UNIVERSITY OF LODZ
DEPARTMENT OF LOGIC

BULLETIN OF THE SECTION OF LOGIC

VOLUME 54, NUMBER 2

ŁÓDŹ, JUNE 2025

UNIVERSITY OF LODZ

DEPARTMENT OF LOGIC

BULLETIN

OF THE SECTION OF LOGIC

VOLUME 54, NUMBER 2

ŁÓDŹ, JUNE 2025

C O P E

Member since 2018

JM13707

Special issue editor
Michał Zawidzki

Layout
Michał Zawidzki

Initiating Editor
Katarzyna Smyczek

Printed directly from camera-ready materials provided
to the Lodz University Press

Published by Lodz University Press

First edition. W.11656.25.0.C

Printing sheets 11.25

Lodz University Press
90-237 Łódź, 34A Jana Matejki St.
www.wydawnictwo.uni.lodz.pl
e-mail: ksiegarnia@uni.lodz.pl
+48 42 635 55 77

The **Bulletin of the Section of Logic** (*BSL*) is a quarterly peer-reviewed journal published with the support from the University of Lodz. Its aim is to act as a forum for a wide and timely dissemination of new and significant results in logic through rapid publication of relevant research papers. *BSL* publishes contributions on topics dealing directly with logical calculi, their methodology, and algebraic interpretation.

Papers may be submitted through the *BSL* online editorial platform at https://czasopisma.uni.lodz.pl/bulletin. While preparing the manuscripts for publication please consult the Submission Guidelines.

* * *

Editorial Office: Department of Logic, University of Lodz
ul. Lindleya 3/5, 90-131 Łódź, Poland
e-mail: bulletin@uni.lodz.pl

Homepage:
https://czasopisma.uni.lodz.pl/bulletin

TABLE OF CONTENTS

Bulletin of the **S**ection of **L**ogic
Volume 54/2 (2025), pp. 153–156
https://doi.org/10.18778/0138-0680.2025.11

PREFACE: NON-CLASSICAL LOGICS. THEORY AND APPLICATIONS (PART I)

The articles in the present and forthcoming issues are revised and extended versions of papers presented at the conference *Non-Classical Logics. Theory and Applications*, held in Łódź on 4–8 September 2024.[1]

Non-Classical Logics. Theory and Applications (NCL) is an international conference series devoted to novel results and survey work in broadly understood non-classical logics and their applications. The first two editions took place in Łódź, Poland, in 2008 and 2009. Subsequently, the conference was held alternately in Toruń (2010, 2012, 2015, 2018) and Łódź (2011, 2013, 2016, 2022). The tenth edition, organised by the University of Lodz in 2022, was the first to publish its proceedings in *Electronic Proceedings in Theoretical Computer Science*. This practice was continued in the most recent, eleventh edition, with all accepted long papers again appearing in an EPTCS volume. The 2024 edition was supported by the European Research Council as part of the project *Coming to Terms: Proof Theory for Definite Descriptions and Other Terms* (ExtenDD), and featured four

[1]Due to the high number of accepted post-conference submissions, the editors decided to divide them into two sets, to be published in two separate issues.

invited talks, eighteen contributed talks, and eighteen short presentations accepted through a light reviewing process.

Given that non-classical logics form a broad and diverse area of research within logic, the contributions collected in this issue address a wide range of topics. They include, among others, proof-theoretic properties of connexive implication, term-modalities, until-free fragments of linear temporal logic, and subminimal intuitionistic negation.

The article "Cut-Elimination and Normalization Theorems for Connexive Logics over Wansing's C1" by Norihiro Kamide develops a unified Gentzen-style proof-theoretical framework for a family of connexive logics based on Wansing's constructive connexive logic C. Within this framework, the author introduces sequent calculi and natural deduction systems (including variants with general elimination rules) for C and its extensions—C3, MC, and CN—obtained by adding the law of excluded middle, Peirce's law, or both. The paper proves cut-elimination theorems for the sequent calculi, normalisation theorems for the corresponding natural deduction systems, and establishes their equivalence. In addition, similar results are obtained for a family of paraconsistent logics (N-family) over Nelson's four-valued logic N4. Compared to earlier work, the article extends and refines proofs, corrects errors from a prior conference version, and provides detailed technical developments that yield an integrated proof-theoretical treatment of connexive and related paraconsistent logics. The article by Takahiro Sawasaki, "Semantic Incompleteness of Liberman et al. (2020)'s Hilbert-Style Systems for Term-Modal Logics with Equality and Non-Rigid Terms", establishes that the Hilbert-style systems proposed by Liberman et al. [1] for term-modal logics are semantically incomplete when extended with standard modal axioms (T, D, 4, 5). Term-modal logic, which allows modal operators indexed by first-order terms, is important in epistemic and deontic contexts, but the author shows that certain formulas valid in the intended Kripke semantics are unprovable in these systems. In particular, the validity of the formula $x = c \to (P(x) \to P(c))$ highlights the gap. To demonstrate this, the paper develops a non-standard Kripke semantics in which the interpretation of constants and function symbols depends on the relations they occur with, thereby exposing the systems' limitations.

The paper also corrects a mischaracterised frame correspondence in Liberman et al.'s paper, providing refined technical results on the relationship between syntax, semantics, and completeness in this family of logics.

Norihiro Kamide and Sara Negri's paper "Unified Sequent Calculi and Natural Deduction Systems for Until-free Linear-time Temporal Logics" introduces a unified Gentzen-style proof-theoretic framework for until-free propositional linear-time temporal logic (LTL) and its intuitionistic variant. It develops both single-succedent sequent calculi and natural deduction systems that extend Gentzen's classical (LK, NK) and intuitionistic (LI, NI) calculi in a uniform way. The main results establish the equivalence between the proposed sequent calculi and natural deduction systems, prove cut-elimination theorems for the calculi, and show normalisation theorems for the deduction systems. By doing so, the article provides a modular, consistent, and proof-theoretically robust foundation for reasoning in these temporal logics, clarifying their structural properties and relations to classical proof theory.

Finally, the article "Continua of Logics Related to Intuitionistic and Minimal Logics" by Kaito Ichikura investigates the landscape of logical systems lying between and around intuitionistic and minimal logics. Building on Vakarelov's work on co-minimal and subminimal logics [2], the paper reformulates earlier approaches in a uniform framework and introduces a simpler characterisation of the intersection of minimal and co-minimal logics. Using algebraic semantics (Wroński's method) rather than neighborhood semantics, the author demonstrates the existence of continua of distinct logics situated between these systems, thereby extending classical results on the cardinality of intermediate logics. The study not only clarifies the relations among various subminimal systems (such as SUBMIN, CO-MIN, and their fragments), but also provides simpler proofs of known results and shows how algebraic tools can yield new insights into the fine-grained structure of the logical space below intuitionistic logic.

References

[1] A. O. Liberman, A. Achen, R. K. Rendsvig, *Dynamic Term-modal Logics for First-order Epistemic Planning*, **Artificial Intelligence**, vol. 286 (2020), p. 103305, DOI: https://doi.org/10.1016/j.artint.2020.103305.

[2] D. Vakarelov, *Nelson's negation on the base of weaker versions of intuitionistic negation*, **Studia Logica**, vol. 80 (2005), pp. 393–430, DOI: https://doi.org/10.1007/s11225-005-8476-5.

Michał Zawidzki
University of Lodz
Department of Logic
Lindleya 3/5
91-131 Łódź, Poland
e-mail: michal.zawidzki@filhist.uni.lodz.pl

Bulletin of the **S**ection of **L**ogic
Volume 54/2 (2025), pp. 157–205
https://doi.org/10.18778/0138-0680.2025.04

Norihiro Kamide

CUT-ELIMINATION AND NORMALIZATION THEOREMS FOR CONNEXIVE LOGICS OVER WANSING'S C[1]

Abstract

Gentzen-style sequent calculi and Gentzen-style natural deduction systems are introduced for a family (C-family) of connexive logics over Wansing's basic constructive connexive logic C. The C-family is derived from C by incorporating Peirce's law, the law of excluded middle, and the generalized law of excluded

[1]This paper is an extended and refined version of the preliminary short conference paper [16], which was published in the proceedings of the 11th International Conference on Non-Classical Logics – Theory and Applications (NCL'24). The results in Sections 4 and 5, concerning natural deduction systems with general elimination rules and certain results on the N-family, are new contributions in this paper. Additionally, this paper includes newly added detailed proofs and further remarks in various sections, including a new concluding remarks section, Section 6. Some errors in the normalization proofs presented in [16] have also been corrected in this paper.

Presented by: Michał Zawidzki
Received: December 22, 2024, **Received in revised form**: April 17, 2025,
Accepted: June 11, 2025, **Published online**: July 2, 2025

middle. Natural deduction systems with general elimination rules are also introduced for the C-family. Theorems establishing the equivalence between the proposed sequent calculi and natural deduction systems are demonstrated. Cut-elimination and normalization theorems are established for the proposed sequent calculi and natural deduction systems, respectively. Additionally, similar results are obtained for a family (N-family) of paraconsistent logics over Nelson's constructive four-valued logic N4.

Keywords: connexive logic, cut-elimination theorem, normalization theorem.

2020 Mathematical Subject Classification: 03B50, 03B53.

1. Introduction

Connexive logics are recognized as philosophically plausible paraconsistent contradictory logics [3, 22, 40, 43]. A distinguishing feature of connexive logics is their validation of the so-called Aristotle's theses: $\sim(\alpha\to\sim\alpha)$ and $\sim(\sim\alpha\to\alpha)$, and the so-called Boethius' theses: $(\alpha\to\beta)\to\sim(\alpha\to\sim\beta)$ and $(\alpha\to\sim\beta)\to\sim(\alpha\to\beta)$. On the one hand, the roots of connexive logics can be traced back to Aristotle and Boethius. On the other hand, modern perspectives on connexive logics were established by Angell [3] and McCall [22].

A basic constructive connexive logic referred to as C, considered a variant of Nelson's constructive four-valued logic N4 [2, 24, 19], was introduced by Wansing in [40]. Furthermore, C was extended by Wansing in [40] to introduce a constructive connexive modal logic, serving as a constructive connexive analogue of the smallest normal modal logic K. For further details on connexive logics, including C, refer to, for example, [3, 22, 40, 4, 17, 20, 43, 30, 26] and the references therein.

In this study, a unified Gentzen-style framework for proving cut-elimination and normalization theorems is employed to investigate several connexive logics over Wansing's C. The term "unified Gentzen-style framework" means that we can handle cut-elimination theorems in Gentzen-style sequent calculi and normalization theorems in Gentzen-style natural deduction systems uniformly, with equivalences between these calculi and systems. Additionally, natural deduction systems with general elimination rules are added to this framework.

The logics under consideration include Omori and Wansing's connexive logic C3 [30], Wansing's material connexive logic MC [43], and Cantwell's connexive logic CN [4]. The relationships among these logics can be summarized as follows: C3 is obtained from C by adding the law of excluded middle $\neg\alpha\vee\alpha$, MC is obtained from C by adding Peirce's law $((\alpha\rightarrow\beta)\rightarrow\alpha)\rightarrow\alpha$, and CN is obtained from C3 by adding Peirce's law.

On the one hand, Gentzen-style or G3-style sequent calculi for C, C3, CN and some intermediate logics between C and C3 have been introduced and investigated [40, 30, 6, 25], along with a Gentzen-style natural deduction system for the implicational fragment of C [13]. On the other hand, a unified Gentzen-style framework for C, C3, MC, and CN has not been established. Therefore, we construct such a framework in this study. This framework enables an integrated proof-theoretical treatment of these logics and establishes a natural correspondence between sequent calculi and natural deduction systems for them.

We now discuss some related works on sequent calculi for connexive logics. The cut-elimination theorem for a Gentzen-style sequent calculus, referred to as sC, was proved by Wansing in [40], although the name sC was not used by him. The cut-elimination theorems for G3-style sequent calculi, namely G3C and G3C3at for C and C3, respectively, were established by Omori and Wansing in [30]. In this context, G3C3at is a sequent calculus that incorporates the rule of atomic excluded middle (at-ex-middle) in place of the rule of excluded middle (ex-middle). The admissibility of (ex-middle) in G3C3at was also demonstrated by them. Consequently, the cut-elimination theorem for a G3-style sequent calculus, referred to as G3C3, which is obtained from G3C3at by replacing (at-ex-middle) with (ex-middle), was also demonstrated by them in [26]. Additionally, the first-order extensions of G3C, G3C3at, and G3C3 were also introduced and investigated by them. The systems G3C, G3C3at, and G3C3 were also used by Niki and Wansing in [26] to explore the provable contradictions of C and C3.

Several sequent calculi for some intermediate logics between C and C3 have recently been studied by Niki in [25]. A three-sided sequent calculus for CN, under the name CC/TTm, has recently been introduced and investigated by Égré et al. in [6]. A natural deduction system, NC2, and a two-sorted typed λ-calculus, 2λ, were introduced and investigated by Wansing in [42] for the bi-connexive propositional logic 2C. Natural deduction systems for two variants of connexive logics concerning non-classical interpretations of a certain kind between negation and implication were studied by Francez in [9]. In addition, some extensions of C were studied by Olkhovikov in [27, 28] and by Omori in [29], although these studies are not concerned with sequent calculus or natural deduction system.

The structure of this paper is as follows.

In Section 2, we introduce Gentzen-style sequent calculi for C, C3, MC, and CN, referred to as sC, sC3, sMC, and sCN, respectively. Furthermore, we prove the cut-elimination theorems for these calculi. The calculi sC3, sMC, and sCN are obtained from sC by adding the excluded middle rule (ex-middle), the Peirce rule (Peirce), and both (ex-middle) and (Peirce), respectively. Moreover, we introduce alternative Gentzen-style sequent calculi for MC and CN, referred to as sMC* and sCN*, respectively. These calculi are obtained from sC by adding the generalized excluded middle rule (g-ex-middle) and both (ex-middle) and (g-ex-middle), respectively. We then prove a theorem establishing the cut-free equivalence between sMC* (sCN*) and sMC (sCN, resp.), along with presenting the cut-elimination theorems for sMC* and sCN*.

In Section 3, we introduce Gentzen-style natural deduction systems for C, C3, MC, and CN, referred to as nC, nC3, nMC, and nCN, respectively. We then prove a theorem establishing equivalence between nC, (nC3, nMC, and nCN), and sC, (sC3, sMC*, and sCN*, resp.). Furthermore, we prove the normalization theorems for nC, nC3, nMC, and nCN.

In Section 4, we introduce natural deduction systems with general elimination rules [33, 37], for C, C3, MC, and CN, referred to as gC, gC3, gMC, and gCN, resp. We then prove a theorem establishing equivalence between gC, (gC3, gMC, and gCN), and sC, (sC3, sMC*, and sCN*, resp.). Furthermore, we prove the normalization theorems for gC, gC3, gMC, and gCN.

In Section 5, we show some similar results on a family (N-family) of paraconsistent logics over Nelson's constructive four-valued logic N4 [2, 24]. The N-family is obtained from N4 by adding Peirce's law, the law of excluded middle, and/or the law of generalized excluded middle.

In Section 6, we conclude this study and provide some final remarks.

2. Gentzen-style sequent calculi and cut-elimination theorems

Formulas of connexive logics are constructed using countably many propositional variables, the logical connectives $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (implication), and $\sim$ (connexive negation). We use small letters $p, q, ...$ to denote propositional variables, Greek small letters $\alpha, \beta, ...$ to denote formulas, and Greek capital letters $\Gamma, \Delta, ...$ to denote finite (possibly empty) sets of formulas. A *sequent* is an expression of the form $\Gamma \Rightarrow \gamma$. We use the expression $L \vdash S$ to represent the fact that a sequent S is provable in a sequent calculus L. We say that "a rule R of inference is *admissible* in a sequent calculus L" if the following condition is satisfied: For any instance

$$\frac{S_1 \cdots S_n}{S}$$

of R, if $L \vdash S_i$ for all i, then $L \vdash S$. Furthermore, we say that "R is *derivable* in L" if there is a derivation from $S_1, \cdots, S_n$ to S in L.

We introduce Gentzen-style sequent calculi LJ$^+$ [34], sC [40], sC3, sMC, and sCN for positive intuitionistic logic, C [40], C3 [30], MC [43], and CN [4], respectively.

Definition 2.1 (LJ$^+$, sC, sC3, sMC, and sCN).

1. LJ$^+$ is defined by the initial sequents and structural and logical inference rules of the following form, for any propositional variable p:

$$p, \Gamma \Rightarrow p \ (\text{init1})$$

$$\frac{\Gamma \Rightarrow \alpha \quad \alpha, \Sigma \Rightarrow \gamma}{\Gamma, \Sigma \Rightarrow \gamma} \ (\text{cut})$$

$$\frac{\Gamma \Rightarrow \alpha \quad \beta, \Delta \Rightarrow \gamma}{\alpha{\to}\beta, \Gamma, \Delta \Rightarrow \gamma}\ (\to\text{left}) \qquad \frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha{\to}\beta}\ (\to\text{right})$$

$$\frac{\alpha, \beta, \Gamma \Rightarrow \gamma}{\alpha\wedge\beta, \Gamma \Rightarrow \gamma}\ (\wedge\text{left}) \qquad \frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha\wedge\beta}\ (\wedge\text{right})$$

$$\frac{\alpha, \Gamma \Rightarrow \gamma \quad \beta, \Gamma \Rightarrow \gamma}{\alpha\vee\beta, \Gamma \Rightarrow \gamma}\ (\vee\text{left})$$

$$\frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha\vee\beta}\ (\vee\text{right1}) \qquad \frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha\vee\beta}\ (\vee\text{right2}).$$

2. sC is obtained from LJ^+ by adding the negated initial sequents and logical inference rules of the following form, for any propositional variable p:

$${\sim}p, \Gamma \Rightarrow {\sim}p\ (\text{init2})$$

$$\frac{\alpha, \Gamma \Rightarrow \gamma}{{\sim}{\sim}\alpha, \Gamma \Rightarrow \gamma}\ ({\sim}\text{left}) \qquad \frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow {\sim}{\sim}\alpha}\ ({\sim}\text{right})$$

$$\frac{\Gamma \Rightarrow \alpha \quad {\sim}\beta, \Delta \Rightarrow \gamma}{{\sim}(\alpha{\to}\beta), \Gamma, \Delta \Rightarrow \gamma}\ ({\sim}{\to}\text{left}) \qquad \frac{\alpha, \Gamma \Rightarrow {\sim}\beta}{\Gamma \Rightarrow {\sim}(\alpha{\to}\beta)}\ ({\sim}{\to}\text{right})$$

$$\frac{{\sim}\alpha, \Gamma \Rightarrow \gamma \quad {\sim}\beta, \Gamma \Rightarrow \gamma}{{\sim}(\alpha\wedge\beta), \Gamma \Rightarrow \gamma}\ ({\sim}\wedge\text{left})$$

$$\frac{\Gamma \Rightarrow {\sim}\alpha}{\Gamma \Rightarrow {\sim}(\alpha\wedge\beta)}\ ({\sim}\wedge\text{right1}) \qquad \frac{\Gamma \Rightarrow {\sim}\beta}{\Gamma \Rightarrow {\sim}(\alpha\wedge\beta)}\ ({\sim}\wedge\text{right2})$$

$$\frac{{\sim}\alpha, {\sim}\beta, \Gamma \Rightarrow \gamma}{{\sim}(\alpha\vee\beta), \Gamma \Rightarrow \gamma}\ ({\sim}\vee\text{left}) \qquad \frac{\Gamma \Rightarrow {\sim}\alpha \quad \Gamma \Rightarrow {\sim}\beta}{\Gamma \Rightarrow {\sim}(\alpha\vee\beta)}\ ({\sim}\vee\text{right}).$$

3. sC3 is obtained from sC by adding the excluded middle rule of the form:

$$\frac{{\sim}\alpha, \Gamma \Rightarrow \gamma \quad \alpha, \Gamma \Rightarrow \gamma}{\Gamma \Rightarrow \gamma}\ (\text{ex-middle}).$$

4. sMC is obtained from sC by adding the Peirce rule of the form:

$$\frac{\alpha{\rightarrow}\beta, \Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha}\ \text{(Peirce)}.$$

5. sCN is obtained from sC3 by adding (Peirce).

Remark 2.2.

1. It is known that single-succedent Gentzen-style sequent calculi for classical logic are obtained from Gentzen's sequent calculus LJ (or other variants such as the G3-style sequent calculus G3ip) for intuitionistic logic by adding one of (ex-middle), (Peirce), and their variants. These single-succeddent calculi have been studied by several researchers [5, 7, 10, 1, 36, 23, 12, 15]. For a survey on these calculi, see, for example, [12, 15].

2. (ex-middle), which corresponds to the law of excluded middle $\sim\alpha\vee\alpha$, was introduced and investigated by von Plato [36, 23], although the name (ex-middle) was not used by him. He showed that (ex-middle) can be restricted to the inference rule of the form:

$$\frac{\sim p, \Gamma \Rightarrow \gamma \quad p, \Gamma \Rightarrow \gamma}{\Gamma \Rightarrow \gamma}\ \text{(at-ex-middle)}$$

   where p is a propositional variable. Namely, (at-ex-middle) and (ex-middle) are equivalent over intuitionistic logic. He proved the cut-elimination theorems for some sequent calculi with (at-ex-middle) or (ex-middle).

3. (Peirce), which corresponds to the Peirce law $((\alpha{\rightarrow}\beta){\rightarrow}\alpha){\rightarrow}\alpha$, was introduced and investigated by Curry [5], Felscher [7], Gordeev [10], and Africk [1]. The cut-elimination theorem for LJ + (Peirce) was proved by them. Specifically, Africk [1] obtained a simple embedding-based proof of the cut-elimination theorem for LJ + (Peirce). The subformula property for a version of LJ + (Peirce) without the falsity

constant $\perp$ was shown by Gordeev. Specifically, he proved in [10] that β in (Peirce) can be restricted to a subformula of some formulas in (Γ, α).

4. Gentzen's LK for classical logic, LJ + (ex-middle), and LJ + (Peirce) are theorem-equivalent within the language $\{\wedge, \vee, \rightarrow, \neg, \perp\}$. However, sC3, sMC, and sCN (and their corresponding logics C3, MC, and CN) are not logically-equivalent. This fact will be formally shown in Theorem 2.7.

Proposition 2.3. Let L be LJ$^+$, sC, sC3, sMC, or sCN. For any formula α and any set Γ of formulas, we have: $L \vdash \alpha, \Gamma \Rightarrow \alpha$.

Proof: By induction on α. □

Proposition 2.4. Let L be LJ$^+$, sC, sC3, sMC, or sCN. The following rule is admissible in cut-free L:

$$\frac{\Gamma \Rightarrow \gamma}{\alpha, \Gamma \Rightarrow \gamma} \text{ (we)}.$$

Proof: By induction on the proofs P of $\Gamma \Rightarrow \gamma$ of (we) in cut-free L. □

The following cut-elimination theorems for LJ$^+$ and sC are well-known.

Theorem 2.5 (Cut-elimination for LJ$^+$ and sC [34, 40]). *Let L be* LJ$^+$ *or* sC. *The rule* (cut) *is admissible in cut-free L.*

We now show the cut-elimination theorems for sC3, sMC, and sCN.

Theorem 2.6 (Cut-elimination for sC3, sMC, and sCN). *Let L be* sC3*,* sMC*, or* sCN*. The rule* (cut) *is admissible in cut-free L.*

Proof (Sketch): We give a sketch of the proof.

• First, we show the cut-elimination theorem for sC3. It is known that the cut-elimination theorem for the G3-style sequent calculus G3C3 for C3, which has (ex-middle), holds [30]. Then, we can show the cut-free equivalence between G3C3 and sC3. Thus, from this equivalence and the cut-elimination theorem for G3C3, we obtain the cut-elimination theorem for sC3.

• Second, we show the cut-elimination theorem for sMC. It is known that the cut-elimination theorem for LJ + (Peirce) holds. This theorem was proved directly and indirectly by using the methods by Gordeev [10] and Africk [1]. Thus, the cut-elimination theorem for the negation-less fragment (i.e., LJ^+ + (Peirce)) of LJ + (Peirce) holds because LJ + (Peirce) is a conservative extension of LJ^+ + (Peirce) by the cut-elimination theorem for LJ + (Peirce). Then, we can show a theorem for embedding (cut-free) sMC into (cut-free) LJ^+ + (Peirce), and by using this theorem, we can show the cut-elimination theorem for sMC. We will show this in the following.

Prior to showing the embedding theorem, we introduce a translation of sMC to LJ^+ + (Peirce). Let Φ be a set of propositional variables and Φ' be the set $\{p' \mid p \in \Phi\}$ of propositional variables. Then, the language $\mathcal{L}_{\mathrm{MC}}$ of sMC is defined using Φ, $\wedge$, $\vee$, $\rightarrow$, and $\sim$. The language $\mathcal{L}_{\mathrm{Int+}}$ of LJ^+ is obtained from $\mathcal{L}_{\mathrm{MC}}$ by replacing $\sim$ with Φ'. A mapping f from $\mathcal{L}_{\mathrm{MC}}$ to $\mathcal{L}_{\mathrm{Int+}}$ is defined inductively by:

1. for any $p \in \Phi$, $f(p) := p$ and $f(\sim p) := p' \in \Phi'$,
2. $f(\alpha \sharp \beta) := f(\alpha) \sharp f(\beta)$ with $\sharp \in \{\wedge, \vee, \rightarrow\}$,
3. $f(\sim\sim\alpha) := f(\alpha)$,
4. $f(\sim(\alpha\wedge\beta)) := f(\sim\alpha)\vee f(\sim\beta)$,
5. $f(\sim(\alpha\vee\beta)) := f(\sim\alpha)\wedge f(\sim\beta)$,
6. $f(\sim(\alpha\rightarrow\beta)) := f(\alpha)\rightarrow f(\sim\beta)$.

An expression $f(\Gamma)$ denotes the result of replacing every occurrence of a formula α in Γ by an occurrence of $f(\alpha)$. We remark that a similar translation defined as above has been used by Gurevich [11], Rautenberg [32] and Vorob'ev [38] to embed Nelson's constructive logic [2, 24] into positive intuitionistic logic.

We then obtain the following theorem for embedding sMC into LJ^+ + (Peirce):

1. sMC $\vdash \Gamma \Rightarrow \gamma$ iff LJ^+ + (Peirce) $\vdash f(\Gamma) \Rightarrow f(\gamma)$,
2. sMC $-$ (cut) $\vdash \Gamma \Rightarrow \gamma$ iff LJ^+ + (Peirce) $-$ (cut) $\vdash f(\Gamma) \Rightarrow f(\gamma)$.

The proof of this theorem is almost the same as that for the theorem for embedding sC or a Gentzen-style sequent calculus for Nelson's paraconsistent four-valued logic N4 into LJ^+. For more information on these embedding theorems, see, for example, [18, 19, 17, 20, 14].

We are ready to prove of the cut-elimination theorem for sMC. Suppose that $\mathrm{sMC} \vdash \Gamma \Rightarrow \gamma$. Then, we have $\mathrm{LJ}^+ + (\mathrm{Peirce}) \vdash f(\Gamma) \Rightarrow f(\gamma)$ by the statement (1) of the theorem, and hence $\mathrm{LJ}^+ + (\mathrm{Peirce}) - (\mathrm{cut}) \vdash f(\Gamma) \Rightarrow f(\gamma)$ by the cut-elimination theorem for $\mathrm{LJ}^+ + (\mathrm{Peirce})$. Then, by the statement (2) of the theorem, we obtain $\mathrm{sMC} - (\mathrm{cut}) \vdash \Gamma \Rightarrow \gamma$.

• Finally, the cut-elimination theorem for sCN can be proved in a similar way as for sMC. □

THEOREM 2.7 (Separation of C, C3, MC, and CN). *The logics* C*,* C3*,* MC*, and* CN *are not logically-equivalent.*

PROOF: To show this, we use sC, sC3, sMC, sCN, and Theorem 2.6. Let p and q be distinct propositional variables. Then, we consider only the following facts:

1. $\Rightarrow ((p{\rightarrow}q){\rightarrow}p){\rightarrow}p$ is provable in cut-free sMC, but not provable in cut-free sC3,
2. $\Rightarrow {\sim}p{\vee}p$ is provable in cut-free sC3, but not provable in cut-free sMC.

The unprovabilities of these sequents are guaranteed by Theorem 2.6. We thus show the case for $\mathrm{sMC} - (\mathrm{cut}) \vdash \Rightarrow ((p{\rightarrow}q){\rightarrow}p){\rightarrow}p$ by:

$$\dfrac{\dfrac{\dfrac{\dfrac{p \Rightarrow p\ (\mathrm{init1}) \quad q \Rightarrow q\ (\mathrm{init1})}{p{\rightarrow}q, p \Rightarrow q}\,(\rightarrow\mathrm{left})}{p{\rightarrow}q \Rightarrow p{\rightarrow}q}\,(\rightarrow\mathrm{right}) \quad p \Rightarrow p\ (\mathrm{init1})}{\dfrac{p{\rightarrow}q, (p{\rightarrow}q){\rightarrow}p \Rightarrow p}{(p{\rightarrow}q){\rightarrow}p \Rightarrow p}\,(\mathrm{Peirce})}\,(\rightarrow\mathrm{left})}{\Rightarrow ((p{\rightarrow}q){\rightarrow}p){\rightarrow}p}\,(\rightarrow\mathrm{right})$$

and the case for $\mathrm{sC3} - (\mathrm{cut}) \vdash \Rightarrow {\sim}p{\vee}p$ by:

$$\dfrac{\dfrac{{\sim}p \Rightarrow {\sim}p\ (\mathrm{init2})}{{\sim}p \Rightarrow {\sim}p{\vee}p}\,(\vee\mathrm{right1}) \quad \dfrac{p \Rightarrow p\ (\mathrm{init1})}{p \Rightarrow {\sim}p{\vee}p}\,(\vee\mathrm{right2})}{\Rightarrow {\sim}p{\vee}p}\,(\mathrm{ex\text{-}middle}).$$

□

Next, we introduce alternative Gentzen-style sequent calculi sMC* and sCN* for MC and CN, respectively. These calculi will be used to prove the normalization theorems for the natural deduction systems nMC and nCN for MC and CN, respectively.

DEFINITION 2.8 (sMC* and sCN*).

1. sMC* is obtained from sC by adding the generalized excluded middle rule of the form:

$$\frac{\alpha{\rightarrow}\beta, \Gamma \Rightarrow \gamma \quad \alpha, \Gamma \Rightarrow \gamma}{\Gamma \Rightarrow \gamma}\ \text{(g-ex-middle)}.$$

2. sCN* is obtained from sC3 by adding (g-ex-middle).

Remark 2.9.

1. (g-ex-middle), which corresponds to the generalized law of excluded middle $(\alpha{\rightarrow}\beta)\vee\alpha$, was introduced and investigated by Kamide in [12], although the name (g-ex-middle) was not used by him. He proved the cut-elimination theorem for LJ + (g-ex-middle) using the method by Africk [1].

2. LJ + (g-ex-middle) is regarded as a sequent calculus for classical logic. Actually, (g-ex-middle) and (ex-middle) are equivalent over positive intuitionistic logic. (g-ex-middle) is regarded as a generalization of (ex-middle) if we assume the falsity constant $\perp$ and the definition $\sim\alpha := \alpha{\rightarrow}\perp$. (g-ex-middle) is also regarded as a generalization of (Peirce) and it was referred to as generalized Peirce rule (named (g-Peirce)) in [12].

3. The following is an example proof of $\Rightarrow (p{\rightarrow}q)\vee p$ in cut-free sMC*:

$$\dfrac{\dfrac{\dfrac{\dfrac{p \Rightarrow p\ \text{(init1)} \quad q \Rightarrow q\ \text{(init1)}}{p{\rightarrow}q, p \Rightarrow q}\ (\rightarrow\text{left})}{p{\rightarrow}q \Rightarrow p{\rightarrow}q}\ (\rightarrow\text{right})}{p{\rightarrow}q \Rightarrow (p{\rightarrow}q)\vee p}\ (\vee\text{right1}) \quad \dfrac{p \Rightarrow p\ \text{(init1)}}{p \Rightarrow (p{\rightarrow}q)\vee p}\ (\vee\text{right2})}{\Rightarrow (p{\rightarrow}q)\vee p}\ \text{(g-ex-middle)}.$$

Proposition 2.10. Let L be sMC* or sCN*. For any formula α and any set Γ of formulas, we have: $L \vdash \alpha, \Gamma \Rightarrow \alpha$.

Proof: By induction on α. □

Proposition 2.11. Let L be sMC* or sCN*. The rule (we) is admissible in cut-free L.

Proof: Similar to the proof of Proposition 2.4. □

Theorem 2.12 (Equivalence between sMC (sCN) and sMC* (sCN*)). *Let L_1 and L_2 be the sequent calculi* sMC *and* sCN*, respectively. Let L_1^* and L_2^* be the sequent calculi* sMC* *and* sCN**, respectively. For any $i \in \{1, 2\}$, we have:*

1. $L_i \vdash \Gamma \Rightarrow \gamma$ *iff* $L_i^* \vdash \Gamma \Rightarrow \gamma$,
2. $L_i - (\text{cut}) \vdash \Gamma \Rightarrow \gamma$ *iff* $L_i^* - (\text{cut}) \vdash \Gamma \Rightarrow \gamma$.

Proof: We show only (2). The fact that $L_i - (\text{cut}) \vdash \Gamma \Rightarrow \gamma$ implies $L_i^* - (\text{cut}) \vdash \Gamma \Rightarrow \gamma$ is obvious because (Peirce) is an instance of (g-ex-middle). Thus, we show that $L_i^* - (\text{cut}) \vdash \Gamma \Rightarrow \gamma$ implies $L_i - (\text{cut}) \vdash \Gamma \Rightarrow \gamma$ by induction on the proofs P of $\Gamma \Rightarrow \gamma$ in $L_i^* - (\text{cut})$. We distinguish the cases according to the last inference of P and show only the following case.

Case (g-ex-middle): The last inference of P is of the form:

$$\dfrac{\begin{array}{cc} \vdots & \vdots \\ \alpha{\to}\beta, \Gamma \Rightarrow \gamma & \alpha, \Gamma \Rightarrow \gamma \end{array}}{\Gamma \Rightarrow \gamma}\ \text{(g-ex-middle)}.$$

By induction hypotheses, we have: $L_i - (\text{cut}) \vdash \alpha{\to}\beta, \Gamma \Rightarrow \gamma$ and $L_i - (\text{cut}) \vdash \alpha, \Gamma \Rightarrow \gamma$. Then, we obtain the required fact:

$$\dfrac{\dfrac{\begin{array}{cc} \vdots\ \mathit{Ind.hyp.} & \vdots\ \mathit{Ind.hyp.} \\ \alpha{\to}\beta, \Gamma \Rightarrow \gamma & \alpha, \Gamma \Rightarrow \gamma \end{array}}{\gamma{\to}\alpha, \alpha{\to}\beta, \Gamma \Rightarrow \gamma}\ ({\to}\text{left})}{\alpha{\to}\beta, \Gamma \Rightarrow \gamma}\ \text{(Peirce)}.$$

□

THEOREM 2.13 (Cut-elimination for sMC* and sCN*). *Let L be* sMC* *or* sCN*. *The rule* (cut) *is admissible in cut-free L.*

PROOF: By Theorems 2.6 and 2.12. □

3. Gentzen-style natural deduction systems and normalization theorems

We now define Gentzen-style natural deduction systems NJ$^+$, nC, nC3, nMC, and nCN for positive intuitionistic logic, C, C3, MC, and CN, respectively. We use the notation $[\alpha]$ in the definitions of natural deduction systems to denote the discharged assumption (i.e., the formula α is a discharged assumption by the underlying logical inference rule).

DEFINITION 3.1 (NJ$^+$, nC, nC3, nMC, and nCN).

1. NJ$^+$ is defined as the logical inference rules of the form: [2]

$$\dfrac{\begin{matrix}[\alpha]\\ \vdots\\ \beta\end{matrix}}{\alpha{\to}\beta}\ (\to\mathrm{I}) \qquad \dfrac{\alpha{\to}\beta \quad \alpha}{\beta}\ (\to\mathrm{E})$$

$$\dfrac{\alpha \quad \beta}{\alpha\wedge\beta}\ (\wedge\mathrm{I}) \qquad \dfrac{\alpha\wedge\beta}{\alpha}\ (\wedge\mathrm{E1}) \qquad \dfrac{\alpha\wedge\beta}{\beta}\ (\wedge\mathrm{E2})$$

$$\dfrac{\alpha}{\alpha\vee\beta}\ (\vee\mathrm{I1}) \qquad \dfrac{\beta}{\alpha\vee\beta}\ (\vee\mathrm{I2}) \qquad \dfrac{\alpha\vee\beta \quad \begin{matrix}[\alpha]\\ \vdots\\ \gamma\end{matrix} \quad \begin{matrix}[\beta]\\ \vdots\\ \gamma\end{matrix}}{\gamma}\ (\vee\mathrm{E}).$$

[2]The discharge in $(\to\mathrm{I})$ can be vacuous. Namely, the following rule is an instance of $(\to\mathrm{I})$:

$$\dfrac{\beta}{\alpha{\to}\beta.}$$

2. nC is obtained from NJ^+ by adding the negated logical inference rules of the form:

$$\frac{\alpha}{{\sim}{\sim}\alpha}\ ({\sim}{\sim}\mathrm{I}) \qquad \frac{{\sim}{\sim}\alpha}{\alpha}\ ({\sim}{\sim}\mathrm{E})$$

$$\frac{\begin{matrix}[\alpha]\\ \vdots \\ {\sim}\beta\end{matrix}}{{\sim}(\alpha{\to}\beta)}\ ({\sim}{\to}\mathrm{I}) \qquad \frac{{\sim}(\alpha{\to}\beta) \quad \alpha}{{\sim}\beta}\ ({\sim}{\to}\mathrm{E})$$

$$\frac{{\sim}\alpha}{{\sim}(\alpha\wedge\beta)}\ ({\sim}\wedge\mathrm{I1}) \qquad \frac{{\sim}\beta}{{\sim}(\alpha\wedge\beta)}\ ({\sim}\wedge\mathrm{I2})$$

$$\frac{{\sim}(\alpha\wedge\beta) \quad \begin{matrix}[{\sim}\alpha]\\ \vdots \\ \gamma\end{matrix} \quad \begin{matrix}[{\sim}\beta]\\ \vdots \\ \gamma\end{matrix}}{\gamma}\ ({\sim}\wedge\mathrm{E})$$

$$\frac{{\sim}\alpha \quad {\sim}\beta}{{\sim}(\alpha\vee\beta)}\ ({\sim}\vee\mathrm{I}) \qquad \frac{{\sim}(\alpha\vee\beta)}{{\sim}\alpha}\ ({\sim}\vee\mathrm{E1}) \qquad \frac{{\sim}(\alpha\vee\beta)}{{\sim}\beta}\ ({\sim}\vee\mathrm{E2}).$$

3. nC3 is obtained from nC by adding the rule of excluded middle of the form:

$$\frac{\begin{matrix}[{\sim}\alpha]\\ \vdots \\ \gamma\end{matrix} \quad \begin{matrix}[\alpha]\\ \vdots \\ \gamma\end{matrix}}{\gamma}\ (\mathrm{EM}).$$

4. nMC is obtained from nC by adding the generalized rule of excluded middle of the form:

$$\frac{\begin{matrix}[\alpha{\to}\beta]\\ \vdots \\ \gamma\end{matrix} \quad \begin{matrix}[\alpha]\\ \vdots \\ \gamma\end{matrix}}{\gamma}\ (\mathrm{GEM}).$$

5. nCN is obtained from nC3 by adding (GEM).

Remark 3.2.

1. (EM) and its restricted version (EM-at) with the propositional variable discharged assumptions were originally introduced by von Plato [36], and called there *Gem* (for *generalized excluded middle*) and *Gem-at*, respectively. He proved normalization theorems for systems with (EM) or (EM-at).

2. Using (EM) and (GEM), we can prove the formulas $\sim\alpha\vee\alpha$ and $(\alpha\to\beta)\vee\alpha$ by:

$$\dfrac{\dfrac{[\sim\alpha]^1}{\sim\alpha\vee\alpha}\,(\vee\mathrm{I}1) \quad \dfrac{[\alpha]^1}{\sim\alpha\vee\alpha}\,(\vee\mathrm{I}2)}{\sim\alpha\vee\alpha}\,(\mathrm{EM})^1$$

$$\dfrac{\dfrac{[\alpha\to\beta]^1}{(\alpha\to\beta)\vee\alpha}\,(\vee\mathrm{I}1) \quad \dfrac{[\alpha]^1}{(\alpha\to\beta)\vee\alpha}\,(\vee\mathrm{I}2)}{(\alpha\to\beta)\vee\alpha}\,(\mathrm{GEM})^1.$$

3. Using (GEM), we can prove the formula $((\alpha\to\beta)\to\alpha)\to\alpha$ by:

$$\dfrac{\dfrac{\dfrac{[(\alpha\to\beta)\to\alpha]^1 \quad [\alpha\to\beta]^2}{\alpha}\,(\to\mathrm{E}) \quad [\alpha]^2}{\alpha}\,(\mathrm{GEM})^2}{((\alpha\to\beta)\to\alpha)\to\alpha}\,(\to\mathrm{I}^*)^1.$$

4. The following Peirce rule was introduced by Curry [5] and studied by Zimmermann [44]:

$$\begin{array}{c}[\alpha\to\beta]\\ \vdots\\ \dfrac{\alpha}{\alpha}\,(\mathrm{PE}).\end{array}$$

   Natural deduction systems with (PE) were considered by them for classical logic and the normalization theorems for these systems were proved by them.

5. (PE) is regarded as an instance of (GEM), and using (PE), we can prove the formula $((\alpha{\to}\beta){\to}\alpha){\to}\alpha$ by:

$$\dfrac{\dfrac{\dfrac{[(\alpha{\to}\beta){\to}\alpha]^1 \quad [\alpha{\to}\beta]^2}{\alpha}\ (\to\mathrm{E})}{\alpha}\ (\mathrm{PE})^2}{((\alpha{\to}\beta){\to}\alpha){\to}\alpha}\ (\to\mathrm{I})^1.$$

6. In this study, we do not consider the natural deduction systems nC + (PE) and nC + (EM) + (PE) because reduction conditions for nC + (PE) and nC + (EM) + (PE) cannot be defined in a similar and uniform way as for nMC and nCN. Moreover, we do not know if nC + (PE) (nC + (EM) + (PE)) and nMC (nCN, respectively) are logically-equivalent or not. Namely, we have not yet proved the equivalence (or difference) of nC + (PE) (nC + (EM) + (PE)) and nMC (nCN, respectively).

7. The $\{\to, \sim\}$-fragment of nC was introduced and investigated by Kamide in [13], wherein the strong normalization theorem for the fragment was proved.

Next, we define some notions for the natural deduction systems.

Definition 3.3. The inference rules $(\to\mathrm{I})$, $(\wedge\mathrm{I})$, $(\vee\mathrm{I1})$, $(\vee\mathrm{I2})$, $(\sim\sim\mathrm{I})$, $(\sim\to\mathrm{I})$, $(\sim\wedge\mathrm{I1})$, $(\sim\wedge\mathrm{I2})$, $(\sim\vee\mathrm{I})$, (EM), and (GEM) are called *introduction rules*, and the inference rules $(\to\mathrm{E})$, $(\wedge\mathrm{E1})$, $(\wedge\mathrm{E2})$, $(\vee\mathrm{E})$, $(\sim\sim\mathrm{E})$, $(\sim\to\mathrm{E})$, $(\sim\wedge\mathrm{E})$, $(\sim\vee\mathrm{E1})$, and $(\sim\vee\mathrm{E2})$ are called *elimination rules*. The notions of *major* and minor premises of the inference rules without (EM) and (GEM) are defined as usual. The notions of *derivation*, *(open and discharged) assumptions* of derivation, and *end-formula* of derivation are also defined as usual. Any derivation starts with an assumption α can be considered a derivation of α from itself. For a derivation $\mathcal{D}$, we use the expression $\mathrm{oa}(\mathcal{D})$ to denote the set of open assumptions of $\mathcal{D}$ and the expression $\mathrm{end}(\mathcal{D})$ to denote the end-formula of $\mathcal{D}$. A formula α is said to be *provable* in a natural deduction system N if there exists a derivation in N with no open assumptions whose end-formula is α.

Remark 3.4. There are no notions of major and minor premises of (EM) and (GEM). Namely, both the premises of (EM) and (GEM) are neither major nor minor premise. In this study, (EM) and (GEM) are treated as introduction rules.

Next, we define a reduction relation $\rhd$ on the set of derivations in the natural deduction systems. Prior to defining $\rhd$, we define some notions concerning $\rhd$.

Definition 3.5. Let L be nC, nC3, nMC, or nCN. Let α be a formula occurring in a derivation $\mathcal{D}$ in L. Then, α is called a *maximum formula* in $\mathcal{D}$ if α satisfies the following conditions:

1. α is the conclusion of an introduction rule, ($\vee$E), or ($\sim\wedge$E),
2. α is the major premise of an elimination rule.

A derivation is said to be *normal* if it contains no maximum formula. The notion of substitution of derivations to assumptions is defined as usual. We assume that the set of derivations is closed under substitution.

Definition 3.6 (Reduction relation). Let γ be a maximum formula in a derivation that is the conclusion of an inference rule R.

1. The definition of the reduction relation $\rhd$ at γ in nC is obtained by the following conditions.
   (a) R is ($\to$I) and γ is $\alpha\to\beta$:

$$\dfrac{\dfrac{\begin{matrix}[\alpha]\\ \vdots\ \mathcal{D}\\ \beta\end{matrix}}{\alpha\to\beta}\,(\to\mathrm{I}) \qquad \begin{matrix}\vdots\ \mathcal{E}\\ \alpha\end{matrix}}{\beta}\,(\to\mathrm{E}) \qquad \rhd \qquad \begin{matrix}\vdots\ \mathcal{E}\\ \alpha\\ \vdots\ \mathcal{D}\\ \beta.\end{matrix}$$

   (b) R is ($\wedge$I) and γ is $\alpha_1\wedge\alpha_2$:

$$\dfrac{\dfrac{\begin{matrix}\vdots\ \mathcal{D}_1\\ \alpha_1\end{matrix} \qquad \begin{matrix}\vdots\ \mathcal{D}_2\\ \alpha_2\end{matrix}}{\alpha_1\wedge\alpha_2}\,(\wedge\mathrm{I})}{\alpha_i}\,(\wedge\mathrm{E}i) \qquad \rhd \qquad \begin{matrix}\vdots\ \mathcal{D}_i\\ \alpha_i\end{matrix}$$

   where i is 1 or 2.

(c) R is ($\vee$I1) or ($\vee$I2) and γ is $\alpha_1\vee\alpha_2$:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\mathcal{D}\\ \alpha_i\end{array}}{\alpha_1\vee\alpha_2}\,(\vee\mathrm{I}i) \quad \begin{array}{c}[\alpha_1]\\ \vdots\,\mathcal{E}_1\\ \delta\end{array} \quad \begin{array}{c}[\alpha_2]\\ \vdots\,\mathcal{E}_2\\ \delta\end{array}}{\delta}\,(\vee\mathrm{E}) \qquad \rhd \qquad \begin{array}{c}\vdots\,\mathcal{D}\\ \alpha_i\\ \vdots\,\mathcal{E}_i\\ \delta\end{array}$$

where i is 1 or 2.

(d) R is ($\vee$E):

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\mathcal{D}_1\\ \alpha\vee\beta\end{array} \quad \begin{array}{c}[\alpha]\\ \vdots\,\mathcal{D}_2\\ \gamma\end{array} \quad \begin{array}{c}[\beta]\\ \vdots\,\mathcal{D}_3\\ \gamma\end{array}}{\gamma}\,(\vee\mathrm{E}) \quad \begin{array}{c}\vdots\,\mathcal{E}_1\\ \delta_1\end{array} \quad \begin{array}{c}\vdots\,\mathcal{E}_2\\ \delta_2\end{array}}{\delta}\,R'$$

$$\rhd \qquad \dfrac{\begin{array}{c}\vdots\,\mathcal{D}_1\\ \alpha\vee\beta\end{array} \quad \dfrac{\begin{array}{c}[\alpha]\\ \vdots\,\mathcal{D}_2\\ \gamma\end{array} \quad \begin{array}{c}\vdots\,\mathcal{E}_1\\ \delta_1\end{array} \quad \begin{array}{c}\vdots\,\mathcal{E}_2\\ \delta_2\end{array}}{\delta}\,R' \quad \dfrac{\begin{array}{c}[\beta]\\ \vdots\,\mathcal{D}_3\\ \gamma\end{array} \quad \begin{array}{c}\vdots\,\mathcal{E}_1\\ \delta_1\end{array} \quad \begin{array}{c}\vdots\,\mathcal{E}_2\\ \delta_2\end{array}}{\delta}\,R'}{\delta}\,(\vee\mathrm{E})$$

where R' is an arbitrary inference rule, and both $\mathcal{E}_1$ and $\mathcal{E}_2$ are derivations of the minor premises of R' if they exist.

(e) R is (${\sim}{\sim}$I), and γ is ${\sim}{\sim}\alpha$:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\mathcal{D}\\ \alpha\end{array}}{{\sim}{\sim}\alpha}\,({\sim}{\sim}\mathrm{I})}{\alpha}\,({\sim}{\sim}\mathrm{E}) \qquad \rhd \qquad \begin{array}{c}\vdots\,\mathcal{D}\\ \alpha.\end{array}$$

(f) R is (${\sim}{\to}$I) and γ is ${\sim}(\alpha\to\beta)$:

$$\dfrac{\dfrac{\begin{array}{c}[\alpha]\\ \vdots\,\mathcal{D}\\ {\sim}\beta\end{array}}{{\sim}(\alpha\to\beta)}\,({\sim}{\to}\mathrm{I}) \quad \begin{array}{c}\vdots\,\mathcal{E}\\ \alpha\end{array}}{{\sim}\beta}\,({\sim}{\to}\mathrm{E}) \qquad \rhd \qquad \begin{array}{c}\vdots\,\mathcal{E}\\ \alpha\\ \vdots\,\mathcal{D}\\ {\sim}\beta.\end{array}$$

(g) R is $({\sim}\wedge\mathrm{I1})$ or $({\sim}\wedge\mathrm{I2})$ and γ is ${\sim}(\alpha_1\wedge\alpha_2)$:

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\mathcal{D}\\ {\sim}\alpha_i\end{matrix}}{{\sim}(\alpha_1\wedge\alpha_2)}\,({\sim}\wedge\mathrm{I1}) \qquad \begin{matrix}[{\sim}\alpha_1]\\ \vdots\,\mathcal{E}_1\\ \delta\end{matrix} \qquad \begin{matrix}[{\sim}\alpha_2]\\ \vdots\,\mathcal{E}_2\\ \delta\end{matrix}}{\delta}\,({\sim}\wedge\mathrm{E}) \qquad \rhd \qquad \begin{matrix}\vdots\,\mathcal{D}\\ {\sim}\alpha_i\\ \vdots\,\mathcal{E}_i\\ \delta\end{matrix}$$

where i is 1 or 2.

(h) R is $({\sim}\wedge\mathrm{E})$:

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\mathcal{D}_1\\ {\sim}(\alpha\wedge\beta)\end{matrix} \qquad \begin{matrix}[{\sim}\alpha]\\ \vdots\,\mathcal{D}_2\\ \gamma\end{matrix} \qquad \begin{matrix}[{\sim}\beta]\\ \vdots\,\mathcal{D}_3\\ \gamma\end{matrix}}{\gamma}\,({\sim}\wedge\mathrm{E}) \qquad \begin{matrix}\vdots\,\mathcal{E}_1\\ \delta_1\end{matrix} \qquad \begin{matrix}\vdots\,\mathcal{E}_2\\ \delta_2\end{matrix}}{\delta}\,R'$$

$$\rhd \quad \dfrac{\begin{matrix}\vdots\,\mathcal{D}_1\\ {\sim}(\alpha\wedge\beta)\end{matrix} \qquad \dfrac{\begin{matrix}[{\sim}\alpha]\\ \vdots\,\mathcal{D}_2\\ \gamma\end{matrix} \quad \begin{matrix}\vdots\,\mathcal{E}_1\\ \delta_1\end{matrix} \quad \begin{matrix}\vdots\,\mathcal{E}_2\\ \delta_2\end{matrix}}{\delta}\,R' \qquad \dfrac{\begin{matrix}[{\sim}\beta]\\ \vdots\,\mathcal{D}_3\\ \gamma\end{matrix} \quad \begin{matrix}\vdots\,\mathcal{E}_1\\ \delta_1\end{matrix} \quad \begin{matrix}\vdots\,\mathcal{E}_2\\ \delta_2\end{matrix}}{\delta}\,R'}{\delta}\,({\sim}\wedge\mathrm{E})$$

where R' is an arbitrary inference rule, and both $\mathcal{E}_1$ and $\mathcal{E}_2$ are derivations of the minor premises of R' if they exist.

(i) R is $({\sim}\vee\mathrm{I})$ and γ is ${\sim}(\alpha_1\vee\alpha_2)$:

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\mathcal{D}_1\\ {\sim}\alpha_1\end{matrix} \qquad \begin{matrix}\vdots\,\mathcal{D}_2\\ {\sim}\alpha_2\end{matrix}}{{\sim}(\alpha_1\vee\alpha_2)}\,({\sim}\vee\mathrm{I})}{{\sim}\alpha_i}\,({\sim}\vee\mathrm{E}i) \qquad \rhd \qquad \begin{matrix}\vdots\,\mathcal{D}_i\\ {\sim}\alpha_i\end{matrix}$$

where i is 1 or 2.

(j) The set of derivations is closed under $\rhd$.

2. The definition of the reduction relation $\rhd$ at γ in nC3 is obtained from the conditions for the reduction relation $\rhd$ at γ in nC by adding the following condition.

(a) R is (EM) and γ is $\gamma_1{\to}\gamma_2$, $\gamma_1{\wedge}\gamma_2$, $\gamma_1{\vee}\gamma_2$, ${\sim}{\sim}\gamma'$, ${\sim}(\gamma_1{\to}\gamma_2)$, ${\sim}(\gamma_1{\wedge}\gamma_2)$, or ${\sim}(\gamma_1{\vee}\gamma_2)$:

$$\dfrac{\dfrac{\begin{array}{c}[{\sim}\alpha]\\ \vdots\ \mathcal{D}_1\\ \gamma\end{array}\quad \begin{array}{c}[\alpha]\\ \vdots\ \mathcal{D}_2\\ \gamma\end{array}}{\gamma}\,(\mathrm{EM})\quad \begin{array}{c}\vdots\ \mathcal{E}_1\\ \delta_1\end{array}\quad \begin{array}{c}\vdots\ \mathcal{E}_2\\ \delta_2\end{array}}{\delta}\,R'$$

$$\rhd\quad \dfrac{\dfrac{\begin{array}{c}[{\sim}\alpha]\\ \vdots\ \mathcal{D}_1\\ \gamma\end{array}\quad \begin{array}{c}\vdots\ \mathcal{E}_1\\ \delta_1\end{array}\quad \begin{array}{c}\vdots\ \mathcal{E}_2\\ \delta_2\end{array}}{\delta}\,R' \qquad \dfrac{\begin{array}{c}[\alpha]\\ \vdots\ \mathcal{D}_2\\ \gamma\end{array}\quad \begin{array}{c}\vdots\ \mathcal{E}_1\\ \delta_1\end{array}\quad \begin{array}{c}\vdots\ \mathcal{E}_2\\ \delta_2\end{array}}{\delta}\,R'}{\delta}\,(\mathrm{EM})$$

where R' is (→E), (∧E1), (∧E2), (∨E), (∼∼E) (∼→E), (∼∧E), (∼∨E1), or (∼∨E2), and both $\mathcal{E}_1$ and $\mathcal{E}_2$ are derivations of the minor premises of R' if they exist.

3. The definition of the reduction relation $\rhd$ at γ in nMC is obtained from the conditions for the reduction relation $\rhd$ at γ in nC by adding the following condition.

(a) R is (GEM) and γ is $\gamma_1{\to}\gamma_2$, $\gamma_1{\wedge}\gamma_2$, $\gamma_1{\vee}\gamma_2$, ${\sim}{\sim}\gamma'$, ${\sim}(\gamma_1{\to}\gamma_2)$, ${\sim}(\gamma_1{\wedge}\gamma_2)$, or ${\sim}(\gamma_1{\vee}\gamma_2)$:

$$\dfrac{\dfrac{\begin{array}{c}[\alpha{\to}\beta]\\ \vdots\ \mathcal{D}_1\\ \gamma\end{array}\quad \begin{array}{c}[\alpha]\\ \vdots\ \mathcal{D}_2\\ \gamma\end{array}}{\gamma}\,(\mathrm{GEM})\quad \begin{array}{c}\vdots\ \mathcal{E}_1\\ \delta_1\end{array}\quad \begin{array}{c}\vdots\ \mathcal{E}_2\\ \delta_2\end{array}}{\delta}\,R'$$

$$\rhd\quad \dfrac{\dfrac{\begin{array}{c}[\alpha{\to}\beta]\\ \vdots\ \mathcal{D}_1\\ \gamma\end{array}\quad \begin{array}{c}\vdots\ \mathcal{E}_1\\ \delta_1\end{array}\quad \begin{array}{c}\vdots\ \mathcal{E}_2\\ \delta_2\end{array}}{\delta}\,R' \qquad \dfrac{\begin{array}{c}[\alpha]\\ \vdots\ \mathcal{D}_2\\ \gamma\end{array}\quad \begin{array}{c}\vdots\ \mathcal{E}_1\\ \delta_1\end{array}\quad \begin{array}{c}\vdots\ \mathcal{E}_2\\ \delta_2\end{array}}{\delta}\,R'}{\delta}\,(\mathrm{GEM})$$

where R' is ($\to$E), ($\wedge$E1), ($\wedge$E2), ($\vee$E), ($\sim\sim$E) ($\sim\to$E), ($\sim\wedge$E), ($\sim\vee$E1), or ($\sim\vee$E2), and both $\mathcal{E}_1$ and $\mathcal{E}_2$ are derivations of the minor premises of R' if they exist.

4. The definition of the reduction relation $\rhd$ at γ in nCN is obtained from the conditions for the reduction relation $\rhd$ at γ in nC3 by adding the other conditions of nMC. Namely, it is defined as all the conditions for both nC3 and nMC.

Prior to proving the normalization theorems for nC, nC3, nMC, and nCN, we need the following lemma.

LEMMA 3.7. *Let N_1, N_2, N_3, and N_4 be* nC*,* nC3*,* nMC*, and* nCN*, respectively. Let S_1, S_2, S_3, and S_4 be* sC*,* sC3*,* sMC**, and* sCN**, respectively. For any $i \in \{1, 2, 3, 4\}$, the following hold.*

1. *If $\mathcal{D}$ is a derivation in N_i such that* $\mathrm{oa}(\mathcal{D}) = \Gamma$ *and* $\mathrm{end}(\mathcal{D}) = \beta$*, then $S_i \vdash \Gamma \Rightarrow \beta$,*

2. *If $S_i -$ (cut) $\vdash \Gamma \Rightarrow \beta$, then we can obtain a derivation $\mathcal{D}'$ in N_i such that*

   (a) $\mathrm{oa}(\mathcal{D}') \subseteq \Gamma$,

   (b) $\mathrm{end}(\mathcal{D}') = \beta$,

   (c) $\mathcal{D}'$ is normal.

PROOF:

1. We prove 1 by induction on the derivations $\mathcal{D}$ of N_i such that $\mathrm{oa}(\mathcal{D}) = \Gamma$ and $\mathrm{end}(\mathcal{D}) = \beta$. We distinguish the cases according to the last inference of $\mathcal{D}$. We show some cases.

   (a) Case (EM): $\mathcal{D}$ is of the form:

$$\dfrac{\begin{matrix}[\sim\alpha]\Gamma_1 \\ \vdots\ \mathcal{D}_1 \\ \gamma\end{matrix} \qquad \begin{matrix}[\alpha]\Gamma_2 \\ \vdots\ \mathcal{D}_2 \\ \gamma\end{matrix}}{\gamma}\ (\mathrm{EM})$$

where $\mathrm{oa}(\mathcal{D}) = \Gamma_1 \cup \Gamma_2$ and $\mathrm{end}(\mathcal{D}) = \gamma$. By induction hypothesis, we have $S_i \vdash {\sim}\alpha, \Gamma_1 \Rightarrow \gamma$ and $S_i \vdash \alpha, \Gamma_2 \Rightarrow \gamma$. Then, we obtain the required fact $S_i \vdash \Gamma_1, \Gamma_2 \Rightarrow \gamma$:

$$\dfrac{\begin{array}{c} \vdots\ \textit{Ind. hyp.} \\ {\sim}\alpha, \Gamma_1 \Rightarrow \gamma \\ \vdots\ (\mathrm{we}) \\ {\sim}\alpha, \Gamma_1, \Gamma_2 \Rightarrow \gamma \end{array} \qquad \begin{array}{c} \vdots\ \textit{Ind. hyp.} \\ \alpha, \Gamma_2 \Rightarrow \gamma \\ \vdots\ (\mathrm{we}) \\ \alpha, \Gamma_1, \Gamma_2 \Rightarrow \gamma \end{array}}{\Gamma_1, \Gamma_2 \Rightarrow \gamma}\ (\text{ex-middle})$$

where (we) is admissible in $S_i -$ (cut) by Propositions 2.4 and 2.11.

(b) Case (${\sim}{\sim}$E): $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c} \Gamma \\ \vdots\ \mathcal{D}_1 \\ {\sim}{\sim}\alpha \end{array}}{\alpha}\ ({\sim}{\sim}\mathrm{E})$$

where $\mathrm{oa}(\mathcal{D}) = \Gamma$ and $\mathrm{end}(\mathcal{D}) = \alpha$. By induction hypothesis, we have $S_i \vdash \Gamma \Rightarrow {\sim}{\sim}\alpha$. Then, we obtain the required fact $S_i \vdash \Gamma \Rightarrow \alpha$:

$$\dfrac{\begin{array}{c} \vdots\ \textit{Ind. hyp.} \\ \Gamma \Rightarrow {\sim}{\sim}\alpha \end{array} \qquad \dfrac{\begin{array}{c} \vdots\ \textit{Prop.}2.3 \\ \alpha \Rightarrow \alpha \end{array}}{{\sim}{\sim}\alpha \Rightarrow \alpha}\ ({\sim}{\sim}\text{left})}{\Gamma \Rightarrow \alpha}\ (\text{cut}).$$

(c) Case (${\sim}{\to}$I): We divide this case into two subcases.

i. Subcase 1: $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c} \Gamma \\ \vdots\ \mathcal{D}' \\ {\sim}\beta \end{array}}{{\sim}(\alpha{\to}\beta)}\ ({\sim}{\to}\mathrm{I})$$

where $\mathrm{oa}(\mathcal{D}) = \Gamma$ and $\mathrm{end}(\mathcal{D}) = {\sim}(\alpha{\to}\beta)$. By induction hypothesis, we have $S_i \vdash \Gamma \Rightarrow {\sim}\beta$. Then, we obtain that $S_i \vdash \Gamma \Rightarrow {\sim}(\alpha{\to}\beta)$:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \textit{Ind. hyp.}\\ \Gamma \Rightarrow {\sim}\beta\end{array}}{\alpha, \Gamma \Rightarrow {\sim}\beta}\ (\text{we})}{\Gamma \Rightarrow {\sim}(\alpha{\rightarrow}\beta)}\ ({\sim}{\rightarrow}\text{right})$$

where (we) is admissible in S_i − (cut) by Propositions 2.4 and 2.11.

ii. Subcase 2: $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c}[\alpha]\ \Gamma\\ \vdots\ \mathcal{D}'\\ {\sim}\beta\end{array}}{{\sim}(\alpha{\rightarrow}\beta)}\ ({\sim}{\rightarrow}\text{I})$$

where $\text{oa}(\mathcal{D}) = \Gamma$ and $\text{end}(\mathcal{D}) = {\sim}(\alpha{\rightarrow}\beta)$. By induction hypothesis, we have $S_i \vdash \alpha, \Gamma \Rightarrow {\sim}\beta$. Then, we obtain the required fact $S_i \vdash \Gamma \Rightarrow {\sim}(\alpha{\rightarrow}\beta)$:

$$\dfrac{\begin{array}{c}\vdots\ \textit{Ind. hyp.}\\ \alpha, \Gamma \Rightarrow {\sim}\beta\end{array}}{\Gamma \Rightarrow {\sim}(\alpha{\rightarrow}\beta)}\ ({\sim}{\rightarrow}\text{right}).$$

(d) Case (${\sim}{\rightarrow}$E): $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c}\Gamma_1\\ \vdots\ \mathcal{D}_1\\ {\sim}(\alpha{\rightarrow}\beta)\end{array} \quad \begin{array}{c}\Gamma_2\\ \vdots\ \mathcal{D}_2\\ \alpha\end{array}}{{\sim}\beta}\ ({\sim}{\rightarrow}\text{E})$$

where $\text{oa}(\mathcal{D}) = \Gamma_1 \cup \Gamma_2$ and $\text{end}(\mathcal{D}) = {\sim}\beta$. By induction hypotheses, we have $S_i \vdash \Gamma_1 \Rightarrow {\sim}(\alpha{\rightarrow}\beta)$ and $S_i \vdash \Gamma_2 \Rightarrow \alpha$. Then, we obtain the required fact $S_i \vdash \Gamma_1, \Gamma_2 \Rightarrow {\sim}\beta$:

$$\dfrac{\begin{array}{c}\vdots\ \textit{Ind. hyp.}\\ \Gamma_2 \Rightarrow \alpha\end{array} \quad \dfrac{\begin{array}{c}\vdots\ \textit{Ind. hyp.}\\ \Gamma_1 \Rightarrow {\sim}(\alpha{\rightarrow}\beta)\end{array} \quad \dfrac{\begin{array}{c}\vdots\ \textit{Prop.}2.3\\ \alpha \Rightarrow \alpha\end{array} \quad \begin{array}{c}\vdots\ \textit{Prop.}2.3\\ {\sim}\beta \Rightarrow {\sim}\beta\end{array}}{{\sim}(\alpha{\rightarrow}\beta), \alpha \Rightarrow {\sim}\beta}\ ({\sim}{\rightarrow}\text{left})}{\alpha, \Gamma_1 \Rightarrow {\sim}\beta}\ (\text{cut})}{\Gamma_1, \Gamma_2 \Rightarrow {\sim}\beta}\ (\text{cut}).$$

(e) Case ($\sim\wedge$E): $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c}\Gamma_1\\ \vdots\ \mathcal{D}_1\\ \sim(\alpha\wedge\beta)\end{array}\quad \begin{array}{c}[\sim\alpha]\Gamma_2\\ \vdots\ \mathcal{D}_2\\ \gamma\end{array}\quad \begin{array}{c}[\sim\beta]\Gamma_3\\ \vdots\ \mathcal{D}_3\\ \gamma\end{array}}{\gamma}\ (\sim\wedge\mathrm{E})$$

where $\mathrm{oa}(\mathcal{D}) = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ and $\mathrm{end}(\mathcal{D}) = \gamma$. By induction hypotheses, we have $S_i \vdash \Gamma_1 \Rightarrow \sim(\alpha\wedge\beta)$, $S_i \vdash \sim\alpha, \Gamma_2 \Rightarrow \gamma$, and $S_i \vdash \sim\beta, \Gamma_3 \Rightarrow \gamma$. Then, we obtain the required fact $S_i \vdash \Gamma_1, \Gamma_2, \Gamma_3 \Rightarrow \gamma$:

$$\dfrac{\begin{array}{c}\vdots\ \mathit{Ind.hyp.}\\ \Gamma_1 \Rightarrow \sim(\alpha\wedge\beta)\end{array} \quad \dfrac{\begin{array}{c}\vdots\ \mathit{Ind.hyp.}\\ \sim\alpha, \Gamma_2 \Rightarrow \gamma\\ \vdots\ (\mathrm{we})\\ \sim\alpha, \Gamma_2, \Gamma_3 \Rightarrow \gamma\end{array} \quad \begin{array}{c}\vdots\ \mathit{Ind.hyp.}\\ \sim\beta, \Gamma_3 \Rightarrow \gamma\\ \vdots\ (\mathrm{we})\\ \sim\beta, \Gamma_2, \Gamma_3 \Rightarrow \gamma\end{array}}{\sim(\alpha\wedge\beta), \Gamma_2, \Gamma_3 \Rightarrow \gamma}\ (\sim\wedge\mathrm{left})}{\Gamma_1, \Gamma_2, \Gamma_3 \Rightarrow \gamma}\ (\mathrm{cut})$$

where (we) is admissible in S_i − (cut) by Propositions 2.4 and 2.11.

(f) Case ($\sim\vee$I): $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c}\Gamma_1\\ \vdots\ \mathcal{D}_1\\ \sim\alpha\end{array}\quad \begin{array}{c}\Gamma_2\\ \vdots\ \mathcal{D}_2\\ \sim\beta\end{array}}{\sim(\alpha\vee\beta)}\ (\sim\vee\mathrm{I})$$

where $\mathrm{oa}(\mathcal{D}) = \Gamma_1 \cup \Gamma_2$ and $\mathrm{end}(\mathcal{D}) = \sim(\alpha\vee\beta)$. By induction hypotheses, we have $S_i \vdash \Gamma_1 \Rightarrow \sim\alpha$ and $S_i \vdash \Gamma_2 \Rightarrow \sim\beta$. Then, we obtain the required fact $S_i \vdash \Gamma_1, \Gamma_2 \Rightarrow \sim(\alpha\vee\beta)$:

$$\dfrac{\begin{array}{c}\vdots\ \mathit{Ind.hyp.}\\ \Gamma_1 \Rightarrow \sim\alpha\\ \vdots\ (\mathrm{we})\\ \Gamma_1, \Gamma_2 \Rightarrow \sim\alpha\end{array} \quad \begin{array}{c}\vdots\ \mathit{Ind.hyp.}\\ \Gamma_2 \Rightarrow \sim\beta\\ \vdots\ (\mathrm{we})\\ \Gamma_1, \Gamma_2 \Rightarrow \sim\beta\end{array}}{\Gamma_1, \Gamma_2 \Rightarrow \sim(\alpha\vee\beta)}\ (\sim\vee\mathrm{right})$$

where (we) is admissible in S_i − (cut) by Propositions 2.4 and 2.11.

2. We prove 2 by induction on the derivations $\mathcal{D}$ of $\Gamma \Rightarrow \beta$ in S_i − (cut). We distinguish the cases according to the last inference of $\mathcal{D}$. We show some cases.

   (a) Case (init2): $\mathcal{D}$ is of the form:

   $$\sim p, \Gamma \Rightarrow \sim p \text{ (init2)}.$$

   In this case, we obtain a required normal derivation $\mathcal{D}'$ by:

   $$\sim p$$

   where $\mathrm{oa}(\mathcal{D}') = \{\sim p\} \subseteq \{\sim p\} \cup \Gamma$ and $\mathrm{end}(\mathcal{D}') = \sim p$.

   (b) Case (ex-middle): $\mathcal{D}$ is of the form:

   $$\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1 \\ \sim\alpha, \Gamma \Rightarrow \gamma\end{array} \quad \begin{array}{c}\vdots\ \mathcal{D}_2 \\ \alpha, \Gamma \Rightarrow \gamma\end{array}}{\Gamma \Rightarrow \gamma} \text{ (ex-middle)}.$$

   By induction hypotheses, we have normal derivations $\mathcal{E}_1$ and $\mathcal{E}_2$ in N_i of the form:

   $$\begin{array}{c}(\sim\alpha, \Gamma)^* \\ \vdots\ \mathcal{E}_1 \\ \gamma\end{array} \qquad \begin{array}{c}(\alpha, \Gamma)^* \\ \vdots\ \mathcal{E}_2 \\ \gamma\end{array}$$

   where $\mathrm{oa}(\mathcal{E}_1) = (\{\sim\alpha\} \cup \Gamma)^* \subseteq \{\sim\alpha\} \cup \Gamma$, $\mathrm{oa}(\mathcal{E}_2) = (\{\alpha\} \cup \Gamma)^* \subseteq \{\alpha\} \cup \Gamma$, $\mathrm{end}(\mathcal{E}_1) = \gamma$, and $\mathrm{end}(\mathcal{E}_2) = \gamma$. Then, we distinguish the cases according to $(\{\sim\alpha\} \cup \Gamma)^*$ and $(\{\alpha\} \cup \Gamma)^*$. We consider the following cases: (1) $\sim\alpha \notin (\{\sim\alpha\} \cup \Gamma)^*$, (2) $\alpha \notin (\{\alpha\} \cup \Gamma)^*$, and (3) $\sim\alpha \in (\{\sim\alpha\} \cup \Gamma)^*$ and $\alpha \in (\{\alpha\} \cup \Gamma)^*$.

      i. Subcase (1): We obtain a required normal derivation $\mathcal{D}'$ in N_i by:

      $$\begin{array}{c}\Gamma^* \\ \vdots\ \mathcal{E}_1 \\ \gamma\end{array}$$

      where $\mathrm{oa}(\mathcal{D}') = \Gamma^* \subseteq \Gamma$ and $\mathrm{end}(\mathcal{D}') = \gamma$.

ii. Subcase (2): We obtain a required normal derivation $\mathcal{D}'$ in N_i by:

$$\begin{array}{c} \Gamma^* \\ \vdots\ \mathcal{E}_2 \\ \gamma \end{array}$$

where $\mathrm{oa}(\mathcal{D}') = \Gamma^* \subseteq \Gamma$ and $\mathrm{end}(\mathcal{D}') = \gamma$.

iii. Subcase (3): We obtain a required normal derivation $\mathcal{D}'$ in N_i by:

$$\dfrac{\begin{array}{c} ([{\sim}\alpha]\ \Gamma)^* \\ \vdots\ \mathcal{E}_1 \\ \gamma \end{array} \quad \begin{array}{c} ([\alpha]\ \Gamma)^* \\ \vdots\ \mathcal{E}_2 \\ \gamma \end{array}}{\gamma}\ (\mathrm{EM})$$

where $\mathrm{oa}(\mathcal{D}') = \Gamma^* \subseteq \Gamma$ and $\mathrm{end}(\mathcal{D}') = \gamma$.

(c) Case (${\sim}{\sim}$left): $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c} \vdots\ \mathcal{E} \\ \alpha, \Gamma \Rightarrow \gamma \end{array}}{{\sim}{\sim}\alpha, \Gamma \Rightarrow \gamma}\ ({\sim}{\sim}\mathrm{left}).$$

By induction hypothesis, we have a normal derivation $\mathcal{E}'$ in N_i of the form:

$$\begin{array}{c} (\alpha, \Gamma)^* \\ \vdots\ \mathcal{E}' \\ \gamma \end{array}$$

where $\mathrm{oa}(\mathcal{E}') = (\{\alpha\} \cup \Gamma)^* \subseteq \{\alpha\} \cup \Gamma$ and $\mathrm{end}(\mathcal{E}') = \gamma$. In what follows, we show only the case for $(\{\alpha\} \cup \Gamma)^* \equiv \{\alpha\} \cup \Gamma$. In this case, we obtain a required normal derivation $\mathcal{D}'$ in N_i by:

$$\begin{array}{c} \dfrac{{\sim}{\sim}\alpha}{\alpha}\ ({\sim}{\sim}\mathrm{E}) \quad \Gamma \\ \vdots\ \mathcal{E}' \\ \gamma \end{array}$$

where $\mathrm{oa}(\mathcal{D}') = \{{\sim}{\sim}\alpha\} \cup \Gamma$ and $\mathrm{end}(\mathcal{D}') = \gamma$.

(d) Case ($\sim\!\to$left): $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \Gamma\Rightarrow\alpha\end{array}\quad\begin{array}{c}\vdots\ \mathcal{D}_2\\ {\sim}\beta,\Delta\Rightarrow\gamma\end{array}}{{\sim}(\alpha\to\beta),\Gamma,\Delta\Rightarrow\gamma}\ ({\sim}\!\to\!\text{left}).$$

By induction hypotheses, we have normal derivations $\mathcal{E}_1$ and $\mathcal{E}_2$ in N_i of the form:

$$\begin{array}{c}\Gamma^*\\ \vdots\ \mathcal{E}_1\\ \alpha\end{array}\qquad\qquad\begin{array}{c}({\sim}\beta,\Delta)^*\\ \vdots\ \mathcal{E}_2\\ \gamma\end{array}$$

where $\text{oa}(\mathcal{E}_1)=\Gamma^*\subseteq\Gamma$, $\text{end}(\mathcal{E}_1)=\alpha$, $\text{oa}(\mathcal{E}_2)=(\{\beta\}\cup\Delta)^*\subseteq\{\beta\}\cup\Delta$, and $\text{end}(\mathcal{E}_2)=\gamma$. In what follows, we show only the case for $\Gamma^*\equiv\Gamma$ and $(\{\beta\}\cup\Delta)^*\equiv\{\beta\}\cup\Delta$. In this case, we obtain a required normal derivation $\mathcal{D}'$ in N_i by:

$$\begin{array}{c}\dfrac{{\sim}(\alpha\to\beta)\quad\begin{array}{c}\Gamma\\ \vdots\ \mathcal{E}_1\\ \alpha\end{array}}{{\sim}\beta}\ ({\sim}\!\to\!\text{E})\qquad\Delta\\ \vdots\ \mathcal{E}_2\\ \gamma\end{array}$$

where $\text{oa}(\mathcal{D}')=\{{\sim}(\alpha\to\beta)\}\cup\Gamma\cup\Delta$ and $\text{end}(\mathcal{D}')=\gamma$.

(e) Case (${\sim}\wedge$left): $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ {\sim}\alpha,\Gamma\Rightarrow\gamma\end{array}\quad\begin{array}{c}\vdots\ \mathcal{D}_2\\ {\sim}\beta,\Gamma\Rightarrow\gamma\end{array}}{{\sim}(\alpha\wedge\beta),\Gamma\Rightarrow\gamma}\ ({\sim}\wedge\text{left}).$$

By induction hypotheses, we have normal derivations $\mathcal{E}_1$ and $\mathcal{E}_2$ in N_i of the form:

$$\begin{array}{c}({\sim}\alpha,\Gamma)^*\\ \vdots\ \mathcal{E}_1\\ \gamma\end{array}\qquad\qquad\begin{array}{c}({\sim}\beta,\Gamma)^*\\ \vdots\ \mathcal{E}_2\\ \gamma\end{array}$$

where $\mathrm{oa}(\mathcal{E}_1) = (\{{\sim}\alpha\} \cup \Gamma)^* \subseteq \{{\sim}\alpha\} \cup \Gamma$, $\mathrm{oa}(\mathcal{E}_2) = (\{{\sim}\beta\} \cup \Gamma)^* \subseteq \{{\sim}\beta\} \cup \Gamma$, and $\mathrm{end}(\mathcal{E}_1) = \mathrm{end}(\mathcal{E}_2) = \gamma$. In what follows, we show only the case for $(\{{\sim}\alpha\} \cup \Gamma)^* \equiv \{{\sim}\alpha\} \cup \Gamma$ and $(\{{\sim}\beta\} \cup \Gamma)^* \equiv \{{\sim}\beta\} \cup \Gamma$. In this case, we obtain a required normal derivation $\mathcal{D}'$ in N_i by:

$$\dfrac{{\sim}(\alpha\wedge\beta) \quad \begin{array}{c}[{\sim}\alpha]\Gamma \\ \vdots\ \mathcal{E}_1 \\ \gamma\end{array} \quad \begin{array}{c}[{\sim}\beta]\Gamma \\ \vdots\ \mathcal{E}_2 \\ \gamma\end{array}}{\gamma}\ ({\sim}\wedge\mathrm{E})$$

where $\mathrm{oa}(\mathcal{D}') = \{{\sim}(\alpha\wedge\beta)\} \cup \Gamma$ and $\mathrm{end}(\mathcal{D}') = \gamma$.

(f) Case (${\sim}\vee$right): $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1 \\ \Gamma \Rightarrow {\sim}\alpha\end{array} \quad \begin{array}{c}\vdots\ \mathcal{D}_2 \\ \Gamma \Rightarrow {\sim}\beta\end{array}}{\Gamma \Rightarrow {\sim}(\alpha\vee\beta)}\ ({\sim}\vee\mathrm{right}).$$

By induction hypotheses, we have normal derivations $\mathcal{E}_1$ and $\mathcal{E}_2$ in N_i of the form:

$$\begin{array}{c}\Gamma^* \\ \vdots\ \mathcal{E}_1 \\ {\sim}\alpha\end{array} \qquad \begin{array}{c}\Gamma^* \\ \vdots\ \mathcal{E}_2 \\ {\sim}\beta\end{array}$$

where $\mathrm{oa}(\mathcal{E}_1) = \mathrm{oa}(\mathcal{E}_2) = \Gamma^* \subseteq \Gamma$, $\mathrm{end}(\mathcal{E}_1) = {\sim}\alpha$, and $\mathrm{end}(\mathcal{E}_2) = {\sim}\beta$. In what follows, we show only the case for $\Gamma^* \equiv \Gamma$. In this case, we obtain a required normal derivation $\mathcal{D}'$ in N_i by:

$$\dfrac{\begin{array}{c}\Gamma \\ \vdots\ \mathcal{E}_1 \\ {\sim}\alpha\end{array} \quad \begin{array}{c}\Gamma \\ \vdots\ \mathcal{E}_2 \\ {\sim}\beta\end{array}}{{\sim}(\alpha\vee\beta)}\ ({\sim}\vee\mathrm{I})$$

where $\mathrm{oa}(\mathcal{D}') = \Gamma$ and $\mathrm{end}(\mathcal{D}') = {\sim}(\alpha\vee\beta)$. □

We then obtain the following theorems.

THEOREM 3.8 (Equivalence between nC-family and sC-family). *Let N_1, N_2, N_3, and N_4 be* nC*,* nC3*,* nMC*, and* nCN*, respectively. Let S_1, S_2, S_3, and S_4 be* sC*,* sC3*,* sMC**, and* sCN**, respectively. For any formula α and any $i \in \{1, 2, 3, 4\}$, $S_i \vdash \Rightarrow \alpha$ iff α is provable in N_i.*

PROOF: Taking $\emptyset$ as Γ in Lemma 3.7, we obtain the claim. □

THEOREM 3.9 (Normalization for nC, nC3, nMC, and nCN). *Let N be* nC*,* nC3*,* nMC*, or* nCN*. All derivations in N are normalizable. More precisely, if a derivation $\mathcal{D}$ in N is given, then we can obtain a normal derivation $\mathcal{D}'$ in N such that* $\mathrm{oa}(\mathcal{D}') \subseteq \mathrm{oa}(\mathcal{D})$ *and* $\mathrm{end}(\mathcal{D}') = \mathrm{end}(\mathcal{D})$.

PROOF: Let N_1, N_2, N_3, and N_4 be nC, nC3, nMC, and nCN, respectively. Let S_1, S_2, S_3, and S_4 be sC, sC3, sMC*, and sCN*, respectively. Let i be 1, 2, 3, or 4. Suppose that a derivation $\mathcal{D}$ in N_i is given, and assume that $\mathrm{oa}(\mathcal{D}) = \Gamma$ and $\mathrm{end}(\mathcal{D}) = \beta$. Then, by Lemma 3.7 (1), we obtain $S_i \vdash \Gamma \Rightarrow \beta$. By the cut-elimination theorem for S_i (i.e., Theorems 2.5, 2.6, and 2.13), we obtain $S_i - (\mathrm{cut}) \vdash \Gamma \Rightarrow \beta$. Then, by Lemma 3.7 (2), we can obtain a normal derivation $\mathcal{D}'$ in N_i such that $\mathrm{oa}(\mathcal{D}') \subseteq \mathrm{oa}(\mathcal{D})$ and $\mathrm{end}(\mathcal{D}') = \mathrm{end}(\mathcal{D})$. □

4. Natural deduction systems with general elimination rules and normalization theorems

We define natural deduction systems with general elimination rules, referred to as gNJ$^+$, gC, gC3, gMC, and gCN, for positive intuitionistic logic, C, C3, MC, and CN, respectively. For these systems, we use the same notations and notions as those for the previously introduced systems.

DEFINITION 4.1 (gNJ$^+$, gC, gC3, gMC, and gCN).

1. gNJ$^+$ is obtained from NJ$^+$ by replacing ($\to$E), ($\wedge$E1), and ($\wedge$E2) with the general elimination rules of the form:

$$\dfrac{\alpha{\to}\beta \quad \alpha \quad \begin{matrix}[\beta]\\ \vdots \\ \gamma\end{matrix}}{\gamma}\ ({\to}\mathrm{GE}) \qquad \dfrac{\alpha{\wedge}\beta \quad \begin{matrix}[\alpha,\beta]\\ \vdots \\ \gamma\end{matrix}}{\gamma}\ ({\wedge}\mathrm{GE}).$$

2. gC is obtained from gNJ$^+$ by adding ($\sim\sim$E), ($\sim\to$E), ($\sim\vee$E1), ($\sim\vee$E2), and the negated general elimination rules of the form:

$$\dfrac{{\sim}{\sim}\alpha \quad \begin{matrix}[\alpha]\\ \vdots \\ \gamma\end{matrix}}{\gamma}\ ({\sim}{\sim}\mathrm{GE}) \qquad \dfrac{{\sim}(\alpha{\to}\beta) \quad \alpha \quad \begin{matrix}[{\sim}\beta]\\ \vdots \\ \gamma\end{matrix}}{\gamma}\ ({\sim}{\to}\mathrm{GE})$$

$$\dfrac{{\sim}(\alpha \vee \beta) \quad \begin{matrix}[{\sim}\alpha,{\sim}\beta]\\ \vdots \\ \gamma\end{matrix}}{\gamma}\ ({\sim}{\vee}\mathrm{GE}).$$

3. gC3 is obtained from gC by adding (EM).
4. gMC is obtained from gC by adding (GEM).
5. gCN is obtained from gC3 by adding (GEM).

Remark 4.2. The rules ($\to$GE), ($\wedge$GE), ($\sim\sim$GE), ($\sim\to$GE), and ($\sim\vee$GE) are referred to as general elimination rules. These rules are considered as elimination rules. For more information on general elimination rules and systems incorporating them, see [33, 37, 23].

Next, we define reduction relations on the set of derivations in natural deduction systems with general elimination rules. We employ the same notions, such as the maximum formula and normal derivation, as those used for nC, nC3, nMC, or nCN.

DEFINITION 4.3 (Reduction relation). Let γ be a maximum formula in a derivation that is the conclusion of an inference rule R.

1. The definition of the reduction relation $\rhd$ at γ in gC is obtained by the following conditions.

(a) R is $(\to\mathrm{I})$ and γ is $\alpha\to\beta$:

$$\dfrac{\dfrac{\begin{matrix}[\alpha]\\ \vdots\ \mathcal{D}\\ \beta\end{matrix}}{\alpha\to\beta}\,(\to\mathrm{I}) \quad \begin{matrix}\vdots\ \mathcal{E}_1\\ \alpha\end{matrix} \quad \begin{matrix}[\beta]\\ \vdots\ \mathcal{E}_2\\ \gamma\end{matrix}}{\gamma}\,(\to\mathrm{GE}) \qquad \rhd \qquad \begin{matrix}\vdots\ \mathcal{E}_1\\ \alpha\\ \vdots\ \mathcal{D}\\ \beta\\ \vdots\ \mathcal{E}_2\\ \gamma.\end{matrix}$$

(b) R is $(\wedge\mathrm{I})$ and γ is $\alpha_1\wedge\alpha_2$:

$$\dfrac{\dfrac{\begin{matrix}\vdots\ \mathcal{D}_1\\ \alpha\end{matrix} \quad \begin{matrix}\vdots\ \mathcal{D}_2\\ \beta\end{matrix}}{\alpha\wedge\beta}\,(\wedge\mathrm{I}) \quad \begin{matrix}[\alpha,\beta]\\ \vdots\ \mathcal{E}\\ \gamma\end{matrix}}{\gamma}\,(\wedge\mathrm{GE}) \qquad \rhd \qquad \begin{matrix}\begin{matrix}\vdots\ \mathcal{D}_1\\ \alpha\end{matrix} \quad \begin{matrix}\vdots\ \mathcal{D}_2\\ \beta\end{matrix}\\ \vdots\ \mathcal{E}\\ \gamma.\end{matrix}$$

(c) R is $(\vee\mathrm{I1})$ or $(\vee\mathrm{I2})$ and γ is $\alpha_1\vee\alpha_2$: The same condition as the one defined in Definition 3.6.

(d) R is $(\vee\mathrm{E})$: The same condition as the one defined in Definition 3.6.

(e) R is $({\sim}{\sim}\mathrm{I})$, and γ is ${\sim}{\sim}\alpha$:

$$\dfrac{\dfrac{\begin{matrix}\vdots\ \mathcal{D}\\ \alpha\end{matrix}}{{\sim}{\sim}\alpha}\,({\sim}{\sim}\mathrm{I}) \quad \begin{matrix}[\alpha]\\ \vdots\ \mathcal{E}\\ \gamma\end{matrix}}{\gamma}\,({\sim}{\sim}\mathrm{GE}) \qquad \rhd \qquad \begin{matrix}\vdots\ \mathcal{D}\\ \alpha\\ \vdots\ \mathcal{E}\\ \gamma.\end{matrix}$$

(f) R is $({\sim}{\to}\mathrm{I})$ and γ is ${\sim}(\alpha\to\beta)$:

$$\dfrac{\dfrac{\begin{matrix}[\alpha]\\ \vdots\ \mathcal{D}\\ {\sim}\beta\end{matrix}}{{\sim}(\alpha\to\beta)}\,({\sim}{\to}\mathrm{I}) \quad \begin{matrix}\vdots\ \mathcal{E}_1\\ \alpha\end{matrix} \quad \begin{matrix}[{\sim}\beta]\\ \vdots\ \mathcal{E}_2\\ \gamma\end{matrix}}{\gamma}\,({\sim}{\to}\mathrm{GE}) \qquad \rhd \qquad \begin{matrix}\vdots\ \mathcal{E}_1\\ \alpha\\ \vdots\ \mathcal{D}\\ {\sim}\beta\\ \vdots\ \mathcal{E}_2\\ \gamma.\end{matrix}$$

(g) R is $({\sim}\wedge\mathrm{I1})$ or $({\sim}\wedge\mathrm{I2})$ and γ is ${\sim}(\alpha_1\wedge\alpha_2)$: The same condition as the one defined in Definition 3.6.

(h) R is (${\sim}\wedge$E): The same condition as the one defined in Definition 3.6.

(i) R is (${\sim}\vee$I) and γ is ${\sim}(\alpha_1\vee\alpha_2)$:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ {\sim}\alpha\end{array}\quad \begin{array}{c}\vdots\ \mathcal{D}_2\\ {\sim}\beta\end{array}}{{\sim}(\alpha\vee\beta)}\,({\sim}\vee\mathrm{I}) \qquad \begin{array}{c}[{\sim}\alpha,{\sim}\beta]\\ \vdots\ \mathcal{E}\\ \gamma\end{array}}{\gamma}\,({\sim}\vee\mathrm{GE}) \quad \rhd \quad \begin{array}{c} \begin{array}{cc}\vdots\ \mathcal{D}_1 & \vdots\ \mathcal{D}_2\\ {\sim}\alpha & {\sim}\beta\end{array}\\ \vdots\ \mathcal{E}\\ \gamma.\end{array}$$

(j) The set of derivations is closed under $\rhd$.

2. The definition of the reduction relation $\rhd$ at γ in gC3 is obtained from the conditions for the reduction relation $\rhd$ at γ in gC by adding the following condition.

(a) R is (EM) and γ is $\gamma_1{\to}\gamma_2$, $\gamma_1\wedge\gamma_2$, $\gamma_1\vee\gamma_2$, ${\sim}{\sim}\gamma'$, ${\sim}(\gamma_1{\to}\gamma_2)$, ${\sim}(\gamma_1\wedge\gamma_2)$, or ${\sim}(\gamma_1\vee\gamma_2)$:

$$\dfrac{\dfrac{\begin{array}{c}[{\sim}\alpha]\\ \vdots\ \mathcal{D}_1\\ \gamma\end{array}\quad \begin{array}{c}[\alpha]\\ \vdots\ \mathcal{D}_2\\ \gamma\end{array}}{\gamma}\,(\mathrm{EM}) \qquad \begin{array}{c}\vdots\ \mathcal{E}_1\\ \delta_1\end{array}\quad \begin{array}{c}\vdots\ \mathcal{E}_2\\ \delta_2\end{array}}{\delta}\,R'$$

$$\rhd \quad \dfrac{\dfrac{\begin{array}{c}[{\sim}\alpha]\\ \vdots\ \mathcal{D}_1\\ \gamma\end{array}\quad \begin{array}{c}\vdots\ \mathcal{E}_1\\ \delta_1\end{array}\quad \begin{array}{c}\vdots\ \mathcal{E}_2\\ \delta_2\end{array}}{\delta}\,R' \qquad \dfrac{\begin{array}{c}[\alpha]\\ \vdots\ \mathcal{D}_2\\ \gamma\end{array}\quad \begin{array}{c}\vdots\ \mathcal{E}_1\\ \delta_1\end{array}\quad \begin{array}{c}\vdots\ \mathcal{E}_2\\ \delta_2\end{array}}{\delta}\,R'}{\delta}\,(\mathrm{EM})$$

where R' is (${\to}$GE), ($\wedge$GE), ($\vee$E), (${\sim}{\sim}$GE) (${\sim}{\to}$GE), (${\sim}\wedge$E), or (${\sim}\vee$GE), and both $\mathcal{E}_1$ and $\mathcal{E}_2$ are derivations of the minor premises of R' if they exist.

3. The definition of the reduction relation $\rhd$ at γ in gMC is obtained from the conditions for the reduction relation $\rhd$ at γ in gC by adding the following condition.

(a) R is (GEM) and γ is $\gamma_1{\to}\gamma_2$, $\gamma_1{\wedge}\gamma_2$, $\gamma_1{\vee}\gamma_2$, ${\sim}{\sim}\gamma'$, ${\sim}(\gamma_1{\to}\gamma_2)$, ${\sim}(\gamma_1{\wedge}\gamma_2)$, or ${\sim}(\gamma_1{\vee}\gamma_2)$:

$$\dfrac{\dfrac{\begin{array}{c}[\alpha{\to}\beta]\\ \vdots\ \mathcal{D}_1\\ \gamma\end{array}\quad \begin{array}{c}[\alpha]\\ \vdots\ \mathcal{D}_2\\ \gamma\end{array}}{\gamma}\,(\mathrm{GEM}) \quad \begin{array}{c}\vdots\ \mathcal{E}_1\\ \delta_1\end{array}\quad \begin{array}{c}\vdots\ \mathcal{E}_2\\ \delta_2\end{array}}{\delta}\,R'$$

$$\rhd \quad \dfrac{\dfrac{\begin{array}{c}[\alpha{\to}\beta]\\ \vdots\ \mathcal{D}_1\\ \gamma\end{array}\quad \begin{array}{c}\vdots\ \mathcal{E}_1\\ \delta_1\end{array}\quad \begin{array}{c}\vdots\ \mathcal{E}_2\\ \delta_2\end{array}}{\delta}\,R' \quad \dfrac{\begin{array}{c}[\alpha]\\ \vdots\ \mathcal{D}_2\\ \gamma\end{array}\quad \begin{array}{c}\vdots\ \mathcal{E}_1\\ \delta_1\end{array}\quad \begin{array}{c}\vdots\ \mathcal{E}_2\\ \delta_2\end{array}}{\delta}\,R'}{\delta}\,(\mathrm{GEM})$$

where R' is (${\to}$GE), (${\wedge}$GE), (${\vee}$E), (${\sim}{\sim}$GE) (${\sim}{\to}$GE), (${\sim}{\wedge}$E), or (${\sim}{\vee}$GE), and both $\mathcal{E}_1$ and $\mathcal{E}_2$ are derivations of the minor premises of R' if they exist.

4. The definition of the reduction relation $\rhd$ at γ in gCN is obtained from the conditions for the reduction relation $\rhd$ at γ in gC3 by adding the other conditions of gMC.

LEMMA 4.4. *Let G_1, G_2, G_3, and G_4 be* gC, gC3, gMC, *and* gCN, *respectively. Let S_1, S_2, S_3, and S_4 be* sC, sC3, sMC*, *and* sCN*, *respectively. For any $i \in \{1,2,3,4\}$, the following hold.*

1. *If $\mathcal{D}$ is a derivation in G_i such that* $\mathrm{oa}(\mathcal{D}) = \Gamma$ *and* $\mathrm{end}(\mathcal{D}) = \beta$, *then* $S_i \vdash \Gamma \Rightarrow \beta$,

2. *If $S_i -$* (cut) $\vdash \Gamma \Rightarrow \beta$, *then we can obtain a derivation $\mathcal{D}'$ in G_i such that*

   (a) $\mathrm{oa}(\mathcal{D}') \subseteq \Gamma$,

   (b) $\mathrm{end}(\mathcal{D}') = \beta$,

   (c) $\mathcal{D}'$ is normal.

Proof:

1. We prove 1 by induction on the derivations $\mathcal{D}$ of G_i such that oa($\mathcal{D}$) = Γ and end($\mathcal{D}$) = β. We distinguish the cases according to the last inference of $\mathcal{D}$. We show some cases.

   (a) Case ($\sim\sim$E): $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c}\Gamma_1\\ \vdots\ \mathcal{D}_1\\ \sim\sim\alpha\end{array} \qquad \begin{array}{c}[\alpha]\ \Gamma_2\\ \vdots\ \mathcal{D}_2\\ \gamma\end{array}}{\gamma}\ (\sim\sim\mathrm{GE})$$

   where oa($\mathcal{D}$) = $\Gamma_1 \cup \Gamma_2$ and end($\mathcal{D}$) = γ.

   i. Subcase (1): The discharge of ($\sim\sim$GE) is vacuous. In this case, by induction hypothesis, we have $S_i \vdash \Gamma_2 \Rightarrow \gamma$. We then obtain the required fact by:

$$\begin{array}{c}\vdots\ \textit{Ind. hyp.}\\ \Gamma_2 \Rightarrow \gamma\\ \vdots\ (\mathrm{we})\\ \sim\sim\alpha, \Gamma_1, \Gamma_2 \Rightarrow \gamma\end{array}$$

   where (we) is admissible in cut-free S_i by Proposition 2.4.

   ii. Subcase (2): α is the discharged assumption of ($\sim\sim$GE). In this case, by induction hypothesis, we have $S_i \vdash \alpha, \Gamma_2 \Rightarrow \gamma$. We then obtain the required fact by:

$$\dfrac{\begin{array}{c}\vdots\ \textit{Ind. hyp.}\\ \alpha, \Gamma_2 \Rightarrow \gamma\\ \vdots\ (\mathrm{we})\\ \alpha, \Gamma_1, \Gamma_2 \Rightarrow \gamma\end{array}}{\sim\sim\alpha, \Gamma_1, \Gamma_2 \Rightarrow \gamma}\ (\sim\sim\mathrm{left})$$

   where (we) is admissible in cut-free S_i by Proposition 2.4.

(b) Case ($\sim\!\rightarrow$GE): $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c}\Gamma_1\\ \vdots\ \mathcal{D}_1\\ \sim(\alpha\rightarrow\beta)\end{array} \quad \begin{array}{c}\Gamma_2\\ \vdots\ \mathcal{D}_2\\ \alpha\end{array} \quad \begin{array}{c}[\sim\beta]\ \Gamma_3\\ \vdots\ \mathcal{D}_3\\ \gamma\end{array}}{\gamma}\ (\sim\!\rightarrow\mathrm{GE})$$

where $\mathrm{oa}(\mathcal{D}) = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ and $\mathrm{end}(\mathcal{D}) = \gamma$.

i. Subcase (1): The discharge of ($\sim\!\rightarrow$GE) is vacuous. In this case, by induction hypothesis, we have $S_i \vdash \Gamma_3 \Rightarrow \gamma$. We then obtain the required fact by:

$$\begin{array}{c}\vdots\ \textit{Ind. hyp.}\\ \Gamma_3 \Rightarrow \gamma\\ \vdots\ (\mathrm{we})\\ \Gamma_1, \Gamma_2, \Gamma_3 \Rightarrow \gamma\end{array}$$

where (we) is admissible in cut-free S_i by Proposition 2.4.

ii. Subcase (2): $\sim\beta$ is the discharged assumption of ($\sim\!\rightarrow$GE). In this case, by induction hypotheses, we have the following: $S_i \vdash \Gamma_1 \Rightarrow \sim(\alpha\rightarrow\beta)$, $S_i \vdash \Gamma_2 \Rightarrow \alpha$, and $S_i \vdash \sim\beta, \Gamma_3 \Rightarrow \gamma$. We then obtain the required fact by:

$$\dfrac{\begin{array}{c}\vdots\ \textit{Ind. hyp.}\\ \Gamma_1 \Rightarrow \sim(\alpha\rightarrow\beta)\end{array} \quad \dfrac{\begin{array}{c}\vdots\ \textit{Ind. hyp.}\\ \Gamma_2 \Rightarrow \alpha\end{array} \quad \begin{array}{c}\vdots\ \textit{Ind. hyp.}\\ \sim\beta, \Gamma_3 \Rightarrow \gamma\end{array}}{\sim(\alpha\rightarrow\beta), \Gamma_2, \Gamma_3 \Rightarrow \gamma}\ (\sim\!\rightarrow\mathrm{left})}{\Gamma_1, \Gamma_2, \Gamma_3 \Rightarrow \gamma}\ (\mathrm{cut}).$$

(c) Case ($\sim\vee$GE): $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c}\Gamma_1\\ \vdots\ \mathcal{D}_1\\ \sim(\alpha\vee\beta)\end{array} \quad \begin{array}{c}[\sim\alpha, \sim\beta]\ \Gamma_2\\ \vdots\ \mathcal{D}_2\\ \gamma\end{array}}{\gamma}\ (\sim\vee\mathrm{GE})$$

where $\mathrm{oa}(\mathcal{D}) = \Gamma_1 \cup \Gamma_2$ and $\mathrm{end}(\mathcal{D}) = \gamma$.

i. Subcase (1): The discharge of ($\sim\vee$GE) is vacuous. In this case, by induction hypothesis, we have $S_i \vdash \Gamma_2 \Rightarrow \gamma$. We then obtain the required fact by:

$$\dfrac{\dfrac{\vdots\ Ind.\ hyp.}{\Gamma_2 \Rightarrow \gamma}}{\Gamma_1, \Gamma_2 \Rightarrow \gamma}\ \vdots\ (\text{we})$$

where (we) is admissible in cut-free S_i by Proposition 2.4.

ii. Subcase (2): $\sim\alpha$ and/or $\sim\beta$ is (are) the discharged assumption(s) of ($\sim\vee$GE). We show only the case that $\sim\beta$ is only the discharged assumption of ($\sim\vee$GE). In this case, by induction hypotheses, we have $S_i \vdash \Gamma_1 \Rightarrow \sim(\alpha\vee\beta)$ and $S_i \vdash \sim\beta, \Gamma_2 \Rightarrow \gamma$. We then obtain the required fact by:

$$\dfrac{\begin{matrix}\vdots\ Ind.\ hyp.\\ \Gamma_1 \Rightarrow \sim(\alpha\vee\beta)\end{matrix} \qquad \dfrac{\dfrac{\begin{matrix}\vdots\ Ind.\ hyp.\\ \sim\beta, \Gamma_2 \Rightarrow \gamma\end{matrix}}{\sim\alpha, \sim\beta, \Gamma_2 \Rightarrow \gamma}\ (\text{we})}{\sim(\alpha\vee\beta), \Gamma_2 \Rightarrow \gamma}\ (\sim\vee\text{left})}{\Gamma_1, \Gamma_2 \Rightarrow \gamma}\ (\text{cut})$$

where (we) is admissible in cut-free S_i by Proposition 2.4.

2. We prove 2 by induction on the derivations $\mathcal{D}$ of $\Gamma \Rightarrow \beta$ in S_i − (cut). We distinguish the cases according to the last inference of $\mathcal{D}$. We show some cases.

   (a) Case ($\sim\sim$left): $\mathcal{D}$ is of the form:

$$\dfrac{\begin{matrix}\vdots\ \mathcal{E}\\ \alpha, \Gamma \Rightarrow \gamma\end{matrix}}{\sim\sim\alpha, \Gamma \Rightarrow \gamma}\ (\sim\sim\text{left}).$$

   By induction hypothesis, we have a normal derivation $\mathcal{E}'$ in G_i of the form:

$$\begin{matrix}(\alpha, \Gamma)^*\\ \vdots\ \mathcal{E}'\\ \gamma\end{matrix}$$

where $\mathrm{oa}(\mathcal{E}') = (\{\alpha\} \cup \Gamma)^* \subseteq \{\alpha\} \cup \Gamma$ and $\mathrm{end}(\mathcal{E}') = \gamma$. In what follows, we show only the case for $(\{\alpha\} \cup \Gamma)^* \equiv \{\alpha\} \cup \Gamma$. In this case, we obtain a required normal derivation $\mathcal{D}'$ in G_i by:

$$\dfrac{{\sim}{\sim}\alpha \qquad \begin{array}{c}[\alpha]\ \Gamma\\ \vdots\ \mathcal{E}'\\ \gamma\end{array}}{\gamma}\ ({\sim}{\sim}\mathrm{GE})$$

where $\mathrm{oa}(\mathcal{D}') = \{{\sim}{\sim}\alpha\} \cup \Gamma$ and $\mathrm{end}(\mathcal{D}') = \gamma$.

(b) Case (${\sim}{\to}$left): $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \Gamma \Rightarrow \alpha\end{array} \qquad \begin{array}{c}\vdots\ \mathcal{D}_2\\ {\sim}\beta, \Delta \Rightarrow \gamma\end{array}}{{\sim}(\alpha{\to}\beta), \Gamma, \Delta \Rightarrow \gamma}\ ({\sim}{\to}\mathrm{left}).$$

By induction hypotheses, we have normal derivations $\mathcal{D}'_1$ and $\mathcal{D}'_2$ in G_i of the form:

$$\begin{array}{c}\Gamma^*\\ \vdots\ \mathcal{D}'_1\\ \alpha\end{array} \qquad\qquad \begin{array}{c}({\sim}\beta, \Delta)^*\\ \vdots\ \mathcal{D}'_2\\ \gamma\end{array}$$

where $\mathrm{oa}(\mathcal{E}_1) = \Gamma^* \subseteq \Gamma$, $\mathrm{end}(\mathcal{E}_1) = \alpha$, $\mathrm{oa}(\mathcal{E}_2) = (\{{\sim}\beta\} \cup \Delta)^* \subseteq \{{\sim}\beta\} \cup \Delta$, and $\mathrm{end}(\mathcal{E}_2) = \gamma$. In what follows, we show only the case for $\Gamma^* \equiv \Gamma$ and $(\{{\sim}\beta\} \cup \Delta)^* \equiv \{{\sim}\beta\} \cup \Delta$. In this case, we obtain a required normal derivation $\mathcal{D}'$ in G_i by:

$$\dfrac{{\sim}(\alpha{\to}\beta) \qquad \begin{array}{c}\Gamma\\ \vdots\ \mathcal{D}'_1\\ \alpha\end{array} \qquad \begin{array}{c}[{\sim}\beta]\ \Delta\\ \vdots\ \mathcal{D}'_2\\ \gamma\end{array}}{\gamma}\ ({\sim}{\to}\mathrm{GE})$$

where $\mathrm{oa}(\mathcal{D}') = \{{\sim}(\alpha{\to}\beta)\} \cup \Gamma \cup \Delta$ and $\mathrm{end}(\mathcal{D}') = \gamma$.

(c) Case (${\sim}{\vee}$left): $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c}\vdots\ \mathcal{E}\\ {\sim}\alpha, {\sim}\beta, \Gamma \Rightarrow \gamma\end{array}}{{\sim}(\alpha{\vee}\beta), \Gamma \Rightarrow \gamma}\ ({\sim}{\vee}\mathrm{left}).$$

By induction hypothesis, we have a normal derivation $\mathcal{E}'$ in G_i of the form:

$$\begin{array}{c} (\sim\alpha, \sim\beta, \Gamma)^* \\ \vdots\ \mathcal{E}' \\ \gamma \end{array}$$

where $\mathrm{oa}(\mathcal{E}') = (\{\sim\alpha, \sim\beta\} \cup \Gamma)^* \subseteq \{\sim\alpha, \sim\beta\} \cup \Gamma$ and $\mathrm{end}(\mathcal{E}') = \gamma$. In what follows, we show only the case for $(\{\sim\alpha, \sim\beta\} \cup \Gamma)^* \equiv \{\sim\alpha, \sim\beta\} \cup \Gamma$. In this case, we obtain a required normal derivation $\mathcal{D}'$ in G_i by:

$$\dfrac{\sim(\alpha\vee\beta) \qquad \begin{array}{c} [\sim\alpha, \sim\beta]\ \Gamma \\ \vdots\ \mathcal{E}' \\ \gamma \end{array}}{\gamma}\ (\sim\vee\mathrm{GE})$$

where $\mathrm{oa}(\mathcal{D}') = \{\sim(\alpha\vee\beta)\} \cup \Gamma$ and $\mathrm{end}(\mathcal{D}') = \gamma$. □

THEOREM 4.5 (Equivalence between gC-family and sC-family). *Let* G_1, G_2, G_3, *and* G_4 *be* gC, gC3, gMC, *and* gCN, *respectively. Let* S_1, S_2, S_3, *and* S_4 *be* sC, sC3, sMC*, *and* sCN*, *respectively. For any formula* α *and any* $i \in \{1, 2, 3, 4\}$, $S_i \vdash \Rightarrow \alpha$ *iff* α *is provable in* G_i.

PROOF: By Lemma 4.4. □

THEOREM 4.6 (Equivalence between gC-family and nC-family). *Let* G_1, G_2, G_3, *and* G_4 *be* gC, gC3, gMC, *and* gCN, *respectively. Let* N_1, N_2, N_3, *and* N_4 *be* nC, nC3, nMC, *and* nCN, *respectively. For any formula* α *and any* $i \in \{1, 2, 3, 4\}$, α *is provable in* G_i *iff* α *is provable in* N_i.

PROOF: By Theorems 3.8 and 4.5. □

THEOREM 4.7 (Normalization for gC, gC3, gMC, and gCN). *Let* G *be* gC, gC3, gMC, *or* gCN. *All derivations in* G *are normalizable.*

PROOF: Similar to the proof of Theorem 3.9. We use Lemma 4.4 and Theorems 2.5, 2.6, and 2.13. □

5. Systems and theorems for N-family

5.1. Gentzen-style sequent calculi and cut-elimination theorems

A Gentzen-style sequent calculus sN4 for N4 is obtained from sC by replacing ($\sim\to$left) and ($\sim\to$right) with the negated logical inference rules of the form:

$$\frac{\alpha, \sim\beta, \Gamma \Rightarrow \gamma}{\sim(\alpha\to\beta), \Gamma \Rightarrow \gamma}\ (\sim\to\text{left}^{\star}) \qquad \frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \sim\beta}{\Gamma \Rightarrow \sim(\alpha\to\beta)}\ (\sim\to\text{right}^{\star})$$

which correspond to the axiom scheme $\sim(\alpha\to\beta) \leftrightarrow \alpha\wedge\sim\beta$. For more information on this calculus, see, for example, [18, 19].

We then obtain the following Gentzen-style sequent calculi for the N-family in a similar way as for the C-family:

1. sN4e = sN4 + (ex-middle),
2. sN4p = sN4 + (Peirce),
3. sN4ep = sN4 + (ex-middle) + (Peirce),
4. sN4g = sN4 + (g-ex-middle),
5. sN4eg = sN4 + (ex-middle) + (g-ex-middle).

Then, we obtain the cut-elimination theorems for these calculi, the cut-free equivalence between sN4p (sN4ep) and sN4g (sN4eg, resp.), and the separation theorem for the N4-based logics that correspond to sN4, sN4e, sN4p, and sN4ep.

Remark 5.1. As presented in [15], a single-succedent Gentzen-style sequent calculus for classical logic is obtained from a single-succedent Gentzen-style sequent calculus for N4 by adding (ex-middle) and the structural and logical inference rules of the following form:

$$\frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \alpha}\ (\text{we-right}) \qquad \frac{\Gamma \Rightarrow \sim\alpha \quad \Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \gamma}\ (\text{explosion}).$$

Namely, sN4e + (we-right) + (explosion) becomes a sequent calculus for classical logic, although the definition of sequent should be modified as $\Gamma \Rightarrow \gamma$ where γ is a formula or the empty set. Additionally, we remark that sC + (explosion) becomes a sequent calculus for trivial logic (i.e., it is a meaningless logic).

5.2. Gentzen-style natural deduction systems and normalization theorems

A Gentzen-style natural deduction system nN4 for N4 is obtained from nC by replacing ($\sim\!\to$I), and ($\sim\!\to$E) with the negated logical inference rules of the form:

$$\frac{\alpha \quad \sim\beta}{\sim(\alpha\to\beta)}\ (\sim\!\to\mathrm{I}^\star) \qquad \frac{\sim(\alpha\to\beta)}{\alpha}\ (\sim\!\to\mathrm{E1}^\star) \qquad \frac{\sim(\alpha\to\beta)}{\sim\beta}\ (\sim\!\to\mathrm{E2}^\star)$$

which correspond to the axiom scheme $\sim(\alpha\to\beta) \leftrightarrow \alpha\wedge\sim\beta$. For more information on this system, see, for example, [31, 39, 13].

We then obtain the following Gentzen-style natural deduction systems for the N-family in a similar way as for the C-family:

1. nN4e = nN4 + (EM),
2. nN4g = nN4 + (GEM),
3. nN4eg = nN4 + (EM) + (GEM).

Of course, we must introduce the appropriate reduction relations with respect to these systems, modifying the reduction relations with respect to nC, nC3, nMC, and nCN. For example, we must introduce the following reduction conditions for the case when R is ($\sim\!\to$I) and γ is $\sim(\alpha\to\beta)$, instead of the condition for the same case in nC, nC3, nMC, and nCN:

1. Subcase 1:

$$\dfrac{\dfrac{\begin{matrix}\vdots\ \mathcal{D}_1 \\ \alpha\end{matrix} \quad \begin{matrix}\vdots\ \mathcal{D}_2 \\ \sim\beta\end{matrix}}{\sim(\alpha\to\beta)}\ (\sim\!\to\mathrm{I}^\star)}{\alpha}\ (\sim\!\to\mathrm{E1}^\star) \qquad \rhd \qquad \begin{matrix}\vdots\ \mathcal{D}_1 \\ \alpha.\end{matrix}$$

2. Subcase 2:

$$\dfrac{\dfrac{\begin{matrix}\vdots\ \mathcal{D}_1 \\ \alpha\end{matrix} \quad \begin{matrix}\vdots\ \mathcal{D}_2 \\ {\sim}\beta\end{matrix}}{{\sim}(\alpha{\to}\beta)}\ ({\sim}{\to}\mathrm{I}^{\star})}{{\sim}\beta}\ ({\sim}{\to}\mathrm{E2}^{\star}) \qquad \rhd \qquad \begin{matrix}\vdots\ \mathcal{D}_2 \\ {\sim}\beta.\end{matrix}$$

Then, we obtain the normalization theorems for these systems and the equivalence between nN4-family and sN4-family.

Remark 5.2. Natural deduction systems for a family of many-valued logics including Nelson's constructive three-valued logic N3 [2, 24] has recently been introduced and investigated by Kürbis and Petrukhin in [21]. For more information on natural deduction systems for N4, N3, and related logics, see, for example, [31, 39, 35, 13, 41].

5.3. Natural deduction systems with general elimination rules and normalization theorems

A natural deduction system with general elimination rules, gN4, for N4 is obtained from nN4 by replacing (→E), (∧E1), (∧E2), (∼∼E), (∼∨E1), (∼∨E2), (∼→E1*), and (∼→E2*) with (→GE), (∧GE), (∼∼GE), (∼∨GE), and the negated general elimination rule of the form:

$$\dfrac{{\sim}(\alpha{\to}\beta) \qquad \begin{matrix}[\alpha, {\sim}\beta] \\ \vdots \\ \gamma\end{matrix}}{\gamma}\ ({\sim}{\to}\mathrm{GE}^{\star}).$$

For more information on this system, see, [13].

We then obtain the following natural deduction systems with general elimination rules, for the N-family in a similar way as for the C-family:

1. gN4e = gN4 + (EM),
2. gN4g = gN4 + (GEM),
3. gN4eg = gN4 + (EM) + (GEM).

Of course, we must introduce the appropriate reduction relations with respect to these systems, modifying the reduction relations with respect to nN4, nN4e, nN4g, and nN4eg. For example, we must introduce the following reduction condition for the case when R is $(\sim\!\to\!\mathrm{I})$ and γ is $\sim(\alpha\to\beta)$, instead of the condition for the same case in nN4, nN4e, nN4g, and nN4eg:

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\mathcal{D}_1\\ \alpha\end{matrix}\quad\begin{matrix}\vdots\,\mathcal{D}_2\\ \sim\beta\end{matrix}}{\sim(\alpha\to\beta)}\,(\sim\!\to\!\mathrm{I}^\star)\qquad\begin{matrix}[\alpha,\sim\beta]\\ \vdots\,\mathcal{E}\\ \gamma\end{matrix}}{\gamma}\,(\sim\!\to\!\mathrm{GE}^\star)\qquad\rhd\qquad\begin{matrix}\begin{matrix}\vdots\,\mathcal{D}_1\\ \alpha\end{matrix}\quad\begin{matrix}\vdots\,\mathcal{D}_2\\ \sim\beta\end{matrix}\\ \vdots\,\mathcal{E}\\ \gamma.\end{matrix}$$

Then, we obtain the normalization theorems for these systems and the equivalence between gN4-family and sN4-family. Additionally, we also obtain the equivalence between gN4-family and nN4-family.

6. Concluding remarks

In this study, we introduced the Gentzen-style sequent calculi sC, sC3, sMC, and sCN for the C-family: C, C3, MC, and CN, respectively. We proved the cut-elimination theorems for these calculi. We also introduced alternative Gentzen-style sequent calculi sMC* and sCN* for MC and CN, respectively. We then proved the theorem for cut-free equivalence between sMC* (sCN*) and sMC (sCN, respectively) and the cut-elimination theorem for sMC* and sCN*.

Furthermore, we introduced the Gentzen-style natural deduction systems nC, nC3, nMC, and nCN for the C-family. We then proved the normalization theorems for nC, nC3, nMC, and nCN. Additionally, we introduced the natural deduction systems with general elimination rules, gC, gC3, gMC, and gCN for the C-family. We then proved the normalization theorems for gC, gC3, gMC, and gCN.

Additionally, we have shown similar results for the N-family of paraconsistent logics based on Nelson's constructive four-valued logic N4. Thus, we have demonstrated that the proposed proof-theoretic framework can handle a wide range of non-classical logics, including connexive and paraconsistent logics. We have established a unified method for proving the cut-elimination and normalization theorems for these logics. In particular, we have obtained computational interpretations for the standard connexive logics C, C3, MC, and CN using the proposed natural deduction systems.

We remark that similar results can also be obtained for the family of paraconsistent logics with the conflation connective [8]. Specifically, we can consider a Gentzen-style sequent calculus for such a logic, derived from sC by replacing ($\sim\!\to$left) and ($\sim\!\to$right) with negated logical inference rules of the form:

$$\frac{\Gamma \Rightarrow {\sim}\alpha \quad {\sim}\beta, \Delta \Rightarrow \gamma}{{\sim}(\alpha{\to}\beta), \Gamma, \Delta \Rightarrow \gamma}\ ({\sim}{\to}\mathrm{left}^{c}) \qquad \frac{{\sim}\alpha, \Gamma \Rightarrow {\sim}\beta}{\Gamma \Rightarrow {\sim}(\alpha{\to}\beta)}\ ({\sim}{\to}\mathrm{right}^{c})$$

where ($\sim\!\to$left^c) and ($\sim\!\to$right^c) correspond to the axiom ${\sim}(\alpha{\to}\beta) \leftrightarrow {\sim}\alpha{\to}{\sim}\beta$ of conflation. This conflation axiom is generally denoted as $-(\alpha{\to}\beta) \leftrightarrow -\alpha{\to}-\beta$ using the conflation connective $-$ [8], which serves as the logical counterpart of the conflation operator used in the algebraic structure of bilattices.

In future work, we intend to obtain similar results for extended intermediate connexive logics with the intuitionistic negation connective $\neg$ or the absurdity constant $\perp$. These extended connexive logics include $\mathrm{C}^{\perp}$, $\mathrm{C}_3^{\perp}$, and Niki's intermediate connexive logics, C_{po}^{ab} and C_{we}^{ab} [25]. As discussed by Niki, Gentzen-style sequent calculi for C_{po}^{ab} and C_{we}^{ab} are considered to include logical inference rules, referred to as the potential omniscience rule and weak negation rules, respectively, of the form:

$$\frac{{\sim}\alpha, \Gamma \Rightarrow \quad \alpha, \Gamma \Rightarrow}{\Gamma \Rightarrow}\ (\text{po-omni}) \qquad \frac{{\sim}\alpha, \Gamma \Rightarrow \gamma \quad \alpha, \Gamma \Rightarrow}{\Gamma \Rightarrow \gamma}\ (\text{we-neg})$$

which correspond to the axiom $\neg\neg({\sim}\alpha \vee \alpha)$ of potential omniscience and the axiom $\neg\alpha{\to}{\sim}\alpha$ of weak negation, respectively. The corresponding natural deduction rules for (po-omni) and (we-neg) can be considered respectively of the form:

$$\dfrac{\begin{array}{c}[{\sim}\alpha]\\ \vdots\\ \bot\end{array} \quad \begin{array}{c}[\alpha]\\ \vdots\\ \bot\end{array}}{\bot}\ (\mathrm{PO}) \qquad \dfrac{\begin{array}{c}[{\sim}\alpha]\\ \vdots\\ \gamma\end{array} \quad \begin{array}{c}[\alpha]\\ \vdots\\ \bot\end{array}}{\gamma}\ (\mathrm{WN}).$$

where $\neg\alpha$ can be defined as $\neg\alpha := \alpha\to\bot$ using $\bot$.

In other future work, from a technical perspective, we aim to prove the strong normalization and Church-Rosser theorems for the proposed natural deduction systems nC, nC3, nMC, nCN, gC, gC3, gMC, gCN, nN4e, nN4g, nN4eg, gN4e, gN4g, gN4eg, as well as for natural deduction systems for the intermediate connexive logics, including $\mathrm{C}^{\bot}$, $\mathrm{C}_3^{\bot}$, C_{po}^{ab}, and C_{wn}^{ab}. However, these issues remain unresolved at present.

Acknowledgements. I would like to thank the anonymous referees for their valuable comments. I would also like to thank Heinrich Wansing and the members of the Contradictory Logics Seminar at Ruhr University Bochum for their valuable comments and discussions, especially during the 1st Workshop on Contradictory Logics, held in Bochum from December 6 to 8, 2023. I would also like to thank the attendees of the 11th International Conference on Non-Classical Logics – Theory and Applications (NCL'24) and the 9th Workshop on Connexive Logics, both held in conjunction in Lodz from September 5 to 9, 2024, for their valuable discussions. Additionally, I would like to thank the anonymous referees of NCL'24 for their valuable comments on the early version of this paper.

References

[1] H. Africk, *Classical logic, intuitionistic logic, and the Peirce rule*, **Notre Dame Journal of Formal Logic**, vol. 33(2) (1992), pp. 229–235, DOI: https://doi.org/10.1305/ndjfl/1093636101.

[2] A. Almukdad, D. Nelson, *Constructible falsity and inexact predicates*, **Journal of Symbolic Logic**, vol. 49(1) (1984), pp. 231–233, DOI: https://doi.org/10.2307/2274105.

[3] R. Angell, *A propositional logics with subjunctive conditionals*, **Journal of Symbolic Logic**, vol. 27 (1962), pp. 327–343, DOI: https://doi.org/10.2307/2964651.

[4] J. Cantwell, *The logic of conditional negation*, **Notre Dame Journal of Formal Logic**, vol. 49 (2008), pp. 245–260, DOI: https://doi.org/10.1215/00294527-2008-010.

[5] H. B. Curry, **Foundations of mathematical logic**, McGraw-Hill, New York (1963).

[6] P. Égré, L. Rossi, J. Sprenger, *De finettian logics of indicative conditionals Part II: Proof theory and algebraic semantics*, **Journal of Philosophical Logic**, vol. 50 (2021), pp. 215–247, DOI: https://doi.org/10.1007/S10992-020-09572-7.

[7] W. Felscher, *Kombinatorische Konstruktionen mit Beweisen und Schnittelimination*, [in:] J. Diller, G. H. Müller (eds.), ⊨ **ISILC Proof Theory Symposion**, Springer Berlin Heidelberg, Berlin, Heidelberg (1975), pp. 119–151.

[8] M. Fitting, *Bilattices and the semantics of logic programming*, **The Journal of Logic Programming**, vol. 11(2) (1991), pp. 91–116, DOI: https://doi.org/10.1016/0743-1066(91)90014-G.

[9] N. Francez, *Natural deduction for two connexive logics*, **IfCoLog Journal of Logics and their Applications**, vol. 3(3) (2016), pp. 479–504.

[10] L. Gordeev, *On cut elimination in the presence of Peirce rule*, **Archiv für Mathematische Logik und Grundlagenforsch**, vol. 26 (1987), pp. 147–164, DOI: https://doi.org/10.1007/BF02017499.

[11] Y. Gurevich, *Intuitionistic logic with strong negation*, **Studia Logica 36**, vol. 36 (1977), pp. 49–59.

[12] N. Kamide, *Cut-free single-succedent systems revisited*, **Bulletin of the Section of Logic**, vol. 34(3) (2005), pp. 165–175.

[13] N. Kamide, *Natural deduction systems for Nelson's paraconsistent logic and its neighbors*, **Journal of Applied Non-Classical Logics**, vol. 15(4) (2005), pp. 405–435, DOI: https://doi.org/10.3166/JANCL.15.405-435.

[14] N. Kamide, *Embedding friendly first-order paradefinite and connexive logics*, **Journal of Philosophical logic**, vol. 51(5) (2022), pp. 1055–1102, DOI: https://doi.org/10.1007/S10992-022-09659-3.

[15] N. Kamide, *Rules of explosion and excluded middle: Constructing a unified single-succedent Gentzen-style framework for classical, paradefinite, paraconsistent, and paracomplete logics*, **Journal of Logic, Language and Information**, vol. 33(2) (2024), pp. 143–178, DOI: https://doi.org/10.1007/s10849-024-09416-6.

[16] N. Kamide, *Unified Gentzen approach to connexive logics over Wansing's C*, [in:] A. Indrzejczak, M. Zawidzki (eds.), **Proceedings of the 11th International Conference on Non-Classical Logics – Theory and Applications (NCL'24)**, vol. 415 of Electronic Proceedings in Theoretical Computer Science (2024), pp. 214–228, DOI: https://doi.org/10.4204/EPTCS.415.19.

[17] N. Kamide, H. Wansing, *Connexive modal logic based on positive S4*, [in:] J.-Y. Beziau, M. E. Coniglio (eds.), **Logic without Frontiers: Festschrift for Walter Alexandre Carnielli on the occasion of his 60th Birthday**, vol. 17 of Tribute Series, College Publications, London (2011), pp. 389–410.

[18] N. Kamide, H. Wansing, *Proof theory of Nelson's paraconsistent logic: A uniform perspective*, **Theoretical Computer Science**, vol. 415 (2012), pp. 1–38, DOI: https://doi.org/10.1016/J.TCS.2011.11.001.

[19] N. Kamide, H. Wansing, **Proof theory of N4-related paraconsistent logics**, vol. 54 of Studies in Logic, College Publications, London (2015), DOI: https://doi.org/10.1007/s11225-017-9729-9.

[20] N. Kamide, H. Wansing, *Completeness of connexive Heyting-Brouwer logic*, **IFCoLog Journal of Logics and their Applications**, vol. 3(3) (2016), pp. 441–466.

[21] N. Kürbis, Y. Petrukhin, *Normalisation for some quite interesting many-valued logics*, **Logic and Logical Philosophy**, vol. 30(3) (2021), pp. 493–534.

[22] S. McCall, *Connexive implication*, **Journal of Symbolic Logic**, vol. 31 (1966), pp. 415–433, DOI: https://doi.org/10.2307/2270458.

[23] S. Negri, J. von Plato, **Structural Proof Theory**, Cambridge University Press, Cambridge (2001), DOI: https://doi.org/10.1017/CBO9780511527340.

[24] D. Nelson, *Constructible falsity*, **Journal of Symbolic Logic**, vol. 14 (1949), pp. 16–26, DOI: https://doi.org/10.2307/2268973.

[25] S. Niki, *Intuitionistic views on connexive constructible falsity*, **Journal of Applied Logics**, vol. 11(2) (2024), pp. 125–157.

[26] S. Niki, H. Wansing, *On the provable contradictions of the connexive logics C and C3*, **Journal of Philosophical Logic**, vol. 52(5) (2023), pp. 1355–1383, DOI: https://doi.org/10.1007/S10992-023-09709-4.

[27] G. K. Olkhovikov, *On a new three-valued paraconsistent logic*, [in:] **Logic of Law and Tolerance**, Ural State University Press, Yekaterinburg (2002), pp. 96–113, it was translated by T.M. Ferguson in IfCoLog Journal of Logics and their Applications 3(3), pp. 317–334, 2016.

[28] G. K. Olkhovikov, *A complete, correct, and independent axiomatization of the first-order fragment of a three-valued paraconsistent logic*, **IfCoLog Journal of Logics and their Applications**, vol. 3(3) (2016), pp. 335–340.

[29] H. Omori, *From paraconsistent logic to dialetheic logic*, [in:] H. Andreas, P. Verdee (eds.), **Logical Studies of Paraconsistent Reasoning in Science and Mathematics**, Springer, Berlin (2016), pp. 111–134, DOI: https://doi.org/10.1007/978-3-319-40220-8_8.

[30] H. Omori, H. Wansing, *An extension of connexive logic C*, [in:] N. Olivetti, R. Verbrugge, S. Negri, G. Sandu (eds.), **Advances in Modal Logic 13**, College Publications, London (2020).

[31] D. Prawitz, **Natural deduction: a proof-theoretical study**, Almqvist and Wiksell, Stockholm (1965), DOI: https://doi.org/10.2307/2271676.

[32] W. Rautenberg, **Klassische und nicht-klassische Aussagenlogik**, Vieweg+Teubner Verlag, Wiesbaden (1979), DOI: https://doi.org/10.1007/978-3-322-85796-5.

[33] P. Schroeder-Heister, *A natural extension of natural deduction*, **Journal of Symbolic Logic**, vol. 49(4) (1984), pp. 1284–1300, DOI: https://doi.org/10.2307/2274279.

[34] M. E. Szabo (ed.), **Collected papers of Gerhard Gentzen**, Studies in Logic and the Foundations of Mathematics, North-Holland (1969), DOI: https://doi.org/10.2307/2272429.

[35] A. M. Tamminga, K. Tanaka, *A natural deduction system for first degree entailment*, **Notre Dame Journal of Formal Logic**, vol. 40(2) (1999), pp. 258–272, DOI: https://doi.org/10.1305/NDJFL/1038949541.

[36] J. von Plato, *Proof theory of full classical propositional logic* (1999), manuscript, 16 pages.

[37] J. von Plato, *Natural deduction with general elimination rules*, **Archive for Mathematical Logic**, vol. 40(7) (2001), pp. 541–567, DOI: https://doi.org/10.1007/s001530100091.

[38] N. Vorob'ev, *A constructive propositional calculus with strong negation (in Russian)*, **Doklady Akademii Nauk SSSR**, vol. 85 (1952), pp. 465–468.

[39] H. Wansing, **The logic of information structures**, vol. 681 of Lecture Notes in Artificial Intelligence, Springer, Berlin, Heidelberg (1993), DOI: https://doi.org/10.1007/3-540-56734-8.

[40] H. Wansing, *Connexive modal logic*, [in:] R. Schmidt, I. Pratt-Hartmann, M. Reynolds, H. Wansing (eds.), **Advances in Modal Logic 5**, King's College Publications, London (2005), pp. 367–385.

[41] H. Wansing, *Falsification, natural deduction and bi-intuitionistic logic*, **Journal of Logic and Computation**, vol. 26(1) (2016), pp. 425–450, DOI: https://doi.org/10.1093/LOGCOM/EXT035.

[42] H. Wansing, *Natural deduction for bi-connexive logic and a two-sorted typed λ-calculus*, **IfCoLog Journal of Logics and their Applications**, vol. 3(3) (2016), pp. 413–440.

[43] H. Wansing, *Connexive Logic*, [in:] E. N. Zalta, U. Nodelman (eds.), **The Stanford Encyclopedia of Philosophy**, Summer 2023 ed., Metaphysics Research Lab, Stanford University (2023).

[44] E. Zimmermann, *Peirce's rule in natural deduction*, **Theoretical Computer Science**, vol. 275 (2002), pp. 561–574, DOI: https://doi.org/10.1016/S0304-3975(01)00296-1.

Norihiro Kamide

Nagoya City University
Graduate School of Data Science
Yamanohata 1, Mizuho-cho, Mizuho-ku, Nagoya
467-8501 Aichi, Japan

e-mail: drnkamide08@kpd.biglobe.ne.jp

Funding information: This research was supported by JSPS KAKENHI Grant Number 23K10990.

Conflict of interests: None.

Ethical considerations: The Author assures of no violations of publication ethics and takes full responsibility for the content of the publication.

Declaration regarding the use of GAI tools: Not used.

Bulletin of the **S**ection of **L**ogic
Volume 54/2 (2025), pp. 207–226
https://doi.org/10.18778/0138-0680.2025.07

Takahiro Sawasaki

SEMANTIC INCOMPLETENESS OF LIBERMAN ET AL. (2020)'S HILBERT-STYLE SYSTEMS FOR TERM-MODAL LOGICS WITH EQUALITY AND NON-RIGID TERMS

Abstract

In this paper, we prove the semantic incompleteness of some expansions of the Hilbert-style system for the minimal normal term-modal logic with equality and non-rigid terms that were proposed in Liberman et al. (2020) "Dynamic Term-modal Logics for First-order Epistemic Planning." Term-modal logic is a family of first-order modal logics having term-modal operators indexed with terms in the first-order language. While some first-order formula is valid over the corresponding class of frames in the involved Kripke semantics, it is not provable in those expansions. We show this fact by introducing a non-standard Kripke semantics which makes the meanings of constants and function symbols relative to the meanings of relation symbols combined with them. We also address an incorrect frame correspondence result given in Liberman et al. (2020).

Keywords: incompleteness, term-modal logic, first-order modal logic.

Presented by: Michał Zawidzki
Received: December 31, 2024, **Received in revised form**: June 17, 2025,
Accepted: July 2, 2025, **Published online**: September 18, 2025

1. Introduction

This paper is an extended version of Sawasaki [10]. In [10] we proved the semantic incompleteness of Liberman et al. [8]'s Hilbert-style system $\mathsf{H}\mathbf{K}$ for the minimal normal term-modal logic $\mathbf{K}$ with equality and non-rigid terms. In this paper, we further prove the semantic incompleteness of all the expansions that [8] has obtained from $\mathsf{H}\mathbf{K}$ by adding some of axioms $\mathsf{T}, \mathsf{D}, \mathsf{4}$ and $\mathsf{5}$. This paper also addresses an incorrect frame correspondence result given in [8].

Term-modal logic, developed by Thalmann [12] and Fitting et al. [4], is a family of first-order modal logics having term-modal operators $[t]$ indexed with terms t in the first-order language. In the language of term-modal logic, for example, $[x]P(x)$, $[f(x)]P(x)$ and $\forall x[f(x)]P(x)$ are formulas. Term-modal logic is more expressive than multi-modal propositional logic and has been applied to epistemic logic in e.g. [7, 8, 13] and deontic logic in e.g. [11, 5, 6]. Some other developments of term-modal logic have been overviewed e.g. in [8, pp. 22–24] and [5, pp. 48–50].

The logics developed in Liberman et al. [8] are first-order dynamic epistemic logics for epistemic planning, and term-modal logic is invoked as its underlying logic. Technically speaking, their term-modal logics are two-sorted normal term-modal logics of the constant domain with equality and non-rigid terms. They make their logics two-sorted because, while letting the domain of a model include both agents and objects, they read an epistemically interpreted term-modal operator K_t as "*agent* t knows." The language defined in [8] allows $K_t\varphi$ to be a formula only if t is a term for an agent, and thereby excludes the possibility that terms denoting objects appear in the argument of the term-modal operator. In [8, p. 17], $\mathsf{H}\mathbf{K}$ and its expansions with $\mathsf{T}, \mathsf{D}, \mathsf{4}$ and $\mathsf{5}$ are claimed to be strongly complete with respect to the class of all the corresponding frames. Most of these results were originally presented in [1]. Later, two issues concerning action models and reduction axioms were fixed in the erratum [9] of [8].

Unfortunately, $\mathsf{H}\mathbf{K}$ and its expansions are semantically incomplete. In particular, the valid first-order formula $x = c \to (P(x) \to P(c))$ is unprovable. To be more precise, for a set $\Gamma \subseteq \{\, \mathsf{T}, \mathsf{D}, \mathsf{4}, \mathsf{5} \,\}$, we can prove that this formula is valid over the class of all frames to which Γ corresponds but is not provable in the system $\mathsf{HK}\Gamma$ obtained from HK by adding Γ. To this end, in Section 3 we introduce a non-standard Kripke semantics which makes the meanings of constants and function symbols relative to the meanings of relation symbols combined with them.

This paper will proceed as follows. In Section 2 we first introduce the syntax in [8]. Since there are some minor defects on the definitions for type, we do this with some modifications. Then we introduce the Kripke semantics and the Hilbert-style systems given in [8], addressing an incorrect frame correspondence result. In Section 3 we prove the semantic incompleteness of the expansions of HK by introducing a non-standard Kripke semantics.

2. Syntax, Semantics and Hilbert-style Systems

We will first introduce the syntax presented in [8, pp. 3–4] with some modifications. The idea there is to define the notions of term and formula while assigning (sequences of) types "`agt`", "`obj`" or "`agt_or_obj`" to all symbols like variables or relation symbols. It is basically the same idea as in Enderton [2, Section 4.3], but there is an important difference. In the syntax of [8], not only `agt` or `obj` but also `agt_or_obj` may be assigned to the arguments of function symbols and relation symbols, so that $P(x)$ seems to be intended to become a formula even when x has type `agt` and P takes type `agt_or_obj`.

However, the original definitions 1–3 for the syntax seem to have two minor defects. First, the original definition 1 for type assignment and the original definition 2 for term are dependent upon one another, thus they are circular definitions. Second, while $P(x)$ seems to be intended to become a formula when x has type `agt` and P takes type `agt_or_obj`, it does not actually become a formula since the original definition 3 for formula requires that the type of x and the type of the argument of P must be the same. Accordingly, for example, $x = x$ cannot be a formula in any signature since the type of x is either `agt` or `obj` but the type of the arguments of $=$ is always `agt_or_obj`.

To amend the above two defects, we redefine the syntax in [8, pp. 3–4] as follows.

Definition 2.1 (Signature). Let $\mathtt{Var}$ be a countably infinite set of *variables*, $\mathtt{Cn}$ a countable set of *constants*, $\mathtt{Fn}$ a countable set of *function symbols*, and $\mathtt{Rel}$ a countable set of *relation symbols* containing the *equality symbol* $=$. Let $\langle \mathtt{TYPE}, \preccurlyeq \rangle$ be also the ordered set of *types* where $\mathtt{TYPE} = \{ \mathtt{agt}, \mathtt{obj}, \mathtt{agt\_or\_obj} \}$ and $\preccurlyeq$ is the reflexive ordering on $\mathtt{TYPE}$ with $\mathtt{agt} \preccurlyeq \mathtt{agt\_or\_obj}$ and $\mathtt{obj} \preccurlyeq \mathtt{agt\_or\_obj}$, i.e.,

$$\preccurlyeq \,:=\, \{ \langle \tau, \tau \rangle \mid \tau \in \mathtt{TYPE} \} \cup \{ \langle \mathtt{agt}, \mathtt{agt\_or\_obj} \rangle, \langle \mathtt{obj}, \mathtt{agt\_or\_obj} \rangle \}.$$

A *type assignment* $\mathtt{t}\colon \mathtt{Var} \cup \mathtt{Cn} \cup \mathtt{Fn} \cup \mathtt{Rel} \to \bigcup_{n \in \mathbb{N}} \mathtt{TYPE}^n$ is an assignment mapping

1. a variable x to a type $\mathtt{t}(x) \in \{ \mathtt{agt}, \mathtt{obj} \}$ such that both $\mathtt{Var} \cap \mathtt{t}^{-1}[\{ \mathtt{agt} \}]$ and $\mathtt{Var} \cap \mathtt{t}^{-1}[\{ \mathtt{obj} \}]$ are countably infinite, where $\mathtt{t}^{-1}[X]$ is the inverse image of a set X;
2. a constant c to a type $\mathtt{t}(c) \in \{ \mathtt{agt}, \mathtt{obj} \}$;
3. a function symbol f to a sequence of types $\mathtt{t}(f) \in \mathtt{TYPE}^n \times \{ \mathtt{agt}, \mathtt{obj} \}$ for some $n \in \mathbb{N}$;
4. the equality symbol $=$ to the sequence of types $\mathtt{t}(=) = \langle \mathtt{agt\_or\_obj}, \mathtt{agt\_or\_obj} \rangle$;
5. a relation symbol P distinct from $=$ to a sequence of types $\mathtt{t}(P) \in \mathtt{TYPE}^n$ for some $n \in \mathbb{N}$.

The tuple $\langle \mathtt{Var}, \mathtt{Cn}, \mathtt{Fn}, \mathtt{Rel}, \mathtt{t} \rangle$ is called a *signature*.

Definition 2.2 (Term of Type). Let $\langle \mathtt{Var}, \mathtt{Cn}, \mathtt{Fn}, \mathtt{Rel}, \mathtt{t} \rangle$ be a signature. The set of *terms of types* is defined as follows.

1. any variable $x \in \mathtt{Var}$ is a term of type $\mathtt{t}(x)$.
2. any constant $c \in \mathtt{Cn}$ is a term of type $\mathtt{t}(c)$.
3. If $t_1, \ldots, t_n$ are terms of types $\tau_1, \ldots, \tau_n$ and f is a function symbol in $\mathtt{Fn}$ such that $\mathtt{t}(f) = \langle \tau'_1, \ldots, \tau'_n, \tau'_{n+1} \rangle$ and $\tau_i \preccurlyeq \tau'_i$, then $f(t_1, \ldots, t_n)$ is a term of type τ'_{n+1}.

For convenience, henceforth we use a type assignment $\mathtt{t}$ to mean its uniquely extended assignment by letting $\mathtt{t}(f(t_1, \ldots, t_n)) = \tau$ for each term of the form $f(t_1, \ldots, t_n)$ of type τ.

Definition 2.3 (Language). Let $\langle \mathtt{Var}, \mathtt{Cn}, \mathtt{Fn}, \mathtt{Rel}, \mathtt{t} \rangle$ be a signature. The *language* is the set of formulas φ defined in the following BNF.

$$\varphi ::= P(t_1, \ldots, t_n) \mid \neg\varphi \mid \varphi \wedge \varphi \mid K_s\varphi \mid \forall x\varphi,$$

where $t_1, \ldots, t_n, s$ are terms with $\mathtt{t}(s) = \mathtt{agt}$ and $P \in \mathtt{Rel}$ such that $\mathtt{t}(P) = \langle \tau_1, \ldots, \tau_n \rangle$ and $\mathtt{t}(t_i) \preccurlyeq \tau_i$. Note here that P can be $=$.

As usual, we use the notations $t \neq s := \neg(t = s)$, $\varphi \to \psi := \neg(\varphi \wedge \neg\psi)$, $\exists x\varphi := \neg\forall x\neg\varphi$, $\bot := P \wedge \neg P$ for some fixed nullary relation symbol P, and $\top := \neg\bot$.

We believe that our definitions successfully capture what was intended in the original definitions 1–3. On top of these definitions, we will follow [8, p. 4] to define the notions of *free variable* and *bound variable* in a formula as usual, where the set of free variables in $K_t\varphi$ is defined as the union of the set of variables in t and the set of free variables in φ. For a variable x, terms t, s and a formula φ such that $\mathtt{t}(x) = \mathtt{t}(s)$ and no variables in s are bound variables in φ, we also define *substitutions* $t(s/x)$ and $\varphi(s/x)$ of s for x in t and φ in a usual manner, except that $(K_t\varphi)(s/x) = K_{t(s/x)}\varphi(s/x)$. Whenever we write $t(s/x)$ or $\varphi(s/x)$, we tacitly assume that $\mathtt{t}(x) = \mathtt{t}(s)$ and no variables in s are bound variables in φ. We also define the lengths of term and formula as usual.

Let us now introduce the Kripke semantics presented in [8, pp. 5–6].

Definition 2.4 (Frame, [8, Def. 4]). A *frame* is a tuple $F = \langle D, W, R \rangle$ where

1. $D := D_{\mathtt{agt\_or\_obj}} := D_{\mathtt{agt}} \sqcup D_{\mathtt{obj}}$ is the disjoint union of a non-empty set $D_{\mathtt{agt}}$ of *agents* and a non-empty set $D_{\mathtt{obj}}$ of *objects*;

2. W is a non-empty set of *worlds*;

3. R is a mapping that assigns to each agent $i \in D_{\mathtt{agt}}$ a binary relation R_i on W, i.e., $R\colon D_{\mathtt{agt}} \to \mathcal{P}(W \times W)$.

Definition 2.5 (Model, [8, Def. 5]). Let $\langle \mathtt{Var}, \mathtt{Cn}, \mathtt{Fn}, \mathtt{Rel}, \mathtt{t} \rangle$ be a signature. A *model* is a tuple $M = \langle D, W, R, I \rangle$ where $\langle D, W, R \rangle$ is a frame and I is an *interpretation* that maps

1. a pair $\langle c, w \rangle$ of some $c \in \mathtt{Cn}$ and $w \in W$ to an element $I(c, w) \in D_{\mathtt{t}(c)}$;
2. a pair $\langle f, w \rangle$ of some $f \in \mathtt{Fn}$ and $w \in W$ to a function $I(f, w) \colon (D_{\tau_1} \times \cdots \times D_{\tau_n}) \to D_{\tau_{n+1}}$, where $\mathtt{t}(f) = \langle \tau_1, \ldots, \tau_n, \tau_{n+1} \rangle$;
3. a pair $\langle =, w \rangle$ of the equality symbol $=$ and some $w \in W$ to the set $I(=, w) = \left\{ \langle d, d \rangle \mid d \in D_{\mathtt{agt\_or\_obj}} \right\}$;
4. a pair $\langle P, w \rangle$ of some $P \in \mathtt{Rel} \setminus \{=\}$ and $w \in W$ to a subset $I(P, w)$ of $D_{\tau_1} \times \cdots \times D_{\tau_n}$, where $\mathtt{t}(P) = \langle \tau_1, \ldots, \tau_n \rangle$.

Definition 2.6 (Valuation, [8, Def. 6, 7]). Let $\langle \mathtt{Var}, \mathtt{Cn}, \mathtt{Fn}, \mathtt{Rel}, \mathtt{t} \rangle$ be a signature. A *valuation* is a mapping $\sigma \colon \mathtt{Var} \to D$ such that $\sigma(x) \in D_{\mathtt{t}(x)}$ and the valuation $\sigma[x \mapsto d]$ is the same valuation as σ except for assigning to a variable x an element $d \in D_{\mathtt{t}(x)}$. Given a valuation σ, a world w and an interpretation I in a model, the extension $[\![t]\!]_w^{I,\sigma}$ of a term t is defined by $[\![x]\!]_w^{I,\sigma} = \sigma(x)$, $[\![c]\!]_w^{I,\sigma} = I(c, w)$, and $[\![f(t_1, \ldots, f_n)]\!]_w^{I,\sigma} = I(f, w)([\![t_1]\!]_w^{I,\sigma}, \ldots, [\![t_n]\!]_w^{I,\sigma})$.

Definition 2.7 (Satisfaction, [8, Def. 8]). The *satisfaction* $M, w \models_\sigma \varphi$ of a formula φ at a world w in a model M under a valuation σ is defined as follows.

$$
\begin{array}{lll}
M, w \models_\sigma P(t_1, \ldots, t_n) & \text{iff} & \langle [\![t_1]\!]_w^{I,\sigma}, \ldots, [\![t_n]\!]_w^{I,\sigma} \rangle \in I(P, w) \quad (P \text{ can be } =) \\
M, w \models_\sigma \neg\varphi & \text{iff} & M, w \not\models_\sigma \varphi \\
M, w \models_\sigma \varphi \wedge \psi & \text{iff} & M, w \models_\sigma \varphi \quad \text{and} \quad M, w \models_\sigma \psi \\
M, w \models_\sigma \forall x \varphi & \text{iff} & M, w \models_{\sigma[x \mapsto d]} \varphi \quad \text{for all } d \in D_{\mathtt{t}(x)} \\
M, w \models_\sigma K_t \varphi & \text{iff} & M, w' \models_\sigma \varphi \quad \text{for all } w' \in W \text{ such that} \\
 & & \langle w, w' \rangle \in R_{[\![t]\!]_w^{I,\sigma}}
\end{array}
$$

Definition 2.8 (Validity, [8, p. 25]). A formula φ is *valid over a frame* F if for all models M based on F, all worlds $w \in W$ and all valuations σ, it holds that $M, w \models_\sigma \varphi$. A formula φ is *valid over a class* $\mathbb{F}$ *of frames* if for all frames $F \in \mathbb{F}$, φ is valid over F.

Remark 2.9. Instead of the x-variant of a valuation σ used in [8], we adopted the notion of $\sigma[x \mapsto d]$ to give the satisfaction for $\forall x\varphi$. This change is just for the clarity of our proof and does not affect the satisfiability of formulas.

For ease of reference, henceforth we call this semantics *TML-semantics*.

To state precisely our result on the semantic incompleteness in the next section, we import the frame correspondence results in [8, p. 18] with some modifications. Let us define the notion of frame correspondence as usual and say that a frame $\langle W, D, R\rangle$ is reflexive, serial, transitive or euclidean if R_i is so for all $i \in D_{\mathtt{agt}}$. We then have the following frame correspondences.

Proposition 2.10. Let $\langle \mathtt{Var}, \mathtt{Cn}, \mathtt{Fn}, \mathtt{Rel}, \mathtt{t}\rangle$ be a signature and $x \in \mathtt{Var}$ with $\mathtt{t}(x) = \mathtt{agt}$.

1. $\mathsf{T} := \forall x(K_x\varphi \to \varphi)$ corresponds to the class of all reflexive frames.
2. $\mathsf{D} := \forall x \neg K_x \bot$ corresponds to the class of all serial frames.
3. $\mathsf{4} := \forall x(K_x\varphi \to K_x K_x \varphi)$ corresponds to the class of all transitive frames.
4. $\mathsf{5} := \forall x(\neg K_x\varphi \to K_x \neg K_x \varphi)$ corresponds to the class of all euclidean frames.

We can find this proposition in [8, p. 18] if $\mathtt{t}(x)$ is supposed to be $\mathtt{agt}$.

In addition to Proposition 2.10, however, [8] also makes the following claims at the same page, where $\#X$ denotes the cardinality of a set X:[1]

(a) $\mathsf{AO}_y^{\vec{x}_n} := \exists x_1 \cdots x_n ((\bigwedge_{i<j\leq n} x_i \neq x_j) \wedge \forall y \bigvee_{i \leq n} y = x_i)$ corresponds to the class of all frames such that $\#D = n$;

(b) $\mathsf{A}_y^{\vec{x}_n} := \exists x_1 \cdots x_n ((\bigwedge_{i \leq n} K_{x_i} \top) \wedge (\bigwedge_{i<j\leq n} x_i \neq x_j) \wedge \forall y (K_y \top \to \bigvee_{i \leq n} y = x_i))$ corresponds to the class of all frames such that $\#D_{\mathtt{agt}} = n$,

[1] In [8, p. 18], $\mathsf{AO}_y^{\vec{x}_n}$ and $\mathsf{A}_y^{\vec{x}_n}$ are called M and N, respectively.

Amongst these, (a) is false in the current two-sorted language. Consider a signature such that all of $x_1, \ldots, x_n, y$ have type $\mathtt{agt}$. Then, it is easy to check that the validity of $\mathsf{AO}_y^{\vec{x}_n}$ over a frame entails $\#D_{\mathtt{agt}} = n$. Hence it follows from $D_{\mathtt{obj}} \neq \emptyset$ that $n < \#D_{\mathtt{agt}} + \#D_{\mathtt{obj}} = \#D$, which contradicts (a). The frame property to which $\mathsf{AO}_y^{\vec{x}_n}$ corresponds is actually as follows, where $|y|$ denotes the set $\{\, x_i \mid \mathtt{t}(y) = \mathtt{t}(x_i), 1 \leq i \leq n \,\}$ and $\overline{\tau}$ denotes the converse type of a type τ.

Proposition 2.11. Let $\langle \mathtt{Var}, \mathtt{Cn}, \mathtt{Fn}, \mathtt{Rel}, \mathtt{t} \rangle$ be a signature such that x_1, …, x_n, y are pairwise distinct variables. Then $\mathsf{AO}_y^{\vec{x}_n}$ corresponds to the class of all frames such that $\#D_{\mathtt{t}(y)} = \#|y|$ and $\#D_{\overline{\mathtt{t}(y)}} \geq (n - \#|y|)$.

Proof: Suppose that $\mathsf{AO}_y^{\vec{x}_n}$ is valid over a frame F. Taking a world w, an interpretation I and a valuation σ arbitrarily, we obtain $\langle F, I \rangle, w \models_\sigma \mathsf{AO}_y^{\vec{x}_n}$. We then have some pairwise distinct elements $d_1, \ldots, d_{\#|y|} \in D_{\mathtt{t}(y)}$ and $d_{\#|y|+1}, \ldots, d_n \in D_{\overline{\mathtt{t}(y)}}$ such that for all $d \in D_{\mathtt{t}(y)}$ the following disjunction holds: $d = d_1$ or $\cdots$ or $d = d_{\#|y|}$ or $d = d_{\#|y|+1}$ or $\cdots$ or $d = d_n$. Since $d \in D_{\mathtt{t}(y)}$ and $d_{\#|y|+1}, \ldots, d_n \in D_{\overline{\mathtt{t}(y)}}$, this disjunction is equivalent to the disjunction that $d = d_1$ or $\cdots$ or $d = d_{\#|y|}$. Thus, $\#D_{\mathtt{t}(y)} = \#|y|$ and $\#D_{\overline{\mathtt{t}(y)}} \geq (n - \#|y|)$.

For the other direction, suppose $\#D_{\mathtt{t}(y)} = \#|y|$ and $\#D_{\overline{\mathtt{t}(y)}} \geq (n - \#|y|)$. We show that $\langle F, I \rangle, w \models_\sigma \mathsf{AO}_y^{\vec{x}_n}$ for any interpretation I, any world w and any valuation σ. By our supposition we have some pairwise distinct elements $d_1, \ldots, d_{\#|y|} \in D_{\mathtt{t}(y)}$ and $d_{\#|y|+1}, \ldots, d_n \in D_{\overline{\mathtt{t}(y)}}$ such that $D_{\mathtt{t}(y)} = \{\, d_1, \ldots, d_{\#|y|} \,\}$. Note here that $\mathtt{t}(x_{k_1}) = \cdots = \mathtt{t}(x_{k_{\#|y|}}) = \mathtt{t}(y)$ and $\mathtt{t}(x_{k_{\#|y|+1}}) = \cdots = \mathtt{t}(x_{k_n}) = \overline{\mathtt{t}(y)}$ for some variables $x_{k_1}, \ldots, x_{k_n}$ coming from $x_1, \ldots, x_n$. Letting $\sigma' = \sigma[x_{k_1} \mapsto d_1] \cdots [x_{k_n} \mapsto d_n]$ and $\sigma'' = \sigma[x_1 \mapsto \sigma'(x_1)] \cdots [x_n \mapsto \sigma'(x_n)]$, we have $\langle F, I \rangle, w \models_{\sigma''} \bigwedge_{i<j\leq n} x_i \neq x_j$. On top of that, we have $\langle F, I \rangle, w \models_{\sigma''} \forall y \bigvee_{i \leq \#|y|} y = x_{k_i}$ hence $\langle F, I \rangle, w \models_{\sigma''} \forall y \bigvee_{i \leq n} y = x_i$. Thus, $\langle F, I \rangle, w \models_\sigma \mathsf{AO}_y^{\vec{x}_n}$. $\square$

On the other hand, (b) is true on some reading, but then just a corollary of Proposition 2.11. For, (b) can be true only if we presuppose on the meta-level that $\mathsf{A}_y^{\vec{x}_n}$ is a well-formed formula under a given signature Σ, which means after all to presuppose at the meta-level that all of $x_1, \ldots, x_n$ have type $\mathtt{agt}$ under Σ. Then $\mathsf{A}_y^{\vec{x}_n}$ is just an equivalent variant of $\mathsf{AO}_y^{\vec{x}_n}$ corresponding to the class of all frames such that $\#D_{\mathtt{agt}} = n$, as shown below.

Corollary 2.12. Let $\langle \mathtt{Var}, \mathtt{Cn}, \mathtt{Fn}, \mathtt{Rel}, \mathtt{t}\rangle$ be a signature such that $x_1, \ldots,$ x_n, y are pairwise distinct variables with $\mathtt{t}(x_1) = \cdots = \mathtt{t}(x_n) = \mathtt{t}(y) = \mathtt{agt}$. Then each of $\mathsf{AO}_y^{\vec{x}_n}$ and $\mathsf{A}_y^{\vec{x}_n}$ corresponds to the class of all frames such that $\#D_{\mathtt{agt}} = n$.

Thus we may treat $\mathsf{A}_y^{\vec{x}_n}$ as $\mathsf{AO}_y^{\vec{x}_n}$ with $\mathtt{t}(x_1) = \cdots = \mathtt{t}(x_n) = \mathtt{t}(y) = \mathtt{agt}$.

Axioms

	all propositional tautologies		
UE	$\forall x\varphi \to \varphi(y/x)$	K	$K_t(\varphi \to \psi) \to (K_t\varphi \to K_t\psi)$
Id	$t = t$	BF†	$\forall x K_t\varphi \to K_t\forall x\varphi$
PS	$x = y \to (\varphi(x/z) \to \varphi(y/z))$	KNI	$x \neq y \to K_t x \neq y$
∃Id	$c = c \to \exists x(x = c)$		
DD	$x \neq y$ if $\mathtt{t}(x) \neq \mathtt{t}(y)$		

Inference rules

MP	From φ and $\varphi \to \psi$, infer ψ
KG	From φ, infer $K_t\varphi$
UG‡	From $\varphi \to \psi$, infer $\varphi \to \forall x\psi$

†: x does not occur in t and ‡: x is not free in φ.

Table 1: The Hilbert-style system $\mathsf{H}\mathbf{K}$ for term-modal logic $\mathbf{K}$

Finally, we introduce the Hilbert-style system $\mathsf{H}\mathbf{K}$ for the minimal normal term-modal logic $\mathbf{K}$ and some expansions of it presented in [8, pp. 17–18]. The Hilbert-style system $\mathsf{H}\mathbf{K}$ is defined as in Table 1. For its expansions, put $\mathrm{AX} = \{\mathsf{T}, \mathsf{D}, \mathsf{4}, \mathsf{5}\}$ throughout the paper. Given a set $\Gamma \subseteq \mathrm{AX}$, the Hilbert-style system $\mathsf{H}\mathbf{K}\Gamma$ is defined to be the system obtained from $\mathsf{H}\mathbf{K}$ by adding all axioms in Γ. The notion of proof in $\mathsf{H}\mathbf{K}\Gamma$

is defined as usual. It is easy to check that $\mathsf{H}\mathbf{K}\mathrm{AX}$ and $\mathsf{H}\mathbf{S5} \coloneqq \mathsf{H}\mathbf{K}\{\,\mathsf{T},\mathsf{5}\,\}$ have the same provability.

Remark 2.13. In [8, Theorem 3], the strong completeness of some expansions of $\mathsf{H}\mathbf{K}\Gamma$ with $\mathsf{AO}_{y}^{\vec{x}_n}$ and $\mathsf{A}_{y'}^{\vec{x'}_{n'}}$ is also claimed. Since there are some signatures and some $n, n' \geq 1$ such that the addition of $\mathsf{AO}_{y}^{\vec{x}_n}$ and $\mathsf{A}_{y'}^{\vec{x'}_{n'}}$ make $\mathsf{H}\mathbf{K}\Gamma$ inconsistent, in those cases we cannot prove the semantic incompleteness of those expansions. On the other hand, the side conditions of $\mathsf{AO}_{y}^{\vec{x}_n}$ and $\mathsf{A}_{y'}^{\vec{x'}_{n'}}$ to keep the consistency of $\mathsf{H}\mathbf{K}\Gamma \cup \{\,\mathsf{AO}_{y}^{\vec{x}_n}, \mathsf{A}_{y'}^{\vec{x'}_{n'}}\,\}$ are somewhat complicated.[2] Thus, for simplicity we confine ourselves to the expansions with axioms in AX.

Before going to the next section, we note that UE and PS are particularly relevant to the semantic incompleteness of $\mathsf{H}\mathbf{K}\Gamma$. As remarked in Fagin et al. [3, pp. 88–89], the ordinary first-order axioms $\forall x\varphi \to \varphi(t/x)$ and $t = s \to (\varphi(t/z) \to \varphi(s/z))$ are not valid in Kripke semantics for first-order modal logic where constants or function symbols are interpreted as non-rigid. Probably because of this reason, [8] instead adopted UE and PS that are the variable-restricted versions of these ordinary first-order axioms. The problem is that UE and PS or their combinations with other axioms are not sufficient to derive a valid formula $x = c \to (P(x) \to P(c))$ over the class of all frames.

3. Semantic Incompleteness

In this section, for all $\Gamma \subseteq \mathrm{AX}$ we prove the semantic incompleteness of $\mathsf{H}\mathbf{K}\Gamma$ by showing that $x = c \to (P(x) \to P(c))$ is valid over the class of all frames to which Γ corresponds in the TML-semantics but not provable in $\mathsf{H}\mathbf{K}\Gamma$. For the former, since any valid formula over the class of all frames is also valid over the class of all frames to which Γ corresponds, it is sufficient to show that $x = c \to (P(x) \to P(c))$ is valid over the class of all frames in the TML-semantics. As expected, it is straightforward to show this fact.

[2] The following two side conditions are at least necessary: (1) if $\mathtt{t}(y) = \mathtt{t}(y')$ then $\#|y| = \#|y'|$; (2) if $\mathtt{t}(y) \neq \mathtt{t}(y')$ then $\#|y| \geq (n' - \#|y'|)$ and $\#|y'| \geq (n - \#|y|)$.

Proposition 3.1. Let $\langle \mathtt{Var}, \mathtt{Cn}, \mathtt{Fn}, \mathtt{Rel}, \mathtt{t} \rangle$ be a signature, $x \in \mathtt{Var}$, $c \in \mathtt{Cn}$ and $P \in \mathtt{Rel}$ with $\mathtt{t}(P) = \langle \mathtt{agt\_or\_obj} \rangle$. A formula $x = c \to (P(x) \to P(c))$ is valid over the class of all frames in the TML-semantics.

For the latter, since any unprovable formula in **HS5** is also unprovable in $\mathbf{HK}\Gamma$, it is sufficient to show that $x = c \to (P(x) \to P(c))$ is not provable in **HS5**. To this end, we introduce a new semantics in which **HS5** is sound with respect to the class of all reflexive, symmetric and transitive frames but $x = c \to (P(x) \to P(c))$ is not valid over this class of frames.

Definition 3.2 (Non-standard Model). Let $\langle \mathtt{Var}, \mathtt{Cn}, \mathtt{Fn}, \mathtt{Rel}, \mathtt{t} \rangle$ be a signature. A *non-standard model* is a tuple $N = \langle D, W, R, J \rangle$ where $\langle D, W, R \rangle$ is a frame in the sense of Definition 2.4 and J is an interpretation that maps

1. a triple $\langle c, w, X \rangle$ of some $c \in \mathtt{Cn}$, some $w \in W$ and some $X \subseteq D^n$ for some $n \in \mathbb{N}$ to an element $J(c, w, X) \in D_{\mathtt{t}(c)}$;
2. a triple $\langle f, w, X \rangle$ of some $f \in \mathtt{Fn}$, some $w \in W$ and some $X \subseteq D^n$ for some $n \in \mathbb{N}$ to a function $J(f, w, X)\colon (D_{\tau_1} \times \cdots \times D_{\tau_n}) \to D_{\tau_{n+1}}$, where $\mathtt{t}(f) = \langle \tau_1, \ldots, \tau_{n+1} \rangle$;
3. a pair $\langle =, w \rangle$ of the equality symbol $=$ and some $w \in W$ to the set $J(=, w) = \{\, \langle d, d \rangle \mid d \in D_{\mathtt{agt\_or\_obj}} \,\}$;
4. a pair $\langle P, w \rangle$ of some $P \in \mathtt{Rel} \setminus \{=\}$ and some $w \in W$ to a subset $J(P, w)$ of $D_{\tau_1} \times \cdots \times D_{\tau_n}$, where $\mathtt{t}(P) = \langle \tau_1, \ldots, \tau_n \rangle$.

Here is the intuition. A subset X of D^n is a set of sequences consisting of either/both of agents and objects. Thus, the set X mentioned in the meanings $J(c, w, X)$ and $J(f, w, X)$ of a constant c and a function symbol f can serve as the meaning of a relation symbol. This trick enables us to make the meanings of constants and function symbols relative to the meanings of relation symbols combined with them.

We then define the notion of satisfaction of formulas in non-standard models. In what follows, we use the same notion of valuation as in the TML-semantics and define the extension $[\![t]\!]^{J,\sigma}_{w,X}$ of a term t in a given non-standard model similarly by letting $[\![x]\!]^{J,\sigma}_{w,X} = \sigma(x)$, $[\![c]\!]^{J,\sigma}_{w,X} = J(c, w, X)$ and $[\![f(t_1, \ldots, t_n)]\!]^{J,\sigma}_{w,X} = J(f, w, X)([\![t_1]\!]^{J,\sigma}_{w,X}, \ldots, [\![t_n]\!]^{J,\sigma}_{w,X})$.

Definition 3.3 (Satisfaction in Non-standard Model). The *satisfaction* $N, w \models_\sigma \varphi$ of a formula φ at a world w in a non-standard model N under a valuation σ is defined as follows.

$$\begin{array}{lll}
N, w \models_\sigma P(t_1, \ldots, t_n) & \text{iff} & \langle [\![t_1]\!]^{J,\sigma}_{w,J(P,w)}, \ldots, [\![t_n]\!]^{J,\sigma}_{w,J(P,w)} \rangle \in J(P, w) \\
 & & (P \text{ can be } =) \\
N, w \models_\sigma \neg\varphi & \text{iff} & N, w \not\models_\sigma \varphi \\
N, w \models_\sigma \varphi \wedge \psi & \text{iff} & N, w \models_\sigma \varphi \quad \text{and} \quad N, w \models_\sigma \psi \\
N, w \models_\sigma \forall x \varphi & \text{iff} & N, w \models_{\sigma[x \mapsto d]} \varphi \quad \text{for all } d \in D_{\mathtt{t}(x)} \\
N, w \models_\sigma K_t \varphi & \text{iff} & N, w' \models_\sigma \varphi \quad \text{for all } w' \in W \text{ such that} \\
 & & \langle w, w' \rangle \in R_{[\![t]\!]^{J,\sigma}_{w,\emptyset}}
\end{array}$$

What we should pay attention here is the satisfactions of atomic formula $P(t_1, \ldots, t_n)$ and term-modal formula $K_t\varphi$. In the satisfaction of $P(t_1, \ldots, t_n)$ in non-standard models, the meaning $[\![t_i]\!]^{J,\sigma}_{w,J(P,w)}$ of each t_i in $P(t_1, \ldots, t_n)$ is determined by the interpretation J, the valuation σ, the world w and *the meaning* $J(P, w)$ *of the relation symbol* P *combined with terms* $t_1, \ldots, t_n$. Thus, as explained in the following Example 3.4, the meaning of a constant c occurring in $P(c)$ could be different from that of c occurring in $Q(c)$.

Example 3.4. Let $lewis \in \mathtt{Cn}$ with $\mathtt{t}(lewis) = \mathtt{agt}$ and $SL, CF \in \mathtt{Rel}$ with $\mathtt{t}(SL) = \mathtt{t}(CF) = \langle \mathtt{agt} \rangle$, and consider a non-standard model such that $J(SL, w) = \{i \in D_{\mathtt{agt}} \mid i \text{ is one of the authors of } \textit{Symbolic Logic}\}$, $J(CF, w) = \big\{ i \in D_{\mathtt{agt}} \mid i \text{ is the author of } \textit{Counterfactuals} \big\}$, $J(lewis, w, J(SL, w))$ is C. I. Lewis and $J(lewis, w, J(CF, w))$ is D. Lewis. Then, the meaning $J(lewis, w, J(SL, w))$ of *lewis* occurring in $SL(lewis)$ is different from the meaning $J(lewis, w, J(CF, w))$ of *lewis* occurring in $CF(lewis)$. Note that, although $J(lewis, w, J(SL, w)) \in J(SL, w)$ in the above non-standard model, we can technically assign to $J(lewis, w, J(SL, w))$ D. Lewis to have a non-standard model such that $J(lewis, w, J(SL, w)) \notin J(SL, w)$.

On the other hand, because the meaning $[\![t]\!]^{J,\sigma}_{w,\emptyset}$ of t in K_t is determined independently of the meaning of any relation symbol, the satisfaction of $K_t\varphi$ in non-standard model is in effect the same as the satisfaction of $K_t\varphi$ in models of the TML-semantics. By this fact we can validate axioms K and BF in this semantics.

The notions of validity is defined as in the TML-semantics. For ease of reference, henceforth we call this semantics *non-standard semantics*.

Now it is easy to see that $x = c \to (P(x) \to P(c))$ is not valid over the class of all reflexive, symmetric and transitive frames in the non-standard semantics.

PROPOSITION 3.5. Let $\langle \mathtt{Var}, \mathtt{Cn}, \mathtt{Fn}, \mathtt{Rel}, \mathtt{t}\rangle$ be a signature, $x \in \mathtt{Var}$, $c \in \mathtt{Cn}$ with $\mathtt{t}(x) = \mathtt{t}(c)$ and $P \in \mathtt{Rel}$ with $\mathtt{t}(P) = \langle \mathtt{agt\_or\_obj}\rangle$. A formula $x = c \to (P(x) \to P(c))$ is not valid over the class of all reflexive, symmetric and transitive frames in the non-standard semantics.

PROOF: We may assume $\mathtt{t}(x) = \mathtt{t}(c) = \mathtt{agt}$ without loss of generality. Let $N = \langle D, W, R, J\rangle$ be a non-standard model such that $w \in W$, R_i is reflexive, symmetric and transitive for all $i \in D_{\mathtt{agt}}$, $D_{\mathtt{agt}} = \{\alpha, \beta\}$, $J(c, w, \{\langle d, d\rangle \mid d \in D_{\mathtt{agt\_or\_obj}}\}) = \alpha$, $J(c, w, \{\alpha\}) = \beta$ and $J(P, w) = \{\alpha\}$. Let σ be also a valuation such that $\sigma(x) = \alpha$. Since $[\![x]\!]^{J,\sigma}_{w,J(=,w)} = \sigma(x) = \alpha = J(c, w, \{\langle d, d\rangle \mid d \in D_{\mathtt{agt\_or\_obj}}\}) = J(c, w, J(=, w)) = [\![c]\!]^{J,\sigma}_{w,J(=,w)}$, we have $N, w \models_\sigma x = c$. It is also easy to see $N, w \models_\sigma P(x)$. However, since $[\![c]\!]^{J,\sigma}_{w,J(P,w)} = J(c, w, J(P, w)) = J(c, w, \{\alpha\}) = \beta$, it fails that $N, w \models_\sigma P(c)$. Therefore $x = c \to (P(x) \to P(c))$ is not valid over the class of all reflexive, symmetric and transitive frames in the non-standard semantics. $\square$

On the other hand, we can use the following lemmas to prove the soundness of $\mathsf{HS5}$ in the non-standard semantics, and thereby obtain the unprovability of $x = c \to (P(x) \to P(c))$ in $\mathsf{HS5}$.

LEMMA 3.6. *Let $\langle \mathtt{Var}, \mathtt{Cn}, \mathtt{Fn}, \mathtt{Rel}, \mathtt{t}\rangle$ be a signature and $x, y \in \mathtt{Var}$ with $\mathtt{t}(x) = \mathtt{t}(y)$. Let $\langle D, W, R, J\rangle$ be also a non-standard model, w a world, X a subset of D^n for some $n \in \mathbb{N}$ and σ a valuation. For all terms t,*

$$[\![t(y/x)]\!]^{J,\sigma}_{w,X} \quad = \quad [\![t]\!]^{J,\sigma[x \mapsto \sigma(y)]}_{w,X}.$$

Proof: By induction on the length of terms. □

Lemma 3.7. *Let* $\langle \mathtt{Var}, \mathtt{Cn}, \mathtt{Fn}, \mathtt{Rel}, \mathtt{t}\rangle$ *be a signature,* $x, y \in \mathtt{Var}$ *with* $\mathtt{t}(x) = \mathtt{t}(y)$ *and* $N = \langle D, W, R, J\rangle$ *a non-standard model. For all worlds* w*, all valuations* σ *and all formulas* φ*,*

$$N, w \models_\sigma \varphi(y/x) \quad \textit{iff} \quad N, w \models_{\sigma[x \mapsto \sigma(y)]} \varphi.$$

Proof: By induction on the length of formulas. □

Theorem 3.8 (Soundness). *If* φ *is provable in* $\mathsf{HS5}$*, then* φ *is valid over the class of all reflexive, symmetric and transitive frames in the non-standard semantics.*

Proof: It is sufficient to prove that all axioms in $\mathsf{HS5}$ are valid and that all inference rules preserve validity. Since the proof of the latter is done as usual, we see only the former.

- For any propositional tautology, its validity is obvious since the non-standard semantics gives the ordinary satisfactions for $\neg$ and $\wedge$.
- For UE, i.e., $\forall x\varphi \to \varphi(y/x)$, suppose $N, w \models_\sigma \forall x\varphi$. Then we have $N, w \models_{\sigma[x\mapsto\sigma(y)]} \varphi$. Thus by Lemma 3.7 $N, w \models_\sigma \varphi(y/x)$ holds, as required.
- For Id, i.e., $t = t$, its validity is obvious.
- For PS, i.e., $x = y \to (\varphi(x/z) \to \varphi(y/z))$, its validity is shown by induction on φ.
  - For φ being of the form $P(t_1, \dots, t_n)$, suppose $N, w \models_\sigma x = y$ and $N, w \models_\sigma P(t_1, \dots, t_n)(x/z)$. Since

    $$\langle [\![t_1(x/z)]\!]^{J,\sigma}_{w,J(P,w)}, \dots, [\![t_n(x/z)]\!]^{J,\sigma}_{w,J(P,w)}\rangle \in J(P, w),$$

    we can use $\sigma(x) = \sigma(y)$ and Lemma 3.6 to obtain

    $$\langle [\![t_1(y/z)]\!]^{J,\sigma}_{w,J(P,w)}, \dots, [\![t_n(y/z)]\!]^{J,\sigma}_{w,J(P,w)}\rangle \in J(P, w).$$

    Thus $N, w \models_\sigma P(t_1, \dots, t_n)(y/z)$.

- For φ being of the forms $\neg\psi$ or $\psi\wedge\gamma$, the proof is straightforward.
- For φ being of the form $\forall z'\psi$, notice first that $z' \neq x$ and $z' \neq y$ since x, y are assumed not to be bound in φ whenever we write $\varphi(x/z)$ and $\varphi(y/z)$. Suppose $N, w \models_\sigma x = y$ and $N, w \models_\sigma (\forall z'\psi)(x/z)$. If $z' = z$, obviously $N, w \models_\sigma (\forall z'\psi)(y/z)$. If $z' \neq z$, then we have $N, w \models_\sigma \forall z'\psi(x/z)$ thus $N, w \models_{\sigma[z'\mapsto d]} \psi(x/z)$ for all $d \in D_{\mathtt{t}(z')}$. Since we have $N, w \models_{\sigma[z'\mapsto d]} x = y$ for all $d \in D_{\mathtt{t}(z')}$, by the inductive hypothesis we obtain $N, w \models_{\sigma[z'\mapsto d]} \psi(y/z)$ for all $d \in D_{\mathtt{t}(z')}$. Therefore, $N, w \models_\sigma (\forall z'\psi)(y/z)$.
- For φ being of the form $K_t\psi$, suppose $N, w \models_\sigma x = y$ and $N, w \models_\sigma (K_t\psi)(x/z)$. Then $N, w' \models_\sigma \psi(x/z)$ for all $w' \in W$ such that $\langle w, w'\rangle \in R_{[\![t(x/z)]\!]^{J,\sigma}_{w,\emptyset}}$. Now, we have $N, w' \models_\sigma x = y$ for all $w' \in W$, and $[\![t(x/z)]\!]^{J,\sigma}_{w,\emptyset} = [\![t(y/z)]\!]^{J,\sigma}_{w,\emptyset}$ by $\sigma(x) = \sigma(y)$ and Lemma 3.6. So by the inductive hypothesis we obtain $N, w' \models_\sigma \psi(y/z)$ for all $w' \in W$ such that $\langle w, w'\rangle \in R_{[\![t(y/z)]\!]^{J,\sigma}_{w,\emptyset}}$. Thus, $N, w \models_\sigma (K_t\psi)(y/z)$.

• For $\exists$Id, i.e., $c = c \to \exists x(x = c)$, notice that $N, w \models_{\sigma[x\mapsto J(c,w,J(=,w))]} x = c$. Then we have $N, w \models_\sigma \exists x(x = c)$ hence $N, w \models_\sigma c = c \to \exists x(x = c)$, as required.

• For DD, i.e., $x \neq y$ if $\mathtt{t}(x) \neq \mathtt{t}(y)$, suppose $\mathtt{t}(x) \neq \mathtt{t}(y)$ and let N, w and σ be arbitrary. By the definition of valuation, each of $\sigma(x)$ and $\sigma(y)$ is in $D_{\mathtt{t}(x)}$ and $D_{\mathtt{t}(y)}$, respectively. Since $\mathtt{t}(x) \neq \mathtt{t}(y)$, $D_{\mathtt{t}(x)}$ and $D_{\mathtt{t}(y)}$ must be disjoint. Thus $N, w \models_\sigma x \neq y$, as required.

• For K, i.e., $K_t(\varphi \to \psi) \to (K_t\varphi \to K_t\psi)$, suppose $N, w \models_\sigma K_t(\varphi \to \psi)$ and $N, w \models_\sigma K_t\varphi$. Let w' be any world such that $\langle w, w'\rangle \in R_{[\![t]\!]^{J,\sigma}_{w,\emptyset}}$. Then we have $N, w' \models_\sigma \varphi \to \psi$ and $N, w' \models_\sigma \varphi$. Thus $N, w' \models_\sigma \psi$, as required.

• For BF, i.e., $\forall x K_t\varphi \to K_t\forall x\varphi$ for x not occurring in t, suppose $N, w \models_\sigma \forall x K_t\varphi$. To show $N, w \models_\sigma K_t\forall x\varphi$, let w' be any world such that $\langle w, w'\rangle \in R_{[\![t]\!]^{J,\sigma}_{w,\emptyset}}$ and take any $d \in D_{\mathtt{t}(x)}$. By our supposition, we have $N, w \models_{\sigma[x\mapsto d]} K_t\varphi$. Now $[\![t]\!]^{J,\sigma}_{w,\emptyset} = [\![t]\!]^{J,\sigma[x\mapsto d]}_{w,\emptyset}$ holds since x does not occur in t. Thus by $\langle w, w' \rangle \in R_{[\![t]\!]^{J,\sigma[x\mapsto d]}_{w,\emptyset}}$, we have $N, w' \models_{\sigma[x\mapsto d]} \varphi$, as required.

- For KNI, i.e., $x \neq y \to K_t x \neq y$, suppose $N, w \models_\sigma x \neq y$. By definition, obviously $N, w' \models_\sigma x \neq y$ for all worlds w'. Thus $N, w \models K_t x \neq y$, as required.
- For T, i.e., $\forall x(K_x\varphi \to \varphi)$, suppose $N, w \models_{\sigma[x\mapsto d]} K_x\varphi$ for any $d \in D_{\mathtt{t}(x)}$. Since $\langle w, w \rangle \in R_{\sigma[x\mapsto d](x)}$ by the reflexivity of the frame of N, we have $N, w \models_{\sigma[x\mapsto d]} \varphi$, as required.
- For $\mathsf{5}$, i.e., $\forall x(\neg K_x\varphi \to K_x\neg K_x\varphi)$, suppose $N, w \models_{\sigma[x\mapsto d]} \neg K_x\varphi$ for any $d \in D_{\mathtt{t}(x)}$. To show $N, w \models_{\sigma[x\mapsto d]} K_x\neg K_x\varphi$, it is sufficient to show $N, v \models_{\sigma[x\mapsto d]} \neg K_x\varphi$ for any world v such that $\langle w, v \rangle \in R_{\sigma[x\mapsto d](x)}$. Now by our supposition we have some world u such that $\langle w, u \rangle \in R_{\sigma[x\mapsto d](x)}$ and $N, u \not\models_{\sigma[x\mapsto d]} \varphi$. Since $\langle v, u \rangle \in R_{\sigma[x\mapsto d](x)}$ by the euclideaness of the frame of N, we have $N, v \models_{\sigma[x\mapsto d]} \neg K_x\varphi$, as required.

By the above argument the proof has completed. □

THEOREM 3.9. *Let* $\Sigma = \langle \mathtt{Var}, \mathtt{Cn}, \mathtt{Fn}, \mathtt{Rel}, \mathtt{t} \rangle$ *be a signature,* $x \in \mathtt{Var}$, $c \in \mathtt{Cn}$ *with* $\mathtt{t}(x) = \mathtt{t}(c)$ *and* $P \in \mathtt{Rel}$ *with* $\mathtt{t}(P) = \langle \mathtt{agt\_or\_obj} \rangle$. *A formula* $x = c \to (P(x) \to P(c))$ *is not provable in* $\mathsf{HS5}$.

PROOF: If $x = c \to (P(x) \to P(c))$ is provable in $\mathsf{HS5}$, then by the soundness (Theorem 3.8) it must be valid over the class of all reflexive, symmetric and transitive frames in the non-standard semantics, which contradicts Proposition 3.5. □

We can now get the semantic incompleteness of $\mathsf{HK}\Gamma$ contradicting Theorem 3 in [8], as follows.

THEOREM 3.10 (Semantic Incompleteness of $\mathsf{HK}\Gamma$). *Let* $\Gamma \subseteq \mathrm{AX}$. *The Hilbert-style system* $\mathsf{HK}\Gamma$ *is semantically incomplete with respect to the class of all frames to which* Γ *corresponds in the TML-semantics, i.e., there exists some formula* φ *such that* φ *is valid over the class of all frames to which* Γ *corresponds in the TML-semantics, but not provable in* $\mathsf{HK}\Gamma$.

Proof: By Proposition 3.1 it follows that $x = c \to (P(x) \to P(c))$ is valid over the class of all frames to which Γ corresponds in the TML-semantics. On the other hand, by Theorem 3.9 it follows that $x = c \to (P(x) \to P(c))$ is not provable in $\mathsf{H}\mathbf{K}\Gamma$. $\square$

Corollary 3.11 ([10, Corollary 1]). The Hilbert-style system $\mathsf{H}\mathbf{K}$ is semantically incomplete with respect to the class of all frames in the TML-semantics.

4. Conclusion

In this paper, for a set $\Gamma \subseteq \mathrm{AX} = \{\mathsf{T}, \mathsf{D}, \mathsf{4}, \mathsf{5}\}$, we proved that Liberman et al.[8]'s Hilbert-style system $\mathsf{H}\mathbf{K}\Gamma$ for the term-modal logic $\mathbf{K}\Gamma$ with equality and non-rigid terms is semantically incomplete with respect to the class of all frames to which Γ corresponds (Theorem 3.10). We also corrected the frame property to which [8] claims that axiom $\mathsf{AO}_y^{\vec{x}_n}$ corresponds (Proposition 2.11).

We make two remarks here. The first remark is that [8] also fails to prove the decidability of $\mathsf{H}\mathbf{K}\{\mathsf{AO}_y^{\vec{x}_n}, \mathsf{A}_{y'}^{\vec{x'}_{n'}}\}$ with $n' < n$ and $\mathtt{t}(x'_1) = \cdots = \mathtt{t}(x'_{n'}) = \mathtt{t}(y') = \mathtt{agt}$ ([8, item 1 of Proposition 7]). Let $\mathbb{F}$ be the class of all frames to which $\{\mathsf{AO}_y^{\vec{x}_n}, \mathsf{A}_{y'}^{\vec{x'}_{n'}}\}$ corresponds. The proof in [8] depends on a claim that $D_{\mathtt{obj}}$ is finite for any frame in $\mathbb{F}$. However, this is not the case in some signatures. A simple counterexample is the case in which $\mathtt{t}(x_1) = \cdots = \mathtt{t}(x_{n'}) = \mathtt{t}(y) = \mathtt{agt}$ and $\mathtt{t}(x_{n'+1}) = \cdots = \mathtt{t}(x_n) = \mathtt{obj}$. Then Proposition 2.11 tells us only that every frame in $\mathbb{F}$ satisfies $\#D_{\mathtt{agt}} = n'$ and $\#D_{\mathtt{obj}} \geq (n - n')$, so we can find a frame in $\mathbb{F}$ such that $D_{\mathtt{obj}}$ is infinite.

The second remark is that, as the problematic first-order formula $x = c \to (P(x) \to P(c))$ suggests, the semantic incompleteness of $\mathsf{H}\mathbf{K}\Gamma$ is irrelevant to its term-modal or two-sorted aspects. To make the point clear, let $\mathcal{L}$ be a first-order modal language having equality, constants, function symbols and only the ordinary non-indexed modal operator $\Box$ as its modal operators. Say that the semantics for first-order modal logic (the FOML-semantics for short) is the Kripke semantics of the constant domain given to $\mathcal{L}$ in which the accessibility relation is a binary relation on worlds, and constants and function symbols are interpreted relative to worlds. Using a semantics similar to the non-standard semantics introduced in Section 3, we can in fact prove that the Hilbert-style system naturally obtained from $\mathsf{HK}\Gamma$ by changing from the two-sorted term-modal language to $\mathcal{L}$ becomes semantically incomplete with respect to the corresponding class of frames in the FOML-semantics. The question to be asked here is how we can obtain a sound and complete Hilbert-style system with respect to this class of frames in the FOML-semantics. To the best of our knowledge, such a Hilbert-style system seems to have never been provided together with a detailed proof in the literature.[3]

A further direction to be pursued is to give sound and complete Hilbert-style systems for term-modal logics including **K** with equality and non-rigid terms. Such systems, for example, might be obtained as slight modifications of the system given in Fagin et al. [3, p. 90]. Another further direction that might be worth studying is to apply the non-standard semantics to the analysis of natural language. As Example 3.4 suggests, it is reasonable to see $J(P, w)$ in $J(c, w, J(P, w))$ as a kind of *context* uniquely determining the denotation of a constant c at a world w. Thus, the non-standard semantics could be used for a semantics capturing the context-dependency of the denotations of nouns in natural language.

Acknowledgements. I would like to thank an anonymous reviewer for this journal for their comments on an earlier version of this paper.

[3] As for a sound and complete proof system with respect to the multi-modal FOML-semantics with the epistemic accessibility relation for each agent, Fagin et al. [3, p. 90] offered a Hilbert-style system having two first-order principles $A(t/x) \to \exists x A$ and $t = s \to (A(t/z) \leftrightarrow A(s/z))$ as axioms with a restriction that t, s must be variables if A has any occurrence of an (non term-modal) epistemic operator K_a. However, the proof of this system's completeness is omitted.

References

[1] A. Achen, **Putting the Agents Back in the Domain: A Two-Sorted Term-Modal Logic**, Master's thesis, University of Copenhagen (2017), DOI: https://doi.org/10.48550/arXiv.1811.01590, bachelor thesis.

[2] H. Enderton, **A Mathematical Introduction to Logic**, 2nd ed., Academic Press (2001), DOI: https://doi.org/10.1016/C2009-0-22107-6.

[3] R. Fagin, Y. Moses, J. Y. Halpern, M. Y. Vardi, **Reasoning about Knowledge**, MIT press (2003), DOI: https://doi.org/10.7551/mitpress/5803.001.0001.

[4] M. Fitting, L. Thalmann, A. Voronkov, *Term-Modal Logics*, **Studia Logica**, vol. 69 (2001), pp. 133–169, DOI: https://doi.org/10.1023/A:1013842612702.

[5] S. Frijters, **All Doctors Have an Obligation to Care for Their Patients: Term-modal Logics for Ethical Reasoning with Quantified Deontic Statements**, Ph.D. thesis, Ghent University (2021), URL: https://biblio.ugent.be/publication/8698101.

[6] S. Frijters, *An Andersonian-Kangerian Reduction of Term-Modal Deontic Logics*, [in:] **Deontic Logic and Normative Systems 16th International Conference, DEON 2023**, College publications (2023), pp. 159–176.

[7] B. Kooi, *Dynamic Term-modal Logic*, [in:] J. van Benthem, S. Ju, F. Veltman (eds.), **A Meeting of the Minds. Proceedings of the Workshop on Logic, Rationality and Interaction, Beijing, 2007, Texts in Computing Computer Science 8**, College Publications (2008), pp. 173–185.

[8] A. O. Liberman, A. Achen, R. K. Rendsvig, *Dynamic Term-modal Logics for First-order Epistemic Planning*, **Artificial Intelligence**, vol. 286 (2020), p. 103305, DOI: https://doi.org/10.1016/j.artint.2020.103305.

[9] A. O. Liberman, A. Achen, R. K. Rendsvig, *Erratum to "Dynamic Term-Modal Logics for First-Order Epistemic Planning" [Artif. Intell. 286 (2020) 103305]*, **Artificial Intelligence**, vol. 323 (2023), p. 103969, DOI: https://doi.org/10.1016/j.artint.2023.103969.

[10] T. Sawasaki, *Semantic Incompleteness of Liberman et al. (2020)'s Hilbert-style System for Term-modal Logic* **K** *with Equality and Non-rigid Terms*, [in:] A. Indrzejczak, M. Zawidzki (eds.), **Proceedings Eleventh International Conference on Non-Classical Logics. Theory and Applications Łódź, Poland, 5–8 September 2024** (2024), pp. 66–76, DOI: https://doi.org/10.4204/EPTCS.415.9.

[11] T. Sawasaki, K. Sano, *Term-Sequence-Dyadic Deontic Logic*, [in:] F. Liu, A. Marra, P. Portner, F. V. D. Putte (eds.), **Deontic Logic and Normative Systems: 15th International Conference, DEON 2020/2021**, College Publication (2021), pp. 376–393.

[12] L. Thalmann, **Term-Modal Logic and Quantifier-free Dynamic Assignment Logic**, Ph.D. thesis, Uppsala University (2000), URL: https://www.diva-portal.org/smash/get/diva2:170775/FULLTEXT01.pdf.

[13] Y. Wang, Y. Wei, J. Seligman, *Quantifier-free Epistemic Term-modal Logic with Assignment Operator*, **Annals of Pure and Applied Logic**, vol. 173(3) (2022), p. 103071, DOI: https://doi.org/10.1016/j.apal.2021.103071.

Takahiro Sawasaki

Kanazawa University
Institute of Liberal Arts and Science
920-1192, Kakuma-machi
Kanazawa-shi, Ishikawa-ken, Japan

e-mail: tsawasaki@staff.kanazawa-u.ac.jp

Funding information: This work was supported by JSPS KAKENHI Grant Number JP23KJ2146.

Conflict of interests: None.

Ethical considerations: The Author assures of no violations of publication ethics and takes full responsibility for the content of the publication.

Declaration regarding the use of GAI tools: Not used.

Bulletin of the **S**ection of **L**ogic
Volume 54/2 (2025), pp. 227–282
https://doi.org/10.18778/0138-0680.2025.09

Norihiro Kamide
Sara Negri

UNIFIED SEQUENT CALCULI AND NATURAL DEDUCTION SYSTEMS FOR UNTIL-FREE LINEAR-TIME TEMPORAL LOGICS[1]

Abstract

A unified Gentzen-style proof-theoretic framework for until-free propositional linear-time temporal logic and its intuitionistic variant is introduced. The framework unifies Gentzen-style single-succedent sequent calculi and natural deduction

[1]This paper extends the conference paper [28], which was published in the Proceedings of the International Conference on Non-Classical Logics: Theory and Applications 2024 (NCL'24). Compared to Sections 2 and 3 of [28], there is a significant difference: new G3-style sequent calculi with direct cut-elimination proofs are introduced, replacing the sequent calculi and their indirect cut-elimination proofs presented in [28]. Specifically, in the new calculi, sequents are treated as multisets rather than sets, allowing for the adoption of a distinct and direct cut-elimination strategy. Section 4 presents new findings on intuitionistic variants that were not included in [28]. Additionally, Section 5 provides new insights into the next-time fragments and related works, which were also not covered in [28].

Presented by: Michał Zawidzki
Received: December 29, 2024, **Received in revised form**: August 3, 2025,
Accepted: September 17, 2025, **Published online**: November 27, 2025

systems for both the classical and intuitionistic versions of these temporal logics. Theorems establishing the equivalence between the proposed sequent calculi and natural deduction systems are proved. Furthermore, the cut-elimination theorems for the proposed sequent calculi and the normalization theorems for the proposed natural deduction systems are established.

Keywords: linear-time temporal logic, intuitionistic linear-time temporal logic, sequent calculus, natural deduction, cut-elimination theorem, normalization theorem.

2020 Mathematical Subject Classification: 03B44, 03F05.

1. Introduction

1.1. Until-free LTL and its intuitionistic variant

Linear-time temporal logic (LTL) and its fragments and variants have been extensively studied [45, 33, 14, 3, 4, 6, 18, 15, 10, 8, 9, 32, 23, 17, 11].[2] The fragment of LTL without the until operator U is referred to as *until-free LTL*. Numerous Gentzen-style sequent calculi for LTL and until-free LTL have been introduced and investigated [33, 43, 44, 49, 4, 18, 15, 10, 23]. Several natural deduction systems for LTL and until-free LTL have also been explored [3, 6].

This study focuses on until-free LTL and its intuitionistic variant as the main target logics.[3] One reason for this focus is its high compatibility

[2]LTL was traditionally studied, for example, in [45, 14]. The fragment of LTL without the until operator U was investigated, for example, in [33, 3, 4, 18, 32, 23].

[3]The until operator U in LTL presents a certain difficulty in constructing a simple cut-free, two-sided, LK-compatible Gentzen-style sequent calculus. An extension of Kawai's Gentzen-style sequent calculus LT_ω, referred to as $\mathrm{LT}_\omega^{\mathrm{U}}$, with the addition of U, was considered in [25], although it is unknown whether the cut-elimination and completeness theorems for $\mathrm{LT}_\omega^{\mathrm{U}}$ hold or not. A few alternative cut-free and complete sequent calculi extended by adding U were developed in [15, 10]. However, we cannot use these calculi in this study, as they are not compatible with the present approach, which treats both sequent calculi and natural deduction systems in a uniform manner.

with Gentzen's LK and NK for classical logic and Gentzen's LI and NI[4] for intuitionistic logic [48, 46]. Specifically, the proposed Gentzen-style single-succedent sequent calculus and Gentzen-style natural deduction system for until-free LTL can be seen as natural extensions of LI and NI, respectively.

In addition, the intuitionistic variant of until-free LTL is considered because of its strong compatibility with Gentzen's LI and NI. Specifically, the proposed Gentzen-style single-succedent sequent calculus and Gentzen-style natural deduction system for this intuitionistic variant are subsystems of the corresponding systems for until-free LTL. These subsystems are obtained in a modular way from the systems of until-free LTL by removing the temporal versions of the rules of excluded middle.[5]

1.2. Sequent calculi and natural deduction systems

Gentzen-style sequent calculi for LTL have been previously explored in the literature. Kawai introduced the sequent calculus LT_ω for first-order until-free LTL, proving both cut elimination and completeness [33].[6] Baratella and Masini developed the 2-sequent calculus $2\mathrm{S}\omega$ for first-order until-free LTL, and established the cut-elimination and completeness theorems [4]. Kamide demonstrated an equivalence theorem between the propositional fragments of LT_ω and $2\mathrm{S}\omega$, providing alternative proofs of cut elimination as a consequence of this equivalence [18]. Further, Kamide presented embedding-based proofs of the cut-elimination and completeness theorems for LT_ω and its propositional fragment [23]. This study introduces a single-succedent version, $\mathrm{G3cLT}_\omega$, of LT_ω and its intuitionistic variant, $\mathrm{G3iLT}_\omega$.

[4] We remark that Gentzen designated his intuitionistic calculus by NI, however, his handwriting for capital I was in the old Sütterlin handwriting, that, as explained by von Plato in [55], p. 83, has been rendered by capital J in printing. This practice has been followed, with rare exceptions, to these days, but recent literature shows a return to the originally intended nomenclature, see e.g. [51]. We shall follow the original notation also for the intuitionistic sequent calculus, with LI instead of LJ.

[5] The proposed sequent calculus for until-free LTL includes the sequent calculus version of the rule of excluded middle, referred to as (ex-middle), and the proposed natural deduction system for until-free LTL includes the natural deduction version of the rule of excluded middle, referred to as (EXM).

[6] We have not yet obtained a cut-free and complete extension of LT_ω with the until operator.

The Gentzen-style natural deduction systems PNK and PNJ were introduced by Baratella and Masini [3] for classical and intuitionistic until-free LTLs, respectively. These systems, PNK and PNJ, are regarded as extensions of Gentzen's NK and NI, and were referred to by the authors as the "logics of positions". In a separate development, Bolotov et al. introduced the natural deduction system PLTL_{ND} [6] for full classical propositional LTL, including the until operator U. The system PLTL_{ND} employs labelled formulas of the form $i : \alpha$ and a temporal induction rule that deals with the next-time operator X and the "globally in the future" operator G. It is notable that PNK, PNJ, and PLTL_{ND} utilize an induction rule and do not incorporate infinite premiss rules for handling temporal operators. In contrast, the natural deduction systems proposed in this study employ infinite premiss rules and do not rely on an induction rule. This alternative approach provides a novel natural deduction system, NLT_ω, of LT_ω and its intuitionistic variant, NILT_ω.

1.3. The approach of this study

In this study, we introduce a unified Gentzen-style framework for until-free propositional LTL and its intuitionistic variant. This framework seamlessly integrates both Gentzen-style single-succedent sequent calculi and Gentzen-style natural deduction systems. Specifically, it allows us to establish an equivalence between these systems and to demonstrate that the cut-elimination theorems for the single-succedent sequent calculi imply the normalization theorems for the natural deduction systems.

The primary aim and original contribution of this study lie in providing a unified treatment for both sequent calculi and natural deduction systems within the context of until-free LTL.[7] Until now, these systems have been

[7]The unified treatment or approach also means that we can obtain a natural correspondence between the sequent calculus and the natural deduction system. More specifically, a natural correspondence refers to a correspondence between the cut-elimination theorem for the sequent calculus and the normalization theorem for the natural deduction system. Thus, the unified approach implies that we can handle these fundamental theorems in a uniform way.

studied separately for LTL and its fragments, rather than in a unified manner. This is in contrast to recent work where a unified treatment of sequent calculus and tableaux calculus for branching-time temporal logics has been explored by Abuin et al. [1]. Our unified approach not only bridges the gap (i.e., non-uniformity) between sequent calculi and natural deduction systems in until-free LTL but also facilitates the transfer of meta-results between these formalisms, providing a significant theoretical advantage for their applications.

To address the issue of the correspondence between cut elimination and normalization, we require a Gentzen-style single-succedent sequent calculus. This necessity arises because the cut-elimination theorem for the typical Gentzen-style multiple-succedent sequent calculi in standard classical LTL does not imply the normalization theorem for the corresponding natural deduction system. A similar situation is observed in classical logic when comparing Gentzen's LK and NK. In contrast, it is well-established that the cut-elimination theorem for the single-succedent calculus LI directly implies the normalization theorem for NI in intuitionistic logic.[8] Therefore, our approach involves developing an LI-like single-succedent sequent calculus tailored for our target logic.

1.4. The proposed single-succedent sequent calculi

To obtain a classical single-succedent sequent calculus, we use the following temporal (single-succedent) excluded middle rule:

$$\frac{\mathrm{X}^i\neg\alpha, \Gamma \Rightarrow \gamma \quad \mathrm{X}^i\alpha, \Gamma \Rightarrow \gamma}{\Gamma \Rightarrow \gamma}\ \text{(ex-middle)}$$

where X^i is an i-times nested next-time operator. By employing this rule, we can prove the law of excluded middle, $\alpha\vee\neg\alpha$, for arbitrary formulas α. The non-temporal version of this rule, without X^i, was originally introduced by von Plato [53, 41]. Pursuing the idea of correspondence between cut elimination and normalization, von Plato developed a single-succedent

[8] See, for example, [41] and the references therein.

sequent calculus for classical logic and proved cut elimination and normalization for the corresponding sequent calculus and natural deduction systems. Building on this approach, we aim to extend these concepts to the target temporal logic. In fact, the G3-style single-succedent sequent calculus $\mathrm{G3cLT}_\omega$ proposed in this study can be seen as a temporal extension of von Plato's calculus. Moreover, the cut-elimination result for $\mathrm{G3cLT}_\omega$ serves as an extension of his cut-elimination results for classical logic. Additionally, an important advantage of this approach is that a single-succedent sequent calculus for an intuitionistic variant of the target logic can be easily derived from $\mathrm{G3cLT}_\omega$ by just removing (ex-middle).

1.5. The proposed natural deduction systems

To obtain the corresponding natural deduction system for until-free LTL, we use rules of the form:

$$\dfrac{\begin{array}{c}[\mathrm{X}^i\neg\alpha]\\ \vdots\\ \gamma\end{array}\qquad\begin{array}{c}[\mathrm{X}^i\alpha]\\ \vdots\\ \gamma\end{array}}{\gamma}\ (\mathrm{EXM})\qquad \dfrac{\mathrm{X}^i\neg\alpha\quad \mathrm{X}^i\alpha}{\gamma}\ (\mathrm{EXP})\qquad \dfrac{\begin{array}{c}[\mathrm{X}^i\alpha]\\ \vdots\\ \mathrm{X}^j\neg\gamma\end{array}\qquad\begin{array}{c}[\mathrm{X}^i\alpha]\\ \vdots\\ \mathrm{X}^j\gamma\end{array}}{\mathrm{X}^i\neg\alpha}\ (\neg\mathrm{I})$$

where (EXM) corresponds to (ex-middle). As mentioned above, the non-temporal version of (EXM) was originally introduced by von Plato [53, 41]; the non-temporal versions of (EXP) and (¬I) were instead originally introduced by Gentzen. For detailed information on these rules, see [54, 55].

Rule (EXP) has been applied in various contexts: Bolotov and Shangin [7] utilized it to construct the paracomplete logic PCont; Kürbis and Petrukhin [36] developed natural deduction systems for a family of many-valued logics, including N3, using this rule; Kamide and Negri [26, 30] employed it to formalize Gurevich logic [16] and Nelson logic [42, 2]. Additionally, Priest [47] proposed rules similar to (EXP) for creating natural deduction systems for logics in the FDE (First Degree Entailment) family.

Rule (EXP) is considered a counterpart to (EXM) and is particularly useful for handling natural deduction systems where negation is treated as a primitive connective, rather than being defined through implication and the falsity constant. In this study, the proposed natural deduction system NLT_ω can be seen as a modified temporal extension of von Plato's classical system, enhanced by incorporating (EXP) and ($\neg$I). The normalization result for NLT_ω extends von Plato's normalization result for classical logic. Moreover, a significant advantage of this approach is that a natural deduction system for an intuitionistic variant of until-free LTL can be easily obtained from NLT_ω by omitting rule (EXM).

We address some remarks on previous results presented in the papers [26, 30] concerning natural deduction systems for logics of strong negation. The paper [26] introduced natural deduction systems for Gurevich logic, intuitionistic logic, and classical logic using (EXP) and/or (EXM) and proved normalization theorems for the natural deduction systems of Gurevich logic and intuitionistic logic. However, [26] contained errors related to these normalization theorems. These errors arose from inappropriate definitions in the natural deduction system $\mathrm{NI}^\star$ for intuitionistic logic.

These issues were identified by Arnon Avron during the 1^{st} Workshop on Contradictory Logics, held in Bochum on December 6-8, 2023. He pointed out a gap in an earlier version of $\mathrm{NI}^\star$, specifically the absence of the non-temporal version of ($\neg$I). These errors have been corrected in the subsequent papers [30, 29]. The results of this study reflect the corrected findings in [30, 29].

1.6. Paper structure

The paper is structured as follows: In Section 2, we explore Gentzen-style sequent calculi and their cut-elimination theorems for until-free propositional LTL. We begin by presenting Kawai's Gentzen-style sequent calculus LT_ω and introducing the newly proposed Gentzen-style single-succedent sequent calculus $\mathrm{G3cLT}_\omega$. Subsequently, we prove the cut-elimination theorem for $\mathrm{G3cLT}_\omega$ using an extension of the standard methodology for G3-style sequent calculi, namely, by first proving invertibility of (most of) the rules and admissibility of weakening and contraction.

In Section 3, we begin by introducing the newly proposed Gentzen-style natural deduction system NLT_ω for until-free propositional LTL. We also define the reduction relation for NLT_ω. Subsequently, we establish the normalization theorem for NLT_ω by exploiting the equivalence theorem between NLT_ω and $\mathrm{G3cLT}_\omega$.

In Section 4, we introduce and investigate a Gentzen-style sequent calculus $\mathrm{G3iLT}_\omega$ and a Gentzen-style natural deduction system NILT_ω for an intuitionistic variant of the until-free propositional LTL. The systems $\mathrm{G3iLT}_\omega$ and NILT_ω are derived from $\mathrm{G3cLT}_\omega$ and NLT_ω, respectively, by removing (ex-middle), and its corresponding rule (EXM). All the structural results and the cut-elimination theorem for $\mathrm{G3iLT}_\omega$ follow directly from the corresponding results for $\mathrm{G3cLT}_\omega$. Then, we establish the normalization theorem for NILT_ω using the translation that gives the equivalence between NILT_ω and $\mathrm{G3iLT}_\omega$.

Section 5 concludes this study with some additional remarks.

2. Sequent calculus and cut elimination

In this study, we assume standard notions and terminologies regarding Gentzen-style sequent calculus and Gentzen-style natural deduction system, and do not provide precise definitions for some of these familiar notions and terminologies.

Formulas of the logic discussed in this study are constructed using countably many propositional variables, the logical connectives $\to$ (implication), $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), G (globally in the future), F (eventually in the future), and X (next-time). We use small letters $p, q, \ldots$ to denote propositional variables and Greek small letters $\alpha, \beta, \ldots$ to denote formulas.

We use Greek capital letters $\Gamma, \Delta, \ldots$ to denote finite (possibly empty) multisets[9] of formulas. For any $\sharp \in \{\mathrm{G}, \mathrm{F}, \mathrm{X}\}$ and any multisets Γ of formulas, we use an expression $\sharp\Gamma$ to denote the multisets $\{\sharp\gamma \mid \gamma \in \Gamma\}$. The symbol $\equiv$ is used to denote the equality of multisets of symbols. The

[9] For the newly introduced G3-style calculi we shall consider a definition based on multisets. This will also be the case for natural deduction although the practice of multiple-discharge makes the choice less visible. Additionally, for the sake of comparison between our calculus and Kawai's calculus LT_ω, LT_ω is also presented in this study using a multiset-based formulation. This modification is not essential to the results of this study.

symbol ω is used to represent the set of natural numbers. An expression $\mathrm{X}^i\alpha$ for any $i \in \omega$ is defined inductively by $\mathrm{X}^0\alpha \equiv \alpha$ and $\mathrm{X}^{n+1}\alpha \equiv \mathrm{X}^n\mathrm{X}\alpha$. We use lower-case letters i, j and k to denote any natural numbers.

We will define Kawai's Gentzen-style sequent calculus LT_ω [33] and a new alternative Gentzten-style single-succedent sequent calculus $\mathrm{G3cLT}_\omega$. Prior to defining these sequent calculi, we need to define some additional notions and notations.

DEFINITION 2.1. A *sequent* for LT_ω is an expression of the form $\Gamma \Rightarrow \Delta$, and a sequent for $\mathrm{G3cLT}_\omega$ is an expression of the form $\Gamma \Rightarrow \gamma$ where γ is a formula or the empty set. We use the expression $L \vdash S$ to represent the fact that a sequent S is derivable in a sequent calculus L. We say that "a rule R is *admissible* in a sequent calculus L" if the following condition is satisfied: For any instance $\frac{S_1 \cdots S_n}{S}$ of R, if $L \vdash S_i$ for all i, then $L \vdash S$.

Additionally, we have to define the notion of height of a derivation. This notion is formulated in a general way and is applicable to all the sequent calculi here presented. Derivations built using these rules are thus (in general) infinite trees, with countable branching but where (as may be proved by induction on the definition of derivation) each branch has finite length. The *leaves* of the trees are the initial sequents. To make this precise, we give a formal definition of the notion of *derivation* $\mathcal{D}$ and the associated notions of its *height* $ht(\mathcal{D})$ and its *end-sequent*.

DEFINITION 2.2.

1. Any sequent $\Gamma \Rightarrow \Delta$, where some formula $\mathrm{X}^i p$ occurs in both Γ and Δ, is a derivation of *height* 0 and with *end-sequent* $\Gamma \Rightarrow \Delta$.

2. If each $\mathcal{D}_n$ is a derivation of height α_n, with end-sequent $\Gamma_n \Rightarrow \Delta_n$ and

$$\frac{\ldots \quad \Gamma_n \Rightarrow \Delta_n \quad \ldots}{\Gamma \Rightarrow \Delta}\ R$$

is an inference (i.e. an instance of a rule), then

$$\dfrac{\cdots \quad \dfrac{\mathcal{D}_n}{\Gamma_n \Rightarrow \Delta_n} \quad \cdots}{\Gamma \Rightarrow \Delta} \; R$$

is a derivation, its height is the countable ordinal $sup_n(\alpha_n) + 1$, and its end-sequent is $\Gamma \Rightarrow \Delta$.

3. Thus, each derivation $\mathcal{D}$ has a countable ordinal height, denoted $ht(\mathcal{D})$, which is the successor of the supremum of the heights of its immediate subderivations. It follows that, if $\mathcal{D}'$ is a sub-derivation of $\mathcal{D}$, then $ht(\mathcal{D}') < ht(\mathcal{D})$.

4. We say that "a rule R of the form $\frac{S_1 \cdots S_n}{S}$ is *height-preserving admissible* in a sequent calculus L" if the following condition is satisfied: If the premisses are derivable with height at most n then also the conclusion is derivable with the same bound on the derivation height. Furthermore, we say that "R is *derivable* in L" if there is a derivation from $S_1, \cdots, S_n$ to S in L.

First, we introduce LT_ω.

Definition 2.3 (LT_ω). In the following definitions, i and k represent arbitrary natural numbers (i.e., $i, k \in \omega$).

The initial sequents of LT_ω are of the following form for any propositional variable p:

$$\mathrm{X}^i p \Rightarrow \mathrm{X}^i p \text{ (init)}$$

The structural rules of LT_ω are of the form:

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \alpha, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi} \text{ (cut)}$$

$$\frac{\Gamma \Rightarrow \Delta}{\alpha, \Gamma \Rightarrow \Delta} \text{ (we-left)} \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \alpha} \text{ (we-right)}$$

$$\frac{\alpha, \alpha, \Gamma \Rightarrow \Delta}{\alpha, \Gamma \Rightarrow \Delta} \text{ (co-left)} \quad \frac{\Gamma \Rightarrow \Delta, \alpha, \alpha}{\Gamma \Rightarrow \Delta, \alpha} \text{ (co-right).}$$

The logical rules of LT_ω are of the form:

$$\frac{\Gamma \Rightarrow \Delta, \mathrm{X}^i\alpha \quad \mathrm{X}^i\beta, \Gamma \Rightarrow \Delta}{\mathrm{X}^i(\alpha{\rightarrow}\beta), \Gamma \Rightarrow \Delta}\ (\rightarrow\text{left}) \qquad \frac{\mathrm{X}^i\alpha, \Gamma \Rightarrow \Delta, \mathrm{X}^i\beta}{\Gamma \Rightarrow \Delta, \mathrm{X}^i(\alpha{\rightarrow}\beta)}\ (\rightarrow\text{right})$$

$$\frac{\Gamma \Rightarrow \Delta, \mathrm{X}^i\alpha}{\mathrm{X}^i\neg\alpha, \Gamma \Rightarrow \Delta}\ (\neg\text{left}) \qquad \frac{\mathrm{X}^i\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \mathrm{X}^i\neg\alpha}\ (\neg\text{right})$$

$$\frac{\mathrm{X}^i\alpha, \mathrm{X}^i\beta, \Gamma \Rightarrow \Delta}{\mathrm{X}^i(\alpha\wedge\beta), \Gamma \Rightarrow \Delta}\ (\wedge\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \mathrm{X}^i\alpha \quad \Gamma \Rightarrow \Delta, \mathrm{X}^i\beta}{\Gamma \Rightarrow \Delta, \mathrm{X}^i(\alpha\wedge\beta)}\ (\wedge\text{right})$$

$$\frac{\mathrm{X}^i\alpha, \Gamma \Rightarrow \Delta \quad \mathrm{X}^i\beta, \Gamma \Rightarrow \Delta}{\mathrm{X}^i(\alpha\vee\beta), \Gamma \Rightarrow \Delta}\ (\vee\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \mathrm{X}^i\alpha, \mathrm{X}^i\beta}{\Gamma \Rightarrow \Delta, \mathrm{X}^i(\alpha\vee\beta)}\ (\vee\text{right})$$

$$\frac{\mathrm{X}^{i+k}\alpha, \Gamma \Rightarrow \Delta}{\mathrm{X}^i\mathrm{G}\alpha, \Gamma \Rightarrow \Delta}\ (\text{Gleft}) \qquad \frac{\{\ \Gamma \Rightarrow \Delta, \mathrm{X}^{i+j}\alpha\ \}_{j\in\omega}}{\Gamma \Rightarrow \Delta, \mathrm{X}^i\mathrm{G}\alpha}\ (\text{Gright})$$

$$\frac{\{\ \mathrm{X}^{i+j}\alpha, \Gamma \Rightarrow \Delta\ \}_{j\in\omega}}{\mathrm{X}^i\mathrm{F}\alpha, \Gamma \Rightarrow \Delta}\ (\text{Fleft}) \qquad \frac{\Gamma \Rightarrow \Delta, \mathrm{X}^{i+k}\alpha}{\Gamma \Rightarrow \Delta, \mathrm{X}^i\mathrm{F}\alpha}\ (\text{Fright}).$$

Remark 2.4.

1. The calculus LT_ω introduced here is a modified propositional version of Kawai's sequent calculus [33] for until-free first-order linear-time temporal logic. Kawai's original sequent calculus was developed as a first-order system incorporating the Barcan formula. In that system, the next-time operator was not used as a modal operator but rather as a special symbol.

2. Note that (Gright) and (Fleft) have infinitely many premises and can also be represented as:

$$\frac{\{\ \Gamma \Rightarrow \Delta, \mathrm{X}^{i+j}\alpha \mid j\in\omega\ \}}{\Gamma \Rightarrow \Delta, \mathrm{X}^i\mathrm{G}\alpha}\ (\text{Gright}) \qquad \frac{\{\ \mathrm{X}^{i+j}\alpha, \Gamma \Rightarrow \Delta \mid j\in\omega\ \}}{\mathrm{X}^i\mathrm{F}\alpha, \Gamma \Rightarrow \Delta}\ (\text{Fleft}).$$

3. The following cut-elimination theorem holds for LT_ω. Rule (cut) is admissible in cut-free LT_ω. For the details, see [33, 18, 23].

4. The cut-elimination theorem for the original first-order LT_ω was proved by Kawai in [33] and was indirectly re-proved for a slightly modified version of LT_ω by Kamide in [18], via the cut-free equivalence between LT_ω and Baratella-Masini's cut-free 2-sequent calculus $2\mathrm{S}\omega$ [4]. Additionally, Kamide provided an alternative proof for the cut-elimination theorem for the slightly modified LT_ω in [23]. This alternative proof was based on a theorem that embeds LT_ω into a Gentzen-style sequent calculus for infinitary logic. For more information on the cut-elimination theorem for LT_ω, see [33, 18, 23].

Next, we introduce $\mathrm{G3cLT}_\omega$. We use the same names for the rules of $\mathrm{G3cLT}_\omega$ as those of LT_ω.

Definition 2.5 ($\mathrm{G3cLT}_\omega$). In the following definitions, i and k denote arbitrary natural numbers and γ denotes either a formula or the empty multiset.

The initial sequents of $\mathrm{G3cLT}_\omega$ are of the following form for any propositional variable p:

$$\mathrm{X}^i p, \Gamma \Rightarrow \mathrm{X}^i p \text{ (init).}$$ [10]

The structural rules of $\mathrm{G3cLT}_\omega$ are of the form: [11]

$$\frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \alpha} \text{ (we-right).}$$

The logical rules of $\mathrm{G3cLT}_\omega$ are of the form:

$$\frac{\mathrm{X}^i(\alpha{\to}\beta), \Gamma \Rightarrow \mathrm{X}^i\alpha \quad \mathrm{X}^i\beta, \Gamma \Rightarrow \gamma}{\mathrm{X}^i(\alpha{\to}\beta), \Gamma \Rightarrow \gamma} \text{ (}{\to}\text{left)} \quad \frac{\mathrm{X}^i\alpha, \Gamma \Rightarrow \mathrm{X}^i\beta}{\Gamma \Rightarrow \mathrm{X}^i(\alpha{\to}\beta)} \text{ (}{\to}\text{right)}$$

[10] The context Γ is required in (init), which distinguishes it from LT_ω.

[11] As will be shown, weakening-left, contraction-left, and cut rules are admissible in $\mathrm{G3cLT}_\omega$.

$$\frac{\mathrm{X}^i\neg\alpha, \Gamma \Rightarrow \mathrm{X}^i\alpha}{\mathrm{X}^i\neg\alpha, \Gamma \Rightarrow}\ (\neg\text{left}) \qquad \frac{\mathrm{X}^i\alpha, \Gamma \Rightarrow}{\Gamma \Rightarrow \mathrm{X}^i\neg\alpha}\ (\neg\text{right})$$

$$\frac{\mathrm{X}^i\neg\alpha, \Gamma \Rightarrow \gamma \quad \mathrm{X}^i\alpha, \Gamma \Rightarrow \gamma}{\Gamma \Rightarrow \gamma}\ (\text{ex-middle})$$

$$\frac{\mathrm{X}^i\alpha, \mathrm{X}^i\beta, \Gamma \Rightarrow \gamma}{\mathrm{X}^i(\alpha \wedge \beta), \Gamma \Rightarrow \gamma}\ (\wedge\text{left}) \qquad \frac{\Gamma \Rightarrow \mathrm{X}^i\alpha \quad \Gamma \Rightarrow \mathrm{X}^i\beta}{\Gamma \Rightarrow \mathrm{X}^i(\alpha \wedge \beta)}\ (\wedge\text{right})$$

$$\frac{\mathrm{X}^i\alpha, \Gamma \Rightarrow \gamma \quad \mathrm{X}^i\beta, \Gamma \Rightarrow \gamma}{\mathrm{X}^i(\alpha \vee \beta), \Gamma \Rightarrow \gamma}\ (\vee\text{left})$$

$$\frac{\Gamma \Rightarrow \mathrm{X}^i\alpha}{\Gamma \Rightarrow \mathrm{X}^i(\alpha \vee \beta)}\ (\vee\text{right1}) \qquad \frac{\Gamma \Rightarrow \mathrm{X}^i\beta}{\Gamma \Rightarrow \mathrm{X}^i(\alpha \vee \beta)}\ (\vee\text{right2})$$

$$\frac{\mathrm{X}^i\mathrm{G}\alpha, \mathrm{X}^{i+k}\alpha, \Gamma \Rightarrow \gamma}{\mathrm{X}^i\mathrm{G}\alpha, \Gamma \Rightarrow \gamma}\ (\text{Gleft}) \qquad \frac{\{\ \Gamma \Rightarrow \mathrm{X}^{i+j}\alpha\ \}_{j\in\omega}}{\Gamma \Rightarrow \mathrm{X}^i\mathrm{G}\alpha}\ (\text{Gright})$$

$$\frac{\{\ \mathrm{X}^{i+j}\alpha, \Gamma \Rightarrow \gamma\ \}_{j\in\omega}}{\mathrm{X}^i\mathrm{F}\alpha, \Gamma \Rightarrow \gamma}\ (\text{Fleft}) \qquad \frac{\Gamma \Rightarrow \mathrm{X}^{i+k}\alpha}{\Gamma \Rightarrow \mathrm{X}^i\mathrm{F}\alpha}\ (\text{Fright}).$$

Remark 2.6.

1. Similar to Gentzen's LK and the single-succedent sequent calculus for classical logic, a theorem can be established to demonstrate the equivalence between LT_ω and $\mathrm{G3cLT}_\omega$, assuming the admissibility of structural rules, including (cut). However, a proof is omitted here, as it was presented in [28] for similar systems. For details on the proof of such an equivalence theorem, see [28], where the equivalence between a slightly modified version of LT_ω and a non-G3-style version, SLT_ω, of $\mathrm{G3cLT}_\omega$ was established.

2. Alternatively, the equivalence between LT_ω and $\mathrm{G3cLT}_\omega$ can be established through the equivalence among LT_ω, SLT_ω, and NLT_ω. Here, SLT_ω refers to the non-G3-style single-succedent sequent

calculus introduced in [28], and NLT_ω is introduced both in the present study and in [28]. Since NLT_ω and (a slightly modified version of) LT_ω are common to both studies, the required equivalence theorem can be obtained.

3. Rule (ex-middle) is a characteristic feature of $\mathrm{G3cLT}_\omega$. By utilizing this rule, we can formalize a single-succedent sequent calculus. This rule is a temporal generalization of the original rule introduced by von Plato [53, 41]. Von Plato originally developed a single-succedent sequent calculus for classical logic using this rule and proved the cut-elimination theorem for it. Therefore, $\mathrm{G3cLT}_\omega$ can be viewed as a temporal extension of his calculus, and the cut-elimination result for $\mathrm{G3cLT}_\omega$ extends his cut-elimination result for classical logic.

4. In [53, 41], (ex-middle) and the following rule were introduced:

$$\frac{\neg p, \Gamma \Rightarrow \gamma \quad p, \Gamma \Rightarrow \gamma}{\Gamma \Rightarrow \gamma} \text{ (ex-middle-at)}$$

where p is a propositional variable. In [53, 41], the following results were presented. The cut rule and rule (ex-middle) are admissible in certain versions of cut-free LI that include (ex-middle-at). As a consequence of these results, these versions possess a weak subformula property that allows for propositional variables and their negations.

Proposition 2.7. Let L be LT_ω or $\mathrm{G3cLT}_\omega$. The sequents of the form $\mathrm{X}^i\alpha, \Gamma \Rightarrow \mathrm{X}^i\alpha$ for any formula α, any multiset Γ of formulas, and any natural number i are derivable in L.

Proof: By induction on α. We show some cases.

1. Case $\alpha \equiv \neg\beta$:

$$\dfrac{\dfrac{\begin{array}{c}\vdots \; \mathit{Ind.\,hyp.} \\ \mathrm{X}^i\neg\beta, \mathrm{X}^i\beta, \Gamma \Rightarrow \mathrm{X}^i\beta\end{array}}{\mathrm{X}^i\beta, \mathrm{X}^i\neg\beta, \Gamma \Rightarrow} \text{ } (\neg\text{left})}{\mathrm{X}^i\neg\beta, \Gamma \Rightarrow \mathrm{X}^i\neg\beta} \text{ } (\neg\text{right}).$$

2. Case $\alpha \equiv \beta{\to}\gamma$:

$$\dfrac{\dfrac{\overset{\vdots\ \textit{Ind. hyp.}}{\mathrm{X}^i(\beta{\to}\gamma), \mathrm{X}^i\beta \Rightarrow \mathrm{X}^i\beta} \quad \overset{\vdots\ \textit{Ind. hyp.}}{\mathrm{X}^i\gamma, \Gamma \Rightarrow \mathrm{X}^i\gamma}}{\mathrm{X}^i\beta, \mathrm{X}^i(\beta{\to}\gamma), \Gamma \Rightarrow \mathrm{X}^i\gamma}\ (\to\text{left})}{\mathrm{X}^i(\beta{\to}\gamma), \Gamma \Rightarrow \mathrm{X}^i(\beta{\to}\gamma)}\ (\to\text{right}).$$

3. Case $\alpha \equiv \mathrm{G}\beta$:

$$\dfrac{\dfrac{\overset{\vdots\ \textit{Ind. hyp.}}{\{\ \mathrm{X}^i\mathrm{G}\beta, \mathrm{X}^{i+j}\beta, \Gamma \Rightarrow \mathrm{X}^{i+j}\beta\ \}_{j\in\omega}}}{\{\ \mathrm{X}^i\mathrm{G}\beta, \Gamma \Rightarrow \mathrm{X}^{i+j}\ \}_{j\in\omega}}\ (\text{Gleft})}{\mathrm{X}^i\mathrm{G}\beta, \Gamma \Rightarrow \mathrm{X}^i\mathrm{G}\beta}\ (\text{Gright}).$$

□

PROPOSITION 2.8. The rule of left weakening

$$\dfrac{\Gamma \Rightarrow \gamma}{\alpha, \Gamma \Rightarrow \gamma}\ (\text{we-left})$$

is height-preserving admissible in $\mathrm{G3cLT}_\omega$.

PROOF: By straightforward induction on the height of the derivation since weakening is in-built in initial sequents and all the rules have an arbitrary context on the left. □

LEMMA 2.9. *All the logical rules of* $\mathrm{G3cLT}_\omega$ *with the exception of* ($\to$left), ($\vee$right1), ($\vee$right2), (Fright) *are hp-invertible. Rule* ($\to$left) *is hp-invertible with respect to the right premiss.*

PROOF: By induction on the height of the derivation of the conclusion of each rule. We show the case of invertibility of ($\to$left) with respect to the right premiss, all the other cases being similar. Assume $\vdash_0 X^i(\alpha{\to}\beta), \Gamma \Rightarrow \gamma$. Then we have an initial sequent with $\gamma \in \Gamma$ and γ is of the form X^jp, i.e. $\vdash_0 X^i(\alpha{\to}\beta), X^jp, \Gamma' \Rightarrow X^jp$. Then also $X^i\beta, X^jp, \Gamma' \Rightarrow X^jp$ is an initial sequent, i.e., $\vdash_0 X^i\beta, \Gamma \Rightarrow \gamma$.

Let us assume the statement to be true for n and prove it for $n+1$. We consider the last rule applied in the derivation. If it is ($\rightarrow$left) with $X^i(\alpha\rightarrow\beta)$ principal, the right premiss gives the desired conclusion. If some other formula is principal in the last step, assume for example that the last step is ($\rightarrow$left) with another principal formula. So we have $\vdash_{n+1} X^i(\alpha\rightarrow\beta), X^j(\epsilon\rightarrow\delta), \Gamma' \Rightarrow \gamma$ and the premisses give $\vdash_n X^i(\alpha\rightarrow\beta), \Gamma' \Rightarrow X^j\epsilon$ and $\vdash_n X^i(\alpha\rightarrow\beta), X^j\delta, \Gamma' \Rightarrow \gamma$. By inductive hypothesis we thus obtain $\vdash_n X^i\beta, \Gamma' \Rightarrow X^j\epsilon$ and $\vdash_n X^i\beta, X^j\delta, \Gamma' \Rightarrow \gamma$ and a step of ($\rightarrow$left) gives the conclusion $\vdash_{n+1} X^i\beta, X^j(\epsilon\rightarrow\delta), \Gamma' \Rightarrow \gamma$. □

Lemma 2.10. *The rule of left contraction*

$$\frac{\alpha, \alpha, \Gamma \Rightarrow \gamma}{\alpha, \Gamma \Rightarrow \gamma}\ \text{(co-left)}$$

is hp-admissible in G3cLT$_\omega$.

Proof: By induction on the height of the derivation of the premiss. In the base case, with height zero, the premiss is an initial sequent and clearly also the conclusion is an initial sequent. Otherwise, we assume that the premiss has derivation height $n+1$ and assume the statement true for derivation height n. We proceed by cases on the rule used to derive $\alpha, \alpha, \Gamma \Rightarrow \gamma$. If it is derived by (we-right) we have $\vdash_n \alpha, \alpha, \Gamma \Rightarrow$, so by induction hypothesis $\vdash_n \alpha, \Gamma \Rightarrow$ and by (we-right) $\vdash_{n+1} \alpha, \Gamma \Rightarrow \gamma$. We proceed in a similar way if $\alpha, \alpha, \Gamma \Rightarrow \gamma$ is the conclusion of a right rule: we apply the induction hypothesis to the premiss(es) of the rule, and then obtain the required fact. If instead $\alpha, \alpha, \Gamma \Rightarrow \gamma$ is the conclusion of a left rule, we distinguish two cases: either α is principal in the last rule, or it is not. In this latter case, we apply the induction hypothesis to the premiss(es) of the last rule and then obtain the required fact. Otherwise, in the former case, we consider the last rule applied and distinguish two sub-cases, depending on whether the last rule is invertible. We observe that all the left rules with the exception of ($\rightarrow$left) are invertible. Additionally, ($\rightarrow$left), ($\neg$left), and (Gleft) have been made invertible with the repetition in the premiss of the principal formula. These cases are straightforward because we find the duplication in the premiss of the rule and the induction hypothesis applies. In the case of an invertible rule, say we have a derivation of $\vdash_{n+1} \mathrm{X}^i(\alpha \vee \beta), \mathrm{X}^i(\alpha \vee \beta), \Gamma \Rightarrow \gamma$ and the premisses of the last rule give $\vdash_n$ $\mathrm{X}^i\alpha, \mathrm{X}^i(\alpha \vee \beta), \Gamma \Rightarrow \gamma$ and $\vdash_n \mathrm{X}^i\beta, \mathrm{X}^i(\alpha \vee \beta), \Gamma \Rightarrow \gamma$. By hp-invertibility of ($\vee$left) we obtain $\vdash_n \mathrm{X}^i\alpha, \mathrm{X}^i\alpha, \Gamma \Rightarrow \gamma$ and $\vdash_n \mathrm{X}^i\beta, \mathrm{X}^i\beta, \Gamma \Rightarrow \gamma$, so by induction hypothesis we have $\vdash_n \mathrm{X}^i\alpha, \Gamma \Rightarrow \gamma$ and $\vdash_n \mathrm{X}^i\beta, \Gamma \Rightarrow \gamma$. Application of ($\vee$left) gives the conclusion $\vdash_{n+1} \mathrm{X}^i(\alpha \vee \beta), \Gamma \Rightarrow \gamma$. If the last rule is ($\rightarrow$left), we have $\vdash_{n+1} \mathrm{X}^i(\alpha\rightarrow\beta), \mathrm{X}^i(\alpha\rightarrow\beta), \Gamma \Rightarrow \gamma$, and the premisses of the rule give $\vdash_n \mathrm{X}^i(\alpha\rightarrow\beta), \mathrm{X}^i(\alpha\rightarrow\beta), \Gamma \Rightarrow \mathrm{X}^i\alpha$ and $\vdash_n \mathrm{X}^i(\alpha\rightarrow\beta), \mathrm{X}^i\beta, \Gamma \Rightarrow \gamma$. By inductive hypothesis the former gives $\vdash_n \mathrm{X}^i(\alpha\rightarrow\beta), \Gamma \Rightarrow \mathrm{X}^i\alpha$ and the letter by (partial) hp-invertibility of ($\rightarrow$left) gives $\vdash_n \mathrm{X}^i\beta, \mathrm{X}^i\beta, \Gamma \Rightarrow \gamma$, so again by inductive hypothesis we obtain $\vdash_n \mathrm{X}^i\beta, \Gamma \Rightarrow \gamma$, and thus by ($\rightarrow$left), $\vdash_{n+1} \mathrm{X}^i(\alpha\rightarrow\beta), \Gamma \Rightarrow \gamma$. □

For G3c [41], the proof of cut elimination eliminates a topmost cut by induction on the complexity of the cut formula and subinduction on the sum of the heights of the derivations of the premisses of cuts. To adapt the proof to a G3-style calculus with infinitary rules, we proceed as in [39]: heights are given by ordinals, and we shall employ the standard notion of (natural or Hessenberg) addition $\alpha\#\beta$ for countable ordinals α and β (cf. e.g. 10.1.2B in [52] for the definition). We recall that $\#$ is commutative and that if $\alpha < \alpha'$ then $\alpha\#\beta < \alpha'\#\beta$.

The *rank* $\pi(I)$ of an instance I of (cut) with cut-free premisses $\mathcal{D}$ and $\mathcal{D}'$ is the pair comprising the depth $d(A)$ of the cut formula and the natural sum $h(\mathcal{D})\#h(\mathcal{D}')$ of the heights of the premisses. We will call the second component the *total height* of the cut. Pairs are ordered lexicographically.

Ordinals are well-ordered, so we can reason by (transfinite) induction; since we actually do it for pairs, we call this *transfinite lexicographic induction.* It can be converted to ordinary transfinite induction by turning pairs into ordinals, e.g. the pair (δ, σ) can be converted to $\delta \cdot \epsilon_0 + \sigma$, where ϵ_0 has the useful property of being greater than any possible value of σ; but pairs are conceptually clearer.

Lemma 2.11. *In*

$$\frac{\Gamma \Rightarrow \alpha \quad \alpha, \Sigma \Rightarrow \gamma}{\Gamma, \Sigma \Rightarrow \gamma}\ (\text{cut})$$

if the premisses admit cut-free derivations in $\mathrm{G3cLT}_\omega$*, then the conclusion also admits a cut-free derivation.*

Proof: By transfinite lexicographic induction on the rank of instances of *cut* and case analysis. We first show the reduction steps for cuts with cut formula principal in both premisses, i.e. *principal cuts.* Then we show how non-principal cuts are reduced by permutation, maintaining the cut formula but reducing the sum of heights. We give the details only of the permutations of cuts into the first premiss; permutations into the second premiss are covered generically.

1. If the cut formula is principal in each premiss for instances of initial sequents, then the conclusion is already an initial sequent, so the cut can be eliminated.

2. If the first premiss is an instance of an initial sequent with the atom $\mathrm{X}^i p$ principal and $\mathrm{X}^i p$ is the cut formula, then the conclusion may be obtained by admissible (we-left) from the second premiss, regardless of the rule used in the second premiss, as in

$$\frac{\Gamma, \mathrm{X}^i p \Rightarrow \mathrm{X}^i p \quad \mathrm{X}^i p, \Gamma' \Rightarrow \gamma}{\Gamma, \mathrm{X}^i p, \Gamma' \Rightarrow \gamma}\ (\text{cut})$$

3. If the cut formula $\mathrm{X}^i\mathrm{F}\alpha$ is principal in each premiss, then we consider the cut

$$\frac{\dfrac{\Gamma \Rightarrow \mathrm{X}^{i+k}\alpha}{\Gamma \Rightarrow \mathrm{X}^i\mathrm{F}\alpha}\ (\text{Fright}) \quad \dfrac{\{\mathrm{X}^{i+j}\alpha, \Gamma' \Rightarrow \gamma\}_{j\in\omega}}{\mathrm{X}^i\mathrm{F}\alpha, \Gamma' \Rightarrow \gamma}\ (\text{Fleft})}{\Gamma, \Gamma' \Rightarrow \gamma}\ (\text{cut})$$

which we transform into

$$\dfrac{\Gamma \Rightarrow \mathrm{X}^{i+k}\alpha \quad \mathrm{X}^{i+k}\alpha, \Gamma' \Rightarrow \gamma}{\Gamma, \Gamma' \Rightarrow \gamma}\ (\mathrm{cut})$$

a cut on a smaller formula. The case with cut formula $\mathrm{X}^i\mathrm{G}\alpha$ is treated in a dual way.

4. Principal cuts with formulas with binary connectives and quantifiers as outermost logical constant are reduced as in the standard proof for G3c (cf. [41]).

5. If the second premiss is an instance of an initial sequent with the atom $\mathrm{X}^i p$ principal and $\mathrm{X}^i p$ is the cut formula, then the conclusion may be obtained by (we-left) from the first premiss, regardless of the rule used in the first premiss.

6. If the second premiss is an instance of an initial sequent with the formula $X^i p$ principal but $X^i p$ not the cut formula, then the conclusion is already an initial sequent, regardless of the rule used in the first premiss.

7. If the cut formula α is not principal in the left premiss, we reason by cases on the last rule used to derive it. Since the calculus is single succedent, the rule cannot be a right rule.

8. It is (Fleft), we have

$$\dfrac{\dfrac{\{\Gamma, \mathrm{X}^{i+j}\beta \Rightarrow \alpha\}_{j\in\omega}}{\Gamma, \mathrm{X}^i\mathrm{F}\beta \Rightarrow \alpha}\ (\mathrm{Fleft}) \qquad \alpha, \Gamma' \Rightarrow \gamma}{\Gamma, \Gamma', \mathrm{X}^i\mathrm{F}\beta \Rightarrow \gamma}\ (\mathrm{cut})$$

which can be transformed to

$$\dfrac{\ldots \qquad \dfrac{\Gamma, \mathrm{X}^{i+j}\beta \Rightarrow \Delta, \alpha \quad \alpha, \Gamma' \Rightarrow \gamma}{\Gamma, \Gamma', \mathrm{X}^{i+j}\beta \Rightarrow \gamma}\ (\mathrm{cut}) \qquad \ldots}{\Gamma, \Gamma', \mathrm{X}^i\mathrm{F}\beta \Rightarrow \gamma}\ (\mathrm{Fleft})$$

i.e., the cut is "permuted upwards" to each of the premisses of (Fleft), with unchanged cut formula γ and reduced total height. All the other cases of non-principal cuts with finitary rules are treated in a similar way.

9. If the cut formula α is not principal in the second premiss, and that premiss is not an initial sequents, then a standard permutation into the second premiss is applicable, with resulting cut(s) of reduced height.

Observe that in each case we have reduced the rank of the cut. □

THEOREM 2.12. *Rule* (cut) *is admissible in* G3cLT$_\omega$.

PROOF: It remains to show that an arbitrary derivation using instances (possibly infinite in number) of rule (cut) can be transformed to a cut-free derivation. Since this number may be infinite, we argue by transfinite induction on the height of the derivation. Consider a derivation $\mathcal{D}$; if it does not end with (cut), but with a step by rule R, then, by inductive hypothesis, each premiss (which has height less than $ht(\mathcal{D})$) can be transformed to a cut-free derivation (with conclusion unchanged), and thus so, by adding an R-step, can $\mathcal{D}$. Otherwise, if $\mathcal{D}$ ends with a cut, the derivations of its premisses both have height less than $ht(\mathcal{D})$; by inductive hypothesis, each can be transformed to a cut-free derivation (with conclusion unchanged). We now use the Lemma to obtain a cut-free derivation of the conclusion of $\mathcal{D}$. □

3. Natural deduction and normalization

3.1. Natural deduction

Next, we define a Gentzen-style natural deduction system NLT$_\omega$ for until-free propositional linear-time temporal logic. As usual in a definition of a natural deduction system we use the notation $[\alpha]$ to denote a discharged assumption (i.e., the formula α is a discharged assumption by the underlying logical rule).

Definition 3.1 (NLT_ω). Let i and k be arbitrary natural numbers.[12] The logical rules of NLT_ω are of the following form, where the discharge in rules that discharge assumptions can be simple, vacuous, or multiple:[13]

$$\dfrac{\begin{array}{c}[\mathrm{X}^i\alpha]\\ \vdots \\ \mathrm{X}^i\beta\end{array}}{\mathrm{X}^i(\alpha{\to}\beta)}\ (\to\mathrm{I}) \qquad \dfrac{\mathrm{X}^i(\alpha{\to}\beta) \quad \mathrm{X}^i\alpha}{\mathrm{X}^i\beta}\ (\to\mathrm{E})$$

$$\dfrac{\mathrm{X}^i\neg\alpha \quad \mathrm{X}^i\alpha}{\gamma}\ (\mathrm{EXP}) \qquad \dfrac{\begin{array}{c}[\mathrm{X}^i\neg\alpha]\\ \vdots \\ \gamma\end{array} \quad \begin{array}{c}[\mathrm{X}^i\alpha]\\ \vdots \\ \gamma\end{array}}{\gamma}\ (\mathrm{EXM}) \qquad \dfrac{\begin{array}{c}[\mathrm{X}^i\alpha]\\ \vdots \\ \mathrm{X}^j\neg\gamma\end{array} \quad \begin{array}{c}[\mathrm{X}^i\alpha]\\ \vdots \\ \mathrm{X}^j\gamma\end{array}}{\mathrm{X}^i\neg\alpha}\ (\neg\mathrm{I})$$

$$\dfrac{\mathrm{X}^i\alpha \quad \mathrm{X}^i\beta}{\mathrm{X}^i(\alpha\wedge\beta)}\ (\wedge\mathrm{I}) \qquad \dfrac{\mathrm{X}^i(\alpha\wedge\beta)}{\mathrm{X}^i\alpha}\ (\wedge\mathrm{E1}) \qquad \dfrac{\mathrm{X}^i(\alpha\wedge\beta)}{\mathrm{X}^i\beta}\ (\wedge\mathrm{E2})$$

$$\dfrac{\mathrm{X}^i\alpha}{\mathrm{X}^i(\alpha\vee\beta)}\ (\vee\mathrm{I1}) \qquad \dfrac{\mathrm{X}^i\beta}{\mathrm{X}^i(\alpha\vee\beta)}\ (\vee\mathrm{I2}) \qquad \dfrac{\mathrm{X}^i(\alpha\vee\beta) \quad \begin{array}{c}[\mathrm{X}^i\alpha]\\ \vdots \\ \gamma\end{array} \quad \begin{array}{c}[\mathrm{X}^i\beta]\\ \vdots \\ \gamma\end{array}}{\gamma}\ (\vee\mathrm{E})$$

$$\dfrac{\{\ \mathrm{X}^{i+j}\alpha\ \}_{j\in\omega}}{\mathrm{X}^i\mathrm{G}\alpha}\ (\mathrm{GI}) \qquad \dfrac{\mathrm{X}^i\mathrm{G}\alpha}{\mathrm{X}^{i+k}\alpha}\ (\mathrm{GE})$$

---

[12] In this definition, α, β, and γ represent arbitrary formulas. In particular, compared with the definition in the sequent calculus, γ is treated as a formula rather than as a formula or the empty multiset.

[13] Observe that, for example, as an instance of ($\to$I) we include a rule of the form:

$$\dfrac{\mathrm{X}^i\beta}{\mathrm{X}^i(\alpha{\to}\beta)}\ (\mathrm{Wk}).$$

$$\frac{\mathrm{X}^{i+k}\alpha}{\mathrm{X}^{i}\mathrm{F}\alpha}\ (\mathrm{FI}) \qquad \frac{\mathrm{X}^{i}\mathrm{F}\alpha \quad \left\{ \begin{array}{c} [\mathrm{X}^{i+j}\alpha] \\ \vdots \\ \gamma \end{array} \right\}_{j\in\omega}}{\gamma}\ (\mathrm{FE}).$$

Remark 3.2.

1. Rules (EXP), (EXM), and (¬I) are characteristic features of NLT_ω. Rules (EXP) and (¬I) are temporal generalizations of the original rules introduced by Gentzen. Rule (EXM) is a temporal generalization of the original rule introduced by von Plato [53, 41]. The non-temporal versions of (EXP), (EXM), and (¬I) were also used by Kamide and Negri in [30] for constructing natural deduction systems for logics of strong negation.

2. An extended intuitionistic natural deduction system with the following restricted version (EXM-at) of (the original non-temporal version of) (EXM) was introduced by von Plato who also proved a normalization theorem for this system [53]:

$$\frac{\begin{array}{c} [\neg p] \\ \vdots \\ \gamma \end{array} \quad \begin{array}{c} [p] \\ \vdots \\ \gamma \end{array}}{\gamma}\ (\text{EXM-at})$$

   where p is a propositional variable. This system was introduced as a natural deduction system for classical propositional logic. It was thus shown in [53] that (EXM) can be restricted to (EXM-at) without changing the provability in classical propositional logic.

3. Using (EXP) and (EXM), we can prove the formulas of the form $(\neg\alpha\wedge\alpha)\to\gamma$ and $\neg\alpha\vee\alpha$, respectively:

$$\dfrac{\dfrac{\dfrac{[\neg\alpha\wedge\alpha]^1}{\neg\alpha}\ (\wedge\mathrm{E1}) \quad \dfrac{[\neg\alpha\wedge\alpha]^1}{\alpha}\ (\wedge\mathrm{E2})}{\gamma}\ (\mathrm{EXP})}{(\neg\alpha\wedge\alpha)\to\gamma}\ (\to\mathrm{I})^1 \qquad \dfrac{\dfrac{[\neg\alpha]^1}{\neg\alpha\vee\alpha}\ (\vee\mathrm{I1}) \quad \dfrac{[\alpha]^1}{\neg\alpha\vee\alpha}\ (\vee\mathrm{I2})}{\neg\alpha\vee\alpha}\ (\mathrm{EXM})^1.$$

4. Using $(\neg\mathrm{I})$ and (EXP), we can prove the formulas of the form $\alpha{\to}\neg\neg\alpha$ and $\neg\neg(\alpha{\to}\alpha)$:

$$\dfrac{\dfrac{\dfrac{[\neg\alpha]^2 \quad [\alpha]^1}{\neg\alpha}\,(\mathrm{EXP}) \qquad \dfrac{[\neg\alpha]^2 \quad [\alpha]^1}{\alpha}\,(\mathrm{EXP})}{\neg\neg\alpha}\,(\neg\mathrm{I})^2}{\alpha{\to}\neg\neg\alpha}\,({\to}\mathrm{I})^1$$

and

$$\dfrac{\dfrac{\dfrac{[\alpha]^3}{\alpha{\to}\alpha}\,({\to}\mathrm{I})^3 \quad [\neg(\alpha{\to}\alpha)]^1}{\alpha{\to}\alpha}\,(\mathrm{EXP}) \qquad \dfrac{\dfrac{[\alpha]^2}{\alpha{\to}\alpha}\,({\to}\mathrm{I})^2 \quad [\neg(\alpha{\to}\alpha)]^1}{\neg(\alpha{\to}\alpha)}\,(\mathrm{EXP})}{\neg\neg(\alpha{\to}\alpha)}\,(\neg\mathrm{I})^1$$

5. (GI) has infinitely many premisses and is also represented as:

$$\dfrac{\mathrm{X}^i\alpha \quad \mathrm{X}^{i+1}\alpha \quad \mathrm{X}^{i+2}\alpha \quad \cdots \quad \mathrm{X}^{i+n}\alpha \quad \cdots}{\mathrm{X}^i\mathrm{G}\alpha}\ (\mathrm{GI}).$$

6. (FE) has infinitely many premisses and is also represented as:

$$\dfrac{\mathrm{X}^i\mathrm{F}\alpha \quad \begin{matrix}[\mathrm{X}^i\alpha]\\ \vdots\\ \gamma\end{matrix} \quad \begin{matrix}[\mathrm{X}^{i+1}\alpha]\\ \vdots\\ \gamma\end{matrix} \quad \begin{matrix}[\mathrm{X}^{i+2}\alpha]\\ \vdots\\ \gamma\end{matrix} \quad \cdots \quad \begin{matrix}[\mathrm{X}^{i+n}\alpha]\\ \vdots\\ \gamma\end{matrix} \quad \cdots}{\gamma}\ (\mathrm{FE}).$$

Next, we define some notions for NLT_ω.

DEFINITION 3.3. Rules $({\to}\mathrm{I})$, $(\wedge\mathrm{I})$, $(\vee\mathrm{I1})$, $(\vee\mathrm{I2})$, $(\neg\mathrm{I})$, (GI), (FI), and (EXM) are called *introduction rules*, and rules $({\to}\mathrm{E})$, $(\wedge\mathrm{E1})$, $(\wedge\mathrm{E2})$, $(\vee\mathrm{E})$, (GE), (FE), and (EXP) are called *elimination rules*. The notions of *major* and minor premisses of the rules without (EXM) and (EXP) are defined as usual. If $\mathrm{X}^i\neg\alpha$ and $\mathrm{X}^i\alpha$ are both premisses of (EXP), then $\mathrm{X}^i\neg\alpha$ and $\mathrm{X}^i\alpha$ are called the major and minor premisses of (EXP), respectively. The notions of *derivation*, *(open and discharged) assumptions* of a derivation, and *end-formula* of a derivation are also defined as usual. For a derivation $\mathcal{D}$, we use the expression $\mathrm{oa}(\mathcal{D})$ to denote the multiset of open assumptions of $\mathcal{D}$ and the expression $\mathrm{end}(\mathcal{D})$ to denote the end-formula of $\mathcal{D}$. A formula α is said to be *provable* in a natural deduction system L if there exists a derivation of L with no open assumption whose end-formula is α.

Remark 3.4. There are no notions of the major and minor premisses of (EXM) and ($\neg$I). Namely, the premisses of (EXM) and ($\neg$I) are neither major nor minor premisses. In this study, (EXP) is treated as an elimination rule, and (EXM) is treated as an introduction rule.

In order to define the notion of normal derivations in NLT_ω, we define a reduction relation $\rhd$ on the set of derivations in NLT_ω. Before defining $\rhd$, we introduce some notions related to $\rhd$.

Definition 3.5. Let α be a formula occurring in a derivation $\mathcal{D}$ in NLT_ω. Then, α is called a *maximum formula* in $\mathcal{D}$ if α satisfies the following conditions:

1. α is the conclusion of an introduction rule, ($\vee$E), or (EXP),
2. α is the major premiss of an elimination rule.

A derivation is said to be *normal* if it contains no maximum formula. The notion of substitution of derivations for assumptions is defined as usual. We assume that the set of derivations is closed under substitution.

Definition 3.6 (Reduction relation). Let γ be a maximum formula in a derivation that is the conclusion of a rule R.

The definition of the reduction relation $\rhd$ at γ in NLT_ω is obtained by the following conditions.

1. R is $(\to\mathrm{I})$ and γ is $\mathrm{X}^i(\alpha\to\beta)$:

$$\dfrac{\dfrac{\begin{matrix}[\mathrm{X}^i\alpha]\\ \vdots\,\mathcal{D}\\ \mathrm{X}^i\beta\end{matrix}}{\mathrm{X}^i(\alpha\to\beta)}\,(\to\mathrm{I}) \qquad \begin{matrix}\vdots\,\mathcal{E}\\ \mathrm{X}^i\alpha\end{matrix}}{\mathrm{X}^i\beta}\,(\to\mathrm{E}) \qquad \rhd \qquad \begin{matrix}\vdots\,\mathcal{E}\\ \mathrm{X}^i\alpha\\ \vdots\,\mathcal{D}\\ \mathrm{X}^i\beta.\end{matrix}$$

2. R is (EXP):

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\mathcal{D}_1\\ \mathrm{X}^i\neg\delta\end{matrix} \quad \begin{matrix}\vdots\,\mathcal{D}_2\\ \mathrm{X}^i\delta\end{matrix}}{\gamma}\,(\mathrm{EXP}) \qquad \begin{matrix}\vdots\,\mathcal{E}_1\\ \pi_1\end{matrix} \quad \begin{matrix}\vdots\,\mathcal{E}_2\\ \pi_2\end{matrix}}{\pi}\,R' \qquad \rhd \qquad \dfrac{\begin{matrix}\vdots\,\mathcal{D}_1\\ \mathrm{X}^i\neg\delta\end{matrix} \quad \begin{matrix}\vdots\,\mathcal{D}_2\\ \mathrm{X}^i\delta\end{matrix}}{\pi}\,(\mathrm{EXP})$$

   where R' is an arbitrary rule, and both $\mathcal{E}_1$ and $\mathcal{E}_2$ are derivations of the minor premisses of R' if they exist.

3. R is $(\neg\mathrm{I})$, γ is $\mathrm{X}^i\neg\alpha$, and β is the conclusion of (EXP):

$$\dfrac{\dfrac{\begin{matrix}[\mathrm{X}^i\alpha]\\ \vdots\,\mathcal{D}_1\\ \mathrm{X}^j\neg\delta\end{matrix} \quad \begin{matrix}[\mathrm{X}^i\alpha]\\ \vdots\,\mathcal{D}_2\\ \mathrm{X}^j\delta\end{matrix}}{\mathrm{X}^i\neg\alpha}\,(\neg\mathrm{I}) \qquad \begin{matrix}\vdots\,\mathcal{E}\\ \mathrm{X}^i\alpha\end{matrix}}{\beta}\,(\mathrm{EXP}) \qquad \rhd \qquad \dfrac{\begin{matrix}\vdots\,\mathcal{E}\\ \mathrm{X}^i\alpha\\ \vdots\,\mathcal{D}_1\\ \mathrm{X}^j\neg\delta\end{matrix} \quad \begin{matrix}\vdots\,\mathcal{E}\\ \mathrm{X}^i\alpha\\ \vdots\,\mathcal{D}_2\\ \mathrm{X}^j\delta\end{matrix}}{\beta}\,(\mathrm{EXP}).$$

4. R is ($\neg$I), γ is $\mathrm{X}^i\neg\delta$, and $\mathrm{X}^i\delta$ is the conclusion of (EXP):

$$\dfrac{\dfrac{\begin{matrix}[\mathrm{X}^i\delta]\\ \vdots\,\mathcal{D}_1\\ \mathrm{X}^j\neg\beta\end{matrix}\qquad \begin{matrix}[\mathrm{X}^i\delta]\\ \vdots\,\mathcal{D}_2\\ \mathrm{X}^j\beta\end{matrix}}{\mathrm{X}^i\neg\delta}\,(\neg\mathrm{I})\qquad \begin{matrix}\vdots\,\mathcal{E}\\ \mathrm{X}^i\delta\end{matrix}}{\mathrm{X}^i\delta}\,(\mathrm{EXP}) \qquad\rhd\qquad \begin{matrix}\vdots\,\mathcal{E}\\ \mathrm{X}^i\delta.\end{matrix}$$

5. R is (EXM) and γ is $\mathrm{X}^i(\gamma_1{\rightarrow}\gamma_2)$, $\mathrm{X}^i(\gamma_1{\wedge}\gamma_2)$, or $\mathrm{X}^i(\gamma_1{\vee}\gamma_2)$:

$$\dfrac{\dfrac{\begin{matrix}[\mathrm{X}^i\neg\alpha]\\ \vdots\,\mathcal{D}_1\\ \gamma\end{matrix}\qquad \begin{matrix}[\mathrm{X}^i\alpha]\\ \vdots\,\mathcal{D}_2\\ \gamma\end{matrix}}{\gamma}\,(\mathrm{EXM})\qquad \begin{matrix}\vdots\,\mathcal{E}_1\\ \delta_1\end{matrix}\qquad \begin{matrix}\vdots\,\mathcal{E}_2\\ \delta_2\end{matrix}}{\delta}\,R'$$

$$\rhd\qquad \dfrac{\dfrac{\begin{matrix}[\mathrm{X}^i\neg\alpha]\\ \vdots\,\mathcal{D}_1\\ \gamma\end{matrix}\qquad \begin{matrix}\vdots\,\mathcal{E}_1\\ \delta_1\end{matrix}\qquad \begin{matrix}\vdots\,\mathcal{E}_2\\ \delta_2\end{matrix}}{\delta}\,R'\qquad \dfrac{\begin{matrix}[\mathrm{X}^i\alpha]\\ \vdots\,\mathcal{D}_2\\ \gamma\end{matrix}\qquad \begin{matrix}\vdots\,\mathcal{E}_1\\ \delta_1\end{matrix}\qquad \begin{matrix}\vdots\,\mathcal{E}_2\\ \delta_2\end{matrix}}{\delta}\,R'}{\delta}\,(\mathrm{EXM})$$

where R' is ($\rightarrow$E), ($\wedge$E1), ($\wedge$E2), or ($\vee$E), and both $\mathcal{E}_1$ and $\mathcal{E}_2$ are derivations of the minor premisses of R' if they exist.

6. R is (EXM), γ is $\mathrm{X}^i\neg\delta$, and $\mathrm{X}^i\delta$ is the conclusion of (EXP):

$$\dfrac{\dfrac{\begin{matrix}[\mathrm{X}^i\neg\alpha]\\ \vdots\,\mathcal{D}_1\\ \mathrm{X}^i\neg\delta\end{matrix}\qquad \begin{matrix}[\mathrm{X}^i\alpha]\\ \vdots\,\mathcal{D}_2\\ \mathrm{X}^i\neg\delta\end{matrix}}{\mathrm{X}^i\neg\delta}\,(\mathrm{EXM})\qquad \begin{matrix}\vdots\,\mathcal{E}\\ \mathrm{X}^i\delta\end{matrix}}{\mathrm{X}^i\delta}\,(\mathrm{EXP}) \qquad\rhd\qquad \begin{matrix}\vdots\,\mathcal{E}\\ \mathrm{X}^i\delta.\end{matrix}$$

7. R is ($\wedge$I) and γ is $\mathrm{X}^i(\alpha_1 \wedge \alpha_2)$:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\mathcal{D}_1\\ \mathrm{X}^i\alpha_1\end{array} \quad \begin{array}{c}\vdots\,\mathcal{D}_2\\ \mathrm{X}^i\alpha_2\end{array}}{\mathrm{X}^i(\alpha_1\wedge\alpha_2)}\,(\wedge\mathrm{I})}{\mathrm{X}^i\alpha_i}\,(\wedge\mathrm{E}i) \qquad \triangleright \qquad \begin{array}{c}\vdots\,\mathcal{D}_i\\ \mathrm{X}^i\alpha_i\end{array}$$

where i is 1 or 2.

8. R is ($\vee$I1) or ($\vee$I2) and γ is $\mathrm{X}^i(\alpha_1 \vee \alpha_2)$:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\mathcal{D}\\ \mathrm{X}^i\alpha_i\end{array}}{\mathrm{X}^i(\alpha_1\vee\alpha_2)}\,(\vee\mathrm{I}i) \quad \begin{array}{c}[\mathrm{X}^i\alpha_1]\\ \vdots\,\mathcal{E}_1\\ \delta\end{array} \quad \begin{array}{c}[\mathrm{X}^i\alpha_2]\\ \vdots\,\mathcal{E}_2\\ \delta\end{array}}{\delta}\,(\vee\mathrm{E}) \qquad \triangleright \qquad \begin{array}{c}\vdots\,\mathcal{D}\\ \mathrm{X}^i\alpha_i\\ \vdots\,\mathcal{E}_i\\ \delta\end{array}$$

where i is 1 or 2.

9. R is ($\vee$E):

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\mathcal{D}_1\\ \mathrm{X}^i(\alpha\vee\beta)\end{array} \quad \begin{array}{c}[\mathrm{X}^i\alpha]\\ \vdots\,\mathcal{D}_2\\ \pi\end{array} \quad \begin{array}{c}[\mathrm{X}^i\beta]\\ \vdots\,\mathcal{D}_3\\ \pi\end{array}}{\pi}\,(\vee\mathrm{E}) \quad \begin{array}{c}\vdots\,\mathcal{E}_n\\ \{\ \delta_n\ \}\end{array}}{\delta}\,R'$$

$$\triangleright \qquad \dfrac{\begin{array}{c}\vdots\,\mathcal{D}_1\\ \mathrm{X}^i(\alpha\vee\beta)\end{array} \quad \dfrac{\begin{array}{c}[\mathrm{X}^i\alpha]\\ \vdots\,\mathcal{D}_2\\ \pi\end{array} \quad \begin{array}{c}\vdots\,\mathcal{E}_n\\ \{\ \delta_n\ \}\end{array}}{\delta}\,R' \quad \dfrac{\begin{array}{c}[\mathrm{X}^i\beta]\\ \vdots\,\mathcal{D}_3\\ \pi\end{array} \quad \begin{array}{c}\vdots\,\mathcal{E}_n\\ \{\ \delta_n\ \}\end{array}}{\delta}\,R'}{\delta}\,(\vee\mathrm{E})$$

where R' is an arbitrary rule, and $\mathcal{E}_1$, $\mathcal{E}_2$, ... , $\mathcal{E}_n$, ... are derivations of the minor premisses of R' if they exist.

10. R is (GI) and γ is $\mathrm{X}^i\mathrm{G}\alpha$:

$$\dfrac{\dfrac{\{\ \overset{\vdots\ \mathcal{D}_j}{\mathrm{X}^{i+j}\alpha}\ \}_{j\in\omega}}{\mathrm{X}^i\mathrm{G}\alpha}\ (\mathrm{GI})}{\mathrm{X}^{i+k}\alpha}\ (\mathrm{GE}) \qquad \rhd \qquad \begin{array}{c}\vdots\ \mathcal{D}_k\\ \mathrm{X}^{i+k}\alpha\end{array}$$

where $k \in \omega$.

11. R is (FI) and γ is $\mathrm{X}^i\mathrm{F}\alpha$:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_k\\ \mathrm{X}^{i+k}\alpha\end{array}}{\mathrm{X}^i\mathrm{F}\alpha}\ (\mathrm{FI}) \qquad \begin{array}{c}[\mathrm{X}^{i+j}\alpha]\\ \vdots\ \mathcal{E}_j\\ \{\ \delta\ \}_{j\in\omega}\end{array}}{\delta}\ (\mathrm{FE}) \qquad \rhd \qquad \begin{array}{c}\vdots\ \mathcal{D}_k\\ \mathrm{X}^{i+k}\alpha\\ \vdots\ \mathcal{E}_k\\ \delta\end{array}$$

where $k \in \omega$.

12. R is (FE):

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathcal{D}\\ \mathrm{X}^i\mathrm{F}\alpha\end{array} \qquad \begin{array}{c}[\mathrm{X}^{i+j}\alpha]\\ \vdots\ \mathcal{D}_j\\ \{\ \pi\ \}_{j\in\omega}\end{array}}{\pi}\ (\mathrm{FE}) \qquad \begin{array}{c}\vdots\ \mathcal{E}_n\\ \{\ \delta_n\ \}\end{array}}{\delta}\ R'$$

$$\rhd \qquad \dfrac{\begin{array}{c}\vdots\ \mathcal{D}\\ \mathrm{X}^i\mathrm{F}\alpha\end{array} \qquad \dfrac{\begin{array}{c}[\mathrm{X}^{i+j}\alpha]\\ \vdots\ \mathcal{D}_j\\ \pi\end{array} \qquad \begin{array}{c}\vdots\ \mathcal{E}_n\\ \{\ \delta_n\ \}\end{array}}{\{\ \delta\ \}_{j\in\omega}}\ R'}{\delta}\ (\mathrm{FE})$$

where R' is an arbitrary rule, and $\mathcal{E}_1$, $\mathcal{E}_2$, ... , $\mathcal{E}_n$, ... are derivations of the minor premisses of R' if they exist.

13. The set of derivations is closed under $\rhd$.

Remark 3.7. We could consider some other reduction conditions that can reduce a redundant derivation to a simpler derivation. The following are examples of such conditions. However, in this study, we do not introduce these conditions. If we did, we would have to appropriately change the notion of normal form according to these additional reduction conditions.

1. R is (EXP) and γ is in the premisses of (EXM):

$$\dfrac{\dfrac{[\mathrm{X}^i\neg\alpha]^1 \quad \begin{array}{c}\vdots\ \mathcal{D}_1\\ \mathrm{X}^i\alpha\end{array}}{\gamma}\,(\mathrm{EXP}) \qquad \dfrac{\begin{array}{c}\vdots\ \mathcal{D}_2\\ \mathrm{X}^i\neg\alpha\end{array} \quad [\mathrm{X}^i\alpha]^1}{\gamma}\,(\mathrm{EXP})}{\gamma}\,(\mathrm{EXM})^1$$

$$\rhd \qquad \dfrac{\begin{array}{c}\vdots\ \mathcal{D}_2\\ \mathrm{X}^i\neg\alpha\end{array} \quad \begin{array}{c}\vdots\ \mathcal{D}_1\\ \mathrm{X}^i\alpha\end{array}}{\gamma}\,(\mathrm{EXP}).$$

2. R is (EXM), γ is the minor premiss of (EXP), and $\neg\gamma$ is the conclusion of (EXP):

$$\dfrac{\dfrac{\begin{array}{c}[\mathrm{X}^i\neg\alpha]\\ \vdots\ \mathcal{D}_1\\ \gamma\end{array} \quad \begin{array}{c}[\mathrm{X}^i\alpha]\\ \vdots\ \mathcal{D}_2\\ \gamma\end{array}}{\gamma}\,(\mathrm{EXM}) \quad \begin{array}{c}\vdots\ \mathcal{E}\\ \neg\gamma\end{array}}{\neg\gamma}\,(\mathrm{EXP}) \qquad \rhd \qquad \begin{array}{c}\vdots\ \mathcal{E}\\ \neg\gamma.\end{array}$$

DEFINITION 3.8. If $\mathcal{D}'$ is obtained from $\mathcal{D}$ by the reduction relation defined in Definition 3.6 then this fact is denoted by $\mathcal{D} \rhd \mathcal{D}'$. A sequence $\mathcal{D}_0, \mathcal{D}_1, \ldots$ of derivations is called a *reduction sequence* if it satisfies the following conditions:

1. $\mathcal{D}_i \rhd \mathcal{D}_{i+1}$ for all $i \geq 0$,
2. the last derivation in the sequence is normal if the sequence is finite.

A derivation $\mathcal{D}$ is called *normalizable* if there is a finite reduction sequence starting from $\mathcal{D}$.

3.2. Equivalence and normalization

In the following discussion, a derivation of $\Gamma \Rightarrow$ in $\mathrm{G3cLT}_\omega$ is interpreted as a derivation $\mathcal{D}$ in NLT_ω such that $\mathrm{oa}(\mathcal{D}) = \Gamma$ and $\mathrm{end}(\mathcal{D}) = \neg p \wedge p$.

Definition 3.9. A multiset Δ of formulas is called a *multiset reduct* of a multiset Γ of formulas if Δ is obtained from Γ by multiplying formulas in Γ, where zero multiplicity is also permitted.[14] For example, $\{\alpha, \alpha, \alpha, \beta\}$ is a multiset reduct of $\{\alpha, \beta, \gamma\}$. Note that the relation of being a multiset reduct is reflexive and transitive.

We use an expression Γ^* to denote a multiset reduct of Γ. We also use an expression $\Gamma \subseteq^* \Delta$ to denote the fact that Γ is a multiset reduct of Δ.

Lemma 3.10. *The following hold:*

1. *If $\mathcal{D}$ is a derivation in NLT_ω such that $\mathrm{oa}(\mathcal{D}) = \Gamma$ and $\mathrm{end}(\mathcal{D}) = \beta$, then $\mathrm{G3cLT}_\omega \vdash \Gamma \Rightarrow \beta$,*

2. *If $\mathrm{G3cLT}_\omega \vdash \Gamma \Rightarrow \beta$, then we can obtain a derivation $\mathcal{D}'$ in NLT_ω such that*

   (a) $\mathrm{oa}(\mathcal{D}') \subseteq^* \Gamma$,

   (b) $\mathrm{end}(\mathcal{D}') = \beta$,

   (c) *$\mathcal{D}'$ is normal.*

Proof:

1. We prove this by induction on the height of the derivations $\mathcal{D}$ of NLT_ω such that $\mathrm{oa}(\mathcal{D}) = \Gamma$ and $\mathrm{end}(\mathcal{D}) = \beta$. We distinguish the cases according to the last rule of $\mathcal{D}$. We show some cases.

   (a) Case ($\rightarrow$I): This case is divided into three cases.

[14] The notion of multiset reduct was introduced in [40, pp. 1805, Definition 1]. It is sometimes thought that natural deduction would not be able to express the rule of weakening and therefore derivability in natural deduction is defined as: γ is derivable from Γ if there is a derivation with open assumptions contained in Γ. Instead of this, the notion of multiset reduct was introduced.

i. $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c}[\mathrm{X}^i\alpha]\ \Gamma \\ \vdots\ \mathcal{E} \\ \mathrm{X}^i\gamma\end{array}}{\mathrm{X}^i(\alpha\to\gamma)}\ (\to\mathrm{I})$$

where $\mathrm{oa}(\mathcal{D}) = \{\mathrm{X}^i\alpha\} \cup \Gamma$ and $\mathrm{end}(\mathcal{D}) = \gamma$. By induction hypothesis, we have $\mathrm{G3cLT}_\omega \vdash \mathrm{X}^i\alpha, \Gamma \Rightarrow \mathrm{X}^i\gamma$. Then, we obtain that $\mathrm{G3cLT}_\omega \vdash \Gamma \Rightarrow \mathrm{X}^i(\alpha\to\gamma)$:

$$\dfrac{\begin{array}{c}\vdots\ \mathit{Ind.\,hyp.} \\ \mathrm{X}^i\alpha, \Gamma \Rightarrow \mathrm{X}^i\gamma\end{array}}{\Gamma \Rightarrow \mathrm{X}^i(\alpha\to\gamma)}\ (\to\text{right}).$$

ii. $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c}\Gamma \\ \vdots\ \mathcal{E} \\ \mathrm{X}^i\gamma\end{array}}{\mathrm{X}^i(\alpha\to\gamma)}\ (\to\mathrm{I})$$

where $\mathrm{oa}(\mathcal{D}) = \Gamma$ and $\mathrm{end}(\mathcal{D}) = \gamma$. By induction hypothesis, we have $\mathrm{G3cLT}_\omega \vdash \Gamma \Rightarrow \mathrm{X}^i\gamma$. Then, we obtain that $\mathrm{G3cLT}_\omega \vdash \Gamma \Rightarrow \mathrm{X}^i(\alpha\to\gamma)$:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathit{Ind.\,hyp.} \\ \Gamma \Rightarrow \mathrm{X}^i\gamma\end{array}}{\mathrm{X}^i\alpha, \Gamma \Rightarrow \mathrm{X}^i\gamma}\ (\text{we-left})}{\Gamma \Rightarrow \mathrm{X}^i(\alpha\to\gamma)}\ (\to\text{right})$$

where (we-left) is admissible in $\mathrm{G3cLT}_\omega$ by Proposition 2.8.

iii. $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c}[\mathrm{X}^i\alpha, \mathrm{X}^i\alpha]\ \Gamma \\ \vdots\ \mathcal{E} \\ \mathrm{X}^i\gamma\end{array}}{\mathrm{X}^i(\alpha{\to}\gamma)}\ ({\to}\mathrm{I})$$

where $\mathrm{oa}(\mathcal{D}) = \{\mathrm{X}^i\alpha, \mathrm{X}^i\alpha\} \cup \Gamma$ and $\mathrm{end}(\mathcal{D}) = \gamma$. By induction hypothesis, we have $\mathrm{G3cLT}_\omega \vdash \mathrm{X}^i\alpha, \mathrm{X}^i\alpha, \Gamma \Rightarrow \mathrm{X}^i\gamma$. Then, by applying an admissible step of contraction we obtain that $\mathrm{G3cLT}_\omega \vdash \Gamma \Rightarrow \mathrm{X}^i(\alpha{\to}\gamma)$:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathit{Ind.\,hyp.} \\ \mathrm{X}^i\alpha, \mathrm{X}^i\alpha, \Gamma \Rightarrow \mathrm{X}^i\gamma\end{array}}{\mathrm{X}^i\alpha, \Gamma \Rightarrow \mathrm{X}^i\gamma}\ (\text{co-left})}{\Gamma \Rightarrow \mathrm{X}^i(\alpha{\to}\gamma)}\ ({\to}\text{right})$$

where (co-left) is admissible in $\mathrm{G3cLT}_\omega$ by Lemma 2.10. All the other cases where multiple discharge of assumptions is used in natural deduction are handled in a similar way via admissible contraction steps and we shall not detail them further.

(b) Case ($\neg$I): $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c}[\mathrm{X}^i\alpha]\Gamma_1 \\ \vdots\ \mathcal{D}_1 \\ \mathrm{X}^j\neg\gamma\end{array} \qquad \begin{array}{c}[\mathrm{X}^i\alpha]\Gamma_2 \\ \vdots\ \mathcal{D}_2 \\ \mathrm{X}^j\gamma\end{array}}{\mathrm{X}^i\neg\alpha}\ (\neg\mathrm{I})$$

where $\mathrm{oa}(\mathcal{D}) = \Gamma_1 \cup \Gamma_2$ and $\mathrm{end}(\mathcal{D}) = \mathrm{X}^i\neg\alpha$. By induction hypotheses, we have $\mathrm{G3cLT}_\omega \vdash \mathrm{X}^i\alpha, \Gamma_1 \Rightarrow \mathrm{X}^j\neg\gamma$ and $\mathrm{G3cLT}_\omega \vdash \mathrm{X}^i\alpha, \Gamma_2 \Rightarrow \mathrm{X}^j\gamma$. Then, we obtain that $\mathrm{G3cLT}_\omega \vdash \Gamma_1, \Gamma_2 \Rightarrow \mathrm{X}^i\neg\alpha$:

$$
\dfrac{\dfrac{\vdots\ \textit{Ind. hyp.}}{X^i\alpha, \Gamma_1 \Rightarrow X^j\neg\gamma} \qquad \dfrac{\dfrac{\dfrac{\vdots\ \textit{Ind. hyp.}}{X^i\alpha, \Gamma_2 \Rightarrow X^j\gamma}}{X^i\neg\gamma, X^i\alpha, \Gamma_2 \Rightarrow X^j\gamma}\ (\text{we-left})}{X^j\neg\gamma, X^i\alpha, \Gamma_2 \Rightarrow}\ (\neg\text{left})}{\dfrac{\dfrac{X^i\alpha, X^i\alpha, \Gamma_1, \Gamma_2 \Rightarrow}{X^i\alpha, \Gamma_1, \Gamma_2 \Rightarrow}\ (\text{co-left})}{\Gamma_1, \Gamma_2 \Rightarrow X^i\neg\alpha}\ (\neg\text{right})}\ (\text{cut})
$$

where (we-left) and (co-left) are admissible in $\mathrm{G3cLT}_\omega$ by Proposition 2.8 and Lemma 2.10, respectively.

(c) Case (EXP): $\mathcal{D}$ is of the form:

$$
\dfrac{\begin{array}{c}\Gamma_1 \\ \vdots\ \mathcal{E}_1 \\ X^i\neg\alpha\end{array} \qquad \begin{array}{c}\Gamma_2 \\ \vdots\ \mathcal{E}_2 \\ X^i\alpha\end{array}}{\beta}\ (\text{EXP})
$$

where $\mathrm{oa}(\mathcal{D}) = \Gamma_1 \cup \Gamma_2$ and $\mathrm{end}(\mathcal{D}) = \beta$. By induction hypotheses, we have $\mathrm{G3cLT}_\omega \vdash \Gamma_1 \Rightarrow X^i\neg\alpha$ and $\mathrm{G3cLT}_\omega \vdash \Gamma_2 \Rightarrow X^i\alpha$. Then, we obtain the required fact that $\mathrm{G3cLT}_\omega \vdash \Gamma_1, \Gamma_2 \Rightarrow \beta$:

$$
\dfrac{\dfrac{\vdots\ \textit{Ind. hyp.}}{\Gamma_2 \Rightarrow X^i\alpha} \qquad \dfrac{\dfrac{\vdots\ \textit{Ind. hyp.}}{\Gamma_1 \Rightarrow X^i\neg\alpha} \qquad \dfrac{\dfrac{\vdots\ \textit{Prop. 2.7}}{X^i\neg\alpha, X^i\alpha \Rightarrow X^i\alpha}}{X^i\neg\alpha, X^i\alpha \Rightarrow}\ (\neg\text{left})}{X^i\alpha, \Gamma_1 \Rightarrow}\ (\text{cut})}{\dfrac{\Gamma_1, \Gamma_2 \Rightarrow}{\Gamma_1, \Gamma_2 \Rightarrow \beta}\ (\text{we-right})}\ (\text{cut})
$$

where (cut) is admissible in $\mathrm{G3cLT}_\omega$ by Theorem 2.12.

(d) Case (EXM): $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{cc} [\mathrm{X}^i\neg\alpha]\Gamma_1 & [\mathrm{X}^i\alpha]\Gamma_2 \\ \vdots\ \mathcal{E}_1 & \vdots\ \mathcal{E}_2 \\ \gamma & \gamma \end{array}}{\gamma}\ (\mathrm{EXM})$$

where $\mathrm{oa}(\mathcal{D}) = \Gamma_1 \cup \Gamma_2$ and $\mathrm{end}(\mathcal{D}) = \gamma$. By induction hypotheses, we have $\mathrm{G3cLT}_\omega \vdash \mathrm{X}^i\neg\alpha, \Gamma_1 \Rightarrow \gamma$ and $\mathrm{G3cLT}_\omega \vdash \mathrm{X}^i\alpha, \Gamma_2 \Rightarrow \gamma$. Then, we obtain the required fact that $\mathrm{G3cLT}_\omega \vdash \Gamma_1, \Gamma_2 \Rightarrow \gamma$:

$$\dfrac{\begin{array}{cc} \vdots\ \mathit{Ind.\,hyp.} & \vdots\ \mathit{Ind.\,hyp.} \\ \mathrm{X}^i\neg\alpha, \Gamma_1 \Rightarrow \gamma & \mathrm{X}^i\alpha, \Gamma_2 \Rightarrow \gamma \\ \vdots\ (\text{we-left}) & \vdots\ (\text{we-left}) \\ \mathrm{X}^i\neg\alpha, \Gamma_1, \Gamma_2 \Rightarrow \gamma & \mathrm{X}^i\alpha, \Gamma_1, \Gamma_2 \Rightarrow \gamma \end{array}}{\Gamma_1, \Gamma_2 \Rightarrow \gamma}\ (\text{ex-middle})$$

where (we-left) is admissible in $\mathrm{G3cLT}_\omega$ by Proposition 2.8.

(e) Case (GI): $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c} \Gamma_j \\ \vdots\ P_j \\ \{\ \mathrm{X}^{i+j}\alpha\ \}_{j\in\omega} \end{array}}{\mathrm{X}^i\mathrm{G}\alpha}\ (\mathrm{GI})$$

where $\mathrm{oa}(\mathcal{D}) = \Gamma = \bigcup\limits_{j\in\omega} \Gamma_j$ and $\mathrm{end}(\mathcal{D}) = \mathrm{X}^i\mathrm{G}\alpha$. By induction hypotheses, we have $\mathrm{G3cLT}_\omega \vdash \Gamma_j \Rightarrow \mathrm{X}^{i+j}\alpha$ for all $j \in \omega$. Then, we obtain the required fact $\mathrm{G3cLT}_\omega \vdash \Gamma \Rightarrow \mathrm{X}^i\mathrm{G}\alpha$:

$$\dfrac{\begin{array}{c} \vdots\ \textit{Ind. hyp.} \\ \Gamma_j \Rightarrow \mathrm{X}^{i+j}\alpha \\ \vdots\ \text{(we-left)} \\ \{\ \Gamma \Rightarrow \mathrm{X}^{i+j}\alpha\ \}_{j\in\omega} \end{array}}{\Gamma \Rightarrow \mathrm{X}^{i}\mathrm{G}\alpha}\ \text{(Gright)}$$

where (we-left) is admissible in $\mathrm{G3cLT}_\omega$ by Proposition 2.8. Note that the induction hypothesis is applied for each of the denumerable set of premisses.

(f) Case (GE): $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c} \Gamma \\ \vdots\ \mathcal{D}' \\ \mathrm{X}^{i}\mathrm{G}\alpha \end{array}}{\mathrm{X}^{i+k}\alpha}\ \text{(GE)}$$

where $\mathrm{oa}(\mathcal{D}) = \Gamma$ and $\mathrm{end}(\mathcal{D}) = \mathrm{X}^{i+k}\alpha$. By induction hypothesis, we have $\mathrm{G3cLT}_\omega \vdash \Gamma \Rightarrow \mathrm{X}^{i}\mathrm{G}\alpha$. Then, we obtain the required fact that $\mathrm{G3cLT}_\omega \vdash \Gamma \Rightarrow \mathrm{X}^{i+k}\alpha$:

$$\dfrac{\begin{array}{c} \vdots\ \textit{Ind. hyp.} \\ \Gamma \Rightarrow \mathrm{X}^{i}\mathrm{G}\alpha \end{array} \qquad \dfrac{\begin{array}{c} \vdots\ \textit{Prop. } 2.7 \\ \mathrm{X}^{i}\mathrm{G}\alpha, \mathrm{X}^{i+k}\alpha \Rightarrow \mathrm{X}^{i+k}\alpha \end{array}}{\mathrm{X}^{i}\mathrm{G}\alpha \Rightarrow \mathrm{X}^{i+k}\alpha}\ \text{(Gleft)}}{\Gamma \Rightarrow \mathrm{X}^{i+k}\alpha}\ \text{(cut)}$$

where (cut) is admissible in $\mathrm{G3cLT}_\omega$ by Theorem 2.12.

(g) Case (FI): $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c} \Gamma \\ \vdots\ \mathcal{D}' \\ \mathrm{X}^{i+k}\alpha \end{array}}{\mathrm{X}^{i}\mathrm{F}\alpha}\ \text{(FI)}$$

where $\mathrm{oa}(\mathcal{D}) = \Gamma$ and $\mathrm{end}(\mathcal{D}) = \mathrm{X}^{i}\mathrm{F}\alpha$. By induction hypothesis, we have $\mathrm{G3cLT}_\omega \vdash \Gamma \Rightarrow \mathrm{X}^{i+k}\alpha$. Then, we obtain the required fact that $\mathrm{G3cLT}_\omega \vdash \Gamma \Rightarrow \mathrm{X}^i\mathrm{F}\alpha$:

$$\dfrac{\begin{array}{c}\vdots\ \textit{Ind. hyp.}\\ \Gamma \Rightarrow \mathrm{X}^{i+k}\alpha\end{array}}{\Gamma \Rightarrow \mathrm{X}^i\mathrm{F}\alpha}\ (\mathrm{Fright}).$$

(h) Case (FE):
$\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c}\Gamma'\\ \vdots\ \mathcal{D}'\\ \mathrm{X}^i\mathrm{F}\alpha\end{array}\qquad \left\{\begin{array}{c}[\mathrm{X}^{i+j}\alpha]\Gamma_j\\ \vdots\ \mathcal{D}_j\\ \gamma\end{array}\right\}_{j\in\omega}}{\gamma}\ (\mathrm{FE})$$

where $\mathrm{oa}(\mathcal{D}) = \Gamma' \cup \Gamma$ with $\Gamma = \bigcup_{j\in\omega}\Gamma_j$ and $\mathrm{end}(\mathcal{D}) = \gamma$. By induction hypotheses, we have $\mathrm{G3cLT}_\omega \vdash \Gamma' \Rightarrow \mathrm{X}^i\mathrm{F}\alpha$ and $\mathrm{G3cLT}_\omega \vdash \mathrm{X}^{i+j}\alpha, \Gamma_j \Rightarrow \gamma$ for all $j \in \omega$. Then, we obtain the required fact, that $\mathrm{G3cLT}_\omega \vdash \Gamma', \Gamma \Rightarrow \gamma$ by the following derivation where the induction hypothesis is applied for each of the denumerable set of premisses:

$$\dfrac{\begin{array}{c}\vdots\ \textit{Ind. hyp.}\\ \Gamma' \Rightarrow \mathrm{X}^i\mathrm{F}\alpha\end{array}\qquad \dfrac{\left\{\begin{array}{c}\vdots\ \textit{Ind. hyp.}\\ \mathrm{X}^{i+j}\alpha, \Gamma_j \Rightarrow \gamma\\ \vdots\ (\text{we-left})\\ \mathrm{X}^{i+j}\alpha, \Gamma \Rightarrow \gamma\end{array}\right\}_{j\in\omega}}{\mathrm{X}^i\mathrm{F}\alpha, \Gamma \Rightarrow \gamma}\ (\mathrm{Fleft})}{\Gamma', \Gamma \Rightarrow \gamma}\ (\mathrm{cut})$$

where (cut) and (we-left) are admissible in $\mathrm{G3cLT}_\omega$ by Theorem 2.12 and Proposition 2.8, respectively.

2. We prove this by induction on the derivations $\mathcal{D}$ of $\Gamma \Rightarrow \beta$ in $\mathrm{G3cLT}_\omega$.

We distinguish the cases according to the last rule of $\mathcal{D}$. We show some cases.

(a) Case (init): $\mathcal{D}$ is of the form:

$$\begin{array}{c} \vdots\ \mathcal{D} \\ \mathrm{X}^i p, \Gamma \Rightarrow \mathrm{X}^i p. \end{array}$$

Then, we have a normal derivation $\mathcal{E}$ in NLT_ω of the form:

$$\begin{array}{c} \vdots\ \mathcal{E} \\ \mathrm{X}^i p \end{array}$$

where $\mathrm{oa}(\mathcal{E}) = \{\mathrm{X}^i p\} \subseteq^* \{\mathrm{X}^i p\} \cup \Gamma$ and $\mathrm{end}(\mathcal{E}) = \mathrm{X}^i p$.

(b) Case (we-right): $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c} \vdots\ \mathcal{D}' \\ \Gamma \Rightarrow \end{array}}{\Gamma \Rightarrow \alpha}\ (\text{we-right})$$

By induction hypothesis, we have a normal derivation $\mathcal{E}'$ in NLT_ω of the form:

$$\begin{array}{c} \Gamma^* \\ \vdots\ \mathcal{E}' \\ \neg p \wedge p \end{array}$$

where $\mathrm{oa}(\mathcal{E}') = \Gamma^* \subseteq^* \Gamma$ and $\mathrm{end}(\mathcal{E}') = \neg p \wedge p$. Then, we obtain a required normal derivation $\mathcal{E}$ by:

$$\dfrac{\dfrac{\begin{array}{c} \Gamma^* \\ \vdots\ \mathcal{E}' \\ \neg p \wedge p \end{array}}{\neg p}\ (\wedge\mathrm{E1}) \qquad \dfrac{\begin{array}{c} \Gamma^* \\ \vdots\ \mathcal{E}' \\ \neg p \wedge p \end{array}}{p}\ (\wedge\mathrm{E2})}{\alpha}\ (\mathrm{Exp})$$

where $\mathrm{oa}(\mathcal{E}) = \Gamma^* \subseteq^* \Gamma$ and $\mathrm{end}(\mathcal{E}) = \alpha$.

(c) Case ($\neg$left): $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c}\vdots\ \mathcal{D}'\\ \mathrm{X}^i\neg\alpha, \Gamma \Rightarrow \mathrm{X}^i\alpha\end{array}}{\mathrm{X}^i\neg\alpha, \Gamma \Rightarrow}\ (\neg\text{left}).$$

By induction hypothesis, we have a normal derivation $\mathcal{E}'$ in NLT_ω of the form:

$$\begin{array}{c}(\mathrm{X}^i\neg\alpha, \Gamma)^*\\ \vdots\ \mathcal{E}'\\ \mathrm{X}^i\alpha\end{array}$$

where $\mathrm{oa}(\mathcal{E}') = (\mathrm{X}^i\neg\alpha, \Gamma)^* \subseteq^* \{\mathrm{X}^i\neg\alpha\} \cup \Gamma$ and $\mathrm{end}(\mathcal{E}') = \mathrm{X}^i\alpha$. Then, we obtain a required normal derivation $\mathcal{E}$ by:

$$\dfrac{\mathrm{X}^i\neg\alpha \qquad \begin{array}{c}(\mathrm{X}^i\neg\alpha, \Gamma)^*\\ \vdots\ \mathcal{E}'\\ \mathrm{X}^i\alpha\end{array}}{\neg p \wedge p}\ (\mathrm{EXP})$$

where $\mathrm{oa}(\mathcal{E}) = (\mathrm{X}^i\neg\alpha, \mathrm{X}^i\neg\alpha, \Gamma)^* \subseteq^* \{\mathrm{X}^i\neg\alpha\} \cup \Gamma$ and $\mathrm{end}(\mathcal{E}) = \neg p \wedge p$ (i.e., $\bot$). We remark that $\{\mathrm{X}^i\neg\alpha, \mathrm{X}^i\neg\alpha\} \cup \Gamma$ is a multiset reduct of $\{\mathrm{X}^i\neg\alpha\} \cup \Gamma$. We also remark that the last rule (EXP) in $\mathcal{E}$ cannot be replaced with ($\to$E), because using ($\to$E) entails a possibility of developing a non-normal derivation. Namely, there is a possibility of the case that the last rule of $\mathcal{E}'$ is ($\to$I).

(d) Case ($\neg$right): $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c}\vdots\ \mathcal{D}'\\ \mathrm{X}^i\alpha, \Gamma \Rightarrow\end{array}}{\Gamma \Rightarrow \mathrm{X}^i\neg\alpha}\ (\neg\text{right}).$$

By induction hypothesis, we have a normal derivation $\mathcal{E}'$ in NLT_ω of the form:

$$\begin{array}{c} (\mathrm{X}^i\alpha, \Gamma)^* \\ \vdots\ \mathcal{E}' \\ \neg p \wedge p \end{array}$$

where $\mathrm{oa}(\mathcal{E}') = (\mathrm{X}^i\alpha, \Gamma)^* \subseteq^* \{\mathrm{X}^i\alpha\} \cup \Gamma$ and $\mathrm{end}(\mathcal{E}') = \neg p \wedge p$. Then, we obtain a required normal derivation $\mathcal{E}$ by:

$$\dfrac{\dfrac{\begin{array}{c} [\mathrm{X}^i\alpha]^1\ \Gamma^* \\ \vdots\ \mathcal{E}' \\ \neg p \wedge p \end{array}}{\neg p}\,(\wedge\mathrm{E}1) \qquad \dfrac{\begin{array}{c} [\mathrm{X}^i\alpha]^1\ \Gamma^* \\ \vdots\ \mathcal{E}' \\ \neg p \wedge p \end{array}}{p}\,(\wedge\mathrm{E}2)}{\mathrm{X}^i\neg\alpha}\,(\neg\mathrm{I})^1$$

where $\mathrm{oa}(\mathcal{E}) = \Gamma^* \subseteq^* \Gamma$ and $\mathrm{end}(\mathcal{E}) = \mathrm{X}^i\neg\alpha$.

(e) Case (ex-middle): $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c} \vdots\ \mathcal{D}_1 \\ \mathrm{X}^i\neg\alpha, \Gamma \Rightarrow \gamma \end{array} \quad \begin{array}{c} \vdots\ \mathcal{D}_2 \\ \mathrm{X}^i\alpha, \Gamma \Rightarrow \gamma \end{array}}{\Gamma \Rightarrow \gamma}\,(\text{ex-middle}).$$

By induction hypotheses, we have normal derivations $\mathcal{E}_1$ and $\mathcal{E}_2$ in NLT_ω of the form:

$$\begin{array}{c} (\mathrm{X}^i\neg\alpha, \Gamma)^* \\ \vdots\ \mathcal{E}_1 \\ \gamma \end{array} \qquad \begin{array}{c} (\mathrm{X}^i\alpha, \Gamma)^* \\ \vdots\ \mathcal{E}_2 \\ \gamma \end{array}$$

where $\mathrm{oa}(\mathcal{E}_1) = (\mathrm{X}^i\neg\alpha, \Gamma)^* \subseteq^* \{\mathrm{X}^i\neg\alpha\} \cup \Gamma$, $\mathrm{oa}(\mathcal{E}_2) = (\mathrm{X}^i\alpha, \Gamma)^* \subseteq^* \{\mathrm{X}^i\alpha\} \cup \Gamma$, $\mathrm{end}(\mathcal{E}_1) = \gamma$, and $\mathrm{end}(\mathcal{E}_2) = \gamma$. Then, we obtain a required normal derivation $\mathcal{E}$ by:

$$\dfrac{\begin{array}{c} [\mathrm{X}^i\neg\alpha]\Gamma^* \\ \vdots\ \mathcal{E}_1 \\ \gamma \end{array} \quad \begin{array}{c} [\mathrm{X}^i\alpha]\Gamma^* \\ \vdots\ \mathcal{E}_2 \\ \gamma \end{array}}{\gamma}\,(\mathrm{EXM})$$

where $\mathrm{oa}(\mathcal{E}) = \Gamma^* \subseteq^* \Gamma$ and $\mathrm{end}(\mathcal{E}) = \gamma$.

(f) Case (Gleft): $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c}\vdots\ \mathcal{D}'\\ \mathrm{X}^i\mathrm{G}\alpha, \mathrm{X}^{i+k}\alpha, \Gamma \Rightarrow \gamma\end{array}}{\mathrm{X}^i\mathrm{G}\alpha, \Gamma \Rightarrow \gamma}\ (\mathrm{Gleft}).$$

By induction hypothesis, we have a normal derivation $\mathcal{E}'$ in NLT_ω of the form:

$$\begin{array}{c}(\mathrm{X}^i\mathrm{G}\alpha \quad \mathrm{X}^{i+k}\alpha \quad \Gamma)^*\\ \vdots\ \mathcal{E}'\\ \gamma\end{array}$$

where $\mathrm{oa}(\mathcal{E}') = (\mathrm{X}^i\mathrm{G}\alpha, \mathrm{X}^{i+k}\alpha, \Gamma)^* \subseteq^* \{\mathrm{X}^i\mathrm{G}\alpha, \mathrm{X}^{i+k}\alpha\} \cup \Gamma$ and $\mathrm{end}(\mathcal{E}') = \gamma$. Then, we obtain a required normal derivation $\mathcal{E}$ by:

$$\begin{array}{ccc}(\mathrm{X}^i\mathrm{G}\alpha & \dfrac{\mathrm{X}^i\mathrm{G}\alpha}{\mathrm{X}^{i+k}\alpha}\ (\mathrm{GE}) & \Gamma)^*\\ \vdots & & \vdots\\ & \vdots\ \mathcal{E}' & \\ & \gamma & \end{array}$$

where $\mathrm{oa}(\mathcal{E}) = (\mathrm{X}^i\mathrm{G}\alpha, \mathrm{X}^i\mathrm{G}\alpha, \Gamma)^* \subseteq^* \{\mathrm{X}^i\mathrm{G}\alpha\} \cup \Gamma$ and $\mathrm{end}(\mathcal{E}) = \gamma$.

(g) Case (Gright): $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c}\vdots\ \mathcal{D}'\\ \{\ \Gamma \Rightarrow \mathrm{X}^{i+j}\alpha\ \}_{j\in\omega}\end{array}}{\Gamma \Rightarrow \mathrm{X}^i\mathrm{G}\alpha}\ (\mathrm{Gright}).$$

By induction hypotheses, we have normal derivations $\mathcal{E}_j$ for all $j \in \omega$ in NLT_ω of the form:

$$\begin{array}{c} \Gamma_j^* \\ \vdots\ \mathcal{E}_j \\ \mathrm{X}^{i+j}\alpha \end{array}$$

where $\mathrm{oa}(\mathcal{E}_j) = \Gamma_j^* \subseteq^* \Gamma_j$ with $\Gamma^* = \bigcup_{j\in\omega} \Gamma_j^* \subseteq^* \Gamma$, and $\mathrm{end}(\mathcal{E}_j) = \mathrm{X}^{i+j}\alpha$. Then, we obtain a required normal derivation $\mathcal{E}$ by:

$$\dfrac{\begin{array}{c} \Gamma_j^* \\ \vdots\ \mathcal{E}_j \\ \{\ \mathrm{X}^{i+j}\alpha\ \}_{j\in\omega} \end{array}}{\mathrm{X}^i\mathrm{G}\alpha}\ (\mathrm{GI})$$

where $\mathrm{oa}(\mathcal{E}) = \Gamma^* \subseteq^* \Gamma$ and $\mathrm{end}(\mathcal{E}) = \mathrm{X}^i\mathrm{G}\alpha$.

(h) Case (Fleft): $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c} \vdots\ \mathcal{D}' \\ \{\ \mathrm{X}^{i+k}\alpha, \Gamma \Rightarrow \gamma\ \}_{j\in\omega} \end{array}}{\mathrm{X}^i\mathrm{F}\alpha, \Gamma \Rightarrow \gamma}\ (\mathrm{Fleft}).$$

By induction hypotheses, we have normal derivations $\mathcal{E}_j$ for all $j \in \omega$ in NLT_ω of the form:

$$\begin{array}{c} (\mathrm{X}^{i+j}\alpha \quad \Gamma_j)^* \\ \vdots\ \mathcal{E}_j \\ \gamma \end{array}$$

where $\mathrm{oa}(\mathcal{E}_j) = (\mathrm{X}^{i+j}\alpha, \Gamma_j)^* \subseteq^* \{\mathrm{X}^{i+j}\alpha\} \cup \Gamma_j$ with $\Gamma^* = \bigcup_{j\in\omega} \Gamma_j^* \subseteq^* \Gamma$ and $\mathrm{end}(\mathcal{E}_j) = \gamma$. Then, we obtain a required normal derivation $\mathcal{E}$ by:

$$
\dfrac{\mathrm{X}^i\mathrm{F}\alpha \quad \left\{ \begin{array}{c} [\mathrm{X}^{i+j}\alpha]^1 \quad \Gamma_j^* \\ \vdots\ \mathcal{E}_j \\ \gamma \end{array} \right\}_{j\in\omega}}{\gamma}\ (\mathrm{FE})^1
$$

where $\mathrm{oa}(\mathcal{E}) = (\mathrm{X}^i\mathrm{F}\alpha, \Gamma)^* \subseteq^* \{\mathrm{X}^i\mathrm{F}\alpha\} \cup \Gamma$ and $\mathrm{end}(\mathcal{E}) = \gamma$.

(i) Case (Fright): $\mathcal{D}$ is of the form:

$$
\dfrac{\begin{array}{c} \vdots\ \mathcal{D}' \\ \Gamma \Rightarrow \mathrm{X}^{i+k}\alpha \end{array}}{\Gamma \Rightarrow \mathrm{X}^i\mathrm{F}\alpha}\ (\text{Fright}).
$$

By induction hypothesis, we have a normal derivations $\mathcal{E}'$ in NLT_ω of the form:

$$
\begin{array}{c} \Gamma^* \\ \vdots\ \mathcal{E}' \\ \mathrm{X}^{i+k}\alpha \end{array}
$$

where $\mathrm{oa}(\mathcal{E}') = \Gamma^* \subseteq^* \Gamma$ and $\mathrm{end}(\mathcal{E}') = \mathrm{X}^{i+k}\alpha$. Then, we obtain a required normal derivation $\mathcal{E}$ by:

$$
\dfrac{\begin{array}{c} \Gamma^* \\ \vdots\ \mathcal{E}' \\ \mathrm{X}^{i+k}\alpha \end{array}}{\mathrm{X}^i\mathrm{F}\alpha}\ (\mathrm{FI})
$$

where $\mathrm{oa}(\mathcal{E}) = \Gamma^* \subseteq^* \Gamma$ and $\mathrm{end}(\mathcal{E}) = \mathrm{X}^i\mathrm{F}\alpha$. □

We obtain the following required theorems.

Theorem 3.11 (Equivalence between NLT_ω and $\mathrm{G3cLT}_\omega$). *For any formula α, $\mathrm{G3cLT}_\omega \vdash\ \Rightarrow \alpha$ iff α is derivable in NLT_ω.*

PROOF: Taking $\emptyset$ as Γ in Lemma 3.10, we obtain the required fact. $\square$

THEOREM 3.12 (Normalization for NLT_ω). *All derivations in* NLT_ω *are normalizable. More precisely, if a derivation* $\mathcal{D}$ *in* NLT_ω *is given, then we can obtain a normal derivation* $\mathcal{E}$ *in* NLT_ω *such that* $\mathrm{oa}(\mathcal{E}) \subseteq^* \mathrm{oa}(\mathcal{D})$ *and* $\mathrm{end}(\mathcal{E}) = \mathrm{end}(\mathcal{D})$.

PROOF: Suppose that a derivation $\mathcal{D}$ in NLT_ω is given, and suppose that $\mathrm{oa}(\mathcal{D}) = \Gamma$ and $\mathrm{end}(\mathcal{D}) = \beta$. Then, by Lemma 3.10 (1), we obtain $\mathrm{G3cLT}_\omega \vdash \Gamma \Rightarrow \beta$. Then, by Lemma 3.10 (2), we can obtain a normal derivation $\mathcal{E}$ in NLT_ω such that $\mathrm{oa}(\mathcal{E}) \subseteq^* \mathrm{oa}(\mathcal{D})$ and $\mathrm{end}(\mathcal{E}) = \mathrm{end}(\mathcal{D})$. $\square$

4. Intuitionistic variant

4.1. Sequent calculus and cut elimination

The language of $\mathrm{G3iLT}_\omega$ is the same as defined in Section 2 for $\mathrm{G3cLT}_\omega$. A sequent for $\mathrm{G3iLT}_\omega$ is an expression of the form $\Gamma \Rightarrow \gamma$ where Γ is a (possibly empty) multiset of formulas and γ is a formula or the empty multiset. The same notions and notations as introduced and presented in Section 2 are used for $\mathrm{G3iLT}_\omega$.

We now define $\mathrm{G3iLT}_\omega$.

DEFINITION 4.1 ($\mathrm{G3iLT}_\omega$). $\mathrm{G3iLT}_\omega$ is obtained from $\mathrm{G3cLT}_\omega$ by deleting (ex-middle).

We have the following propositions.

PROPOSITION 4.2. The sequents of the form $\mathrm{X}^i\alpha, \Gamma \Rightarrow \mathrm{X}^i\alpha$ for any formula α, any set Γ of formulas, and any natural number i are derivable in $\mathrm{G3iLT}_\omega$.

PROOF: Similar to the proof of Proposition 2.7. By induction on α. $\square$

PROPOSITION 4.3. Rule (we-left) is height-preserving admissible in $\mathrm{G3iLT}_\omega$.

PROOF: Similar to the proof of Proposition 2.8. $\square$

PROPOSITION 4.4. Rule (co-left) is height-preserving admissible in $\mathrm{G3iLT}_\omega$.

PROOF: Similar to the proof of Proposition 2.10. $\square$

THEOREM 4.5. *The rule* (cut) *is admissible in* $\mathrm{G3iLT}_\omega$.

PROOF: Similar to the proof of Theorem 2.12. □

We also obtain the following constructive property for $\mathrm{G3iLT}_\omega$.

THEOREM 4.6 (Timed disjunction property for $\mathrm{G3iLT}_\omega$). *For any formulas* α *and* β *and any* $i \in \omega$, *if* $\mathrm{G3iLT}_\omega \vdash \Rightarrow \mathrm{X}^i(\alpha \vee \beta)$, *then either* $\mathrm{G3iLT}_\omega \vdash \Rightarrow \mathrm{X}^i\alpha$ *or* $\mathrm{G3iLT}_\omega \vdash \Rightarrow \mathrm{X}^i\beta$.

PROOF: Immediate by Theorem 4.5. □

4.2. Natural deduction and normalization

The same notions and notations as introduced and presented in Section 3 are used for NILT_ω.

First, we define NILT_ω.

DEFINITION 4.7 (NILT_ω). NILT_ω is obtained from NLT_ω by deleting (EXM).

Next, we define the reduction relation on NILT_ω.

DEFINITION 4.8 (Reduction relation). The definition of the reduction relation on NILT_ω is obtained form Definition 3.6 in NLT_ω by deleting the conditions concerning (EXM).

We then obtain the following lemma.

LEMMA 4.9. *We have the following statements.*

1. *If* $\mathcal{D}$ *is a derivation in* NILT_ω *such that* $\mathrm{oa}(\mathcal{D}) = \Gamma$ *and* $\mathrm{end}(\mathcal{D}) = \beta$, *then* $\mathrm{G3iLT}_\omega \vdash \Gamma \Rightarrow \beta$,
2. *If* $\mathrm{G3iLT}_\omega \vdash \Gamma \Rightarrow \beta$, *then we can obtain a derivation* $\mathcal{D}'$ *in* NILT_ω *such that*
   (a) $\mathrm{oa}(\mathcal{D}') \subseteq^* \Gamma$,
   (b) $\mathrm{end}(\mathcal{D}') = \beta$,
   (c) $\mathcal{D}'$ *is normal.*

PROOF: Similar to the proof of Lemma 3.10. □

We then obtain the following required theorems.

THEOREM 4.10 (Equivalence between NILT_ω and $\mathrm{G3iLT}_\omega$). *For any formula α, $\mathrm{G3iLT}_\omega \vdash \Rightarrow \alpha$ iff α is derivable in NILT_ω.*

PROOF: Similar to the proof of Theorem 3.11. We use Lemma 4.9. □

THEOREM 4.11 (Normalization for NILT_ω). *All derivations in NILT_ω are normalizable. More precisely, if a derivation $\mathcal{D}$ in NILT_ω is given, then we can obtain a normal derivation $\mathcal{E}$ in NILT_ω such that $\mathrm{oa}(\mathcal{E}) \subseteq^* \mathrm{oa}(\mathcal{D})$ and $\mathrm{end}(\mathcal{E}) = \mathrm{end}(\mathcal{D})$.*

PROOF: Similar to the proof of Theorem 3.12. We use Lemma 4.9. □

5. Conclusion and remarks

5.1. Conclusion

In this study, we introduced a unified Gentzen-style proof-theoretic framework for until-free propositional linear-time temporal logic (LTL) and its intuitionistic variant.

First, we proposed the Gentzen-style single-succedent sequent calculus $\mathrm{G3cLT}_\omega$ for until-free propositional LTL. Subsequently, we proved the cut-elimination theorem for $\mathrm{G3cLT}_\omega$ following the methodology for G3-style sequent calculi with infinitary rules, as in [39].

Second, we introduced the Gentzen-style natural deduction system NLT_ω for until-free propositional LTL, along with a reduction relation for NLT_ω. Following this, we established the normalization theorem for NLT_ω by utilizing the equivalence theorem between NLT_ω and $\mathrm{G3cLT}_\omega$.

Third, we introduced and investigated a Gentzen-style sequent calculus, $\mathrm{G3iLT}_\omega$, and a Gentzen-style natural deduction system, NILT_ω, for an intuitionistic variant of the until-free propositional LTL. The systems $\mathrm{G3iLT}_\omega$ and NILT_ω are derived from $\mathrm{G3cLT}_\omega$ and NLT_ω by omitting the rules (ex-middle) and (EXM), respectively. The cut-elimination theorem for $\mathrm{G3iLT}_\omega$ is then immediate as a subcase of the cut-elimination theorem for $\mathrm{G3cLT}_\omega$. Subsequently, we established the normalization theorem for NILT_ω by utilizing the equivalence theorem between NILT_ω and $\mathrm{G3iLT}_\omega$.

5.2. Remarks on the merits of our approach

We now highlight the merits of our approach. In particular, we emphasize the advantages of the proposed infinitary systems, which incorporate logical inference rules with infinitely many premises. These systems exhibit three key features: uniformity, modularity, and compatibility.

Regarding uniformity, by employing inference rules with infinitely many premises, we can treat both Gentzen-style sequent calculi and Gentzen-style natural deduction systems in a uniform manner. In particular, we establish a natural correspondence between the Gentzen-style single-succedent sequent calculi G3cLT$_\omega$ and G3iLT$_\omega$ and the Gentzen-style natural deduction systems NLT$_\omega$ and NILT$_\omega$, respectively.

Regarding modularity, the systems with infinitary rules can be extended in a modular way. In particular, G3cLT$_\omega$ and NLT$_\omega$ are obtained from G3iLT$_\omega$ and NILT$_\omega$ simply by adding the rules *(ex-middle)* and *(EXM)*, respectively. This modularity, together with uniformity, is a distinctive advantage not available in previously proposed systems.

By using rules with infinitely many premises, we also gain an advantage in establishing smoothly a Glivenko theorem for G3cLT$_\omega$ and G3iLT$_\omega$. This result is an analogue of the Glivenko theorem for Gentzen's LK and LI in classical and intuitionistic logics. This theorem is formally presented as follows: For any formula α, G3cLT$_\omega \vdash \Rightarrow \alpha$ if and only if G3iLT$_\omega \vdash \Rightarrow \neg\neg\alpha$. The proof of this theorem can be given in a similar way as presented in [22].

In addition to these merits, by using rules with infinitely many premises, we can obtain certain theorems for embedding G3cLT$_\omega$ and G3iLT$_\omega$ into a Gentzen-style sequent calculus LK$_\omega$ for infinitary classical logic and a Gentzen-style sequent calculus LI$_\omega$ for infinitary intuitionistic logic, respectively. These theorems can be proved in the same way as presented in [19, 23].

Further, we can construct finite fragments of G3cLT$_\omega$ and G3iLT$_\omega$, in which the infinite domain ω of rules with infinitely many premises is restricted to a finite domain $\omega_l = \{n \in \omega \mid n \leq l\}$ for a fixed positive integer l. A system based on LT$_\omega$ of this kind was studied, for example, in [21]. These systems have been shown to be embeddable into LK or LI, and hence are decidable.

The above-mentioned merits imply that our framework is highly compatible with the traditional frameworks of classical logic, intuitionistic logic, infinitary classical logic, and infinitary intuitionistic logic. This naturally extends the traditional proof theory for these standard logics. This was the basic aim of this study.

5.3. Remarks on next-time fragments

The next-time fragments (i.e., the $\{\mathrm{G}, \mathrm{F}\}$-less fragments) of the proposed systems possess several desirable properties. To fix the terminology, let XT_ω, SXT_ω, SIXT_ω, NXT_ω, and NIXT_ω denote the next-time fragments of LT_ω, $\mathrm{G3cLT}_\omega$, $\mathrm{G3iLT}_\omega$, NLT_ω, and NILT_ω, respectively.

Then, the cut-elimination theorems for XT_ω, SXT_ω, and SIXT_ω hold by virtue of the cut-elimination theorems for XT_ω, $\mathrm{G3cLT}_\omega$, and $\mathrm{G3iLT}_\omega$ and their conservativeness. We can demonstrate theorems for embedding XT_ω and $\mathrm{G3iLT}_\omega$ into Gentzen-style sequent calculi for classical logic and intuitionistic logic, respectively. Such a Gentzen-style sequent calculus, referred to here as LK, for classical logic is the X-less fragment of XT_ω (i.e., LK is obtained from XT_ω by deleting all occurrences of X^i). Similarly, such a Gentzen-style sequent calculus, referred to here as LI, for intuitionistic logic is the X-less fragment of SIXT_ω (i.e., LI is obtained from SIXT_ω by deleting all occurrences of X^i).

The equivalence between NXT_ω (or NIXT_ω) and SXT_ω (or SIXT_ω, respectively) can also be established. The normalization theorems for NXT_ω and NIXT_ω can be demonstrated similarly to those for NLT_ω and NILT_ω, since NXT_ω and NIXT_ω are proper subsystems of NLT_ω and NILT_ω, respectively.

As demonstrated above, we can derive the theorems for embedding XT_ω and SIXT_ω into LK and LI, respectively. By virtue of these theorems, we can also establish the decidability of XT_ω and SIXT_ω, as well as the Craig interpolation theorems for XT_ω and SIXT_ω. These results, based on the embedding theorems into LK and LI, cannot be obtained for LT_ω and $\mathrm{G3iLT}_\omega$ because these systems are not embeddable into LK and LI, respectively. Instead, they can be embedded into Gentzen-style sequent calculi

LK_ω and LI_ω for infinitary logic and infinitary intuitionistic logic, respectively, which are known to be undecidable. Additionally, it is well-known that the Craig interpolation theorem does not hold for LTL. For more information on Craig interpolation theorem for the next-time fragment of LTL, see [24]. In conclusion, XT_ω and $SIXT_\omega$ are analogous to classical logic and intuitionistic logic, respectively, while LT_ω and $G3iLT_\omega$ are analogous to infinitary logic and infinitary intuitionistic logic, respectively.

5.4. Related and future works

Gentzen-style sequent calculi and natural deduction systems for some extended intuitionistic variants of until-free propositional LTL with paraconsistent negation were examined by Kamide and Wansing in [31], where the corresponding display sequent calculi were also discussed. Kamide clarified the relationship among until-free propositional LTL, first-order monadic omega-logic, propositional generalized definitional reflection logic, and propositional infinitary logic in [25], using Gentzen-style sequent calculi for the investigation. Recently, Kamide proposed and investigated refutation-aware Gentzen-style sequent calculi for until-free propositional LTL in [27], although their intuitionistic variants and Gentzen-style natural deduction systems were not studied.

Gentzen-style natural deduction systems and related typed λ-calculi for various fragments of LTL and related modal logics have been extensively studied [3, 5, 6, 12, 13, 34, 37, 38, 35, 50, 56, 20] to establish a foundation for staged computation in multi-level programming. Gentzen-style natural deduction systems and sequent calculi for variants of the next-time fragment of LTL were surveyed and investigated in [20], where Davies' logic for binding-time analysis was also discussed.

From an application perspective, Davies [12] proposed a typed λ-calculus λ° (including a next-time operator $\bigcirc$ insted of X) for a fragment of intuitionistic LTL to discuss multi-level binding-time analysis. Taha et al. [50] introduced an extension of λ° called MetaML, which incorporates properties like run-time generation and persistent code. Moggi et al. [37] further developed an extension of MetaML called AIM (an idealized MetaML), and Benaissa et al. [5] proposed a refinement of AIM known as λ^{BN}.

Davies and Pfenning [13] introduced an alternative typed λ-calculus $\lambda^{[]}$ (incorporating an S4-type modal operator $[]$) for intuitionistic S4-modal logic, aimed at analyzing staged computation. Nanevski [38] and Kim et al. [34] explored various type systems based on $\lambda^{[]}$, while Yuse and Igarashi [56] introduced $\lambda^{\circ[]}$, a type system combining λ° and $\lambda^{[]}$, designed to manage both persistent code (using $[]$) and ephemeral code (using $\bigcirc$).

In future work, we aim to prove the strong normalization and Church-Rosser theorems for NLT_ω and NILT_ω, as well as for their first-order extensions. Additionally, we plan to introduce the corresponding typed λ-calculi for NLT_ω and NILT_ω with the Curry-Howard correspondence, and to apply these calculi to the analysis of staged computation in multi-level programming.

Acknowledgements. We would like to thank the anonymous referees for their valuable comments.

References

[1] A. Abuin, A. Bolotov, M. Hermo, P. Lucio, *Tableaux and sequent calculi for CTL and ECTL: Satisfiability test with certifying proofs and models*, **Journal of Logical and Algebraic Methods in Programming 130, 100828**, (2023), DOI: https://doi.org/10.1016/J.JLAMP.2022.100828.

[2] A. Almukdad, D. Nelson, *Constructible falsity and inexact predicates*, **Journal of Symbolic Logic**, vol. 49(1) (1984), pp. 231–233, DOI: https://doi.org/10.2307/2274105.

[3] S. Baratella, A. Masini, *A proof-theoretic investigation of a logic of positions*, **Annals of Pure and Applied Logic**, vol. 123 (2003), pp. 135–162, DOI: https://doi.org/10.1016/S0168-0072(03)00021-6.

[4] S. Baratella, A. Masini, *An approach to infinitary temporal proof theory*, **Archive for Mathematical Logic**, vol. 43(8) (2004), pp. 965–990, DOI: https://doi.org/10.1007/S00153-004-0237-Z.

[5] Z.-E.-A. Benaissa, E. Moggi, W. Taha, T. Sheard, *Logical modalities and multi-stage programming*, [in:] **Proceedings of Workshop on Intuitionistic Modal Logics and Applications (IMLA'99)** (1999), 15 pp.

[6] A. Bolotov, A. Basukoski, O. M. Grigoriev, V. Shangin, *Natural deduction calculus for linear-time temporal logic*, [in:] **Proceedings of the 10th European Conference on Logics in Artificial Intelligence (JELIA 2006)**, vol. 4160 of Lecture Notes in Computer Science (2006), pp. 56–68, DOI: https://doi.org/10.1007/11853886_7.

[7] A. Bolotov, V. Shangin, *Natural deduction system in paraconsistent setting: Proof search for PCont*, **Journal of Intelligent Systems**, vol. 21(1) (2012), pp. 1–24, DOI: https://doi.org/10.1515/JISYS-2011-0021.

[8] B. Boretti, S. Negri, *Decidability for Priorean linear time using a fixed-point labelled calculus*, [in:] **Proceedings of the 18th International Conference on Automated Reasoning with Analytic Tableaux and Related Methods (TABLEAUX)**, vol. 5607 of Lecture Notes in Computer Science (2009), pp. 108–122, DOI: https://doi.org/10.1007/978-3-642-02716-1_9.

[9] B. Boretti, S. Negri, *On the finitization of Priorean linear time*, [in:] M. D'Agostino, G. Giorello, F. Laudisa, T. Pievani, C. Sinigaglia (eds.), **New Essays in Logic and Philosophy of Science**, College Publications, London (2010), pp. 67–79.

[10] K. Brünnler, M. Lange, *Cut-free sequent systems for temporal logic*, **Journal of Logical and Algebraic Methods in Programming**, vol. 76(2) (2008), pp. 216–225, DOI: https://doi.org/10.1016/J.JLAP.2008.02.004.

[11] S. Cerrito, V. Goranko, S. Paillocher, *Partial model checking and partial model synthesis in LTL using a Tableau-based approach*, [in:] **Proceedings of the 8th International Conference on Formal Structures for Computation and Deduction (FSCD)** (2023), pp. 23:1–23:21, DOI: https://doi.org/10.4230/LIPICS.FSCD.2023.23.

[12] R. Davies, *A Temporal-Logic Approach to Binding-Time Analysis*, [in:] **Proceedings of the 11th Annual IEEE Symposium on Logic in Computer Science (LICS)**, IEEE Computer Society (1996), pp. 184–195, DOI: https://doi.org/10.1109/LICS.1996.561317.

[13] R. Davies, F. Pfenning, *A modal analysis of staged computation*, **Journal of the ACM**, vol. 48(3) (2001), pp. 555–604, DOI: https://doi.org/10.1145/382780.382785.

[14] E. A. Emerson, *Temporal and modal logic*, Elsevier and MIT Press (1990), pp. 995–1072, DOI: https://doi.org/10.1016/B978-0-444-88074-1.50021-4.

[15] J. Gaintzarain, M. Hermo, P. Lucio, M. Navarro, F. Orejas, *A cut-free and invariant-free sequent calculus for PLTL*, vol. 4646 of Lecture Notes in Computer Science, Springer (2007), pp. 481–495, DOI: https://doi.org/10.1007/978-3-540-74915-8_36.

[16] Y. Gurevich, *Intuitionistic logic with strong negation*, **Studia Logica**, vol. 36 (1977), pp. 49–59.

[17] S. Huang, R. Cleaveland, *A tableau construction for finite linear-time temporal logic*, **Journal of Logical and Algebraic Methods in Programming**, vol. 125 (2022), p. 100743, DOI: https://doi.org/10.1016/J.JLAMP.2021.100743.

[18] N. Kamide, *An equivalence between sequent calculi for linear-time temporal logic*, **Bulletin of the Section of Logic**, vol. 35(4) (2006), pp. 187–194.

[19] N. Kamide, *Embedding Linear-Time Temporal Logic into Infinitary Logic: Application to Cut-Elimination for Multi-agent Infinitary Epistemic Linear-Time Temporal Logic*, [in:] M. Fisher, F. Sadri, M. Thielscher (eds.), **Computational Logic in Multi-Agent Systems**, Springer Berlin Heidelberg, Berlin, Heidelberg (2009), pp. 57–76, DOI: https://doi.org/10.1007/978-3-642-02734-5_5.

[20] N. Kamide, *Notes on an extension of Davies' logic for binding-time analysis*, **Far East Journal of Applied Mathematics**, vol. 44(1) (2010), pp. 37–57.

[21] N. Kamide, *Bounded linear-time temporal logic: A proof-theoretic investigation*, **Annals of Pure and Applied Logic**, vol. 163(4) (2012), pp. 439–466, DOI: https://doi.org/10.1016/J.APAL.2011.12.002.

[22] N. Kamide, *Temporal Gödel-Gentzen and Girard translations*, **Mathematical Logic Quarterly**, vol. 59(1–2) (2013), pp. 66–83, DOI: https://doi.org/10.1002/MALQ.201100083.

[23] N. Kamide, *Embedding theorems for LTL and its variants*, **Mathematical Structures in Computer Science**, vol. 25(1) (2015), pp. 83–134, DOI: https://doi.org/10.1017/S0960129514000048.

[24] N. Kamide, *Interpolation theorems for some variants of LTL*, **Reports on Mathematical Logic**, vol. 50 (2015), pp. 3–30, URL: https://rml.tcs.uj.edu.pl/rml-50/1-kamide.pdf.

[25] N. Kamide, *Relating first-order monadic omega-logic, propositional linear-time temporal logic, propositional generalized definitional reflection logic and propositional infinitary logic*, **Journal of Logic and Computation**, vol. 27(7) (2017), pp. 2271–2301, DOI: https://doi.org/10.1093/LOGCOM/EXX006.

[26] N. Kamide, *Natural deduction with explosion and excluded middle*, [in:] **Proceedings of the 53rd IEEE International Symposium on Multiple-valued Logic (ISMVL 2023)** (2023), pp. 24–29, DOI: https://doi.org/10.1109/ISMVL57333.2023.00016.

[27] N. Kamide, *Refutation-Aware Gentzen-Style Calculi for Propositional Until-Free Linear-Time Temporal Logic*, **Studia Logica**, vol. 111(6) (2023), pp. 979–1014, DOI: https://doi.org/10.1007/S11225-023-10052-7.

[28] N. Kamide, S. Negri, *A unified Gentzen-style framework for until-free LTL*, vol. 415 of Electronic Proceedings in Theoretical Computer Science (2024), pp. 165–179, DOI: https://doi.org/10.4204/EPTCS.415.16.

[29] N. Kamide, S. Negri, *Proof theory for extended Belnap–Dunn and intuitionistic logics*, **Studia Logica**, (2025), DOI: https://doi.org/10.1007/s11225-025-10203-y, online first.

[30] N. Kamide, S. Negri, *Unified natural deduction for logics of strong negation*, **Notre Dame Journal of Formal Logic**, vol. 66(4) (2025), pp. 543–580, DOI: https://doi.org/10.1215/00294527-2025-0016.

[31] N. Kamide, H. Wansing, *Combining linear-time temporal logic with constructiveness and paraconsistency*, **Journal of Applied Logic**, vol. 8(1) (2011), pp. 33–61, DOI: https://doi.org/10.1016/J.JAL.2009.06.001.

[32] N. Kamide, H. Wansing, *A paraconsistent linear-time temporal logic*, **Fundamenta Informaticae**, vol. 106(1) (2011), pp. 1–23, DOI: https://doi.org/10.3233/FI-2011-374.

[33] H. Kawai, *Sequential calculus for a first order infinitary temporal logic*, **Zeitschrift für Mathematische Logik und Grundlagen der Mathematik**, vol. 33 (1987), pp. 423–432, DOI: https://doi.org/10.1002/MALQ.19870330506.

[34] I. Kim, K. Yi, C. Calcagno, *A polymorphic modal type system for lisp-like multi-staged languages*, [in:] J. G. Morrisett, S. L. P. Jones (eds.), **Proceedings of the 33rd ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages (POPL)**, ACM (2006), pp. 257–268, DOI: https://doi.org/10.1145/1111037.1111060.

[35] K. Kojima, A. Igarashi, *On constructive linear-time temporal logic*, [in:] **Proceedings of Intuitionistic Modal Logics and Applications Workshop (IMLA'08)** (2008), 12 pp.

[36] N. Kürbis, Y. Petrukhin, *Normalisation for some quite interesting many-valued logics*, **Logic and Logical Philosophy**, vol. 30(3) (2021), pp. 493–534.

[37] E. Moggi, W. Taha, Z.-E.-A. Benaissa, T. Sheard, *An Idealized MetaML: Simpler, and More Expressive*, [in:] S. D. Swierstra (ed.), **Proceedings of the 8th European Symposium on Programming (ESOP'99)**, vol. 1576 of Lecture Notes in Computer Science, Springer (1999), pp. 193–207, DOI: https://doi.org/10.1007/3-540-49099-X_13.

[38] A. Nanevski, *Meta-programming with names and necessity*, [in:] M. Wand, S. L. P. Jones (eds.), **Proceedings of the Seventh ACM SIGPLAN International Conference on Functional Programming (ICFP '02)**, ACM (2002), pp. 206–217, DOI: https://doi.org/10.1145/581478.581498.

[39] S. Negri, *Geometric rules in infinitary logic*, [in:] O. Arieli, A. Zamansky (eds.), **Arnon Avron on Semantics and Proof Theory of Non-Classical Logics, Outstanding Contributions to Logic**, Springer (2021), pp. 265–293.

[40] S. Negri, J. von Plato, *Sequent calculus in natural deduction style*, **Journal of Symbolic Logic 66 (4)**, (2001), pp. 1803–1816, DOI: https://doi.org/10.2307/2694976.

[41] S. Negri, J. von Plato, **Structural Proof Theory** (2001), DOI: https://doi.org/10.1017/CBO9780511527340.

[42] D. Nelson, *Constructible falsity*, **Journal of Symbolic Logic**, vol. 14 (1949), pp. 16–26, DOI: https://doi.org/10.2307/2268973.

[43] B. Paech, *Gentzen-Systems for propositional temporal logics*, [in:] E. Börger, H. K. Büning, M. M. Richter (eds.), **CSL '88**, Springer Berlin Heidelberg, Berlin, Heidelberg (1989), pp. 240–253, DOI: https://doi.org/10.1007/BFb0026305.

[44] R. Pliuškevičius, *Investigation of finitary calculus for a discrete linear time logic by means of infinitary calculus*, [in:] J. Bārzdinš, D. Bjørner (eds.), **Baltic Computer Science**, Springer Berlin Heidelberg, Berlin, Heidelberg (1991), pp. 504–528, DOI: https://doi.org/10.1007/BFb0019366.

[45] A. Pnueli, *The temporal logic of programs* (1977), pp. 46–57, DOI: https://doi.org/10.1109/SFCS.1977.32.

[46] D. Prawitz, **Natural deduction: a proof-theoretical study** (1965), DOI: https://doi.org/10.2307/2271676.

[47] G. Priest, *Natural Deduction Systems for Logics in the FDE Family*, [in:] H. Omori, H. Wansing (eds.), **New Essays on Belnap-Dunn Logic**, Springer International Publishing, Cham (2019), pp. 279–292, DOI: https://doi.org/10.1007/978-3-030-31136-0_16.

[48] M. E. Szabo, *Collected papers of Gerhard Gentzen*, [in:] M. E. Szabo (ed.), **Studies in Logic and the Foundations of Mathematics**, North-Holland (1969), DOI: https://doi.org/10.2307/2272429, english translation.

[49] M. E. Szabo, *A sequent calculus for Kröger logic*, [in:] A. Salwicki (ed.), **Logics of Programs and Their Applications**, Springer Berlin Heidelberg, Berlin, Heidelberg (1983), pp. 295–303, DOI: https://doi.org/10.1007/3-540-11981-7_21.

[50] W. Taha, T. Sheard, *Multi-Stage Programming with Explicit Annotations*, [in:] J. P. Gallagher, C. Consel, A. M. Berman (eds.), **Proceedings of the ACM SIGPLAN Symposium on Partial Evaluation and Semantics-Based Program Manipulation (PEPM '97)**, ACM (1997), pp. 203–217, DOI: https://doi.org/10.1145/258993.259019.

[51] L. Tranchini, *Natural deduction for bi-intuitionistic logic*, **Journal of Applied Logic**, vol. 25(Supplement) (2017), pp. S72–S96, DOI: https://doi.org/10.1016/J.JAL.2017.12.001.

[52] A. S. Troelstra, H. Schwichtenberg, **Basic Proof Theory**, vol. 45 of Cambridge Tracts in Theoretical Computer Science, Cambridge University Press, (2000), DOI: https://doi.org/10.1017/CBO9781139168717.

[53] J. von Plato, *Proof theory of full classical propositional logic* (1999), manuscript, 16 pp.

[54] J. von Plato, **Elements of Logical Reasoning** (2014), DOI: https://doi.org/10.1017/CBO9781139567862.

[55] J. von Plato, **Saved from the Cellar: Gerhard Gentzen's Shorthand Notes on Logic & Foundations of Mathematics** (2017), DOI: https://doi.org/10.1007/978-3-319-42120-9.

[56] Y. Yuse, A. Igarashi, *A modal type system for multi-level generating extensions with persistent code*, [in:] A. Bossi, M. J. Maher (eds.), **Proceedings of the 8th International ACM SIGPLAN Conference on Principles and Practice of Declarative Programming**, ACM (2006), pp. 201–212, DOI: https://doi.org/10.1145/1140335.1140360.

Norihiro Kamide
Nagoya City University
School of Data Science
Yamanohata 1, Mizuho-cho, Mizuho-ku, Nagoya
467-8501, Aichi, Japan
e-mail: drnkamide08@kpd.biglobe.ne.jp

Sara Negri

University of Genoa
Department of Mathematics
via Dodecaneso 35
16146 Genoa, Italy

e-mail: sara.negri@unige.it

Funding information: This research was supported by JSPS KAKENHI Grant Number 23K10990, the project "Infinity and Intensionality: Towards A New Synthesis" funded by the Research Council of Norway, and the project "Modalities in Substructural Logics: Theory, Methods and Applications MOSAIC", funded by the Community Research and Development Information Service (CORDIS) of the European Commission. The second author also acknowledges the MIUR Excellence Department Project awarded to Dipartimento di Matematica, Università di Genova, CUP D33C23001110001 and the "Gruppo Nazionale per le Strutture Algebriche, Geometriche e le loro Applicazioni" (GNSAGA) of the Istituto Nazionale di Alta Matematica (INdAM).

Conflict of interests: None.

Ethical considerations: The Authors assure of no violations of publication ethics and take full responsibility for the content of the publication.

The percentage share of the author in the preparation of the work: Norihiro Kamide %, Sara Negri 50%

Declaration regarding the use of GAI tools: Not used.

Bulletin of the **S**ection of **L**ogic
Volume 54/2 (2025), pp. 283–323
https://doi.org/10.18778/0138-0680.2025.06

Kaito Ichikura

CONTINUA OF LOGICS RELATED TO INTUITIONISTIC AND MINIMAL LOGICS

Abstract

We analyze the relationship between logics around intuitionistic logic and minimal logic. We characterize the intersection of minimal logic and co-minimal logic introduced by Vakarelov, and reformulate logics given in the previous studies by Vakarelov, Bezhanishvili, Colacito, de Jongh, Vargas, and Niki in a uniform language. We also compare the new logic with other known logics in terms of the cardinalities of logics between them. Specifically, we apply Wronski's algebraic semantics, instead of neighborhood semantics used in the previous studies, to show the existence of continua of logics between known logics and the new logic. This result is an extension of the conventional results, and the proof is given in a simpler way.

Keywords: intuitionistic logic, minimal logic, subminimal logic, co-minimal logic, Yankov formula.

Presented by: Michał Zawidzki
Received: January 25, 2025, **Received in revised form**: April 18, 2025,
Accepted: June 6, 2025, **Published online**: July 7, 2025

1. Introduction

In this paper, we explore the relation between known and new logics with a focus on subminimal logics, which were introduced by [13, 12]. Since we have two backgrounds, we will first describe the backgrounds and then clarify the aims of the paper.

1.1. Vakarelov's logics and Minimal logic

Intuitionistic logic contains *ex contradictione quodlibet* (ECQ for short), which is the inference rule deriving any conclusion B from A and $\neg A$. Minimal logic, which is introduced by [6], is the logic excluding ECQ from Intuitionistic logic. When we treat classical, intuitionistic and minimal logics, negation is usually defined by making use of the absurdity constant and implication, i.e., $\neg A := A \to \bot$. We call this kind of negation *intuitionistic negation* by following the terminology used by [13, 12]. Using intuitionistic negation, minimal logic is the weakest logic with respect to the strength of negation, on the assumption that the implication is at least intuitionistic.

However, there is another way to treat negation in classical/intuitionistic/minimal logics. That is, to take negation as a *primitive* logical connective with one argument. We call this kind of negation *subminimal negation* by following the terminology used in [13] again. In [13], subminimal negation was introduced to analyze strong negation in a more general framework. By using subminimal negation, we can analyze properties of negation in a more detailed way than intuitionistic negation, and we can define logics weaker than minimal logic, which we shall call *subminimal logics*.

In this paper, we are interested in two systems introduced in [13]. One is the $\bot$-free fragment of **SUBMIN** (Definition 6.1), which is one of the subminimal logics. The other is co-minimal logic (**CO-MIN** for short) (Definition 2.7), which has ECQ. The positive fragment of **CO-MIN** coincides with that of intuitionistic logic, but **CO-MIN** is neither weaker nor stronger than minimal logic. Moreover, **CO-MIN** is stronger than **SUBMIN**. The relationship between the four systems discussed by Vakarelov can be summarized as follows.

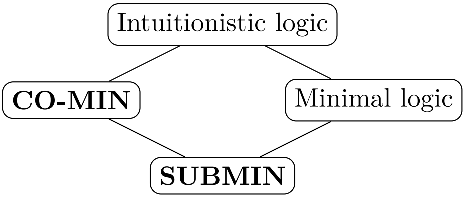


Figure 1: Main systems discussed by Vakarelov

Note here that the systems above the lines are strictly stronger than the lower systems.

1.2. Subminimal logics

Let us now move on to the second background. In [3, 4], several subminimal logics that are closely related to **SUBMIN** were introduced. By using subminimal negation instead of intuitionistic negation, we can treat negation separately from implication and absurdity. Then, the following properties of negation do not hold automatically.

(Co) $(A \to B) \to (\neg B \to \neg A)$ (Contraposition);

(NECQ) $(A \wedge \neg A) \to \neg B$ (Negative Ex Contradiction Quodibet);

(N) $(A \leftrightarrow B) \to (\neg A \leftrightarrow \neg B)$ (Congruence).

Hence we can obtain systems by adding these as axioms to the positive fragment of intuitionistic logic with subminimal negation and the following hierarchy.
Furthermore, [3] established the soundness and completeness theorems for the newly introduced subminimal logics using neighborhood semantics.

The relationship between logics has been analyzed by measuring of the cardinality of logics between them. In [17], the existence of a continuum of logics between classical and intuitionistic logics is proved, by using algebraic semantics. As a related result, [9] showed the existence of a continuum of logics between some intutitionistic modal logics. Another result that is

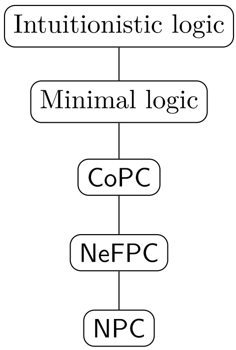


Figure 2: Systems discussed by Colacito, Bezhanishvili and de Jongh

directly related to the above systems can be found in [1] in which the existence of continua of logics between the systems included in the above figure is established.

Although **SUBMIN** was mentioned in [3] its relationship to the logics introduced in [13] was not clarified. It was Niki who revealed, in [8], the relations between **SUBMIN** and subminimal logics included in Figure 2. As a result, the systems are related as summarized in the following figure. Note that Niki refers to the $\bot$-free system **SUBMIN** (**SUBMIN**$^-$ for short) as $\mathsf{An}^-\mathsf{PC}$.

1.3. The aims of the paper

Building on the two backgrounds, we have two aims for this paper. First, we investigate the systems introduced by Vakarelov in some further detail. More specifically, we observe that there is a simpler characterization of **CO-MIN** and introduce the intersection of **CO-MIN** and minimal logic. Second, we establish some new results concerning the existence of continua of logics between systems that are not discussed so far in the literature. The results are established by making use of the techniques introduced in [15], and this strategy has the advantage of simplifying the proofs for the previous results established in [1].

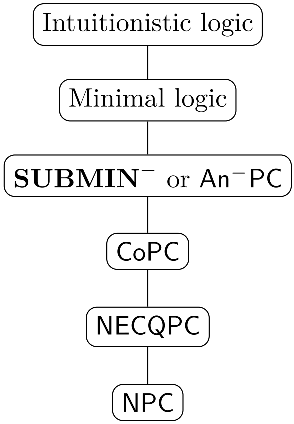


Figure 3: Systems discussed by Colacito, Bezhanishvili and de Jongh, after Niki's clarification

The remainder of this paper is structured as follows. In Section 2, we introduce proof systems for subsystems of intuitionistic logic and establish some results related to the first aim. In Section 3 we define algebraic semantics for the systems defined in Section 2, as a preparation for the main result. In Section 4, we show the existence of continua of logics between the logics discussed in this paper, and this will be related to the second aim. In Section 5, we conclude the paper by summarizing the main findings of the paper and pointing out some directions for further research. In the appendix, we prove the soundness and completeness theorems for the main logics of this paper using the Kripke semantics introduced by Vakarelov in [13].

2. Proof system

We shall use the language $\mathcal{L}_\neg$ consisting of denumerable propositional variables, $\wedge, \vee, \to$ and $\neg$. The set of propositional variables is denoted by $\mathcal{P}$. We define formulas of $\mathcal{L}_\neg$ as follows: $A \coloneqq p \mid \neg A \mid (A \wedge A) \mid (A \vee A) \mid (A \to A)$, and employ the abbreviation $A \leftrightarrow B := (A \to B) \wedge (B \to A)$. The formula $A[p_1/B_1, \ldots, p_n/B_n]$ is obtained by replacing all occurrences of p_i in A with B_i for each $i = 1, \ldots, n$ and leaving all other variables fixed. We call this operation *substitution*.

Definition 2.1 (NPC [3]). We define the Hilbert system NPC by adding a negative axioms to the positive axiom and rules of intuitionistic/minimal logic.

Axioms

- Ax1: $p \to (q \to p)$;
- Ax2: $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$;
- Ax3: $p \to (p \vee q)$;
- Ax4: $q \to (p \vee q)$;
- Ax5: $(p \to r) \to [(q \to r) \to ((p \vee q) \to r)]$;
- Ax6: $(p \wedge q) \to p$; Ax7 : $(p \wedge q) \to q$;
- Ax8: $p \to (q \to (p \wedge q))$;
- (N): $(p \leftrightarrow q) \to (\neg p \leftrightarrow \neg q)$.

Inference rules

- (MP): If A and $A \to B$, then B;
- (Sub): If A, then $A[p_1/B_1, \ldots, p_n/B_n]$, where $p_1, \ldots, p_n$ are propositional variables in A and $B_1, \ldots, B_n$ are formulas.

For a formula A, a sequence $A_1, \ldots, A_n$ is a *proof* of A in NPC, if A_i satisfies one of the following for any $1 \leq i \leq n$:

1. A_i is an axiom;
2. A_i is the result of applying (MP) to formulas A_j and A_k for some $j, k < i$;
3. A_i is the result of applying (Sub) to a formula A_j for some $j < i$;
4. $A_n = A$.

$\mathsf{NPC} \vdash A$ denotes that there is a *proof* of A in NPC. Unless there is a risk of misunderstanding, the set $\{A \mid \mathsf{NPC} \vdash A\}$ is also denoted by NPC.

For a set of formulas Γ, a formula A, a sequence $A_1, \ldots, A_n$ is a *deduction* of A from Γ in NPC, if A_i satisfies one of the following for any $1 \leq i \leq n$:

1. A_i is in NPC or in Γ;
2. A_i is the result of applying (MP) to formulas A_j and A_k for some $j, k < i$;
3. $A_n = A$.

$\Gamma \vdash_{\mathsf{NPC}} A$ denotes there is a deduction of A from Γ in NPC.

In considering deductions with hypothesis, (Sub) is not allowed. If (Sub) were allowed, $p \vdash q$ would be derived for any p and q.

We consider the following axioms related to negation.

Definition 2.2 (Additional Axioms).

- $(\mathsf{An}): \ (p \to \neg p) \to \neg p$;
- $(\mathsf{An}^-): \ (p \to \neg p) \to (\neg q \to \neg p)$;
- $(\mathsf{Co}): \ (p \to q) \to (\neg q \to \neg p)$;
- $(\mathsf{NECQ}): \ (p \wedge \neg p) \to \neg q$;
- $(\mathsf{ECQ}): \ (p \wedge \neg p) \to q$;
- $(\mathsf{AVQ}): \ \neg\neg(\neg(p \to p) \to q)$;
- $(\mathsf{An} \cap \mathsf{ECQ}): \ \neg\neg(p \to p) \vee (\neg(q \to q) \to r)$;
- $(\mathsf{CoECQ}): \ \neg p \to \neg(q \wedge \neg q)$.

By adding the above axioms, we obtain the following Hilbert systems.

- NECQPC is the Hilbert system adding (NECQ) to NPC. NECQ is an abbreviation of *negative ex contradiction quodibet*.
- CoPC is the Hilbert system adding (Co) to NPC (in this case, (N) is redundant [3]). Co is an abbreviation of *contraposition*.
- An^-PC is the Hilbert system adding (An^-) to NPC.
- CoECQPC is the Hilbert system adding (CoECQ) to NPC. CoECQ is an abbreviation of *contraposition of ex contradiction quodibet*.
- $\mathsf{An} \cap \mathsf{ECQ}$PC is the Hilbert system adding $(\mathsf{An} \cap \mathsf{ECQ})$ to An^-PC.

- $\mathsf{MPC}_\neg$ is the Hilbert system adding (An) to NPC. An is an abbreviation for *<u>a</u>bsorption of <u>n</u>egation*.
- ECQPC is the Hilbert system adding (ECQ) to NPC. ECQ is an abbreviation of *<u>e</u>x <u>c</u>ontradiction <u>q</u>uodibet*.
- AVQPC is the Hilbert system adding (AVQ) and (An) to NPC. AVQ is an abbreviation of *<u>av</u>oidability of <u>q</u>uodibet*.
- IPC is the Hilbert system adding (ECQ) and (An) to NPC. (In this case, (N) is redundant, see Lemma 2.5.)

Note that the system AVQPC was introduced in [14].[1] For each Hilbert system, the deducibility relation "$\Gamma \vdash A$" is defined as in the NPC case.

In [3, 8] the systems $\mathsf{NeF}, \mathsf{CoPC}, \mathsf{An}^-\mathsf{PC}$ and $\mathsf{MPC}_\neg$ are defined in the language with $\top$. The systems $\mathsf{NECQPC}, \mathsf{CoPC}, \mathsf{An}^-\mathsf{PC}$ and $\mathsf{MPC}_\neg$ we defined above are the $\top$-free fragments of them, respectively. This can be proved in the same way as **CO-MIN** and $\textbf{CO-MIN}^{-\top,\bot}$ in Appendix.

Unless there is a risk of misinterpretation, for any Hilbert system H above, we denote the set $\{A \mid \mathsf{H} \vdash A\}$ by H in the same way as NPC.

For each Hilbert system H defined above, the set $\{A \mid \mathsf{H} \vdash A\}$ can be summed up in one concept. Because we use it later, we define it as follows.

Definition 2.3. A super-N-logic (sN-logic, for short) in the language $\mathcal{L}_\neg$ is any set L of $\mathcal{L}_\neg$-formulas satisfying the conditions:

- $\mathsf{NPC} \subseteq L$;
- L is closed under modus ponens;
- L is closed under substitution.

As mentioned, for each Hilbert system H defined above, the set $\{A \mid \mathsf{H} \vdash A\}$ is an sN-logic.

Lemma 2.4 (Deduction theorem). *Let $\Gamma \cup \{A, B\}$ be a set of formulas. The following holds:*

$$\Gamma \cup \{A\} \vdash_{\mathsf{NPC}} B \;\Leftrightarrow\; \Gamma \vdash_{\mathsf{NPC}} A \to B.$$

[1] This formulation was given by [7].

To prove Deduction theorem, we need only $\mathsf{Ax1}$ and $\mathsf{Ax2}$ and the fact that (MP) is the unique inference rule in deductions. So, it holds for all Hilbert systems defined above.

We will use Deduction theorem without a mentioning in what follows.

Lemma 2.5. *Let* $\mathsf{IPC}^{-(\mathsf{N})}$ *be the Hilbert system removing* (N) *from* IPC. $\mathsf{IPC}^{-(\mathsf{N})} \vdash (\mathsf{N})$.

Proof: We can show $\mathsf{IPC}^{-(\mathsf{N})} \vdash (\neg p \wedge q \wedge (p \leftrightarrow q)) \to (p \wedge \neg p)$ as follows: We can prove $\neg p, q$ and $p \leftrightarrow q$ from $\neg p \wedge q \wedge (p \leftrightarrow q)$ by using $\mathsf{Ax6}, 7$. Since we can show p from q and $p \leftrightarrow q$ by (MP), we can show $p \wedge \neg p$ from $\neg p \wedge q \wedge (p \leftrightarrow q)$ by $\mathsf{Ax8}$. Then we have the following:

1. $\mathsf{IPC}^{-(\mathsf{N})} \vdash (\neg p \wedge q \wedge (p \leftrightarrow q)) \to (p \wedge \neg p)$
2. $\mathsf{IPC}^{-(\mathsf{N})} \vdash (\neg p \wedge q \wedge (p \leftrightarrow q)) \to \neg q$ ((ECQ) and 1)
3. $\mathsf{IPC}^{-(\mathsf{N})} \vdash (\neg p \wedge (p \leftrightarrow q)) \to (q \to \neg q)$ (From 2)
4. $\mathsf{IPC}^{-(\mathsf{N})} \vdash (\neg p \wedge (p \leftrightarrow q)) \to \neg q$ ((An) and 3)
5. $\mathsf{IPC}^{-(\mathsf{N})} \vdash (p \leftrightarrow q) \to (\neg p \to \neg q)$ (From 4)
6. $\mathsf{IPC}^{-(\mathsf{N})} \vdash (p \leftrightarrow q) \to (\neg q \to \neg p)$ (The same way as 1-5)
7. $\mathsf{IPC}^{-(\mathsf{N})} \vdash (p \leftrightarrow q) \to (\neg p \leftrightarrow \neg q)$ ($\mathsf{Ax8}, 5$ and 6). $\square$

Lemma 2.6. $\mathsf{NPC} \subseteq \mathsf{NECQPC} \subseteq \mathsf{CoPC} \subseteq \mathsf{An^{-}PC} \subseteq \mathsf{MPC}_{\neg}$.

Proof: By combining the results of [3, Page 12] and [8, Page 970], we have $\mathsf{NPC} \subseteq \mathsf{NeFPC} \subseteq \mathsf{CoPC} \subseteq \mathsf{An^{-}PC} \subseteq \mathsf{MPC}_{\neg}$. In their proofs, the axiom for $\top$ is not used. Since $\mathsf{NPC}, \mathsf{NECQPC}, \mathsf{CoPC}, \mathsf{An^{-}PC}, \mathsf{MPC}_{\neg}$ are $\top$-free fragment of them. $\square$

Hereinafter Lemma 2.6 is used without reference.

We shall show that *co-minimal logic* (**CO-MIN** for short (cf. [13])) is a conservative extension of ECQPC.

To define **CO-MIN**, we shall use the language $\mathcal{L}_{\neg,\top,\bot}$ consisting of $\mathcal{L}_{\neg}$, $\top$, $\bot$. We define formulas of $\mathcal{L}_{\neg,\top,\bot}$ as follows: $A := p \mid \top \mid \bot \mid \neg A \mid (A \wedge A) \mid (A \vee A) \mid (A \to A)$.

Definition 2.7 (cf. [13]). **CO-MIN** is the Hilbert system obtained by adding the axioms $\bot \to p$, $p \to \top$, $\neg p \to \neg\neg\top$ and $\neg\top \to p$ to CoPC in $\mathcal{L}_{\neg,\top,\bot}$.

We define $\mathbf{CO\text{-}MIN}^{-\top,\bot}$ as the Hilbert system adding the axioms $\neg p \to \neg\neg(q \to q)$ and $\neg(p \to p) \to q$ to CoPC in $\mathcal{L}_{\neg}$.

The following lemma shows that **CO-MIN** is a conservative extension for $\mathbf{CO\text{-}MIN}^{-\top,\bot}$.

Lemma 2.8. *Given a formula A in the language of $\mathcal{L}_{\neg}$,*

$$\mathbf{CO\text{-}MIN} \vdash A \;\Leftrightarrow\; \mathbf{CO\text{-}MIN}^{-\top,\bot} \vdash A.$$

Proof: See Appendix. □

Next, we show the equivalence between $\mathbf{CO\text{-}MIN}^{-\top,\bot}$ and ECQPC.

Lemma 2.9. ECQPC *contains* $\mathsf{An^{-}PC}$.

Proof: It suffices to show that $\mathsf{ECQPC} \vdash (p \to \neg p) \to (\neg q \to \neg p)$.

We can show $\mathsf{ECQPC} \vdash ((p \to \neg p) \wedge \neg q) \to (q \leftrightarrow p)$ as follows:

We can show $p \to \neg p$ and $\neg q$ from $(p \to \neg p) \wedge \neg q$ by $\mathsf{Ax}6, 7$. Assume q, then we can show p from $\neg q$ by (ECQ). Assume p, then we can show $\neg p$ from $p \to \neg p$, and so q by (ECQ) again.

Then we have the following:

1. $\mathsf{ECQPC} \vdash ((p \to \neg p) \wedge \neg q) \to (q \leftrightarrow p)$
2. $\mathsf{ECQPC} \vdash ((p \to \neg p) \wedge \neg q) \to (\neg q \to \neg p)$ ((N) and 1)
3. $\mathsf{ECQPC} \vdash (p \to \neg p) \to (\neg q \to \neg p)$ (From 2). □

Lemma 2.10. ECQPC *is equivalent to* $\mathbf{CO\text{-}MIN}^{-\top,\bot}$.

Proof: Since $p \to p$ is derivable, $\neg(p \to p) \to q$ is equivalent to the instance $((p \to p) \wedge \neg(p \to p)) \to q$ of ECQ. Then it suffices to show that $\mathsf{ECQPC} \vdash \neg p \to \neg\neg(q \to q)$.

1. $\mathsf{ECQPC} \vdash \neg(q \to q) \to \neg\neg(q \to q)$
2. $\mathsf{ECQPC} \vdash (\neg(q \to q) \to \neg\neg(q \to q)) \to (\neg p \to \neg\neg(q \to q))$ (An^-)
3. $\mathsf{ECQPC} \vdash \neg p \to \neg\neg(q \to q)$ (1 and 2).

For the other direction, it suffices to show $\mathbf{CO\text{-}MIN}^{-\top,\perp} \vdash (p \wedge \neg p) \to q$.

1. $\mathbf{CO\text{-}MIN}^{-\top,\perp} \vdash (p \wedge \neg p) \to (p \leftrightarrow (p \to p))$
2. $\mathbf{CO\text{-}MIN}^{-\top,\perp} \vdash (p \wedge \neg p) \to (\neg p \to \neg(p \to p))$ ((N) and 1)
3. $\mathbf{CO\text{-}MIN}^{-\top,\perp} \vdash (p \wedge \neg p) \to \neg(p \to p)$ (From 2)
4. $\mathbf{CO\text{-}MIN}^{-\top,\perp} \vdash \neg(p \to p) \to q$ (Ax of $\mathbf{CO\text{-}MIN}^{-\top,\perp}$)
5. $\mathbf{CO\text{-}MIN}^{-\top,\perp} \vdash (p \wedge \neg p) \to q$ (3 and 4). $\square$

In order to show $\mathsf{An} \cap \mathsf{ECQPC}$ is the intersection of ECQPC and $\mathsf{MPC}_\neg$, we show the following lemma.

Lemma 2.11. *The Hilbert system* $\mathsf{An}^-\mathsf{PC}^+$ *consisting of* $\mathsf{An}^-\mathsf{PC}$ *plus the axiom* $\neg\neg(p \to p)$ *is equivalent to* $\mathsf{MPC}_\neg$.

Proof: $\mathsf{An}^-\mathsf{PC}^+ \vdash (p \to \neg p) \to \neg p$ can be shown as follows.

1. $\mathsf{An}^-\mathsf{PC}^+ \vdash (p \to \neg p) \to (\neg\neg(p \to p) \to \neg p)$ (An^-)
2. $\mathsf{An}^-\mathsf{PC}^+ \vdash \neg\neg(p \to p) \to ((p \to \neg p) \to \neg p)$ (From 1)
3.; $\mathsf{An}^-\mathsf{PC}^+ \vdash (p \to \neg p) \to \neg p$ (2 and Ax of $\mathsf{An}^-\mathsf{PC}^+$).

For the converse, it suffices to show $\mathsf{MPC}_\neg \vdash \neg\neg(p \to p)$, since $\mathsf{MPC}_\neg \vdash (\mathsf{An}^-)$.

1. $\mathsf{MPC}_\neg \vdash \neg(p \to p) \to \neg\neg(p \to p)$ (NECQ)
2. $\mathsf{MPC}_\neg \vdash (\neg(p \to p) \to \neg\neg(p \to p)) \to \neg\neg(p \to p)$ (An)
3. $\mathsf{MPC}_\neg \vdash \neg\neg(p \to p)$ (1 and 2). $\square$

Lemma 2.12. $\mathsf{An} \cap \mathsf{ECQPC}$ *is the intersection of* ECQPC *and* $\mathsf{MPC}_\neg$.

PROOF: By Lemma 2.9, to prove $\mathsf{An} \cap \mathsf{ECQPC} \subseteq \mathsf{ECQPC}$ and $\mathsf{An} \cap \mathsf{ECQPC} \subseteq \mathsf{MPC}_{\neg}$, it suffices to show that $\mathsf{ECQPC} \vdash \neg\neg(p \to p) \vee (\neg(q \to q) \to r)$ and $\mathsf{MPC}_{\neg} \vdash \neg\neg(p \to p) \vee (\neg(q \to q) \to r)$, but it is immediate from Lemma 2.10 and Lemma 2.11.

For the other direction, note that $\mathsf{ECQPC} = \mathbf{CO\text{-}MIN}^{-\top,\bot}$ and $\mathsf{MPC}_{\neg} = \mathsf{An}^{-}\mathsf{PC}^{+}$, so $\mathsf{An}^{-}\mathsf{PC}$ is contained in the intersection of ECQPC and $\mathsf{MPC}_{\neg}$. Suppose that A is in the intersection of ECQPC and $\mathsf{MPC}_{\neg}$. Since $\mathsf{MPC}_{\neg} \vdash A$, $\mathsf{An}^{-}\mathsf{PC} \vdash B \to A$, where B is a conjunction of substitution instances of $\neg\neg(p \to p)$. Since $\mathsf{ECQPC} \vdash A$, $\mathsf{An}^{-}\mathsf{PC} \vdash C \to A$, where C is a conjunction of substitution instances of $\neg(q \to q) \to r$. By $\mathsf{Ax5}$, $\mathsf{An}^{-}\mathsf{PC} \vdash (B \vee C) \to A$. Since $B \vee C$ is equivalent to a conjunction of instances of $\mathsf{An} \cap\ \mathsf{ECQ}$, we obtain $\mathsf{An} \cap \mathsf{ECQPC} \vdash A$. $\square$

It will be seen below that **SUBMIN** can be further formalized differently.

LEMMA 2.13. $\mathsf{CoECQPC}$ *is equivalent to* $\mathsf{An}^{-}\mathsf{PC}$.

PROOF: We show that $\mathsf{CoECQPC} \vdash (p \to \neg p) \to (\neg q \to \neg p)$.

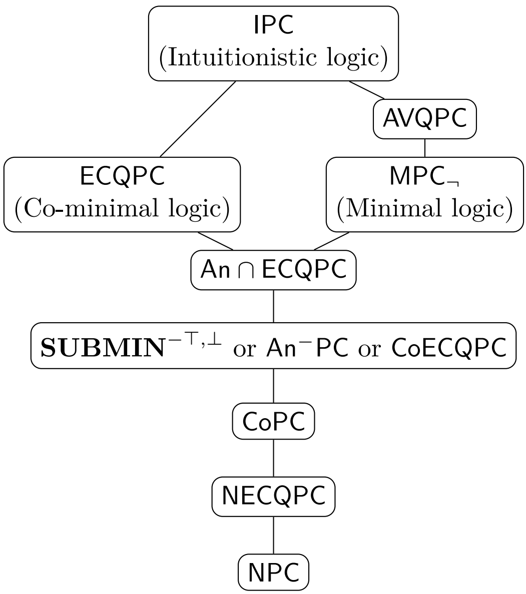
We can infer $\mathsf{CoECQPC} \vdash (p \to \neg p) \to ((p \wedge \neg p) \leftrightarrow p)$ by using $\mathsf{Ax6}, 7$.

1. $\mathsf{CoECQPC} \vdash (p \to \neg p) \to ((p \wedge \neg p) \leftrightarrow p)$
2. $\mathsf{CoECQPC} \vdash (p \to \neg p) \to (\neg(p \wedge \neg p) \leftrightarrow \neg p)$ ((N) and 1)
3. $\mathsf{CoECQPC} \vdash \neg(p \wedge \neg p) \to ((p \to \neg p) \to \neg p)$ (From 2)
4. $\mathsf{CoECQPC} \vdash \neg q \to \neg(p \wedge \neg p)$ ((CoECQ))
5. $\mathsf{CoECQPC} \vdash \neg q \to ((p \to \neg p) \to \neg p)$ (3 and 4)
6. $\mathsf{CoECQPC} \vdash (p \to \neg p) \to (\neg q \to \neg p)$ (From 5)

For the other direction, we show that $\mathsf{An}^{-}\mathsf{PC} \vdash \neg p \to \neg(q \wedge \neg q)$.

1. $\mathsf{An^-PC} \vdash \neg q \to \neg(q \wedge \neg q)$ ($\mathsf{Ax7}$ and (Co))
2. $\mathsf{An^-PC} \vdash (q \wedge \neg q) \to \neg q$ ($\mathsf{Ax7}$)
3. $\mathsf{An^-PC} \vdash (q \wedge \neg q) \to \neg(q \wedge \neg q)$ (1 and 2)
4. $\mathsf{An^-PC} \vdash ((q \wedge \neg q) \to \neg(q \wedge \neg q)) \to (\neg p \to \neg(q \wedge \neg q))$ (($\mathsf{An^-}$))
5. $\mathsf{An^-PC} \vdash \neg p \to \neg(q \wedge \neg q)$ (3 and 4) □

The results concerning the relations between the systems so far can be summarized as follows.



See the Appendix for $\mathbf{SUBMIN}^{-\top,\bot}$. The systems other than $\mathsf{An} \cap \mathsf{ECQPC}$ were defined using different languages and the relations were already known. The relation of $\mathsf{An} \cap \mathsf{ECQPC}$ to other systems is a new result of this paper, but at this point, it is not yet clear whether it can be separated from other systems. In Section 4, using algebraic semantics, we separate $\mathsf{An} \cap \mathsf{ECQPC}$ from the others. Before presenting the main results, we will first prepare some tools that we will use in the proofs of the main results.

3. Algebraic semantics

We now turn to introduce algebraic semantics for logics in the previous section.

Definition 3.1 (N-algebra (cf. [3])). Let $\langle |\mathfrak{A}|, \wedge_{\mathfrak{A}}, \vee_{\mathfrak{A}} \rangle$ be a lattice with the greatest element $1_{\mathfrak{A}}$. The order $\leq_{\mathfrak{A}}$ in the lattice is defined by

$$a \leq_{\mathfrak{A}} b :\Leftrightarrow a \wedge_{\mathfrak{A}} b = a$$

for any $a, b \in |\mathfrak{A}|$. The N-algebra $\langle |\mathfrak{A}|, 1_{\mathfrak{A}}, \wedge_{\mathfrak{A}}, \vee_{\mathfrak{A}}, \rightarrow_{\mathfrak{A}}, \neg_{\mathfrak{A}} \rangle$ is given by defining a binary operator $\rightarrow_{\mathfrak{A}}$ and an unary operator $\neg_{\mathfrak{A}}$ over $|\mathfrak{A}|$ as follows:

$$a \rightarrow_{\mathfrak{A}} b := \max\{c \in |\mathfrak{A}| \mid a \wedge_{\mathfrak{A}} c \leq_{\mathfrak{A}} b\};$$

$$(a \leftrightarrow_{\mathfrak{A}} b) \rightarrow_{\mathfrak{A}} (\neg_{\mathfrak{A}} a \leftrightarrow_{\mathfrak{A}} \neg_{\mathfrak{A}} b) = 1_{\mathfrak{A}},$$

where $a \leftrightarrow_{\mathfrak{A}} b$ is an abbreviation of $(a \rightarrow_{\mathfrak{A}} b) \wedge_{\mathfrak{A}} (b \rightarrow_{\mathfrak{A}} a)$.

For any N-algebra $\mathfrak{A}$, the following holds in the same manner as Heyting algebras:

$$a \leq_{\mathfrak{A}} b \Longleftrightarrow a \rightarrow_{\mathfrak{A}} b = 1_{\mathfrak{A}}.$$

This relation will be used without notice in what follows.

We now define the following algebraic conditions that correspond to the axioms.

Definition 3.2 (Additional conditions for negation). The following conditions corresponding to the additional inference rules in Definition 2.2 are defined as follows:

1. $(\mathsf{An})^E :\ (x \rightarrow \neg x) \rightarrow \neg x = 1$;

2. $(\mathsf{An}^-)^E$: $(x \to \neg x) \to (\neg y \to \neg x) = 1$;
3. $(\mathsf{Co})^E$: $(x \to y) \to (\neg y \to \neg x) = 1$;
4. $(\mathsf{NECQ})^E$: $(x \wedge \neg x) \to \neg y = 1$;
5. $(\mathsf{ECQ})^E$: $(x \wedge \neg x) \to y = 1$;
6. $(\mathsf{AVQ})^E$: $\neg\neg(\neg(x \to x) \to y) = 1$;
7. $(\mathsf{An} \cap \mathsf{ECQ})^E$: $\neg\neg(x \to x) \vee (\neg(y \to y) \to z) = 1$;
8. $(\mathsf{CoECQ})^E$: $\neg x \to \neg(y \wedge \neg y) = 1$.

These conditions are also denoted without E unless there is a risk of misunderstanding.

In what follows, we fix $\Psi = \{\wedge, \vee, \to\}$. Recall that $\mathcal{P}$ is the set of propositional variables.

Definition 3.3 (Valuation of N-algebra). For any N-algebra and any map v from $\mathcal{P}$ to $|\mathfrak{A}|$, the *valuation* $\overline{v}$ in $\mathfrak{A}$ is defined as follows:

1. If A is a propositional variable, then $\overline{v}(A) := v(A)$;
2. If $A = \neg B$, then $\overline{v}(\neg B) := \neg_{\mathfrak{A}} \overline{v}(B)$;
3. If $A = B \otimes C$, then $\overline{v}(B \otimes C) := \overline{v}(B) \otimes_{\mathfrak{A}} \overline{v}(C)$, for $\otimes \in \Psi$.

Since $\overline{v}$ is uniquely determined for any v, by this definition the valuation $\overline{v}$ is also written as v unless there is a risk of misunderstanding.

For given a valuation v and any formula A, if $v(A) = 1_{\mathfrak{A}}$, then we say that A is *true in* v. If A is true in v for any valuation v in $\mathfrak{A}$, then we say that A is *true in* $\mathfrak{A}$, which is denoted by $\mathfrak{A} \models A$. For a class $\mathcal{C}$ of N-algebras, if $\mathfrak{A} \models A$ for any $\mathfrak{A}$ in $\mathcal{C}$, then we say that A is *true in* $\mathcal{C}$. For any set Δ of formulas, any formula A and any class $\mathcal{C}$ of N-algebras, if, for any N-algebra $\mathfrak{A} \in \mathcal{C}$ and valuation v in $\mathfrak{A}$, A is true in v whenever B is true in v for any $B \in \Delta$, then we say A is true under Δ in $\mathcal{C}$ which is denoted by $\Delta \models_{\mathcal{C}} A$. If Δ is empty, it is denoted by $\models_{\mathcal{C}} A$.

We can prove the completeness theorem for the logics defined above and these N-algebras. In particular, here we show the completeness theorem regarding ECQPC.

THEOREM 3.4. *Let $\mathcal{C}_{\mathsf{ECQ}}$ be the class of* N*-algebras that validate* ECQ*. For any formula A, the following holds:*

$$\mathsf{ECQPC} \vdash A \;\Leftrightarrow\; \models_{\mathcal{C}_{\mathsf{ECQ}}} A.$$

PROOF: With regard to the forward implication, it can be proved by induction on the length of the deduction. For the other direction, it can be proved in the same manner as Heyting algebras (cf. [2, page 195]). □

Thus, completeness theorems can be proved in the same manner for the remaining logics. In more general, we can show the following.

THEOREM 3.5 (Completeness theorem). *For s*N*-logic L, let $\mathcal{C}_L$ be the class of* N*-algebras in which all theorems of L are true. For any formula A, the following holds:*

$$L \vdash A \;\Leftrightarrow\; \models_{\mathcal{C}_L} A.$$

PROOF: We can show this theorem in the same way as [3, page 51] by replacing $\top$ with $p \to p$. □

4. The existence of continua of logics between pairs of logics below intuitionistic logic

In this section, we show the main theorem: there are continua of logics between logics introduced in Section 2. In order to show it, we introduce the following definitions and lemmas. Most of them are introduced by [15]. The Heyting algebras are defined as usual.

4.1. Preliminaries

DEFINITION 4.1 (Second greatest element (cf. [15])). Given a N-algebra $\mathfrak{A}$, we say that $\mathfrak{A}$ has *the second greatest element* if the greatest element exists in $|\mathfrak{A}| \setminus \{1_{\mathfrak{A}}\}$. We write the second greatest element in $\mathfrak{A}$ as $\star_{\mathfrak{A}}$ if it exists.

The role of the second greatest element will be explained when we introduce Yankov formulas.

Recall that $\Psi = \{\wedge, \vee, \to\}$.

DEFINITION 4.2 (Ψ-reduct, Ψ-embedding (cf. [15])). For any Heyting algebra $\mathfrak{A}$, we define the following:

1. The algebra $\langle |\mathfrak{A}|, 1_{\mathfrak{A}}, 0_{\mathfrak{A}}, \langle \otimes_{\mathfrak{A}} \mid \otimes \in \Psi \rangle \rangle$ is the Ψ-*reduct*.
2. For a Heyting algebra $\mathfrak{B}$, a Ψ-*embedding* from $\mathfrak{A}$ to $\mathfrak{B}$ is an embedding from the Ψ-reduct of $\mathfrak{A}$ to the Ψ-reduct of $\mathfrak{B}$.

The following is a generalization of Yankov formulas (cf. [16]).

DEFINITION 4.3 (Yankov formula of N-algebra). In what follows, for any function g and $a \in dom(g)$, g_a denotes the value of g at a. For any finite N-algebra $\mathfrak{A}$ with the second greatest element, any injection g from $|\mathfrak{A}|$ to $\mathcal{P}$, and any formula A, the *Yankov formula* $\chi^g_A(\mathfrak{A})$ with $\mathfrak{A}$, g and A is defined as follows:

$$\chi^g_A(\mathfrak{A}) := \bigwedge\{(g_a \otimes g_b) \to g_{a \otimes_{\mathfrak{A}} b} \mid \otimes \in \Psi, a, b \in |\mathfrak{A}|\} \cup \{g_{a \otimes_{\mathfrak{A}} b} \to (g_a \otimes g_b) \mid \otimes \in \Psi, a, b \in |\mathfrak{A}|\} \to (g_{\star_{\mathfrak{A}}} \vee A).$$

The antecedent of $\chi^g_A(\mathfrak{A})$ is denoted by $\theta(\chi^g_A(\mathfrak{A}))$.

Yankov formula will be used when we separate logics and make continua of logics. For any finite N-algebra $\mathfrak{A}$ with the second greatest element, the antecedent of Yankov formula asserts that algebraic operators except for negation are rewritten by corresponding logical connectives. The succedent of Yankov formula is defined by using the second greatest element. In a valuation v making the antecedent of Yankov formula true, $v(g_{\star_{\mathfrak{A}}})$ is interpreted as the second greatest element in the model. If we choose N-algebra appropriately, the succedent of Yankov formula is interpreted as the second greatest element, then the N-algebra refutes the Yankov formula.

The difference from the conventional definition of the Yankov formula is in the succedent of the formula: We added the formula A. In order to separate logics, we take an instance of the axioms concerning negation as A in the proof of the main theorem.

In the following, we will prove some lemmata to prove the main theorem.

An N-algebra with the minimum element can be reformed into a Heyting algebra by modifying the negation.

Definition 4.4 (Heyting algebraization of N-algebra). For any N-algebra $\mathfrak{A} = \langle |\mathfrak{A}|, 1_{\mathfrak{A}}, \wedge_{\mathfrak{A}}, \vee_{\mathfrak{A}}, \to_{\mathfrak{A}}, \neg_{\mathfrak{A}} \rangle$ with the minimum element $0_{\mathfrak{A}}$, the Heyting algebra $\mathfrak{A}^H = \langle |\mathfrak{A}|, 1_{\mathfrak{A}^H}, 0_{\mathfrak{A}^H}, \wedge_{\mathfrak{A}^H}, \vee_{\mathfrak{A}^H}, \to_{\mathfrak{A}^H}, \neg_{\mathfrak{A}^H} \rangle$ is defined as follows:

$|\mathfrak{A}^H| := |\mathfrak{A}|, 1_{\mathfrak{A}^H} := 1_{\mathfrak{A}}, 0_{\mathfrak{A}^H} := 0_{\mathfrak{A}}, \wedge_{\mathfrak{A}^H} := \wedge_{\mathfrak{A}}, \vee_{\mathfrak{A}^H} := \vee_{\mathfrak{A}}, \to_{\mathfrak{A}^H} := \to_{\mathfrak{A}}$, and

$$\neg_{\mathfrak{A}^H} a := a \to_{\mathfrak{A}} 0_{\mathfrak{A}}.$$

Definition 4.5 (Filter). For any lattice H, a *filter* F of H is a nonempty subset of H satisfying the following 1 and 2:

1. If $a, b \in F$, then $a \wedge_H b \in F$;
2. If $a \in F$ and $a \leq_H b$, then $b \in F$.

Given $a \in H$, the filter $\{b \in H \mid a \leq b\}$ is the smallest filter containing a. We refer to this as the *filter generated by* a.

Proposition 4.6. For any Heyting algebra $\mathfrak{A}$ and any subset F of $|\mathfrak{A}|$, F is a filter of $\mathfrak{A}$ if and only if F satisfies both of the following 1 and 2:

1. $1_{\mathfrak{A}} \in F$;
2. If $a \in F$ and $a \to_{\mathfrak{A}} b \in F$, then $b \in F$.

Definition 4.7 (Quotient algebra of Heyting algebra). For any Heyting algebra $\mathfrak{A}$ and any filter F of $\mathfrak{A}$, the binary relation $\sim_F$ over $\mathfrak{A}$ is defined as follows:

$$a \sim_F b :\Longleftrightarrow a \to_{\mathfrak{A}} b \in F \text{ and } b \to_{\mathfrak{A}} a \in F \text{ for any } a, b \in |\mathfrak{A}|.$$

It is easy to show that this binary relation $\sim_F$ is a congruence relation over $\mathfrak{A}$. We define the congruence class of $a \in |\mathfrak{A}|$ with respect to $\sim_F$ by $[a]_F := \{b \in |\mathfrak{A}| \mid a \sim_F b\}$. The set of congruence classes $\{[a]_F \mid a \in |\mathfrak{A}|\}$ is denoted by $|\mathfrak{A}|/F$.

The *quotient algebra*
$\mathfrak{A}/F = \langle |\mathfrak{A}|/F, 1_{\mathfrak{A}/F}, 0_{\mathfrak{A}/F}, \wedge_{\mathfrak{A}/F}, \vee_{\mathfrak{A}/F}, \to_{\mathfrak{A}/F}, \neg_{\mathfrak{A}/F} \rangle$ is defined as follows: for any $[a]_F, [b]_F \in |\mathfrak{A}|/F$

1. $1_{\mathfrak{A}/F} := [1_{\mathfrak{A}}]_F$;
2. $0_{\mathfrak{A}/F} := [0_{\mathfrak{A}}]_F$;

3. $[a]_F \wedge_{\mathfrak{A}/F} [b]_F := [a \wedge_{\mathfrak{A}} b]_F$;
4. $[a]_F \vee_{\mathfrak{A}/F} [b]_F := [a \vee_{\mathfrak{A}} b]_F$;
5. $[a]_F \to_{\mathfrak{A}/F} [b]_F := [a \to_{\mathfrak{A}} b]_F$;
6. $\neg_{\mathfrak{A}/F}[a]_F := [\neg_{\mathfrak{A}} a]_F$.

It is easy to show that $\mathfrak{A}/F$ is a Heyting algebra.

The following is a generalization of Lemma 2 in [15] to Yankov formula of N-algebra.

LEMMA 4.8. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be finite N-algebras with the second greatest elements such that, for each filter F of $\mathfrak{B}^H$, there is no Ψ-embedding from $\mathfrak{A}^H$ to $\mathfrak{B}^H/F$. Then, for any formula A and any injection $g : |\mathfrak{A}| \to \mathcal{P}$, $\mathfrak{B} \models \chi^g_A(\mathfrak{A})$ holds.*

PROOF: Recall that any Heyting algebra is an N-algebra. We first show the statement for finite Heyting algebras $\mathfrak{C}$ and $\mathfrak{D}$ with the second greatest elements.

To show the contraposition, assume $\mathfrak{D} \not\models \chi^g_A(\mathfrak{C})$ for some $g : |\mathfrak{C}| \to \mathcal{P}$ and a formula A. Then there is a valuation v such that $v(\theta(\chi^g_A(\mathfrak{C}))) \not\leq_{\mathfrak{D}} v(g_{\star_\mathfrak{C}} \vee A)$, where $\theta(\chi^g_A(\mathfrak{C}))$ is the antecedent of $\chi^g_A(\mathfrak{C})$. Consider another Yankov formula $\chi^g_{g\star_\mathfrak{C}}(\mathfrak{C})$ for a propositional variable $g_{\star_\mathfrak{C}}$. Since $v(\theta(\chi^g_{g\star_\mathfrak{C}}(\mathfrak{C}))) = v(\theta(\chi^g_A(\mathfrak{C})))$ and $v(g_{\star_\mathfrak{C}}) \leq_{\mathfrak{D}} v(g_{\star_\mathfrak{C}} \vee A)$, $v(\theta(\chi^g_{g\star_\mathfrak{C}}(\mathfrak{C}))) \leq_{\mathfrak{D}} v(g_{\star_\mathfrak{C}})$ implies a contradiction, namely $v(\theta(\chi^g_A(\mathfrak{C}))) \leq_{\mathfrak{D}} v(g_{\star_\mathfrak{C}} \vee A)$. Therefore $v(\theta(\chi^g_{g_{\star_\mathfrak{C}}}(\mathfrak{C}))) \not\leq_{\mathfrak{D}} v(g_{\star_\mathfrak{C}})$ and so $\mathfrak{D} \not\models \chi^g_{g_{\star_\mathfrak{C}}}(\mathfrak{C})$ holds.

Take the filter G of $\mathfrak{D}$ generated by $v(\theta(\chi^g_{g_{\star_\mathfrak{C}}}(\mathfrak{C})))$. From the construction of G, we have the following:

$$v(g_{\star_\mathfrak{C}}) \notin G; \tag{9}$$

$$[v(g_{c\otimes d})]_G = [v(g_c \otimes g_d)]_G \text{ for any } c, d \in |\mathfrak{C}| \text{ and } \otimes \in \Psi. \tag{10}$$

Let $v' : \mathfrak{C} \to \mathfrak{D}/G$ be the map defined by $v'(c) := [v(g_c)]_G$ for any $c \in |\mathfrak{C}|$. We prove v' is a Ψ-embedding.

We see that each operators in Ψ is preserved by v'. For $\wedge$, we have the following equation for each $c, d \in |\mathfrak{C}|$.

$$\begin{aligned}
v'(c \wedge_{\mathfrak{C}} d) &= [v(g_{c \wedge_{\mathfrak{C}} d})]_G \\
&= [v(g_c \wedge g_d)]_G \\
&= [v(g_c) \wedge_{\mathfrak{D}} v(g_d)]_G \\
&= [v(g_c)]_G \wedge_{\mathfrak{D}/G} [v(g_b)]_G \\
&= v'(c) \wedge_{\mathfrak{D}/G} v'(d).
\end{aligned}$$

The cases $\vee$ and $\rightarrow$ can be shown in the same manner.

We can see that v' is an injection as follows.

Assume $v'(c) = v'(d)$. Then $[v(g_c)]_G = [v(g_d)]_G$ holds, and so for any $c, d \in |\mathfrak{C}|$,

$$\begin{aligned}
v(g_c) \leftrightarrow_{\mathfrak{D}} v(g_d) &= (v(g_c) \rightarrow_{\mathfrak{D}} v(g_d)) \wedge_{\mathfrak{D}} (v(g_d) \rightarrow_{\mathfrak{D}} v(g_c)) \\
&= v(g_c \rightarrow g_d) \wedge_{\mathfrak{D}} v(g_d \rightarrow g_c)
\end{aligned}$$

By $[v(g_c)]_G = [v(g_d)]_G$, we have $v(g_c \rightarrow g_d) \wedge_{\mathfrak{D}} v(g_d \rightarrow g_c) \in G$, and so $v(g_{c \rightarrow_{\mathfrak{C}} d}), v(g_{d \rightarrow_{\mathfrak{C}} c}) \in G$ from (10).

Furthermore, the equation $1_{\mathfrak{D}} \leftrightarrow_{\mathfrak{D}} v(g_{1_{\mathfrak{C}}}) = v(g_c \rightarrow g_c) \leftrightarrow_{\mathfrak{D}} v(g_{c \rightarrow_{\mathfrak{C}} c})$ holds. Since we have $v(g_c \rightarrow g_c) \leftrightarrow_{\mathfrak{D}} v(g_{c \rightarrow_{\mathfrak{C}} c}) \in G$, we obtain that $v(g_{1_{\mathfrak{C}}}) \in G$.

If we assume $c \neq d$, then $c \rightarrow_{\mathfrak{C}} d \neq 1_{\mathfrak{C}}$ or $d \rightarrow_{\mathfrak{C}} c \neq 1_{\mathfrak{C}}$ hold. Since $\mathfrak{C}$ has the second greatest element $\star_{\mathfrak{C}}$, we have $c \rightarrow_{\mathfrak{C}} d \leq_{\mathfrak{C}} \star_{\mathfrak{C}}$ or $d \rightarrow_{\mathfrak{C}} c \leq_{\mathfrak{C}} \star_{\mathfrak{C}}$. We assume $c \rightarrow_{\mathfrak{C}} d \leq_{\mathfrak{C}} \star_{\mathfrak{C}}$. Then the equation $v(g_{c \rightarrow_{\mathfrak{C}} d} \rightarrow g_{\star_{\mathfrak{C}}}) \leftrightarrow_{\mathfrak{D}} v(g_{(c \rightarrow_{\mathfrak{C}} d) \rightarrow_{\mathfrak{C}} \star_{\mathfrak{C}}}) = (v(g_{c \rightarrow_{\mathfrak{C}} d}) \rightarrow_{\mathfrak{D}} v(g_{\star_{\mathfrak{C}}})) \leftrightarrow_{\mathfrak{D}} v(g_{1_{\mathfrak{C}}})$ holds.

Since we have $v(g_{c \rightarrow_{\mathfrak{C}} d} \rightarrow g_{\star_{\mathfrak{C}}}) \leftrightarrow_{\mathfrak{D}} v(g_{(c \rightarrow_{\mathfrak{C}} d) \rightarrow_{\mathfrak{C}} \star_{\mathfrak{C}}}), v(g_{1_{\mathfrak{C}}}), v(g_{c \rightarrow_{\mathfrak{C}} d}) \in G$, we obtain $v(g_{\star_{\mathfrak{C}}}) \in G$. But this contradicts (9). If $d \rightarrow_{\mathfrak{C}} c \leq_{\mathfrak{C}} \star_{\mathfrak{C}}$ we can derive a contradiction in the same manner. Therefore we obtain $c = d$, and v' is an embedding. This completes a proof of the statements for Heyting algebras $\mathfrak{C}$ and $\mathfrak{D}$.

Next we show the statement for any finite N-algebra $\mathfrak{A}$ and $\mathfrak{B}$ with the second greatest elements. Assume that, for each filter F of $\mathfrak{B}^H$, there is no Ψ-embedding from $\mathfrak{A}^H$ to $\mathfrak{B}^H/F$. Then, from the above argument, $\mathfrak{B}^H \models \chi^g_{\star_{\mathfrak{A}}}(\mathfrak{A})$ holds for any injection $g : |\mathfrak{A}| \rightarrow \mathcal{P}$. Since $\chi^g_{\star_{\mathfrak{A}}}(\mathfrak{A})$ does not contain the negation $\neg$, it implies $\mathfrak{B} \models \chi^g_{\star_{\mathfrak{A}}}(\mathfrak{A})$. Then, for any valuation v, any formula A and any $g : |\mathfrak{A}| \to \mathcal{P}$, we have $v(\theta(\chi^g_A(\mathfrak{A}))) = v(\theta(\chi^g_{\star_{\mathfrak{A}}}(\mathfrak{A}))) \leq_{\mathfrak{B}} v(g_{\star_{\mathfrak{A}}}) \leq_{\mathfrak{B}} v(g_{\star_{\mathfrak{A}}} \vee A)$, and so $\mathfrak{B} \models \chi^g_A(\mathfrak{A})$. □

Definition 4.9. For any natural number n, subsets of natural numbers a_n, r_n and s_n are defined as follows:

$$a_n := \{i \mid i < n\}, r_n := a_n \cup \{n+1\}, s_n := r_n \cup \{n+3\}$$

Lemma 4.10. *For any Heyting algebras $\mathfrak{A}$, let $\underline{\mathfrak{A}}$ be the algebra obtained by adding a new minimum element $0_{\underline{\mathfrak{A}}}$ to $\mathfrak{A}$ and changing the definition of negation $\neg$ into $\neg_{\underline{\mathfrak{A}}} a := a \to_{\underline{\mathfrak{A}}} 0_{\underline{\mathfrak{A}}}$. Then $\underline{\mathfrak{A}}$ is a Heyting algebra.*

Proof: It is immediate that $\underline{\mathfrak{A}}$ is a lattice by the definition. For any $a \in |\underline{\mathfrak{A}}|$, $a \to_{\underline{\mathfrak{A}}} 0_{\underline{\mathfrak{A}}}$ and $0_{\underline{\mathfrak{A}}} \to_{\underline{\mathfrak{A}}} a$ exist as follows. If $a = 0_{\underline{\mathfrak{A}}}$, then both of them are equal to $1_{\underline{\mathfrak{A}}}$. If $a \neq 0_{\underline{\mathfrak{A}}}$, then $c \neq 0_{\underline{\mathfrak{A}}}$ implies $0_{\mathfrak{A}} \leq_{\underline{\mathfrak{A}}} a \wedge_{\underline{\mathfrak{A}}} c$ and so $a \wedge_{\underline{\mathfrak{A}}} c \not\leq 0_{\underline{\mathfrak{A}}}$. Therefore, $a \to_{\underline{\mathfrak{A}}} 0_{\underline{\mathfrak{A}}} = 0_{\underline{\mathfrak{A}}}$. The equation $0_{\underline{\mathfrak{A}}} \to_{\underline{\mathfrak{A}}} a = 1_{\underline{\mathfrak{A}}}$ is immediate from the definition of $\to$. □

In order to distinct logics, we construct countably many algebras in the following.

For any natural number n, let us put $|\mathfrak{A}_n| = \{a_0, \ldots, a_{n+8}\} \cup \{r_0, \ldots, r_{n+6}\} \cup \{s_0, s_{n+4}, \mathbb{N}\}$ and for any $x, y \in |\mathfrak{A}_n|, x \to_{\mathfrak{A}_n} y = \bigcup\{z \in |\mathfrak{A}_n| \mid x \cap z \subseteq y\}, \neg_{\mathfrak{A}_n} x = x \to_{\mathfrak{A}_n} a_0$.

The set $\mathfrak{A}_n$ forms a Heyting algebra by taking algebraic operations $\vee$ and $\wedge$ as set operations $\cup$ and $\cap$.

Lemma 4.11 (cf. Lemma 5.7 in [15]). *For any natural number n, $\mathfrak{A}_n = \langle |\mathfrak{A}_n|, \mathbb{N}, a_0, \cap, \cup, \to_{\mathfrak{A}_n}, \neg_{\mathfrak{A}_n} \rangle$ is a Heyting algebra.*

Hence $\underline{\mathfrak{A}_n}$ is a Heyting algebra with the second greatest element a_{n+8} for any natural number n by Lemma 4.11.

$\underline{\mathfrak{A}_0}, \underline{\mathfrak{A}_1}$ and $\underline{\mathfrak{A}_2}$ are represented with Hasse diagrams as in Figure 4.

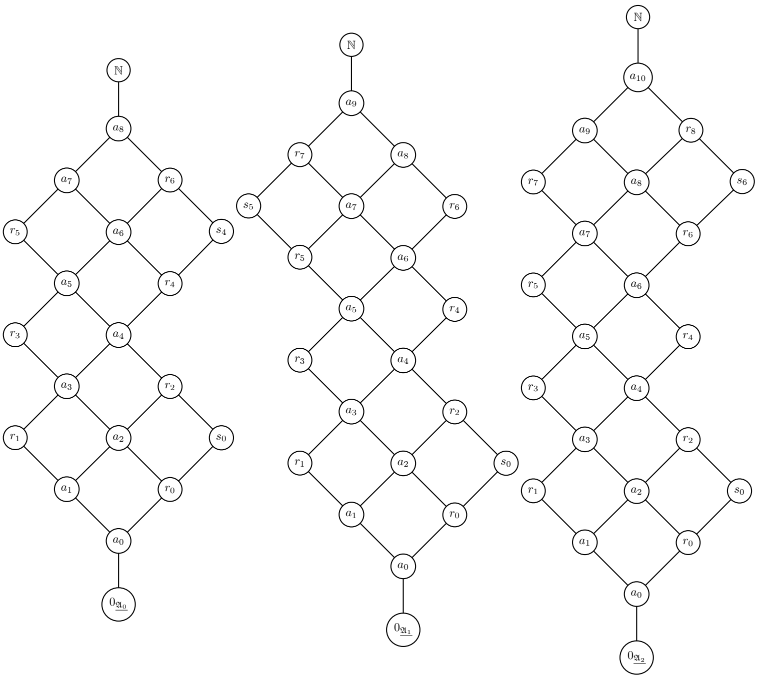


Figure 4: Heyting algebras $\underline{\mathfrak{A}_0}, \underline{\mathfrak{A}_1}$ and $\underline{\mathfrak{A}_2}$

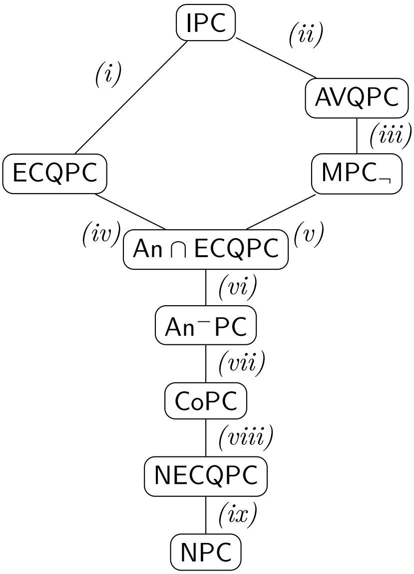
Consider an $\{\to\}$ embedding from $\mathfrak{A}_n$ to $\mathfrak{A}_m$. Since it preserves the order $\leq_{\mathfrak{A}_n}$, r_1, a_2, s_0 must be mapped into themselves respectively, and $r_{n+5}, a_{n+6}, s_{n+4}$ must be mapped into $r_{m+5}, a_{m+6}, s_{m+4}$, via $\{\to\}$-embedding. This is the key point of the lemma below.

LEMMA 4.12 (cf. [15]). *For any natural numbers n and m, the following statement holds: If $n \neq m$, then $\underline{\mathfrak{A}_n}$ cannot be $\{\to\}$-embedded into a quotient algebra $\underline{\mathfrak{A}_m}/F$ of $\underline{\mathfrak{A}_m}$ for any filter F of $\underline{\mathfrak{A}_m}$.*

4.2. Main Results

The main results of this paper are summarized as the following theorem.

THEOREM 4.13. *There exist continua of logics in each line in the figure below.*



Hereafter, when the number in the above figure is used as anything other than a subscript, it is used to indicate a proof of the existence of continua of logics between the two logics.

LEMMA 4.14. *There is a continuum of logics between* IPC *and* ECQPC*. The same holds between* $\mathsf{MPC}_\neg$ *and* $\mathsf{An} \cap \mathsf{ECQPC}$*.*

PROOF: We prove this lemma as follows.

Step 1. We construct countably many models $N^1(\underline{\mathfrak{A}_n})$ of ECQPC based on $\underline{\mathfrak{A}_n}$ in Lemma 4.11.

Step 2. For each natural number n and each set $I \subset \mathbb{N}$ of natural numbers, we define Yankov formula $\mathcal{A}_n$ and logic $L^1(I)$.

Step 3. We show $N^1(\underline{\mathfrak{A}_k}) \models \mathcal{A}_l$ if $k \neq l$.

Step 4. We show $\mathcal{A}_n \in L^1(I)$ iff $n \in I$ by proving $N^1(\underline{\mathfrak{A}_n}) \not\models \mathcal{A}_n$.

Step 5. We show that there is continuum of logics between IPC and ECQPC.

Step 1. For any natural number n, the algebra

$$N^1(\underline{\mathfrak{A}_n}) = \langle |N^1(\underline{\mathfrak{A}_n})|, 1_{N^1(\underline{\mathfrak{A}_n})}, \wedge_{N^1(\underline{\mathfrak{A}_n})}, \vee_{N^1(\underline{\mathfrak{A}_n})}, \rightarrow_{N^1(\underline{\mathfrak{A}_n})}, \neg_{N^1(\underline{\mathfrak{A}_n})} \rangle$$

is defined as follows:

$$\begin{aligned} |N^1(\underline{\mathfrak{A}_n})| &:= |\underline{\mathfrak{A}_n}|, \\ 1_{N^1(\underline{\mathfrak{A}_n})} &:= \mathbb{N}, \\ \wedge_{N^1(\underline{\mathfrak{A}_n})} &:= \wedge_{\underline{\mathfrak{A}_n}}, \\ \vee_{N^1(\underline{\mathfrak{A}_n})} &:= \vee_{\underline{\mathfrak{A}_n}}, \\ \rightarrow_{N^1(\underline{\mathfrak{A}_n})} &:= \rightarrow_{\underline{\mathfrak{A}_n}}, \end{aligned}$$

and

$$\neg_{N^1(\underline{\mathfrak{A}_n})} a := \begin{cases} a_{n+8} & \text{if } a = 0_{\underline{\mathfrak{A}_n}}; \\ 0_{\underline{\mathfrak{A}_n}} & \text{otherwise,} \end{cases} \text{ for any } a \in |N^1(\underline{\mathfrak{A}_n})|.$$

We show that $N^1(\underline{\mathfrak{A}_n})$ validates the conditions (N), (ECQ) and $(\mathsf{An} \cap \mathsf{ECQ})$.

For (N), let $a, b \in |N^1(\underline{\mathfrak{A}_n})|$. If $a \neq 0_{\underline{\mathfrak{A}_n}}$ and $b \neq 0_{\underline{\mathfrak{A}_n}}$, then

$$\begin{aligned} (a \leftrightarrow_{N^1(\underline{\mathfrak{A}_n})} b) &\rightarrow_{N^1(\underline{\mathfrak{A}_n})} (\neg_{N^1(\underline{\mathfrak{A}_n})} a \leftrightarrow_{N^1(\underline{\mathfrak{A}_n})} \neg_{N^1(\underline{\mathfrak{A}_n})} b) \\ &= (a \leftrightarrow_{N^1(\underline{\mathfrak{A}_n})} b) \rightarrow_{N^1(\underline{\mathfrak{A}_n})} (0_{\underline{\mathfrak{A}_n}} \leftrightarrow_{N^1(\underline{\mathfrak{A}_n})} 0_{\underline{\mathfrak{A}_n}}) \\ &= (a \leftrightarrow_{N^1(\underline{\mathfrak{A}_n})} b) \rightarrow_{N^1(\underline{\mathfrak{A}_n})} \mathbb{N} \\ &= \mathbb{N}. \end{aligned}$$

Hence (N) holds for $a \neq 0_{\underline{\mathfrak{A}_n}}$ and $b \neq 0_{\underline{\mathfrak{A}_n}}$. If $a = 0_{\underline{\mathfrak{A}_n}}$ and $b \neq 0_{\underline{\mathfrak{A}_n}}$, then

$$\begin{aligned}(0_{\underline{\mathfrak{A}_n}} \leftrightarrow_{N^1(\underline{\mathfrak{A}_n})} b) \to_{N^1(\underline{\mathfrak{A}_n})} (\neg_{N^1(\underline{\mathfrak{A}_n})} 0_{\underline{\mathfrak{A}_n}} \leftrightarrow_{N^1(\underline{\mathfrak{A}_n})} \neg_{N^1(\underline{\mathfrak{A}_n})} b) &\\ = (0_{\underline{\mathfrak{A}_n}} \leftrightarrow_{N^1(\underline{\mathfrak{A}_n})} b) \to_{N^1(\underline{\mathfrak{A}_n})} (a_{n+8} \leftrightarrow_{N^1(\underline{\mathfrak{A}_n})} 0_{\underline{\mathfrak{A}_n}}) &\\ = 0_{\underline{\mathfrak{A}_n}} \to_{N^1(\underline{\mathfrak{A}_n})} 0_{\underline{\mathfrak{A}_n}} &\\ = \mathbb{N}. &\end{aligned}$$

The equality $0_{\underline{\mathfrak{A}_n}} \leftrightarrow_{N^1(\underline{\mathfrak{A}_n})} b = 0_{\underline{\mathfrak{A}_n}}$ follows from $0_{\underline{\mathfrak{A}_n}} \leftrightarrow_{N^1(\underline{\mathfrak{A}_n})} b = (0_{\underline{\mathfrak{A}_n}} \to_{N^1(\underline{\mathfrak{A}_n})} b) \wedge_{N^1(\underline{\mathfrak{A}_n})} (b \to_{N^1(\underline{\mathfrak{A}_n})} 0_{\underline{\mathfrak{A}_n}})$ and $b \to_{N^1(\underline{\mathfrak{A}_n})} 0_{\underline{\mathfrak{A}_n}} = 0_{\underline{\mathfrak{A}_n}}$. Hence (N) holds for $a = 0_{\underline{\mathfrak{A}_n}}$ and $b \neq 0_{\underline{\mathfrak{A}_n}}$. The remaining case can be proven in the same manner. So (N) holds in $N^1(\underline{\mathfrak{A}_n})$.

For (ECQ), let $a, b \in |N^1(\underline{\mathfrak{A}_n})|$. Then

$$\begin{aligned}(a \wedge_{N^1(\underline{\mathfrak{A}_n})} \neg_{N^1(\underline{\mathfrak{A}_n})} a) \to_{N^1(\underline{\mathfrak{A}_n})} b &= 0_{\underline{\mathfrak{A}_n}} \to_{N^1(\underline{\mathfrak{A}_n})} b\\ &= \mathbb{N}.\end{aligned}$$

So $N^1(\underline{\mathfrak{A}_n})$ validates (ECQ).

It is immediate that $(\mathsf{An} \cap \mathsf{ECQ})$ follows from (ECQ).

Step 2. Take a nonempty proper subset I of natural numbers and an injection $g : |N^1(\underline{\mathfrak{A}_n})| \to \mathcal{P}$, and let $\mathcal{A}_n$ be $\chi^g_{\underline{(g_{0_{\underline{\mathfrak{A}_n}}} \to \neg g_{0_{\underline{\mathfrak{A}_n}}}) \to \neg g_{0_{\underline{\mathfrak{A}_n}}}}}(N^1(\underline{\mathfrak{A}_n}))$. Let $L^1(I)$ be the logic adding axioms $\mathcal{A}_n$ into ECQPC for $n \in I$. Note that $L^1(I)$ is a sN-logic.

Step 3. We show that $N^1(\underline{\mathfrak{A}_k}) \models \mathcal{A}_l$ for natural numbers k, l with $k \neq l$. Note that $N^1(\underline{\mathfrak{A}_l})^H = \underline{\mathfrak{A}_l}$. There is no $\{\to\}$-embedding from $\underline{\mathfrak{A}_l}$ to $\underline{\mathfrak{A}_k}/F$ for any natural number k with $k \neq l$ and any filter F of $\underline{\mathfrak{A}_k}$ by Lemma 4.12, and so there is no Ψ-embedding $\underline{\mathfrak{A}_l}$ to $\underline{\mathfrak{A}_k}/F$ for any natural number k and l with $k \neq l$ and any filter F of $\underline{\mathfrak{A}_k}$. Therefore for any natural number k and l with $k \neq l$, $N^1(\underline{\mathfrak{A}_k}) \models \mathcal{A}_l$ holds from Lemma 4.8.

Step 4. We can also prove the following.

$$\mathcal{A}_n \in L^1(I) \Leftrightarrow n \in I.$$

Since $\Leftarrow$ is obvious, we show $\Rightarrow$. We assume $n \notin I$ and show that $\mathcal{A}_n \notin L^1(I)$. Since $N^1(\underline{\mathfrak{A}_n})$ validates (N), (ECQ) and $\mathcal{A}_m$ for each $m \neq n$, it is enough to show $N^1(\underline{\mathfrak{A}_n}) \not\models \mathcal{A}_n$. Let g^{-1} be the inverse map of g, i.e., $g^{-1} : \{g_a \in \mathcal{P} \mid a \in |N^1(\underline{\mathfrak{A}_n})|\} \to |N^1(\underline{\mathfrak{A}_n})|$ such that $g^{-1}(g(a)) = a$. Take a valuation v into $|N^1(\underline{\mathfrak{A}_n})|$ which is an extension of g^{-1}. Then $v(\theta(\chi^g_{(g_{0_{\underline{\mathfrak{A}_n}}} \to \neg g_{0_{\underline{\mathfrak{A}_n}}}) \to \neg g_{0_{\underline{\mathfrak{A}_n}}}}(N^1(\underline{\mathfrak{A}_n})))) = \mathbb{N}$ since $v(g_a \otimes g_b) = a \otimes_{N^1(\underline{\mathfrak{A}_n})} b = v(g_{a \otimes_{N^1(\underline{\mathfrak{A}_n})} b})$ for any $a, b \in |N^1(\underline{\mathfrak{A}_n})|$ and any $\otimes \in \Psi$. On the other hand, $v((g_{0_{\underline{\mathfrak{A}_n}}} \to \neg g_{0_{\underline{\mathfrak{A}_n}}}) \to \neg g_{0_{\underline{\mathfrak{A}_n}}}) \neq \mathbb{N}$ and so $v(((g_{0_{\underline{\mathfrak{A}_n}}} \to \neg g_{0_{\underline{\mathfrak{A}_n}}}) \to \neg g_{0_{\underline{\mathfrak{A}_n}}}) \vee g_{\star_{\mathfrak{A}}}) = \star_{\mathfrak{A}} \neq \mathbb{N}$. Therefore, $N^1(\underline{\mathfrak{A}_n}) \not\models \chi^g_{(g_{0_{\underline{\mathfrak{A}_n}}} \to \neg g_{0_{\underline{\mathfrak{A}_n}}}) \to \neg g_{0_{\underline{\mathfrak{A}_n}}}}(N^1(\underline{\mathfrak{A}_n}))$. Then

$$
\begin{aligned}
&v((g_{0_{\underline{\mathfrak{A}_n}}} \to \neg g_{0_{\underline{\mathfrak{A}_n}}}) \to \neg g_{0_{\underline{\mathfrak{A}_n}}})\\
&\qquad = (v(g_{0_{\underline{\mathfrak{A}_n}}}) \to_{N^1(\underline{\mathfrak{A}_n})} \neg_{N^1(\underline{\mathfrak{A}_n})} v(g_{0_{\underline{\mathfrak{A}_n}}})) \to_{N^1(\underline{\mathfrak{A}_n})} \neg_{N^1(\underline{\mathfrak{A}_n})} v(g_{0_{\underline{\mathfrak{A}_n}}}))\\
&\qquad = (0_{\underline{\mathfrak{A}_n}} \to_{N^1(\underline{\mathfrak{A}_n})} a_{n+8}) \to_{N^1(\underline{\mathfrak{A}_n})} a_{n+8}\\
&\qquad = \mathbb{N} \to_{N^1(\underline{\mathfrak{A}_n})} a_{n+8},
\end{aligned}
$$

and so $v((g_{0_{\underline{\mathfrak{A}_n}}} \to \neg g_{0_{\underline{\mathfrak{A}_n}}}) \to \neg g_{0_{\underline{\mathfrak{A}_n}}}) \neq \mathbb{N}$. Therefore $N^1(\underline{\mathfrak{A}_n}) \not\models \mathcal{A}_n$ from the construction of $\mathcal{A}_n$. Since $L^1(I)$ is the minimum logic containing each axioms of ECQPC and $\mathcal{A}_m$ for $m \in I$, we have $L^1(I) \subseteq \{A \mid N^1(\underline{\mathfrak{A}_n}) \models A\}$. Thus we obtain $\mathcal{A}_n \notin L^1(I)$. This is the end of Step 4.

Step 5. By the result of Step 4, it follows that $L^1(I) \neq L^1(J)$ for any nonempty proper subsets I and J of natural numbers with $I \neq J$.

Since $\mathsf{IPC} \vdash (p \to \neg p) \to \neg p$, IPC proves succedent of $\mathcal{A}_n$ for any natural number n, and so $\mathsf{IPC} \vdash \mathcal{A}_n$.

Then, for any nonempty proper subset I of natural numbers, $\mathsf{ECQPC} \subsetneq L^1(I) \subsetneq \mathsf{IPC}$ hold. Since the choice of I is continuum, there is continuum of logics between IPC and ECQPC. We can show (v) by letting $L^5(I)$ be the logic adding axioms $\mathcal{A}_n$ into $\mathsf{An} \cap \mathsf{ECQPC}$ for $n \in I$ in the same manner as (i). $\square$

LEMMA 4.15. *There is a continuum of logics between* IPC *and* AVQPC. *The same holds between* ECQPC *and* $\mathsf{An} \cap \mathsf{ECQPC}$.

Proof: For any natural number n, the algebra

$$N^2(\underline{\mathfrak{A}_n}) = \langle |N^2(\underline{\mathfrak{A}_n})|, 1_{N^2(\underline{\mathfrak{A}_n})}, \wedge_{N^2(\underline{\mathfrak{A}_n})}, \vee_{N^2(\underline{\mathfrak{A}_n})}, \rightarrow_{N^2(\underline{\mathfrak{A}_n})}, \neg_{N^2(\underline{\mathfrak{A}_n})} \rangle$$

is defined in the same way as Lemma 4.14 except for negation. $\neg_{N^2(\underline{\mathfrak{A}_n})}$ is defined as follows:

$$\neg_{N^2(\underline{\mathfrak{A}_n})} a := \mathbb{N} \text{ for any } a \in |N^2(\underline{\mathfrak{A}_n})|.$$

First, we see that $N^2(\underline{\mathfrak{A}_n})$ validates the conditions $(\mathsf{N}), (\mathsf{An}), (\mathsf{AVQ})$ and $(\mathsf{An} \cap \mathsf{ECQ})$.

For (N), take any $a, b \in |N^2(\underline{\mathfrak{A}_n})|$. Then

$$\begin{aligned}
(a \leftrightarrow_{N^2(\underline{\mathfrak{A}_n})} b) \rightarrow_{N^2(\underline{\mathfrak{A}_n})} (\neg_{N^2(\underline{\mathfrak{A}_n})} a \leftrightarrow_{N^2(\underline{\mathfrak{A}_n})} \neg_{N^2(\underline{\mathfrak{A}_n})} b)& \\
= (a \leftrightarrow_{N^2(\underline{\mathfrak{A}_n})} b) \rightarrow_{N^2(\underline{\mathfrak{A}_n})} (\mathbb{N} \leftrightarrow_{N^2(\underline{\mathfrak{A}_n})} \mathbb{N})& \\
= (a \leftrightarrow_{N^2(\underline{\mathfrak{A}_n})} b) \rightarrow_{N^2(\underline{\mathfrak{A}_n})} \mathbb{N}& \\
= \mathbb{N}.&
\end{aligned}$$

So $N^2(\underline{\mathfrak{A}_n})$ validates (N).

For (An), let $a \in |N^2(\underline{\mathfrak{A}_n})|$. Then

$$\begin{aligned}
(a \rightarrow_{N^2(\underline{\mathfrak{A}_n})} \neg_{N^2(\underline{\mathfrak{A}_n})} a) \rightarrow_{N^2(\underline{\mathfrak{A}_n})} \neg_{N^2(\underline{\mathfrak{A}_n})} a &= (a \rightarrow_{N^2(\underline{\mathfrak{A}_n})} \mathbb{N}) \rightarrow_{N^2(\underline{\mathfrak{A}_n})} \mathbb{N} \\
&= \mathbb{N}.
\end{aligned}$$

So $N^2(\underline{\mathfrak{A}_n})$ validates (An). The case (AVQ) can be shown in the same way.

It is immediate that $(\mathsf{An} \cap \mathsf{ECQ})$ holds from (An).

For any natural number n, we define $\mathcal{B}_n := \chi^g_{(g_{\mathbb{N}} \wedge \neg g_{\mathbb{N}}) \rightarrow g_{0_{\underline{\mathfrak{A}_n}}}}(N^2(\underline{\mathfrak{A}_n}))$. Using $N^2(\underline{\mathfrak{A}_n})$, $\mathcal{B}_n$ and (ECQ) instead of $N^1(\underline{\mathfrak{A}_n})$, $\mathcal{A}_n$ and (An), respectively, we can show (ii) and (iv) in the same manner as (i). □

We can show the remining cases in the same way as Lemma 4.14.

Lemma 4.16. *There is a continuum of logics between* AVQPC *and* $\mathsf{MPC}_{\neg}$.

Proof: For any natural number n, the algebra

$$N^3(\underline{\mathfrak{A}_n}) = \langle |N^3(\underline{\mathfrak{A}_n})|, 1_{N^3(\underline{\mathfrak{A}_n})}, \wedge_{N^3(\underline{\mathfrak{A}_n})}, \vee_{N^3(\underline{\mathfrak{A}_n})}, \rightarrow_{N^3(\underline{\mathfrak{A}_n})}, \neg_{N^3(\underline{\mathfrak{A}_n})} \rangle$$

is defined in the same way as Lemma 4.14 except for negation. $\neg_{N^3(\underline{\mathfrak{A}_n})}$ is defined as follows:

$$\neg_{N^3(\underline{\mathfrak{A}_n})} a := a \rightarrow_{N^3(\underline{\mathfrak{A}_n})} a_0 \text{ for any } a \in |N^3(\underline{\mathfrak{A}_n})|.$$

First, we see that $N^3(\underline{\mathfrak{A}_n})$ validates conditions (N) and (An).

For (N), take any $a, b \in |N^3(\underline{\mathfrak{A}_n})|$. It is immediate that (N) holds in $N^3(\underline{\mathfrak{A}_n})$ in view of the following.

$(a \rightarrow_{N^3(\underline{\mathfrak{A}_n})} b) \wedge_{N^3(\underline{\mathfrak{A}_n})} (b \rightarrow_{N^3(\underline{\mathfrak{A}_n})} a_0) \leq_{N^3(\underline{\mathfrak{A}_n})} a \rightarrow_{N^3(\underline{\mathfrak{A}_n})} a_0$,

$(b \rightarrow_{N^3(\underline{\mathfrak{A}_n})} a) \wedge_{N^3(\underline{\mathfrak{A}_n})} (a \rightarrow_{N^3(\underline{\mathfrak{A}_n})} a_0) \leq_{N^3(\underline{\mathfrak{A}_n})} b \rightarrow_{N^3(\underline{\mathfrak{A}_n})} a_0$.

For (An), take any $a \in |N^3(\underline{\mathfrak{A}_n})|$. Then

$$\begin{aligned}
&(a \rightarrow_{N^3(\underline{\mathfrak{A}_n})} \neg_{N^3(\underline{\mathfrak{A}_n})} a) \rightarrow_{N^3(\underline{\mathfrak{A}_n})} \neg_{N^3(\underline{\mathfrak{A}_n})} a \\
&\qquad = (a \rightarrow_{N^3(\underline{\mathfrak{A}_n})} (a \rightarrow_{N^3(\underline{\mathfrak{A}_n})} a_0)) \rightarrow_{N^3(\underline{\mathfrak{A}_n})} (a \rightarrow_{N^3(\underline{\mathfrak{A}_n})} a_0) \\
&\qquad = (a \rightarrow_{N^3(\underline{\mathfrak{A}_n})} a_0) \rightarrow_{N^3(\underline{\mathfrak{A}_n})} (a \rightarrow_{N^3(\underline{\mathfrak{A}_n})} a_0) \\
&\qquad = \mathbb{N}.
\end{aligned}$$

Hence (An) hold in $N^3(\underline{\mathfrak{A}_n})$.

For any natural number n, we define $\mathcal{D}_n := \chi^g_{\neg\neg(\neg(g_{\mathbb{N}} \rightarrow g_{\mathbb{N}}) \rightarrow g_{0_{\underline{\mathfrak{A}_n}}})}(N^3(\underline{\mathfrak{A}_n}))$. Using $N^3(\underline{\mathfrak{A}_n})$, $\mathcal{D}_n$ and (AVQ) instead of $N^1(\underline{\mathfrak{A}_n})$, $\mathcal{A}_n$ and (An), respectively, we can show (iii) in the same way as (i). $\square$

Lemma 4.17. *There is a continuum of logics between* $\mathsf{An} \cap \mathsf{ECQPC}$ *and* $\mathsf{An}^-\mathsf{PC}$.

Proof: For any natural number n, the algebra

$$N^6(\underline{\mathfrak{A}_n}) = \langle |N^6(\underline{\mathfrak{A}_n})|, 1_{N^6(\underline{\mathfrak{A}_n})}, \wedge_{N^6(\underline{\mathfrak{A}_n})}, \vee_{N^6(\underline{\mathfrak{A}_n})}, \rightarrow_{N^6(\underline{\mathfrak{A}_n})}, \neg_{N^6(\underline{\mathfrak{A}_n})} \rangle$$

is defined in the same way as Lemma 4.14 except for negation. $\neg_{N^4(\underline{\mathfrak{A}_n})}$ is defined as follows:

$$\neg_{N^6(\underline{\mathfrak{A}_n})} a := a_{n+8} \text{ for any } a \in |N^6(\underline{\mathfrak{A}_n})|.$$

We can show $N^6(\underline{\mathfrak{A}_n})$ validates (N) and (An^-) in the same way as (ii).

For any natural number n, we define

$$\mathcal{F}_n := \chi^g_{\neg\neg(g_{0_{\underline{\mathfrak{A}_n}}} \to g_{0_{\underline{\mathfrak{A}_n}}}) \vee (\neg(g_{\mathbb{N}} \to g_{\mathbb{N}}) \to g_{0_{\underline{\mathfrak{A}_n}}})}(N^6(\underline{\mathfrak{A}_n})).$$

Using $N^6(\underline{\mathfrak{A}_n})$, $\mathcal{F}_n$ and $(\mathsf{An} \cap \mathsf{ECQ})$ instead of $N^1(\underline{\mathfrak{A}_n})$, $\mathcal{A}_n$ and (An) respectively, we can show (vi) in the same manner as (i). □

LEMMA 4.18. *There is a continuum of logics between* $\mathsf{An}^-\mathsf{PC}$ *and* CoPC.

PROOF: For any natural number n, the algebra

$$N^7(\underline{\mathfrak{A}_n}) = \langle |N^7(\underline{\mathfrak{A}_n})|, 1_{N^7(\underline{\mathfrak{A}_n})}, \wedge_{N^7(\underline{\mathfrak{A}_n})}, \vee_{N^7(\underline{\mathfrak{A}_n})}, \to_{N^7(\underline{\mathfrak{A}_n})}, \neg_{N^7(\underline{\mathfrak{A}_n})} \rangle$$

is defined in the same way as Lemma 4.14 except for negation. $\neg_{N^7(\underline{\mathfrak{A}_n})}$ is defined as follows:

$$\neg_{N^7(\underline{\mathfrak{A}_n})} a := \begin{cases} \mathbb{N} & \text{if } a = 0_{\underline{\mathfrak{A}_n}}; \\ a_0 & \text{otherwise ,} \end{cases} \text{ for any } a \in |N^7(\underline{\mathfrak{A}_n})|.$$

First, we see that $N^7(\underline{\mathfrak{A}_n})$ validates conditions (Co) and (N).
We see that (Co). Let $a, b \in N^7(\underline{\mathfrak{A}_n})$. If $a \neq 0_{\underline{\mathfrak{A}_n}}$ and $b \neq 0_{\underline{\mathfrak{A}_n}}$, then

$$\begin{aligned}(a \to_{N^7(\underline{\mathfrak{A}_n})} b) \to_n (\neg_{N^7(\underline{\mathfrak{A}_n})} b \to_{N^7(\underline{\mathfrak{A}_n})} \neg_{N^7(\underline{\mathfrak{A}_n})} a) \\ &= (a \to_{N^7(\underline{\mathfrak{A}_n})} b) \to_{N^7(\underline{\mathfrak{A}_n})} (a_0 \to_{N^7(\underline{\mathfrak{A}_n})} a_0) \\ &= (a \to_{N^7(\underline{\mathfrak{A}_n})} b) \to_{N^7(\underline{\mathfrak{A}_n})} \mathbb{N} \\ &= \mathbb{N}.\end{aligned}$$

Hence (Co) holds for $a \neq 0_{\underline{\mathfrak{A}_n}}$ and $b \neq 0_{\underline{\mathfrak{A}_n}}$. If $a \neq 0_{\underline{\mathfrak{A}_n}}$ and $b = 0_{\underline{\mathfrak{A}_n}}$, then

$$\begin{aligned}(0_{\underline{\mathfrak{A}_n}} \to_{N^7(\underline{\mathfrak{A}_n})} b) \to_{N^7(\underline{\mathfrak{A}_n})} (\neg_{N^7(\underline{\mathfrak{A}_n})} b \to_{N^7(\underline{\mathfrak{A}_n})} \neg_{N^7(\underline{\mathfrak{A}_n})} 0_{\underline{\mathfrak{A}_n}}) \\ &= \mathbb{N} \to_{N^7(\underline{\mathfrak{A}_n})} (a_0 \to_{N^7(\underline{\mathfrak{A}_n})} \mathbb{N}) \\ &= \mathbb{N}.\end{aligned}$$

Hence (Co) holds for $a \neq 0_{\underline{\mathfrak{A}_n}}$ and $b = 0_{\underline{\mathfrak{A}_n}}$. If $a \neq 0_{\underline{\mathfrak{A}_n}}$ and $b \neq 0_{\underline{\mathfrak{A}_n}}$, then

$$\begin{aligned}(a \to_{N^7(\underline{\mathfrak{A}_n})} 0_{\underline{\mathfrak{A}_n}}) \to_{N^7(\underline{\mathfrak{A}_n})} (\neg_{N^7(\underline{\mathfrak{A}_n})} 0_{\underline{\mathfrak{A}_n}} \to_{N^7(\underline{\mathfrak{A}_n})} \neg_{N^7(\underline{\mathfrak{A}_n})} a) \\ &= (a \to_{N^7(\underline{\mathfrak{A}_n})} 0_{\underline{\mathfrak{A}_n}}) \to_{N^7(\underline{\mathfrak{A}_n})} (\mathbb{N} \to_{N^7(\underline{\mathfrak{A}_n})} a_0) \\ &= 0_{\underline{\mathfrak{A}_n}} \to_{N^7(\underline{\mathfrak{A}_n})} a_0 \\ &= \mathbb{N}.\end{aligned}$$

Hence (Co) holds for $a \neq 0_{\underline{\mathfrak{A}_n}}$ and $b \neq 0_{\underline{\mathfrak{A}_n}}$. The remaining case can be proven in the same manner. So (Co) holds in $N^7(\underline{\mathfrak{A}_n})$. The cases (N) can be shown in the same manner.

For any natural number n, we define $\mathcal{G}_n := \chi^{g}_{(g_{a_0} \to \neg g_{a_0}) \to (\neg g_{0_{\underline{\mathfrak{A}_n}}} \to \neg g_{a_0})}(N^7(\underline{\mathfrak{A}_n}))$. Using $N^7(\underline{\mathfrak{A}_n})$, $\mathcal{G}_n$ and (An$^-$) instead of $N^1(\underline{\mathfrak{A}_n})$, $\mathcal{A}_n$ and (An) respectively, we can show (vii) in the same manner as (i). □

LEMMA 4.19. *There is a continuum of logics between* CoPC *and* NECQPC.

PROOF: For any natural number n, the algebra

$$N^8(\underline{\mathfrak{A}_n}) = \langle |N^8(\underline{\mathfrak{A}_n})|, 1_{N^8(\underline{\mathfrak{A}_n})}, \wedge_{N^8(\underline{\mathfrak{A}_n})}, \vee_{N^8(\underline{\mathfrak{A}_n})}, \to_{N^8(\underline{\mathfrak{A}_n})}, \neg_{N^8(\underline{\mathfrak{A}_n})} \rangle$$

is defined in the same way as Lemma 4.14 except for negation. $\neg_{N^8(\underline{\mathfrak{A}_n})}$ is defined as follows:

$$\neg_{N^8(\underline{\mathfrak{A}_n})} a := \begin{cases} \mathbb{N} & \text{if } a = a_{n+8}; \\ a_{n+8} & \text{otherwise ,} \end{cases} \quad \text{for any } a \in |N^8(\underline{\mathfrak{A}_n})|.$$

First, we see that $N^8(\underline{\mathfrak{A}_n})$ validates the conditions (N) and (NECQ). For (N), let $a, b \in N^8(\underline{\mathfrak{A}_n})$. If $a \neq a_{n+8}$ and $b \neq a_{n+8}$, then

$$\begin{aligned}(a \leftrightarrow_{N^8(\underline{\mathfrak{A}_n})} b) \to_{N^8(\underline{\mathfrak{A}_n})} (\neg_{N^8(\underline{\mathfrak{A}_n})} a \leftrightarrow_{N^8(\underline{\mathfrak{A}_n})} \neg_{N^8(\underline{\mathfrak{A}_n})} b) \\ &= (a \leftrightarrow_{N^8(\underline{\mathfrak{A}_n})} b) \to_{N^8(\underline{\mathfrak{A}_n})} (a_{n+8} \leftrightarrow_{N^8(\underline{\mathfrak{A}_n})} a_{n+8}) \\ &= (a \leftrightarrow_{N^8(\underline{\mathfrak{A}_n})} b) \to_{N^8(\underline{\mathfrak{A}_n})} \mathbb{N} \\ &= \mathbb{N}.\end{aligned}$$

sHence (N) holds for $a \neq a_{n+8}$ and $b \neq a_{n+8}$. If $a = a_{n+8}$ and $b \neq a_{n+8}$, then

$$
\begin{aligned}
(a_{n+8} \leftrightarrow_{N^8(\underline{\mathfrak{A}_n})} b) \to_{N^8(\underline{\mathfrak{A}_n})} (\neg_{N^8(\underline{\mathfrak{A}_n})} a_{n+8} \leftrightarrow_{N^8(\underline{\mathfrak{A}_n})} \neg_{N^8(\underline{\mathfrak{A}_n})} b) \\
&= (a_{n+8} \leftrightarrow_{N^8(\underline{\mathfrak{A}_n})} b) \to_{N^8(\underline{\mathfrak{A}_n})} (\mathbb{N} \leftrightarrow_{N^8(\underline{\mathfrak{A}_n})} a_{n+8}) \\
&= (a_{n+8} \leftrightarrow_{N^8(\underline{\mathfrak{A}_n})} b) \to_{N^8(\underline{\mathfrak{A}_n})} a_{n+8} \\
&= \mathbb{N}.
\end{aligned}
$$

The equality $\mathbb{N} \leftrightarrow_{N^8(\underline{\mathfrak{A}_n})} a_{n+8} = a_{n+8}$ follows from $\mathbb{N} \leftrightarrow_{N^8(\underline{\mathfrak{A}_n})} a_{n+8} = (\mathbb{N} \to_{N^8(\underline{\mathfrak{A}_n})} a_{n+8}) \wedge_{N^8(\underline{\mathfrak{A}_n})} (a_{n+8} \to_{N^8(\underline{\mathfrak{A}_n})} \mathbb{N})$ and $\mathbb{N} \to_{N^8(\underline{\mathfrak{A}_n})} a_{n+8} = a_{n+8}$. Hence (N) holds for $a = a_{n+8}$ and $b \neq a_{n+8}$. The remaining case can be proven in the same manner.

For (NECQ), let $a, b \in |N^8(\underline{\mathfrak{A}_n})|$, then

$$
\begin{aligned}
(a \wedge_{N^8(\underline{\mathfrak{A}_n})} \neg_{N^8(\underline{\mathfrak{A}_n})} a) \to_{N^8(\underline{\mathfrak{A}_n})} \neg_{N^8(\underline{\mathfrak{A}_n})} b &= a_{n+8} \to_{N^8(\underline{\mathfrak{A}_n})} \neg_{N^8(\underline{\mathfrak{A}_n})} b \\
&= \mathbb{N}.
\end{aligned}
$$

Hence (NECQ) holds in $N^8(\underline{\mathfrak{A}_n})$.

For any natural number n, we define $\mathcal{H}_n := \chi^g_{(g_{a_0} \to g_{a_{n+8}}) \to (\neg g_{a_{n+8}} \to \neg g_{a_0})}(N^8(\underline{\mathfrak{A}_n}))$. Using $N^8(\underline{\mathfrak{A}_n})$, $\mathcal{H}_n$ and (Co) instead of $N^1(\underline{\mathfrak{A}_n})$, $\mathcal{A}_n$ and (An) respectively, we can show (viii) in the same manner as (i). $\square$

Lemma 4.20. *There is a continuum of logics between* NECQPC *and* NPC.

Proof: For any natural number n, the algebra

$$N^9(\underline{\mathfrak{A}_n}) = \langle |N^9(\underline{\mathfrak{A}_n})|, 1_{N^9(\underline{\mathfrak{A}_n})}, \wedge_{N^9(\underline{\mathfrak{A}_n})}, \vee_{N^9(\underline{\mathfrak{A}_n})}, \to_{N^9(\underline{\mathfrak{A}_n})}, \neg_{N^9(\underline{\mathfrak{A}_n})} \rangle$$

is defined in the same way as Lemma 4.14 except for negation. $\neg_{N^9(\underline{\mathfrak{A}_n})}$ is defined as follows:

$$\neg_{N^9(\underline{\mathfrak{A}_n})} a := \begin{cases} \mathbb{N} & \text{if } a = \mathbb{N}; \\ a_{n+8} & \text{otherwise} \end{cases}, \quad \text{for any } a \in |N^9(\underline{\mathfrak{A}_n})|.$$

As in the case of (viii), we can show that $N^9(\underline{\mathfrak{A}_n})$ validates (N).

For any natural number n, we define $\mathcal{H}_n := \chi^g_{(g_{\mathbb{N}} \wedge \neg g_{\mathbb{N}}) \to \neg g_{a_{n+8}}}(N^9(\underline{\mathfrak{A}_n}))$. Using $N^9(\underline{\mathfrak{A}_n})$, $\mathcal{H}_n$ and (NECQ) instead of $N^1(\underline{\mathfrak{A}_n})$, $\mathcal{A}_n$ and (An) respectively, we can show (ix) in the same manner as (i). $\square$

Lemma 4.19 and Lemma 4.20 are already proved in [1, Proposition 3.5] using neighborhood semantics.

5. Concluding remarks

5.1. Summary of the main results

In [13], it was clear that **SUBMIN** is a subsystem of both co-minimal logic and minimal logic, but it was not clear if **SUBMIN** is the strongest logic among the systems that are contained in both co-minimal logic and minimal logic. We identified that **SUBMIN** is not the strongest logic, but $\mathsf{An} \cap \mathsf{ECQPC}$ is the strongest logic. Moreover, we presented new formalizations of co-minimal logic and **SUBMIN** that clarify their relationships with ECQ. Furthermore, we applied the method used in [15] and simplified the proofs of the results offered in [1] and obtained new results that are not included in [1]. Figure 5 summarizes the main results of this paper.
Note here that there exist continua of logics in each line in the figure above, as we proved in Theorem 4.13.

5.2. Future Directions

There is still much work to be done in this area of research. Some directions that seem to be worth exploring are described here.

ECQ and substructural logics For proof systems, this paper focused on Hilbert systems. But we can also define sequent calcului for these logics (cf. [8, 3, 11]). Then, given the deep connections between substructural logics and algebraic semantics, it will be interesting to explore the substructural versions of LN, which is a sequent calculus version of NPC, and its extensions. Note that the existence of continua of logics between pairs of substructural logics is explored in [10]. Therefore, it would be interesting to examine whether the method used in this paper can be applied to substructural versions of LN and related systems.

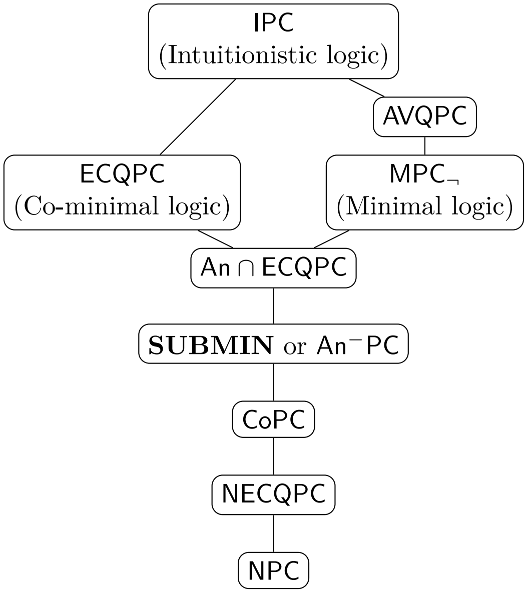


Figure 5: Summary of the results presented in the paper

Another method We used algebraic semantics for showing the existence of continua of logics between pairs of logics related to intuitionistic logic and minimal logic. It is known that there is a duality ("Priestley duality") between the class of Priestley spaces and the class of bounded distributive lattices. Since N-algebras are distributive lattices, we might be able to use topological semantics to obtain yet another proof for the existence of continua of logics.[2]

[2]This idea was pointed out by Prof. Hanamantagouda P. Sankappanavar after the presentation based on an earlier draft of this paper at *Non-Classical Logics: Theory and Applications 2024*. I would like to thank Prof. Sankappanavar for this interesting comment.

No gap between logics This paper focused on the existence of continua of logics between logics related to intuitionistic logic and minimal logic. However, there are pairs of intermediate logics where no logic exists between the pair, as shown in [5]. More specifically, if we consider the family of extensions of intuitionistic logic, known as the n-valued Gödel logic G_n, then for $n \geq 3$, no logic exists between G_n and G_{n+1}. In view of this result, it will be interesting to see if extensions of the logics we considered in this paper will have similar results.

First-order logics We discussed only propositional logics in this paper, but of course it will be interesting to consider first-order logics. If we add universal and existential quantifiers to these logics with the usual axioms and rules, *without* the additional axioms such as the constant domain axiom, then we can show immediately the existence of continua of logics between first-order expansions of logics discussed in this paper by considering propositional logics to be predicate logics with only zero argument predicate symbols. However, other cases remain to be explored in further detail.

Appendix: Kripke semantics for logics above $\mathsf{An^-PC}$

Definition 6.1. We define the following Hilbert systems:

- **SUBMIN** is the Hilbert system obtained by adding axioms $\bot \to p$, $p \to \top$ and $\neg p \to \neg\neg\top$ to CoPC in $\mathcal{L}_{\neg,\top,\bot}$.
- $\mathbf{SUBMIN}^{-\top,\bot}$ as the Hilbert system adding axioms $\neg p \to \neg\neg(q \to q)$ to CoPC in $\mathcal{L}_{\neg}$.
- **MIN** is the Hilbert system obtained by adding axioms $\bot \to p$, $p \to \top$, $\neg p \to \neg\neg\top$ and $\neg\neg\top$ to CoPC in $\mathcal{L}_{\neg,\top,\bot}$.
- $\mathbf{MIN}^{-\top,\bot}$ as the Hilbert system adding axioms $\neg p \to \neg\neg(q \to q)$ and $\neg\neg(p \to p)$ to CoPC in $\mathcal{L}_{\neg}$.

Kripke semantics for the logic **SUBMIN** ((F, G)-semantics for short) was introduced in [13].

DEFINITION 6.2 (cf. [13]). An (F, G)-frame $\mathcal{F}$ for $\mathsf{An^{-}PC}$ is a quadruple $(W, \leq, F, G)$ satisfying the following:

- $(W, \leq)$ is a quasi-order, i.e., $\leq$ is a reflexive and transitive relation on W (we call an element of W a world);
- F, G are upward closed subsets of W such that F is a subset of G.

An (F, G)-model $\mathcal{M}$ is a pair $(\mathcal{F}, \mathcal{V})$ satisfying the following:

- $\mathcal{F}$ is an (F, G)-frame;
- $\mathcal{V}$ is a mapping assigning an upward closed set of worlds to each propositional variable.

A valuation of formulas in $\mathcal{L}_{\neg,\top,\bot}$ is inductively defined as follows:

- $\mathcal{M}, w \Vdash_{(F,G)} \top$ for all $w \in W$;
- $\mathcal{M}, w \nVdash_{(F,G)} \bot$ for all $w \in W$;
- $\mathcal{M}, w \Vdash_{(F,G)} p \iff w \in \mathcal{V}(p)$;
- $\mathcal{M}, w \Vdash_{(F,G)} A \wedge B \iff \mathcal{M}, w \Vdash_{(F,G)} A$ and $\mathcal{M}, w \Vdash_{(F,G)} B$;
- $\mathcal{M}, w \Vdash_{(F,G)} A \vee B \iff \mathcal{M}, w \Vdash_{(F,G)} A$ or $\mathcal{M}, w \Vdash_{(F,G)} B$;
- $\mathcal{M}, w \Vdash_{(F,G)} A \to B \iff \forall v \geq w[\mathcal{M}, v \Vdash_{(F,G)} A \Rightarrow \mathcal{M}, v \Vdash_{(F,G)} B]$;
- $\mathcal{M}, w \Vdash_{(F,G)} \neg A \iff \forall v \geq w[\mathcal{M}, v \Vdash_{(F,G)} A \Rightarrow v \in F]$ and $w \in G$.

$\mathcal{F} \models_{(F,G)} A$ denotes that $(\mathcal{F}, \mathcal{V}), w \Vdash_{(F,G)} A$ for all $\mathcal{V}$ and $w \in W$. $\mathcal{M} \models_{(F,G)} A$ denotes that $\mathcal{M}, w \Vdash_{(F,G)} A$ for all $w \in W$. For any set of formulas Γ, $\Gamma \models_{(F,G)} A$ denotes $\mathcal{M}, w \Vdash_{(F,G)} B$ for any $B \in \Gamma$ implies $\mathcal{M}, w \Vdash_{(F,G)} A$, for any $\mathcal{M}$ and $w \in W$, where W is the set of worlds of $\mathcal{M}$.

By imposing conditions on (F, G)-frame, the soundness and completeness theorems holds with some already defined logics.

FACT 6.3 (cf. [13]).

1. $\Gamma \vdash_{\mathbf{SUBMIN}} A \iff \Gamma \models_{(F,G)} A$.
2. $\Gamma \vdash_{\mathbf{MIN}} A \iff \Gamma \models_{(F,G)} A$ for the class of (F, G)-semantics with $G = W$.
3. $\Gamma \vdash_{\mathbf{CO\text{-}MIN}} A \iff \Gamma \models_{(F,G)} A$ for the class of (F, G)-semantics with $F = \varnothing$.
4. $\Gamma \vdash_{\mathbf{IPC}} A \iff \Gamma \models_{(F,G)} A$ for the class of (F, G)-semantics with $G = W$ and $F = \varnothing$, where **IPC** is intuitionistic logic.

By excluding the conditions on $\top$ and $\bot$, (F,G)-semantics for formulas in $\mathcal{L}_\neg$ can be given. Then we can show the soundness and completeness in the same way.

Fact 6.4.

1. $\Gamma \vdash_{\mathbf{SUBMIN}^{-\top,\bot}} A \iff \Gamma \models_{(F,G)} A$.
2. $\Gamma \vdash_{\mathbf{MIN}^{-\top,\bot}} A \iff \Gamma \models_{(F,G)} A$ for the class of (F,G)-semantics with $G = W$.
3. $\Gamma \vdash_{\mathbf{CO\text{-}MIN}^{-\top,\bot}} A \iff \Gamma \models_{(F,G)} A$ for the class of (F,G)-semantics with $F = \varnothing$.
4. $\Gamma \vdash_{\mathbf{IPC}^{-\top,\bot}} A \iff \Gamma \models_{(F,G)} A$ for the class of (F,G)-semantics with $G = W$ and $F = \varnothing$.

Hence, for $\top$ and $\bot$ free formula A, $\mathbf{CO\text{-}MIN} \vdash A \Leftrightarrow \mathbf{CO\text{-}MIN}^{-\top,\bot} \vdash A$ is immediate from the above facts. The same argument holds for the remaining cases. Furthermore, $\mathbf{SUBMIN}^{-\top,\bot}$ is equivalent to $\mathsf{An}^{-}\mathsf{PC}$, and $\mathbf{MIN}^{-\top,\bot}$ is equivalent to $\mathsf{MPC}_\neg$ (cf. [8]).

The soundness and completeness theorems can be shown for $\mathsf{An} \cap \mathsf{ECQPC}$ and AVQPC by adding similar conditions for F and G.

Definition 6.5 ($\mathsf{An} \cap \mathsf{ECQ}$-frame and AVQ-frame). For an (F,G)-frame without $\top$ and $\bot$, F' denotes $:= \{w \in W \mid \forall v \geq w (v \notin F)\}$. We define the following conditions:

- $\mathsf{An} \cap \mathsf{ECQ}$-frames are (F,G)-frames with $F' \cup G = W$.
- AVQ-frames are (F,G)-frames such that $\forall u \geq w[u \notin F']$ implies $w \in F$ for all $w \in W$ and $G = W$ hold.

We write $\Vdash_{\mathsf{An}\cap\mathsf{ECQ}}, \models_{\mathsf{An}\cap\mathsf{ECQ}}, \Vdash_{\mathsf{AVQ}}$ and $\models_{\mathsf{AVQ}}$ for validity with respect to the classes of $\mathsf{An} \cap \mathsf{ECQ}$-frames and AVQ-frames.

Definition 6.6 (cf. [14]). A sub-normal Kripke frame is a tuple $(W, \leq, F)$ satisfying the following:

- $(W, \leq)$ is a quasi-order, i.e., $\leq$ is a reflexive and transitive relation on W (we call an element of W a world);
- F is an upward closed subset of W such that for every $w \notin F$, there is $v \in W$ such that $w \leq v$ and for every $u \in W$, if $v \leq u$, then $u \notin F$.

For any AVQ-frame $(W, \leq, F, G)$, G is uniquely determined only by W and F, and so the AVQ-frame can be rewritten as $(W, \leq, F)$. In [14], *sub-normal Kripke frame* $(W, \leq, F)$ was introduced. Because the condition for F in AVQ-frane is the contraposition of the condition for F in sub-normal Kripke frame, AVQ-frame is sub-normal frame, and vice versa. Because our definition of AVQ-frame is more convenient for the proof for soundness, we use the definition of AVQ-frame.

We show the soundness and completeness of $\mathsf{An} \cap \mathsf{ECQPC}$ for the class of $\mathsf{An} \cap \mathsf{ECQPC}$-frames.

THEOREM 6.7 (Soundness of $\mathsf{An} \cap \mathsf{ECQPC}$). *If* $\Gamma \vdash_{\mathsf{An} \cap \mathsf{ECQPC}} A$, *then* $\Gamma \models_{\mathsf{An} \cap \mathsf{ECQ}} A$.

PROOF: We show this statement by induction on the length of the deduction. Here we show only the cases for the negative axiom ($\mathsf{An} \cap \mathsf{ECQ}$), since the remaining case can be proven in the same manner as [8]. Let $(\mathcal{F}, \mathcal{V})$ and $w \in W$ be arbitrary.

Suppose $(\mathcal{F}, \mathcal{V}), w \not\Vdash_{\mathsf{An} \cap \mathsf{ECQ}} \neg\neg(p \to p)$. First, we observe that $(\mathcal{F}, \mathcal{V}), v \Vdash_{\mathsf{An} \cap \mathsf{ECQ}} \neg\neg(p \to p)$ iff $v \in G$ for any $v \in W$. We show $(\mathcal{F}, \mathcal{V}), u \not\Vdash_{\mathsf{An} \cap \mathsf{ECQ}} \neg(q \to q)$ for every $u \geq w$, which implies $w \Vdash_{\mathsf{An} \cap \mathsf{ECQ}} \neg(q \to q) \to r$. Since $w \notin G$, $w \in F'$ and so $u \notin F$ for every $u \geq w$. By the definition of the valuation, $(\mathcal{F}, \mathcal{V}), u \not\Vdash_{\mathsf{An} \cap \mathsf{ECQ}} \neg(q \to q)$ for every $u \geq w$. $\Box$

In what follows, we call a set of formulas Δ saturated if the following conditions hold,

- There is a formula A such that $A \notin \Delta$ (nontriviality);
- $\Delta \vdash_{\mathsf{An} \cap \mathsf{ECQPC}} A \Rightarrow A \in \Delta$;
- $\Delta \vdash_{\mathsf{An} \cap \mathsf{ECQPC}} A \vee B \Rightarrow \Delta \vdash_{\mathsf{An} \cap \mathsf{ECQPC}} A$ or $\Delta \vdash_{\mathsf{An} \cap \mathsf{ECQPC}} B$.

THEOREM 6.8 (Completeness of $\mathsf{An} \cap \mathsf{ECQPC}$). *If* $\Gamma \models_{\mathsf{An} \cap \mathsf{ECQ}} A$, *then* $\Gamma \vdash_{\mathsf{An} \cap \mathsf{ECQPC}} A$.

PROOF: Given $\Gamma \not\models_{\mathsf{An} \cap \mathsf{ECQ}} A$, we construct a saturated set $\Gamma_0 \supset \Gamma$ such that $\Gamma_0 \not\models_{\mathsf{An} \cap \mathsf{ECQ}} A$. Then the canonical model $\mathcal{M} = (W, \leq, F, G, \mathcal{V})$ with respect to Γ_0 is defined standardly. For F and G, we define $F := \{\Delta \mid \neg B \in \Delta \text{ for all } B\}$and $G := \{\Delta \mid \neg B \in \Delta \text{ for some } B\}$. It is sufficient to show $F' \cup G = W$, since the remaining case can be proven in the same way as [8].

Suppose $\Delta \notin G$, and so $\neg B \notin \Delta$ for any B. Hence $\neg\neg(p \to p) \notin \Delta$. By $\mathsf{An} \cap \mathsf{ECQPC} \vdash \neg\neg(p \to p) \vee (\neg(q \to q) \to C)$ for any formula C, $\neg(q \to q) \to C \in \Delta$. By nontriviality of saturated set, $\neg(q \to q) \notin \Delta'$ for any $\Delta' \geq \Delta$, and so we obtain $\Delta' \notin F$ for any $\Delta' \geq \Delta$, and so $\Delta \in F'$. □

We next show the soundness and completeness of AVQPC for the class of AVQPC-frames.

THEOREM 6.9 (Soundness of AVQPC). *If* $\Gamma \vdash_{\mathsf{AVQPC}} A$, *then* $\Gamma \models_{\mathsf{AVQPC}} A$.

PROOF: We show this statement by induction on the length of the deduction. Here we show only the cases for the negative axiom (AVQPC), since the remaining case can be proven in the same manner as [8]. Let $(\mathcal{F}, \mathcal{V})$ and $w \in W$ be arbitrary.

Suppose $(\mathcal{F}, \mathcal{V}), u \Vdash_{\mathsf{AVQ}} \neg(\neg(p \to p) \to q)$ for $u \geq w$. We want to show $u \in F$. By assumption, $(\mathcal{F}, \mathcal{V}), v \Vdash_{\mathsf{AVQ}} \neg(p \to p) \to q$ implies $v \in F$ for any $v \geq u$. If $(\mathcal{F}, \mathcal{V}), v \Vdash_{\mathsf{AVQ}} \neg(p \to p) \to q$, then $v \in F$. Otherwise, $(\mathcal{F}, \mathcal{V}), v' \Vdash_{\mathsf{AVQ}} \neg(p \to p)$ and $(\mathcal{F}, \mathcal{V}), v' \nVdash_{\mathsf{AVQ}} q$ for some $v' \geq v \geq u$. By the definition of AVQ-frame, $u \in F$. □

For the completeness, we need the following lemma.

LEMMA 6.10. $\mathsf{MPC}_{\neg} \vdash \neg A \leftrightarrow (A \to \neg(p \to p))$.

PROOF: $\mathsf{MPC}_{\neg} \vdash \neg A \to (A \to \neg(p \to p))$ is immediate from (NECQ). For the other direction, by (NECQ), $\mathsf{MPC}_{\neg} \vdash ((A \to \neg(p \to p)) \wedge A) \to (A \to \neg A)$, and so $\mathsf{MPC}_{\neg} \vdash (A \to \neg(p \to p)) \to (A \to \neg A)$. By ($\mathsf{An}$), $\mathsf{MPC}_{\neg} \vdash (A \to \neg A) \to \neg A$, and so $\mathsf{MPC}_{\neg} \vdash (A \to \neg(p \to p)) \to \neg A$. □

THEOREM 6.11 (Completeness of AVQPC). *If* $\Gamma \models_{\mathsf{AVQ}} A$, *then* $\Gamma \vdash_{\mathsf{AVQPC}} A$.

PROOF: Given $\Gamma \nvdash_{\mathsf{AVQ}} A$, we construct a saturated set $\Gamma_0 \supset \Gamma$ such that $\Gamma_0 \nvdash_{\mathsf{AVQ}} A$. Then the canonical model $\mathcal{M} = (W, \leq, F, G, \mathcal{V})$ with respect to Γ_0 is defined standardly. For F and G, we define $F := \{\Delta \mid \neg B \in \Delta \text{ for all } B\}$ and $G := \{\Delta \mid \neg B \in \Delta \text{ for some } B\}$. It is sufficient to show $\forall \Delta' \geq \Delta[\Delta' \notin F']$ implies $\Delta \in F$ for all $\Delta \in W$, since the remaining case can be proven in the same manner as [8], and $G = W$ is obvious: Take any $\Delta \in W$ and suppose $\Delta' \notin F'$ for any $\Delta' \geq \Delta \cdots (\star)$. We want to show $\Delta \in F$. Suppose $\Delta \notin F$, and so $\neg B \notin \Delta$ for some B. By (Co), AVQPC $\vdash (B \to (p \to p)) \to (\neg(p \to p) \to \neg B)$. Hence $\neg(p \to p) \notin \Delta$. Let $C_0, C_1, C_2 \ldots$ be an enumeration of all formulas. By (AVQ) and the previous lemma, AVQPC $\vdash ((\neg(p \to p) \to C_0) \to \neg(p \to p)) \to \neg(p \to p)$, and so $(\neg(p \to p) \to C_0) \to \neg(p \to p) \notin \Delta$. Hence, there is a saturated $\Delta^0 \supseteq \Delta$ such that $\neg(p \to p) \to C_0 \in \Delta^0$ and $\neg(p \to p) \notin \Delta^0$. By repeating this operation, we can take a sequence of saturated set $\Delta^0 \subseteq \Delta^1 \subseteq \Delta^2 \subseteq \ldots$ and a saturated set $\bigcup_{i \in \omega} \Delta^i$. By the nontriviality of saturated set, $\neg(p \to p) \notin \Delta'$ for any $\Delta' \geq \bigcup_{i \in \omega} \Delta^i$, and so $\Delta' \notin F$ and $\cup_{i \in \omega} \Delta^i \in F'$. But this contradicts $(\star)$, and so $\Delta \in F$. □

Acknowledgements. This paper is based on the author's master thesis written under the supervision of Nobu-Yuki Suzuki. I am grateful to Nobu-Yuki Suzuki for collaboration at the early stage of this work. I would also like to thank Takako Nemoto, Hitoshi Omori and Satoru Niki for useful discussions.

References

[1] N. Bezhanishvili, A. Colacito, D. de Jongh, *A study of subminimal logics of negation and their modal companions*, [in:] **Language, Logic, and Computation 12th International Tbilisi Symposium, TbiLLC 2017, Lagodekhi, Georgia, September 18-22, 2017, Revised Selected Papers 12**, Springer (2019), pp. 21–41, DOI: https://doi.org/10.1007/978-3-662-59565-7_2.

[2] A. Chagrov, M. Zakharyaschev, **Modal logic**, Oxford University Press (1997).

[3] A. Colacito, **Minimal and Subminimal Logic of Negation**, Master's thesis, University of Amsterdam (2016), URL: https://eprints.illc.uva.nl/id/eprint/986.

[4] A. Colacito, D. de Jongh, A. L. Vargas, *Subminimal negation*, **Soft Computing**, vol. 21 (2017), pp. 165–174, DOI: https://doi.org/10.1007/s00500-016-2391-8.

[5] T. Hosoi, H. Ono, *Intermediate propositional logics (a survey)*, **Journal of Tsuda College**, vol. 5 (1973), pp. 67–82.

[6] I. Johansson, *Der Minimalkalkül, ein reduzierter intuitionistischer Formalismus*, **Compositio Mathematica**, vol. 4 (1937), pp. 119–136, URL: http://eudml.org/doc/88648.

[7] S. Niki, *Decidable variables for constructive logics*, **Mathematical Logic Quarterly**, vol. 66(4) (2020), pp. 484–493, DOI: https://doi.org/10.1002/malq.202000022.

[8] S. Niki, *Subminimal logics in light of Vakarelov's logic*, **Studia Logica**, vol. 108(5) (2020), pp. 967–987, DOI: https://doi.org/10.1007/s11225-019-09884-z.

[9] H. Ono, *On some intuitionistic modal logics*, **Publications of the Research Institute for Mathematical Sciences**, vol. 13(3) (1977), pp. 687–722.

[10] N. Suzuki, *The existence of 2^{ω} logics lacking the weakening rule below the intuitionistic logic*, **Reports on Mathematical Logic**, vol. 21 (1987), pp. 85–95.

[11] M. Takano, *A Syntactical Study of the Subminimal Logic with Nelson Negation*, **Nihonkai Mathematical Journal**, vol. 18(1–2) (2007), pp. 17–31, URL: https://projecteuclid.org/journals/nihonkai-mathematical-journal/volume-18/issue-1-2/A-Syntactical-Study-of-the-Subminimal-Logic-with-Nelson-Negation/nihmj/1273587757.full.

[12] D. Vakarelov, *Consistency, Completeness and Negation*, [in:] G. Priest, R. Routley, J. Norman (eds.), **Paraconsistent Logic: Essays on the inconsistent**, Philosophia Verlag (1989), pp. 328–363.

[13] D. Vakarelov, *Nelson's negation on the base of weaker versions of intuitionistic negation*, **Studia Logica**, vol. 80 (2005), pp. 393–430, DOI: https://doi.org/10.1007/s11225-005-8476-5.

[14] P. W. Woodruff, *A note on JP*, **Theoria**, vol. 36 (2008), pp. 183–184, URL: https://api.semanticscholar.org/CorpusID:145545536.

[15] A. Wroński, *The degree of completeness of some fragments of the intuitionistic propositional logic*, **Reports on Mathematical Logic**, vol. 2 (1974).

[16] V. Yankov, *On the relation between deducibility in intuitionistic propositional calculus and finite implicative structures*, **Doklady Akademii Nauk SSSR**, vol. 151(6) (1963), pp. 1293–1294.

[17] V. Yankov, *The construction of a sequence of strongly independent superintuitionistic propositional calculi*, **Doklady Akademii Nauk SSSR**, vol. 181 (1968), pp. 33–34.

Kaito Ichikura
Tohoku University
Graduate School of Information Sciences
6-3-09 Aoba, Aramaki-aza Aoba-ku
980-8579 Sendai, Japan
e-mail: ichikura.kaito.t7@dc.tohoku.ac.jp

Funding information: This work was partially supported by JST SPRING, Grant Number JPMJSP2114.

Conflict of interests: None.

Ethical considerations: The Author assures of no violations of publication ethics and takes full responsibility for the content of the publication.

Declaration regarding the use of GAI tools: Not used.

Submission Guidelines

Manuscripts Papers submitted to the *BSL* should be formatted using the `BSLstyle` LaTeX class with the manuscript option loaded, which can be downloaded at https://czasopisma.uni.lodz.pl/bulletin/libraryFiles/downloadPublic/603. All prospective authors should read the "Instructions for authors" file included in the style files folder and follow the guidelines included there. Abstract and keywords are compulsory parts of each submission as they will be used in the *BSL* online search tools. Mind that an abstract should contain no references and the list of keywords should consist of at least 3 items. It is also recommended that each author having an ORCID number provides it in the `.tex` source file. Authors who are unable to comply with these requirements should contact the Editorial Office in advance.

Paper Length There is no fixed limit imposed on the length of submitted papers, however one can expect that for shorter papers, up to 18 pages long, the Editorial Board will be able to reduce the time needed for the reviewing process.

Footnotes should be avoided as much as possible, however it is not disallowed to use them if necessary.

Bibliography should be formatted using BibTeX and the `BSLbibstyle` bibliography style (to be found in the style files folder). It is essential that to each bibliography item a plain DOI number (i.e., not a full link) is attached whenever applicable. If a submitted paper is accepted for publication, the author(s) should provide the bibliography file in the `.bib` format among other source files. For more details on bibliography processing the authors are referred to the "Instructions for authors". Authors unfamiliar with BibTeX are advised to familiarize themselves with the short tutorial or video tutorial on managing bibliographies with BibTeX.

Affiliation and mailing addresses of all the authors should be included in the `\Affiliation` and `\AuthorEmail` fields, respectively, in the source `.tex` file.

Submission When the manuscript is ready, it should be submitted through our editorial platform, using the the «Make a Submission» button. If the paper is meant to be included in a special issue, the appropriate section name should be selected before submitting it. If the paper is regular, the authors can indicate the editor they would like to supervise the editorial process or leave this decision to the Editorial Office by leaving the "Comments for the Editor" section blank. For the duration of the whole editorial process of the manuscript it must not be submitted for review to any other venue.

Publication Once the manuscript has been accepted for publication and the galley proof has been revised by the authors, the article is given a DOI number and published in the *Early View* section, where articles accepted for publication and awaiting assignment to an issue are made available to the public. The authors will be notified when their article is assigned to an issue.

Copyright permission It is the authors' responsibility to obtain the necessary copyright permission from the copyright owner(s) of the submitted paper or extended abstract to publish the submitted material in the *BSL*.