UNIVERSITY OF LODZ
DEPARTMENT OF LOGIC

BULLETIN OF THE SECTION OF LOGIC

VOLUME 53, NUMBER 3

ŁÓDŹ, SEPTEMBER 2024

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Layout
Michał Zawidzki

Initiating Editor
Katarzyna Smyczek

Printed directly from camera-ready materials provided
to the Lodz University Press

Published by Lodz University Press

First edition. W.11325.24.0.C

Printing sheets 9.5

Lodz University Press
90-237 Łódź, 34A Jana Matejki St.
www.wydawnictwo.uni.lodz.pl
e-mail: ksiegarnia@uni.lodz.pl
+48 42 635 55 77

The **Bulletin of the Section of Logic** (*BSL*) is a quarterly peer-reviewed journal published with the support from the University of Lodz. Its aim is to act as a forum for a wide and timely dissemination of new and significant results in logic through rapid publication of relevant research papers. *BSL* publishes contributions on topics dealing directly with logical calculi, their methodology, and algebraic interpretation.

Papers may be submitted through the *BSL* online editorial platform at https://czasopisma.uni.lodz.pl/bulletin. While preparing the munuscripts for publication please consult the Submission Guidelines.

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Homepage: https://czasopisma.uni.lodz.pl/bulletin

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Volume 53/3 (2024), pp. 275–299
https://doi.org/10.18778/0138-0680.2024.05

Leonard Kupś
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MEANING IS USE: THE CASE OF PROPOSITIONAL IDENTITY

Abstract

We study natural deduction systems for a fragment of intuitionistic logic with propositional identity from the point of view of proof-theoretic semantics. We argue that the identity connective is a natural operator to be treated under the *elimination rules as basic* approach.

Keywords: intuitionistic Logic, non-Fregean logic, proof-theoretic semantics.

2020 Mathematical Subject Classification: 03B20, 03B60, 03F03.

1. Introduction

The main idea behind proof-theoretic semantics is to view the meaning of a logical connective as given by the conditions under which a corresponding proposition *can be asserted*[1]. This approach is related to the Wittgenstein's slogan that *meaning is use*, contrary to the traditional view that meaning is given by the *truth conditions*. From the point of view of natural deduction, there are two kinds of rules: introduction rules and elimination rules. Let us start with the famous observation by Gentzen:

[1] A thorough exposition of this approach is presented in the work of [12], [4] and [14] among many others.

Presented by: Nils Kürbis
Received: October 11, 2022
Published online: May 20, 2024

> ”The introductions represent, as it were, the ’definitions’ of the symbol concerned, and the eliminations are no more, in the final analysis, than the consequences of those definitions.” [8, p. 80]

Introduction rules specify the conditions under which a proposition of a certain form can be asserted, while elimination rules state the requirements pertaining to what can be deduced from a given proposition. Most intuitionistic connectives, such as implication, have both introduction and elimination rules. The *Falsum* constant is an exception here—it has only an elimination rule. The reason is that Falsum cannot be asserted under any conditions, but hypothetically we need to know what can be deduced from it (in our case, every proposition). Thus, the meaning of the Falsum constant is established only by the elimination rule. In this paper we will try to show that a propositional identity connective can be treated in a similar manner.

In the next section, fundamental notions will be introduced, concerning both the logic used throughout the paper and proof-theoretic semantics in general. Then we shall turn to the definition of validity based on introductory rules. In the following subsection, a number of examples will be examined. The third section concerns validity with elimination rules as basic, and it is structured analogically to the previous one: firstly we define such validity then we provide examples. The final section is a brief conclusion.

2. Intuitionistic Logic with Identity

In intuitionistic terms, we are not interested in propositions being true or false but in *constructions* which prove them. Equivalence of two formulae, A and B, means that every proof of A can be transformed into a proof of B and *vice versa*. Thus, whenever A is provable B is provable as well. However, it is interesting to consider a stronger notion which says that the classes of constructions proving A and B are *exactly* the same. This is the intended interpretation of the propositional identity connective on the grounds of intuitionistic logic.

2.1. BHK

What follows is a version of the BHK-interpretation of Falsum ($\bot$), intuitionistic implication ($\supset$) and propositional identity ($\equiv$).

there is no proof of $\bot$	
a is a proof of $A \supset B$	a is a construction that converts each proof a_1 of A into a proof $a_2(a_1)$ of B
a is a proof of $A \equiv B$	a is the identity function

Here a formula A may be thought of as representation of the set of its own proofs.

The additional condition for identity can be interpreted in the following ways:

- a proof of $A \equiv B$ is the identity function transforming a given proof of A into a proof of B;
- a proof of $A \equiv B$ establishes the fact that two sets of proofs are equal.

Both interpretations use the notion of identity function, but differently typed—one of them transforms proofs, the other one sets of proofs. We do not claim that every proof of A can be transformed into a proof of B by the identity function, since we want formula $A \equiv A$ to be valid under our interpretation of $\equiv$ (naturally there may be non-normal proofs of A which are not identical to a normal one).

Naturally, identity is stronger than implication: if we have an arbitrary proof of $A \equiv B$ it will also count as a proof of $A \supset B$ and $B \supset A$ (due to symmetry of $\equiv$).

2.2. Hilbert-style formalization

The logic we are going to consider can be thought of as an intuitionistic variant of basic non-fregean logic (SCI—*Sentential Calculus with Identity*) introduced by [1]. We call it ISCI—*Intuitionistic Sentential Calculus with Identity.*

The language $\mathcal{L}_{\mathsf{ISCI}}$ of the logic ISCI is defined by the following grammar:

$$A \ ::= \ V \mid \bot \mid \ A\&A \mid A \vee A \mid A \supset A \mid A \equiv A$$

where V is a denumerable set of propositional variables. The axiom system for ISCI can be obtained from any such system for INT by the addition of

$\equiv$-specific axioms (see Table 1). The first axiom underlines that identity is reflexive; the second axiom shows identity as a stronger connective than implication; and the third axiom expresses the fact that $\equiv$ is a congruence relation. The axioms are valid under the proposed interpretation of the identity connective, as it has been shown in [3]. The only rule of inference is *modus ponens*.

Table 1. Axioms for propositional identity; $\otimes \in \{\&, \vee, \supset, \equiv\}$

1. $A \equiv A$
2. $(A \equiv B) \supset (A \supset B)$
3. $(A \equiv B) \supset ((C \equiv D) \supset ((A \otimes C) \equiv (B \otimes D)))$

2.3. Natural deduction—synthetic approach

Following the prevailing *meaning is use* paradigm, throughout the paper we use the framework of natural deduction.

The first natural deduction system for ISCI we consider closely follows the corresponding axiom system. We use standard natural deduction rules for intuitionistic logic adding three specific rules from Table 2. The notation $[A \equiv A^n]^j$ indicates that the assumption $A \equiv A$ is discharged, n indicates the number of instances of a formula that are closed and j is the discharge label. This system was shown to be complete with respect to Hilbert-style system and enjoys normalisation [2].

According to the well-known Gentzen's idea the meaning of each connective is fixed by its introduction rule(s) and corresponding elimination rules are somehow justified by means of introduction rules. Here all rules are *general elimination rules* [11][2]. They are formulated in a general form: conclusions do not have specified logical form. Thus, there are possible applications of elimination rules for a given connective which introduce a formula with the same connective as a main sign.

[2]Althought rule $\equiv_1$ would be considered *general introduction rule* in [10], we prefer to consider it as a general elimination rule, since the formula introduced in the conclusion does not have specified logical form. But certainly there are *introductory applications* of this rule, that is, application which introduces identity.

Table 2. Identity specific rules in the system $\mathbf{ND}_{\mathsf{ISCI}}$ $(0 \leq n)$

$$\cfrac{\begin{array}{c}[A \equiv A^n]^j \\ \vdots \\ C\end{array}}{C}\ \equiv_1, j \qquad\qquad \cfrac{A \equiv B \qquad \begin{array}{c}[A \supset B^n]^j \\ \vdots \\ C\end{array}}{C}\ \equiv_2, j$$

$$\cfrac{A \equiv C \quad B \equiv D \qquad \begin{array}{c}[(A \otimes B) \equiv (C \otimes D)^n]^j \\ \vdots \\ F\end{array}}{F}\ \equiv_3, j$$

However, these rules reflect an important feature of propositional identity: it is intensional and it cannot be established solely on the fact that both of its components are provable. Thus, we cannot *synthesise* propositional identity and we do not know how to introduce it with one exception—one can safely assume reflexive identity, since assumption of this form can always be closed. On the other hand there is a specific rule for synthesizing more complex identities from simpler ones (due to the importance of this rule we call it *synthetic approach* to identity). However, it does not entail that this connective has no meaning: these rules give us hints on how to proceed when we have already established that some identity holds. Thus, we know how to use it and, according to Wittgenstein's slogan, it has meaning.

2.4. Natural deduction—analytic approach

The system $\mathsf{ND}_{\mathsf{ISCI}}$ introduced in the previous section is closely related to the axiomatic formulation of ISCI. Natural deduction rules in this system correspond to axioms. Yet, since we know that the symbol $\equiv$ is semantically interpreted as equality (in the classical version, the SCI system) or as identity function in ISCI, we can treat identity in a similar manner as equality is treated in First-Order Logic (FOL). The rules are presented in Table 3.

Table 3. Identity specific rules in the system $\mathbf{ND}^*_{\mathsf{ISCI}}$ $(0 \leq n)$

$$\dfrac{[A]^u}{A \equiv A} \equiv I, u \qquad \dfrac{A \equiv B \quad \phi(A) \quad \begin{matrix}[\phi(B/A)^n]^j \\ \vdots \\ F\end{matrix}}{F} \equiv E, j$$

On the left-hand side we have the introduction rule for identity: having established A we can conclude that $A \equiv A$ at the same time discharging the open assumption A[3]. According to the elimination rule, if we have established that $A \equiv B$ and we have a formula ϕ with at least one occurrence of the formula A (we indicate that there exists such occurrence by $\phi(A)$), we can conclude formula $\phi(B/A)$, that is the formula ϕ with at least one occurrence of A replaced by B. Due to the central character of the elimination rule, which enables replacing identical subformulas in a given formula, we call this approach to identity *analytic*.

Contrary to the synthetic approach to ISCI, the present set of rules is compatible with Gentzen's analysis of logical connectives: each connective has both introduction and elimination rule. Thus, a new detour is possible: the elimination rule for the identity connective has been applied just after the introduction rule for that connective:

$$\dfrac{\dfrac{[A]}{A \equiv A} \equiv I \qquad \begin{matrix}\vdots \\ \phi(A)\end{matrix}}{\phi(A/A)} \equiv E$$

[3] We choose this form of the introduction rule for $\equiv$ to exhibit the similarity between BHK-interpretations of implication and identity: both the former and the latter denote a function, but in case of identity it is a very specific one. Other possibility, since the assumption is immediately discharged, is to consider a no-premiss rule:

$$\dfrac{}{A \equiv A} \equiv I$$

This derivation can easily be transformed in such a way that an occurrence of $A \equiv A$ disappears from the derivation:

$$\begin{array}{c} \vdots \\ \phi(A) \end{array}$$

The system is complete and enjoys normalisation, see [2].

3. Validity based on introduction rules

Let us now recall some basic concepts of proof-theoretic semantics to serve as a rudiment of our further inquiry. First and foremost, it is convenient to think of proof-theoretic semantics in contrast to standard *model-theoretic semantics*. In standard semantics we start with some names and sentences which are represented by terms and formulae. Then we assign meaning to these objects and we specify truth conditions. Having done that we are finally able to define the notions of *validity* and *entailment*. In proof-theoretic semantics the starting point is the notion of an argument which can be represented as a formal object, most often as a *derivation* in a natural deduction system. The next step is to define the notion of validity of derivations and arguments which they represent. So, contrary to model-theoretic semantics, we build up semantic notions from an inferential point of view. Note that the validity of concrete natural deduction rules is established in terms of validity of derivations:

> (...) rules or consequences are regarded as steps which preserve the validity of arguments (...) [14, p. 529]

We shall start with some terminological remarks. By *derivation structure* (*proof skeleton* in Prawitz terms) we mean a logical representation of a certain type of arguments. It can be depicted as a natural deduction derivation, with a conclusion as root and formulae called assumptions as leaves, built from arbitrary rules of the form:

$$\dfrac{\begin{array}{c}[\Gamma_1]^i \\ \vdots \\ A_1\end{array} \quad \dots \quad \begin{array}{c}[\Gamma_n]^i \\ \vdots \\ A_n\end{array}}{B}\; R, i$$

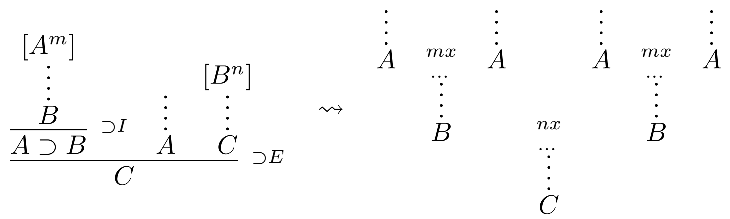
Note that derivation structures may not be properly built derivations in one of the natural deduction systems we have just defined. If all assumptions within a given derivation structure are discharged then it is *closed* [14, p. 530]. Otherwise, a derivation structure is *open*. A *canonical* derivation structure ends with an application of an introduction rule [6, p. 36].

The notion of validity is relativised to an *atomic system* S and a *reduction system* $\mathcal{J}$. By the *atomic system* S we understand a logic-free system with production rules for atomic formulae [14, p. 542], which correspond to production systems of grammars [16]. In our case the atomic formulae are propositional variables and identities. By the *reduction system* we mean a system of meta-rules enabling transformation of one derivation structure into another. Look at normalisation of derivations as an example of a reduction system. The *detour convertibility* serves to exclude, in the given derivation, consecutive pairs of introduction and elimination rules applications for the same connective. The *permutation convertibility* allows a rearrangement of assumptions if an instance of an elimination rule has a major premiss that is a conclusion of another elimination rule application. Examples of both normalisations are shown in Examples 2 and 3.

$$\dfrac{B \equiv A}{A \equiv B}\,{\scriptstyle sym} \quad \rightsquigarrow \quad \dfrac{\begin{array}{c}\vdots\\ B \equiv A\end{array} \quad \dfrac{[B]^1}{B \equiv B}\,{\scriptstyle \equiv I.1} \quad [A \equiv B]^2}{A \equiv B}\,{\scriptstyle \equiv E.2}$$

Example 1. Rule *sym* cannot be justified using I-validity

There are two main approaches to the definition of validity of derivations (and some combinations of them; for an in-depth classification see [7]). One of them, which we will address first, closely follows Gentzen and assumes that introduction rules are meaning-giving and elimination rules need to be somehow justified based on introduction rules. Another approach, which we believe is more appropriate for our treatment of propositional identity, is based on the primacy of elimination rules. According to Schroeder-Heister the distinction between these two paradigms reflects the duality between *verificationism* and *falsificationism* [15].



Example 2. Detour convertibility example

The definition of the validity of a derivation is given below. Further, we are going to introduce yet another notion of validity and thenceforth, we are going to refer to this type of validity as *I-validity*.

1. Every closed proof in the underlying atomic system is valid.
2. A closed canonical proof is considered valid, if its immediate subproofs are valid.
3. A closed non-canonical proof is considered valid, if it reduces to a valid closed canonical proof or to a closed proof in the atomic system.
4. An open proof is considered valid, if every closed proof obtained by replacing its open assumptions with closed proofs and its open variables with closed terms is valid [16].

The exact definition of validity based on introduction rules is formulated below. The S in the following definition refers to an arbitrary atomic system, $\mathcal{J}$ is a *justification*, that is a reduction system. Validity depends on the underlying atomic system S and on the type of reduction procedures used as well. S' is an *extension* of the system S if S' is S or S' results from adding further production rules to S.

$$\frac{\dfrac{\begin{matrix}\vdots\\ A\&B\end{matrix}\quad \begin{matrix}[A^m,B^n]\\ \vdots\\ C\&D\end{matrix}}{C\&D}\,\&E \qquad \begin{matrix}[C^k,D^l]\\ \vdots\\ E\end{matrix}}{E}\,\&E \quad\rightsquigarrow\quad \frac{\begin{matrix}\vdots\\ A\&B\end{matrix}\qquad \dfrac{\begin{matrix}[A^m,B^n]\\ \vdots\\ C\&D\end{matrix}\quad\begin{matrix}[C^k,D^l]\\ \vdots\\ E\end{matrix}}{E}\,\&E}{E}\,\&E$$

Example 3. Permutation convertibility example

DEFINITION 3.1 (I-validity).

1. Every closed derivation structure in S is S-valid with respect to $\mathcal{J}$ (for every $\mathcal{J}$).
2. A closed canonical derivation structure is S-valid with respect to $\mathcal{J}$, if its immediate substructure $\begin{matrix}A\\ \vdots\\ B\end{matrix}$ is S-valid with respect to $\mathcal{J}$.
3. A closed non-canonical derivation structure is S-valid with respect to $\mathcal{J}$, if it reduces, with respect to $\mathcal{J}$, to a canonical derivation structure, which is S-valid, with respect to $\mathcal{J}$.
4. An open derivation structure

$$\begin{matrix}A_1,\ldots,A_n\\ \vdots\\ B\end{matrix}$$

where all open assumptions are among $A_1,\ldots,A_n$ is S-valid with respect to $\mathcal{J}$, if for every extension S' of S and every extension $\mathcal{J}'$ of $\mathcal{J}$, and for every list of i closed derivation structures $\begin{matrix}\vdots\\ A_i\end{matrix}$ $(1 \leq i \leq n)$, which are S'-valid with respect to $\mathcal{J}'$,

$$\begin{matrix}\vdots & & \vdots\\ A_1, & \ldots, & A_n\\ & \vdots & \\ & B & \end{matrix}$$

is S'-valid with respect to $\mathcal{J}'$ [15, pp. 162-163].

$$\frac{A \equiv B}{(A \supset B)\&(B \supset A)}\, R_1 \quad \rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{A \equiv B \quad [A]^2 \quad [B]^1}{B}\,{\scriptstyle \equiv E.1}}{A \supset B}\,{\scriptstyle \supset I.2} \qquad \dfrac{\dfrac{\dfrac{\begin{matrix}\vdots \\ A \equiv B\end{matrix} \quad \dfrac{[A]^3}{A \equiv A}\,{\scriptstyle \equiv I.3} \quad [B \equiv A]^4}{B \equiv A}\,{\scriptstyle \equiv E.4} \quad [B]^6 \quad [A]^5}{A}\,{\scriptstyle \equiv E.5}}{B \supset A}\,{\scriptstyle \supset I.6}}{(A \supset B)\&(B \supset A)}\,{\scriptstyle \& I}$$

Example 4. Justification of R_1 using I-validity

Validity is first and foremost a feature of a derivation structure, but when we say that a natural deduction rule is valid, what we mean is that the corresponding one-step derivation structure is valid.

Examples of I-valid derivations In this section, we are going to examine validity as defined in the definition 3.1. Each example starts with a rule which is reduced to a valid derivation structure in the arbitrary underlying atomic system S. Every exemplary rule that we are going to address in this paper is an instance of an open derivation structure with exactly one open assumption. According to point (4) of definition 3.1 we are going to treat those open assumptions as follows: we extend S in such a way that the assumption in question can be derived from (at least one) valid derivation, and we proceed examining the given example as an instance of closed derivation structure.

We are going to focus on examples based on the analytic approach to ISCI that includes an introduction rule for the $\equiv$ connective. The example 1 shows a reduction with no open assumptions (excluding the initial one), yet impossible to be reduced to a canonical form—i.e. to introduce the $\equiv$ connective in the last step of the derivation. Thus, this derivation does not meet the point (iii) of definition 3.1 and therefore is not I-valid.

However, in the case of the derivation structure that includes the $\equiv$ connective but not as the main connective in the conclusion, as rule R_1 (see Ex. 4, p. 285), it is possible to reduce the derivation to the canonical form. Therefore, rule R_1 is I-valid, even though it contains *sym* derivation structure.

As we have seen the I-validity fails to recognise valid derivations that include the $\equiv$ connective as a primary connective in the conclusion. Thus, an approach based on introduction rules is unsatisfactory in the case of ISCI. Therefore, we are going to turn to the elimination rules based alternative in the next section.

4. Validity based on elimination rules

There exists a notion of e-canonicity of derivations [5, 7] that is closely related to the I-validity (3.1), which is unsatisfactory in our case for the same reasons that we have described in the previous section. In what follows, we are going to define e-validity in a different manner.

In the introductory section, we quoted Gentzen (p. 1), who perceived elimination rules as consequences of definitions given by the introduction rules for the given connective. Therefore, validity with elimination rules as basic views a derivation as valid if all immediate logical consequences, that can be derived from that derivation, are valid as well. As I-validity examines whether all the steps taken in the derivation to this point are legitimate, acting retrospectively in a sense, the E-validity is prospective, investigating the legitimacy of the steps that can be taken from the conclusion of the derivation: if all applications of elimination rules to the end formula of some derivation structure $\mathcal{D}$ result in E-valid derivation structures, then the initial derivation structure $\mathcal{D}$ is considered to be E-valid.

Validity based on elimination rules (*E-validity*) is defined as follows (adapted from [15, pp. 164–166]) for the synthetic approach to ISCI:

Definition 4.1 (E-validity).

1. Every closed derivation in S is *E-valid* with respect to $\mathcal{J}$, (for every $\mathcal{J}$).

2. (&) A closed derivation structure $\begin{array}{c}\vdots\\ A\& B\end{array}$ is E-valid in S with respect to $\mathcal{J}$, if the closed derivation structure

$$\dfrac{\begin{array}{c}\vdots\\ A\& B\end{array} \qquad \begin{array}{c}[A,B]^1\\ \vdots\\ C\end{array}}{C}\ {\scriptstyle \&E.1}$$

is E-valid in S with respect to $\mathcal{J}$, or reduces to derivation structures which are E-valid in S with respect to $\mathcal{J}$.

($\supset$) A closed derivation structure $\begin{array}{c}\vdots\\ A\supset B\end{array}$ is E-valid in S with respect to $\mathcal{J}$, if for every extension S' of S and every extension $\mathcal{J}'$ of $\mathcal{J}$ and for every closed derivation structure $\begin{array}{c}\vdots\\ A\end{array}$ which is E-valid in S with respect to $\mathcal{J}'$, the (closed) derivation structure

$$\dfrac{\begin{array}{c}\vdots\\ A\supset B\end{array}\quad \begin{array}{c}\vdots\\ A\end{array}\quad \begin{array}{c}[A]^1\\ \vdots\\ C\end{array}}{C}\ {\supset}E.1$$

is E-valid in S' with respect to $\mathcal{J}'$, or reduces to derivation structures which are E-valid in S' with respect tp $\mathcal{J}'$.

($\vee$) A closed derivation structure $\begin{array}{c}\vdots\\ A\vee B\end{array}$ is E-valid in S with respect to $\mathcal{J}$, if for every extension S' of S and every extension $\mathcal{J}'$ of $\mathcal{J}$, and for all derivation structures $\begin{array}{c}A\\ \vdots\\ C\end{array}$ and $\begin{array}{c}B\\ \vdots\\ C\end{array}$ with atomic C, which are E-valid in S' with respect to $\mathcal{J}'$ and which depend on no assumptions beyond A and B, respectively, the (closed) derivation structure

$$\dfrac{A\vee B\quad \begin{array}{c}[A]^1\\ \vdots\\ C\end{array}\quad \begin{array}{c}[B]^1\\ \vdots\\ C\end{array}}{C}\ {\vee}E.1$$

is E-valid in S' with respect to $\mathcal{J}'$, or reduces to derivation structure, which is E-valid in S' with respect tp $\mathcal{J}'$.

(≡) i. A closed derivation structure $\begin{array}{c}\vdots\\ A \equiv B\end{array}$ is E-valid in S with respect to $\mathcal{J}$, if for every extension S' of S and every extension $\mathcal{J}'$ of $\mathcal{J}$, and for every closed derivation structure $\begin{array}{c}A \supset B\\ \vdots\\ C\end{array}$ with atomic C, which is E-valid in S' with respect to $\mathcal{J}'$ and which depends on no assumptions beyond $A \supset B$, the (closed) derivation structure

$$\dfrac{\begin{array}{c}\vdots\\ A \equiv B\end{array} \quad \begin{array}{c}[A \supset B]^1\\ \vdots\\ C\end{array}}{C} \equiv_2 .1$$

is E-valid in S' with respect to $\mathcal{J}'$, or reduces to derivation structure, which is E-valid in S' with respect to $\mathcal{J}'$.

ii. A closed derivation structure $\begin{array}{c}\vdots\\ A \equiv C\end{array} \begin{array}{c}\vdots\\ B \equiv D\end{array}$ is E-valid in S with respect to $\mathcal{J}$, if for every extension S' of S and every extension $\mathcal{J}'$ of $\mathcal{J}$, and for every closed derivation structure $\begin{array}{c}(A \otimes B) \equiv (C \otimes D)\\ \vdots\\ F\end{array}$ with atomic F, which is E-valid in S' with respect to $\mathcal{J}'$ and which depends on no assumptions beyond $(A \otimes B) \equiv (C \otimes D)$, the (closed) derivation structure

$$\dfrac{\begin{array}{c}\vdots\\ A \equiv C\end{array} \quad \begin{array}{c}\vdots\\ B \equiv D\end{array} \quad \begin{array}{c}[(A \otimes B) \equiv (C \otimes D)]^1\\ \vdots\\ F\end{array}}{F} \equiv_3 .1$$

is E-valid in S' with respect to $\mathcal{J}'$, or reduces to derivation structure, which is E-valid in S' with respect to $\mathcal{J}'$.

3. A closed derivation structure $\begin{array}{c}\vdots\\ A\end{array}$ of an atomic formula A, which is not a derivation in S, is E-valid in S with respect to $\mathcal{J}$, if it reduces with respect to $\mathcal{J}$ to a derivation in S.

4. An open derivation structure

$$\begin{array}{c} A_1, \dots, A_n \\ \vdots \\ B \end{array}$$

where all open assumptions are among $A_1, \dots, A_n$ is E-valid in S with respect to $\mathcal{J}$, if for every extension S' of S and every extension $\mathcal{J}'$ of $\mathcal{J}$, and for every list of i closed derivation structures $\begin{array}{c} \vdots \\ A_i \end{array}$ $(1 \leq i \leq n)$, which are valid in S' with respect to $\mathcal{J}'$,

$$\begin{array}{ccc} \vdots & & \vdots \\ A_1, & \dots, & A_n \\ & \vdots & \\ & B & \end{array}$$

is valid in S' with respect to $\mathcal{J}'$.

The definition based on the analytic approach to ISCI is the same as def. 4.1 with different rule for $\equiv$:

DEFINITION 4.2. ($\equiv^*$) A closed derivation structure $\begin{array}{c} \vdots \\ A \equiv B \end{array}$ is E-valid in S with respect to $\mathcal{J}$, if for every extension S' of S and every extension $\mathcal{J}'$ of $\mathcal{J}$, and for all closed derivation structures $\begin{array}{c} \vdots \\ \phi(A) \end{array}$ which are E-valid in S with respect to $\mathcal{J}$, the (closed) derivation structure

$$\dfrac{\begin{array}{c} \vdots \\ A \equiv B \end{array} \quad \begin{array}{c} \vdots \\ \phi(A) \end{array} \quad \begin{array}{c} [\phi(B/A)] \\ \vdots \\ F \end{array}}{F} \equiv E$$

is E-valid in S' with respect to $\mathcal{J}'$, or reduces to derivation structures, which are E-valid in S' with respect to $\mathcal{J}'$.

$$\dfrac{B \equiv A}{A \equiv B}\,sym \quad \rightsquigarrow \quad \dfrac{\dfrac{B\equiv A}{A\equiv B}\,sym \qquad \begin{matrix}\vdots\\ \phi(A)\end{matrix} \qquad \begin{matrix}[\phi(B/A)]^1\\ \vdots \\ C\end{matrix}}{C}\,{\equiv}E.1 \quad \rightsquigarrow$$

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ B\equiv A\end{matrix} \qquad \dfrac{[B]^1}{B\equiv B}\,{\equiv}I.1 \qquad [A\equiv B]^2}{A\equiv B}\,{\equiv}E.2 \qquad \begin{matrix}\vdots\\ \phi(A)\end{matrix} \qquad \begin{matrix}[\phi(B/A)]^3\\ \vdots\\ C\end{matrix}}{C}\,{\equiv}E.3$$

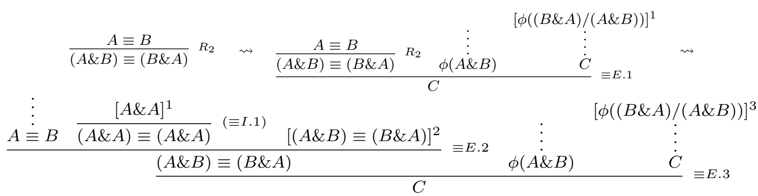
Example 5. Justification of *sym* using E-validity and analytic approach to identity.

Examples of E-valid derivations In this section, we will analyse E-validity of some natural deductions rules by providing appropriate justifications. Just as in the case of examples for I-validity, rules are instances of open derivations with exactly one open assumption each. According to point (4) of definition 4.1 we are going to proceed with reductions of those derivations assuming that there exists (at least) one valid derivation for the open assumption in question, analogically to the I-validity examples.

When examining the validity concerning definition 4.1 we start with an assumption that the rule in question is valid. Then from the conclusion of that rule an atomic formula is derived, with an application of an elimination rule for the main connective in the conclusion. In the next step, the formula C is derived from the premiss of the given rule. If the last reduction is successful the rule is valid according to E-validity.

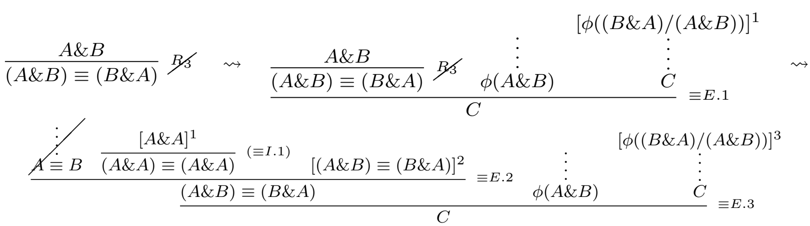
The example of rule *sym* in the analytic approach to the ISCI was not I-valid (see Ex. 1). However, it proves to be E-valid (see Ex. 5). In the first step we assume that the rule *sym* is valid and we apply the elimination rule for the $\equiv$ connective: therefore, we assume that (1) there is a valid closed derivation from which we can conclude a formula ϕ with at least one occurrence of formula A, and (2) there is a valid close derivation with formula $\phi(B)$ (that is a formula ϕ with at least one occurrence of A replaced by B) as a (discharged) assumption and C as a conclusion; and we derive a conclusion C. In the next step, analogically to previous examples we derive C solely from the premiss of the rule *sym*. There are no additional open assumptions, therefore rule *sym* is valid.

An analogical method is applied in the case of E-valid rule and R_2 (see Ex. 6).



Example 6. Justification of R_2 using E-validity and analytic approach to identity.

In the case of invalid rule R_3 (see Ex. 7), to conclude C the formula $A \equiv B$ is needed but we cannot discharge it as an assumption. Thus, the derivation structure is no longer closed and the rule R_3 is E-invalid.



Example 7. R_3 cannot be justified using E-validity and analytic approach to identity.

We can also analyse these rules in the synthetic approach. The steps taken in the reductions are very similar to the analytic approach. In the case of the rule *sym* (see Ex. 8), we begin by assuming that the rule in question is valid and apply the $\equiv .2$ elimination rule to the conclusion: thus, we assume that there is at least one closed, valid derivation from which we

$$\dfrac{B \equiv A}{A \equiv B}\,sym \quad \rightsquigarrow \quad \dfrac{\dfrac{B \equiv A}{A \equiv B}\,sym \qquad \begin{array}{c}[A \supset B]^1\\ \vdots\\ C\end{array}}{C}\,{\equiv}2.1 \quad \rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{B \equiv A \quad \dfrac{[B \equiv B]^1}{B \equiv B}\,{\equiv}1.1 \quad [(B \equiv B) \equiv (A \equiv B)]^2}{(B \equiv B) \equiv (A \equiv B)}\,{\equiv}3.2 \quad [(B \equiv B) \supset (A \equiv B)]^3}{(B \equiv B) \supset (A \equiv B)}\,{\equiv}2.3 \quad \dfrac{[B \equiv B]^4}{B \equiv B}\,{\equiv}1.4 \quad \dfrac{[A \equiv B]^6 \quad \begin{array}{c}[A \supset B]^5\\ \vdots\\ C\end{array}}{C}\,{\equiv}2.5}{C}\,{\supset}.6$$

Example 8. Justification of *sym* using E-validity and synthetic approach to identity.

can conclude atomic C from (closed) assumption $A \supset B$, and we conclude C. Then, we attempt to conclude C from the premiss of the *sym* rule. Even though the derivation is rather complex we are successful—there are no open assumptions, excluding the initial one—the rule *sym* is E-valid.

Analogically, we can prove that the rule R_2 is E-valid (Ex. 9 p. 293).

Interestingly, the rule R_3 in the synthetic approach (Ex. 10) fails to meet the criteria of E-validity for the same reasons as in the analytic approach. In the last step of the reduction, one of the assumptions is open. Therefore, the rule R_3 cannot be reduced to a closed derivation structure of required form and is not E-valid.

5. Object identity and propositional identity

Since we are interested in proof-theoretical treatment of *propositional identity* connective it would be helpful to look into, at first sight analogous, characterization of *object identity*. There is an ongoing debate about the proof-theoretical characterization of it. Usually, one can add two rules for a given natural deduction system for First-Order Logic (see [11]).

$$\dfrac{}{a = a}\,ref$$

$$\dfrac{a = b \quad Pa}{Pb}\,rep$$

$$\frac{A \equiv B}{(A\&B) \equiv (B\&A)}\,R_2 \quad \rightsquigarrow \quad \frac{\dfrac{A \equiv B}{(A\&B) \equiv (B\&A)}\,R_2 \qquad \begin{matrix}[(A\&B) \supset (B\&A)]^1 \\ \vdots \\ C\end{matrix}}{C}\,{\equiv}2.1 \quad \rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \equiv B \quad \dfrac{[A \equiv A]^1}{A \equiv A}\,{\equiv}1.1 \quad [(A \equiv A) \equiv (B \equiv A)]^2}{(A \equiv A) \equiv (B \equiv A)}\,{\equiv}3.2 \quad [(A \equiv A) \supset (B \equiv A)]^3}{(A \equiv A) \supset (B \equiv A)}\,{\equiv}2.3 \quad \dfrac{[A \equiv A]^4}{A \equiv A}\,{\equiv}1.4 \quad [B \equiv A]^5}{B \equiv A}\,{\supset}.5 \quad A \equiv B \quad [(A\&B) \equiv (B\&A)]^6}{(A\&B) \equiv (B\&A)}\,{\equiv}3.6 \quad [(A\&B) \supset (B\&A)]^7}{C}\,{\equiv}2.7$$

Example 9. Justification of R_2 using E-validity e-validity and synthetic approach to identity.

$$\frac{A\&B}{(A\&B)\equiv(B\&A)}\,\cancel{R_3} \quad\rightsquigarrow\quad \dfrac{\dfrac{A\&B}{(A\&B)\equiv(B\&A)}\,\cancel{R_3} \qquad \begin{array}{c}[(A\&B)\supset(B\&A)]^1\\ \vdots\\ C\end{array}}{C}\,{\equiv}2.1 \quad\rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[A\equiv A]^1}{A\equiv A}{\equiv}1.1 \quad \dfrac{[B\equiv B]^2}{B\equiv B}{\equiv}1.2 \quad [(A\equiv B)\equiv(A\equiv B)]^3}{(A\equiv B)\equiv(A\equiv B)}{\equiv}3.3 \quad [(A\equiv B)\supset(A\equiv B)]^4}{(A\equiv B)\supset(A\equiv B)}{\equiv}2.4 \quad \begin{array}{c}\vdots\\ \cancel{A\equiv B}\end{array} \quad \dfrac{[(A\equiv B)]^5}{(A\&B)\equiv(B\&A)}}{(A\&B)\equiv(B\&A)}{\supset}.5 \quad \dfrac{[(A\&B)\supset(B\&A)]^6}{C}}{C}{\supset}.6$$

Example 10. R_3 cannot be justified using E-validity and synthetic approach to identity.

The argument against such a treatment of identity is that it is against Gentzen's dictum that introduction rules for a given operator justify elimination rule(s) for it. It seems that *ref* does not justify *rep*, at least in a way analogous to the way the rules for the connectives like conjunction or implication do—introduction rule produces only reflexive identities, but elimination rule assume an arbitrary identity.

One of the rival propositions is to go back to Leibnizian laws of identity:

(P1) $\forall P \forall x, y((Px \supset\subset Py) \supset x = y)$—*identity of indiscernibles*

(P2) $x = y \supset (Px \supset\subset Py)$—*indiscernibility of identicals*

Intuitively, P1 gives us grounds for asserting identities while P2 enables us to infer something from it, when it has been already established [13]. The problem is that P1 requires Second-Order Logic to bind predicate variables. But we can somehow encode it in a natural deduction rule by means of a restriction of its use. The following rule:

$$\frac{\begin{array}{c}[Pa]\\ \vdots\\ Pb\end{array}}{a = b}\ \mathsf{P1}$$

can be used, provided P does not occur free in any assumption other than Pa. Then elimination rule follows from the introduction rule:

$$\frac{a = b \quad Pa}{Pb}\ \mathsf{P2}$$

and standard detour conversions can be applied—the following derivation

$$\frac{\dfrac{\begin{array}{c}[Pa]\\ \vdots\\ Pb\end{array}}{a = b}\ \mathsf{P1} \qquad Pa}{Pb}\ \mathsf{P2}$$

reduces to

$$\begin{array}{c}Pa\\ \vdots\\ Pb\end{array}$$

The thoughtful discussion of these two approaches can be found in [9].

Unfortunately, Leibnizian approach seems not to work in the context of propositional identity. Assume we were to accept the following rule (where $C(A)$ denotes a formula C having a formula A as a subformula):

$$\dfrac{\begin{array}{c}[C(A)]\\ \vdots\\ C(B)\end{array}}{A\equiv B}\,I$$

with strong side condition that the rule can be applied if formula C (and any of its subformulas) does not occur in any assumption other than the one specified in the rule scheme. Then we would be able to prove $A\&B\equiv B\&A$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[(A\&B)\&C]}{A\&B}}{B} \qquad \dfrac{\dfrac{[(A\&B)\&C]}{A\&B}}{A}}{B\&A} \qquad \dfrac{[(A\&B)\&C]}{C}}{(B\&A)\&C}}{A\&B\equiv B\&A}\,I$$

which should not be provable in the basic logic ISCI (it is considered valid in some of its extensions though).

6. Conclusions

In the context of pure intuitionistic logic proof-theoretic semantics based on elimination rules can be equivalent to semantics based on introduction rules. The differences arising from these two approaches are mostly of philosophical and procedural nature. However, it is more complicated in the context of ISCI. Propositional identity is different than other intuitionistic connectives. We can introduce intuitionistic disjunction having proved one of its disjuncts. Similarly, one can introduce intuitionistic implication when a derivation of the consequent from the antecedent is given. Yet, no formula of the form $A\equiv B$ can be obtained from its subformulae only, with the exception of $A\equiv A$. As we have seen on examples in this paper, it follows that the approach based on elimination rules works well in the extension of intuitionistic logic we have considered. It is also philosophically plausible—the fact that identity cannot be synthesised from its subformulae does not mean that we cannot hypothetically reason about identities and establish some of the desired properties, such as symmetry or transitivity. Moreover, the approach based on elimination rules is naturally compatible with Wittgenstein's dictum. When we are inside a certain *Sprachspiel* sometimes only decomposition rules (elimination rules) for certain operators exists. Consider equality between real numbers. There is no effective way of establishing that two reals are equal, but we can still claim that equality between real numbers is an equivalence relation. It would be interesting to compare these two paradigms using some other intensional propositional operators.

Acknowledgements. The work of the second author has been founded by National Science Centre, Poland, grant no. 2017/26/E/HS1/00127.

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Bulletin of the **S**ection of **L**ogic
Volume 53/3 (2024), pp. 301–319
https://doi.org/10.18778/0138-0680.2024.10

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POSITIVE COMPLETE THEORIES AND POSITIVE STRONG AMALGAMATION PROPERTY

Abstract

We introduce the notion of positive strong amalgamation property and we investigate some universal forms and properties of this notion.
Considering the close relationship between the amalgamation property and the notion of complete theories, we explore the fundamental properties of positively complete theories, and we illustrate the behaviour of this notion by bringing changes to the language of the theory through the groups theory.

Keywords: h-inductive theory, existentially closed, complete theory, positive amalgamation, positive strong amalgamation, mathematical model, semantics.

2020 Mathematical Subject Classification: 03C07, 03C48, 03C52, 03C95.

1. Positive complete theories

1.1. Positive logic

The positive model theory in its present form was introduced by Ben Yaacov and Poizat [5] following the line of research of Hrushovski [3] and Pillay [4]. It is considered as a part of the eastern model theory introduced by Abraham Robinson, wich is concerned essentially with the study of existentially closed models and model-complete theories in the context of incomplete inductive theories. The main tools in the study of incomplete inductive theories are embedding, existential formulas and inductive sentences. Keep in consideration homomorphisms and positive formulas, the

Presented by: María Manzano
Received: December 25, 2022
Published online: June 4, 2024

positive logic offers a wider and simpler framework as compared to the eastern model theory.

In this subsection we summarize the basic concepts of positive logic which will be used throughout the paper.

Let L be a first order language. we stipulate that L includes the symbol of equality and the constant $\perp$ denoting the antilogy.
The quantifier-free positive formulas are built from atomics by using the connectives $\wedge$ and $\vee$. The positive formulas are of the form: $\exists \bar{x}\varphi(\bar{x}, \bar{y})$, where φ is quantifier-free positive formula. The variables $\bar{x}$ in the expression of the φ are said to be free.
The simple h-inductive sentences are the formulas without free variables that can be written in the form:

$$\forall \bar{x}(\exists \bar{y}\varphi(\bar{x}, \bar{y}) \rightarrow \exists \bar{z}\psi(\bar{x}, \bar{z})).$$

where φ and ψ are quantifier-free positive formulas.
A sentence is said to be h-inductive if it is a finite conjunction of simple h-inductive sentences.
The h-universal sentences are the sentences that can be written as negation of a positive sentence. Note that the conjunction (resp, disjunction) of two h-universal sentences is equivalent to an h-universal sentence.

Let A and B be two L-structures and f a mapping from A into B. f is said to be

- a homomorphism, if for every tuple $\bar{a}$ from A and for every atomic formula ϕ, $A \models \phi(\bar{a})$ implies $B \models \phi(f(\bar{a}))$. In this case we say that B is a continuation of A.
- an embedding, if f is a homomorphism such that for every atomic formula ϕ; $A \models \phi(\bar{a})$ if and only if $B \models \phi(f(\bar{a}))$.
- an immersion whenever $\bar{a} \in A$ and $f(\bar{a})$ satisfy the same L-positive formulas, for every $\bar{a} \in A$.

For every L-structure A, we denote by $L(A)$ the language obtained from L by adjoining the element of A as constants. Let $Diag(A)$ (resp. $Diag^{+}(A)$) the set of atomic and negated atomic (resp. positive quantifier-free) sentences satisfied by A in the language $L(A)$.
We denote by $Diag^{+\star}(A)$ the set of L-sentences $\exists \bar{x}\ \varphi(\bar{x})$ satisfied by A where $\varphi(\bar{x})$ is a quantifier-free positive formula.

Definition 1.1. A model M of an h-inductive theory T is said to be positively closed (in short; pc) if every homomorphism from M to a model of T is an immersion.

A class of L-structures is said to be h-inductive if it is closed with respect to the inductive limit of homomorphisms. For more details on the notion of h-inductive sequences and limits, the reader is invited to [5].
In [5, Théorème 1, lemme 12] it is shown that every member of an h-inductive class is continued in a pc member of the class, and the pc models of an h-inductive theory form an h-inductive class.

1.2. Positive complete and T-complete theories

Definition 1.2. Two h-inductive L-theories are said to be companion if they have the same pc models.

Every h-inductive theory T has a maximal companion denoted $T_k(T)$, called the Kaiser's hull of T which is the h-inductive theory of the pc models of T. Likewise, T has a minimal companion denoted $T_u(T)$, formed by its h-universal consequences sentences.

Note that if T' is an h-inductive theory such that $T_u(T) \subseteq T' \subseteq T_k(T)$ then T' and T are companion theories.

Definition 1.3. Let T be an h-inductive theory.

- T is said to be model-complete if every model of T is a pc model of T.
- We say that T has a model-companion whenever $T_k(T)$ is model-complete.

Let A be a L-structure and B a subset of A. We shall use the following notations:

- $T_i(A)$ (resp. $T_u(A)$) denote the set of h-inductive (resp. h-universal) $L(A)$-sentences satisfied by A.
- $T_i^\star(A)$ (resp. $T_u^\star(A)$) denote the set of h-inductive (resp. h-universal) L-sentences satisfied by A.
- $T_k(A)$ (resp. $T_k^\star(A)$) denote the Kaiser's hull of $T_i(A)$ (resp. of $T_i^\star(A)$).

- $T_i(A|B)$ (resp. $T_u(A|B)$) denote the set of h-inductive (resp. h-universal) $L(B)$-sentences satisfied by A.

Definition 1.4. Let A and B two L-structures and f a homomorphism from A into B. f is said to be a strong immersion if B is a model of $T_i(A)$ in the language $L(A)$.

Definition 1.5.

- An h-inductive theory T is said to be positively complete or it has the joint continuation (in short JC) property if any two models of T have a common continuation in a model of T.
- Let T_1, T_2 and T three h-inductive L-theories. T_1 and T_2 are said to be T-complete if for every models A of T_1 and B of T_2, there is C a common continuation of A and B such that $C \vdash T$.

The following remark lists some simple properties which will be useful in the rest of the paper.

Remark 1.6. Let A and B two L-structures and T an h-inductive L-theory.

1. $T_u(A) \cup Diag^+(A) \subseteq T_i(A)$ and $T_u^\star(A) \cup Diag^{+\star}(A) \subseteq T_i^\star(A)$.
2. A is a pc model of $T_i(A)$, and $T_i(A) = T_k(A)$.
3. $T_u(T)$ (resp. $T_u(A)$) is the h-universal part of $T_k(T)$ (resp. of $T_i(A)$). The same is true for $T_u^\star(A)$ and $T_i^\star(A)$.
4. $T_i(A) \subseteq T_i(B) \Rightarrow T_u(A) \subseteq T_u(B)$.
5. $T_i^\star(A) \subseteq T_i^\star(B) \Rightarrow T_u^\star(A) \subseteq T_u^\star(B)$.
6. If T is has the JC property, then for every pc model A of T we have; $T_k(T) = T_i^\star(A)$ and $T_u(T) = T_u^\star(A)$.
7. If A and B are pc models of T and B is a continuation of A then $T_i^\star(A) = T_i^\star(B)$.
8. If A is continued in B then $T_u^\star(B) \subseteq T_u^\star(A)$, and $T_u(B|A) \subseteq T_u(A)$ in the language $L(A)$.
9. If A is immersed in B then $T_u^\star(A) = T_u^\star(B)$, $T_i^\star(B) \subseteq T_i^\star(A)$ and $T_u(B|A) = T_u(A)$ in the language $L(A)$.
10. $T_u^\star(A) = \{\neg\exists\bar{x}\varphi(\bar{x}) \mid \exists\bar{x}\varphi(\bar{x}) \notin Diag^{+\star}(A)\}$.

11. $Diag^{+\star}(A) \subseteq Diag^{+\star}(B) \Leftrightarrow T_u^\star(B) \subseteq T_u^\star(A)$.

12. If $T_u^\star(A) \subseteq T_u^\star(B)$ (resp. $T_i^\star(A) \subseteq T_i^\star(B)$), then $T \cup Diag^+(A) \cup Diag^+(B)$ is consistent in the language $L(A \cup B)$. Indeed, if the set $T \cup Diag^+(A) \cup Diag^+(B)$ is inconsistent, then there are $\varphi(\bar{a}) \in Diag^+(A)$ and $\psi(\bar{b}) \in Diag^+(B)$ such that $T \vdash \neg\exists\bar{x}, \bar{y}(\varphi(\bar{x}) \wedge \psi(\bar{y}))$. So $\neg\exists\bar{x}, \bar{y}(\varphi(\bar{x}) \wedge \psi(\bar{y})) \in T_u^\star(A)$. Given that $A \models \varphi(\bar{a})$ then $\neg\exists\bar{y}\psi(\bar{y}) \in T_u^\star(A)$. By hypothesis $\neg\exists\bar{y}\psi(\bar{y}) \in T_u^\star(B)$, which contradicts the fact that $B \models \psi(\bar{b})$.

13. If $T_u^\star(A) \subseteq T_u^\star(B)$ then $Diag^+(A) \cup Diag^+(B)$ is consistent in the language $L(A \cup B)$.

14. For every pc models A and B of T, if $T_u^\star(A) = T_u^\star(B)$ then $T_i^\star(A) = T_i^\star(B)$.

15. T_1 and T_2 are T-complete if and only if for every $A \vdash T_1$ and $B \vdash T_2$, $Diag^+(A) \cup Diag^+(B) \cup T$ is $L(A \cup B)$-consistent.

16. Let (T_1, T_2) be a pair of T-complete theories. For every T_1', T_2' and T' companion theories of T_1, T_2 and T respectively, the pair (T_1', T_2') is T'-complete.

Lemma 1.7. *Let A be a pc model of an h-inductive L-theory T, then*

1. *$T_u^\star(A)$ is minimal in the set $\{T_u^\star(B) \mid B \models T\}$.*

2. *$T_i^\star(A)$ is maximal in the set $\{T_i^\star(B) \mid B \models T\}$.*

Proof:

1. Let B a model of T such that $T_u^\star(B) \subseteq T_u^\star(A)$. By the property 12 of the Remark 1.6, there exists C a model of T that is a common continuation of A and B. Given that A is a pc model, by the properties 8 and 9 of the Remark 1.6 we obtain:

$$T_u^\star(A) = T_u^\star(C) \subseteq T_u^\star(B).$$

2. Let B a model of T such that $T_i^\star(A) \subseteq T_i^\star(B)$. We claim that $Diag^+(A) \cup T_i(B)$ is consistent in the language $L(A \cup B)$. Indeed, if not, by compactness there exists $\psi(\bar{a}) \in Diag^+(A)$ such that

$T_i(B) \models \neg\exists\bar{x}\psi(\bar{x})$. Given that $T_i^\star(B)$ is the part of $T_i(B)$ without parameters of B, then $T_i^\star(B) \models \neg\exists\bar{x}\psi(\bar{x})$. On the other hand since

$$\exists\bar{x}\psi(\bar{x}) \in Diag^{+\star}(A) \subset T_i^\star(A) \subseteq T_i^\star(B),$$

a contradiction. Thereby $Diag^+(A) \cup T_i(B)$ is consistent in the language $L(A \cup B)$, which implies the existence of a model D of $T_i(B)$ in the language $L(A \cup B)$, such that

$$A \xrightarrow{f} D \xleftarrow{g} B.$$

where f is an homomorphism and g an immersion.
Given that D is also a model of T and A pc model of T, then f is an immersion. By the property 9 of the Remark 1.6 we obtain

$$T_i^\star(B) \subseteq T_i^\star(D) \subseteq T_i^\star(A) \subseteq T_i^\star(B). \qquad \square$$

LEMMA 1.8. *Let T_1, T_2 and T three h-inductive L-theories. T_1 and T_2 are T-complete if and only if one of the following holds:*

1. *For every free-quantifier positive formulas $\varphi(\bar{x})$, If $T \vdash \neg\exists\bar{x}\varphi(\bar{x})$ then $T_1 \vdash \neg\exists\bar{x}\varphi(\bar{x})$ and $T_1 \vdash \neg\exists\bar{x}\varphi(\bar{x})$.*

2. $T_u(T) \subseteq T_u(T_1) \cap T_u(T_2)$.

PROOF:

1. Suppose that T_1, T_2 and T satisfy the property 1 of the Lemma. Let A and B models of T_1 and T_2 respectively. We claim that $Diag^+(A) \cup Diag^+(B) \cup T$ is $L(A \cup B)$-consistent. If not, there are $\varphi(\bar{a}) \in Diag^+(A)$ and $\psi(\bar{b}) \in Diag^+(B)$ such that $T \vdash \neg(\exists\bar{x}\varphi(\bar{x}) \wedge \exists\bar{y}\psi(\bar{y}))$. Thereby $T_1 \vdash \neg(\exists\bar{x}\varphi(\bar{x}) \wedge \exists\bar{y}\psi(\bar{y}))$ and $T_2 \vdash \neg(\exists\bar{x}\varphi(\bar{x}) \wedge \exists\bar{y}\psi(\bar{y}))$, a contradiction.

2. Suppose that T_1 and T_2 are T-complete. Since every model of T_1 or T_2 can be continued in a model of T then $T_u(T) \subseteq T_u(T_1) \cap T_u(T_2)$.

3. It is clear that if $T_u(T) \subseteq T_u(T_1) \cap T_u(T_2)$ then T, T_1 and T_2 satisfy the property 1. $\square$

LEMMA 1.9. *An h-inductive T theory has the JC property if and only if it satisfies one of the following properties:*

1. *For any free-quantifier positive formulas $\varphi(\bar{x})$ and $\psi(\bar{y})$, if $T \vdash \neg\exists\bar{x}\varphi(\bar{x}) \vee \neg\exists\bar{y}\psi(\bar{y})$ then $T \vdash \neg\exists\bar{x}\varphi(\bar{x})$ or $T \vdash \neg\exists\bar{y}\psi(\bar{y})$.*
2. *$T_u(T) = T_u^\star(A)$ for some model A of T.*
3. *$T_k(T) = T_i^\star(A)$ for some model A of T.*
4. *For every pc models A and B of T we have $T_u^\star(A) = T_u^\star(B)$.*

PROOF:

1. Clear
2. Let T be an h-inductive theory and A a model of T such that $T_u(T) = T_u^\star(A)$. Let B and C two pc models of T. Given that $T_u(T) = T_u^\star(A) \subseteq T_u^\star(B) \cap T_u^\star(C)$, by the minimality of the h-universal theory of the pc models (Lemma 1.7), we obtain

$$T_u^\star(A) = T_u^\star(B) = T_u^\star(C).$$

From the property 13 of the Remark 1.6, it follows that there is a common continuation of B and C by a model of T. Thereby T has the JC property.
The other direction follows from the property 6 of the Remark 1.6.

3. Let A be a model of T such that $T_k(T) = T_i^\star(A)$. Let B and C be two pc models of T. Since

$$T_i^\star(A) = T_k(T) \subseteq T_i^\star(B) \cap T_i^\star(C)$$

then $T_u^\star(A) \subseteq T_u^\star(B) \cap T_u^\star(C)$. By Lemma 1.7 we obtain

$$T_u^\star(A) = T_u^\star(B) = T_u^\star(C).$$

By the property 13 of the Remark 1.6, we get a common continuation of B and C by a model of T. Thereby T has the JC property.
The other direction follows from the property 6 of the Remark 1.6.

4. Clear. □

Lemma 1.10. *Let A be a L-structure. The theories $T_u^\star(A)$ and $T_i^\star(A)$ are companion and positively completes.*

Proof: It is clear that every model of $T_i^\star(A)$ is a model of $T_u^\star(A)$.
Now, we will show that every model of $T_u^\star(A)$ is continued into a model of $T_i^\star(A)$. Let B be a model of $T_u^\star(A)$, we claim that $Diag^+(B) \cup T_i^\star(A)$ is consistent in the language $L(B)$. Indeed, otherwise, there exists $\psi(\bar{b}) \in Diag^+(B)$ such that $T_i^\star(A) \models \neg\exists\bar{x}\psi(\bar{x})$, so $\neg\exists\bar{x}\psi(\bar{x}) \in T_u^\star(A)$. Given that $T_u^\star(A) \subseteq T_u^\star(B)$ and $\exists\bar{x}\psi(\bar{x}) \in Diag^{+\star}(B)$, a contradiction. Thereby $Diag^+(B) \cup T_i^\star(A)$ is consistent, so B is continued in a model of $T_i^\star(A)$.

The second part of the lemma results from the properties 2 and 3 of the lemma 1.9, since $T_u(T_u^\star(A)) = T_u^\star(A)$ and $T_i(T_i^\star(A)) = T_i^\star(A)$. □

Remark 1.11.

- We have the same results of the lemma 1.10 for the theories $T_u(A|B)$ and $T_i(A|B)$, where B is a subset of A.
- Let A_e be a pc model of an h-inductive theory T. Let A be a subset of A_e. Every pc model of $T_u(A_e|A)$ in the language $L(A)$ is a pc model of T in the language L.
  Indeed, Let B_e be a pc model of $T_u(A_e|A)$, since $T_u(A_e|A)$ is positively complete, there is a common continuation C of A_e and B_e in the language $L(A)$ which in turn can be continued in a pc model C_e of T. As A_e is immersed in C_e, so C_e is a model of $T_u(A_e|A)$, then B_e is immersed in C_e, which implies that B_e is a pc model of T.
- Let A and B be two models of an h-inductive theory T. If A is immersed in B then B is continued in a pc model of $T_i(A)$ in the language $L(A)$. Indeed, Since A is immersed in B then B is a model of $T \cup Diag^+(A) \cup T_u(A)$ in the language $L(A)$. let C be a pc model of $T_u(A)$ in which B is continued, then C is a pc model of $T_i(B)$ (first bullet of the Remark 1.11) and B is continued in C.

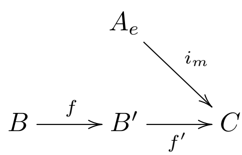
Lemma 1.12. *Let T be a positively complete h-inductive L-theory and A_e a pc model of T that is also a pc model of an h-inductive L-theory T'. Then every pc model of T is a pc model of T', and every pc model of T' that is a model of T is a pc model of T.*

Proof: Given that A_e is a pc model of T', then $T' \subset T_k(T) = T_i^\star(A_e)$.
Let B be a pc model of T, since $T_i^\star(A_e) = T_i^\star(B)$ then B is a model of T'.

Let f be a homomorphism from B into B' a pc model of T'. By the property 8 of the Remark 1.6, we have

$$T_u^\star(B') \subseteq T_u^\star(B) = T_u(T) = T_u^\star(A_e).$$

Now by the property 12 of the Remark 1.6, we obtain the consistency of $T' \cup Diag^+(A_e) \cup Diag^+(B')$,then we get the following diagram:



where C is a model of T' that we can suppose a pc model of T'. We deduce the following equalities:

$$T_k(T) = T_i^\star(B) = T_i^\star(A_e) = T_i^\star(C) = T_i^\star(B').$$

Thereby f is an immersion, and B is a pc model of T'.

For the second part of the lemma. Let B_e be a pc model of T' such that $B_e \vdash T$, let f be a homomorphism from B_e into a pc model B of T. Given that B is also a pc model of T', then f is an immersion, and so B_e is a pc model of T. □

Corollary 1.13. Let T be an h-inductive theory and A a pc model of T. Every pc model of the L-theory $T_i^\star(A)$ is a pc model of T, and every pc model of T which is a model of $T_i^\star(A)$ is a pc model of $T_i^\star(A)$.

Proof: The corollary follows directly from the fact that $T_i^\star(A)$ is positively complete and A is a common pc of T and $T_i^\star(A)$. □

Remark 1.14. Let be A_e a pc model of T and $A \subseteq A_e$. Let $< A >$ be the L-substructure of A_e generated by A. Given that $T_u(A_e| < A >)$ and $T_u(A_e|A)$ are positively complete theories and A_e is a common pc model of $T_u(A_e| < A >)$ and $T_u(A_e|A)$, it follows from Lemma 1.12 that $T_u(A_e| < A >)$ and $T_u(A_e|A)$ are companion theories.

The following example list some anomaly situations in the positive logic that we will try to deal by some changes focused on the language and the theories.

Example 1.15.

1. Let T_{pos} the h-inductive theory of posets in the relational language $L = \{\leq\}$. T_{pos} is positively complete and has only one pc model which is the trivial structure $(\{x\}, \leq)$.
2. Let $L = \{f\}$ be the language formed by 1-ary function symbol f.
   (a) For every integer n, let T_n be the h-inductive theory $\{\exists x\ f^n(x) = x\}$. For every n, the theory T_n is positively complete and has only one pc model which is the structure $(\{x\}, f)$ such that $f(x) = x$.
   (b) For every integer n, let T'_n be the h-inductive theory $\{\neg\exists x\ f^n(x) = x\}$. We can consider the models of T'_n as directed graphs such that the vertexes of the graph are the element of the structure, and two vertexes a and b are jointed by an edge pointed from a into b if $f(a) = b$. The theory T'_n is positively complete and has only one pc model that is the graph G_n such that, for every prime p that not divide n, G_n contains one cycles of length p.
3. Let T_g the h-inductive theory of groups in the usual language L_g of groups. T_g is complete and the trivial group is the unique pc model of T_g.
4. Let $L^* = L_g \cup \{R\}$ where L_g is the language of groups and R a symbol of binary relation interpreted by $R(a, b) \leftrightarrow a \neq b$. Let T^*_g the usual theory of groups over the language L_1. Since the L^*-homomorphism are the L_g-embeddings then the pc models of T^*_g are the existentially closed groups in the context of logic with negation, so T^*_g is positively complete.
5. Let $L^+ = L_g \cup \{a\}$ where a is a symbol of constant and let $T^+_g = T_g \cup \{a \neq e\}$. Let p and q two prime numbers. Since the groups $\mathbb{Z}_p$ and $\mathbb{Z}_q$ (where the constant a is interpreted by $\bar{1}$) cannot be L^+-continued in a L^+-group, then the theory T^+_g is not positively complete.
   Let G^+ be a pc group of the theory T^+_g. We claim that G^+ is either simple or the intersection of all nontrivial normal subgroups of G^+ is nonempty. Indeed, suppose that G^+ is not simple and let N be a normal subgroups of G^+. Given that the natural L_g-homomorphism $\pi : G^+ \to G^+/N$ is not an immersion then π is not an L^+-homomorphism, which implies that $\pi(a) = \bar{e}$, so $a \in N$.

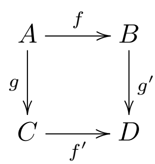
Thereby a belongs to the intersection of all normal subgroups of G^+. Note that if G^+ is simple then G^+ is an existentially closed groups and the constant $a \in L^+$ can be interpreted by any element of $G^+ - \{e\}$. In the case where G^+ is not simple then the constant a can be interpreted by any element of $N - \{e\}$ where N in the intersection of the nontrivial normal subgroups of G^+, and we have $N =< a >^{G^+}$ the normal subgroup generated by a.

2. General forms of positive amalgamation

In this section we will use the letters h, e, i, s to abbreviate the terms respectively of homomorphisms, embeddings, immersions and strong immersions.

Definition 2.1. Let Γ be a class of L-structures and A a member of Γ. We say that A is an:

- $[h, e, i, s]$-amalgamation basis of Γ; if for every B, C in Γ, f an homomorphism from A into B and g an embedding from A into C, there exist $D \in \Gamma$, f' an immersion from B into D and g' a strong immersion from C into D such that the following diagram commutes:



  We say that Γ has the $[h, e, i, s]$-amalgamation property if every element of Γ is an $[h, e, i, s]$-amalgamation basis of Γ.
  By the same way we define all the other possible forms of amalgamation properties.

- $[h, e]$-asymmetric amalgamation basis of Γ, if A is $[h, e, h, e]$-amalgamation basis of Γ.
  By the same way we define all forms of asymmetric amalgamation properties.

- $[h]$- amalgamation basis of Γ, if A is $[h, h, h, h]$-amalgamation basis of Γ.

By the same way we define all the other possible forms of $[x]$-amalgamation properties.

- $[h, e, i, s]$-strong amalgamation basis of Γ, if for every B, C members of Γ such that A is continued into B by an homomorphisms f and embedded in C by an embedding g, then there exist $D \in \Gamma$, f' an immersion from C into D, and g' a strong immersion from B into D such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ C & \xrightarrow[f']{} & D \end{array}$$

  and $\forall (b, c) \in B \times C$, if $g'(b) = f'(c)$ then there is $a \in A$ such that $b = f(a)$ and $c = g(a)$.
  We say that A is an $[h]$- strong amalgamation basis of Γ, if A is a $[h, h, h, h]$- strong amalgamation basis.
  By the same way we define all other possible forms of strong amalgamation properties.

In the following remark, we observe that the most forms of the amalgamations property defined above can be characterized by the notions of completeness and positive completeness defined in the previous section.

Remark 2.2. Let T be an h-inductive L-theory and A a model of T. We have the following properties:

1. A is an $[h]$-amalgamation basis of T if and only if $T \cup Diag^+(A)$ is positively $L(A)$-complete theory.
2. A is an $[e, e, h, h]$-amalgamation basis of T if and only if $T \cup Diag(A)$ is positively $L(A)$-complete theory.
3. A is an $[i, e, h, h]$-amalgamation basis of T if and only if $T \cup Diag^+(A)$ and $T_u(A)$ is are T-complete in the language $L(A)$.

By the same way we can characterize all other forms of amalgamation except the strong amalgamation forms.

In the following example we will list some facts on amalgamation property with the notations and terms given in the definition 2.1.

Example 2.3.

1. Every L-structure A is an $[i, h, s, h]$-amalgamation basis in the class of L-structures (lemma 4, [1]). Since every strong immersion is an immersion, it follows that every L-structure A is an $[s, h]$-asymmetric amalgamation basis in the class of L-structures.

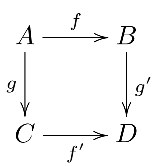
2. Every L-structure A is an $[s, i]$-asymmetric amalgamation basis in the class of L-structures (lemma 5, [1]).

3. Every L-structure A is an $[e, s]$-asymmetric amalgamation basis in the class of L-structures (lemma 4, [2]).

4. Every L-structure A is an $[i, h]$-asymmetric amalgamation basis in the class of L-structures (lemma 8, [5]).

5. Every pc model of an h-inductive theory T is an $[h]$-amalgamation basis in the class of model of T.

Lemma 2.4. *Let I be a totally ordered set and let $(A_i, f_{i,j})_{i,j \in I}$ be an h-inductive sequence of $[h]$-strong amalgamation basis of an h-inductive theory T. Then the inductive limit of $(A_i, f_{i,j})_{i,j \in I}$ is an h-amalgamation basis of T that satisfies the following property:*
For every models B and C of T, if $f \in Hom(A, B)$ and $g \in Hom(A, C)$ then there is D a model of T such that the following diagram commutes:



where f' and g' are homomorphisms, and $\forall (b, c) \in B \times C$, if $g'(b) = f'(c)$ then there exist $a, a' \in A$ such that $b = f(a)$ and $c = g(a')$.

Proof: Let A be the h-inductive limit of the sequence $(A_i, f_{i,j})_{i,j \in I}$, let B and C two continuation of A in the class of models of T. We claim that the following set in $L(B \cup C)$-consistent,

$$T \cup Diag^+(B) \cup Diag^+(C) \cup \{b \neq c |\ b \in B - A, c \in C - A\},$$

where every elements of A is interpreted by the same symbols of constant in B and C. Indeed, otherwise there exist $\varphi(\bar{a},\bar{b}) \in Diag^+(B)$ and $\psi(\bar{a},\bar{c}) \in Diag^+(B)$ where $\bar{b} \in B - A$ (ie, if $\bar{b} = (b_1, \cdots, b_n)$ then $\forall 1 \leqslant i \leqslant n, b_i \in B - A$) and $\bar{c} \in C - A$ such that

$$T \vdash \forall \bar{x}, \bar{y}, \bar{z} \ \ ((\varphi(\bar{x},\bar{y}) \wedge \psi(\bar{x},\bar{z})) \rightarrow \bigvee_{i,j} y_i = z_j) \tag{2.1}$$

Now, let $i \in I$ such that $\bar{a} \in A_i$ and let $f_i = f_{|A_i}$ and $g_i = f_{|A_i}$. Since A_i is a $[h]$-strong amalgamation basis, there are D a model of T, $f' \in Hom(B,D)$ and $g' \in Hom(C,D)$ such that

$$\forall (b,c) \in B \times C, \ f'(b) = g'(c) \rightarrow \exists a \in A \ f_i(a) = b \wedge g_i(a) = c. \tag{2.2}$$

By 2.1 and 2.2, there is $a \in A$ such that $f_i(a) = f(a) = b_i$ and $g_i(a) = g(a) = c_i$, a contradiction. □

Theorem 2.5. *Every L-structure A is a $[s,i,s,i]$-strong amalgamation basis in the class of L-structures.*

Proof: Let A, B and C be three L-structures such that A is immersed in B and strongly immersed in C. Suppose that the set

$$T_i(B) \cup T_u(C) \cup Diag^+(B) \cup Diag^+(C) \cup \{b \neq c | \ b \in B - A, c \in C - A\}$$

is $L(B \cup C)$-inconsistent. Then there are $\neg\psi(\bar{a},\bar{c}) \in T_u(C)$, $\varphi_1(\bar{a},\bar{b}) \in Diag^+(B)$ and $\varphi_2(\bar{a},\bar{c}) \in Diag^+(C)$ where ψ is a positive formula, and φ_1, φ_2 quantifier-free positive formulas, such that:

$$T_i(B) \cup \{\neg\psi(\bar{a},\bar{c}), \varphi_1(\bar{a},\bar{b}), \varphi_2(\bar{a},\bar{c}), \bigwedge_{i,j} b_i \neq c_j\}$$

is $L(B \cup C)$-inconsistent, thereby

$$T_i(B) \vdash \forall \bar{y}((\varphi_1(\bar{a},\bar{b}) \wedge \varphi_2(\bar{a},\bar{y})) \rightarrow (\psi(\bar{a},\bar{y}) \vee \bigvee_{i,j} b_i = y_j)). \tag{2.3}$$

Now, since $C \nvDash \psi(\bar{a},\bar{c})$ and $C \vDash \varphi_2(\bar{a},\bar{c})$, then there is $\bar{a}' \in A$ such that $A \nvDash \psi(\bar{a},\bar{a}')$ and $A \vDash \varphi_2(\bar{a},\bar{a}')$, because otherwise we obtain

$$A \vdash \forall \bar{x}(\varphi_2(\bar{a},\bar{x}) \rightarrow \psi(\bar{a},\bar{x})).$$

and given that $C \vdash T_i(A)$, we get a contradiction.

So, we obtain $B \not\models \psi(\bar{a}, \bar{a}')$ and $B \models \varphi_2(\bar{a}, \bar{a}')$. From (2.3) we obtain $B \models \bigvee_{i,j} b_i = a'_j$, a contradiction. Then

$$T' = T_i(B) \cup T_u(C) \cup Diag^+(B) \cup Diag^+(C) \cup \{b \neq c |\ b \in B - A, c \in C - A\}$$

is $L(B \cup C)$-consistent. Let D a model of T', then the following diagram commutes

$$\begin{array}{ccc} A & \longrightarrow & B \\ \big\downarrow & & \big\downarrow{\scriptstyle s} \\ C & \underset{i}{\longrightarrow} & D \end{array}$$

where s in a strong immersion and i an immersion. Let $b \in B$ and $c \in C$ such that $s(b) = i(c)$, then there exists $a, a' \in A$ such that $a = b$ and $a' = c$. Thus

$$i(a) = s(a) = s(b) = i(c) = i(a'),$$

So $a = a'$ and $b = c = a$. □

Theorem 2.6. *Let T be an h-inductive theory. Every model A of T is a $[i, i, h, h]$-strong amalgamation basis of T.*

Proof: Let A, B and C be models of T. Let f and g two immersions from A to B and C respectively. We claim that the set

$$T_u(A) \cup Diag^+(B) \cup Diag^+(C) \cup \{b \neq c |\ b \in B - A, c \in C - A\}$$

is $L(B \cup C)$-consistent. Indeed, otherwise, there are $\bar{a} \in A, \bar{b} \in B - A, \bar{c} \in C - A$, $\varphi(\bar{a}, \bar{b}) \in Diag^+(B)$, and $\psi(\bar{a}, \bar{c}) \in Diag^+(C)$ such that

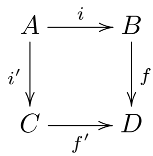
$$T_u(A) \cup \{\varphi(\bar{a}, \bar{b}), \psi(\bar{a}, \bar{c}), \bigwedge_{i,j} b_i \neq c_j\}$$

is $L(B \cup C)$-inconsistent, which implies that;

$$T_u(A) \vdash \forall \bar{y}, \bar{z} \ \ ((\varphi(\bar{a}, \bar{y}) \wedge \psi(\bar{a}, \bar{z})) \to \bigvee_{i,j} y_i = z_j). \tag{2.4}$$

Now, since $C \models \psi(\bar{a}, \bar{c})$ and A is immersed in C, then there is $\bar{a}' \in A$ such that $A \models \psi(\bar{a}, \bar{a}')$, so $B \models \psi(\bar{a}, \bar{a}') \wedge \varphi(\bar{a}, \bar{b})$. On the other hand, given that $\bar{b} \in B - A$ and B is a model of $T_u(A)$ then $B \models \bigvee_{i,j} b_i = a'_j$, a contradiction.

Let D be a model of $T_u(A) \cup Diag^+(B) \cup Diag^+(C) \cup \{b \neq c |\ b \in B - A, c \in C - A\}$, then the following diagram commutes:



where i, i' are immersions and f, f' are homomorphisms. Considering that D is a model of $T_u(A)$ then $f \circ i$ and $f' \circ i'$ are immersions.
Now, let $b \in B$ and $c \in C$ such that $f(b) = f'(c)$, then there exist $a, a' \in A$ such that $i(a) = b$ and $i'(a') = c$. So,

$$f \circ i(a') = f' \circ i'(a') = f(b) = f \circ i(a)$$

then $a = a'$. Thereby A is a $[i, i, h, h]$-strong amalgamation basis of T. □

Corollary 2.7. Every pc model of an h-inductive theory T is an $[h]$-strong amalgamation basis of T.

Lemma 2.8. *Every model of an h-inductive theory T is a $[i, h, s, h]$-strong amalgamation basis of T.*

Proof: Let A, B and C three models of T such that A is immersed in B and continued in C by a homomorphism f. The proof consists in showing the $L(B \cup C)$-consistency of the set

$$T' = T_i(C) \cup Diag^+(B) \cup Diag^+(C) \cup \{b \neq c |\ b \in B - A, c \in C - f(A)\}.$$

Suppose that is not the case, then there are $\varphi(\bar{a}, \bar{c}) \in Diag^+(C)$ and $\psi(\bar{a}, \bar{b}) \in Diag^+(B)$ where $\bar{b} \in B - A$ and $\bar{c} \in C - A$, such that;

$$T_i(C) \vdash \forall \bar{y} \ \ ((\varphi(\bar{a}, \bar{c}) \wedge \psi(\bar{a}, \bar{y})) \rightarrow \bigvee_{i,j} y_i = c_j).$$

Given that $B \models \psi(\bar{a}, \bar{b})$ and A is immersed B, there is $\bar{a}' \in A$ such that $A \models \psi(\bar{a}, \bar{a}')$. Which implies $C \models \varphi(\bar{a}, \bar{c}) \wedge \psi(\bar{a}, \bar{f}(\bar{a}'))$, thereby $C \models \bigvee_{i,j} f(\bar{a}')_i = c_j$, a contradiction.
Let D be a model of T', let f' be the natural homomorphism defined from B into D and i' the natural strong immersion defined from C into D. Let

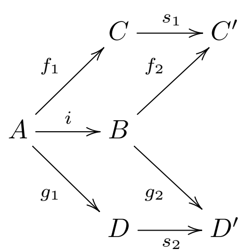
$b \in B$ and $c \in C$ such that $f'(b) = i'(c)$, so there are $a, a' \in A$ such that $i(a) = b$ and $f(a') = c$. Then

$$i' \circ f(a) = f' \circ i(a) = i' \circ f(a'),$$

thus $f(a) = c$ and $i(a) = b$. □

Lemma 2.9. *Let B be a $[h]$-strong amalgamation basis of T and A a model of T that is immersed in B, then A is a $[h]$-strong amalgamation basis of T.*

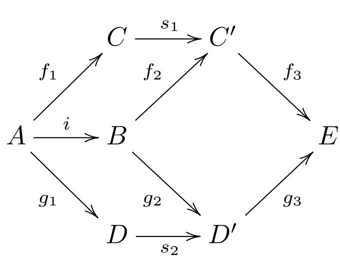
Proof: Let A, C and D be models of T. Let $f_1 \in Hom(A, B)$ and $f_2 \in Hom(A, D)$. Given that A is immersed in B and every L-structure is an $[i, h, s, h]$-strong amalgamation basis in the class of models of T (Lemma 2.8), we obtain the following commutative diagram:



where i_1 and i_2 are immersions, f_2 and g_2 homomorphisms and C', D' two models of T that satisfy:

$$\begin{cases} \forall (b, c) \in B \times C,\ f_2(b) = s_1(c) \to \exists a \in A,\ b = i(a) \wedge c = f_1(a) \\ \forall (b, d) \in B \times D,\ g_2(b) = s_2(d) \to \exists a \in A,\ b = i(a) \wedge d = g_1(a). \end{cases} \tag{2.5}$$

Now, since B is a $[h]$-strong amalgamation basis of T, we complete the previous diagram and we get the following commutative diagram:



where E is a model of T, f_3 and g_3 two homomorphisms such that: $\forall c' \in C', \forall d' \in D'$, if $f_3(c') = g_3(d')$ then there is $b \in B$ such that $f_2(b) = c'$ and $g_2(b) = d'$.
Let $c \in C$ and $d \in D$ such that $f_3 \circ s_1(c) = g_3 \circ s_2(d)$, then there is $b \in B$ such that $f_2(b) = s_1(c)$ and $g_2(b) = s_2(d)$. So there are $a, a' \in A$ such that:

$$\begin{cases} f_1(a) = c & i(a) = b \\ g_1(a') = d & i(a') = b, \end{cases}$$

and given that i is an immersion we have $f_1(a) = c$ and $g_1(a) = d$. So, A is a $[h]$-strong amalgamation basis. □

Acknowledgements. The Researchers would like to thank the Deanship of Graduate Studies and Scientific Research at Qassim University for financial support (QU-APC-2024-9/1).

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Bulletin of the **S**ection of **L**ogic
Volume 53/3 (2024), pp. 321–344
https://doi.org/10.18778/0138-0680.2024.11

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ON COMBINING INTUITIONISTIC AND S4 MODAL LOGIC

Abstract

We address the problem of combining intuitionistic and S4 modal logic in a non-collapsing way inspired by the recent works in combining intuitionistic and classical logic. The combined language includes the shared constructors of both logics namely conjunction, disjunction and falsum as well as the intuitionistic implication, the classical implication and the necessity modality. We present a Gentzen calculus for the combined logic defined over a Gentzen calculus for the host S4 modal logic. The semantics is provided by Kripke structures. The calculus is proved to be sound and complete with respect to this semantics. We also show that the combined logic is a conservative extension of each component. Finally we establish that the Gentzen calculus for the combined logic enjoys cut elimination.

Keywords: combination of logics, intuitionistic logic, modal logic, cut elimination.

2020 Mathematical Subject Classification: 03B62, 03F05, 03B20, 03B45.

1. Introduction

Prawitz was the first to recognize the relevance of tolerance when combining intuitionistic and classical first-order logic [12] (see also [13, 2]). Therein Prawitz proposes a combined logic where the intuitionistic logician accepts that the *tertium non datur* $A \vee_c \neg A$ holds even when A is an intuitionistic

Presented by: Luiz Carlos Pereira
Received: June 7, 2023
Published online: June 5, 2024

formula. On the other hand, the classical logician must also accept that the *tertium non datur* $A \vee_i \neg A$ does not hold even when A is a classical formula.

This non-collapsing combination of intuitionistic and classical logic was obtained by enriching intuitionistic logic with classical constructors while sharing falsum, conjunction, negation and the universal quantifier. This logic was endowed with a natural deduction calculus. An equivalent sequent calculus presentation was discussed in [11], under the name Ecumenical sequent calculus system (using indirect translations via cuts, see [17, 9]).

The interest on combining intuitionistic and classical logic has been around namely in fibring of logics (see [3, 4]). Fibring is a combination technique that given two logics defines another one by putting together the deductive components of each logic while sharing or not some constructors. Soon after its initial proposal, the collapsing problem of intuitionistic into classical logic was identified and a proposal for avoiding this problem appeared in [1]. Later on in [16] a general solution called modulated fibring was proposed for avoiding any such collapse. Furthermore, it is worthwhile to refer to the unified calculus LU presented in [5] where a common non-collapsing single sequent calculus for classical, intuitionistic and linear logics is proposed.

Inspired by the tolerance principle identified by Prawitz in [12], we propose a non-collapsing combination between propositional intuitionistic and propositional classical modal logic S4 sharing $\perp$, $\wedge$ and $\vee$. The idea is to embed intuitionistic logic into modal logic S4 in such a way that intuitionistic logic does not loose its identity (following [6, 8]). The properties of false, conjunction and disjunction are the same for both logics and so they share these constructors. On the other hand we have an intuitionistic implication and a classical implication because these constructors have different properties. There are also an intuitionistic and a classical negation defined by abbreviation from intuitionistic and classical implication, respectively. We consider a set of (classical) propositional variables that are also used to define (intuitionistic) propositional constructors in such a way that hereditariness (necessity) holds. In this way we work with pure Kripke structures for S4 and accommodate intuitionistic constructors in this framework.

As far as we know there are no efforts on fibring intuitionistic and modal logic S4. Nevertheless we expect that such a combination would lead to a collapse of the intuitionistic part into the classical propositional part of

S4. In [7] a intuitionistic modal logic (the host) is enriched with classical constructors. This approach is different from the one we adopt herein namely because the host of our combination is classical modal logic S4.

The paper is organized as follows. In Section 2 we present the language and the Gentzen calculus for the combination of intuitionistic and S4 modal logic. We show that reasoning in this combination extends reasoning in the components. In Section 3 we prove that the Gentzen calculus for the combined logic enjoys cut elimination. We introduce the Kripke semantics for the combination in Section 4 and prove that the combination is conservative over the combined logics. In Section 5 we establish soundness and completeness of the Gentzen calculus with respect to the semantics. Finally in Section 6 we give an overview of the paper and discuss future work.

2. Gentzen calculus

The main objective of this section is to present a Gentzen calculus for the combination of the propositional intuitionistic logic J and propositional modal logic S4 that we denote by $\mathsf{J} \sqcup \mathsf{S4}$. We start by presenting the language $L_{\mathsf{J} \sqcup \mathsf{S4}}$ and then the sequent calculus rules and axioms. After presenting the notion of derivation we provide some examples and establish that reasoning over the combined logic extends reasoning over each component.

We consider fixed a denumerable set P. Let $P_s = \{p_s : p \in P\}$ be the set of (classical) propositional variables. The combined logic has the following sets of constructors $C_0 = \{\bot\} \cup P_i$ where P_i is the set $\{p_i : p \in P\}$, $C_1 = \{\Box_s\}$ and $C_2 = \{\wedge, \vee, \supset_i, \supset_s\}$. We denote by $L_{\mathsf{J} \sqcup \mathsf{S4}}$ the set of formulas inductively defined by the constructors in C_1 and C_2 over $C_0 \cup P_s$. We may use $\neg_i \varphi$ and $\neg_s \varphi$ as abbreviations of $\varphi \supset_i \bot$ and $\varphi \supset_s \bot$, respectively. Moreover we use $\Diamond_s \varphi$ as an abbreviation of $\neg_s \Box_s \neg_s \varphi$. We denote by L_{J} the set of formulas inductively generated by $\wedge$, $\vee$ and $\supset_i$ over $\{\bot\} \cup P_i$. Similarly we denote by L_{S4} the set of formulas inductively generated by $\Box_s$, $\wedge$, $\vee$ and $\supset_s$ over $\{\bot\} \cup P_s$.

A *sequent* is a pair (Γ, Δ), denoted by $\Gamma \to \Delta$, where Γ and Δ are finite multisets of formulas in $L_{\mathsf{J} \sqcup \mathsf{S4}}$. The Gentzen calculus $\mathrm{G}_{\mathsf{J} \sqcup \mathsf{S4}}$ is composed of the following rules for constructors:

$$(\mathrm{LP}_i) \quad \frac{\Box_s p_s, \Gamma \to \Delta}{p_i, \Gamma \to \Delta} \qquad\qquad (\mathrm{RP}_i) \quad \frac{\Gamma \to \Delta, \Box_s p_s}{\Gamma \to \Delta, p_i}$$

$$(\mathrm{L}\wedge)\quad \frac{\beta_1, \beta_2, \Gamma \to \Delta}{\beta_1 \wedge \beta_2, \Gamma \to \Delta} \qquad (\mathrm{R}\wedge)\quad \frac{\Gamma \to \Delta, \beta_1 \quad \Gamma \to \Delta, \beta_2}{\Gamma \to \Delta, \beta_1 \wedge \beta_2}$$

$$(\mathrm{L}\vee)\quad \frac{\beta_1, \Gamma \to \Delta \quad \beta_2, \Gamma \to \Delta}{\beta_1 \vee \beta_2, \Gamma \to \Delta} \qquad (\mathrm{R}\vee)\quad \frac{\Gamma \to \Delta, \beta_1, \beta_2}{\Gamma \to \Delta, \beta_1 \vee \beta_2}$$

$$(\mathrm{L}\supset_s)\quad \frac{\Gamma \to \Delta, \beta_1 \qquad \beta_2, \Gamma \to \Delta}{\beta_1 \supset_s \beta_2, \Gamma \to \Delta} \qquad (\mathrm{R}\supset_s)\quad \frac{\beta_1, \Gamma \to \Delta, \beta_2}{\Gamma \to \Delta, \beta_1 \supset_s \beta_2}$$

$$(\mathrm{L}\supset_i)\quad \frac{\Box_s(\beta_1 \supset_s \beta_2), \Gamma \to \Delta}{\beta_1 \supset_i \beta_2, \Gamma \to \Delta} \qquad (\mathrm{R}\supset_i)\quad \frac{\Gamma \to \Delta, \Box_s(\beta_1 \supset_s \beta_2)}{\Gamma \to \Delta, \beta_1 \supset_i \beta_2}$$

$$(\mathrm{L}\Box_s)\quad \frac{\beta, \Box_s\beta, \Gamma \to \Delta}{\Box_s\beta, \Gamma \to \Delta} \qquad (\mathrm{R}\Box_s)\quad \frac{\Box_s\Gamma \to \Diamond_s \Delta, \beta}{\Gamma', \Box_s\Gamma \to \Diamond_s \Delta, \Delta', \Box_s\beta}$$

the following axioms

$$(\mathrm{Ax})\quad p_s, \Gamma \to \Delta, p_s \qquad (\mathrm{L}\bot)\quad \bot, \Gamma \to \Delta$$

and

$$(\mathrm{Cut})\quad \frac{\Gamma \to \Delta, \beta \quad \beta, \Gamma \to \Delta}{\Gamma \to \Delta}$$

known as the *cut rule*.

A *derivation* for $\Psi \to \Lambda$ is a sequence $\Psi_1 \to \Lambda_1 \dots \Psi_n \to \Lambda_n$ such that $\Psi_1 \to \Lambda_1$ is $\Psi \to \Lambda$ and for $j = 1, \dots, n$

- either $\Psi_j \to \Lambda_j$ is an axiom
- or $\Psi_j \to \Lambda_j$ is the conclusion of a rule and the premises appear from $j+1$ to n.

When there is a derivation for $\Psi \to \Lambda$ we may write

$$\vdash_{\mathsf{G}_{\mathsf{J} \sqcup \mathsf{S4}}} \Psi \to \Lambda.$$

We say that φ is a *theorem* in $\mathsf{J} \sqcup \mathsf{S4}$, written $\vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi$, whenever $\vdash_{G_{\mathsf{J} \sqcup \mathsf{S4}}} \rightarrow \varphi$.

Observe that the rules applied in a derivation are such that the premiss(es) is (are) below the line of inference.

We now establish useful proof-theoretical results concerning weakening, cut, inversion and contraction.

Proposition 2.1. If there is a derivation $\mathcal{D}$ for $\Psi \rightarrow \Lambda$ in $\mathrm{G}_{\mathsf{J} \sqcup \mathsf{S4}}$ then there is a derivation $\mathcal{D}[\Psi' \rightarrow \Lambda']$ for $\Psi', \Psi \rightarrow \Lambda, \Lambda'$ in $\mathrm{G}_{\mathsf{J} \sqcup \mathsf{S4}}$ using the same rules by the same order over the same formulas.

The previous result follows immediately by a straightforward induction. We also omit the proof of the following proposition because it follows straightforwardly.

Proposition 2.2. The multiplicative cut rule

$$\frac{\Gamma \rightarrow \Delta, \beta \quad \beta, \Gamma' \rightarrow \Delta'}{\Gamma, \Gamma' \rightarrow \Delta, \Delta'}$$

is derivable in $\mathrm{G}_{\mathsf{J} \sqcup \mathsf{S4}}$.

The following result is needed for proving that the contraction rules are derivable.

Proposition 2.3. The inversion lemma holds for all rules of $\mathrm{G}_{\mathsf{J} \sqcup \mathsf{S4}}$.

We now state that the left and right contraction rules are derivable in $\mathrm{G}_{\mathsf{J} \sqcup \mathsf{S4}}$. The proof is similar to the proof of Proposition 2.17 of [14].

Proposition 2.4. If there is a derivation for $\varphi, \varphi, \Psi \rightarrow \Lambda$ in $\mathrm{G}_{\mathsf{J} \sqcup \mathsf{S4}}$ then there is a derivation for $\varphi, \Psi \rightarrow \Lambda$ in $\mathrm{G}_{\mathsf{J} \sqcup \mathsf{S4}}$ with at most the same length and with the same cut formulas. Moreover, if there is a derivation for $\Psi \rightarrow \Lambda, \varphi, \varphi$ in $\mathrm{G}_{\mathsf{J} \sqcup \mathsf{S4}}$ then there is a derivation for $\Psi \rightarrow \Lambda, \varphi$ in $\mathrm{G}_{\mathsf{J} \sqcup \mathsf{S4}}$ with at most the same length and with the same cut formulas.

We now provide derived rules for the negations $\neg_s$ and $\neg_i$.

Proposition 2.5. *Let* $\beta \in L_{\mathsf{J} \sqcup \mathsf{S4}}$. *Then*

$$(L\neg_i) \quad \frac{\Box_s(\neg_s \beta), \Gamma \rightarrow \Delta}{\neg_i \beta, \Gamma \rightarrow \Delta} \qquad (R\neg_i) \quad \frac{\Gamma \rightarrow \Delta, \Box_s(\neg_s \beta)}{\Gamma \rightarrow \Delta, \neg_i \beta}$$

and

$$(L\neg_s)\quad \frac{\Gamma \to \Delta, \beta}{\neg_s \beta, \Gamma \to \Delta} \qquad (R\neg_s)\quad \frac{\beta, \Gamma \to \Delta}{\Gamma \to \Delta, \neg_s \beta}$$

PROOF: The rules for $\neg_i$ follow from the following sequences:

1.	$\beta \supset_i \bot, \Gamma \to \Delta$	L$\supset_i$ 2
2.	$\Box_s(\beta \supset_s \bot), \Gamma \to \Delta$	

and

1.	$\Gamma \to \Delta, \beta \supset_i \bot$	R$\supset_i$ 2
2.	$\Gamma \to \Delta, \Box_s(\beta \supset_s \bot)$	

using the abbreviations of $\neg_i$ and $\neg_s$. Similarly for the rules for $\neg_s$. □

Observe that

$$\vdash_{\mathsf{G}_{\mathsf{J}\sqcup\mathsf{S4}}} \beta, \Gamma \to \Delta, \beta$$

and so we use this fact when presenting derivations under the name gAx.

The reader may wonder whether the rules of a sequent calculus for J are derivable in the Gentzen calculus for the combination $\mathsf{J} \sqcup \mathsf{S4}$. The answer is that it is not always the case. For instance the usual intuitionistic rule for introducing $\supset_i$ in the succedent

$$\frac{\Gamma, \beta_1 \to \beta_2}{\Gamma \to \beta_1 \supset_i \beta_2}$$

is not always derivable in $\mathrm{G}_{\mathsf{J}\sqcup\mathsf{S4}}$. It is true that if Γ is empty, we could obtain

1	$\to \beta_1 \supset_i \beta_2$
2	$\to \Box_s(\beta_1 \supset_s \beta_2)$
3	$\to \beta_1 \supset_s \beta_2$
4	$\beta_1 \to \beta_2$

But if Γ is not empty, the application of (R$\Box_s$) would not be possible in general.

Example 2.6. The following derivation

1.	$\to p_i \supset_i (\Box_s p_i)$	$R\supset_i$ 2
2.	$\to \Box_s(p_i \supset_s (\Box_s p_i))$	$R\Box_s$ 3
3.	$\to p_i \supset_s (\Box_s p_i)$	$R\supset_s$ 4
4.	$p_i \to \Box_s p_i$	LP_i 5
5.	$\Box_s p_s \to \Box_s p_i$	$R\Box_s$ 6
6.	$\Box_s p_s \to p_i$	RP_i 7
7.	$\Box_s p_s \to \Box_s p_s$	gAx

shows that $\vdash_{\mathsf{J}\sqcup\mathsf{S4}} p_i \supset_i (\Box_s p_i)$ expressing that hereditariness holds for any constructor p_i in P_i. The derivation

1.	$\to \varphi \vee (\neg_s \varphi)$	$R\vee$ 2
2.	$\to \varphi, \neg_s \varphi$	$R\neg_s$ 3
3.	$\varphi \to \varphi$	gAx

proves that $\vdash_{\mathsf{J}\sqcup\mathsf{S4}} \varphi \vee_s (\neg_s \varphi)$ asserting that *tertium non datur* holds when using classical negation. Finally, the derivation

1.	$\to (\varphi_1 \supset_i \varphi_2) \supset_i (\varphi_1 \supset_s \varphi_2)$	$R\supset_i$ 2
2.	$\to \Box_s((\varphi_1 \supset_i \varphi_2) \supset_s (\varphi_1 \supset_s \varphi_2))$	$R\Box_s$ 3
3.	$\to (\varphi_1 \supset_i \varphi_2) \supset_s (\varphi_1 \supset_s \varphi_2)$	$R\supset_s$ 4
4.	$\varphi_1 \supset_i \varphi_2 \to \varphi_1 \supset_s \varphi_2$	$R\supset_s$ 5
5.	$\varphi_1, \varphi_1 \supset_i \varphi_2 \to \varphi_2$	$L\supset_i$ 6
6.	$\varphi_1, \Box_s(\varphi_1 \supset_s \varphi_2) \to \varphi_2$	$L\Box_s$ 7
7.	$\varphi_1, \varphi_1 \supset_s \varphi_2, \Box_s(\varphi_1 \supset_s \varphi_2) \to \varphi_2$	$L\supset_s$ 8,9
8.	$\varphi_1, \varphi_2, \Box_s(\varphi_1 \supset_s \varphi_2) \to \varphi_2$	gAx
9.	$\varphi_1, \Box_s(\varphi_1 \supset_s \varphi_2) \to \varphi_2, \varphi_1$	gAx

proves that $\vdash_{\mathsf{J}\sqcup\mathsf{S4}} (\varphi_1 \supset_i \varphi_2) \supset_i (\varphi_1 \supset_s \varphi_2)$ expressing the intuitionistic relationship between $\supset_i$ and $\supset_s$.

Next result shows that the combined logic is an extension of intuitionistic logic, that is, every theorem in intuitionistic logic J is a theorem in the combination $\mathsf{J}\sqcup\mathsf{S4}$.

PROPOSITION 2.7. Let $\varphi \in L_{\mathsf{J}}$ and H_{J} be the Hilbert calculus for intuitionistic logic presented in [15] over L_{J}. Then $\vdash_{\mathrm{H}_{\mathsf{J}}} \varphi$ in H_{J} implies $\vdash_{\mathrm{G}_{\mathsf{J}\sqcup\mathsf{S4}}} \varphi$.

Proof: We start by proving that if φ is an axiom of $\mathrm{H_J}$ then $\vdash_{\mathrm{G_{J \sqcup S4}}} \varphi$. We just consider the axiom

$$(\varphi_1 \supset_i \varphi_2) \supset_i ((\varphi_1 \supset_i (\neg_i \varphi_2)) \supset_i (\neg_i \varphi_1)).$$

The sequence

1.	$\to (\varphi_1 \supset_i \varphi_2) \supset_i ((\varphi_1 \supset_i (\neg_i \varphi_2)) \supset_i (\neg_i \varphi_1))$	$\mathrm{R}{\supset_i}$ 2
2.	$\to \Box_s((\varphi_1 \supset_i \varphi_2) \supset_s ((\varphi_1 \supset_i (\neg_i \varphi_2)) \supset_i (\neg_i \varphi_1)))$	$\mathrm{R}\Box_s$ 3
3.	$\to (\varphi_1 \supset_i \varphi_2) \supset_s ((\varphi_1 \supset_i (\neg_i \varphi_2)) \supset_i (\neg_i \varphi_1))$	$\mathrm{R}{\supset_s}$ 4
4.	$\varphi_1 \supset_i \varphi_2 \to (\varphi_1 \supset_i (\neg_i \varphi_2)) \supset_i (\neg_i \varphi_1)$	$\mathrm{L}{\supset_i}$ 5
5.	$\Box_s(\varphi_1 \supset_s \varphi_2) \to (\varphi_1 \supset_i (\neg_i \varphi_2)) \supset_i (\neg_i \varphi_1)$	$\mathrm{R}{\supset_i}$ 6
6.	$\Box_s(\varphi_1 \supset_s \varphi_2) \to \Box_s((\varphi_1 \supset_i (\neg_i \varphi_2)) \supset_s (\neg_i \varphi_1))$	$\mathrm{R}\Box_s$ 7
7.	$\Box_s(\varphi_1 \supset_s \varphi_2) \to (\varphi_1 \supset_i (\neg_i \varphi_2)) \supset_s (\neg_i \varphi_1)$	$\mathrm{R}{\supset_s}$ 8
8.	$\varphi_1 \supset_i (\neg_i \varphi_2), \Box_s(\varphi_1 \supset_s \varphi_2) \to \neg_i \varphi_1$	$\mathrm{L}{\supset_i}$ 9
9.	$\Box_s(\varphi_1 \supset_s (\neg_i \varphi_2)), \Box_s(\varphi_1 \supset_s \varphi_2) \to \neg_i \varphi_1$	$\mathrm{R}\neg_i$ 10
10.	$\Box_s(\varphi_1 \supset_s (\neg_i \varphi_2)), \Box_s(\varphi_1 \supset_s \varphi_2) \to \Box_s \neg_s \varphi_1$	$\mathrm{R}\Box_s$ 11
11.	$\Box_s(\varphi_1 \supset_s (\neg_i \varphi_2)), \Box_s(\varphi_1 \supset_s \varphi_2) \to \neg_s \varphi_1$	$(\mathrm{L}\Box_s)^2$ 12
12.	$\varphi_1 \supset_s (\neg_i \varphi_2), \varphi_1 \supset_s \varphi_2, \Box_s(\varphi_1 \supset_s (\neg_i \varphi_2)),$ $\Box_s(\varphi_1 \supset_s \varphi_2) \to \neg_s \varphi_1$	$\mathrm{R}\neg_s$ 13
13.	$\varphi_1, \varphi_1 \supset_s (\neg_i \varphi_2), \varphi_1 \supset_s \varphi_2, \Box_s(\varphi_1 \supset_s (\neg_i \varphi_2)),$ $\Box_s(\varphi_1 \supset_s \varphi_2) \to$	$\mathrm{L}{\supset_s}$ 14,15
14.	$\varphi_1, \varphi_1 \supset_s (\neg_i \varphi_2), \Box_s(\varphi_1 \supset_s (\neg_i \varphi_2)),$ $\Box_s(\varphi_1 \supset_s \varphi_2) \to \varphi_1$	gAx
15.	$\varphi_1, \varphi_1 \supset_s (\neg_i \varphi_2), \varphi_2, \Box_s(\varphi_1 \supset_s (\neg_i \varphi_2)),$ $\Box_s(\varphi_1 \supset_s \varphi_2) \to$	$\mathrm{L}{\supset_s}$ 16,19
16.	$\varphi_1, \neg_i \varphi_2, \varphi_2, \Box_s(\varphi_1 \supset_s (\neg_i \varphi_2)), \Box_s(\varphi_1 \supset_s \varphi_2) \to$	$\mathrm{L}\neg_i$ 17
17.	$\varphi_1, \Box_s \neg_s \varphi_2, \varphi_2, \Box_s(\varphi_1 \supset_s (\neg_i \varphi_2)), \Box_s(\varphi_1 \supset_s \varphi_2) \to$	$\mathrm{L}\Box_s$ 18
18.	$\varphi_1, \neg_s \varphi_2, \Box_s \neg_s \varphi_2, \varphi_2, \Box_s(\varphi_1 \supset_s (\neg_i \varphi_2)),$ $\Box_s(\varphi_1 \supset_s \varphi_2) \to$	$\mathrm{L}\neg_s$ 20
19.	$\varphi_1, \varphi_2, \Box_s(\varphi_1 \supset_s (\neg_i \varphi_2)), \Box_s(\varphi_1 \supset_s \varphi_2) \to \varphi_1$	gAx
20.	$\varphi_1, \Box_s \neg_s \varphi_2, \varphi_2, \Box_s(\varphi_1 \supset_s (\neg_i \varphi_2)),$ $\Box_s(\varphi_1 \supset_s \varphi_2) \to \varphi_2$	gAx

is a derivation for $\vdash_{\mathrm{G_{J \sqcup S4}}} (\varphi_1 \supset_i \varphi_2) \supset_i ((\varphi_1 \supset_i (\neg_i \varphi_2)) \supset_i (\neg_i \varphi_1))$.

It remains to show that $\vdash_{\mathrm{G}_{\mathsf{J}\sqcup\mathsf{S4}}} \varphi_1, \varphi_1 \supset_i \varphi_2 \to \varphi_2$. Indeed consider the sequence

1.	$\varphi_1, \varphi_1 \supset_i \varphi_2 \to \varphi_2$	L$\supset_i$ 2
2.	$\varphi_1, \Box_s(\varphi_1 \supset_s \varphi_2) \to \varphi_2$	L$\Box_s$ 3
3.	$\varphi_1, \varphi_1 \supset_s \varphi_2, \Box_s(\varphi_1 \supset_s \varphi_2) \to \varphi_2$	L$\supset_s$ 4,5
4.	$\varphi_1, \varphi_2, \Box_s(\varphi_1 \supset_s \varphi_2) \to \varphi_2$	gAx
5.	$\varphi_1, \Box_s(\varphi_1 \supset_s \varphi_2) \to \varphi_2, \varphi_1$	gAx

The fact that there is a derivation for $\mathrm{G}_{\mathsf{J}\sqcup\mathsf{S4}}$ for $\to \varphi$ follows by a straightforward induction on the length of a derivation for φ in H_{J}. □

We provide an example of the use of *Modus Ponens* (MP) in the context of a derivation in $\mathrm{G}_{\mathsf{J}\sqcup\mathsf{S4}}$. We now show that

$$\begin{cases} \vdash_{\mathrm{G}_{\mathsf{J}\sqcup\mathsf{S4}}} \to \varphi_1 \\ \vdash_{\mathrm{G}_{\mathsf{J}\sqcup\mathsf{S4}}} \to \varphi_1 \supset_i \varphi_2 \end{cases} \quad \text{implies} \quad \vdash_{\mathrm{G}_{\mathsf{J}\sqcup\mathsf{S4}}} \to \varphi_2.$$

We start by observing that there are derivations for

$$\begin{cases} (\dagger) & \vdash_{\mathrm{G}_{\mathsf{J}\sqcup\mathsf{S4}}} \to \varphi_1, \varphi_2 \\ (\ddagger) & \vdash_{\mathrm{G}_{\mathsf{J}\sqcup\mathsf{S4}}} \varphi_1 \to \varphi_2, \varphi_1 \supset_i \varphi_2 \end{cases}$$

using Proposition 2.1. Then the sequence

1.	$\to \varphi_2$	Cut 2,3
2.	$\varphi_1 \to \varphi_2$	Cut 4,5
3.	$\to \varphi_1, \varphi_2$	(†)
4	$\varphi_1 \supset_i \varphi_2, \varphi_1 \to \varphi_2$	MP
5.	$\varphi_1 \to \varphi_2, \varphi_1 \supset_i \varphi_2$	(‡)

is a derivation for $\vdash_{\mathrm{G}_{\mathsf{J}\sqcup\mathsf{S4}}} \to \varphi_2$.

Similarly to the previous result it is straightforward to show that reasoning over the combined logic is an extension of the reasoning in S4 modal logic.

Proposition 2.8. Let $\varphi \in L_{\mathsf{S4}}$. Then φ is a theorem of $\mathsf{J} \sqcup \mathsf{S4}$ when φ is a theorem of $\mathsf{S4}$.

The next example shows that *tertium non datur* holds in the combined logic $\mathsf{J} \sqcup \mathsf{S4}$ with respect to the L_{S4} fragment.

Example 2.9. Let $\varphi \in L_{\mathsf{S4}}$. Then $\varphi \vee (\neg_s \varphi)$ is a theorem in $\mathsf{J} \sqcup \mathsf{S4}$, by Proposition 2.8, since $\varphi \vee (\neg_s \varphi)$ is a theorem in $\mathsf{S4}$.

3. Cut elimination

The main goal of this section is to prove the *Gentzen's Hauptsatz* for $\mathrm{G}_{\mathsf{J} \sqcup \mathsf{S4}}$. We follow the strategy of the proof in [14].

We start by introducing the notion of branch of a derivation. A *branch* of a derivation $\Psi_1 \to \Lambda_1 \ \cdots \ \Psi_n \to \Lambda_n$ starting at sequent $\Psi_i \to \Lambda_i$ is a finite subsequence $\Psi_{i_1} \to \Lambda_{i_1} \ \cdots \ \Psi_{i_m} \to \Lambda_{i_m}$ of the derivation such that:

- $\Psi_{i_1} \to \Lambda_{i_1}$ is $\Psi_i \to \Lambda_i$;
- for each $1 \leq j < m$, $\Psi_{i_j} \to \Lambda_{i_j}$ is the conclusion of a rule in the derivation and $\Psi_{i_{j+1}} \to \Lambda_{i_{j+1}}$ is a premise of that rule in the derivation;
- $\Psi_{i_m} \to \Lambda_{i_m}$ is either Ax or L$\perp$.

Moreover, the *depth* of a branch is the number of sequents in the branch minus 1.

Let $\mathcal{D}$ be a derivation in $\mathrm{G}_{\mathsf{J} \sqcup \mathsf{S4}}$ where the cut rule was applied in step i from premises at steps j and k.

The *level* of this cut application at i is the sum of the maximum depth of a branch starting at the premise in j with the maximum depth of a branch starting at the premise in k. The *complexity* of a formula φ denoted by $|\varphi|$ is inductively defined as follows.

- $|p_s| = |\perp| = 0$ for every $p_s \in P_s$
- $|p_i| = 2$ for every $p_i \in P_i$
- $|\varphi_1 \wedge \varphi_2| = |\varphi_1 \vee \varphi_2| = |\varphi_1 \supset_s \varphi_2| = \max(|\varphi_1|, |\varphi_2|) + 1$

- $|\varphi_1 \supset_i \varphi_2| = \max(|\varphi_1|, |\varphi_2|) + 3$
- $|\Box_s \varphi_1| = |\varphi_1| + 1$.

The *rank* of a cut application in $\mathcal{D}$ is the complexity of the respective cut formula plus 1. The *cutrank* of $\mathcal{D}$ is the maximum of the ranks of the cut applications in $\mathcal{D}$ (the cutrank of a derivation with no cut applications is 0).

PROPOSITION 3.1. Given a derivation $\mathcal{D}$ for $\vdash_{\mathsf{G}_{\mathsf{J} \sqcup \mathsf{S4}}} \Psi \to \Lambda$ where $\Psi \to \Lambda$ is obtained by a cut from derivations with a lower cutrank than $\mathcal{D}$ then there is a derivation $\mathcal{D}^\bullet$ for $\vdash_{\mathsf{G}_{\mathsf{J} \sqcup \mathsf{S4}}} \Psi \to \Lambda$ with a lower cutrank than $\mathcal{D}$.

PROOF: Let $\mathcal{D}$ be

1	$\Psi \to \Lambda$	Cut 2,n
2	$\Psi \to \Lambda, \varphi$	
	$\mathcal{D}_1$	
n	$\varphi, \Psi \to \Lambda$	
	$\mathcal{D}_2$	

The proof follows by induction on the level of the cut. The base cases are straightforward (see [17] and [14]). With respect to the inductive step we only consider the case where the lengths of $\mathcal{D}_1$ and $\mathcal{D}_2$ are greater than 1. We start by considering the case where φ is principal in both premises of the cut. There are several subcases to consider depending on the main constructor of φ. We omit the subcases where the main constructor is from S4 (see [17]).

(1) φ is $p_i \in P_i$. Then $\mathcal{D}$ is the sequence

1.	$\Psi \to \Lambda$	Cut 2,n
2.	$\Psi \to \Lambda, p_i$	RP$_i$ 3
3.	$\Psi \to \Lambda, \Box_s p_s$	
	$\mathcal{D}'_1$	
n.	$p_i, \Psi \to \Lambda$	LP$_i$ $n+1$
$n+1$.	$\Box_s p_s, \Psi \to \Lambda$	
	$\mathcal{D}'_2$	

Hence the target $\mathcal{D}^\bullet$ can be of the form

$$\begin{array}{lll}
1. & \Psi \to \Lambda & \text{Cut } 2,n-1 \\
2. & \Psi \to \Lambda, \Box_s p_s & \\
 & \mathcal{D}'_1 & \\
n-1. & \Box_s p_s, \Psi \to \Lambda & \\
 & \mathcal{D}'_2 &
\end{array}$$

since this derivation has lower cutrank than $\mathcal{D}$ and it is for the same goal.
(2) φ is the formula $\varphi_1 \supset_i \varphi_2$. Then $\mathcal{D}$ is the sequence

$$\begin{array}{lll}
1. & \Psi \to \Lambda & \text{Cut } 2,n \\
2. & \Psi \to \Lambda, \varphi_1 \supset_i \varphi_2 & \text{R}\supset_i 3 \\
3. & \Psi \to \Lambda, \Box_s(\varphi_1 \supset_s \varphi_2) & \\
 & \mathcal{D}'_1 & \\
n. & \varphi_1 \supset_i \varphi_2, \Psi \to \Lambda & \text{L}\supset_i n+1 \\
n+1. & \Box_s(\varphi_1 \supset_s \varphi_2), \Psi \to \Lambda & \\
 & \mathcal{D}'_2 &
\end{array}$$

Thus the target $\mathcal{D}^\bullet$ can be of the form

$$\begin{array}{lll}
1. & \Psi \to \Lambda & \text{Cut } 2,n-1 \\
2. & \Psi \to \Lambda, \Box_s(\varphi_1 \supset_s \varphi_2) & \\
 & \mathcal{D}'_1 & \\
n-1. & \Box_s(\varphi_1 \supset_s \varphi_2), \Psi \to \Lambda & \\
 & \mathcal{D}'_2 &
\end{array}$$

because this derivation has lower cutrank than $\mathcal{D}$ and it is for the same goal.

We now consider the case where the cut formula is not principal in the premise at step 2. Moreover we assume that the rule applied at step 2 is L$\supset_i$. So $\mathcal{D}$ is of the following form:

1.	$\varphi_1 \supset_i \varphi_2, \Psi_1 \to \Lambda$	Cut 2,n
2.	$\varphi_1 \supset_i \varphi_2, \Psi_1 \to \Lambda, \varphi$	L$\supset_i$ 3
3.	$\Box_s(\varphi_1 \supset_s \varphi_2), \Psi_1 \to \Lambda, \varphi$	
	$\mathcal{D}_1'$	
n.	$\varphi, \varphi_1 \supset_i \varphi_2, \Psi_1 \to \Lambda$	
	$\mathcal{D}_2$	

Thus Cut can be applied to the premise of $\supset_i$ taking into account Proposition 2.1:

1.	$\Box_s(\varphi_1 \supset_s \varphi_2), \varphi_1 \supset_i \varphi_2, \Psi_1 \to \Lambda$	Cut 2,n
2.	$\Box_s(\varphi_1 \supset_s \varphi_2), \varphi_1 \supset_i \varphi_2, \Psi_1 \to \Lambda, \varphi$	
	$\mathcal{D}_1'[\varphi_1 \supset_i \varphi_2 \to]$	
n.	$\varphi, \Box_s(\varphi_1 \supset_s \varphi_2), \varphi_1 \supset_i \varphi_2, \Psi_1 \to \Lambda$	
	$\mathcal{D}_2[\Box_s(\varphi_1 \supset_s \varphi_2) \to]$	

Then by the induction hypothesis on the level of the cut there is the following derivation

1.	$\Box_s(\varphi_1 \supset_s \varphi_2), \varphi_1 \supset_i \varphi_2, \Psi_1 \to \Lambda$
	$\mathcal{D}_1^\bullet$

with less cutrank than the original one. Hence we have the following derivation

1.	$\varphi_1 \supset_i \varphi_2, \varphi_1 \supset_i \varphi_2, \Psi_1 \to \Lambda$	L$\supset_i$ 2
2.	$\Box_s(\varphi_1 \supset_s \varphi_2), \varphi_1 \supset_i \varphi_2, \Psi_1 \to \Lambda$	
	$\mathcal{D}_1^\bullet$	

and the thesis follows by Proposition 2.4. □

The next result follows straightforwardly by induction on the number of cuts with the greatest cutrank taking into account Proposition 3.1.

PROPOSITION 3.2. Given a derivation for $\vdash_{G_{J \sqcup S4}} \Psi \to \Lambda$ with non null cutrank then there is a derivation for $\vdash_{G_{J \sqcup S4}} \Psi \to \Lambda$ with a lower cutrank than the given one.

Finally, we are ready to establish *Gentzen's Hauptsatz* for $G_{J \sqcup S4}$. The proof follows immediately by induction on the cutrank of the given derivation taking into account Proposition 3.2.

PROPOSITION 3.3. Given a derivation for $\vdash_{G_{J \sqcup S4}} \Psi \to \Lambda$, then there is a derivation with no cut applications for $\vdash_{G_{J \sqcup S4}} \Psi \to \Lambda$.

4. Kripke semantics

The objective of this section is to introduce the main semantic concepts for $J \sqcup S4$. Then we prove that the combined logic is conservative with respect to each component.

A *Kripke structure* for the combined logic $J \sqcup S4$ is a triple $M = (W, R, V)$ such that (W, R) is a Kripke frame where R is a reflexive and transitive relation and $V : P_s \times W \to \{0, 1\}$ is a valuation map. We denote by $\mathcal{M}_{J \sqcup S4}$ the class of all Kripke structures for $J \sqcup S4$.

We define that $M \in \mathcal{M}_{J \sqcup S4}$ and $w \in W$ *locally satisfies* φ written

$$M, w \Vdash_{J \sqcup S4} \varphi$$

by induction on φ as follows:

- $M, w \nVdash_{J \sqcup S4} \perp$
- $M, w \Vdash_{J \sqcup S4} p_s$ whenever $V(p_s, w) = 1$
- $M, w \Vdash_{J \sqcup S4} p_i$ whenever $V(p_s, w') = 1$ for every $w' \in W$ such that wRw'
- $M, w \Vdash_{J \sqcup S4} \varphi_1 \wedge \varphi_2$ whenever $M, w \Vdash_{J \sqcup S4} \varphi_j$ for each $j = 1, 2$
- $M, w \Vdash_{J \sqcup S4} \varphi_1 \vee \varphi_2$ whenever $M, w \Vdash_{J \sqcup S4} \varphi_j$ for some $j = 1, 2$
- $M, w \Vdash_{J \sqcup S4} \varphi_1 \supset_s \varphi_2$ whenever $M, w \Vdash_{J \sqcup S4} \varphi_1$ implies $M, w \Vdash_{J \sqcup S4} \varphi_2$
- $M, w \Vdash_{J \sqcup S4} \varphi_1 \supset_i \varphi_2$ whenever $M, w' \Vdash_{J \sqcup S4} \varphi_1$ implies $M, w' \Vdash_{J \sqcup S4} \varphi_2$ for every $w' \in W$ such that wRw'

- $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \Box_s \varphi_1$ whenever $M, w' \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi_1$ for all $w' \in W$ such that wRw'.

Following the abbreviations we also have

- $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \neg_s \varphi$ whenever $M, w \not\Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi$
- $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \neg_i \varphi$ whenever $M, w' \not\Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi$ for every $w' \in W$ such that wRw'.

We extend local satisfaction to sets of formulas as follows: $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \Psi$ whenever $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \psi$ for every $\psi \in \Psi$.

Moreover we say that M *satisfies* φ, written

$$M \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi$$

whenever $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi$ for every $w \in W$ and $M \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \Psi$ whenever $M \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \psi$ for every $\psi \in \Psi$. Finally, we say that Ψ *entails* φ, written

$$\Psi \vDash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi$$

if $M \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi$ whenever $M \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \Psi$ for every $M \in \mathcal{M}_{\mathsf{J} \sqcup \mathsf{S4}}$. When $\Psi = \emptyset$ we say that φ is *valid* and write $\vDash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi$.

We now show that the combined logic $\mathsf{J} \sqcup \mathsf{S4}$ is conservative with respect to intuitionistic logic J. We assume that J is endowed with a Kripke semantics (see [15]) and denote by $\mathcal{M}_{\mathsf{J}}$ the class of all Kripke structures for J.

Proposition 4.1. Let $\varphi \in L_{\mathsf{J}}$. Then $\vDash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi$ implies $\vDash_{\mathsf{J}} \varphi$.

Proof: Let $M \in \mathcal{M}_{\mathsf{J}}$ where $M = (W, R, V)$. We denote by M' the Kripke structure (W, R, V') with $V' : P_s \times W \to \{0, 1\}$ such that $V'(p_s, w) = 1$ whenever $V(p_i, w) = 1$ and $V'(p_s, w) = 0$ otherwise. Thus M' is a Kripke structure for $\mathsf{J} \sqcup \mathsf{S4}$. We start by proving by induction on φ that

$$M, w \Vdash_{\mathsf{J}} \varphi \text{ if and only if } M', w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi.$$

(Base) φ is p_i. Thus $M, w \Vdash_{\mathsf{J}} p_i$ iff $V(p_i, w) = 1$ iff $V(p_i, w') = 1$ for every $w' \in W$ such that wRw' iff $V'(p_s, w') = 1$ for every $w' \in W$ such that wRw' iff $M', w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} p_i$.

(Step) We only consider the case where φ is $\varphi_1 \supset_i \varphi_2$. Hence $M, w \Vdash_{\mathsf{J}} \varphi_1 \supset_i \varphi_2$ iff for every $w' \in W$ such that wRw' if $M, w' \Vdash_{\mathsf{J}} \varphi_1$ then $M, w' \Vdash_{\mathsf{J}}$

φ_2 iff (IH) for every $w' \in W$ such that wRw' if $M', w' \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi_1$ then $M', w' \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi_2$ iff $M', w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi_1 \supset_i \varphi_2$.

So $M' \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi$ if and only if $M \Vdash_{\mathsf{J}} \varphi$.

Finally we are ready to prove the thesis. Assume that $\vDash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi$ and let $M \in \mathcal{M}_{\mathsf{J}}$. Then M' as defined above is in $\mathcal{M}_{\mathsf{J} \sqcup \mathsf{S4}}$. Hence $M' \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi$. Thus, as shown above $M \Vdash_{\mathsf{J}} \varphi$. □

The next example shows that *tertium non datur* does not hold in the L_{J} fragment of the combined logic $\mathsf{J} \sqcup \mathsf{S4}$.

Example 4.2. Let $\varphi \in L_{\mathsf{J}}$. Then $\nvDash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi \vee (\neg_i \varphi)$ by Proposition 4.1 because $\nvDash_{\mathsf{J}} \varphi \vee (\neg_i \varphi)$.

It is straightforward to show that validity over the combined logic is a conservative extension with respect to validity in $\mathsf{S4}$ modal logic.

PROPOSITION 4.3. Let $\varphi \in L_{\mathsf{S4}}$. Then φ is valid in $\mathsf{J} \sqcup \mathsf{S4}$ if and only if φ is valid in $\mathsf{S4}$.

5. Soundness and completeness

The main objective of this section is to prove that the Gentzen calculus $\mathrm{G}_{\mathsf{J} \sqcup \mathsf{S4}}$ for the combination of intuitionistic logic J and modal logic $\mathsf{S4}$ defined in Section 2 is sound and complete with respect to the Kripke semantics introduced in Section 4

We begin by extending the semantic notions to sequents. We say that $M = (W, R, V) \in \mathcal{M}_{\mathsf{J} \sqcup \mathsf{S4}}$ *locally satisfies* in $w \in W$ the sequent $\Psi \to \Lambda$, written

$$M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \Psi \to \Lambda$$

whenever $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \Psi$ implies that there is $\lambda \in \Lambda$ such that $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \lambda$. Moreover, we say that M *satisfies* the sequent $\Psi \to \Lambda$, written

$$M \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \Psi \to \Lambda$$

whenever $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \Psi \to \Lambda$ for every $w \in W$. Furthermore we say $\Psi \to \Lambda$ is *valid*, written $\vDash_{\mathsf{J} \sqcup \mathsf{S4}} \Psi \to \Lambda$, whenever $M \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \Psi \to \Lambda$ for every $M \in \mathcal{M}_{\mathsf{J} \sqcup \mathsf{S4}}$.

In the sequel we need two properties. The first one states that satisfaction of boxed formulas is preserved by the Kripke relation. The second one states that for diamond formulas non-satisfiability is preserved.

PROPOSITION 5.1. Let $M \in \mathcal{M}_{\mathsf{J} \sqcup \mathsf{S4}}$, $w \in W$ and $\varphi \in L_{\mathsf{J} \sqcup \mathsf{S4}}$. Then

- if $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \Box_s \varphi$, $w' \in W$ and wRw' then $M, w' \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \Box_s \varphi$ by transitivity of R
- if $M, w \nVdash_{\mathsf{J} \sqcup \mathsf{S4}} \Diamond_s \varphi$, $w' \in W$ and wRw' then $M, w' \nVdash_{\mathsf{J} \sqcup \mathsf{S4}} \Diamond_s \varphi$ by transitivity of R.

Soundness brings to light that the host of the combination is S4 modal logic. Hence we need to translate formulas in $L_{\mathsf{J} \sqcup \mathsf{S4}}$ to equivalent formulas in L_{S4}. For that we need the following map inspired by the Gödel-McKinsey-Tarski translation [15, 8].

Let $\tau_{\mathsf{J} \sqcup \mathsf{S4}} : L_{\mathsf{J} \sqcup \mathsf{S4}} \to L_{\mathsf{S4}}$ be the map inductively defined as follows:

- $\tau_{\mathsf{J} \sqcup \mathsf{S4}}(p_s) = p_s$
- $\tau_{\mathsf{J} \sqcup \mathsf{S4}}(p_i) = \Box_s p_s$
- $\tau_{\mathsf{J} \sqcup \mathsf{S4}}(\bot) = \bot$
- $\tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi * \psi) = \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi) * \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\psi)$ where $* \in \{\wedge, \vee\}$
- $\tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi_1 \supset_s \varphi_2) = \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi) \supset_s \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\psi)$
- $\tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi_1 \supset_i \varphi_2) = \Box_s(\tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi) \supset_s \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\psi))$
- $\tau_{\mathsf{J} \sqcup \mathsf{S4}}(\Box_s \varphi_1) = \Box_s \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi_1)$.

Observe that $\tau_{\mathsf{J} \sqcup \mathsf{S4}}(\neg_i \varphi) = \Box_s(\neg_s \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi))$ and $\tau_{\mathsf{J} \sqcup \mathsf{S4}}(\neg_s \varphi) = \neg_s \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi)$. We extend the definition of $\tau_{\mathsf{J} \sqcup \mathsf{S4}}$ as follows:

$$\tau_{\mathsf{J} \sqcup \mathsf{S4}}(\Psi) = \{\tau_{\mathsf{J} \sqcup \mathsf{S4}}(\psi) : \psi \in \Psi\}.$$

The following result shows that the translation of a formula is locally equivalent to the original formula.

PROPOSITION 5.2. Let $\varphi \in L_{\mathsf{J} \sqcup \mathsf{S4}}$, M be a Kripke structure and $w \in W$. Then,

$$M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi \text{ if and only if } M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi).$$

PROOF: The proof is by induction on the structure of φ.

(Base) There are three cases.

(1) φ is $p_s \in P_s$. The result is immediate.

(2) φ is $p_i \in P_i$. Thus $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} p_i$ iff $V(p_s, w') = 1$ for every $w' \in W$ such that wRw' iff $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \Box_s p_s$ iff $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \tau_{\mathsf{J} \sqcup \mathsf{S4}}(p_i)$.

(3) φ is $\perp$. The result is immediate.

(Step) There are five cases.

(1) φ is $\varphi_1 \wedge \varphi_2$. Then $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi_1 \wedge \varphi_2$ iff $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi_j$ for $j = 1, 2$ iff (by IH) $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi_j)$ for $j = 1, 2$ iff $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi_1) \wedge \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi_2)$ iff $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi_1 \wedge \varphi_2)$.

(2) φ is $\varphi_1 \vee \varphi_2$. Similar to case (1) of step.

(3) φ is $\varphi_1 \supset_s \varphi_2$. So $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi_1 \supset_s \varphi_2$ iff if $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi_1$ then $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi_2$ iff (by IH) if $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi_1)$ then $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi_2)$ iff $M, w \Vdash_{\mathsf{JS4}} \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi_1) \supset_s \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi_2)$ iff $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi_1 \supset_s \varphi_2)$.

(4) φ is $\varphi_1 \supset_i \varphi_2$. Thus $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi_1 \supset_i \varphi_2$ iff if $M, w' \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi_1$ then $M, w' \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi_2$ for every $w' \in W$ such that wRw' iff (by IH) if $M, w' \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi_1)$ then $M, w' \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi_2)$ for every $w' \in W$ such that wRw' iff $M, w' \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi_1) \supset_s \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi_2)$ for every $w' \in W$ such that wRw' iff $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \Box_s(\tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi_1) \supset_s \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi_2))$ iff $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi_1 \supset_i \varphi_2)$.

(5) φ is $\Box_s \varphi_1$. Thus $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \Box_s \varphi_1$ iff $M, w' \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi_1$ for every $w' \in W$ such that wRw' iff (by IH) $M, w' \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi_1)$ for every $w' \in W$ such that wRw' iff $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \Box_s \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi_1)$ iff $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\Box_s \varphi_1)$. $\Box$

The next result extends to entailment the equivalence between a formula and its translation. We omit the proof since it follows straightforwardly from Propositions 4.3 and 5.2.

Proposition 5.3. Let $\varphi \in L_{\mathsf{J} \sqcup \mathsf{S4}}$. Then $\vDash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi$ if and only if $\vDash_{\mathsf{S4}} \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi)$.

We are now ready to prove the soundness of $G_{\mathsf{J} \sqcup \mathsf{S4}}$. We start by proving that the rules are sound.

A rule is said to be *sound* whenever for every Kripke structure $M \in \mathcal{M}_{\mathsf{J} \sqcup \mathsf{S4}}$, if M satisfies the premises of the rule then M also satisfies the conclusion of the rule.

PROPOSITION 5.4. The rules of $G_{J \sqcup S4}$ are sound.

PROOF: Let $M \in \mathcal{M}_{J \sqcup S4}$.
(LP$_i$) Suppose that $M \Vdash_{J \sqcup S4} \Box_s p_s, \Gamma \to \Delta$. Let $w \in W$. Assume that $M, w \Vdash_{J \sqcup S4} \Gamma$ and $M, w \Vdash_{J \sqcup S4} p_i$. Then $M, w \Vdash_{J \sqcup S4} \tau_{J \sqcup S4}(p_i)$ by Proposition 5.2 and so $M, w \Vdash_{J \sqcup S4} \Box_s p_s$. Hence $M, w \Vdash_{J \sqcup S4} \delta$ for some $\delta \in \Delta$ using the hypothesis.

(RP$_i$) Assume that $M \Vdash_{J \sqcup S4} \Gamma \to \Delta, \Box_s p_s$. Let $w \in W$. Suppose that $M, w \Vdash_{J \sqcup S4} \Gamma$. There are two cases. (1) $M, w \Vdash_{J \sqcup S4} \delta$ for some $\delta \in \Delta$ and so $M, w \Vdash_{J \sqcup S4} \Gamma \to \Delta, p_i$. (2) $M, w \Vdash_{J \sqcup S4} \Box_s p_s$. Hence $M, w \Vdash_{J \sqcup S4} \tau_{J \sqcup S4}(p_i)$ and so $M, w \Vdash_{J \sqcup S4} p_i$ by Proposition 5.2.

(L$\supset_i$) Suppose that $M \Vdash_{J \sqcup S4} \Box_s(\beta_1 \supset_s \beta_2), \Gamma \to \Delta$. Let $w \in W$. Assume that $M, w \Vdash_{J \sqcup S4} \beta_1 \supset_i \beta_2$ and $M, w \Vdash_{J \sqcup S4} \Gamma$. Thus $M, w \Vdash_{J \sqcup S4} \tau_{J \sqcup S4}(\beta_1 \supset_i \beta_2)$ by Proposition 5.2 and so $M, w \Vdash_{J \sqcup S4} \Box_s(\tau_{J \sqcup S4}(\beta_1) \supset_s \tau_{J \sqcup S4}(\beta_2))$. Thus, $M, w' \Vdash_{J \sqcup S4} \tau_{J \sqcup S4}(\beta_1) \supset_s \tau_{J \sqcup S4}(\beta_2)$ for every $w' \in W$ such that wRw' and so $M, w' \Vdash_{J \sqcup S4} \tau_{J \sqcup S4}(\beta_1 \supset_s \beta_2)$ for every $w' \in W$ such that wRw'. Therefore, again by Proposition 5.2 $M, w' \Vdash_{J \sqcup S4} \beta_1 \supset_s \beta_2$ for every $w' \in W$ such that wRw'. So $M, w \Vdash_{J \sqcup S4} \Box_s(\beta_1 \supset_s \beta_2)$. Hence, there is $\delta \in \Delta$ such that $M, w \Vdash_{J \sqcup S4} \delta$ using the hypothesis.

(R$\supset_i$) Assume that $M \Vdash_{J \sqcup S4} \Gamma \to \Delta, \Box_s(\beta_1 \supset_s \beta_2)$. Let $w \in W$. Suppose that $M, w \Vdash_{J \sqcup S4} \Gamma$. There are two cases. (1) There is $\delta \in \Delta$ such that $M, w \Vdash_{J \sqcup S4} \delta$ and therefore $M, w \Vdash \Gamma \to \Delta, \beta_1 \supset_i \beta_2$. (2) $M, w \Vdash_{J \sqcup S4} \Box_s(\beta_1 \supset_s \beta_2)$. Hence $M, w' \Vdash_{J \sqcup S4} \beta_1 \supset_s \beta_2$ for every $w' \in W$ such that wRw' and so, by Proposition 5.2, $M, w' \Vdash_{J \sqcup S4} \tau_{J \sqcup S4}(\beta_1 \supset_s \beta_2)$ for every $w' \in W$ such that wRw'. Hence $M, w' \Vdash_{J \sqcup S4} \tau_{J \sqcup S4}(\beta_1) \supset_s \tau_{J \sqcup S4}(\beta_2)$ for every $w' \in W$ such that wRw' Thus, $M, w \Vdash_{J \sqcup S4} \Box_s(\tau_{J \sqcup S4}(\beta_1) \supset_s \tau_{J \sqcup S4}(\beta_2))$ and so $M, w \Vdash_{J \sqcup S4} \tau_{J \sqcup S4}(\beta_1 \supset_i \beta_2)$. Finally, by Proposition 5.2, $M, w \Vdash_{J \sqcup S4} \beta_1 \supset_i \beta_2$.

(L$\Box_s$) Suppose that $M \Vdash_{J \sqcup S4} \beta, \Box_s \beta, \Gamma \to \Delta$. Let $w \in W$ and assume that $M, w \Vdash_{J \sqcup S4} \Box_s \beta$ and $M, w \Vdash_{J \sqcup S4} \Gamma$. Then $M, w' \Vdash_{J \sqcup S4} \beta$ for every $w' \in W$ such that wRw'. Hence, $M, w \Vdash_{J \sqcup S4} \beta$ by reflexivity and so there is $\delta \in \Delta$ such that $M, w \Vdash_{J \sqcup S4} \delta$.

(R$\Box_s$) Assume that $M \Vdash_{J \sqcup S4} \Box_s \Gamma \to \Diamond_s \Delta, \beta$. Let $w \in W$ and suppose that $M, w \Vdash_{J \sqcup S4} \Box_s \Gamma$ and $M, w \Vdash_{J \sqcup S4} \Gamma'$. There are two cases to consider. (1) $M, w \Vdash_{J \sqcup S4} \Diamond_s \delta$ for some $\delta \in \Delta$ and the thesis follows. (2) Otherwise let $w' \in W$ be such that wRw'. Observe that $M, w' \Vdash_{J \sqcup S4} \Box_s \Gamma$ by Proposition 5.1. Moreover, $M, w' \not\Vdash_{J \sqcup S4} \Diamond_s \delta$ for for every $\delta \in \Delta$ by

Proposition 5.1. So, $M, w' \Vdash_{\mathsf{J}\sqcup\mathsf{S4}} \beta$ using the hypothesis.
The other cases follow in a similar way. □

The next step is to show that the axioms of $\mathrm{G}_{\mathsf{J}\sqcup\mathsf{S4}}$ are sound. We say that an axiom is *sound* whenever it is satisfied by every Kripke structure in $\mathcal{M}_{\mathsf{J}\sqcup\mathsf{S4}}$. The following result is straightforward.

Proposition 5.5. The axioms of $\mathrm{G}_{\mathsf{J}\sqcup\mathsf{S4}}$ are sound.

Finally we have the soundness result.

Proposition 5.6. Let $\varphi \in L_{\mathsf{J}\sqcup\mathsf{S4}}$. Then $\vdash_{\mathsf{J}\sqcup\mathsf{S4}} \varphi$ implies $\models_{\mathsf{J}\sqcup\mathsf{S4}} \varphi$.

Proof: We must start by proving that

$$(\dagger) \qquad \vdash_{\mathrm{G}_{\mathsf{J}\sqcup\mathsf{S4}}} \Psi \to \Lambda \text{ implies } \models_{\mathsf{J}\sqcup\mathsf{S4}} \Psi \to \Lambda.$$

The proof follows by a straightfoward induction on the length of a derivation for $\Psi \to \Lambda$ using Proposition 5.5 and Proposition 5.4. Hence assuming $\vdash_{\mathsf{J}\sqcup\mathsf{S4}} \varphi$ then $\vdash_{\mathrm{G}_{\mathsf{J}\sqcup\mathsf{S4}}} \to \varphi$. Thus, by (†), $\models_{\mathsf{J}\sqcup\mathsf{S4}} \to \varphi$. Therefore, $\models_{\mathsf{J}\sqcup\mathsf{S4}} \varphi$. □

Completeness We start by showing that the sequent derivation in $\mathrm{G}_{\mathsf{J}\sqcup\mathsf{S4}}$ is a conservative extension of the sequent derivation in G_{S4} modulo the translation $\tau_{\mathsf{J}\sqcup\mathsf{S4}}$ (see [17] for the Gentzen calculus for $\mathsf{S4}$). The strategy of proving completeness that we follow is similar to the one in [10].

Proposition 5.7. Let $\Psi \cup \Lambda \subseteq L_{\mathsf{J}\sqcup\mathsf{S4}}$. Then

$$\vdash_{\mathrm{G}_{\mathsf{J}\sqcup\mathsf{S4}}} \Psi \to \Lambda \quad \text{if and only if} \quad \vdash_{\mathrm{G}_{\mathsf{S4}}} \tau_{\mathsf{J}\sqcup\mathsf{S4}}(\Psi) \to \tau_{\mathsf{J}\sqcup\mathsf{S4}}(\Lambda).$$

Proof:
($\to$) Let $\Psi_1 \to \Lambda_1 \ldots \Psi_n \to \Lambda_n$ be a derivation for $\Psi \to \Lambda$ in $\mathrm{G}_{\mathsf{J}\sqcup\mathsf{S4}}$. The proof follows by induction on n.

(Basis) $n = 1$. There are two cases. (1) $\Psi_1 \to \Lambda_1$ is justified by (Ax), that is, it is of the form $p_s, \Gamma \to \Delta, p_s$. Hence $\tau_{\mathsf{J}\sqcup\mathsf{S4}}(p_s), \tau_{\mathsf{J}\sqcup\mathsf{S4}}(\Gamma) \to \tau_{\mathsf{J}\sqcup\mathsf{S4}}(\Delta), \tau_{\mathsf{J}\sqcup\mathsf{S4}}(p_s)$ is also justified by (Ax) in G_{S4}. (2) $\Psi_1 \to \Lambda_1$ is justified by (L⊥), that is, it is of the form $\bot, \Gamma \to \Delta$ and so $\tau_{\mathsf{J}\sqcup\mathsf{S4}}(\bot), \tau_{\mathsf{J}\sqcup\mathsf{S4}}(\Gamma) \to \tau_{\mathsf{J}\sqcup\mathsf{S4}}(\Delta)$ is also justified by (L⊥) in G_{S4} because $\tau_{\mathsf{J}\sqcup\mathsf{S4}}(\bot)$ is $\bot$.

(Step) There are several cases. We only present the proof for (LP$_i$) and (R$\supset_i$). The other proofs follow in a similar way.

(1) $\Psi_1 \to \Lambda_1$ is the conclusion of rule (LP$_i$), that is, is of the form $p_i, \Psi_1' \to \Lambda_1$ and so there is $j = 2, \ldots, n$ such that $\Psi_j \to \Lambda_j$ is $\Box_s p_s, \Psi_1' \to \Lambda_1$. Hence $\vdash_{\mathrm{G}_{\mathsf{J} \sqcup \mathsf{S4}}} \Box_s p_s, \Psi_1' \to \Lambda_1$ and so by (IH) $\vdash_{\mathrm{G}_{\mathsf{S4}}} \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\Box_s p_s), \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\Psi_1') \to \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\Lambda_1)$. So there is a derivation in G_{S4} for $\Box_s p_s, \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\Psi_1') \to \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\Lambda_1)$. The thesis follows since $\tau_{\mathsf{J} \sqcup \mathsf{S4}}(p_i)$ is $\Box_s p_s$.

(2) $\Psi_1 \to \Lambda_1$ is the conclusion of rule (R$\supset_i$), that is, is of the form $\Psi_1 \to \Lambda_1', \varphi_1 \supset_i \varphi_2$ and therefore there is $j = 2, \ldots, n$ such that $\Psi_1 \to \Lambda_1', \Box_s(\varphi_1 \supset_s \varphi_2)$. Thus $\vdash_{\mathrm{G}_{\mathsf{J} \sqcup \mathsf{S4}}} \Psi_1 \to \Lambda_1', \Box_s(\varphi_1 \supset_s \varphi_2)$ and so $\vdash_{\mathrm{G}_{\mathsf{S4}}} \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\Psi_1) \to \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\Lambda_1'), \Box_s(\tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi_1) \supset_s \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi_2))$ by (IH). The thesis follows because $\Box_s(\tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi_1) \supset_s \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi_2))$ is $\tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi_1 \supset_i \varphi_2)$. □

The previous result can be extended straightforwardly to derivation of formulas.

Proposition 5.8. Let $\varphi \in L_{\mathsf{J} \sqcup \mathsf{S4}}$. Then $\vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi$ if and only if $\vdash_{\mathsf{S4}} \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi)$.

We are ready to prove completeness of $\mathrm{G}_{\mathsf{J} \sqcup \mathsf{S4}}$ with respect to $\mathcal{M}_{\mathsf{J} \sqcup \mathsf{S4}}$.

Proposition 5.9. Let $\varphi \in L_{\mathsf{J} \sqcup \mathsf{S4}}$. Then $\vDash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi$ implies $\vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi$.

Proof: Suppose that $\vDash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi$. Hence $\vDash_{\mathsf{S4}} \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi)$ by Proposition 5.3. Thus $\vdash_{\mathsf{S4}} \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi)$ by completeness of S4 (see [17]) and so, by Proposition 5.8, $\vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi$. □

6. Concluding remarks

Inspired by the works of [12] and [11], we propose a logic combining intuitionistic and S4 modal logic in a tolerant way. That is, the intuitionistic logician accepts that the classical principles are present for the modal language fragment of the logic and the modal logician accepts that the intuitionistic principles hold in the intuitionistic language fragment of the logic.

We endow the logic with a Gentzen calculus and with a Kripke semantics and show that the combined logic is sound and complete. We prove that the combined logic extends conservatively intutionistic and S4 modal logic. Moreover we show that the cut rule can be eliminated.

We want to study other metaproperties of the combined logic namely decidability, Craig interpolation and definability. Moreover, we would like to investigate combinations of intuitionistic and other modal logics.

Furthermore, it would be interesting to leave the realm of Kripke semantics and analyze for example the combination of paraconsistent logics with intuitionistic or classical logic.

Acknowledgements. The authors would like to thank an anonymous referee for the careful comments and suggestions. They also acknowledge the support of Instituto de Telecomunicações Research Unit reference UIDB/50008/2020 funded by Fundação para a Ciência e a Tecnologia (FCT) and the Department of Mathematics of Instituto Superior Técnico, Universidade de Lisboa.

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Bulletin of the **S**ection of **L**ogic
Volume 53/3 (2024), pp. 345–364
https://doi.org/10.18778/0138-0680.2024.07

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ON PRE-HILBERT AND POSITIVE IMPLICATIVE PRE-HILBERT ALGEBRAS

Abstract

In the paper, pre-Hilbert algebras are defined as a generalization of Hilbert algebras (namely, a Hilbert algebra is just a pre-Hilbert algebra satisfying the property of antisymmetry). Pre-Hilbert algebras have been inspired by Henkin's Positive Implicative Logic. Their properties and characterizations are investigated. Some important results and examples are given. Moreover, positive implicative pre-Hilbert algebras are introduced and studied, their connections with some algebras of logic are presented. The hierarchies existing between the classes of algebras considered here are shown.

Keywords: Hilbert algebra, pre-Hilbert algebra, BCK-algebra, BCC-algebra, BE-algebra, positive implicativity.

1. Introduction

L. Henkin [5] introduced the notion of "implicative model", as a model of positive implicative propositional calculus. In 1960, A. Monteiro [14] has given the name "Hilbert algebras" to the dual algebras of Henkin's implicative models. In 1966, K. Iséki [7] introduced a new notion called a BCK algebra. It is an algebraic formulation of the BCK-propositional calculus system of C. A. Meredith [13], and generalize the concept of implicative algebras (see [1]). To solve some problems on BCK algebras, Y. Komori [12] introduced BCC algebras. These algebras (also called BIK^{+}-algebras) are an algebraic model of BIK^{+}-logic. In [10], as a generalization of BCK

Presented by: Janusz Ciuciura
Received: September 5, 2023
Published online: May 20, 2024

algebras, H. S. Kim and Y. H. Kim defined BE algebras. In 2008, A. Walendziak [15] defined commutative BE algebras and proved that they are BCK algebras. Later on, in 2010, D. Buşneag and S. Rudeanu [3] introduced the notion of pre-BCK algebra. A BCK algebra is just a pre-BCK with the antisymmetry. In 2016, A. Iorgulescu [6] introduced new generalizations of BCK and Hilbert algebras (RML, aBE, pi-BE, pimpl-RML algebras and many others).

In the paper, we define pre-Hilbert algebras in such a way that a Hilbert algebra is just a pre-Hilbert algebra satisfying the property of antisymmetry. It is a solution to Open problem 6.30 of [6]. We give basic properties and examples of pre-Hilbert algebras. We also give some characterizations of these algebras. Moreover, we introduce and investigate positive implicative pre-Hilbert algebras and present their connections with some algebras of logic. We show the hierarchies existing between all classes of algebras considered here.

The motivation of this study consists algebraic and logical arguments. Pre-Hilbert algebras introduced and investigated in the paper belong to a wide class of algebras of logic, they are a natural generalization of well-known Hilbert algebras. The definition of a pre-Hilbert algebra presented here is inspired by Henkin's Positive Implicative Logic [5].

2. Preliminaries

Let $\mathcal{A} = (A, \to, 1)$ be an algebra of type $(2, 0)$. We define the binary relation $\leq$ by: for all $x, y \in A$,

$$x \leq y \Longleftrightarrow x \to y = 1.$$

We consider the following list of properties ([6]) that can be satisfied by $\mathcal{A}$:

(An) (Antisymmetry) $x \to y = 1 = y \to x \Longrightarrow x = y$,

(An') (Antisymmetry) $(x \leq y$ and $y \leq x) \Longrightarrow x = y$,

(B) $(y \to z) \to [(x \to y) \to (x \to z)] = 1$,

(B') $y \to z \leq (x \to y) \to (x \to z)$,

(BB) $(y \to z) \to [(z \to x) \to (y \to x)] = 1$, (BB') $y \to z \leq (z \to x) \to (y \to x)$,

(C) $[x \to (y \to z)] \to [y \to (x \to z)] = 1$,

(C') $x \to (y \to z) \leq y \to (x \to z)$,

(D) $y \to ((y \to x) \to x) = 1$,

(D') $y \leq (y \to x) \to x$,

(Ex) (Exchange) $x \to (y \to z) = y \to (x \to z)$,

(K) $x \to (y \to x) = 1$,

(K') $x \leq y \to x$,

(L) (Last element) $x \to 1 = 1$,

(L') (Last element) $x \leq 1$,

(M) $1 \to x = x$,

(Re) (Reflexivity) $x \to x = 1$,

(Re') (Reflexivity) $x \leq x$,

(Tr) (Transitivity) $x \to y = 1 = y \to z \Longrightarrow x \to z = 1$,

(Tr') (Transitivity) $(x \leq y \text{ and } y \leq z) \Longrightarrow x \leq z$,

(*) $y \to z = 1 \Longrightarrow (x \to y) \to (x \to z) = 1$,

(*') $y \leq z \Longrightarrow x \to y \leq x \to z$,

(**) $y \to z = 1 \Longrightarrow (z \to x) \to (y \to x) = 1$,

(**') $y \leq z \Longrightarrow z \to x \leq y \to x$.

Remark 2.1. The properties in the list are the most important properties satisfied by a BCK algebra.

Lemma 2.2 ([6]). *Let* $\mathcal{A} = (A, \to, 1)$ *be an algebra of type* $(2, 0)$*. Then the following hold:*

(i) (M) + (K) $\Longrightarrow$(L);

(ii) (M) + (B) $\Longrightarrow$(*), (**);

(iii) (M) + (*) $\Longrightarrow$ (Tr);

(iv) (M) + (**) $\Longrightarrow$ (Tr);

(v) (M) + (BB) $\Longrightarrow$ (Re), (B), (C);

(vi) (C) + (An) $\Longrightarrow$ (Ex);

(vii) (M) + (L) + (**) $\Longrightarrow$ (K);

(viii) (M) + (B) + (C) $\Longrightarrow$ (BB).

Proof: (i)–(vii) follow from Proposition 2.1 and Theorem 2.7 of [6].

(viii) Let $x, y, z \in A$. By (B) and (C), $1 = (z \to x) \to ((y \to z) \to (y \to x)) \leq (y \to z) \to ((z \to x) \to (y \to x))$. From (M) we conclude that $(y \to z) \to ((z \to x) \to (y \to x)) = 1$, that is, (BB) holds in $\mathcal{A}$. □

Following Iorgulescu [6], we say that $(A, \to, 1)$ is an *RML algebra* if it verifies the axioms (Re), (M), (L). We introduce now the following definition.

Definition 2.3. ([6]) Let $\mathcal{A} = (A, \to, 1)$ be an RML algebra. The algebra $\mathcal{A}$ is said to be:

1. an *aRML algebra* if it verifies (An),
2. a *pre-BCC algebra* if it verifies (B),
3. a *pre-BBBCC algebra* if it verifies (BB),
4. a *BCC algebra* if it verifies (B), (An), that is, it is a pre-BCC algebra with (An),
5. a *BE algebra* if it verifies (Ex),
6. an *aBE algebra* if it verifies (Ex), (An), that is, it is a BE algebra with (An),

7. a *pre-BCK algebra* if it verifies (B), (Ex), that is, it is a pre-BCC algebra with (Ex) or, equivalently, it is a BE algebra with (B),

8. a *BCK algebra* if it is a pre-BCK algebra verifying (An).

Denote by **RML**, **aRML**, **pre-BCC**, **pre-BBBCC**, **BCC**, **BE**, **aBE**, **pre-BCK**, **BCK** the classes of RML, aRML, pre-BCC, pre-BBBCC, BCC, BE, aBE, pre-BCK, BCK algebras respectively. By definitions, we have
pre-BCC = **RML** + (B), **pre-BBBCC** = **RML** + (BB),
BE = **RML** + (Ex), **pre-BCK** = **pre-BCC** + (Ex) = **BE** + (B),
aRML = **RML** + (An), **BCC** = **pre-BCC** + (An),
aBE = **BE** + (An) = **aRML** + (Ex), **BCK** = **pre-BCK** + (An).

Remark 2.4. By Lemma 2.2 (v), (viii), **pre-BBBCC** = **pre-BCC** + (C). Since (C) + (An) $\Longrightarrow$ (Ex), we have **BCK** = **BCC** + (Ex) = **pre-BCC** + (Ex) + (An) = **pre-BCC** + (C) + (An) = **pre-BBBCC** + (An).

The interrelationships between the classes of algebras mentioned before are visualized in Figure 1.
It is known that $\leq$ is an order relation in BCC and BCK algebras. By definition, in RML and BE algebras, $\leq$ is a reflexive relation; in aRML and aBE algebras, $\leq$ is reflexive and antisymmetric. By Lemma 2.2 (ii)–(iv), in pre-BCC, pre-BBBCC and pre-BCK algebras, $\leq$ is reflexive and transitive (i.e., it is a pre-order relation).

3. Definition and properties of pre-Hilbert algebras

Let $\mathcal{A} = (A, \rightarrow, 1)$ be an algebra of type $(2, 0)$. Now, we consider the following properties:

(pi) $x \rightarrow (x \rightarrow y) = x \rightarrow y$,

(p-1) $x \rightarrow (y \rightarrow z) \leq (x \rightarrow y) \rightarrow (x \rightarrow z)$,

(p-2) $(x \rightarrow y) \rightarrow (x \rightarrow z) \leq x \rightarrow (y \rightarrow z)$,

(pimpl) $x \rightarrow (y \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z)$.

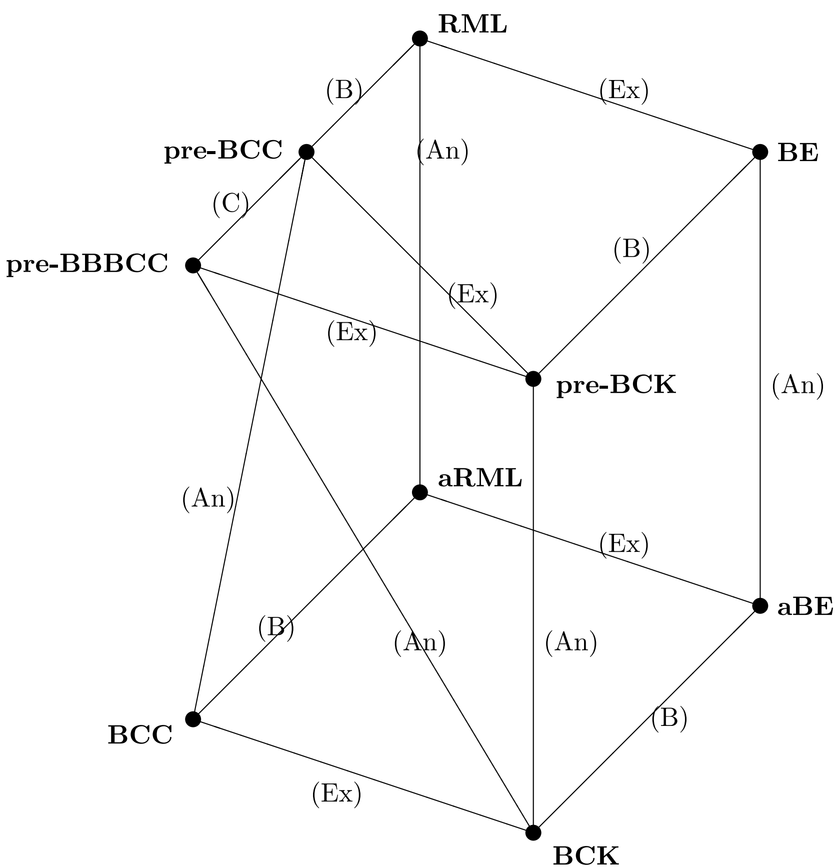
RML
(B)
(Ex)
pre-BCC
(An)
BE
(C)
pre-BBBCC
(B)
(Ex)
(Ex)
pre-BCK
(An)
aRML
(An)
(Ex)
aBE
(B)
(An)
(An)
BCC
(B)
(Ex)
BCK

Figure 1.

Remark 3.1. The properties above are the most important properties satisfied by Hilbert algebras. Recall that an algebra $(A, \rightarrow, 1)$ *is called a* Hilbert algebra if it verifies the axioms (An), (K), (p-1). In [4], A. Diego proved that Hilbert algebras satisfy (Re), (M), (L), (pi), (p-2), (pimpl). Moreover, he showed that the class of all Hilbert algebras is a variety.

Proposition 3.2. Let $(A, \rightarrow, 1)$ be an algebra of type $(2, 0)$. Then the following are true:

(i) (Re) + (M) + (pimpl) $\Longrightarrow$ (pi),

(ii) (p-1) + (p-2) + (An) $\Longrightarrow$ (pimpl),

(iii) (Re) + (pi) $\Longrightarrow$ (L).

Proof: (i) By Proposition 6.4 of [6].

(ii) Obvious.

(iii) Let $x \in A$. We have $x \rightarrow 1 \overset{\text{(Re)}}{=} x \rightarrow (x \rightarrow x) \overset{\text{(pi)}}{=} x \rightarrow x \overset{\text{(Re)}}{=} 1$, thus (L) holds in $(A, \rightarrow, 1)$. □

Remark 3.3. From Proposition 3.2 (i) it follows that in RML algebras, (pimpl) implies (pi). For BCK algebras, (pimpl) and (pi) are equivalent (cf. Theorem 8 of [8]).

Proposition 3.4. Let $(A, \rightarrow, 1)$ be an algebra of type $(2, 0)$. Then the following are true:

(i) (M) + (K) + (p-1) $\Longrightarrow$ (Re),

(ii) (M) + (L) + (p-1) $\Longrightarrow$ (*),

(iii) (K) + (Tr) + (p-1) $\Longrightarrow$ (B),

(iv) (M) + (K) + (**) + (p-1) $\Longrightarrow$ (C),

(v) (Re) + (M) + (C) $\Longrightarrow$ (D),

(vi) (M) + (K) + (**) + (C) $\Longrightarrow$ (p-2).

Proof: (i) Let $x \in A$. We have $1 \overset{\text{(K)}}{=} x \rightarrow ((x \rightarrow x) \rightarrow x) \overset{\text{(p-1)}}{\leq} (x \rightarrow (x \rightarrow x)) \rightarrow (x \rightarrow x) \overset{\text{(K)}}{=} 1 \rightarrow (x \rightarrow x) \overset{\text{(M)}}{=} x \rightarrow x$.

(ii) Let $x, y, z \in A$ and suppose that $y \leq z$. We obtain $1 \overset{\text{(L)}}{=} x \to (y \to z) \overset{\text{(p-1)}}{\leq} (x \to y) \to (x \to z)$. Hence, by (M), $x \to y \leq x \to z$.

(iii) Let $x, y, z \in A$. By (K) and (p-1), $y \to z \leq x \to (y \to z)$ and $x \to (y \to z) \leq (x \to y) \to (x \to z)$. Applying (Tr), we get $y \to z \leq (x \to y) \to (x \to z)$.

(iv) Let $x, y, z \in A$. From (p-1) we obtain

$$x \to (y \to z) \leq (x \to y) \to (x \to z). \tag{3.1}$$

By (K), $y \leq x \to y$ and hence, by (**),

$$(x \to y) \to (x \to z) \leq y \to (x \to z). \tag{3.2}$$

From (M) and (**) it follows that (Tr) holds in $\mathcal{A}$. Using (Tr), from (3.1) and (3.2) we have $x \to (y \to z) \leq y \to (x \to z)$.

(v) We have $1 \overset{\text{(Re)}}{=} (y \to x) \to (y \to x) \overset{\text{(C)}}{\leq} y \to ((y \to x) \to x)$. Applying (M), we get (D).

(vi) Conditions (K) and (**) imply (3.2), see the proof of (iv). By (C), $y \to (x \to z) \leq x \to (y \to z)$. Then $(x \to y) \to (x \to z) \leq x \to (y \to z)$, by (Tr). □

We introduce the following notion:

Definition 3.5. A *pre-Hilbert algebra* is an algebra $(A, \to, 1)$ of type $(2, 0)$ satisfying (M), (K) and (p-1).

Let us denote by **pre-H** and **H** the classes of pre-Hilbert and Hilbert algebras, respectively.

Remark 3.6. Since (An) + (K) + (p-1) imply (M) (see [4]), a Hilbert algebra is in fact a pre-Hilbert algebra verifying (An), that is, **H** = **pre-H** + (An).

Remark 3.7. A motivation for the definition of pre-Hilbert algebra is Positive (Implicative) Logic given by L. Henkin [5]. This logic is the part of intuitionistic logic corresponding to formulas in which implication occurs as the only connective. The propositional calculus of Henkin system of positive logic is specified by the following two axiom schemes:

(H1) $\alpha \to (\beta \to \alpha)$,

(H2) $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$.

and the modus ponens inference rule. Conditions (K) and (p-1) of Definition 3.5 are inspired by axioms (H1) and (H2), respectively. Moreover, (M) is inspired by the modus ponens (indeed, from (M) it follows that if $x = 1$ and $x \to y = 1$, then $y = 1$).

Remark 3.8. Note that Definition 3.5 is a solution to Open problem 6.30 of [6].

THEOREM 3.9. *Pre-Hilbert algebras satisfy* (Re), (M), (L), (K), (*), (**), (Tr), (B), (C), (D), (BB), (p-1), (p-2).

PROOF: Let $\mathcal{A}$ be a pre-Hilbert algebra. By definition, $\mathcal{A}$ satisfies (M), (K) and (p-1). By Proposition 3.4 (i), (M) + (K) + (p-1) imply (Re); thus (Re) holds in $\mathcal{A}$. By Lemma 2.2 (i), (M) + (K) imply (L); thus (L) holds. From Proposition 3.4 (ii) we conclude that $\mathcal{A}$ satisfies (*), hence it also satisfies (Tr) by Lemma 2.2 (iii). Applying Proposition 3.4 (iii), we deduce that (B) holds in A. Then (**) also holds, see Lemma 2.2 (ii). By Proposition 3.4 (iv), (M) + (K) + (p-1) + (**) imply (C); thus (C) holds. From Proposition 3.4 (v) and (vi) it follows that (D) and (p-2) hold. By Lemma 2.2 (viii), (M) + (B) + (C) imply (BB); thus (BB) holds. □

THEOREM 3.10. *Let* $\mathcal{A} = (A, \to, 1)$ *be an algebra of type* $(2, 0)$. *The following are equivalent:*

(i) $\mathcal{A}$ *is a pre-Hilbert algebra;*

(ii) $\mathcal{A}$ *is a pre-BCC algebra satisfying (C) and (p-1);*

(iii) $\mathcal{A}$ *satisfies (M), (L), (B), (C) and (p-1);*

(iv) $\mathcal{A}$ *satisfies (M), (L), (BB) and (p-1);*

(v) $\mathcal{A}$ *is a pre-BBBCC algebra satisfying (p-1).*

PROOF: (i) $\Longrightarrow$ (ii) Follows from Theorem 3.9.

(ii) $\Longrightarrow$ (iii) By definition.

(iii) $\Longrightarrow$ (iv) By Lemma 2.2 (viii).

(iv) $\Longrightarrow$ (v) Since (M) + (BB) imply (Re), we conclude that $\mathcal{A}$ is a pre-BBBCC algebra. Then (v) holds.

(v) $\Longrightarrow$ (i) Pre-BBBCC algebras satisfy (M), (L), (B), hence also (**) and (K) (by Lemma 2.2 (ii), (vii)). Then $\mathcal{A}$ satisfies (M), (K) and (p-1). Thus $\mathcal{A}$ is a pre-Hilbert algebra. □

Example 3.11. ([6], 9.24) Let $A = \{a, b, c, d, 1\}$ and $\to$ be given by the following table:

$\to$	a	b	c	d	1
a	1	a	c	c	1
b	1	1	d	c	1
c	a	b	1	1	1
d	a	b	1	1	1
1	a	b	c	d	1

.

Then $(A, \to, 1)$ verifies (Re), (M), (L), (BB). It does not verify (An) for $x = c$, $y = d$; (Ex) for $x = a, y = b, z = c$; (pi) for $x = a, y = b$ and (p-1) for $x = y = a, z = b$. Therefore, $(A, \to, 1)$ is a pre-BBBCC algebra without (An), (Ex) and (p-1).

Remark 3.12. Pre-Hilbert algebras do not have to satisfy (An), (Ex), (pi); see example below.

Example 3.13. Consider the set $A = \{a, b, c, d, 1\}$ and the operation $\to$ given by the following table:

$\to$	a	b	c	d	1
a	1	c	b	d	1
b	a	1	1	d	1
c	a	1	1	d	1
d	a	c	c	1	1
1	a	b	c	d	1

.

We can observe that the properties (M), (K), (p-1) (hence (Re), (L), (B), (BB), (C), (D), (*), (**), (Tr), (p-2)) are satisfied. Then, $(A, \to, 1)$ is a pre-Hilbert algebra. It does not satisfy (An) for $(x, y) = (b, c)$; (Ex) and (pimpl) for $(x, y, z) = (a, d, b)$; (pi) for $(x, y) = (a, b)$.

DEFINITION 3.14. If $\mathcal{A}$ is a pre-Hilbert algebra not satisfying (An), (Ex) and (pi), then we say that $\mathcal{A}$ *is* proper.

Remark 3.15. The algebra given in Example 3.13 is a proper pre-Hilbert algebra.

Remark 3.16. By Theorem 3.10, **pre-H** = **pre-BBBCC** + (p-1). Hence **H** = **pre-H** + (An) = **pre-BBBCC** + (An) + (p-1) = **BCK** + (p-1). From

Example 3.11 it follows that **pre-H** is a proper subclass of **pre-BBBCC**, that is, **pre-H** $\subset$ **pre-BBBCC**.

By Remark 3.16 and Figure 1, we can draw now the hierarchy between **pre-BBC** and **H**, in the next Figure 2.

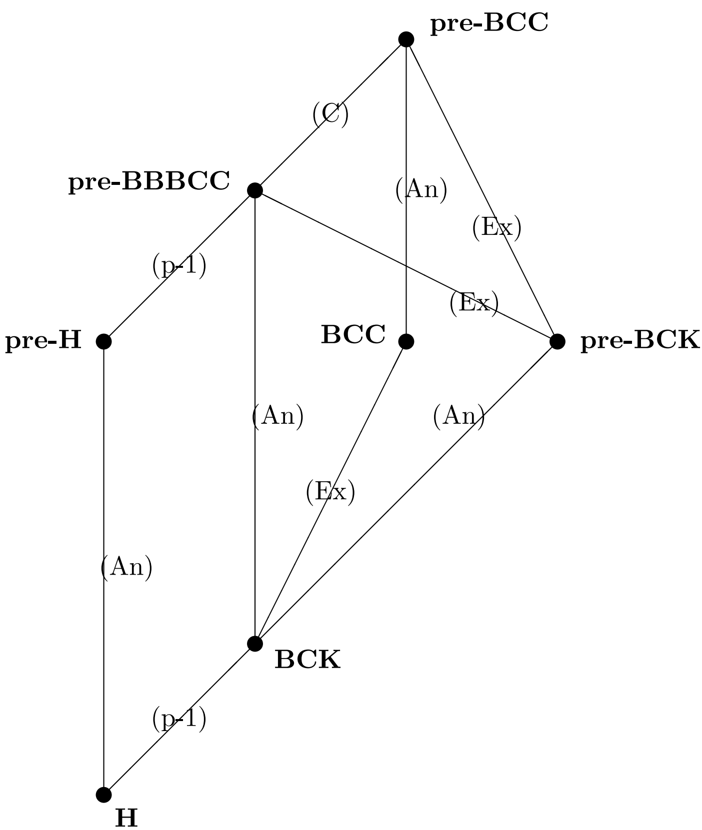

Figure 2.

PROPOSITION 3.17. Let $\mathcal{A} = (A, \rightarrow, 1)$ be a pre-Hilbert algebra. Then $\mathcal{A}$ induces a pre-order $\leq$ on A, defined by: $x \leq y \Longleftrightarrow x \rightarrow y = 1$ and 1 is the element of A satisfying the following conditions:

(L1) $x \leq 1$,

(L2) $1 \leq x \Longrightarrow x = 1$.

Proof: Straightforward. □

Proposition 3.18. Let A be a non-void set of elements and $\leq$ be a pre-order relation on A and 1 be the element of A satisfying (L1) and (L2). We define the operation $\to$ by

$$x \to y = \begin{cases} 1, & \text{if} \quad x \leq y \\ y, & \text{if} \quad \text{otherwise.} \end{cases}$$

Then $\mathcal{A} = (A, \to, 1)$ is a pre-Hilbert algebra.

Proof: It is easy to see that $\mathcal{A}$ satisfies (Re), (M), (L), (Tr) and (K). Observe that $\mathcal{A}$ also satisfies (p-1). Let $x, y, z \in A$. We shall consider three cases.

Case 1: $x \leq z$. Then $(x \to y) \to (x \to z) = (x \to y) \to 1 = 1$. Since $\mathcal{A}$ satisfies (L), we conclude that (p-1) holds for $x \leq z$.

Case 2: $x \not\leq z$ (that is, $x \leq z$ is false) and $x \leq y \to z$. In this case, we have $y \leq z$ and $x \not\leq y$. We obtain $x \to (y \to z) = 1 = y \to z = (x \to y) \to (x \to z)$.

Case 3: $x \not\leq z$ and $x \not\leq y \to z$. Then $y \not\leq z$. Therefore, $x \to (y \to z) = x \to z = z$ and $(x \to y) \to (x \to z) = (x \to y) \to z = z$, since $x \to y \not\leq z$. Thus (p-1) holds in $\mathcal{A}$. Consequently, $\mathcal{A}$ is a pre-Hilbert algebra. □

In particular, we have the following

Example 3.19. Let $\mathbb{Z}$ be the set of integers and let for $x, y \in \mathbb{Z}$ the symbol $x \mid y$ means that x divides y. Then the relation $\mid$ is a pre-order on $\mathbb{Z}$ which is not an order (for example, $1 \mid -1$ and $-1 \mid 1$ but $1 \neq -1$). Moreover, $x \mid 0$ for each $x \in \mathbb{Z}$ and if $0 \mid x$, then $x = 0$. If we define the operation $\to$ by

$$x \to y = \begin{cases} 0, & \text{if} \quad x \mid y \\ y, & \text{if} \quad \text{otherwise,} \end{cases}$$

then $(\mathbb{Z}, \mid, 0)$ is a pre-Hilbert algebra.

Remark 3.20. The class of all pre-Hilbert algebras is a variety. Therefore, if $\mathcal{A}_1$ and $\mathcal{A}_2$ are two pre-Hilbert algebras, then the direct product $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ is also a pre-Hilbert algebra.

Let T be be any set and, for each $t \in T$, let $\mathcal{A}_t = (A_t, \to_t, 1)$ be a pre-Hilbert algebra. Suppose that $A_s \cap A_t = \{1\}$ for $s \neq t$, $s, t \in T$. Set $A = \bigcup_{t \in T} A_t$ and define the binary operation $\to$ on A via

$$x \to y = \begin{cases} x \to_t y & \text{if } x, y \in A_t;\ t \in T, \\ y & \text{if } x \in A_s,\ y \in A_t;\ s, t \in T,\ s \neq t. \end{cases}$$

It is easy to check that $\mathcal{A} = (A, \to, 1)$ is a pre-Hilbert algebra. The algebra $\mathcal{A}$ will be called the *disjont union of* $(\mathcal{A}_t)_{t \in T}$.

Proposition 3.21. Any (proper) pre-Hilbert algebra can be extended to a (proper) pre-Hilbert algebra containing one element more.

Proof: Let $\mathcal{A} = (A, \to, 1)$ be a *pre-Hilbert* algebra and let $\delta \notin A$. On the set $B = A \cup \{\delta\}$ consider the operation:

$$x \to' y = \begin{cases} x \to y & \text{if} \quad x, y \in A, \\ \delta & \text{if} \quad x \in A \text{ and } y = \delta, \\ 1 & \text{if} \quad x = \delta \text{ and } y \in B. \end{cases}$$

Obviously, $\mathcal{B} := (B, \to', 1)$ satisfies the axioms (M) and (K). Further, the axiom (p-1) is easily satisfied for all $x, y, z \in A$. Moreover, by routine calculation we can verify it in the case when at least one of x, y, z is equal to δ. Thus, by definition, $\mathcal{B}$ is a pre-Hilbert algebra. Clearly, if $\mathcal{A}$ is a proper pre-Hilbert algebra, then $\mathcal{B}$ is also a proper pre-Hilbert algebra. □

4. Positive implicative pre-Hilbert algebras

Recall that any Hilbert algebra satisfies (pi) and (pimpl), but pre-Hilbert algebras do not have to satisfy these properties (see Example 3.13). From [6] we have the following definitions:

Definition 4.1 ([6]).

1. A pi-*RML algebra* is an RML algebra verifying (pi).
2. A positive implicative RML algebra, or a *pimpl-RML algebra* for short, is a RML algebra verifying (pimpl).

Remark 4.2. Note that pimpl-RML algebras are also called generalized Tarski algebras (see [11], [9], [6]).

First we give some characterizations of pi-pre-Hilbert algebras.

THEOREM 4.3. *Let* $\mathcal{A} = (A, \to, 1)$ *be an algebra of type* $(2, 0)$*. The following are equivalent:*

(i) $\mathcal{A}$ *is a pi-pre-Hilbert algebra;*

(ii) $\mathcal{A}$ *satisfies (M), (K), (p-1), (pi);*

(iii) $\mathcal{A}$ *satisfies (M), (BB) and (pi);*

(iv) $\mathcal{A}$ *is a pi-pre-BBBCC algebra;*

(v) $\mathcal{A}$ *satisfies (M), (B), (C) and (pi);*

(vi) $\mathcal{A}$ *is a pi-pre-BCC algebra with (C).*

PROOF: (i) $\Longrightarrow$ (ii) By definition.

(ii) $\Longrightarrow$ (iii) Follows from Theorem 3.9.

(iii) $\Longrightarrow$ (iv) By Lemma 2.2 (v) and Proposition 3.2 (iii), $\mathcal{A}$ satisfies (Re) and (L). Then $\mathcal{A}$ is a pi-pre-BBBCC algebra.

(iv) $\Longrightarrow$ (v) Follows from Lemma 2.2 (v).

(v) $\Longrightarrow$ (vi) By Lemma 2.2 (viii), $\mathcal{A}$ satisfies (BB). Applying Lemma 2.2 (v), we conclude that (Re) holds in $\mathcal{A}$. From Lemma 3.2 (iii) it follows that (L) also holds in $\mathcal{A}$. Thus (vi) is satisfied.

(vi) $\Longrightarrow$ (i) Let $\mathcal{A}$ be a pi-pre-BCC algebra with (C). Then $\mathcal{A}$ satisfies (Re), (M), (L), (B) (hence, by Lemma 2.2, (*), (**), (Tr), (K)), (pi), (C). To prove (p-1), let $x, y, z \in A$. From (B) we conclude that $y \to z \leq (x \to y) \to (x \to z)$. Using (*), we get

$$x \to (y \to z) \leq x \to ((x \to y) \to (x \to z)). \tag{4.1}$$

By (C),

$$x \to ((x \to y) \to (x \to z)) \leq (x \to y) \to (x \to (x \to z)). \tag{4.2}$$

Applying (Tr) and (pi), we obtain $x \to (y \to z) \leq (x \to y) \to (x \to (x \to z)) = (x \to y) \to (x \to z)$. Consequently, $\mathcal{A}$ is a pi-pre-Hilbert algebra. □

Example 4.4 ([6, 10.17]). Let $A = \{a, b, c, d, 1\}$ and $\to$ be defined as follows:

$\to$	a	b	c	d	1
a	1	b	b	d	1
b	a	1	1	d	1
c	a	1	1	d	1
d	a	c	c	1	1
1	a	b	c	d	1

.

Properties (M), (BB) and (pi) are satisfied, as is easy to check. From Therem 4.3 we conclude that $(A, \to, 1)$ is a pi-pre-Hilbert algebra. It does not satisfy (An) for $x = b$, $y = c$; (Ex) and (pimpl) for $x = a, y = d, z = b$.

Remark 4.5. (1) Example 3.13 shows that there exists a pre-Hilbert algebra which is not a pi-pre-Hilbert algebra. Therefore, **pi-pre-H** $\subset$ **pre-H**.
(2) From Therem 4.3 we deduce that **pi-pre-H** = **pi-pre-BBBCC** = **pi-pre-BCC** + (C).
(3) By definitions,
pi-RML = **RML** + (pi),
pi-pre-BCC = **pre-BCC** + (pi) = **pi-RML** + (B),
pi-BE = **BE** + (pi) = **pi-RML** + (Ex),
pi-pre-BCK = **pre-BCK** + (pi) = **pi-BE** + (B) and
pi-pre-BCK = **pi-pre-BCC** + (Ex) = **pi-pre-H** + (Ex).

By Remark 4.5, we can draw the hierarchy between classes **RML** and **pi-pre-BCK**, in the next Figure 3.

Now we give several characterizations of positive implicative pre-Hilbert algebras. We will use the following lemma:

LEMMA 4.6 ([6]). *Let $\mathcal{A} = (A, \to, 1)$ be an algebra of type $(2, 0)$ satisfy (Re), (M) and (pimpl). Then $\mathcal{A}$ satisfies (L), (BB), (hence (B), (*), (**), (Tr)), (K), (C), (p-1), (p-2), (pi).*

From Lemma 4.6 we obtain

PROPOSITION 4.7. Let $\mathcal{A} = (A, \to, 1)$ be an algebra of type $(2, 0)$. The following are equivalent:

(i) $\mathcal{A}$ is a pimpl-pre-Hilbert algebra;

(ii) $\mathcal{A}$ satisfies (Re), (M), (pimpl);

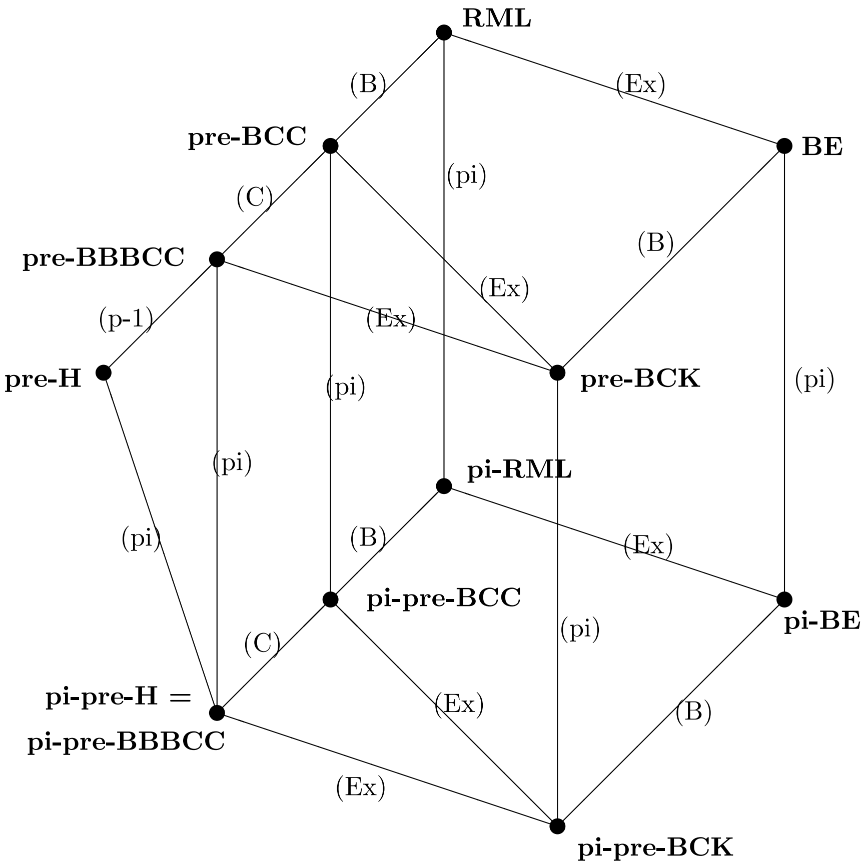
RML
(B)
(Ex)
pre-BCC
BE
(pi)
(C)
(B)
pre-BBBCC
(Ex)
(p-1)
(Ex)
pre-H
pre-BCK
(pi)
(pi)
(pi)
pi-RML
(pi)
(B)
(Ex)
pi-pre-BCC
pi-BE
(pi)
(C)
pi-pre-H =
pi-pre-BBBCC
(Ex)
(B)
(Ex)
pi-pre-BCK

Figure 3.

(iii) $\mathcal{A}$ is a pimpl-RML algebra, that is, it is a generalized Tarski algebra;

(iv) $\mathcal{A}$ is a pimpl-pre-BCC algebra;

(v) $\mathcal{A}$ is a pimpl-pre-BBBCC algebra.

Example 4.8 ([6, 10.18]). Consider the set $A = \{a, b, c, d, 1\}$ and the operation $\to$ given by the following table:

$\to$	a	b	c	d	1
a	1	b	b	1	1
b	a	1	1	a	1
c	a	1	1	a	1
d	1	c	c	1	1
1	a	b	c	d	1

.

We can observe that the properties (Re), (M), (pimpl) (hence (L), (B), (BB), (C), (D), (*), (**), (Tr), (p-1), (p-2)) are verified. Then, $(A, \to, 1)$ is a pimpl-pre-Hilbert algebra. It does not verify (An) for $(x, y) = (b, c)$; (Ex) for $(x, y, z) = (a, d, b)$. Hence, it is not a pimpl-BE algebra.

Remark 4.9. (1) By Proposition 4.7, **pimpl-pre-H** = **pimpl-RML** = **pimpl-pre-BCC** = **pimpl-pre-BBBCC**. Since (Re) + (M) + (pimpl) imply (B), we conclude that **pimpl-BE** = **pimpl-pre-BCK**.
(2) From (1) we have **pimpl-pre-H** = **pi-pre-H** + (pimpl) = **pi-RML** + (pimpl) = **pi-pre-BCC** + (pimpl) and **pimpl-BE** = **pi-BE** + (pimpl) = **pi-pre-BCK** + (pimpl), because (Re) + (M) + (pimpl) imply (pi).
(3) Moreover, **pimpl-pre-H** + (Ex) = **pimpl-RML** + (Ex) = **pimpl-BE**.

Remark 4.10. By Remarks 6.19 and 6.19 of [6], we get **H** = **pimpl-aRML** = **pimpl-BCC** = **pimpl-BCK** = **pimpl-aBE** = **pimpl-BE** + (An).

Remark 4.11. Note that a self-distributive BE algebra (see [10]) is in fact our pimpl-BE algebra.

Example 4.12. Let $(\mathbb{Z}, |, 0)$ be the algebra given in Example 3.19. It is easy to see that $(\mathbb{Z}, |, 0)$ satisfies (Re), (M), (Ex) and (pimpl). Then, it is a pimpl-BE algebra. Since $1 \to -1 = 0 = -1 \to 1$ but $1 \neq -1$, and $(2 \to 1) \to 2 = 1 \to 2 = 0 \neq 2$ we deduce that $(\mathbb{Z}, |, 0)$ does not satisfy (An). Therefore, it is not a Hilbert algebra.

Remark 4.13. Examples 4.4, 4.8 and 4.12 show that *the inclusions below are proper.*

pi-pre-H ⊃ **pimpl-pre-H** ⊃ **pimpl-BE** ⊃ **H**.

From Remarks 4.9 and 4.10 we obtain Figure 4.

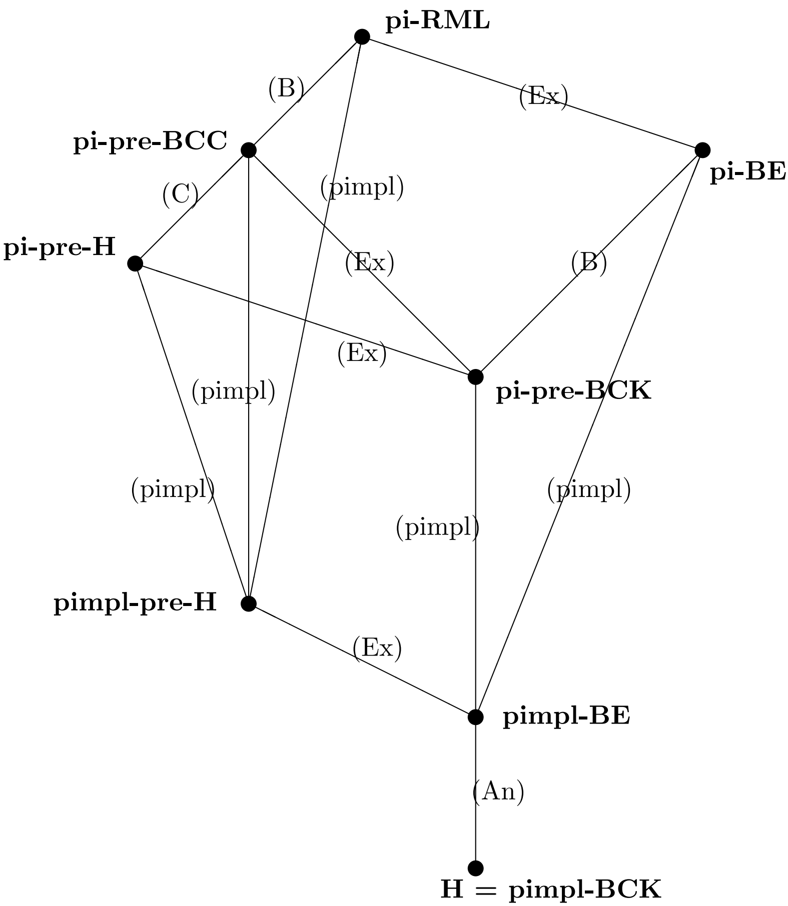

Figure 4.

5. Summary and future work

In this paper, we introduced pre-Hilbert algebras as a generalization of well-known Hilbert algebras. We investigated basic properties of pre-Hilbert algebras and presented some examples and characterizations of these algebras. We defined and studied positive implicative pre-Hilbert algebras and obtained their connections with some other algebras of logic considered here. In particular, we proved that the class of positive implicative pre-Hilbert algebras coincides with the class of generalized Tarski algebras. Finally, we showed the interrelationships between some subclasses of the class of pi-RML algebras.

The results obtained in the paper can be a starting point for future research. We suggest the following topics:

(1) Studying pre-Hilbert algebras with the implicative property, that is, verifying the identity $(x \to y) \to x = x$.

(2) Describing the deductive systems, the congruences, the quotient algebras, etc. of pre-Hilbert algebras.

(3) Investigating the connections between pre-Hilbert algebras and GE algebras (generalized exchange algebras) introduced in 2021 by R. Bandaru et al. [2].

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Bulletin of the **S**ection of **L**ogic
Volume 53/3 (2024), pp. 365–397
https://doi.org/10.18778/0138-0680.2024.04

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ABOUT LOGICALLY PROBABLE SENTENCES

Abstract

The starting point of this paper is the empirically determined ability to reason in natural language by employing probable sentences. A sentence is understood to be logically probable if its schema, expressed as a formula in the language of classical propositional calculus, takes the logical value of truth for the majority of Boolean valuations, i.e., as a logically probable formula. Then, the formal system **P** is developed to encode the set of these logically probable formulas. Based on natural semantics, a strong completeness theorem for **P** is proved. Alternative notions of consequence for logically probable sentences are also considered.

Keywords: probable sentences, majority, logically probable formula, Boolean valuation.

Intuitive motivation

Natural language reasoning can occasionally lead from true premises to false conclusions, which is incorrect from the standpoint of classical logic. Most of the time, the formulas of the classical propositional calculus (PC) that correlate to such erroneous inferences are not particularly interesting from a logical point of view. Consider the inference: "If it is raining, the roadway will be wet. Therefore (the conclusion): If it is not raining, then the roadway will not be wet." Similarly: "If it is raining, the roadway will be wet. Therefore (conclusion): If the roadway is wet, then it was raining." People routinely employ similar reasoning in their daily lives, even though the results of these inferences are not logically certain or do not follow logically from the premises. One non-trivial explanation for why this occurs

Presented by: Andrzej Indrzejczak
Received: November 6, 2023
Published online: April 23, 2024

is because, in the majority of real-world situations, the roadway simply will not be wet if it has not rained. In other words, determining if it has rained recently usually suffices to determine how wet the road is. We will not investigate whether the water accidentally leaked from somewhere because of a malfunction of a passing truck transporting mineral water or because a Zeppelin flying nearby dropped a massive water-filled balloon. Extreme situations, i.e., those that do not follow the ordinary course of things, are disregarded in our predictions. We are only interested in what is normal or typical, or what happens in most everyday situations. The precise sense in which the conclusions of the above two inferences follow from the premises will be given later in the paper.

We will provide one additional, perhaps distant analogy, which could be helpful in the intuitive grounding of the research undertaken in the next paragraphs. Consider a random device that is being utilized in a particular manner for a specific purpose. Typically, such a device will continue to function effectively until it wears out or malfunctions. As long as it is operated in accordance with the instruction manual, the device will function fairly effectively. In other words, the device will work if the manufacturer's requirements are satisfied, but it will not function well if the manufacturer's conditions are not met. The previous sentence contains two conditional assertions that one will undoubtedly run into in everyday life. Both are only probable, and we believe that is why we should try to find the rules for employing probable statements.

In order to summarize the overall issue, the key problem is how logical rules govern sentences that are merely probable, since it is known beforehand that they do not generally hold true yet—at the same time—are true in a limited number of or in *most* cases.

The observations outlined above and similar facts lead us to interest in reasoning that involves sentences based on patterns (formulas) that are true for most Boolean valuations.

1. Introduction

Let us take a typical propositional language based on an alphabet that comprises: (a) a countable set of propositional variables $V = \{p_1, p_2, p_3, \ldots\}$; (b) connectives: $\neg, \rightarrow, \vee, \wedge, \equiv$, respectively called negation, implication, disjunction, conjunction and equivalence; (c) the) and (brackets, i.e., respectively, the closing bracket and the opening bracket. For the sake of convenience, we shall represent the variables with the symbols: $p, q, r, s, t, \ldots$. The meaning of the connectives is characterized by the so-called truth tables for classical logic. The set of all well-formed formulas that are based on the aforementioned alphabet is denoted by the symbol $Form_{PC}$. The symbol $Form_{\rightarrow}$ denotes a proper subset of the set $Form_{PC}$ and contains formulas built only with the use of the variables, the sign of implication, and the brackets. The set of all subsets of the set X is denoted by the symbol 2^X, and the set of all finite subsets of the set X is denoted by the symbol $FinX$. The derivability relation for the language of PC will be denoted by $\vdash_{PC}$, and the corresponding consequence operation will be denoted by—C_{PC}. If X is a set, the symbol $|X|$ denotes the cardinality of X. If $A \in Form_{PC}$, the set of propositional variables in the formula A is denoted by the symbol $Var(A)$, and the symbol $|Var(A)|$ denotes the cardinality of this set, e.g., $Var((p \rightarrow q) \vee r)) = \{p, q, r\}$ and $|Var((p \rightarrow q) \vee r))| = |\{p, q, r\}| = 3$.

From the definition of the set of all Boolean valuations, which we denote by the symbol Val, we know that it has a power of continuum. Each element of the set Val is an extension of the *valuation of propositional variables* $v : V \rightarrow \{0, 1\}$. We will use the same symbol v for valuations of propositional variables and valuations of formulae, as this should not cause any confusion and is convenient.

Definition 1.1. Let $A \in Form_{PC}$. For every $v, v' \in Val$: vR_Av' iff $v(p) = v'(p)$, for any $p \in Var(A)$.

The equivalence classes of the relation R_A will be denoted by $[v]_{R_A}$. Each valuation $v' \in [v]_{R_A}$ can be uniquely assigned the restriction of the valuation v' to the propositional variables occuring in A, denoted further by $v' \lceil Var(A)$. We have:

Fact 1.2. Let $A \in Form_{PC}$. For every valuation $v' \in [v]_{R_A}$ holds:

$$v' \lceil Var(A) = v \lceil Var(A).$$

Proof: For every $p \in Var(A)$, $(v' \lceil Var(A))(p) = (v \lceil Var(A))(p)$, which gives $v' \lceil Var(A) = v \lceil Var(A)$. □

We will call each such restriction $v \lceil Var(A)$: *significantly different Boolean valuation of the formula A or significantly different Boolean valuation* for short, when it is clear what formula is involved. For the established formula $A \in Form_{PC}$, there is a mutually one-to-one correspondence between equivalence classes and valuation restrictions.

This gives the following:

FACT 1.3. For any formula A, if $|Var(A)| = n$, then $|\{v\lceil Var(A) : v \in Val\}| = 2^n$ (the number of *significantly different Boolean valuations* is 2^n).

DEFINITION 1.4. A formula $A \in Form_{PC}$ is called a tautology (or a PC tautology) iff $v(A) = 1$ for every $v \in Val$.

The set of all tautologies of Classical Propositional Calculus (CPC) will be further denoted by $TAUT_{PC}$, with or without the index PC.

The obvious fact holds:

FACT 1.5. If A is a PC tautology, then $(v\lceil Var(A))(A) = 1$, for every valuation v.

2. Logically probable formulas

DEFINITION 2.1 (Logical probability function). We will call the function $m : Form_{PC} \to [0, 1]$ into a closed interval of real (rational) numbers **the logical probability function** if for any $A \in Form_{PC}$, $m(A) = |\{v\lceil Var(A) : v(A) = 1\}|/2^{|Var(A)|}$.

DEFINITION 2.2.[1] A formula $A \in Form_{PC}$ will be called a **logically probable formula** iff $m(A) > 1/2$.

DEFINITION 2.3. A set $X \subset Form_{PC}$ is called contradictory iff $A \in X$ and $\neg A \in X$ for some $A \in Form_{PC}$. A set of formulas of PC is called non-contradictory if it is not contradictory.

For the propositional language $Form_{PC}$ (recall the formulas inside the set $Form_{PC}$ can use the negation sign) and a consequence operation C defined for that language, the notion of a *contradictory* set of formulas is not equivalent to the notion of an *inconsistent* set of formulas because a

[1] According to the meaning, a 'probable sentence' is one whose probability is greater than 1/2, in the range of real numbers from 0 to 1. This sense of sentence probability was considered by [2, p. 7]. I owe the Reviewer a significant simplification of these two basic definitions.

contradictory set must simultaneously contain a formula and its negation, while an inconsistent set need not.

Recall that a set X is (*simply*) *consistent* (under C) iff there is no formula A such that both $A \in C(X)$ and $\neg A \in C(X)$.

A set X is absolutely consistent (under C) iff $C(X) \neq Form_{PC}$.[2]

In general, if $X \subset Form_{PC}$ is *simply consistent*, then X is *absolute consistent*.

Using Definition 2.3, we can see that: if $C(X)$ is non-contradictory, then X is consistent; but if X is contradictory, then X is inconsistent because $X \subset C(X)$.

We cannot say much about the consistency of the whole set P because we do not have a relevant operation of the consequence for the set P defined, and that goes beyond our present work. However, there is a sense in which the set P is provably consistent, namely by virtue of Lemma 2.4, under the *idle consequence* Id ($Id(X) = X$, for any set X); cf. [8, p. 38]: $Id(P) = P$ and P is non-contradictory.

Our further considerations (from Section 3 onwards) will concern the set P but they will be restricted to purely implicational language. We will then revisit the issue of consistency.

Lemma 2.4. *The set P is non-contradictory.*

Proof: Assume that P is contradictory, which means that some A and $\neg A$ are members of the set P. This means that $1/2 < m(A)$ and $1/2 < m(\neg A)$, but $m(\neg A) = (1 - m(A))$, hence $m(A) < 1/2$, which gives a contradiction. □

Lemma 2.5. *The following statements hold: (a) P is decidable, i.e., there exists an algorithm to determine in a finite number of steps whether any formula $A \in Form_{PC}$ belongs to the set P or not. (b) P is not closed under substitution. (c) P is not closed under modus ponens. (d) $TAUT \subset P$. (e) it exists such $A \in P$, that $A \notin P$ and $\neg A \notin P$. (f) the formulas built only from propositional variables, brackets and the disjunction connective belong to P. (g) P is inconsistent in propositional logic. (h) $P \neq Form_{PC}$. (i) P is not closed with respect to the rule with the schema $(A \to B), (B \to C)//(A \to C)$.*

[2] Such an understanding of consistency is also suitable for languages without the negation sign.

Proof: (a) Use the method of truth tables. (b) $(p \to p) \to q \notin P$ while $p \to q \in P$. (c) $(p \vee q) \in P$ and $(p \vee q) \to q \in P$. (d) if $A \in TAUT$, then $m(A) = 1$. (e) $m(p) = m(\neg p) = 1/2$. (f) $m(p \vee q) = 3/4$. (g) $q \in C_{PC}(P)$ and $\neg q \in C_{PC}(P)$ with *modus ponens* and *substitution*. (h) $m(p) = 1/2$. (i) $((p \to (q \to p)) \to (p \to q))$, $((p \to q) \to r) \in P$, but $((p \to (q \to p)) \to r) \notin P$. □

Lemma 2.6. *Let $A, B \in Form_{PC}$, the variable $p \notin Var(A)$, $|Var(A)| = n$ and $Var(B) = Var(A) \cup \{p\}$, then the following hold:*

1. $|Var(B)| = (n+1)$;
2. $R_B \subset R_A$;
3. $[v']_{R_A} = [v'']_{R_B} \cup [v''']_{R_B}$; *where* $v''(p) = 1$ *and* $v'''(p) = 0$.

Proof: Ad. 1. Case 1 is obvious.

Ad. 2. Suppose $< v, v' >\in R_B$, i.e., for any variable $q \in Var(B)$, $v(q) = v'(q)$. The set $Var(B)$ is a superset of $Var(A)$ ($Var(B) \supset Var(A)$), for each variable $r \in Var(A)$, $v(r) = v'(r)$, which results in vR_Av'.

Ad. 3. For the proof, take the pair $< v, v' >\in R_A$. Then, $v(q) = v'(q)$ for every $q \in Var(A)$. Any valuations v'', v''' that belong to R_B take the same logical value for the variables belonging to the set $Var(A)$ as the valuations v and v'. However, the only difference between the valuations v'' and v''' is the value they assign to the variable r, i.e., $v''(r) = 0$ and $v'''(r) = 1$, or reversely. The variable has a value of 0 in one of these valuations and a value of 1 in the other, yet both valuations fall under the class $[v']_{R_A}$. □

To explain it in another way and perhaps more intuitively, let us observe that significantly different valuations of the formula A i.e. each $v \lceil Var(A)$, for $v \in Val$, can be represented as finite sequences of 0s and 1s. If x represents such a string of length n, then the strings x0 and x1 represent strings of length $n+1$. Up to each finite height (level) n, the full binary tree contains all such zero-one sequences with n-elements.[3]

We know from the previous lemma that if we add a new propositional variable to formula A by means of any of the binary connectives, the number of equivalence classes of the new formula will double. On the other

[3] A binary tree consisting of a root alone has a height of 0.

hand, if by equating two different variables we reduce the number of variables in formula A (for example, if $p, q \in Var(A)$ and $|Var(A)| = n$), and we substitute p for q in each place, we get formula B. Then, of course, $s = |Var(B)| = |Var(A)| - 1 = (n - 1)$ and the number of equivalence classes will decrease from $2n$ to 2^{n-1}, that is, it will be halved. If, on the other hand, we increase the number of variables appearing in formula A by combining it with the binary connective $* \in \{\vee, \wedge, \to, \equiv\}$ with any formula B and obtain $(A * B)$, then as long as $|Var(A) \cap Var(B)| = r$, the number of equivalence classes $R_{(A*B)}$ will be 2^{n+s-r}. This is also the number of all significantly different Boolean valuations of the formula $(A * B)$ i.e. $|\{v\lceil Var(A * B) \colon v \in Val\}| = 2^{n+s-r}$.

Let us pay attention to the following important lemma with a somewhat complex formulation:

LEMMA 2.7. *Let $A, B, (A*B) \in Form_{PC}$, where $* \in \{\to, \wedge, \vee, \equiv\}$, $A \in P$, $Var(A) = n$, $Var(B) = s$, $|Var(A) \cap Var(B)| = r$, $m(A) = k/2^n$, and let us denote with t the number of those valuations of the subformula A for which it takes the value 1 in the set of all significantly different Boolean valuations of the formula $(A * B)$ i.e. $|\{v\lceil(A * B) \colon v(A) = 1\}| = t$, then $k/2^n = t/2^{n+s-r}$.*

PROOF: Suppose that the number of all Boolean valuations of the subformula A in $(A * B)$ for which it takes the value 1 is t, i.e., $|\{v\lceil Var(A * B) : v(A) = 1\}| = t$. We know that $2^{n+s-r} = (2^n \cdot 2^{s-r})$, therefore $(2^{n+s-r}/2^n) = 2^{s-r}$. From here we can see that the valuations of the subformula A have been repeated 2^{s-r} times without change in the set of all valuations of the formula $(A * B)$, which means $t = (k \cdot 2^{s-r})$. Now $t/2^{n+s-r} = t/(2^n \cdot 2^{s-r}) = (k \cdot 2^{s-r})/(2^n \cdot 2^{s-r}) = k/2^n$. □

The preceding lemmas should perhaps clarify the understanding of the following lemmas and their proofs.

LEMMA 2.8. *The set P is closed with respect to each of the following rules of conjunction elimination: $(A \wedge B)//A$; and $(A \wedge B)//B$.*

PROOF: Suppose that the formula $(A \wedge B)$ belongs to the set P. Hence, the majority of rows in the last column of its truth table contain 1. The truth table of this formula has 1 in some row of the last column iff the truth tables for each of formulas A and B have 1 in that row. □

Lemma 2.9. *The set P is closed with respect to each of the following rules of disjunction introduction*: $A//(A \vee B)$ *and* $B//(A \vee B)$.

Proof: If the truth table for formula A has 1 in the majority of rows in the last column, then the last column of the truth table for the formula $(A \vee B)$ contains 1 in at least the same rows as formula A; the same holds for formula B. □

Lemma 2.10. *The set P is closed with respect to the rule given by* $A//(B \to A)$.

Proof: Let A be a member of the set P. The truth table for formula A contains 1 in most of the rows in the last column. There will also be 1 in the same rows of the truth table for $(B \to A)$ since an implication takes the value 1 if its successor takes the value 1[4]. □

Lemma 2.11. *The subset D of the set P (i.e., $D \subset P$) of formulas which contain just one propositional variable is a proper subset of the set $TAUT$ ($D \subset TAUT$).*

Proof: The truth table for any formula $A \in D$ contains only two valuations of the single variable. There is only one majority for a two-element set, which is both elements of the set or all of them. □

Lemma 2.12. *For the set of countertautologies of PC, i.e., the set* $CTAUT := \{A : \neg A \in TAUT\}$, $(CTAUT \cap P) = \emptyset$ *holds.*

Proof: The last column of the truth table for the countertautologies contains only 0s. □

Theorem 2.13. *The set P is closed with respect to a weakened form of the rule of detachment of the scheme: if $(A \to B) \in P$ and $A \in TAUT$, then $B \in P$.*

Proof: Suppose that $(A \to B) \in P$ and $A \in TAUT$. If the formula B were not an element of the set P, then at least half of the rows in the last column of the truth table for formula $(A \to B)$ would contain 0, since every row in the last column of the truth table for the formula A would

[4]This is just a sketch of the proof as the exact proof requires longer presentation, but this should be clear enough.

contain 1, and then the whole implication would not be a member of the set P, which contradicts the assumption. □

THEOREM 2.14. *The set P is closed with respect to a weakened form of the detachment rule of the scheme: if $(A \to B) \in TAUT$ and $A \in P$, then $B \in P$.*

PROOF: As in the proof of Theorem 2.13. □

LEMMA 2.15. *The set P is not closed with respect to a rule of the scheme: $A, B//(A \wedge B)$.*

PROOF: The formulas $A = (p \to q)$ and $B = (q \to p)$ both belong to the set P, but $((p \to q) \wedge (q \to p)) \notin P$. □

3. A system P of logically probable formulas in an implicational language

Now, we will focus on the set $Form_{\to}$ of well-formed formulas built using only propositional variables, brackets, and the implication sign; we shall limit our consideration to the implicational part of PC, unless we explicitly indicate otherwise or it is clear from the context. Strictly speaking, we will consider the set $P_{\to} = (Form_{\to} \cap P)$; however, for the sake of convenience, we will continue to use the P symbol as long as this does not lead to confusion. We shall now define the syntactic consequence operation and the corresponding derivability relationship $\vdash_{\mathbf{P}}$. As is already known, the set of PC tautologies in a language with a single connective of implication can be axiomatized into the following system T:

($T1$) $((A \to B) \to ((B \to C) \to (A \to C)))$ (hypothetical syllogism);

($T2$) $(A \to (B \to A))$ (simplification);

($T3$) $(((A \to B) \to A) \to A)$ (Peirce's law);

(MP) $A, (A \to B)//B$ (rule of detachment).

The set $TAUT_{\to} := TAUT_{PC} \cap Form_{\to}$ is axiomatizable by means of rule schemes T in the sense that all classical PC tautologies in our language can be derived using formulas falling under the ($T1$)–($T3$) schemes and (MP), i.e., $C_T(\emptyset) = TAUT_{\to}$, where C_T is a consequence determined by T. In addition to these formulas, which are derivable in T, we still have *strictly probable formulas* in the set P which are true for most but not for all valuations (cf. Definition 3.1). We already know that the set of such formulas is not closed with respect to the rule of detachment or the substitution. The following question then arises:

[The Key Question] Is the entire set $P_{\rightarrow}$, and in particular the set of strictly probable formulas (see Definition 3.1 below), axiomatizable i.e., defining an effective set of axioms, being a proper subset of the set $P_{\rightarrow}$, when closed under the finite set of effective rules, gives the whole set $P_{\rightarrow}$?

Definition 3.1. We will call formula A a *strictly logically probable formula* when $A \in P$ and $A \notin C_T(\emptyset) = TAUT_{\rightarrow}$. We will denote the set of all such formulas by the symbol P'.

The set P' is closed under the following version of the non-standard rule, called the *Successor Rule* (RN):

Lemma 3.2. *If $A \in P', m(B) = 1/2$, and $((A \rightarrow B) \rightarrow B) \notin TAUT_{\rightarrow}$, then $(A \rightarrow B) \in P'$.*

Proof: Following the assumption of this lemma, if $((A \rightarrow B) \rightarrow B) \notin TAUT_{\rightarrow}$, then for a certain valuation $v, v(A) = v(B) = 0$ and $v(A \rightarrow B) = 1$. Since $m(B) = 1/2$, i.e., $B \notin P$, then in the worst case exactly half of the truth table for formula $(A \rightarrow B)$ will contain zeros, and there will be $m(A \rightarrow B) = 1/2$. That is, half of the last column of the truth table for the whole formula will then contain zeros in those rows where formula B takes the value zero. The valuation v gives us the guarantee that $|\{v\lceil Var(A \rightarrow B) \colon v(A \rightarrow B) = 1\}| > |\{v\lceil Var(A \rightarrow B) \colon v(A \rightarrow B) = 0\}|$ will occur: the whole implication will have at least one valuation v (which assigns the whole formula the value) more than the number of valuations assigning the value 0 to the implication. □

The above objections are exemplified by the following formulas: $(p \rightarrow q) \rightarrow ((r \rightarrow r) \rightarrow s)$ (satisfies the assumptions of the lemma and belongs to P') and $(p \rightarrow q) \rightarrow ((p \rightarrow p) \rightarrow p)$ (does not satisfy the assumptions of the lemma and does not belong to the set P').

Lemma 3.3. *If $A \in P'$ and the variable $p \notin Var(A)$, then $(A \rightarrow p) \in P'$.*

Proof: Directly from Lemma 3.2. for $B = p$. □

At the same time we have, in a sense, a dual to Lemma 3.2.:

Lemma 3.4. *If $A \in P$, then $(B \to A) \in P$.*

Proof: The proof is straightforward. □

Lemma 3.5. *Let $A \in Form_{\to}$ and $|Var(A)| = n$. Then $2^{n-1} \leq |\{v\lceil Var(A)\colon v(A) = 1\}|$ i.e. $m(A) \geq 1/2$.*

Proof: Since Lemma 3.5 is very general, to demonstrate its validity we shall use structural induction by the number of instances of connectives in formula A. Let us assume the assumptions of the lemma. Base step: A is a single variable p. Hence, there is only one Boolean valuation for which p takes the value 1, and $2^0 = 1 \leq 1$. If A is a simple implication $(p \to q)$, then the cardinality of the set of Boolean valuations for which this implication takes the Boolean value of truth is obviously 3 and is greater than $2^1 = 2$. Inductive step: suppose that the lemma holds for formulas B and C, and we want to prove that it holds for $A = (B \to C)$. Suppose $|Var(B) \cap Var(C)| = k$. We then have to consider cases where $k = 0$, i.e., $Var(B) \cap Var(C) = \emptyset$, and where $k > 0$, i.e., $Var(B) \cap Var(C) \neq \emptyset$. In the first case, assuming that $|Var(B)| = n$ and $|Var(C)| = m$, the truth table for formula $(B \to C)$ has 2^{n+m} rows. The column under formula C will contain 1 in half or more of the rows. The last column of such a table will contain 1 in at least the same rows since an implication with a true successor takes the logical value of truth. We will consider the second case, where formulas B and C share at least one propositional variable, i.e., $k > 0$. In this case, formula $(B \to C)$ will have $(n + m - k)$ different propositional variables, and its truth table will have 2^{n+m-k} rows. The number of valuations of the output formula is decreased by 2^k times because some valuations of the $n + m$ variables are discarded as a result of the equivocation of the shared variables. Let us take formula C as a starting point for consideration and assume that it has 2^m significantly different valuations, while formula B has 2^{n-k} such valuations. In this case, the final truth table T_A for the whole formula $A = (B \to C)$ will have 2^{n+m-k} valuations and will have, as a fragment, a 2^{n-k}-fold repetition of the table for C with 2^m rows. In each such fragment for C, at least half of the rows will contain 1 (based on the inductive assumption), therefore the same rows under formula A will also contain 1 since implication A with the true successor is true. Since this is the case in every occurrence of the fragment for C, so it is the same in the whole T_A table for A. □

Lemma 3.6. *Let* $Form_2 = \{A \colon A \in Form_{\to}$ *and* $|Var(A)| = 2\}$. *Then* $Form_2 = (X \cup Y) \cup Z$, *where:* $X = \{A \colon m(A) = 1\}$, $Y = \{A \colon m(A) = 1/2\}$ *and* $Z = \{A \colon m(A) = 3/4\}$.

Proof: This follows from Lemma 3.5. because $2^{2-1} = 2^1 = 2$ for $n = 2$. The number of all significantly different Boolean valuations of A is equal to 4, so the numbers k for which $2 \leq k \leq 4$ are only $2, 3$ and 4. □

Some known theorems on classical propositional calculus can be applied to our considerations on the set of exclusively implicational formulas. The compactness theorem is one of these.

Lemma 3.7. *The set* P *is closed with respect to a weakened form of the detachment rule of the scheme:* $A \in P$ *and* $(A \to B) \in P$ *and* $m(A \wedge B) > 1/2$, *then* $B \in P$.

Proof: Let $A \in P$ and $(A \to B) \in P$ and $m(A \wedge B) > 1/2$. Then it is straightforward that $m(B) > 1/2$. □

Lemma 3.8. *Every subset of the set of formulas* $X \subset Form_{\to}$ *is satisfiable.*

Proof: Straightforward from Lemma 3.5. □

Definition 3.9. A rule of the form $A//e(A)$, where e is an endomorphism, will be called a *restricted substitution rule* (RSu).

To avoid going into the technical details, it suffices to say that $e(A)$ is the result of substituting only propositional variables for propositional variables in the formula A, with the following caveats:

- a propositional variable has been substituted for a propositional variable;
- one and the same variable is substituted for a particular variable in all places where it occurs;
- the cardinality of the set $Var(A)$, cannot change as a result of the substitution i.e., $|Var(A)| = |Var(e(A))|$.

Let $h \colon V \to V$ be a permutation of the set V onto V. The set of all such permutations will be denoted by $Perm := \{h : h$ is a permutation of the set $V\}$. Each such permutation can be extended uniquely to the substitution e, which maps $Form_{PC}$ to $Form_{PC}$ and acts as a substitution in our system[5]. Strictly speaking, $e(A)$ is the value which is taken by a substitution e applied to the formula A. We will use the letter h, possibly with a subscript to denote arbitrary permutation of the set V, and we will use e for the substitution of the entire set $Form_{PC}$[6].

LEMMA 3.10. *The set P is closed under the rule (RSu) of the scheme: $A//e(A)$, for any substitution e.*

PROOF: Suppose $A \in P'$, since the case is obvious for $A \in TAUT_{\to}$. Therefore, $m(A) > 1/2$. It is clear that we can always find such endomorphisms e_1, e_2 that $A//e_1(A)$ and $e_1(A)//e_2(e_1(A)) = e(A)$ when $Var(A) \cap Var(e_1(A)) = \emptyset$ and $Var(e_1(A)) \cap Var(e_2(e_1(A))) = \emptyset$. For a better understanding of what happens when performing substitutions such as e_1 or e_2, let us imagine a truth table for the formula A. The substitution consists solely of respectively replacing, in the row describing the table, some sentence variables with others, as a result of which we get a truth table for $e_1(A)$, and similarly for $e_2(e_1(A))$. The places in the table where the logical values of the subformulas occur remain unchanged; in particular, the number $m(A)$ remains unchanged. Hence, $m(e(A)) > 1/2$. □

a. Axiom schemas and inference rules for the system P

We are now in a position to answer the Key Question posed above about the axiomatic system.

DEFINITION 3.11. The system **P** is defined by the axioms arising from the following schemes:

$(T1)$ $((A \to B) \to ((B \to C) \to (A \to C)))$;

$(T2)$ $(A \to (B \to A))$;

$(T3)$ $(((A \to B) \to A) \to A)$;

$(T4)$ formula $(p \to q)$;[7]

[5] This designation of e has its own tradition; cf. [8, pp. 18–22].

[6] Such substitutions, also called automorphisms, form a group and satisfies the conditions of composition, associativity, identity and inverse. To denote substitution in our sense we will use the signs: e, (possible with a subscript) and α, β.

[7] The variables p and q are distinct. By pure coincidence, a system with our axioms, specially $(T4)$, is mentioned in the paper [4, p. 193].

and the following inference rules:

(RO) if $\vdash_{\mathbf{P}} A$ and $\vdash_{\mathbf{P}} (A \to B)$ and $m(A \wedge B) > 1/2$, then $\vdash_{\mathbf{P}} B$;

(RN) if $\vdash_{\mathbf{P}} A$ and $m((A \to B) \to B) \neq 1$, then $\vdash_{\mathbf{P}} (A \to B)$;

(RSu) if $\vdash_{\mathbf{P}} A$, then $\vdash_{\mathbf{P}} e(A)$, for any substitution e.

Note. The Reviewer of the paper has requested to explain why in the rule RO above is allowed to use the semantical information in the form $m(A \wedge B) > 1/2$. Firstly, it should be noted that the system presented differs from the usual semantic presentation of classical logic by the 'quantifier' binding the set of logical valuations of the formula. Usually metalogic allows only two types of the quantifiers in its semantic description: 'for each' and 'it exists'. Our description additionally allows 'for most'. Secondly, the properties of such systems are not well recognized and studied, and our quantifier can be introduced into the metalogic of various non-classical systems. Third, the questionable condition can be thrown out beyond the formulation of the rule itself to the form:

(RO) if $\vdash_{\mathbf{P}} A$ and $\vdash_{\mathbf{P}} (A \to B)$, then $\vdash_{\mathbf{P}} B$; provided that $m(A \wedge B) > 1/2$.

And fourth, some conditions of a semantic nature are excluded because they are too general and trivialize the Key Question of axiomatizability, as in the example: $\vdash_{\mathbf{P}} A$, provided that $m(A) > 1/2$.

LEMMA 3.12. *The rules $RO1$ and $RO2$ are derivable in* $\mathbf{P}$*; the scheme of $RO1$ is: if $A \in TAUT_{\to}$ and $\vdash_{\mathbf{P}} (A \to B)$, then $\vdash_{\mathbf{P}} B$, and the scheme of $RO2$ is: $(A \to B) \in TAUT_{\to}$, then $\vdash_{\mathbf{P}} B$.*

PROOF: We will give the proof for $RO1$ only because the proof for $RO2$ is similar. Let us suppose that $A \in TAUT_{\to}$ and $\vdash_{\mathbf{P}} (A \to B)$, then $m(A \wedge (A \to B)) = m(A \wedge B) > 1/2$, so $m(B) > 1/2$, and $\vdash_{\mathbf{P}} B$. □

LEMMA 3.13. *In the system* $\mathbf{P}$*, there is a derivable rule of the scheme: $A//((A \to B) \to B)$.*

PROOF: Suppose that: $\vdash_{\mathbf{P}} A$. The formula $(A \to ((A \to B) \to B))$ is a tautology of PC. Hence, by virtue of the (RO) rule, we have: $((A \to B) \to B)$. □

This rule is important because it corresponds in our system to the rule described in Lemma 3.2.

In order to clarify the restrictions imposed on the (RN) rule, let us note that the formulas $((p \to q) \to p)$ and $((p \to p) \to q)$ are not elements of the set P, but the formula $((q \to p) \to p)$ is. However, since we cannot use the RN rule for the derivation of this formula, we derive it slightly differently. Here is the Hilbert-style proof in the system **P**, which is characterized in Definition 3.11:

1. $(p \to q)$ from axiom $(T4)$;
2. $(q \to r)$ from $(T4)$ by rule (RSu);
3. $((q \to r) \to p)$ by virtue of Lemma 3.3;
4. $((q \to r) \to p) \to ((q \to p) \to p)$ the PC-tautology;
5. $((q \to p) \to p)$ by virtue of Theorem 2.14. from 4. and 3.

Recall the following properties that a consequence operator $C : 2^{FormPC} \to 2^{FormPC}$ might satisfy for any set $X, Y \subset Form_{PC}$:

(i) (Reflexivity) $X \subset C(X)$;

(ii) (Monotonicity) $X \subset Y \Rightarrow C(X) \subset C(Y)$;

(iii) (Idempotency) $CC(X) \subset C(X)$;

(iv) (Structurality) $eC(X) \subset C(eX)$ for each endomorphism e;

(v) (Finite) $C(X) = \cup\{C(Y) : Y \subset X \wedge Y \in FinX\}$.

If C satisfies the conditions (i)–(iv), it is called *structural*; if it satisfies conditions (i)–(iii), (v) it is called *finitistic*; and if it satisfies all the above conditions, it is called *standard*. If we restrict the number of endomorphisms[8] to the class of all automorphisms of the set $Form_{PC}$, we obtain a more detailed notion of the structural consequence operation. When $C_{\mathbf{P}}$ is the consequence that corresponds to our system **P**, then the following occurs:

[8] In general, an endomorphism of $Form_{PC}$ need not be a function from $Form_{PC}$ onto $Form_{PC}$. In our case, it is always so because automorphisms are the unique extensions of permutations of the set V.

Lemma 3.14. *The consequence operation $C_{\mathbf{P}}$ satisfies conditions (i.)-(v.), except that condition (iv.) holds not for each endomorphism but for any substitution e that is a unique extension of the certain permutation $h \in Perm$.*

Proof: The proof of condition (iv) requires special attention. Suppose that $A \in \alpha C_{\mathbf{P}}(X)$. This means that such B exists that $B \in C_{\mathbf{P}}(X)$ and $A = \alpha B$. Hence, there also exists a proof of αB based on the set αX. □

Definition 3.15. A non-atomic formula A of the PC language is called quasi-Horn if the following conditions are met: a) A is in canonical conjunctive normal form ($CCNF$); b) each literal clause (disjunction) contains **at least** one positive literal.

The original *Horn clause* is supposed to contain at most one positive literal, while our *quasi-Horn* formulas are supposed to contain at least one positive literal.

Theorem 3.16 (weak completeness theorem). *The consequence operation $C_{\mathbf{P}}$ that corresponds to the system* **P** *has the property: $C_{\mathbf{P}}(\emptyset) = P$.*

Proof: ($\Rightarrow$) Let us first prove the implications from left to right. Suppose that the formula $A \in C_{\mathbf{P}}(\emptyset)$; thus, it has a proof based on a set of axioms of the system **P** of the form $< D_1, D_2, D_3, \ldots, D_n >$, where each $D_i (0 < i \leq n)$ is either an axiom of the system or has been obtained from prior expressions of this sequence by means of any of the rules of the system. We will give a sketch of a well-known inductive proof, the essence of which consists in showing that the property being proved is preserved by the rules of the system. When the formula is an axiom, the matter is evident because each axiom belongs to P. Also, for the rules (RO) and (RN), the proof is straightforward by virtue of the corresponding theorems: Theorem 2.13, Theorem 2.14., and Lemma 3.2. Consequently, we will concentrate on the case of the restricted substitution rule (RSu). Therefore, let us assume that $A \in P'$, i.e., A, is a strictly probable formula. Let us take any bijection $\alpha : V \to V$. Let $Var(A) = \{p_1, \ldots, p_k\}$, hence $Var(e(A)) = \{e(p_1), \ldots, e(p_k)\}$. According to the definition, $v(A) = 1$ for the majority of the 2^k valuations. If $v_j (0 < j \leq 2^k)$ is such a valuation that $v_j(A) = 1$, then for each such valuation v_j the variables will take corresponding logical values. Let us define the new valuation v_m in the form $v_j(p_i) = v_m(h(p_i))$, hence we have $v_j(A) = v_m(e(A))$, and so on for every j.

($\Leftarrow$) We now want to prove the converse implication. Let us therefore assume that $A \in P$ and that $Var(A) = \{p_1, \ldots, p_k\}$. If A is a tautology, then the case is obvious. Let be $A \in P'$, then $m(A) > 1/2$. By virtue of the relevant metatheorems, there is a formula of canonical conjunctive normal form for A, which we will denote by A_{CCNF}. Such a formula is a conjunction of the clauses $A_i (1 \leq i < 2^{k-1})$, of disjunctive form whose members, called literals, are single propositional variables or their negations occurring in the formula A. Let us note that if A is a purely implicational formula, then A_{CCNF} is *quasi-Horn*. It can easily be seen by virtue of Lemma 2.12. that A_{CCNF} is a conjunction of at most 2^{k-1} conjuncts. But the disjunction in the form $(\neg p_1 \vee \neg p_2 \vee \ldots \vee \neg p_k)$ is excluded as a such conjunct because, for the valuation v, if $v(p_1) = v(p_2) = \ldots = v(p_k) = 1$, then $v(A) = 1$. So, each conjunct (elementary disjunction) of the formula A_{CCNF} contains at least one positive literal, and the number of all conjuncts is less than 2^{k-1}. It is a fact that $(A_{CCNF} \equiv A) \in TAUT_{PC}$, and this is provable in PC in the functionally complete language, which, by virtue of the law of exportation of the scheme $(B \wedge C \to D) \equiv (B \to (C \to D))$ and the rule of equivalence elimination, we can transform to the form $(A_1 \to (A_2 \to (\ldots (A_{((2^{k-1})-1)} \to A) \ldots) \in TAUT_{\to}$, that is $\vdash_{\mathbf{P}} (A_1 \to (A_2 \to (\ldots (A_{((2^{k-1})-1)} \to A) \ldots)$. By applying the (RO) rule 2^{k-1} times and detaching in each step the subsequent formula A_i, we get $\vdash_{\mathbf{P}} A$. Applying the RO rule is in any case permissible since each formula A_i and formula A satisfy the constraint imposed on its application. Note also that every clause A_i is quasi-Horn, therefore part of such a formula has always one of the following formulas or an equivalent formula: $\neg p \vee q$; $p \vee \neg q$; $p \vee q$, where $p, q \in X$.[9] The corresponding formulas in the form of an implication $p \to q$, $q \to p$, and $((p \to q) \to q)$ are equivalent to each of the preceding formulas and are derivable in $\mathbf{P}$. If the A_{CCNF} clause is more complex, then by the rule in Lemma 3.13. we can include further disjunctive members of each clause of formula A. This completes the proof. □

LEMMA 3.17. *If* $A \vdash_{\mathbf{P}} e(A)$, *then* $\vdash_{\mathbf{P}} (A \to e(A))$.

PROOF: Suppose that $|Var(A)| = m$ and $|Var(e(A))| = n$. If $A \in TAUT_{\to}$, then $e(A)$ is also a tautology. So, suppose that $A \notin TAUT$ and $A \vdash_{\mathbf{P}} e(A)$.

[9]Case: $\neg p \vee \neg q$, is excluded as not being a quasi-Horn.

If we show that $m(A \to e(A)) > 1/2$, then by virtue of the weak completeness theorem $\vdash_{\mathbf{P}} (A \to e(A))$. We know that $m(A) = m(e(A)) \geq 1/2$. The table for the whole formula $(A \to e(A))$, when m(A)=m(e(A))=1/2, will have ones in the rows where its predecessor, formula A, has zeros, and this will be exactly half of all the values of the whole implication. In addition to this, the whole formula will have ones in the rows in which formula A and $e(A)$ take the value 1. Such rows certainly exist, hence the last column of the table for the whole formula will be more than half full of ones. We perform analogous reasoning for the case when $m(A) = m(e(A)) > 1/2$. So, $\vdash_{\mathbf{P}} (A \to e(A))$ by virtue of Theorem 3.16. □

To prove the weak deduction theorem, we need to weaken the **P** system to a system, which we will tentatively denote by the symbol $\mathbf{P}_{-}$. In this new system, we will abandon the RSu rule, but we will close the axioms of the system $\mathbf{P}_{-}$ to any bijections and their endomorphisms, that is, to our substitution.

Theorem 3.18 (weak deduction theorem). *If* $\{A\} \vdash_{\mathbf{P}_{-}} B$, *then* $\vdash_{\mathbf{P}_{-}} (A \to B)$.

Proof: We will base our reasoning on the principle of ordinal induction, which is equivalent to normal induction. We will prove precisely the following: $\forall k(\forall i(i < k \to W(i)) \to W(k))$. Then, we will consider a general sentence of the form: $\forall n W(n)$. In our case, the formula $W(n)$ will have the following meaning: "a proof of $B(< D_1, \ldots, D_n = B >)$ based on A, having length n, can be transformed into a certain proof of $(A \to B)$ based on the set $\emptyset$". The theorem holds in the case where A is a tautology, so we assume that $A \notin TAUT_{\to}$. If $k = 1$, then it must be shown that there is $(\forall i(i < 1 \to W(i)) \to W(1)$. Since the antecedent is true, it is equivalent to the sentence $W(1)$, i.e., a proof of length 1 has the property W. Thus, two cases must be considered: (i) $B = A$; (ii) B is an axiom. If $B = A$, then $(A \to B) = (A \to A)$, and this is the theorem of the system $\mathbf{P}_{-}$. When B is an axiom of $\mathbf{P}_{-}$, and if we take an axiom of the form $(B \to (A \to B))$, then we can detach B (using RO), and we have $(A \to B)$. Now, if we suppose that the theorem of the system $\mathbf{P}_{-}$ holds for $\forall i(i < k \to W(i))$, then we need to show that it holds for a proof of length k, i.e., $W(k)$. Our system has three inference rules, so we need to examine each of the three cases. If formula D_k is an axiom of the system or is identical to formula A, then we repeat the reasoning for $k = 1$. For rule RO, suppose that there exist such indices $i, j < k$ that $D_i = (D_j \to D_k) = (D_j \to B)$ and $m(D_j \wedge B) > 1/2$. By virtue of the induction assumption, both $A \to D_i$ and $A \to D_j$ are theorems of the system. Let us now take a tautology of the form $(A \to (D_j \to D_k)) \to ((A \to D_j) \to (A \to D_k))$. We can detach $(A \to (D_j \to D_k))$ from it because it is a theorem of the system; next, from the formula $((A \to D_j) \to (A \to D_k))$, which is also a theorem of the system, we can detach $(A \to D_j)$ because $m(D_i \wedge D_k) > 1/2$, and also $(A \to (D_i \wedge D_k)) > 1/2$. Let us now consider the RN rule. Let us assume that $D_k = (D_j \to C)$ for some formula $D_j = B$ (which has a proof shorter than k), and let us assume that $((B \to C) \to C) \notin TAUT_{\to}$. By virtue of the induction assumption, we have a proof of $(A \to D_j) = (A \to B)$. Now, we need a proof for $(A \to (B \to C))$, where $A \notin TAUT_{\to}$. By the RN rule, we have $\vdash_{\mathbf{P}\_} ((A \to B) \to C)$, because $\vdash_{\mathbf{P}\_} (A \to B)$ and $(((A \to B) \to C) \to C) \notin TAUT_{\to}$, i.e., there exists such a valuation v that $v(C) = v(B) = 0$ and $v(A) = 1$. From the tautology $((A \to B) \to C) \to (A \to (B \to C))$ and its predecessor, we also have the successor $\vdash_{\mathbf{P}\_} (A \to (B \to C))$. □

The converse of the above theorem does not hold for **P**.

Theorem 3.19. *It is not true that if* $\vdash_{\mathbf{P}} (A \to B)$*, then* $\{A\} \vdash_{\mathbf{P}} B$*.*

Proof: The theorem $\vdash_{\mathbf{P}} ((q \to p) \to p)$, whose proof is provided above, serves as the counterexample to the implication from Theorem 3.20. The thesis of the system **P** is also $\vdash_{\mathbf{P}} (q \to p)$; we obtain this thesis from the axiom $(T4)(p \to q)$ by applying RSu. On the other hand, because the rule of detachment is not a rule of **P**, the single variable is not a **P**-theorem by virtue of Theorem 3.16. and the proof of Lemma 2.5. □

Lemma 3.20. *The set P is absolutely consistent under the consequence $C_{\mathbf{P}}$.*

Proof: Directly from Theorem 3.16. and that, for example, $((p \to p) \to p) \notin P$. □

4. Three propositions for the definition of the entailment relation

Definition 4.1 (entailment_1). The formula A follows from a set of formulas X (symbolically: $X \models_1 A$) iff for every $B \in X$, if $B \in P$, then $A \in P$ (or else: if $X \subset P$, then $A \in P$).

Definition 4.2 (entailment_2). Formula A follows from a finite set of formulas X (symbolically: $X \models_2 A$) iff $(\wedge X \rightarrow A) \in P$, where $\wedge X$ is a generalized conjunction of the elements of the set X, i.e., $\wedge X := (A_1 \wedge \ldots \wedge A_n)$.

Definition 4.3 (entailment_3). The formula A follows from a finite set of formulas X (symbolically: $X \models_3 A$) iff for most Boolean valuations v of the formulas which belong to X—if these valuations have been assigned the value 1—the value 1 has also been assigned to the formula A.

5. Some remarks on the family of all majorities

We should bear in mind that the proper object of the present work is some notion of **majority**, which here we have fortunately managed to *insert* into the consideration of the classical propositional calculus. So, the general work on the notion of majority is still to be done. In this section, we will assume that the family $\pi(X)$ of the subsets of the set X is a majority in the set of all valuations of the set X. In this section, we will try to give suggestions for applying some typical algebraic concepts to the family $\pi(X)$. We do so because the findings of this section might, for someone, form the basis of possible further investigations.

Definition 5.1. Let X be any finite set of propositional variables such that $|X| = n$, and let V_X denote the set of all valuations of variables of the set X of the form $v : X \rightarrow \{0, 1\}$. Then, by the symbol $\pi(X)$ we denote a subset of the set 2^{V_X} of the form $\pi(X) = \{Y : Y \subset V_X \wedge 2^{n-1} < |Y|\}$.

Each element of the set V_X can be naturally assigned to the formulas of the language of the classical propositional calculus, but this is only possible for a functionally complete language. In the case of a functionally incomplete language, there are valuations of variables to which the formulas of such a language do not correspond. On the other hand, a certain set V_X can be assigned to every formula A, where $X = Var(A)$. Among formulas with two different propositional variables in a purely implicational language, there are infinitely many tautologies, e.g., $(p \rightarrow (q \rightarrow p))$, $((p \rightarrow p) \rightarrow (q \rightarrow q))$, $((p \rightarrow (p \rightarrow q)) \rightarrow (p \rightarrow q))$, and many others. The same is true for other formulas from the set P, especially P'. As already mentioned, when a language is not functionally complete, such as a purely implicational language, then the set $\pi(X)$ may have less cardinality since there may be no formulas in a functionally incomplete language that define certain Boolean valuations. For example, in the case of formulas of the $Form_{\rightarrow}$ language that are built with only two propositional variables, p and q, we have only four elements of the set $V_{\{p,q\}}$, and five elements of the set $\pi(\{p,q\})$, respectively:

- tautologies, i.e., formulas that are true for all valuations;
- $v(p) = 1$ and $v(q) = 1$, or $v(p) = 1$ and $v(q) = 0$, or $v(p) = 0$ and $v(q) = 1$, and the corresponding formula is $(p \rightarrow q) \rightarrow q$;
- $v(p) = 1$ and $v(q) = 1$, or $v(p) = 1$ and $v(q) = 0$, or $v(p) = 0$ and $v(q) = 0$, and the corresponding formula is $(q \rightarrow p)$;
- $v(p) = 1$ and $v(q) = 1$, or $v(p) = 0$ and $v(q) = 0$, or $v(p) = 0$ and $v(q) = 1$, and the corresponding formula is $(p \rightarrow q)$.

LEMMA 5.2. *For the valuations $v(p) = 1$ and $v(q) = 0$, or $v(p) = 0$ and $v(q) = 0$, or $v(p) = 0$ and $v(q) = 1$, there is no purely implicational formula that defines them.*

PROOF: If there were such a purely implicational formula with two variables $A(p,q)$ that were true only for these cases, then $p/q := A(p,q)$ could serve as the definition of a Sheffer stroke by the implication alone. This would make it possible to define all binary connectives of the classical logic solely by implication, which is not possible. □

The set $\pi(X)$ from Definition 5.1. is particularly interesting because of its cardinality. For example, $|\pi(X)| = 5$ for $|X| = 2$; while $|\pi(X)| = 94$ when $|X| = 3$. The general formula for determining the cardinality of this set looks like this: $|\pi(X)| = \binom{2^n}{2^{(n-1)}+1} + \ldots + \binom{2^n}{2^{(n-1)}+2^{(n-1)}}$, when $|X| = n$. Note that this function can be composed with the Var function and extended to the set of all $Form_{PC} : \pi'(A) := \pi(Var(A))$. For the sake of emphasis, let us observe that for a language with implication alone there are only four sets of valuations that the implication formulas correspond to.

DEFINITION 5.3. For a finite set of variables X, the family $\pi(X)$ forms a certain algebra $< \pi(X), \cup, ' >$, which satisfies the following conditions:

i. $X \in \pi(X)$;

ii. If $Y \subset Z$ and $Y \in \pi(X)$, then $Z \in \pi(X)$;

iii. If $Y \in \pi(X)$, then $Y' \notin \pi(X)$.

We will now apply the definition of a filter to the family $\pi(X)$.

DEFINITION 5.4. A family of sets $F \subset \pi(X)$ is called a filter in the family $\pi(X)$ if the following conditions are satisfied:

I. $X \in F$;

II. $Y \subset Z$ and $Y \in F$, then $Z \in F$;

III. If $Z \in F$ and $Y \in F$, then $Z \cap Y \in F$;

IV. $\emptyset \notin F$.

As can be seen, the family F being the filter significantly reduces the cardinality of the family $\pi(X)$. For example, let us take the already considered case of $|X| = 2$ and $|\pi(X)| = 5$. We then have $V_{\{p,q\}} = \{11, 10, 01, 00\}$ and $\pi(\{p, q\}) = \{X_1 = \{11, 10, 01, 00\}$, $X_2 = \{11, 10, 01\}$, $X_3 = \{11, 10, 00\}$, $X_4 = \{11, 00, 01\}$, $X_5 = \{10, 01, 00\}\}$. There is only one filter F in this family and it is non-proper: $F = \{X_1\}$.

Based on the presented facts, we shall determine how we will understand the semantic model of our system **P**. Since the traditional semantic model that we are familiar with suffices for our purposes, we have so far been able to understand it intuitively.

DEFINITION 5.5. A structure $M_P =<< \{1, 0\}; f_{\rightarrow} >; D = \{1\} >$ is called a **normal model** for **P**.[10]

Functions, which are valuations, assign logical values of truth (1) or false (0) to all propositional variables and formulas.

We have the following typology of formulas:

A. Formula A is *satisfiable* iff there is a valuation for which A takes the logical truth value;

B. Formula A is *logically probable* iff it takes the logical truth value for 'most' Boolean valuations of A i.e. for most restrictions $v\lceil Var(A)$ of Boolean valuations of A.

C. Formula A is a *tautology* iff it takes the logical value of true for all Boolean valuations.

[10] Of course, usually there are more functions: $f_{\neg}, f_{\vee}, f_{\wedge}, f_{\equiv}$.

6. The strong completeness of the system P

Let us now consider Definition 4.1. as the fundamental definition of the entailment. This definition is similar to Tarski's definition of the entailment, which preserves truthfulness in the sense that a proposition A does not follow from the set of true propositions X when A itself is not true. A formula A does not follow from the set of logically probable formulas X when it is not itself logically probable, as stated in Definition 4.1. Additionally, unlike other definitions that only permit finite sets X, this definition admits any cardinality of the set X. It should be noted that, from the perspective of natural language, the requirement that the set of premises is finite is not very unreasonable. Let us see, therefore, whether the consequence that results from Definition 4.1. has the properties of a consequence in the Tarskian (classical) sense. Since the so-understood consequence is a relation, $\models_1 \subset 2^{Form} \times Form$, we need to determine the properties of this relation. We will show that it has the properties of Tarski's consequence, namely reflexivity, cut and monotonicity [5, p. 5].

Lemma 6.1.

a. *If $A \in X$, then $X \models_1 A$; (reflexivity).*

b. *If $X \models_1 B$, for any $B \in Y$ and $X \cup Y \models_1 C$, then $X \models_1 C$; (cut).*[11]

c. *If $X \models_1 A$ and $X \subset Y$, then $Y \models_1 A$; (monotonicity).*

Proof: Ad a. Suppose that $A \in X$ and $X \subset P$ holds, then, of course, by the definition of inclusion we have $A \in P$.

Ad b. Suppose $X \models_1 B$, for every $B \in Y$ and $X \cup Y \models_1 C$ and $X \subset P$. From these presumptions, we aim to demonstrate that $X \models_1 C$. We therefore have for every $B \in Y$, $B \in P$, that is $Y \subset P$. If $X \subset P$ and $Y \subset P$, then $(X \cup Y) \subset P$. And from this, we straightforwardly obtain $C \in P$.

Ad c. Suppose that $X \models_1 A$ and $Y \subset P$. Hence, we have $X \subset P$, and by the first assumption we have $A \in P$. □

[11] Makinson [5] calls cut also a cumulative transitivity and characterizes it in the terms of the consequence operator as follows: if $X \subset Y \subset C(X)$, then $C(Y) \subset C(X)$. Cut, given the above definition and in the presence of monotonicity, is equivalent to the idempotence condition $C(C(X)) \subset C(X)$.

It seems natural to ask about the completeness of the system **P**, that is, more precisely, whether every true formula of the language is simultaneously provable in this system. Such a theorem has the following strong version: if $X \models_1 A$, then $X \vdash_{\mathbf{P}} A$. We will attempt to provide a proof of this important theorem later in this paper. On the other hand, the converse implication if $X \vdash_{\mathbf{P}} A$, then $X \models_1 A$ is called the *soundness theorem* for the system P, and we shall attempt to prove it first. The proof is similar to the left-to-right implication of Theorem 3.16.

THEOREM 6.2 (strong soundness of the system **P**). *For any $X \subset Form_{\rightarrow}$ and $A \in Form_{\rightarrow}$, if $X \vdash_{\mathbf{P}} A$, then $X \models_1 A$.*

PROOF: Suppose that $X \vdash_{\mathbf{P}} A$ holds for given X and A, which we will write in an equivalent way: $A \in C_{\mathbf{P}}(X)$. Thus, A has a proof that is based on the set of formulas X and the set of axioms of the system **P** of the form $d =< D_1, D_2, D_3, \ldots, D_n = A >$, where each $D_i(1 \leq i \leq n)$ is either a member of the set X, an axiom of the system, or has been obtained from the prior expressions of this sequence using any of the four rules of the system. The sequence that is a proof of a formula A based on the set of formulas X will only contain a finite subset Y of the elements of the set X, and for this set we have $Y \vdash_{\mathbf{P}} A$. This is due to the finite length of the proof of a formula A. We will give a sketch of the well-known proof of the inductive hypothesis $W(n)$, which states that if d is a proof of A based on the finite set $Y \subset X$, then $Y \models_1 A$, which we shall demonstrate for any n. To do so, it suffices to prove $\forall n(\forall k(k < n \rightarrow W(k)) \rightarrow W(n))$. Let us assume that the antecedent of the implication holds for any $k < n$ and $W(k)$. The formula $D_n = A$ in the proof d must appear as a result of any of the following steps. When the formula A is an axiom or an element of the set Y, then, of course, $Y \models_1 A$. Also, for rules (RO) and (RN), the case is evident by virtue of the corresponding theorems: Theorem 2.13., Theorem 2.14. and Lemma 3.2. Therefore, only the proof for the case of the restricted substitution rule (RSu) is needed. Let us therefore assume that $A \in P'$, i.e., A, is a strictly probable formula. Let us take any bijection $h : V \rightarrow V$. Let $Var(A) = \{p_1, \ldots, p_m\}$. Hence, $Var(e(A)) = \{h(p_1), \ldots, h(p_m)\}$. By definition of the 2^m valuations of formula A, for most of them $v(A) = 1$. Let $v_j(0 < j < 2^m)$ be such that $v_j(A) = 1$, then for each valuation v_j of the propositional variables of formula A, this formula will take the corresponding logical values. Let us define $v_j(p_i) = v_j(h(p_i))$, hence $v_j(A) = v_j(e(A))$ for every j. That is, if $X \models_1 A$, then $\{A\} \models_1 e(A)$, and finally $X \models_1 e(A)$. $\square$

We now proceed to the proof of completeness theorem: this is the most important theorem for the system **P**; the proof is akin to the proof of the right-to-left implication of Theorem 3.16. But first we draw the corollary, the proof of which is based on Theorem 6.2 and Lemma 3.20:

COROLLARY 6.3. The system **P** is absolutely consistent, i.e., $C_{\mathbf{P}}(\emptyset) \neq Form_{\to}$.

THEOREM 6.4 (strong completeness theorem for **P**). *For any $X \subset Form_{\to}$ and $A \in Form_{\to}$: if $X \models_1 A$, then $X \vdash_{\mathbf{P}} A$.*

PROOF: We prove this theorem using some variant of Lindenbaum's lemma for our language that has no negation. Suppose then that $X \models_1 A$, i.e., that if $X \subset P$, then $A \in P$. For an indirect proof, suppose that $A \notin C_{\mathbf{P}}(X)$, hence the set X is consistent in the Post sense. Using Lindenbaum's lemma, we can extend the set X to a maximal and consistent set X^* for a purely implicational language. $X^* = \cup_{m \in N} X_m$, where $X_0 = X$; $X_{m+1} = X_m \cup \{w_m\}$, if $X_m \cup \{W_m\}$ is consistent; and $X_{m+1} = X_m$, if $X_m \cup \{W_m\}$ is inconsistent. It is known that all formulas of the set $Form_{\to}$ can be put on an infinite list: $w_0, w_1, w_2, \ldots$ We must now prove the auxiliary lemmas concerning the set X^*. $\square$

LEMMA 6.5. *Let a formula $A \in Form_{\to}$ be such that $Var(A) = \{p_1, p_2, \ldots, p_n\}$: if for any Boolean valuation v and for any $n \geq i > 0$, $v(p_i) = 1$, then $v(A) = 1$.*

PROOF: Induction by the complexity degree of a formula A. If $A = p$ and $v(p) = 1$, $v(A) = 1$. Suppose $A = (B \to C)$ and let formulas B, C satisfy the assumptions of the theorem, i.e., $v(B) = v(C) = 1$, then $v(A) = v(B \to C) = v(B) \to v(C) = 1 \to 1 = 1$. $\square$

LEMMA 6.6. *For any $A \in Form_{\to} : A \in P$ iff $A \in X^*$.*

PROOF: Suppose that $A \in P$. By virtue of Theorem 3.16, $\vdash_{\mathbf{P}} A$, and from the monotonicity of the consequence operation (Lemma 6.1(c)), we obtain $X^* \vdash_{\mathbf{P}} A$, hence $A \in X^*$. If $A \notin X^*$ and the set X^* is maximal, then $X^* \cup \{A\}$ is inconsistent. Conversely, suppose that $A \in X^*$, hence $X^* \vdash_{\mathbf{P}} A$. For the purpose of an indirect proof, we will assume that

$A \notin P$. By virtue of Lemma 3.5, in exactly half of the Boolean valuations the formula A takes the Boolean value 1, and it takes the Boolean value 0 for the other half. Then, for such an A, $\{A\} \vdash_{\mathbf{P}} p$, where p is some variable belonging to $Var(A)$.[12] By further applying the rule (RSu) $\{p\} \vdash_{\mathbf{P}} q$, we get $\{A\} \vdash_{\mathbf{P}} q$ for any variable q. This, in turn, leads us to assert that X^* is inconsistent, which contradicts the assumption. □

One final step still needs to be proven:

Lemma 6.7. *If $A \in Form_{\rightarrow}$ and $m(A) = 1/2$, then $\{A\} \vdash_{\mathbf{P}} p$ for some variable $p \in Var(A)$.*

Proof: Assume that the assumptions of the lemma hold and let $|Var(A)| = n$. Thus, exactly 2^{n-1} possible Boolean valuations of formula A take the value 1, and the other half of the valuations obviously take the value 0. If the formula $A = p$, the lemma obviously holds. Suppose, then, that A is a compound formula and has the form $A = (A_1 \rightarrow B_1)$ for some A_1 and $B_1 \in Form_{\rightarrow}$. Then, $v(A) = 0$ iff $v(A_1) = 1$ and $v(B_1) = 0$. Assuming that $B_1 = (A_2 \rightarrow B_2)$, then $v(B_1) = 0$ iff $v(A_2 \rightarrow B_2) = 0$ iff $v(A_2) = 1$ and $v(B_2) = 0$. Thus, $v(A) = 0$ iff $v(A_1) = 1$ and $v(B_1) = 0$; then $v(A_2) = 1$ and $v(B_2) = 0$. Following this pattern, after $k > 1$ steps we arrive at $B_{k-1} = (A_k \rightarrow p)$, where $p \in Var(A)$; iff $(v(A_1) = 1) \wedge (v(A_2) = 1) \wedge \ldots \wedge (v(A_k) = 1) \wedge (v(B_k) = v(p) = 0)$. From transitivity, we have if $v(A) = 0$, then $v(p) = 0$, for some variable $p \in Var(A)$. By contraposition for this variable, if $v(p) = 1$, then $v(A) = 1$. For the proof of the converse implication, the key issue is whether it would be possible that $v(p) = 0$, while $v(A) = 1$ for some valuation v. This case is ruled out since the table for the formula A has 2^n rows, half of which contain 0 and half of which contain 1 in the last column. In the column under the variable p, also half of the cells contain 0 and half contain 1. Consequently, if $v(A) = 0$, then $v(p) = 0$, so 0 occurs at least in those rows of the column under the variable p where the valuation of formula A equals 0, and exactly half of the valuations equal 0. According to Definition 2.1, none of the other rows in the column under the variable p can contain 0 because there are 2^{n-1} rows containing 1. Let us now construct a disjunctive normal form of the formula A, that is A_{APN}. This formula is a disjunction of 2^{n-1} conjunctions of literals with n members. In each such conjunction, there is as its

[12]The proof of this claim is given in Lemma 6.6 below.

member a variable p. We can move this variable to the front of the conjunction by applying the law of distributivity of the disjunction over the conjunction to A_{APN}. This allows us to derive p within PC: $A_{APN} \vdash_{PC} p$. Also, in PC it holds that $A_{APN} \equiv A$ and $A_{APN} \vdash_{PC} A$. By virtue of the extensionality rule, we have $A \vdash_{PC} p$. From the deduction theorem for PC, we have $\vdash_{PC} (A \to p)$; and from the definition of the system **P**, we have $\vdash_{P} (A \to p)$. Hence from A, by rule $(RO1)$, we will obtain p, and finally we will also obtain $p \vdash_{P} q$, for any variable q, by rule (RSu). □

LEMMA 6.8. *The set $X^* \cup \{A\}$ is inconsistent when $A \in Form_{\to}$ and exactly half of its Boolean valuations take the value* 1 *(as in Lemma* 6.6*).*

PROOF: The set $\{A\}$ is inconsistent, as follows from Lemma 6.6. So, by virtue of monotonicity, also $X^* \cup \{A\}$ is inconsistent.

This completes the proof of Lemmas 6.6 and 6.7, and also the proof of Theorem 6.4 (the strong completeness theorem). □

As an illustration, let us observe that within the formulas of the PC language written with three different variables (p, q, r), there are 70 formulas in the disjunctive normal form that have exactly half of the rows occupied by 1s in the last column of their respective truth tables. Only three of these formulas have equivalents that are simply implicational formulas. Such formulas are characterized by the fact that applying the Quine-McCluskey method—also known as Karnaugh's method—to minimize Boolean functions yields a single variable. The formulas in disjunctive normal form used in the example are:

- $(p \wedge q \wedge r) \vee (p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge r) \vee (\neg p \wedge \neg q \wedge r)$;
- $(p \wedge q \wedge r) \vee (p \wedge \neg q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge \neg r)$;
- $(p \wedge q \wedge r) \vee (\neg p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge \neg r)$.

When understood as sets, the consequence relation $\models_1$ and the relation $\models_{PC}$ are distinct since they intersect. This is because $\{q \to p\} \models_1 (p \to q)$, yet $\{q \to p\} \models_{PC} (p \to q)$ does not hold since, for valuations $v(p) = 1$ and $v(q) = 0$, $v(q \to p) = 1$, while $v(p \to q) = 0$. Both relations hold when premises and conclusions are tautologies. On the other hand, $\{(p \to q) \to r, p \to q\} \models_{PC} r$, but $\{(p \to q) \to r, p \to q\} \models_1 r$ does not hold because the premises are elements of the set P, while the single variable r is not.

A study of the consequence relation $\models_1$ reveals that it has some unexpected properties, such as $\{q \to p\} \models_1 p \to q$ or—even more contentious—$\{q \to p\} \models_1 r \to s$. Due to the proven completeness theorem, $\{q \to p\} \vdash_P r \to s$ holds because the derivation is allowed by the substitution rule (RSu). This observation can serve as the starting point for consideration of a system without this rule, which can be a challenging issue.

7. Further considerations regarding the entailment relation

Since this paragraph concerns only purely implicational language, we need to adapt Definition 4.2, which is of the logical consequence in the second sense we have given, i.e., $\models_2$.

LEMMA 7.1 (entailment_2′). *The formula A follows from a finite set of formulas $X = \{A_1, A_2, \ldots, A_n\}$ (i.e., symbolically, $\{A_1, A_2, \ldots, A_n\} \models_2 A$) iff $(A_1 \to (A_2 \to (\ldots(A_n \to A))\ldots) \in P$.*

PROOF: This transformation is made possible by the equivalence that is the theorem of the PC: $(A \wedge B) \to C) \equiv (A \to (B \to C))$. □

We will now demonstrate how the above understanding of the consequence relation does not meet the classical properties of the consequence.

LEMMA 7.2. *The relation $\models_2$ satisfies the conditions a. and c., but it does not satisfy the condition b.:*

a. *If $A \in X$, then $X \models_2 A$; (reflexivity).*

b. *It is not true that if for all $B \in Y$, $X \models_2 B$, and $X \cup Y \models_2 C$, then $X \models_2 C$; (cut).*

c. *If $X \models_2 A$ and $X \subset Y$, then $Y \models_2 A$; (monotonicity).*

PROOF: Ad a. Suppose that there exists $A \in X = \{A_1, A_2, \ldots, A_n\}$. Therefore, $A = A_i$, for some $0 < i < n + 1$. Hence, $\{A_1, A_2, \ldots, A_n\} = \{A_1, A_2, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n, A_i\} \models_2 A$, and since $(A_1 \to (A_2 \to (\ldots (A_{i-1} \to (A_{i+1} \to (\ldots(A_n \to (A_i \to A)))\ldots)$ is a tautology; therefore, by virtue of Definition 2.2, $\{A_1, A_2, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n, A_i\} \models_2 A$ because $A_i = A$.

Ad b. Suppose that for every $B \in Y : X \models_2 B$, and $X \cup Y \models_2 C$, where $Y = \{(q \to p)\}$; $B = (q \to p)$; $X = \{(p \to q)\}$; $C = p$. We have $(q \to p) \in Y$; $\{(p \to q)\} \models_2 (q \to p)$, because $((p \to q) \to (q \to p)) \in P$; $\{(p \to q)\} \cup \{(q \to p)\} = \{(p \to q), (q \to p)\} \models_2 p$, because $(((p \to q) \wedge (q \to p)) \to p) \in P$; but $X = \{(p \to q)\} \models_2 p$ does not hold because $((p \to q) \to p) \notin P$.

Ad c. Suppose $X \models_2 A$ and $X \subset Y$ and $|Y| = n$ for some n. If Y is not finite, then the consequence holds vacuously. By Definition 4.2., we have $(\wedge X \to A) \in P$ and $(\wedge Y \to \wedge X) \in TAUT$, and from the transitivity $(\wedge Y \to \wedge X) \to ((\wedge X \to A) \to (\wedge Y \to A))$. After commutating and detaching (using $(RO1)$), we get $(\wedge Y \to \wedge X) \to (\wedge Y \to A)$; then by detaching the antecedent which is a tautology (using $(RO2)$), we get $(\wedge Y \to A)$. This formula is a member of P. This, in turn, by virtue of Lemma 7.1, gives $Y \models_2 A$. □

Let us now examine the properties of the third relation of consequence. Note that here we are dealing with richer language since there is at least a conjunction in addition to the implication. Nevertheless, we will try to remove it. As we know, the set P itself is not closed with respect to the rule of conjunction introduction (cf. Lemma 2.15). Let us note that the following holds:

Lemma 7.3. *Let $X \subset Form_{PC}$ and $|X| = n$. For any Boolean valuation $v : v(X) = 1$ iff $v(\wedge X) = 1$.*

Proof: Suppose $v(X) = 1$. This is so iff $v(A) = 1$ for any $A \in X$. Then, of course, $v(\wedge X) = 1$. Conversely, if $v(\wedge X) = 1$, then $v(A) = 1$, for every member A of the conjunction $\wedge X$. □

Based on above consideration, we can reformulate the definition of the consequence $\models_3$ and use it for the following lemma:

Lemma 7.4. *Let $X \subset Form_{\to}$ be a finite set and $A \in Form_{\to}$. Then, $X \models_3 A$ iff every Boolean valuation v from the set of the majority of Boolean valuations satisfying the set X, i.e., $v(X) = 1$, also satisfies A, i.e., $v(A) = 1$.*

Proof: From Definition 4.3 and Lemma 7.3. □

The idea behind this term is that we want every valuation which belongs to a majority of valuations and assigns a logical truth value to the conjunction of all premises to assign a logical truth value to the conclusion as well. This term is different from both $\models_1$ (even for finite sets of premises) and $\models_2$. For example, let us take the formulas: $(p \to q)$, $(q \to p) \in P$. Then, $((p \to q) \wedge (q \to p)) \to p \in P$, that is $\{(p \to q), (q \to p)\} \models_2 p$, but $\{(p \to q), (q \to p)\} \not\models_1 p$. To distinguish $\models_3$ from $\models_2$, note that $\{(p \to q)\} \models_2 (q \to p)$ because $((p \to q) \to (q \to p)) \in P$; however, $\{(p \to q)\} \models_3 (q \to p)$ does not hold because, for a valuation where $v(p) = 0$ and $v(q) = 1$, we have $v(p \to q) = 1$ and $v(q \to p) = 0$.

Lemma 7.5. *For any finite $X, Y \subset Form_{PC}$, and $A, B, C \in Form_{PC}$:*

A. *If $A \in X$, then $X \models_3 A$; (reflexivity)*

B. *If for all $B \in Y : X \models_3 B$, and $X \cup Y \models_3 C$, then $X \models_3 C$; (cut)*

C. *If $X \models_3 A$ and $X \subset Y$, then $Y \models_3 A$; (monotonicity).*

Proof: Ad A. Let $A \in X$, $|X| = n$. Suppose also that $\wedge X \in P$, i.e., it takes the value 1 for most Boolean valuations of this formula. For each such Boolean valuation v, if $v(\wedge X) = 1$, then $v(A) = 1$. For if $v(A) = 0$, then there would also be $v(X) = 0$. Thus $X \models_3 A$.

Ad B. Suppose that for every $B \in Y, X \models_3 B$, and $X \cup Y \models_3 C$. Therefore, any valuation v of the conjunction $\wedge X$ that assigns it the value 1 assigns the value 1 to any formula $B \wedge Y$, therefore any such valuation assigns the value 1 to the conjunction $\wedge Y$. Hence, if $v(\wedge X) = 1$, then $v(\wedge Y) = 1$. Thus, the set of such valuations that $v(X) = 1$ is contained in the set of valuations such that $v(Y) = 1$. With $X \cup Y \models_3 C$, by Definition 4.3 we can assert that each valuation (belonging to the majority of valuations) that assigns value 1 to the conjunction $\wedge(X \cup Y) = (\wedge X) \wedge (\wedge Y)$ assigns the same value to the formula C. Based on the corresponding PC tautology, we have $(\wedge X) \wedge (\wedge Y) \equiv (\wedge X) \wedge (\wedge X \to \wedge Y)$. So, there is an inclusion of the set of Boolean valuations v such that $v((\wedge X) \wedge (\wedge Y)) = 1$ in the set of valuations such that $v(\wedge X) = 1$. Hence, $X \models_3 C$.

Ad C. Suppose that $|X| = n$, $|Y| = m$, $X \models_3 A$, $X \subset Y$ and, for an indirect proof, for any $Y : Y \not\models_3 A$. The majority of valuations in the set of all Boolean valuations of the conjunction $\wedge X$ for which $v(\wedge X) = 1$, under the assumption $X \subset Y$, is the superset of the set of those Boolean valuations for which $v(\wedge Y) = 1$. Thus, $Y \models_3 A$, which is a contradiction. □

Because of the finiteness condition, it is debatable whether Definition 4.3 of the consequence is adequate for the realm of logically probable formulations.

8. Conclusions

In our deliberations, the notion of majority plays an important role. There are other research fields for which this concept is also important. For example, there are studies focused on decision-making in the fields of social choice theory, political sciences and economics, whose authors have examined how "[...] individual preferences and interests can be combined into a collective decision." [3, p. 1]. These studies are synthetically described in [3], and the latest results are presented in [7]. This fact alone indicates some conceptual affinity between these studies and considerations presented in this paper. Moreover, the notion of majority in both my theory and the theory of decision making is analysed in the context of inferences and logic, although from different angles. In studies devoted to group decision-making, this notion refers to the majority in a certain group of subjects as a whole. In this case, the majority is a result of individual choices made by decision-makers, and the decision of the group as a whole (i.e., a collective judgement) is obtained through aggregation functions. In the case analysed in this paper, the majority is formed only by the logical valuations, and—unlike in the decision-making process—we do not need to involve any extralogical apparatus. In short, group decision-making theory uses means outside the arsenal of logic, while in our conception we remain within purely logical concepts and the standard language of PC. We cannot rule out combining our take on a majority within the framework of group choice theory.

Another related line of research can be found in the field of logical probability, which stems from Carnap and was developed by, for example, [1]. This probability is defined for first-order sentences as the fraction of the set of finite models for which a sentence is true in the set of all finite models of the sentence. However, this study focused on logical probability, unlike the study described in this paper; cf. [1].

The basic ideas of our paper were developed in the post-graduate thesis of Olszewski, later summarized in a paper [6]. As I have already mentioned, the real *powerbroker* of these considerations is the notion of *majority*. While in the presented paper *majority* appears in some tricky way, it seems that the main further considerations should focus on the abstract notion of majority. Here by way of example, we will give five proposals of the definitions of majority for some set U to show the richness of this concept. Below we define the families of subsets of the universe U, denoted by $\pi^k(U)$; the elements of these families are subsets which are the *majorities* in the set U; cf. [6]:

Def. A. Let U be any set, finite or infinite: $\pi^1(U) = \{Y \subset U\colon |Y| > |Y'|\}$.

Def. B. Let U be an infinite set of any cardinality: $\pi^2(U) = \{Y \subset U\colon |Y'| = n\}$.

Def. C. Let U be an infinite set of any cardinality: $\pi^3(X) = \{Y \subset U\colon |Y| > |Y'|\}$.

Def. D. Let U be a metric space with its metric $d : \pi^4(U) = \{Y \subset U\colon |Y| > |Y'| \text{ and } dY > dY'\}$.

Def. E. Let U be a topological space: $\pi^5(U) = \{Y \subset U\colon |Y| > |Y'| \text{ and if } Y \text{ a } \textit{dense} \text{ subset of } U\}$.

A quite natural direction for further research is to extend the main results, including the completeness theorems, to the whole language of propositional calculus. In a sense, the concept presented can be extended to the concept of logical probability (majority) of first order formulas for a finite universe. The concepts of the *probability of the propositional language formulas* and the *majority*, understood as a form of modality operator, seem equally promising.

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Bulletin of the **S**ection of **L**ogic
Volume 53/3 (2024), pp. 399–418
https://doi.org/10.18778/0138-0680.2024.08

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FREE SPECTRA OF EQUIVALENTIAL ALGEBRAS WITH CONJUNCTION ON DENSE ELEMENTS

Abstract

We construct free algebras in the variety generated by the equivalential algebra with conjunction on dense elements and compute the formula for the free spectrum of this variety. Moreover, we describe the decomposition of free algebras into directly indecomposable factors.

Keywords: Fregean varieties, equivalential algebras, dense elements, free algebra, free spectra.

2020 Mathematical Subject Classification: 08B20, 03G25.

1. Introduction

The equivalential algebra with conjunction on dense elements was introduced in [5]. This algebra turned out to be one of the four polynomially nonequivalent three-element algebras, that generates a congruence permutable Fregean variety. The other algebras are as follows. The first one is the three-element equivalential algebra (without any additional operation). It is very well known, also when it comes to the construction of the n-generated free algebras, as well as the cardinality of these algebras for small n and for some subvarieties (see [2], [9], [6], [7]). The other one is the three-element Brouwerian semilattice. This algebra also has been very well

*Partially supported by the NCN OPUS grant 2022/45/B/HS1/00606.

Presented by: Janusz Czelakowski
Received: December 4, 2023
Published online: May 20, 2024

researched ([3]). The third of these algebras is the three-element equivalential algebra with conjunction on the regular elements. It was studied in [4]. The mentioned work contains the description of this algebra, its most important properties, the representation theorem, the construction of the free algebra and the free spectrum.

Whereas, when it comes to the equivalential algebra with conjunction on dense elements, we proved in [5] the representation theorem and we give a sketch of the construction the finitely generated free algebras. The aim of this paper is to extend the results of [5] by providing the formula for the free spectrum (Section 4). In this way we complete the full description (with accuracy to the polynomially equivalence) of the free algebras in congruence permutable Fregean varieties generated by three-element algebras.

The second aim of this article is to describe the directly indecomposability of the free algebras in the variety generated by the equivalential algebra with conjunction on dense elements (Theorem 3.6).

2. Equivalential algebras with conjunction on the dense elements

Preliminary facts can be found in [5], but for the convenience of the reader we recall some basic information.

Definition 2.1. **An equivalential algebra with conjunction on the dense elements** is an algebra $\mathbf{D} := (\{0, *, 1\}, \cdot, d, 1)$ of type $(2, 2, 0)$, where $(\{0, *, 1\}, \cdot, 1)$ is an equivalential algebra and d is a binary commutative operation presented in the table below (on the right):

$\cdot$	1	$*$	0
1	1	$*$	0
$*$	$*$	1	0
0	0	0	1

d	1	$*$	0
1	1	$*$	1
$*$	$*$	$*$	$*$
0	1	$*$	1

The interpretation of the name is given in [5, Definition 4.1]. We denote by $\mathcal{V}(\mathbf{D})$ the variety generated by $\mathbf{D}$.

A crucial role in the construction of the finitely generated free algebras is played by the subdirectly irreducible algebras in $\mathcal{V}(\mathbf{D})$.

PROPOSITION 2.2. [5, Proposition 4.6] There are only three (up to isomorphism) nontrivial subdirectly irreducible algebras in $\mathcal{V}(\mathbf{D})$: $\mathbf{D}, \mathbf{2}, \mathbf{2}^{\wedge}$, where:

$$\mathbf{2} := \{\{0,1\}, \cdot, d, 1\}, \text{ where } d \equiv 1,$$

$$\mathbf{2}^{\wedge} := \{\{*,1\}, \cdot, d, 1\}, \text{ where } \ d(x,y) := x \wedge y.$$

Whatsmore, $\mathbf{2}$ and $\mathbf{2}^{\wedge}$ are subalgebras of $\mathbf{D}$.

Let $\mathbf{A} \in \mathcal{V}(\mathbf{D})$. We denote by $\mathrm{Cm}(\mathbf{A})$ the set of all completely meet-irreducible congruences on $\mathbf{A}$.

We define an order $\leq$ on $\mathrm{Cm}(\mathbf{A})$ as follows:

$$\varphi \leq \psi \text{ iff } \varphi \subseteq \psi, \text{ for } \varphi, \psi \in \mathrm{Cm}(\mathbf{A}).$$

We use the following notation:

$$\overline{L} := \{\mu \in \mathrm{Cm}(\mathbf{A}) : \mathbf{A}/\mu \cong \mathbf{2}\},$$

$$\underline{L} := \{\mu \in \mathrm{Cm}(\mathbf{A}) : \mathbf{A}/\mu \cong \mathbf{D}\},$$

$$P := \{\mu \in \mathrm{Cm}(\mathbf{A}) : \mathbf{A}/\mu \cong \mathbf{2}^{\wedge}\},$$

$$L := \overline{L} \cup \underline{L}.$$

In our case it turns out that

$$\varphi \leq \psi \text{ iff } (\varphi \in \underline{L}, \psi \in \overline{L}, \varphi < \psi) \text{ or } \varphi = \psi. \tag{O1}$$

Moreover, if $\varphi < \psi$, then $\psi = \varphi^{+}$.

Let $\varphi, \psi \in \mathrm{Cm}(\mathbf{A})$. We introduce an equivalence relation on $\mathrm{Cm}(\mathbf{A})$ as follows (see [1, p. 51]):

$$\varphi \sim \psi \text{ iff the intervals } [\varphi, \varphi^{+}] \text{ and } [\psi, \psi^{+}] \text{ are projective.}$$

DEFINITION 2.3. Let $\mathbf{A} \in \mathcal{V}(\mathbf{D})$. The structure $\mathbf{Cm}(\mathbf{A}) := (\mathrm{Cm}(\mathbf{A}), \leq, \sim)$ is called a frame of $\mathbf{A}$.

From [5, Proposition 5.4, Theorem 5.5] we get that the equivalence classes of the relation $\sim$ on $\mathrm{Cm}(\mathbf{A})$ take the following form:

1. $\overline{L} \in \mathrm{Cm}(\mathbf{A})/_{\sim}$,
2. $\mu/_{\sim} = \{\mu\}$ for all $\mu \in \underline{L} \cup P$.

Moreover, $(\overline{L} \cup \{1_{\mathbf{A}}\}, \bullet, 1_{\mathbf{A}})$ forms a Boolean group, where $\mu_1 \bullet \mu_2 := (\mu_1 \div \mu_2)'$ for $\mu_1, \mu_2 \in \overline{L}$ ($\div$ denotes the symmetric difference and $'$ denotes the complement of a set).

Now, we recall that every finite algebra from $\mathcal{V}(\mathbf{D})$ can be naturally decomposed as the direct product of two algebras:

Proposition 2.4. Let $\mathbf{A} \in \mathcal{V}(\mathbf{D})$ be finite. Then:

$$\mathbf{A} \cong \mathbf{A}/_{\bigwedge L} \times \mathbf{A}/_{\bigwedge P}.$$

To construct the free algebras in $\mathcal{V}(\mathbf{D})$ we need the notion of the hereditary sets [5, Definition 6.1] and the representation theorem [5, Theorem 6.2].

Definition 2.5. Let $\mathbf{A} \in \mathcal{V}(\mathbf{D})$ and $Z \subseteq \mathrm{Cm}(\mathbf{A})$. A set Z is **hereditary** if:

1. $Z = Z\uparrow$,
2. $\overline{L} \subseteq Z$ or $((\overline{L} \cap Z) \cup \{\mathbf{1_A}\}, \bullet)$ is a hyperplane in $(\overline{L} \cup \{\mathbf{1_A}\}, \bullet)$.

We will denote by $\mathcal{H}(\mathbf{A})$ the set of all hereditary subsets of $\mathrm{Cm}(\mathbf{A})$.

Theorem 2.6. *Let* $\mathbf{A} \in \mathcal{V}(\mathbf{D})$ *and let* $\mathbf{A}$ *be finite. Then the map* $M : A \ni a \to M(a) := \{\mu \in \mathrm{Cm}(\mathbf{A}) : a \in 1/\mu\}$ *is the isomorphism between* $\mathbf{A}$ *and* $(\mathcal{H}(\mathbf{A}), \leftrightarrow, \mathsf{d}, \mathit{1})$, *where*

$$Z \leftrightarrow Y := ((Z \div Y)\downarrow)'$$

$$\mathsf{d}(Z, Y) := [Z \cup ((Z\downarrow)' \cap L)] \cap [Y \cup ((Y\downarrow)' \cap L)],$$

$$\mathit{1} := \mathrm{Cm}(A),$$

for $Z, Y \in \mathcal{H}(\mathbf{A})$.

Using the above theorem we can build up elements of algebra $\mathbf{A}$ in $\mathcal{V}(\mathbf{D})$ from the set $\mathrm{Cm}(\mathbf{A})$ with the order and the partial Boolean operation, i.e. from the structure $(\mathrm{Cm}(\mathbf{A}), \leq, \bullet)$.

3. Free algebras

Let $n \in \mathbb{N}$ and let X be an n-element set of free generators of $\mathbf{F}_{\mathbf{D}}(n)$, where $D = \{0, *, 1\}$ is ordered by $0 < * < 1$. In [5] (Section 7) we give only the sketch of the construction of $\mathbf{F}_{\mathbf{D}}(n)$, which was based on the observation that we can identify any element of $\mathrm{Cm}(\mathbf{F}_{\mathbf{D}}(n))$ with a certain map, which sends free generators in some subdirectly irreducible algebra in $\mathcal{V}(\mathbf{D})$. Now, we will give a more detailed description of this construction, however, based on a slightly different approach, using the fact that the only subdirectly irreducible algebras in $\mathcal{V}(\mathbf{D})$ are $\mathbf{D}$ and its subalgebras $\mathbf{2}^{\wedge}$ and $\mathbf{2}$ given by sets $\{*, 1\}$ and $\{0, 1\}$ (2.2). Recall that

$$\underline{L} = \{\mu \in \mathrm{Cm}(\mathbf{F}_{\mathbf{D}}(n)) : \mathbf{F}_{\mathbf{D}}(n)/\mu \cong \mathbf{D}\},$$

$$\overline{L} = \{\mu \in \mathrm{Cm}(\mathbf{F}_{\mathbf{D}}(n)) : \mathbf{F}_{\mathbf{D}}(n)/\mu \cong \mathbf{2}\},$$

$$L = \underline{L} \cup \overline{L},$$

$$P = \{\mu \in \mathrm{Cm}(\mathbf{F}_{\mathbf{D}}(n)) : \mathbf{F}_{\mathbf{D}}(n)/\mu \cong \mathbf{2}^{\wedge}\}.$$

We denote by e the map $e : \{0, *, 1\} \to \{0, 1\}$ given by $e(0) = 0$ and $e(*) = e(1) = 1$. Clearly, such defined e is a homomorphism of $\mathbf{D}$ onto $\mathbf{2}$.

Put now $S(n) := \big\{f : X \to D : f^{-1}(\{0, *\}) \neq \emptyset\big\}$. As $\mathbf{F}_{\mathbf{D}}(n)$ is the free algebra, every $f \in S(n)$ can be uniquely extended to a homomorphism $\overline{f}$ from $\mathbf{F}_{\mathbf{D}}(n)$ to $\mathbf{D}$ with $\mathrm{Im}\,\overline{f}$ equal to one of three algebras: $\mathbf{D}$, $\mathbf{2}^{\wedge}$, or $\mathbf{2}$. Thus $\ker \overline{f} \in \mathrm{Cm}(\mathbf{F}_{\mathbf{D}}(n))$. In $S(n)$ we introduce an order relation $\preceq$ and a partial binary operation $\cdot$ in the following way. For $f, g \in S(n)$ we put $f \preceq g$ if and only if $f = g$ or $g = e \circ f$, and if $* \notin \mathrm{Im}\, f \cup \mathrm{Im}\, g$ we define $(f \cdot g)(x) := 1$ if $f(x) = g(x)$ and $(f \cdot g)(x) := 0$ if $f(x) \neq g(x)$ for $x \in X$.

The following theorem allows us to identify the structures $(S(n), \preceq, \cdot)$ and $(\mathrm{Cm}(\mathbf{F}_{\mathbf{D}}(n)), \leq, \bullet)$, where '$\bullet$' is the partial Boolean operation on $\overline{L}$.

THEOREM 3.1. *The map $\varphi : S(n) \ni f \to \ker \overline{f} \in$ is an isomorphism of the structures $(S(n), \preceq, \cdot)$ and $(\mathrm{Cm}(\mathbf{F}_{\mathbf{D}}(n)), \leq, \bullet)$.*

PROOF: (1) φ is onto. Let $\mu \in \mathrm{Cm}(\mathbf{F}_{\mathbf{D}}(n))$. Then $\mathbf{F}_{\mathbf{D}}(n)/\mu$ is isomorphic to $\mathbf{K} \in \{\mathbf{D}, \mathbf{2}, \mathbf{2}^{\wedge}\}$. In all three cases we denote the isomorphism by ι. Put $\pi_\mu(t) = t/\mu$ for $t \in F_{\mathbf{D}}(n)$. Then $\iota \circ \pi_\mu : \mathbf{F}_{\mathbf{D}}(n) \to \mathbf{K}$ is a surjective homomorphism. Hence $\iota \circ \pi_\mu|_X \in S(n)$ and $\varphi\,(\iota \circ \pi_\mu|_X) = \ker(\iota \circ \pi_\mu) = \ker(\pi_\mu) = \mu$, as desired.

(2) φ is one-to-one. Suppose, on the contrary, that $f, g \in S(n)$, $f \neq g$ and $\ker \overline{f} = \ker \overline{g}$. There is no loss of generality in assuming that there exists $x \in X$ such that $f(x) < g(x)$. Clearly, $f(x) = 0$ and $g(x) = *$. Then from $\operatorname{Im} \overline{f} \cong \mathbf{F_D}(n)/\ker \overline{f} = \mathbf{F_D}(n)/\ker \overline{g} \cong \operatorname{Im} \overline{g}$ we deduce that $\operatorname{Im} \overline{f} = \operatorname{Im} \overline{g}$. As $0 \in \operatorname{Im} \overline{f}$ and $* \in \operatorname{Im} \overline{g}$, we have $\mathbf{D} = \operatorname{Im} \overline{f} = \operatorname{Im} \overline{g}$. In consequence, there exists $y \in X$ such that $f(y) = *$, and hence $\overline{f}(yxx) = 1 = \overline{f}(1)$, and so $1 = \overline{g}(yxx) = g(y) * * = g(y) = f(y)$, a contradiction.

(3) φ nad φ^{-1} are monotone. Let $f, g \in S(n)$. If $f \prec g$, then $g = e \circ f$. Since e is a homomorphism, we get $\overline{g} = e \circ \overline{f}$, and so $\ker \overline{f} \leq \ker \overline{g}$. Conversely, assume that $\ker \overline{f} < \ker \overline{g}$. From (O1) we have $\operatorname{Im} \overline{g} \cong \mathbf{F_D}(n)/\ker \overline{g} \cong \mathbf{2}$ and $\operatorname{Im} \overline{f} \cong \mathbf{F_D}(n)/\ker \overline{f} \cong \mathbf{D}$, and, in consequence, $\operatorname{Im} \overline{f} = \mathbf{D}$. Then there exists $x \in X$ such that $f(x) = *$, and so $e(f(x)) = 1$. Hence $\ker \overline{f} < \ker(\overline{e \circ f}) \in \operatorname{Cm}(\mathbf{F_D}(n))$. Thus, using (O1) we obtain $\ker \overline{g} = \ker(\overline{e \circ f})$, which implies $g = e \circ f$, as required.

(4) φ preserves the partial operations. Let $f, g \in S(n)$, $f \neq g$, and $* \notin \operatorname{Im} f \cup \operatorname{Im} g$. Then $\mathbf{F_D}(n)/\ker \overline{f} \cong \operatorname{Im} \overline{f} = \mathbf{2}$ and $\mathbf{F_D}(n)/\ker \overline{g} \cong \operatorname{Im} \overline{g} = \mathbf{2}$. Moreover, $\ker \overline{f} \bullet \ker \overline{g} = (\ker \overline{f} \div \ker \overline{g})' \in \operatorname{Cm}(\mathbf{F_D}(n))$ and $\operatorname{Cm}(\mathbf{F_D}(n))/(\ker \overline{f} \bullet \ker \overline{g}) \cong \mathbf{2}$. Put $h := \varphi^{-1}(\ker \overline{f} \bullet \ker \overline{g})$. For $x \in X$ we have $h(x) = 1$ iff $(x, 1) \in \ker \overline{f} \bullet \ker \overline{g}$ iff $(f(x) = 1$ and $g(x) = 1)$ or $(f(x) = 0$ and $g(x) = 0)$ iff $(f \cdot g)(x) = 1$. Thus $\ker \overline{f} \bullet \ker \overline{g} = \varphi(f \cdot g)$, which completes the proof. □

From the above theorem we get the following corollaries.

Corollary 3.2. $|\operatorname{Cm}(\mathbf{F_D}(n))| = 3^n - 1$.

Corollary 3.3.

1. $\overline{L} = \{\ker \overline{f} : f \in S(n) \text{ and } * \notin \operatorname{Im} f\}$;

   $\underline{L} = \{\ker \overline{f} : f \in S(n) \text{ and } \{0, *\} \subseteq \operatorname{Im} f\}$;

   $P = \{\ker \overline{f} : f \in S(n) \text{ and } 0 \notin \operatorname{Im} f\}$.

Moreover, for $f, g \in S(n)$ we have

2. $\ker \overline{f} \leq \ker \overline{g}$ if and only if $\{0, *\} \subseteq \operatorname{Im} f$, $* \notin \operatorname{Im} g$ and $f^{-1}(\{*, 1\}) = g^{-1}(\{1\})$;
3. the Boolean operation on $\overline{L} \cup \{\mathbf{1}_{\mathbf{F_D}(n)}\}$ is defined by $(1, x) \in \ker \overline{f} \bullet \ker \overline{g}$ if and only if $f(x) = g(x)$ for $x \in X$.

Theorem 3.1 allows us to identify elements from $\mathrm{Cm}(\mathbf{F_D}(n))$ with maps f from X to D. In the diagram, we will label these maps by the set of generators belonging to the kernel of f.

Observe that the construction of the frame $\mathrm{Cm}(\mathbf{F_D}(n))$ is similar to the construction of the frame of the equivalential algebras with conjunction on the regular elements, described in [4]. The number of elements of the frame is the same in both cases, but the equivalence classes of relation $\sim$ are different.

This construction proceeds as follows:

1. Each $\mu \in \mathrm{Cm}(\mathbf{F_D}(n))$ is labelled by the set of indices $\{i : x_i \in X \cap (1/\mu)\} \subseteq \{1, \ldots, n\}$.
2. $\overline{L}$ has $2^n - 1$ elements labelled by all proper subsets of $\{1, \ldots, n\}$ and these elements form only one equivalence class.
3. P has $2^n - 1$ elements labelled by all proper subsets of $\{1, \ldots, n\}$, but in this case each element forms a one-element equivalence class.
4. If $\mu \in \overline{L}$ is labelled by $S \subsetneq \{1, \ldots, n\}$, so below μ (i. e. in $\underline{L}$) there are elements labelled by all proper subsets of S.
5. Each $\mu \in \underline{L}$ forms a one-element equivalence class.

We will also use the following designations in the figures:

1. Each dot denotes an element of the frame.
2. Straight lines denote a partial ordering directed upwards.
3. The equivalence class with more than one element is marked with an ellipse.
4. Each dot that does not lie in an ellipse denotes a one-element equivalence class.

3.1. $\mathbf{F_D}(2)$

$\mathrm{Cm}(\mathbf{F_D}(2))$ has 8 elements (Fig. 1): So, there are 9 hereditary sets on the left side and 8 hereditary sets on the right side. Finally, $|\mathbf{F_D}(2)| = 9 \cdot 8 = 72$.

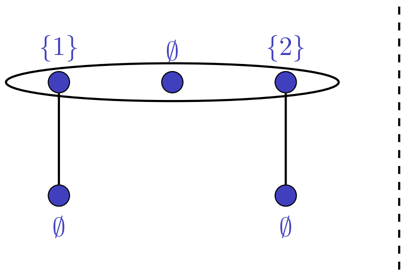

Figure 1. $\mathrm{Cm}(\mathbf{F_D}(2))$

3.2. $\mathbf{F_D}(3)$

$\mathrm{Cm}(\mathbf{F_D}(3))$ has 26 elements (Fig. 2): On the left side there are 4536 hereditary sets, and on the right side there are 128 hereditary sets. Consequently, $|\mathbf{F_D}(3)| = 4536 \cdot 128 = 580608$.

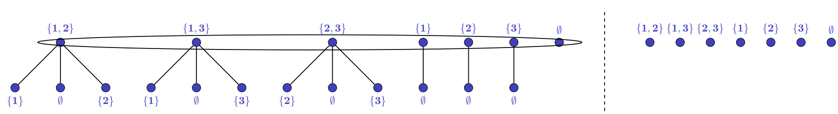

Figure 2. $\mathrm{Cm}(\mathbf{F_D}(3))$

3.3. Direct indecomposability of $\mathbf{F_D}(n)$

Let us start from the following observation.

PROPOSITION 3.4. Let $g \equiv 0 \in S(n)$. Then $\ker \overline{g}$ is the only minimal element of $(\mathrm{Cm}(\mathbf{F_D}(n)), \leq)$ lying in $\overline{L}$.

PROOF: Assume to the contrary that $f \in S(n)$ and $\ker \overline{f} < \ker \overline{g}$. Then $* \in \mathrm{Im}\, f$ and $g^{-1}(\{1\}) = f^{-1}(\{*, 1\}) \neq \emptyset$, a contradiction.

To prove the uniqueness take $h \in S(n)$ such that $* \notin \mathrm{Im}\, h$, and $h \neq g$. Take $x_0 \in X$ such that $h(x_0) = 1$. Define $f_0 : X \to D$ given by $f_0(x) = h(x)$ for $x \neq x_0$ and $f_0(x) = *$ for $x = x_0$. Since $0 \in \mathrm{Im}\, h$, we get $\{0,*\} \subseteq \mathrm{Im}\, f_0$ and $f_0^{-1}(\{*,1\}) = h^{-1}(\{1\})$. From Corollary 3.3 we obtain $\ker \overline{f_0} < \ker \overline{h}$. $\square$

Example 3.5.

1. $|X| = 1$. Then $\mathrm{Cm}(\mathbf{F_D}(1)) = L \cup P$, where $L := \{\ker \overline{f} : f(x_1) = 0\}$ and $P := \{\ker \overline{f} : f(x_1) = *\}$. From Proposition 2.4 we get $\mathbf{F_D}(1) \cong \mathbf{2} \times \mathbf{2}^{\wedge}$.
2. $|X| = 2$. From Fig. 1 we see that $\underline{L} = \{\mu_1, \mu_2\}$, $\mu_1 \leq \ker \overline{f}$, where $f(x_1) = 1$, $f(x_2) = 0$, and $\mu_2 \leq \ker \overline{g}$, where $g(x_1) = 0$, $g(x_2) = 1$. Then $\mu_1 \wedge \mu_2 \leq \ker \overline{f} \wedge \ker \overline{g} \leq \ker \overline{f} \bullet \ker \overline{g} = \ker \overline{h}$, where $h(x_1) = 0$, $h(x_2) = 0$. Hence $\bigwedge L = \mu_1 \wedge \mu_2$. Moreover $\ker \overline{f} < \mu_1 \vee \mu_2$, and so $\mu_1 \vee \mu_2 = \mathbf{1}_{\mathbf{F_D}(2)}$. Thus $\mathbf{F_D}(2)/\bigwedge L \cong \mathbf{F_D}(2)/\mu_1 \times \mathbf{F_D}(2)/\mu_2 \cong \mathbf{D} \times \mathbf{D}$, and finally, from [Proposition 2.4] $\mathbf{F_D}(2) \cong \mathbf{D}^2 \times (\mathbf{2}^{\wedge})^3$.

Unfortunately, for $n \geq 3$, the situation is not so easy, since $\mathbf{F_D}(n)/\bigwedge L$ is not directly decomposable.

THEOREM 3.6. *$\mathbf{F_D}(n)/\bigwedge L$ is directly indecomposable for $n \geq 3$.*

PROOF: Let $n \geq 3$. For contradiction assume that $\mathbf{F_D}(n)/\bigwedge L$ is directly decomposable. Then, since $\mathrm{Con}(\mathbf{F_D}(n)/\bigwedge L) = \{\varphi/\bigwedge L : \varphi \in \mathrm{Con}\,\mathbf{F_D}(n)$ and $\bigwedge L \leq \varphi\}$, we can find $\alpha_1, \alpha_2 \in \mathrm{Con}\,\mathbf{F_D}(n)$ such that $\bigwedge L < \alpha_i$ for $i = 1,2$, $\bigwedge L = \alpha_1 \wedge \alpha_2$ and $\alpha_1 \vee \alpha_2 = \mathbf{1}_{\mathbf{F_D}(n)}$. For $i = 1,2$ define $M(\alpha_i) := \{\mu \in \mathrm{Cm}(\mathbf{F_D}(n)) : \alpha_i \leq \mu\}$. First, we show that $M(\alpha_i) \cap L \neq \emptyset$. For this purpose, we deduce from properties of α_i ($i = 1,2$) that $\alpha_i \neq \mathbf{1}_{\mathbf{F_D}(n)}$. Thus $M(\alpha_i) \neq \emptyset$ and if $M(\alpha_i) \cap L = \emptyset$, then we would find $\mu \in M(\alpha_i)$ such that $\mu \in P$, and so $\bigwedge L < \mu$. From [5, Proposition 5.4] we know that $|\mu/\sim| = 1$, and this means, by [1, Lemma 22], that there exists $\gamma \in L$ such that $\gamma \leq \mu$, which contradicts (O1). Thus, we get $M(\alpha_i) \cap L \neq \emptyset$, and, consequently, $M(\alpha_i) \cap \overline{L} \neq \emptyset$.

Moreover, using [8, Theorem 4.4.(1)], we deduce that $M(\alpha_i) \cap (\overline{L} \cup \{\mathbf{1}_{\mathbf{F_D}(n)}\})$ ($i = 1,2$) is a Boolean subgroup of $(\overline{L} \cup \{\mathbf{1}_{\mathbf{F_D}(n)}\}, \bullet)$. Boolean groups can be treated as vector spaces over $\mathbb{Z}_2$. We show that $\overline{L} \cup \{\mathbf{1}_{\mathbf{F_D}(n)}\}$ can be split as the direct sum of its vector subspaces $M(\alpha_i) \cap (\overline{L} \cup \{\mathbf{1}_{\mathbf{F_D}(n)}\})$ ($i = 1,2$). Firstly, we observe that $M(\alpha_1) \cap M(\alpha_2) = \emptyset$, since otherwise there exists $\mu \in M(\alpha_1) \cap M(\alpha_2)$, and so $\mathbf{1}_{\mathbf{F_D}(n)} = \alpha_1 \vee \alpha_2 \leq \mu$, a contradiction. Take now $\mu \in \overline{L}$ such that $\mu \notin M(\alpha_1) \cup M(\alpha_2)$. Then, applying

[5, Proposition 5.5] and [1, Lemma 22] again, by $\alpha_1 \wedge \alpha_2 \leq \mu$ we deduce that there exist $\mu_1, \mu_2 \in \overline{L}$ such that $\alpha_1 \leq \mu_1$, $\alpha_2 \leq \mu_2$ and $\mu_1 \wedge \mu_2 \leq \mu$. Now from [8, Lemma 3.10] we get that $\mu = \mu_1 \bullet \mu_2$, $\mu_1 \in M(\alpha_1)$, and $\mu_2 \in M(\alpha_2)$. Hence $M(\alpha_i) \cap (\overline{L} \cup \{\mathbf{1}_{\mathbf{F_D}(n)}\})$ $(i = 1, 2)$ form a direct sum equal $\overline{L} \cup \{\mathbf{1}_{\mathbf{F_D}(n)}\}$ of dimension n. Hence $\left|\{\mu \in \overline{L} : \mu \notin M(\alpha_1) \cup M(\alpha_2)\}\right| > 2$ for $n \geq 3$. We know from Proposition 3.4 that in $\overline{L}$ there is a unique minimal element of $(\mathrm{Cm}(\mathbf{F_D}(n)), \leq)$. Take $\mu \in \overline{L}$ such that $\alpha_1 \nleq \mu$, $\alpha_2 \nleq \mu$, and μ is not minimal in $(\mathrm{Cm}(\mathbf{F_D}(n)), \leq)$. Then there is $\gamma \in \underline{L}$ such that $\gamma < \mu$. As $\alpha_1 \wedge \alpha_2 \leq \gamma$ and, by [5, Proposition 5.4], $|\gamma/_{\sim}| = 1$, we obtain, using [1, Lemma 22] again, $\alpha_1 \leq \gamma$ or $\alpha_2 \leq \gamma$. Thus $\alpha_1 \leq \mu$ or $\alpha_2 \leq \mu$, a contradiction. □

4. Free spectrum

In this section we compute the cardinality of the free algebras in $\mathbf{F_D}(n)$, which is a laborious task. However, it is finally possible to find the explicit formula on the free spectrum.

From the definition of L and P, property O1 and Definition 2.5 it follows that:

Proposition 4.1.

$$|\mathbf{F_D}(n)| = |\mathcal{H}(L)| \cdot |\mathcal{H}(P)|,$$

where $\mathcal{H}(L) := \{Z \cap L : Z \in \mathcal{H}(\mathbf{F_D}(n))\}$ and $\mathcal{H}(P) := \{Z \cap P : Z \in \mathcal{H}(\mathbf{F_D}(n))\}$.

We first compute the right factor of this product.

Proposition 4.2. Let $P = \{\mu \in \mathrm{Cm}(\mathbf{F_D}(n)) : \mathbf{F_D}(n)/\mu \cong \mathbf{2}^{\wedge}\}$. Then:

$$|\mathcal{H}(P)| = 2^{2^n - 1}.$$

Proof: The set P contains $2^n - 1$ elements and every subset of the P is a hereditary set. Therefore, the number of the hereditary sets is equal to $2^{2^n - 1}$. □

Next, we compute the left factor. For this, we will use the following lemma. This fact has been used in the proof of [6, Theorem 10]. However, it was given without proof. We will denote by $P(n)$ the family of all subsets of the set $\{1, \ldots, n\}$.

LEMMA 4.3. *[6, p. 1352] The map*

$$S : P(n) \ni A \to S(A) := \{C \in P(n) : |A \setminus C| \ \ \textit{is even}\}$$

gives a one-to-one correspondence between $P(n)$ *and* $\{H \subseteq P(n) : (H, \bullet)$ *is a hyperplane* $(P(n), \bullet)$ *or* $H = P(n)\}$, *where* $(P(n), \bullet)$ *is a Boolean group with the operation* $\bullet$ *defined as follows:* $B \bullet C := (B \div C)'$ *for* $B, C \in P(n)$.

PROOF: We give only the main ideas of the proof (we will skip the tedious details). First, we note that $(S(A), \bullet)$ is a subgroup $(P(n), \bullet)$ for $A \in P(n)$. This is because $\{1, \ldots, n\} \in S(A)$ is a neutral element of$(P(n), \bullet)$ and it is easy to check (by considering parity) that $C_1 \bullet C_2 \in S(A)$ for $C_1, C_2 \in S(A)$.

In the same manner we can see that if D_1, $D_2 \notin S(A)$, so $D_1 \bullet D_2 \in S(A)$. Therefore, if $A \neq \emptyset$, so $S(A)$ is a maximal subgroup of $(P(n), \bullet)$ (if $A = \emptyset$, so it is obvious that $S(A) = P(n)$).

It remains to prove that S is bijective. Since the sets $\{H \subseteq P(n) : H$ jest is a hyperplane or $H = P(n)\}$ and $P(n)$ have the same cardinality (equal to 2^n), thus it is sufficient to prove that S is injective. Let $A, B \in P(n)$ such that $A \neq B$. We give the proof only for the case $|A|$—is odd, $|B|$—is odd; the other cases are left to the reader. Thus there exists $x \in P(n)$, such that $x \in A$ and $x \notin B$ (or, conversely). Then $|A \setminus \{x\}|$ is even, so $\{x\} \in S(A)$ and $|B \setminus \{x\}|$ is odd. In consequence, $\{x\} \notin S(B)$. Therefore $S(A) \neq S(B)$. □

Next, we use [6, Theorem 10.1], which we adapted to our case.

THEOREM 4.4. *We have*

$$|\mathcal{H}(L)| = \sum_{k=0}^{n} \binom{n}{k} \prod_{m=1}^{n-1} (l(m))^{\alpha_k(n,n-m)} \tag{4.1}$$

where

$$\alpha_k(n,j) := \sum_{p} \binom{n-k}{j-2p} \binom{k}{2p} \tag{4.2}$$

for $p \in \mathbb{N}$, *such that* $\max(0, k+j-n) \leq 2p \leq \min(k,j)$ *for* $k, j \in \mathbb{N}$, $0 \leq k \leq n$ *and* $1 \leq j \leq n$, *where*

$$l(m) := 2^{2^m - 1}.$$

Proof: Let $A \in P(n)$. Therefore, from Lemma 4.3 it follows that $(S(A), \bullet)$ is hyperplane in $(P(n), \bullet)$ or $(S(A), \bullet)$=$(P(n), \bullet)$. Write $\mathcal{H}(A) := \{Z \in \mathcal{H}(L) : Z \cap \overline{L} = S(A)\}$. Then $\mathcal{H}(L) = \bigcup\{\mathcal{H}(A) : A \in P(n)\}$. Similarly to [6, p. 1352], every $\mathcal{H}(A)$ can be identified with the Cartesian product of the family subsets of C, such that $C \in S(A) \setminus \{1, \ldots, n\}$. However, in our case the number of such subsets is $2^{2^m - 1}$, where $m = |C|$.

From this we deduce that:

$$|\mathcal{H}(L)| = \sum_{A \in P(n)} |\mathcal{H}(A)| =$$

$$\sum_{A \in P(n)} \prod_{m=0}^{n-1} l(m)^{|\{C \in S(A) : |C| = m\}|}.$$

Next, note that if $|A| = |B|$, so:

$$|\{C \in S(A) : |C| = m\}| = |\{C \in S(B) : |C| = m\}|,$$

for $A, B \in P(n)$. Therefore:

$$|\mathcal{H}(L)| = \sum_{k=0}^{n} \binom{n}{k} \prod_{m=1}^{n-1} l(m)^{\beta_k(n,m)},$$

where

$$\beta_k(n, m) := |\{C \in S(A) : |C| = m\}|,$$

for $A \in P(n)$, such that $|A| = k$.

Now, we calculate $\beta_k(n, m)$. If $C \in S(A)$ and $|C| = m$, so $|A \setminus C| = 2p$, for $p \in \mathbb{N}$. Since $|A| = k$, so $|A \cap C| = k - 2p$. Thus $|C \setminus A| = m - (k - 2p)$. Consequently, $|\{C \in S(A) : |C| = m\}| = \binom{k}{2p} \cdot \binom{n-k}{m-(k-2p)}$. Finally we get that

$$\begin{aligned}\beta_k(n,m) &= \sum_p \binom{n-k}{m-(k-2p)}\binom{k}{2p}\\ &= \sum_p \binom{n-k}{(n-k)-(m-(k-2p))}\binom{k}{2p}\\ &= \sum_p \binom{n-k}{n-m-2p}\binom{k}{2p},\end{aligned}$$

where $p \in \mathbb{N}$, such that $\max(0, k-m) \le 2p \le \min(k, n-m)$, for $k, m \in \mathbb{N}$, $0 \le k \le n$ and $0 \le m \le n-1$. Taking $\alpha_k(n,j) := \beta_k(n, n-j)$ (then $j = n-m$) we get (4.1). □

To get the explicit formula, we use the following Lemmas.

LEMMA 4.5 ([6, Proposition 11]). *The functions α_k ($k \in \mathbb{N}$) fulfill:*

1. *The recurrence equation*

$$\alpha_k(n+1, j) = \alpha_k(n,j) + \alpha_k(n, j-1),$$

for $n \ge k$ and $1 \le j \le n$.

2. *The boundary conditions:*

$$\alpha_k(k,j) = \begin{cases} \binom{k}{j}, & j - \textit{even} \\ 0, & j - \textit{odd} \end{cases} \quad : 0 \le j \le k.$$

3.
$$\alpha_k(n,n) = \begin{cases} 1, & k - \textit{even} \\ 0, & k - \textit{odd} \end{cases} \quad : n \ge k.$$

4.
$$\alpha_k(n, 0) = 1 : n \ge k.$$

LEMMA 4.6. *[6, Lemma 12] Let $n, k \in \mathbb{N}$ and $n \ge k$. Let us consider the generating functions for the coefficients $\alpha_k(n,j)$ ($j = 0, \dots, n$) given by*

$$t_{n,k}(z) := \sum_{j=0}^{n} \alpha_k(n,j) z^j.$$

Then:

$$t_{k,k}(z) = \sum_{\substack{j=0 \\ j-parzyste}}^{k} \binom{k}{j} z^j \tag{4.3}$$

where

$$t_{n,k}(z) = (z+1)^{n-k} t_{k,k}(z). \tag{4.4}$$

Next, we prove the following Lemma.

LEMMA 4.7. *Let $k, j \in \mathbb{N}$. Then:*

1)

$$\sum_{\substack{j=0 \\ j-even}}^{k} \binom{k}{j} 2^{k-j} = \frac{3^k+1}{2},$$

2)

$$\sum_{\substack{j=0 \\ j-even}}^{k} \binom{k}{j} = 2^{k-1}.$$

PROOF: Ad. 1. Let:

$$a := \sum_{\substack{j=0 \\ j-even}}^{k} \binom{k}{j} 2^{k-j} \text{ and } b := \sum_{\substack{j=0 \\ j-odd}}^{k} \binom{k}{j} 2^{k-j}.$$

Then $a + b = \sum_{j=0}^{k} \binom{k}{j} 2^{k-j} \cdot 1^j = (2+1)^k = 3^k$.
In turn

$$a - b = \sum_{\substack{j=0 \\ j-even}}^{k} \binom{k}{j} 2^{k-j} - \sum_{\substack{j=0 \\ j-odd}}^{k} \binom{k}{j} 2^{k-j} = \sum_{j=0}^{k} \binom{k}{j} 2^{k-j} \cdot (-1)^j = (2-1)^k = 1.$$

From the system of equations: $a + b = 3^k$ and $a - b = 1$ we get $a = \frac{3^k+1}{2}$.

Ad. 2. Let:

$$a := \sum_{\substack{j=0 \\ j-even}}^{k} \binom{k}{j} \text{ and } b := \sum_{\substack{j=0 \\ j-odd}}^{k} \binom{k}{j}.$$

Hence $a + b = \sum_{j=0}^{k} \binom{k}{j} = 2^k$,
and

$$a - b = \sum_{\substack{j=0 \\ j-even}}^{k} \binom{k}{j} - \sum_{\substack{j=0 \\ j-odd}}^{k} \binom{k}{j}$$

$$= \sum_{j=0}^{k} \binom{k}{j} 1^{k-j} \cdot (-1)^j = (1-1)^k = 0.$$

From the system of equations: $a + b = 2^k$ and $a - b = 0$ it follows that $a = 2^{k-1}$. □

Now, we prove the following result:

Corollary 4.8.

$$t_{n,k}(1) = \begin{cases} 2^n & k = 0, \\ 2^{n-1} & k \neq 0. \end{cases} \tag{4.5}$$

Proof: From (4.4) and then from (4.3), we have for $k = 0$:

$$t_{n,0}(1) = 2^n t_{0,0}(1) = 2^n \sum_{\substack{j=0 \\ j-even}}^{0} \binom{0}{j} 1^j = 2^n.$$

Now, let $k \neq 0$:

$$t_{n,k}(1) = 2^{n-k} t_{k,k}(1) = 2^{n-k} \sum_{\substack{j=0 \\ j-even}}^{k} \binom{k}{j} 1^j.$$

Lemma 4.7 shows that

$$\sum_{\substack{j=0 \\ j-even}}^{k} \binom{k}{j} = 2^{k-1}.$$

Hence

$$t_{n,k}(1) = 2^{n-k} \cdot 2^{k-1} = 2^{n-k+k-1} = 2^{n-1}.$$ □

Finally we prove the following theorem.

THEOREM 4.9.

$$|\mathcal{H}(L)| = 2^{3^n-2^{n+1}+1} + (2^{-2^{n-1}-2^n+1})\sum_{k=1}^{n}\binom{n}{k}2^{\frac{3^{n-k}(3^k+1)}{2}}.$$

PROOF: From (4.1) it follows that:

$$|\mathcal{H}(L)| = \sum_{k=0}^{n}\binom{n}{k}\prod_{m=1}^{n-1}(2^{2^m-1})^{\alpha_k(n,n-m)}. \tag{4.6}$$

Replacing $n-m$ by j, we get:

$$|\mathcal{H}(L)| = \sum_{k=0}^{n}\binom{n}{k}2^{\sum_{j=1}^{n-1}(2^{n-j}-1)\alpha_k(n,j)}. \tag{4.7}$$

We calculate separately the above exponent.
Let $W_k := \sum_{j=1}^{n-1}(2^{n-j}-1)\alpha_k(n,j)$. Then:

$$\begin{aligned}W_k = \sum_{j=1}^{n-1}(2^{n-j}-1)\alpha_k(n,j) &= \sum_{j=1}^{n-1}2^{n-j}\alpha_k(n,j) - \sum_{j=1}^{n-1}\alpha_k(n,j)\\ &= \sum_{j=0}^{n}2^{n-j}\alpha_k(n,j) - 2^{n-0}\alpha_k(n,0) - 2^{n-n}\alpha_k(n,n)\\ &\quad -(\sum_{j=0}^{n}\alpha_k(n,j) - \alpha_k(n,0) - \alpha_k(n,n))\\ &= \sum_{j=0}^{n}2^{n-j}\alpha_k(n,j) - 2^n\alpha_k(n,0) - \alpha_k(n,n)\\ &\quad -\sum_{j=0}^{n}\alpha_k(n,j) + \alpha_k(n,0) + \alpha_k(n,n).\end{aligned}$$

Simplifying and applying Lemma 4.5(4), we get:

$$W_k = \sum_{j=0}^{n}2^{n-j}\alpha_k(n,j) - \sum_{j=0}^{n}\alpha_k(n,j) - 2^n + 1. \tag{4.8}$$

We now compute the first sum in (4.8). We denote it by S_k. Then

$$S_k = \sum_{j=0}^{n} 2^{n-j}\alpha_k(n,j) = \sum_{j=0}^{n} 2^n \cdot 2^{-j}\alpha_k(n,j) = 2^n \sum_{j=0}^{n} 2^{-j}\alpha_k(n,j).$$

On account of Lemma 4.6, we have:

$$S_k = 2^n \cdot t_{n,k}(\frac{1}{2}) = 2^n \cdot (\frac{1}{2}+1)^{n-k} \cdot t_{k,k}(\frac{1}{2}) = 2^n \frac{3^{n-k}}{2^{n-k}} \sum_{\substack{j=0 \\ j-even}}^{k} \binom{k}{j} 2^{-j} =$$

$$3^{n-k} \cdot 2^k \sum_{\substack{j=0 \\ j-even}}^{k} \binom{k}{j} 2^{-j} = 3^{n-k} \sum_{\substack{j=0 \\ j-even}}^{k} \binom{k}{j} 2^{k-j}.$$

From Lemma 4.7 we conclude that

$$S_k = 3^{n-k}\frac{3^k+1}{2}.$$

It follows from Lemma 4.6 that the second sum in (4.8) is equal to $t_{n,k}(1)$. Hence:

$$W_k = 3^{n-k}\frac{3^k+1}{2} - t_{n,k}(1) - 2^n + 1.$$

Applying Corollary 4.8 and Lemma 4.6 we deduce that:

$$W_0 = 3^n - 2^{n+1} + 1,$$

In turn for $k \neq 0$ we get:

$$W_k = 3^{n-k}\frac{3^k+1}{2} - 2^{n-1} - 2^n + 1.$$

We can now return to (4.7). We get:

$$|\mathcal{H}(L)| = 2^{3^n-2^{n+1}+1} + \sum_{k=1}^{n} \binom{n}{k} 2^{\frac{3^{n-k}(3^k+1)}{2} - 2^{n-1} - 2^n + 1} =$$

$$2^{3^n-2^{n+1}+1} + (2^{-2^{n-1}-2^n+1}) \sum_{k=1}^{n} \binom{n}{k} 2^{\frac{3^{n-k}(3^k+1)}{2}}.$$

□

We can now formulate our main result.

Theorem 4.10. *Let $n \in \mathbb{N}$. Then*

$$|\mathbf{F_D}(n)| = 2^{3^n - 2^n} + \sum_{k=1}^{n} \binom{n}{k} 2^{\frac{3^n + 3^{n-k}}{2} - 2^{n-1}}.$$

Proof: Combining Theorem 4.9 with Proposition 4.2 we deduce that

$$|\mathbf{F_D}(n)| = 2^{2^n - 1}(2^{3^n - 2^{n+1} + 1} + (2^{-2^{n-1} - 2^n + 1}) \sum_{k=1}^{n} \binom{n}{k} 2^{\frac{3^{n-k}(3^k+1)}{2}}) =$$

$$2^{3^n - 2^{n+1} + 1 + 2^n - 1} + (2^{-2^{n-1} - 2^n + 1 + 2^n - 1}) \sum_{k=1}^{n} \binom{n}{k} 2^{\frac{3^{n-k}(3^k+1)}{2}} =$$

$$2^{3^n - 2^n} + 2^{-2^{n-1}} \sum_{k=1}^{n} \binom{n}{k} 2^{\frac{3^{n-k}(3^k+1)}{2}} = 2^{3^n - 2^n} + \sum_{k=1}^{n} \binom{n}{k} 2^{\frac{3^n + 3^{n-k}}{2} - 2^{n-1}}.$$

□

Corollary 4.11. Let $n \in \mathbb{N}$. Then $|\mathbf{F_D}(n)|$ is asymptotically equal to $2^{3^n - 2^n}$.

Proof: According to the above theorem, it is sufficient to show that:

$$\frac{\sum_{k=1}^{n} \binom{n}{k} 2^{\frac{3^n + 3^{n-k}}{2} - 2^{n-1}}}{2^{3^n - 2^n}} \xrightarrow{n \to +\infty} 0.$$

First observe that:

$$0 \leq \frac{\sum_{k=1}^{n} \binom{n}{k} 2^{\frac{3^n + 3^{n-k}}{2} - 2^{n-1}}}{2^{3^n - 2^n}} \leq \frac{2^n \cdot 2^{\frac{3^n + 3^{n-1}}{2} - 2^{n-1}}}{2^{3^n - 2^n}} \leq \frac{2^{n + \frac{3^n + 3^{n-1}}{2} - 2^{n-1}}}{2^{3^n - 2^n}}.$$

We next show that: $\frac{2^{n + \frac{3^n + 3^{n-1}}{2} - 2^{n-1}}}{2^{3^n - 2^n}} \xrightarrow{n \to +\infty} 0$.
Since $n + \frac{3^n + 3^{n-1}}{2} - 2^{n-1} = n + \frac{4 \cdot 3^{n-1}}{2} - 2^{n-1} = n + 2 \cdot 3^{n-1} - 2^{n-1}$, it follows that:

$$\begin{aligned} \frac{2^{n + \frac{3^n + 3^{n-1}}{2} - 2^{n-1}}}{2^{3^n - 2^n}} &= \frac{2^{n + 2 \cdot 3^{n-1} - 2^{n-1}}}{2^{3^n - 2^n}} \\ &= 2^{n + 2 \cdot 3^{n-1} - 2^{n-1} - 3^n + 2^n} \\ &= 2^{n - 3^{n-1} + 2^{n-1}}. \end{aligned}$$

Now that $n - 3^{n-1} + 2^{n-1} = 3^{n-1}(\frac{n}{3^{n-1}} - 1 + (\frac{2}{3})^{n-1}) \xrightarrow{n\to+\infty} -\infty$, we have

$$\frac{2^{n+\frac{3^n+3^{n-1}}{2}-2^{n-1}}}{2^{3^n-2^n}} \xrightarrow{n\to+\infty} 0,$$

and the proof is complete. □

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