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# EXPLORING THE DEFINITION OF NON-MONOTONICITY – LOGICAL AND PSYCHOLOGICAL CONSIDERATIONS

**Abstract**

When humans reason, they are able to revise their beliefs in light of new information and abandon obsolete conclusions. Logicians argued, that in some cases, such reasonings appear to be non-monotonic. Thus, many different, seemingly non-monotonic systems were created to formally model such cases. The purpose of this article is to re-examine the definition of non-monotonicity and its implementation in non-monotonic logics and in examples of everyday human reasoning. We will argue that many non-monotonic logics employ some weakened versions of the definitions of non-monotonicity, since in-between different steps of reasoning they either: a) allow previously accepted premises to be removed, or b) change the rules of inference. Of the two strategies, the second one seems downright absurd, since changing the rules of a given logic is a mere replacement of that logic with the rules of another. As a consequence we obtain two logics, whereas the definition of a non-monotonic logic is supposed to define one. The definition of non-monotonicity does not permit either of these cases, which means that such logics are monotonic.

*Keywords*: non-monotonicity, reasoning, belief revision.

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## 1. Introduction

Humans are relatively proficient at revising their beliefs and behavior based on incoming new information. They are able to abandon old conclusions while seemingly retaining all old premises. For example, I could easily say that *'My apartment is located 5 minutes from the main station by car, we will get there in that time,'* and then upon seeing the traffic correct myself by saying: *'Well, now it is going to take us at least 10 minutes.'* This appears to be a reasoning in which I have learned a new premise *'There is a heavy traffic'* and based on that I have rejected a previously accepted conclusion *'It will take us 5 minutes.'* Thus, I realize that the premise 'My apartment is located 5 minutes from the main station by car,' was actually: 'My apartment is located 5 minutes from the main station by car, with average traffic volume.' This is remarkable because in classical logic all reasonings are *monotonic*, which means that adding new premises to an existing set can never cause a previously accepted conclusion to be abandoned. If something was true based on past information, then adding new information (but not removing old information!) can never render it *untrue*. In fact, any deductive inference appears to be necessarily monotonic [24, p. 223]. Because of that, for some logicians it appeared that monotonicity is not a property of human everyday thinking. To address that, they created *non-monotonic* logics, which were, among other things, intended to model how humans adapt to new information.

The purpose of this article is to critically examine the claim that human reasoning is non-monotonic. In order to do that, we will analyze the relations between: the definition of non-monotonicity, the rules of some non-monotonic logics and psychological data about human belief revision and reasoning. In short, we will argue that: a) human reasoning could **appear to be** non-monotonic at a first glance, b) the analyzed non-monotonic logics do not satisfy the definition of non-monotonicity, c) in empirical studies it is extremely difficult to address the question if humans reason non-monotonically due to formal constraints and the requirements posed by the definition of non-monotonicity.

## 2. The understanding of 'reasoning' in logic and psychology

In this article we will analyze if it is justified to use the term *non-monotonic* when describing human reasoning.[1] Therefore, we must start from addressing what *reasoning* is. Both psychologists and logicians will agree that reasoning is a process of reaching conclusions from premises. Unfortunately, the details of what that definition entails are going to differ [40]. For logicians, that process is typically going to entail formal manipulations on the truth valuations of propositions. These truth valuations may be expressed just as simple '0's and '1,'s (in classical logic and most other logics), but also other values (e.g., probabilities) [12, 45]. All of these variants consider reasoning to be a manipulation of 'truth-functional' operators. In this sense, contemporary cognitive psychology differs from logic. Psychologist P. N. Johnson-Laird [14, p. 1], one of the most esteemed researchers of logic within human reasoning, wrote:

> *'Thirty years ago psychologists believed that human reasoning depended on formal rules of inference akin to those of a logical calculus. This hypothesis ran into difficulties, which led to an alternative view: reasoning depends on envisaging the possibilities consistent with the starting point – a perception of the world, a set of assertions, a memory, or some mixture of them. We construct mental models of each distinct possibility and derive a conclusion from them. (...) On this account, reasoning is a simulation of the world fleshed out with our knowledge, not a formal rearrangement of the logical skeletons of sentences.'*

In this quote, Johnson-Laird [14] signals that reasoning in cognitive science must necessarily entail processing entire contents of sentences and not just operations on their truth-values. In other words, truth-functional logics were ill-equipped to model human reasoning, because they stripped it of its essential property: processing information [37]. This shift in paradigm that Johnson-Laird [14] describes here resulted in cognitive science moving

[1] The word "reasoning" normally does not have a plural form in the English language. However, in this work "reasoning" is understood more broadly than the traditional use of the word in logic. Namely, we also refer to specific instances of human thought processes, which we call: "reasonings."

away from using truth-functional logics as the proper notation for mental models. In response to that, logicians started developing some non-truth-functional logics (for examples see: [21, 25]. Fortunately, even when reasonings are not truth-functional and cannot be easily expressed with formulaic relations between a handful of truth values, they can still be either monotonic or non-monotonic. Both logic and cognitive science usually agree that *reasoning* involves reaching conclusions from premises. The difference is, that in truth-functional logics this process will just involve applying rules to truth-values of sentences. In contrast, in cognitive science *reasoning* will involve exploration of the content and discovery of information entailed by it [37].

## 3. Fundamental problems with non-monotonicity

### 3.1. Definition of monotonicity and non-monotonicity

Inference $\vdash$ is monotonic if and only if for any $\alpha \in For$ and $X, Y \subseteq For$: if $X \vdash \alpha$, then $X \cup Y \vdash \alpha$, where $For$ is the set of formulas of the language. Thus, an inference $\vdash$ is non-monotonic if and only if for some $\alpha \in For$ and $X, Y \subseteq For$: $X \vdash \alpha$ and $X \cup Y \nvdash \alpha$. Equivalently, an inference $\vdash$ is non-monotonic if and only if there are such $\alpha \in For$ and $X, Y \subseteq For$ that $X \vdash \alpha$ but $X \cup Y \nvdash \alpha$. In other words, in monotonic inference/reasoning, if a conclusion $\alpha$ follows from a set of premises $X$, then it must also follow from any superset of $X$. The addition of new premises is not able to invalidate anything in the set of the old conclusions. In contrast, in non-monotonic inference it is possible that a conclusion $\alpha$ follows from the set $X$ but does not follow from some superset $X \cup Y$. In that case, adding a new premise may invalidate some of the previously accepted conclusions. Thus, two obvious facts are clear from the definition of non-monotonic inference [20].

The first fact is that the set of premises $X$ must be identical at both of the steps: $X \vdash \alpha$ and $X \cup Y \nvdash \alpha$. It means that even after new premises are added and create a new superset $Z = X \cup Y$, the $X$ symbol must denote exactly the same set of premises in both steps.

The second fact is that the inference '$\vdash$' must also be the same in both of the reasoning steps, which means it is defined by the same set of axioms and inference rules. Otherwise, the definition is not a definition of one inference, but two, since it contains symbols of two different inferences: $X \vdash_1 \alpha$ and $X \cup Y \nvdash_2 \alpha$.

In this work we will point out that most of the allegedly non-monotonic logics violate these conditions from a purely formal perspective [20]. For example, the fundamental works of Makinson [23] on non-monotonic extensions of classical logic have that problem, as well as the popular *System P* [12, 15] or adaptive logics [1]. The creation of these non-monotonic systems is often motivated by the desire to better capture the fact that human everyday reasonings are flexible and deal well with uncertainty [16]. For example, Makinson [23, p. 5] directly claims that: *'We are all non-monotonic'*. Therefore, these systems are good candidates for analyzing if non-monotonic logics really exist.

### 3.2. Is there such a thing as a 'non-monotonic consequence relation'?

The conditions of reflexivity, monotonicity and idempotence were used by Alfred Tarski (1935, 1936) to define the concept of a consequence relation. Let, as above, *For* be the set of all formulas of some formal language $\mathcal{L}$, then: $C : 2^{For} \to 2^{For}$[2] is a consequence operation if and only if for any $X \subseteq For$ :

1. $X \subseteq C(X)$ **reflexivity** of operation $C$ in language $\mathcal{L}$
2. $C(C(X)) \subseteq C(X)$ **idempotence** of operation $C$ in language $\mathcal{L}$
3. if $X \subseteq Y$ then $C(X) \subseteq C(Y)$ **monotonicity** of operation $C$ in language $\mathcal{L}$

$X$ is a set of premises, while $C(X)$ is a set of consequences – conclusions following from $X$ on the basis of operation $C$. The condition of reflexivity means that all the premises follow from themselves on the basis of $C$. The condition of idempotence means that repeated application of the operation $C$ on the set $C(X)$ will not expand $C(X)$ with any new conclusions. The condition of monotonicity means that expanding the set of premises cannot shrink the set $C(X)$. Therefore, if someone would like to assume the condition of **non-monotonicity** then they negate the monotonicity condition: for some $X, Y \subseteq For$,

$$X \subseteq Y \text{ and it is not true that } C(X) \subseteq C(Y)$$

[2] $2^X$ denotes the set of all subsets of the set $X$.

Thus, the consequence operation by definition assumed monotonicity. As a result, any construction which abandons that condition is not a consequence operation in the Tarskian sense. Instead, it is some other kind of operation. By analogy, the equivalence relation is by definition reflexive, symmetric and transitive. It is not permissible to say that some equivalence relations could be non-transitive, because that would constitute a different type of relation, not equivalence. Similarly, the term *"non-monotonic consequence operation"* violates the basic definition of consequence operation. Non-monotonic operations are operations, but not consequence operations. However, as *'operations'* they can be further analyzed in good faith.

In formal logic it is more common to write about consequence relations than operations. A consequence relation $\vdash$ is equivalent to some consequence operation $C$: for any $\alpha \in For$ and $X \subseteq For$,

$$X \vdash \alpha \text{ iff } \alpha \in C(X)$$

In that notation, the condition of non-monotonicity takes the, already used above, form: for some $\alpha \in For$ and $X, Y \subseteq For$,

$$X \vdash \alpha \text{ and } X \cup Y \nvdash \alpha$$

## 4. Hunting for an example of a non-monotonic reasoning

### 4.1. Common examples from philosophical literature

After all the concepts have been properly defined we can move on to analyzing claims made about non-monotonicity. The inspiration for developing non-monotonic logics was human everyday reasoning. Some examples of such everyday reasoning have become classic in the literature. Namely, *"Tweety the Ostrich"*, *"medical diagnosis"* and *"meeting in the pub."*

**Tweety the Ostrich**. We know that *Tweety is a bird* ($1^{st}$ premise). Based on the $2^{nd}$ premise: *birds fly* we conclude that Tweety flies. However, when we later learn that *Tweety is an ostrich* (new premise) we abandon the previous conclusion that Tweety flies.

**Medical diagnosis**. While observing a patient we notice symptoms $a, b$ and $c$ ($1^{st}$ premise). Based on these symptoms and the medical knowledge about these symptoms ($2^{nd}$ and other premises) we conclude that the patient suffers from the disease $z$. However, when we later learn that the patient suffers also from the symptom $d$ (new premise) we abandon the previous diagnosis and decide that the patient suffers from the disease $y$ and does not suffer from the disease $z$.

**Meeting in the Pub**. John has an appointment with Thomas at a certain time in the pub ($1^{st}$ premise). When the time is near John leaves his house and goes to the pub. However, on his way he receives a message that Thomas had an accident and was taken to the hospital (new premise). Because of that John abandons his plan of going to the pub.

An analysis of these three examples reveals similarities between them. In each of them, reasoning leads to the most *probable* or *expected* conclusion, which could become false if something stands in the way of its truthfulness. There are many possible scenarios in which conclusions like the ones presented could become falsified. Tweety flies, unless it is not an ostrich, penguin, kiwi, has a broken wing, etc. Patient with symptoms $a, b, c$ suffers from the disease $z$ unless they do not also suffer from other symptoms $d, e, f$. John is going to meet Thomas in the pub unless one of them has an accident, one of them forgets about the meeting, etc. [19].

When formalizing the structure of each of these reasonings let us assume that $X$ is the set of premises, $\alpha$ is a conclusion and $\beta$ is a premise containing information that *nothing stood in the way.* $\beta$ is true when every condition for the falsity of $\alpha$ is false, where: $\delta_i$ (for $i \in I$), are those conditions. We accept $\beta$ as long as we do not accept any $\delta_i$. This is necessary because the set $\{\beta, \delta_i\}$ is inconsistent. Therefore, each of the examples presented above can be represented with two steps:

Step one: $X \cup \{\beta\} \vdash \alpha$

Step two: $X \cup \{\delta_i\} \nvdash \alpha$

Such a reasoning cannot be considered non-monotonic because the set of premises $X \cup \{\beta\}$ is not a subset of $X \cup \{\delta_i\}$. The premise $\beta$ only belongs to the first one and not to the second one. Therefore, the original set of premises was not really expanded with a premise, but instead replaced with a new one, that was not a superset of the old one. Unfortunately, "non-monotonic logicians do NOT want to explicitly represent the "nothing-stood-in-the-way" information given here by the formula $\beta$ (nor do they believe it is possible)."[3] Meanwhile, it is not difficult to see

[3]The quoted sentence comes from one of the anonymous reviews of this article, hence the capitalization of the word "NOT." The inclusion of the enthymematic condition "unless..." is the essence of the so-called default logics proposed by Reiter [31]. Indeed,

that the enthymematic use of the aforementioned beta condition in the first step of reasoning is obvious and natural, as long as our goal is to represent human everyday thinking.

After taking a brief look at these common examples of alleged human non-monotonic reasonings, it is time to systematically analyze the problem.

### 4.2. In formal logic

Let us start with the most general issue – can a formal logical system be non-monotonic? Logic can be defined syntactically and semantically. Let us consider both cases. Let $\mathcal{L}$ be a formal language and $For$ a set of all formulas of $\mathcal{L}$.

#### 4.2.1. Logic in syntactic form

Let us consider a logic $S$ with a formal language $\mathcal{L}$ given by a set of axioms $A$ and a set of rules $R$. The relation $\vdash_S$ of syntactic derivability on the grounds of $S$ we define in a standard way: for any $\alpha \in For$ and $X \subseteq For, X \vdash_S \alpha$ if and only if there exists $(\gamma_1, ..., \gamma_n)$ a finite sequence of language $\mathcal{L}$ formulas so that the last element of that sequence is $\alpha$: $\gamma_n = \alpha$, and moreover, for any $i \in \{1, ..., n\}, \gamma_i$ is:

1. a formula from the set $X$ (i.e., an assumption), or
2. a formula from the set $A$ (i.e., an axiom), or
3. a formula which is the result of using a rule from the set $R$ on formulas appearing earlier in the sequence than $\gamma_i$ – which means that they belong to the subsequence $(\gamma_1, ..., \gamma_{i-1})$

The sequence of formulas $(\gamma_1, ..., \gamma_n)$ is the proof of $\alpha$ from set $X$ on the grounds of $S$. It appears that the inference $\vdash_S$ must be monotonic. Let us assume that $\alpha \in For$ and $X, Y \subseteq For$ fulfill two criteria:

1. $X \subseteq Y$
2. $X \vdash_S \alpha$

the mentioned condition is not explicitly represented in default logics. Taking such enthymematic conditions into account in some hidden form is present in almost all supposedly non-monotonic systems.

From the second criterion it follows that there is a sequence $(\gamma_1, ..., \gamma_n)$ that is the proof of $\alpha$ from the set $X$ on the grounds of $S$. From the first criterion it follows that every formula belonging to $X$ which appears in the proof $(\gamma_1, ..., \gamma_n)$ must also belong to the set $Y$. That means that exactly the same sequence of formulas $(\gamma_1, ..., \gamma_n)$, that is the proof of $\alpha$ from $X$ is also a proof of $\alpha$ from $Y$. Therefore, $Y \vdash_S \alpha$.

This demonstrates that the concept of non-monotonicity is incompatible with the standard Hilbertian concept of *proof* (see [21]).

#### 4.2.2. Logic in semantic form

Let us consider logic $S$ with a formal language $\mathcal{L}$ given with a set $V$ of valuations: $v : For \to \{1, 0\}$. The relation $\models_S$ of semantic consequence on the grounds of $S$ is defined in a standard way: for any $\alpha \in For$ and $X \subseteq For, X \models_S \alpha$ if and only if, for any valuation $v \in V$, if $v$ fulfils the set $X$, then $v(\alpha) = 1$. The valuation $v$ fulfils the set of formulas from $X$ if and only if $v(\beta) = 1$, for any formula $\beta \in X$.

It appears that the inference $\models_S$ must be monotonic. Let us assume that $\alpha \in For$ and $X, Y \subseteq For$ fulfil the criteria: $X \subseteq Y, X \models_S \alpha$. Let us also assume that $X \neq \emptyset$.[4] Let $v(X) = \{v(\beta) : \beta \in X\}$. Then, because of the second condition, for any $v \in V, v(\alpha) = 1$, if $v(X) = \{1\}$. Now, let us assume that for some $v \in V, v(Y) = \{1\}$. Because of the first condition $v(X) = \{v(\beta) : \beta \in X\} \subseteq \{v(\beta) : \beta \in Y\} = v(Y) = \{1\}$. Therefore, $x(X) = \{1\}$, and because of the third condition: $v(\alpha) = 1$. This means that for any $v \in V, v(\alpha) = 1$, if $v(Y) = \{1\}$. Therefore, $Y \models_S \alpha$. When $X = \emptyset$, then $\alpha$ is a tautology of $S$, so it follows from any set, also from $Y$.

This demonstrates that the concept of non-monotonicity is incompatible with the concept of semantic consequence. Such proof can be replicated also for logics which semantic interpretation is given by the more general notion of models (see [21]).

The definition of non-monotonicity is extremely hard to fulfil, no matter if a logic is defined syntactically or semantically. Many of the so-called non-monotonic logics are *de facto* monotonic logics that tinker with some of the premises. This tinkering takes various forms, which sometimes gives the impression that old premises are not really removed. However, barring

[4] In other words, the value of a set is either a set with a singleton 1 or a singleton 0 or a set with 0 and 1.

us from using a premise, no matter the reason for it, is equivalent to removing it. Sometimes, instead of blocking premises, the rules of inference change or the "application of rules of inference" changes (e.g., adaptive logics). However, the two steps of non-monotonicity require us to keep all the old premises, as well as all the rules of inference intact. This fundamental incompatibility between the concept of logicality and the concept of non-monotonicity is sometimes acknowledged when researchers point out that the term *non-monotonic logic* should not be used since it is an oxymoron [7].

### 4.3. Everyday reasoning: the rationale $\Longrightarrow$ succession conditionals

After discussing the most general formal systems, let us move on to everyday reasoning and consider if non-monotonicity is indeed present there. After all, the fact that non-monotonicity is not easily formalized is not an argument against its presence in real-life human reasonings.

Some fundamental everyday reasonings are traditionally classified into four types: inference, proving, explaining and verification [35, 5, 18]. They are based on the conditional "*rationale* $\Longrightarrow$ *succession*", which expresses the empirical, analytical, structural, tetical, logical or mixed relations. The decision to accept or reject truthfulness of some of these conditionals is arbitrary, but usually they represent the most commonly accepted ways of forming beliefs. Let us analyze if any of these reasonings can be non-monotonic.

#### 4.3.1. Inference

In the case of inference we accept some sentence $R$ as true and we wonder about its consequences. We find a succession $N$ of $R$ so that the conditional: $R \Longrightarrow N$ is true. As long as $R \Longrightarrow N$ and $R$ are accepted as true, we are forced to accept $N$ as true too. However, expanding the set of premises could possibly make us determine that ¬N, in which case the set of conclusions would become inconsistent. Such inconsistency forces us to revise the set of premises and resign from the truth of either $R \Longrightarrow N$ or $R$. Such a procedure does not violate monotonicity. In fact, staying with inconsistent conclusions also does not.

### 4.3.2. Proving

In the case of proving, we wonder if some sentence $N$ is true. To find that out we look for some $R$ which is a true rationale for $N$, so that the conditional $R \implies N$ is true. If we demonstrate the truth of $R$, then just like in inference, we are bound to accept $N$. Similarly, expanding the set of premises can produce inconsistency, but not abandonment of conclusions. After considering the infallible conditionals, let us move on to the fallible ones.

### 4.3.3. Verification

In the case of verification, we wonder if some sentence $R$ is true. To find that out we look for some true succession $N$ for our $R$ so that the conditional: $R \implies N$ is true. Such a procedure is fallible, since the truth of $R \implies N$ and $N$ does not guarantee the truth of $R$. Because of that fallibility verification can be an element of a two-step reasoning that generates the impression of non-monotonicity.

*Example*

Let us imagine that the teacher wants to check if Eve read the book assigned in class. In order to check that, she asks Eve three questions about the content of the book and Eve answers them all correctly. The teacher concludes that Eve read the book. However, guided by intuition, the teacher asks one more question about a very central point of the plot and Eve does not know the answer. The teacher changes her opinion and concludes that Eve did not read the book (maybe she just watched the TV adaptation that changed the plot).

Let us denote:

$R$ = *Eve read the book*

$N_1$ = *Eve answered 3 questions correctly*

$N_2$ = *Eve answered 4 questions correctly*

The teacher accepts the truth of relations $R \implies N_1$ and $R \implies N_2$, and also knows $N_1$ to be true. In a fallible way of verification she concludes that $R$. However, after asking the fourth question, she learns that $N_2$ is false and changes her conclusion to $\neg R$. The trick is that in this second step she is no longer using verification, which is fallible, but instead infallible deduction. From $R \implies N_2$ and $N_2$ infallibly follows $\neg R$.

Step one: $\{R \implies N_1, R \implies N_2, N_1\} \approx_v R$.[5]

Step two: $\{R \implies N_1, R \implies N_2, N_1, \neg N_2\} \models \neg R$

As we see, such an example does not fulfil the criteria of non-monotonicity and the same principle can be applied to other examples of verification. Even though expanding the set of premises causes abandonment of previously accepted conclusions, the two steps of reasoning employ different relations of consequence – fallible verification $\approx_v$ versus infallible deduction $\models$.

#### 4.3.4. Explaining

In the case of explaining we know that some sentence $N$ is true and we wonder why. To explain the truth of $N$ we look for some $R$ so that the conditional $R \implies N$ is true. Knowing only the truth of the conditional and the truth of the succession we conclude that $R$ is true too. However, this is a fallible reasoning. In most reasoning that employ explaining there are multiple rationales for a true succession. Most often there are more than one conditional, $R_1 \implies N, \ldots, R_n \implies N$ and we choose one.

*Example*

Let us imagine that we work at the office and our colleague John is still not at his desk. We wonder about his absence and conclude that he must be sick. After a while John calls us and tells us that he is stuck in traffic. After the phone call we abandon our earlier conclusion about his illness. Let us denote:

$R_1$ = *John is sick*

$R_2$ = *John is stuck in traffic*

$N$ = *John is absent at work*

We accept the truth of relations $R_1 \implies N$ and $R_2 \implies N$. In this example, we also accept that $R_1 \implies \neg R_2$ and $R_2 \implies \neg R_1$. Moreover, we know N to be true. Because we have a choice between $R_1$ and $R_2$, we arbitrarily choose $R_1$ as the rationale for N. However, after talking to John we know $R_2$ to be true, which falsifies $R_1$. Just like in the previous example, in the first step we used fallible explaining, and in the second step infallible deduction: from $R_2$ it follows that $R_2$.

[5] Fallible reasonings are denoted with $\approx$, since the symbol $\models$ is used for infallible deduction.

Step one: $\{R_1 \implies N, R_2 \implies N, N, R_1 \implies \neg R_2, R_2 \implies \neg R_1\} \approx_E R_1$, (also $\neg R_2$).

Step two: $\{R_1 \implies N, R_2 \implies N, N, R_1 \implies \neg R_2, R_2 \implies \neg R_1, R_2\} \models R_2$, (also $\neg R_1$) and because we do not tolerate contradictions:

$$\{R_1 \implies N, R_2 \implies N, N, R_1 \implies \neg R_2, R_2 \implies \neg R_1, R_2\} \not\models R_1.$$

Similarly as in the case of verification, the condition of non-monotonicity cannot be fulfilled, because we are changing between different relations of consequence. That means that we still perform monotonic reasonings according to the well-known definition.

At their inception, non-monotonic logics were intended to encompass infallible deductive reasonings [11, 41]. These attempts were not as fruitful as originally envisioned, which led David Makinson – the progenitor of non-monotonicity to write that non-monotonicity may only hold in fallible reasonings [24, p. 223]. However, even that seems to encounter difficulties, given that verification and explaining seem to have no inherent need for claims of non-monotonicity. However, let us delve deeper into other types of reasoning in search for non-monotonicity.

## 4.4. Other types of reasoning

The abovementioned verification and explaining are fallible, but it is still difficult to formulate any example that demonstrates undisputable non-monotonicity. In both examples we presented, the fallible reasoning performed initially is replaced with infallible deduction when new premises arise. Because of that, even though fallible, they are subjected to some logical rigor, since they accept the rule governing an implication, that it is not possible for a true implication to have a true rationale and false succession. This separates them from even more loose types of reasoning, some of which are known as heuristics. In this sense heuristics are reasoning patterns based on fallible rules, which are otherwise known to *usually* provide accurate conclusions.[6] Here we will discuss one in particular, because it shows that non-monotonicity can actually be observed, if one loosens the rigor of reasoning enough.

[6] Rules of reasoning in living organisms are subjected to adaptive pressures, like every other biological trait those organisms may have. As a result, these rules reflect their *utility* for the organism – trade-off between accuracy, speed and resource consumption – not their accuracy alone.

#### 4.4.1. Analogy

Reasoning with analogy is a heuristic based on perceived similarity. From logical perspective it entails ascribing two objects with a shared property and inductively deriving some other properties which these objects are supposed to share [22]. From cognitive perspective it is an act of comparing mental representations which involves their retrieval from long-term memory, identifying elements shared between those representations and inductively deriving new information [13]. Its fallibility is particularly high, even though it is an extremely widespread phenomenon. Therefore, its popularity is not caused by its accuracy, but rather its remarkable ability to start reasoning from a scratch when we have very little information about the subject at hand. In general analogy is a reasoning that takes the form:

It was the case in the past, in some situation $S_1$ that $a, b, c$, $d$.

It is the case now, in some situation $S_2$ that $a, b, c$.

Therefore I conclude that in the current situation $S_2$ it is $d$ as well.

Due to the particularly loose structure of the analogy, we are able to construct reasoning that is indeed non-monotonic.

*Example*

I remember that in city A the town hall is in the city center and that A is an old city. Therefore, I conclude that in B, which is also an old city, the town hall also has to be in the city center. However, I then also remember that A was never damaged during the war, while B was. I also know that in C, which was damaged during the war, the town hall was moved away from the historical center, to a more modern area. As a result, by analogy between B and C I now conclude that B has its town hall in the modern area and abandon my previous conclusion.

---

Step one:
$a =$ A is an old city.
$b =$ Town hall in A is in the city center.
$a' =$ B is an old city.
Therefore:
$b' =$ Town hall in B is in the city center.

---

Step two:
$a =$ A is an old city.
$b =$ Town hall in A is in the city center.
$c =$ A was not damaged during the war.

$a'$ = B is an old city.
$c'$ = B was damaged during the war.
Therefore:
No analogy is made, no conclusion is reached.

---

Because no analogy was found, step two arguably does not even exist. However, the next step does, given the new information that we learned about the city C. Steps two and three could me merged together, but we keep them separate here for clarity purposes.

---

Step three:
$a$ = A is an old city.
$b$ = Town hall in A is in the city center.
$c$ = A was not damaged during the war.
$e$ = C is an old city.
$f$ = C was damaged during the war.
$g$ = Town hall in C is in a modern district.
$a'$ = B is an old city.
$c'$ = B was damaged during the war.
Therefore:
$g'$ = Town hall in B is in a modern district.

---

To summarize, in step one through analogy we have:

$$\{a, b, a'\} \approx_A b'$$

In step three through analogy we have:

$$\{a, b, c, a', e, f, g, e', c'\} \approx_A g'$$

Among the premises of step three there are all the premises from step 1. No premises are abandoned or blocked. Then, because $b'$ and $g'$ are contradictory, we also obtain:

$$\{a, b, c, a', e, f, g, e', c'\} \not\approx_A b'$$

as well as,

$$\{a, b, c, a', e, f, g, e', c'\} \approx_A \neg b'$$

It seems to be a reasonable assumption that heuristics are good candidates for possible reasonings that involve non-monotonicity.

### 4.5. The three constructions of David Makinson

#### 4.5.1. The first construction

In the first example of allegedly non-monotonic reasoning in this paper we have said that seeing a lot of road traffic may change our assessment of the time needed to arrive somewhere. One could say that the original statement *'My apartment is located 5 minutes from the main station'* could be complemented with a hidden assumption: *'Unless something unusual happens.'* To account for such unspoken premises in reasonings, Makinson [23] proposed an additional set $K$ of background assumptions, which he called *the set of expectations.* Such an idea was known since antiquity where philosophers worked with the concept of *enthymemes,* the premises that are not explicitly stated due to their obviousness.

The non-monotonicity of an inference that employs the set $K$ of expectations was defined in the following way: first, we must define a new consequence relation. Let $\mathcal{L}$ be some language with $For$ a set of all formulas,, where $K \subseteq For$, and $Cn$ be the classical consequence operation. Then, $C_K$ will be the *consequence relation of the axial assumptions* $K$ and $\vdash_K$ the *relation of the axial assumptions* $K$, if for any $X \subseteq For, \alpha \in For$:

$$\alpha \in C_K(X) \text{ iff } \alpha \in Cn(K \cup X)$$
$$X \vdash_K \alpha \text{ iff } (K \cup X) \vdash \alpha.$$

Then:

$$Cn_K(X) = \{\cap Cn(K' \cup X) : K' \subseteq K \text{ and } K' \text{ is maximally consistent with } X\}$$
$$X \mid\!\sim_K \text{ iff } (K' \cup X) \vdash \alpha, \text{ for any } K' \subseteq K, \text{ maximally consistent with } X$$

The relations $\mid\!\sim_K$ are called the *background assumptions consequences.* Based on the way they were just defined, Makinson [23] argues that they are non-monotonic in the following way: let us assume the following set $K = \{p \to q, q \to r\}$. Because $K \cup \{p\}$ is consistent, then just the whole $K$ is the only one maximally consistent with $\{p\}$ subset of $K$. Therefore, $r \in C_K(\{p\})$, because $r \in Cn(K \cup \{p\}) \neq L$, while at the same time: $r \notin C_K(\{p, \neg q\})$. In fact $K \cup \{p, \neg q\}$ is inconsistent. Moreover, there is only one subset of $K$, which is maximally consistent with $\{p, \neg q\}$. It is $K' = \{q \to r\}$. It is easy to notice that $r \notin Cn(\{q \to r, p, \neg q\})$. Thus, $r \in C_K(\{p\})$

and $r \notin C_K(\{p, \neg q\})$, although $\{p\} \subseteq \{p, \neg q\}$. As a result we obtain an apparently non-monotonic reasoning where:

$r \in C_K(\{p\})$ (i.e., Proposition $r$ belongs to the set of conclusions following from $\{p\}$ by the rules of $C_K$)

But at the same time:

$r \notin C_K(\{p, \neg q\})$ (i.e., Proposition $r$ does not belong to the set of conclusions following from $\{p, \neg q\}$ by the rules of $C_K$)

At first glance it appears that the defined consequence relation is non-monotonic. After all, expanding the set of premises with $\neg q$ just shrunk the possible list of conclusions. However, this construction unfortunately does not satisfy the definition of non-monotonicity. Namely, that the set of premises cannot be changed in other ways than adding new premises to it. In the procedure outlined above for every reasoning step a new *set of expectations* (i.e., the hidden assumptions) is selected. The used notation seems to suggest that the whole *set of expectations* $K$ is used at every step by using the same: $C_K$ everywhere, whereas in fact various subsets of the set $K$ are used. This problem was pointed out by [20] and puts into question the usage of the term: *'non-monotonicity'* for this and similar constructs. However, despite terminological confusions, this logical construction is heavily grounded in our current understanding of human cognition. It was an attempt at capturing one of the many ways in which our everyday reasonings deviate from the predictions of classical logic. Namely, the fact that our beliefs, attitudes, memories and any other construct expressible with propositions does not form a single unified set, but is instead partitioned based on various criteria [32, 42].

The existence of such a partitioning mechanism in cognition is highly useful, since it conserves resources when communicating and when processing information on your own. It would be highly inefficient (if not impossible) from the point of energy expenditure if humans explicitly stated all the premises they used in every reasoning. In fact, cognitive scientists postulate the existence of a hierarchy of beliefs [10]. This hierarchy can take many forms. For example, in many models of thinking, conscious (or *language based*) information processing is considered to run *'on top of'* unconscious stimuli-based processing [38]. Despite being non-verbalized, all the information processed by those evolutionarily older systems can potentially be expressed in symbolic form compatible with formal logic and they certainly influence the way in which humans reason. As a result, any logic that intends to model human thinking should be aware of the existence of information that influences how we reason, but remains unspoken. However, even the hierarchies of beliefs within linguistic cognitive systems are enough to justify the utility of the *set of expectations* by [23], without having to rely on the stimuli-based ones.

In cognitive psychology, researchers use the term *cognitive schema* to describe *'the basic structural components of cognitive organization through which humans come to identify, interpret, categorize and evaluate their experiences'* [34, p. 129]. From a logical perspective, schemas can be seen as sets of beliefs expressed as propositions, partitioned on the basis of their utility in given situations. The same sentence can be understood completely differently in two different contexts, because different *enthymemes/expectations/schemas* are active in those contexts. If a friend calls us *'an idiot'* in a pub, we are significantly less likely to be offended than when a random person on the street does the same. That is because in the cognitive schema relevant for interpreting communications with friends, insults are considered playful and bonding, which is not the case with strangers.

Cognitive schemas are organized hierarchically [10]. At the top of the hierarchy are the *core beliefs*, which are the most basic, central and unquestionable convictions we hold about reality. Researchers believe that people very rarely articulate them, even to themselves. In simple words, these core beliefs describe how we think that the world really *'is'* [4]. From these beliefs, other, more detailed convictions and attitudes are derived and separated into schemas for different situations. Psychotherapists also tend to categorize them into: beliefs about the self, the others and the external world. As a result, it appears that the idea that different premises from the *set of expectations* $K$ should be used depending on the situation, agrees with the current view on how unconscious cognitive schemas guide our reasoning. However, such a procedure/strategy has nothing to do with non-monotonicity.

### 4.5.2. The second construction

The second construction of Makinson [23] through which he intended to introduce non-monotonicity to modeling human thinking relies on selective usage of Boolean valuations ($V$). Let $W \subseteq V, X \subseteq For, \alpha \in For$. Then,

$$\alpha \in C_W(X)(X \vdash_W \alpha) \text{ iff for any } v \in W, \text{ if } v(X) = 1, \text{ then } v(\alpha) = 1.$$

Here, the consequence relations $C_W$ (also $(\vdash_W)$) are called *axial-valuation consequences*. These are monotonic, and non-monotonicity is achieved through the introduction of the so-called *preferential model*, which is a set $(W)$ ordered by $<$, an irreflexive and transitive relation on $W$. If we let $\langle W, < \rangle$ be a preferential model, then

$$X \mathrel{|\!\sim}_{<} \alpha \text{ iff } v(\alpha) = 1, \text{ for any } v \in W \text{ minimal among all valuations from } W \text{ satisfying } X.$$

Here, '$\mathrel{|\!\sim}_{<}$' are called *preferential consequences* or *default-valuations consequences*. They are shown to be non-monotonic in the following way: assume a language containing only three sentences: $p, q, r$ and let $W = \{v_1, v_2\}$ such that $v_1(p) = v_2(p) = 1, v_1(q) = 0, v_2(q) = 1, v_1(r) = 1, v_2(r) = 0$. The relation $<$ orders the set $W$ as follows: $v_1 < v_2$. Then, $\{p\} \mathrel{|\!\sim}_{<} r$. That is because $\{v_1\}$ is the set of all elements minimal among all valuations satisfying $\{p\}$ and $v_1(r) = 1$. However, it is not the case that $\{p, q\} \mathrel{|\!\sim}_{<} r$. That is because, it is $\{v_2\}$ that is the set of all elements minimal among all valuations satisfying $\{p, q\}$, and $v_2(r) = 0$. Thus, for some $\langle W, < \rangle, X, Y \subseteq For, \alpha \in For$, it is the case that: $X \mathrel{|\!\sim}_{<} \alpha$, but not $X \cup Y \mathrel{|\!\sim}_{<} \alpha$ (see: [20] for an in-depth analysis).

Within the sound and complete semantics designed by Makinson, the second construction allows the user of a language to select different rules of inference at different steps of reasoning. This is possible since the set $W$ as well the order on that set is arbitrary. As a result, the second construction does not define one non-monotonic logic, but rather a whole class of them without stable rules and with varying forms of implication. Therefore, just like the first construction, it violates the definition of non-monotonicity because the rules of inference change. The attempt at modeling human thinking via dynamic, changeable sets of premises and rules of inference corresponds to how sensitive our cognition is to different contexts in which reasonings happen.

Context dictates which unspoken schema will guide our inferences and psychologists have shown that it can be easily influenced, creating an impression of non-monotonicity. In an extensive field of research, spanning decades, researchers have shown a robust *framing effect* in risky decision making [36, 6]. It is a phenomenon, where presenting people with logically equivalent information, but expressed in a slightly different way, may completely change the conclusions they derive from it. A famous example was given by Tversky and Kahneman [39] and dubbed *"the Asian disease problem."* In their experiment two groups of people were examined. Both groups were informed that due to an outbreak of a deadly disease 600 people may die. The task of the participants was to choose one of the treatment programs to combat the disease based on the expected number of saved lives. The first group had to make a choice between the following options:

A: *'200 people will be saved'*

B: *'There is a 1/3 probability that 600 people will be saved, and a 2/3 probability that no people will be saved'*

The second group of participants had to choose between:

C: *'400 people will die'*

D: *'There is a 1/3 probability that nobody will die, and a 2/3 probability that 600 people will die'*

Despite the fact that options A and C are logically equivalent, presenting the treatment program in a positive language (A) makes 72% of participants choose it, while presenting it in a negative language (C) makes that number only 22%. This problem persists through rigorous methodological control of the ambiguity of the used language, to keep the options presented to participants as undeniably equivalent as possible [6]. According to the allegedly non-monotonic constructions of Makinson [23] as well as the schema theory [27] this *framing effect* and dynamically changing inference principles (i.e., the cognitive schema, for extensive examples see: [17, 9]) can be successfully modeled with a change in the underlying set of expectations $K$.

### 4.5.3. The third construction

The third construction of Makinson [23] is intended to capture the human ability of changing the understanding of a sentence at successive stages of reasoning. Such a change might be a minor correction to how we interpret a word, but it is always dictated by some previously accepted premises. The transformation of a proposition is achieved by applying the so-called *rules of sentence conversion*. Every rule has the form of $\langle \alpha, \beta \rangle$ and together they form a set $R \subseteq For^2$. Applying the rules of $R$ to sentences in $X$ yields an *image of X closed on R* set: $R(X) = \{\beta \in For : \langle \alpha, \beta \rangle \in R$ and $\alpha \in X\}$. Because applying some rules could potentially result in introducing inconsistency to the set of premises, they can be used selectively. That selectivity is expressed by ordering the set $R$ and indexing every rule: $\langle R \rangle = \{\langle \alpha_i, \beta_i \rangle : i < \omega\}$. With the help of this set we can now define the *axial-rules consequence* relation for any $X \subseteq For$:

$$Cn_{\langle R \rangle}(X) = \cup\{X_n : n < \omega\}, \text{ where } X_0 = Cn(X) \text{ and}$$
$$X_{n+1} = Cn(X_n \cup \{\beta\}).$$

where $\langle \alpha, \beta \rangle$ is the first rule in $\langle R \rangle$ such that $\alpha \in X_n, \beta \notin X_n$ and $\beta$ are not inconsistent with $X_n$ (see: [20] for an in-depth analysis).

The fact that rules in $\langle R \rangle$ are ordered means that premises can be effectively changed at different steps in the reasoning. From a psychological perspective this could be another example of the *framing effect* mentioned above alongside the second construction. Furthermore, the fact that change in the interpretation of a proposition happens mid-reasoning reminds of a process known in psychology as *cognitive reappraisal*. Cognitive reappraisal is a *'flexible regulatory strategy that draws on cognitive control and executive functioning to reframe stimuli or situations within the environment to change their meaning and emotional valence'* [43, p. 390]. In other words, cognitive reappraisal happens when we consciously try to reinterpret a situation in the light of new information. For example, when a person is devastated after being fired from their job, they may reappraise the situation and instead of seeing it as a failure, see it as the beginning of a new opportunity to grow. Cognitive reappraisal is different from just simply changing our conclusions based on new information, because it necessarily entails changing the interpretation of some old information. The authors of the allegedly non-monotonic systems focused a lot on the conclusions that change in reasonings, but they failed to see the premises that also change with them. This does not mean their constructions are altogether wrong or useless. In fact, they are useful for modelling, for example, cognitive reappraisal. Something that more classical approaches could not do. However, they are not non-monotonic.

### 4.6. Adaptive logics

One of the examples of modern logics that were not afraid of addressing the reality of actual human reasonings are adaptive logics. Despite the fact that many logicians hold *psychologism* in low regard, Diderik Batens developed a whole family of logics that were *'intended to explicate actual forms of reasoning'* [1, p. 47] and *'both everyday reasoning and scientific reasoning'* [2, p. 222]. Adaptive logics are defined as logics that *adapt* specifically to the premises of reasonings. Adaptation, from a semantic perspective means that some models of the premises are selected preferentially, depending on the *abnormalities* of those premises. From a proof-theoretic perspective it means that some rules of inference apply depending on the presence or absence of some consequences derived from the set of premises [1]. Defining logic this way is potentially very useful from the perspective of its accuracy in representing human reasoning but raises doubts about meeting the definition of non-monotonicity. This is important, because adaptive logics were created with the intention of being non-monotonic and put non-monotonicity forward as one of their central concepts [1, 2].

The rationale for making adaptive logics allegedly non-monotonic is based on the existence of *external* and *internal* dynamics in reasonings [1]. External dynamics are the concept that has been discussed in this paper many times already: the fact that when new premises become known, old conclusions can be withdrawn. In contrast, internal dynamics describe that even if premises do not change, conclusions can change at different stages of reasoning.

Let us consider the way adaptive logics are formalized and then identify the specific points in which they become unwillingly monotonic. A so-called *flat* adaptive logic is characterized by:

1. A *Lower limit logic* – any monotonic logic
2. A *set of abnormalities* – a set of formulas characterized by a logical form
3. An *adaptive strategy* – a description of how to interpret the premises

The first part of the adaptive logic – its *lower limit logic* defines the part that does not adapt itself to the premises. It could be classical logic but also any other logic, for example, some paraconsistent logic. From a semantic perspective, the rules of the adaptive logic $AL$ are therefore a superset of rules from the lower limit logic, $Cn_{LLL}(X) \subseteq Cn_{AL}(X)$. The *set of abnormalities* $\Omega$ *'comprises the formulas that are presupposed to be false, unless and until proven otherwise.'* [1, p. 48]. This is extremely reminiscent of the *set of expectations* in the first construction by Makinson [23]. Both contain formulas which are going to be blocked under some specific circumstances. However, instead of classically understood presuppositions, $\Omega$ deals with formulas that are in force if and only if they are not contradicted by the set of premises. This is explained by introducing another concept: the *upper limit logic*. The upper limit logic is obtained by extending the lower limit logic with the requirement that no abnormalities from the set $\Omega$ are logically possible. The upper limit logic requires premise sets to be free from *abnormalities* and if there are any, it trivializes the set of conclusions (i.e., the principle of explosion). For example, an adaptive logic can be constructed so that if the lower limit logic is set to be the classical logic and the set of abnormalities $\Omega$ contains formulas of the form: $\exists\alpha \wedge \exists\neg\alpha$ ($\exists\alpha$ is an abbreviation of the existential closure of $\alpha$), then the upper limit logic is classical logic extended with the axiom: $\exists\alpha \supset \forall\alpha$ [1].

In consequence, if the set of premises does not contain any abnormality, then the conclusions derived with adaptive logic ($\vdash_{AL}$) are going to be identical to the conclusions derived with the upper limit logic ($\vdash_{ULL}$). However, as soon as a new premise is added, so that it satisfies one of the abnormality formulas from $\Omega$, then adaptive logic is going to deviate from the upper limit logic. The author states that *'it avoids abnormalities 'in as far as' the premises permit'* [1, p. 49]. However, this means that if during reasoning we add a new premise that satisfies a formula from $\Omega$, we change the rules of inference. The change in those rules follows a pattern, which is described by the adaptive strategy of the logic $AL$, but the change happens nonetheless.

For example, if the lower limit-logic is set to be some paraconsistent logic $PL$, which is a fragment of $CL$ and the set $\Omega$ consists of formulas that take the form: $\exists(\alpha \wedge \neg\alpha)$, then the upper limit logic of that adaptive logic $AL$ will be $CL$. This means that if the premise set $X$ contains some formulas that take the form of some elements of set of abnormalities $\Omega$ then the adaptive logic $AL$ will deliver more consequences than the lower limit logic. Namely, all the consequences from the upper limit logic that are not *blocked* by the abnormalities from $\Omega$ [3]. The key term here is the word *blocked*, because in order to block a consequence that was previously derived it necessarily means to block a rule of inference that was previously used. The claim to non-monotonicity in this case comes from the fact that the *adaptive strategy* of a given adaptive logic $AL$ is specified upfront and defines how and when some rules of inference will be used or not. As a result, if we denote the dynamic, non-monotonic nature of adaptive logics by saying that if there are some: $X, Y$ and $\alpha$ such that: $X \vdash_{AL} \alpha$ and $X \cup Y \nvdash_{AL} \alpha$, then it is questionable whether indexing the $\vdash$ with $AL$ means the same thing at both stages. Thanks to the fact that we specify $Y$ we can reconstruct which rules of inference does $\vdash_{AL}$ use at each stage[7], but they are going to be different depending on the stage. In fact, while at a first glance $Y$ appears to be merely a set of premises added to the $X$, in reality it also alters the logic operating behind $\vdash_{AL}$. By *alter* we mean here that it defines a selection of the rules of inference that are allowed or disallowed. It selects them in accordance with the *adaptive strategy*, but the resulting set of rules is different nonetheless. **We understand that barring a rule from being applied or premise from being used is the same as removing that rule or premise altogether.** The relevance of that postulate is most visible when confronted with the way adaptive logics describe the effects of using the set of abnormalities. Rule of inference can be barred from applying: *"Put differently, that the premises have certain consequences may prevent a rule of inference to be applicable to some other consequences of the premises"* [1, p. 46], and premises can be removed: *"The set of abnormalities (...) $\Omega$ comprises the formulas that are presupposed to be false, unless and until proven otherwise"* [1, p. 48]. If a formula is *"presupposed"* to be either true or false, then that formula is effectively a premise in reasoning, even if not explicitly named that way. If the original presupposition changes at some later stage of reasoning, then that premise has changed and the reasoning cannot satisfy the definition of non-monotonicity.

This means that the definition of monotonicity is satisfied here, because dynamically changing rules of inference and/or dynamically changing premises are still used in a monotonic manner – every new rule defines a new logic which is clearly monotonic. As a result, any given $AL$ can be seen not as a single logic, but as a formal system of some kind, from which one logic

[7] In adaptive logics, the term "stage" has a strict meaning as "stage of proof," which is different from the common language "stage of reasoning," see: *"(...) if $A$ is "derivable at a stage" from $Y$, there is a proof from $\Gamma$ and a stage $s$ such that $A$ is derived at stage $s$ of that proof"* [1, p. 58].

is being selected based on $Y$. Both upper and lower-limit logics are monotonic, but the impression of non-monotonicity is created via the addition of the *adaptive strategy*. However, it is important to note that this issue is merely terminological and does not question neither the utility of adaptive systems nor their construction. Instead, we argue that adaptive systems achieve the appearance of non-monotonicity using only monotonic logics.

### 4.7. System P

One of the most widely endorsed allegedly non-monotonic systems is the *System P* proposed by Kraus et al. [15]. System P is thought to achieve non-monotonicity through the use of preferential models, which provide a formal framework for reasoning about plausibility and normality among possible worlds. In this framework, conclusions are drawn based not on the entirety of possible worlds that satisfy a given set of premises, but rather on a dynamically determined subset of these worlds – the so-called "preferred" worlds. The preference ordering among worlds, denoted as $\prec$, captures the relative normality or plausibility of different scenarios.

In preferential models, a conditional assertion $\alpha \sim \beta$ is interpreted to mean that in all of the most preferred worlds satisfying $\alpha$, the formula $\beta$ holds. The key feature of this system is that the set of most preferred worlds satisfying $\alpha$ may shift when additional premises are introduced. This mechanism permits the invalidation of previously drawn conclusions, seemingly enabling non-monotonic reasoning. Specifically, while $\alpha \sim \beta$ may hold under a given set of premises, the introduction of new information, such as $\gamma$, can alter the set of preferred worlds satisfying $\alpha$ and, consequently, the validity of $\beta$ in this revised context.

To illustrate this formally, let $W = \langle S, l, \prec \rangle$ denote a preferential model, where $S$ is the set of states (each corresponding to a possible world), $l : S \to U$ is a labeling function mapping states to worlds, and $\prec$ is a strict partial order representing the preference relation. A state $s \in S$ is said to satisfy a formula $\alpha$ if and only if the world $l(s)$ satisfies $\alpha$. The conclusion $\alpha \sim \beta$ holds if and only if for all $s$ that are minimal with respect to $\prec$ in the set of states satisfying $\alpha$, $l(s)$ also satisfies $\beta$. When an additional premise $\gamma$ is introduced, the set of minimal states satisfying $\alpha \wedge \gamma$ may exclude some states that were previously minimal for $\alpha$. If these excluded states were crucial for supporting the conclusion $\beta$, the inference $\alpha \sim \beta$ will no longer hold.

Similarly to adaptive logics and to the three constructions of David Makinson, the dynamic nature of the preferential consequence relation hides the fact that it should be indexed with the underlying preferential model. A conclusion $\alpha \mid\!\sim_{W_1} \beta$ is valid only within the specific preferential model $W_1$. When new premises lead to a shift in the model (e.g., from $W_1$ to $W_2$), the consequence relation should change accordingly (i.e., to $\mid\!\sim_{W_1}$), and previously valid conclusions may no longer hold. Thus, the apparent non-monotonicity is, in fact, a result of changing the rules of inference via a shift in the model, rather than a violation of monotonicity within any fixed model.

The correspondence between this approach and our everyday reasoning is expressed in the way authors interpret the non-monotonic inference relation $'\mid\!\sim'$. If we write: $a \mid\!\sim c$ then we read it as: 'if a then *normally* c.' In our example, 'if my apartment is reachable in 5 minutes from the station by car then we *normally* will get there in time.' On a practical note, the probability semantics by Gilio [12] interpret the conditional $'\mid\!\sim'$ with probability intervals, making the phrase *'normally'* easier to study empirically. *'Normally'* then means *with high probability*. The required probability $(x)$ is arbitrary and expressed with an interval $[x_*, x^*]$, creating a probabilistic consequence relation:

$$a \mid\!\sim_x c \text{ is interpreted as } P(c \mid a) \in [x_*, x^*]$$

For an extensive overview of the probabilistic interpretations of non-monotonic logics see: [26, 29].

## 5. Discussion

So far we have established several clues on the way to determining if the label of non-monotonicity can be assigned to human everyday reasoning. We know that numerous inferences that humans routinely perform violate the predictions of classical logic. Humans are context sensitive and within one reasoning they are able to switch between different sets of premises, different rules of inference and revise previously accepted conclusions. As a result, logics that call themselves non-monotonic perform well at predicting the outcomes of some human inferences. For example, their rules make it so that the conjunction fallacy stops seeming like a fallacy and presents itself as a rational decision making process in an uncertain environment [28]. However, we have also established that removing or changing premises as well as changing rules of inference cannot be a part of a non-monotonic system. Monotony is violated if and only if addition of a new premise invalidates past conclusions.[8]

When summarizing the selected non-monotonic systems it is important to note that many of them were created with the intention of modeling deductive reasonings [15, 23]. However, due to frequent problems these systems have with following the definition of non-monotonicity [20] and their use of defeasible inference, it appears that they are better suited to model solely abductive reasonings and resign from the ambition of deductivity. For example, Makinson [24, p. 223] consciously noted that: *'While monotony holds for deductive inference, (...) it is quite unacceptable for non-deductive reasoning, whether probabilistic or expressed in qualitative terms,'* thus admitting that monotony holds for deduction.

In the face of these issues with allegedly non-monotonic systems, how can we answer the question: is human reasoning non-monotonic? To make a claim: *'We are non-monotonic'* we cannot just rely on the fact that some allegedly non-monotonic logics are better than monotonic logics at predicting some heuristics. Science already knows many examples of phenomena that are convieniently modeled with some paradigm, even though we know that its rules do not correspond well with reality. For example, Newtonian physics is still the most useful way of predicting physical phenomena on medium-size scale, even though our understanding of physics has moved way past beyond them. Given that many allegedly non-monotonic logics struggle to satisfy the definition of non-monotonicity, we are facing a very difficult conundrum in trying to answer if humans reason non-monotonically. Some non-monotonic logicians are very aware of that fact. For example, Pfeifer and Douven [30, p. 108] summarized their experimental results that showed agreement between System P and empirical data by saying: *'It would be misleading, though, to speculate that our subjects have a 'nonmonotonic inference engine' in their minds that processes incomplete uncertain information. Even if human subjects were perfect in handling the axioms and some elementary theorems of System P, they would not necessarily be able to handle more complex tasks.'*

[8] Abandoning a conclusion after nothing new was added would also constitute a violation of monotony as it can be expressed via an addition of an empty set.

Despite those issues, in this article we have identified a very promising candidate for strict non-monotonicity: reasoning with analogy. However, a question remains: is such a reasoning typical or rather an outlier? This question is very difficult to answer even when using the state-of-the-art neuroscientific tools which are able to track information processing when it unfolds (i.e., functional magnetic resonance, electroencephalography, magnetoencephalography). These techniques are not able to track the neuronal symbolic representation well enough to say which premise was used and which is not when people reach a conclusion. That means that we would not be able to tell if a reasoning analyzed with these techniques satisfied the definition of non-monotonicity. Naturally, by saying that, we admit that we consider reasoning to be a phenomenon of brain activity and that the structure of reasoning is represented by patterns of that activity.

It might seem suspicious that we say that neuroscientific tools are ill-equipped to answer if humans reason non-monotonically. After all, are not most studies in cognitive science and experimental philosophy performed by analyzing participants' responses to carefully crafted questions or stimuli [44]? Why would non-monotonicity be different? The answer lies once again in the fact that we are now concerned with the definition of non-monotonicity. We are specifically interested in identifying a reasoning where absolutely all original premises are fixed throughout the whole reasoning. That means that our neuroscientific tools would need spatio-temporal resolution high enough to locate and track every single individual premise, as they are represented by the brain. Such technology does not exist yet. Unfortunately, including traditional methods and just asking people about their reasoning does not help either. The *'hidden/enthymematic'* premises that people use at different stages of reasoning are mostly unconscious and unspoken. As a result, to identify them we cannot just rely on what people say, but have to analyze the neurophysiological trace of their unconscious reasoning.

Not every property of reasoning has to fulfill such steep empirical requirements to be falsified. For example, paraconsistency (i.e., a property characterizing reasonings that tolerate contradictory premises) is easier to investigate because it only requires a single pair of inconsistent premises to exist within one reasoning [33]. Examining a single pair of premises gives experimentalists the ability to forcibly present them to research participants as experimental stimuli. Then, the brain activity in response to these two particular inconsistent premises may be examined. In the case of non-monotonicity such an approach is impossible, because we need to track every single premise that may or may not have been used in a reasoning.

However, despite the fact that we are currently unable to investigate the question if our reasoning is non-monotonic, we are able to investigate some predictions of allegedly non-monotonic logics. For example, Da Silva et al. [8, p. 110] justify their empirical investigations of the rules of system P by saying: *'No current experimental device can provide relevant and direct observation of the human inferential system 'at work'. Yet, we are able to observe the conclusions derived by human participants in the context of a given set of premises.'* This is true because we do not have to keep track of all the premises to test the effects of some particular logical rules. Only monotonicity itself poses an exceptional challenge. In fact, Da Silva et al. [8, p. 110] were already partially aware of that problem since they remarked immediately afterwards: *'In other words, we do not see these patterns as direct inference rules(...), but as general emerging properties of the inferential apparatus.'*, indicating that they are not studying how the human reasoning *really* works, but instead what patterns emerge from the answers of participants.

The answer to the question: *do humans reason non-monotonically?* is entirely open. It will remain that way until our neuroscientific tools become even more accurate. Despite that, the existing allegedly non-monotonic systems have done an excellent work at pointing out the differences between the classical logic and the way in which humans reason. However, the definition of non-monotonicity appears to be so fundamental that creating a logic which would completely satisfy it, is hard to achieve.

## 6. Conclusion

There are many very interesting formal systems that have been/are being developed under the banner of non-monotonicity. There are many examples that illustrate peoples' abandonment of previously derived conclusions and are also classified as examples of non-monotonicity in our thinking. The purpose of this work is not to question the value of the systems discussed here – their value is indisputable. Nor is the purpose of this paper to question the fact that people sometimes reject beliefs that they themselves once arrived at. The purpose of this paper is to show that both systems considered non-monotonic and examples of supposed non-monotonicity in human thinking do not satisfy the definition of non-monotonicity. It seems that the condition of the monotonicity expresses such a fundamental property of our thinking that:

1. We still do not have any formal system satisfying the definition of non-monotonicity.
2. The examples of human reasoning widely cited in logical literature as being non-monotonic also do not satisfy the definition of non-monotonicity.

Naturally, the fact that we "still" do know neither a non-monotonic system nor a well-established case of non-monotonic thinking does not mean that we will never know one. However, the scale of attempts to construct non-monotonic logics, as well as the multitude of examples of allegedly non-monotonic human thinking, may suggest that monotonicity is an unassailable principle of our thinking. Perhaps we should start using a more appropriate term to describe reasoning that abandons previously deduced conclusions. Such a change in nomenclature would be advisable, as the current common use of the term "non-monotonic" is misleading in suggesting something that might be not realizable. Perhaps "self-corrective" would be a good candidate to replace the unfortunate "non-monotonic"?

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# OPEN FILTERS AND CONGRUENCE RELATIONS ON SELF-DISTRIBUTIVE WEAK HEYTING ALGEBRAS

**Abstract**

In this paper, we study (open) filters and deductive systems of self-distributive weak Heyting algebras (SDWH-algebras) and obtain some results which determine the relationship between them. We show that the variety of SDWH-algebras is not weakly regular and every open filter is the kernel of at least one congruence relation. Finally, we characterize those SDWH-algebras which are weakly regular by using some properties involving principal congruence relations.

*Keywords*: SDWH-algebra, open filter, deductive system, congruence kernel, weakly regular.

*2020 Mathematical Subject Classification*: 06D20, 06B10, 18A15.

## 1. Introduction

Celani and Jansana introduced the concept of weak Heyting algebras in 2005 ([4]). A WH-algebra is a bounded distributive lattice with a binary operation $\to$ satisfying the properties of the strict implication in the modal logic K. These algebras are a generalization of Heyting algebras. Alizadeh and Joharizadeh ([1]) presented an algorithm to construct and count all

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nonisomorphic finite WH-algebras. San Martín ([14]) studied the compatible operations in some subvarieties of the variety of WH-algebras. He studied Principal congruences in WH-algebras in [15] and characterized the congruences in weak implicative semi-lattices in [16]. The notion of multipliers in weak Heyting algebras was defined in [10] and the relations between multipliers, closure operators, and homomorphisms in weak Heyting algebras were obtained.

Some of the known subvarieties of the variety WH-algebras are SRL, B, RWH and TWH. In 1976, years before the definition of WH-algebras, the subresiduated lattices were defined and studied in a different way by George Epstein and Alfred Horn [9]. In the mentioned paper the authors proved that the Lindenbaum-Tarski algebra of the calculus R4 is a subresiduated lattice. They also introduced several subvarieties of SRL and counterpart logic.

Another subvariety of WH-algebras is the variety of basic algebras, first studied by Mohammad Ardeshir and Wim Rutenberg in 1998 ([2]). The counterpart logic of this variety, also called Basic logic, was first introduced by Albert Visser in 1981 ([17]) and then by Wim Ruitenberg in 1992 ([14]).

As mentioned in [4], variety RWH corresponds to the logic defined by the class of reflexive Kripke models, and the variety TWH corresponds to the logic defined by the class of transitive Kripke models.

These five varieties (WH, SRL, B, RWH and TWH) are Archimedean varieties with congruence extension properties (CEP), but they are not locally finite either.

A self-distributive operation is distributive over itself. They have an important role in mathematics because of their connection with many fields such as knot theory, algebraic combinatorics, quantum groups ([7]), quandles ([11]) and Hilbert algebra ([8]). Also, self-distributive operations provide solutions of the Yang–Baxter equation.

Recently, we introduced self-distributive WH-algebras and obtained some of their properties. SDWH-algebras of orders 3 and orders 4 were characterized. Finally, we obtained the relation between SDWH-algebras and known subvarieties of WH-algebras, like TWH-algebras, RWH-algebras, SRL-algebras and Basic algebras ([13]). The relations between these subvarieties of WH-algebra are depicted in Figure 1.

Birkhoff studied the relation between congruence relations and ideals of lattices in [3]. He proposed in:

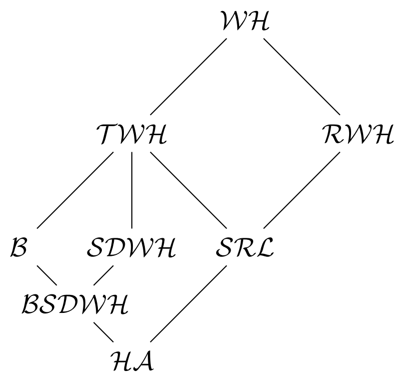


**Figure 1.** The order of $\mathcal{WH}$ subvarieties

**Problem 73.** Find necessary and sufficient conditions, in order that the correspondence between the congruence relations and ideals of a lattice be one-to-one.

Historically ideal theory for lattices was developed by Hashimoto ([12]). He established that there is a one-to-one correspondence between ideals and congruence relations of a lattice $L$ under which the ideal corresponding to a congruence relation is a whole congruence class under it if and only if $L$ is a generalized Boolean algebra. An algebra with a constant 1 is weakly regular if every two congruence relations coincide whenever they have the same congruence class containing 1 ([7]). An interesting problem is to find weakly regular algebras in varieties that are not varieties of weakly regular algebras (see [5]).

In this paper, we study the (generated) open filters of SDWH-algebras and prove that the lattice of open filters is a complete Heyting algebras such that the compact elements are principal open filters. Then the notion of deductive systems of an SDWH-algebra is introduced and the relations between deductive systems, open filters, and filters of SDWH-algebras are obtained. It is shown that every open filter is a kernel of at least one congruence relation on an SDWH-algebra. Moreover, the variety of SDWH-algebras is not weakly regular. We use the concepts of deductive systems and open filters to define two congruence relations on every SDWH-algebra and obtain the relation between them. Finally, we obtain the necessary and sufficient conditions for which an SDWH-algebra is weakly regular.

## 2. Preliminaries

In this section, we recall the basic definitions and some properties of weak Heyting-algebras which we will need in the next sections.

Definition 2.1 ([4]). An algebra $\mathcal{H} = (H, \wedge, \vee, \rightarrow, 0, 1)$ of type $(2, 2, 2, 0, 0)$ is called a weak Heyting algebra (or WH-algebra) if $(H, \vee, \wedge, 0, 1)$ is a bounded distributive lattice and the following conditions hold for all $x, y, z \in H$:

(WH1) $(x \rightarrow y) \wedge (x \rightarrow z) = x \rightarrow (y \wedge z)$,

(WH2) $(x \rightarrow z) \wedge (y \rightarrow z) = (x \vee y) \rightarrow z$,

(WH3) $(x \rightarrow y) \wedge (y \rightarrow z) \leq x \rightarrow z$,

(WH4) $x \rightarrow x = 1$.

The following proposition provides some properties of WH-algebras.

Proposition 2.2. ([1, 4]) Let $\mathcal{H}$ be a WH-algebra. Then the following hold for all $x, y, z \in H$:

(W1) if $x \leq y$, then $y \rightarrow z \leq x \rightarrow z$ and $z \rightarrow x \leq z \rightarrow y$,

(W2) if $x \leq y$, then $x \rightarrow y = 1$,

(W3) if $x \leq y \leq z$, then $z \rightarrow x = (z \rightarrow y) \wedge (y \rightarrow x)$,

(W4) $x \rightarrow y = x \rightarrow (x \wedge y)$,

(W5) $(x \rightarrow y) \rightarrow (y \rightarrow z) \leq (x \rightarrow y) \rightarrow (x \rightarrow z)$.

Definition 2.3 ([4, 13]). Let $\mathcal{H}$ be a WH-algebra.

(1) $\mathcal{H}$ is a Basic algebra iff satisfies the inequality $x \leq 1 \rightarrow x$ (I),

(2) $\mathcal{H}$ is a RWH-algebra iff satisfies the inequality $x \wedge (x \rightarrow y) \leq y$ (R),

(3) $\mathcal{H}$ is a TWH-algebra iff satisfies the inequality $x \rightarrow y \leq z \rightarrow (x \rightarrow y)$ (T),

(4) $\mathcal{H}$ is a subresiduated lattice, or sr-lattice iff satisfies the inequalities (T) and (R),

(5) $\mathcal{H}$ is an SDWH-algebra iff satisfies $x \to (y \to z) = (x \to y) \to (x \to z)$ $(SD)$.

Proposition 2.4 ([13]). Let $\mathcal{H}$ be a WH-algebra.

(1) $\mathcal{H}$ is a Heyting algebra if and only if $x = 1 \to x$, for all $x \in H$,

(2) $\mathcal{H}$ is an SDWH-algebra if and only if $x \to (y \to z) = y \to (x \to z)$, for all $x, y, z \in H$.

Proposition 2.5 ([13]). Let $\mathcal{H}$ be an SDWH-algebra. Then the following hold, for all $x, y, z \in H$,

(1) $x \to (y \to x) = 1$,

(2) $x \to (x \to y) = 1 \to (x \to y) = x \to (1 \to y)$,

(3) $x \to (y \to (x \wedge y)) = 1$,

(4) $y \to z \leq x \to (y \to z)$,

(5) $x \to y \leq (z \to x) \to (z \to y)$,

(6) $x \to y \leq (y \to z) \to (x \to z)$.

Definition 2.6 ([4]). Let $\mathcal{L}$ be a lattice. A non-empty subset $F$ of $L$ is called a filter of $\mathcal{L}$, if it is satisfies the following conditions, for all $x, y \in L$

(F1) If $x, y \in F$, then $x \wedge y \in F$,

(F2) If $x \in F$ and $x \leq y$, then $y \in F$.

A filter $F$ of a WH-algebra $\mathcal{H}$ is called an open filter of $\mathcal{H}$, if it is satisfies the following condition, for all $x \in H$.

(OF) If $x \in F$, then $1 \to x \in F$.

We denote by $OF(\mathcal{H})$ the set of all open filters of $\mathcal{H}$.

Proposition 2.7 ([3]). Let $(L, \wedge, \vee, 0, 1)$ be a bounded distributive lattice. If $\langle a \rangle$ is the filter generated by element $a \in L$, we have

(1) $\langle a \rangle = \{x \in L |\ a \leq x\ \}$,

(2) $a \leq b$, then $\langle b \rangle \subseteq \langle a \rangle$,

(3) $\langle a\rangle \vee \langle b\rangle = \langle a\wedge b\rangle$,

(4) $\langle a\rangle \cap \langle b\rangle = \langle a\vee b\rangle$.

Proposition 2.8 ([13, 14]). Let $\mathcal{H}$ be an SDWH-algebra. Given an integer $n \geq 1$, we define inductively

$$\Box^0(x) = x, \quad \Box^1(x) = 1 \to x, \quad \Box^n(x) = 1 \to (\Box^{n-1}(x)),$$
$$x \to^0 y = y, \quad x \to^n y = x \to (x \to^{n-1} y).$$

Then the following hold for all $x, y, z \in H$,

(N1) $x \to^{n+1} y = \Box^n(x \to y)$,

(N2) $\Box^n(x\wedge y) = \Box^n(x) \wedge \Box^n(y)$,

(N3) $n \leq m$ implies $\Box^n(x) \leq \Box^m(x)$,

(N4) $\Box^n(x \to (y \to z)) = \Box^{n+1}(x \to y) \to \Box^{n+1}(x \to z)$.

Let $\mathcal{H}$ be WH-algebra and $a, b \in H$. By $\Phi(a, b)$, we denote the principal congruence relation of $\mathcal{H}$ generated by $(a, b)$, i.e., the smallest congruence relation that contains $(a, b)$.

Proposition 2.9 ([16]). Let $\mathcal{H}$ be WH-algebra. The binary term is defined

$$t_n(a, b) = (a \leftrightarrow b) \wedge \Box(a \leftrightarrow b) \wedge \cdots \wedge \Box^n(a \leftrightarrow b),$$

where $a \leftrightarrow b = (a \to b) \wedge (b \to a)$. Then $(x, y) \in \Phi(a, b)$ if and only if there exists $n \in \mathbb{N}$ satisfying:

(C1) $x \wedge a \wedge b \wedge t_n(a, b) = y \wedge a \wedge b \wedge t_n(a, b)$,

(C2) $(x \vee a \vee b) \wedge t_n(a, b) = (y \vee a \vee b) \wedge t_n(a, b)$,

(C3) $t_n(a, b) \leq x \leftrightarrow y$.

Definition 2.10 ([6]). An algebra $\mathcal{A}$ with a constant 1 is called weakly regular iff for each congruence relations $\theta, \phi$ on $\mathcal{A}$, we have $\theta = \phi$ whenever $[1]_\theta = [1]_\phi$.
A variety $V$ is weakly regular if every $\mathcal{A} \in V$ has this property.

## 3. Open filters and deductive systems

In this section, we study the structure of open filters and deductive systems of SDWH-algebras.

Let $S$ be a non-empty subset of a WH-algebra $\mathcal{H}$. The smallest open filter of $\mathcal{H}$ containing $S$, (i.e. $\cap\{F \in OF(\mathcal{H}) | S \subseteq F\}$), is called the open filter generated by $S$ and it will be denoted by $\langle S \rangle_O$. If $S = \{a\}$, we write $\langle a \rangle_O$ instead of $\langle\ \{a\}\ \rangle_O$ and it is called principal open filter.

Proposition 3.1. Let $\mathcal{H}$ be an SDWH-algebra. Then the following statements are hold, for all $a, b \in H$:

1. $\langle a \rangle_O = \{x \in H | \Box(a) \wedge a \leq x\} = \langle \Box(a) \wedge a \rangle$,
2. if $a \leq 1 \to a$, then $\langle a \rangle_O = \langle a \rangle$,
3. $a \leq b$ implies $\langle b \rangle_O \subseteq \langle a \rangle_O$,
4. $\langle a \rangle_O \vee \langle b \rangle_O = \langle a \wedge b \rangle_O$,
5. $\langle a \vee b \rangle_O \subseteq \langle a \rangle_O \cap \langle b \rangle_O = \langle (\Box(a) \wedge a) \vee (\Box(b) \wedge b) \rangle_O$,
6. if $\langle a \rangle_O = \langle b \rangle_O$, then $x \to a = x \to b$ for all $x \in H$,
7. $\langle a \to b \rangle_O = \langle a \to b \rangle$.

Proof: (1) By Proposition 2.7 part (1), we have $\langle a \wedge (1 \to a) \rangle = \{x \in H | a \wedge (1 \to a) \leq x\}$. Thus $F = \{x \in H | a \wedge (1 \to a) \leq x\}$ is a filter. We will prove that $F$ is open. Let $x \in F$. Since $1 \to a \leq 1 \to (1 \to a)$ by Proposition 2.5 part (4), then

$$a \wedge (1 \to a) \leq (1 \to a) \wedge (1 \to (1 \to a)) = 1 \to (a \wedge (1 \to a)) \leq 1 \to x$$

by (WH1) and (W1). Then $1 \to x \in F$. Hence $F$ is open filter containing $a$. But $\langle a \rangle_O$ is the smallest open filter containing $a$, therefore $\langle a \rangle_O \subseteq F$. On the other hand, since $a, 1 \to a \in \langle a \rangle_O$, then $a \wedge (1 \to a) \in \langle a \rangle_O$ by (F1). For any $x \in F$, we get $x \in \langle a \rangle_O$ by (F2). Hence $F \subseteq \langle a \rangle_O$.
(2) It follows from part (1).
(3) Since $a \leq b$, then $a \wedge (1 \to a) \leq b \wedge (1 \to b)$ by (W1). Using Proposition 2.7 part (2), we get $\langle b \wedge (1 \to b) \rangle \subseteq \langle a \wedge (1 \to a) \rangle$. Hence $\langle b \rangle_O \subseteq \langle a \rangle_O$ by part (1).
(4) Using part (1), (WH1) and Proposition 2.7 part (3), we have

$$\begin{aligned}\langle a \wedge b \rangle_O =& \langle a \wedge b \wedge (1 \to (a \wedge b)) \rangle = \langle a \wedge (1 \to a) \wedge b \wedge (1 \to b) \rangle \\ =& \langle a \wedge (1 \to a) \rangle \vee \langle b \wedge (1 \to b) \rangle = \langle a \rangle_O \vee \langle b \rangle_O.\end{aligned}$$

(5) Using part (1) and then Proposition 2.7 part (4), we have
$\langle a\rangle_O \cap \langle b\rangle_O = \langle \Box(a) \wedge a\rangle \cap \langle \Box(b) \wedge b\rangle = \langle (\Box(a) \wedge a) \vee (\Box(b) \wedge b)\rangle$.

Put $u := (\Box(a) \wedge a) \vee (\Box(b) \wedge b) = (a \wedge (1 \to a)) \vee (b \wedge (1 \to b))$. We will show that $u \leq 1 \to u$. By (W1), (WH1) and Proposition 2.5 part (4), we obtain

$$\begin{aligned}
1 \to u &= 1 \to [(a \wedge (1 \to a)) \vee (b \wedge (1 \to b))] \\
&\geq [1 \to (a \wedge (1 \to a))] \vee [1 \to (b \wedge (1 \to b))] \\
&= [(1 \to a) \wedge (1 \to (1 \to a))] \vee [(1 \to b) \wedge (1 \to (1 \to b))] \\
&\geq (1 \to a) \vee (1 \to b) \\
&\geq (a \wedge (1 \to a)) \vee (b \wedge (1 \to b)) = u.
\end{aligned}$$

So $\langle u\rangle_O = \langle u\rangle$ by part (2). Hence $\langle a\rangle_O \cap \langle b\rangle_O = \langle (\Box(a) \wedge a) \vee (\Box(b) \wedge b)\rangle_O$. Also, $\langle a \vee b\rangle_O \subseteq \langle a\rangle_O \cap \langle b\rangle_O$ by part (3).
(6) Let $\langle a\rangle_O = \langle b\rangle_O$. Then $\langle a \wedge (1 \to a)\rangle = \langle b \wedge (1 \to b)\rangle$. We get $a \wedge (1 \to a) = b \wedge (1 \to b)$. So $x \to (a \wedge (1 \to a)) = x \to (b \wedge (1 \to b))$. Using Proposition 2.5 parts (2), (4) and (WH1) we get $x \to a = x \to b$.
(7) Using part (1) and then Proposition 2.5 part (4), we get
$\langle a \to b\rangle_O = \{x \in H | (a \to b) \wedge (1 \to (a \to b)) \leq x\} = \{x \in H | a \to b \leq x\} = \langle a \to b\rangle$. $\Box$

In an SDWH-algebra $\langle a\rangle_O \cap \langle b\rangle_O, \langle a \vee b\rangle_O$ may not be equal in general. See the following example:

*Example* 3.2. Let $H = \{0, a, b, 1\}$ where $0 < a, b < 1$ such that $a, b$ are not comparable. Consider the following binary operation:

| $\to$ | 0 | a | b | 1 |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 |
| a | 0 | 1 | 0 | 1 |
| b | 1 | 1 | 1 | 1 |
| 1 | 0 | 1 | 0 | 1 |

It is easy to see that $\mathcal{H} = (H, \vee, \wedge, \to, 0, 1)$ is an SDWH-algebra and
$\langle a\rangle_O = \{c \in H | x \geq a \wedge (1 \to a)\} = \{x \in H | x \geq a\} = \{1, a\}$,
$\langle b\rangle_O = \{c \in H | x \geq b \wedge (1 \to b)\} = \{x | \in H | x \geq 0\} = \{1, b, a, 0\}$.
Then $\langle a\rangle_O \cap \langle b\rangle_O = \{1, a\}$, but $\langle a \vee b\rangle_O = \langle 1\rangle_O = \{1\}$. Therefore $\langle a \vee b\rangle_O \subsetneq \langle a\rangle_O \cap \langle b\rangle_O$.

Lemma 3.3. *Let $F$ be an open filter of an SDWH-algebra $\mathcal{H}$ and $y \in F$. Then $x \to y \in F$ for all $x \in H$.*

Proof: Let $x \in H$ be arbitrary. We have $1 \to y \in F$ by (OF). Since $1 \to y \leq x \to y$ by (W1), then $x \to y \in F$ by (F2). □

The next proposition gives a concrete description of the open filter generated by a subset of an SDWH-algebra.

Proposition 3.4. Let $\{F_i\}_{i\in I}$ be a family of open filters of an SDWH-algebra $\mathcal{H}$, $S \subseteq H$ and $a \in H \backslash S$. Then

(1) $\langle S\rangle_O = \{x \in H |\ s_1 \wedge \cdots \wedge s_n \wedge \Box(s'_1) \wedge \cdots \wedge \Box(s'_m) \leq x$ for some $m, n \in N, s_1, \ldots, s_n, s'_1, \ldots, s'_m \in S\} = \langle S \cap \Box(\langle S\rangle)\rangle$,

(2) $\langle S \cup \{a\}\rangle_O = \{x \in H | a \wedge s_1 \wedge \cdots \wedge s_n \wedge \Box(a) \wedge \Box(s'_1) \wedge \cdots \wedge \Box(s'_m) \leq x$ for some $m, n \in N, s_1, \ldots, s_n, s'_1, \ldots, s'_m \in S\}$,

(3) $\langle \cup_{i\in I} F_i\rangle_O = \{x \in H | f_{i_1} \wedge f_{i_2} \wedge ... \wedge f_{i_m} \leq x$ for some $j = 1, ..., m$ and $f_{i_j} \in F_{i_j}\}$.

Proof: (1) We denote by $F$ the set from the right part of equality from announce (above). It is easy to prove that $F$ is a filter containing $S$. We will show that $F$ is open. Let $x \in F$. Then there exist $m, n \in N$ and $s_1, \ldots, s_n, s'_1, \ldots, s'_m \in S$ such that $s_1 \wedge \cdots \wedge s_n \wedge \Box(s'_1) \wedge \cdots \wedge \Box(s'_m) \leq x$. Since $1 \to (1 \to s'_i) \geq 1 \to s'_i$ by Proposition 2.5 part (4), then

$$\begin{aligned} 1 \to x &\geq (1 \to s_1) \wedge \cdots \wedge (1 \to s_n) \wedge (1 \to (1 \to s'_1)) \wedge \cdots \wedge (1 \to (1 \to s'_m)) \\ &\geq (1 \to s_1) \wedge \cdots \wedge (1 \to s_n) \wedge (1 \to s'_1) \wedge \cdots \wedge (1 \to s'_m). \end{aligned}$$

by (WH1). Hence $1 \to x \in F$ by (F2). But $\langle S\rangle_O$ is smallest open filter containing $S$, therefore $\langle S\rangle_O \subseteq F$.
Now, we have $s_i, 1 \to s'_i \in \langle S\rangle_O$. Thus $s_1 \wedge \cdots \wedge s_n \wedge (1 \to s'_1) \wedge \cdots \wedge (1 \to s'_n) \in \langle S\rangle_O$ by (F1). So for any $x \in F$, we have $x \in \langle S\rangle_O$. Hence $F \subseteq \langle S\rangle_O$.
(2) and (3) are a direct consequence of (1). □

Proposition 3.5. Let $\mathcal{H}$ be an SDWH-algebra. Then $(OF(\mathcal{H}),\wedge,\vee,\{1\},H)$ is a complete distributive lattice.

Proof: Suppose that $\{F_i\}_{i\in I}$ is a family of open filters of $\mathcal{H}$. It is easy to check that the infimum of of this family is $\wedge_{i\in I} F_i = \cap_{i\in I} F_i$ and the supermum is $\vee_{i\in I} F_i = \langle \cap_{i\in I} F_i\rangle_O$ Therefore $(OF(\mathcal{H}), \wedge, \vee, \{1\}, H)$ is a complete lattice. We will show that for every open filter $F$ and every family $\{F_i\}_{i\in I}$ of open filters, $F \wedge (\vee_{i\in I} F_i) = \vee_{i\in I}(F \wedge F_i)$. Clearly, $\vee_{i\in I}(F \wedge F_i) \subseteq F \wedge (\vee_{i\in I} F_i)$. Conversely, suppose that $x \in F \wedge (\vee_{i\in I} F_i)$. Then $x \in F$ and $x \geqslant f_{i_1} \wedge f_{i_2} \wedge ... \wedge f_{i_m}$ for some $j = 1, ..., m$ and $f_{i_j} \in F_{i_j}$. Since $(H, \vee, \wedge, 0, 1)$ is a distributive lattice, then $x = x \vee (f_{i_1} \wedge f_{i_2} \wedge ... \wedge f_{i_m}) \geqslant (x \vee f_{i_2}) \wedge ... \wedge (x \vee f_{i_m})$. We have $x \vee f_{i_j} \in F \cap F_{i_j}$, for every $1 \leq j \leq m$. So $x \in \vee_{i\in I}(F \wedge F_i)$ by Proposition 3.4 part (3). Hence $F \wedge (\vee_{i\in I} F_i) \subseteq \vee_{i\in I}(F \wedge F_i)$. □

Proposition 3.6. Let $F_1, F_2$ be open filters of an SDWH-algebra $\mathcal{H}$. Put $F_1 \to F_2 := \{x \in H | \langle x \rangle_O \cap F_1 \subseteq F_2\}$. Then $F_1 \to F_2 = \{x \in H | (x \wedge \Box(x)) \vee y \in F_2$, for all $y \in F_1\} \in OF(\mathcal{H})$.

Proof: Put $F := \{x \in H | (x \wedge \Box(x)) \vee y \in F_2, \text{for all } y \in F_1\}$. We will prove that $F_1 \to F_2 = F$. Suppose that $x \in F_1 \to F_2$. Then $\langle x \rangle_O \cap F_1 \subseteq F_2$. Let $y \in F_1$ be arbitrary. We get that $(x \wedge \Box(x)) \vee y \in \langle x \rangle_O \cap F_1$. So $(x \wedge \Box(x)) \vee y \in F_2$. Therefore $x \in F$. Hence $F_1 \to F_2 \subseteq F$.
Conversely, suppose that $x \in F$ and $y \in \langle x \rangle_O \cap F_1$. Then $(x \wedge \Box(x)) \leq y$ and $y \in F_1$. We get that $y = (x \wedge \Box(x)) \vee y \in F_2$. Thus $x \in F_1 \to F_2$. Hence $F \subseteq F_1 \to F_2$.
Now, we will prove that $F_1 \to F_2$ is an open filter. Since $(1 \wedge \Box(1)) \vee y = 1 \in F_2$ for all $y \in F_1$, then $1 \in F_1 \to F_2$ and $F_1 \to F_2$ is a non-empty subset of $H$. Let $x, y \in H$ such that $x \leq y$ and $x \in F_1 \to F_2$. So $\langle x \rangle_O \cap F_1 \subseteq F_2$ and $\langle y \rangle_O \subseteq \langle x \rangle_O$ by Proposition 3.1 part (3). Then $\langle y \rangle_O \cap F_1 \subseteq \langle x \rangle_O \cap F_1 \subseteq F_2$. Hence $y \in F_1 \to F_2$.
Let $x, y \in H$ such that $x, y \in F_1 \to F_2$. Then $\langle x \rangle_O \cap F_1 \subseteq F_2$ and $\langle y \rangle_O \cap F_1 \subseteq F_2$. Using Proposition 3.1 part (4) and Proposition 3.4, we have $\langle x \wedge y \rangle_O \cap F_1 = (\langle x \rangle_O \vee \langle y \rangle_O) \cap F_1 \subseteq F_2$. Therefore $x \wedge y \in F_1 \to F_2$. Hence $F_1 \to F_2$ is a filter.
Let $x \in F_1 \to F_2$. Then $(x \wedge \Box(x)) \vee y \in F_2$, for all $y \in F_1$. Since $\Box(x) \leq \Box^2(x)$ by (N3), then $(x \wedge \Box(x)) \vee y \subseteq \Box(x) \vee y = (\Box(x) \wedge \Box^2(x)) \vee y$. So $(\Box(x) \wedge \Box^2(x)) \vee y \in F_2$. Hence $F_1 \to F_2$ is open. □

In the next proposition, we will prove that $OF(\mathcal{H})$ forms a complete Heyting algebra with respect to inclusion.

Proposition 3.7. Let $\mathcal{H}$ be an SDWH-algebra. Define binary operations $\wedge$, $\vee$ and $\to$ on $OF(\mathcal{H})$ as follows: for all $F_1, F_2 \in OF(\mathcal{H})$, $F_1 \wedge F_2 = F_1 \cap F_2$,

$F_1 \vee F_2 = \langle F_1 \cup F_2 \rangle_O$, $F_1 \to F_2 = \{x \in H | \langle x \rangle_O \cap F_1 \subseteq F_2\}$. Then $(OF(\mathcal{H}), \wedge, \vee, \to, \{1\}, H)$ is a complete Heyting algebra.

PROOF: By Proposition 3.5, $(OF(\mathcal{H}), \wedge, \vee, \{1\}, H)$ is a complete lattice. Next, we will prove that $F_1 \wedge F_2 \subseteq F_3$ if and only if $F_1 \subseteq F_2 \to F_3$. Suppose $F_1 \wedge F_2 \subseteq F_3$ and $x \in F_1$. Then $\langle x \rangle_O \subseteq F_1$, hence $\langle x \rangle_O \wedge F_2 \subseteq F_1 \wedge F_2 \subseteq F_3$. Thus $x \in F_2 \to F_3$.
Conversely, suppose that $F_1 \subseteq F_2 \to F_3$ and $x \in F_1 \wedge F_2$. Then $x \in F_1$. So $x \in F_2 \to F_3$. We get $\langle x \rangle_O \wedge F_2 \subseteq F_3$. Then $x \in F_3$. Hence $F_1 \wedge F_2 \subseteq F_3$. □

PROPOSITION 3.8. Let $F$ be an open filter of an SDWH-algebra $\mathcal{H}$. Then $F$ is a compact element of $(OF(\mathcal{H}), \wedge, \vee, \to, \{1\}, H)$ if and only if $F$ is a principal open filter of $\mathcal{H}$.

PROOF: Suppose that $F$ is a compact element of $(OF(\mathcal{H}), \wedge, \vee, \to, \{1\}, H)$. Since $F = \vee_{x \in F} \langle x \rangle_O$, then there exist $x_1, x_2, ..., x_n \in F$ such that $F = \langle x_1 \rangle_O \vee \langle x_2 \rangle_O \vee ... \vee \langle x_n \rangle_O$. Using Proposition 3.1 part (4), we have $F = \langle x_1 \wedge x_2 \wedge ... \wedge x_n \rangle_O$. Hence $F$ is a principal open filter of $\mathcal{H}$.
Conversely, let $F$ be a principal open filter of of $\mathcal{H}$. Then there exists $x \in F$ such that $F = \langle x \rangle_O$. Suppose that $\{F_i\}_{i \in I}$ is a family of open filters of $\mathcal{H}$ such that $F \subseteq \vee_{i \in I} F_i$. Then $x \in \langle \cup_{i \in I} F_i \rangle_O$. Then there exist $i_j \in I$, $f_{i_j} \in F_{i_j}$ $(j = 1, ..., m)$ such that $f_{i_1} \wedge f_{i_2} \wedge ... \wedge f_{i_m} \leq x$ by Proposition 3.4 part (3). So $x \in \langle F_{i_1} \cup F_{i_2} \cup ... \cup F_{i_m} \rangle_O$. Hence $F = \langle x \rangle_O \subseteq F_{i_1} \vee F_{i_2} \vee ... \vee F_{i_m}$. □

We define deductive system of an SDWH algebra in a usual way:

DEFINITION 3.9. A subset $D$ is called a deductive system of an SDWH algebra $\mathcal{H}$ if it is satisfies the following conditions, for all $x, y \in H$:

(D1) $1 \in D$,

(D2) $x, x \to y \in D$ imply $y \in D$.

The set of all deductive system of $\mathcal{H}$ is denoted by $Ds(\mathcal{H})$.

PROPOSITION 3.10. Let $\mathcal{H}$ be an SDWH algebra. Then $Ds(\mathcal{H}) \subseteq OF(\mathcal{H}) \subseteq F(\mathcal{H})$.

PROOF: Let $D \in Ds(\mathcal{H})$. We will show $D$ is an open filter.
(F1) Let $x \in D$, $y \in H$ and $x \leq y$. Then $x \to y = 1 \in D$ by (W2). So $y \in D$ by (D2).

(F2) Let $x, y \in D$. By Proposition 2.5 part (3), we have $x \to (y \to (x \wedge y)) = 1 \in D$. Then $y \to x \wedge y \in D$. Hence $x \wedge y \in D$ by (D2).
(OF3) Let $x \in D$. We have $x \to (1 \to x) = 1 \in D$ by Proposition 2.5 part (1). Thus $1 \to x \in D$ by (D2). Therefore $D \in OF(\mathcal{H})$.
It is clear that every open filter is a filter of $\mathcal{H}$. $\square$

In the following example, we will see that every open filter may not be a deductive system of an SDWH-algebra and there exists a filter that is not an open filter.

*Example* 3.11. Let $H = \{0, a, b, 1\}$ with $0 < a, b < 1$, such that $a, b$ are not comparable. Consider the following binary operation:

| $\to$ | 0 | a | b | 1 |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 |
| a | 1 | 1 | 1 | 1 |
| b | b | b | 1 | 1 |
| 1 | b | b | 1 | 1 |

It is easy to see that $\mathcal{H} = (H, \vee, \wedge, \to, 0, 1)$ is an SDWH-algebra and
$F(\mathcal{H}) = \{\{1\}, \{1, b\}, \{1, a\}, H\}$,
$OF(\mathcal{H}) = \{\{1\}, \{1, b\}, H\}$,
$Ds(\mathcal{H}) = \{H\}$.
So $Ds(\mathcal{H}) \subsetneqq OF(\mathcal{H}) \subsetneqq F(\mathcal{H})$ .

Theorem 3.12. *Let $\mathcal{H}$ be an SDWH-algebra. The following are equivalent:*

(1) $1 \to x \leq x$, *for all* $x \in H$,

(2) $OF(\mathcal{H}) = Ds(\mathcal{H})$.

Proof: (1) $\Rightarrow$ (2) By Proposition 3.10, we have $Ds(\mathcal{H}) \subseteq OF(\mathcal{H})$. Let $F \in OF(\mathcal{H})$ and $x, x \to y \in F$. We will show that (D2) is true. By $y \leq x \vee y \leq 1$ and (W3) we have:

$$1 \to y = (1 \to (x \vee y)) \wedge ((x \vee y) \to y).$$

But $(x \vee y) \to y = x \to y \in F$ by (WH2). Since $1 \to x \leq 1 \to (x \vee y)$ and $1 \to x \in F$ by (F3), then $1 \to (x \vee y) \in F$ by (F2). So $1 \to y \in F$ by (F1). Thus $y \in F$ by assumption and (F2). Therefore $y \in Ds(\mathcal{H})$ and $OF(\mathcal{H}) = Ds(\mathcal{H})$.

(2) $\Rightarrow$ (1) Let $x \in H$. Then open filter $F_x := \langle 1 \to x\rangle_O$ is a deductive system by assumption. Obviously, $1, 1 \to x \in F_x$. So $x \in F_x$ by (D2). Hence $1 \to x \leq x$ by Proposition 3.1 part (7). $\square$

PROPOSITION 3.13. Let $\mathcal{H}$ be an SDWH-algebra. The following are equivalent:

(1) $x \leq 1 \to x$, for all $x \in H$,

(2) $F(\mathcal{H}) = OF(\mathcal{H})$.

PROOF: (1) $\Rightarrow$ (2) By Proposition 3.10, we have $OF(\mathcal{H}) \subseteq F(\mathcal{H})$. Let $F \in F(\mathcal{H})$. We will show that $F$ is open. Let $x \in F$. By assumption, we have $x \leq 1 \to x$. Hence $1 \to x \in F$ that is, $F \in OF(\mathcal{H})$.

(2) $\Rightarrow$ (1) Let $x \in H$. Then the filter $F_x = \{y \in H | x \leq y\}$ is an open filter by assumption. Thus $1 \to x \in F_x$. So $x \leq 1 \to x$. $\square$

COROLLARY 3.14. An SDWH-algebra $\mathcal{H}$ is a basic algebra if and only if $F(\mathcal{H}) = OF(\mathcal{H})$.

COROLLARY 3.15. Let $\mathcal{H}$ be an SDWH-algebra. The following are equivalent:

(1) $x = 1 \to x$, for all $x \in H$,

(2) $\mathcal{H}$ is Heyting algebra,

(3) $F(\mathcal{H}) = OF(\mathcal{H}) = Ds(\mathcal{H})$,

(4) $F(\mathcal{H}) = Ds(\mathcal{H})$.

The smallest deductive system of an SDWH-algebra $\mathcal{H}$ containing $S$, (i.e. $\cap\{D \in Ds(\mathcal{H}) | S \subseteq D\}$), is called the deductive system generated by $S$ and it will be denoted by $\langle S\rangle_D$ ($\langle a\rangle_D$ is called principal deductive system.)

PROPOSITION 3.16. Let $\mathcal{H}$ be an SDWH-algebra. If $a, b \in H$, then

(1) $\langle a\rangle_D = \{x \in H \mid \Box^n(a \to x) = 1, \text{ for some } n \in \mathbb{N}\}$
$= \{x \in H \mid a \to^n x = 1, \text{ for some } n \in \mathbb{N}\}$,

(2) $a \leq b$ implies $\langle b\rangle_D \subseteq \langle a\rangle_D$,

(3) $\langle a \vee b\rangle_D = \langle a\rangle_D \cap \langle b\rangle_D$,

(4) $\langle a \wedge b\rangle_D = \langle a\rangle_D \vee \langle b\rangle_D$.

PROOF: (1) We will show $D = \{x \in H \mid \Box^n(a \to x) = 1,$ for some $n \in \mathbb{N}\}$ is a deductive system of $\mathcal{H}$. We have $a \to 1 = 1$, so $1 \in D$. Let $x, x \to y \in D$. Then there exist $m, n \in \mathbb{N}$ such that $\Box^n(a \to x) = 1$ and $\Box^m(a \to (x \to y)) = 1$. Then $\Box^{n+m}(a \to x) = \Box^m(1) = 1$ and $\Box^{m+n}(a \to (x \to y)) = \Box^n(1) = 1$ by (N3). So we have

$$\begin{aligned}\Box^{m+n+1}(a \to y) &= 1 \to (\Box^{m+n}(a \to y)) \\ &= \Box^{m+n}(a \to x) \to \Box^{m+n}(a \to y) \\ &= \Box^{m+n}(a \to (x \to y)) \\ &= \Box^n(1) = 1.\end{aligned}$$

by (N4). Thus $y \in D$. Hence $D \in Ds(\mathcal{H})$. Also, we have $\Box^1(a \to a) = 1$. Hence $a \in D$. Then there exists $n \in \mathbb{N} \cup \{0\}$ such that $\Box^n(a \to x) = 1 \in \langle a \rangle_D$. Since $1 \to \Box^{n-1}(a \to x) = 1 \in \langle a \rangle_D$ and $1 \in \langle a \rangle_D$, then $\Box^{n-1}(a \to x) = 1 \in \langle a \rangle_D$ by (DS2). By inductively, we obtain $a \to x \in \langle a \rangle_D$. But $a \in \langle a \rangle_D$. So $x \in \langle a \rangle_D$ by (DS2). Hence $D \subseteq \langle a \rangle_D$. Since $\langle a \rangle_D$ is the smallest deductive system containing $a$, we obtain $D = \langle a \rangle_D$. Using (N1), we have $a \to^n x = \Box^{n-1}(a \to x)$. So it is easy to prove that $\langle a \rangle_D = \{x \in H \mid a \to^n x = 1,$ for some $n \in \mathbb{N}\}$.
(2) Let $x \in \langle b \rangle_D$. Then there exists $n \in \mathbb{N}$ such that $\Box^n(b \to x) = 1$ by part (1). By assumption $a \leq b$. So $b \to x \leq a \to x$ by (W1). Using (N5), we obtain $\Box^n(b \to x) \leq \Box^n(a \to x)$. Therefore $\Box^n(a \to x) = 1$. So $x \in \langle a \rangle_D$ by part (1). Hence $\langle b \rangle_D \subseteq \langle a \rangle_D$.
(3) Let $x \in \langle a \rangle_D \cap \langle a \rangle_D$. Then there exist $n, m \in \mathbb{N}$ such that we have $\Box^n(a \to x) = 1$ and $\Box^m(b \to x) = 1$ by part (2). Put $p := max\{m, n\}$. By (N3), we obtain $\Box^p(a \to x) \geq \Box^n(a \to x) = 1$. Similarly $\Box^p(b \to x) = 1$. Using (WH3) and then (N2), we get $\Box^p((a \vee b) \to x) = \Box^p((a \to x) \wedge (b \to x)) = \Box^p(a \to x) \wedge \Box^p(b \to x) = 1$. Hence $x \in \langle a \vee b \rangle_D$. Therefore $\langle a \rangle_D \cap \langle b \rangle_D \subseteq \langle a \vee b \rangle_D$.
Conversely, we have $a, b \leq a \vee b$. By part (2), we obtain $\langle a \vee b \rangle_D \subseteq \langle a \rangle_D, \langle b \rangle_D$. Hence $\langle a \vee b \rangle_D \subseteq \langle a \rangle_D \cap \langle b \rangle_D$. □

## 4. Congruence relations on SDWH algebras

In this section, we study some properties that establish some connections among the congruence relations, the open filters, and the deductive systems of an SDWH-algebra $\mathcal{H}$.

We denote by $Con(\mathcal{H})$ the congruence lattice of an SDWH-algebra $\mathcal{H}$. As usual, for a $\theta \in Con(\mathcal{H})$ denote by $[1]_\theta$ its congruence class containing the element 1, so-called kernel of $\theta$.

Definition 4.1. Let $F$ be an open filter of an SDWH-algebra $\mathcal{H}$. Define two binary relations $\Theta_F$ and $\Gamma_F$ on $H$ as follows:
$\Theta_F = \{(x, y) \in H \times H | x \wedge f \leq y \text{ and } y \wedge f \leq x \text{ for some} f \in F\}$,
$\Gamma_F = \{(x, y) \in H \times H | x \to y, y \to x \in F\}$.

Proposition 4.2. Let $F$ be an open filter of an SDWH-algebra $\mathcal{H}$. Then $\Theta_F$ is the least congruence relation on $H$ such that $[1]_{\Theta_F} = F$.

Proof: Clearly, $\Theta_F$ is reflexive and symmetric. In order to prove transivity, let $x, y, z \in H$ such that $(x, y), (y, z) \in \Theta_F$. We have $x \wedge f_1 \leq y$, $y \wedge f_1 \leq x$, $y \wedge f_2 \leq z$ and $z \wedge f_2 \leq y$ for some $f_1, f_2 \in F$. Then $x \wedge (f_1 \wedge f_2) \leq z$, $z \wedge (f_1 \wedge f_2) \leq x$ and $f_1 \wedge f_2 \in F$ by (F1). So $(x, z) \in \Theta_F$. Therefore $\Theta_F$ is an equivalence relation on $H$.
Now, we will prove that $\Theta_F$ compatible with $\wedge, \vee, \to$. Let $(x, y), (a, b) \in \Theta_F$, then $x \wedge f_1 \leq y$, $y \wedge f_1 \leq x$, $a \wedge f_2 \leq b$ and $b \wedge f_2 \leq a$, for some $f_1, f_2 \in F$. Put $f = f_1 \wedge f_2$. Then $f \in F$ by (F1). Thus $x \wedge a \wedge f \leq y \wedge b$, $y \wedge b \wedge f \leq x \wedge a$, $(x \vee a) \wedge f \leq (y \vee b)$ and $(y \vee b) \wedge f \leq (x \vee a)$ for $f \in F$. So $(x \wedge a, y \wedge b) \in \Theta_F$ and $(x \vee a, y \vee b) \in \Theta_F$. Therefore $\Theta_F$ compatible with $\wedge$ and $\vee$.
We will show that $(a \to x, a \to y) \in \Theta_F$, and $(a \to y, b \to y) \in \Theta_F$ which implies by transitivity of $\Theta_F$ that $(a \to x, b \to y) \in \Theta_F$.
Since $x \wedge f_1 \leq y$, then $(a \to x) \wedge (a \to f_1) = a \to (x \wedge f_1) \leq a \to y$ by (WH1) and (W1). Similarly, we have $(a \to y) \wedge (a \to f_1) \leq a \to x$. By Lemma 3.3, we have $a \to f_1 \in F$ because $f_1 \in F$ and $F$ is an open filter. So $(a \to x, a \to y) \in \Theta_F$.

Since $f_2 \in F$ and $F$ is open, then $1 \to f_2 \in F$. We have $1 \to f_2 \leq a \to f_2 = a \to (a \wedge f_2) \leq a \to b$, so $a \to b \in F$. Similarly, we obtain $b \to a \in F$. Thus $(a \to b) \wedge (b \to a) \in F$ by (F1). We have $(b \to y) \wedge [(a \to b) \wedge (b \to a)] \leq (a \to b) \wedge (b \to y) \leq a \to y$ and $(a \to y) \wedge [(a \to b) \wedge (b \to a)] \leq b \to y$. Hence $(a \to y, b \to y) \in F$. Therefore $\Theta_F$ is compatible with $\to$.
Suppose that $x \in [1]_{\Theta_F}$, so $(x, 1) \in \Theta_F$. Then $x \wedge f \leq 1$ and $1 \wedge f \leq x$ for some $f \in F$. So $x \in F$ by (F2). Conversely, let $x \in F$, we have $x \wedge x \leq 1$ and $1 \wedge x \leq x$ for $x \in F$. Then $(x, 1) \in \Theta_F$, so $x \in [1]_{\Theta_F}$. Therefore $[1]_{\Theta_F} = F$.

Suppose that $\theta \in Con(\mathcal{H})$ such that $[1]_\theta = F$. Let $(x, y) \in \Theta_F$. Then there exists $f \in F$ such that $x \wedge f \leq y$ and $y \wedge f \leq x$. We get $x \wedge f = y \wedge f$. Since $f \in F = [1]_\theta$, then $(f, 1) \in \theta$. So $(x, x \wedge f) = (x \wedge 1, x \wedge f) \in \theta$ and $(y, y \wedge f) = (y \wedge 1, y \wedge f) \in \theta$. Thus $(x, y) \in \theta$ by transitivity. Hence $\Theta_F \subseteq \theta$. □

Corollary 4.3. Let $F$ be an open filter of an SDWH-algebra $\mathcal{H}$. Then $F$ is a congruence kernel in $\mathcal{H}$ that is, $F = [1]_\theta$ for some $\theta \in Con(\mathcal{H})$,

Proposition 4.4. Let $F$ be an open filter of an SDWH-algebra $\mathcal{H}$. Then $\Gamma_F$ is congruence relation on $\mathcal{H}$ such that $F \subseteq [1]_{\Gamma_F}$.

Proof: Obviously, $\Gamma_F$ is reflexive and symmetric. The transitivity follows from (WH3). So $\Gamma_F$ is an equivalence relation on $H$.

Let $(x, y) \in \Gamma_F$, and $(a, b) \in \Gamma_F$. Then $x \to y, y \to x \in F$ and $a \to b, b \to a \in F$. By (W1) and (WH1), we obtain $x \to y \leq (x \wedge a) \to y = (x \wedge a) \to (y \wedge a)$ and $y \to x \leq (y \wedge a) \to x = (y \wedge a) \to (x \wedge a)$. Then $(x \wedge a) \to (y \wedge a) \in F$ and $(y \wedge a) \to (x \wedge a) \in F$ by (F2). So $(x \wedge a, y \wedge a) \in \Gamma_F$. Similarly, we can prove that $(a \wedge y, b \wedge y) \in \Gamma_F$. Then $(x \wedge a, y \wedge b) \in \Gamma_F$ by transitivity. Therefore $\Gamma_F$ compatible with $\wedge$.
Using (W1) and (WH2), we get $x \to y \leq x \to (y \vee a) = (x \vee a) \to (y \vee a)$ and $y \to x \leq y \to (x \vee a) = (y \vee a) \to (x \vee a)$. Thus $(x \vee a, y \vee a) \in \Gamma_F$. Similarly, $(a \vee y, b \vee y) \in \Gamma_F$. Thus $(x \vee a, y \vee b) \in \Gamma_F$ Therefore $\Gamma_F$ compatible with $\vee$.
By Proposition 2.5 part (5), we have $x \to y \leq (a \to x) \to (a \to y)$ and $y \to x \leq (a \to y) \to (a \to x)$. By (F2), we obtain $(a \to x, a \to y) \in \Gamma_F$. By Proposition 2.5 part (6), $b \to a \leq (a \to y) \to (b \to y)$ and $a \to b \leq (b \to y) \to (a \to y)$. Using (F2), we get $(a \to y, b \to y) \in \Gamma_F$. Hence $(a \to x, b \to y) \in \Gamma_F$. So $\Gamma_F$ compatible with $\to$. Therefore $\Gamma_F$ is a congruence relation on $H$. Let $x \in F$ be arbitrary. Since $F$ is open, then $1 \to x \in F$. By (W2) and (F2), $x \to 1 \in F$. Thus $F \subseteq [1]_{\Gamma_F}$. □

In Proposition 4.4, $F$ and $[1]_{\Gamma_F}$ may not be equal in general. See the following example:

*Example* 4.5. Consider the open filter $F = \{1, b\}$ of the SDWH-algebra $\mathcal{H}$ in Example 3.11. Then $F \subsetneq [1]_{\Gamma_F} = H$.

Corollary 4.6. Let $D$ be a deductive system of an SDWH-algebra $\mathcal{H}$. Then $\Gamma_D$ is the greatest congruence relation on $H$ such that $D = [1]_{\Gamma_D}$.

Proof: By Proposition 3.10 and Proposition 4.4, $\Gamma_D$ is congruence relation on $H$ such that $D \subseteq [1]_{\Gamma_D}$. Now, suppose that $x \in [1]_{\Gamma_D}$. Then $1 = x \to 1 \in D$ and $1 \to x \in D$. By (D2), we get $x \in D$. Hence $[1]_{\Gamma_D} \subseteq D$. Suppose that $\theta \in Con(\mathcal{H})$ such that $D = [1]_{\Gamma_D}$. Let $(x, y) \in \theta$. Then $(x \to y, y \to y) = (x \to y, 1)$. So $x \to y \in [1]_\theta = D$. Similarly $y \to x \in D$. Hence $(x, y) \in \Gamma_D$. Therefore $\theta \subseteq \Gamma_D$. □

Proposition 4.7. Let $F$ be an open filter of an SDWH-algebra $\mathcal{H}$. Then $\frac{\mathcal{H}}{\Gamma_F}$ is a Heyting algebra if and only if $F$ is a deductive system of $H$.

Proof: Suppose that $F$ is a deductive system of $H$ and $x \in H$ be arbitrary. We have $1 \to ((1 \to x) \to x) = (1 \to x) \to (1 \to x) = 1 \in F$. Since $F$ is a deductive system and $1 \in F$, then $(1 \to x) \to x \in F$. Also, $x \to (1 \to x) = 1 \to (x \to x) = 1 \in F$. Thus $(x, 1 \to x) \in \Gamma_F$ that is, $[x]_{\Gamma_F} = [1]_{\Gamma_F} \to [x]_{\Gamma_F}$ for all $x \in H$. Hence $\frac{\mathcal{H}}{\Gamma_F}$ is a Heyting algebra.
Conversely, let $x, x \to y \in F$. Then we have $[x] = [x \to y] = [1]$. Since $\frac{\mathcal{H}}{\Gamma_F}$ is a Heyting algebra, then $[x \wedge y] = [x \wedge (x \to y)] = [1]$. We obtain $x \wedge y \in F$. Hence $y \in F$. □

Proposition 4.8. Let $\theta$ be a congruence relation on an SDWH-algebra $\mathcal{H}$. Then

(1) $[1]_\theta \in OF(\mathcal{H})$,

(2) $\Theta_{[1]_\theta} \subseteq \theta \subseteq \Gamma_{[1]_\theta}$.

Proof: (1) Let $x, y \in [1]_\theta$. Then $(x, 1) \in \theta$ and $(y, 1) \in \theta$. By compatibility of $\theta$ with $\wedge$, we have $(x \wedge y, 1) \in \theta$. Thus $x \wedge y \in [1]_\theta$.
Let $x \in [1]_\theta$ such that $x \leq y$. Then $(x, 1) \in \theta$ and $x \vee y = y$. So $(y, 1) = (x \vee y, 1 \vee y) \in \theta$. Hence $[1]_\theta$ is a filter. If $x \in [1]_\theta$, then $(x, 1) \in \theta$. So $(1 \to x, 1 \to 1) \in \theta$. Hence $[1]_\theta$ is an open filter.
(2) Let $(x, y) \in \theta_{[1]_\theta}$. Then there exists $f \in [1]_\theta$ such that $x \wedge f \leq y$ and $y \wedge f \leq x$. We obtain $x \wedge f = y \wedge f$ and $(f, 1) \in \theta$. So $(x \wedge f, x) \in \theta$ and $(y \wedge f, y) \in \theta$. Thus $(x, y) \in \theta$. Hence $\theta_{[1]_\theta} \subseteq \theta$.
Now, suppose that $(x, y) \in \theta$. Then $(x \to y, y \to y) = (x \to y, 1) \in \theta$ and

$(y \to x, x \to x) = (y \to x, 1) \in \theta$. So $x \to y \in [1]_\theta$ and $y \to x \in [1]_\theta$. Hence $(x, y) \in \Gamma_{[1]_\theta}$. Therefore $\theta \subseteq \Gamma_{[1]_\theta}$ □

In an SDWH-algebra $\theta$ and $\Theta_{[1]_\theta}$ may not be equal in general. See the following example.

*Example* 4.9. Let $H = \{0, a, b, 1\}$ where $0 < a < b < 1$. Consider the following binary operation:

| $\to$ | 0 | a | b | 1 |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 |
| a | 1 | 1 | 1 | 1 |
| b | a | a | 1 | 1 |
| 1 | a | a | b | 1 |

Obviously $\mathcal{H} = (H, \vee, \wedge, \to, 0, 1)$ is an SDWH-algebra. Consider the congruence relation $\theta = \{(1,1), (b,b), (1,b), (b,1), (a,a), (0,0), (a,0), (0,a)\}$ on $H$. Then $[1]_\theta = \{1, b\}$ is an open filter of $\mathcal{H}$ and

$$\Theta_{[1]_\theta} = \{(1,1), (b,b), (1,b), (b,1), (a,a), (0,0)\}.$$

Hence $\Theta_{[1]_\theta} \subsetneq \theta$.

For every RWH-algebra $\mathcal{H}$, there is an isomorphism between the lattice of open filters of $\mathcal{H}$ and the lattice congruence relation of $\mathcal{H}$ (see [4, 15]).

If $\mathcal{H}$ is an SDWH-algebra, then he natural map $\theta \mapsto [1]_\theta$ associated with $Con(\mathcal{H})$ and $OF(\mathcal{H})$ is well defined and onto, but not one-to-one in general as you can see in the following example. This example also shows that the open filters of SDWH-algebras can be kernels of more than one congruence relation. Hence the variety of SDWH-algebras is not weakly regular.

*Example* 4.10. Let $H = \{0, a, b, 1\}$ where $0 < a < b < 1$. Consider the following binary operation:

| $\to$ | 0 | a | b | 1 |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 |
| a | 0 | 1 | 1 | 1 |
| b | 0 | 1 | 1 | 1 |
| 1 | 0 | a | a | 1 |

It is easy to see that $\mathcal{H} = (H, \vee, \wedge, \to, 0, 1)$ is an SDWH-algebra and for all $x \in H$ we have $1 \to x \leq x$. We have $OF(\mathcal{H}) = Ds(\mathcal{H}) = \{\{1\}, \{1, b, a\}, H\}$. But we have four congruence relations on $H$ as follows:

$\theta_1 = \Delta = \{(0,0),(a,a),(b,b),(1,1)\},\ [1]_{\theta_1} = F_1 = \{1\},$
$\theta_2 = \Delta \cup \{(a,b),(b,a)\},\ [1]_{\theta_2} = F_1 = \{1\},$
$\theta_3 = \Delta \cup \{(a,b),(b,a),(1,b),(b,1),(1,a),(a,1)\},\ [1]_{\theta_3} = F_2 = \{1,b,a\},$
$\theta_4 = \nabla = \{(x,y)|x,y \in H\},\ [1]_{\theta_4} = F_3 = H.$

*Remark.* Let $\mathcal{H}$ be an SDWH-algebra. Then the natural map $\theta \mapsto [1]_\theta$ is an order isomorphism from $Con(\mathcal{H})$ to $OF(\mathcal{H})$ if and only if $\mathcal{H}$ is a weakly regular algebra.

In the following, we will obtain a characterization of weakly regular SDWH-algebras.

LEMMA 4.11. *Let $\mathcal{H}$ be an SDWH-algebra and $a,b \in H$. Then $(x,y) \in \Phi(a,b)$ if and only if*

(1) $x \wedge a \wedge b \wedge (a \leftrightarrow b) = y \wedge a \wedge b \wedge (a \leftrightarrow b)$,

(2) $(x \vee a \vee b) \wedge (a \leftrightarrow b) = (y \vee a \vee b) \wedge (a \leftrightarrow b)$,

(3) $a \leftrightarrow b \leq x \leftrightarrow y$.

PROOF: For all $n \in \mathbb{N}$, we have $a \to b \leq \Box^n(a \to b)$ and $b \to a \leq \Box^n(b \to a)$ by (N6). Then $a \leftrightarrow b \leq \Box^n(a \leftrightarrow b)$ by (N2). So $t_n(a,b) = a \leftrightarrow b$. The result follows from Proposition 2.9. □

Recall that a variety V has equationally definable principal congruences (EDPC) if there exists a finite family of quaternary terms $\{u_i, v_i\}_{i=1}^r$ such that for every algebra $\mathcal{A}$ in V and every principal congruence $\Phi(a,b)$ of $\mathcal{A}$, if and only if $u_i(a,b,c,d) = v_i(a,b,c,d)$ for each $i = 1,...,r$ ([6]). For algebraizable logics, EDPC corresponds to the deduction-detachment theorem.

COROLLARY 4.12. The variety of SDWH-algebras has EDPC.

PROPOSITION 4.13. Let $c$ be an element of an SDWH-algebra $\mathcal{H}$. Then

(1) $\Phi(1,c) = \Theta_{\langle c\rangle_O}$,

(2) $\Phi(1,c) = \Theta_{[1]_{\Phi(1,c)}}$.

PROOF: (1) Let $(x,y) \in \Phi(1,c)$. We have $x \wedge c \wedge (1 \to c) = y \wedge c \wedge (1 \to c)$ by Lemma 4.11 part (1) and (W2). Since $c \wedge (1 \to c) \in \langle c\rangle_O$, then we obtain $(x,y) \in \Theta_{\langle c\rangle_O}$. So $\Phi(1,c) \subseteq \Theta_{\langle c\rangle_O}$.

Conversely, suppose that $(x, y) \in \Theta_{\langle c\rangle_O}$. Then there exists $f \in \langle c\rangle_O$ such that $x \wedge f \leq y$ and $y \wedge f \leq x$. We get $c \wedge (1 \to c) \leq f$ and $x \wedge f = y \wedge f$. So

$$x \wedge c \wedge (1 \to c) = (x \wedge f) \wedge c \wedge (1 \to c) = (y \wedge f) \wedge c \wedge (1 \to c) = y \wedge c \wedge (1 \to c).$$

Hence $x \wedge c \wedge 1 \wedge (1 \leftrightarrow c) = y \wedge c \wedge 1 \wedge (1 \leftrightarrow c)$. It is obvious that $(x \vee c \vee 1) \wedge (1 \leftrightarrow c) = (y \vee c \vee 1) \wedge (1 \leftrightarrow c)$. We have $x \wedge f = y \wedge f \leq y$, so $x \to f = x \to (x \wedge f) \leq x \to y$ by (W4). Thus

$$1 \to c = 1 \to (c \wedge (1 \to c)) \leq 1 \to f \leq x \to f \leq x \to y.$$

Similarly, we can prove that $1 \to c \leq y \to x$. Thus $1 \leftrightarrow c \leq x \leftrightarrow y$. Hence $(x, y) \in \Phi(1, c)$ by Lemma 4.11.
(2) By Proposition 4.8 part (2), we have $\Theta_{[1]_{\Phi(1,c)}} \subseteq \Phi(1, c)$. Conversely, suppose that $(x, y) \in \Phi(1, c) = \Theta_{\langle c\rangle_O}$. Then there exists $f \in \langle c\rangle_O$ such that $c \wedge (1 \to c) \leq f$ and $x \wedge f = y \wedge f$. We have
$1 \wedge c \wedge 1 \wedge (1 \leftrightarrow c) = f \wedge c \wedge 1 \wedge (1 \leftrightarrow c)$,
$(1 \vee c \vee 1) \wedge (1 \leftrightarrow c) = (f \vee c \vee 1) \wedge (1 \leftrightarrow c)$,
$1 \leftrightarrow c \leq x \leftrightarrow f$.
Therefore $(1, f) \in \Phi(1, c)$ by Lemma 4.11. Thus $f \in [1]_{\Phi(1,c)}$. Hence $(x, y) \in \Theta_{[1]_{\Phi(1,c)}}$. □

Theorem 4.14. *Let $\mathcal{H}$ be an SDWH-algebra. Then $\mathcal{H}$ is weakly regular if and only if $\Phi(a, b) = \Theta_{[1]_{\Phi(a,b)}}$, for all $a, b \in H$.*

Proof: Suppose that $\mathcal{H}$ is weakly regular and $a, b$ are two arbitrary elements of $H$. We have $\Phi(a, b) \in Con(\mathcal{H})$, so $F = [1]_{\Phi(a,b)} \in OF(\mathcal{H})$ by Proposition 4.8. Also, $\Theta_F \in Con(\mathcal{H})$ such that $[1]_{\Theta_F} = F$ by Proposition 4.2. Since $\mathcal{H}$ is weakly regular, then $\Theta_F = \Phi(a, b)$. Hence $\Phi(a, b) = \Theta_{[1]_{\Phi(a,b)}}$.

Conversely, let $\theta_1, \theta_2 \in Con(\mathcal{H})$ such that $[1]_{\theta_1} = [1]_{\theta_2}$. Suppose that $(x, y) \in \theta_1$, then $\Phi(x, y) \subseteq \theta_1$. We obtain $[1]_{\Phi(x,y)} \subseteq [1]_{\theta_1} = [1]_{\theta_2}$. It is easy to show that $\Theta_{[1]_{\Phi(x,y)}} \subseteq \Theta_{[1]_{\theta_2}}$. Using assumption and Proposition 4.8 part (2), we obtain $\Phi(x, y) \subseteq \theta_2$. Thus $(x, y) \in \theta_2$. Hence $\theta_1 \subseteq \theta_2$. Similarly, we can prove that $\theta_2 \subseteq \theta_1$. Therefore $\theta_1 = \theta_2$. □

Corollary 4.15. Let $\mathcal{H}$ be an SDWH-algebra. If for all $a, b \in H$, there exist $c \in H$ such that $\Phi(a, b) = \Phi(1, c)$, then $\mathcal{H}$ is weakly regular.

Proof: It follows from Theorem 4.14 and Proposition 4.13. □

Proposition 4.16. Let $\mathcal{H}$ be an SDWH-algebra such that $H = \{0, a, b, 1\}$, $0 < a, b < 1$, $a, b$ are not comparable. Then $\mathcal{H}$ is weakly regular.

Proof: We will show that $\Phi(a, 0) = \Phi(b, 1)$, $\Phi(b, 0) = \Phi(a, 1)$ and $\Phi(a, b) = \Phi(1, 0)$. We have

$a \wedge b \wedge 1 \wedge (1 \to b) = 0 \wedge b \wedge 1 \wedge (1 \to b)$,
$(a \vee b \vee 1) \wedge (1 \to b) = (0 \vee b \vee 1) \wedge (1 \to b)$,
$1 \wedge a \wedge 0 \wedge (a \to 0) = b \wedge a \wedge 0 \wedge (a \to 0)$,
$(1 \vee a \vee 0) \wedge (a \to b) = (1 \vee a \vee 0) \wedge (a \to 0)$.

Also, we have $a \to 0 = 1 \to b$ by (W5). Then $a \leftrightarrow 0 = 1 \leftrightarrow b$. So $\Phi(a, 0) = \Phi(b, 1)$. Similarly, we can prove $\Phi(b, 0) = \Phi(a, 1)$. We have

$a \wedge 1 \wedge 0 \wedge (1 \to 0) = b \wedge 1 \wedge 0 \wedge (1 \to 0)$,
$(a \vee 1 \vee 0) \wedge (1 \to 0) = (b \vee 1 \vee 0) \wedge (1 \to 0)$,
$1 \wedge a \wedge b \wedge (a \leftrightarrow b) = 0 \wedge a \wedge b \wedge (a \leftrightarrow b)$,
$(1 \vee a \vee b) \wedge (a \leftrightarrow b) = (0 \vee a \vee b) \wedge (a \leftrightarrow b)$.

By (W5), we have $a \to b = 1 \to b$ and $b \to a = 1 \to a$. Then

$a \leftrightarrow b = (1 \to b) \wedge (1 \to a) = 1 \to (a \wedge b) = 1 \leftrightarrow 0$.

So $\Phi(a, b) = \Phi(1, 0)$. Hence $\mathcal{H}$ is weakly regular by Corollary 4.15. □

## 5. Conclusions and future works

In this paper, we have studied SDWH-algebras in the context of Birkhoff's Problem 73, that is we have studied whether or not SDHW-algebras are weakly regular. To do this, we have considered open filters and deductive systems in SDWH-algebras to show that in general, they are not weakly regular, and give necessary and sufficient conditions for an SDWH-algebra to be weakly regular by using principal congruence relations.

In the future, we will introduce and study a corresponding logic to SDWH-algebras and investigate some basic properties of this logic. But here are some open questions still about SDW-algebras to be studied. Is there any representation theorem for SDWH-algebras? Is the class of weakly regular SDWH-algebras a variety? With positive answer to this, we should know the relation between this proper subvariety of SDWH-algebras and the other subvarieties of WH- algebras such as the variety the varieties of RWH, TWH, SRL and B. It would be interesting to find a characterization of the WH-spaces that correspond to the algebras in the subvariety of SDWH-algebras.

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# D-COMPLETE SINGLE AXIOMS FOR THE EQUIVALENTIAL CALCULUS WITH THE RULES $D$ AND $R$

**Abstract**

Ulrich showed that most of the known axiomatisations of the classical equivalence calculus (EC) are D-incomplete, that is, they are not complete with the condensed detachment rule (D) as the primary rule of the proof procedure. He proved that the axiomatisation $EEpEqrErEqp, EEEpppp$ by Wajsberg is D-complete and pointed out a number of D-complete single axioms, including one organic single axiom. In this paper we present new single axioms for EC with the condensed detachment and the reversed condensed detachment rules that form D-complete bases and are organic.

*Keywords*: equivalential calculus, D-complete, single axiom, condensed detachment.

*2020 Mathematical Subject Classification*: 03B05, 03B20.

## 1. Introduction

The main goal of the paper is to present new single axioms for the equivalential calculus (EC) with two rules of the proof procedure: the condensed detachment (D) and the reversed condensed detachement (R). The axioms form with the rules many different D-complete bases for EC. The first part of the article introduces EC and the basic concepts used in the paper. Then the issue of single inorganic axioms for a certain variant of EC calculus is discussed. In the third part of the paper, 8 new organic axioms for EC, unknown so far, are pointed out.

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## 2. Equivalential calculus

The well-formed formulas (wff) of the classical equivalential calculus are the formulas built from a binary connective $E$ and denumerably many sentence letters $p, q, r, \ldots$. Each sentence letter is a wff. If $\alpha$ and $\beta$ are wff, so is $E\alpha\beta$.

The classical equivalential calculus (EC) is the set of all formulas that are tautologies of the standard matrix for the equivalence (E) from the classical propositional calculus. The set is identical to the set of such formulas in which each sentence letter occurs an even number of times Leśniewski [5] was the first to point this out.

In the early days of the study, EC axiomatisations were sets of formulas and two standard rules of the proof procedure: modus ponens for equivalence and substitution. The first axiomatisation was proposed by Leśniewski [5]. The first single shortest axiom was found by Łukasiewicz [12]. Currently, many different axiomatisations of EC are known, and research on EC is focused on the area of finding the single shortest axiom depending on an established set of rules of the proof procedure.

Instead of the modus ponens for equivalence and substitution, the condensed detachment ($D$) rule was introduced, which combines detachment with the best possible substitution. A detailed presentation of the rule $D$ may be found in, for example, [1, 2, 4, 9]. Suppose that $s(\beta)$ is some substitution of formula $\beta$. The rule $D$ allows one to write $s(\beta)$ in the proof if you have formula $E\alpha\beta$ and formula $\gamma$ for which the formulas $s(\alpha)$ and $s(\gamma)$ are identical. Moreover, the substitution $s$ needs to satisfy the condition that it is always a most general unifier (cf. [8]) for the formulas $\alpha$ and $\gamma$ and the formula $s(\beta)$ has the smallest possible number of common sentence letters with $E\alpha\beta$. In short, the result of applying the rule $D$ to the formulas $E\alpha\beta$ and $\gamma$ should be such a formula $s(\beta)$ from which, using only the substitution rule, one can obtain the set of all possible formulas that can be obtained from these formulas using the modus ponens rule for equivalence and the substitution rule. The formula $s(\beta)$ is therefore the most general formula possible.

By analogy, one can define the reversed condensed detachment ($R$). A detailed presentation of the rule $R$ may be found in, for example, [6, 3]. The difference between these rules is that the rule $D$ allows you to detach

$s(\alpha)$ from the formula $s(E\alpha\beta)$ and the result is $s(\beta)$. In contrast, the rule $R$ allows you to detach $s(\beta)$ from the formula $s(E\alpha\beta)$ and the result is $s(\alpha)$. The other conditions are exactly the same as for the rule $D$.

EC built solely with the rule $D$ we will denote as EC+D. EC with the rule $R$ we will denote as EC+R. We will abbreviate EC with the two rules $D$ and $R$ as EC+DR.

We will say of EC that it is complete if and only if it contains all expressions satisfying the standard matrix for the equivalence connective. So EC is complete if and only if it contains all such formulas in which each sentence letter occurs an even number of times. We will say that a theory is D-complete, if an axiomatisation based on the rule $D$ or the rule $R$, as the only rules allowed in the proof, forms a complete theory. We say that a calculus is D-incomplete, if it is based on the rule $D$ or the rule $R$ and there exists at least one formula satisfying the standard matrix for the equivalence functor that cannot be proved in the theory.

The converse formula is a formula in which every subformula of the form $E\alpha\beta$ is replaced with $E\beta\alpha$. E.g. the converse of $Epq$ is $Eqp$, the converse of $EEpqr$ is $ErEqp$. A formula is an organic formula if and only if no proper subformula of this formula is a theorem. Otherwise, we say that the formula is inorganic. E.g. the formula $EEpqEqp$ is organic, but the formula $EEppq$ is inorganic, since its frament $Epp$ is a theorem of EC. The set of all theorems of EC is identical to the set of such formulas in which each sentence letter occurs an even number of times. We say that a formula is two-property if and only if each sentence letter occuring occurs two times in the formula. So every formula of EC can be derived from some two-property formula by the substitution rule. It is interesting to note that if a formula is two-property, then using the rule $D$ or the rule $R$, only formulas with the same property can be derived from it.

We currently know fourteen single shortest (11-character long) axioms for EC+D, the last one found in 2003 [11]. In addition, fourteen corresponding converse formulas [7], which are axioms for EC+R. Furthermore, eleven (11-character long) axioms are known for EC+DR [3]. All these 39 formulas are single shortest axioms for EC+DR and are D-incomplete bases for EC+DR. We use the names of these axioms as in [3].

Ulrich [9] has shown that the axioms $\{EEpEqrErEqp, EEEpppp\}$ [10] form a D-complete base for EC+D. It is easy to show that the converse axioms constitute a D-complete base for EC+R.

Theorem 2.1. *Formulas*

$$EEEpqrEErqp, \tag{2.1}$$

$$EpEpEpp \tag{2.2}$$

*constitute a D-complete base for EC+R.*

Proof: These formulas are converse axioms of Wajsberg's axioms $\{EEpEqrErEqp, EEEpppp\}$. Since Wajsberg's axioms are D-complete with the rule $D$, their converses are D-complete with the rule $R$. □

The calculus EC+DR was investigated by Hodgson [3]. Each of the known classical single shortest axioms for this calculus is D-incomplete. In the following section, we will show the axiomatizations that are D-complete.

## 3. Inorganic single axioms for EC+DR

We will point out some general facts about the EC+DR calculus and discuss the single inorganic axioms for this calculus.

Lemma 3.1. *If the axiomatisation is a D-complete base for EC+D or EC+R, it is a D-complete base for EC+DR.*

Proof: The proof is immediate, it is sufficient to note that EC+DR is formed by adding one of the rules of the proof procedure ($D$ or $R$). Thus, EC+DR is formed from EC+D or EC+R by expanding the set of original rules of the proof procedure by the rule $D$ or the rule $R$, respectively. Monotonicity ensures that if EC+D or EC+R is D-complete, then EC+DR is also D-complete. □

Ulrich [9] proved that any formula of the scheme $EsEsEsEsA$, where $A$ is any single D-incomplete axiom for EC+D and does not contain the variable $s$, is a D-complete base for EC+D.

Theorem 3.2. *Let $A$ be any single D-incomplete axiom for EC+R, such that $s$ does not occur in $A$. Then $EEEEAssss$ is a D-complete base for EC+R.*

PROOF: We conduct a 4-fold detachment using the rule $R$, which results in a single axiom $A$. From this axiom we derive the expression $EEEpqrEErqp$ (2.1), which is a converse of Wajsberg's first axiom, on the same basis we derive the expression $EEzEyExwEEEEAzyxw$. These derivations are possible because the formulas are two-property. From the second formula we detach $EEEEAssss$ using the rule $R$, as a result we get $EpEpEpp$ (2.2), which is the converse of Wajsberg's second axiom. □

The proof is analogous to the one in [9]. The axioms with the schemes $EsEsEsEsA$ and $EEEEAssss$ are each other's converses.

THEOREM 3.3. *Let $A$ be any single D-incomplete axiom for EC+DR, such that $s$ does not occur in $A$. Then $EsEsEsEsA$ and $EEEEAssss$ is a D-complete base for EC+DR.*

PROOF: For all single axioms $A$ D-incomplete for EC+D and EC+R the theorem is true by Lemma 3.1. For the single D-incomplete axioms $A$ of EC+DR, it can be shown that by 4-fold detachment via the rule $D$ or the rule $R$, one can always derive $A$ from $EsEsEsEsA$ or $EEEEAssss$. Since $A$ is a single axiom of EC+DR, it is possible to derive (2.1), $EEzEyExwEEEEAzyxw$ and $EEEEEEEAzyxwEzEyExw$ from it, and by the latter two axiom (2.2) can be derived from the corresponding axiom $A$ by means of an appropriate rule. □

Since we know 39 single 11-character D-incomplete axioms for EC+DR, by virtue of Theorem 3.3 above, 78 single axioms of EC+DR can be identified that constitute D-complete bases.

THEOREM 3.4. *Let $A$ be any single D-incomplete axiom for EC+DR, such that $s$ does not occur in $A$. Then formulas:*

$$EsEEEAsss \tag{3.1}$$

$$EsEsEEAss \tag{3.2}$$

$$EsEsEsEAs \tag{3.3}$$

*are D-complete bases for EC+DR, each separately as a single axiom.*

Proof: As there are two rules available to us, $D$ or $R$, we can apply them as required. By fourfold detachment we always obtain the axiom $A$. Then from $A$ and the given axiom it will always be possible to derive Wajsbegr's axioms (2.1), (2.2), analogous to the proof of Theorem 3.3. □

As a result, we have 117 new axioms. In total, we can generate 195 axioms with these techniques. All these axioms are inorganic. All these axioms are 19 characters long. Whether there is a shorter-than-19-character single D-complete inorganic axiom for EC+D, EC+R, EC+DR remains an open question.

## 4. Organic single axioms EC+DR

We discuss some organic D-complete axioms for EC+DR calculus. The axioms (4.2), (4.3), (4.4), (4.5), (4.7), (4.8), (4.9), (4.10) were previously unknown.

Ulrich [9] has shown that the formula

$$EEpqEEqrEsEsEsEsEpr \tag{4.1}$$

is a D-complete base for EC+D. The converse of this formula,

$$EEEEEErpssssErqEqp, \tag{4.2}$$

constitutes the D-complete base for EC+R. Both of these expressions are organic and constitute, by virtue of Lemma 3.1, each separately, D-complete bases for EC+DR.

In the proofs of Theorems 4.3 and 4.4 we use the standard notation for the rules $D$ or the rule $R$. E.g. the description D1.2 means that the rule $D$ was applied to line 1 and line 2, which in this case were the minor and major premises for the rule $D$. The description D1.1 means that rule D was applied to line 1, which in this case was the minor and major premises for the rule $D$. Similarly, the description DD1.1.1 means the application of the rule $D$ to line 1 and to a certain formula D1.1, which is formed from the application of the rule $D$ to line 1.

THEOREM 4.1. *Formula*

$$EEEpqrEsEsEsEsEErpq \tag{4.3}$$

*is a single organic axiom of EC+DR, which forms a D-complete base.*

PROOF:

$$\begin{aligned}
& 1.\ EEEpqrEsEsEsEsEErpq \\
D1.1 = {} & 2.\ EtEtEtEtEEEsEsEsEsEErpqEpqr \\
DDDD2.1.1.1.1 = {} & 3.\ EEEsEsEsEsEErpqEpqr \\
R3.1 = {} & 4.\ EEwEwEwEwEEEEEpqrEsEsEsEsEErpqtuEtu \\
R4.1 = {} & 5.\ EwEwEwEwEEEEEtuvExExExExEEvtuEEpqr \\
& \quad EsEsEsEsEErpq \\
DDDD5.1.1.1.1 = {} & 6.\ EEEEEtuvExExExExEEvtuEEpqr \\
& \quad EsEsEsEsEErpq \\
R6.1 = {} & 7.\ EEEEstuExExExExEEustEErEErpqEpq \\
D7.1 = {} & 8.\ EErEErpqEpq \\
D1.8 = {} & 9.\ EsEsEsEsEEEpqrEErpq \\
DDDD9.1.1.1.1 = {} & 10.\ EEEpqrEErpq
\end{aligned}$$

Formula $EEEpqrEErpq$ (TN) is a single D-incomplete axiom for EC+R, so any two-property formula can be derived from it, including these two;

11. $EEpEqrErEqp$
12. $EEwExEyEzEEEpqrEErpqEEEwxyz$

Formula 11. is one of the Wajsberg D-complete axioms for EC+D. The second axiom can be derived in one step.

$$D12.9 = 13.\ EEEpppp. \qquad \square$$

THEOREM 4.2. *Formula*

$$EEEEEEqEprssssErEqp \tag{4.4}$$

*is a single organic axiom for EC+DR, which forms a D-complete base.*

Proof:

$$
\begin{aligned}
&1.\ EEEEEEqEprssssErEqp\\
R1.1 = {}&2.\ EEEEErEEqpEEEEEqEprsssstttt\\
RRRR2.1.1.1.1 = {}&3.\ ErEEqpEEEEEqEprssss\\
D3.1 = {}&4.\ EEutEEEEEuEtEEEEEEqEprssss\\
&\phantom{4.\ }ErEqpvvvv\\
D4.1 = {}&5.\ EEEEEEEEEEqEprssssEErEqp\\
&\phantom{5.\ }EEEEEEuEtvwwwwEvEutxxxx\\
RRRR5.1.1.1.1 = {}&6.\ EEEEEEqEprssssEErEqp\\
&\phantom{6.\ }EEEEEEuEtvwwwwEvEut\\
D6.1 = {}&7.\ EEEqpEEqEprrEEEEEEtEsuwwwwEuEts\\
R7.1 = {}&8.\ EEtsEEtEsuu\\
R1.8 = {}&9.\ EEEEEEqEprErEqpssss\\
RRRR9.1.1.1.1 = {}&10.\ EEqEprErEqp
\end{aligned}
$$

Formula $EEpEqrErEpq$ (WN) is a single D-incomplete axiom for EC+D, so any two-property formula can be derived from it, including these two:

$$
\begin{aligned}
&11.\ EEEpqrEErqp\\
&12.\ EEzEyExwEEEEEEqEprErEqpzyxw
\end{aligned}
$$

Formula in the row 11. is the converse of Wajsberg D-complete axiom (2.1) for EC+R. The converse (2.2) of the second axiom can be derived in one step.

$$D12.9 = 13.\ EpEpEpp. \qquad \square$$

Theorem 4.3. *Formula*

$$EEEpqrEsEsEsEsEpEqr \tag{4.5}$$

*is a single organic axiom for EC+DR, which forms a D-complete base.*

Proof:

$$1.\ EEEpqrEsEsEsEsEpEqr$$
$$R.1.1 = 2.\ EEEEpqrpEqr$$

Formula $EEEEpqrpEqr$ (OYJ) is a single D-complete axiom of EC+DR. Two formulas can be derived from it:

$$3.\ EEEpqrEErqp$$
$$4.\ EEEEpqrExEyEzEwEpEqrExEyEzw$$

Formula in the row 3. is the converse of Wajsberg's D-complete axiom (2.1). The second axiom (2.2) can be derived in one step.

$$D4.1 = 5.\ EpEpEpp.$$ □

Formula
$$EEEEEEEpqrssssEpEqr, \tag{4.6}$$
the reverse of (4.5), is a single D-complete axiom of EC+DR as well.

Applying an analogous proof technique to Theorem 4.3, it can be proved that the following formulas are single D-complete axioms for EC+DR:

$$EEpEqrEsEsEsEsErEpq, \tag{4.7}$$
$$EEEpqrEsEsEsEsEqErp \tag{4.8}$$

Axiom 4.7 allows the derivation of the axiom $EpEEqrEqErp$ (XIM), which is a single D-incomplete axiom of EC+DR. Axiom 4.8 allows for a derivation of the axiom $EEEpqEEqrpr$ (HXH), which is a single D-incomplete axiom of EC+R.

The corresponding reverses of these formulas are single D-complete axioms for EC+DR. The reverse of (4.7), a formula

$$EEEEEEEpqrssssEErpq, \tag{4.9}$$

allows for a derivation of the axiom $EEEEpqrEqrp$ (DXN), which is a single D-incomplete axiom of EC+DR. The reverse of (4.8), a formula

$$EEEEEEEpqrssssEqErp, \tag{4.10}$$

allows for a derivation of the axiom $EpEEqEprErq$ (XGF), which is a single D-incomplete axiom EC+D.

Axioms (4.2), (4.3), (4.4), (4.5) (4.7), (4.8), (4.9), (4.10) are organic. All these axioms are 19 characters long. Whether there is a shorter-than-19-character single D-complete axiom for EC+D, EC+R, EC+DR remains an open question.

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Natnael Teshale Amare

# $\mathcal{L}$-WEAKLY 1-ABSORBING PRIME IDEALS AND FILTERS

**Abstract**

In this manuscript, we have presented the concept of $\mathcal{L}$-weakly 1-absorbing prime ideals and $\mathcal{L}$-weakly 1-absorbing prime filters within an ADL. Mainly, we illustrate the connections between $\mathcal{L}$-weakly prime ideals (filters) and $\mathcal{L}$-weakly 1-absorbing prime ideals (filters), as well as between $\mathcal{L}$-weakly 1-absorbing prime ideals (filters) and $\mathcal{L}$-weakly 2-absorbing ideals (filters). Lastly, we have shown that both the image and inverse image of $\mathcal{L}$-weakly 1-absorbing prime ideals (filters) result in $\mathcal{L}$-weakly 1-absorbing prime ideals (filters).

*Keywords*: Weakly 1-absorbing prime ideal, weakly 1-absorbing prime filter, $\mathcal{L}$-weakly 2-absorbing ideal, $\mathcal{L}$-weakly 1-absorbing prime ideal, $\mathcal{L}$-weakly 2-absorbing filter, $\mathcal{L}$-weakly 1-absorbing prime filter.

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## 1. Introduction

The idea of prime ideals(filters) is vital in the study of structure theory of distributive lattices in general and in particular, that of Boolean algebras. Badawi [7] introduced the concept of 2-absorbing ideals in commutative rings, extending the idea of prime ideals from [11]. Chuadhari [9] further extended 2-absorbing ideals to semi-rings. Badawi and Darani [8] introduced weakly 2-absorbing ideals in commutative rings, a generalization of weakly prime ideals by Anderson and Smith [6]. Wasakidar and Gaikerad [24] extended the concepts of 2-absorbing and weakly 2-absorbing ideals to

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lattices. Natnael TA [23, 2, 1] introduced weakly 2-absorbing ideals and weakly 2-absorbing filters, along with 1-absorbing prime filters in an ADL. L.A. Zadeh [25] defined a fuzzy subset of a set $X$ as a function mapping elements to real numbers in $[0, 1]$. Goguen [12] expanded this concept by using a complete lattice $\mathcal{L}$ instead of the valuation set $[0, 1]$, aiming for a more comprehensive exploration of fuzzy set theory through fuzzy sets. Darani and Ghasemi [10], as well as Mandal [14], introduced fuzzy 2-absorbing ideals and 2-absorbing fuzzy ideals for commutative rings, respectively, generalizing the concept of fuzzy prime ideals in rings explored by June [13] and Sharma [18]. Nimbhorkar and Patil [15, 16] introduced fuzzy weakly 2-absorbing ideals in lattices. In our previous work [20, 21], we introduced the concepts of fuzzy ideals and filters within an ADL, serving as the basis for our research. Natnael [5, 2] later expanded on this by introducing the concept of fuzzy 2-absorbing ideals and filters in an ADL.

In this paper, we have introduced the concept of $\mathcal{L}$-weakly $1A$-prime ideals and filters in an ADL, aiming to extend the idea of $\mathcal{L}$-prime ideals and filters in an ADL as presented in [17, 19]. Initially, we define $\mathcal{L}$-weakly $1A$-prime ideals, which are less stringent than $\mathcal{L}$-prime ideals. Also, we study on $\mathcal{L}$-weakly $1A$-prime filters in an ADL which is weaker than that $\mathcal{L}$-prime filters. Our main emphasis is on investigating the connections between $\mathcal{L}$-prime ideals and $\mathcal{L}$-weakly $1A$-prime ideals, as well as the relationships between $\mathcal{L}$-weakly $1A$-prime ideals and $\mathcal{L}$-$2A$-ideals. Also, we investigating the connections between $\mathcal{L}$-prime filters and $\mathcal{L}$-weakly $1A$-prime filters, and $\mathcal{L}$-weakly $1A$-prime filters and $\mathcal{L}$-$2A$-filters. Counter examples are provided to demonstrate that the converses of these relationships do not hold. Furthermore, we demonstrate that the direct product of any two $\mathcal{L}$-prime ideals ($\mathcal{L}$-prime filters) results in an $\mathcal{L}$-weakly $1A$-prime ideal ($\mathcal{L}$-weakly $1A$-prime filter) in an ADL. However, it is important to note that the product of $\mathcal{L}$-weakly $1A$-prime ideals ($\mathcal{L}$-weakly $1A$-prime filters) may not necessarily yield an $\mathcal{L}$-weakly $1A$-prime ideal ($\mathcal{L}$-weakly $1A$-prime filter) in an ADL. Additionally, we establish that both the image and pre-image of any $\mathcal{L}$-weakly $1A$-prime ideals ($\mathcal{L}$-weakly $1A$-prime filters) are again $\mathcal{L}$-weakly $1A$-prime ideals ($\mathcal{L}$-weakly $1A$-prime filters).

## 2. Preliminaries

In this portion, we revisit certain definitions and fundamental findings primarily sourced from [20, 17, 22].

DEFINITION 2.1. An algebra $R = (R, \wedge, \vee, 0)$ of type $(2, 2, 0)$ is referred to as an ADL if it meets the subsequent conditions for all $r, s$ and $t$ in $R$.

1. $0 \wedge r = 0$
2. $r \vee 0 = r$
3. $r \wedge (s \vee t) = (r \wedge s) \vee (r \wedge t)$
4. $r \vee (s \wedge t) = (r \vee s) \wedge (r \vee t)$
5. $(r \vee s) \wedge t = (r \wedge t) \vee (s \wedge t)$
6. $(r \vee s) \wedge s = s$.

Every distributive lattice with a lower bound is categorized as an ADL.

*Example* 2.2. For any nonempty set $A$, it's possible to transform it into an ADL that doesn't constitute a lattice by selecting any element 0 from $A$ and fixing an arbitrary element $u_0 \in R$. For every $u, v \in R$, define $\wedge$ and $\vee$ on $R$ as follows:

$$u \wedge v = \begin{cases} v & \text{if } u \neq u_0 \\ u_0 & \text{if } u = u_0 \end{cases} \qquad \text{and} \qquad u \vee v = \begin{cases} u & \text{if } u \neq u_0 \\ v & \text{if } u = u_0 \end{cases}$$

Then $(A, \wedge, \vee, u_0)$ is an ADL (called the **discrete ADL**) with $u_0$ as its zero element.

DEFINITION 2.3. Consider $R = (R, \wedge, \vee, 0)$ be an ADL. For any $r$ and $s \in R$, establish $r \leq s$ if $r = r \wedge s$ (which is equivalent to $r \vee s = s$). Then $\leq$ is a partial order on $R$ with respect to which 0 is the smallest element in $R$.

THEOREM 2.4. *The following conditions are valid for any $r, s$ and $t$ in an ADL $R$.*

(1) $r \wedge 0 = 0 = 0 \wedge r$ *and* $r \vee 0 = r = 0 \vee r$

(2) $r \wedge r = r = r \vee r$

(3) $r \wedge s \leq s \leq s \vee r$

(4) $r \wedge s = r$ *iff* $r \vee s = s$

(5) $r \wedge s = s$ *iff* $r \vee s = r$

(6) $(r \wedge s) \wedge t = r \wedge (s \wedge t)$ *(in other words, $\wedge$ is associative)*

(7) $r \vee (s \vee r) = r \vee s$

(8) $r \leq s \Rightarrow r \wedge s = r = s \wedge r \ \big(\ \textit{iff}\ r \vee s = s = s \vee r\big)$

(9) $(r \wedge s) \wedge t = (s \wedge r) \wedge t$

(10) $(r \vee s) \wedge t = (s \vee r) \wedge t$

(11) $r \wedge s = s \wedge r$ *iff* $r \vee s = s \vee r$

(12) $r \wedge s = \inf\{r, s\}$ *iff* $r \wedge s = s \wedge r$ *iff* $r \vee s = \sup\{r, s\}$.

Definition 2.5. Let $R$ and $G$ be ADLs and form the set $R \times G = \{(r, g) : r \in R \text{ and } g \in G\}$. For all $(r_1, g_1), (r_2, g_2) \in R \times G$, define $\wedge$ and $\vee$ in $R \times G$ by $(r_1, g_1) \wedge (r_2, g_2) = (r_1 \wedge r_2, g_1 \wedge g_2)$ and $(r_1, g_1) \vee (r_2, g_2) = (r_1 \vee r_2, g_1 \vee g_2)$. Then $(R \times G, \wedge, \vee, 0)$ is an ADL under the pointwise operations and $0 = (0, 0)$ is the zero element in $R \times G$.

Definition 2.6. A non-empty subset, denoted as $F$ in an ADL $R$ is termed an ideal (filter) in $R$ if it satisfies the conditions: if $u$ and $v$ belong to $F$, then $u \vee v$ $(u \wedge v)$ is also in $F$, and for every element $r$ in $R$, the $u \wedge r$ $(r \vee u)$ is in $F$.

Definition 2.7. A proper ideal(filter) $F$ in $R$ is a prime ideal (filter) if for any $u$ and $v$ belongs $R$, $u \wedge v$ $(u \vee v)$ belongs $F$, then either $u$ belongs $F$ or $v$ belongs $F$.

Definition 2.8. Let $R$ and $G$ be ADLs. A mapping $k : R \to G$ is called a homomorphism if the following are satisfied, for any $r, s, t \in R$.

(1) $k(r \wedge s \wedge t) = k(r) \wedge k(s) \wedge k(t)$

(2) $k(r \vee s \vee t) = k(r) \vee k(s) \vee k(t)$

(3) $k(0) = 0$.

Definition 2.9. An $\mathcal{L}$-subset $\Phi^w$ is defined as a mapping from $R$ to a complete lattice $L$ that adheres to the infinite meet distributive law. When the lattice $L$ is represented by the unit interval $[0,1]$ of real numbers, these $\mathcal{L}$-subsets correspond to the conventional notion of $\mathcal{L}$-subsets in $R$.

Definition 2.10. An $\mathcal{L}$-subset $\Phi^w$ is an $\mathcal{L}$-ideal(filter) in $R$, if $\Phi^w(0) = 1\big(\Phi^w(u) = 1$, for any maximal element $u$ in $R\big)$ and $\Phi^w(r \vee s) = \Phi^w(r) \wedge \Phi^w(s)$ $\big(\Phi^w(r \wedge s) = \Phi^w(r) \wedge \Phi^w(s)\big)$, for all $r$ and $s$ belongs to $R$.

Theorem 2.11. *Let $\Phi^w$ be an $\mathcal{L}$-ideal and $\emptyset \neq F \subseteq R$. Then for any $r$ and $s$ belongs to $R$, we have the following:*

1. *If $r \leq s$, then $\Phi^w(s) \leq \Phi^w(r)$*
2. *If $r$ is an associate with $s$, then $\Phi^w(r) = \Phi^w(s)$*
3. *$\Phi^w(r \wedge s) = \Phi^w(s \wedge r)$ and $\Phi^w(r \vee s) = \Phi^w(s \vee r)$*
4. *If $r \in \langle F]$, then $\bigwedge_{i=1}^{n} \Phi^w(x_i) \leq \Phi^w(r)$, for some $x_1, x_2, ..., x_n \in F$*
5. *If $r \in \langle s]$, then $\Phi^w(s) \leq \Phi^w(r)$*
6. *If $u$ is maximal in $R$, then $\Phi^w(u) \leq \Phi^w(r)$*
7. *$\Phi^w(u) = \Phi^w(v)$, for any maximal elements $u$ and $v$ in $R$.*

Theorem 2.12. *Let $\Phi^w$ be an $\mathcal{L}$-filter and $\emptyset \neq F \subseteq R$. Then for any $r, s \in R$, we have the following.*

1. *If $r \leq s$, then $\Phi^w(r) \leq \Phi^w(s)$*
2. *If $r \sim s$, then $\Phi^w(r) = \Phi^w(s)$*
3. *$\Phi^w(r \vee s) = \Phi^w(s \vee r)$*
4. *If $r \in [F\rangle$, then $\bigwedge_{i=1}^{n} \Phi^w(x_i) \leq \Phi^w(r)$, for some $x_1, x_2, ..., x_n \in F$*
5. *If $r \in [s\rangle$, then $\Phi^w(s) \leq \Phi^w(r)$.*

Definition 2.13. A proper $\mathcal{L}$-ideal(filter) $\Phi^w$ is referred to as a prime $\mathcal{L}$-ideal(filter) if $\psi \wedge \eta \leq \Phi^w$ implies either $\psi \leq \Phi^w$ or $\eta \leq \Phi^w$, for any $\mathcal{L}$-ideals(filters) $\psi$ and $\eta$ in $R$.

Definition 2.14. A proper $\mathcal{L}$-ideal(filter) $\Phi^w$ is an $\mathcal{L}$-prime ideal(filter) in $R$ if $\Phi^w(r \wedge s)\Big(\Phi^w(r \vee s)\Big)$ equals either $\Phi^w(r)$ or $\Phi^w(s)$, for any $r$ and $s$ in $R$.

## 3. $\mathcal{L}$-weakly 1$A$-prime ideals

In the subsequent discussion, we present the concepts of $\mathcal{L}$-weakly 1−absorbing prime ideals in an ADL $R$ and their characterizations. Initially, let us revisit the definition outlined in [23], indicating that a proper ideal $H$ in $R$ is a weakly 1−absorbing prime ideal (in short, a weakly 1$A$-prime ideal) in $R$ if, for all elements $r, s$, and $t$ in $R$ such that $r \wedge s \wedge t \neq 0$, the condition $r \wedge s \wedge t$ belonging to $H$ implies either $r \wedge s$ belonging to $H$ or $t$ belonging to $H$. Now, we aim to extend this outcome to the realm of $\mathcal{L}$-weakly 1$A$-prime ideals as elucidated below.

Definition 3.1. A proper $\mathcal{L}$-ideal $\Phi^w$ in $R$ is referred to as an $\mathcal{L}$-weakly 1$A$-prime ideal in $R$ if for any elements $r$,$s$ and $t$ belongs to $R$ such that $r \wedge s \wedge t \neq 0$, the inequality $\Phi^w(r \wedge s \wedge t) \leq \Phi^w(r \wedge s) \vee \Phi^w(t)$ remains valid.

*Example* 3.2. Let $R = \{0, r, s, t\}$ and the chain $L$ consisting of four elements $\{0, \gamma, \beta, 1\}$, where $0 < \gamma < \beta < 1$ and let $\vee$ and $\wedge$ be binary operations on $R$ defined by:

| $\vee$ | 0 | r | s | t |
|---|---|---|---|---|
| 0 | 0 | r | s | t |
| $r$ | r | r | r | r |
| $s$ | s | s | s | s |
| $t$ | t | r | s | t |

| $\wedge$ | 0 | r | s | t |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| $r$ | 0 | r | s | t |
| $s$ | 0 | r | s | t |
| $t$ | 0 | t | t | t |

Define an $\mathcal{L}$-subset $\Phi^w$ in $R$ as follows: $\Phi^w(0) = 1$, $\Phi^w(r) = \gamma = \Phi^w(s)$ and $\Phi^w(t) = \beta$. It is evident that $\Phi^w$ is an $\mathcal{L}$-ideal in $R$. Furthermore, for any elements $r, s$ and $t \in R$ such that $r \wedge s \wedge t = t \neq 0$, we observe that $\Phi^w(r \wedge s \wedge t) = \beta = \gamma \vee \beta = \Phi^w(r \wedge s) \vee \Phi^w(t)$. Consequently, $\Phi^w$ qualifies as an $\mathcal{L}$-weakly 1$A$-prime ideal in $R$.

Following that, we define the concept of an $\mathcal{L}$-weakly 1$A$-prime ideal with respect to $\beta$-cut, where $\Phi^w_\beta = \{r \in R : \beta \leq \Phi^w(r)\}$.

Theorem 3.3. *Let $\Phi^w$ be an $\mathcal{L}$-ideal in $R$. Then an ideal $\Phi^w_\beta$ is a weakly 1$A$-prime ideal in $R$, for all $\beta \in L$ iff $\Phi^w$ is an $\mathcal{L}$-weakly 1$A$-prime ideal in $R$.*

Proof: Assume $\Phi^w_\beta$ is a weakly 1$A$-prime ideal, for all $\beta \in L$. In this case, for any elements $r, s, t \in R$ such that $r \wedge s \wedge t \neq 0$, it is ensured that either $r \wedge s \in \Phi^w_{\Phi^w(r \wedge s \wedge t)}$ or $t \in \Phi^w_{\Phi^w(r \wedge s \wedge t)}$, leading to $\Phi^w(r \wedge s \wedge t) \leq \Phi^w(r \wedge s)$ or

$\Phi^w(t)$. Consequently, $\Phi^w(r \wedge s \wedge t) \leq \Phi^w(r \wedge s) \vee \Phi^w(t)$. Conversely, if $\Phi^w$ is an $\mathcal{L}$-weakly $1A$-prime ideal, consider $r, s, t \in R$ such that $r \wedge s \wedge t \in \Phi^w_\beta$, for all $\beta \in L$. This implies $\beta \leq \Phi^w(r \wedge s \wedge t)$, which further leads to $\beta \leq \Phi^w(r \wedge s) \vee \Phi^w(t)$. Consequently, either $\beta \leq \Phi^w(r \wedge s)$ or $\beta \leq \Phi^w(t)$. Hence, either $r \wedge s \in \Phi^w_\beta$ or $t \in \Phi^w_\beta$. Therefore, $\Phi^w_\beta$ is a weakly $1A$-prime ideal in $R$. □

Corollary 3.4. An ideal $P$ in $R$ is classified as a weakly $1A$-prime ideal in $R$ iff its characteristic set $\chi_P$ is an $\mathcal{L}$-weakly $1A$-prime ideal in $R$.

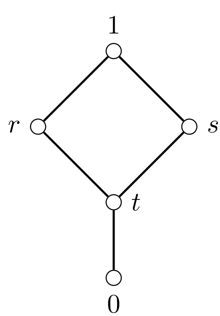
In the upcoming theorems, we establish the connections between $\mathcal{L}$-weakly $1A$-prime ideals and both $\mathcal{L}$-weakly prime ideals and $\mathcal{L}$-weakly $2A$-ideals within the context of an ADL.

Theorem 3.5. *Let $\Phi^w$ be an $\mathcal{L}$-ideal in $R$. Then $\Phi^w$ is an $\mathcal{L}$-weakly $1A$-prime ideal in $R$ only if $\Phi^w$ is an $\mathcal{L}$-weakly prime ideal in $R$.*

Proof: Assume $\Phi^w$ is an $\mathcal{L}$-weakly prime ideal in $R$. For any elements $r, s, t \in R$ such that $r \wedge s \wedge t \neq 0$, it follows that $\Phi^w(r \wedge s \wedge t) \leq \Phi^w(r) \vee \Phi^w(s \wedge t)$, or $\Phi^w(r \wedge s \wedge t) \leq \Phi^w(r \wedge s) \vee \Phi^w(t)$. This establishes the conclusion. □

In the provided example, we demonstrate that every $\mathcal{L}$-weakly $1A$-prime ideal in $R$ does not qualify as an $\mathcal{L}$-weakly prime ideals in $R$.

*Example* 3.6. Let $D = \{0, u, v\}$ be a discrete ADL with 0 as its zero element defined in 2.2 and $L = \{0, r, s, t, 1\}$ be the lattice represented by the Hasse diagram given below:




Examine the set $D \times L = \{(y, z) \mid y \in D \text{ and } z \in L\}$. Then, the structure $(D \times L, \wedge, \vee, 0)$ forms an ADL, employing pointwise operations $\wedge$ and $\vee$ on

$D \times L$, where 0 is defined as $(0,0)$. Consider $P = \{0, t\}$. It is evident that $P$ is an ideal in $L$. Now define $\Phi^w : D \times L \to [0,1]$ by

$$\Phi^w(y,z) = \begin{cases} 1 & \text{if } (y,z) = (0,0) \\ 3/4 & \text{if } y \neq 0 \textit{ and } z \in P \\ 0 & \text{otherwise} \end{cases}$$

for all $(y,z) \in D \times L$. Moreover, $\Phi^w$ is identified as an $\mathcal{L}$-ideal. Consequently, $\Phi^w$ qualifies as an $\mathcal{L}$-weakly $1A$-prime ideal, while $\Phi^w$ does not meet the criteria for an $\mathcal{L}$-weakly prime ideal in $D \times L$. This distinction arises from the fact that $\Phi^w((u,r) \wedge (v,s)) = 3/4 \nleq 0$ whereas $\Phi^w(u,r) \vee \Phi^w(v,s)$ results in 0.

Definition 3.7 ([4]). A proper $\mathcal{L}$-ideal $\Phi^w$ in $R$ is an $\mathcal{L}$-weakly $2A$-ideal in $R$ if for any elements $r$,$s$ and $t \in R$ such that $r \wedge s \wedge t \neq 0$, $\Phi^w(r \wedge s \wedge t) \leq \Phi^w(r \wedge s) \vee \Phi^w(r \wedge t) \vee \Phi^w(s \wedge t)$.

Theorem 3.8. *Let $\Phi^w$ be an $\mathcal{L}$-ideal in $R$. If $\Phi^w$ is an $\mathcal{L}$-weakly $1A$-prime ideal in $R$, then $\Phi^w$ is an $\mathcal{L}$-weakly $2A$-ideal in $R$. The converse of this result is not true.*

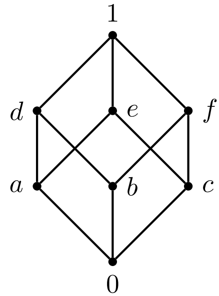
Proof: Assume $\Phi^w$ is an $\mathcal{L}$-weakly $1A$-prime ideal in $R$. Then for all $r, s, t \in R$ such that $r \wedge s \wedge t \neq 0$, it follows that $\Phi^w(r \wedge s \wedge t) \leq \Phi^w(r \wedge s) \vee \Phi^w(t)$. By theorem 2.11(1) and (3), we deduce $\Phi^w(t) \leq \Phi^w(t \wedge s) = \Phi^w(s \wedge t)$ and $\Phi^w(t) \leq \Phi^w(t \wedge r) = \Phi^w(r \wedge t)$. Consequently, $\Phi^w(t) \leq \Phi^w(s \wedge t) \vee \Phi^w(r \wedge t)$. This implies, $\Phi^w(r \wedge s \wedge t) \leq \Phi^w(r \wedge s) \vee \Phi^w(s \wedge t) \vee \Phi^w(r \wedge t)$. Hence, $\Phi^w$ qualifies as an $\mathcal{L}$-weakly $2A$-ideal in $R$. □

*Example* 3.9. Let $D = \{0, u, v\}$ be a discrete ADL with 0 as its zero element defined in 2.2 and $L = \{0, a, b, c, d, e, f, 1\}$ be a lattice whose Hasse diagram is given below. Let $Q = \{0, b, c, f\}$. Clearly $Q$ is an ideal in $L$. Define $\mathcal{L}$-subset $\Phi^w : R \to [0, 1]$ by

$$\Phi^w(x,y) = \begin{cases} 1 & \text{if } x = 0 \textit{ and } y \in Q \\ 1/3 & \text{otherwise} \end{cases}$$

for all $(x,y) \in D \times L$. It is evident that $\Phi^w$ qualifies as an $\mathcal{L}$-ideal in $R$. Consequently, $\Phi^w$ is an $\mathcal{L}$-weakly $2A$-ideal in $R$. However, it does not meet the criteria for being an $\mathcal{L}$-weakly $1A$-prime ideal in $D \times L$, as illustrated by the instance

$$\begin{aligned}\Phi^w((0,d)\wedge(u,e)\wedge(v,f)) &= 1\\ &\nleq 1/3\\ &= \Phi^w((0,d)\wedge(u,e))\vee\Phi^w(v,f).\end{aligned}$$




The product of $\mathcal{L}$-subsets $\Phi^w$ and $\Psi^w$ in $R$ and $G$ respectively is denoted by $\Phi^w \times \Psi^w$ and defined by $(\Phi^w \times \Psi^w)(a,b) = \Phi^w(a) \wedge \Psi^w(b)$, for all $(a,b) \in R \times G$.

Theorem 3.10. *Let $\Phi^w$ and $\Psi^w$ be $\mathcal{L}$-ideals in $R$ and $G$ respectively. If $\Phi^w \times \Psi^w$ is an $\mathcal{L}$-weakly $1A$-prime ideal of $R \times G$, then $\Phi^w$ and $\Psi^w$ are $\mathcal{L}$-weakly $1A$-prime ideals in $R$ and $G$ respectively.*

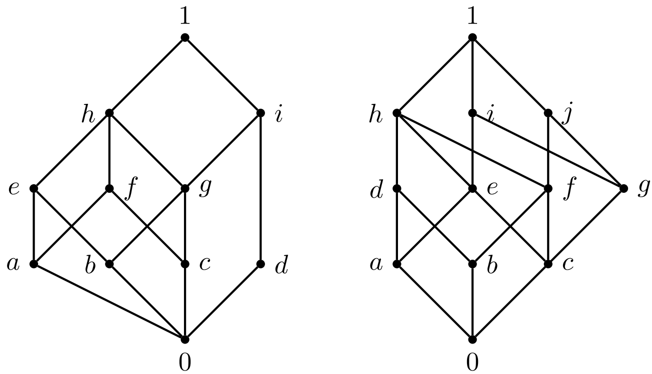
Proof: Suppose that $\Phi^w \times \Psi^w$ is an $\mathcal{L}$-weakly $1A$-prime ideal of $R \times G$. Let $r,s,t \in R$ and $x,y,z \in G$ such that $r\wedge s\wedge t \neq 0$ and $x\wedge y\wedge z \neq 0$. Consider,

$$\begin{aligned}\Phi^w(r\wedge s\wedge t)\wedge\Psi^w(x\wedge y\wedge z) &= (\Phi^w\times\Psi^w)(r\wedge s\wedge t, x\wedge y\wedge z)\\ &= (\Phi^w\times\Psi^w)\big((r,x)\wedge(s,y)\wedge(t,z)\big)\\ &\leq (\Phi^w\times\Psi^w)\big((r,x)\wedge(s,y)\big)\vee(\Phi^w\times\Psi^w)(t,z)\\ &= \big(\Phi^w(r\wedge s)\wedge\Psi^w(x\wedge y)\big)\vee\big(\Phi^w(t)\wedge\Psi^w(z)\big)\\ &= \Big(\Phi^w(r\wedge s)\vee\big(\Phi^w(t)\wedge\Psi^w(z)\big)\Big)\\ &\quad\wedge\Big(\Psi^w(x\wedge y)\vee\big(\Phi^w(t)\wedge\Psi^w(z)\big)\Big)\\ &= \big(\Phi^w(r\wedge s)\vee\Phi^w(t)\big)\wedge\big(\Phi^w(r\wedge s)\vee\Psi^w(z)\big)\\ &\quad\wedge\big(\Psi^w(x\wedge y)\vee\Phi^w(t)\big)\wedge\big(\Psi^w(x\wedge y)\vee\Psi^w(z)\big)\\ &\leq \big(\Phi^w(r\wedge s)\vee\Phi^w(t)\big)\wedge\big(\Psi^w(x\wedge y)\vee\Psi^w(z)\big).\end{aligned}$$

Hence the result. □

The direct product of any two $\mathcal{L}$-weakly $1A$-prime ideals in $R$ may not result in an $\mathcal{L}$-weakly $1A$-prime ideal in $R$; an illustrative example can be considered.

*Example* 3.11. Let $R = \{0, a, b, c, d, e, f, g, h, i, 1\}$ and $G = \{0, a, b, c, d, 6e, f, g, h, i, j, 1\}$ be the lattice represented by the Hasse diagram respectively given below:




Define $\mathcal{L}$-subsets $\Phi^w : R \to [0,1]$ and $\Psi^w : G \to [0,1]$, respectively as follows: $\Phi^w(0) = \Phi^w(b) = \Phi^w(c) = \Phi^w(g) = 1, \Phi^w(a) = 0.5, \Phi^w(d) = \Phi^w(e) = \Phi^w(f) = \Phi^w(h) = \Phi^w(i) = \Phi^w(1) = 0$ and $\Psi^w(0) = \Psi^w(a) = \Psi^w(b) = 1, \Psi^w(c) = \Psi^w(e) = 0.75, \Psi^w(d) = \Psi^w(f) = \Psi^w(g) = \Psi^w(h) = \Psi^w(i) = \Psi^w(j) = \Psi^w(1) = 0$. Clearly both $\Phi^w$ and $\Psi^w$ are $\mathcal{L}$-weakly $1A$-prime ideals in $R$ and $G$ respectively. However, $\Phi^w \times \Psi^w$ is not $\mathcal{L}$-weakly $1A$-prime ideal in $R \times G$. This is demonstrated by considering,

$$\begin{aligned}(\Phi^w \times \Psi^w)(e \wedge f \wedge g, h \wedge i \wedge j) &= (\Phi^w \times \Psi^w)(0, c)\\ &= \Phi^w(0) \wedge \Psi^w(c)\\ &= 0.75\\ &\nleq 0.5\\ &= (\Phi^w \times \Psi^w)(e \wedge f, h \wedge i) \vee (\Phi^w \times \Psi^w)(g, j).\end{aligned}$$

COROLLARY 3.12. Let $\Phi^w$ and $\Psi^w$ be $\mathcal{L}$-ideals in $R$ and $G$ respectively. Then $\Phi^w$ is an $\mathcal{L}$-weakly $1A$-prime ideal in $R$ if and only if $\Phi^w_\beta = \Psi^w_\beta \times G$ or $\Phi^w_\beta = R \times \Psi^w_\beta$, for all $\beta \in L$.

THEOREM 3.13. *Assume $R$ and $G$ are ADLs, and $k : R \to G$ is a lattice homomorphism. If $\Psi^w$ represents an $\mathcal{L}$-weakly $1A$-prime ideal in $G$, then $k^{-1}(\Psi^w)$ is an $\mathcal{L}$-weakly $1A$-prime ideal in $R$. Additionally, in the case of $k$ being an epimorphism and $\Phi^w$ being an $\mathcal{L}$-weakly $1A$-prime ideal in $R$, it follows that $k(\Phi^w)$ is an $\mathcal{L}$-weakly $1A$-prime ideal in $G$.*

PROOF: Suppose that $\Psi^w$ is an $\mathcal{L}$-weakly $1A$-prime ideal in $G$ and let $k$ be a lattice homomorphism. Then, for all $r, s, t \in G$ such that $r \wedge s \wedge t \neq 0$,

$$\begin{aligned} k^{-1}(\Psi^w)(r \wedge s \wedge t) &= \Psi^w\big(k(r \wedge s \wedge t)\big) \\ &= \Psi^w\big(k(r) \wedge k(s) \wedge k(t)\big) \\ &\leq \Psi^w\big(k(r) \wedge k(s)\big) \vee \Psi^w(k(t)) \\ &= \Psi^w\big(k(r \wedge s)\big) \vee \Psi^w(k(t)) \\ &= k^{-1}(\Psi^w)(r \wedge s) \vee k^{-1}(\Psi^w)(t). \end{aligned}$$

Thus $k^{-1}(\Psi^w)$ is an $\mathcal{L}$-weakly $1A$-prime ideal in $R$. Also, let $k$ be an isomorphism and suppose that $\Phi^w$ is an $\mathcal{L}$-weakly $1A$-prime ideal in $R$. Let $a, b, c \in R$ such that $a \wedge b \wedge c \neq 0$. Now, consider,

$$\begin{aligned} k(\Phi^w)(a \wedge b) \vee k(\Phi^w)(c) &= \Big[ \bigvee_{a \wedge b \in k^{-1}(x \wedge y)} \Phi^w(a \wedge b) \Big] \vee \Big[ \bigvee_{c \in k^{-1}(z)} \Phi^w(c) \Big] \\ &\geq \Big[ \bigvee_{a \wedge b \wedge c \in k^{-1}(x \wedge y \wedge z)} \Phi^w(a \wedge b \wedge c) \Big] \\ &= k(\Phi^w)(a \wedge b \wedge c). \end{aligned}$$

Thus, $k(\Phi^w)$ is an $\mathcal{L}$-weakly $1A$-prime ideal in $G$. □

## 4. $\mathcal{L}$-weakly 1A-Prime Filters

In the subsequent discussion, we present the concepts of $\mathcal{L}$-weakly 1-absorbing prime filters and their characterizations. To begin with, let's review the definition provided in [1], stating that a proper filter $H$ in $R$ is a 1-absorbing prime filter (referred to as a weakly $1A$-prime filter) if, for all elements $r, s, t \in R$ such that $r \vee s \vee t \neq 1$, the condition $r \vee s \vee t$ belonging to $H$ implies either $r \vee s$ belonging to $H$ or $t$ belonging to $H$. Now, we aim to extend this outcome to the realm of L-weakly $1A$-prime filters as elaborated below.

Definition 4.1. A proper $\mathcal{L}$-filter $\Phi^w$ in $R$ is an $\mathcal{L}$-weakly $1A$-prime filter in $R$ when, for any elements $r, s$ and $t$ in $R$ such that $r \vee s \vee t \neq 1$, the condition $\Phi^w(r \vee s \vee t) \leq \Phi^w(r \vee s) \vee \Phi^w(t)$ is satisfied.

*Example* 4.2. Let $R$ be an ADL defined in example 3.2 with elements $\{0, r, s, t\}$, and $L = [0,1]$. Define an $\mathcal{L}$-subset $\Phi^w : R \to L$ as follows: $\Phi^w(0) = 0$, $\Phi^w(r) = 1, \Phi^w(s) = 3/4$ and $\Phi^w(t) = 1/2$. It is evident that $\Phi^w$ is an $\mathcal{L}$-filter. Now, consider any elements $a, b, c \in R$ such that $a \vee b \vee c \neq 1$. Then $\Phi^w(a \vee b \vee c) \leq \Phi^w(a \vee b) \vee \Phi^w(c)$. Consequently, $\Phi^w$ qualifies as an $\mathcal{L}$-weakly $1A$-prime filter in $R$.

Subsequently, we elaborate on the notion of an $\mathcal{L}$-weakly $1A$-prime filter concerning the $\gamma$-cut.

Theorem 4.3. *Suppose $\Phi^w$ is an $\mathcal{L}$-filter in $R$. A filter $\Phi^w_\gamma$ is a weakly $1A$-prime filter in $R$, for all $\gamma \in L$ if and only if $\Phi^w$ qualifies as an $\mathcal{L}$-weakly $1A$-prime filter in $R$.*

Proof: Assume that $\Phi^w_\gamma$ is a weakly $1A$-prime filter for all $\gamma \in L$. In this case, for any elements $r, s, t \in R$ such that $r \vee s \vee t \neq 1$, it follows that either $r \vee s$ is an element of $\Phi^w_{\Phi^w(r \vee s \vee t)}$ or $t$ is an element of $\Phi^w_{\Phi^w(r \vee s \vee t)}$. This implies $\Phi^w(r \vee s \vee t) \leq \Phi^w(r \vee s)$ or $\Phi^w(t)$. Consequently, $\Phi^w(r \vee s \vee t) \leq \Phi^w(r \vee s) \vee \Phi^w(t)$, leading to the desired result. Conversely, assume $\Phi^w$ is an $\mathcal{L}$-weakly $1A$-prime filter. Consider $r, s, t \in R$ such that $r \vee s \vee t \neq 1$. If $r \vee s \vee t$ is an element of $\Phi^w_\gamma$, then $\gamma \leq \Phi^w(r \vee s \vee t) \leq \Phi^w(r \vee s) \vee \Phi^w(t)$, which implies that either $\gamma \leq \Phi^w(r \vee s)$ or $\gamma \leq \Phi^w(t)$. This, in turn, means that either $r \vee s \in \Phi^w_\gamma$ or $t \in \Phi^w_\gamma$. Therefore, $\Phi^w_\gamma$ is a weakly $1A$-prime filter in $R$. □

Corollary 4.4. A filter $F$ in $R$ is classified as a weakly $1A$-prime filter in $R$ iff $\chi_F$ is an $\mathcal{L}$-weakly $1A$-prime filter in $R$.

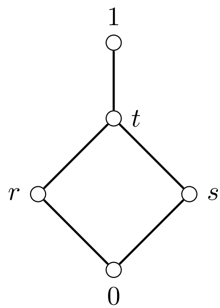
In the following discourse, we clarify the relationships between $\mathcal{L}$-weakly prime filters and $\mathcal{L}$-weakly $1A$-prime filters within an ADL.

Theorem 4.5. *Suppose $\Phi^w$ is an $\mathcal{L}$-filter in $R$. Then $\Phi^w$ is an $\mathcal{L}$-weakly $1A$-prime filter in $R$ only if $\Phi^w$ is an $\mathcal{L}$-weakly prime filter in $R$.*

Proof: It is clear. □

In the forthcoming example, we illustrate the presence of $\mathcal{L}$-weakly $1A$-prime filters in an ADL $R$ that do not meet the criteria for being $\mathcal{L}$-weakly prime filters in $R$.

*Example* 4.6. Consider the discrete ADL $D = \{0, u, v\}$ with 0 as its zero element, as defined in 2.2. Let $L = \{0, r, s, t, 1\}$ represent the lattice depicted in the given Hasse diagram:




Consider $D \times L = \{(d, e) \mid d \in D \text{ and } e \in L\}$. Then, the structure $(D \times L, \wedge, \vee, 0)$ forms an ADL through point-wise operations $\wedge$ and $\vee$ on $D \times L$, where 0 is represented by $(0, 0)$, the zero element in $D \times L$. Define $F = \{t, 1\}$. It is evident that $F$ is a filter in $L$. Now define $\Phi^w : D \times L \to [0, 1]$ by

$$\Phi^w(d, e) = \begin{cases} 0 & \text{if } (d, e) = (0, 0) \\ 1 & \text{if } d \neq 0 \textit{ and } e \in F \\ 0.55 & \text{otherwise} \end{cases}$$

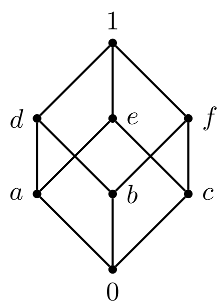
for all $(d, e) \in D \times L$. Additionally, $\Phi^w$ is an $\mathcal{L}$-filter of $D \times L$. Then $\Phi^w_1 = \{(u, t), (v, t), (u, 1), (v, 1)\}$. Consequently, $\Phi^w$ emerges as an $\mathcal{L}$-weakly $1A$-prime filter of $D \times L$. However, $\Phi^w$ does not qualify as an $\mathcal{L}$-weakly prime filter of $D \times L$, as $\Phi^w_1$ is a weakly $1A$-prime filter of $D \times L$ but not weakly prime filter. This is demonstrated by considering, $(u, r), (v, s)$ in $D \times L$, where $(u, r) \vee (v, s) = (v, t)$ belongs to $\Phi^w_1$ implying $(u, r) \notin \Phi^w_1$ and $(v, s) \notin \Phi^w_1$.

Definition 4.7 ([3]). A proper $\mathcal{L}$-filter $\Phi^w$ in $R$ is an $\mathcal{L}$-weakly $2A$-filter in $R$ if for any elements $r$,$s$ and $t \in R$ such that $r \vee s \vee t \neq 1$, $\Phi^w(r \vee s \vee t) \leq \Phi^w(r \vee s) \vee \Phi^w(r \vee t) \vee \Phi^w(s \vee t)$.

Theorem 4.8. *Suppose $\Phi^w$ is an $\mathcal{L}$-filter in $R$. If $\Phi^w$ is an $\mathcal{L}$-weakly $1A$-prime filter in $R$, then $\Phi^w$ is an $\mathcal{L}$-weakly $2A$-filter in $R$. The converse of this result is not true.*

Proof: Let $\Phi^w$ be an $\mathcal{L}$-weakly $1A$-prime filter in $R$. Then, for all $r, s, t \in R$ such that $r \vee s \vee t \neq 1$, it holds that $\Phi^w(r \vee s \vee t) \leq \Phi^w(r \vee s) \vee \Phi^w(t)$. By utilizing Theorem 2.12(1) and (3), we can deduce that $\Phi^w(t) \leq \Phi^w(t \vee s) = \Phi^w(s \vee t)$ and $\Phi^w(t) = \Phi^w(t \vee r) = \Phi^w(r \vee t)$, given that $t \leq t \vee s$ and $t \leq t \vee r$. Consequently, $\Phi^w(t) \leq \Phi^w(r \vee t) \vee \Phi^w(s \vee t)$. This leads to the conclusion that $\Phi^w(r \vee s \vee t) \leq \Phi^w(r \vee s) \vee \Phi^w(r \vee t) \vee \Phi^w(s \vee t)$, thus establishing the desired result. □

*Example* 4.9. Let $D = \{0, u, v\}$ be a discrete ADL with 0 as its zero element defined in 2.2 and $L = \{0, a, b, c, d, e, f, 1\}$ be a lattice whose Hasse diagram is given below:




Define $\mathcal{L}$-filter $\Phi^w : R \to [0, 1]$ by

$$\Phi^w(y, z) = \begin{cases} 0 & \text{if } (y, z) = (0, 0) \\ 3/4 & \text{if } y = u \textit{ and } z = 1 \\ 1/2 & \text{otherwise} \end{cases}$$

for all $(y, z) \in D \times L$. It is evident that $\Phi^w$ qualifies as an $\mathcal{L}$-weakly filter of $D \times L$. Let $H = \Phi^w_{3/4} = \{(u, 1)\}$. Notably, $H$ emerges as a filter in $D \times L$. Consequently, $\Phi^w$ identified as an $\mathcal{L}$-weakly $2A$-filter of $D \times L$, albeit not $\mathcal{L}$-weakly $1A$-prime filter. This is demonstrated by considering any elements $(0, a), (u, c), (v, b) \in D \times L$, where $(0, a) \vee (u, c) \vee (v, b)$ belongs to $H$, implying $(0, a) \vee (u, c) = (u, e) \notin H$ and $(v, b) \notin H$.

Theorem 4.10. *Consider $\mathcal{L}$-weakly filters $\Phi^w$ and $\Psi^w$ be in $R$ and $G$, respectively. If the product $\Phi^w \times \Psi^w$ forms an $\mathcal{L}$-weakly $1A$-prime filter in*

*$R \times G$, then both $\Phi^w$ and $\Psi^w$ individually constitute $\mathcal{L}$-weakly $1A$-prime filters in $R$ and $G$, respectively.*

PROOF: Assume that $\Phi^w \times \Psi^w$ is an $\mathcal{L}$-weakly $1A$-prime filter. Take $r, s, t \in R$ and $x, y, z \in G$ such that $r \vee s \vee t \neq 1$ and $x \vee y \vee z \neq 1$. Then,

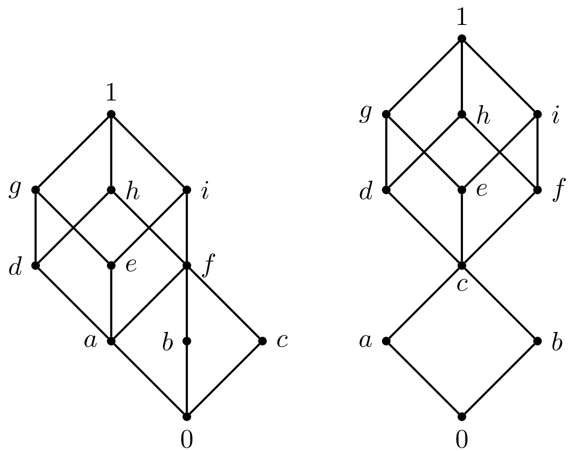
$$\begin{aligned}
\Phi^w(r \vee s \vee t) \wedge \Psi^w(x \vee y \vee z) &= (\Phi^w \times \Psi^w)(r \vee s \vee t, x \vee y \vee z) \\
&= (\Phi^w \times \Psi^w)\big((r,x) \vee (s,y) \vee (t,z)\big) \\
&\leq (\Phi^w \times \Psi^w)\big((r,x) \vee (s,y)\big) \vee (\Phi^w \times \Psi^w)(t,z) \\
&= \Big(\Phi^w(r \vee s) \wedge \Psi^w(x \vee y)\Big) \vee \Big(\Phi^w(t) \wedge \Psi^w(z)\Big) \\
&= \Big(\Phi^w(r \vee s) \vee \big(\Phi^w(t) \wedge \Psi^w(z)\big)\Big) \\
&\quad \wedge \Big(\Psi^w(x \vee y) \vee \big(\Phi^w(t) \wedge \Psi^w(z)\big)\Big) \\
&= \big(\Phi^w(r \vee s) \vee \Phi^w(t)\big) \wedge \big(\Phi^w(r \vee s) \vee \Psi^w(z)\big) \\
&\quad \wedge \big(\Psi^w(x \vee y) \vee \Phi^w(t)\big) \wedge \big(\Psi^w(x \vee y) \vee \Psi^w(z)\big) \\
&\leq \big(\Phi^w(r \vee s) \vee \Phi^w(t)\big) \wedge \big(\Psi^w(x \vee y) \vee \Psi^w(z)\big).
\end{aligned}$$

Hence the result. □

The presence of $\mathcal{L}$-weakly $1A$-prime filters does not guarantee that their direct product will be an $\mathcal{L}$-weakly $1A$-prime filter. An example demonstrating this is provided below.

*Example* 4.11. Let $R = \{0, a, b, c, d, e, f, g, h, i, 1\}$ and $G = \{0, a, b, c, d, e, f, g, h, i, 1\}$ be the lattice represented by the Hasse diagram respectively given below:

Define $\mathcal{L}$-subsets $\Phi^w$ and $\Psi^w$ in $R$ and $G$, respectively such that for $\Phi^w$: $\Phi^w(0) = \Phi^w(a) = 0$, $\Phi^w(b) = 1/3$, $\Phi^w(c) = 0$, $\Phi^w(d) = \Phi^w(e) = \Phi^w(g) = 3/5, \Phi^w(f) = 1, \Phi^w(h) = 3/5, \Phi^w(i) = 3/5, \Phi^w(1) = 1$ and for $\Psi^w$: $\Psi^w(0) = \Psi^w(a) = \Psi^w(b) = 0$, $\Psi^w(c) = \Psi^w(d) = \Psi^w(e) = \Psi^w(f) = 1/2, \Psi^w(i) = \Psi^w(g) = \Psi^w(h) = \Psi^w(1) = 1$. Clearly, both $\Phi^w$ and $\Psi^w$ are $\mathcal{L}$-weakly $1A$-prime filters in $R$ and $G$, respectively. However, the direct product $\Phi^w \times \Psi^w$ is not $\mathcal{L}$-weakly $1A$-prime filter in $R \times G$, as evidenced by the example where




$$\begin{aligned}(\Phi^w\times\Psi^w)(d\vee e\vee f, d\vee e\vee f) &= (\Phi^w\times\Psi^w)(1,1)\\ &= 1\\ &\nleq 3/5\\ &= (\Phi^w\times\Psi^w)(d\vee e, d\vee e)\vee(\Phi^w\times\Psi^w)(f,f).\end{aligned}$$

Corollary 4.12. Let $\Phi^w$ and $\Psi^w$ be $\mathcal{L}$-filters in $R$ and $G$, respectively, and for all $\beta\in L$. Then $\Phi^w$ is an $\mathcal{L}$-weakly $1A$-prime filter in $R$ if and only if $\Phi^w_\beta=\Psi^w_\beta\times G$ or $\Phi^w_\beta=R\times\Psi^w_\beta$, where $\Phi^w_\beta$ and $\Psi^w_\beta$ are weakly $1A$-prime filter in $R$ and $G$ respectively.

Lastly, we explore the homomorphism of $\mathcal{L}$-weakly $1A$-prime filters in ADLs.

Theorem 4.13. *Consider ADLs $R$ and $G$, with a lattice homomorphism $k:R\to G$. Then $k^{-1}(\Psi^w)$ is an $\mathcal{L}$-weakly $1A$-prime filter in $R$ only if $\Psi^w$ is an $\mathcal{L}$-weakly $1A$-prime filter in $G$. Additionally, if $k$ is an epimorphism and $\Phi^w$ is an $\mathcal{L}$-weakly $1A$-prime filter in $R$, then $k(\Phi^w)$ is an $\mathcal{L}$-weakly $1A$-prime filter in $G$.*

Proof: Let $k:R\to G$ be a lattice homomorphism. Suppose that $\Psi^w$ is an $\mathcal{L}$-weakly $1A$-prime filter in $G$. For all $r,s,t\in G$ such that $r\vee s\vee t\neq 1$. Then

$$\begin{aligned} k^{-1}(\Psi^w)(r \vee s \vee t) &= \Psi^w\big(k(r \vee s \vee t)\big) \\ &= \Psi^w\big(k(r) \vee k(s) \vee k(t)\big) \\ &\leq \Psi^w\big(k(r) \vee k(s)\big) \vee \Psi^w(k(t)) \\ &= \Psi^w\big(k(r \vee s)\big) \vee \Psi^w(k(t)) \\ &= k^{-1}(\Psi^w)(r \vee s) \vee k^{-1}(\Psi^w)(t). \end{aligned}$$

Thus $k^{-1}(\Psi^w)$ is an $\mathcal{L}$-weakly $1A$-prime filter in $R$. Let $k$ be an isomorphism and suppose that $\Phi^w$ be an $\mathcal{L}$-weakly $1A$-prime filter in $R$. For all $a, b, c \in R$ such that $a \vee b \vee c \neq 1$. Now, consider,

$$\begin{aligned} k(\Phi^w)(a \vee b) \vee k(\Phi^w)(c) &= \Big[\bigvee_{a \vee b \in k^{-1}(x \wedge y)} \Phi^w(a \vee b)\Big] \vee \Big[\bigvee_{c \in k^{-1}(z)} \Phi^w(c)\Big] \\ &\geq \Big[\bigvee_{a \vee b \vee c \in k^{-1}(x \wedge y \wedge z)} \Phi^w(a \vee b \vee c)\Big] \\ &= k(\Phi^w)(a \vee b \vee c). \end{aligned}$$

Thus, $g(\Phi^w)$ is an $\mathcal{L}$-weakly $1A$-prime filter in $G$. □

## 5. Conclusion

This study concentrates on investigating $\mathcal{L}$-weakly $1A$-prime ideals and filters within an ADL, constituting a pivotal aspect of our research. We delve into the characteristics of these elements, exploring their properties. Furthermore, we elucidate the connection between $\mathcal{L}$-weakly prime filters (ideals) and $\mathcal{L}$-weakly $1A$-prime filters (ideals) in ADLs. Notably, we offer examples to illustrate instances where the converse relationship may not be applicable.

**Author contribution statement.** I affirm that I am the exclusive author of this work, and I have not consulted any sources other than those explicitly cited in the references. Additionally, I confirm that this manuscript has not been submitted to any other journal for publication.

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# SOME ADDITIONAL AXIOMS FOR T-NORMAL LOGICS. DEFINING K45, KB4, KD45 AND S5 WITHOUT USING MODAL RULES

**Abstract**

The paper studies extensions of t-normal logics $\mathbf{S0.5}^{\circ}$ and **S0.5** obtained by means of some axioms of normal logics. We will prove determination theorems for these extensions by appropriate Kripke-style models. It will allow us to obtain the determinations of the logics **K45**, **KB4** (= **KB5**), **KD45** and **S5** without using modal rules.

*Keywords*: modal logic, t-normal logics, Kripke-style semantics.

## Introduction

The definition of modal t-normal logics differs from the definition of normal logics in that we only take the necessity of classical tautologies instead of the rule of necessitation. The first such logic, **S0.5**, was defined by E. J. Lemmon in [3]. The smallest t-normal logic, $\mathbf{S0.5}^{\circ}$, was studied by R. Routley in [8]. In [5, 6, 7], we explored various types of t-normal logics and their location in the lattice of modal logics. The Lemmon's logic **S0.5** is the extension of $\mathbf{S0.5}^{\circ}$ by the following formula:

$$\Box p \supset p \tag{T}$$

The following formulas are theses of **S0.5**:

$$p \supset \Diamond p \tag{$\mathtt{T_d}$}$$

$$\Box p \supset \Diamond p \tag{D}$$

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This paper studies extensions of t-normal logics $\mathbf{S0.5}^\circ$ and $\mathbf{S0.5}$ using axioms known from normal logics: ($\mathtt{D}$), ($\mathtt{T}$) and the following[1]

$$\Box p \supset \Box\Box p \tag{4}$$

$$\Diamond\Diamond p \supset \Diamond p \tag{$4_d$}$$

$$p \supset \Box\Diamond p \tag{B}$$

$$\Diamond\Box p \supset p \tag{$B_d$}$$

$$\Diamond p \supset \Box\Diamond p \tag{5}$$

$$\Diamond\Box p \supset \Box p \tag{$5_d$}$$

It is known that dual versions are not needed in normal logics, i.e., formulas without the lower subscript 'd' are sufficient (or vice versa). For t-normal logics, the dual and non-dual versions of a given formula are independent.

As additional axioms for $\mathbf{S0.5}^\circ$ and $\mathbf{S0.5}$, we will also use the following formulas:

$$\Diamond\Box p \supset \Diamond\Diamond p \tag{$D^m$}$$

$$\Box\Box p \supset \Box\Diamond p \tag{$D^m_d$}$$

$$\Diamond\Box p \supset \Diamond p \tag{$T^m$}$$

$$\Box p \supset \Box\Diamond p \tag{$T^m_d$}$$

The names of the above formulas say that we obtain them from ($\mathtt{D}$), ($\mathtt{T}$) and ($\mathtt{T}_\mathtt{d}$), respectively, through the monotonicity rule and duality used for normal logics. So ($\mathtt{T}^\mathtt{m}$), ($\mathtt{T}^\mathtt{m}_\mathtt{d}$) $\in$ $\mathbf{KT}$ and ($\mathtt{D}^\mathtt{m}$), ($\mathtt{D}^\mathtt{m}_\mathtt{d}$) $\in \mathbf{KD} \subsetneq \mathbf{KT}$. These formulas are independent for t-normal logics.

Section 1 provides the necessary facts about modal logic. Following [4], we write that the normal logics **K45**, **KB4** (= **KB5**) and **KD45** are determined by the suitable classes of simplified Kripke-style models (which refers to the known fact that the class of universal Kripke models determines the logic **S5**). We end this section with a definition of t-normal modal logics, distinguishing very weak t-normal logics as those that are not closed under the replacement of tautological equivalents. We will notice that, unlike for normal logics, there is a significant difference between t-normal logics that are built in the set of formulas with two primary modal connectives '$\Box$' and '$\Diamond$' and that are built in the set with only the first of them (i.e. $\Diamond := \neg\Box\neg$).

[1] In [5, 6, 7] were explored various kinds of t-normal logics with additional axioms from sets $\Box\Phi$, where $\Phi \subseteq \mathbf{S0.5}$.

In Section 2, we present a syntactic and semantic analysis of four basic very weak t-normal logics: $\mathbf{S0.5}^\circ$, $\mathbf{S0.5}^\circ[\mathtt{D}]$, $\mathbf{S0.5}^\circ[\mathtt{T^q}]$, **S0.5**. Unlike previous papers [5, 6, 7], this research will be presented in the set For, i.e., with two primitive modal connectives: '$\Box$' and '$\Diamond$'. We will use specific examples to show the difference that occurs when these logics are built in the set $\mathrm{For}_\Box$. Furthermore, following [5], in the Appendix, we will present an analysis of canonical models and completeness theorems for $\mathbf{S0.5}^\circ$, $\mathbf{S0.5}^\circ[\mathtt{D}]$, $\mathbf{S0.5}^\circ[\mathtt{T^q}]$ and **S0.5** built-in For with respect to suitable classes of Kripke-style models. This will also be used in the next section, where we analyze extensions of these logics with additional axioms.

In Section 3, we explore other t-normal logics with additional axioms, which we provided on page 512. For these logics, we give determination theorems with respect to the suitable classes of Kripke-style models. Thanks to this, we find the dependencies between the considered extensions of **S0.5** and $\mathbf{S0.5}^\circ$. We also provide what the equivalents of these logics in the set $\mathrm{For}_\Box$ would look like.

In [4] for the logics **K45**, **KB4** (= **KB5**) and **KD45** are given the determination theorems by suitable classes of simplified Kripke-style models. Using these theorems, the determination of the logic **S5** by the class of universal Kripke models, and the facts obtained in Section 3, in Section 4 we will prove that $\mathbf{K45} = \mathbf{S0.5}^\circ[\mathtt{4},\mathtt{4_d},\mathtt{5},\mathtt{5_d}]$, $\mathbf{KB4} = \mathbf{S0.5}^\circ[\mathtt{B},\mathtt{4},\mathtt{4_d},\mathtt{5},\mathtt{5_d}]$, $\mathbf{KD45} = \mathbf{S0.5}^\circ[\mathtt{D},\mathtt{4},\mathtt{4_d},\mathtt{5},\mathtt{5_d}]$ and $\mathbf{S5} = \mathbf{S0.5}^\circ[\mathtt{T},\mathtt{4},\mathtt{4_d},\mathtt{5},\mathtt{5_d}]$. Thus, we will show that these normal logics are definable without modal rules.

# 1. Normal and t-normal modal logics

## 1.1. Formulas, PL-tautologies and modal logics

**Formulas.** Modal propositional formulas with two modal connectives are built in the standard way from propositional letters (or *atoms*) from the set $\mathrm{At} := \{p, q, p_1, p_2, p_3, \ldots\}$), the Boolean propositional connectives '$\neg$', '$\vee$', '$\wedge$', '$\supset$' and '$\equiv$' (for negation, conjunction, disjunction, and material implication and equivalence, respectively) the modal connectives '$\Box$' ('It is necessary that') and '$\Diamond$' ('It is possible that'), and brackets. Let For be the set of all modal propositional formulas.

Often, modal logics are examined in a set $\mathrm{For}_\Box$ of formulas built in a standard way without using the possibility sign '$\Diamond$'. This sign is just an abbreviation for '$\neg\Box\neg$'. Of course, $\mathrm{For}_{\Box} \subsetneq \mathrm{For}$. In both case, we put $\Box\Phi := \{\Box\varphi : \varphi \in \Phi\}$ for any subset $\Phi$ of formulas.

Moreover, let $\mathrm{For}_{\mathrm{cl}}$ be the set of all classical propositional formulas built without modal connectives.

**PL-tautologies.** Let $\mathrm{Taut}_{\mathrm{cl}}$ be the set of all tautologies from $\mathrm{For}_{\mathrm{cl}}$ and **PL** be the set of all their instances from For, which we will call *PL-tautologies*. Following [1], we say that a formula is *propositionally atomic* iff it is either atomic in the ordinary sense (i.e., it belongs to At) or modal (i.e., it has the form $\ulcorner\Box\varphi\urcorner$ or $\ulcorner\Diamond\varphi\urcorner$). Let PAt be the set of all propositionally atomic formulas. Moreover, let $\mathsf{Val}^{\mathsf{cl}}$ be the set of all valuations $V\colon \mathrm{For} \to \{0,1\}$ which preserve classical conditions for Boolean connectives. Of course, $V \in \mathsf{Val}^{\mathsf{cl}}$ iff for some assignment $v\colon \mathrm{PAt} \to \{0,1\}$, $V$ is the unique extension of $v$ by classical truth conditions for Boolean connectives. It is obvious:

Lemma 1.1. *For any* $\varphi \in \mathrm{For}$: $\varphi \in \mathbf{PL}$ *iff* $V(\varphi) = 1$ *for any* $V \in \mathsf{Val}^{\mathsf{cl}}$.

A subset $\Psi$ of For is *PL-consistent* iff that there is a $V \in \mathsf{Val}^{\mathsf{cl}}$ such that $V[\Psi] = \{1\}$. Moreover, for $\varphi \in \mathrm{For}$, we put $\Psi \models_{\mathbf{PL}} \varphi$ iff the set $\Psi \cup \{\neg\varphi\}$ is not PL-consistent. We have: $\Psi \models_{\mathbf{PL}} \varphi$ iff either $\varphi \in \mathbf{PL}$ or there are $n > 0$, $\psi_1, \ldots, \psi_n \in \Psi$ such that $\ulcorner(\psi_1 \wedge \cdots \wedge \psi_n) \supset \varphi\urcorner \in \mathbf{PL}$. So $\emptyset \models_{\mathbf{PL}} \varphi$ iff $\varphi \in \mathbf{PL}$.

**Modal logics.** Following [1, p. 46], we say that a subset $\boldsymbol{L}$ of For is a *modal logic* iff $\boldsymbol{L}$ is closed under uniform substitution and the following rule for all $\Psi \subseteq \mathrm{For}$ and $\varphi \in \mathrm{For}$:[2]

(RPL) if $\Psi \subseteq \boldsymbol{L}$ and $\Psi \models_{\mathbf{PL}} \varphi$, then $\varphi \in \boldsymbol{L}$.

So $\boldsymbol{L}$ is a modal logic iff $\boldsymbol{L}$ includes $\mathrm{Taut}_{\mathrm{cl}}$ and is closed under substitution and *detachment*, i.e., for all $\varphi, \psi \in \mathrm{For}$:

(det) if $\ulcorner\varphi \supset \psi\urcorner \in \boldsymbol{L}$ and $\varphi \in \boldsymbol{L}$, then $\psi \in \boldsymbol{L}$.

All members of $\boldsymbol{L}$ are called its *theses*. We say that $\boldsymbol{L}$ is *consistent* iff $\boldsymbol{L} \neq \mathrm{For}$.

The set **PL** is the smallest modal logic. So all modal logic include **PL**.

We say that $\varphi$ is *deducible* from a subset $\Psi$ in $\boldsymbol{L}$ (written: $\Psi \vdash_{\boldsymbol{L}} \varphi$) iff either $\varphi \in \boldsymbol{L}$ or there are $n > 0$, $\psi_1, \ldots, \psi_n \in \Psi$ such that $\ulcorner(\psi_1 \wedge \cdots \wedge \psi_n) \supset \varphi\urcorner \in \boldsymbol{L}$. Notice that:

[2] In [1], Chellas considers *systems of modal logic*, which do not have to be closed under uniform substitution.

- if $\Psi \models_{\mathbf{PL}} \varphi$ then $\Psi \vdash_{\boldsymbol{L}} \varphi$.
- $\varphi \in \boldsymbol{L}$ iff $\emptyset \vdash_{\boldsymbol{L}} \varphi$ iff $\boldsymbol{L} \vdash_{\boldsymbol{L}} \varphi$.

Moreover, we say that formulas $\varphi$ and $\psi$ are ***L**-equivalent* iff both $\varphi \vdash_{\boldsymbol{L}} \psi$ and $\psi \vdash_{\boldsymbol{L}} \varphi$, i.e. $\ulcorner \varphi \equiv \psi \urcorner \in \boldsymbol{L}$.

For any $\Phi \subseteq \mathrm{For}$, let $\boldsymbol{L}[\Phi]$ be the smallest modal logic including $\boldsymbol{L} \cup \Phi$.

### 1.2. Normal modal logics

**Definition.** A modal logic $\boldsymbol{L}$ is *normal* iff $\boldsymbol{L}$ contains the formulas:

$$\Diamond\varphi \equiv \neg\Box\neg\varphi \tag{df$\Diamond$}$$

$$\Box(p \supset q) \supset (\Box p \supset \Box q) \tag{K}$$

and is closed under the *rule of necessitation*, i.e., for any $\varphi \in \mathrm{For}$:

(nec) if $\varphi \in \boldsymbol{L}$ then $\ulcorner \Box\varphi \urcorner \in \boldsymbol{L}$.

Any normal logic $\boldsymbol{L}$ includes $\Box\mathbf{PL}$ and is closed under the following rules for all $\Psi \subseteq \mathrm{For}$ and $\varphi, \psi, \chi \in \mathrm{For}$:

(rk) if $\Psi \vdash_{\boldsymbol{L}} \varphi$ then $\Box\Psi \vdash_{\boldsymbol{L}} \Box\varphi$;
($\mathrm{rk_d}$) if $\Psi, \psi \vdash_{\boldsymbol{L}} \varphi$ then $\Box\Psi \cup \{\Diamond\psi\} \vdash_{\boldsymbol{L}} \Diamond\varphi$;
(cgr) if $\ulcorner \varphi \equiv \psi \urcorner \in \boldsymbol{L}$, then $\ulcorner \Box\varphi \equiv \Box\psi \urcorner \in \boldsymbol{L}$;
(rep) $\ulcorner \varphi \equiv \psi \urcorner \in \boldsymbol{L}$, then $\ulcorner \chi \equiv \chi[{}^{\varphi}/\!/_{\psi}] \urcorner \in \boldsymbol{L}$.

where $\chi[{}^{\varphi}/\!/_{\psi}]$ is any formula that results from $\chi$ by replacing zero or more occurrences of $\varphi$, in $\chi$, by $\psi$. Hence $\boldsymbol{L}$ is also closed under *replacement of tautological equivalents* iff for all $\chi, \varphi, \psi \in \mathrm{For}$ we have:

(rte) if $\ulcorner \varphi \equiv \psi \urcorner \in \mathbf{PL}$, then $\ulcorner \chi \equiv \chi[{}^{\varphi}/\!/_{\psi}] \urcorner \in \boldsymbol{L}$.

So the following formulas are theses of any normal logic:

$$\Box p \equiv \neg\Diamond\neg p \tag{df$\Box$}$$

$$\Box(p \supset q) \supset (\Diamond p \supset \Diamond q) \tag{$\mathrm{K_d}$}$$

$$\Box(p \wedge q) \equiv (\Box p \wedge \Box q) \tag{R}$$

$$\Diamond(p \vee q) \equiv (\Diamond p \vee \Diamond q) \tag{$\mathrm{R_d}$}$$

$$\Diamond(p \supset q) \equiv (\Box p \supset \Diamond q) \tag{$\mathrm{R'_d}$}$$

A modal logic is normal iff it is closed under (cgr) and contains (K) and $\Box\top$ (for some $\top \in \mathrm{Taut_{cl}}$).

*Remark* 1.2. 1. If we consider a given normal logic in the set $\mathrm{For}_\Box$, then (df$\Diamond$) is unnecessary because it is just a shortcut on one side of the PL-tautology '$\neg\Box\neg p \equiv \neg\Box\neg p$'. Moreover, (df$\Box$) is a shortcut of the thesis '$\Box p \equiv \neg\neg\Box\neg\neg p$'.

2. For normal logics, it does not matter whether we examine them in For or their versions in $\mathrm{For}_\Box$. Namely, assume that for any formula $\varphi$ of For, the formula $\varphi^\Box$ from $\mathrm{For}_\Box$ is its copy created by replacing each occurrence of '$\Diamond$' with '$\neg\Box\neg$'. Then $\varphi$ is a thesis of a normal logic $\boldsymbol{L}$ iff $\varphi^\Box$ is its thesis in the $\mathrm{For}_\Box$-version denoted by $\boldsymbol{L}_\Box$. Moreover, $\boldsymbol{L}_\Box = \boldsymbol{L} \cap \mathrm{For}_\Box$ and $\boldsymbol{L}_\Box \subsetneq \boldsymbol{L}$. $\Box$

**Selected normal logics.** The smallest normal logic is denoted by **K**. Other known normal logics are build using (D), (T), (4), (B), (5) and the following:

$$\Diamond\top \supset (\mathtt{T}), \tag{$\mathtt{T^q}$}$$

where $\top$ is an arbitrary tautology of propositional classical logic.[3] Using the names of the above formulas, to simplify the naming of normal logics, we write $\mathbf{KX}_1 \ldots \mathbf{X}_n$ to denote the smallest normal logic containing formulas $(\mathtt{X}_1)$, …, $(\mathtt{X}_n)$. We put $\mathbf{S5} := \mathbf{KT5}$ and $\mathbf{S4} := \mathbf{KT4}$. Since $\ulcorner\Diamond\top\urcorner \in \mathbf{KD}$, we have $\mathbf{KT} = \mathbf{KDT^q}$. Moreover, $\mathbf{KT^q} \subsetneq \mathbf{K4T^q} \subsetneq \mathbf{KB4} = \mathbf{KB5} = \mathbf{K5T^q} \subsetneq \mathbf{S5}$, $\mathbf{KBT^q} \subsetneq \mathbf{KB4}$, $\mathbf{S5} = \mathbf{KTB4} = \mathbf{KDB4} = \mathbf{KDB5} = \mathbf{KD5T^q}$, $\mathbf{KD} \subsetneq \mathbf{KT} \subsetneq \mathbf{S4} \subsetneq \mathbf{S5}$, $\mathbf{KT^q} \subsetneq \mathbf{KT} \subsetneq \mathbf{KTB}$, $\mathbf{KB} \subsetneq \mathbf{KTB}$, $\mathbf{K4} \subsetneq \mathbf{K45} \subsetneq \mathbf{KB4}$ and $\mathbf{K5} \subsetneq \mathbf{K45} \subsetneq \mathbf{KD45} \subsetneq \mathbf{S5}$.

**Simplified Kripke-style semantics for K45, KB4, KD45 and S5.** Following [4], for logics **K45**, **KB4** (= **KB5**) and **KD45** – instead of relational Kripke models – we can use *simplified models* of the form $\langle W, A, V\rangle$, where $W$ is a non-empty set of worlds, $A \subseteq W$ ($A$ is a set of *common alternatives* to all worlds from $W$), and $V$ is a *valuation* as a function $V\colon \mathrm{For} \times W \to \{0,1\}$ which for any $x \in W$ gives $V(\cdot, x) \in \mathsf{Val}^{\mathsf{cl}}$ and, moreover, for any $\varphi \in \mathrm{For}$ we have:

($V_\Box$) $V(\Box\varphi, x) = 1$ iff for each $y \in A$ we have $V(\varphi, y) = 1$;
($V_\Diamond$) $V(\Diamond\varphi, x) = 1$ iff for some $y \in A$ we have $V(\varphi, y) = 1$.

[3] The name '$\mathtt{T^q}$' is an abbreviation for 'quasi-T', because (T) and ($\mathtt{T^q}$) are valid in all reflexive and quasi-reflexive Kripke frames, respectively. In a given quasi-reflexive Kripke frame, an *accessibility* relation $R$ on a set $W$ of worlds satisfies (see [1, p. 92, Exercise 3.51]): $\forall_{x\in W}(\exists_{y\in W}\, x\,R\,y \Rightarrow x\,R\,x)$.

We say that a simplified model $\langle W, A, V\rangle$ is *universal* (resp. *empty*, *non-empty*) iff $A = W$ (resp. $A = \emptyset$, $A \neq \emptyset$). Of course, a universal model $\langle W, W, V\rangle$ can be simplified to $\langle W, V\rangle$.[4] Commonly, such universal models are applied to **S5**.

We say that a formula $\varphi$ is *true* in a model $\langle W, A, V\rangle$ iff $V(\varphi, x) = 1$ for each $x \in W$. We say that a formula is *valid* in a class $\boldsymbol{M}$ of models iff it is true in all models from $\boldsymbol{M}$. A class $\boldsymbol{M}$ *determines* a given logic if its theses are all those and only those formulas valid in $\boldsymbol{M}$.

The following fact is known:

THEOREM 1.3 ([1]). **S5** *is determined by the class of all universal models.*

Moreover, we have (see [4, Theorem 1.1]):

THEOREM 1.4. 1. **K45** *is determined by the class of all simplified models.*
2. **KB4** *is determined by the class of empty or universal models.*
3. **KD45** *is determined by the class of non-empty simplified models.*

### 1.3. T-normal modal logics

**Definition.** Following [5], a modal logic is *t-normal* iff it includes the set $\Box\mathrm{Taut}_{\mathrm{cl}}$ and contains (df$\Diamond$), (K). Every t-normal logic also includes $\Box$**PL** and contains (df$\Box$), ($\mathrm{K_d}$), (R), ($\mathrm{R_d}$), ($\mathrm{R'_d}$). All normal logics are t-normal.[5] Every modal logic that extends a given t-normal logic is also t-normal.

Let $\boldsymbol{L}$ be a t-normal logic. Using $\Box$**PL**, (K), ($\mathrm{K_d}$), (R), ($\mathrm{R_d}$), we obtain:

(pk) if $\Psi \models_{\mathbf{PL}} \varphi$ then $\Box\Psi \vdash_{\boldsymbol{L}} \Box\varphi$;
($\mathrm{pk_d}$) if $\Psi, \psi \models_{\mathbf{PL}} \varphi$ then $\Box\Psi \cup \{\Diamond\psi\} \vdash_{\boldsymbol{L}} \Diamond\varphi$.
(pe) if $\ulcorner\varphi \equiv \psi\urcorner \in \mathbf{PL}$ then $\ulcorner\Box\varphi \equiv \Box\psi\urcorner \in \boldsymbol{L}$;
($\mathrm{pe_d}$) if $\ulcorner\varphi \equiv \psi\urcorner \in \mathbf{PL}$ then $\ulcorner\Diamond\varphi \equiv \Diamond\psi\urcorner \in \boldsymbol{L}$.

*Remark* 1.5. As for normal logics, if we consider a given t-normal logic in $\mathrm{For}_{\Box}$, then (df$\Diamond$) is unnecessary (see Remark 1.2(1)). Also (df$\Box$) is a shortcut of the thesis '$\Box p \equiv \neg\neg\Box\neg\neg p$'. But, there may be some confusion regarding the two approaches to t-normal logics. We will show differences between both approaches in Section 2.4. $\Box$

[4] A universal model $\langle W, A\rangle$ also corresponds to the following relational model $\langle W, W \times W, V\rangle$ with the universal relation $R = W \times W$ *accessibility of worlds.*

[5] The term 't-normal' means that the rule of necessity from normal logics is limited to PL-tautologies, i.e., we have only $\Box\mathbf{PL} \subseteq \boldsymbol{L}$ instead of the rule (nec).

**Very weak t-normal logics.** If a t-normal logic is not closed under (rte), it will be called *very weak t-normal* (briefly: *vwt-normal*). In this paper, we will deal with such logics.[6]

## 2. The first four t-normal logics

### 2.1. Definitions and basic properties

Following [8], we denote the smallest t-normal logic by $\mathbf{S0.5}^{\circ}$. Following [3], by $\mathbf{S0.5}$, we denote the smallest t-normal logic containing (T). We have $\mathbf{S0.5} = \mathbf{S0.5}^{\circ}[\mathtt{T}]$; so the sign '$\mathbf{S0.5}^{\circ}$' means: $\mathbf{S0.5}$ without (T).

Notice that ($\mathtt{T}^{\mathtt{q}}$) is $\mathbf{S0.5}^{\circ}$-equivalent to each of the following formulas:

$$\Box p \supset (p \vee \Box q)$$
$$\Diamond q \supset (\Box p \supset p)$$
$$(\mathtt{D}) \supset (\mathtt{T})$$

Formulas (D) and $\ulcorner \Diamond \top \urcorner$ are $\mathbf{S0.5}^{\circ}$-equivalent. They and ($\mathtt{T}_{\mathtt{d}}$) belong to $\mathbf{S0.5}$. We have $\mathbf{S0.5}^{\circ}[\mathtt{D}] \subsetneq \mathbf{S0.5}$, $\mathbf{S0.5}^{\circ}[\mathtt{T}^{\mathtt{q}}] \subsetneq \mathbf{S0.5}$ and $\mathbf{S0.5} = \mathbf{S0.5}^{\circ}[\mathtt{D}, \mathtt{T}^{\mathtt{q}}]$.

*Remark* 2.1. Lemmon [3] and Routley [8] investigated $\mathbf{S0.5}$ and $\mathbf{S0.5}^{\circ}$, respectively, in the set $\mathrm{For}_{\Box}$ (see Remark 1.5). The such version of $\mathbf{S0.5}$ was also presented in [2]. Moreover, the versions of $\mathbf{S0.5}^{\circ}$, $\mathbf{S0.5}^{\circ}[\mathtt{D}]$, $\mathbf{S0.5}^{\circ}[\mathtt{T}^{\mathtt{q}}]$ and $\mathbf{S0.5}$ in $\mathrm{For}_{\Box}$ was studied in [5, 6, 7]. □

### 2.2. Kripke-style semantics for $\mathbf{S0.5}^{\circ}$ and $\mathbf{S0.5}$. Soundness and completeness

Let $w$ be any object and $A$ be any set. A *t-normal Kripke-style model* (briefly: *tn-model*) is any triple $\langle w, A, V\rangle$ such that $V$ is a *valuation* as a function $V\colon \mathrm{For} \times (\{w\} \cup A) \to \{0,1\}$ which for any $x \in A \cup \{w\}$ gives $V(\cdot, x) \in \mathsf{Val}^{\mathsf{cl}}$ and for any $\varphi \in \mathrm{For}$ we have:

($V^w_{\Box}$) $V(\Box\varphi, w) = 1$ iff for each $x \in A$ we have $V(\varphi, x) = 1$;

($V^w_{\Diamond}$) $V(\Diamond\varphi, w) = 1$ iff for some $x \in A$ we have $V(\varphi, x) = 1$.

We say that $w$ is a *distinguished world*, $A$ is a set of *alternative worlds to* $w$ and $\langle w, A, V\rangle$ based on $w$ and $A$. Moreover, we say that a tn-model is *self-associate* (resp. *empty*, *non-empty*) iff $w \in A$ (resp. $A = \emptyset$, $A \neq \emptyset$).

[6] In [5, 6, 7] various kinds of t-normal logics closed under (rte) were studied.

We say that a formula $\varphi$ is *true* (resp. *false*) in a tn-model $\langle w, A, V\rangle$ iff $V(\varphi, w) = 1$ (resp. $V(\varphi, w) = 0$). We say that a formula is *valid* in a class $\boldsymbol{M}$ of tn-models (or $\boldsymbol{M}$-*valid*) iff it is true in all models from $\boldsymbol{M}$.

The following lemma shows how tn-models can be constructed:

LEMMA 2.2. *Let $w$ be an object, $A$ be a set, $v_w\colon \mathrm{At} \to \{0,1\}$ and $V_x \in \mathsf{Val}^{\mathsf{cl}}$ for each $x \in A\setminus\{w\}$. Then there is the unique $V\colon \mathrm{For}\times(A\cup\{w\}) \to \{0,1\}$ such that $\langle w, A, V\rangle$ is a tn-model.*

PROOF: For any $\alpha \in \mathrm{At}$ we put $V(\alpha, w) := v_w(\alpha)$ and for any $\varphi \in \mathrm{PAt}$ and $x \in A \setminus \{w\}$ we put $V(\varphi, x) := V_x(\varphi)$. Using truth conditions for Boolean connectives and ($V^w_\Box$), ($V^w_\Diamond$), we uniquely extend $V$. □

The following facts are also obvious:

FACT 2.3.

1. The rules (RPL) and (det) preserve the truth in each tn-model.
2. All instances of formulas (K) and (df$\Diamond$), and all formulas of $\mathbf{PL}\cup\Box\mathbf{PL}$ are valid in the class of all tn-models.

FACT 2.4. Let $w$ be any object and $A$ be any set. Then:

1. For any tn-model $\mathfrak{M}$ based on $w$ and $A$: (D) is true in $\mathfrak{M}$ iff $A \neq \emptyset$.
2. (T) are true in all tn-models based on $w$ and $A$ iff $w \in A$.
3. ($\mathtt{T}^{\mathtt{q}}$) are true in all tn-models based on $w$ and $A$ iff either $A = \emptyset$ or $w \in A$.

THEOREM 2.5 (Soundness).

1. *All theses of* $\mathbf{S0.5}^\circ$ *are valid in the class of all tn-models.*
2. *All theses of* $\mathbf{S0.5}^\circ$[D] *are valid in the class of all non-empty tn-models.*
3. *All theses of* $\mathbf{S0.5}^\circ$[$\mathtt{T}^{\mathtt{q}}$] *are valid in the class of all tn-models which are empty or self-associate.*
4. *All theses of* $\mathbf{S0.5}$ *are valid in the class of all self-associate tn-models.*

Given the above theorem, we can assume that the classes of models mentioned in the following items are suitable for the logics $\mathbf{S0.5}^\circ$, $\mathbf{S0.5}^\circ$[D], $\mathbf{S0.5}^\circ$[$\mathtt{T}^{\mathtt{q}}$] and $\mathbf{S0.5}$, respectively. We denote this classes by $\mathbf{M}_{\mathrm{S0.5}^\circ}$, $\mathbf{M}_{\mathrm{S0.5}^\circ[\mathtt{D}]}$, $\mathbf{M}_{\mathrm{S0.5}^\circ[\mathtt{T}^{\mathtt{q}}]}$ and $\mathbf{M}_{\mathrm{S0.5}}$. For all models of these classes we can assume that for all worlds from $A \setminus \{w\}$, all modal propositionally atomic formulas have arbitrary values.

Finally, Theorem A.7 in Appendix give the completeness of the logics $\mathbf{S0.5}^{\circ}$, $\mathbf{S0.5}^{\circ}[\mathtt{D}]$, $\mathbf{S0.5}^{\circ}[\mathtt{T^q}]$ and $\mathbf{S0.5}$.

THEOREM 2.6 (Completeness). *All formulas valid in the class* $\mathsf{M}_{\mathbf{S0.5}^{\circ}}$ (*resp.* $\mathsf{M}_{\mathbf{S0.5}^{\circ}[\mathtt{D}]}$, $\mathsf{M}_{\mathbf{S0.5}^{\circ}[\mathtt{T^q}]}$, $\mathsf{M}_{\mathbf{S0.5}}$) *are theses of* $\mathbf{S0.5}^{\circ}$ (*resp.* $\mathbf{S0.5}^{\circ}[\mathtt{D}]$, $\mathbf{S0.5}^{\circ}[\mathtt{T^q}]$, $\mathbf{S0.5}$).

### 2.3. Some conclusions

By Fact 2.4 and Theorem 2.5, we get:

FACT 2.7.

1. ($\mathtt{T^q}$), ($\mathtt{D}$) and any formula of the form $\ulcorner\Diamond\varphi\urcorner$ do not belong to $\mathbf{S0.5}^{\circ}$.
2. ($\mathtt{D}$) and any formula of the form $\ulcorner\Diamond\varphi\urcorner$ do not belong to $\mathbf{S0.5}^{\circ}[\mathtt{T^q}]$.
3. ($\mathtt{T^q}$) does not belong to $\mathbf{S0.5}^{\circ}[\mathtt{D}]$.
4. ($\mathtt{T}$) belong neither to $\mathbf{S0.5}^{\circ}[\mathtt{T^q}]$ nor $\mathbf{S0.5}^{\circ}[\mathtt{D}]$.
5. $\mathbf{S0.5}^{\circ} \subsetneq \mathbf{S0.5}^{\circ}[\mathtt{D}] \subsetneq \mathbf{S0.5}$ and $\mathbf{S0.5}^{\circ} \subsetneq \mathbf{S0.5}^{\circ}[\mathtt{T^q}] \subsetneq \mathbf{S0.5}$.

FACT 2.8. The following implications are not theses of $\mathbf{S0.5}$:

$$\Box\Box p \supset \Box\Box\neg\neg p \qquad\qquad \Box\Box\neg\neg p \supset \Box\Box p$$
$$\Box\Diamond p \supset \Box\neg\Box\neg p \qquad\qquad \Box\neg\Box\neg p \supset \Box\Diamond p$$
$$\Box\Box p \supset \Box\neg\Diamond\neg p \qquad\qquad \Box\neg\Diamond\neg p \supset \Box\Box p$$

So $\mathbf{S0.5}^{\circ}$, $\mathbf{S0.5}^{\circ}[\mathtt{D}]$, $\mathbf{S0.5}^{\circ}[\mathtt{T^q}]$ and $\mathbf{S0.5}$ are not closed under (rte).[7]

PROOF: It is easy to point out suitable self-associate tn-models in which the above formulas are false. Hence, by Theorem 2.5(4) and Fact 2.7, $\mathbf{S0.5}^{\circ}$, $\mathbf{S0.5}^{\circ}[\mathtt{D}]$, $\mathbf{S0.5}^{\circ}[\mathtt{T^q}]$ and $\mathbf{S0.5}$ are not closed under (rte). □

The theorems below concern modal propositionally atomic formulas.[8]

THEOREM 2.9. *For any* $\boldsymbol{L} \in \{\mathbf{S0.5}^{\circ}, \mathbf{S0.5}^{\circ}[\mathtt{T^q}], \mathbf{S0.5}^{\circ}[\mathtt{D}], \mathbf{S0.5}\}$ *and* $\varphi \in \mathrm{For}$:

$$\ulcorner\Box\varphi\urcorner \in \boldsymbol{L} \quad \textit{iff} \quad \varphi \in \mathbf{PL}.$$

PROOF: Firstly, $\Box\mathbf{PL} \subsetneq \mathbf{S0.5}^{\circ}[\mathtt{T^q}] \subsetneq \mathbf{S0.5}$ and $\Box\mathbf{PL} \subsetneq \mathbf{S0.5}^{\circ}[\mathtt{D}] \subsetneq \mathbf{S0.5}$. Secondly, let $\varphi \notin \mathbf{PL}$, $w \neq a$, $A := \{w, a\}$. Then, by Lemma 1.1, for some $V_a \in \mathsf{Val}^{\mathsf{cl}}$ we have that $V_a(\varphi) = 0$. By Lemma 2.2, for $V_a$ and any

[7] In [5, 6, 7], t-normal logics closed under (rte) in versions built-in $\mathrm{For}_{\Box}$ are examined.

[8] [7, Facts 3.8 and 3.9] provides these theorems in versions for logics built-in $\mathrm{For}_{\Box}$.

assignment $v_w\colon \mathrm{At} \to \{0,1\}$ there is a self-associate tn-model $\langle w, A, V\rangle$ such that $V(\Box\varphi, w) = 0$. Hence $\ulcorner\Box\varphi\urcorner \notin \mathbf{S0.5}$, by Theorem 2.5(4). Moreover, we use Fact 2.7. □

THEOREM 2.10. *For any* $\boldsymbol{L} \in \{\mathbf{S0.5}^\circ, \mathbf{S0.5}^\circ[\mathtt{D}]\}$, $\Psi \subseteq \mathrm{For}$ *and* $\varphi \in \mathrm{For}$:

$$\Box\Psi \vdash_{\boldsymbol{L}} \Box\varphi \quad \textit{iff} \quad \Psi \models_{\mathbf{PL}} \varphi.$$

PROOF: Firstly, by (pk), if $\Psi \models_{\mathbf{PL}} \varphi$, then $\Box\Psi \vdash_{\mathbf{S0.5}^\circ} \Box\varphi$ and it entails $\Box\Psi \vdash_{\mathbf{S0.5}^\circ[\mathtt{D}]} \Box\varphi$. Secondly, suppose that $\Psi \not\models_{\mathbf{PL}} \varphi$ and $w \neq a$. Then, by Lemma 1.1, for some $V_a \in \mathsf{Val}^{\mathsf{cl}}$ we have $V_a[\Psi] = \{1\}$ and $V_a(\varphi) = 0$. By Lemma 2.2, for $V_a$ and any $v_w\colon \mathrm{At} \to \{0,1\}$ there is a non-empty tn-model $\langle w, \{a\}, V\rangle$ such that $V[\Box\Psi] = \{1\}$ and $V(\Box\varphi, w) = 0$. Hence $\Box\Psi \nvdash_{\mathbf{S0.5}^\circ[\mathtt{D}]} \Box\varphi$, by Theorem 2.5. □

*Remark* 2.11. For $\mathbf{S0.5}^\circ[\mathtt{T^q}]$ and $\mathbf{S0.5}$, the "⇒"-part of Theorem 2.10 does not hold. Indeed, '$\Box\Box p \supset \Box p$' belong to $\mathbf{S0.5}^\circ[\mathtt{T^q}]$ ( $\subsetneq \mathbf{S0.5}$). Therefore, $\Box\Box p \vdash_{\mathbf{S0.5}^\circ[\mathtt{T^q}]} \Box p$ and $\Box\Box p \vdash_{\mathbf{S0.5}} \Box p$, but $\Box\Box p \not\models_{\mathbf{PL}} \Box p$. □

## 2.4. Similarities and differences between the two approaches

Versions of t-normal logic built-in the set $\mathrm{For}_\Box$ include $\mathrm{Taut}_{\mathrm{cl}}$ and $\Box\mathrm{Taut}_{\mathrm{cl}}$, contain (K) and are closed under (det) and uniform substitutions. All such versions include $\mathbf{PL}_\Box$ ($:= \mathbf{PL} \cap \mathrm{For}_\Box$) and $\Box\mathbf{PL}_\Box$. We use the sign '$\Diamond$' as an abbreviation for '$\neg\Box\neg$'. As theses of such versions of t-normal logics, we obtain these formulas whose shortcuts are ($\mathtt{df}\Diamond$), ($\mathtt{df}\Box$), ($\mathtt{K_d}$), ($\mathtt{R_d}$), ($\mathtt{R'_d}$) (see Remark 1.2(1)).

Let us denote by $\mathbf{S0.5}^\circ_\Box$ the smallest t-normal logic built-in $\mathrm{For}_\Box$. Moreover, let $\mathbf{S0.5}_\Box$ be the smallest t-normal logic built-in $\mathrm{For}_\Box$ containing (T) (see Remark 2.1). The formulas for which ($\mathtt{T_d}$), (D), $\ulcorner\Diamond\top\urcorner$ and all $\mathbf{S0.5}^\circ$-equivalents to ($\mathtt{T^q}$) are shortcuts belong to $\mathbf{S0.5}_\Box$.

Let $\mathbf{S0.5}^\circ_\Box[\mathtt{T^q}]$ be the smallest t-normal logic built-in $\mathrm{For}_\Box$ containing ($\mathtt{T^q}$). As theses of $\mathbf{S0.5}^\circ_\Box[\mathtt{T^q}]$, we obtain these formulas whose shortcuts are $\mathbf{S0.5}^\circ$-equivalents to ($\mathtt{T^q}$). Moreover, let $\mathbf{S0.5}^\circ_\Box[\mathtt{D}]$ be the smallest t-normal logic built-in $\mathrm{For}_\Box$ containing '$\Box p \supset \neg\Box\neg p$', whose shortcut is (D).

Let $\boldsymbol{L} \in \{\mathbf{S0.5}^\circ, \mathbf{S0.5}^\circ[\mathtt{D}], \mathbf{S0.5}^\circ[\mathtt{T^q}], \mathbf{S0.5}\}$. For $\boldsymbol{L}_\Box$ we use tn-models, which we define in the same way as tn-models for $\boldsymbol{L}$ with the only difference that the set For is replaced by $\mathrm{For}_\Box$, and we only use ($V^w_\Box$). We have

(∗) All formulas from $\mathrm{For}_\Box$ true in all tn-models for $\boldsymbol{L}$ are also true in all tn-models for $\boldsymbol{L}_\Box$.

In [5, Theorem 4.8], an appropriate version of the completeness theorem for $\boldsymbol{L}_{\Box}$ is given.[9] We can prove:

THEOREM 2.12. $\boldsymbol{L}_{\Box} = \boldsymbol{L} \cap \mathrm{For}_{\Box}$. *So* $\boldsymbol{L}_{\Box} \subsetneq \boldsymbol{L}$.

PROOF: It is obvious that $\boldsymbol{L}_{\Box} \subseteq \boldsymbol{L} \cap \mathrm{For}_{\Box}$. Suppose that $\varphi \in \boldsymbol{L} \cap \mathrm{For}_{\Box}$. We take any tn-models for $\boldsymbol{L}_{\Box}$. By (∗) and Theorem 2.5, $\varphi$ is true in this model. From the completeness theorem for $\boldsymbol{L}_{\Box}$, we obtain that $\varphi \in \boldsymbol{L}_{\Box}$. □

From Theorems 2.9, 2.10 and 2.12 we obtain:

COROLLARY 2.13 ([5]). For all $\varphi \in \mathrm{For}_{\Box}$ and $\Psi \subseteq \mathrm{For}_{\Box}$:

1. For $\boldsymbol{L} \in \{\mathbf{S0.5}^{\circ}, \mathbf{S0.5}^{\circ}[\mathtt{D}], \mathbf{S0.5}^{\circ}[\mathtt{T^q}], \mathbf{S0.5}\}$, $\ulcorner\Box\varphi\urcorner \in \boldsymbol{L}_{\Box}$ iff $\varphi \in \mathbf{PL}_{\Box}$.
2. For $\boldsymbol{L} \in \{\mathbf{S0.5}^{\circ}, \mathbf{S0.5}^{\circ}[\mathtt{D}]\}$, $\Box\Psi \vdash_{\boldsymbol{L}_{\Box}} \Box\varphi$ iff $\Psi \models_{\mathbf{PL}_{\Box}} \varphi$.

As we mentioned in Remark 1.5, there may be some confusion regarding the two approaches to these logics. The difference between them is visible from Fact 2.8. Namely, the implications below are not theses of **S0.5** but even are theses even of $\mathbf{S0.5}^{\circ}_{\Box}$:

$$\Box\Diamond p \supset \Box\neg\Box\neg p \qquad\qquad \Box\neg\Box\neg p \supset \Box\Diamond p$$

Indeed, in $\mathbf{S0.5}^{\circ}_{\Box}$ the above implications are just shortcuts on one side of the PL-tautology '$\Box\neg\Box\neg p \supset \Box\neg\Box\neg p$'. Hence also, '$\Box(\Box\Diamond p \equiv \Box\neg\Box\neg p)$' belongs to $\mathbf{S0.5}^{\circ}_{\Box}$. However, it does not contradict Corollary 2.13 because, in $\mathbf{S0.5}^{\circ}_{\Box}$, these three forms are just abbreviations of suitable formulas from $\mathbf{PL}_{\Box}$ and $\Box\mathbf{PL}_{\Box}$, respectively.

Finally, note that the following implications are also not theses of $\mathbf{S0.5}_{\Box}$:

$$\Box\Box p \supset \Box\neg\Diamond\neg p \qquad\qquad \Box\neg\Diamond\neg p \supset \Box\Box p$$

Indeed, for $\mathbf{S0.5}_{\Box}$, these formulas are just abbreviations of the following:

$$\Box\Box p \supset \Box\neg\neg\Box\neg\neg p \qquad\qquad \Box\neg\neg\Box\neg\neg p \supset \Box\Box p$$

which are $\mathbf{S0.5}^{\circ}_{\Box}$-equivalent to '$\Box\Box p \supset \Box\Box\neg\neg p$' and '$\Box\Box\neg\neg p \supset \Box\Box p$', respectively. Fact 2.8 and Theorem 2.12 say that the last formulas are not theses of $\mathbf{S0.5}^{\circ}_{\Box}$.

[9] Its proof is an appropriate version of the proof of Theorem A.7.

# 3. Other t-normal logics with additional axioms

## 3.1. Additional axioms

Theorem 2.5(4) shows that none of formulas ($\mathtt{D^m}$), ($\mathtt{D^m_d}$), ($\mathtt{T^m}$), ($\mathtt{T^m_d}$), ($\mathtt{4}$), ($\mathtt{4_d}$), ($\mathtt{B}$), ($\mathtt{B_d}$), ($\mathtt{5}$), ($\mathtt{5_d}$) belongs to **S0.5**. The formulas listed here are additional axioms with which we will extend $\mathbf{S0.5}^\circ$ and **S0.5**. It is evident that:

- ($\mathtt{D^m}$) $\in$ **S0.5**[$\mathtt{T^m}$], ($\mathtt{T^m}$) $\in$ **S0.5**[$\mathtt{B_d}$] and ($\mathtt{B}$) $\in \mathbf{S0.5}^\circ$[$\mathtt{5}$];
- ($\mathtt{D^m_d}$) $\in$ **S0.5**[$\mathtt{T^m_d}$], ($\mathtt{T^m_d}$) $\in$ **S0.5**[$\mathtt{B}$] and ($\mathtt{B_d}$) $\in$ **S0.5**[$\mathtt{5_d}$];
- ($\mathtt{T^m}$) $\in \mathbf{S0.5}^\circ$[$\mathtt{D^m}$,$\mathtt{4_d}$] and ($\mathtt{T^m_d}$) $\in \mathbf{S0.5}^\circ$[$\mathtt{4}$,$\mathtt{D^m_d}$];
- ($\mathtt{T^m}$) $\in \mathbf{S0.5}^\circ$[$\mathtt{5_d}$,$\mathtt{D}$] and ($\mathtt{T^m_d}$) $\in \mathbf{S0.5}^\circ$[$\mathtt{D}$,$\mathtt{5}$];
- ($\mathtt{D^m}$), ($\mathtt{D^m_d}$) $\in \mathbf{S0.5}^\circ$[$\mathtt{D}$,$\mathtt{5_d}$,$\mathtt{T^m_d}$] and ($\mathtt{D^m}$), ($\mathtt{D^m_d}$) $\in \mathbf{S0.5}^\circ$[$\mathtt{D}$,$\mathtt{5}$,$\mathtt{T^m}$];
- ($\mathtt{D^m}$), ($\mathtt{D^m_d}$) $\in \mathbf{S0.5}^\circ$[$\mathtt{D}$,$\mathtt{5}$,$\mathtt{5_d}$].

Further, we will show that there are no other dependencies between additional axioms.

We are interested in such t-normal logics, which have a given additional axiom and its dual form. To simplify naming of logics, we will write $\mathbf{S0.5}^\circ.\mathbf{X}_1 \dots \mathbf{X}_n$ to denote the smallest t-normal logic containing formulas ($\mathtt{X}_1$), ..., ($\mathtt{X}_n$) and their dual forms. Moreover, the notation $\mathbf{S0.5}.\mathbf{X}_1 \dots \mathbf{X}_n$ will indicate the suitable smallest extension of **S0.5**. For example:

- $\mathbf{S0.5}^\circ.\mathbf{4T}^\mathrm{m} \subseteq \mathbf{S0.5}^\circ.\mathbf{4D}^\mathrm{m}$ and $\mathbf{S0.5.4D}^\mathrm{m} = \mathbf{S0.5.4T}^\mathrm{m}$.

Further, we will show that the following combinations of additional axioms give normal logics (see Theorem 4.1):

(†) $\mathbf{S0.5}^\circ.\mathbf{45} = \mathbf{K45}$, $\mathbf{S0.5}^\circ.\mathbf{D45} = \mathbf{KD45}$ and $\mathbf{S0.5}^\circ.\mathbf{B45} = \mathbf{KB4}$ (= **KB5**);

(‡) $\mathbf{S0.5.45} = \mathbf{S5}$.

Moreover, we will show that the remaining combinations of additional axioms give vwt-normal logics (see Fact 3.7).

The following fact and results obtained in Section 2.3 will show differences between the logics thus obtained and the logics $\mathbf{S0.5}^\circ$ and **S0.5**.

FACT 3.1. For all $\varphi, \psi \in \mathrm{For}$:

1. (a) If $\ulcorner \varphi \equiv \psi \urcorner \in \mathbf{PL}$ then $\ulcorner \Box\Box\varphi \equiv \Box\Box\psi \urcorner \in \mathbf{S0.5}[\mathtt{4}]$.
   (b) $\ulcorner \Diamond\varphi \equiv \Diamond\neg\Box\neg\varphi \urcorner \in \mathbf{S0.5}[\mathtt{4}]$.
2. (a) If $\ulcorner \varphi \equiv \psi \urcorner \in \mathbf{PL}$ then $\ulcorner \Diamond\Diamond\varphi \equiv \Diamond\Diamond\psi \urcorner \in \mathbf{S0.5}[\mathtt{4_d}]$.
   (b) $\ulcorner \Box\varphi \equiv \Box\neg\Diamond\neg\varphi \urcorner \in \mathbf{S0.5}[\mathtt{4_d}]$.

3. (a) $\ulcorner\Box\Box\varphi \equiv \Box\neg\Diamond\neg\varphi\urcorner$ and (b) $\ulcorner\Diamond\Diamond\varphi \equiv \Diamond\neg\Box\neg\varphi\urcorner$ belong to **S0.5.4**.

4. (a) If $\ulcorner\varphi \equiv \psi\urcorner \in \mathbf{PL}$ then $\ulcorner\Box\Diamond\varphi \equiv \Box\Diamond\psi\urcorner \in \mathbf{S0.5}[5] \cap \mathbf{S0.5}^{\circ}[\mathtt{D},4_{\mathtt{d}},5]$.
(b) $\ulcorner\Box\varphi \equiv \Diamond\neg\Diamond\neg\varphi\urcorner \in \mathbf{S0.5}[5] \cap \mathbf{S0.5}^{\circ}[\mathtt{D},4_{\mathtt{d}},5]$.

5. (a) If $\ulcorner\varphi \equiv \psi\urcorner \in \mathbf{PL}$ then $\ulcorner\Diamond\Box\varphi \equiv \Diamond\Box\psi\urcorner \in \mathbf{S0.5}[5_{\mathtt{d}}] \cap \mathbf{S0.5}^{\circ}[\mathtt{D},4,5_{\mathtt{d}}]$.
(b) $\ulcorner\Diamond\varphi \equiv \Box\neg\Box\neg\varphi\urcorner \in \mathbf{S0.5}[5_{\mathtt{d}}] \cap \mathbf{S0.5}^{\circ}[\mathtt{D},4,5_{\mathtt{d}}]$.

6. (a) $\ulcorner\Box\Diamond\varphi \equiv \Box\neg\Box\neg\varphi\urcorner$ and (b) $\ulcorner\Diamond\Box\varphi \equiv \Diamond\neg\Diamond\neg\varphi\urcorner$ belong to **S0.5.5** and **S0.5°.D45**.

7. (a) $\Box(\mathtt{T_d}) \in \mathbf{S0.5}^{\circ}[5]$, (b) $\Box(\mathtt{T}) \in \mathbf{S0.5}^{\circ}[5_{\mathtt{d}}]$ and (c) $\Box(\mathtt{D}) \in \mathbf{S0.5}^{\circ}.\mathbf{5}$.

Proof: *Ad* 1. (a) By (T), (4), (pe); (b) By (T), (4), (pe), (df$\Diamond$).

*Ad* 2. (a) By ($\mathtt{T_d}$), ($4_{\mathtt{d}}$), ($\mathrm{pe_d}$). (b) By ($\mathtt{T_d}$), ($4_{\mathtt{d}}$), ($\mathrm{pe_d}$), (df$\Box$).

*Ad* 3. (a) By (4), (T) and item 2(b). (b) By ($4_{\mathtt{d}}$), ($\mathtt{T_d}$) and item 1(b).

*Ad* 4. (a) From (T), (5), ($\mathrm{pe_d}$) we have: $\Box\Diamond\varphi \equiv \Diamond\varphi \equiv \Diamond\psi \equiv \Box\Diamond\psi$. Moreover, by (D), ($4_{\mathtt{d}}$), ($\mathrm{pe_d}$), (5): $\Box\Diamond\varphi \supset \Diamond\Diamond\varphi \supset \Diamond\varphi \equiv \Diamond\psi \supset \Box\Diamond\psi$. (b) From (T), (5), (pe) we have: $\Diamond\neg\varphi \equiv \Box\Diamond\neg\varphi \equiv \Box\neg\neg\Diamond\neg\varphi$. Hence and (df$\Diamond$), (df$\Box$) we have: $\Box\varphi \equiv \Diamond\neg\Diamond\neg\varphi$. Moreover, by (D), ($4_{\mathtt{d}}$), (5) we have: $\Box\Diamond\neg\varphi \supset \Diamond\Diamond\neg\varphi \supset \Diamond\neg\varphi \supset \Box\Diamond\neg\varphi$. So we use (df$\Diamond$), (df$\Box$) and (pe).

*Ad* 5. (a) By ($\mathtt{T_d}$), ($5_{\mathtt{d}}$) and ($\mathrm{pe_d}$) we have: $\Diamond\Box\varphi \equiv \Box\varphi \equiv \Box\psi \equiv \Diamond\Box\psi$. Moreover, by ($5_{\mathtt{d}}$), (pe), (4), (D): $\Diamond\Box\varphi \supset \Box\varphi \supset \Box\psi \equiv \Box\Box\psi \supset \Diamond\Box\psi$. (b) From ($\mathtt{T_d}$), ($5_{\mathtt{d}}$) and (pe): $\Box\neg\varphi \equiv \Diamond\Box\neg\varphi \equiv \Diamond\neg\neg\Box\neg\varphi$. Hence and (df$\Diamond$), (df$\Box$) we have: $\Diamond\varphi \equiv \Box\neg\Box\neg\varphi$. Moreover, by ($5_{\mathtt{d}}$), (4), (D) we have: $\Diamond\Box\neg\varphi \supset \Box\neg\varphi \supset \Box\Box\neg\varphi \supset \Diamond\Box\neg\varphi$. So we use (df$\Diamond$), (df$\Box$) and (pe).

*Ad* 6. (a) By (T), (5): $\Box\Diamond\varphi \equiv \Diamond\varphi$. Moreover, by (D), ($4_{\mathtt{d}}$), (5), we have: $\Box\Diamond\varphi \supset \Diamond\Diamond\varphi \supset \Diamond\varphi \supset \Box\Diamond\varphi$. So in both cases we use item 5(b). (b) By ($\mathtt{T_d}$), ($5_{\mathtt{d}}$): $\Diamond\Box\varphi \equiv \Box\varphi$. Moreover, by ($5_{\mathtt{d}}$), (4), (D), we have: $\Diamond\Box\varphi \supset \Box\varphi \supset \Box\Box\varphi \supset \Diamond\Box\varphi$. So in both cases we use item 4(b).

*Ad* 7. (a) By (pk), $\Box\Diamond p \vdash_{\boldsymbol{L}} \Box(p \supset \Diamond p)$ and $\Box\neg p \vdash_{\boldsymbol{L}} \Box(p \supset \Diamond p)$ for any t-normal logic $\boldsymbol{L}$. Moreover, $\Diamond p \vdash_{\mathbf{S0.5}^{\circ}[5]} \Box\Diamond p$ and $\neg\Diamond p \vdash_{\mathbf{S0.5}^{\circ}} \Box\neg p$. Thus, '$\Box(p \supset \Diamond p)$' $\in \mathbf{S0.5}^{\circ}[5]$.

(b) By (pk), $\Box p \vdash_{\boldsymbol{L}} \Box(\Box p \supset p)$ and $\Box\neg\Box p \vdash_{\boldsymbol{L}} \Box(\Box p \supset p)$ for any t-normal logic $\boldsymbol{L}$. Moreover, $\neg\Box p \vdash_{\mathbf{S0.5}^{\circ}[5_{\mathtt{d}}]} \Box\neg\Box p$. Therefore, '$\Box(\Box p \supset p)$' belongs to $\mathbf{S0.5}^{\circ}[5_{\mathtt{d}}]$.

(c) By (pk), $\Box(\mathtt{T}), \Box(\mathtt{T_d}) \vdash_{\boldsymbol{L}} \Box(\mathtt{D})$ for any t-normal logic $\boldsymbol{L}$. So we use (a) and (b). □

### 3.2. Kripke-style semantics for additional axioms. Soundness

To use tn-models for additional axioms, we must assume an appropriate condition for a given axiom. In a tn-model $\langle w, A, V\rangle$, every one of these conditions will apply to any formula $\varphi$:

$$\exists_{x\in A}\, V(\Box\varphi, x) = 1 \implies \exists_{y\in A}\, V(\Diamond\varphi, y) = 1, \tag{cD$^{\mathtt{m}}\varphi$}$$

$$\forall_{x\in A}\, V(\Box\varphi, x) = 1 \implies \forall_{y\in A}\, V(\Diamond\varphi, y) = 1, \tag{cD$^{\mathtt{m}}_{\mathtt{d}}\varphi$}$$

$$\exists_{x\in A}\, V(\Box\varphi, x) = 1 \implies \exists_{y\in A}\, V(\varphi, y) = 1, \tag{cT$^{\mathtt{m}}\varphi$}$$

$$\forall_{x\in A}\, V(\varphi, x) = 1 \implies \forall_{y\in A}\, V(\Diamond\varphi, y) = 1, \tag{cT$^{\mathtt{m}}_{\mathtt{d}}\varphi$}$$

$$V(\varphi, w) = 1 \implies \forall_{y\in A}\, V(\Diamond\varphi, y) = 1, \tag{cB$\varphi$}$$

$$V(\varphi, w) = 0 \implies \forall_{y\in A}\, V(\Box\varphi, y) = 0, \tag{cB$_{\mathtt{d}}\varphi$}$$

$$\forall_{x\in A}\big(\exists_{y\in A}\, V(\varphi, y) = 1 \implies V(\Diamond\varphi, x) = 1\big), \tag{c5$\varphi$}$$

$$\forall_{x\in A}\big(V(\Box\varphi, x) = 1 \implies \forall_{y\in A}\, V(\varphi, x) = 1\big), \tag{c5$_{\mathtt{d}}\varphi$}$$

$$\forall_{x\in A}\big(\forall_{y\in A}\, V(\varphi, y) = 1 \implies V(\Box\varphi, x) = 1\big), \tag{c4$\varphi$}$$

$$\forall_{x\in A}\big(V(\Diamond\varphi, x) = 1 \implies \exists_{y\in A}\, V(\varphi, y) = 1\big). \tag{c4$_{\mathtt{d}}\varphi$}$$

Moreover, for (T), (D) and ($\mathtt{T}^{\mathtt{q}}$) we use the conditions '$w \in A$', '$A \neq \emptyset$' and 'either $w \in A$ or $A = \emptyset$', respectively.

*Remark* 3.2. (i) In all self-associate tn-models: (cT$^{\mathtt{m}}\varphi$) entails (cD$^{\mathtt{m}}\varphi$); (cT$^{\mathtt{m}}_{\mathtt{d}}\varphi$) entails (cD$^{\mathtt{m}}_{\mathtt{d}}\varphi$); (cB$_{\mathtt{d}}\varphi$) entails (cT$^{\mathtt{m}}\varphi$); (cB$\varphi$) entails (cT$^{\mathtt{m}}_{\mathtt{d}}\varphi$); (c5$\varphi$) entails (cB$\varphi$); (c5$_{\mathtt{d}}\varphi$) entails (cB$_{\mathtt{d}}\varphi$).

(ii) Apart from the above, no other dependencies exist between the given conditions. □

The following lemma is easy to prove:

LEMMA 3.3. *Let* $\chi$ *is an additional axiom,* $\varphi \in \mathrm{For}$ *and* $\mathfrak{M}$ *be a tn-model. We put* $\chi^{\varphi} := \chi[p/\varphi]$*. Then:*

$$\chi^{\varphi} \textit{ is true in } \mathfrak{M} \textit{ iff } \varphi \textit{ satisfies the condition } (\mathrm{c}\chi\varphi) \textit{ in } \mathfrak{M}.$$

Let $\Phi$ be a non-empty set of formulas which contains some or all of the formulas used as additional axioms (including (T), (D) and ($\mathtt{T}^{\mathtt{q}}$)). Then we will call $\mathbf{S0.5}^{\circ}[\Phi]$-*model* all those and only those tn-models in which conditions for all instances of the formulas in $\Phi$ are satisfied.

THEOREM 3.4 (Soundness). *All theses of* $\mathbf{S0.5}^{\circ}[\Phi]$ *are valid in the class of all* $\mathbf{S0.5}^{\circ}[\Phi]$*-models.*

We will further use the following lemma:

LEMMA 3.5. *Let* $\{(4), (4_{\mathtt{d}}), (5), (5_{\mathtt{d}})\} \subseteq \Phi$*,* $\langle w, A, V\rangle$ *be an* $\mathbf{S0.5}^{\circ}[\Phi]$*-model and* $W := \{w\} \cup A$*. Then:*

1. $\langle W, A, V\rangle$ *is a simplified Kripke-style model.*
2. *If also* (B) $\in \Phi$ *then* $\langle W, A, V\rangle$ *is an empty or universal Kripke model.*
3. *If also* (D) $\in \Phi$ *then* $\langle W, A, V\rangle$ *is a non-empty simplified model.*
4. *If also* (T) $\in \Phi$ *then* $\langle W, V\rangle$ *is a universal Kripke model.*

PROOF: *Ad* 1. Let $\varphi \in$ For. By ($V^w_{\Box}$), (c4$\varphi$) and (c5$_{\mathtt{d}}\varphi$), for any $x \in W$: $V(\Box\varphi, x) = 1$ iff $V(\varphi, y) = 1$ for each $y \in A$. By ($V^w_{\Diamond}$), (c4$_{\mathtt{d}}\varphi$) and (5$_{\mathtt{d}}$), for any $x \in W$: $V(\Diamond\varphi, x) = 1$ iff $V(\varphi, y) = 1$ for some $y \in A$. Thus, $\langle W, A, V\rangle$ satisfies conditions ($V_{\Box}$) and ($V_{\Diamond}$) from p. 516.

*Ad* 2. By item 1, $\langle W, A, V\rangle$ satisfies ($V_{\Box}$) and ($V_{\Diamond}$). Assume that $A \neq \emptyset$.

For ($V_{\Box}$) with $A = W$: Let $\varphi \in$ For. By (cB$_{\mathtt{d}}\varphi$), we have:

(i) for any $x \in A$: if $V(\Box\varphi, x) = 1$ then $V(\varphi, w) = 1$).

Moreover, assume that $V(\varphi, w) = 0$. Then, by (cB$_{\mathtt{d}}\varphi$), $V(\Box\varphi, x) = 0$ for each $x \in A$. So $V(\Box\varphi, x_0) = 0$ for some $x_0 \in A$ because $A \neq \emptyset$. Hence $V(\Box\varphi, w) = 0$, by ($V^w_{\Box}$). So we obtain:

(ii) if $V(\Box\varphi, w) = 1$ then $V(\varphi, w) = 1$.

Thus, using (i), (ii), ($V^w_{\Box}$) and ($V_{\Box}$), we obtain:

($V_{\Box}$) for any $x \in W$: $V(\Box\varphi, x) = 1$ iff $\forall_{y \in W}\ V(\varphi, y) = 1$.

For ($V_{\Diamond}$) with $A = W$: Let $\varphi \in$ For. By (cB$\varphi$), we have:

(i′) for any $x \in A$: $V(\varphi, w) = 1 \Rightarrow V(\Diamond\varphi, x) = 1$.

Moreover, using (ii) and (df$\Diamond$) for $\neg\varphi$, we obtain:

(ii′) if $V(\varphi, w) = 1$ then $V(\Diamond\varphi, w) = 1$.

Thus, using (i′), (ii′), ($V^w_{\Diamond}$) and ($V_{\Diamond}$), we obtain:

($V_{\Box}$) for any $x \in W$: $V(\Diamond\varphi, x) = 1$ iff $\exists_{y \in W}\ V(\varphi, y) = 1$.

*Ad* 3. $A \neq \emptyset$, by Fact 2.4(1).

*Ad* 4. Suppose that (T) $\in \boldsymbol{L}$. Then (D), (B) and (B$_{\mathtt{d}}$) belong to $\boldsymbol{L}$. Hence, by item 3, $A \neq \emptyset$. So $\langle W, V\rangle$ is a universal Kripke model, by item 2. $\Box$

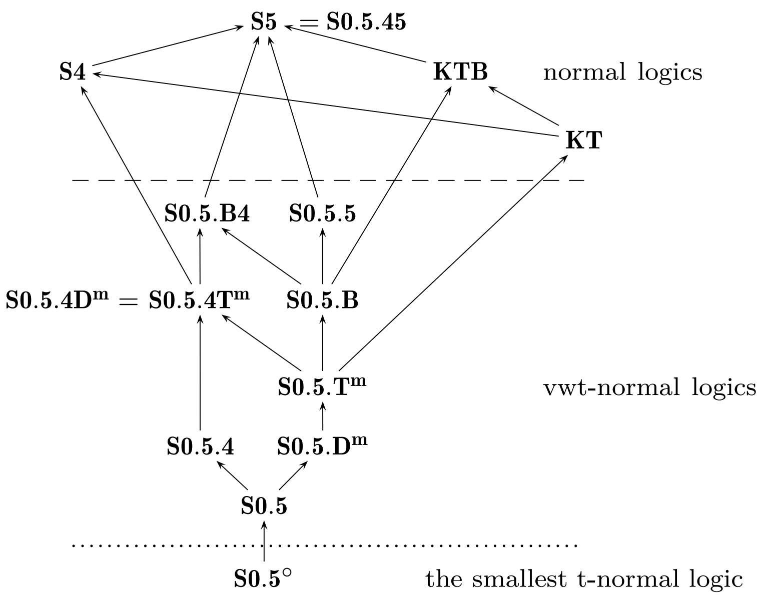


**Figure 1.** The dependencies between the considered extensions of **S0.5**

### 3.3. Some conclusions

By constructing appropriate countermodels, by Theorem 3.4, we have the following facts (cf. Remark 3.2(ii)):

FACT 3.6.

1. $(D^m) \notin$ **S0.5**$[D^m_d]$ and $(D^m_d) \notin$ **S0.5**$[D^m]$.
2. $(T^m)(T^m_d) \notin$ **S0.5.D**$^m$.
3. Neither $(B)$ nor $(B_d)$ belongs to neither **S0.5.T**$^m$ nor **S0.5.4**.
4. $(4) \notin$ **S0.5**$[4_d]$ and $(4_d) \notin$ **S0.5**$[4]$.
5. $(B) \notin$ **S0.5**$[B_d]$ and $(B_d) \notin$ **S0.5**$[B]$.
6. $(5) \notin$ **S0.5**$[5_d]$ and $(5_d) \notin$ **S0.5**$[5]$.
7. $(4), (4_d) \notin$ **S0.5.5** and $(5), (5_d) \notin$ **S0.5.B4**.

The dependencies between the considered extensions of the logics **S0.5** and **S0.5**$^\circ$ are presented in Figures 1 and 2, respectively.

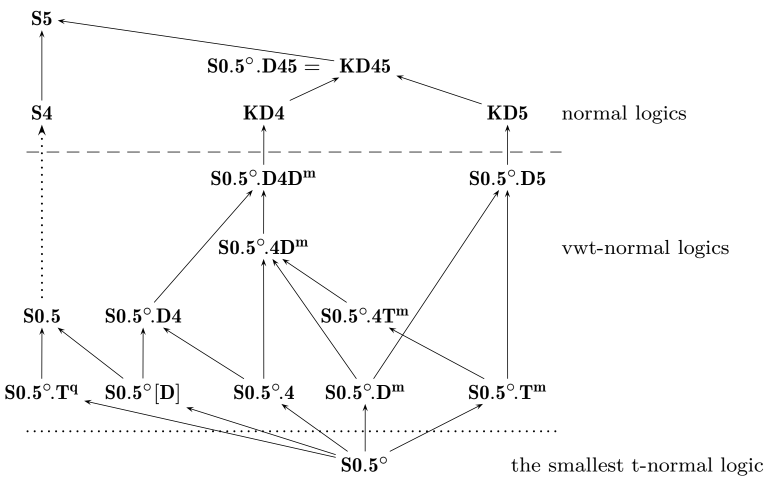


**Figure 2.** The dependencies between the considered extensions of **S0.5**°

Also, by constructing appropriate two-element self-associate counter-models and using Theorem 3.4, we obtain the following fact, which shows that the logics **S0.5.B4** and **S0.5.5** (and all others included therein) are not closed under (rte) (cf. Fact 3.1).

FACT 3.7. 1. The formulas '$\Box\Diamond p \supset \Box\Diamond\neg\neg p$', '$\Box\Diamond\neg\neg p \supset \Box\Diamond p$', '$\Diamond\Box p \supset \Diamond\Box\neg\neg p$' and '$\Diamond\Box\neg\neg p \supset \Diamond\Box p$' do not belong to **S0.5.B4**. So '$\Box(\Diamond p \supset \Diamond\neg\neg p)$' and '$\Box(\Diamond\neg\neg p \supset \Diamond p)$' too.

2. The formulas '$\Box\Box p \supset \Box\Box\neg\neg p$', '$\Box\Box\neg\neg p \supset \Box\Box p$', '$\Diamond\Diamond p \supset \Diamond\Diamond\neg\neg p$' and '$\Diamond\Diamond\neg\neg p \supset \Diamond\Diamond p$' do not belong to **S0.5.5**. So '$\Box(\Box p \supset \Box\neg\neg p)$' and '$\Box(\Box\neg\neg p \supset \Box p)$' too.

Moreover, we have (cf. Fact 3.1(3,5)):

FACT 3.8. Neither '$\Box(\Diamond p \supset \neg\Box\neg p)$', '$\Box(\neg\Box\neg p \supset \Diamond p)$', '$\Box(\Box p \supset \neg\Diamond\neg p)$' nor '$\Box(\neg\Diamond\neg p \supset \Box p)$' belongs to either **S0.5.5** or **S0.5.B4**.

*Remark* 3.9. Logics considered here can also be built in the set $\mathrm{For}_\Box$. Facts 3.8 and 3.8 show the differences between the two approaches. Moreover, we will show that for versions built in the set $\mathrm{For}_\Box$, we can omit abbreviations of (5), (B) and ($\mathtt{T}^{\mathtt{m}}_{\mathtt{d}}$).

Indeed, ($5_{\mathtt{d}}$) is an abbreviation of '$\neg\Box\neg\Box p \supset \Box p$'. From it, by **PL** and the substitution $p/\neg p$, we have '$\neg\Box\neg p \supset \Box\neg\Box\neg p$', an abbreviation of (5). Therefore, this last shortcut belongs to $\mathbf{S0.5}^{\circ}_{\Box}[5_{\mathtt{d}}]$.

($\mathtt{B}_{\mathtt{d}}$) is an abbreviation of '$\neg\Box\neg\Box p \supset p$'. From it, by **PL** and the substitution $p/\neg p$, we have '$p \supset \Box\neg\Box\neg p$', an abbreviation of (B). Therefore, this last shortcut belongs to $\mathbf{S0.5}^{\circ}_{\Box}[\mathtt{B}_{\mathtt{d}}]$.

($\mathtt{T}^{\mathtt{m}}$) is an abbreviation of '$\neg\Box\neg\Box p \supset \neg\Box\neg p$'. From it, by **PL** and the substitution $p/\neg p$, we have '$\Box\neg\neg p \supset \Box\neg\Box\neg p$'. Hence, by (pe), we have '$\Box p \supset \Box\neg\Box\neg p$, an abbreviation of ($\mathtt{T}^{\mathtt{m}}_{\mathtt{d}}$). Therefore, this last shortcut belongs to $\mathbf{S0.5}^{\circ}_{\Box}[\mathtt{T}^{\mathtt{m}}]$. □

### 3.4. Completeness

Let $\boldsymbol{L}$ be a t-normal logic and $\langle w_\Gamma, A_\Gamma, V_\Gamma\rangle$ be the canonical model for $\boldsymbol{L}$ and $\Gamma \in \mathrm{Max}_{\boldsymbol{L}}$ (see Appendix A.2).

LEMMA 3.10. *Let $\chi$ be a formula from* (4), ($4_{\mathtt{d}}$), (B), ($\mathtt{B}_{\mathtt{d}}$), (5), ($5_{\mathtt{d}}$), ($\mathtt{D}^{\mathtt{m}}$), ($\mathtt{D}^{\mathtt{m}}_{\mathtt{d}}$), ($\mathtt{T}^{\mathtt{m}}$), ($\mathtt{T}^{\mathtt{m}}_{\mathtt{d}}$). *If $\boldsymbol{L}$ contains $\chi$, then any formula $\varphi$ satisfies condition* (c$\chi\varphi$) *in $\langle w_\Gamma, A_\Gamma, V_\Gamma\rangle$.*

PROOF: For any $\varphi \in$ For, using the definition of canonical models and Lemmas A.1 and A.5, and conditions ($V^w_{\Box}$) and ($V^w_{\Diamond}$) for $V_\Gamma$, we obtain that $\varphi$ satisfies condition (c$\chi\varphi$) in $\langle w_\Gamma, A_\Gamma, V_\Gamma\rangle$. □

Let $\Phi$ be a non-empty set of formulas which contains some or all of the formulas used as additional axioms (including (T), (D) and ($\mathtt{T}^{\mathtt{q}}$)). We put $\boldsymbol{L} := \mathbf{S0.5}^{\circ}[\Phi]$. Let $\mathsf{M}_{\boldsymbol{L}}$ be the class of all $\boldsymbol{L}$-models. From Lemmas A.5, A.6 and 3.10 we have:

FACT 3.11. All canonical models for $\boldsymbol{L}$ belong to $\mathsf{M}_{\boldsymbol{L}}$.

We can show that $\boldsymbol{L}$ is complete with respect to $\mathsf{M}_{\boldsymbol{L}}$.

THEOREM 3.12. *All formulas valid in the class $\mathsf{M}_{\boldsymbol{L}}$ are theses of $\boldsymbol{L}$.*

PROOF: Let $\varphi$ be valid in $\mathsf{M}_{\boldsymbol{L}}$ and $\Gamma \in \mathrm{Max}_{\boldsymbol{L}}$. By Fact 3.11, the canonical model $\langle w_\Gamma, A_\Gamma, V_\Gamma\rangle$ for $\boldsymbol{L}$ and $\Gamma$ belongs to $\mathsf{M}_{\boldsymbol{L}}$. So $V_\Gamma(\varphi, w_\Gamma) = 1$. Hence $\varphi \in \Gamma$. Therefore, $\varphi$ belongs to all $\boldsymbol{L}$-maximal sets. Hence $\varphi \in \boldsymbol{L}$, by Lemma A.3(2). □

## 4. Determining K45, KB4, KD45 and S5 without using modal rules

Using Lemma 3.5 and Theorems 1.3, 1.4 and 3.12, we obtain (†) and (‡), i.e., **K45**, **KB4**, **KD45** and **S5** can be defining without using modal rules.

THEOREM 4.1. *(†) and (‡) hold.*

PROOF: It is obvious that $\mathbf{S0.5^{\circ}.45} \subseteq \mathbf{K45}$, $\mathbf{S0.5^{\circ}.B45} \subseteq \mathbf{KB4}$ (= **KB5**), $\mathbf{S0.5^{\circ}.D45} \subseteq \mathbf{KD45}$ and $\mathbf{S0.5.45} \subseteq \mathbf{S5}$. We will show that we also have the reverse inclusions.

For $\mathbf{S5} \subseteq \mathbf{S0.5.45}$: Suppose that $\varphi \in \mathbf{S5}$. We will prove that $\varphi$ is valid in $\mathsf{M}_{\text{S0.5.45}}$. Let $\langle w, A, V\rangle$ be any **S0.5.45**-model. Then, by Lemma 3.5, $\langle W, V\rangle$ is a universal Kripke model. So, by the assumption and Theorem 1.3, for any $x \in W$ we have $V(\varphi, x) = 1$. So also $V(\varphi, w) = 1$; i.e., $\varphi$ is true in $\langle w, A, V\rangle$. Therefore, $\varphi$ is valid in $\mathsf{M}_{\text{S0.5.45}}$. Hence $\varphi \in \mathbf{S0.5.45}$, by Theorem 3.12.

Similarly, using Lemma 3.5 and Theorems 1.4 and 3.12, we obtain that $\mathbf{K45} \subseteq \mathbf{S0.5^{\circ}.45}$, $\mathbf{KB4} \subseteq \mathbf{S0.5^{\circ}.B45}$ and $\mathbf{KD45} \subseteq \mathbf{S0.5^{\circ}.D45}$. □

## A. Completeness of S0.5°, S0.5°[D], S0.5°[Tq], S0.5

The results reported here are adapted for $\mathbf{S0.5^{\circ}}$, $\mathbf{S0.5^{\circ}}[\mathtt{D}]$, $\mathbf{S0.5^{\circ}}[\mathtt{T^q}]$ and **S0.5** built-in For from those obtained in [5] (where these logics are analyzed in $\mathrm{For}_{\Box}$ and the broader class of t-regular logics is analyzed).

### A.1. Notions and facts concerning maximal consistent sets

Let $\boldsymbol{L}$ be a consistent t-normal logic. A set $\Psi$ is $\boldsymbol{L}$-*consistent* iff for some $\varphi \in \mathrm{For}$ we have $\Psi \nvdash_{\boldsymbol{L}} \varphi$; equivalently in the light of **PL**, iff $\Psi \nvdash_{\boldsymbol{L}} p \wedge \neg p$. Every $\boldsymbol{L}$-consistent set is **PL**-consistent.

We say that $\Gamma$ is $\boldsymbol{L}$-*maximal* iff $\Gamma$ is $\boldsymbol{L}$-consistent and $\Gamma$ has only $\boldsymbol{L}$-inconsistent proper extensions. By changing $\boldsymbol{L}$ to **PL**, we will obtain the definition of **PL**-*maximal sets*. Let $\mathrm{Max}_{\boldsymbol{L}}$ and $\mathrm{Max}_{\text{PL}}$ be the sets of all $\boldsymbol{L}$-maximal and **PL**-maximal sets, respectively.

We will use the following lemmas (which can be proven as in [1]).

LEMMA A.1. *Let* $\Gamma \in \mathrm{Max}_{\boldsymbol{L}}$. *Then* $\boldsymbol{L} \subseteq \Gamma$ *and for all* $\varphi, \psi \in \mathrm{For}$:

1. $\Gamma \vdash_{\boldsymbol{L}} \varphi$ *iff* $\varphi \in \Gamma$.

2. $\ulcorner\neg\varphi\urcorner \in \Gamma$ *iff* $\varphi \notin \Gamma$.
3. $\ulcorner\varphi \wedge \psi\urcorner \in \Gamma$ *iff both* $\varphi \in \Gamma$ *and* $\psi \in \Gamma$.
4. $\ulcorner\varphi \vee \psi\urcorner \in \Gamma$ *iff either* $\varphi \in \Gamma$ *or* $\psi \in \Gamma$.
5. $\ulcorner\varphi \supset \psi\urcorner \in \Gamma$ *iff either* $\varphi \notin \Gamma$ *or* $\psi \in \Gamma$.
6. $\ulcorner\varphi \equiv \psi\urcorner \in \Gamma$ *iff either* $\varphi, \psi \in \Gamma$ *or* $\varphi, \psi \notin \Gamma$.

Notice that from Lemma A.1(2) we obtain:

FACT A.2. Every $\boldsymbol{L}$-maximal set is **PL**-maximal.

LEMMA A.3. *For all* $\Psi \subseteq \mathrm{For}$ *and* $\varphi \in \mathrm{For}$:

1. $\Psi \vdash_{\boldsymbol{L}} \varphi$ *iff* $\varphi \in \Gamma$ *for each* $\Gamma \in \mathrm{Max}_{\boldsymbol{L}}$ *such that* $\Psi \subseteq \Gamma$.
2. $\varphi \in \boldsymbol{L}$ *iff* $\varphi \in \Gamma$ *for each* $\Gamma \in \mathrm{Max}_{\boldsymbol{L}}$.

LEMMA A.4. *For all* $\Gamma \in \mathrm{Max}_{\boldsymbol{L}}$ *and* $\varphi \in \mathrm{For}$ *the following conditions are equivalent*:

(a) $\ulcorner\Box\varphi\urcorner \in \Gamma$.
(b) $\Gamma \vdash_{\boldsymbol{L}} \Box\varphi$.
(c) $\{\psi : \ulcorner\Box\psi\urcorner \in \Gamma\} \vdash_{\mathbf{PL}} \varphi$.
(d) $\varphi \in \Delta$ *for each* $\Delta \in \mathrm{Max}_{\mathbf{PL}}$ *such that* $\{\, \psi : \ulcorner\Box\psi\urcorner \in \Gamma \,\} \subseteq \Delta$.

PROOF: "(a) $\Rightarrow$ (d)" It is trivial. "(d) $\Leftrightarrow$ (c)" By Lemma A.3(1).

"(c) $\Rightarrow$ (b)" Ether $\varphi \in \mathbf{PL}$ or for some $\psi_1, \ldots, \psi_n \in \{\psi : \ulcorner\Box\psi\urcorner \in \Gamma\}$, $n > 0$, we have $\ulcorner(\psi_1 \wedge \cdots \wedge \psi_n) \supset \varphi\urcorner \in \mathbf{PL}$. But the first case entails the second case. Hence $\ulcorner(\Box\psi_1 \wedge \cdots \wedge \Box\psi_n) \supset \Box\varphi\urcorner \in \boldsymbol{L}$, by (pk). But $\Gamma$ contains each of $\ulcorner\Box\psi_1\urcorner, \ldots, \ulcorner\Box\psi_n\urcorner$ since $\Box\mathbf{PL} \subseteq \Gamma$. So $\Gamma \vdash_{\boldsymbol{L}} \Box\varphi$.

"(a) $\Leftrightarrow$ (b)" By Lemma A.1(1). □

## A.2. Canonical models. Completeness

Let $\boldsymbol{L}$ be a t-normal logic and $\Gamma \in \mathrm{Max}_{\boldsymbol{L}}$. We say that $\langle w_\Gamma, A_\Gamma, V_\Gamma\rangle$ is the *canonical model for* $\boldsymbol{L}$ *and* $\Gamma$ iff it satisfies the following conditions:

- $w_\Gamma := \Gamma$,
- $A_\Gamma := \{\, \Delta \in \mathrm{Max}_{\mathbf{PL}} : \forall_{\psi \in \mathrm{For}}(\ulcorner\Box\psi\urcorner \in \Gamma \Rightarrow \psi \in \Delta\,) \,\}$,
- $V_\Gamma\colon \mathrm{For} \times (\{w_\Gamma\} \cup A_\Gamma) \to \{0, 1\}$ is the valuation such that for all $\varphi \in \mathrm{For}$ and $\Delta \in \{w_\Gamma\} \cup A_\Gamma$

$$V_\Gamma(\varphi,\Delta) \coloneqq \begin{cases} 1 & \text{if } \varphi \in \Delta \\ 0 & \text{otherwise} \end{cases}$$

We need the following lemmas to prove the completeness of **S0.5**$^\circ$, **S0.5**$^\circ$[D], **S0.5**$^\circ$[T$^q$] and **S0.5**.

Lemma A.5. *$\langle w_\Gamma, A_\Gamma, V_\Gamma\rangle$ is a tn-model.*

Proof: Thanks to properties of maximal sets (see Lemma A.1), for every $\Delta \in \{w_\Gamma\} \cup A_\Gamma$ the assignment $V_\Gamma(\cdot,\Delta)$ belongs to $\mathsf{Val}^{\mathsf{cl}}$. Moreover, we prove that $V_\Gamma(\cdot, w_\Gamma)$ satisfies ($V^w_\Box$) and ($V^w_\Diamond$) for each $\varphi \in$ For.

Firstly, $V_\Gamma(\Box\varphi, w_\Gamma) = 1$ iff $\ulcorner\Box\varphi\urcorner \in \Gamma$ iff $\varphi \in \Delta$ for each $\Delta \in \mathrm{Max}_{\mathrm{PL}}$ such that $\{\psi \in \mathrm{For} : \ulcorner\Box\psi\urcorner \in \Gamma\} \subseteq \Delta$ (by Lemma A.4) iff $\varphi \in \Delta$ for each $\Delta \in A_\Gamma$ iff $V_\Gamma(\varphi,\Delta) = 1$ for each $\Delta \in A_\Gamma$.

Secondly, since $\boldsymbol{L} \subseteq \Gamma$, $\ulcorner\Diamond\varphi \equiv \neg\Box\neg\varphi\urcorner \in \Gamma$. Hence, by Lemma A.1, $V_\Gamma(\Diamond\varphi, w_\Gamma) = 1$ iff $\ulcorner\Diamond\varphi\urcorner \in \Gamma$ iff $\ulcorner\Box\neg\varphi\urcorner \notin \Gamma$ iff $V_\Gamma(\neg\varphi,\Delta) = 0$ for some $\Delta \in A_\Gamma$ iff $V_\Gamma(\varphi,\Delta) = 1$ for some $\Delta \in A_\Gamma$. □

Lemma A.6. 1. *If $\boldsymbol{L}$ contains (T) then $w_\Gamma \in A_\Gamma$.*

2. *If $\boldsymbol{L}$ contains (D) then $A_\Gamma \neq \emptyset$.*

3. *If $\boldsymbol{L}$ contains (T$^q$) then either $A_\Gamma = \emptyset$ or $w_\Gamma \in A_\Gamma$.*

Proof: By Lemma A.1, $\boldsymbol{L} \subseteq \Gamma$. So in any specific case we have:

1. For any $\psi \in$ For, $\ulcorner\Box\psi \supset \psi\urcorner \in \Gamma$. So, if $\ulcorner\Box\psi\urcorner \in \Gamma$ then $\psi \in \Gamma$, by Lemma A.1(5). Hence $\Gamma \in A_\Gamma$. Moreover, $\Gamma \in \mathrm{Max}_{\mathrm{PL}}$, by Fact A.2.

2. For any $\tau \in \mathrm{Taut}_{\mathrm{cl}}$ we have $\ulcorner\Box\tau\urcorner$ and $\ulcorner\Box\tau \supset \Diamond\tau\urcorner$ belong to $\Gamma$. So, $\Diamond\tau \in \Gamma$, by Lemma A.1(5). Hence $V(\Diamond\tau,\Gamma) = 1$. So, by Lemma A.5, for some $\Delta \in A_\Gamma$ we have $V(\tau,\Delta) = 1$. Therefore, $A_\Gamma \neq \emptyset$.

3. For any $\psi \in$ For we have $\ulcorner(\mathrm{D}) \supset (\Box\psi \supset \psi)\urcorner \in \Gamma$. Suppose that $A_\Gamma \neq \emptyset$. Then (D) $\in \Gamma$, by Fact 2.4(1) and Lemma A.5. Thus, $\ulcorner\Box\psi \supset \psi\urcorner \in \Gamma$. Therefore, as in item 1, we can show that $\Gamma \in A_\Gamma$. □

For $\boldsymbol{L} \in \{\mathbf{S0.5}^\circ, \mathbf{S0.5}^\circ[\mathrm{D}], \mathbf{S0.5}^\circ[\mathrm{T}^q], \mathbf{S0.5}\}$. Let $\mathbf{M}_{\boldsymbol{L}}$ be the class of all $\boldsymbol{L}$-models. We can show that $\boldsymbol{L}$ is complete with respect to $\mathbf{M}_{\boldsymbol{L}}$.

Theorem A.7 (Completeness). *All formulas valid in $\mathbf{M}_{\boldsymbol{L}}$ are theses of $\boldsymbol{L}$.*

Proof: For **S0.5**$^\circ$: Suppose that $\varphi$ is valid in $\mathbf{M}_{\mathbf{S0.5}^\circ}$ and $\Gamma \in \mathrm{Max}_{\mathbf{S0.5}^\circ}$. By Lemma A.5, $\langle w_\Gamma, A_\Gamma, V_\Gamma\rangle$ belongs to $\mathbf{M}_{\mathbf{S0.5}^\circ}$. Thus, $V_\Gamma(\varphi, w_\Gamma) = 1$. Hence $\varphi \in \Gamma$. So, we have shown that $\varphi$ belongs to all **S0.5**$^\circ$-maximal sets. Hence $\varphi \in \mathbf{S0.5}^\circ$, by Lemma A.3(2).

For $\boldsymbol{L} \in \{\mathbf{S0.5}^{\circ}[\mathtt{D}], \mathbf{S0.5}^{\circ}[\mathtt{T^q}], \mathbf{S0.5}\}$: Same as above, taking $\boldsymbol{L}$ instead of $\mathbf{S0.5}^{\circ}$. By Lemmas A.5 and A.6, $\langle w_\Gamma, A_\Gamma, V_\Gamma\rangle$ belongs to $\mathsf{M}_{\boldsymbol{L}}$. □

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# HILBERT ALGEBRAS WITH HILBERT-GALOIS CONNECTIONS II

**Abstract**

Hilbert algebra with a Hilbert-Galois connection, or HilGC-algebra, is a triple $(A, f, g)$ where $A$ is a Hilbert algebra, and $f$ and $g$ are unary maps on $A$ such that $f(a) \leq b$ iff $a \leq g(b)$, and $g(a \to b) \leq g(a) \to g(b)$ for all $a, b \in A$. In this paper, we are going to prove that some varieties of HilGC-algebras are characterized by first-order conditions defined in the dual space and that these varieties are canonical. Additionally, we will also study and characterize the congruences of an HilGC-algebra through specific closed subsets of the dual space. This characterization will be applied to determine the simple algebras and subdirectly irreducible HilGC-algebras.

*Keywords*: Hilbert algebras, modal operators, Galois connection, canonical varieties, congruences.

*2020 Mathematical Subject Classification*: 03B45, 03B60.

## 1. Introduction

This paper can be read as a continuation of [6] where we defined the notion of Hilbert-Galois algebra. Recall that an order-preserving connection in a Hilbert algebra $A$ is a pair $(f, g)$, where $f, g : A \to A$ are order-preserving maps such that $a \leq (g \circ f)(a)$ and $(f \circ g)(a) \leq a$, for $a \in A$ (see Definition 2.6). A Hilbert-Galois connection on $A$ is an order-preserving connection $(f, g)$ such that $g$ is a Hilbert semi-homomorphism, i.e., $g(a \to b) \leq g(a) \to$

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$g(b)$, for all $a, b \in A$. A Hilbert algebra with a Hilbert-Galois connection, or HilGC-algebra, is a triple $(A, f, g)$ where $A$ is a Hilbert algebra and the pair $(f, g)$ is a Hilbert-Galois connection. As shown in [6], the class of HilGC-algebras is a variety. Moreover, it was proved that there exists a topological duality between the category of HilGC-algebras and the class of Hilbert-Galois spaces. A Hilbert-Galois space is a structure $(X, \mathcal{T}_{\mathcal{K}}, R)$ where $(X, \mathcal{T}_{\mathcal{K}})$ is a Hilbert space (the dual space of a Hilbert algebra), and $R$ is a binary relation on $X$ satisfying certain conditions (see Definition 2.12).

In this paper we applied the representation developed in [6] to characterize some subvarieties of HilGC-algebras in terms of first-order conditions defined in the dual space. As consequence of this characterization, we show that these varieties are canonical. The duality given in [6] is applied to study the congruences of HilGC-algebras. We prove that the lattice of congruences of a HilGC-algebra $(A, f, g)$ is isomorphic to the lattice of Galois implicative filters (Definition 4.1), and dually isomorphic to the lattice of certain closed subsets of the dual space of $(A, f, g)$ called $G$-closed (Definition 4.3). The characterization is applied to study the simple and subdirectly irreducible HilGC-algebras.

## 2. Preliminaries

We assume that the reader is familiar with basic concepts with Hilbert algebras and with the duality between the category of Hilbert algebras and Hilbert homomorphisms, and the category of Hilbert spaces and $H$-functional relations [2, 3, 4, 5, 8]. Nevertheless, in this section we will recall the definitions, results and notations that will be needed in the rest of this paper.

Let $\langle X, \leq \rangle$ be a poset and consider the powerset $\mathcal{P}(X)$. Let $Y \subseteq X$. We say that $Y$ is an *upset* (resp. *downset*) if $Y = \{x \in X : \exists y \in Y\ (y \leq x)\} = [Y)$ (resp. $Y = \{x \in X : \exists y \in Y\ (x \leq y)\} = (Y]$) . The set of *all upset* subsets of $X$ is denoted by $\mathrm{Up}\,(X)$. The set complement of a subset $Y \subseteq X$ is denoted by $Y^c$.

The purely implicational subreducts of Heyting algebras are known in the literature as Hilbert algebras, or (positive) implication algebras [7, 8, 9].

Definition 2.1. A *Hilbert algebra* is an algebra $A = \langle A, \rightarrow, 1\rangle$ of type $(2, 0)$ such that the following axioms hold in $A$:

1. $a \rightarrow a = 1$,
2. $1 \rightarrow a = a$,
3. $a \rightarrow (b \rightarrow c) = (a \rightarrow b) \rightarrow (a \rightarrow c)$,
4. $(a \rightarrow b) \rightarrow ((b \rightarrow a) \rightarrow a) = (b \rightarrow a) \rightarrow ((a \rightarrow b) \rightarrow b)$.

Hilbert algebras form a variety denoted by Hil. Every Hilbert algebra $A$ has a natural order $\leq$ defined by $a \leq b$ iff $a \rightarrow b = 1$. Given a Hilbert algebra $A$ and a sequence $a, a_1, \ldots, a_n \in A$, we define:

$$(a_1, \ldots, a_n; a) = \begin{cases} a_1 \rightarrow a & \text{if} \quad n = 1, \\ a_1 \rightarrow (a_2, \ldots, a_n; a) & \text{if} \quad n > 1. \end{cases}$$

A nonempty subset $F \subseteq A$ is an *implicative filter* of $A$ if $1 \in F$, and if $a, a \rightarrow b \in F$ then $b \in F$. The set of all implicative filters of $A$ is denoted by $\mathrm{Fi}(A)$. Note that every implicative filter of $A$ is an upset of $A$. Let $S \subseteq A$. The implicative filter generated by $S$ is $\langle S\rangle = \bigcap\{F \in \mathrm{Fi}(A) : S \subseteq F\}$. The deductive system generated by a subset $S \subseteq A$ can be characterized as the set

$$\langle S\rangle = \{a \in A : \exists\, \{a_1, \ldots, a_n\} \subseteq S : (a_1, \ldots, a_n; a) = 1\}\,.$$

The following result is proved in [2] and [9] and we will be useful in this paper:

Lemma 2.2. *Let* $A \in$ Hil. *Let* $F \in \mathrm{Fi}(A)$ *and* $a \in A$. *Then,*

$$F \vee \langle a\rangle = \langle F \cup \{a\}\rangle = \{b \in A : a \rightarrow b \in F\ \}\,.$$

Let $F \in \mathrm{Fi}(A) - \{A\}$. We will say that $F$ is *irreducible* if for any $F_1, F_2 \in \mathrm{Fi}(A)$ such that $F = F_1 \cap F_2$, it follows that $F = F_1$ or $F = F_2$. The set of all irreducible implicative filters of a Hilbert algebra $A$ is denoted by $X(A)$. A downset $I$ of $A$ is called an *order-ideal of* $A$ if for all $a, b \in I$ there exists $c \in I$ such that $a \leq c$ and $b \leq c$. The set of all order-ideals of $A$ is denoted by $\mathrm{Ido}(A)$.

The following is a Hilbert algebra analogue of Birkhoff's Prime Filter Theorem and it is proved in [3].

Theorem 2.3. *Let* $A \in \text{Hil}$. *Let* $F \in \text{Fi}(A)$ *and let* $I \in \text{Ido}(A)$ *such that* $F \cap I = \emptyset$. *Then, there exists* $x \in X(A)$ *such that* $F \subseteq x$ *and* $x \cap I = \emptyset$.

Corollary 2.4. Let $A \in \text{Hil}$. Then,

1. for all $a, b \in A$, if $a \nleq b$, then there exists $x \in X(A)$ such that $a \in x$ and $b \notin x$.
2. If $x \in X(A)$ and $a, b \notin x$, there exists $c \notin x$ such that $a, b \leq c$.
3. If $x \in X(A)$, then $a \to b \notin x$ iff there exists $y \in X(A)$ such that $x \subseteq y$, $a \in y$ and $b \notin y$.

Let $\langle X, \mathcal{T} \rangle$ be a topological space. We recall that *the specialization dual order of* $\langle X, \mathcal{T} \rangle$ is the binary relation $\leq \subseteq X \times X$ defined by:

$$x \leq y \text{ iff } \forall W \in \mathcal{T} (x \notin W \text{ then } y \notin W). \tag{2.1}$$

If $\langle X, \mathcal{T} \rangle$ is $T_0$, then $\langle X, \leq \rangle$ is a poset. Now we define the Hilbert spaces as special $T_0$ topological spaces $\langle X, \mathcal{T}_{\mathcal{K}} \rangle$ having a base of compact sets $\mathcal{K}$. Recall also that a subset $Y \subseteq X$ is said to be irreducible when, for all closed sets $Y_1, Y_2 \subseteq X$, we have that $Y = Y_1 \cup Y_2$ entails $Y = Y_1$ or $Y = Y_2$. A space is *sober* when, for every irreducible closed set $Y \subseteq X$ there exists a unique $x \in X$ such that $Y = \text{cl}(x)$. Let $D(X) := \{U : U^c \in \mathcal{K}\}$.

Definition 2.5. [4] An *H-space* is a topological space $\langle X, \mathcal{T}_{\mathcal{K}} \rangle$ such that:

(H1) $\mathcal{K}$ is a base of open and compact subsets for the topology $\mathcal{T}_{\mathcal{K}}$,

(H2) $U \Rightarrow V = (U \cap V^c]^c \in D(X)$, for all $U, V \in D(X)$,

(H3) $\langle X, \mathcal{T}_{\mathcal{K}} \rangle$ is sober.

If $\langle X, \mathcal{T}_{\mathcal{K}} \rangle$ is an $H$-space, then it is easy to see that $D(X) = \langle D(X), \Rightarrow, X \rangle$ is a Hilbert algebra.

Let $A \in \text{Hil}$. Then $A$ is isomorphic to the subalgebra $D(X(A)) = \{\varphi(a) : a \in A\}$ of the Hilbert algebra $\langle \text{Up}(X(A)), \Rightarrow, X(A) \rangle$ via the map $\varphi : A \to \text{Up}(X(A))$ defined by $\varphi(a) = \{x \in X(A) : a \in x\}$. From the results on representation for Hilbert algebras in [4] we have that $\langle X(A), \mathcal{T}_{\mathcal{K}_A} \rangle$ is an $H$-space where the family $\mathcal{K}_A = \{\varphi(a)^c : a \in A\}$ is a base of compact subsets.

Next we will review definitions and properties of Hilbert algebras with Hilbert-Galois connections introduced in [6].

Definition 2.6. [6] Let $A \in$ Hil. A pair $(f, g)$, where $f : A \to A$ and $g : A \to A$ are maps, is called *Hilbert-Galois connection* between in $A$ if

1. $f(a) \leq b$ iff $a \leq g(b)$, for all $a, b \in A$.
2. $g(b \to c) \leq g(b) \to g(c)$, for all $b, c \in A$, i.e., $g$ is a semi-homomorphism on $A$.

A triple $\langle A, f, g\rangle$ is a *Hilbert algebra with a Hilbert-Galois connection*, or a HilGC-algebra for short, if $(f, g)$ is a Hilbert-Galois connection defined in $A$.

In [6] we give the following equational characterization of HilGC-algebras.

Theorem 2.7. *[6] Let $A \in$ Hil and let $f : A \to A$ and $g : A \to A$ two maps. Then $\langle A, f, g\rangle$ is a HilGC-algebra if and only if the maps $f$ and $g$ satisfy the following conditions for all $a, b \in A$:*

(HilGC1) $g(1) = 1$.

(HilGC2) $f(a) \leq f((a \to b) \to b)$.

(HilGC3) $g(a \to b) \to (g(a) \to g(b)) = 1$.

(HilGC4) $a \to g(f(a)) = 1$.

(HilGC5) $f(g(a)) \to a = 1$.

In [6] we prove that $f, g$ are monotonic maps. We denote by HilGC the variety of HilGC-algebras.

Proposition 2.8. Let $\langle A, f, g\rangle$ be a HilGC-algebra. Then:

(1) If $F \in \mathrm{Fi}(A)$ then $g^{-1}(F) \in \mathrm{Fi}(A)$,

(2) If $x \in X(A)$ then $f^{-1}(x^c), (g(x^c)] \in \mathrm{Ido}(A)$.

Proof: Item (1) and the affirmation $f^{-1}(x^c) \in \mathrm{Ido}(A)$, for each $x \in X(A)$, are proved in Proposition 14 of [6]. We prove that $(g(x^c)] \in \mathrm{Ido}(A)$. Assume that $x \in X(A)$ and let $a, b \in (g(x^c)]$. Then there exist $c, d \notin x$ such that $a \leq g(c)$ and $b \leq g(d)$, or equivalently, $f(a) \leq c$ and $f(b) \leq d$. Since $c, d \notin x$, by Corollary 2.4, there exists $e \notin x$ such that $c, d \leq e$. Thus, $f(a) \leq e$ and $f(b) \leq e$ and consequently, $a \leq g(e)$ and $b \leq g(e)$. Since $e \in x^c$ results, $g(e) \in (g(x^c)]$, and thus $(g(x^c)] \in \mathrm{Ido}(A)$. □

Lemma 2.9. *[6, Lemma 21] Let* $\langle A, f, g\rangle \in \mathsf{HilGC}$*. Then*

1. *Let* $x \in X(A)$*. For all* $a \in A$*,* $g(a) \notin x$ *iff there exists* $y \in X(A)$ *such that* $g^{-1}(x) \subseteq y$ *and* $a \notin y$*,*
2. *Let* $x \in X(A)$*. For all* $a \in A$*,* $f(a) \in x$ *iff there exists* $y \in X(A)$ *such that* $y \subseteq f^{-1}(x)$ *and* $a \in y$*.*

We recall that a *IntGC-frame* is a relational structure $\langle X, \leq, R\rangle$ where $\langle X, \leq\rangle$ is a poset and $R \subseteq X \times X$ is a relation satisfying the condition

$$\leq^{-1} \circ R \circ \leq^{-1} \subseteq R. \tag{2.2}$$

We note that by the condition (2.2) and the reflexivility of $\leq^{-1}$, we have that $\leq^{-1} \circ R = R$ and $R \circ \leq^{-1} = R$.

Lemma 2.10. *Let* $\mathcal{F} = \langle X, \leq, R\rangle$ *be an IntGC-frame. Then,*

$$R^{-1}(x), R(x)^c \in \mathrm{Up}(X), \text{ for each } x \in X.$$

Proof: Let $x \in X$. We prove that $R(x)^c \in \mathrm{Up}(X)$. Let $y \leq z$ and $y \in R(x)^c$. Suppose that $z \in R(x)$. As $(z, y) \in \leq^{-1}$ results $(x, y) \in R$ and so, $y \in R(x)$, which is an absurd. Thus, $z \in R(x)^c$. Similarly, we can prove that $R^{-1}(x) \in \mathrm{Up}(X)$. □

It is know that if $\langle X, \leq\rangle$ is a poset, then $\langle \mathrm{Up}(X), \cup, \cap, \Rightarrow, \emptyset, X\rangle$ is a Heyting algebra. Moreover, we define the operators $f_R : \mathrm{Up}(X) \to \mathrm{Up}(X)$, and $g_R : \mathrm{Up}(X) \to \mathrm{Up}(X)$ as

$$f_R(U) = \{x \in X : R(x) \cap U \neq \emptyset\} = R^{-1}(U), \tag{2.3}$$

and

$$g_R(U) = \left\{x \in X : R^{-1}(x) \subseteq U\right\}, \tag{2.4}$$

for each $U \in \mathrm{Up}(X)$, respectively. The condition (2.2) ensures that $A(\mathcal{F}) = \langle \mathrm{Up}(X), \cup, \cap, \Rightarrow, f_R, g_R, \emptyset, X\rangle$ is a Heyting-Galois algebra, and in particular is a Hilbert-Galois algebra (see [6] Example 19).

If $\langle A, f, g\rangle \in \mathsf{HilGC}$, then $\mathcal{F}(A) = \langle X(A), \subseteq, R_A\rangle$ is an IntGC-frame, where the relation $R_A \subseteq X(A) \times X(A)$ is defined by

$$(x, y) \in R_A \text{ iff } y \subseteq f^{-1}(x).$$

By [6, Lemma 24], the relation $R_A$ can be also defined as

$$(x, y) \in R_A \text{ iff } g^{-1}(y) \subseteq x.$$

The following representation theorem for HilGC-algebras follows from the results given in [6].

THEOREM 2.11 (of Representation). *Let $A = \langle A, f, g\rangle \in \mathsf{HilGC}$. Then the map $\varphi : A \to A(\mathcal{F}(A))$ is an embedding. Thus, $A$ is isomorphic to some subalgebra of $A(\mathcal{F}(A))$.*

Now we recall the dual topological spaces of HilGC-algebras.

DEFINITION 2.12. [6, Def. 22] $\langle X, \mathcal{T}_{\mathcal{K}}, R\rangle$ is a *Hilbert-Galois space*, or *HG-space*, if $\langle X, \mathcal{T}_{\mathcal{K}}\rangle$ is an $H$-space and

1. $R^{-1}(U) \in D(X)$, for all $U \in D(X)$,
2. $R(U^c)^c \in D(X)$, for all $U \in D(X)$,
3. $R^{-1}(x)$ is a closed subset of $\langle X, \mathcal{T}_{\mathcal{K}}\rangle$, for all $x \in X$.

In [6] was proved that if $\langle X, \mathcal{T}_{\mathcal{K}}, R\rangle$ is an $HG$-space, then $\langle D(X), \Rightarrow, f_R, g_R, X\rangle$ is a HilGC-algebra where the operators $f_R : D(X) \to D(X)$ and $g_R : D(X) \to D(X)$ are defined by (2.3) and (2.4), respectively. Moreover, the map $\varepsilon_X : X \to X(D(X))$ given by $\varepsilon(x) = \{U \in D(X) : x \in U\}$ is a homeomorphism such that $(x, y) \in R$ iff $(\varepsilon(x), \varepsilon(y)) \in R_{D(X)}$, for all $x, y \in X$. If $\langle A, f, g\rangle \in \mathsf{HilGC}$, then $\langle X(A), \mathcal{T}_{\mathcal{K}_A}, R_A\rangle$ is an HG-space such that the map $\varphi : A \to D(X(A))$ given by $\varphi(a) = \{x \in X(A) : a \in x\}$ is an isomorphism of HilGC-algebras. For more details on the duality between HG-spaces and HilGC-algebras see [6].

## 3. Some canonical subvarieties of HGC-algebras

By Theorem 2.11, any HilGC-algebra $A$ is a subalgebra of the HilGC-algebra $A(\mathcal{F}(A))$. The algebra $A(\mathcal{F}(A))$ is known as the canonical extension or canonical embedding algebra of $A$. We shall say that a variety $\mathsf{V}$ of HilGC-algebras is *canonical* it it is closed under canonical extensions, i.e., if $A \in \mathsf{V}$ then $A(\mathcal{F}(A)) \in V$. The notion of canonical varieties is an algebraic formulation of the notion of canonical logics ([1]). In this section we prove that certain varieties of HilGC-algebras are canonical.

*Remark* 3.1. Let $\mathsf{V}$ be a variety of HilGC-algebras. Let

$$Fr(\mathsf{V}) = \{\mathcal{F}(A) : A \in \mathsf{V}\}$$

be the class of IntGC-frames associated to $\mathsf{V}$. Let F be a class of IntGC-frames. Let $\mathrm{Alg}(\mathrm{F}) = \{A(\mathcal{F}) : \mathcal{F} \in \mathrm{F}\}$ be class of HilGC-algebras associated to $\mathsf{F}$. We note that if F is a class of IntGC-frames such that $\mathrm{Alg}(\mathrm{F}) \subseteq \mathsf{V}$ and $Fr(\mathsf{V}) \subseteq \mathrm{F}$, then $\mathsf{V}$ is canonical. Indeed. If $A \in \mathsf{V}$, then $\mathcal{F}(A) \in Fr(\mathsf{V}) \subseteq \mathrm{F}$. So, $A(\mathcal{F}(A)) \in Alg(\mathrm{F}) \subseteq \mathsf{V}$, i.e. $\mathsf{V}$ is canonical.

Let $A$ be a HilGC-algebra . We will write $A \models \alpha \leq \beta$ when the equation $\alpha \wedge \beta \approx \alpha$ is valid in $A$. In the following Theorem 3.2, we characterize some classes of IntGC-frames. In the Theorem 3.3 we prove that some varieties of HilGC-algebras are canonical.

THEOREM 3.2. *Let $\mathcal{F} = \langle X, \leq, R\rangle$ be an IntGC-frame. Then*

1. $A(\mathcal{F}) \models a \leq g(a)$ *iff* $A(\mathcal{F}) \models f(a) \leq a$ *iff* $R \subseteq \leq^{-1}$.
2. $A(\mathcal{F}) \models g(a) \leq a$ *iff* $A(\mathcal{F}) \models a \leq f(a)$ *iff $R$ is reflexive.*
3. $A(\mathcal{F}) \models g(a) \leq g^2(a)$ *iff* $A(\mathcal{F}) \models f^2(a) \leq f(a)$ *iff $R$ is transitive.*
4. $A(\mathcal{F}) \models g^2(a) \leq g(a)$ *iff* $A(\mathcal{F}) \models f(a) \leq f^2(a)$ *iff $R$ is weakly dense, i.e.,* $R \subseteq R^2$.
5. $A(\mathcal{F}) \models f(a) \leq g(a)$ *iff* $A(\mathcal{F}) \models a \leq g^2(a)$ *iff* $A \models f^2(a) \leq a$ *iff*
$$\forall x \forall y \forall z ((x, y) \in R \wedge (y, z) \in R \Rightarrow z \leq x).$$
6. $A(\mathcal{F}) \models g(a) \leq f(a)$ *iff* $R(x) \cap R^{-1}(x) \neq \emptyset$ *for all* $x \in X$.

PROOF: We prove (1), (4), and (6). The others items are left to the reader.

(1) $\Rightarrow$) Let $(x, y) \in R$. Let $U = [y) \in \mathrm{Up}(X)$. As $y \in U \subseteq g_R(U)$, we get $R^{-1}(y) \subseteq U$. So, $x \in [y)$ ,i.e., $y \leq x$.

$\Leftarrow$) Assume that $R \subseteq \leq^{-1}$. Let $x \in U$ and let $y \in R^{-1}(x)$. So, $x \leq y$ and as $U \in \mathrm{Up}(X)$, $y \in U$. Thus, $U \subseteq g_R(U)$.

(4) $\Rightarrow$) Let $(x, y) \in R$. Suppose that $z \notin R(x)$ for all $z \in R^{-1}(y)$. Let $U = (x]^c \in \mathrm{Up}(X)$. We prove that $y \in g_R(g_R(U))$, i.e., $R^{-1}(y) \subseteq g_R(U)$. Let $w \in R^{-1}(y)$. We need to prove that $R^{-1}(w) \subseteq U = (x]^c$, i.e.,

$R^{-1}(w) \cap (x] = \emptyset$. On the contrary, we suppose that there exists $u \in R^{-1}(w)$ such that $u \leq x$. Thus, $(x, w) \in R$, which contradicts our assumption because $w \in R^{-1}(y)$. So, $y \in g_R(g_R(U)) \subseteq g_R(U)$ and consequently, $R^{-1}(y) \subseteq U = (x]^c$. Contradiction, because $x \in R^{-1}(y)$. The direction $\Leftarrow$) is easy and left to the reader.

(6) $\Rightarrow$) Let $x \in X$ and consider $U = R(x)^c$. Suppose that $R(x) \cap R^{-1}(x) = \emptyset$, then $R^{-1}(x) \subseteq R(x)^c$. So, $x \in g_R(U) \subseteq f_R(U)$, i.e., $R(x) \cap R^{-1}(x) \neq \emptyset$, which is a contradiction. Thus, $R(x) \cap R^{-1}(x) \neq \emptyset$.

$\Leftarrow$) Let $U \in \mathrm{Up}(X)$ and let $x \in g_R(U)$, i.e., $R^{-1}(x) \subseteq U$. By assumption, $R(x) \cap R^{-1}(x) \neq \emptyset$. So, there exists $y \in X$ such that $y \in R(x)$ and $y \in R^{-1}(x)$. Thus, $y \in R(x)$ and $y \in U$, and consequently, $x \in f_R(U)$. □

THEOREM 3.3. *Let $A \in \mathsf{HilGC}$. Let $\langle X, \leq, R\rangle$ be the IntGC-frame of $A$. Then,*

1. $A \models a \leq g(a)$ *iff* $A \models f(a) \leq a$ *iff* $R \subseteq \subseteq^{-1}$.
2. $A \models g(a) \leq a$ *iff* $A \models a \leq f(a)$ *iff $R$ is reflexive.*
3. $A \models g(a) \leq g^2(a)$ *iff* $A \models f^2(a) \leq f(a)$ *iff $R$ is transitive.*
4. $A \models g^2(a) \leq g(a)$ *iff* $A \models f(a) \leq f^2(a)$ *iff $R$ is weakly dense, i.e.,* $R \subseteq R^2$.
5. $A \models f(a) \leq g(a)$ *iff* $A \models a \leq g^2(a)$ *iff* $A \models f^2(a) \leq a$ *iff*
$$\forall x \forall y \forall z\, ((x, y) \in R \wedge (y, z) \in R \Rightarrow z \subseteq x)\,.$$
6. $A \models g(a) \leq f(a)$ *iff* $R(x) \cap R^{-1}(x) \neq \emptyset$ *for all* $x \in X$.

PROOF: We will prove only the assertions (2), (4) and (6). The other proofs are analogous.

(2) Assume that $g(a) \leq a$, for all $a \in A$. In particular, for $f(a) \in A$ : $g(f(a)) \leq f(a)$ and by (HilGC4),

$$a \leq g(f(a)) \leq f(a).$$

Similarly, we prove that $A \models a \leq f(a)$ implies $A \models g(a) \leq a$.

To prove that $R$ is reflexive showing that $g^{-1}(x) \subseteq x$, for every $x \in X$. Let $a \in g^{-1}(x)$, then $g\,(a) \in x$. By assumption, $a \in x$. Conversely, suppose that there exists $a \in A$ such that $g(a) \not\leq a$. So, there exists $x \in X$ such that $g(a) \in x$ and $a \notin x$. That is, $g^{-1}(x) \not\subseteq x$ or equivalently, $(x, x) \notin R$.

(4) Assume that $g^2(a) \leq g(a)$ for all $a \in A$. Let $x, y \in X$ such that $(x, y) \in R$, i.e., $g^{-1}(y) \subseteq x$. By Proposition 2.8, $g^{-1}(y) \in \mathrm{Fi}(A)$ and $(g(x^c)] \in \mathrm{Ido}(A)$. Suppose that there exists $a \in g^{-1}(y) \cap (g(x^c)]$. So, $g(a) \in y$ and there exists $b \notin x$ such that $a \leq g(b)$. Thus, $g(a) \leq g^2(b) \leq g(b)$ and consequently, $g(b) \in y$, i.e., $b \in g^{-1}(y)$. By assumption, $b \in x$, which is impossible. So, $g^{-1}(y) \cap (g(x^c)] = \emptyset$ and by Theorem 2.3, there exists $z \in X$ such that $g^{-1}(y) \subseteq z$ and $z \cap (g(x^c)] = \emptyset$. Consequently, $g^{-1}(z) \cap g^{-1}(g(x^c)) = \emptyset$ and as $x^c \subseteq g^{-1}(g(x^c))$, we get $g^{-1}(z) \cap x^c = \emptyset$, i.e., $g^{-1}(z) \subseteq x$. Thus, we have that there exists $z \in X$ such that $(x, z) \in R$ and $(z, y) \in R$, this is, $(x, y) \in R^2$.

Conversely, suppose that there exists $a \in A$ such that $g^2(a) \nleq g(a)$. So, there exists $y \in X(A)$ such that $g^2(a) \in y$ and $g(a) \notin y$. By Lemma 2.9, there exists $x \in X$ such that $(x, y) \in R$ and $a \notin x$. By assumption, $(x, y) \in R^2$, this is, there exists $z \in A$ such that $(x, z) \in R$ and $(z, y) \in R$. So, $g^{-1}(z) \subseteq x$ and $g^{-1}(y) \subseteq z$. As $a \notin x$, we get that $g^2(a) \notin y$, which is a contradiction.

Now, assume that $f(a) \leq f^2(a)$ for all $a \in A$. Let $x, y \in X$ such that $(x, y) \in R$. We will prove that the implicative filter $g^{-1}(y)$ and the order-ideal $f^{-1}(x^c)$ of $A$ are disjoints. On the contrary, suppose that there exists $a \in A$ such that $a \in g^{-1}(y)$ and $a \in f^{-1}(x^c)$, that is, $g(a) \in y$ and $f(a) \notin x$. As $y \subseteq f^{-1}(x)$, we have $f(g(a)) \in x$. By assumption, $f^2(g(a)) \in x$. On the other hand, by (HilGC5), $f(g(a)) \leq a$ and by monotony of $f$, $f^2(g(a)) \leq f(a)$. Since $x$ is an upset of $A$, $f(a) \in x$, which is a contradiction. Thus, $g^{-1}(y) \cap f^{-1}(x^c) = \emptyset$ and so, there exists $z \in X$ such that $g^{-1}(y) \subseteq z$ and $f^{-1}(x^c) \cap z = \emptyset$, i.e., $z \subseteq f^{-1}(x)$, that is, $(z, y) \in R$ and $(x, z) \in R$. Conversely, suppose that there exists $a \in A$ such that $f(a) \nleq f^2(a)$. There exists $x \in X$ such that $f(a) \in x$ and $f^2(a) \notin x$. By Lemma 2.9, there exists $y \in X$ such that $(x, y) \in R$ and $a \in y$. By assumption, there exists $z \in X$ such that $(x, z) \in R$ and $(z, y) \in R$, i.e., $z \subseteq f^{-1}(x)$ and $y \subseteq f^{-1}(z)$. As $a \in y$, results $f(a) \in z$ and consequently, $f^2(a) \in x$, a contradiction.

(6) Let $g(a) \leq f(a)$ for all $a \in A$. Let $x \in X$. We will prove that $g^{-1}(x) \cap f^{-1}(x^c) = \emptyset$. Suppose the contrary. Let $a \in A$ such that $a \in g^{-1}(x)$ and $a \in f^{-1}(x^c)$. As $g(a) \in x$, by assumption we obtain $f(a) \in x$, which is impossible. Thus, there exists $y \in X$ such that $g^{-1}(x) \subseteq y$ and $y \subseteq f^{-1}(x)$. Consequently, $y \in R^{-1}(x) \cap R(x)$.

Now, assume that $R^{-1}(x) \cap R(x) \neq \emptyset$ for all $x \in X$ and suppose that there exists $a \in A$ such that $g(a) \nleq f(a)$. So, there exists $x \in X$ such that $g(a) \in x$ and $f(a) \notin x$. By assumption, there exists $y \in X$ such that $g^{-1}(x) \subseteq y$ and $y \subseteq f^{-1}(x)$. As $a \in g^{-1}(x)$, we obtain $f(a) \in x$, which is contradiction. □

We denote by $\mathsf{V}_\Gamma$ be the variety of HilGC-algebras generated by the set of equations $\Gamma$. Let us consider the set of equations . By Theorem 3.2 and Theorem 3.3 we have the following result.

THEOREM 3.4. *Any variety of HilGC-algebras $\mathsf{V}_{\Gamma_0}$ generated by a finite subset $\Gamma_0$ of the set of equations $\Gamma = \{\phi \wedge g(\phi) \approx \phi,\ \phi \wedge g(\phi) \approx g(\phi),\ g(\phi) \wedge g^2(\phi) \approx g(\phi),\ g^2(\phi) \wedge g(\phi) \approx g^2(\phi),\ f(\phi) \wedge g(\phi) \approx f(\phi), f(\phi) \wedge g(\phi) \approx g(\phi)\}$ is canonical.*

## 4. Congruences of HilGC-algebras

Let $A$ be a Hilbert algebra. Let $\mathrm{Con}(A)$ be the lattice of congruences of $A$. It is known that the equivalence class

$$[1]_\theta = \{a \in A : (1, a) \in \theta\},$$

is an implicative filter. Moreover, if $F \in \mathrm{Fi}(A)$, then the binary relation $\theta_F$ defined by

$$(a, b) \in \theta_F \quad \text{iff} \quad a \to b, b \to a \in F$$

is a congruence of $A$. A well-known result given by A. Diego [8] (see also [7] or [9]) ensures that $\mathrm{Con}(A)$ is isomorphic to the lattice of the implicative filters of $A$ under inverse mappings $\theta \to [1]_\theta$ and $F \to \theta_F$.

There exists a bijective correspondence between implicative filters of a Hilbert algebra and closed subsets of the dual space of $A$ ([4]). Let $A$ be a Hilbert algebra and let $\langle X, \mathcal{T}\rangle$ its dual $H$-space. We denote by $\mathcal{C}(X)$ the lattice of closed subsets of $\langle X, \mathcal{T}\rangle$. If $F \in \mathrm{Fi}(A)$, then

$$\delta(F) = \{x \in X : F \subseteq x\} \in \mathcal{C}(X).$$

If $Y \in \mathcal{C}(X)$, then

$$\pi(Y) = \{a \in A : Y \subseteq \varphi(a)\} \in \mathrm{Fi}(A).$$

Moreover, if $Y \in \mathcal{C}(X)$ and $F \in \mathrm{Fi}(A)$ then, $\delta(\pi(Y)) = Y$ and $\pi(\delta(F)) = F$. Thus, there is a dual isomorphism between $\mathrm{Fi}(A)$ and $\mathcal{C}(X)$. Note that if $Y \in \mathcal{C}(X)$ then

$$\sigma(Y) = \left\{(a,b) \in A^2 : a \to b, b \to a \in \pi(Y)\right\}$$

is a congruence of $A$.

If $L$ is a lattice, we denote by $L^d$ the lattice with the dual order. To denote that two lattices $L_1$ and $L_2$ are isomorphic we will write $L_1 \cong L_2$. By the results given by A. Diego [8] (see also [9]) and the results given in [4], we have the following lattice isomorphisms

$$\mathrm{Con}(A) \cong \mathrm{Fi}(A) \cong \mathcal{C}(X)^d.$$

Let $A \in \mathsf{HilGC}$. An Hilbert congruence $\theta$ is called *G-congruence* if it is compatible with $f$ and $g$, i.e., if $(a,b), (c,d) \in \theta$, then $(f(a), f(b)) \in \theta$, and $(g(a), g(b)) \in \theta$. We denote by $\mathrm{Con}_G(A)$ the set of all $G$-congruences of $A$.

Now, we will study the particular class of implicative filters in a HilGC-algebra $A$ that are in bijective correspondence with its $G$-congruences.

Definition 4.1. Let $\langle A, f, g\rangle$ be a HilGC-algebra. Let $F \in \mathrm{Fi}(A)$. We said that $F$ is a *Galois implicative filter, or* $G$-filter for short, if $F$ satisfies the following proprieties:

(GF1) $a \in F$ implies $g(a) \in F$, i.e., $F \subseteq g^{-1}(F)$,

(GF2) $a \to b \in F$ implies $f(a) \to f(b) \in F$.

The set of all Galois implicative filters of a HilGC-algebra $A$ ordered by inclusion will be denoted by $\mathrm{Fi}_G(A)$. It is almost trivial to prove that $\bigcap \{F_i : F_i \in \mathrm{Fi}_G(A)\} \in \mathrm{Fi}_G(A)$. Consequently, for every $S \subseteq A$ there exists the least $G$ -filter containing $S$. Thus, given $S \subseteq A$, the set

$$\langle S\rangle_G = \bigcap \{F \in \mathrm{Fi}_G(A) : S \subseteq F\}$$

is called the $G$-filter generated by $S$. Note that $\langle \emptyset\rangle_G = \{1\}$ is the trivial $G$-filter. Moreover, since $\mathrm{Fi}_G(A)$ is closed under arbitrary intersections and contains the whole $A$, it is a complete lattice with respect to set inclusion whose meets coincide with set intersections and joins are $G$-filter generated by set unions of given $G$-filters.

Proposition 4.2. Let $A = \langle A, f, g\rangle$ be a HilGC-algebra. Then,

$$\mathrm{Fi}_G(A) \cong \mathrm{Con}_G(A).$$

Proof: We need to prove that $[1]_\theta \in \mathrm{Fi}_G(A)$, for each $\theta \in \mathrm{Con}_G(A)$ and that $\theta_F$ is a $G$-congruence of $A$, for each $F \in \mathrm{Fi}_G(A)$. Let $\theta \in \mathrm{Con}_G(A)$. So, $[1]_\theta \in \mathrm{Fi}(A)$. We prove that $[1]_\theta$ satisfies the conditions of Definition 4.1. Let $a, b \in A$.

(GF1) Let $a \in [1]_\theta$, i.e., $(1, a) \in \theta$. As $\theta \in \mathrm{Con}_G(A)$, $(g(1), g(a)) = (1, g(a)) \in \theta$. Thus, $g(a) \in [1]_\theta$.

(GF2) Let $a \to b \in [1]_\theta$, i.e., $(1, a \to b) \in \theta$. Thus,

$$(1 \to b, (a \to b) \to b) = (b, (a \to b) \to b) \in \theta.$$

As $\theta \in \mathrm{Con}_G(A)$, $(f(b), f((a \to b) \to b)) \in \theta$ and so,

$$(f(a) \to f(b), f(a) \to f((a \to b) \to b)) \in \theta.$$

By (HilGC2), $(f(a) \to f(b), 1) \in \theta$, that is, $f(a) \to f(b) \in [1]_\theta$.

Now, assume that $F \in \mathrm{Fi}_G(A)$. Then $\theta_F$ is a Hilbert congruence. Let $(a, b) \in \theta_F$, that is, $a \to b, b \to a \in F$. By (GF2), $f(a) \to f(b), f(b) \to f(a) \in F$ and consequently, $(f(a), f(b)) \in \theta_F$. On the other hand, by (GF1), $g(a \to b), g(b \to a) \in F$. Since $g$ is a semi-homomorphism, $g(a \to b) \leq g(a) \to g(b)$ and as $F$ is an upset of $A$, we get $g(a) \to g(b) \in F$. Analogously, we have $g(b) \to g(a) \in F$ and so, $(g(a), g(b)) \in \theta_F$. Thus, $\theta_F$ is an $G$-congruence of $A$. □

## 4.1. *G*-closed

Now we are going to prove that the lattice of $G$-filters of a HilGC-algebra $\langle A, f, g\rangle$ is dually isomorphic to the lattice of certain closed sets of the dual space of $\langle A, f, g\rangle$.

Let $X$ be a set and $R$ a binary relation defined on $X$. Let $Y$ be a subset of $X$. Let $R^{-1}(Y) = \bigcup \left\{R^{-1}(y) : y \in Y\right\}$.

We recall that if $\langle X, \mathcal{T}_{\mathcal{K}}\rangle$ is an $H$-space, then every closed subset of $X$ is an upset of $X$, i.e., for $Y \in \mathcal{C}(X)$ we have that $x \leq y$ and $x \in Y$ implies $y \in Y$, where we recall that the order $\leq$ is given by (2.1).

Definition 4.3. Let $\langle X, \mathcal{T}_{\mathcal{K}}, R\rangle$ be an $HG$-space and let $Y \in \mathcal{C}(X)$. We shall say that $Y$ is a *$G$-closed* if $Y$ satisfies the following conditions:

(G1) $R^{-1}(Y) \subseteq Y$.

(G2) $\max(R(x)) \subseteq Y$, for all $x \in Y$.

The family of all $G$-closed subsets of an $HG$-space $\langle X, \mathcal{T}_{\mathcal{K}}, R\rangle$ is denoted by $\mathcal{C}_G(X)$. It is clear that $X$ and $\emptyset$ are trivially $G$-closed subsets of an $HG$-space $\langle X, \mathcal{T}_{\mathcal{K}}, R\rangle$ and it is easy to check from the above definition that the $\mathcal{C}_G(X)$ is closed under arbitrary intersections and that the union of any finite family of $G$-closed subsets is again an $G$-closed set. So, we can conclude that the set of all $G$-closed subsets of $\langle X, \mathcal{T}_{\mathcal{K}}, R\rangle$ is a complete sublattice of $\mathcal{P}(X)$ which shall be denoted also by $\mathcal{C}_G(X)$.

Lemma 4.4. *Let $\langle A, f, g\rangle$ be a HilGC-algebra and let $\langle X, \mathcal{T}, R\rangle$ be its dual $HG$-space. Then, for all $x \in X$ and $a \in A$ such that $f(a) \in x$ there exists $z \in \max(R(x))$ such that $a \in z$*

Proof: Let $x \in X$ and let $a \in A$ such that $f(a) \in x$ . We consider the following family of implicative filters of $A$:

$$\mathcal{F} = \left\{D \in \mathrm{Fi}(A) : D \subseteq f^{-1}(x) \text{ and } a \in D\right\}.$$

We prove that $\mathcal{F} \neq \emptyset$. As $f(a) \in x$, by Lemma 2.9 there exists $y \in X$ such that $y \subseteq f^{-1}(x)$ and $a \in y$. We see that every chain in $\mathcal{F}$ has an upper bound in $\mathcal{F}$. Let $C = \{D_i\}_{i \in I}$ be a chain of elements of $\mathcal{F}$. Consider $P = \bigcup_{i \in I} \{D_i : D_i \in C\}$. As $D_i \subseteq P$ for every $i \in I$, $P$ is an upper bound of $C$. We will prove that $P \in \mathcal{F}$. As $D_i \in \mathrm{Fi}(A)$ for all $i \in I$, $1 \in D_i$ for all $i \in I$ and so, $1 \in P$. Let $b, b \to c \in P$. So, there are $i, j \in I$ such that $b \in D_i$ and $b \to c \in D_j$. Without loss of generality, we may assume that $i \leq j$ and so, that $D_i \subseteq D_j$. Thus, $b, b \to c \in D_j$ and as $D_j \in \mathrm{Fi}(A)$, $c \in D_j$. Thus, $c \in P$ and consequently, $P \in \mathrm{Fi}(A)$. On the other hand. As for all $i \in I$, we get $D_i \subseteq f^{-1}(x)$ and $a \in D_i$ , so we have

$$P = \bigcup_{i \in I} \{D_i : D_i \in C\} \subseteq f^{-1}(x) \text{ and } a \in P,$$

and so, $P \in \mathcal{F}$. Thus, every chain in $\mathcal{F}$ has an upper bound in $\mathcal{F}$ and by Zorn's Lemma, there is $m \in \max(\mathcal{F})$ and so, $m \subseteq f^{-1}(x)$ and $a \in m$. Now, we shall prove that $m \in X$. Let $a, b \in A$ such that $a, b \notin m$. We consider the implicative filters $F_a = \langle m \cup \{a\} \rangle$ and $F_b = \langle m \cup \{b\} \rangle$. As $m$ is maximal of $\mathcal{F}$ and $m \subset F_a$, we get that $F_a \nsubseteq f^{-1}(x)$. Analogously, $F_b \nsubseteq f^{-1}(x)$. So, there exist $c, d \in A$ such that $c \in F_a$, $d \in F_b$ and $c, d \notin f^{-1}(x)$. By Lemma 2.2, $a \to c, b \to d \in m$. As $f(c), f(d) \notin x$ and $x \in X$, by Corollary 2.4, there exists $k \notin x$ such that $f(c) \leq k$ and $f(d) \leq k$, or equivalently, $c \leq g(k)$ and $d \leq g(k)$. Thus, $a \to c \leq a \to g(k)$ and $b \to d \leq b \to g(k)$, and consequently, $a \to g(k), b \to g(k) \in m$. Now, we will prove that $g(k) \notin m$. On the contrary. Suppose that $g(k) \in m \subseteq f^{-1}(x)$. So, $f(g(k)) \in x$ and by (HilGC5), results $k \in x$, which is impossible. So, for $a, b \notin m$ there exists $g(k) \notin m$ such that $a \to g(k), b \to g(k) \in m$. Thus, $m \in X$ and consequently, $m \in R(x)$.

We have proved that for all implicative filters of $A$ belonging to $\mathcal{F}$ there exists $m \in X$ such that $m$ is a maximal element of them. In particular, we can affirm that this happens if we consider only irreducible filters. This is, for all irreducible implicative filters of $A$ belonging to $\mathcal{F}$ there exists $m \in X$ such that $m$ is a maximal element of them. Thus, if $f(a) \in x$ then there exists $z \in \max(R(x))$ such that $a \in z$. □

The next result gives a characterization of the $G$-congruences applying the duality given in [6] for the HilGC-algebras.

Proposition 4.5. Let $\langle A, f, g \rangle$ be a HilGC-algebra and let $\langle X, \mathcal{T}, R \rangle$ its dual $HG$-space. Then,

$$\mathcal{C}_G(X)^d \cong \mathrm{Con}_G(A).$$

Proof: Consider the map $\sigma : \mathcal{C}_G(X) \to \mathrm{Con}_G(A)$ given by:

$$\sigma(Y) = \{(a, b) \in A^2 : a \to b, b \to a \in \pi(Y)\}.$$

We recall that $\sigma(Y)$ is a Hilbert congruence. Let $a, b \in A$ such that $(a, b) \in \sigma(Y)$, i.e., $a \to b, b \to a \in \pi(Y)$, i.e., $Y \subseteq \varphi(a \to b) = \varphi(a) \Rightarrow \varphi(b)$ and $Y \subseteq \varphi(b) \Rightarrow \varphi(a)$. We prove that $(f(a), f(b)) \in \sigma(Y)$, i.e.,

$$Y \subseteq \varphi(f(a)) \Rightarrow \varphi(f(b)) \text{ and } Y \subseteq \varphi(f(b)) \Rightarrow \varphi(f(a)).$$

First, we take $x \in Y$ and we will show that $[x) \cap \varphi(f(a)) \subseteq \varphi(f(b))$. Let $y \in X$ such that $y \in [x) \cap \varphi(f(a))$. So, $x \subseteq y$ and $f(a) \in y$. Since $f(a) \in y$, by Lemma 4.4, there exists $z \in \max(R(y))$ such that $a \in z$. On the other hand, as $Y \in \mathcal{C}(X)$, $Y$ is an upset of $X$ and so, $y \in Y$. Consequently, $\max(R(y)) \subseteq Y$ because $Y \in \mathcal{C}_G(X)$. Thus, $z \in Y \subseteq \varphi(a) \Rightarrow \varphi(b)$, that is, $[z) \cap \varphi(a) \subseteq \varphi(b)$. As $z \in [z) \cap \varphi(a)$, we obtain $z \in \varphi(b)$ and so, $f(b) \in y$. We have proved that $Y \subseteq \varphi(f(a)) \Rightarrow \varphi(f(b))$. By a similar argument we can prove that $Y \subseteq \varphi(f(b)) \Rightarrow \varphi(f(a))$.

To prove that $(g(a), g(b)) \in \sigma(Y)$, we show that

$$Y \subseteq \varphi(g(a)) \Rightarrow \varphi(g(b)) \text{ and } Y \subseteq \varphi(g(b)) \Rightarrow \varphi(g(a)).$$

Suppose that $Y \nsubseteq \varphi(g(a)) \Rightarrow \varphi(g(b))$. So, there exists $x \in Y$ such that $x \notin \varphi(g(a)) \Rightarrow \varphi(g(b))$, i.e., $[x) \cap \varphi(g(a)) \nsubseteq \varphi(g(b))$. Hence, there exists $z \in X$ such that $z \in [x) \cap \varphi(g(a))$ and $z \notin \varphi(g(b))$. As $g(b) \notin z$, by Lemma 2.9, there exists $w \in X$ such that $g^{-1}(z) \subseteq w$ and $b \notin w$. As $g(a) \in z$, we have that $a \in w$, i.e., $w \in \varphi(a)$. Moreover, since $x \in Y$ and $Y \in \mathrm{Up}(X)$, we have $z \in Y$. Thus, $w \in R^{-1}(Y)$ and as $Y \in \mathcal{C}_G(X)$, $w \in Y$. By assumption, $Y \subseteq \varphi(a) \Rightarrow \varphi(b)$, and so, $[w) \cap \varphi(a) \subseteq \varphi(b)$. Since $w \in [w) \cap \varphi(a)$, we have $w \in \varphi(b)$, which is a contradiction. Then, we have proved that $Y \subseteq \varphi(g(a)) \Rightarrow \varphi(g(b))$. Analogously, we prove that $Y \subseteq \varphi(g(b)) \Rightarrow \varphi(g(a))$. Thus, $\sigma(Y) \in \mathrm{Con}_G(A)$, for each $Y \in \mathcal{C}_G(X)$ and consequently $\sigma$ is well defined.

Let $Y, W \in \mathcal{C}_G(X)$. It is clear that if $Y \subseteq W$ then $\pi(W) \subseteq \pi(Y)$ and consequently, $\sigma(W) \subseteq \sigma(Y)$. To prove that $\sigma$ is one-to-one, assume that $\sigma(W) = \sigma(Y)$ and suppose that $Y \neq W$. Without loss of generality, we assume that $Y \nsubseteq W$, i.e., there exists $x \in Y$ such that $x \notin W$. As $W$ is a closed subset of $\langle X, \mathcal{T} \rangle$, there exists $a \in A$ such that $W \subseteq \varphi(a)$ and $x \notin \varphi(a)$. Thus, $a = 1 \to a, 1 = a \to 1 \in \pi(W)$ and consequently, $(1, a) \in \sigma(W) = \sigma(Y)$. So, $a \to 1, 1 \to a \in \pi(Y)$. Thus $a \in \pi(Y)$ and so, $Y \subseteq \varphi(a)$. As $x \in Y$, we have $a \in x$, which is a contradiction.

It remains to prove that $\sigma$ is onto. Let $\theta \in \mathrm{Con}_G(A)$. We recall that $Y = \delta([1]_\theta) \in \mathcal{C}(X)$. We prove that $Y$ satisfies the two conditions of Definition 4.3.

(G1) Let $x \in X$ such that $x \in R^{-1}(Y)$. So, there exists $y \in Y$ such that $(x, y) \in R$, that it, $[1]_\theta \subseteq y$ and $g^{-1}(y) \subseteq x$. We prove that $x \in Y$ showing that $[1]_\theta \subseteq x$. Let $a \in [1]_\theta$. As $\theta \in \mathrm{Con}_G(A)$, by Proposition 4.2, $[1]_\theta \in \mathrm{Fi}_G(A)$ and thus, $g(a) \in [1]_\theta \subseteq y$. Consequently, $a \in x$. We have proved that $R^{-1}(Y) \subseteq Y$.

(G2) Suppose that there exists $x \in Y$ such that $\max(R(x)) \nsubseteq Y$. So, there exist $x, z \in X$ such that $[1]_\theta \subseteq x$, $z \in \max(R(x))$ and $z \notin Y$, i.e., $[1]_\theta \nsubseteq z$. So, there exists $a \in A$ such that $a \in [1]_\theta$ and $a \notin z$. We consider the ideal $f^{-1}(x^c)$ and the implicative filter $\langle z \cup \{a\} \rangle$ of $A$, and we will prove that there are disjoint. Conversely, suppose that there exists $b \in A$ such that $b \in \langle z \cup \{a\} \rangle \cap f^{-1}(x^c)$. By Lemma 2.2, $a \to b \in z$ and $f(b) \notin x$. By the assumption, $z \in R(x)$, i.e., $z \subseteq f^{-1}(x)$ and so, $f(a \to b) \in x$. On the other hand, as $(1, a) \in \theta$, we obtain $(b, a \to b) \in \theta$. As $\theta \in \mathrm{Con}_G(A)$, we have $(f(b), f(a \to b)) \in \theta$ and so, $(1, f(a \to b) \to f(b)) \in \theta$. Thus, $f(a \to b) \to f(b) \in [1]_\theta \subseteq x$ and since $f(a \to b) \in x$, we get $f(b) \in x$, which is a contradiction. Thus, $\langle z \cup \{a\} \rangle \cap f^{-1}(x^c) = \emptyset$ and consequently, there exists $y \in X$ such that $z \subseteq y$, $a \in y$ and $y \cap f^{-1}(x^c) = \emptyset$, that is, $y \subseteq f^{-1}(x)$, i.e., $y \in R(x)$. As $z \in \max(R(x))$ results $y = z$. Thus, $a \in z$, which contradicts our assumption. So, $\max(R(x)) \subseteq Y$ for all $x \in Y$.

Finally, we prove that $\sigma(Y) = \theta$. Let $a, b \in A$. Then,

$$\begin{aligned} (a,b) \in \sigma(Y) \quad &\text{iff} \quad a \to b, b \to a \in \pi(Y) = \pi(\delta([1]_\theta)) = [1 \\ &\text{iff} \quad (a,b) \in \theta_{[1]_\theta} = \theta \end{aligned}$$

□

By Propositions 4.2 and 4.5, we have the following result.

Corollary 4.6. Let $\langle A, f, g \rangle$ be a HilGC-algebra and let let $\langle X, \mathcal{T}, R \rangle$ its dual $HG$-space. Then,

$$\mathrm{Con}_G(A) \cong \mathrm{Fi}_G(A) \cong \mathcal{C}_G(X)^d.$$

### Simple and subdirectly irreducibles algebras

We are going to apply the topological characterization of the $G$-congruences to give a characterization of the simple algebras and subdirectly irreducible algebras.

Let us recall that an algebra $A$ is *subdirectly irreducible* if and only if there exists the smallest non trivial congruence relation $\theta$ in $A$. A particular case are the simple algebras, $A$ is *simple* if and only if $A$ has only two congruence relations.

Let $\langle A, f, g \rangle$ be a HilGC-algebra and let $\langle X, \mathcal{T}, R \rangle$ its dual space. By Propositions 4.5 and 4.2, we can affirm that a HilGC-algebra $\langle A, f, g \rangle$ is subdirectly irreducible if and only if there exists the smallest non-trivial Galois implicative filter of $A$ iff in the dual $HG$-space $\langle X, \mathcal{T}, R \rangle$ there exists the largest $Y \in \mathcal{C}_G(X) - \{X, \emptyset\}$. Moreover, $\langle A, f, g \rangle$ is simple iff $\mathrm{Fi}_G(A) = \{\{1\}, A\}$ iff $\mathcal{C}_G(X) = \{\emptyset, X\}$.

Let $\langle X, \mathcal{T}, R\rangle$ be an $HG$-space. As the family $\mathcal{C}_G(X)$ is closed under arbitrary intersections, we can define for each $x \in X$ the set

$$Y_x = \bigcap \{Y \in \mathcal{C}_G(X) : x \in Y\} \in \mathcal{C}_G(X).$$

Note that $Y_x$ is the smallest $G$-closed set containing the element $x$.

Now, we can characterize the simple and subdirectly irreducible HilGC-algebras.

THEOREM 4.7. *Let $A$ be a HilGC-algebra and $\langle X, \mathcal{T}, R\rangle$ its dual $HG$-space. Then:*

1. *$A$ is simple iff $Y_x = X$, for each $x \in X$.*
2. *$A$ is subdirectly irreducible iff $\{x \in X : Y_x \neq X\} \in \mathcal{C}_G(X) - \{X\}$.*

PROOF: (1) Assume that $A$ is simple. So, $\mathcal{C}_G(X) = \{\emptyset, X\}$. Let $x \in X$. As $x \in Y_x$, $Y_x \neq \emptyset$ and since $Y_x \in \mathcal{C}_G(X)$, we have that $Y_x = X$. Reciprocally. Let $Z \in \mathcal{C}_G(X)$ and suppose that $Z \neq \emptyset$. So, there exists $x \in X$ such that $x \in Z$. Thus, $X = Y_x \subseteq Z \subseteq X$. So, $Z = X$ and consequently, $A$ is simple.

(2) Consider the set

$$W = \{x \in X : Y_x \neq X\}.$$

Assume that $A$ is subdirectly irreducible and let $V$ be the largest element of $\mathcal{C}_G(X) - \{X\}$. We will prove that $V = W$. Let $x \in X$ such that $x \in V$. As $Y_x$ is the smallest $G$-closed set containing the element $x$, $Y_x \subseteq V \neq X$ and hence, $x \in W$. To prove the other inclusion, we take $x \in W$, i.e., $Y_x \neq X$. Thus, $Y_x \in \mathcal{C}_G(X) - \{X\}$, and so, $Y_x \subseteq V$. As $x \in Y_x$, we obtain $x \in V$. Thus, $W = V \in \mathcal{C}_G(X) - \{X\}$.

Reciprocally, assume that $W \in \mathcal{C}_G(X) - \{X\}$. We will prove that $W$ is the largest element of $\mathcal{C}_G(X) - \{X\}$. Suppose that there exists $Z \in \mathcal{C}_G(X)$ such that $Z \nsubseteq W$. So, there exists $x \in Z$ such that $x \notin W$, this is, $Y_x = X$. Thus, $X = Y_x \subseteq Z$ and so, $Z = X$. □

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