is said to go *from* $\mathscr{D}_1^1$ *to* $\mathscr{D}_2^n$.

Observe in passing that I put no constraint on reduction systems. In particular, they may contain reduction sequences which send one and the same argument structure onto distinct values, e.g., $\langle \mathscr{D}, \mathscr{D}^* \rangle$ and $\langle \mathscr{D}, \mathscr{D}^{**} \rangle$, with $\mathscr{D}^* \neq \mathscr{D}^{**}$—what Schroeder-Heister calls *alternative justifications* [29]. Also, a reduction system has no complexity bound—e.g., the system might be very "big", say, the set $\mathbb{S}$ of *all* the reductions is a reduction system. I shall come back to this later on.

Definition 2.11. $\mathscr{D}$ reduces to $\mathscr{D}^*$ *modulo* a reduction system $\mathfrak{S}$, written $\mathscr{D} \leq_{\mathfrak{S}} \mathscr{D}^*$, iff $\mathfrak{S}$ contains a reduction sequence from $\mathscr{D}$ to $\mathscr{D}^*$.

Definition 2.12. $\langle \mathscr{D}, \mathfrak{S} \rangle$ is *valid* on $\mathfrak{B}$ iff

- $\mathscr{D}$ is closed for $A \in \mathtt{ATOM}_{\mathscr{L}} \Longrightarrow \mathscr{D} \leq_{\mathfrak{S}} \mathscr{D}^*$ where $\mathscr{D}^*$ witnesses $\vdash_{\mathfrak{B}} A$
- $\mathscr{D}$ is closed for $A \notin \mathtt{ATOM}_{\mathscr{L}} \Longrightarrow \mathscr{D} \leq_{\mathfrak{S}} \mathscr{D}^*$ where $\mathscr{D}^*$ is canonical with immediate sub-structures $\mathscr{D}^{**}$ such that $\langle \mathscr{D}^{**}, \mathfrak{S} \rangle$ is valid on $\mathfrak{B}$
- $\mathscr{D}$ is open from $A_1, ..., A_n \Longrightarrow$ for every σ, every $\mathfrak{T} \supseteq \mathfrak{S}$, if $\langle \sigma(A_i), \mathfrak{T} \rangle$ is valid on $\mathfrak{B}$ with $\sigma(A_i)$ closed, then $\langle \mathscr{D}^\sigma, \mathfrak{T} \rangle$ is valid on $\mathfrak{B}$.

Definition 2.13. $\langle \mathscr{D}, \mathfrak{S} \rangle$ is *logically valid* iff it is valid on every $\mathfrak{B}$.

From the notion of argumental (logical) validity we can extract a notion of (logical) consequence.

Definition 2.14. $\Gamma \models_{\mathfrak{B}} A$ iff there is $\langle \mathscr{D}, \mathfrak{S} \rangle$ valid on $\mathfrak{B}$, with $\mathscr{D}$ from Γ to A.

Definition 2.15. $\Gamma \models A$ iff there is $\langle \mathscr{D}, \mathfrak{S} \rangle$ logically valid, with $\mathscr{D}$ from Γ to A.

2.3. nB-eS

As said, the nB-eS approach does without argument structures and reductions. The nB-eS notion of (logical) consequence is defined, not as existence of a (logically) valid argument, but directly by induction on the complexity of formulas.

Definition 2.16. $\Gamma \Vdash_{\mathfrak{B}} A$ iff

- $\Gamma = \emptyset \Longrightarrow$
  - $A \in \mathtt{ATOM}_{\mathscr{L}} \Longrightarrow \vdash_{\mathfrak{B}} A$
  - $A = B \wedge C \Longrightarrow \Vdash_{\mathfrak{B}} B$ and $\Vdash_{\mathfrak{B}} C$
  - $A = B \vee C \Longrightarrow \Vdash_{\mathfrak{B}} B$ or $\Vdash_{\mathfrak{B}} C$
  - $A = B \rightarrow C \Longrightarrow B \Vdash_{\mathfrak{B}} C$
- $\Gamma \neq \emptyset \Longrightarrow (\Vdash_{\mathfrak{B}} \Gamma \Longrightarrow \Vdash_{\mathfrak{B}} A)$

(where $\Vdash_{\mathfrak{B}} \Gamma$ means $\Vdash_{\mathfrak{B}} B$ for every $B \in \Gamma$).

DEFINITION 2.17. $\Gamma \Vdash A$ iff, for every $\mathfrak{B}$, $\Gamma \Vdash_{\mathfrak{B}} A$.

2.4. Relations with CL

Schroeder-Heister has proved in [30] that CL is sound and complete over $\Vdash$ with classical meta-logic.

THEOREM 2.18. $\Gamma \vdash_{\mathtt{CL}} A \Longleftrightarrow \Gamma \Vdash A$ *with classical meta-logic.*

PROOF: ($\Longrightarrow$) Since we know IL is sound over $\Vdash$, see e.g. [18], it is enough if we prove $\Vdash A \vee \neg A$. For any $\mathfrak{B}$, by classical meta-logic, either $\Vdash_{\mathfrak{B}} A$ or $\nVdash_{\mathfrak{B}} A$. In both cases, $\Vdash_{\mathfrak{B}} A \vee \neg A$. ($\Longleftarrow$) By classical meta-logic, the right-hand side condition for $\Gamma \Vdash_{\mathfrak{B}} A$ with $\Gamma \neq \emptyset$, namely, $\Vdash_{\mathfrak{B}} \Gamma \Longrightarrow \Vdash_{\mathfrak{B}} A$, can be understood as ($\nVdash_{\mathfrak{B}} \Gamma$ or $\Vdash_{\mathfrak{B}} A$), where $\nVdash_{\mathfrak{B}} \Gamma$ means that, for some $B \in \Gamma$, $\nVdash_{\mathfrak{B}} B$. So, on every $\mathfrak{B}$, $A \to B$ can be interpreted as $\neg A \vee B$. Put $[\![A]\!] = \{\mathfrak{B} \mid \Vdash_{\mathfrak{B}} A\}$ and, when $\mathbb{B}$ is the set of all the $\mathfrak{B}$-s, put $\overline{[\![A]\!]} = \mathbb{B} - [\![A]\!]$. By Convention 2.5, we have $\overline{[\![A]\!]} = [\![\neg A]\!]$. By the same convention, $[\![\bot]\!] = \emptyset$. So $\langle \wp(\mathbb{B}), \emptyset, \mathbb{B}, \cap, \cup, \overline{[\![\ ]\!]} \rangle$, i.e., the algebra of the semantic values of formulas via $\Vdash_{\mathfrak{B}}$, is a Boolean algebra. □

The situation with LnP-tV seems to be similar since, as showed in [12], a proof of soundness of CL with classical meta-logic is available also for $\models$.

PROPOSITION 2.19. For every $\mathfrak{B}$, $\models_{\mathfrak{B}} A \vee \neg A$ with classical meta-logic.

PROOF: For any $\mathfrak{B}$, by classical meta-logic, $\models_{\mathfrak{B}} A$ or $\not\models_{\mathfrak{B}} A$. If $\models_{\mathfrak{B}} A$, then there is $\langle \mathscr{D}, \mathfrak{S}_{\mathfrak{B}} \rangle$ valid on $\mathfrak{B}$ with $\mathscr{D}$ for A closed. But then, if we put

$$\mathscr{D}^*_{\mathfrak{B}} = \frac{\begin{array}{c}\mathscr{D}\\ A\end{array}}{A \vee \neg A}$$

we have that $\langle \mathscr{D}^*_{\mathfrak{B}}, \mathfrak{S}_{\mathfrak{B}} \rangle$ is valid on $\mathfrak{B}$. If $\not\models_{\mathfrak{B}} A$, then the one-step argument structure

$$\frac{A}{\bot}$$

is (vacuously) valid on $\mathfrak{B}$ when paired with $\emptyset$. Hence, when we put

$$\mathscr{D}^{**} = \dfrac{\dfrac{\dfrac{[A]_1}{\bot}}{\neg A}\,1}{A \vee \neg A}$$

we have that $\langle \mathscr{D}^{**}, \emptyset \rangle$ is valid on $\mathfrak{B}$. □

THEOREM 2.20. $\Gamma \vdash_{\mathtt{CL}} A \Longrightarrow \Gamma \models A$ *with classical meta-logic.*

PROOF: Since IL is known to be sound over $\models$, see [23, 29], it is enough if we prove $\models A \vee \neg A$. Let

$$\mathtt{EM} = \dfrac{}{A \vee \neg A}$$

and

$$\mathfrak{T} = \{\langle \mathtt{EM}, \mathscr{D}^{*}_{\mathfrak{B}} \rangle \mid \mathfrak{B} \in \|A\|\} \cup \{\langle \mathtt{EM}, \mathscr{D}^{**} \rangle\}$$

where $\|A\| = \{\mathfrak{B} \mid \ \models_{\mathfrak{B}} A\}$, and where the $\mathscr{D}^{*}_{\mathfrak{B}}$-s and $\mathscr{D}^{**}$ are as in the proof of Proposition 2.19, and

$$\mathfrak{U} = \mathfrak{T} \cup \bigcup_{\mathfrak{B} \in \|A\|} \mathfrak{S}_{\mathfrak{B}}$$

where the $\mathfrak{S}_{\mathfrak{B}}$-s are as in the proof of Proposition 2.19. It is then easy to see that $\langle \mathtt{EM}, \mathfrak{U} \rangle$ is logically valid. □

Clearly, in view of Proposition 2.19, the proof of Theorem 2.20 would have gone through also by taking $\mathbb{S}$ as above, i.e., the set of all the reductions, in place of $\mathfrak{U}$. However, $\mathfrak{U}$ constitutes a more informative choice.

3. Schroeder-Heister's remark and uniformity

From the above we can obviously conclude that, if one takes the meta-logic to be classical, then completeness of intuitionistic logic just fails for nB-eS and LnP-tV. A natural question one may now ask is whether it makes sense to use classical meta-logic to prove or disprove completeness of IL in a proof-based framework. This is of course relevant for Prawitz's original nP-tV too, since one may want to extend to it the incompleteness theorems for nB-eS and LnP-tV proved above.

If we limit ourselves to nB-eS, one cannot expect that, in view of Theorem 2.18, an intuitionistic proof of completeness of `IL` over $\Vdash$ can ever be found. This is because of a remark put forward by Schroeder-Heister:

> claiming that completeness can nonetheless be proved by intuitionistic (and hence also by classical) means implies claiming a classical contradiction. Given that these proofs can be coded in first-order arithmetic and that classical arithmetic and Heyting arithmetic are equiconsistent, such a claim cannot be upheld. In simpler terms, inconsistency is a negative result, and on the negative side classical and intuitionistic logics coincide. [30, p. 501]

Can we use the remark to settle (in)completeness of `IL` also relative to nP-tV? Classical meta-logic implies soundness of `CL` (and hence incompleteness of `IL`) over $\models$, via Theorem 2.20. But this is not enough for applying Schroeder-Heister's remark directly to nP-tV. While certainly Prawitzian in spirit, due to the presence of argument structures and reductions, and to the idea that (logical) consequence means existence of a (logically) valid argument, LnP-tV is, as said, only a *variant* of nP-tV. However, it is now important to establish where LnP-tV and nP-tV precisely differ since, based on this, a better assessment of whether Schroeder-Heister's remark holds for nP-tV can be achieved.

Via Definition 2.10, reduction systems in LnP-tV can be seen as broad proof-rewriting systems where, as said above, one and the same argument structure can be transformed in different alternative ways, and where there is essentially no upper bound on the computational complexity of the proof-rewriting itself. In nP-tV, reduction sequences are instead given in a much more constrained way, namely, as induced by functions ϕ, also called reductions, which go from and to argument structures, and which are defined on a sub-set $\mathbb{D}$ of an inference rule (i.e., a set of argument structures) in such a way that, for every $\mathscr{D} \in \mathbb{D}$ and every σ as in Definition 2.8: $\mathscr{D}$ is from Γ to $A \Longrightarrow \phi(\mathscr{D})$ is from $\Delta \subseteq \Gamma$ to A; $\mathscr{D}^\sigma \in \mathbb{D}$; $\phi(\mathscr{D}^\sigma) = \phi(\mathscr{D})^\sigma$. Also, it seems to be part of Prawitz's understanding in [23] that ϕ, and the reduction sequences induced by a given set of reductions, are *uniform*, i.e., the outputs that ϕ produces for given inputs, as well as the reduction sequences made up of given inputs and outputs of reductions, can be specified independently of the potential validity of the inputs or outputs relative to specific sets of reductions or specific atomic bases. The prototype reduction of nP-tV is hence the kind of function used for removing detours in proofs of normalisation for Natural Deduction calculi, and the prototype reduction sequences of nP-tV are the sequences of Natural Deduction derivations induced by functions of that kind.

The validity of an argument structure in nP-tV is not relative to a reduction system as in Definition 2.10, but to a set of reductions $\mathfrak{J}$ as specified above. Apart from that, the definitions of validity of $\langle \mathscr{D}, \mathfrak{J} \rangle$ on $\mathfrak{B}$ and of logical validity of $\langle \mathscr{D}, \mathfrak{J} \rangle$ are in all ways similar, *mutatits mutandis*, to what happens in Definitions 2.12 and 2.13.

It is easy to see that, if $\langle \mathscr{D}, \mathfrak{J} \rangle$ is valid (logically or on some $\mathfrak{B}$) in nP-tV, then there is $\mathfrak{S}$ such that $\langle \mathscr{D}, \mathfrak{S} \rangle$ is valid (logically or on $\mathfrak{B}$) in LnP-tV: take $\mathfrak{S}$ to be the set of the reduction sequences induced by $\mathfrak{J}$ on $\mathscr{D}$. Generally, however, the inverse fails: $\mathfrak{S}$ gives rise to a $\mathfrak{J}$ only when special conditions obtain.

E.g., the presence in $\mathfrak{S}$ of what, following Schroeder-Heister [29], I have called above alternative justifications, may speak against $\mathfrak{S}$ being the graph of a function. Suppose for example that

$$\mathfrak{S} = \{\langle \mathscr{D}, \mathscr{D}^* \rangle, \langle \mathscr{D}, \mathscr{D}^{**} \rangle\}$$

where $\langle \mathscr{D}, \mathscr{D}^* \rangle$ and $\langle \mathscr{D}, \mathscr{D}^{**} \rangle$ are reductions, $\mathscr{D}$ is closed, and $\mathscr{D}^* \neq \mathscr{D}^{**}$. So, $\mathfrak{S}$ is a reduction system, but clearly $\mathfrak{S}$ cannot be the graph of a function defined on some sub-set of the rule which $\mathscr{D}$ belongs to. However, this may not be a big problem, thus I shall not discuss it further in what follows. One potential solution might be, roughly, that of considering the sub-sets of a given reduction system which coincide with reduction sequences induced by a set of reductions in Prawitz's sense, and then associating to each such sub-set as many functions from and to argument structures as are required for generating the reduction sequences themselves. In our example above, the sub-sets are of course $\{\langle \mathscr{D}, \mathscr{D}^* \rangle\}$ and $\{\langle \mathscr{D}, \mathscr{D}^{**} \rangle\}$, and to them we associate the functions $\phi_1(\mathscr{D}) = \mathscr{D}^*$ and $\phi_2(\mathscr{D}) = \mathscr{D}^{**}$, so to have $\mathfrak{J} = \{\phi_1, \phi_2\}$.

Much more seriously, reduction systems in the sense of Definition 2.10 may violate the uniformity constraint, i.e., from a reduction system we may not be able to extract a set of reductions, nor reduction sequences induced by these reductions, which are uniform in the sense hinted at above.

This is best seen with the reduction system $\mathfrak{U}$, especially its sub-set $\mathfrak{T}$, that I used for Theorem 2.20 for proving classically soundness of CL over $\models$. $\mathfrak{T}$ contains the (unproblematic) reduction $\langle \mathtt{EM}, \mathscr{D}^{**} \rangle$, where EM is as in the proof of Theorem 2.20, while $\mathscr{D}^{**}$ is as in the proof of Proposition 2.19, plus the pairs $\langle \mathtt{EM}, \mathscr{D}^{*}_{\mathfrak{B}} \rangle$ for every $\mathfrak{B} \in \|A\|$, where $\|A\|$ is as in the proof of Theorem 2.20, while $\mathscr{D}^{*}_{\mathfrak{B}}$ is as in the proof of Proposition 2.19. Had we to extract from $\mathfrak{U}$ a reduction for EM in Prawitz's sense, this would have to be a function ϕ onto argument structures from argument structures *and atomic bases*, such that

$$\phi(\mathfrak{B}, \mathtt{EM}) = \begin{cases} \mathscr{D}^{**} & \mathfrak{B} \notin \|A\| \\ \mathscr{D}^{*}_{\mathfrak{B}} & \mathfrak{B} \in \|A\| \end{cases}$$

Alternatively—with a strategy similar to the one hinted above for alternative justifications—we may consider the set of reductions for EM in Prawitz's sense

$$\mathfrak{F} = \phi_{\perp} \cup \bigcup_{\mathfrak{B} \in \|A\|} \phi_{\mathfrak{B}}$$

where

$$\phi_{\perp}(\mathtt{EM}) = \mathscr{D}^{**}$$

and

$$\phi_{\mathfrak{B}}(\mathtt{EM}) = \mathscr{D}^{*}_{\mathfrak{B}}.$$

If we assume that each $\mathscr{D}^{*}_{\mathfrak{B}}$ is obtained by appending introduction of disjunction to an argument structure which is valid on $\mathfrak{B}$ relative to some set of reductions $\mathfrak{J}_{\mathfrak{B}}$ in Prawitz's sense, we would have then to associate to EM either the set of reductions

$$\{\phi\} \cup \bigcup_{\mathfrak{B} \in \|A\|} \mathfrak{J}_{\mathfrak{B}}$$

or the set of reductions

$$\mathfrak{F} \cup \bigcup_{\mathfrak{B} \in \|A\|} \mathfrak{I}_{\mathfrak{B}}.$$

But neither of these solutions works in nP-tV. As concerns the first, ϕ is non-uniform (and so too are the reduction sequences it generates), because its outputs depend on the atomic base relative to which the validity of EM is to be evaluated. As for the second solution, although each element of

$$\bigcup_{\mathfrak{B} \in \|A\|} \phi_{\mathfrak{B}}$$

might well be seen as uniform, the reduction sequences generated by

$$\mathfrak{F} \cup \bigcup_{\mathfrak{B} \in \|A\|} \mathfrak{I}_{\mathfrak{B}}$$

are also non-uniform, as we cannot describe the output values without referring to the actual $\mathscr{D}^*_{\mathfrak{B}}$-s which exist on each $\mathfrak{B} \in \|A\|$—where of course there might be a distinct $\mathscr{D}^*_{\mathfrak{B}}$ for each distinct $\mathfrak{B} \in \|A\|$.

4. nB-eS and nP-tV

Schroeder-Heister's remark points out that a classical proof of incompleteness of IL over $\Vdash$ is enough for ruling out the existence of an intuitionistic proof of completeness of IL over $\Vdash$. On the other hand, as I already said, LnP-tV is much in the spirit of nP-tV, as it uses argument structures and reductions, and it defines (logical) consequence as existence of a (logically) valid argument; also, we know that in the case of LnP-tV too, we can prove classically the incompleteness of IL over $\models$. From the latter two facts, one may be tempted to infer that Schroeder-Heister's remark might also apply to nP-tV.

We saw this is wrong, though, since classical meta-logic is not the only thing needed to prove the incompleteness of IL for a version of nPTS with argument structures, reductions, and (logical) consequence defined as existence of a (logically) valid argument. For, we must also use non-uniform reductions and reduction sequences, which are *not* allowed in nP-tV.

The issue of uniformity of reductions and reduction sequences is crucial for understanding the difference, not only between nP-tV and LnP-tV, but also and above all between nP-tV and nB-eS. Since we are seeking whether a remark formulated for the nB-eS picture is applicable to nP-tV, it is thus time we turn to this latter issue.

4.1. Admissibility and choice

In [14], nB-eS and LnP-tV are proved to be actually equivalent, both relative to consequence over an atomic base, and relative to logical consequence.[4]

THEOREM 4.1. $\Gamma \Vdash_{\mathfrak{B}} A \Longleftrightarrow \Gamma \models_{\mathfrak{B}} A$.

THEOREM 4.2. $\Gamma \Vdash A \Longleftrightarrow \Gamma \models A$.[5]

Although the proofs of these result do not require classical meta-logic, the results cannot be used as such to establish a direct connection between nB-eS and nP-tV since, again, they force a reading of reductions and reduction sequences which breaks the uniformity constraint. The results also suggest, though, that the difference between LnP-tV and nP-tV must somehow be at play when comparing nB-eS and nP-tV too.

This is actually the case, but is easily overlooked as nB-eS does not involve argument structures and reductions at all. In the case of consequence over an atomic base, however, the issue lurks out when trying to prove that, if $\Gamma \Vdash_{\mathfrak{B}} A$ with $\Gamma = \{A_1, ..., A_n\}$ $(n > 0)$, then there is an argument structure from Γ to A which is valid on $\mathfrak{B}$ modulo some set of reductions—under a suitable reading of reductions and reduction sequences.

[4]Actually, for the equivalence to hold, sets of assumptions in nB-eS must always be finite—which is not surprising, nor harmful, since so are they in LnP-tV. Moreover, in [14] I did not use the kind of reductions that, following [29], I am employing here. Instead, I introduced a version of nP-tV where reductions and reduction sequences generated by them, understood in a strictly Prawitzian sense, are allowed to be non-constructive and non-uniform. In view of this, however, those results are easily adapted to the LnP-tV framework developed in this paper.

[5]Observe that, via Theorem 2.18, from this result it also follows that, besides soundness, we also have completeness of `CL` over $\models$. I could have proved soundness and completeness of `CL` over $\models$ as a corollary of Theorem 2.18 plus Theorem 4.2, but the proof of soundness I gave in Theorem 2.20 is to my mind more informative about the kind of non-uniform reductions and reduction sequences we need, and will be thus more relevant when I will prove the logical validity of WEM over LnP-tV (and nP-tV) in Section 5.

Let us see this more clearly. Assume that $\Vdash_{\mathfrak{B}} \Gamma \Longrightarrow \Vdash_{\mathfrak{B}} A$, and concede that $\Vdash_{\mathfrak{B}} \Gamma$ and $\Vdash_{\mathfrak{B}} A$ imply the existence of closed argument structures $\mathscr{D}_1, ..., \mathscr{D}_n, \mathscr{D}$ for $A_1, ..., A_n, A$ valid on $\mathfrak{B}$ modulo some set of reductions—under a suitable reading of reductions and reduction sequences. Whenever we are given $\mathscr{D}_1, ..., \mathscr{D}_n$ for $A_1, ..., A_n$ of the kind described, we can find, depending on these, a closed argument structure for A of the kind described, call it $f(\mathscr{D}_1, ..., \mathscr{D}_n)$. To prove that from this we can infer the existence of an open argument structure from Γ to A as required, we seem to have no other means than associating the one-step argument structure

$$\mathscr{D} = \frac{A_1 \qquad \dots \qquad A_n}{A}$$

to a mapping ϕ which sends any instance $\mathscr{D}^\sigma$ of $\mathscr{D}$ obtained by replacing the unbound assumptions $A_1, ..., A_n$ by closed argument structures $\mathscr{D}_1, ..., \mathscr{D}_n$ for them of the kind described, onto $f(\mathscr{D}_1, ..., \mathscr{D}_n)$, i.e.,

$$\phi(\mathscr{D}^\sigma) = f(\mathscr{D}_1, ..., \mathscr{D}_n).$$

So, $\mathscr{D}$ is valid on $\mathfrak{B}$ modulo ϕ, plus the sets of reductions relative to which the images of f are valid on $\mathfrak{B}$—this is a simplified version of the full proof, which can be found in [14, Proposition 3].

The point is now that ϕ is non-uniform, as its outputs cannot be specified independently of the validity of the inputs on the given atomic base, that is, independently of the specific closed valid argument for A which exists by assumption on the atomic base, in connection with specific closed valid arguments for $A_1, ..., A_n$ whose validity on the atomic base is assumed. More generally, the problem concerns the nB-eS clause for consequence on an atomic base with non-empty set of assumptions, i.e.,

(A) $\Gamma \Vdash_{\mathfrak{B}} A \Longleftrightarrow (\Vdash_{\mathfrak{B}} \Gamma \Longrightarrow \Vdash_{\mathfrak{B}} A)$, with $\Gamma \neq \emptyset$.

Observe that, with argument structures and reductions, (A) says that every inference from Γ to A which is *admissible* on $\mathfrak{B}$, is also *derivable* on $\mathfrak{B}$, i.e., if the existence of a proof for every element of Γ on $\mathfrak{B}$ implies the existence of a proof for A on $\mathfrak{B}$, then A is provable from Γ on $\mathfrak{B}$.

In an approach *à la* nP-tV, i.e., with argument structures, and uniform reductions and reduction sequences, only the left-to-right direction of (A) can be taken for granted. In fact, one can show that the following conditions are all satisfied by a notion of consequence over an atomic base extracted from the nP-tV notion of argumental validity over an atomic base hinted at above—which I shall write $\models^u$ to highlight that it is in all ways similar to $\models$, except for the requirement of uniformity on reductions and reduction sequences.

PROPOSITION 4.3. The following facts hold for $\models^u_{\mathfrak{B}}$:

- $A \in \mathtt{ATOM}_{\mathscr{L}} \Longrightarrow (\models^u_{\mathfrak{B}} A \Longleftrightarrow \vdash_{\mathfrak{B}} A)$
- $A = B \wedge C \Longrightarrow (\models^u_{\mathfrak{B}} A \Longleftrightarrow \models^u_{\mathfrak{B}} B \text{ and } \models^u_{\mathfrak{B}} C)$
- $A = B \vee C \Longrightarrow (\models^u_{\mathfrak{B}} A \Longleftrightarrow \models^u_{\mathfrak{B}} B \text{ or } \models^u_{\mathfrak{B}} C)$
- $A = B \rightarrow C \Longrightarrow (\models^u_{\mathfrak{B}} A \Longleftrightarrow B \models^u_{\mathfrak{B}} C)$
- $\Gamma \models^u_{\mathfrak{B}} A$ with $\Gamma \neq \emptyset \Longrightarrow (\models^u_{\mathfrak{B}} \Gamma \Longrightarrow \models^u_{\mathfrak{B}} A)$

With Proposition 4.3, and classical meta-logic again, we can actually prove that the nB-eS and nP-tV notions of consequence on an atomic base coincide. This is interesting in itself, but also for introducing the next topic I want to deal with.

PROPOSITION 4.4. $\Gamma \models^u_{\mathfrak{B}} A \Longleftrightarrow \Gamma \Vdash_{\mathfrak{B}} A$ with classical meta-logic.

PROOF: Suppose $\Gamma = \emptyset$. We proceed by induction on A:

- $A \in \mathtt{ATOM}_{\mathscr{L}} \Longrightarrow (\models^u_{\mathfrak{B}} A \Longleftrightarrow \vdash_{\mathfrak{B}} A \Longleftrightarrow \Vdash_{\mathfrak{B}} A)$
- $A = B \wedge C \Longrightarrow (\models^u_{\mathfrak{B}} B \text{ and } \models^u_{\mathfrak{B}} C \overset{\text{i.h.}}{\Longleftrightarrow} \Vdash_{\mathfrak{B}} B \text{ and } \Vdash_{\mathfrak{B}} C)$
- $A = B \vee C \Longrightarrow (\models^u_{\mathfrak{B}} B \text{ or } \models^u_{\mathfrak{B}} C \overset{\text{i.h.}}{\Longleftrightarrow} \Vdash_{\mathfrak{B}} B \text{ or } \Vdash_{\mathfrak{B}} C)$
- $A = B \rightarrow C \Longrightarrow$ we split the two cases. Suppose $B \models^u_{\mathfrak{B}} C$ and $B \nVdash_{\mathfrak{B}} C$. By classical meta-logic, $\Vdash_{\mathfrak{B}} B$ and $\nVdash_{\mathfrak{B}} C$. By i.h., $\models^u_{\mathfrak{B}} B$ hence, by Proposition 4.3, $\models^u_{\mathfrak{B}} C$. By i.h., we also have $\not\models^u_{\mathfrak{B}} C$. Contradiction. By classical meta-logic, $B \models^u_{\mathfrak{B}} C \Longrightarrow B \Vdash_{\mathfrak{B}} C$. Suppose $B \Vdash_{\mathfrak{B}} C$ and $B \not\models^u_{\mathfrak{B}} C$. By classical meta-logic, the following

obtains: for every $\mathscr{D}$ from B to C, for every $\mathfrak{J}$, there is σ as in Definition 2.8 such that, for some $\mathfrak{H} \supseteq \mathfrak{J}$, $\langle \sigma(B), \mathfrak{H} \rangle$ is valid on $\mathfrak{B}$ with $\sigma(B)$ closed, and $\langle \mathscr{D}^\sigma, \mathfrak{H} \rangle$ is not valid on $\mathfrak{B}$. Now, from the fact that $\langle \sigma(B), \mathfrak{H} \rangle$ is valid on $\mathfrak{B}$, i.e., $\models^u_{\mathfrak{B}} B$, by i.h., we have $\Vdash_{\mathfrak{B}} B$, hence $\Vdash_{\mathfrak{B}} C$, since we were assuming $B \Vdash_{\mathfrak{B}} C$. From $\Vdash_{\mathfrak{B}} C$ we infer by i.h. again that $\models^u_{\mathfrak{B}} C$, i.e., there is $\langle \mathscr{D}^*, \mathfrak{J}^* \rangle$ valid on $\mathfrak{B}$, with $\mathscr{D}^*$ closed for C. Consider now

$$\mathscr{D}^{**} = \frac{B}{C}$$

and consider

$$\mathfrak{J}^{**} = \{\phi\} \cup \mathfrak{J}^*$$

where ϕ is such that, for every σ,

$$\phi((\mathscr{D}^{**})^\sigma) = \mathscr{D}^*$$

Then, $\langle \mathscr{D}^{**}, \mathfrak{J}^{**} \rangle$ is valid on $\mathfrak{B}$, contradicting the assumption $B \not\models^u_{\mathfrak{B}} C$. Hence, $B \Vdash_{\mathfrak{B}} C \Longrightarrow B \models^u_{\mathfrak{B}} C$.

The case with $\Gamma \neq \emptyset$ is proved in a way similar to the implication case with $\Gamma = \emptyset$. □

Note that ϕ in the proof of Proposition 4.4 is perfectly uniform—it is just a constant function. Now, since we have proved that $\Vdash$ and $\models^u$ coincide over atomic bases when the meta-logic is classical, we may be led to think that the same holds for the respective notions of logical consequence. But this is not so again, because of another difference between nP-tV and nB-eS.

Logical consequence is defined in nB-eS simply as consequence over all atomic bases, namely, as per Definition 2.16,

(C) $\Gamma \Vdash A$ iff, for every $\mathfrak{B}$, $\Gamma \Vdash_{\mathfrak{B}} A$.

If argument structures and reductions are brought in, (C) becomes: A is a logical consequence of Γ iff, for every $\mathfrak{B}$, there is $\langle \mathscr{D}, \mathfrak{J} \rangle$ valid on $\mathfrak{B}$ with $\mathscr{D}$

from Γ to A. But this is not how things go in nP-tV. The quantifiers of the previous reading are here inverted, i.e., we do not require a valid argument for each atomic base, but a logically valid argument, namely, an argument which is valid over all atomic bases.

By Definition 2.15, this is how things go in LnP-tV too. As said, though, if the uniformity constraint on reductions and reduction sequences is given up, one can prove (constructively) Theorem 4.2, that is, the equivalence of $\Vdash$ and $\models$. As before, let us begin by assuming that, for every $\mathfrak{B}$, $\Gamma \Vdash_{\mathfrak{B}} A$, with $\Gamma = \{A_1, ..., A_n\}$ $(n > 0)$, and let us concede that this implies that, for every $\mathfrak{B}$, there is $\mathscr{D}_{\mathfrak{B}}$ valid on $\mathfrak{B}$ modulo some set of reductions—under a suitable reading of reductions and reduction sequences—with $\mathscr{D}_{\mathfrak{B}}$ from Γ to A, i.e.,

$$\begin{array}{ccc} A_1 & \ldots & A_n \\ & \mathscr{D}_{\mathfrak{B}} & \\ & A & \end{array}$$

To infer from this that there is a logically valid argument from Γ to A, we seem to have again no other means than associating the one-step argument structure

$$\mathscr{D} = \frac{A_1 \qquad \ldots \qquad A_n}{A}$$

to a mapping ϕ as follows. Let $\mathscr{D}^\sigma$ be any instance of $\mathscr{D}$ obtained by replacing the unbound assumptions $A_1, ..., A_n$ by closed argument structures $\mathscr{D}_1, ..., \mathscr{D}_n$ for them which are valid on $\mathfrak{B}$ modulo some set of reductions—under a suitable reading of reductions and reduction sequences. Then

$$\phi(\mathscr{D}^\sigma) = \begin{array}{ccc} \mathscr{D}_1 & & \mathscr{D}_n \\ A_1 & \ldots & A_n \\ & \mathscr{D}_{\mathfrak{B}} & \\ & A & \end{array}$$

So, $\mathscr{D}$ is valid on every $\mathfrak{B}$ modulo ϕ, plus the sets of reductions relative to which the $\mathscr{D}_{\mathfrak{B}}$-s are valid—this is a simplified version of the full proof, which can be found in [14, Proposition 4].

Note that ϕ behaves like a sort of choice-function, which picks out the right argument structure relative to each $\mathfrak{B}$. So, it is similar to the reduction system $\mathfrak{U}$ as in the proof of Theorem 2.20, or to the function ϕ extracted from it in Section 3. As I observed in [12] already, the fact that we do not need classical meta-logic to prove the result above shows that the non-uniform character of reductions has, on the general constructivist spirit of the approach, essentially the same effects as classical meta-logic. Conversely, if we just require uniform reductions, *even without giving up* classical meta-logic, we can no longer prove Proposition 2.19, hence Theorem 2.20—the most we can obtain is the equivalence of nP-tV and nB-eS relative to consequence on an atomic base. There seems to be here another interesting, potential connection with the work of Barroso Nascimento, Pereira and Pimentel [2], as they show that, in one of the proof-theoretic semantics for (a variant of) Prawitz's ecumenical logic [25] they develop[6], metalinguistic excluded middle is valid in a "local monotonic" sense when classical proofs are defined in the "Hilbertian" sense specified in Section 2.1. Once again, this topic can be investigated in future works.

4.2. Completeness of the implication-free fragment on nP-tV

Due to (A) and (C) from the previous section, it is therefore difficult to see how to prove (in)completeness results for nP-tV via the results that we have for nB-eS. One could wonder how far one can go by allowing only for the left-to-right direction of (C), i.e.,

$$\Gamma \models^u_{\mathfrak{B}} A \Longrightarrow (\models^u_{\mathfrak{B}} \Gamma \Longrightarrow \models^u_{\mathfrak{B}} A)$$

and by reading (A) with quantifiers inverted, i.e., there is $\mathscr{D}$ from Γ to A logically valid modulo some set of uniform reductions.

By way of example, I show in this section that one can go as far as establishing completeness of the implication-free fragment of `IL`, written `IL`*—a similar result has been proved by Humberstone for a similar notion

[6]Again, the weak kind with a distinction between local and global validity on an atomic base in the sense of Cobreros [4].

of validity [8, Theorem 4.13.3]. In the next section, however, I shall also show that, in certain relevant cases, one *cannot* go further than that.

Let us first of all introduce a preliminary definition and some preliminary, though simple results—as said, in what follows, up to the end of this section but not beyond, (sets of) formulas have to be taken as implication-free.

DEFINITION 4.5. To any A we associate the inductively defined set $\mathtt{E}_A$:

- $A \in \mathtt{ATOM}_{\mathscr{L}} \Longrightarrow \mathtt{E}_A = \{A\}$
- $A = B \wedge C \Longrightarrow \mathtt{E}_A = \{x \wedge y \mid \langle x, y \rangle \in \mathtt{E}_B \times \mathtt{E}_C\}$
- $A = B \vee C \Longrightarrow \mathtt{E}_A = \{x \mid x \in \mathtt{E}_B \cup \mathtt{E}_C\}$.

PROPOSITION 4.6. $A \vdash_{\mathtt{IL}^*} \bigvee_{x \in \mathtt{E}_A} x$ and $\bigvee_{x \in \mathtt{E}_A} x \vdash_{\mathtt{IL}^*} A$.

PROPOSITION 4.7. $\vdash_{\mathtt{IL}^* \cup \mathfrak{B}} A \Longleftrightarrow \vdash_{\mathfrak{B}} A$ with $A \in \mathtt{ATOM}_{\mathscr{L}}$.

PROOF: Apply standard results from normalisation theory for Natural Deduction, see [3, 20, 22]. □

PROPOSITION 4.8. $\models^u_{\mathfrak{B}} A \Longleftrightarrow \vdash_{\mathtt{IL}^* \cup \mathfrak{B}} A$.

PROOF: By induction on A—using the disjunction property. □

Now some additional definitions—inspired by the *import* and *export* functions that de Campos Sanz, Piecha and Schroeder-Heister defined for atomic rules of any level, see [18, 19].

DEFINITION 4.9. For any atomic rule $\mathfrak{R}$ of the form

$$\frac{}{A}$$

we set $\mathfrak{R}^* = A$ and, for any set of atomic rules $\mathfrak{S} = \{\mathfrak{R}_1, ..., \mathfrak{R}_n, ...\}$ of level 0, we set $\mathfrak{S}^* = \{\mathfrak{R}_1^*, ..., \mathfrak{R}_n^*, ...\}$.

DEFINITION 4.10. For any disjunction-free formula A we set:

- $A \in \mathtt{ATOM}_{\mathscr{L}} \Longrightarrow A^\circ = \{\mathfrak{R}\}$, where $\mathfrak{R}$ is the atomic rule

$$\frac{}{A}$$

- $A = \bigwedge_{i \leq n} B_i \Longrightarrow A^\circ = \bigcup\{B_i^\circ \mid i \leq n\}$.

For any disjunction-free set $S = \{A_1, ..., A_n, ...\}$, we set $S^\circ = \bigcup\{A_1^\circ, ..., A_n^\circ, ...\}$. Observe that every $x \in \mathtt{E}_A$ is disjunction-free, therefore x° is defined, and is a set of atomic rules of level 0—so $(x^\circ)^*$ is defined too.

Proposition 4.11. With $x \in \mathtt{E}_A$, $(x^\circ)^* \vdash_{\mathtt{IL}^*} x$ and, for every $y \in (x^\circ)^*$, $x \vdash_{\mathtt{IL}^*} y$.

Proof: By induction on A:

- $A \in \mathtt{ATOM}_{\mathscr{L}} \Longrightarrow$ trivial
- $A = B \wedge C \Longrightarrow x$ is of the form $z \wedge w$ with $z \in \mathtt{E}_B$ and $w \in \mathtt{E}_C$, while

$$(x^\circ)^* = ((z \wedge w)^\circ)^* = \{\bigcup\{z^\circ, w^\circ\}\}^* = \bigcup\{(z^\circ)^*, (w^\circ)^*\}$$

By i.h., $(z^\circ)^* \vdash_{\mathtt{IL}^*} z$ and $(w^\circ)^* \vdash_{\mathtt{IL}^*} w$, so $\bigcup\{(z^\circ)^*, (w^\circ)^*\} \vdash_{\mathtt{IL}^*} z \wedge w$. Vice versa, by i.h., for every $y \in (z^\circ)^*$, $z \vdash_{\mathtt{IL}^*} y$ so, because $z \wedge w \vdash_{\mathtt{IL}^*} z$, $z \wedge w \vdash_{\mathtt{IL}^*} y$. We prove similarly that, for every $y \in (w^\circ)^*$, $z \wedge w \vdash_{\mathtt{IL}^*} y$. Therefore, for every $y \in \bigcup\{(z^\circ)^*, (w^\circ)^*\}$, $z \wedge w \vdash_{\mathtt{IL}^*} y$.

The case with $A = B \vee C$ is in all ways similar to the conjunction case. □

We moreover have this result—proved in [13] for full IL and atomic bases of any level.

Proposition 4.12. $\Gamma \vdash_{\mathtt{IL}^* \cup \mathfrak{B}} A \Longleftrightarrow (\Gamma, \Delta \vdash_{\mathtt{IL}^*} A$, for some finite $\Delta \subseteq \mathfrak{B}^*)$.

Finally, the following holds trivially.

Proposition 4.13. $\Gamma \models^u A \Longrightarrow$ for every $\mathfrak{B}, (\models^u_{\mathfrak{B}} \Gamma \Longrightarrow \models^u_{\mathfrak{B}} A)$.

So, we have the following.

Theorem 4.14. $\Gamma \models^u A \Longrightarrow \Gamma \vdash_{\mathtt{IL}^*} A$.

Proof: Suppose $\Gamma = \emptyset$. By Proposition 4.13, for every $\mathfrak{B}$, $\models^u_{\mathfrak{B}} A$. Hence, by Proposition 4.8, for every $\mathfrak{B}$, $\vdash_{\mathtt{IL}^* \cup \mathfrak{B}} A$. With $\mathfrak{B} = \emptyset$, $\vdash_{\mathtt{IL}^*} A$. Suppose then $\Gamma = \{A_1, ..., A_n\}$ $(n > 0)$. By Proposition 4.13, for every $\mathfrak{B}$, $\models^u_{\mathfrak{B}}$

$\Gamma \Longrightarrow \models^{u}_{\mathfrak{B}} A$. Consider any $S = \{x_1, ..., x_n\}$ with $x_i \in \mathtt{E}_{A_i}$ $(i \leq n)$—there might be some $x_j \in \mathtt{E}_{A_j}$ such that $x_i = x_j$. As S is disjunction-free, S° is defined, and is a set of atomic rules of level 0—so $(S^\circ)^*$ is defined too. Suppose now that S is also $\bot$-free. Then, S° is an atomic base, so we have $\models^{u}_{S^\circ} \Gamma \Longrightarrow \models^{u}_{S^\circ} A$. We apply Proposition 4.8 again, and we obtain $\vdash_{\mathtt{IL}^* \cup S^\circ} \Gamma \Longrightarrow \vdash_{\mathtt{IL}^* \cup S^\circ} A$, where $\vdash_{\mathtt{IL}^* \cup S^\circ} \Gamma$ means $\vdash_{\mathtt{IL}^* \cup S^\circ} B$ for every $B \in \Gamma$. By Proposition 4.12,

$$(S^\circ)^* \vdash_{\mathtt{IL}^*} \Gamma \Longrightarrow (S^\circ)^* \vdash_{\mathtt{IL}^*} A.$$

The same implication holds trivially when S is not $\bot$-free, since in this case $\bot \in (S^\circ)^*$. Now, spelling the implication out, we have

$$\bigcup\{(x_1^\circ)^*, ..., (x_n^\circ)^*\} \vdash_{\mathtt{IL}^*} \Gamma \Longrightarrow \bigcup\{(x_1^\circ)^*, ..., (x_n^\circ)^*\} \vdash_{\mathtt{IL}^*} A$$

By Proposition 4.11, we have then

$$\{x_1, ..., x_n\} \vdash_{\mathtt{IL}^*} \Gamma \Longrightarrow \{x_1, ..., x_n\} \vdash_{\mathtt{IL}^*} A.$$

By arbitrariness of the choice of S, we thus have

$$\bigvee_{x \in \mathtt{E}_{A_1}} x, ..., \bigvee_{x \in \mathtt{E}_{A_n}} x \vdash_{\mathtt{IL}^*} \Gamma \Longrightarrow \bigvee_{x \in \mathtt{E}_{A_1}} x, ..., \bigvee_{x \in \mathtt{E}_{A_n}} x \vdash_{\mathtt{IL}^*} A.$$

But the antecedent of this implication holds by Proposition 4.6, so we have

$$\bigvee_{x \in \mathtt{E}_{A_1}} x, ..., \bigvee_{x \in \mathtt{E}_{A_n}} x \vdash_{\mathtt{IL}^*} A$$

which, again by Proposition 4.6, just means $\Gamma \vdash_{\mathtt{IL}^*} A$. □

5. Validity of WEM and analysis of the result

The existence of a (constructive) proof of completeness of $\mathtt{IL}^*$ over nP-tV might lead one to hope that the result extends (possibly, in a constructive way) to full $\mathtt{IL}$. However, this cannot be the case, unless we require reductions and reduction sequences to undergo more constraints than the kind of uniformity I have dealt with above. This is because of Theorem 5.1 and Corollary 5.2 below.

The proofs of these result will make crucial use of two principles. The first is Convention 2.5 above, i.e., the requirement that atomic bases never prove $\bot$ categorically, that is, that atomic bases are always consistent. The second is that I shall allow reductions (both in the LnP-tV and in the nP-tV sense) to be defined on canonical argument structures. I shall comment upon these choices in the concluding remarks.

THEOREM 5.1. $\models \neg A \vee \neg\neg A$ *when* WEM *holds in the meta-logic.*

PROOF: Since WEM holds in the meta-logic, for every $\mathfrak{B}$, either $\not\models_{\mathfrak{B}} A$ or (not $\not\models_{\mathfrak{B}} A$). If the first, then the one-step argument structure

$$\frac{A}{\bot}$$

is vacuously valid on $\mathfrak{B}$ when paired with $\emptyset$, and so is

$$\mathscr{D}^* = \cfrac{\cfrac{\cfrac{[A]_1}{\bot}}{\neg A}\,1}{\neg A \vee \neg\neg A}$$

If (not $\not\models_{\mathfrak{B}} A$), then $\not\models_{\mathfrak{B}} \neg A$. For, suppose $\models_{\mathfrak{B}} \neg A$. Then, by Convention 2.5, $\not\models_{\mathfrak{B}} A$, which contradicts our assumption. So, the one-step argument structure

$$\frac{\neg A}{\bot}$$

is vacuously valid on $\mathfrak{B}$ when paired with $\emptyset$, and so is

$$\mathscr{D}^{**} = \cfrac{\cfrac{\cfrac{[\neg A]_1}{\bot}}{\neg\neg A}\,1}{\neg A \vee \neg\neg A}$$

Let now

$$\mathtt{WEM} = \frac{}{\neg A \vee \neg\neg A}$$

and consider the reduction system

$$\mathfrak{S} = \{\langle \texttt{WEM}, \mathscr{D}^* \rangle, \langle \mathscr{D}^*, \mathscr{D}^{**} \rangle\}.$$

With WEM in the meta-logic, for every $\mathfrak{B}$, $\langle \texttt{WEM}, \mathfrak{S} \rangle$ is valid on $\mathfrak{B}$. □

Observe that the reduction system $\mathfrak{S}$ in the proof of Theorem 5.1 is specified in a completely base-independent way. We do not need to bring in specific argument structures whose shape may change depending on specific atomic bases which `WEM` is to be evaluated over. The argument structures $\mathscr{D}^*$ and $\mathscr{D}^{**}$ remain invariant throughout all atomic bases. In fact, $\mathfrak{S}$ is even finite (for every instance of `WEM`), *contra* the potentially infinite cardinality of $\mathfrak{U}$ in the proof of Theorem 2.20.

That $\mathfrak{S}$ is uniform in the sense specified above (and is even the graph of a composite function) can be seen by observing that the proof of Theorem 5.1 can be given also for nP-tV, by associating `WEM` to a finite set of reductions—where the latter are understood now in Prawitz's sense, i.e., as functions from and to argument structures which respect the conditions stated in Section 3.

Corollary 5.2. $\models^u \neg A \vee \neg\neg A$ when WEM holds in the meta-logic.

Proof: Set

$$\phi_1(\texttt{WEM}) = \mathscr{D}^* \text{ and } \phi_2(\mathscr{D}^*) = \mathscr{D}^{**}.$$

Then, $\langle \texttt{WEM}, \{\phi_1, \phi_2\} \rangle$ is logically valid in the sense of nP-tV. □

Of course, while the definitions of $\mathfrak{S}$ in the proof of Theorem 5.1 and of $\{\phi_1, \phi_2\}$ in the proof of Corollary 5.2 are completely uniform, the computation process they give rise to for validating `WEM` on each atomic base is non-constructive. The proofs work only because we are assuming WEM in the meta-logic. Hence, one might say, the question of the completeness of `IL` over nP-tV is not settled yet, since we may still hope for a constructive proof of completeness.

But we can now make appeal to Schroeder-Heister's remark that a classical proof of incompleteness of `IL` is enough for ruling out an intuitionistic proof of completeness of `IL`, insofar as both proofs can be coded in first-order arithmetic. The only reason we had for claiming that we could not exploit the classical proof of incompleteness of IL on LnP-tV for applying Schroeder-Heister's remark directly to nP-tV, was that for LnP-tV we did not need classical meta-logic only, but non-uniform reductions and reductions sequences too. Now we have a meta-logic which is non-intuitionistic (and weaker than classical meta-logic), and uniform reductions and reduction sequences. So, provided we accept that the proofs of Theorem 5.1 and of Corollary 5.2 can be coded in first-order arithmetic, Schroeder-Heister's remark does apply, and we can rule out the existence of an intuitionistic proof of completeness of IL over nP-tV.

6. Concluding remarks

To conclude, I would like to make some remarks concerning Theorem 5.1 and Corollary 5.2. There are certain changes one can make to the nP-tV and LnP-tV pictures as provided in this paper. Two of these do not affect the validity of the proofs of Theorem 5.1 and Corollary 5.2, whereas other two turn out to be crucial. Let me start from the former:

- Prawitz's original approach in [23] is limited to atomic rules of level ≤ 1. Theorem 5.1 and Corollary 5.2 also hold in this case;
- as I defined it, a single-node atomic derivation will always be a derivation of an atom depending on no assumption-formulas, but only on an assumed rule of level 0. Therefore, atomic derivations never contain assumption-formulas, and we cannot speak of derivability of an atom from assumptions-formulas in an atomic base. It is possible, of course, to define atomic derivations so that they contain both assumption-formulas and assumed rules of level 0, or take assumption-formulas to be identical with assumed rules of level 0. Theorem 5.1 and Corollary 5.2 also hold in these cases.

Let us now discuss the changes which may affect Theorem 5.1 and Corollary 5.2:

- rather than reading $\bot$ as an atomic formula, one could take it to be a nullary connective. This issue is connected with how $\bot$ is semantically dealt with. I have assumed atomic bases to be always consistent, and this fact is crucially used in the proofs of Theorem 5.1 and of Corollary 5.2. But there are other options. One can, e.g., allow atomic bases to be inconsistent, but require them to contain all the atomic instances of *ex falso*—see, e.g., [18, 19]—or, especially when $\bot$ is seen as a nullary connective, explain $\bot$ at the semantic level by equating its validity on an atomic base with the validity of all atoms on that base—see, e.g., [27]. A thorough discussion of this alternatives, and of their connection with the assumption of consistency of atomic bases, is as said to be found in [2]. Be that as it may, a proof of incompleteness of IL over nP-tV which does not rely at all on how $\bot$ is understood (and so applies to minimal logic too) has been recently provided in [16]. The proof uses an instance of excluded middle (thus, a stronger meta-logic than the one used in the proofs of Theorem 5.1 and Corollary 5.2), and shows $p \to (q \vee r) \models^u (p \to q) \vee (p \to r)$ with p, q and r atoms, without however generalising to formulas obtained by replacing p, q and r with any $A, B, C \in \texttt{FORM}_{\mathscr{L}}$. Via Schroeder-Heister's remark, this result is however enough for refuting Prawitz's conjecture. Since nP-tV is a semantics of proofs (as opposed to a semantic of formulas), there might be, more in general, reasons to stay content with a notion of logical consequence which is *not* closed under replacements. These topics can be discussed in future works;

- the reduction system $\mathfrak{S}$ used in the proof of Theorem 5.1, and the set of reductions $\mathfrak{J}$ used in the proof of Corollary 5.2, use reductions defined on $\mathscr{D}^*$. Now, one may not like the idea of applying reductions to argument structures in canonical form. If so, then one can set in the proof of Theorem 5.1

$$\mathfrak{S} = \{\langle \texttt{WEM}, \mathscr{D}^* \rangle, \langle \texttt{WEM}, \mathscr{D}^{**} \rangle\}$$

and in the proof of Corollary 5.2

$$\phi^1(\texttt{WEM}) = \mathscr{D}^*,\ \phi^2(\texttt{WEM}) = \mathscr{D}^{**}.$$

This solution would work perfectly fine in the context of LnP-tV, where alternative justifications are allowed, but may be problematic in the case of nP-tV, as here reductions might be required to be deterministic. However, a set of both uniform and deterministic reductions in the sense of nP-tV can be given in such a way as to have $\neg\neg p \to q \vee r \models^u (\neg\neg p \to q) \vee (\neg\neg p \to r)$—with Convention 2.5 and WEM in the meta-logic [9]. Once again, via Schroeder-Heister's remark, this refutes completeness of `IL` over nP-tV but, similarly to the incompleteness proof of [16] (which incidentally also uses uniform and deterministic reductions and reduction sequences), it does not hold under replacement of p, q, r with any formulas. The above remarks thus also apply here and, as said, these topics can be discussed in future works.

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BRIDGING CLASSICAL AND MODERN APPROACHES TO THALES' THEOREM

Abstract

In this paper, we reconstruct Euclid's theory of similar triangles, as developed in Book VI of the *Elements*, along with its 20th-century counterparts, formulated within the systems of Hilbert, Birkhoff, Borsuk and Szmielew, Millman and Parker, as well as Hartshorne. In the final sections, we present recent developments concerning non-Archimedean fields and mechanized proofs.

Thales' theorem (VI.2) serves as the reference point in our comparisons. It forms the basis of Euclid's system and follows from VI.1 – the only proposition within the theory of similar triangles that explicitly applies the definition of proportion.

Instead of the ancient proportion, modern systems adopt the arithmetic of line segments or real numbers. Accordingly, they adopt other propositions from Euclid's Book VI, such as VI.4, VI.6, or VI.9, as a basis.

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In § 10, we present a system that, while meeting modern criteria of rigor, reconstructs Euclid's theory and mimics its deductive structure, beginning with VI.1. This system extends to automated proofs of Euclid's propositions from Book VI.

Systems relying on real numbers provide the foundation for trigonometry as applied in modern mathematics. In § 9, we prove Thales' theorem in geometry over the hyperreal numbers. Just as Hilbert managed to prove Thales' theorem without referencing the Archimedean axiom, so do we by applying the arithmetic of the non-Archimedean field of hyperreal numbers.

Keywords: Thales' theorem, 20th-century foundations of geometry, the *Elements*, mechanical proofs.

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1. Interpreting Thales' Theorem and Euclidean Proportion

1. Thales' theorem, also known as the intercept theorem or the fundamental theorem of proportionality, plays a central role in Euclid's theory of similar figures, as developed in Book VI of the *Elements* [17]. The first proposition of this book, VI.1, states that triangles with the same height are to each other as their bases. Proposition VI.2 is what has been referred to as

Thales' theorem since the 19th century. Propositions VI.4–7 outline the criteria for similar triangles, with VI.4 stating that in equiangular triangles, corresponding sides are proportional.

In modern reconstructions of Euclid's theory of similar figures, Proposition VI.4 is crucial because it forms the foundation of trigonometry. However, for the reasons explained below, modern geometry has abandoned the ancient concept of proportion. Therefore, the general objective is to reestablish VI.4 on grounds independent of Euclidean proportion as defined in Book V of the *Elements*.

2. Euclidean proportion is the most significant ancient Greek theory transmitted to early modern mathematics. In contrast to Euclidean rigor, early modern mathematicians applied it in unorthodox ways. Although governed by the Archimedean axiom, the theory was applied to both standard and infinitesimal triangles.

It laid the foundations for early modern optics, mechanics, and the advancements of 17th-century calculus. Viewed from the perspective of mathematical techniques, Newton's *Principia* represents a synthesis of Euclidean proportion and infinitesimals.

In contemporary mathematics, trigonometry encodes the Euclidean theory of similar triangles. Contrary to the widespread view advanced by 20th-century structuralism in the philosophy of mathematics, trigonometry is a part of modern calculus that does not derive solely from the axioms of real numbers.

3. Euclidean proportion and calculus converge in determining the derivative of $\sin x$. To this end, the following trigonometric identity is essential:

$$\sin(x+h) - \sin x = 2\sin\frac{h}{2}\cos(x+\frac{h}{2}),$$

which rests on Euclidean principles of similar triangles.

Furthermore, evaluating the limit

$$\lim_{h\to 0}\frac{\sin h}{h}$$

also necessitates reference to Euclidean geometry.

Moreover, trigonometry, as applied in calculus, requires the radian measure of angles, which, in turn, relies on the final proposition of Euclid's Book VI.

Based on these foundations, calculus techniques, such as derivatives and power series, are applied to trigonometric functions [7].

4. Nowadays, Euclidean proportion, in a more accessible form, is commonly encountered in school mathematics or historical contexts. It is typically expressed using fractions rather than its original formulation. In fact, the foundational achievement of Descartes' *La Géométrie* (1637) [16] was the transformation of proportion into the arithmetic of line segments through the implicit rule [6]:

$$a : b :: c : d \Rightarrow a = b \cdot \frac{c}{d}.$$

In a definition of sorts, Descartes introduced the product and division of line segments on the very first pages of his essay through a diagram: given $AB = 1$, the line EB is the product of DB and CB; similarly, CB is the result of dividing EB by DB; see Fig. 1.

Obviously, these definitions are not rigorous by modern standards. Rather, they can be interpreted as an application of Thales' theorem: instead of the proportion $DB : 1 :: EB : CB$, Descartes introduces a novel operations, namely $EB = DB \cdot CD$ and $BC = \frac{EB}{DB}$.

Throughout *La Géométrie*, these operations, along with the addition of line segments, satisfy the laws of an ordered field. These rules were applied in mathematics implicitly until the end of the 19th century when Hilbert introduced the axioms of an ordered field [21, 22].

Thus, alongside trigonometry – presented either in elementary form or as power series – Euclidean proportion resonates in modern calculus through the laws of an ordered field.

4. Moritz Pasch's *Vorlesungen über Neuere Geometrie* (1882) [30] initiated the process of establishing Euclidean geometry on new foundations. Hilbert's *Grundlagen der Geometrie* and other 20th-century systems that followed also sought to reconstruct the Euclidean theory of similar figures within these new frameworks.

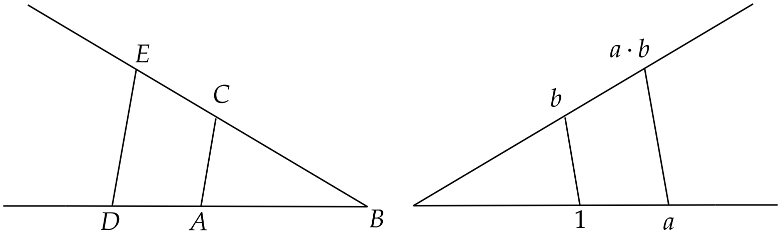

Figure 1: *La Géom*étrie, p. 298 (left). An interpretation of Descartes' definition (right).

The ancient concept of proportion was abandoned, giving rise to two general strategies: one based on the arithmetic of line segments and the other on the properties of real numbers, understood as an ordered field with completeness.

The arithmetic of line segments is inspired by Descartes' arithmetic. Indeed, instead of Descartes' arbitrary angle $\angle DBE$, one can use a right angle while adopting Descartes' definitions; see Fig. 10 or 13. In this formulation, the rules of an ordered field can be justified within Euclidean proportion; see § 3 below.

However, while Descartes introduced ordered field arithmetic based on the ancient concept of proportion, namely,

$$a \cdot d = c \cdot b \Leftrightarrow_{df} a : b :: c : d,$$

modern geometry seeks to replicate ancient proportions through the laws of an ordered field. In contemporary frameworks, Euclidean proportion is recovered as the product of line segments:

$$a : b = c : d \Leftrightarrow_{df} a \cdot d = c \cdot b,$$

or within the arithmetic of real numbers:

$$a : b = c : d \Leftrightarrow_{df} \frac{a}{b} = \frac{c}{d}.$$

Below, we examine variations of the first approach by discussing the systems of Hilbert and Hartshorne in §§ 4 and 5, respectively. In this approach, as in Euclid and Descartes, a, b, c, d stand for line segments.

The second strategy relies on real numbers and considers relationships between the lengths of line segments, that is, real numbers assigned to line segments based on axioms or sophisticated arguments.

Geometry developed in mainstream mathematics interprets Euclidean geometry within the so-called Euclidean spaces $\mathbb{R}^n$, with $\mathbb{R}^2$ serving as the model example of the Euclidean plane. This approach either incorporates a form of completeness for real numbers into the axioms of geometry or assumes a bijection between a geometric line and the real numbers. To be clear, this approach seeks to justify, through foundational studies, what really happened when 20th-century mathematics established its foundations on real numbers.

Below, we address real-numbers approaches by discussing the systems of Birkhoff and Millman–Parker in §§ 7 and 8, respectively. In § 6, we discuss the system of Borsuk–Szmielew, which forms a bridge between synthetic, Hilbert-style geometry and geometry based on real numbers.

Euclid's theory of proportion relies on the Archimedean axiom, which is explicitly included as Definition 4 in Book V. Hilbert's arithmetic of line segments does not reference the Archimedean axiom. On the other hand, approaches based on real numbers do rely on this axiom, as the real numbers form the largest Archimedean field. In § 9, we present a proof of Thales' theorem based on the arithmetic of hyperreal numbers, which constitute a non-Archimedean ordered field.

2. Book VI of the *Elements*

2.1. Definition of proportion

1. Thales' theorem is connected to Euclidean proportion through Proposition VI.1 – the only proposition in Book VI, except for the last one (VI.33), that explicitly references the definition of proportion, Defintion 5, Book V. We interpret this definition using the following formula:

$$a : b :: c : d \Leftrightarrow_{df} (\forall m, n \in \mathbb{N})[(na >_1 mb \Rightarrow nc >_2 md) \wedge$$
$$\wedge (na = mb \rightarrow nc = md) \wedge (na <_1 mb \Rightarrow nc <_2 md)].$$

Pairs a, b and c, d are to be of the same *kind.* This assumption is formalized by $a, b \in \mathfrak{M}_1 = (M_1, +, <_1)$, and $c, d \in \mathfrak{M}_2 = (M_2, +, <_2)$, which means that magnitudes of the same kind can be added and compared in terms of greater-than relationship. In the context of the Proposition VI.I, a, b are line segments, and c, d are triangles, or parallelograms. Specifically, based on this foundational assumption, triangles, somehow, can be added and compared as *greater* and *lesser* [6].

In Proposition VI.33, Euclid states the proportion between angles in a circle, on the one hand, and respective arcs, on the other.

THEOREM 2.1 (*Elements*, VI.1 [17]). *Let ABC and ACD be triangles, and EC and CF parallelograms, of the same height AC. I say that as base BC is to base CD, so triangle ABC (is) to triangle ACD, and parallelogram EC to parallelogram CF.*

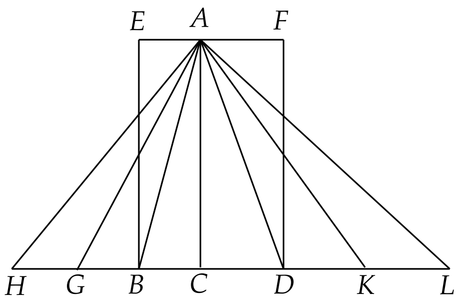

Figure 2: *Elements*, VI.1.

PROOF: By construction: $BC = GB = HG$ and $CD = DK = KL$. Then, by I.37, the equality of triangles holds: $\triangle AHC = 3\triangle ABC$, and $\triangle ALC = 3\triangle ADC$; see Fig. 2

Finally, Euclid applies these data to the definition of proportion:

$$3\triangle ABC \gtreqless 3\triangle ADC \Rightarrow 3BC \gtreqless 3DC. \tag{2.1}$$

This formula interprets Euclid's words: "And if base HC is equal to base CL, then triangle AHC is also equal to triangle ACL. And if base HC exceeds base CL, then triangle AHC also exceeds triangle ACL. And if less (than) less."

We express this as:

$$\triangle ABC : \triangle ADC :: BC : DC.$$

Note that, instead of *equal multiples* (na, nc, mb, md) referred to in Definition V.5, Euclid sets $n = m = 3$. □

2. The *Elements* do not explicitly justify the reasoning behind why the triangles are equal, greater, or lesser, i.e., (2.1). Euclid assumes that the equality or inequality of line segments directly translates to the equality or inequality of the triangles formed on these segments. Indeed, arguments relying on areas play a role in the theory of similar figures. Modern systems, however, seek to eliminate them.

On the other hand, some crucial results in Book VI concern the relationship between the areas of similar figures, yet modern systems do not recover them.

In Proposition VI.33, Euclid states the proportion between angles in a circle (inscribed or central) and their corresponding arcs. The proof similarly applies to triples of angles and triples of arcs; see Fig. 3.

Modern geometers, except for Birkhoff, do not address this issue at all, but it still echoes in calculus.

2.2. The role of VI.2 in Euclid's system and modern mathematics

1. Thales' theorem forms the foundation of the theory of similar figures, which is developed in the subsequent propositions of Book VI of the *Elements*.

Proposition VI.3 states that in a triangle, the bisector of an angle divides the opposite side into segments proportional to the other two sides.

Proposition VI.4 establishes that equiangular triangles are similar, meaning the sides about equal angles are proportional.

The following propositions introduce criteria for similar triangles analogous to the congruence criteria: Side-Side-Side, Side-Angle-Side and Side-Angle-Angle:

VI.5: Triangles with proportional sides are equiangular.

VI.6: If two triangles have equal angles and the sides about these angles are proportional, then the triangles are equiangular, and their corresponding sides are proportional.

VI.7: If two triangles have equal angles and the sides about another pair of equal angles are proportional, then the triangles are similar, provided these angles are either both less than $\pi/2$ or both greater than or equal to $\pi/2$.

The next two theorems are the most well-known consequences of Thales' Theorem that are not directly related to trigonometry.

VI.8: In a right-angled triangle, the altitude dropped from the right angle divides the triangle into two smaller triangles, each similar to the original triangle and to each other.

VI.9 shows how to divide a line segment AB into n equal parts. The procedure is as follows: Let lines AB and AC form an angle. Along the arm AC, place the same arbitrary segment n times. Connect the endpoint of the last segment to B. Then, draw n lines parallel to this segment. These parallel lines will intersect AB, dividing it into n equal parts.

In Proposition VI.12, Euclid demonstrates how to find the so-called fourth proportional line segment.

Moreover, Proposition VI.15 forms the basis for the modern formula for the area of a triangle $\frac{1}{2}ab\sin\alpha$, where α is an angle between sides a and b.

Proposition VI.19 introduces the formula – in modern terms – stating that the areas of similar triangles are proportional to the square of the similarity scale, while VI.20 extends this result to similar polygons.

Proposition VI.31 establishes the addition of similar figures. Since the Pythagorean Theorem (*Elements*, I.47) enables the addition of squares, and all squares are similar, we view VI.31 as a generalization of I.47.

2. In discussing modern attempts to prove Thales' theorem, we will show that some of the above theorems are assumed in advance, either as axioms or propositions based on geometric or analytic grounds.

Specifically, Hilbert proves VI.4, and then, with the use of the arithmetic of line segments, proves VI.2.

Birkhoff adopts VI.6 as an axiom, and then, with the use of the arithmetic of reals numbers, VI.2 follows easily.

Borsuk and Szmielew, as well as Millman and Parker, prove VI.9, and then, using the results concerning measures (Borsuk and Szmielew) or arithmetic of real numbers (Millman and Parker), they can prove VI.2.

In § 10, we present a system in which VI.1 is an axiom and show that this system allows us to prove VI.2 – indeed, all propositions of Book VI (except VI.33) – in an Euclidean fashion. This system is also related to a method for automated proofs of propositions from Book VI.

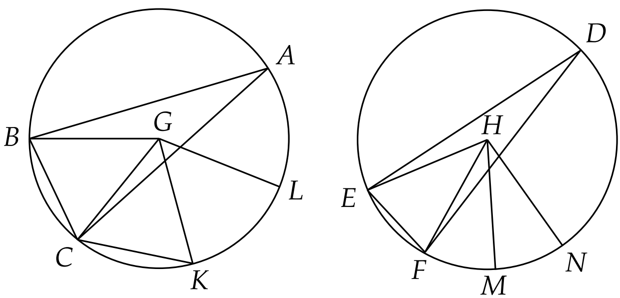

Figure 3: *Elements*, VI.33.

2.3. Proposition VI.33

1. In Proposition VI.33, Euclid shows that: *In equal circles, angles have the same ratio as the circumferences on which they stand.*

The accompanying diagram (see Fig. 3) represents various magnitudes referred to in the proposition: the angle $\angle BGC$, the sector of the circle

$sec\, BGC$, the $arc\, BC$, as well as the triangle $\triangle BGC$, and the segment BC – Ptolemy called it the chord, and we call it the sine of the angle $\angle BGC$.

With this notation, Euclid's proposition can be phrased as follows:

$$\angle BGC :: \angle EHF :: arc\, BC : arc\, EF.$$

Since circles are equal, angles and arcs can be taken in the same circle. Indeed, in modern mathematics, this is typically the unit circle.

2. Establishing the relationships between $arc\, BC$ and the sine BC was one of the most difficult problems in the history of mathematics.

Ptolemy managed to determine the relationship between the ratio of two arcs and the ratio of two sines. In accordance with Euclidean theory, he compared the ratio of two arcs, which are magnitudes of one kind, with the ratio of two line segments (sines), which are magnitudes of a different kind. This result laid the foundation for tables of chords, which had been in use from antiquity until modern times.

Given that sine $= x$ and arc $= z$ correspond to the angle α, Newton determined the arc in terms of sine, that is, the series for $\arcsin x$, as well as the sine in terms of arc, that is, the series of $\sin z$.

Euler managed to combine the series for $\sin x$ and $\cos x$ with the exponential function e^{ix}, where x stands for an arc of the unit circle [11].

3. In modern mathematics, the identification of the arc $arc\, BC$ and the angle $\angle BGC$ is established through the concept of radian measure. This identification is achieved as follows: converting degrees α to radians x is based on the formula:

$$\frac{x}{2\pi} = \frac{\alpha}{360}.$$

Assuming the unit circle, this formula corresponds to the relationship: the length of the arc is to the circumference of the circle as the measure of the angle in degrees is to 360°. That is the straightforward application of Proposition VI.33.

The length of the arc l corresponding to the angle α is determined in a similar way:

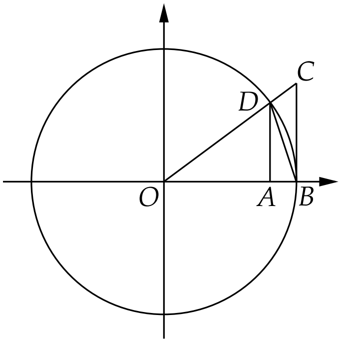

Figure 4: Determining inequalities $\sin x < x < \tan x$.

$$\frac{l}{2\pi} = \frac{\alpha}{360}.$$

The area P of the sector of the circle corresponding to the angle α is given by the formula:

$$\frac{P}{\pi} = \frac{\alpha}{360}.$$

Hence, we obtain that the length of the arc is equal to the measure of the angle in radians, and the area of the circular sector corresponding to angle x is $\frac{x}{2}$:

$$x = 2\pi\frac{\alpha}{360}, \quad l = 2\pi\frac{\alpha}{360}, \quad P = \pi\frac{\alpha}{360}.$$

4. Here is how these relate to determining the limit $\frac{\sin x}{x}$ at 0.

In calculus, the inequalities

$$\sin x < x < \tan x \tag{2.2}$$

are derived by comparing the areas of figures represented in Fig. 4:

$$\text{area of } \triangle ODB < \text{area of the sector of circle } ODB < \text{area of } \triangle OCB.$$

Substituting the formulas for the areas, we obtain (2.2).

2.4. Proving VI.2

1. Below, we reconstruct Euclid's proof of the Thales' theorem.

THEOREM 2.2 (*Elements*, VI.2 [17]). *If some straight line is drawn parallel to one of the sides of a triangle, then it will cut the sides of the triangle proportionally. And if the sides of a triangle are cut proportionally then the straight line joining the cutting will be parallel to the remaining side of the triangle.*

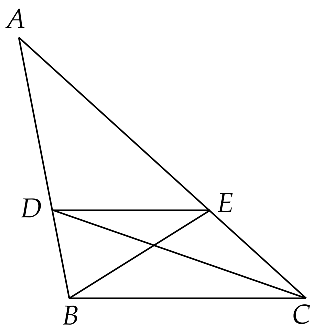

Figure 5: *Elements*, VI.2.

We present Euclid's proof in a schematized form, using modern symbolic conventions to enhance clarity. Specifically, we employ standard notations such as $\|$ to denote parallel lines. Additionally, we introduce specialized symbols, such as $\xrightarrow[I.38]{}$, where the arrow indicates a connective "for" rather than a formal logical implication, and the subscript "I.38" references Euclid's Proposition I.38.

The proof is as follows:

$$DE \| BC \quad \xrightarrow[I.37]{} \quad \triangle BDE = \triangle CDE$$

$$\begin{aligned}
&\xrightarrow[V.7]{} \quad \triangle BDE : \triangle ADE :: \triangle CDE : \triangle ADE \\
&\xrightarrow[VI.1]{} \quad \triangle BDE : \triangle ADE :: BD : DA \\
&\xrightarrow[VI.1]{} \quad \triangle CDE : \triangle ADE :: CE : EA \\
&\xrightarrow[V.11]{} \quad BD : DA :: CE : EA. \qquad (2.3)
\end{aligned}$$

The second part goes like that.

$$\begin{aligned}
BD : DA :: CE : EA,& \\
BD : DA :: \triangle BDE : \triangle ADE,& \\
CE : EA :: \triangle CDE : \triangle ADE & \xrightarrow[V.11]{} \quad \triangle BDE : \triangle ADE :: \\
& \qquad\quad :: \triangle CDE : \triangle ADE \\
& \xrightarrow[V.9]{} \quad \triangle BDE = \triangle CDE \\
& \xrightarrow[I.39]{} \quad DE \| BC. \qquad (2.4)
\end{aligned}$$

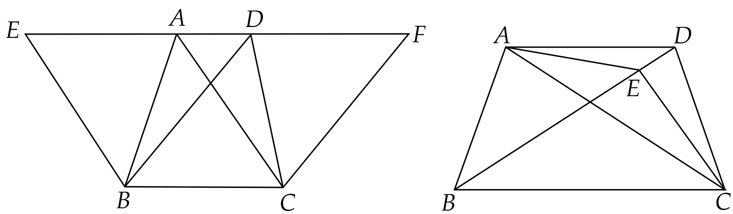

Figure 6: *Elements* I.37 and I.39. □

Propositions on equal figures referenced in this proof include the following (see Fig. 6):

THEOREM 2.3 (*Elements*, I.37 [17]). *Triangles which are on the same base and between the same parallels are equal to one another.*

That is,

$$AD \parallel BC \Rightarrow \triangle ABC = \triangle DBC.$$

THEOREM 2.4 (*Elements*, I.39 [17]). *Equal triangles which are on the same base, and on the same side, are also between the same parallels.*

That is,

$$\triangle ABC = \triangle DBC \Rightarrow AD \parallel BC.$$

2. In fact, VI.2 consists of two propositions: VI.2a, which moves from parallelism to proportion, and VI.2b, which moves from proportion to parallelism.

In a more synthetic manner, supported by Fig. 7, the first part of Euclid's proof is as follows:

$$l \parallel p \xrightarrow[I.37]{} T_1 = T_2 \xrightarrow[V.7]{} \frac{T_1}{T} = \frac{T_2}{T} \xrightarrow[VI.1]{} \frac{b}{a} = \frac{d}{c}.$$

And the second:

$$\frac{b}{a} = \frac{d}{c} \xrightarrow[VI.1]{} \frac{T_1}{T} = \frac{T_2}{T} \xrightarrow[V.9]{} T_1 = T_2 \xrightarrow[I.39]{} l \parallel p.$$

In these formulas, T, T_1, and T_2 represent triangles, and instead of the proportion $a : b :: c : d$, we use the equality of fractions.

These schemes emphasize the role of three sub-theories in Euclid's proof: the theory of proportion, Proposition VI.1 (along with the definition of proportion), and the concept of parallel lines related to equal figures through Propositions I.37 and I.39.

3. Comparing Euclid's approach with that of modern mathematics, the key issue lies in the ancient concept of proportion. In Euclid's system, it applies to triangles and line segments, while the crucial move relies on the relationship:

$$\frac{T_1}{T} = \frac{T_2}{T} \Leftrightarrow \frac{b}{a} = \frac{d}{c}.$$

In contrast, Hilbert and Hartshorne reconstruct the proportion of line segments based on the arithmetic of line segments. Using geometric prin-

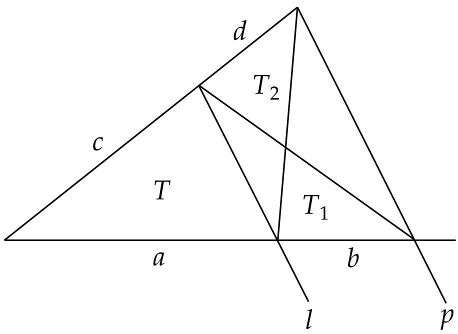

Figure 7: Proof of *Elements*, VI.2 schematized.

ciples, they prove Euclid's Proposition VI.4. Then, in proving VI.2a, the assumption of parallel lines $l \parallel p$ implies that the respective triangles are equiangular, and, due to VI.4, the proportion $a : c = b : d$ holds; see Fig. 8 (left).

Borsuk and Szmielew, as well as Millman and Parker, derive VI.2a from VI.9 – indeed, it can be proved without referring to Thales' theorem.

4. Although the derivation of VI.2 in these systems differs, they enable us to prove VI.2b in the same manner: Supposing that $\frac{a}{b} = \frac{c}{d}$ and that l is not parallel to p, a line q parallel to l is introduced, which intersects a line segment d'. Due to VI.2a, the proportion $\frac{a}{b} = \frac{c}{d'}$ holds. By the arithmetic of line segments, it follows that $d = d'$, leading to a contradiction; see Fig. 8 (right).

5. In Birkhoff's system, the deductive structure of VI.2 is quite different. Birkhoff adopts VI.6 as an axiom, and from proportionality of sides $\frac{b}{a} = \frac{d}{c}$, it follows that $b = ka$ and $d = kc$, where $k = \frac{b}{a}$. By VI.6, the respective triangles are equiangular, leading to the conclusion $l \parallel p$, that is VI.2b; see Fig. 8 (left).

For VI.2a, suppose $l \parallel p$ and $\frac{b}{a} \neq \frac{d}{c}$. For some d', the equality holds $\frac{b}{a} = \frac{d'}{c}$.[1] Then, $b = ka$ and $d' = kc$, where $k = \frac{b}{a}$, and the respective triangles are equiangular, which means $l \parallel q$, contradicting Playfair Axiom; see Fig. 8 (right).

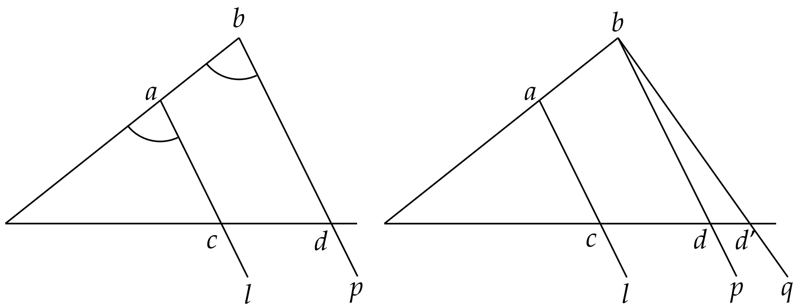

Figure 8: Modern proofs of VI.2a (left) and VI.2b (right) schematized.

After this overview of the techniques applied in proving Thales' theorem, we proceed to a more detailed presentation of specific approaches.

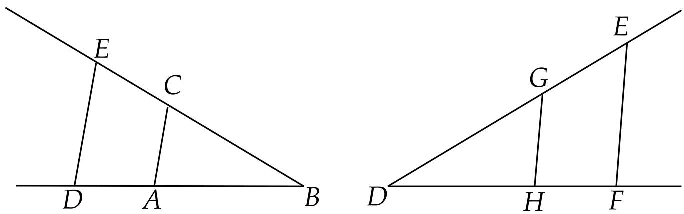

Figure 9: Product of line segments: Descartes' *La Géom*étrie, p. 298 (left), Euclid's *Elements*, VI.12 (right).

[1]In Greek mathematics, this property is called the fourth proportional. It is employed implicitly in Euclid's Book V.

3. Descartes' arithmetic

3.1. *Elements*, Book V

Descartes' arithmetic of line segments is based on Proposition VI.12 of Euclid's *Elements*. Descartes adopts a unit line segment $AB = 1$ as well as an arbitrary angle $\angle DBE$; see Fig. 9. To simplify the setup, we assume $\angle DBE = \frac{\pi}{2}$ and demonstrate that the rules of arithmetic can be justified by the laws of proportion developed in Book V of the *Elements*.

Below, we include Propositions 7 to 25 of Book V. Although they are stylized in algebraic form, this modern formulation serves only to highlight the similarities between proportions and the arithmetic of fractions. Here, equality denotes equal figures [10].

V.7	$a = b \to a : c :: b : c, \;\; a = b \Rightarrow c : a :: c : b.$
V.8	$a > c \Rightarrow a : d \succ c : d, \;\; a > c \Rightarrow d : c \succ d : a.$
V.9	$a : c :: b : c \Rightarrow a = b.$
V.10	$a : c \succ b : c \Rightarrow a > b, \;\; c : b \succ c : a \Rightarrow b < a.$
V.11	$a : b :: c : d, \; c : d :: e : f \Rightarrow a : b :: e : f$
V.12	$a : b :: c : d, \; a : b :: e : f \Rightarrow a : b :: (a + c + f) : (b + d + f).$
V.13	$a : b :: c : d, \; c : d \succ e : f \Rightarrow a : b \succ e : f.$
V.14	$a : b :: c : d, \; a > c \Rightarrow b > d.$
V.15	$a : b :: na : nb.$
V.16	$a : b :: c : d \Rightarrow a : c :: b : d.$
V.17	$(a + b) : b :: (c + d) : d \Rightarrow a : b :: c : d.$
V.18	$a : b :: c : d \Rightarrow (a + b) : b :: (c + d) : d.$
V.19	$(a + b) : (c + d) :: a : c \Rightarrow b : d :: (a + b) : (c + d).$
V.22	$a : b :: d : e, \; b : c :: e : f \Rightarrow a : c :: d : f.$
V.23	$(a : b :: e : f, \; b : c :: d : e) \Rightarrow a : c :: d : f.$
V.24	$a : c :: d : f, \; b : c :: e : f \Rightarrow (a + b) : c :: (d + e) : f.$
V.25	$(a : c :: e : f, \; a > c > f, \; a > e > f) \Rightarrow a + f > c + e.$

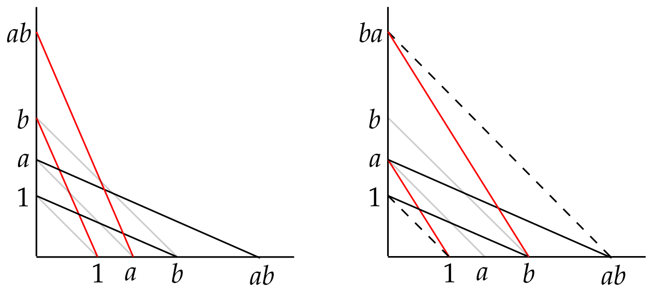

Figure 10: Product of line segments (left). Commutativity of the product (right).

3.2. Arithmetic of line segments

First, we demonstrate the commutativity of the product. By the definition of the product, we obtain (see red lines in Fig. 10):

$$\frac{a}{1} = \frac{ab}{b}, \quad \frac{ba}{a} = \frac{b}{1}.$$

Then, by Proposition V.23, the following proportion holds:

$$\frac{a}{1} = \frac{ab}{b}, \ \frac{ba}{a} = \frac{b}{1} \xrightarrow[V.23]{} \frac{ba}{1} = \frac{ab}{1}.$$

By Proposition V.9, it follows that

$$ba = ab. \qquad \square$$

In Fig. 10 (left), we represent product ab on two axes, given the congruence of the respective right-angled triangles. In Fig. 10 (right), we represent the product ab as a result of this modification. Note that the dashed lines are parallel due to a result derived from the theorems in Book V: since $ba = ab$, the respective triangle is isosceles.

Conversely, in Hilbert's arithmetic of line segments and other modern approaches to the arithmetic of line segments, geometric arguments establish that certain lines are parallel, thereby justifying that $ab = ba$.

Second, the distributive law requires more careful attention. The left diagram in Fig. 11 illustrates the products $c \cdot a$ and $c \cdot b$; this representation assumes that the product is commutative. By drawing the parallel to the line $c1$ through $a + b$, we apply Thales' theorem to obtain

$$\frac{x}{a} = \frac{ca}{a},$$

where the segment a in the ratio $x : a$ corresponds to the difference $(a+b)-b$.

Applying Euclid's Proposition V.9, we conclude that $x = ca$.

Ultimately, this leads to

$$x + cb = ca + cb.$$

The continuous lines in the right diagram in Fig. 11 represent the following proportions, expressed as fractions (the first follows from the definition of the product, and the second from the above argument):

$$\frac{c}{1} = \frac{c(a+b)}{a+b}, \qquad \frac{ca+cb}{c} = \frac{a+b}{1}.$$

Applying Proposition V.23, we obtain:

$$\frac{ca+cb}{1} = \frac{c(a+b)}{1}.$$

Finally, applying Proposition V.9 once again, we arrive at:

$$ca + cb = c(a+b),$$

where ca, cb, and $c(a+b)$ are constructed based on Descartes' definition. □

Fig. 10 and 11 depict the respective relations between parallel lines. However – let us emphasize – the arguments presented above do not rely on geometry but rather on propositions from Book V. Although modern geometers adopt Descartes' definition of the product, they seek to justify the laws of arithmetic through geometric principles.

Dashed lines in diagrams Fig. 10 and 11 are parallel due to the theory of proportions. Based on the above arguments, Euclid's Proposition V.23 encodes the commutativity and distributivity of Descartes' product.

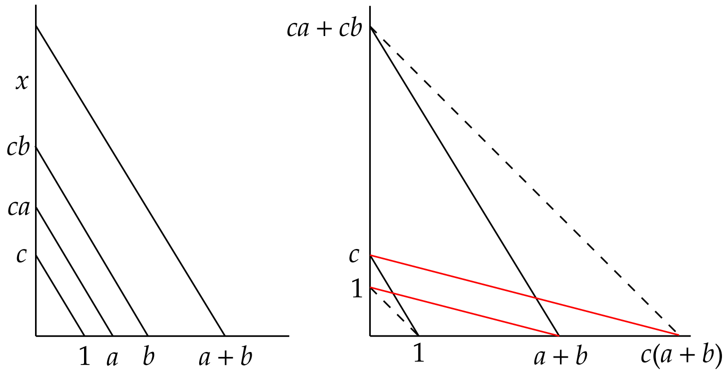

Figure 11: Distribuitive law.

4. Hilbert

Hilbert's axioms for synthetic geometry were first presented in *Grundlagen der Geometrie* (1899) [21]. Although referring to Euclid's *Elements*, they are based on a different methodology. Hilbert does not include a straightedge and compass as constructive tools. From the set of constructions developed by Euclid, he selects two specific ones: the transportation of line segments (*Elements*, I.2, *Grundlagen*, C1) and the transportation of angles (*Elements*, I.23, *Grundlagen*, C4). Accordingly, the deductive structures of the *Elements* and *Grundlagen* differ [5].

Moreover, a new approach, unknown in Greek mathematics, is based on the uniqueness referenced in Axioms I.1, C.1, and C.4; the Parallel Axiom also implies uniqueness.

In what follows, we will point out only the differences related to Thales' theorem.

Below, we present Hilbert's axioms following Marvin Greenberg's concise version [19].

4.1. Hilbert system of axioms

Hilbert adopts primitive concepts: point, line, and plane, as well as primitive relationships: point B lies between A and C, denoted here as $A - B - C$, congruence of line segments and angles. He also defines a half-line (ray), an angle, and a triangle.

Axioms of Incidence

I1. For any two distinct points A, B, there exists a unique line l containing A, B.

I2. Every line contains at least two points.

I3. There exist three noncollinear points (that is, three points not all contained in a single line).

Axioms of Betweenness

B1. If B is between A and C, (written $A - B - C$), then A, B, C are three distinct points on a line, and also $C - B - A$.

B2. For any two distinct points A, B, there exist points C, D, E such that $A - B - C$, $A - D - B$, and $E - A - B$.

B3. Given three distinct points on a line, one and only one of them is between the other two.

B4. (Pasch). Let A, B, C be three non collinear points, and let l be a line not containing any of A, B, C. If l contains a point D lying between A and B, then it must also contain either a point lying between A and C or a point lying between B and C.

Axioms of Congruence for Line Segments

C1. Given a line segment AB, and given a ray (half-line) r originating at a point C, there exists a unique point D on the ray r such that $AB \equiv CD$.

C2. If $AB \equiv CD$ and $AB \equiv EF$, then $CD \equiv EF$. Every line segment is congruent to itself.

C3. (Addition). Given three points A, B, C on a line satisfying $A-B-C$, and three further points D, E, F on a line satisfying $D-E-F$, if $AB \equiv DE$

and $BC \equiv EF$, then $AC \equiv DF$.

Axioms of congruence for Angles

C4. Given an angle $\angle BAC$ and given a ray $\overrightarrow{DF}$, there exists a unique ray $\overrightarrow{DE}$, on a given side of the line DF, such that $\angle BAC \equiv \angle EDF$.

CS. For any three angles α, β, γ, if $\alpha \equiv \beta$ and $\alpha \equiv \gamma$, then $\beta \equiv \gamma$. Every angle is congruent to itself.

C6. (SAS) Given triangles ABC and DEF, suppose that $AB \equiv DE$ and $AC \equiv DF$, and $\angle BAC \equiv \angle EDF$. Then the two triangles are congruent, namely, $BC \equiv EF$, $\angle ABC \equiv \angle DEF$ and $\angle ACB \equiv \angle DFE$.

Archimedes' axiom

Given line segments AB and CD, there is a natural number n such that n copies of AB added together will be greater than CD.

Parallel axiom

For each point A and each line l, there is at most one line containing A that is parallel to l.

4.2. Hilbert's arithmetic of line segments

1. Addition and the greater-than relation between line segments and angles are defined in Hilbert's system, whereas in the *Elements*, these are a primitive operation and a primitive relationship, respectively. Today, Hilbert-style definitions are standard. Here is a reminder:

Definition 4.1. [23, p. 30] We say that $c = AC$ is the sum of the two segments $a = AB$ and $b = BC$ if B lies between A and C. In other words

$$c = a + b \Leftrightarrow_{df} A - B - C.$$

The segments a and b are said to be smaller than c, which we indicate by writing

$$a < c,\ b < c \Leftrightarrow_{df} A - B - C.$$

Addition is both associative and commutative.

2. Thales' theorem is presented in *Grundlagen* as the final result in the chapter entitled *Theory of Proportion*. Hilbert begins this chapter by saying:

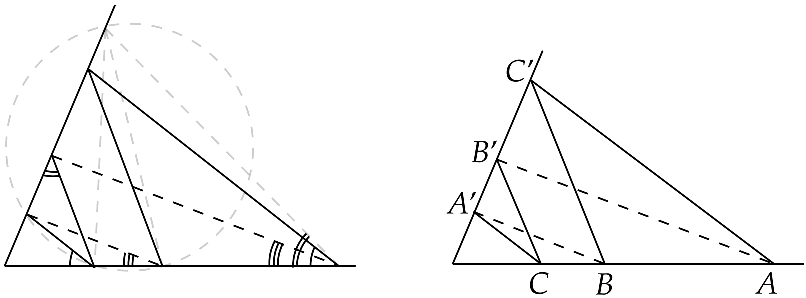

Figure 12: Proof of Pascal's theorem.

"At the beginning of this chapter, we shall present briefly certain preliminary ideas concerning complex number systems, which will later be of service to us in our discussion." He then enumerates 17 properties related to addition, multiplication, and the greater-than relationship. In fact, he provides the first-ever axioms for an ordered field.

In the following, Hilbert clarifies: "In the present chapter, we propose, by the aid of these axioms, to establish Euclid's theory of proportion; that is, we shall establish it for the plane and that independently of the axiom of Archimedes".

Indeed, the definition of the product of line segments, as well as its properties such as commutativity and distributivity, is based on the properties of congruent triangles and angles inscribed in a circle, as presented in Books I and III of *The Elements.* The Archimedean axiom is introduced as Definition 4 of Book V, meaning that Euclid's theory of proportion and similar triangles depends on it.

THEOREM 4.2 (Pascal 's theorem [23, p. 25]). *Let A, B, C and A', B', C' be two sets of points on the arms of an angle. If CB' is parallel to BC' and CA' is parallel to AC', then BA' is parallel to AB'.*

The proof of this theorem is quite intricate, prompting Hilbert to dedicate an entire subsection to its discussion. Figure 12 illustrates the proof. Essentially, Hilbert explores the reverse of Euclid's III.22 on a quadrilateral inscribed in a circle.

The definition of the product of line segments aligns with Descartes' approach, using the right angle instead of any angle. Moreover, Hilbert modifies Descartes' approach setting a and ab on the same arm of the angle; see Fig.13 (left) and Fig. 10.

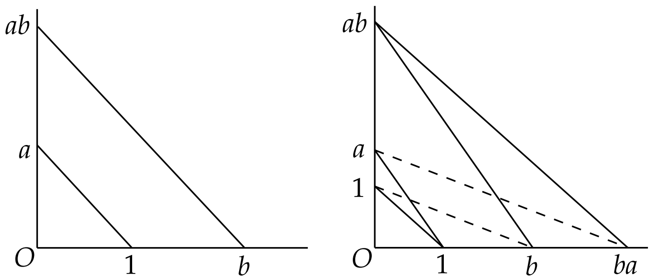

Figure 13: Hilbert's definition of the product (left). Commutativity of the product (right).

Hilbert shows that this product is commutative, associative, and distributive; the proofs rely on Pappus' theorem (which Hilbert calls Pascal's theorem). Figure 13 (right) illustrates the proof of the commutativity law: Since triangles $\triangle 1O1$ and $\triangle(ab)O(ba)$ are isosceles, the equality $ab = ba$ follows.

3. Hilbert axioms for an ordered field include the following: For $a \neq 0$, and b, there exists the unique element x such that

$$ax = b.$$

Figure 14 (left) illustrates the construction of the line $\frac{b}{a}$. The same figure shows that $\frac{b}{a}a = b$. Since multiplication is commutative, it follows that $\frac{b}{a}$ solves the equation $ax = b$.

Setting $b = 1$ in this axiom implies the existence of the inverse a^{-1}; see Fig. 14 (right).

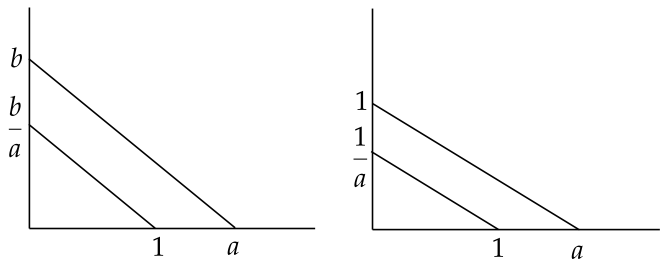

Figure 14: Division of line segments (left). The inverse of a line segment (right).

4.3. Thales' theorem and similar triangles

1. In Descartes' *La Geometrie*, we identify the implicit rule that transforms proportion into an equality of quotients [6]:

$$a : b :: c : d \Leftrightarrow \frac{a}{b} = \frac{c}{d}.$$

However, just as Descartes' arithmetic is based on Thales' theorem, so is this rule. Hilbert arrived at a similar result without referring to Thales.

Definition 4.3. [23, p. 34] Segments a, b, a', b' are in proportion if and only if the equality of products ab' and ba' holds:

$$a : b = a' : b' \Leftrightarrow_{df} ab' = a'b.$$

Since Hilbert's arithmetic includes division, we can rephrase this as follows:

$$a : b = a' : b' \Leftrightarrow \frac{a}{b} = \frac{a'}{b'}.$$

DEFINITION 4.4 ([23, p. 34]). Equiangular triangles are called similar.

2. Before addressing Thales' theorem, Hilbert proves the counterpart of Euclid's Proposition VI.4.

THEOREM 4.5. *[23, p. 34] In equiangular triangles sides about equal angles are proportional.*

PROOF: Hilbert first considers right-angled triangles and then, in the general case, examines triangles composed of right-angled triangles; see Fig. 15 and Fig. 16.

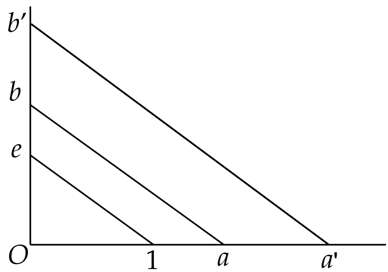

Figure 15: The first part of proposition 4.5.

The assumption about equiangular triangles $\triangle bOa$ and $\triangle b'Oa'$ implies that lines through a, b and a', b' are parallel. A point e is chosen so that the line passing through $e, 1$ is parallel to these lines.

By the definition of the product, we have $b = ea$ and $b' = ea'$. Then, $a'b = a'ea$, $ea'a = b'a$. Due to commutativity, $a'b = b'a$, that is, $a : b = a' : b'$.

In the general case, Hilbert proceeds as follows: intersecting the bisectors in triangles T and T' determines points S and S'. Perpendiculars dropped from S and S' onto the sides of the triangles decompose T and T' into right-angled triangles such that respective pairs are equiangular. This approach allows Hilbert to apply the previous result related to right-angled triangles; see Fig. 16.

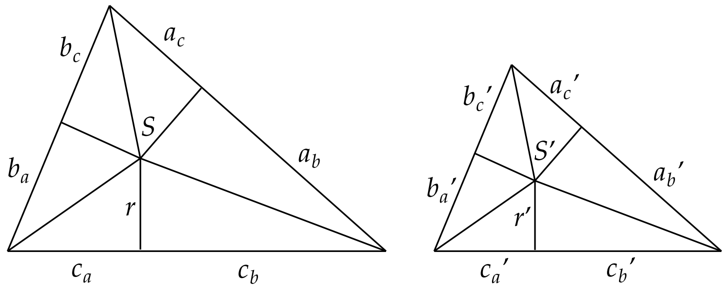

Figure 16: Decomposition of equiangular triangles.

Consequently, proportions relating to respective triangles making T and T' follow:

$$a_b : r = a'_b : r', \quad a_c : r = a'_c : r', \quad b_c : r = b'_c : r', \quad b_a : r = b'_a : r'.$$

Turning proportions into equalities of products gives:

$$a_b r' = a'_b r, \quad a_c r' = a'_c r, \quad b_c r' = b'_c r, \quad b_a r' = b'_a r.$$

By the distributive law, one obtains:

$$(a_b + a_c)r' = (a'_b + a'_c)r', \qquad (b_c + b_a)r' = (b'_c + b'_a)r',$$

Then:

$$ar' = a'r, \qquad br' = b'r,$$

and also:

$$b'ar' = b'a'r, \qquad a'br' = a'b'r.$$

Finally, dividing the equality $b'ar' = a'br'$ by r' gives

$$b'a = a'b,$$

or, in the equivalent form:

$$a : b = a' : b'.$$

Note that, to divide $b'ar' = a'br'$ by r', Hilbert had to show that multiplication of line segments is an associative operation. □

Furthermore Hilbert writes: "From the theorem just demonstrated, we can easily deduce the fundamental theorem in the theory of proportion".

THEOREM 4.6. *[23, p. 35] If two parallel lines cut segments a, b from one side of an angle and segments a', b' from the other side, then the proportion $a : b = a' : b'$ holds. Conversely, if four segments a, b, a', b' satisfy this proportion and a, a' and b, b' are laid off along the two sides of an angle respectively, then the straight lines joining the extremities of a and b and of a' and b' are parallel.*

Hilbert leaves this without proof. Indeed, the proof proceeds straightforwardly, if instead of proportion $a : b = a' : b'$, we consider the equality:

$$\frac{a}{b} = \frac{a'}{b'}.$$

Then, we can apply the rule of arithmetic, specifically:[2]

$$\frac{a}{a'} = \frac{a+b}{a'+b'} \Rightarrow \frac{a}{a'} = \frac{b}{b'}. \tag{4.1}$$

As triangles $\triangle BAC$ and $\triangle BDE$ are equiangular, it follows from the previous theorem and (4.1) that

$$\frac{a}{a'} = \frac{b}{b'}.$$

For the second part, suppose $\frac{a}{b} = \frac{a'}{b'}$ and $DE \nparallel AC$. Let the line AC' parallel to DE cut on the side BC line segment b''. By the Playfair's axiom, the segment b" is unique.[3] Now, let $b'' < b'$; see Fig. 17.

[2] In fact, it is an arithmetic interpretation of Euclid's Proposition V.17.

[3] For clarity, $b' = BC$ and $b'' = BC'$.

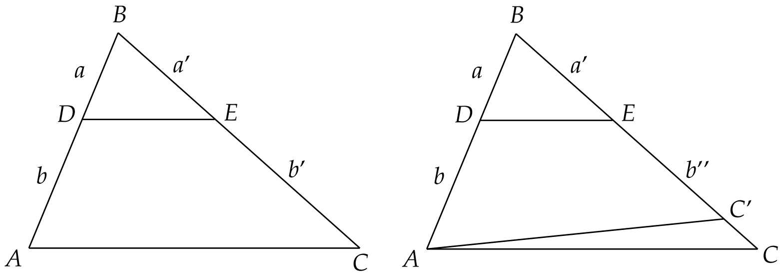

Figure 17: Proposition 4.6.

By the first part, the equalitites hold:

$$\frac{a'}{b'} = \frac{a}{b} = \frac{a'}{b''}.$$

This leads to a contradiction, as it implies $b' = b''$. □

Note that in Euclid's system, the implication

$$a : b :: a : b'' \Rightarrow b' = b''$$

also holds but is based on Proposition V.9, which requires Archimedes axiom.

On the other hand,

$$\frac{a'}{b'} = \frac{a'}{b''} \Rightarrow b = b''$$

obtains in any ordered field (Archimedean or non-Archimedean) and, therefore, in Hilbert's arithmetic of line segments.

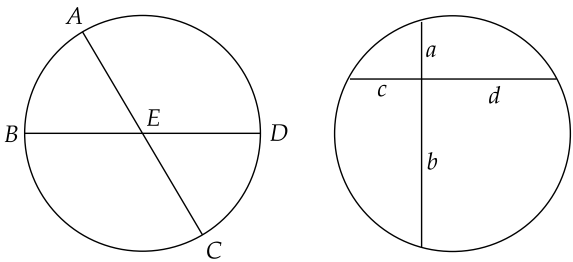

Figure 18: *Elements*, III.35 (left) and its simplified version (right).

5. Harsthorne

5.1. Two ways of introducing product

1. There are two propositions in Euclid's *Elements* that may serve as a basis for the product of line segments: VI.12 and III.35. Descartes and later Hilbert followed the first path, while Robin Hartshorne [20] chose the other.

Hilbert's axioms do not include circles; however, his arithmetic employs properties of quadrilaterals inscribed in a circle. Hartshorne simplified these arguments by introducing the arithmetic of line segments based on Euclid's Proposition III.35.

Euclid states III.35 in terms of equal areas: Chords in a circle intersect in such a way that the respective rectangles are equal,

$$AE.EC = BE.ED$$

In a simplified form, we can consider perpendicular chords, giving[4]

$$ab = cd;$$

see Fig. 18. When one of the lines a, b, c, d is set to 1, say $c = 1$, we may

[4] Here, in Euclidean context, $AE.BE$ or ab denotes a rectangle with sides AE, EC or a, b.

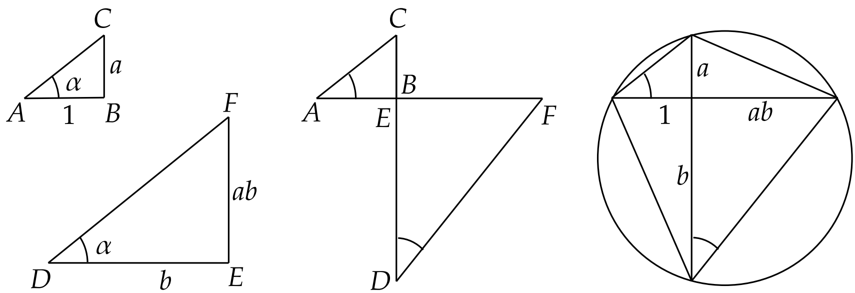

Figure 19: Hartshorne's definition of the product of line segments.

treat $d = ab$ as a definition of a product.

Hartshorne's definition of a product is as follows ([20], p. 170): Taking two equiangular right-angled triangles — the first with legs 1 and a, the second with leg b, the line EF is the product of CB and ED; see Fig. 19 (left). Indeed, given that these triangles are arranged as shown in Fig. 19 (middle), vertices A, C, D, and E form a cyclic quadrilateral, and we can interpret the lines AF and CD as intersecting chords; see Fig. 19 (right).

2. In Fig. 20, we compare two ways of introducing the product based on Euclid's propositions. The assumption $l \parallel p$, translates into the equality of angles, $\alpha = \beta$, and then, by VI.2a, into the proportion $a : c :: d : b$. On the other hand, the proportion $a : c :: d : b$, by VI.2b, translates into the condition $l \parallel p$. We summarize this in the following formula:

$$\alpha = \beta \underset{VI.2}{\Longleftrightarrow} \frac{a}{c} = \frac{d}{b}.$$

In the second approach, if $\alpha = \beta$, then the vertices of the triangles form a cyclic quadrilateral, and by III.35, $ac = bd$. The implication can also be reversed, but the corresponding argument requires a new setting involving equal areas. We summarize this in the following formula:

$$\alpha = \beta \underset{III.35}{\Longleftrightarrow} ac = bd.$$

The definition of multiplication of segments in the above manner allows Hartshorne to simplify the proof of the properties of multiplication.

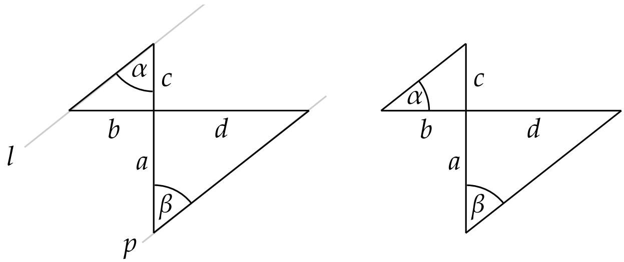

Figure 20: Two ways of defining product: proportion (left) and intersecting chords (right).

5.2. Arithmetic and Thales' theorem

Hartshorne's approach simplifies proofs of properties such as commutativity and distributivity. For example, to show that $ab = ba$, we consider a triangle with angle α and determine the product ab. In the same circle, if we consider a triangle with angle β, we obtain the product ba. It turns out to be the same line segment. Therefore, $ab = ba$; see Fig. 21 (left).

The inverse of a line segment is also easily determined within this approach. An isosceles triangle with height b and base 2 determines a circle, which in turn determines the segment $\frac{1}{b}$; Fig. 21 (middle and right).

A similar technique enables Hartshorne to prove that the multiplication of line segments is an associative operation [20, p. 172]. Therefore, his arithmetic enables one to find the fourth proportional. Indeed, his proof of Thales' theorem follows Hilbert's theorems 4.5 and 4.6, presented above.

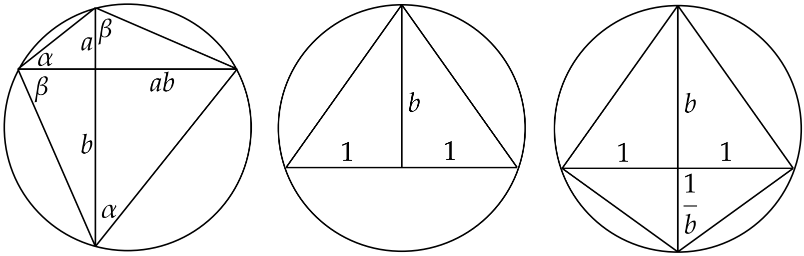

Figure 21: Hartshorne's arithmetic of line segments.

6. Borsuk–Szmielew

6.1. Introducing the measure of line segments

1. In [13], an absolute plane geometry is defined as the system $(S, P, L, \mathbf{B}, \mathbf{D})$, where S (space) represents the set of points, P the set of planes, L the set of straight lines, $\mathbf{B}$ the ternary relation of betweenness, and $\mathbf{D}$ the quaternary relation of equidistance – both in the space S. These primitive concepts and relations are governed by axioms of incidence, betweenness, and equidistance, and the congruence of line segments.

Presenting this approach to Thales theorem, we consider only line segments, therefore, for simplicity, we adopt the notation $a \equiv b$ to denote the congruence of line segments.[5]

Generally, Borsuk and Szmielew adopt Hilbert's axioms. However, unlike Hilbert's system, the Borsuk-Szmielew axioms do not include the concept of an angle. Thus, instead of Hilbert's axiom C4 (a transportation of angles), they include an axiom on the transportation of triangles, and instead of C6 (the Side-Angle-Side congruence criteria), they include the so-called

[5] $\mathbf{D}$ is the relation between four points and defines a congruence of line segments: $AB \equiv CD$ iff $\mathbf{D}(A, B, C, D)$. Strictly speaking, a line segment a is an equivalence class of congruent segments determined by two points.

five-segments axiom. The parallel axiom remains the same as in Hilbert's system, namely, the Playfair's axiom.

2. The crucial novelty is the continuity axiom expressed in terms of Dedekind cuts – specifically, in terms of total order, or more precisely, the relation **B** ([13], p. 140). This axiom enables the demonstration of the existence of a finitely additive measure on the set of line segments.

THEOREM 6.1. *[13, pp. 156–157] For a given line segment a and a real number r, there exists a unique measure φ on the set of line segments, satisfying $\varphi(a) = r$ and the following conditions:*

$$(1) \quad a \equiv b \Rightarrow \varphi(a) = \varphi(b), \quad (2) \quad \varphi(a + b) = \varphi(a) + \varphi(b). \tag{6.1}$$

The measure $\varphi(a)$, or the length of the line segment a, is denoted by $|a|_\varphi$. Since there are various measures, for simplicity, we distinguish one that assigns the value 1 to a given line segment, say u. We denote this measure as $|a|$. Thus

$$|u| = 1.$$

The crucial property of a measure is as follows: given its value on one line segment, for example, $\varphi(u)$, it determines the value for any other line segment. In other words, if the measure φ is determined for some line segment a, then φ extends uniquely on the entire set of line segments.

Clearly, in Euclidean geometry, the choice of the unit line segment is based on convention. In contrast, in hyperbolic geometry, there exists a unique line segment that is distinguished on a geometric basis ([13], p. 242).

Note also that the real number unit is itself a matter of convention. Given that $(\mathbb{R}, +, \cdot, 0, 1, <)$ is the field of real numbers – an ordered field equipped with the completeness axiom – the field $(\mathbb{R}_+, +_1, \circ, 1, e, <_1)$ is also the field of real numbers. In this new field, the (standard) product plays the role of addition, and a new product is introduced via the exponential map:

$$x +_1 y =_{df} x \cdot y, \quad x \circ y =_{df} e^{\log x \cdot \log y};$$

while the total order $<_1$ is a restriction of $<$ to the set $\mathbb{R}_+$.

Amid these conventions, the total order of real numbers holds a special position, as it is the unique total order compatible with both addition and multiplication of real numbers.

3. The existence measure theorem is quite involved and essentially explores a property of real numbers, specifically, that the set of dyadic numbers is dense in $(\mathbb{R}, <)$.[6]

Similarly, due to the density of dyadic numbers in $(\mathbb{R}, <)$, the following theorem is established, which, as we demonstrate below, essentially encodes Thales' theorem.

THEOREM 6.2. *[13, p. 156] For any two measures φ_0 and φ_1, there exists a real number λ such that*

$$\varphi_0 = \lambda \varphi_1. \tag{6.2}$$

The proof of Theorem 6.2 involves a special operation on segments introduced by Borsuk and Szmielew. Specifically, for any dyadic number $\mathfrak{w}$ and any line segment a, they define a new operation:

$$\mathfrak{w} a,$$

where the line segment $\mathfrak{w} a$ is constructed through additions and bisections. Moreover, for any measure function φ, the following holds:

$$\varphi(\mathfrak{w} a) = \mathfrak{w} \varphi(a), \quad \varphi(a - b) = \varphi(a) - \varphi(b),$$

as well as the compatibility with the order:[7]

$$a < b \Rightarrow \varphi(a) < \varphi(b).$$

Since a value on a line segment a determines the measure of any other line segment, we can rephrase Theorem 6.2 as follows: For any measure φ, there exists a real number λ such that

$$\varphi(a) = \lambda |a|. \tag{6.3}$$

[6] In the context of ordered fields, this is equivalent to the Archimedean axiom.

[7] Note, however, that in modern systems, the *greater-than* relationship between line segments is defined through the relation $\mathbf{B}$, and its uniqueness is not demonstrated.

Roughly speaking, given two measures φ_0, φ_1 such that $\varphi_0(a) = r$, and $\varphi_1(a) = s$, the relationship between φ_0 and φ_1 is given by

$$\varphi_0 = \tfrac{r}{s}\varphi_1.$$

Finally [13, pp. 156–157], Borsuk and Szmielew demonstrate that a line l is isometric to the line of real numbers $(\mathbb{R}, <)$. In another words, starting from scratch – using only the axioms of synthetic geometry along with continuity – they show that any geometric line is isomorphic to the real number line, where the metric on $(\mathbb{R}, <)$ is given by the absolute value of real numbers:

$$\varrho(r, s) = |r - s|, \qquad r, s \in \mathbb{R}.$$

Since the real numbers form a real-closed field, there exists the unique order compatible with both addition and multiplication, and consequently, the unique absolute value.

The systems of Birkhoff, as well as Millman and Parker, which we discuss below, take this theorem for granted, that is, they adopt as an axiom the existence of a bijection between a geometric line and the real number line $(\mathbb{R}, <)$.

6.2. Thales' theorem

1. In the context of Thales' theorem, the crucial proposition is that parallel projection f of segments lying on a line l onto a line p is a similarity map; see Fig. 22.

$$f : l \mapsto p.$$

Theorem 6.3 ([13, pp. 158, 216]). *Parallel projection of one line onto another line is a similarity map.*

Proof: First, we show that the parallel projection defines a measure, that is, it satisfies the conditions (6.1). We focus on the first condition, as it relates to a geometric insight that we will explore further in our paper. The second condition consists in showing that the parallel projection is compatible with the relation **B**.

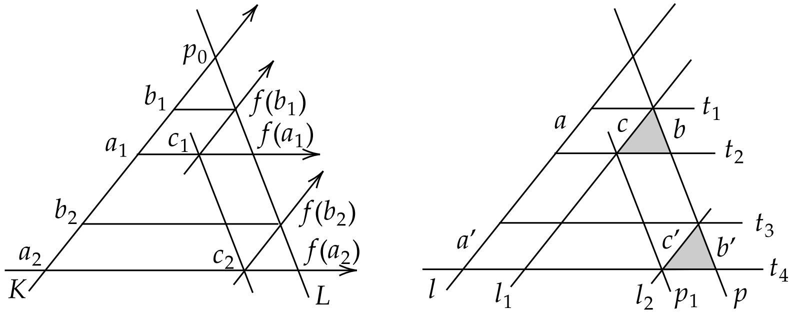

Figure 22: Properties of parallel projections, [13], p. 215 (left), schematized proof (right).

Fig. 22 (left) represents Borsuk and Szmielew's diagram, where f maps points. In the neighboring diagram, we interpret f as mapping line segments.

Let

$$t_1 \parallel t_2 \parallel t_3 \parallel t_4, \quad l \parallel l_1 \parallel l_2, \quad p \parallel p_1, \quad f(a) = b, \; f(a') = b'.$$

The aim is to show the relationship:

$$a \equiv a' \Rightarrow b \equiv b'.$$

In Fig. 22 (left), the quadrilateral $a_1 b_1 f(b_1) c_1$ is a parallelogram. Thus, $a \equiv c$. Similarly, we show that $a' \equiv c'$. Therefore, $c \equiv c'$.[8]

Triangles $\triangle c_1 f(b_1) f(a_1)$ and $\triangle c_2 f(b_2) f(a_2)$, or the grey triangles in the right diagram, are equiangular. By the Side-Angle-Side congruence criterion, we have, $f(b_1)f(a_1) \equiv f(b_2)f(a_2)$, or in the right diagram $b \equiv b'$.

Second, the parallel projection f satisfies conditions (6.1), one can define a measure:

$$\varphi(a) = |f(a)|.$$

[8]The properties of parallelograms applied in this argument are stated in Euclid's Propositions I.33 and I.34.

As observed earlier, given a value on one line segment, a measure is determined, which means that φ extends uniquely to the set of all line segments.

Now, since φ is a measure, by Theorem 6.2, there exists a real number λ such that

$$\varphi(a) = |f(a)| = \lambda|a|.$$

This means that φ is a similarity map, with λ being the similarity scale. □

2. Thales' theorem is phrased in terms of similarity mappings and reduces to the statement that the parallel projection f is a similarity map [13, p. 216].

The substance of this theorem is as follows: For parallel projection f of l onto p there exists a real number λ such that for any line segments a, b on l obtains (see Fig. 23):

$$f(a) = \lambda a, \quad f(b) = \lambda b.$$

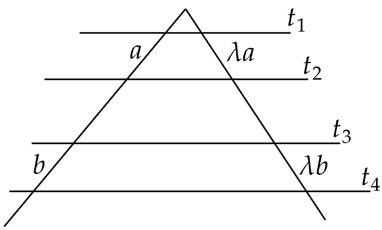

Figure 23: Thales' theorem by Borsuk and Szmielew.

6.3. Similar triangles

1. As a consequence of Thales' theorem, Borsuk and Szmielew obtain the following: ([13, pp. 216–217]):

In a triangle $\triangle ABC$, if lines p_1, l_1, parallel to AB and AC respectively, intersect the sides of triangle $\triangle ABC$, cutting line segments a on side AC, b on side BC, and c on side AB, then the following equalities hold:

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'},$$

where $AC = c'$, $BC = b'$, and $AB = c'$; see Fig. 24.

Indeed, taking into account parallels p, p_1, by 6.3, we obtain

$$b = \lambda a, \quad b' = \lambda a'.$$

Similarly, considering the parallels l, l_1, we obtain

$$b = \mu c, \quad b' = \mu c'.$$

Finally, due to the arithmetic of real numbers, it follows that:

$$\frac{a}{a'} = \frac{\lambda a}{\lambda a'} = \frac{b}{b'} = \frac{\mu c}{\mu c'} = \frac{c}{c'}.$$

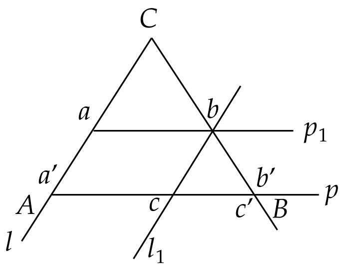

Figure 24: Between Thales' theorem and Euclid's VI.4.

2. At the top of this train of propositions, Borsuk and Szmielew place their interpretation of Euclid's Proposition VI.4 ([13], p. 217): *In equiangular triangles, the sides about equal angles are proportional*; in this context, proportional means the equality of quotients within the arithmetic of real numbers.

Indeed, if the triangles T, with sides a, b, c, and T', with sides a', b', c', are equiangular, then by cutting a copy of side a on side a' and drawing a parallel l to side p, we construct a copy of the triangle T inside triangle T'; see Fig. 25. Then, by the previous result,

$$\frac{a}{a'} = \frac{b}{b'},$$

or, as required

$$\frac{a}{b} = \frac{a'}{b'}$$

The same holds for the pair c, c'.

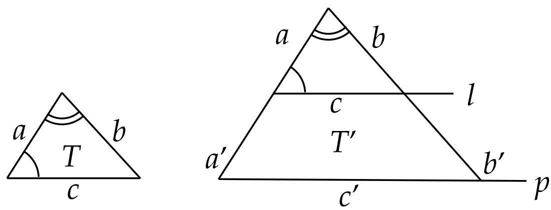

Figure 25: Euclid' Proposition VI.4 by Borsuk and Szmielew.

7. Birkhoff

1. Birkhoff [4] designed a system that enables one to "present the simplest geometric facts". As usual in 20th-century systems, he adopts point and line as undefined terms. Furthermore, a real number $d(A, B)$, called the distance between two points, is an undefined relation between two points. Similarly, a real number $\angle AOB \, (mod \, 2\pi)$, called an angle formed by points A, O, B, is an undefined relation between three points. In fact, Birkhoff combines the concept of congruence with an interpretation in terms of real numbers.

The system is based on four axioms.

1. Postulate of Line Measure. A set of points $\{A, B, ...\}$ on any line can be put into one–to–one correspondence with the real numbers $\{a, b, ...\}$ so that $|b - a| = d(A, B)$ for all points A and B.

This postulate enables the transportation of line segments.

2. Point-Line Postulate. There is one and only one line, l, that contains any two given distinct points P and Q.

Non-intersecting lines are called parallel.

3. Postulate of Angle Measure. A set of rays $\{l, m, n...\}$ through any point O can be put into one-to-one correspondence with the real numbers $a(mod 2\pi)$ so that if A and B are points (not equal to O) of l and m, respectively, the difference $a_m - a_l(mod 2\pi)$ of the numbers associated with the lines l and m is $\angle AOB$.

This postulate enables the transportation of angles. By the following convention, it also introduces the radian measure of angles: "Two half lines l, m through O are said to form a *straight angle* if $\angle lOm = \pi$". ([4], p. 332)

Interestingly, in the *Elements*, this corresponds to Proposition I.13, where, instead of π, it states "two right angles".

4. Postulate of Similarity. Given two triangles ABC and $A'B'C'$ and some constant $k > 0$, $d(A', B') = kd(A, B)$, $d(A', C') = kd(A, C)$ and $\angle B'A'C' = \angle BAC$, then $d(B', C') = kd(B, C)$, $\angle C'B'A' = \angle CBA$, and $\angle A'C'B' = \angle ACB$

This postulate implies Wallis's axiom, which, in standard systems, is proven to be equivalent to Euclid's Parallel Postulate or the Playfair Axiom.

By definition, similar triangles are equiangular, and their corresponding sides are proportional.

Fig. 26 represents Postulate IV schematically. It is Euclid's Proposition VI.6.

2. Birkhoff shows that from his postulates follow Euclid' Proposition VI.4 and VI.5. He also derives other propositions, such as I.5, I.32, I.33–34, I.47, and Playfair's version of the parallel axiom.

Ultimately, Birkhoff seeks to prove Euclid's VI.33 (see Fig.27):

$$\frac{\angle POR}{arc\, PR} = \frac{\angle POQ}{arc\, PQ}.$$

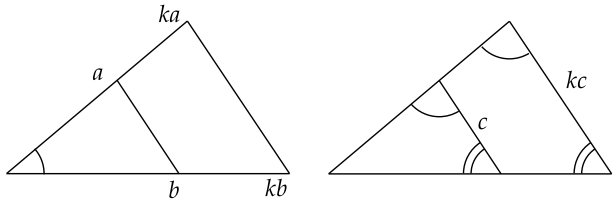

Figure 26: Birkhoff's Postulate IV.

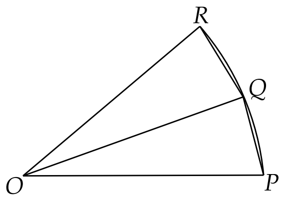

Figure 27: Birkhoff's version of Euclid's VI.33.

To this end, he applies continuity arguments and a simplified proportion. Finally, he writes: "Hence the (sensed) angle $\angle POQ$ coincides with the (sensed) arc lengthen PQ subtended on the unit circle" [4, p. 345].

However, this argument relies on the following supposition: "On the basis of the preceding theorem, Euclidean arc length can be defined in the usual manner and the angle POQ (P, Q on the circle) may be defined as the arc PQ on the unit circle ($r = 1$)" [4, p. 344].

The usual definition is based on the Riemann integral, which assumes the radian measure. Clearly, Postulate III explicitly adopts the radian measure of angles.

3. Birkhoff does not prove Thales' theorem; therefore, we present our version based on his postulates.

Thales' theorem, specifically part VI.2b, follows from Postulate IV. Indeed, if $\frac{a}{c} = \frac{b}{d}$, than $\frac{d}{c} = \frac{b}{a}$. Taking $k = \frac{b}{a}$ and applying Postulate IV, we conclude that the respective trianlges are equiangular, which means the lines l and p are parallel; see Fig. 8 (left).

Now, for part VI.2a, suppose the lines l and p are parallel, but the required proportion does not hold, i.e., $\frac{a}{c} \neq \frac{b}{d}$. By the arithmetic of real numbers, there exists d' such that the equality holds: $\frac{a}{c} = \frac{b}{d'}$. Applying the previous part of the proof to this proportion, we obtain that the lines l and q are parallel. However, this contradicts Playfair's Axiom and the uniqueness of parallel to l passing through the endpoint of line segment b; see Fig. 8 (right).

Note that the argument relies on a simple rule of the arithmetic of real numbers, which states that for three numbers a, b, c, there exists a fourth number d' such that the following equality holds: $\frac{a}{c} = \frac{b}{d'}$. Its ancient Greek counterpart is called the fourth proportional. Indeed, we include it as an axiom for the theory developed in Book V. However, it is employed implicitly, and Euclid's proof of VI.2 does not make use of it [10, pp. 47–51].

8. Millman–Parker

1. Millman-Parker's system [28] is a mixture of Hilbert's synthetic approach and the metric space technique applied to the plane G; in effect, G turns out to be $\mathbb{R} \times \mathbb{R}$. They take *point*, *line* and *plane* as a primitive concepts, and introduce incidence axioms:

(i) For every two points A, B there is a line l with $A \in l$ and $B \in l$.

(ii) Every line has at least two points.

Instead of congruence of line segments, they introduce a distance map.

Definition 8.1 ([28, p. 28]). A distance is a function $d : G \times G \to \mathbb{R}$ such that for all $A, B \in G$:

1. $d(A, B) \geq 0$;
2. $d(A, B) = 0$ if and only if $A = B$;
3. $d(A, B) = d(B, A)$.

Then, the congruence is interpreted through the distance map:

$$AB \equiv CD \Leftrightarrow d(A, B) = d(C, D).$$

The relationship *lying between*, similarly, is defined by the distance map:

$$A - B - C \Leftrightarrow d(AB) + d(BC) = d(AC).$$

Borsuk and Szmielew established a bridge between synthetic and metric geometry: starting with Hilbert-style axioms, they introduced a metric and a coordinate system. Millman and Parker, on the other hand, simply introduce a coordinate system by definition.

Definition 8.2 ([28, p. 30]). A function $f : l \to \mathbb{R}$ is a ruler (or coordinate system) for the line l if:

1. f is a bijection;
2. for each pair of points A and B on l

$$|f(A) - f(B)| = d(A, B).$$

In other words, f is an isometry, given that a metric on the line $(\mathbb{R}, <)$ is introduced by the absolute value.

Let us reiterate, since none of the geometers pay attention to this fact: as the real numbers form a real-closed field, there exists the unique order compatible with addition and multiplication, and consequently, the unique absolute value on the real numbers.

2. In Millman-Parker's system, the proof of Thales' theorem is based on the division of a line segment into equal parts.

Theorem 8.3 ([28, p. 59]). *Any segment AB can be divided into n equal parts.*

The proof employs the bijection between the straight line and the real numbers. Let A and B lie on the line l. Setting A as the origin of the coordinate system, we divide the length of the segment, the real number $d = d(A, B)$, by n, obtaining the real number $\frac{d}{n}$. Let AA_1 be preimage of $\frac{d}{n}$. By successively marking segments of length $\frac{d}{n}$, we obtain points $A_1, A_2, ..., A_n$ such that $d(A_i, A_{i+1}) = \frac{d}{n}$. □

In [28], it is given as an exercise; however, from the perspective of the techniques employed, it is crucial to understand how, in a given system of geometry, one can divide a line segment. Note that in the *Elements*, the counterpart of this theorem, namely VI.9, is based on Thales' theorem. More precisely, in VI.9, Euclid marks off equal segments on one arm of an angle and, using VI.2, proves that parallel lines through the endpoints of these segments determine equal parts on the other arm. This is the key argument in Millman-Parker's proof of Thales' theorem.

The next theorem provides a method for dividing a segment into equal parts without referencing real numbers.

THEOREM 8.4 ([28, p. 231]). *Let l_1, l_2, l_3 be distinct parallel lines. Let t_1 intersect l_1, l_2, l_3 at A, B, C, respectively, and let t_2 intersect l_1, l_2, l_3 at D, E i F, respectively. If $AB \equiv BC$, then $DE \equiv EF$.*

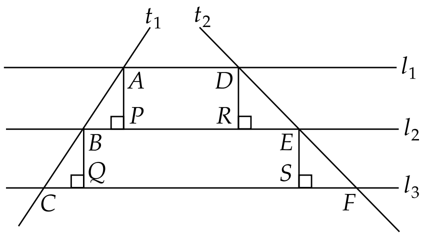

Figure 28: Theorem 8.4, [28], p. 231.

PROOF: Let P and R be feet of perpendiculars from A and D onto l_2, respectively, and Q and S be feet of perpendiculars from B and E onto l_3, respectively; see Fig. 28. Since $AB \equiv BC$ and $\angle A = \angle B$, the right-angled triangles $\triangle APB$ and $\triangle BQC$ are congruent.

The assumption on parallel lines implies

$$AP \equiv DR, \quad BQ \equiv ES.$$

Since $AP \equiv BQ$, it follows that $DR \equiv BQ$.

Since $\angle D = \angle E$, the right-angled triangles $\triangle DRE$ and $\triangle ESF$ are congruent, which implies that $DE \equiv EF$. $\square$

From this point on, we apply the symbol AB for the line segment or its length. This convention simplifies notation, while the context always determines the correct meaning.

Here is Millman-Parker's proof of Thales' theorem.

THEOREM 8.5. *[[28, p. 231]] Let l_1, l_2 i l_3 be parallel lines. Let t_1 and t_2 be two transversals which intersect l_1, l_2, l_3 at A, B, C and D, E, F with $A - B - C$ as shown in Fig. 29. Then*

$$\frac{BC}{AB} = \frac{EF}{DE}.$$

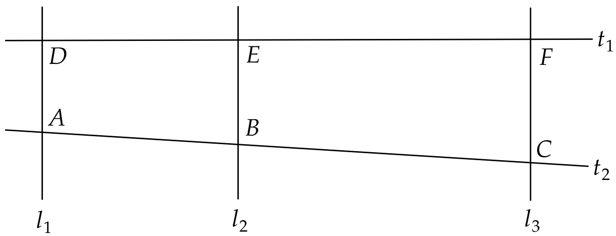

Figure 29: Thales' theorem by Millman and Parker.

PROOF: For the proof, it is shown that numbers $\frac{BC}{AB}$ and $\frac{EF}{DE}$ cannot differ, that is:

$$(\forall n \in \mathbb{N}) \left| \frac{BC}{AB} - \frac{EF}{DE} \right| < \frac{1}{n}.$$

This identity criterion is equivalent to the Archimedean axiom formulated in terms of an ordered $(\mathbb{F}, +, \cdot, 0, 1, <)$:

$$(\forall r \in \mathbb{F}_+)(\exists n \in \mathbb{N}) \left(\tfrac{1}{n} < r\right).$$

or simply

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

(1) For any n, an integer p is defined by:

$$p = \max\left\{k \in \mathbb{N} : k \leq \frac{nBC}{AB}\right\}. \tag{8.1}$$

From this definition, it follows that:

$$\frac{p}{n} \leq \frac{BC}{AB} < \frac{p+1}{n}, \tag{8.2}$$

or, in an equivalent form that carries more geometrical significance:

$$p\frac{AB}{n} \leq BC(p+1)\frac{AB}{n}. \tag{8.3}$$

Now, by Theorem 8.3, the segment AB is divided into n segments each of length $\frac{AB}{n}$, determining the points $A_1, ..., A_{n-1}$.

Then, laying off $p+1$ segments of length $\frac{AB}{n}$ along BC, determines points $B_1, B_2, ..., B_p$ on BC, with B_{p+1} lying beyond C; see Fig. 30.

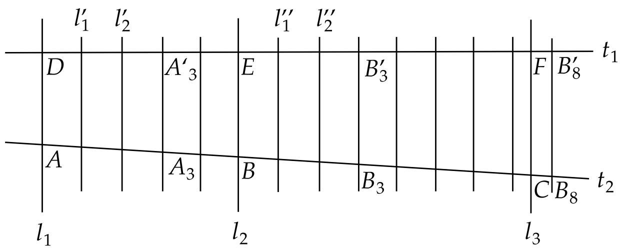

Figure 30: Approximating BC using segments of length $\frac{AB}{n}$ ($n = 5$, $p = 7$).

(2) Let l'_i be the parallel to l_1 passing through A_i; it meets t_1 at a point A'_i.

Similarly, let l''_j be the parallel to l_1 passing through B_j; it meets t_1 at a point B'_j.

By Theorem 8.4, the points A'_i divide the DE into equal segments each of length $\frac{DE}{n}$.

The segments $B'_jB'_{j+1}$ approximating the line segment EF, are all of the of length $\frac{DE}{n}$. Moreover, inequalities analogous to 8.3 hold:

$$p\frac{DE}{n} \leq EF < (p+1)\frac{DE}{n}. \tag{8.4}$$

(3) Due to the arithmetic of real numbers, inequalities 8.4 can be we rewritten in the following form:

$$-\frac{p+1}{n} < -\frac{EF}{DE} \leq -\frac{p}{n}.$$

Adding this to the inequality 8.2 gives

$$-\frac{1}{n} < \frac{BC}{AB} - \frac{EF}{DE} < \frac{1}{n},$$

which implies that

$$\left|\frac{BC}{AB} - \frac{EF}{DE}\right| < \frac{1}{n}.$$

□

Theorem 8.5 relates to Euclid's version VI.2a. Version VI.2b could proceed in the same way as in Hilbert's or Birkhoff's system.

9. From nonstandard approach to the hyperreal plane

9.1. Generalizing previous results

1. The geometrical argument for Thales' theorem, as presented in the approaches of Borsuk and Szmielew, as well as Millman and Parker, relies on the observation that parallel lines cutting equal segments on one arm of an angle also cut equal segments on the other. We can summarize it as follows:

Let the arms p, l of an angle be intersected by the parallel lines t_1, t_2, t_3, t_4, which cut segment a, a' on p and b, b' on l. When $a \equiv a'$, it follows that

$b \equiv b'$; see Fig. 31 (left):

$$t_1 \parallel t_2 \parallel t_3 \parallel t_4,\ a \equiv a' \Rightarrow b \equiv b'.$$

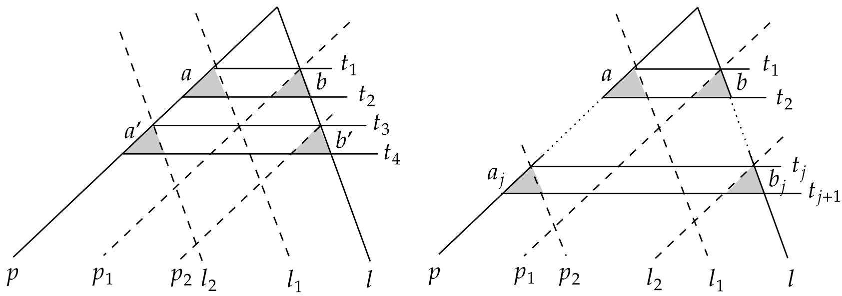

Figure 31: Cutting equal segments on arms of an angle.

The proof is based on the properties of parallelogram stated in Euclid's Propositions I.33–34 [5, p. 79]. Millman and Parker, instead of using parallelograms, apply rectangles; see Fig. 28.

Note, however, that this argument also applies to infinitely many parallel lines $\{t_j\}_{j\in J}$. That is, if parallel lines t_j, cutting arms p, l of an angle in such way that segments a_j on the arm p are equal, then the corresponding segments b_j on the arm l are also equal; see Fig. 31 (right).

2. With this construction, we can prove Euclid's VI.9 without referencing Thales' theorem. Suppose we want to divide the line segment CB into m equal parts; see Fig. 32 (left). To achieve this, lay off m segments of length ε along ray AC, and enumerate them as $(\varepsilon_j)_{j\leq m}$. Join A, the endpoint of ε_m, to B. By drawing parallels to AB through the endpoints of ε_j, we obtain equal parts on segment CB.

3. We can also reverse this argument. In Fig. 32 (left), ε represents equal segments on side AC, while δ represents equal segments on side BC. Let us enumerate these segments as $(\varepsilon_j)_{j\leq m}$ and $(\delta_j)_{j\leq m}$, with $m \in \mathbb{N}$, and let t_1 join the endpoints of ε_1 and δ_1.

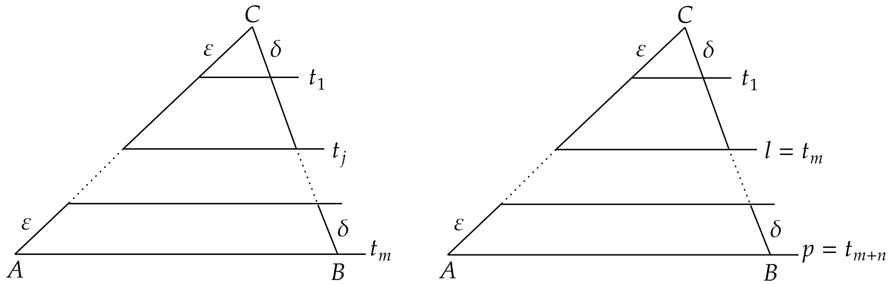

Figure 32: Reversing Euclid's VI.9 (left). Co-measurable case of VI.2 (right).

Lines $(t_j)_{j \leq m}$ parallel to t_1 passing through endpoints of ε_j determine equal segments on side BC. Since the first of these segments equals δ, it follows that all segments on BC are equal to δ. Moreover, since AB runs through the endpoints of ε_m, δ_m, the side AB and the line t_m coincide.

9.2. Thales' theorem on the real plane $\mathbb{R} \times \mathbb{R}$

In the plane $\mathbb{R} \times \mathbb{R}$, a segment AC has length d, and we assume there exist points $A_1, ..., A_n$ such that segments $AA_1, ..., A_iA_{i+1}, ..., A_nC$ are equal, each having length $\frac{d}{n}$.

It what follows, we assume $a, b, c, d, \varepsilon, \delta$ denote both segments and their lengths, initially as real numbers.

1. Let us first consider a commensurable case of Thales' theorem:

Theorem 9.1. *In the triangle $\triangle ACB$, if parallel lines l, p cut line segments a, b and c, d on the sides AC and BC of the triangle, respectively, and there exists a segment ε such that $\frac{a}{b} = \frac{m\varepsilon}{n\varepsilon}$, with $m, n \in \mathbb{N}$, then it follows that $\frac{a}{b} = \frac{c}{d}$:*

$$l \parallel p, \quad \frac{a}{b} = \frac{m\varepsilon}{n\varepsilon} \Rightarrow \frac{a}{b} = \frac{c}{d}, \quad \textit{given} \quad a, b, c, d, \varepsilon \in \mathbb{R}_+.$$

PROOF: Marking off $m+n$ segments ε on side AC, we draw $m+n$ lines through the endpoints of these segments, parallel to l. The first parallel determines a segment δ, and BC is divided into $m+n$ segments, each of length δ.

Since $c = m\delta$ and $d = n\delta$, it follows that

$$\frac{a}{b} = \frac{m\varepsilon}{n\varepsilon} = \frac{m\delta}{n\delta} = \frac{c}{d}.$$

□

Fig. 32 (right) illustrates this proof.

Although the proof is simple, in the next section, we will generalize it by taking hyperintegers K, L, instead of integers m, n, that is, special infinite numbers.

2. The above observations enable us to apply techniques from nonstandard analysis to prove Thales' theorem while remaining within the real plane $\mathbb{R} \times \mathbb{R}$. To this end, we briefly sketch the basics of hyperreal numbers in this section. The crucial concept is that of hyperfinite integers, which enable us to reconstruct arguments 8.1–8.4, as well as the argument developed in § 9.1.

The ordered field of hyperreals, or nonstandard real numbers, $(\mathbb{R}^*, +, \cdot, 0, 1, <)$ is the extension of real numbers, where $\mathbb{R}^* = \mathbb{R}^{\mathbb{N}}/\mathcal{U}$, with $\mathcal{U}$ being a non-principial ultrafilter on $\mathbb{N}$ [5, 18]. Thus, a hyperreal number is represented by an equivalence class determined by a sequence of real numbers,

$$[(r_1, r_2, ...)] \in \mathbb{R}^*.$$

A constant sequence $[(r, r, ..)]$ represents the standard real number r.

The absolute value is defined in the same way as in any ordered field.

We define the class of infinitely small, infinitely large, and limited numbers as follows (where n ranges over $\mathbb{N}$):

$$x \in \Omega \Leftrightarrow (\forall n)(|x| < \tfrac{1}{n}), \quad x \in \Psi \Leftrightarrow (\forall n)(|x| > n), \quad x \in \mathbb{L} \Leftrightarrow (\exists n)(|x| < n).$$

One can easily verify the following relationships:

$$\Omega + \Omega,\ \Omega \cdot \Omega \subset \Omega, \quad \Omega \cdot \mathbb{L} \subset \Omega, \quad \text{and} \ \ x \in \Omega \Leftrightarrow x^{-1} \in \Psi, \ \ x \neq 0.$$

Due to these definitions, any positive real number r is greater than any infinitesimal hyperreal number.

The set Ψ includes the set of hyperintegers $\mathbb{N}^*$, which extends the set of natural numbers. The structure $(\mathbb{N}^*, +, \cdot, 0, 1)$ forms a nonstandard (uncountable) model of Peano arithmetic. The elements of $\mathbb{N}^*$, denoted below as K, L, are represented by equivalence classes of sequences of natural numbers, such as $[(n_1, n_2, ...)]$.

3. Returning to geometry, let the angle $\angle ACB$ be placed in the real plane $\mathbb{R} \times \mathbb{R}$. Let l and p intersect its arms, forming the line segments a, b on AC and c, d on BC, respectively. We assume that $a, b, c, d \in \mathbb{R}$, and we use these symbols to denote both the segments and their lengths. Our goal is to prove the following implication (see Fig. 33):

$$l \parallel p \Rightarrow \frac{a}{b} = \frac{c}{d}.$$

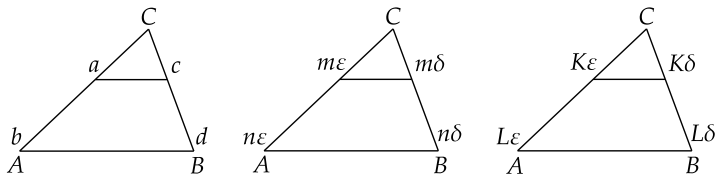

Figure 33: Thales' theorem from co-measurable case (middle) to nonstandard approach (left).

We consider two cases: (a) co-measurable and (b) non-comeasurable.

(Ad a) By co-measurable line segments in nonstandard sense, we mean that there exists an infinitesimal ε such that

$$a = K\varepsilon, \quad b = L\varepsilon,$$

where $K, L \in \mathbb{N}^*$, and $\varepsilon \in \Omega$.

First, on the arm AC, we set $K+L$ segments ε. Let us enumerate them as in the section § 9.1(2) above $\{\varepsilon_j : j \leq K+L\}$.

Let t_1 be parallel to l, passing through the endpoint of the first one, ε_1. It determins the triangle with vertex C and sides of lengths ε and δ; see Fig. 34 (left). By the previous considerations, these parallels cut $K+L$ equal segments on BC, each with length δ.

As in the section § 9.2(1), due to the arithmetic of an ordered field, we obtain

$$\frac{a}{b} = \frac{K\varepsilon}{L\varepsilon} = \frac{K\delta}{L\delta} = \frac{c}{d}. \tag{9.1}$$

(Ad b) Non-commessurable case. Suppose a and b are not co-measurable. Let K be any hyperinteger, and set

$$\varepsilon = \frac{a}{K}.$$

Since a standard real number a is a limited hyperreal, $a \in \mathbb{L}$, the number ε is infinitesimal, $\varepsilon \in \Omega$, and we can express a as $a = K\varepsilon$.

Then, for some hyperinteger L, the following inequalities hold:[9]

$$L\varepsilon \leq b < (L+1)\varepsilon. \tag{9.2}$$

Similarly to 8.1, within the hyperintegers, such an L exists.

Indeed, given $K = [(k_1, k_2, ...)]$ and a is represented by the equivalence class $[(a, a, ...)]$, we have

$$\varepsilon = [(\tfrac{a}{k_1}, \tfrac{a}{k_2}, ...)].$$

For each j, there exists an integer l_j, such that

$$l_j \frac{a}{k_j} \leq b < (l_j + 1)\frac{a}{k_j}, \quad l_j \in \mathbb{N}.$$

Setting $L = [(l_1, l_2, ...)]$, we obtain 9.2. □

Now, suppose the equality $\frac{a}{b} = \frac{c}{d}$ does not hold. Then the equality $\frac{b}{a} = \frac{d}{c}$ also does not hold. Suppose $\frac{b}{a}$ is greater:

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[9] By an additional argument, we could show that $L\varepsilon < b$; however, this is not necessary.

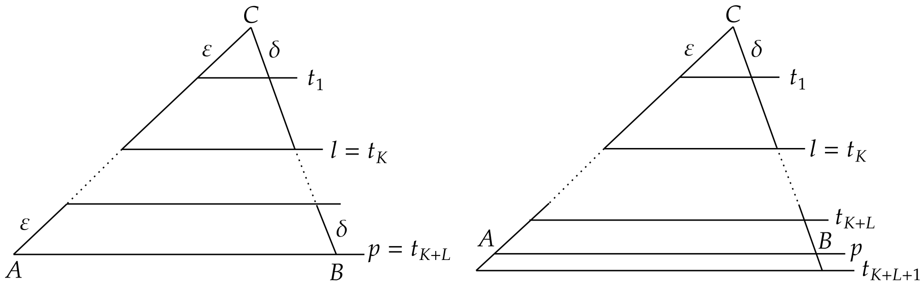

Figure 34: Thales' theorem in nonstandard approach.

$$\frac{b}{a} - \frac{d}{c} = \sigma, \quad \sigma \in \mathbb{R}_+. \tag{9.3}$$

On the ray AC, we set $K + L + 1$ segments ε. Let us enumerate them as $\{\varepsilon_j : j \leq K + L + 1\}$.

Let t_1 be parallel to l, passing through the endpoint of ε_1. It determins the line segment δ on side BC.

Lines t_j parallel to t_1, are drawn through the endpoints of the segments ε_j, determining the line segments δ_j on arm BC, each equal to δ; see Fig. 34.

It follows that:

$$(K + L)\varepsilon < AC < (K + L + 1)\varepsilon,$$

accordingly,

$$(K + L)\delta < BC < (K + L + 1)\delta.$$

Moreover,

$$a = K\varepsilon, \quad L\varepsilon < b < (L + 1)\varepsilon, \quad c = K\delta, \quad L\delta < d < (L + 1)\delta. \tag{9.4}$$

From 9.3 and 9.4, it follows:

$$\sigma = \frac{b}{a} - \frac{d}{c} < \frac{L+1}{K} - \frac{L}{K} = \frac{1}{K}. \tag{9.5}$$

Since $\frac{1}{K}$ is infinitesimal, it contradicts the assumption that σ is a positive real number.

□

9.3. Thales' theorem on the hyperreal plane $\mathbb{R}^* \times \mathbb{R}^*$

1. The arguments developed in the previous subsection translate to the hyperreal plane $\mathbb{R}^* \times \mathbb{R}^*$.

Suppose a, b, c, d are positive hyperreal numbers, $a, b, c, d \in \mathbb{R}^*_+$.

To prove Thales' theorem (VI.2a), we consider two cases: (a) line segments a, b are co-measurable (in the nonstandard sense), (b) line segments a, b are non-comeasurable.

(Ad a) The co-measurable case proceeds analogously to the previous one and concludes with formula 9.1.

(Ad b) Suppose the equality $\frac{b}{a} = \frac{d}{c}$ does not hold and $\frac{b}{a}$ is greater:

$$\frac{b}{a} - \frac{d}{c} = \sigma, \quad \sigma \in \mathbb{R}^*_+. \tag{9.6}$$

Whether σ is infinitesimal or limited, we can find an infinitely large number K such that the inequality holds:

$$\frac{1}{K} < \sigma. \tag{9.7}$$

Indeed, given

$$\sigma = [(r_1, r_2, ..)],$$

for each index j, due to the Archimedean axiom, there exists an integer k_j such that the inequality holds

$$\frac{1}{r_j} < k_j.$$

Setting

$$K = [(k_1, k_2, ...)],$$

we obtain

$$\frac{1}{\sigma} < K,$$

or 9.7. Then, with this K, we reiterate arguments 9.2 to 9.5.

Specifically, given

$$a = [(a_1, a_2, ...)], \quad b = [(b_1, b_2, ...)], \quad \varepsilon = [(\tfrac{a_1}{k_1}, \tfrac{a_2}{k_2}, ...)],$$

we find l_j such that

$$l_j \tfrac{a_j}{k_j} \leq b_j < (l_j + 1) \tfrac{a_j}{k_j},$$

and set

$$L = [(l_1, l_2, ...)].$$

As a result, the contradiction follows:

$$\sigma = \frac{b}{a} - \frac{d}{c} < \frac{1}{K} < \sigma.$$

□

2. Arguments developed above in § 9 rest, in part, on arithmetic, in part, on geometry. Let us summarize them.

Suppose $l \parallel p$ and $\frac{a}{b} \neq \frac{c}{d}$. Let

$$0 < \sigma = \frac{b}{a} - \frac{d}{c}.$$

There exists an integer k such that

$$\frac{b}{a} - \frac{d}{c} < \frac{1}{k} < \sigma.$$

This leads to a contradiction:

$$\sigma < \frac{1}{k} < \sigma.$$

Let us note, that we can formalize this as a sentence:

$$(\forall a, b, c, d \in \mathbb{R})(\exists k \in \mathbb{N})(\tfrac{b}{a} - \tfrac{d}{c} < \tfrac{1}{k} < \tfrac{b}{a} - \tfrac{d}{c}). \tag{T}$$

Statement T is true in Euclidean geometry over real plane $\mathbb{R} \times \mathbb{R}$.

By the transfer principle ([18, ch. 4]) it is also true in Euclidean geometry over hyperreal plane $\mathbb{R} \times \mathbb{R}$:

$$(\forall a, b, c, d \in \mathbb{R}^*)(\exists k \in \mathbb{N}^*)(\tfrac{b}{a} - \tfrac{d}{c} < \tfrac{1}{k} < \tfrac{b}{a} - \tfrac{d}{c}). \qquad \text{(T*)}$$

As for the arguments justifying T, they can also be formalized in a similar manner – for example, using Tarski-style geometry – in a form transferable from geometry over $\mathbb{R} \times \mathbb{R}$ to geometry over $\mathbb{R}^* \times \mathbb{R}^*$.

10. Area method

In reconstructing Euclid's theory of similar triangles, the 20th-century systems, instead of Thales' theorem, adopt or prove some other propositions from Book VI.

Hilbert base his theory of proportion on Proposition VI.4, while its proof explored the reverse of Euclid's III.21. Birkhoff adopts VI.6 as an axiom.

Borsuk and Szmielew, as well as Millman and Parker, based their approach on VI.9 and the arithmetic of real numbers. Borsuk and Szmielew, in particular, explore the density of dyadic numbers in $(\mathbb{R}, <)$, while Millman and Parker apply another version of the Archimedean axiom.

In Section 9, we generalize these arguments and show that they can be developed in the non-Archimedean field of hyperreal numbers.

All these systems interpret Euclidean proportion as a equality of divisions. Moreover, whether in the arithmetic of line segments or real numbers, they explore the concept of the fourth proportional. Although the idea of the fourth proportional arises from Greek mathematics, Euclid did not refer to it in Book VI.

In this final section, we present an approach that adopts VI.1 as an axiom. Then, the proof of Proposition VI.9 aligns with Euclid's original proof.

10.1. Area method

The area method, pioneered in [14], is a technique of proving theorems and constructing solutions in Euclidean geometry. [26] provides its axiomatic description. In [10], we presented a model for these axioms.

From the perspective of formal systems, the language of the area method includes one kind of variables, and symbols of a binary, $\overline{}$, and a ternary function, S. We also need the language of a commutative field characteristic 0, that is, symbols of binary functions, $+, \cdot$ (sum and product), and unary functions $-, ^{-1}$ (an opposite and inverse element), as well as constants $0, 1$, and finitely many constants and r.

Less formally, there are three primitive notions in the area method: point, length of a directed segment, and a signed area of a triangle. An ordered pair of points is called a directed segment, an ordered triple – a triangle. In what follows, capital letters A, B, C, etc., stand for points. The length of a directed segment, $\overline{AB}$, in short, is an element of an ordered field. Similarly, the signed area of a triangle, S_{ABC}, in short, is an element of the ordered field. $\overline{AB}$ and S_{ABC} can be positive, negative, or zero and they are processed in the arithmetic of a commutative field.

To model Euclidean geometry, we need some definitions that we apply in axioms.

Definition 10.1. Points A, B, C are collinear iff $S_{ABC} = 0$.

Definition 10.2.

Two segments AD and BC, where $A \neq D$ and $B \neq C$, are parallel, iff $S_{ABC} = S_{DBC}$. For this relation, we adopt the standard symbol $AD \parallel BC$.

Definition 10.3. For three points A, B and C, the Pythagorean difference, denoted by P_{ABC}, is defined by

$$P_{ABC} = \overline{AB}^2 + \overline{BC}^2 - \overline{AC}^2.$$

Definition 10.4. Two segments DB and CA, where $D \neq B$ and $C \neq A$, are perpendicular iff $P_{DCA} = P_{BCA}$. This relation is denoted by $DB \perp CA$.

Here are the axioms for the area method [26].

A1. $\overline{AB} = 0$ if and only if A and B are identical.

A2. $S_{ABC} = S_{CAB}$.

A3. $S_{ABC} = -S_{BAC}$.

A4. If $S_{ABC} = 0$, then $\overline{AB} + \overline{BC} = \overline{AC}$ (Chasles' axiom).

A5. There are points A, B and C such that $S_{ABC} \neq 0$ (not all points are collinear).

A6. $S_{ABC} = S_{DBC} + S_{ADC} + S_{ABD}$ (all points are in the same plane).[10]

A7. For each element r of F, there exists a point P, such that $S_{ABP} = 0$ and $\overline{AP} = r\overline{AB}$ (construction of a point on a line).

A8. If $A \neq B$, $S_{ABP} = 0, \overline{AP} = r\overline{AB}$, $S_{ABP'} = 0$ and $\overline{AP'} = r\overline{AB}$, then $P = P'$.

A9. If $PQ \parallel CD$ and $\frac{\overline{PQ}}{\overline{CD}} = 1$, then $DQ \parallel PC$ (Euclid's proposition I.33).

A10. If $S_{PAC} \neq 0$ and $S_{ABC} = 0$, then $\frac{\overline{AB}}{\overline{AC}} = \frac{S_{PAB}}{S_{PAC}}$ (Euclid's proposition VI.1).

A11. If $C \neq D$ and $AB \perp CD$ and $EF \perp CD$, then $AB \parallel EF$.

A12. If $A \neq B$, $AB \perp CD$ and $AB \parallel EF$, then $EF \perp CD$.

A13. If $FA \perp BC$ and $S_{FBC} = 0$, then $4 \cdot S_{ABC}^2 = \overline{AF}^2\overline{BC}^2$ (formula for the area of a triangle).

The schemes of Euclid's proof and the area method proof are almost identical. The only difference is that within the area method, one must respect the order of the endpoints of line segments and the vertices of triangles, whereas Euclid arbitrarily permutes the names of a triangle's vertices in his proofs.

Below, we present a proof of Thales' theorem within the area method (see Fig. 5) [10].

(1) From the assumption $DE \parallel BC$, by definition, we obtain the equality of signed areas $S_{DEB} = S_{DEC}$.

(2) By the arithmetic of the filed:

[10]The idea of signed area originates from Hilbert's *Foundations of Geometry*. In [25, ch. 5] he proves the theorem that is a counterpart of axiom A6.

$$\frac{S_{DEB}}{S_{DAE}} = \frac{S_{DEC}}{S_{DAE}}.$$

(3) By A2, we can permute the names of vertices. Then by A10, the following equalities hold:

$$\frac{S_{BDE}}{S_{DAE}} = \frac{\overline{BD}}{\overline{DA}}, \quad \frac{S_{CDE}}{S_{DAE}} = \frac{\overline{CE}}{\overline{EA}}.$$

(4) By transitivity of equality, we obtain,

$$\frac{\overline{BD}}{\overline{DA}} = \frac{\overline{CE}}{\overline{EA}}.$$

□

10.2. GCLC prover

The Area Method enables the mechanization of Euclid's propositions. If understanding arguments in synthetic geometry involves grasping the axioms and rules of inference, then in automated proofs, it involves eliminating points. Elimination lemmas specify this procedure. Given that, an automated proof proceeds as follows:

1. The thesis of a theorem is translated into an expression in the Area Method language.
2. Given some starting points, new points are introduced, one by one, through the allowed constructions (construction stage).
3. Each point introduced in the construction stage is eliminated based on elimination lemmas, but in reverse order, i.e., the last constructed is the first in the elimination process, etc. (elimination stage).
4. The process reaches identity $1 = 1$ or $0 = 0$ and stops.

Since the point elimination method is suitable for algorithmization, it is used to create programs for automatic proving theorems, the so-called provers. An example of such a prover is GCLC [27]. It is a tool for visualizing and creating mathematical drawings, and automatically proving geometric theorems. Prover GCLC generates traditional proofs based on the geometric properties of objects: the coordinates of the entered points are not taken into account in the automatic proof. Point, area, and segment are primitive concepts; they do not have any numerical expressions in the automatic proof; they are symbols. In this sense, it automatically produces synthetic proofs for geometric theorems, which justifies its use in proving theorems of Euclid's geometry.

Thales' theorem, as well as most of the propositions from Book VI of Euclid's Elements, have been successfully proven using GCLC. Below is an example of such a proof, illustrating how the Area Method can be applied effectively within the system to automate these classical geometric arguments.

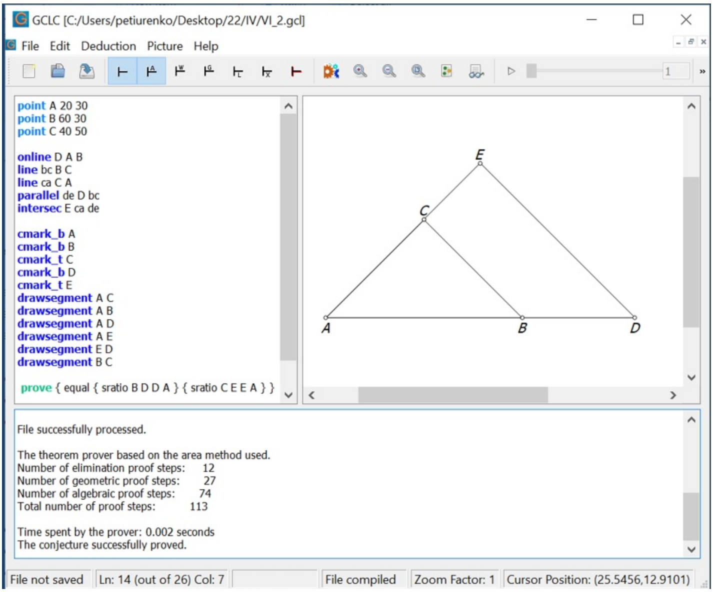

Figure 35: Theorem VI.2, construction at GCLC.

Proof: Case 1. Let DE be drawn parallel to one of the sides BC of triangle ABC. We need to show that as BD is to DA, so CE is to EA.

Automatic proof will be performed only in the case when only allowed constructions are used in the geometric interpretation of the theorem:

- point – defines any point;
- line – defines a straight line that passes through two points;
- intersect – defines the point of intersection of two lines;
- online – defines a point on the line;
- midpoint – defines a point as the middle of the segment;
- parallel – defines a line parallel to the given line passing through the given point;
- towards – division of the line segment in the given ratio;
- translate – translation a line segment;
- foot – defines a line perpendicular to a given line passing through a given point outside the line;
- perp – defines a line perpendicular to the given line passing through a given point on the line.

Construction steps:

```
point A 20 30
point B 60 30
point C 40 50
online D A B
line bc B C
line ca C A
parallel de D bc
intersec E ca de
```

The coordinates of the entered points are not taken into account in the automatic proof. Point, area, and segment are primitive concepts; they do not have any numerical expressions in the automatic proof; they are symbols.

Theorem thesis for case 2 in terms of automatic proof:

```
prove { equal { sratio B D D A } { sratio C E E A } }
```

In ordinary mathematical language, it can be written like this:

$$\frac{\overline{BD}}{\overline{DA}} = \frac{\overline{CE}}{\overline{EA}}.$$

Case 2. Let the sides AB and AC of triangle ABC be cut proportionally such that as BD is to DA, so CE is to EA. Let DE have been joined. Prove that DE is parallel to BC.

This case is a bit more complicated. We have the following construction steps:

```
point A 20 30
point B 60 30
point C 40 50
towards D A B 0.3
towards E A C 0.3
```

Theorem thesis for case 2 in terms of automatic proof:

```
prove { parallel D E B C }
```

In ordinary mathematical language, it means $DE \parallel BC$.

In the towards command we introduce a concrete 0.3 parameter which means that $\frac{AD}{DB} = \frac{AE}{EC} = \frac{1}{3}$. This may mean that the proof we get is not general. Let's analyze the automatic proof GCLC:

$$S_{DBC}=S_{EBC}$$

$$S_{DBC}=S_{BCE}$$

$$S_{DBC}=(S_{BCA} + (0.3 \cdot (S_{BCC} + (-1 \cdot S_{BCA}))))$$

$$S_{BCD}=(S_{BCA} + (0.3 \cdot (0 + (-1 \cdot S_{BCA}))))$$

$$S_{BCD}{=}(0.7 \cdot S_{BCA})$$

$$(S_{BCA} + (0.3 \cdot (S_{BCB} + (-1 \cdot S_{BCA})))){=}(0.7 \cdot S_{BCA})$$

$$(S_{BCA} + (0.3 \cdot (0 + (-1 \cdot S_{BCA})))){=}(0.7 \cdot S_{BCA})$$

$$0 = 0$$

If we change 0.3 to r and 0.7 to $1 - r$, where $r \in \mathbb{R}$, the proof will not change, and hence proof is general. □

Note: If we look at the first line of the proof, we can see that the prover reformulated the parallelity to the equality of two areas of triangles (because this is exactly how the parallelity of two lines is defined in the area method). Hence, the thesis of theorem VI. 1 case 2 can be formulated as

```
proof {equal {signed_area3 DBC} {signed_area3 EBC}}.
```

In ordinary mathematical language, it means $S_{DBC} = S_{EBC}$.

The proof will remain the same if we use the above thesis.

11. Other modern interpretations

Our study is far from being a monograph on interpretations of Thales' theorem. Nevertheless, we can highlight some trends related to this theorem.

In § 6 we present Thales' theorem in the framework of the Borsuk–Szmielew system. Within the Polish mathematical tradition, this system is regarded as an axiomatic foundation of Euclidean geometry and as the basis for the modern treatment of geometry in Euclidean spaces $\mathbb{R}^n$ [32, p. 17], namely, the spaces underlying the development of modern calculus.

The development presented in § 9 is motivated by studies on the mathematical foundations of Newton's *Principia*. As shown in [8], Euclid's geometry combined with infinitesimals plays a fundamental role in Newton's framework. In particular, Newton's subtle use of Thales theorem enabled him to develop a technique that, in modern calculus, corresponds to the notion of the second derivative. We further demonstrate that a synthetic account of the mathematical basis of the *Principia* is possible, as opposed to an analytic reconstruction based on hyperreals.[11]

Hilbert's account, discussed in § 4, shows that he treats Thales theorem by means of the arithmetic of line segments. To this end, he invokes properties of the circle established by Euclid in Book III of the *Elements*. This approach, however, entails a certain inconsistency, since Hilbert's axioms do not include circles. Refinements of Hilbert's perspective, such as [31] and [34], develop the arithmetic of line segments on foundations independent of circle properties.

The system developed in [31] provides the foundations for a program of automated proofs of successive Books of the *Elements*, initiated with [3]. One of many difficulties in this line of research concerns how to treat Propositions I.35–44, which rely on the notion of equal figures. The arithmetic of line segments, whether in Hilbert's or Tarski's formulation, offers a way forward by introducing the concept of the area of a figure defined as the product of line segments. This approach, however, departs from the aim of reconstructing Euclid's geometry in a manner faithful to Euclidean techniques.

In response to this difficulty, Michael Beeson [2] proposes a definition of the equality of figures based on the theory of proportion developed by Paul Bernays (included in [24], pp. 203–206). Bernays refined Hilbert's approach, specifically avoiding reference to the arithmetic of line segments and instead developing an arithmetic of ratios (with no need to refer to a unit segment). To this end, he also applies the cyclic quadrilateral theorem, and, like Hilbert, he managed to prove Euclid's Proposition VI.4 without appealing to the Archimedean axiom.[12]

[11] See [12].

[12] As for the use of this approach in the project of automated proofs, it requires further clarification, since the theory refers to Euclid's Proposition III.20 (the angle at the center is double that at the circumference), which in turn relies on an algebra of angles that requires additional axioms. Beeson, however, does not recognize this dependency; while listing the theorems involved in proving the cyclic quadrilateral theorem, he writes: "Prop. III.21 uses III.20, which uses I.5 and I.32; the point is that the use of equal figures starts with I.34, so III.21 could be reached in two propositions after I.32, without using equal figures" [2, p. 618]. In fact, the actual proof of III.20 requires more than I.5 and

A very recent advance in automated proofs of Euclidean geometry is AlphaGeometry [33]. The system uses AI to prove geometric theorems from *Mathematical Olympiads*. Its methodology mirrors mathematical practice: it adopts a deductive basis of 45 standard propositions, treated as axioms – within the system, these are called rules. Notably, Thales' theorem – listed as the eighth rule [15] – does not require proof.

Finally, we address a wide-ranging project led by Taras Banakh on the foundations of geometry, continuously updated on arXiv [1]. The project approaches geometry through the lens of its basic concepts; accordingly, Banakh distinguishes incidence (linear), order, and metric geometries.[13] In the opening sentence of his manifesto, Banakh writes: "Linear geometry studies geometric properties that can be expressed via the notion of a line. All information about lines is encoded in a ternary relation called a line relation." Within this perspective, Thales' theorem belongs to linear geometry.

Unlike Euclid, Banakh develops a framework in which Thales theorem serves to define proportion. However, because this approach does not include angles, Thales theorem is not linked to Proposition VI.4 (in equiangular triangles, corresponding sides are proportional). That proposition underlies modern trigonometry and therefore also plays a role in modern calculus. Hence, within Banakh's framework, Thales theorem is decoupled from calculus. Moreover, in contrast to Wu's project, which is motivated by the mechanization of geometry, Banakh's program is driven purely by geometric considerations.

12. Conclusions

Let us go back to our schematic reconstruction of Euclid's proof of Thales theorem discussed in § 2.4:

$$l \parallel p \xrightarrow[I.37]{} T_1 = T_2 \xrightarrow[V.7]{} \frac{T_1}{T} = \frac{T_2}{T} \xrightarrow[VI.1]{} \frac{b}{a} = \frac{d}{c},$$

I.32.

[13] For an alternative perspective, see [34].

and

$$\frac{b}{a} = \frac{d}{c} \xrightarrow[VI.1]{} \frac{T_1}{T} = \frac{T_2}{T} \xrightarrow[V.9]{} T_1 = T_2 \xrightarrow[I.39]{} l \parallel p.$$

In a more schematic form, it is a relationship between parallelism and proportion of line segments:

$$l \parallel p \leftrightarrow \frac{b}{a} = \frac{d}{c},$$

or in a Euclidean stylization:

$$l \parallel p \leftrightarrow b : a :: d : c.$$

20th-century systems reinterpreted the concept of proportion $b : a :: d : c$ in terms of the arithmetic of line segments or real numbers. Due to this change, they omit reference to Proposition VI.1 and the mixed proportion involved in that proposition – namely, the proportion between triangles and line segments, as presented in Section § 2.1. They also omit a proportion between triangles, as shown in the above scheme.

Taking VI.1 as an axiom, the Area Method, on one hand, renews the Euclidean technique of proportion, and on the other hand, makes it mechanical, bringing it into 21st-century mathematics. In a way, it realizes the dream of Ian Mueller, a great admirer of Greek mathematics, as he stated: "Since VI.1 is the only important use of Eudoxus' definition in book VI [Definition 5, Book V], it is clear that any theory enabling one to prove VI.1 and standard laws of proportion would suffice as a basis for book VI" [29, p. 156].

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