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Tarek Sayed Ahmed

LIFTING RESULTS FOR FINITE DIMENSIONS TO THE TRANSFINITE IN SYSTEMS OF VARIETIES USING ULTRAPRODUCTS

Abstract

We redefine a system of varieties definable by a schema of equations to include finite dimensions. Then we present a technique using ultraproducts enabling one to lift results proved for every finite dimension to the transfinite. Let **Ord** denote the class of all ordinals. Let $\langle \mathbf{K}_\alpha : \alpha \in \mathbf{Ord} \rangle$ be a system of varieties definable by a schema. Given any ordinal α, we define an operator Nr_α that acts on $\mathbf{K}_\beta$ for any $\beta > \alpha$ giving an algebra in $\mathbf{K}_\alpha$, as an abstraction of taking α-neat reducts for cylindric algebras. We show that for any positive k, and any infinite ordinal α that $\mathbf{S}\mathsf{Nr}_\alpha \mathbf{K}_{\alpha+k+1}$ cannot be axiomatized by a finite schema over $\mathbf{S}\mathsf{Nr}_\alpha \mathbf{K}_{\alpha+k}$ given that the result is valid for all finite dimensions greater than some fixed finite ordinal. We apply our results to cylindric algebras and Halmos quasipolyadic algebras with equality. As an application to our algebraic result we obtain a strong incompleteness theorem (in the sense that validitities are not captured by finitary Hilbert style axiomatizations) for an algebraizable extension of $L_{\omega,\omega}$.

Keywords: algebraic logic, systems of varieties, ultraproducts, non-finite axiomaitizability.

2020 Mathematical Subject Classification: 03G15, 03B45.

1. Introduction

We follow the notation of [1, 2]. Fix $2 < n < \omega$. In [5] Hirsch, Hodkinson and Maddux prove that for any positive $k \geq 1$, $\mathbf{S}\mathsf{Nr}_n \mathsf{CA}_{n+k+1} \subsetneq$

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$\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{n+k}$; in fact this gap between the two varieties cannot be finitely axiomatized [4]. In [3], this result was generalized to other algebras of logic (or cylindric-like algebras) such as Pinter's substitution algebras, Halmos' polyadic algebras with and without equality; all of dimension n and for infinite dimensions for all the aforementioned algebras, together with cylindrc algebras not dealt with in [5]. In [3], the result was proved first for the finite dimensional case, then it was lifted to the transfinite using a lifting technique originating with Monk in proving non-finite axiomatizability of RCA_ω by a finite schema of equations, cf. [2]. The technique involves the use of ultraproducts, to lift results proved for all finite dimensions to the transfinite. Here we show that this technique lends itself to a much wider context.

We generalize this technique to the very general notion of *a system of varieties definable by a schema of equations* introduced by Henkin et al. cf. [2, Definition 5.6.11] to encompass all such aforementioned systems of varieties of algebras and potentially much more. A substantial new addition here is that we allow finite dimensions in our definition of a system of varieties definable by schema. What is basically characteristc of such systems $\langle \mathbf{K}_\alpha : \alpha \in \mathbf{Ord}\rangle$, is that for each ordinal $\alpha \in \mathbf{Ord}$, they define a variety of algebra of dimension α, $\mathbf{K}_\alpha$, such that if $\alpha < \beta$, and $\mathfrak{A} \in \mathbf{K}_\beta$, then the reduct of $\mathfrak{A}$ obtained by discarding the operations indexed by ordinals in β and outside α, call it $\mathfrak{Rd}_\alpha\mathfrak{A}$, is in $\mathbf{K}_\alpha$. Furthermore, one can navigate between various dimensions using more complex operators, like the *neat reduct operator* denoted by Nr_μ, where μ is any ordinal. For $\alpha < \beta$, and $\mathfrak{A} \in \mathbf{K}_\beta$ say, then $\mathsf{Nr}_\alpha\mathfrak{A} \in \mathbf{K}_\alpha$ and $\mathsf{Nr}_\alpha\mathfrak{A} \subseteq \mathfrak{Rd}_\alpha\mathfrak{A}$. Finally, for infinite dimensions the system is captured (defined) uniformy by a single schema of equations. For example the system for $\mathbf{CA} = \langle \mathsf{CA}_\alpha : \alpha \geq \omega\rangle$, the indices i, j in the operations of cylindrifcations and diagonal elements, c_i and d_{ij} $(i, j \in \omega)$ vary according to one finite schema that is finite in a two sorted sense.

2. Preliminarlies

We start with the definition counting in finite dimensions for system of varieties definable by a schema. Counting in finite dimensional algebras is new.

DEFINITION 2.1.

(i) Let $2 \leq m \in \omega$. A finite m type schema is a quadruple $t = (T, \delta, \rho, c)$ such that T is a set, δ and ρ map T into ω, $c \in T$, and $\delta c = \rho c = 1$ and $\delta f \leq m$ for all $f \in T$.

(ii) A type schema as in (i) defines a signature t_n for each $n \geq m$ as follows. The domain T_n of t_n is

$$T_n = \{(f, k_0, \dots, k_{\delta f-1}) : f \in T, k \in {}^{\delta f}n\}.$$

For each $(f, k_0, \dots, k_{\delta f-1}) \in T_n$ we set $t_n(f, k_0, \dots, k_{\delta f-1}) = \rho f$.

(iii) A system $(\mathbf{K}_n : n \geq m)$ of classes of algebras is of type schema t if for each $n \geq m$, $\mathbf{K}_n$ is a class of algebras having signature t_n.

DEFINITION 2.2. Let t be a finite m type schema.

(i) With each $m \leq n \leq \beta$ we associate a language L_n^t in the signature t_n: for each $f \in T$ and $k \in {}^{\delta f}n$, we have a function symbol $f_{k0,\dots,k(\delta f-1)}$ of rank ρf.

(ii) Let $m \leq \beta \leq n$, and let $\eta \in {}^{\beta}n$ be an injection. We associate with each term τ of L_β^t a term $\eta^+\tau$ of L_n^t. For each $\kappa \in \omega, \eta^+ v_\kappa = v_\kappa$. If $f \in T, k \in {}^{\delta f}\alpha$, and $\sigma_1, \dots, \sigma_{\rho f-1}$ are terms of L_β^t, then

$$\eta^+ f_{k(0),\dots,k(\delta f-1)}\sigma_0, \dots, \sigma_{\rho f-1} = f_{\eta(k(0)),\dots,\eta(k(\delta f-1))}\eta^+\sigma_0 \dots \eta^+\sigma_{\rho f-1}.$$

Then we associate with each equation $e := \sigma = \tau$ of L_β^t the equation $\eta^+\sigma = \eta^+\tau$ of L_α^t, which we denote by $\eta^+(e)$. We say that $\eta^+(e)$ is an n instance of e, obtained by applying the injective map η.

(iii) A system $\mathbf{K} = (\mathbf{K}_n : n \geq m)$ of finite m type schema t is a *complete system of varieties definable by a schema*, if there is a system $(\Sigma_n : n \geq m)$ of equations such that $\mathsf{Mod}(\Sigma_n) = \mathbf{K}_n$, and for $n \leq m < \omega$ if $e \in \Sigma_n$ and $\rho : n \to m$ is an injection, then $\rho^+e \in \Sigma_m$; $(\mathbf{K}_\alpha : \alpha \geq \omega)$ is a system of varieties definable by a schema and $\Sigma_\omega = \bigcup_{n \geq m} \Sigma_n$.

DEFINITION 2.3.

(1) Let α, β be ordinals, $\mathfrak{A} \in \mathbf{K}_\beta$ and $\rho : \alpha \to \beta$ be an injection. We assume for simplicity of notation, and without any loss, that in addition to cylindrifiers, we have only one unary function symbol f such that $\rho(f) = \delta(f) = 1$. (The arity is one, and f has only one index.)

Then $\mathfrak{Rd}^\rho_\alpha\mathfrak{A}$ is the α-dimensional algebra obtained for $\mathfrak{A}$ by defining for $i \in \alpha$ $\mathsf{c_i}$ by $\mathsf{c}_{\rho(\mathsf{i})}$, and f_i by $f_{\rho(i)}$. $\mathfrak{Rd}_\alpha\mathfrak{A}$ is $\mathfrak{Rd}^\rho_\alpha\mathfrak{A}$ when ρ is the inclusion map.

(2) As in the first part we assume only the existence of one unary operator with one index. Let $\mathfrak{A} \in \mathbf{K}_\beta$, and $x \in A$. The dimension set of x, denoted by Δx, is the set $\Delta x = \{i \in \alpha : \mathsf{c}_i x \neq x\}$. We assume that if $\Delta x \subseteq \alpha$, then $\Delta f(x) \leq \alpha$. Then $\mathsf{Nr}_\alpha\mathfrak{B}$ is the subuniverse of $\mathfrak{Rd}_\alpha\mathfrak{B}$ consisting only of α dimensional elements.

(3) For $K \subseteq \mathbf{K}_\beta$ and an injection $\rho : \alpha \to \beta$, then $\mathfrak{Rd}^\rho_\alpha K = \{\mathfrak{Rd}^\rho_\alpha\mathfrak{A} : \mathfrak{A} \in K\}$ and $\mathsf{Nr}_\alpha K = \{\mathsf{Nr}_\alpha\mathfrak{A} : \mathfrak{A} \in K\}$.

3. Lifting results to the transfinite using ultraproducts

3.1. Main result

We start with a Definition:

Definition 3.1. Let $(\mathbf{K}_\alpha : \alpha \geq 3)$ be a complete system of varieties definable by a schema. Then for $\alpha \leq \mu \leq \beta$ and $K \subseteq \mathbf{K}_\beta$, $\mathsf{Nr}_\mu K := \{\mathsf{Nr}_\mu\mathfrak{A} : \mathfrak{A} \in K\}$.

The hypothesis in the next theorem presupposes the existence of certain finite dimensional algebras that we know do exist for certain cylindric-like algebras. This will be witnessed in a while, cf. Corollary 3.3. Also, in what follows, the symbol $\mathbf{S}$ stands for the operation of forming subalgebras

Theorem 3.2. *Let $(\mathbf{K}_\alpha : \alpha \geq 3)$ be a complete system of varieties definable by a schema. Assume that for $3 \leq m < n < \omega$, there is an m dimensional algebra $\mathfrak{C}(m,n,r)$ such that*

(1) $\mathfrak{C}(m,n,r) \in \mathbf{S}\mathsf{Nr}_m\mathbf{K}_n$,

(2) $\mathfrak{C}(m,n,r) \notin \mathbf{S}\mathsf{Nr}_m\mathbf{K}_{n+1}$,

(3) $\Pi_{r\in\omega}\mathfrak{C}(m,n,r) \in \mathbf{S}\mathsf{Nr}_m\mathbf{K}_\omega$,

(4) *For $m < n$ and $k \geq 1$, there exists $x_n \in \mathfrak{C}(n, n+k, r)$ such that $\mathfrak{C}(m, m+k, r) \cong \mathfrak{Rl}_{x_n}\mathfrak{C}(n, n+k, r)$.*

Assume that for any $3 < \alpha < \beta$, $\mathsf{SNr}_\alpha\mathbf{K}_\beta$ *is a variety. Then for any ordinal* $\alpha \geq \omega$ *and finite number* $k \geq 1$, *for every ordinal* $l \geq k+1$, $\mathsf{SNr}_\alpha\mathbf{K}_{\alpha+l}$ *is not axiomatizable by a finite schema over* $\mathsf{SNr}_\alpha\mathbf{K}_{\alpha+k}$.

Proof: The proof is divided into 3 parts.

Part 1: Let α be an infinite ordinal. Let X be any finite subset of α and let

$$I = \{\Gamma : X \subseteq \Gamma \subseteq \alpha,\ |\Gamma| < \omega\}.$$

For each $\Gamma \in I$ let $M_\Gamma = \{\Delta \in I : \Delta \supseteq \Gamma\}$ and let F be any ultrafilter over I such that for all $\Gamma \in I$ we have $M_\Gamma \in F$ (such an ultrafilter exists because $M_{\Gamma_1} \cap M_{\Gamma_2} = M_{\Gamma_1 \cup \Gamma_2}$).

For each $\Gamma \in I$ let ρ_Γ be a bijection from $|\Gamma|$ onto Γ. For each $\Gamma \in I$ let $\mathfrak{A}_\Gamma, \mathfrak{B}_\Gamma$ be $\mathbf{K}_\alpha$-type algebras.

We claim that

(*) If for each $\Gamma \in I$ we have $\mathfrak{Rd}^{\rho_\Gamma}\mathfrak{A}_\Gamma = \mathfrak{Rd}^{\rho_\Gamma}\mathfrak{B}_\Gamma$, then we have

$$\Pi_{\Gamma/F}\mathfrak{A}_\Gamma = \Pi_{\Gamma/F}\mathfrak{B}_\Gamma.$$

The proof is standard using Los' theorem.

Indeed, $\Pi_{\Gamma/F}\mathfrak{A}_\Gamma$, $\Pi_{\Gamma/F}\mathfrak{Rd}^{\rho_\Gamma}\mathfrak{A}_\rho$ and $\Pi_{\Gamma/F}\mathfrak{B}_\Gamma$ all have the same universe, by assumption. Also each operator o of $\mathbf{K}_\alpha$ is also the same for both ultraproducts, because $\{\Gamma \in I : \dim(o) \subseteq \mathsf{rng}(\rho_\Gamma)\} \in F$.

Now we claim that

(**) if $\mathfrak{Rd}^{\rho_\Gamma}\mathfrak{A}_\Gamma \in \mathbf{K}_{|\Gamma|}$, for each $\Gamma \in I$, then $\Pi_{\Gamma/F}\mathfrak{A}_\Gamma \in \mathbf{K}_\alpha$. For this, it suffices to prove that each of the defining axioms for $\mathbf{K}_\alpha$ holds for $\Pi_{\Gamma/F}\mathfrak{A}_\Gamma$.

Let $\sigma = \tau$ be one of the defining equations for $\mathbf{K}_\alpha$, the number of dimension variables is finite, say n.

Take any $i_0, i_1, \ldots, i_{n-1} \in \alpha$. We have to prove that

$$\Pi_{\Gamma/F}\mathfrak{A}_\Gamma \models \sigma(i_0, \ldots, i_{n-1}) = \tau(i_0 \ldots, i_{n-1}).$$

Suppose that they are all in $\mathsf{rng}(\rho_\Gamma)$, say $i_0 = \rho_\Gamma(j_0)$, $i_1 = \rho_\Gamma(j_1)$, $\ldots, i_{n-1} = \rho_\Gamma(j_{n-1})$, then $\mathfrak{Rd}^{\rho_\Gamma}\mathfrak{A}_\Gamma \models \sigma(j_0, \ldots, j_{n-1}) = \tau(j_0, \ldots j_{n-1})$, since $\mathfrak{Rd}^{\rho_\Gamma}\mathfrak{A}_\Gamma \in \mathbf{K}_{|\Gamma|}$, so $\mathfrak{A}_\Gamma \models \sigma(i_0, \ldots, i_{n-1}) = \tau(i_0 \ldots, i_{n-1})$.

Hence $\{\Gamma \in I : \mathfrak{A}_\Gamma \models \sigma(i_0, \ldots, i_{n-1}) = \tau(i_0, \ldots, i_{n-1})\} \supseteq \{\Gamma \in I : i_0, \ldots, i_{n-1} \in \mathsf{rng}(\rho_\Gamma\} \in F$. It now easily follows that

$$\Pi_{\Gamma/F}\mathfrak{A}_\Gamma \models \sigma(i_0, \ldots, i_{n-1}) = \tau(i_0, \ldots, i_{n-1}).$$

Thus $\Pi_{\Gamma/F}\mathfrak{A}_\Gamma \in \mathbf{K}_\alpha$, and we are done.

Part 2: Let $k \geq 1$ and $r \in \omega$. Let α, I, F and ρ_Γ be as above and assume the hypothesis of the theorem. Let $\mathfrak{C}^r_\Gamma$ be an algebra similar to $\mathbf{K}_\alpha$ such that

$$\mathfrak{Rd}^{\rho_\Gamma}\mathfrak{C}^r_\Gamma = \mathfrak{C}(|\Gamma|, |\Gamma| + k, r).$$

Let

$$\mathfrak{B}^r = \Pi_{\Gamma/F \in I}\mathfrak{C}^r_\Gamma.$$

Then we have

1. $\mathfrak{B}^r \in \mathbf{S}S\mathsf{Nr}_\alpha\mathbf{K}_{\alpha+k}$ and

2. $\mathfrak{B}^r \notin \mathbf{S}\mathsf{Nr}_\alpha\mathbf{K}_{\alpha+k+1}$.

For each $\Gamma \in I$ let $\mathfrak{C}(|\Gamma|, k)$ be an algebra having the same signature as $\mathbf{K}_{|\Gamma|+k}$ such that $\mathsf{Nr}_{|\Gamma|}\mathfrak{C}(|\Gamma|, k) \cong \mathfrak{C}(|\Gamma|, |\Gamma| + k, r)$. Let σ_Γ be a one to one function $(|\Gamma| + k) \to (\alpha + k)$ such that $\rho_\Gamma \subseteq \sigma_\Gamma$ and $\sigma_\Gamma(|\Gamma| + i) = \alpha + i$ for every $i < k$. Let $\mathfrak{A}_\Gamma$ be an algebra similar to a $\mathbf{K}_{\alpha+k}$ such that $\mathfrak{Rd}^{\sigma_\Gamma}\mathfrak{A}_\Gamma = \mathfrak{C}(|\Gamma|, k)$. By (**) with $\alpha + k$ in place of α, $\{\alpha + i : i < k\}$ in place of X, $\{\Gamma \subseteq \alpha + k : |\Gamma| < \omega,\ X \subseteq \Gamma\}$ in place of I, and with σ_Γ in place of ρ_Γ, we know that $\Pi_{\Gamma/F}\mathfrak{A}_\Gamma \in \mathbf{K}_{\alpha+k}$.

Part 3: Now we prove the third part of the theorem, putting the superscript r to use. Let $l \geq k + 1$, and we can assume that $l \leq \omega$. Recall that $\mathfrak{B}^r = \Pi_{\Gamma/F}\mathfrak{C}^r_\Gamma$, where $\mathfrak{C}^r_\Gamma$ has the type of $\mathbf{K}$ and $\mathfrak{Rd}^{\rho_\Gamma}\mathfrak{C}^r_\Gamma = \mathfrak{C}(|\Gamma|, |\Gamma| + k, r)$. We know that

$$\Pi_{r/U}\mathfrak{Rd}^{\rho_\Gamma}\mathfrak{C}^r_\Gamma = \Pi_{r/U}\mathfrak{C}(|\Gamma|, |\Gamma| + k, r) \subseteq \mathsf{Nr}_{|\Gamma|}\mathfrak{A}_\Gamma,$$

for some $\mathfrak{A}_\Gamma \in \mathbf{K}_{|\Gamma|+\omega}$. Let $\lambda_\Gamma : |\Gamma| + l \to \alpha + l$ extend $\rho_\Gamma : |\Gamma| \to \Gamma\ (\subseteq \alpha)$ and satisfy

$$\lambda_\Gamma(|\Gamma| + i) = \alpha + i$$

for $i < l$. Now in this part, we take the l reduct of $\mathfrak{A}_\Gamma$. Accordingly, let $\mathfrak{F}_\Gamma$ be a $\mathbf{K}_{\alpha+l}$ type algebra such that $\mathfrak{Rd}^{\lambda_\Gamma}\mathfrak{F}_\Gamma = \mathfrak{Rd}_l\mathfrak{A}_\Gamma$. But now as before, $\Pi_{\Gamma/F}\mathfrak{F}_\Gamma \in \mathbf{K}_{\alpha+l}$, and

$$\begin{aligned}\Pi_{r/U}\mathfrak{B}^r &= \Pi_{r/U}\Pi_{\Gamma/F}\mathfrak{C}^r_\Gamma\\ &\cong \Pi_{\Gamma/F}\Pi_{r/U}\mathfrak{C}^r_\Gamma\\ &\subseteq \Pi_{\Gamma/F}\mathsf{Nr}_{|\Gamma|}\mathsf{c}A_\Gamma\\ &= \Pi_{\Gamma/F}\mathsf{Nr}_{|\Gamma|}\mathfrak{Rd}^{\lambda_\Gamma}\mathfrak{F}_\Gamma\\ &\subseteq \mathsf{Nr}_\alpha\Pi_{\Gamma/F}\mathfrak{F}_\Gamma.\end{aligned}$$

We are ready to prove the negative axiomatizability result. It is a Los' argument at heart, modulo some adjustments, because we are dealing with schemes, so that we will not deal with a finite set of equations, but rather α instances of a finite set of equations in the signature of $\mathbf{K}_\omega$. Let $k \geq 1$ and $l \geq k+1$. Assume for contradiction that $\mathbf{S}\mathsf{Nr}_\alpha\mathbf{K}_{\alpha+l}$ is axiomatizable by a finite schema over $\mathbf{S}\mathsf{Nr}_\alpha\mathbf{K}_{\alpha+k}$. We can assume that there is only one equation, such that all its α instances, axiomatize $\mathbf{S}\mathsf{Nr}_\alpha\mathbf{K}_{\alpha+l}$ over $\mathbf{S}\mathsf{Nr}_\alpha\mathbf{K}_{\alpha+k}$. So let σ be an equation in the signature of $\mathbf{K}_\omega$ and let E be its α instances such that for any $\mathfrak{A} \in \mathbf{S}\mathsf{Nr}_\alpha\mathbf{K}_{\alpha+k}$ we have $\mathfrak{A} \in \mathbf{S}\mathsf{Nr}_\alpha\mathbf{K}_{\alpha+l}$ iff $\mathfrak{A} \models E$. Then for all $r \in \omega$, there is an instance of σ, σ_r say, such that $\mathfrak{B}^r$ does not model σ_r. σ_r is obtained from σ by some injective map $\mu_r : \omega \to \alpha$. For $r \in \omega$, let $v_r \in {}^\alpha\alpha$, be an injection such that $\mu_r(i) = v_r(i)$ for $i \in \mathsf{ind}(\sigma_r)$, and let $\mathfrak{A}_r = \mathfrak{Rd}^{v_r}\mathfrak{B}^r$. Now $\Pi_{r/U}\mathfrak{A}_r \models \sigma$. But then

$$\{r \in \omega : \mathfrak{A}_r \models \sigma\} = \{r \in \omega : \mathfrak{B}^r \models \sigma_r\} \in U,$$

contradicting that $\mathfrak{B}^r$ does not model σ_r for all $r \in \omega$. □

3.2. Applications

In this section we lift results proved for all finite dimensions to the transfinite using ultraproducts. Let α be an ordinal. The next result is new:

Corollary 3.3. For any ordinal $\alpha \geq \omega$, any positive $k \geq 1$, and any ordinal $l \geq k+1$, the variety $\mathbf{S}\mathsf{Nr}_\alpha\mathsf{CA}_{\alpha+l}$ is not axiomatizable by a finite schema over the variety $\mathbf{S}\mathsf{Nr}_\alpha\mathsf{CA}_{\alpha+k}$.

Proof: Fix $2 < m < n < \omega$. Let $\mathfrak{C}(m,n,r)$ be the algebra $\mathfrak{Ca}(\mathbf{H})$ where $\mathbf{H} = H_m^{n+1}(\mathfrak{A}(n,r),\omega))$, is the CA_m atom structure consisting of all $n+1$–wide m–dimensional wide ω hypernetworks [4, Definition 12.21] on $\mathfrak{A}(n,r)$ as defined in [4, Definition 15.2]. Then $\mathfrak{C}(m,n,r) \in \mathsf{CA}_m$. Then for any $r \in \omega$ and $3 \leq m \leq n < \omega$, $\mathfrak{C}(m,n,r) \in \mathsf{Nr}_m\mathsf{CA}_n$, $\mathfrak{C}(m,n,r) \notin \mathbf{S}\mathsf{Nr}_m\mathsf{CA}_{n+1}$

and $\Pi_{r/U}\mathfrak{C}(m,n,r) \in \mathsf{RCA}_m$, cf. [4, Corollaries 15.7, 5.10, Exercise 2, pp. 484, Remark 15.13]. Take

$$x_n = \{f \in H_n^{n+k+1}(\mathfrak{A}(n,r),\omega); m \leq j < n \to \exists i < m, f(i,j) = \mathsf{Id}\}.$$

Then $x_n \in \mathfrak{C}(n, n+k, r)$ and $\mathsf{c}_i x_n \cdot \mathsf{c}_j x_n = x_n$ for distinct $i, j < m$. Furthermore (*), $I_n : \mathfrak{C}(m, m+k, r) \cong \mathfrak{Rl}_{x_n}\mathfrak{Rd}_m\mathfrak{C}(n, n+k, r)$ via the map, defined for $S \subseteq H_m^{m+k+1}(\mathfrak{A}(m+k,r),\omega))$, by

$$I_n(S) = \{f \in H_n^{n+k+1}(\mathfrak{A}(n,r),\omega) : f \restriction {}^{\leq m+k+1}m \in S,$$
$$\forall j(m \leq j < n \to \exists i < m, f(i,j) = \mathsf{Id})\}.$$

Applying Theorem 3.2 we get the required. □

We let QEA_α stand for the class of quasipolyadic equality algebras of dimension α as defined in [2]. We use the formalism given in the appendix of [3] following Sain and Thompson [6] where this variety is denoted by FPEA_α short for *finitary polyadic equality algebras of dimension* α. For $\alpha < \omega$, QEA_α is definitionally equivalent to Halmos' polyadic algebras of dimension α denoted in [2] by PEA_α. A *quasi-polyadic equality set algebra* is an algebra of the form $\langle \mathfrak{B}({}^\alpha U), \mathsf{C}_i, \mathsf{S}_{[i|j]}, \mathsf{S}_{[i,j]}, \mathsf{D}_{ij}\rangle_{i,j<\alpha}$ where for $i, j \in \alpha$, $\mathsf{S}_{[i,j]}$ is the unary operation of substitution corresponding to the transposition $[i, j]$ defined for $X \subseteq^\alpha U$ as follows: $S_{[i,j]}X = \{s \in {}^\alpha U : s \circ [i,j] \in X\}$. The abstract variety $\mathsf{QEA}_\alpha(\mathsf{FPEA}_\alpha)$ is defined by a finite schema of equations (in [6]) which holds in the class of quasipolyadic set algebras of the same dimension. This schema is recalled in the appendix of [3].

Fix $2 < m < n < \omega$. Let $\mathfrak{C}(m,n,r)$ be the algebra $\mathfrak{Ca}(\mathbf{H})$ where $\mathbf{H} = H_m^{n+1}(\mathfrak{A}(n,r),\omega))$, is the CA_m atom structure consisting of all $n+1$–wide m–dimensional wide ω hypernetworks [4, Definition 12.21] on $\mathfrak{A}(n,r)$ as defined in [4, Definition 15.2]. Then $\mathfrak{C}(m,n,r) \in \mathsf{CA}_m$, and it can be easily expanded to a QEA_m, since $\mathfrak{C}(m,n,r)$ is 'symmetric', in the sense that it allows a polyadic equality expansion by defining substitution operations corresponding to transpositions. This follows by observing that $\mathbf{H}$ is obviously symmetric in the following exact sense: For $\theta : m \to m$ and $N \in \mathbf{H}$, $N\theta \in \mathbf{H}$, where $N\theta$ is defined by $(N\theta)(x,y) = N(\theta(x),\theta(y))$. Hence, the binary polyadic operations defined on the atom structure $\mathbf{H}$ the obvious way (by swapping co–ordinates) lifts to polyadic operations of its complex algebra $\mathfrak{C}(m,n,r)$. In more detail, for a transposition $\tau : m \to m$, and $X \subseteq \mathbf{H}$, define $\mathsf{s}_\tau(X) = \{N \in \mathbf{H} : N\tau \in X\}$. Furthermore, for any $r \in \omega$

and $3 \leq m \leq n < \omega$, $\mathfrak{C}(m,n,r) \in \mathsf{Nr}_m\mathsf{QEA}_n$, $\mathfrak{Rd}_{ca}\mathfrak{C}(m,n,r) \notin \mathbf{S}\mathsf{Nr}_m\mathsf{CA}_{\mathsf{n+1}}$ and $\Pi_{r/U}\mathfrak{C}(m,n,r) \in \mathsf{RQEA}_m$ by easily adapting [4, Corollaries 15.7, 5.10, Exercise 2, pp. 484, Remark 15.13] to the QEA context.

THEOREM 3.4. *Let $2 < m < n < \omega$. For $\mathbf{K} \in \{\mathsf{CA}, \mathsf{QEA}\}$, any positive $k \geq 1$,for any ordinal $l \geq k+1$, the variety $\mathbf{S}\mathsf{Nr}_m\mathbf{K}_{m+l}$ is not finitely axiomatizable over the variety $\mathbf{S}\mathsf{Nr}_m\mathbf{K}_{m+k}$.*

Now from Theorem 3.2 we get (the known [3, Corollary 14]):

THEOREM 3.5. *For any ordinal $\alpha \geq \omega$, for $\mathbf{K} \in \{\mathsf{CA}, \mathsf{QEA}\}$, for any positive $k \geq 1$, and for any ordinal $l \geq k+1$, the variety $\mathbf{S}\mathsf{Nr}_\alpha\mathbf{K}_{\alpha+l}$ is not finitely axiomatizable over the variety $\mathbf{S}\mathsf{Nr}_\alpha\mathbf{K}_{\alpha+k}$.*

We denote by L_ω the basic algebraizable typeless extension of $L_{\omega,\omega}$ with usual Tarskian square semantics dealt with in [2, § 4.3]. For provability we use the basic proof system in [2, p. 157, § 4.3] which is a natural algebrazable (in the standard Blok-Pigizzi sense) extension of a complete calculas for $L_{\omega,\omega}$ expressed in terms of so-called restricted formulas. A restricted formula is one in which the variables in its atomic subformulas appear only in their natural order. We write $\vdash_{\omega+k}$ for provability using $\omega+k$ variables where k is any positive number. As an immediate corollary to the result proved in Corollary 3.3, we get:

THEOREM 3.6. *For any positive number $k \geq 1$, there is no finite schemata of L_ω whose set Σ of instances satisfies $\Sigma \vdash_{\omega+k} \phi \Longleftrightarrow \vdash_{\omega+k+1} \phi$.*

The last Theorem says that using only one extra variable to proofs adds an 'infinite' strength to the proof system which is certainly an oddity and a telling 'finite-infinite' discrepancy if read only this way. This result (formulated in an entirely abstract form) seems to us centered at the very core of the so many non-finite axiomatizability results of varieties of representable algebras recurring in algebraic logic. This stems from the observation that for CAs (and many cylindric-like algebras such as quasi polyadic algebras with and without equality also dealt with above), we have that for any ordinal α, $\mathbf{S}\mathsf{Nr}_\alpha\mathsf{CA}_{\alpha+\omega} = \mathsf{RCA}_\alpha$, and that for any ordinal $\alpha > 2$, the sequence $\langle \mathbf{S}\mathsf{Nr}_\alpha\mathsf{CA}_{\alpha+k+1} : k \geq 1\rangle_{k\in\omega}$ is a strictly decreasing sequence with respect to class inclusion with the minimum gaps (of length only one, namely, from $\mathbf{S}\mathsf{Nr}_\alpha\mathbf{K}_{\alpha+k+1}$ to $\mathbf{S}\mathsf{Nr}_\alpha\mathbf{K}_{\alpha+k}$ for any positive k and any ordinal $\alpha > 2$) allowing no finite schema axiomatization.

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SUP-HESITANT FUZZY INTERIOR IDEALS IN Γ-SEMIGROUPS

Abstract

In this paper, we defined the concept of $\mathcal{SUP}$-hesitant fuzzy interior ideals in Γ-semigroups, which is generalized of hesitant fuzzy interior ideals in Γ-semigroups. Additionally, we study fundamental properties of $\mathcal{SUP}$-hesitant fuzzy interior ideals in Γ-semigroups. Finally, we investigate characterized properties of those.

Keywords: SUP-hesitant fuzzy interior ideal, hesitant fuzzy interior ideal, interval valued fuzzy interior ideal.

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1. Introduction

The theory of fuzzy sets (FSs), considered by Zadeh in [27] has applications in mathematics, engineering, medical science, and other fields. Torra and Narukawa [25] extended the knowledge of a fuzzy set go to a hesitant fuzzy set (HFS) which is a function from a reference set to a power set of the unit interval and a generalization of intuitionistic fuzzy sets (IFSs) and interval-valued fuzzy sets (IvFSs) [26]. Then in 2015, Jun et al. [14] introduced the concept of HFSs and studied many algebraic structures, such as properties of hesitant fuzzy left (right, generalized bi-, bi-, two-sided) ideals of semigroups. In 1981, Sen introduced the concept of Γ-semigroup as a

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generalization of the plain semigroup and ternary semigroup. The many classical notions and results of (ternary) semigroups have been extended and generalized to Γ-semigroups, by many mathematicians. For instance, Dutta, and Davvaz [7, 8] studied the theory of Γ-semigroups via fuzzy subsets. Siripitukdet and Iampan [22, 23], Siripitukdet and Julatha [24], Dutta and Adhikari [8], Saha and Sen [20, 21], Hila, [10, 11] and Chinram [4, 5], and Uckun et al. [18] studied the theory of Γ-semigroup via intuitionistic fuzzy subsets. Abbasi et al. [1] introduced hesitant fuzzy left (resp., right, bi-, interior, and two-sided) Γ-ideals of Γ-semigroups. Julatha and Iampan [13] introduced a sup-hesitant fuzzy Γ-ideal, which is a general concept of an interval valued fuzzy Γ-ideal and a hesitant fuzzy Γ-ideal, of a Γ-semigroup and studied its properties via level sets, fuzzy sets, interval-valued fuzzy sets, and hesitant fuzzy sets. In 2018, Mosrijai et al., [16] presented the concept from HFSs in UP-algebras, namely $\mathcal{SUP}$-hesitant fuzzy UP-subalgebras (UP-filters, UP-ideals, strong UPideals). In 2019, Muhiuddin and Jun [17] introduced and studied the properties of $\mathcal{SUP}$-hesitant fuzzy subalgebras and their translations and extensions. In 2020, Muhiuddin et al. [17] studied the concept of $\mathcal{SUP}$-hesitant fuzzy ideals in BCK/BCI-algebras. In the same year, Harizavi and Jun [9] introduced $\mathcal{SUP}$-hesitant fuzzy quasi-associative ideal in BCI algebras. Later, Dey et al. [6] developed the concept of hesitant multi-fuzzy sets by combining the hesitant fuzzy set with the multi-fuzzy set. In 2021, Jittburus and Julatha [12] discussed the properties of $\mathcal{SUP}$-hesitant fuzzy ideals of semigroups and studied the characterizations in terms of sets, FSs, HFSs, and IvFSs. In 2022, P. Julatha and A. Iampan [13] studied the $\mathcal{SUP}$-hesitant fuzzy ideal in Γ-semigroup and considered the basic properties of those.

In this paper, we study the definition and properties of $\mathcal{SUP}$-hesitant fuzzy interior ideals in Γ-semigroups and investigate the properties of those.

2. Preliminaries

Throughout this paper, we denote a Γ-semigroup by $\mathcal{S}$.

In this section, we give some fundamental concepts about Γ-semigroups, fuzzy sets, intuitionistic fuzzy sets, interval valued fuzzy sets and hesitant fuzzy sets are presented. These notions will be helpful in later sections.

Let $\mathcal{S}$ and Γ be non-empty sets. Then $\mathcal{S}$ is called a *Γ-semigroup* $\mathcal{S}$ if there exists a function $\mathcal{S} \times \Gamma \times \mathcal{S} \to \mathcal{S}$ written as $(e_1, \alpha, e_2) \mapsto e_1 \alpha e_2$

satisfying the axiom $(e_1\alpha e_2)\beta e_3 = e_1\alpha(e_2\beta e_3)$ for all $e_1, e_2, e_3 \in \mathcal{S}$ and $\alpha, \beta \in \Gamma$. A non-empty subset L of $\mathcal{S}$ is called a *subsemigroup* of $\mathcal{S}$ if $L\Gamma L \subseteq L$. A non-empty subset L of $\mathcal{S}$ is called a *left* (right) ideal of $\mathcal{S}$ if $\mathcal{S}\Gamma L \subseteq L$ ($L\Gamma\mathcal{S} \subseteq L$). By an Γ-*ideal* L of $\mathcal{S}$, we mean a left ideal and a right ideal of $\mathcal{S}$. A subsemigroup L of $\mathcal{S}$ is called a *interior ideal* of $\mathcal{S}$ if $\mathcal{S}\Gamma L\Gamma\mathcal{S} \subseteq L$.

A *fuzzy set* (FS) of a non-empty set $\mathcal{T}$ is a function $\omega : \mathcal{T} \to [0, 1]$.

DEFINITION 2.1 ([15]). A FS ω of $\mathcal{S}$ is said to be a *fuzzy subsemigroup* (FSG) of $\mathcal{S}$ if $\omega(e_1\gamma e_2) \geq \omega(e_1) \wedge \omega(e_2)$ for all $e_1, e_2 \in \mathcal{S}$ and $\gamma \in \Gamma$.

DEFINITION 2.2 ([19]). A FS ω of $\mathcal{S}$ is said to be a *fuzzy left (right) ideal* (FLI(FRI)) of $\mathcal{S}$ if $\omega(e_1\gamma e_2) \geq \omega(e_2)$ ($\omega(e_1\gamma e_2) \geq \omega(e_1)$) for all $e_1, e_2 \in \mathcal{S}$ and $\gamma \in \Gamma$. A FS ω of $\mathcal{S}$ is called an *fuzzy ideal* of $\mathcal{S}$ if it is both a fuzzy left ideal and a fuzzy right ideal of $\mathcal{S}$.

DEFINITION 2.3 ([19]). A FS ω of $\mathcal{S}$ is said to be an *fuzzy interior ideal* (FII) of $\mathcal{S}$ if ω is a FSG and $\omega(e_1\gamma e_2\alpha e_3) \geq \omega(e_2)$ for all $e_1, e_2, e_3 \in \mathcal{S}$ and $\gamma, \alpha \in \Gamma$.

An *intuitionistic fuzzy set* (IFS) $\mathcal{A}$ in $\mathcal{T}$ is the form $\mathcal{A} = \{e, \omega_{\mathcal{A}}, \vartheta_{\mathcal{A}} \mid e \in \mathcal{A}\}$ where $\omega_{\mathcal{A}} : \mathcal{T} \to [0, 1]$ and $\vartheta_{\mathcal{A}} : \mathcal{T} \to [0, 1]$ and where $0 \leq \omega_{\mathcal{A}}(e) + \vartheta_{\mathcal{A}}(e) \leq 1$ for all $e \in \mathcal{A}$ [2].

DEFINITION 2.4 ([18]). An IFS $\mathcal{A} = (\omega_{\mathcal{A}}, \vartheta_{\mathcal{A}})$ in $\mathcal{T}$ is called an intuitionistic fuzzy subemigroup (IFSG) of $\mathcal{S}$ if $\omega_{\mathcal{A}}(e_1\gamma e_2) \geq \max\{\omega_{\mathcal{A}}(e_1), \omega_{\mathcal{A}}(e_2)\}$ and $\vartheta_{\mathcal{A}}(e_1\gamma e_2) \leq \min\{\vartheta_{\mathcal{A}}(e_1), \vartheta_{\mathcal{A}}(e_2)\}$ for all $e_1, e_2 \in \mathcal{S}$ and $\gamma \in \Gamma$.

DEFINITION 2.5 ([18]). An IFS $\mathcal{A} = (\omega_{\mathcal{A}}, \vartheta_{\mathcal{A}})$ in $\mathcal{T}$ is called an intuitionistic fuzzy ideal (IFI) of $\mathcal{S}$ if $\omega_{\mathcal{A}}(e_1\gamma e_2) \leq \max\{\omega_{\mathcal{A}}(e_1), \omega_{\mathcal{A}}(e_2)\}$ and $\vartheta_{\mathcal{A}}(e_1\gamma e_2) \geq \min\{\vartheta_{\mathcal{A}}(e_1), \vartheta_{\mathcal{A}}(e_2)\}$ for all $e_1, e_2 \in \mathcal{S}$ and $\gamma \in \Gamma$.

DEFINITION 2.6 ([18]). An IFS $\mathcal{A} = (\omega_{\mathcal{A}}, \vartheta_{\mathcal{A}})$ in $\mathcal{T}$ is called an intuitionistic interior ideal (IFII) of $\mathcal{S}$ if $\mathcal{A} = (\omega_{\mathcal{A}}, \vartheta_{\mathcal{A}})$ is an IFSG and $\omega_{\mathcal{A}}(e_1\gamma e_2\alpha e_3) \geq \omega_{\mathcal{A}}(e_2)$ and $\vartheta_{\mathcal{A}}(e_1\gamma e_2\alpha e_3) \leq \vartheta_{\mathcal{A}}(e_2)$ for all $e_1, e_2, e_3 \in \mathcal{S}$ and $\gamma, \alpha \in \Gamma$.

Let $\mathcal{C}[0, 1]$ be the set of all closed subintervals of $[0, 1]$, i.e.,

$$\mathcal{C}[0, 1] = \{\tilde{p} = [p^-, p^+] \mid 0 \leq p^- \leq p^+ \leq 1\}.$$

Let $\hat{p} = [p^-, p^+]$ and $\hat{q} = [q^-, q^+] \in \Omega[0, 1]$. Define the operations $\preceq$, $=$, $\curlywedge$ and $\curlyvee$ as follows:

(1) $\hat{p} \preceq \hat{q}$ if and only if $p^- \leq q^-$ and $p^+ \leq q^+$.

(2) $\hat{p} = \hat{q}$ if and only if $p^- = q^-$ and $p^+ = q^+$.

(3) $\hat{p} \curlywedge \hat{q} = [(p^- \wedge q^-), (p^+ \wedge q^+)]$.

(4) $\hat{p} \curlyvee \hat{q} = [(p^- \vee q^-), (p^+ \vee q^+)]$.
If $\hat{p} \succeq \hat{q}$, we mean $\hat{q} \preceq \hat{p}$.

Definition 2.7 ([19]). Let $\mathcal{T}$ be a non-empty set. Then the function $\hat{\omega} : \mathcal{T} \to \mathcal{C}[0,1]$ is called *interval valued fuzzy set* (shortly, IvFS) of $\mathcal{T}$.

Next, we shall give definitions of various types of interval valued fuzzy subsemigroups.

Definition 2.8 ([3]). An IvFS $\hat{\omega}$ of $\mathcal{S}$ is said to be an *interval valued fuzzy subsemigroup* (IvF subsemigroup) of $\mathcal{S}$ if $\hat{\omega}(e_1\gamma e_2) \succeq \hat{\omega}(e_1) \curlywedge \hat{\omega}(e_2)$ for all $e_1, e_2 \in \mathcal{S}$ and $\gamma \in \Gamma$.

Definition 2.9 ([3]). An IvFS $\hat{\omega}$ of $\mathcal{S}$ is said to be an *interval valued fuzzy left (right) ideal* (IvF left (right) ideal) of $\mathcal{S}$ if $\hat{\omega}(e_1\gamma e_2) \succeq \hat{\omega}(e_2)$ ($\hat{\omega}(e_1\gamma e_2) \succeq \hat{\omega}(e_1)$) for all $e_1, e_2 \in \mathcal{S}$ and $\gamma \in \Gamma$. An IvFS $\hat{\omega}$ of $\mathcal{S}$ is called an *IvF ideal* of $\mathcal{S}$ if it is both an IvF left ideal and an IvF right ideal of $\mathcal{S}$.

Definition 2.10 ([3]). An IvFS $\hat{\omega}$ of $\mathcal{S}$ is said to be an *interval valued fuzzy interior ideal* (IvF interior ideal) of $\mathcal{S}$ if $\hat{\omega}$ is an IvF subsemigroup and $\hat{\omega}(e_1\gamma e_2\alpha e_3) \succeq \hat{\omega}(e_2)$ for all $e_1, e_2, e_3 \in \mathcal{S}$ and $\gamma, \alpha \in \Gamma$.

Let L be a non-empty subset of $\mathcal{T}$. *An interval valued characteristic function* ($\hat{\lambda}_L$) of L is defined by

$$\hat{\lambda}_L : \mathcal{T} \to \mathcal{C}[0,1], e \mapsto \begin{cases} \overline{1} & \text{if } eu \in L, \\ \overline{0} & \text{otherwise,} \end{cases}$$

for all $e \in \mathcal{T}$.

For two IvFSs $\hat{\omega}$ and $\hat{\vartheta}$ of $\mathcal{S}$, define the product $\hat{\omega} \circ \hat{\vartheta}$ as follows: for all $e \in \mathcal{S}$,

$$(\hat{\omega} \circ \hat{\vartheta})(e) = \begin{cases} \underset{e=tz}{\curlyvee} \{\hat{\omega}(t) \curlywedge \hat{\vartheta}(z)\} & \text{if } e = tz, \\ \overline{0} & \text{otherwise.} \end{cases}$$

DEFINITION 2.11 ([14]). A *hesitant fuzzy set* (HFS) on a non-emptyset $\mathcal{T}$ is a function $\mathfrak{h} : \mathcal{T} \to \mathcal{P}([0,1])$.

DEFINITION 2.12 ([1]). A HFS $\mathfrak{h}$ on $\mathcal{S}$ is called a *hesitant fuzzy subsemigroup* (HFSG) on $\mathcal{S}$ if it satisfies:

$$\mathfrak{h}(e_1\gamma e_2) \supseteq \mathfrak{h}(e_1) \cap \mathfrak{h}(e_2) \text{ for all } e_1, e_2 \in \mathcal{S} \text{ and } \gamma \in \Gamma.$$

DEFINITION 2.13 ([1]). A HFS $\mathfrak{h}$ on $\mathcal{S}$ is called a *hesitant fuzzy left (resp., right) ideal* on $\mathcal{S}$ if it satisfies:

$$\mathfrak{h}(e_1\gamma e_2) \supseteq \mathfrak{h}(x)(\mathfrak{h}(e_1) \supseteq \mathfrak{h}(e_2)) \text{ for all } e_1, e_2 \in \mathcal{S} \text{ and } \gamma \in \Gamma.$$

An HFS $\mathfrak{h}$ of $\mathcal{S}$ is called an *hesitant fuzzy ideal* of $\mathcal{S}$ if it is both a hesitant fuzzy left ideal and a hesitant fuzzy right ideal of $\mathcal{S}$.

DEFINITION 2.14 ([1]). A HFS $\mathfrak{h}$ on $\mathcal{S}$ is called a *hesitant fuzzy interior ideal* (HFII) on $\mathcal{S}$ if it satisfies:

$$\mathfrak{h} \text{ is a HFs and } \mathfrak{h}(e_1\gamma e_2\alpha e_3) \supseteq \mathfrak{h}(e_2) \text{ for all } e_1, e_2, e_3 \in \mathcal{S} \text{ and } \gamma, \alpha \in \Gamma.$$

Let L be a non-empty subset of $\mathcal{T}$. The *characteristic hesitant fuzzy set* (CH_L) of L is defined by

$$CH_L : \mathcal{T} \to \mathcal{P}([0,1]), x \mapsto \begin{cases} [0,\ 1] & \text{if } e \in L, \\ \emptyset & \text{otherwise,} \end{cases}$$

for all $e \in \mathcal{T}$.

For two HFSs $\mathfrak{h}$ and $\mathfrak{g}$ of $\mathcal{S}$, define the product $\mathfrak{h} \circ \mathfrak{g}$ as follows: for all $e \in \mathcal{S}$,

$$(\mathfrak{h} \circ \mathfrak{g})(e) = \begin{cases} \bigcup_{e=tz} \{\mathfrak{h}(t) \cap \mathfrak{g}(z)\} & \text{if } e = tz, \\ \emptyset & \text{otherwise.} \end{cases}$$

3. $\mathcal{SUP}$-hesitant fuzzy interior ideals in Γ-Semigroups

In this section, we define the concepts of $\mathcal{SUP}$-hesitant fuzzy interior ideals of $\mathcal{S}$ and characterize $\mathcal{SUP}$-hesitant fuzzy interior ideals of $\mathcal{S}$.

For any HFS $\mathfrak{h}$ on $\mathcal{T}$ and $\Theta \in \mathcal{P}[0,1]$, define $\mathcal{SUP}\Theta$ and $\mathcal{S}[\mathfrak{h};\Theta]$ by

$$\mathcal{SUP}\Theta = \begin{cases} \sup \Theta & \text{if } \Theta \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathcal{S}[\mathfrak{h};\Theta] = \{x \in \mathcal{X} \mid \mathcal{SUP}\ \mathfrak{h}(x) \geq \mathcal{SUP}\Theta\}.$$

Then the following assertions are true:

(1) For every IvFS $\tilde{A}$ on $\mathcal{X}, \mathcal{SUP}\tilde{A}(x) = \sup \tilde{A}(x) = A^+(x), \forall x \in \mathcal{X}$.

(2) If $\Theta, \Upsilon \in \mathcal{P}[0,1]$ with $\Theta \subseteq \Upsilon$, then $\mathcal{SUP}\Theta \subseteq \mathcal{SUP}\Psi$ and $\mathcal{S}[\mathfrak{h};\Upsilon] \subseteq \mathcal{S}[\mathfrak{h};\Theta]$.

Definition 3.1. An HFS $\mathfrak{h}$ on $\mathcal{S}$ is called a SUP-hesitant fuzzy interior ideal of $\mathcal{S}$ related to Θ (Θ-$\mathcal{SUP}$-HFI) if the set $\mathcal{S}[\mathfrak{h};\Theta]$ is an interior ideal of $\mathcal{S}$. We call that $\mathfrak{h}$ is a $\mathcal{SUP}$-hesitant fuzzy interior ideal ($\mathcal{SUP}$-HFII) of $\mathcal{S}$ if $\mathfrak{h}$ is a Θ-$\mathcal{SUP}$-HFII of $\mathcal{S}, \forall \Theta \in \mathcal{P}[0,1]$ with $\mathcal{S}[\mathfrak{h};\Theta] \neq \emptyset$.

The following Lemmas are tools to prove Theorem 3.7.

Lemma 3.2. *If $\Theta, \Psi \in \mathcal{P}[0,1]$ with $\mathcal{SUP}\Theta = \mathcal{SUP}\Upsilon$ and $\mathfrak{h}$ is a Θ-$\mathcal{SUP}$-HFI of $\mathcal{S}$, then $\mathfrak{h}$ is a Ψ-$\mathcal{SUP}$-HFI of $\mathcal{S}$.*

Proof: Assume that $\Theta, \Upsilon \in \mathcal{P}[0,1]$ with $\mathcal{SUP}\Theta = \mathcal{SUP}\Upsilon$ and $\mathfrak{h}$ is a Θ-$\mathcal{SUP}$-HFI of $\mathcal{S}$. Then $\mathcal{SUP}\Theta \subseteq \mathcal{SUP}\Psi$ and $\mathcal{S}[\mathfrak{h};\Upsilon] \subseteq \mathcal{S}[\mathfrak{h};\Theta]$. Thus, by Definition 3.1, $\mathfrak{h}$ is a Υ-$\mathcal{SUP}$-HFI of $\mathcal{S}$. □

Lemma 3.3. *Every IvF interior ideal of $\mathcal{S}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$.*

Proof: Assume that $\tilde{A}$ is an IvF interior ideal of $\mathcal{S}$ and let $\Theta \in \mathcal{P}[0,1]$ with $\mathcal{S}[\tilde{A};\Theta] \neq \emptyset$. Let $e_1, e_3 \in \mathcal{S}$, $e_2 \in \mathcal{S}[\tilde{A};\Theta]$ and $\gamma, \alpha \in \Gamma$. Then $\sup \tilde{A}(e_2) \geq \mathcal{SUP}\Theta$. Since $\tilde{A}$ is an IvF interior ideal of $\mathcal{S}$, we have $\mathcal{SUP}\Theta \leq \sup \tilde{A}(e_2) \preceq \tilde{A}(e_1\gamma e_2\alpha e_3)$. Thus, $e_1\gamma e_2\alpha e_3 \in \mathcal{S}[\tilde{A},\Theta]$. Hence, $\tilde{A}$ is an interior ideal of $\mathcal{S}$. So, $\tilde{A}$ is a Θ-$\mathcal{SUP}$-HFII of $\mathcal{S}$. Therefore, $\tilde{A}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$. □

Lemma 3.4. *Every HFII of $\mathcal{S}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$.*

Proof: Assume that $\mathfrak{h}$ is a HFII of $\mathcal{S}$ and let $\Theta \in \mathcal{P}[0,1]$ with $\mathcal{S}[\tilde{A};\Theta] \neq \emptyset$. Let $e_1, e_3 \in \mathcal{S}$ and $e_2 \in \mathcal{S}[\mathfrak{h};\Theta]$ and $\gamma, \alpha \in \Gamma$. Then $\mathfrak{h}(e_1\gamma e_2\alpha e_3) \supseteq \mathfrak{h}(e_2)$. Thus, $\mathcal{SUP}\mathfrak{h}(e_1\gamma e_2\alpha e_3) \geq \mathfrak{h}(e_2) \geq \mathcal{SUP}\Theta$ so $e_1\gamma e_2\alpha e_3 \in \mathcal{S}[\mathfrak{h};\Theta]$.

Hence, $\mathcal{S}[\mathfrak{h};\Theta]$ is an interior ideal of $\mathcal{S}$, and so $\mathfrak{h}$ is a Θ-$\mathcal{SUP}$-HFII of $\mathcal{S}$. Therefore, $\mathfrak{h}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$. □

THEOREM 3.5. *Let S be a regular* Γ*-semigroup $\mathcal{S}$. Then HFS $\mathfrak{h}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$ if and only if $\mathfrak{h}$ is a $\mathcal{SUP}$-HFI of $\mathcal{S}$.*

PROOF: It is a direct result from that a non-empty subset L of a regular Γ-semigroup $\mathcal{S}$ is an interior ideal of $\mathcal{S}$ if and only if L is an ideal of $\mathcal{S}$. □

For every HFS $\mathfrak{h}$ on $\mathcal{T}$ and $\Theta \in \mathcal{P}[0,1]$, we define the HFS $\mathcal{H}(\mathfrak{h};\Theta)$ on $\mathcal{T}$ by $\forall e \in \mathcal{T}$,

$$\mathcal{H}(\mathfrak{h};\Theta)(e) = \{r \in \Theta \mid \mathcal{SUP}\mathfrak{h}(e) \geq r\}.$$

We denote $\mathcal{H}(\mathfrak{h};\bigcup_{e\in\mathcal{T}}\mathfrak{h}(e))$ by $\mathcal{H}_h$ and denote $\mathcal{H}(\mathfrak{h};[0,1])$ by $\mathcal{I}_h$. Then the following assertions are true: for all $e \in \mathcal{T}$;

(1) $\mathcal{I}_h$. is an IvFS on $\mathcal{S}$.

(2) $\mathfrak{h}(e) \subseteq \mathcal{H}_h \subseteq \mathcal{I}$.

(3) $\mathcal{SUP}\mathfrak{h}(e) = \mathcal{SUP}\mathcal{H}_h(x) = \mathcal{SUP}\mathcal{I}_h(e)$.

(4) $\mathcal{H}(\mathfrak{h},\Theta)(e) \subseteq \Theta$.

(5) $\mathcal{H}(\mathfrak{h},\Theta)(e) = \Theta$ if an only if $e \in \mathcal{S}[\mathfrak{h},\Theta]$.

LEMMA 3.6. *An HFS $\mathfrak{h}$ on $\mathcal{S}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$ if and only if $\mathcal{H}(\mathfrak{h};\Theta)$ is an HFII of $\mathcal{S}, \forall\Theta \in \mathcal{P}[0,1]$.*

PROOF: Let $\Theta \in \mathcal{P}[0,1]$, $e_1, e_2, e_3 \in \mathcal{S}$ and $\gamma, \alpha \in \Gamma$. Suppose that $\mathfrak{h}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$, and let $r \in \mathcal{H}(\mathfrak{h};\Theta)(e_2)$. Then $a \in \mathcal{H}(\mathfrak{h};\Theta)(a)$. Thus, $\mathcal{SUP}(\mathfrak{h}(a)) \geq r \in \Theta$. Hence, $e_2 \in \mathcal{S}[\mathfrak{h}(e_2)]$. Since $\mathfrak{h}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$, we have $e_1e_2e_3 \in \mathcal{S}[\mathfrak{h}(a)]$. Thus, $\mathcal{SUP}\mathfrak{h}(e_1e_2e_3) \geq \mathfrak{h}(e_1) \geq r \in \Theta$. Hence, $r \in \mathcal{H}(\mathfrak{h};\Theta)(e_1e_2e_3)$. Therefore, $\mathcal{H}(\mathfrak{h};\Theta)(e_2) \subseteq \mathcal{H}(\mathfrak{h};\Theta)(e_1e_2e_3)$. We conclude that $\mathcal{H}(\mathfrak{h};\Theta)$ is a HFII of $\mathcal{S}$.

In contrat, suppose that $\mathfrak{h}$ is a $\mathcal{H}(\mathfrak{h};\Theta)$ is a HFII of $\mathcal{S}$ and $e_2 \in \mathcal{S}[\mathfrak{h};\Theta]$, $e_1, e_3 \in \mathcal{S}$. Then $\mathcal{H}(\mathfrak{h},\Theta)(e_2) = \Theta$. Since $\mathfrak{h}$ is a $\mathcal{H}(\mathfrak{h};\Theta)$ is a HFII of $\mathcal{S}$ we have $\Theta = \mathcal{H}(\mathfrak{h},\Theta)(e_2) \subseteq \mathcal{H}(\mathfrak{h};\Theta)(e_2) \subseteq \mathcal{H}(\mathfrak{h};\Theta)(e_1e_2e_3)$ and so $\Theta \subseteq \mathcal{H}(\mathfrak{h};\Theta)(e_1e_2e_3)$. Hence $\mathcal{SUP}\mathfrak{h}(e_1e_2e_3) \geq \mathcal{SUP}\Theta$. Thus $e_1e_2e_3 \in \mathcal{S}[\mathfrak{h};\Theta]$.

Therefore $\mathcal{S}[\mathfrak{h};\Theta]$ is an interior ideal of $\mathcal{S}$. This implies that $\mathfrak{h}$ is a $\Theta-\mathcal{SUP}$-HFII of $\mathcal{S}$. Thus $\mathfrak{h}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$. □

The following theorem is a result of Lemma 3.3, 3.4, and 3.6.

THEOREM 3.7. *Let $\mathfrak{h}$ is a HFS in $\mathcal{K}$. Then the following statements are equivalent.*

(1) *$\mathcal{H}_h$ is an HFII of $\mathcal{S}$.*

(2) *$\mathcal{H}_h$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$.*

(3) *$\mathcal{I}_h$ is a $\mathcal{SUP}$-HFII of $\mathcal{K}$.*

(4) *$\mathcal{I}_h$ is an HFII of $\mathcal{S}$.*

(5) *$\mathfrak{h}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$.*

(6) *$\mathcal{I}_h$ is an IvFII of $\mathcal{S}$.*

PROOF: By Lemma 3.4, we get that, $1 \Rightarrow 2$ and $3 \Rightarrow 4$.

By Lemma 3.6, we get that, $5 \Rightarrow 2$ and $5 \Rightarrow 6$.

By Lemma 3.3, we get that, $3 \Rightarrow 6$.

Now, we proof $1 \Rightarrow 5$. Let $\Theta \in \mathcal{P}[0,1], e_1, e_2e_3 \in \mathcal{S}$ and $\gamma, \alpha \in \Gamma$. Then $\mathcal{SUP}\mathcal{H}_h(e_2) = \mathcal{SUP}\mathfrak{h}(e_2) \geq \mathcal{SUP}\Theta$. Thus, $e_2 \in \mathcal{S}[\mathcal{H}_h;\Theta]$. So, $\mathcal{S}[\mathcal{H}_h;\Theta]$ is an interior ideal of $\mathcal{S}$ with $e_1\gamma e_2\alpha e_3 \in \mathcal{S}[\mathcal{H}_h;\Theta]$ which implies that $\mathcal{SUP}\mathfrak{h}(e_1\gamma e_2\alpha e_3) = \mathcal{SUP}\mathcal{H}_h(e_1\gamma e_2\alpha e_3) \geq \mathcal{SUP}\Theta$. Hence, $e_1\gamma e_2\alpha e_3 \in \mathcal{S}[\mathfrak{h};\Theta]$. Therefore $\mathcal{S}[\mathfrak{h};\Theta]$ is an interior ideal of $\mathcal{S}$. We conclude that $\mathfrak{h}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$.

For $1 \Rightarrow 6$, let $e_1, e_2e_3 \in \mathcal{S}$ and $\gamma, \alpha \in \Gamma$. Then $e_2 \in S[\mathfrak{h};\mathfrak{h}(e)]$. Thus, $\mathcal{SUP}\mathfrak{h}(e_2) \leq \mathcal{SUP}\mathfrak{h}(e_1\gamma e_2\alpha e_3)$. Hence, $\mathcal{I}_h(e_2) = [0, \mathcal{SUP}\mathfrak{h}(e_2)]$. So, $\mathcal{I}_h(e_2) \preceq \mathcal{I}_h(e_1\gamma e_2\alpha e_3)$. Therefore, $\mathcal{I}_h$ is an IvFBII of $\mathcal{S}$.

The proof of $2 \Rightarrow 6$ is similar to $1 \Rightarrow 5$. □

In [12], the author define $\mathcal{F}_\mathfrak{h}$ in $\mathcal{T}$ by $\mathcal{F}_\mathfrak{h} = \mathcal{SUP}\mathfrak{h}(x)$ for all $x \in \mathcal{T}$.

THEOREM 3.8. *An HFS $\mathfrak{h}$ on $\mathcal{K}$ is a $\mathcal{SUP}$-HFBII of $\mathcal{S}$ if and only if $\mathcal{F}_h$ is a FII of $\mathcal{S}$.*

PROOF: Let $e_1, e_2e_3 \in \mathcal{S}$ and $\gamma, \alpha \in \Gamma$. Then $\mathfrak{h}(e_2) = \Theta$ for some $\Theta \in \mathcal{P}[0,1]$. Thus, $e_2 \in \mathcal{S}[\mathfrak{h};\Theta]$. By assumption, we have $e_1\gamma e_2\alpha e_3 \in \mathcal{S}[\mathfrak{h};\Theta]$. Hence, $\mathcal{F}_h(e_1\gamma e_2\alpha e_3) = \mathcal{SUP}\mathfrak{h}(e_1\gamma e_2\alpha e_3) \geq \mathcal{SUP}\Theta = \mathcal{SUP}(\mathfrak{h}(e_2)) = \mathfrak{h}(a) = \mathcal{F}_h(e_2)$. Therefore, $\mathcal{F}_h$ is a fuzzy interior ideal of $\mathcal{S}$.

In contrat, let $\Theta \in \mathcal{P}[0,1], e_2 \in \mathcal{S}[\mathfrak{h};\Theta], e_1, e_3 \in \mathcal{S}$. Then $\mathcal{SUP}\mathfrak{h}(e_1e_2e_3) = \mathcal{F}_h(e_1\gamma e_2\alpha e_3) \geq \mathcal{F}_h(e_2) = \mathcal{SUP}h(e_2) \geq \mathcal{SUP}\Theta$. This implies that $e_1\gamma e_2\alpha e_3 \in \mathcal{S}[\mathfrak{h};\Theta]$. Hence, $\mathcal{S}[\mathfrak{h};\Theta]$ is an interior ideal of $\mathcal{S}$. So, $\mathfrak{h}$ is a Θ-$\mathcal{SUP}$-HFII of $\mathcal{S}$. Therefore, $\mathfrak{h}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$. □

The following result is an immediate consequence of Theorem 3.8.

COROLLARY 3.9. An HFS $\mathfrak{h}$ in $\mathcal{S}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$ if and only if $\mathcal{SUP}\mathfrak{h}(e_1\gamma e_2\alpha e_3) = \mathfrak{h}(e_2)$ for all $e_1, e_2e_3 \in \mathcal{S}$ and $\gamma, \alpha \in \Gamma$.

For any IFS $\mathcal{A} = (\omega_{\mathcal{A}}, \vartheta_{\mathcal{A}})$ on $\mathcal{T}$ and $\Theta \in \mathcal{P}[0,1]$, we define the HFS $\mathcal{H}^{\Theta}_{\mathcal{A}}$ on $\mathcal{T}$ and IvFS $\mathcal{I}_{\mathcal{A}}$ in $\mathcal{A}$

$$\mathcal{H}^{\Theta}_{\mathcal{A}}(e) = \left\{ t \in \Theta \mid \frac{\vartheta_{\mathcal{A}}(e)}{2} \leq t \leq \frac{1+\omega_{\mathcal{A}}(e)}{2} \right\}$$

and

$$\mathcal{I}_{\mathcal{A}}(e) = \left[\frac{1-\vartheta_{\mathcal{A}}(e)}{2}, \frac{1+\omega_{\mathcal{A}}(e)}{2} \right]$$

for all $e \in \mathcal{T}$.

THEOREM 3.10. *Suppose that* $\mathcal{A} = (\omega_{\mathcal{A}}, \vartheta_{\mathcal{A}})$ *be an IFS in* $\mathcal{S}$*. The following are equivalent.*

(1) $\mathcal{A}$ *is an IFII of* $\mathcal{S}$*.*

(2) $\mathcal{H}^{\Theta}_{\mathcal{A}}$ *is a HFII of* $\mathcal{S}$ *for all* $\Theta \in \mathcal{P}[0,1]$*.*

(3) $\mathcal{I}_{\mathcal{A}}$ *is an IvFII of* $\mathcal{K}$*.*

PROOF: 1. $\Rightarrow$ 2. Suppose that $\mathcal{A}$ is an IFII of $\mathcal{S}$ and $\Theta \in \mathcal{P}[0,1]$. Let $e_1, e_2, e_3 \in \mathcal{S}$, $\gamma, \alpha \in \Gamma$ and $t \in \mathcal{H}^{\Theta}_{\mathcal{A}}(e_2)$. Then $t \in \Theta$ and $\frac{\vartheta_{\mathcal{A}}}{2} \leq t \leq \frac{1+\omega_{\mathcal{A}}}{2}$. Since $\mathcal{A}$ is an IFII of $\mathcal{S}$, we have

$$\frac{\vartheta_{\mathcal{A}}(e_1\gamma e_2\alpha e_3)}{2} \leq \vartheta_{\mathcal{A}}(e_2) \leq t \leq \frac{1+\omega_{\mathcal{A}}(e_2)}{2} \leq \frac{1+\omega_{\mathcal{A}}(e_1\gamma e_2\alpha e_3)}{2}.$$

Thus, $t \in \mathcal{H}^{\Theta}_{\mathcal{A}}(e_1\gamma e_2\alpha e_3)$. Hence, $\mathcal{H}^{\Theta}_{\mathcal{A}}(e_2) \subseteq \mathcal{H}^{\Theta}_{\mathcal{A}}(e_1\gamma e_2\alpha e_3)$.
Therefore, $\mathcal{H}^{\Theta}_{\mathcal{A}}$ is an HFII of $\mathcal{S}$.

2. $\Rightarrow$ 1. Suppose that $\mathcal{H}^{\Theta}_{\mathcal{A}}$ is am HFII of $\mathcal{S}$, and $\mathcal{A}$ is not an IFII of $\mathcal{S}$. Then there are $e_1, e_2, e_3 \in \mathcal{S}$ and $\gamma, \alpha \in \Gamma$ such that $\omega_{\mathcal{A}}(e_1e_2e_3) < \omega_{\mathcal{A}}(e_2)$. Choose $t = \frac{1}{4}(\omega_{\mathcal{A}}(e_1\gamma e_2\alpha e_3) + \omega_{\mathcal{A}}(e_2))$. We have $\frac{1}{2} + t \in [0,1]$

and $\frac{\omega_{\mathcal{A}}(e_1\gamma e_2\alpha e_3)}{2} < t < \omega_{\mathcal{A}}(e_2)$. Thus, $\frac{\vartheta_{\mathcal{A}}(e_2)}{2} \leq \frac{1}{2} < \frac{1}{2} + t < \frac{1+\omega_{\mathcal{A}}(e_2)}{2}$. So, $\frac{1}{2} + t \in \mathcal{H}^{\Theta}_{\mathcal{A}}(e_2)$ Since $\mathcal{H}^{\Theta}_{\mathcal{A}}$ is an HFII of $\mathcal{S}$, we have $\mathcal{H}^{[0,1]}_{\mathcal{A}}$ is an HFII on $\mathcal{S}$. It implies that $\frac{1}{2} + t \in \mathcal{H}^{[0,1]}_{\mathcal{A}}(e_1\gamma e_2\alpha e_3)$. Hence, $\frac{1}{2} + t \leq \frac{1+\omega_{\mathcal{A}}(e_1\gamma e_2\alpha e_3)}{2}$ and

$$\begin{aligned}\omega_{\mathcal{A}}(e_1\gamma e_2\alpha e_3) &= 2\left(\frac{1+\omega_{\mathcal{A}}(e_1\gamma e_2\alpha e_3)}{2}\right) - 1\\ &\geq 2\left(\frac{1}{2} + t\right)\\ &= 2t\\ &> \omega_{\mathcal{A}} e_1\gamma e_2\alpha e_3).\end{aligned}$$

It is a contradiction. Hence, $\omega_{\mathcal{A}}(e_1\gamma e_2\alpha e_3) \geq \omega_{\mathcal{A}}(e_2)$. Therefore, $\mathcal{A}$ is an IFII of $\mathcal{S}$.

1. $\Rightarrow$ 3. Suppose that $\mathcal{A}$ is an IFII of $\mathcal{S}$. Let $e_1, e_2, e_3 \in \mathcal{S}$ and $\gamma, \alpha \in \Gamma$. Then $\frac{1-\vartheta_{\mathcal{A}}(e_1\gamma e_2\alpha e_3)}{2} \geq \frac{1-\vartheta_{\mathcal{A}}(e_2)}{2} = \frac{1-\vartheta_{\mathcal{A}}(e_2)}{2}$ and $\frac{1+\omega_{\mathcal{A}}(e_1\gamma e_2\alpha e_3)}{2} \geq \frac{1+\omega_{\mathcal{A}}(e_2)}{2} = \frac{1+\omega_{\mathcal{A}}(e_2)}{2}$. Thus, $\mathcal{I}_{\mathcal{A}}(e_1\gamma e_2\alpha e_3) \succeq \mathcal{I}_{\mathcal{A}}(e_2)$. Hence, $\mathcal{I}_{\mathcal{A}}$ is an IvFII of $\mathcal{S}$.

3. $\Rightarrow$ 1. Suppose that $\mathcal{I}_{\mathcal{A}}$ is an IvFII of $\mathcal{K}$, and let $e_1, e_2, e_3 \in \mathcal{S}$. Then $\mathcal{I}_{\mathcal{A}}(e_1\gamma e_2\alpha e_3) \succeq \mathcal{I}_{\mathcal{A}}(e_2)$. Thus, $\frac{1-\vartheta_{\mathcal{A}}(e_1\gamma e_2\alpha e_3)}{2} \geq \frac{1-\vartheta_{\mathcal{A}}(e_2)}{2}$ and $\frac{1+\omega_{\mathcal{A}}(e_1\gamma e_2\alpha e_3)}{2} \geq \frac{1+\omega_{\mathcal{A}}(e_2)}{2}$. Hence, $\omega_{\mathcal{A}}(e_1e_2e_3) \geq \omega_{\mathcal{A}}(e_2)$ and $\vartheta_{\mathcal{A}}(e_1\gamma e_2\alpha e_3) \leq \vartheta_{\mathcal{A}}(e_2)$. Therefore, $\mathcal{A}$ is an IFII of $\mathcal{S}$. □

Corollary 3.11. Suppose that $\mathcal{A} = (\omega_{\mathcal{A}}, \vartheta_{\mathcal{A}})$ be an IFS in $\mathcal{S}$. The followings are equivalent.

(1) $\mathcal{H}^{\Theta}_{\mathcal{A}}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$ for all $\Theta \in \mathcal{P}[0,1]$.

(2) $\mathcal{I}_{\mathcal{A}}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$.

For any HFS $\mathfrak{h}$ on $\mathcal{T}$, the HFS $\mathfrak{h}^*$ is defined by $\mathfrak{h}^*(e) = \{1 - \mathcal{SUP}\mathfrak{h}(e)\}$ for all $e \in \mathcal{T}$. We call $\mathfrak{h}^*$ a **supermum complement** [16] of $\mathfrak{h}$ on $\mathcal{T}$. Then $\mathcal{SUP}\mathfrak{h}^*(e) = 1 - \mathcal{SUP}\mathfrak{h}(e)$ for all $e \in \mathcal{T}$. Hence, $(\mathcal{F}_{\mathfrak{h}}, \mathcal{F}_{\mathfrak{h}^*})$ is an IFS in $\mathcal{T}$.

Theorem 3.12. *An HFS $\mathfrak{h}$ on $\mathcal{S}$ is a $\mathcal{SUP}-$HFII of $\mathcal{S}$ if and only if $(\mathcal{F}_h, \mathcal{F}_h^*)$ is an IFII of $\mathcal{S}$.*

Proof: Suppose that $\mathfrak{h}$ is a $\mathcal{SUP}-$HFII of $\mathcal{S}$, and let $e_1, e_2, e_3 \in \mathcal{S}$, $\gamma, \alpha \in \Gamma$. Then, by Theorem 3.8,

$$\mathcal{SUP}\mathfrak{h}(e_1\gamma e_2\alpha e_3) = \mathcal{F}_{\mathfrak{h}}(e_1\gamma e_2\alpha e_3) \geq \mathfrak{h}(e_2) = \mathcal{SUP}\mathfrak{h}(e_2).$$

and

$$\mathcal{F}_{\mathfrak{h}^*}(e_1\gamma e_2\alpha e_3) = 1 - \mathcal{SUP}\mathfrak{h}(e_1\gamma e_2\alpha e_3) \leq 1 - \mathcal{SUP}\mathfrak{h}(e_2) = \mathcal{F}_{\mathfrak{h}^*}(e_2).$$

Hence, $(\mathcal{F}_h, \mathcal{F}_h^*)$ is an IFII of $\mathcal{S}$.

Conversely, suppose that $(\mathcal{F}_h, \mathcal{F}_h^*)$ is an IFII of $\mathcal{S}$. Then $\mathcal{F}_h$ is FII of $\mathcal{S}$. Thus, by Theorem 3.8, $\mathfrak{h}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$. □

For HFS $\mathfrak{h}$ on $\mathcal{T}$ and $t \in [0,1]$, define

$$\mathcal{U}_{\mathcal{SUP}}(\mathfrak{h};t) = \{e \in \mathcal{T} \mid \mathcal{SUP}\mathfrak{h}(e) \geq t\}$$

and

$$\mathcal{L}_{\mathcal{SUP}}(\mathfrak{h};t) = \{e \in \mathcal{T} \mid \mathcal{SUP}\mathfrak{h}(e) \leq t\}.$$

We call the $\mathcal{U}_{\mathcal{SUP}}$ a *$\mathcal{SUP}$-upper t-level subset* and call the $\mathcal{L}_{\mathcal{SUP}}$ a *$\mathcal{SUP}$-lower t-level subset* [16] of $\mathfrak{h}$.

THEOREM 3.13. *Let $\mathfrak{h}$ is an HFS on $\mathcal{S}$. Then the following statements holds;*

(1) *$\mathfrak{h}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$ if and only if $\mathcal{U}_{\mathcal{SUP}}(\mathfrak{h};t)$ is either empty of an interior ideal of $\mathcal{S}$ for all $t \in [0,1]$.*

(2) *$\mathfrak{h}^*$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$ if and only if $\mathcal{L}_{\mathcal{SUP}}(\mathfrak{h};t)$ is either empty of an interior ideal of $\mathcal{S}$ for all $t \in [0,1]$.*

PROOF:

(1) Suppose that $\mathfrak{h}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$ and $t \in [0,1]$ such that $\mathcal{U}_{\mathcal{SUP}}(\mathfrak{h};t) \neq \emptyset$. Choose $\Theta = \{t\}$. Then $\mathcal{S}[\mathfrak{h},\Theta] = \mathcal{U}_{\mathcal{SUP}}(\mathfrak{h};t) \neq \emptyset$. By assumption, we have $\mathcal{U}_{\mathcal{SUP}}(\mathfrak{h};t) = \mathcal{S}[\mathfrak{h},\Theta]$ is an interior ideal of $\mathcal{S}$.

Conversely, suppose that $\mathcal{U}_{\mathcal{SUP}}(\mathfrak{h};t)$ is either empty of an interior ideal of $\mathcal{S}$ for all $t \in [0,1]$ and $\Theta \in \mathcal{P}[0,1]$ such that $\mathcal{S}[\mathfrak{h},\Theta] \neq \emptyset$. Choose $t = \mathcal{SUP}\Theta$. Then $\mathcal{U}_{\mathcal{SUP}}(\mathfrak{h};t) = \mathcal{S}[\mathfrak{h},\Theta] \neq \emptyset$. By assumption, we have $\mathcal{S}[\mathfrak{h},\Theta] = \mathcal{U}_{\mathcal{SUP}}(\mathfrak{h};t)$ is an interior ideal of $\mathcal{S}$. Thus, $\mathfrak{h}$ is a Θ-$\mathcal{SUP}$-HFII of $\mathcal{S}$. Hence, $\mathfrak{h}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$.

(2) Suppose that $\mathfrak{h}^*$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$ and $t \in [0,1]$ such that $\mathcal{L}_{\mathcal{SUP}}(\mathfrak{h};t) \neq \emptyset$. Choose $\Upsilon = \{1-t\}$. Then $\mathcal{S}[\mathfrak{h}^*,\Upsilon] = \mathcal{L}_{\mathcal{SUP}}(\mathfrak{h};t) \neq \emptyset$. By assumption, we have $\mathcal{L}_{\mathcal{SUP}}(\mathfrak{h};t) = \mathcal{S}[\mathfrak{h}^*,\Upsilon]$ is an interior ideal of $\mathcal{S}$.

Conversely, suppose that $\mathcal{L}_{\mathcal{SUP}}(\mathfrak{h};t)$ is either empty of an interior ideal of $\mathcal{S}$ for all $t \in [0,1]$ and $\Upsilon \in \mathcal{P}[0,1]$ such that $\mathcal{S}[\mathfrak{h}^*,\Upsilon] \neq \emptyset$. Choose $t = 1 - \mathcal{SUP}\Upsilon$. Then

$$\mathcal{L}_{\mathcal{SUP}}(\mathfrak{h};t) = \mathcal{S}[\mathfrak{h}^*,\Upsilon] \neq \emptyset.$$

By assumption, we have $\mathcal{S}[\mathfrak{h}^*,\Upsilon] = \mathcal{L}_{\mathcal{SUP}}(\mathfrak{h};t)$ is an interior ideal of $\mathcal{S}$. Thus, $\mathfrak{h}^*$ is a Ψ-$\mathcal{SUP}$-HFII of $\mathcal{S}$. Hence, $\mathfrak{h}^*$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$. □

For $\Theta, \Psi \in \mathcal{P}[0,1]$ with $\mathcal{SUP}\Theta < \mathcal{SUP}\Psi$, define a function $H_L^{(\Theta,\Upsilon)}$ as follows:

$$H_L^{(\Theta,\Psi)}\mathcal{T} \to \mathcal{P}([0,1]), e \mapsto \begin{cases} \Upsilon & \text{if } e \in I, \\ \Theta & \text{otherwise,} \end{cases}$$

THEOREM 3.14. *Let L be a non-empty subset of $\mathcal{S}$ and $\Theta, \Upsilon \in \mathcal{P}[0,1]$ with $\mathcal{SUP}\Theta < \mathcal{SUP}$. Then L is an interior ideal of $\mathcal{S}$ if and only if $\mathcal{H}_L^{(\Theta,\Upsilon)}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$.*

PROOF: Suppose that L is an interior ideal of $\mathcal{S}$ and $\mathcal{SUP}\mathcal{H}_L^{(\Theta,\Upsilon)}(e_1e_2e_3) < \mathcal{SUP}\mathcal{H}_L^{(\Theta,\Upsilon)}(e_2)$ for some $e_1, e_2, e_3 \in \mathcal{S}$ and $\gamma, \alpha \in \Gamma$. Then $\mathcal{SUP}\mathcal{H}_L^{(\Theta,\Upsilon)}(e_2) = \mathcal{SUP}\Upsilon$, which implies that $e_2 \in L$. Since L is an interior ideal of $\mathcal{S}$, we have $e_1\gamma e_2\alpha e_3 \in L$, and so

$$\mathcal{SUP}\mathcal{H}_L^{(\Theta,\Upsilon)}(e_1\gamma e_2\alpha e_3) = \mathcal{SUP}\Upsilon = \mathcal{SUP}\mathcal{H}_L^{(\Theta,\Upsilon)}(e_2).$$

It is a contradiction. Hence, $\mathcal{SUP}\mathcal{H}_L^{(\Theta,\Upsilon)}(e_1\gamma e_2\alpha e_3) \geq \mathcal{SUP}\mathcal{H}_L^{(\Theta,\Upsilon)}(e_2)$, for all $e_1, e_2, e_3 \in \mathcal{S}$ and $\gamma, \alpha \in \Gamma$. By Theorem 3.8, $\mathcal{H}_L^{(\Theta,\Upsilon)}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$.

Conversely, let $e_1, e_3 \in \mathcal{S}$, $e_2 \in L$ and $\gamma, \alpha \in \Gamma$. Then $\mathcal{H}_L^{(\Theta,\Upsilon)}(e_2) = \Upsilon$. Since $\mathcal{H}_L^{(\Theta,\Upsilon)}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$, by Theorem 3.9, we have $\mathcal{H}_L^{(\Theta,\Upsilon)}(e_1\gamma e_2\alpha e_3) \geq \mathcal{SUP}\mathcal{H}_L^{(\Theta,\Upsilon)}(e_2) = \mathcal{SUP}\Upsilon$, which implies that $e_1\gamma e_2\alpha e_3 \in L$. Hence, L is an interior ideal of $\mathcal{S}$. □

COROLLARY 3.15. Let I be a non-empty subset of $\mathcal{K}$. Then, the following statements are equivalent.

(1) L is an interior ideal of $\mathcal{K}$.

(2) $\tilde{\lambda}_L$ is a $\mathcal{SUP}$-HFII of $\mathcal{K}$.

(3) CH_L is a $\mathcal{SUP}$-HFII of $\mathcal{K}$.

4. $\mathcal{SUP}$-hesitant fuzzy translations

In this section, we define of $\mathcal{SUP}$-hesitant fuzzy translations of $\mathcal{SUP}$-HFIIs of semigroups and discuss the cencepts of extensions and intensions of $\mathcal{SUP}$-HFIIs.

For an HFS $\mathfrak{h}$ on $\mathcal{T}$, let $\mathcal{K}_{\mathfrak{h}} := 1 - \sup\{\mathcal{SUP}\mathfrak{h}(e) \mid e \in \mathcal{T}\}$.

Let $t \in [0, \mathcal{K}_{\mathfrak{h}}]$, and we say that an HFS g on $\mathcal{T}$ is $\mathcal{SUP}$-hesitnat fuzzy t^+-traslation ($\mathcal{SUP}$-HFT$_t^+$) of $\mathfrak{h}$ if $\mathcal{SUP}\mathfrak{h}(e) + t$ for all $e \in \mathcal{T}$. Then $\mathfrak{h}$ is a $\mathcal{SUP}$-HFT$_0^+$ of $\mathfrak{h}$, and in the case that ρ_1 and ρ_2 are $\mathcal{SUP}$-HFT$_t^+$ of $\mathfrak{h}$, we see that $\mathcal{SUP}\rho_1(e) = \mathcal{SUP}\rho_2(e)$ for all $e \in \mathcal{T}$ but ρ_1 may be not equal to ρ_2.

THEOREM 4.1. *Let $\mathfrak{h}$ be a $\mathcal{SUP}$-HFII of $\mathcal{S}$ and $t \in [0, \mathcal{K}_{\mathfrak{h}}]$. Then every $\mathcal{SUP}$-HFT$_t^+$ of $\mathfrak{h}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$.*

PROOF: Suppose that ρ is a $\mathcal{SUP}$-HFT$_t^+$ of $\mathfrak{h}$, and let $e_1, e_2, e_3 \in \mathcal{S}$, $\gamma, \alpha \in \Gamma$. Then

$$\mathcal{SUP}\rho(e_1\gamma e_2\alpha e_3) = \mathcal{SUP}\mathfrak{h}(e_1\gamma e_2\alpha e_3) + t \geq \mathcal{SUP}\mathfrak{h}(e_2) + t = \mathcal{SUP}\mathfrak{h}(e_2).$$

Thus, by Corollary 3.9, ρ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$. □

THEOREM 4.2. *Let $\mathfrak{h}$ be an HFII of $\mathcal{S}$ such that it is a $\mathcal{SUP}$-HFT$_t^+$ is $\mathcal{SUP}$-HFII of $\mathcal{S}$ for some $t \in [0, \mathcal{K}_{\mathfrak{h}}]$. Then $\mathfrak{h}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$.*

PROOF: Suppose that a $\mathcal{SUP}$-HFT$_t^+$ ρ of $\mathfrak{h}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$ when $t \in [0, \mathcal{K}_{\mathfrak{h}}]$. Then for all $e_1, e_2, e_3\mathcal{S}$ and $\gamma, \alpha \in \Gamma$,

$$\mathcal{SUP}\mathfrak{h}(e_1\gamma e_2\alpha e_3) = \mathcal{SUP}\rho(e_1\gamma e_2\alpha e_3) - t \geq \mathcal{SUP}\rho(e_2) - t = \mathcal{SUP}\rho(e_2).$$

Thus, by Corollary 3.9, $\mathfrak{h}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$. □

THEOREM 4.3. *Let $\mathfrak{h}$ be an HFS on $\mathcal{S}$ and $t \in [0, \mathcal{K}_{\mathfrak{h}}]$. Then a $\mathcal{SUP}$-HFT$_t^+$ of $\mathfrak{h}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$ if and only if $\mathcal{U}_{\mathcal{SUP}}(\mathfrak{h}; m - t)$ either empty or an interior ideal of $\mathcal{S}$ for all $m \in [t, 1]$.*

PROOF: (⇒) By Theorem 3.13. 1.

(⇐) Let ρ be a $\mathcal{SUP}$-HFT$_t^+$ of $\mathfrak{h}$ and $e_1, e_2, e_3 \in \mathcal{S}$, $\gamma, \alpha \in \Gamma$. Choose $m := \mathcal{SUP}\rho(e_2)$. Then $m - t = \mathcal{SUP}\rho(e_2) - t = \mathcal{SUP}\mathfrak{h}(e_2)$. Thus, $e_2 \in \mathcal{U}_{\mathcal{SUP}}(\mathfrak{h}; m - t)$. By assumption, $e_1\gamma e_2\alpha e_3 \in \mathcal{U}_{\mathcal{SUP}}(\mathfrak{h}; m - t)$. Hence,

$\mathcal{SUP}\rho(e_1\gamma e_2\alpha e_3) = \mathcal{SUP}\mathfrak{h}(e_1\gamma e_2\alpha e_3) + t \geq m = \mathcal{SUP}\rho(e_2)$. By Corollary 3.9, $\mathfrak{h}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$. □

For an HFS $\mathfrak{h}$ on $\mathcal{S}$, define $\pm_{\mathfrak{h}} := \inf\{\mathcal{SUP}\mathfrak{h}(e) \mid e \in \mathcal{S}\}$.

For $t \in [0, \pm_{\mathfrak{h}}]$ an HFS g of $\mathcal{S}$ is said to be $\mathcal{SUP}$-hesitant fuzzy t^--translation ($\mathcal{SUP}$-HFT$_{t^-}$) of $\mathfrak{h}$ if $\mathcal{SUP}\rho(e) = \mathcal{SUP}\mathfrak{h}(e) - t$ for all $e \in \mathcal{S}$. Then $\mathfrak{h}$ is a $\mathcal{SUP}$-HFT$_{0^-}$ of $\mathfrak{h}$.

THEOREM 4.4. *Let $\mathfrak{h}$ be a $\mathcal{SUP}$-HFII of $\mathcal{S}$ and $t \in [0, \pm_{\mathfrak{h}}]$. Then every $\mathcal{SUP}$-HFT$_{t^-}$ of $\mathfrak{h}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$.*

PROOF: It follows Theorem 4.1. □

THEOREM 4.5. *Let $\mathfrak{h}$ be an HFS on $\mathcal{S}$ such that its $\mathcal{SUP}$-HFT$_{t^-}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$ for some $t \in [0, \pm_{\mathfrak{h}}]$. Then $\mathfrak{h}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$.*

PROOF: It follows Theorem 4.2. □

5. Conclusion

In this paper, we study the results for $\mathcal{SUP}$-hesitant fuzzy interior ideals in Γ-semigroups. Finally, we get the relation of HFBII, $\mathcal{SUP}$-HFII and IvFII in Γ-semigroup in Theorem 3.7. In future work, we can study other results in these algebraic structures.

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SOME LOGICS IN THE VICINITY OF INTERPRETABILITY LOGICS

Abstract

In this paper we shall define semantically some families of propositional modal logics related to the interpretability logic **IL**. We will introduce the logics **BIL** and $\mathbf{BIL}^+$ in the propositional language with a modal operator $\Box$ and a binary operator $\Rightarrow$ such that $\mathbf{BIL} \subseteq \mathbf{BIL}^+ \subseteq \mathbf{IL}$. The logic **BIL** is generated by the relational structures $\langle X, R, N\rangle$, called basic frames, where $\langle X, R\rangle$ is a Kripke frame and $\langle X, N\rangle$ is a neighborhood frame. We will prove that the logic $\mathbf{BIL}^+$ is generated by the basic frames where the binary relation R is definable by the neighborhood relation N and, therefore, the neighborhood semantics is suitable to study the logic $\mathbf{BIL}^+$ and its extensions. We shall also study some axiomatic extensions of **BIL** and we will prove that these extensions are sound and complete with respect to a certain classes of basic frames. Finally, we prove that the logic $\mathbf{BIL}^+$ and some of its extensions are complete respect with the class of neighborhood frames.

Keywords: interpretability logic, Kripke frames, neighbourhood frames, Veltman semantics.

2020 Mathematical Subject Classification: 03B45, 03B60.

1. Introduction

The logic **GL** is known as the logic of provability and it is well known that **GL** is complete with respect to the class of all transitive and conversely well-founded finite Kripke frames [1, 2]. Interpretability logics is a family of classical propositional logics that extends the provability logic **GL** with

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a binary modal operator $\rhd$ used for study formal interpretability . Among these logics, the interpretability logic **IL** plays an important role [5, 6, 9]. The logic **IL** extends the provability logic **GL** by adding the binary modal operator connective $\rhd$ and the following axioms:

J1 $\Box(A \to B) \to (A \rhd B)$;

J2 $(A \rhd B) \wedge (B \rhd C) \to (A \rhd C)$;

J3 $(A \rhd C) \wedge (B \rhd C) \to ((A \vee B) \rhd C)$;

J4 $(A \rhd B) \to (\Diamond A \to \Diamond B)$;

J5 $\Diamond A \rhd A$.

Here the connective $\Diamond$ is defined as $\Diamond A := \neg\Box\neg A$.

The most commonly used semantics for **IL** and its extensions is the Veltman semantics (or ordinary Veltman semantics) [5, 6, 9, 10]. A Veltman frame is a relational structure $\langle X, R, \{S_x : x \in X\}\rangle$, where X is a non-empty set, R is a transitive and converse well-founded binary relation on X, and for each $x \in X$, S_x is a binary relation on $R(x) = \{y \in X : (x, y) \in R\}$ satisfying additional conditions. The relation R is used to interpret modal formulas $\Box A$, and the collection of binary relations $\{S_x : x \in X\}$, together with the binary relation R, is used to interpret formulas of type $A \rhd B$. De Jongh and Veltman proved that the logic **IL** is sound and complete with respect to all Veltman models [5]. Other semantics utilized for the study of **IL** is the called generalized Veltman semantics or Verbrugge semantics [8, 6]. In Verbrugge semantics the modal operator $\Box$ is interpreted as before, but the binary modality $\rhd$ is interpreted by means of a collection of neighborhood relations $\{N_x : N_x \subseteq R(x) \times \mathcal{P}(R(x)) \setminus \{\emptyset\}\}_{x \in X}$ satisfying additional conditions.

One of the main objectives of this paper is to present a family of logics that extends the normal modal logic **K** in the vicinity of the interpretability logic **IL**. We will study a logic, called *basic interpretability logic* (**BIL**), defined semantically by means of structures $\langle X, R, N\rangle$, called basic frames, where X is a non-empty set, R is a binary relation defined on X, and N is a neighborhood relation, i.e, $N \subseteq X \times \mathcal{P}(X)$ [4, 7]. The binary relation R is used to interpret the modal operator $\Box$, and the neighborhood relation N is used to interpret a binary operator $\Rightarrow$ defined as $A \Rightarrow B := \neg B \rhd \neg A$.

An important difference from Verbrugge's semantics is that we will not define a neighborhood relation for each point. We will use a single neighborhood relation for all points. We will treat initially the modalities $\Box$ and $\rhd$ independently. Thus, in principle, there is no connection between the relations R and N. In the interpretability logic **IL** the formulas $\Box A$ and $\bot \rhd \neg A$ are deductively equivalent, that is $\Box A \to (\bot \rhd \neg A)$ and $(\bot \rhd \neg A) \to \Box A$ are theorems of **IL**. In this paper these formulas are theorems in the extension $\mathbf{BIL}^+ = \mathbf{BIL} + \{\mathrm{J1}, \mathrm{J4}\}$. We will see that $\mathbf{BIL}^+$ is complete with respect to special basic frames $\langle X, R, N \rangle$ satisfying the condition: for all $x, y \in X$, $(x, y) \in R$ iff there exists $Y \in N(x)$ such that $y \in Y$. In other words, in the basic frames $\langle X, R, N \rangle$ of $\mathbf{BIL}^+$ the binary relation R is definable by means of the neighborhood relation N as $R(x) := \bigcup\{Y : Y \in N(x)\}$. This condition corresponds precisely to the fact that in this logic the formulas $\Box A$ and $\bot \rhd \neg A$ are deductively equivalents. Therefore to study extensions of $\mathbf{BIL}^+$ is enough to consider neighborhood frames instead of basic frames.

This paper is organized as follows. In Section 2 will define the basic interpretability logic **BIL**, and the basic frames. We will prove that **BIL** is sound with respect to the class of all basic frames. We shall study some axiomatic extensions of **BIL** and we will prove that these extensions are sound with respect to a certain classes of basic frames. In Section 3 we will prove that the logic **BIL** and the extensions defined in section 2 are complete. In Section 5 we shall prove that the logic $\mathbf{BIL}^+$ and some of its extensions are complete respect with the class of neighborhood frames [3, 7].

2. The basic logic BIL and some extensions

We consider a language $\mathcal{L}$ which consists of a set Var of countably many propositional variables $p, q, r, \ldots$, logical constants $\bot, \top$, and propositional connectives $\neg$, $\wedge$, $\vee$, and $\to$. The language $\mathcal{L}(\Box)$ of modal logic consists of the language $\mathcal{L}$ and a unary modal operator $\Box$. The language $\mathcal{L}(\Box, \rhd)$ of interpretability logic is the language $\mathcal{L}$ with a unary modal operator $\Box$, and a binary operator $\rhd$. In the usual interpretability logics the modal operator $\Box$ can be defined as $\neg A \rhd \bot$. But in our basic logic the connectives $\rhd$ and $\Box$ are primitives , i.e, $\Box$ is not definable by $\bot$ and $\rhd$. In the presence of classical negation, we can define a binary connective $\Rightarrow$ as

$A \Rightarrow B := \neg B \rhd \neg A$. We can also work with the language $\mathcal{L}(\Box, \Rightarrow)$, and in this case the connective $\rhd$ is defined as $A \rhd B := \neg B \Rightarrow \neg A$. In view of this interdefinability, it is necessary to consider only one of the connectives. In this paper we are going to work mainly with the language $\mathcal{L}(\Box, \Rightarrow)$. The set of all formulas is denoted by Fm.

We consider the following list of formulas and rules:

C All tautologies of Propositional Calculus;

K $\Box(A \to B) \to (\Box A \to \Box B)$;

L $\Box(\Box A \to A) \to \Box A$;

J1 $\Box(A \to B) \to (A \Rightarrow B)$;

J2 $(A \Rightarrow B) \wedge (B \Rightarrow C) \to (A \Rightarrow C)$;

J3 $(A \Rightarrow B) \wedge (A \Rightarrow C) \to (A \Rightarrow (B \wedge C))$;

J4 $(A \Rightarrow B) \to (\Box A \to \Box B)$;

J5 $A \Rightarrow \Box A$;

M $(A \Rightarrow B) \to ((\Box C \to A) \Rightarrow (\Box C \to B))$;

M0 $(A \Rightarrow B) \to ((\Box C \to A) \Rightarrow (\Box C \to \Box B))$;

P $(A \Rightarrow B) \to \Box(A \Rightarrow B)$;

P0 $(\Box A \Rightarrow B) \to \Box(A \Rightarrow B)$;

MP $\dfrac{A \quad A \to B}{B}$;

N $\dfrac{A}{\Box A}$;

RI $\dfrac{A \to B}{A \Rightarrow B}$.

A basic interpretability logic is any consistent set of formulas Λ of $\mathcal{L}(\Box, \Rightarrow)$ which contains the axioms C, K, J2, J3, and is closed under the rules MP, N and RI, and uniform substitution. The minimal basic interpretability logic is denoted by **BIL**. We also consider the logic $\mathbf{BIL}^+ := \mathbf{BIL} + \{\mathrm{J1}, \mathrm{J4}\}$.

As we will see later, **BIL** is the set of all valid formulas in the basic frames defined in Definition 2.1. The logic defined as $\mathbf{IL} := \mathbf{BIL}^+ + \{\mathrm{L}, \mathrm{J5}\}$ is known as the interpretability logic [9].

Let Λ be a basic interpretability logic. If A is a theorem of Λ we write $A \in \Lambda$ or $\vdash_\Lambda A$. If there is no risk of confusion we will write $\vdash$ instead of $\vdash_\Lambda$. If Γ is a set of formulas we write $\Gamma \vdash A$ iff there exist $A_1, \ldots, A_n \in \Gamma$ such that $\vdash (A_1 \wedge \ldots \wedge A_n) \to A$. We shall say that two formulas A and B are deductively equivalents, in symbols $A \leftrightarrow B$, if $\vdash A \to B$ and $\vdash B \to A$. It is easy to see the following equivalences and derived rules

1. $(A \Rightarrow (B \wedge C)) \leftrightarrow (A \Rightarrow B) \wedge (A \Rightarrow C)$;
2. $\Box(A \wedge B) \leftrightarrow \Box A \wedge \Box B$;
3. $\Box\top \leftrightarrow \top$;
4. $\dfrac{A \to B}{(C \Rightarrow A) \to (C \Rightarrow B)}$;
5. $\dfrac{A \to B}{(B \Rightarrow C) \to (A \Rightarrow C)}$.
6. If $A_1 \leftrightarrow B_1$ and $A_2 \leftrightarrow B_2$, then $(A_1 \Rightarrow A_2) \leftrightarrow (B_1 \Rightarrow B_2)$.

Consider the logic $\mathbf{BIL}^+$. By the axiom J1 we have that $\vdash_{\mathsf{BIL}^+} \Box(\top \to A) \to (\top \Rightarrow A)$, and by the axiom J4, we get $\vdash_{\mathsf{BIL}^+} (\top \Rightarrow A) \to \Box(\top \to A)$. As $(\top \to A) \leftrightarrow A$, we get that $\Box A \leftrightarrow (\top \Rightarrow A)$.

Let X be a non-empty set. The power set of a set X is denoted by $\mathcal{P}(X)$. Given a binary relation R on a set X, let $R(x) = \{y \in X \mid (x, y) \in R\}$, for $x \in X$. For $Y \subseteq X$, let $R[Y] = \bigcup\{R(y) : y \in Y\}$. Define the operator $\Box_R : \mathcal{P}(X) \to \mathcal{P}(X)$ as

$$\Box_R(U) = \{x \in X \mid R(x) \subseteq U\},$$

for each $U \subseteq X$. A *Kripke frame* is a pair $\langle X, R\rangle$ where X is a non-empty set and R is a binary relation on X.

A *neighbourhood frame* is a structure $\mathcal{F} = \langle X, N \rangle$, where X is a non-empty set and $N \subseteq X \times \mathcal{P}(X)$. Neighbourhood frame were initially introduced to define a semantics for non-normal modal logics [7]. The elements of $N(x)$ are called neighbourhoods.

Given a neighborhood frame $\langle X, N \rangle$ we define the binary operator

$$\Rightarrow_N : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathcal{P}(X)$$

as

$$U \Rightarrow_N V := \{x \in X : \forall Y \in N(x)\, (\text{if } Y \subseteq U \text{ then } Y \subseteq V)\}.$$

We note that the structure $\langle \mathcal{P}(X), \cup, \cap, \to, ^c, \Rightarrow_N, \Box_R \rangle$ is a Boolean algebra with a unary modal operator $\Box_R$ and with a binary operator $\Rightarrow_N$, where the boolean negation is defined as $U^c := X \setminus U$, and the implication $\to$ is defined as $U \to V := U^c \cup V$, for all $U, V \in \mathcal{P}(X)$. Thus, $\langle \mathcal{P}(X), \Rightarrow_N, \Box_R \rangle$ is a particular case of Boolean algebras with operators [1].

Definition 2.1. We say that a triple $\mathcal{F} = \langle X, R, N \rangle$ is a *basic interpretability frame* if $\langle X, R \rangle$ is a Kripke frame and $\langle X, N \rangle$ is a neighborhood frame.

Lemma 2.2. *Let $\langle X, R, N \rangle$ be a basic frame. Then the algebra $\langle \mathcal{P}(X), \Rightarrow_N, \Box_R \rangle$ satisfies the following identities*

(1) $\Box_R(X) = X$;

(2) $\Box_R(U \to V) \subseteq \Box_R(U) \to \Box_N(V)$;

(3) $U \Rightarrow_N U = X$;

(4) $(U \Rightarrow_N V) \cap (V \Rightarrow_N W) \subseteq U \Rightarrow_N W$;

(5) $(U \Rightarrow_N V) \cap (U \Rightarrow_N W) = U \Rightarrow_N (V \cap W)$;

(6) *If* $U \subseteq V$, *then* $W \Rightarrow_N U \subseteq W \Rightarrow_N V$ *and* $V \Rightarrow_N W \subseteq U \Rightarrow_N W$.

Proof: As example, we will prove the condition (4). Let $x \in (U \Rightarrow_N V) \cap (V \Rightarrow_N W)$. Let $Y \in N(x)$ and such that $Y \subseteq U$. As $x \in U \Rightarrow_N V$, we get $Y \subseteq V$, and since $x \in V \Rightarrow_N W$ we have $Y \subseteq W$. Thus, $(U \Rightarrow_N V) \cap (V \Rightarrow_N W) \subseteq U \Rightarrow_N W$. $\Box$

Definition 2.3. A *valuation* on a basic frame $\mathcal{F} = \langle X, R, N\rangle$ is a function $V : Var \to \mathcal{P}(X)$. A valuation V can be extended recursively to the set of all formulas Fm by means of the following clauses:

1. $V(\top) = X$, $V(\bot) = \emptyset$,
2. $V(p \wedge q) = V(p) \cap V(q)$,
3. $V(p \vee q) = V(p) \cup V(q)$,
4. $V(p \to q) = V(p)^c \cup V(q)$,
5. $V(\Box p) = \{x \in X \mid R(x) \subseteq V(p)\}$,
6. $V(p \Rightarrow q) = \{x : \forall Y \in N(x)\,(Y \subseteq V(p) \text{ implies } Y \subseteq V(q))\}$.

A *model* is a pair $\mathcal{M} = \langle \mathcal{F}, V\rangle$, where $\mathcal{F}$ is a basic frame and V is a valuation on it.

It is easy to see that $V(\Box A) = \Box_R V(A)$, and $V(A \Rightarrow B) = V(A) \Rightarrow_N V(B)$, for any formulas A and B. A formula A is *valid* in a model $\langle \mathcal{F}, V\rangle$ if $V(A) = X$. A formula A is valid in a basic frame $\mathcal{F}$, in symbols $\mathcal{F} \models A$, if $V(A) = X$, for all valuation V defined on it. A set of formulas Γ is valid in a basic frame $\mathcal{F}$, in symbols $\mathcal{F} \models \Gamma$, if $\mathcal{F} \models A$ for all $A \in \Gamma$. The class of all basic frames validating the set of formulas Γ will be denoted by $\mathrm{Fr}(\Gamma)$. For any class of basic frames $\mathbf{F}$, a formula A is valid in $\mathbf{F}$, notation $\models_{\mathbf{F}} A$, if $\mathcal{F} \models A$ for all $\mathcal{F} \in \mathbf{F}$. The set of all formulas valid in $\mathbf{F}$ is $\mathrm{Th}(\mathbf{F}) = \{A \in Fm : \models_{\mathbf{F}} A\}$. If $\mathbf{F} = \{\mathcal{F}\}$,we write $\mathrm{Th}(\mathcal{F})$ instead of $\mathrm{Th}(\{\mathcal{F}\})$.

Let P be a first or higher-order frame condition in the language $\{R, N\}$. We say that the condition P is *valid* in a basic frame $\mathcal{F}$, in notation $\mathcal{F} \Vdash P$, if it is valid in the sense of a first or higher order structure. We shall that a frame condition P characterizes a formula A if for every basic frame $\mathcal{F}$, $\mathcal{F} \Vdash P$ iff $\mathcal{F} \models A$.

A logic Λ is *sound* with respect to a class of basic frames $\mathbf{F}$ if $\Lambda \subseteq \mathrm{Th}(\mathbf{F})$. A logic Λ is complete with respect to a class of basic frames $\mathbf{F}$ if $\mathrm{Th}(\mathbf{F}) \subseteq \Lambda$. A logic Λ is characterized by a class $\mathbf{F}$ of basic frames or is complete relative to a class of basic frames $\mathbf{F}$ if $\Lambda = \mathrm{Th}(\mathbf{F})$. Moreover, it is frame complete if $\Lambda = \mathrm{Th}(\mathrm{Fr}(\Lambda))$. It is clear that a logic Λ is frame complete if and only if it is characterized by some class of frames.

We first prove that the logic **BIL** is sound with respect to the class of all basic frames.

Proposition 2.4 (Soundness). Let **Fr** be the class of all basic frames. Then **BIL** $\subseteq$ Th(**Fr**) and Fr(**BIL**) = **Fr**.

Proof: By Lemma 2.2 (4) and (5) we have that J2 and J3 are valid in all basic frames, and it is clear that the rules Modus Ponens preserve the validity. Then it suffices to prove that the rule RI preserve the validity. But this also follows from Lemma 2.2 (6). Thus, we have that every theorem of **BIL** is valid in every basic frames, i.e., **BIL** $\subseteq$ Th(**Fr**). On the other hand, it is clear that Fr(**BIL**) = **Fr**. □

Now we are going to introduce certain relational conditions defined in basic frames and we are going to prove soundness of extensions of **BIL** respect to these relational conditions. Let us consider the following relational conditions:

RJ1 If $(x, Y) \in N$, then $Y \subseteq R(x)$.

RJ4 If $(x, y) \in R$, then there exists $Y \subseteq X$ such that $(x, Y) \in N$, $Y \subseteq R(x)$ and $y \in Y$.

RJ5 If $(x, Y) \in N$, then $R(y) \subseteq Y$ for any $y \in Y$.

RP If $(x, y) \in R$ and $(y, Y) \in N$, then $(x, Y) \in N$.

RP0 If $(x, y) \in R$ and $(y, Y) \in N$, then there exists $Z \subseteq X$ such that $y \in Z$, $R[Z] \subseteq Y \subseteq Z$ and $(x, Z) \in N$.

RM If $(x, Y) \in N$ and $y \in Y$, then there exists $Z \subseteq X$ such that $(x, Z) \in N$, $y \in Z \subseteq Y$ and $R[Z] \subseteq R(y)$.

RM0 If $(x, Y) \in N$, $y \in Y$ and $(y, z) \in R$, then there exists $Z \subseteq X$ such that $(x, Z) \in N$, $z \in Z \subseteq Y$ and $R[Z] \subseteq R(y)$.

Theorem 2.5 (Soundness of extensions of **BIL**). *Let $\mathcal{F} = \langle X, R, N\rangle$ be a basic frame.*

1. $\mathcal{F} \Vdash$ RJ1 *iff* $\mathcal{F} \models \Box(A \to B) \to (A \Rightarrow B)$.
2. $\mathcal{F} \Vdash$ RJ4 *iff* $\mathcal{F} \models (A \Rightarrow B) \to (\Box A \to \Box B)$.
3. $\mathcal{F} \Vdash$ RJ5 *iff* $\mathcal{F} \models A \Rightarrow \Box A$.
4. $\mathcal{F} \Vdash$ RP *implies that* $\mathcal{F} \models (A \Rightarrow B) \to \Box(A \Rightarrow B)$.

5. $\mathcal{F} \Vdash$ RP0 *implies that* $\mathcal{F} \models (\Box A \Rightarrow B) \to \Box(A \Rightarrow B)$.

6. $\mathcal{F} \Vdash$ RM *implies that* $\mathcal{F} \models (A \Rightarrow B) \to ((\Box C \to A) \Rightarrow (\Box C \to B))$.

7. $\mathcal{F} \Vdash$ RM0 *implies that* $\mathcal{F} \models (A \Rightarrow B) \to ((\Box C \to A) \Rightarrow (\Box C \to \Box B))$.

PROOF: 1. Suppose that $\mathcal{F} \models \Box(A \to B) \to (A \Rightarrow B)$. Let $Y \in N(x)$. Consider the subset $U = R(x)$. Then $x \in \Box_R(U) = \Box_R(X \to U) \subseteq X \Rightarrow_N U$. As $Y \subseteq X$, we get $Y \subseteq U = R(x)$.

Assume that $\mathcal{F}$ satisfies RJ1. Let $U, V \in \mathcal{P}(X)$. Let $x \in \Box_R(U \to V)$ and $Y \in N(x)$ such that $Y \subseteq U$. Then $Y \subseteq R(x)$. As $Y \subseteq R(x) \cap U \subseteq V$, we have $Y \subseteq V$. Thus, $x \in U \Rightarrow_N V$. Then $\mathcal{F} \models \Box(A \to B) \to (A \Rightarrow B)$.

2. Assume that $\mathcal{F} \models (A \Rightarrow B) \to (\Box A \to \Box B)$. Let $x, y \in X$ such that $(x, y) \in R$. Consider the subsets $U = R(x)$ and $V = \{y\}^c = X - \{y\}$. Then $x \in \Box_R(U)$ and as $R(x) \nsubseteq \{y\}^c$, we get that $x \notin \Box_R(V)$. So, $x \notin \Box_R(U)^c \cup \Box_R(V) = \Box_R(U) \to \Box_R(V)$. As $U \Rightarrow_N V \subseteq \Box_R(U) \to \Box_R(V)$, we have $x \notin U \Rightarrow_N V$. Then there exists $Y \in N(x)$ such that $Y \subseteq U$ and $Y \nsubseteq V = \{y\}^c$, i.e., $Y \subseteq R(x)$ and $y \in Y$.

Assume that $\mathcal{F}$ satisfies RJ4. Let $U, V \in \mathcal{P}(X)$ and $x \in U \Rightarrow_N V$. Suppose that $x \in \Box_R(U)$. We prove that $x \in \Box_R(V)$. Let $y \in R(x)$. By hypothesis there exists $Y \in N(x)$ such that $Y \subseteq R(x)$ and $y \in Y$. As $R(x) \subseteq U$, we have $Y \subseteq U$, and as $x \in U \Rightarrow_N V$, we get $Y \subseteq V$. Thus, $y \in V$, i.e., $x \in \Box_R(V)$.

3. Assume that $\mathcal{F} \models A \Rightarrow \Box A$. Let $Y \in N(x)$ and $y \in Y$. Suppose that $R(y) \nsubseteq Y$. Then there exists $z \in R(y)$ such that $z \notin Y$. Let $U = \{z\}^c$. So, $Y \subseteq U$, and as $x \in X = U \Rightarrow_N \Box_R(U)$, we get $Y \subseteq \Box_R(U)$. Then $R(y) \subseteq U = \{z\}^c$, which is a contradiction. Thus, $R(y) \subseteq Y$.

Assume that $\mathcal{F}$ satisfies RJ5. We prove that $U \Rightarrow_N \Box_R(U) = X$ for any $U \subseteq X$. Let $x \in X$,$U \subseteq X$ and $Y \in N(x)$ such that $Y \subseteq U$. Let $y \in Y$. Then $R(y) \subseteq Y \subseteq U$, i.e., $y \in \Box_R(U)$. Thus, $Y \subseteq \Box_R(U)$.

4. Assume that $\mathcal{F} \models$ RP. Let $U, V \in \mathcal{P}(X)$, $x \in X$, and suppose that $x \in U \Rightarrow_N V$. We prove that $R(x) \subseteq U \Rightarrow_N V$. Let $y \in X$ and $Y \subseteq X$ such that $(x, y) \in R$, $(y, Y) \in N$ and $Y \subseteq U$. Then $(x, Y) \in N$, and as $x \in U \Rightarrow_N V$, we have $Y \subseteq V$. Thus, $x \in \Box_R(U \Rightarrow_N V)$.

5. Assume that $\mathcal{F} \Vdash$ RP0. Let $U, V \in \mathcal{P}(X)$ and $x \in X$. Suppose that $x \in \Box_R U \Rightarrow_N V$. We prove that $x \in \Box_R(U \Rightarrow_N V)$. Let $y \in X$ and $Y \subseteq X$ such that $(x, y) \in R$, $(y, Y) \in N$, and $Y \subseteq U$. By hypothesis there exist $Z \subseteq X$ such that $y \in Z$, $R[Z] \subseteq Y \subseteq Z$ and $(x, Z) \in N$. Since

$R[Z] \subseteq Y \subseteq U$, we have $Z \subseteq \Box_R U$. As $x \in \Box_R U \Rightarrow_N V$, $Z \subseteq V$. Now, by the inclusion $Y \subseteq Z$ we get $Y \subseteq V$. Thus, $y \in U \Rightarrow_N V$.

6. Assume that $\mathcal{F} \Vdash$ RM. Let $U, V, W \in \mathcal{P}(X)$ and $x \in X$ such that $x \in U \Rightarrow_N V$. Let $Y \subseteq X$ such that $(x, Y) \in N$ and $Y \subseteq \Box_R(W) \to U$, i.e., $Y \cap \Box_R(W) \subseteq U$. We prove that $Y \cap \Box_R(W) \subseteq V$. Let $y \in Y \cap \Box_R(W)$. By hypothesis, there exists $Z \subseteq X$ such that $(x, Z) \in N$, $y \in Z \subseteq Y$ and $R[Z] \subseteq R(y)$. As $y \in \Box_R(W)$, we have $R[Z] \subseteq R(y) \subseteq W$, i.e., $Z \subseteq \Box_R(W)$. Then $Z \subseteq Y \cap \Box_R(W) \subseteq U$. Since $(x, Z) \in N$ and $x \in U \Rightarrow_N V$, we get $Z \subseteq V$. Finally, as $y \in Z$, we have $y \in V$.

7. Assume that $\mathcal{F} \Vdash$ RM0. Let $U, V, W \in \mathcal{P}(X)$ and $x \in X$ such that $x \in U \Rightarrow_N V$. Let $Y \subseteq X$ such that $(x, Y) \in N$ and $Y \subseteq \Box_R(W) \to U$, i.e., $Y \cap \Box_R(W) \subseteq U$. We prove that $Y \cap \Box_R(W) \subseteq \Box_R(V)$. Let $y \in Y \cap \Box_R(W)$. We need to prove that $y \in \Box_R(V)$. Let $z \in X$ such that $(y, z) \in R$. By hypothesis, there exists $Z \subseteq X$ such that $(x, Z) \in N$, $z \in Z \subseteq Y$ and $R[Z] \subseteq R(y)$. Since, $y \in \Box_R(W)$, we have $R[Z] \subseteq R(y) \subseteq W$, i.e., $Z \subseteq \Box_R(W)$. Then $Z \subseteq Y \cap \Box_R(W) \subseteq U$. Since $(x, Z) \in N$ and $x \in U \Rightarrow_N V$, we get $Z \subseteq V$. Finally, as $z \in Z$, we have $z \in V$, i.e., $y \in \Box_R(V)$. □

From Theorem 2.5 we have that a logic Λ obtained by extending **BIL** by any subset of formulas of the set {J1, J4, J5, M, M0, P,P0} is sound respect with an adequate class of basic frames.

3. Canonical models and completeness theorem

In this section we introduce the canonical basic frame and model for **BIL** and some its extensions. Throughout this section Λ will denote any logic such that **BIL** $\subseteq \Lambda$.

We follow the standard strategy: in order to prove completeness of a logic Λ with respect to a class of models **M**, we define the canonical frame $\mathcal{F}_\Lambda$ and the canonical model $\langle \mathcal{F}_\Lambda, V_\Lambda \rangle$ and we prove that $\langle \mathcal{F}_\Lambda, V_\Lambda \rangle \in$ **M**, and for any formula A, $A \in \Lambda$ iff A is valid in $\langle \mathcal{F}_\Lambda, V_\Lambda \rangle$. This means that logic Λ is *canonical*. From this fact we have that the completeness of Λ with respect the class **M** immediately follows.

A set of formulas Γ is a theory of Λ, or an Λ-theory, if $\Lambda \subseteq \Gamma$, it is closed under $\vdash_\Lambda$, i.e., $A \in \Gamma$ and $A \vdash_\Lambda B$, then $B \in \Gamma$, and it is closed under $\wedge$, i.e., if $A, B \in \Gamma$, then $A \wedge B \in \Gamma$. A theory Γ is Λ-consistent if $\bot \notin \Gamma$. When there is no risk of confusion, we will directly say that Γ is a theory instead of Γ is a Λ-theory. The set of all theories of Λ is denoted by $\mathcal{T}(\Lambda)$. A theory Γ is complete if it is consistent and for every formula A, $A \in \Gamma$ or $\neg A \in \Gamma$. A consistent theory Γ is maximal if for any consistent theory Δ such that $\Gamma \subseteq \Delta$ we have that $\Gamma = \Delta$. It is clear that a theory Γ is complete if and only if it is maximal if and only if it is consistent and for all formulas A, B, if $A \vee B \in \Gamma$ then $A \in \Gamma$ or $B \in \Gamma$.

Let X_Λ be the set of all maximal Λ-theories. By the Lindenbaum's lemma, for every consistent theory T there exists a maximal theory Γ such that $T \subseteq \Gamma$. Moreover, for each formula A, if $A \notin T$, then there exists a maximal theory Γ such that $T \subseteq \Gamma$ and $A \notin \Gamma$. The set of maximal theories determined by a theory T is the set

$$\hat{T} := \{\Gamma \in X_\Lambda : T \subseteq \Gamma\}.$$

Similarly, the set of maximal theories determined by a formula A is the set $\hat{A} = \{\Gamma \in X_\Lambda : A \in \Gamma\}$. We note that if T and H are two theories, $T \subseteq H$ iff $\hat{H} \subseteq \hat{T}$. This fact will be used in several proofs.

For each $\Gamma \in X_\Lambda$ and for each non-empty set Z of formulas we define the set of formulas:

$$\mathsf{D}_\Gamma(Z) := \{A \in Fm : \exists C_1, \ldots, C_n \in Z \ (C_1 \wedge \ldots\ldots \wedge C_n) \Rightarrow A \in \Gamma\}.$$

Lemma 3.1. *For any $\Gamma \in X_\Lambda$ and for any non-empty set Z of formulas, $\mathsf{D}_\Gamma(Z)$ is a theory such that $Z \subseteq \mathsf{D}_\Gamma(Z)$, and for all $A, B \in Fm$, if $A \Rightarrow B \in \Gamma$ and $A \in \mathsf{D}_\Gamma(Z)$, then $B \in \mathsf{D}_\Gamma(Z)$.*

Proof: Let $\Gamma \in X_\Lambda$ and let Z be a non-empty set of formulas. As $C \Rightarrow C \in \Gamma$, for each $C \in Z$, we get $Z \subseteq \mathsf{D}_\Gamma(Z)$.

Since Z is a non-empty set, there exists $C \in Z$. As $C \to \top \in \Gamma$, we have $C \Rightarrow \top \in \Gamma$. So, $\top \in \mathsf{D}_\Gamma(Z)$.

Let $A, B \in \mathsf{D}_\Gamma(Z)$. We prove that $A \wedge B \in \mathsf{D}_\Gamma(Z)$. Then there exist $C_1, \ldots, C_n, D_1, \ldots D_m \in Z$ such that $(C_1 \wedge \ldots \wedge C_n) \Rightarrow A \in \Gamma$ and $(D_1 \wedge \ldots \wedge D_m) \Rightarrow B \in \Gamma$. Let $C = C_1 \wedge \ldots \wedge C_n$ and $D = D_1 \wedge \ldots \wedge D_m$. Then $(C \wedge D) \Rightarrow C \in \Gamma$ and $(C \wedge D) \Rightarrow D \in \Gamma$. So, by axiom J2 we have $(C \wedge D) \Rightarrow A \in \Gamma$ and $(C \wedge D) \Rightarrow B \in \Gamma$. By J3, $(C \wedge D) \Rightarrow (A \wedge B) \in \Gamma$. Thus, $A \wedge B \in \mathsf{D}_\Gamma(Z)$.

We prove that $\mathsf{D}_\Gamma(Z)$ is closed under $\vdash$. Let $A \in \mathsf{D}_\Gamma(Z)$ and $A \vdash B$. Then there exist $C_1, \ldots, C_n \in Z$ such that $(C_1 \wedge \ldots \wedge C_n) \Rightarrow A \in \Gamma$. As $\vdash A \to B$, by the rule R1 and the axiom J2 we have $\vdash ((C_1 \wedge \ldots \wedge C_n) \Rightarrow A) \to ((C_1 \wedge \ldots \wedge C_n) \Rightarrow B)$. Since Γ is a theory, $(C_1 \wedge \ldots \wedge C_n) \Rightarrow B \in \Gamma$.

Therefore, $B \in \mathsf{D}_\Gamma(Z)$.

Let $A, B \in Fm$ such that $A \Rightarrow B \in \Gamma$ and $A \in \mathsf{D}_\Gamma(Z)$. Then there exist $C_1, \ldots, C_n \in Z$ such that $(C_1 \wedge \ldots \wedge C_n) \Rightarrow A \in \Gamma$. So, $((C_1 \wedge \ldots \wedge C_n) \Rightarrow A) \wedge (A \Rightarrow B) \in \Gamma$. By axiom J2, $(C_1 \wedge \ldots \wedge C_n) \Rightarrow B \in \Gamma$, i.e., $B \in \mathsf{D}_\Gamma(Z)$. □

We are now in position to define the canonical model of any logic Λ that extends the logic **BIL**.

Definition 3.2. The *canonical basic frame* of Λ is the relational structure

$$\mathcal{F}_\Lambda := \langle X_\Lambda, R_\Lambda, N_\Lambda \rangle ,$$

where

1. X_Λ is the set of all maximal theories;
2. R_Λ is a binary relation defined on X_Λ by
$$(\Gamma, \Delta) \in R_\Lambda \text{ iff } \Box^{-1}(\Gamma) \subseteq \Delta,$$
where $\Box^{-1}(\Gamma) = \{A \in Fm : \Box A \in \Gamma\}$;
3. N_Λ is a subset of $X_\Lambda \times \mathcal{P}(X_\Lambda)$ defined by
$$(\Gamma, Y) \in N_\Lambda \text{ iff } \exists T \in \mathcal{T}(\Lambda) \left(Y = \hat{T} \text{ and } \mathsf{D}_\Gamma(T) \subseteq T \right).$$

Since the image of the relation N_Λ is the family

$$\left\{ Y \subseteq X_\Lambda : \exists T \in \mathcal{T}(\Lambda)\, (Y = \hat{T}) \right\},$$

we can also define the relation N_Λ as

$$\left(\Gamma, \hat{T}\right) \in N_\Lambda \text{ iff } \forall A, B \in Fm \; (A \Rightarrow B \text{ and } A \in T \text{ then } B \in T).$$

We define the canonical valuation V_Λ given by $V_\Lambda(p) = \{\Gamma \in X_\Lambda : p \in \Gamma\}$, for every propositional variable p. We note that $V_\Lambda(p) = \hat{p}$, for each variable p.

In the following result we need recall that for any formula A and for any consistent theory T, $A \in T$ iff $A \in \Gamma$, for any $\Gamma \in \hat{T}$.

Lemma 3.3. *Let* $A, B \in Fm$. *Let* Γ *be a maximal theory. Then* $A \Rightarrow B \notin \Gamma$ *iff there exists a consistent theory* T *such that* $(\Gamma, \hat{T}) \in N_\Lambda$, $A \in T$ *and* $B \notin T$.

Proof: Assume that $A \Rightarrow B \notin \Gamma$. Let us consider the theory

$$T = \mathsf{D}_\Gamma(\{A\}) = \{C \in Fm : A \Rightarrow C \in \Gamma\}.$$

By Lemma 3.1 $(\Gamma, \hat{T}) \in N_\Lambda$, $A \in T$, and $B \notin T$. The proof of the other direction is immediate. □

Lemma 3.4. *For every maximal theory* Γ *and for any formula* A,

$$\Gamma \in V_\Lambda(A) \text{ iff } A \in \Gamma.$$

Proof: The proof is by induction on the construction of A. For atomic and propositional formulas the proof is standard. The case of formulas $\Box A$ is usual (see for example [1]). Let $A, B \in Fm$. Let Γ be a maximal theory. Let $A \Rightarrow B \in \Gamma$. We need to show that $\Gamma \in V_\Lambda(A \Rightarrow B)$. Suppose that $\hat{T} \in N_\Lambda(\Gamma)$ and $\hat{T} \subseteq V_\Lambda(A)$. Then, $A \in T$. As $A \Rightarrow B \in \Gamma$, $A \in T$ and $\hat{T} \in N_\Lambda(\Gamma)$, we get $B \in T$. By the induction hypothesis, $\hat{T} \subseteq V_\Lambda(B)$. Thus, $\Gamma \in V_\Lambda(A \Rightarrow B)$.

On the other hand, if $A \Rightarrow B \notin \Gamma$, then by Lemma 3.3 there exists a consistent theory T such that $(\Gamma, \hat{T}) \in N_\Lambda$, $A \in T$ but $B \notin T$. By induction hypothesis, $\hat{T} \subseteq V_\Lambda(A)$ and $\hat{T} \nsubseteq V_\Lambda(B)$, i.e., $\Gamma \notin V_\Lambda(A \Rightarrow B)$. □

Theorem 3.5 (Completeness of **BIL**). *Let* **Fr** *be the class of all basic frames. Then,* $\mathbf{BIL} = \mathsf{Th}(\mathbf{Fr})$.

Proof: If A is a formula such that $A \notin \mathbf{BIL}$, then there exists a maximal theory Γ such that $A \notin \Gamma$. By Lemma 3.4, $\Gamma \notin V_\Lambda(A)$. Then A is not valid in the canonical model $\langle \mathcal{F}_{\mathbf{BIL}}, V_{\mathbf{BIL}} \rangle$ of **BIL**. Thus, A is not valid in the canonical frame $\mathcal{F}_{\mathbf{BIL}}$ of **BIL**. i.e., $A \notin \mathrm{Th}(\mathbf{Fr})$. □

4. Completeness of extensions of BIL

Our next aim is to prove the completeness for several extensions of **BIL**. To prove the completeness of the extensions of **BIL** we will proceed in the usual way. That is, we are going to prove that the canonical basic frame of each logic Λ such that $\mathbf{BIL} \subseteq \Lambda$ is a basic frame of Λ.

Proposition 4.1. Let Λ be a logic such that $\mathbf{BIL} \subseteq \Lambda$. Then

(1) $\Box(A \to B) \to (A \Rightarrow B) \in \Lambda$ iff $\mathcal{F}_\Lambda \Vdash$ RJ1.

(2) $(A \Rightarrow B) \to (\Box A \to \Box B) \in \Lambda$ iff $\mathcal{F}_\Lambda \Vdash$ RJ4.

(3) $A \Rightarrow \Box A \in \Lambda$ iff $\mathcal{F}_\Lambda \Vdash$ RJ5.

Proof: (1) $\Rightarrow$) Let $\Gamma \in X_\Lambda$ and let T be a theory such that $(\Gamma, \hat{T}) \in N_\Lambda$. Let $A \in \Box^{-1}(\Gamma)$. As $\Box(\top \to A) \to (\top \Rightarrow A) \in \Gamma$ and $\Box(\top \to A) \leftrightarrow \Box A$, we have $\top \Rightarrow A \in \Gamma$. Since $\top \in T$ and $(\Gamma, \hat{T}) \in N_\Lambda$, $A \in T$. Thus, $\Box^{-1}(\Gamma) \subseteq T$, and this is equivalent to the inclusion $\hat{T} \subseteq R_\Lambda(\Gamma)$.

$\Leftarrow$). Suppose that $\mathcal{F}_\Lambda \Vdash$ RJ1 and that there exist formulas A and B such that $\Box(A \to B) \to (A \Rightarrow B) \notin \Lambda$. Then there exists a maximal theory Γ such that $\Box(A \to B) \to (A \Rightarrow B) \notin \Gamma$. So, $\Box(A \to B) \in \Gamma$ and $A \Rightarrow B \notin \Gamma$. By Lemma 3.3 there exists a theory T such that $(\Gamma, \hat{T}) \in N_\Lambda$, $A \in T$ but $B \notin T$. By hypothesis, $\Box^{-1}(\Gamma) \subseteq T$. So $A \to B \in T$ and by MP, $B \in T$, which is a contradiction. Thus, $\Box(A \to B) \to (A \Rightarrow B) \in \Lambda$.

(2) $\Rightarrow$) Let $\Gamma, \Delta \in X_\Lambda$. Suppose that $(\Gamma, \Delta) \in R_\Lambda$. Let us consider the theory $T = \Box^{-1}(\Gamma)$. We prove that $(\Gamma, \hat{T}) \in N_\Lambda$. Let $A \Rightarrow B \in \Gamma$ and $A \in \Box^{-1}(\Gamma)$. So $\Box A \to \Box B \in \Gamma$ and $\Box A \in \Gamma$. Then $\Box B \in \Gamma$. Thus, $(\Gamma, \hat{T}) \in N_\Lambda$. It is clear that $\Delta \in \hat{T}$ and $\hat{T} \subseteq R_\Lambda(\Gamma)$.

$\Leftarrow$) Suppose that $\mathcal{F}_\Lambda \Vdash$ RJ4. We suppose that there exist formulas A and B such that $(A \Rightarrow B) \to (\Box A \to \Box B) \notin \Lambda$. Then there exists a maximal theory Γ such that $A \Rightarrow B \in \Gamma$, $\Box A \in \Gamma$ and $\Box B \notin \Gamma$. By Lemma 3.4 there exists $\Delta \in X_\Lambda$ such that $\Delta \in R_\Lambda(\Gamma)$ and $B \notin \Delta$. By hypothesis, there exists a theory T such that $(\Gamma, \hat{T}) \in N_\Lambda$, $\Delta \in \hat{T}$ and $\hat{T} \subseteq R_\Lambda(\Gamma)$. So, $A \in \Box^{-1}(\Gamma) \subseteq T \subseteq \Delta$. As $A \Rightarrow B \in \Gamma$, $A \in T$ and $(\Gamma, \hat{T}) \in N_\Lambda$, we get $B \in T$. So, $B \in \Delta$, which is a contradiction. Therefore, $(A \Rightarrow B) \to (\Box A \to \Box B) \in \Lambda$.

(3) $\Rightarrow$) Let $\Gamma \in X_\Lambda$ and let T be a theory such that $(\Gamma, \hat{T}) \in N_\Lambda$. We prove that for any $\Delta \in \hat{T}$, $R_\Lambda(\Delta) \subseteq \hat{T}$, $T \subseteq \Box^{-1}(\Delta)$. Let $A \in T$. As $A \Rightarrow \Box A \in \Gamma$, and $(\Gamma, \hat{T}) \in N_\Lambda$, we get $\Box A \in T \subseteq \Delta$, i.e., $A \in \Box^{-1}(\Delta)$.

The direction $\Leftarrow$) is easy and left to the reader. $\Box$

Corollary 4.2. Let Λ be any logic such that $\mathbf{BIL}^+ \subseteq \Lambda$. For all $\Gamma, \Delta \in X_\Lambda$,

$(\Gamma, \Delta) \in R_\Lambda$ iff there exists a theory T such that $(\Gamma, \hat{T}) \in N_\Lambda$ and $T \subseteq \Delta$.

According to this result we have that in any extension of the logic $\mathbf{BIL}^+$ the canonical relation R_Λ is definable by means of the canonical neighborhood relation N_Λ. This fact will be used in Section 5 to propose a simplify semantics for extension of $\mathbf{BIL}^+$.

Lemma 4.3. *Let Λ be a logic such that $\mathbf{BIL} \subseteq \Lambda$.*

(1) *If $(A \Rightarrow B) \to ((\Box C \to A) \Rightarrow (\Box C \to B))$ is an axiom schema of Λ, then $((A \wedge \Box C) \Rightarrow B) \to (A \Rightarrow (\Box C \to B)) \in \Lambda$.*

(2) *If $(A \Rightarrow B) \to ((\Box C \to A) \Rightarrow (\Box C \to \Box B))$ is an axiom schema of Λ, then $((A \wedge \Box C) \Rightarrow B) \to (A \Rightarrow (\Box C \to \Box B)) \in \Lambda$.*

Proof: We prove only (1). The proof of (2) is similar.

Suppose that $(A \Rightarrow B) \to (\Box C \to A) \Rightarrow (\Box C \to B)$ is an axiom schema of Λ. Then the following formula is an instance of this axiom

$$((A \wedge \Box C) \Rightarrow B) \to ((\Box C \to (A \wedge \Box C)) \Rightarrow (\Box C \to B)).$$

As $(\Box C \to (A \wedge \Box C)) \leftrightarrow (\Box C \to A)$, we have

$$((A \wedge \Box C) \Rightarrow B) \to ((\Box C \to A) \Rightarrow (\Box C \to B)) \in \Lambda. \tag{4.1}$$

Since $A \to (\Box C \to (A \wedge \Box C)) \in \Lambda$, by rule RI

$$A \Rightarrow (\Box C \to (A \wedge \Box C)) \in \Lambda,$$

and consequently

$$A \Rightarrow (\Box C \to A) \in \Lambda. \tag{4.2}$$

By axiom J2 we get

$$[(A \Rightarrow (\Box C \to A)) \wedge ((\Box C \to A) \Rightarrow (\Box C \to B))] \to (A \Rightarrow (\Box C \to B)) \in \Lambda,$$

and by (4.2) we have

$$((\Box C \to A) \Rightarrow (\Box C \to B)) \to (A \Rightarrow (\Box C \to B)) \in \Lambda. \tag{4.3}$$

Finally, by (4.1), (4.3) and axiom J2 we get

$$((A \wedge \Box C) \Rightarrow B) \to (A \Rightarrow (\Box C \to B)) \in \Lambda. \qquad \Box$$

For each theory T, define $\Box T := \{\Box A : A \in T\}$. The following lemma is necessary in the proof of Theorem 4.5.

Lemma 4.4. *Let H be a consistent theory and let Δ be a maximal theory. Then, $R_\Lambda[\hat{H}] \subseteq R_\Lambda(\Delta)$ iff $\Box^{-1}(\Delta) \subseteq \Box^{-1}(H)$.*

Proof: Suppose that $R_\Lambda[\hat{H}] \subseteq R_\Lambda(\Delta)$ but $\Box^{-1}(\Delta) \nsubseteq \Box^{-1}(H)$. Then there exists $\Box D \in \Delta$ such that $\Box D \notin H$. So, there are maximal theories G and K such that $H \subseteq G$, $\Box D \notin G$, $(G, K) \in R_\Lambda$ and $D \notin K$. Then $G \in \hat{H}$ and $K \in R_\Lambda(G) \subseteq R_\Lambda[\hat{H}]$. Hence, $K \in R_\Lambda(\Delta)$, i.e., $\Box^{-1}(\Delta) \subseteq K$. But this implies that $D \in K$, which is a contradiction. Thus, $\Box^{-1}(\Delta) \subseteq \Box^{-1}(H)$.

Suppose that $\Box^{-1}(\Delta) \subseteq \Box^{-1}(H)$. Let $K \in R_\Lambda[\hat{H}]$. Then there exists $G \in \hat{H}$ such that $(G, K) \in R_\Lambda$, i.e., $H \subseteq G$ and $\Box^{-1}(G) \subseteq K$. So, $\Box^{-1}(H) \subseteq \Box^{-1}(G) \subseteq K$. Thus, $\Box^{-1}(\Delta) \subseteq K$, i.e., $K \in R_\Lambda(\Delta)$. $\Box$

Theorem 4.5. *Let Λ be a logic such that* **BIL** $\subseteq \Lambda$. *Then*

(1) $(A \Rightarrow B) \to ((\Box C \to A) \Rightarrow (\Box C \to B)) \in \Lambda$ *iff* $\mathcal{F}_\Lambda \Vdash$ RM.

(2) $(A \Rightarrow B) \to ((\Box C \to A) \Rightarrow (\Box C \to \Box B)) \in \Lambda$ *iff* $\mathcal{F}_\Lambda \Vdash$ RM0.

Proof: (1) $\Rightarrow$) Let $\Gamma, \Delta \in X_\Lambda$ and let T be a theory such that $(\Gamma, \hat{T}) \in N_\Lambda$ and $T \subseteq \Delta$. Consider the set $\Box(\Box^{-1}(\Delta)) = \{\Box A : A \in \Box^{-1}(\Delta)\}$ and the theory $\mathsf{D}_\Gamma\left(T \cup \Box(\Box^{-1}(\Delta))\right)$. We prove that

$$\mathsf{D}_\Gamma\left(T \cup \Box(\Box^{-1}(\Delta))\right) \subseteq \Delta.$$

Let $B \in \mathsf{D}_\Gamma\left(T \cup \Box(\Box^{-1}(\Delta))\right)$. Then there exists $A \in T$ and there exists $C_1, \ldots, C_n \in \Box^{-1}(\Delta)$ such that

$$(A \wedge \Box C_1 \wedge \ldots \wedge \Box C_n) \Rightarrow B \in \Gamma.$$

Since $\Box C_1 \wedge \ldots \wedge \Box C_n \leftrightarrow \Box(C_1 \wedge \ldots \wedge C_n)$, we get

$$(A \wedge \Box(C_1 \wedge \ldots \wedge C_n)) \Rightarrow B \in \Gamma.$$

By Lemma 4.3 (1) we have $A \Rightarrow (\Box(C_1 \wedge \ldots \wedge C_n) \to B) \in \Gamma$. As $(\Gamma, \hat{T}) \in N_\Lambda$ and $A \in T$, we have $\Box(C_1 \wedge \ldots \wedge C_n) \to B \in T \subseteq \Delta$. Finally, as $\Box(C_1 \wedge \ldots \wedge C_n) \in \Delta$, we get $B \in \Delta$. Thus $Z = \mathsf{D}_\Gamma\left(T \cup \Box(\Box^{-1}(\Delta))\right)$ is a consistent theory such that $Z \subseteq \Delta$ and $(\Gamma, Z) \in N_\Lambda$. By construction, $T \subseteq Z \subseteq \Delta$. Since $\Box(\Box^{-1}(\Delta)) \subseteq Z$, we have $\Box^{-1}(\Delta) \subseteq \Box^{-1}(Z)$, i.e., $R_\Lambda[\hat{Z}] \subseteq R_\Lambda(\Delta)$. As $T \subseteq Z \subseteq \Delta$, we have that $\Delta \in \hat{Z} \subseteq \hat{T}$. Thus, we have found a theory Z such that $(\Gamma, Z) \in N_\Lambda$, $\Delta \in \hat{Z} \subseteq \hat{T}$, and $R_\Lambda[\hat{Z}] \subseteq R_\Lambda(\Delta)$, i.e. $\mathcal{F}_\Lambda \Vdash$ RM.

(1) $\Leftarrow$) Suppose that $\mathcal{F}_\Lambda \Vdash$ RM and there exists formulas A, B and C such that $(A \Rightarrow B) \to ((\Box C \to A) \Rightarrow (\Box C \to B)) \notin \Lambda$. Then there exists a maximal theory Γ such that $(A \Rightarrow B) \to ((\Box C \to A) \Rightarrow (\Box C \to B)) \notin \Gamma$. So, $A \Rightarrow B \in \Gamma$ and $(\Box C \to A) \Rightarrow (\Box C \to B) \notin \Gamma$. Then there exists a theory T such that $(\Gamma, \hat{T}) \in N_\Lambda$, $\Box C \to A \in T$ and $\Box C \to B \notin T$. So, there exists a maximal theory Δ such that $\Box C \in \Delta$ and $B \notin \Delta$. By hypothesis, there exists a theory H such that

$$(\Gamma, \hat{H}) \in N_\Lambda, \Delta \in \hat{H} \subseteq \hat{T} \text{ and } R_\Lambda[\hat{H}] \subseteq R_\Lambda(\Delta).$$

By Lemma 4.4 we have

$$(\Gamma, \hat{H}) \in N_\Lambda, T \subseteq H \subseteq \Delta \text{ and } \Box^{-1}(\Delta) \subseteq \Box^{-1}(H).$$

As $\Box C \in \Delta$, we get $\Box C \in H$. Moreover, $\Box C \to A \in T \subseteq H$, and by Modus Ponens, $A \in H$. Since $A \Rightarrow B \in \Gamma$, $(\Gamma, \hat{H}) \in N_\Lambda$ and $A \in H$, we deduce $B \in H \subseteq \Delta$, i.e., $B \in \Delta$, which is a contradiction. Thus, $(A \Rightarrow B) \to ((\Box C \to A) \Rightarrow (\Box C \to B)) \in \Lambda$.

(2) The proof is very similar to the proof of (1). We prove only the direction $\Rightarrow$). Let $\Gamma, \Delta, \Theta \in X_\Lambda$ and let T be a theory such that $(\Gamma, \hat{T}) \in N_\Lambda$, $\Delta \in \hat{T}$, and $(\Delta, \Theta) \in R_\Lambda$. We prove that there exists a theory H such that $(\Gamma, \hat{H}) \in N_\Lambda$, $\Theta \in \hat{H} \subseteq \hat{T}$ and $R_\Lambda[\hat{H}] \subseteq R_\Lambda(\Delta)$. Consider the theory $H = \mathrm{D}_\Gamma\left(T \cup \Box(\Box^{-1}(\Delta))\right)$. We prove that $H \subseteq \Theta$. Let $B \in H$.

Then there exist $A \in T$ and $C_1, \ldots, C_n \in \Box^{-1}(\Delta)$ such that $(A \wedge \Box C_1 \wedge \ldots \wedge \Box C_n) \Rightarrow B \in \Gamma$. Since $\Box C_1 \wedge \ldots \wedge \Box C_n \leftrightarrow \Box(C_1 \wedge \ldots \wedge C_n)$, we get $(A \wedge \Box(C_1 \wedge \ldots \wedge C_n)) \Rightarrow B \in \Gamma$. Then by Lemma 4.3, $A \Rightarrow (\Box(C_1 \wedge \ldots \wedge C_n) \to \Box B) \in \Gamma$. As $(\Gamma, \hat{T}) \in N_\Lambda$ and $A \in T$, we get $\Box(C_1 \wedge \ldots \wedge C_n) \to \Box B \in T \subseteq \Delta$. Moreover, as $\Box(C_1 \wedge \ldots \wedge C_n) \in \Delta$, we have $\Box B \in \Delta$. Then, $B \in \Box^{-1}(\Delta) \subseteq \Theta$. By construction $T \subseteq H \subseteq \Theta$, and as $\Box(\Box^{-1}(\Delta)) \subseteq H \subseteq \Theta$, we have $\Box^{-1}(\Delta) \subseteq \Box^{-1}(H)$, i.e., $R_\Lambda[\hat{H}] \subseteq R_\Lambda(\Delta)$. $\Box$

PROPOSITION 4.6. Let Λ be a logic such that $\mathbf{BIL} \subseteq \Lambda$. Then

(1) $(A \Rightarrow B) \to \Box(A \Rightarrow B) \in \Lambda$ iff $\mathcal{F}_\Lambda \Vdash$ RP.

(2) $(\Box A \Rightarrow B) \to \Box(A \Rightarrow B) \in \Lambda$ iff $\mathcal{F}_\Lambda \Vdash$ RP0.

PROOF: We prove only (2). The proof of (1) is similar and left to the reader.

$\Rightarrow$) Let $\Gamma, \Delta \in X_\Lambda$and let T be a theory such that $(\Gamma, \Delta) \in R_\Lambda$ and $(\Delta, \hat{T}) \in N_\Lambda$. We consider the theory

$$\mathrm{D}_\Gamma(\Box(T)) = \{B \in Fm : \exists A \in T\ (\Box A \Rightarrow B \in \Gamma)\}.$$

We prove that $\mathrm{D}_\Gamma(\Box(T)) \subseteq T$. If $B \in \mathrm{D}_\Gamma(\Box(T))$ then there exists $A \in T$ such that $\Box A \Rightarrow B \in \Gamma$. So, $\Box(A \Rightarrow B) \in \Gamma$, and as $(\Gamma, \Delta) \in R_\Lambda$, we get $A \Rightarrow B \in \Delta$. Since $(\Delta, \hat{T}) \in N_\Lambda$,we have $B \in T$. Consider the theory $H = \mathrm{D}_\Gamma(\Box(T))$. Then, $\Box(T) \subseteq H \subseteq T$. Now it is easy to see that $R_\Lambda[\hat{H}] \subseteq \hat{T} \subseteq \hat{H}$.

The direction $\Leftarrow$) it is easy and left to the reader. $\Box$

We denote by $\mathbf{BIL}(A_1, \ldots, A_n)$ the basic logic $\mathbf{BIL}$ together with the axioms schemata $A_1, \ldots, A_n$.

THEOREM 4.7. *Any extension of* $\mathbf{BIL}$ *obtained by adding any subset of the following set of formulas*

$$\{\mathrm{J1}, \mathrm{J4}, \mathrm{J5}, \mathrm{M}, \mathrm{M0}, \mathrm{P}, \mathrm{P0}\}$$

is canonical and therefore frame complete.

PROOF: Let $\Lambda_X = \mathbf{BIL}(X)$ be the basic interpretability logic where X is one of these subsets. Consider the properties that characterize its frames stated in Theorem 2.5. Then Propositions 4.1 and 4.6, and Theorem 4.5 establish that the canonical basic frame $\mathcal{F}_{\Lambda_X}$ has these properties. Therefore it is a frame of the logic Λ_X, that is, the logic Λ_X is canonical. $\Box$

5. Pure neighbourhood semantics

Let us consider the class $\mathbf{BFr}^+$ of basic frames satisfying the relational properties RJ1 and RJ4. By Theorem 2.5 and Theorem 4.7 the logic $\mathbf{BIL}^+$ is characterized by the class $\mathbf{BFr}^+$, i.e, $\mathbf{BIL}^+ = \mathsf{Th}(\mathbf{BFr}^+)$.

Consider the language $\mathcal{L}(\Rightarrow)$ and with the modal operator $\Box$ defined by $\Box A := \top \Rightarrow A$. Let $\langle X, N\rangle$ be a neighborhood frame. A valuation on a neighborhood frame $\langle X, N\rangle$ is any function $V : Var \to \mathcal{P}(X)$. A valuation V can be extended recursively to the set of all formulas Fm by means of the same clauses given in Definition 2.3 for the connectives $\top$,$\bot$,$\wedge$, $\vee$ and $\Rightarrow$. As $\Box A := \top \Rightarrow A$, the clause for the modal operator is $V(\Box A) = \{x \in X : \forall Y \in N(x)\ (Y \subseteq U)\}$. A *neighborhood model* is a pair $\mathcal{M} = \langle \mathcal{F}, V\rangle$, where $\mathcal{F}$ is a neighborhood frame and V is a valuation on it. The notions of formula valid in a neighborhood frame and neighborhood model are defined as in the case of basic frames and basic models (for more details see [4, 7, 3]).

Let $\langle X, N\rangle$ be a neighborhood frame. We take the binary relation $R_N \subseteq X \times X$ defined by:

$$(x, y) \in R_N \text{ iff } \exists Y \in N(x) \text{ such that } y \in Y. \tag{5.1}$$

Then it is immediate to see that $\langle X, R_N, N\rangle \in \mathbf{BFr}^+$ and

$$\langle X, N\rangle \models A \text{ iff } \langle X, R_N, N\rangle \models A,$$

for any formula A.

On the other hand, we consider a basic frame $\langle X, R, N\rangle$. We define the binary relation $R_N \subseteq X \times X$ defined by (5.1). We note that $R_N(x) = \bigcup\{Y : Y \in N(x)\}$. If $\langle X, R, N\rangle \in \mathbf{BFr}^+$, then by RJ1 we have $R_N \subseteq R$, and by RJ4 we get that $R \subseteq R_N$. Thus, in the basic frames of $\mathbf{BFr}^+$ the binary relation R and R_N are the same, i.e., R is definable by the relation N. Consequently if we work in the language $\mathcal{L}(\Rightarrow)$ and the modal operator $\Box$ is definable as $\Box A := \top \Rightarrow A$, then

$$\langle X, R, N\rangle \models A \text{ iff } \langle X, N\rangle \models A,$$

for any formula A. Consequently we can study extensions of $\mathbf{BIL}^+$ by means of neighborhood frames $\langle X, N\rangle$ where the operator $\Box$ is interpreted semantically by the relation R_N. Thus, if $\mathbf{NFr}$ is the class of all neighborhood frames and $\mathsf{Th}(\mathbf{NFr})$ is the set of all formulas valid in the class $\mathbf{NFr}$, we have the following result.

Theorem 5.1 (Soundness and Completeness). *$\mathbf{BIL}^+ = \mathsf{Th}(\mathbf{NFr})$.*

Soundness and Completeness for all axiomatic extensions of $\mathbf{BIL}^+$ by means of the formulas RJ5, M0,M,P and P0 is proved in the same way as in the Theorems 2.5, 4.1 and 4.5 but using the auxiliary relation R_N for the modality $\Box$. For example, the logic $\mathbf{BIL}^+ + \{(A \Rightarrow B) \rightarrow \Box(A \Rightarrow B)\}$ is complete with respect to the class of neighborhood frames $\langle X, N\rangle$ satisfying the relational condition RP, where $R = R_N$. For completeness we state the following result whose proof is exactly the same as the case of basic frames.

THEOREM 5.2. *Any extension of* $\mathbf{BIL}^+$ *by any subset of* {RJ5,M0,M,P,P0} *is canonical and therefore frame complete with respect to pure neighborhood frames.*

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FUZZY SUB-EQUALITY ALGEBRAS BASED ON FUZZY POINTS

Abstract

In this paper, by using the notion of fuzzy points and equality algebras, the notions of fuzzy point equality algebra, equality-subalgebra, and ideal were established. Some characterizations of fuzzy subalgebras were provided by using such concepts. We defined the concepts of $(\in, \in)$ and $(\in, \in \vee\, q)$-fuzzy ideals of equality algebras, discussed some properties, and found some equivalent definitions of them. In addition, we investigated the relation between different kinds of (α, β)-fuzzy subalgebras and (α, β)-fuzzy ideals on equality algebras. Also, by using the notion of $(\in, \in)$-fuzzy ideal, we defined two equivalence relations on equality algebras and we introduced an order on classes of X, and we proved that the set of all classes of X by these order is a poset.

Keywords: equality algebra, fuzzy set, fuzzy point, fuzzy ideal, sub-equality algebras, $(\in, \in)$-fuzzy sub-equality algebras, $(\in, \in \vee\, q)$-fuzzy sub-equality algebras, $(q, \in \vee\, q)$-fuzzy sub-equality algebras.

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1. Introduction

EQ-algebras were introduced by Novák et al [15]. Equality algebras were introduced by Jenei [12] by removing the multiplication operation and as

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an extension of EQ-algebras. In [9, 13] the authors investigated the relation between equality algebra and BCK-meet-semilattice. Dvurečenskij et al. in [10] defined pseudo-equality algebra as an extention of equality algebra and study some properties of it. Borzooei et al. [7] introduced some types of filters of equality algebras and studied the relation between them and moreover, they considered relations among equality algebras and some of the other logical algebras such as residuated lattice, MTL-algebra, BL-algebra, MV-algebra, and etc., in [19]. Since ideal theory is an important notion in logical algebras, Paad [16] introduced the notion of the ideal in bounded equality algebras and showed that there is a reciprocal correspondence between ideals and congruence relation.

Fuzzy sets were first introduced by Zadeh [18] and then studied by many mathematicians. Some mathematicians tried to overcome its shortcomings by presenting various extensions of fuzzy sets, and some other mathematicians studied fuzzy sets on various algebraic structures such as logical algebraic structures, groups, and rings. In [8] the notion of fuzzy ideal in bounded equality algebras is defined, and several properties are studied. Fuzzy ideal generated by a fuzzy set is established, and a fuzzy ideal is made by using the collection of ideals. Characterizations of fuzzy ideal were discussed. Conditions for a fuzzy ideal to attained its infimum on all ideals are provided. Homomorphic image and preimage of fuzzy ideal were considered. Quotient structures of equality algebra induced by (fuzzy) ideal were studied. The idea of the quasi-coincidence of a fuzzy point with a fuzzy set has played a very important role in generating fuzzy subalgebras of BCK/BCI-algebras, called (α, β)-fuzzy subalgebras of BCK/BCI-algebras, introduced by Jun [14]. Moreover, $(\in, \in \vee q)$-fuzzy subalgebra is a useful generalization of a fuzzy subalgebra in BCK/BCI-algebras. Many researchers applied the fuzzy structures on logical algebras [2, 1, 3, 4, 5, 6, 11, 17]. Then studied point fuzzy on various algebraic structures, such as hoop, BCK/BCI-algebra, different kinds of hyperstructures, and so on.

In this paper, by using the notion of fuzzy points and equality algebras, the notions of fuzzy point equality algebra, equality-subalgebra, and ideal are established. Some characterizations of fuzzy subalgebras are provided by using such concepts. We define the concepts of $(\in, \in)$ and $(\in, \in \vee q)$-fuzzy ideals of equality algebras, discuss some properties, and find some equivalent definitions of them. In addition, we investigate the relation between different kinds of (α, β)-fuzzy subalgebras and (α, β)-fuzzy ideals on equality algebras. Also, by using the notion of $(\in, \in)$-fuzzy ideal we define two equivalence relations on equality algebras and we introduce an order on classes of X, and we prove that the set of all classes of X by these order is a poset.

2. Preliminaries

This section lists the known default contents that will be used later.

Definition 2.1 ([12]). By an *equality algebra*, we mean an algebraic structure $(X, \wedge, \sim, 1)$ satisfying the following conditions.

(E1) $(X, \wedge, 1)$ is a commutative idempotent integral monoid,

(E2) The operation "$\sim$" is commutative,

(E3) $(\forall a \in X)(a \sim a = 1)$,

(E4) $(\forall a \in X)(a \sim 1 = a)$,

(E5) $(\forall a, b, c \in X)(a \leq b \leq c \;\Rightarrow\; a \sim c \leq b \sim c,\; a \sim c \leq a \sim b)$,

(E6) $(\forall a, b, c \in X)(a \sim b \leq (a \wedge c) \sim (b \wedge c))$,

(E7) $(\forall a, b, c \in X)(a \sim b \leq (a \sim c) \sim (b \sim c))$,

where $a \leq b$ if and only if $a \wedge b = a$.

In an equality algebra $(X, \wedge, \sim, 1)$, we define two operations "$\rightarrow$" and "$\leftrightarrow$" on X as follows:

$$a \rightarrow b := a \sim (a \wedge b), \tag{2.1}$$

$$a \leftrightarrow b := (a \rightarrow b) \wedge (b \rightarrow a). \tag{2.2}$$

Proposition 2.2 ([12]). Let $(X, \wedge, \sim, 1)$ be an equality algebra. Then for all $a, b, c \in X$, the following assertions are valid:

$$a \to b = 1 \;\Leftrightarrow\; a \leq b, \tag{2.3}$$
$$a \to (b \to c) = b \to (a \to c), \tag{2.4}$$
$$1 \to a = a,\; a \to 1 = 1,\; a \to a = 1, \tag{2.5}$$
$$a \leq b \to c \;\Leftrightarrow\; b \leq a \to c, \tag{2.6}$$
$$a \leq b \to a, \tag{2.7}$$
$$a \leq (a \to b) \to b, \tag{2.8}$$
$$a \to b \leq (b \to c) \to (a \to c), \tag{2.9}$$
$$b \leq a \;\Rightarrow\; a \leftrightarrow b = a \to b = a \sim b, \tag{2.10}$$
$$a \sim b \leq a \leftrightarrow b \leq a \to b, \tag{2.11}$$
$$a \leq b \;\Rightarrow \begin{cases} b \to c \leq a \to c, \\ c \to a \leq c \to b \end{cases} \tag{2.12}$$

An equality algebra $(X, \wedge, \sim, 1)$ is said to be *bounded* if there exists an element $0 \in X$ such that $0 \leq a$ for all $a \in X$. In a bounded equality algebra $(X, \wedge, \sim, 1)$, we define the negation "$\neg$" on X by $\neg a = a \to 0 = a \sim 0$ for all $a \in X$.

Definition 2.3 ([16]). Let X be a bounded equality algebra. A subset A of X is called an *ideal* of X if it satisfies:

$$(\forall x, y \in X)(x \leq y,\, y \in A \;\Rightarrow\; x \in A), \tag{2.13}$$
$$\neg x \to y \in A, \text{ for all } x, y \in A. \tag{2.14}$$

Lemma 2.4 ([16]). *Let X be a bounded equality algebra. A subset A of X is an ideal of X if and only if it satisfies in the following conditions:*

$$0 \in A, \tag{2.15}$$
$$(\forall x, y \in X)(\neg(\neg y \to \neg x) \in A,\; y \in A \;\Rightarrow\; x \in A). \tag{2.16}$$

Definition 2.5 ([16]). Let X be a bounded equality algebra and P be an ideal of X. Then P is called a *prime ideal* of X if it satisfies for any $x, y \in X$, $\neg(x \to y) \in P$ or $\neg(y \to x) \in P$.

Let X be a non-empty set. The function $\lambda : X \longrightarrow [0, 1]$ is called a *fuzzy set*.

Let X and Y be sets and $f : X \to Y$ be a function. If μ is a fuzzy set in X, then the *image* of μ under f is denoted by $f(\mu)$ and is defined as follows:

$$f(\mu): Y \to [0,1],\ y \mapsto \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu(x) & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

If ν is a fuzzy set in $f(X)$, then the *preimage* of ν under f is denoted by $f^{-1}(\nu)$ and is defined by

$$f^{-1}(\nu): X \to [0,1],\ x \mapsto \nu(f(x)).$$

Definition 2.6. A fuzzy set λ in X is said to be a *fuzzy ideal of* X if for any $x, y \in X$:

$$\lambda(0) \geq \lambda(x), \quad \text{and} \quad \lambda(x) \geq \min\{\lambda(\neg(\neg y \to \neg x)), \lambda(y)\}.$$

A fuzzy set λ in a set X of the form

$$\lambda(y) := \begin{cases} t \in (0,1] & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases}$$

is said to be a *fuzzy point with support x and value t* and is denoted by x_t.

For a fuzzy point x_t and a fuzzy set λ in a set X, we have the symbol $x_t \alpha \lambda$, where $\alpha \in \{\in, q, \in\vee q, \in\wedge q\}$.

To say that $x_t \in \lambda$ (resp. $x_t q \lambda$) means that $\lambda(x) \geq t$ (resp. $\lambda(x) + t > 1$), and in this case, x_t is said to *belong to* (resp. *be quasi-coincident with*) a fuzzy set λ.

To say that $x_t \in\vee q\, \lambda$ (resp. $x_t \in\wedge q\, \lambda$) means that $x_t \in \lambda$ or $x_t q \lambda$ (resp. $x_t \in \lambda$ and $x_t q \lambda$).

If $x_t \alpha \lambda$ is not established for $\alpha \in \{\in, q\}$, it is written by $x_t \overline{\alpha} \lambda$.

3. $(\in, \in)$-fuzzy sub-equality algebras

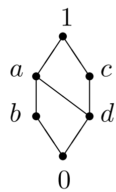
In this section, we define a sub-equality of an equality algebra X and investigate that intersection and union of family of sub-equality algebra of X is a sub-equality algebra. Then, we investigate the properties of the $(\in, \in)$-fuzzy sub-equality algebras.

Note. In what follows, let $(X, \wedge, \sim, 1)$ or X denote as an equality algebra unless otherwise specified.

Definition 3.1. A *sub-equality algebra* of an equality algebra X is a non-empty subset S of X, closed under the operations of X and equipped with the restriction to S at these operations. It means that a subset S of X is called a *sub-equality algebra* of X if $x \sim y \in S$ and $x \wedge y \in S$, for all $x, y \in S$.

Note. Note that every non-empty sub-equality algebra contains the element 1.

Example 3.2. Let $X = \{0, a, b, c, d, 1\}$ be a set with the following Hasse diagram.



Then $(X, \wedge, 1)$ is a meet semilattice with top element 1. Define an operation $\sim$ on X by Table 1.

Table 1. Cayley table for the binary operation "$\sim$"

$\sim$	0	a	b	c	d	1
0	1	d	c	b	a	0
a	d	1	a	d	c	a
b	c	a	1	0	d	b
c	b	d	0	1	a	c
d	a	c	d	a	1	d
1	0	a	b	c	d	1

Then $\mathcal{E} = (X, \wedge, \sim, 1)$ is a bounded equality algebra, and the implication "$\rightarrow$" is given by Table 2. Let $S_1 = \{1, b\}, S_2 = \{1, c\}$, $S_3 = \{1, a, b\}$ and $S_4 = \{1, a, c\}$. Clearly, S_1, S_2 and S_3 are sub-equality algebras of X, but S_4 isn't, since $a \sim c = d \notin S_4$.

Proposition 3.3. Let $\{X_i \mid i \in I\}$ be a family of sub-equality algebras of X. Then $\bigcap_{i \in I} X_i$ is a sub-equality algebras of X.

In the following example, we show that the union of a family of sub-equality algebras may not be a sub-equality algebra, in general.

Table 2. Cayley table for the implication "$\rightarrow$"

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	a	c	c	1
b	c	1	1	c	c	1
c	b	a	b	1	a	1
d	a	1	a	1	1	1
1	0	a	b	c	d	1

Example 3.4. Let X be the equality algebra as in Example 3.2. We show that, S_1 and S_2 are two sub-equality algebras of X, but $S = S_1 \cup S_2 = \{b, c, 1\}$ is not a sub-equality algebra of X, because $b \sim c = 0 \notin S$.

In the following proposition we investigate that under which condition, the union of a family of sub-equality algebras is a sub-equality algebra.

Proposition 3.5. Let $\{X_i \mid i \in I\}$ be a family of sub-equality algebra of X. If for any $i, j \in I$, $X_i \subseteq X_j$ or $X_j \subseteq X_i$, then $\bigcup_{i \in I} X_i$ is a sub-equality algebra of X.

Proposition 3.6. Let S be a sub-equality algebra of X. Then for any $x, y \in S$, $x \rightarrow y \in S$.

In the following example, we show that the reverse of the above proposition may not be true, in general.

Example 3.7. Let X be an equality algebra as in Example 3.2. Obviously, $S = \{1, a, c\}$ is closed under the operation $\rightarrow$. But S is not a sub-equality algebra of X, because $a \wedge c = d \notin S$ and $a \sim c = d \notin S$.

In the following proposition, we investigate that under which condition, close under the operation $\rightarrow$ is equal with property of sub-equality algebra.

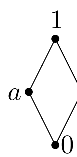
Proposition 3.8. Let X be bounded. If S is an ideal of X which is closed under $\rightarrow$, then S is a sub-equality algebra of X.

Proof: Suppose $x, y \in S$. Since $x \wedge y \leq x$, S is an ideal of X and $x \in S$, we have $x \wedge y \in S$. Also, by (2.11) $x \sim y \leq x \rightarrow y$. Since by assumption,

S is an ideal of X and $x \to y \in S$, we get $x \sim y \in S$. Thus, S is a sub-equality algebra of X. □

In the following example, we show that the ideal of equality algebra is not close under the operation $\to$.

Example 3.9. Let $X = \{0, a, b, 1\}$ be a set with the following Hasse diagram.



We define a binary operation $\sim$ and $\to$ on X by Tables 3 and 4, respectively. Then X is an equality algebra. Clearly, $S = \{0, a\}$ is an ideal of $\mathcal{E}$, but it isn't close under the operation $\to$, because $a \to 0 = b \notin S$.

Table 3. Cayley table for the binary operation "$\sim$"

$\sim$	0	a	b	1
0	1	b	a	0
a	b	1	b	a
b	a	a	1	b
1	0	a	b	1

Table 4. Cayley table for the binary operation "$\to$"

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

Definition 3.10. A fuzzy set λ in X is called an $(\in, \in)$-*fuzzy sub-equality algebra* of X if the following assertion is valid.

$$(\forall x, y \in X)(\forall t, k \in [0,1]) \left(x_t \in \lambda,\ y_k \in \lambda \ \Rightarrow \left\{ \begin{array}{l} (x \sim y)_{\min\{t,k\}} \in \lambda \\ (x \wedge y)_{\min\{t,k\}} \in \lambda) \end{array} \right. \right). \tag{3.1}$$

Example 3.11. Let X be the equality algebra as in Example 3.9. Define a fuzzy set λ in X as follows:

$$\lambda : X \to [0,1],\ x \mapsto \begin{cases} 0.5 & \text{if} \quad x = 0, \\ 0.4 & \text{if} \quad x = a, \\ 0.4 & \text{if} \quad x = b, \\ 0.8 & \text{if} \quad x = 1 \end{cases}$$

Then λ is an $(\in, \in)$-fuzzy sub-equality algebra of X.

We consider characterizations of an $(\in, \in)$-fuzzy sub-equality algebra.

Theorem 3.12. *A fuzzy set λ in X is an $(\in, \in)$-fuzzy sub-equality algebra of X if and only if the following assertion is valid.*

$$(\forall x, y \in X) \left(\begin{array}{l} \lambda(x \sim y) \geq \min\{\lambda(x), \lambda(y)\} \\ \lambda(x \wedge y) \geq \min\{\lambda(x), \lambda(y)\} \end{array} \right). \tag{3.2}$$

Proof: Assume that λ is an $(\in, \in)$-fuzzy sub-equality algebra of X. Note that $x_{\lambda(x)} \in \lambda$ and $y_{\lambda(y)} \in \lambda$ for all $x, y \in X$. By (3.1), we have $(x \sim y)_{\min\{\lambda(x),\lambda(y)\}} \in \lambda$ and $(x \wedge y)_{\min\{\lambda(x),\lambda(y)\}} \in \lambda$. Then $\lambda(x \sim y) \geq \min\{\lambda(x), \lambda(y)\}$ and $\lambda(x \wedge y) \geq \min\{\lambda(x), \lambda(y)\}$ for all $x, y \in X$.

Conversely, suppose λ satisfies the condition (3.2). Let $x, y \in X$ and $t, k \in [0,1]$ such that $x_t \in \lambda$ and $y_k \in \lambda$. Then $\lambda(x) \geq t$ and $\lambda(y) \geq k$, which imply from (3.2) that

$$\lambda(x \sim y) \geq \min\{\lambda(x), \lambda(y)\} \geq \min\{t, k\}$$

and

$$\lambda(x \wedge y) \geq \min\{\lambda(x), \lambda(y)\} \geq \min\{t, k\}.$$

Hence $(x \sim y)_{\min\{t,k\}} \in \lambda$ and $(x \wedge y)_{\min\{t,k\}} \in \lambda$. Therefore λ is an $(\in, \in)$-fuzzy sub-equality algebra of X. □

THEOREM 3.13. *If a fuzzy set* λ *in* X *is an* $(\in,\in)$*-fuzzy sub-equality algebra of* X*, then*

$$(\forall x, y \in X)(\forall t, k \in [0,1]) \left(x_t \in \lambda,\ y_k \in \lambda \ \Rightarrow (x \to y)_{\min\{t,k\}} \in \lambda \right). \tag{3.3}$$

PROOF: Assume that λ is an $(\in,\in)$-fuzzy sub-equality algebra of X. Let $x, y \in X$ and $t, k \in [0,1]$ be such that $x_t \in \lambda$ and $y_k \in \lambda$. Then $\lambda(x) \geq t$ and $\lambda(y) \geq k$, which implies from (3.2) that

$$\begin{aligned}\lambda(x \to y) &= \lambda(x \sim (x \wedge y)) \\ &\geq \min\{\lambda(x), \lambda(x \wedge y)\} \\ &\geq \min\{\lambda(x), \min\{\lambda(x), \lambda(y)\}\} \\ &\geq \min\{\lambda(x), \lambda(y)\} \\ &\geq \min\{t, k\}\end{aligned}$$

Hence, $(x \to y)_{\min\{t,k\}} \in \lambda$. □

In the following example, we show that the converse of the above theorem may not be true, in general.

Example 3.14. Let X be the equality algebra as in Example 3.9. Define a fuzzy set λ in X as follows:

$$\lambda : X \to [0,1],\ x \mapsto \begin{cases} 0.3 & \text{if} \quad x = 0, \\ 0.4 & \text{if} \quad x = a, \\ 0.4 & \text{if} \quad x = b, \\ 0.8 & \text{if} \quad x = 1 \end{cases}$$

Then λ is satisfies in (3.3). But, it isn't an $(\in,\in)$-fuzzy sub-equality algebra of X, since $0.3 = \lambda(a \wedge b) \not\geq \min\{\lambda(a), \lambda(b)\} = 0.4$.

THEOREM 3.15. *If* λ *is a non-zero* $(\in,\in)$*-fuzzy sub-equality algebra of* X*, then the set*

$$X_0 := \{x \in X \mid \lambda(x) \neq 0\} \tag{3.4}$$

is a sub-equality algebra of X*.*

PROOF: Let $x, y \in X_0$. Then $\lambda(x) > 0$ and $\lambda(y) > 0$. Note that $x_{\lambda(x)} \in \lambda$ and $y_{\lambda(y)} \in \lambda$. If $\lambda(x \sim y) = 0$ or $\lambda(x \wedge y) = 0$, then $\lambda(x \sim y) =$

$0 < \min\{\lambda(x), \lambda(y)\}$ or $\lambda(x \wedge y) = 0 < \min\{\lambda(x), \lambda(y)\}$, that is, $(x \sim y)_{\min\{\lambda(x),\lambda(y)\}}\overline{\in}\lambda$ or $(x \wedge y)_{\min\{\lambda(x),\lambda(y)\}}\overline{\in}\lambda$, which is a contradiction. Thus $\lambda(x \sim y) \neq 0$ and $\lambda(x \wedge y) \neq 0$. Hence $x \sim y \in X_0$ and $x \wedge y \in X_0$. Therefore X_0 is a sub-equality algebra of X. □

DEFINITION 3.16. Let X be bounded. A fuzzy set λ in X is called an $(\in, \in)$-*fuzzy ideal* of X if the following assertions are valid.

$$(\forall x \in X)(\forall t \in [0,1])(x_t \in \lambda \;\Rightarrow\; 0_t \in \lambda), \tag{3.5}$$

$$(\forall x, y \in X)(\forall t, k \in [0,1])(x_t \in \lambda,\; \neg(\neg x \to \neg y)_k \in \lambda \;\Rightarrow y_{\min\{t,k\}} \in \lambda). \tag{3.6}$$

Example 3.17. Let X be the equality algebra as in Example 3.9. Define a fuzzy set λ in X by $\lambda(0) = 0.8, \lambda(a) = 0.6$ and $\lambda(b) = \lambda(1) = 0.5$. Then λ is an $(\in, \in)$-fuzzy sub-equality algebra of X.

THEOREM 3.18. *The following are equivalent.*

(i) *A fuzzy set λ is a fuzzy ideal of X.*

(ii) *A fuzzy set λ is an $(\in, \in)$-fuzzy ideal of X.*

PROOF: $(i) \Rightarrow (ii)$: Let λ be a fuzzy ideal of X and $x_t \in \lambda$. Then $\lambda(x) \geq t$. Since by Definition 2.6 $\lambda(0) \geq \lambda(x)$, for any $x \in X$, we have $\lambda(0) \geq \lambda(x) \geq t$ and so $0_t \in \lambda$. Now, suppose $x_t \in \lambda$ and $\neg(\neg x \to \neg y)_s \in \lambda$. Then $\lambda(x) \geq t$ and $\lambda(\neg(\neg x \to \neg y)) \geq s$. Since λ is a fuzzy ideal, we get

$$\lambda(y) \geq \min\{\lambda(x), \lambda(\neg(\neg x \to \neg y))\} \geq \min\{t, s\},$$

Hence $y_{\min\{t,s\}\in\lambda}$. Therefore, λ is an $(\in, \in)$-fuzzy ideal of X.
$(ii) \Rightarrow (i)$: Let λ be an $(\in, \in)$-fuzzy ideal of X and $\lambda(x) = t$, for $x \in X$. Then $x_t \in \lambda$. By (3.5), $0_t \in \lambda$ and so $\lambda(0) \geq t = \lambda(x)$. Hence, $\lambda(0) \geq \lambda(x)$. Let $x, y \in X$ such that $\lambda(x) = t$ and $\lambda(\neg(\neg x \to \neg y)) = s$. Then $x_t \in \lambda$ and $\neg(\neg x \to \neg y)_s \in \lambda$. By (3.6), we have $y_{\min\{t,s\}} \in \lambda$ and so, $\lambda(y) \geq \min\{t, s\} = \min\{\lambda(x), \lambda(\neg(\neg x \to \neg y))\}$. □

PROPOSITION 3.19. Let λ be an $(\in, \in)$-fuzzy ideal of X. Then for all $x, y \in X$, the following assertions are valid:

(1) $\forall x \in X, \quad \lambda(1) \leq \lambda(x)$

(2) $\forall x, y \in X, \quad \lambda(x \to y) \geq \min\{\lambda(x), \lambda(y)\}$

(3) $\forall x, y \in X, \quad$ if $x \leq y$, then $\lambda(x) \geq \lambda(y)$

PROOF: (1), (2) The proof is clear.
(3) Let $x \leq y$. Then $\neg y \leq \neg x$, so $\neg(\neg y \to \neg x) = \neg 1 = 0$. Since λ is an $(\in, \in)$-fuzzy ideal of X by Theorem 3.18 , we have

$$\lambda(x) \geq \min\{\lambda(\neg(\neg y \to \neg x)), \lambda(y)\} = \min\{\lambda(0), \lambda(y)\} = \lambda(y).$$

Thus λ is order reversing. □

PROPOSITION 3.20. If λ is a non-zero $(\in, \in)$-fuzzy ideal of X, then $X_0 = \{x \in X | \lambda(x) \neq 0\}$ is an ideal of X.

PROOF: Since λ is non-zero, there exists $x \in X$ such that $\lambda(x) \neq 0$ and so $X_0 \neq \emptyset$. Suppose $x \in X_0$. Then $\lambda(x) > 0$. By Theorem 3.18, $\lambda(0) \geq \lambda(x) > 0$. Thus, $0 \in X_0$. Now, consider $x, \neg(\neg x \to \neg y) \in X_0$. Then by Theorem 3.18, we have

$$\lambda(y) \geq \min\{\lambda(x), \lambda(\neg(\neg x \to \neg y))\} > 0$$

Hence $\lambda(y) > 0$, and so $y \in \lambda_0$. Therefore, X_0 is an ideal of X. □

In the following theorem, we investigate that under which condition, the converse of Theorem 3.13 is true, in general.

THEOREM 3.21. *Let X be bounded and a fuzzy set λ in X be an $(\in, \in)$-fuzzy ideal of X. If the following assertion is valid,*

$$(\forall x, y \in X)(\forall t, k \in [0, 1]) \left(\ x_t \in \lambda, \ y_k \in \lambda \ \Rightarrow (x \to y)_{\min\{t,k\}} \in \lambda \ \right), \tag{3.7}$$

then, the fuzzy set λ in X is an $(\in, \in)$-fuzzy sub-equality algebra of X.

PROOF: Let $x_t \in \lambda$ and $y_k \in \lambda$. Since for any $x, y \in X$, by (2.11) $x \sim y \leq x \to y$ and λ in X is an $(\in, \in)$-fuzzy ideal, by Proposition 3.19(3), we have

$$\lambda(x \sim y) \geq \lambda(x \to y) \geq \min\{\lambda(x), \lambda(y)\} \geq \min\{t, k\}$$

Thus $(x \sim y)_{\min\{t,k\}} \in \lambda$. Also, we know that $x \wedge y \leq x, y$. Then $\lambda(x), \lambda(y) \leq \lambda(x \wedge y)$, by Proposition 3.19(3). Hence, $\min\{t, k\} \leq \min\{\lambda(x), \lambda(y)\} \leq \lambda(x \wedge y)$ and so, $(x \wedge y)_{\min\{t,k\}} \in \lambda$. Therefore, S is an $(\in, \in)$-fuzzy sub-equality algebra of X □

Given a fuzzy set λ in X, we consider the set

$$U(\lambda; t) := \{x \in X \mid \lambda(x) \geq t\}, \tag{3.8}$$

which is called an $\in$-*level set* of λ (related to t).

THEOREM 3.22. *A fuzzy set λ in X is an $(\in, \in)$-fuzzy sub-equality algebra of X if and only if the non-empty $\in$-level set $U(\lambda; t)$ of λ is a sub-equality algebra of X for all $t \in [0, 1]$.*

PROOF: Let λ be a fuzzy set in X such that $U(\lambda; t)$ is a non-empty sub-equality algebra of X for all $t \in [0, 1]$. Let $x, y \in X$ and $t, k \in [0, 1]$ be such that $x_t \in \lambda$ and $y_k \in \lambda$. Then $\lambda(x) \geq t$ and $\lambda(y) \geq k$, and so $x, y \in U(\lambda; \min\{t, k\})$. By hypothesis, we have $x \sim y \in U(\lambda; \min\{t, k\})$ and $x \wedge y \in U(\lambda; \min\{t, k\})$. Then $(x \sim y)_{\min\{t,k\}} \in \lambda$ and $(x \wedge y)_{\min\{t,k\}} \in \lambda$. Therefore λ is an $(\in, \in)$-fuzzy sub-equality algebra of X.

Conversely, assume that λ is an $(\in, \in)$-fuzzy sub-equality algebra of X. Let $x, y \in U(\lambda; t)$ for all $t \in [0, 1]$. Then $\lambda(x) \geq t$ and $\lambda(y) \geq t$, that is, $x_t \in \lambda$ and $y_t \in \lambda$. By (3.1) we have $(x \sim y)_t \in \lambda$ and $(x \wedge y)_t \in \lambda$. Then $x \sim y \in U(\lambda; t)$ and $x \wedge y \in U(\lambda; t)$. Therefore $U(\lambda; t)$ of λ is a sub-equality algebra of X for all $t \in [0, 1]$. □

COROLLARY 3.23. Consider a fuzzy set λ in X is an $(\in, \in)$-fuzzy sub-equality algebra of X. Then λ is closed under the operation $\rightarrow$ if and only if the non-empty $\in$-level set $U(\lambda; t)$ of λ is closed under the operation $\rightarrow$.

PROOF: Let a fuzzy set λ in X be an $(\in, \in)$-fuzzy sub-equality algebra of X. For any $x, y \in U(\lambda; t)$, $\lambda(x), \lambda(y) \geq t$ and we get $\lambda(x \rightarrow y) = \lambda(x \sim (x \wedge y)) \geq \min\{\lambda(x), \lambda(y)\} \geq t$. Hence $x \rightarrow y \in U(\lambda; t)$.
Conversely, suppose $U(\lambda; t)$ is closed under the operation $\rightarrow$. Let $x, y \in X$ and $t, k \in [0, 1]$ be such that $x_t \in \lambda$ and $y_k \in \lambda$. Then $\lambda(x) \geq t$ and $\lambda(y) \geq k$, and so $\lambda(x), \lambda(y) \geq \min\{t, k\}$. Thus $x, y \in U(\lambda; \min\{t, k\})$. Since $U(\lambda; t)$ is closed under $\rightarrow$, we get $x \rightarrow y \in U(\lambda; \min\{t, k\})$. Hence $\lambda(x \rightarrow y) \geq \min\{t, k\}$. □

THEOREM 3.24. *Let λ be an $(\in, \in)$-fuzzy sub-equality algebra of X. Then the following are equivalent.*

(i) *λ is an $(\in, \in)$-fuzzy ideal of X.*

(ii) *The nonempty set $U(\lambda; t)$ is an ideal of X.*

Proof: $(i) \Rightarrow (ii)$: Let λ be an $(\in, \in)$-fuzzy ideal of X such that $x \in U(\lambda; t)$. Then $\lambda(x) \geq t$. By (i), since $x_t \in \lambda$, we have $0_t \in \lambda$ and so $\lambda(0) \geq t$. Hence, $0 \in U(\lambda; t)$. Now, suppose $x, \neg(\neg x \to \neg y) \in U(\lambda; t)$. Then $\lambda(x) \geq t$ and $\lambda(\neg(\neg x \to \neg y)) \geq t$ and so $x_t \in \lambda$ and $\neg(\neg x \to \neg y)_t \in \lambda$. By (i), we have $y_t \in \lambda$ and so $\lambda(y) \geq t$. Thus, $y \in U(\lambda; t)$. Therefore $U(\lambda; t)$ is an ideal of X.
$(ii) \Rightarrow (i)$: Let $x_t \in \lambda$. Then $\lambda(x) \geq t$ and so $x \in U(\lambda; t)$. By (ii), $0 \in U(\lambda; t)$ and so $\lambda(0) \geq t$. Hence $0_t \in \lambda$. Suppose $x_t \in \lambda$ and $\neg(\neg x \to \neg y)_k \in \lambda$. Then $x, \neg(\neg x \to \neg y) \in U(\lambda; \min\{t, k\})$. By (ii), $y \in U(\lambda; \min\{t, k\})$. Hence $y_{\min\{t,k\}} \in \lambda$. Therefore, λ is an $(\in, \in)$-fuzzy ideal of X. □

Theorem 3.25. *Let S be an ideal of X. For any $t \in [0, 1]$, there exists an $(\in, \in)$-fuzzy ideal λ of X such that $U(\lambda; t) = S$.*

Proof: Let $t \in [0, 1]$ and $\lambda : X \longrightarrow [0, 1]$ is defined by $\lambda(x) = t$, for any $x \in S$ and $\lambda(x) = 0$, otherwise. By definition, clearly $U(\lambda; t) = S$. So it is enough to prove that λ is an $(\in, \in)$-fuzzy ideal of X. Let $x \in X$. Then $\lambda(x) = 0$ or $\lambda(x) = t$. Since S is an ideal of X, we have $0 \in S$ and so $\lambda(0) = t$. Hence, $\lambda(0) \geq \lambda(x)$, for any $x \in X$.
Now, suppose $x_t \in \lambda$ and $\neg(\neg x \to \neg y)_k \in \lambda$. Then, we have the following cases:

Case 1: If $\lambda(x) = \lambda(\neg(\neg x \to \neg y)) = t$. Then $x, \neg(\neg x \to \neg y) \in S$. Since S is an ideal of X, we have $y \in S$ and so $\lambda(y) = t$. Hence, $\lambda(y) \geq \min\{\lambda(x), \lambda(\neg(\neg x \to \neg y))\}$.

Case 2: If $\lambda(x) = t$ and $\lambda(\neg(\neg x \to \neg y)) = 0$. Then $x \in S$ and $\neg(\neg x \to \neg y) \notin S$. Then $\lambda(y) = 0$ or $\lambda(y) = t$ and in both case $\lambda(y) \geq \min\{\lambda(x), \lambda(\neg(\neg x \to \neg y))\} = 0$.

Case 3: If $\lambda(x) = 0$ and $\lambda(\neg(\neg x \to \neg y)) = t$, then similar to Case 2, $\lambda(y) \geq \min\{\lambda(x), \lambda(\neg(\neg x \to \neg y))\}$.

Case 4: If $\lambda(x) = \lambda(\neg(\neg x \to \neg y)) = 0$, then $x, \neg(\neg x \to \neg y) \notin S$. Clearly,

$$\lambda(y) \geq \min\{\lambda(x), \lambda(\neg(\neg x \to \neg y))\} = 0.$$

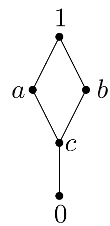
Therefore, λ is an $(\in, \in)$-fuzzy ideal of X. □

Definition 3.26. Let X be bounded. A fuzzy set λ in X is called a *fuzzy prime ideal* of X if the following assertions are valid.

$$(\forall x \in X)(\lambda(0) \geq \lambda(x)) \tag{3.9}$$

$$(\forall x, y \in X) \begin{cases} \lambda(\neg(x \to y)) \geq \min\{\lambda(x), \lambda(y)\} \\ \text{or} \\ \lambda(\neg(y \to x)) \geq \min\{\lambda(y), \lambda(x)\} \end{cases} \tag{3.10}$$

Example 3.27. Let $X = \{0, a, b, c, 1\}$ be a set with the following Hasse diagram.



Then $(X, \wedge, 1)$ is a commutative idempotent integral monoid. We define a binary operation $\sim$ on X by Table 5. Then $(X, \wedge, \sim, 1)$ is an equality

Table 5. Cayley table for the implication "$\sim$"

$\sim$	0	a	b	c	1
0	1	0	0	0	0
a	0	1	b	a	c
b	0	b	1	c	a
c	0	a	c	1	b
1	0	c	a	b	1

algebra, and the implication "$\to$" is given by Table 6. We define a fuzzy set λ in X as follows:

$$\lambda : X \to [0, 1], \ x \mapsto \begin{cases} 0.7 & \text{if } x = 0, \\ 0.6 & \text{if } x = c, \\ 0.5 & \text{if } x = a, \\ 0.3 & \text{if } x \in \{b, 1\}. \end{cases}$$

Then λ is a fuzzy prime ideal of X.

Table 6. Cayley table for the implication "→"

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	b	1	b	1
c	0	a	a	1	1
1	0	c	a	b	1

DEFINITION 3.28. Let X be bounded. A fuzzy set λ in X is called an $(\in, \in)$-*fuzzy prime ideal* of X if the following assertions are valid.

$$(\forall x, y \in X)(\forall t, k \in [0,1]) \left(x_t \in \lambda,\ y_k \in \lambda \ \Rightarrow \left\{ \begin{array}{l} \neg(x \to y)_{\min\{t,k\}} \in \lambda \\ \text{or} \\ \neg(y \to x)_{\min\{t,k\}} \in \lambda \end{array} \right. \right). \tag{3.11}$$

Example 3.29. Let X be an equality algebra in as Example 3.27. Obviously, λ is an $(\in, \in)$-fuzzy prime ideal of X.

THEOREM 3.30. *Let X be bounded. Then, λ is a fuzzy prime ideal of X if and only if $U(\lambda; t)$ is a prime ideal of X.*

PROOF: Let $x \in U(\lambda; t)$. Then $\lambda(x) \geq t$. Since λ is a fuzzy prime ideal of X, we have $\lambda(0) \geq \lambda(x) \geq t$. Thus, $0 \in U(\lambda; t)$. Suppose $x, y \in U(\lambda; t)$. Then $\lambda(x), \lambda(y) \geq t$. Since $\lambda(\neg(x \to y)) \geq \min\{\lambda(x), \lambda(y)\} \geq t$ or $\lambda(\neg(y \to x)) \geq t$, we have $\neg(x \to y) \in U(\lambda; t)$ or $\neg(y \to x) \in U(\lambda; t)$. Hence $U(\lambda; t)$ is a prime ideal of X.
Conversely, assume $\lambda(x) = t$. Then $\lambda(x) \geq t$ and so $x \in U(\lambda; t)$. Since $U(\lambda; t)$ is a prime ideal of X, $0 \in U(\lambda; t)$. Thus $\lambda(0) \geq t = \lambda(x)$. Suppose $x_t \in \lambda$ and $y_k \in \lambda$. Then $x, y \in U(\lambda; \min\{t, k\})$. Since $U(\lambda; \min\{t, k\})$ is a prime ideal, we have $\neg(x \to y) \in U(\lambda; \min\{t, k\})$ or $\neg(y \to x) \in U(\lambda; \min\{t, k\})$. Hence $\neg(x \to y)_{\min\{t,k\}} \in \lambda$ or $\neg(y \to x)_{\min\{t,k\}} \in \lambda$, so λ is a fuzzy prime ideal of X. □

THEOREM 3.31. *A fuzzy set λ in X is an $(\in, \in)$-fuzzy prime ideal of X if and only if λ is a fuzzy prime ideal.*

PROOF: The proof is similar to the proof of Theorem 3.18. □

THEOREM 3.32. *Let λ in X be an $(\in, \in)$-fuzzy ideal on X. Define the relation*

$$x \equiv_\lambda y \Longleftrightarrow \neg(\neg x \to \neg y)_{\lambda(0)} \in \lambda \text{ and } \neg(\neg y \to \neg x)_{\lambda(0)} \in \lambda,$$

for any $x, y \in X$. Then $\equiv_\lambda$ is an equivalence relation on X

PROOF: Let $x, y, z \in X$. Clearly, the relation $\equiv_\lambda$ is reflexive and symmetric. Suppose $x \equiv_\lambda y$ and $y \equiv_\lambda z$. Then $\neg(\neg x \to \neg y)_{\lambda(0)} \in \lambda, \neg(\neg y \to \neg x)_{\lambda(0)} \in \lambda, \neg(\neg y \to \neg z)_{\lambda(0)} \in \lambda$ and $\neg(\neg z \to \neg y)_{\lambda(0)} \in \lambda$. Thus by (2.9) and (2.12) we have

$$\neg y \to \neg z \le (\neg x \to \neg y) \to (\neg x \to \neg z) \le \neg\neg(\neg x \to \neg y) \to \neg\neg(\neg x \to \neg z),$$

and so,

$$\neg(\neg\neg(\neg x \to \neg y) \to \neg\neg(\neg x \to \neg z)) \le \neg(\neg y \to \neg z).$$

Since λ is an $(\in, \in)$-fuzzy ideal of X, by Proposition 3.19(3), we get

$$\begin{aligned} \lambda(0) &= \lambda(\neg(\neg y \to \neg z)) \\ &\le \lambda(\neg(\neg\neg(\neg x \to \neg y) \to \neg\neg(\neg x \to \neg z))) \\ &\le \lambda(0). \end{aligned}$$

Hence $(\neg(\neg\neg(\neg x \to \neg y) \to \neg\neg(\neg x \to \neg z)))_{\lambda(0)} \in \lambda$. In addition by assumption and Theorem 3.18, we have

$$\begin{aligned} \lambda(\neg(\neg x \to \neg z)) &\ge \min\{\lambda(\neg(\neg\neg(\neg x \to \neg y) \to \neg\neg(\neg x \to \neg z))), \\ &\qquad \lambda(\neg(\neg x \to \neg y))\} \\ &= \min\{\lambda(0), \lambda(0)\} = \lambda(0) \end{aligned}$$

Hence $\neg(\neg x \to \neg z)_{\lambda(0)} \in \lambda$. By similar way, $\neg(\neg z \to \neg x)_{\lambda(0)} \in \lambda$, and so $\equiv_{\lambda(0)}$ is transitive. Therefore, $\equiv_{\lambda(0)}$ is an equivalence relation on X. □

Note. Denote by $[x]_\lambda$ the set $\{y \in X | x \equiv_\lambda y\}$ and $\frac{X}{\equiv_\lambda}$ the set $\{[x]_\lambda | x \in X\}$.

PROPOSITION 3.33. Let λ in X be an $(\in, \in)$-fuzzy ideal on X. Then $[0] = \{x \in X | \lambda(x) = \lambda(0)\}$ and $[1] = \{x \in X | \lambda(\neg x) = \lambda(0)\}$.

Proof: Let λ be an $(\in, \in)$-fuzzy ideal on X. Then $[0] = \{x \in X | x \equiv_{\lambda(0)} 0\} = \{x \in X | \lambda(\neg(\neg x \to \neg 0)) = \lambda(0)$ and $\lambda(\neg(\neg 0 \to \neg x)) = \lambda(0)\} = \{x \in X | \lambda(\neg\neg x) = \lambda(0)\}$. Since $x \leq \neg\neg x$ and λ is an $(\in, \in)$-fuzzy ideal of X, by Proposition 3.19(3), $\lambda(0) = \lambda(\neg\neg x) \leq \lambda(x) \leq \lambda(0)$. Hence, $\lambda(x) = \lambda(0)$. So $[0] = \{x \in X | \lambda(\neg\neg x) = \lambda(0)\} = \{x \in X | \lambda(x) = \lambda(0)\}$. The proof of other case is similar. □

Proposition 3.34. Let λ be an $(\in, \in)$-fuzzy ideal of X. Define

$$[x] \leq [y] \Longleftrightarrow \neg(\neg x \to \neg y)_{\lambda(0)} \in \lambda,$$

for any $[x], [y] \in \frac{X}{\equiv_{\lambda(0)}}$. Then $(\frac{X}{\equiv_{\lambda(0)}}, \leq)$ is a poset.

Proof: Let $[x], [y] \in \frac{X}{\equiv_{\lambda(0)}}$. Obviously, $\leq$ is reflexive. Suppose $[x] \leq [y]$ and $[y] \leq [x]$. Then $\neg(\neg x \to \neg y)_{\lambda(0)} \in \lambda$ and $\neg(\neg y \to \neg x)_{\lambda(0)} \in \lambda$. Thus $x \equiv_{\lambda(0)} y$ and so $[x] = [y]$. Assume that $[x] \leq [y]$ and $[y] \leq [z]$ for any $x, y, z \in X$. Then $\neg(\neg x \to \neg y)_{\lambda(0)} \in \lambda$ and $\neg(\neg y \to \neg z)_{\lambda(0)} \in \lambda$. By similar to the proof of Theorem 3.32, we have

$$\lambda(\neg(\neg y \to \neg z)) \leq \lambda(\neg(\neg\neg(\neg x \to \neg y) \to \neg\neg(\neg x \to \neg z))).$$

From $\neg(\neg y \to \neg z)_{\lambda(0)} \in \lambda$, we get $\neg(\neg\neg(\neg x \to \neg y) \to \neg\neg(\neg x \to \neg z))_{\lambda(0)} \in \lambda$, and so by Theorem 3.18, we have

$$\begin{aligned}\lambda(\neg(\neg x \to \neg z)) &\geq \min\{\lambda(\neg(\neg\neg(\neg x \to \neg y) \to \neg\neg(\neg x \to \neg z)),\\ &\qquad \lambda(\neg(\neg x \to \neg y))\}\\ &= \min\{\lambda(0), \lambda(0)\}\\ &= \lambda(0).\end{aligned}$$

Hence $\neg(\neg x \to \neg z)_{\lambda(0)} \in \lambda$ and so $[x] \leq [z]$. Therefore, $(\frac{X}{\equiv_{\lambda(0)}}, \leq)$ is a poset. □

Theorem 3.35. *Let λ be an $(\in, \in)$-fuzzy ideal of X. Define*

$$x \sim_\lambda y \Longleftrightarrow \neg(x \to y)_{\lambda(0)} \in \lambda \text{ and } \neg(y \to x)_{\lambda(0)} \in \lambda$$

Then for any $x, y \in X$, $\sim_\lambda$ is an equivalence relation on X.

Proof: Let λ be an $(\in,\in)$-fuzzy ideal of X. Clearly the relation $\sim_\lambda$ is a reflexive and symetric relation on X. Suppose $x, y, z \in X$ such that $x \sim_\lambda y$ and $y \sim_\lambda z$. Then $\neg(x \to y)_{\lambda(0)} \in \lambda, \neg(y \to x)_{\lambda(0)} \in \lambda, \neg(y \to z)_{\lambda(0)} \in \lambda$ and $\neg(z \to y)_{\lambda(0)} \in \lambda$. Then by (2.11) and (2.9) we have,

$$x \to y \leq (y \to z) \to (x \to z) \leq \neg(x \to z) \to \neg(y \to z).$$

Thus $\neg(\neg(x \to z) \to \neg(y \to z)) \leq \neg(x \to y)$. Since λ is an $(\in,\in)$-fuzzy ideal of X, by Proposition 3.19(3), we get $\lambda(0) = \lambda(\neg(x \to y)) \leq \lambda(\neg(\neg(x \to z) \to \neg(y \to z))) \leq \lambda(0)$. Hence $\neg(\neg(x \to z) \to \neg(y \to z))_{\lambda(0)} \in \lambda$. Since

$$\begin{aligned}\neg(\neg(x \to z) \to \neg(y \to z)) &= \neg(\neg(x \to z) \to \neg\neg\neg(y \to z)) \\ &= \neg(\neg\neg(y \to z) \to \neg\neg(x \to z)),\end{aligned}$$

by Theorem 3.18, we have $\neg(\neg\neg(y \to z) \to \neg\neg(x \to z))_{\lambda(0)} \in \lambda$ and $\neg(y \to z)_{\lambda(0)} \in \lambda$, we get $\neg(x \to z)_{\lambda(0)} \in \lambda$. By similar way, $\neg(z \to x)_{\lambda(0)} \in \lambda$, and so $x \sim_\lambda z$. Therefore $\sim_\lambda$ is an equivalence relation on X. □

Also, similar to Proposition 3.34, we can define an order $\leq$ on X as follows:

$$[x] \leq [y] \Longleftrightarrow \neg(x \longrightarrow y)_{\lambda(0)} \in \lambda,$$

and prove that $(\frac{X}{\sim}, \leq)$ is a poset.

4. $(\in, \in\vee q)$-fuzzy sub-equality algebra

In this section, we define an $(\in, \in\vee q)$-fuzzy sub-equality of an equality algebra X and investigate that the properties of the $(\in, \in\vee q)$-fuzzy sub-equality algebras.

Definition 4.1. A fuzzy set λ in X is called an $(\in, \in\vee q)$-*fuzzy sub-equality algebra* of X if the following assertion is valid.

$$(\forall x, y \in X)(\forall t, k \in [0,1]) \left(x_t \in \lambda,\ y_k \in \lambda\ \Rightarrow \left\{ \begin{array}{l} (x \sim y)_{\min\{t,k\}} \in\vee q\, \lambda \\ (x \wedge y)_{\min\{t,k\}} \in\vee q\, \lambda \end{array} \right. \right). \tag{4.1}$$

Example 4.2. Consider the equality algebra $(X, \sim, \wedge, 1)$ which is described in Example 3.9. Define a fuzzy set λ in X as follows:

$$\lambda : X \to [0,1],\ x \mapsto \begin{cases} 0.8 & \text{if } \ x = 1, \\ 0.3 & \text{if } \ x = a, \\ 0.71 & \text{if } \ x = 0, \\ 0.73 & \text{if } \ x = b. \end{cases}$$

Then λ is an $(\in, \in\vee q)$-fuzzy sub-equality algebra of X.

Note. Every $(\in, \in)$-fuzzy sub-equality algebra is an $(\in, \in\vee q)$-fuzzy sub-equality algebra.

The converse of Note 4 is not true in general as seen in the following example.

Example 4.3. The $(\in, \in\vee q)$-fuzzy sub-equality algebra μ in Example 4.2 is not an $(\in, \in)$-fuzzy sub-equality algebra of X since

$$0.3 = \lambda(a) = \lambda(b \sim 0) \ngeq \min\{\lambda(b), \lambda(0)\} = 0.7$$

We consider characterizations of $(\in, \in\vee q)$-fuzzy sub-equality algebra.

THEOREM 4.4. *A fuzzy set λ in X is an $(\in, \in\vee q)$-fuzzy sub-equality algebra of X if and only if the following assertion is valid.*

$$(\forall x, y \in X) \left(\begin{array}{l} \lambda(x \sim y) \geq \min\{\lambda(x), \lambda(y), 0.5\} \\ \lambda(x \wedge y) \geq \min\{\lambda(x), \lambda(y), 0.5\} \end{array} \right). \tag{4.2}$$

PROOF: Assume λ is an $(\in, \in\vee q)$-fuzzy sub-equality algebra of X and $x, y \in X$. Suppose $\min\{\lambda(x), \lambda(y)\} < 0.5$. If $\lambda(x \sim y) < \min\{\lambda(x), \lambda(y)\}$, then $\lambda(x \sim y) < t \leq \min\{\lambda(x), \lambda(y)\}$ for some $t \in [0, 0.5)$, since $t \leq \min\{\lambda(x), \lambda(y)\} < 0.5$. It follows that $x_t \in \lambda$ and $y_t \in \lambda$. By assumption, $(x \sim y) \in\vee q\lambda$ and so $\lambda(x \sim y) \geq t$ or $\lambda(x \sim y) + t > 1$. If $\lambda(x \sim y) \geq t$, then is a contradiction, since $\lambda(x \sim y) < t$. If $\lambda(x \sim y) + t > 1$, then $\lambda(x \sim y) > 1 - t > 0.5$, is contradiction, since $\lambda(x \sim y) < 0.5$. Hence, $\lambda(x \sim y) \geq \min\{\lambda(x), \lambda(y)\}$, and so $(x \sim y)_t \in\vee q\lambda$. By the similar discussion, we get $\lambda(x \wedge y) \geq \min\{\lambda(x), \lambda(y)\}$ whenever $\min\{\lambda(x), \lambda(y)\} < 0.5$. Assume that $\min\{\lambda(x), \lambda(y)\} \geq 0.5$. Then $x_{0.5} \in \lambda$ and $y_{0.5} \in \lambda$. It follows from (4.1) that $(x \sim y)_{0.5} = (x \sim y)_{\min\{0.5, 0.5\}} \in\vee q\,\lambda$ and $(x \wedge y)_{0.5} = (x \wedge y)_{\min\{0.5, 0.5\}} \in\vee q\,\lambda$. Thus $\lambda(x \sim y) \geq 0.5$ and $\lambda(x \wedge$

$y) \geq 0.5$. Consequently, $\lambda(x \sim y) \geq \min\{\lambda(x), \lambda(y), 0.5\}$ and $\lambda(x \wedge y) \geq \min\{\lambda(x), \lambda(y), 0.5\}$.

Conversely, suppose λ satisfies the condition (4.2). Let $x, y \in X$ and $t, k \in [0,1]$ be such that $x_t \in \lambda$ and $y_k \in \lambda$. Then $\lambda(x) \geq t$ and $\lambda(y) \geq k$. If $\lambda(x \sim y) < \min\{t, k\}$, then $\min\{\lambda(x), \lambda(y)\} \geq 0.5$ because if $\min\{\lambda(x), \lambda(y)\} < 0.5$, then

$$\lambda(x \sim y) \geq \min\{\lambda(x), \lambda(y), 0.5\} \geq \min\{\lambda(x), \lambda(y)\} \geq \min\{t, k\}$$

which is a contradiction. Hence, $\lambda(x \sim y) \geq 0.5$. Similarly, if $\lambda(x \wedge y) < \min\{t, k\}$, then $\min\{\lambda(x), \lambda(y)\} \geq 0.5$. It follows that

$$\lambda(x \sim y) + \min\{t, k\} > 2\lambda(x \sim y) \geq 2\min\{\lambda(x), \lambda(y), 0.5\} \geq 1$$

and

$$\lambda(x \wedge y) + \min\{t, k\} > 2\lambda(x \wedge y) \geq 2\min\{\lambda(x), \lambda(y), 0.5\} = 1.$$

Hence $(x \sim y)_{\min\{t,k\}} q\lambda$ and $(x \wedge y)_{\min\{t,k\}} q\lambda$, and so $(x \sim y)_{\min\{t,k\}} \in\vee q\lambda$ and $(x \wedge y)_{\min\{t,k\}} \in\vee q\lambda$. Therefore, λ is an $(\in, \in\vee q)$-fuzzy sub-equality algebra of X. □

THEOREM 4.5. *If a fuzzy set λ in X is an $(\in, \in\vee q)$-fuzzy sub-equality algebra of X, then the following assertion is valid.*

$$(\forall x, y \in X)(\forall t, k \in [0,1]) \left(x_t \in \lambda,\ y_k \in \lambda \ \Rightarrow (x \to y)_{\min\{t,k\}} \in\vee q\, \lambda \right). \tag{4.3}$$

PROOF: Assume λ is an $(\in, \in\vee q)$-fuzzy sub-equality algebra of X. Let $x, y \in X$ and $t, k \in [0,1]$ be such that $x_t \in \lambda$ and $y_k \in \lambda$. Then $\lambda(x) \geq t$ and $\lambda(y) \geq k$, thus by Theorem 3.13, we have $(x \to y)_{\min\{t,k\}} \in \lambda$. Hence, $(x \to y)_{\min\{t,k\}} \in\vee q\, \lambda$. □

In the following example, we show that the converse of the above theorem may not be true, in general.

Example 4.6. According to Example 3.14, we have $0.3 = \lambda(a \wedge b) = \lambda(0) \not\geq \min\{\lambda(a), \lambda(b)\} = 0.4$, also $\lambda(a \wedge b) + \min\{\lambda(a), \lambda(b)\} \not\geq 1$. Hence, λ is not an $(\in, \in\vee q)$-fuzzy sub-equality.

THEOREM 4.7. *A fuzzy set λ in X is an $(\in, \in\vee q)$-fuzzy sub-equality algebra of X if and only if the non-empty $\in$-level set $U(\lambda; t)$ of λ is a sub-equality algebra of X for all $t \in (0, 0.5]$.*

Proof: Assume that λ is an $(\in, \in \vee q)$-fuzzy sub-equality algebra of X. Let $x, y \in U(\lambda; t)$ for $t \in (0, 0.5]$. Then $\lambda(x) \geq t$ and $\lambda(y) \geq t$. It follows from Theorem 4.4 that $\lambda(x \sim y) \geq \min\{\lambda(x), \lambda(y), 0.5\} \geq \min\{t, 0.5\} = t$ and $\lambda(x \wedge y) \geq \min\{\lambda(x), \lambda(y), 0.5\} \geq \min\{t, 0.5\} = t$. Hence $x \sim y \in U(\lambda; t)$ and $x \wedge y \in U(\lambda; t)$. Therefore, $U(\lambda; t)$ is a sub-equality algebra of X.

Conversely, suppose the non-empty $\in$-level set $U(\lambda; t)$ of λ is a sub-equality algebra of X for all $t \in (0, 0.5]$. If there exists $x, y \in X$ such that $\lambda(x \sim y) < \min\{\lambda(x), \lambda(y), 0.5\}$ or $\lambda(x \wedge y) < \min\{\lambda(x), \lambda(y), 0.5\}$, then $\lambda(x \sim y) < t \leq \min\{\lambda(x), \lambda(y), 0.5\}$ or $\lambda(x \wedge y) < t \leq \min\{\lambda(x), \lambda(y), 0.5\}$ for some $t \in (0, 1]$. Hence $t \leq 0.5$ and $x, y \in U(\lambda; t)$, and by assumption $x \sim y \in U(\lambda; t)$ and $x \wedge y \in U(\lambda; t)$, and so $\lambda(x \sim y) \geq t$ and $\lambda(x \wedge y) \geq t$ which is a contradiction. Hence, $\lambda(x \sim y) \geq \min\{\lambda(x), \lambda(y), 0.5\}$ and $\lambda(x \wedge y) \geq \min\{\lambda(x), \lambda(y), 0.5\}$. Therefore, by Theorem 4.4, λ is an $(\in, \in \vee q)$-fuzzy sub-equality algebra of X. □

We provide a condition for an $(\in, \in \vee q)$-fuzzy sub-equality algebra to be an $(\in, \in)$-fuzzy sub-equality algebra.

Theorem 4.8. *If an $(\in, \in \vee q)$-fuzzy sub-equality algebra λ of X satisfies the condition*

$$(\forall x \in X)(\lambda(x) < 0.5), \tag{4.4}$$

then λ is an $(\in, \in)$-fuzzy sub-equality algebra of X.

Proof: Let $x, y \in X$ and $t, k \in [0, 1]$ be such that $x_t \in \lambda$ and $y_k \in \lambda$. Then $\lambda(x) \geq t$ and $\lambda(y) \geq k$. By assumption, (4.4), and Theorem 4.4, we have

$$\lambda(x \sim y) \geq \min\{\lambda(x), \lambda(y), 0.5\} = \min\{\lambda(x), \lambda(y)\} \geq \min\{t, k\}$$

and

$$\lambda(x \wedge y) \geq \min\{\lambda(x), \lambda(y), 0.5\} = \min\{\lambda(x), \lambda(y)\} \geq \min\{t, k\}.$$

Hence $(x \sim y)_{\min\{t,k\}} \in \lambda$ and $(x \wedge y)_{\min\{t,k\}} \in \lambda$. Therefore λ is an $(\in, \in)$-fuzzy sub-equality algebra of X. □

Proposition 4.9. If λ is a non-zero $(\in, \in \vee q)$-fuzzy sub-equality algebra of X, then $\lambda(1) > 0$.

Proof: Assume that $\lambda(1) = 0$. Since λ is non-zero, there exists $x \in X$ such that $\lambda(x) = t \neq 0$, and so for any $t \in (0, 1]$, $x_t \in \lambda$. Then $\lambda(x \sim x) =$

$\lambda(1) = 0$ and $\lambda(x \sim x) + t = \lambda(1) + t = t \leq 1$, that is, $(x \sim x)_t \overline{\in} \lambda$ and $(x \sim x)_t \, \overline{q} \, \lambda$. Thus $(x \sim x)_t \, \overline{\in \vee q} \, \lambda$, which is a contradiction. Therefore $\lambda(1) > 0$. □

COROLLARY 4.10. If λ is a non-zero $(\in, \in)$-fuzzy sub-equality algebra of X, then $\lambda(1) > 0$.

THEOREM 4.11. *For any sub-equality algebra S of X and $t \in [0, 0.5)$, there exists an $(\in, \in \vee q)$-fuzzy sub-equality algebra λ of X such that $U(\lambda; t) = S$.*

PROOF: Let λ be a fuzzy set in X defined by

$$\lambda : X \to [0,1], \;\; x \mapsto \begin{cases} t & \text{if } \; x \in S, \\ 0 & \text{otherwise,} \end{cases} \tag{4.5}$$

where $t \in [0, 0.5)$. Obviously, $U(\lambda; t) = S$. Suppose that $\lambda(x \sim y) < \min\{\lambda(x), \lambda(y), 0.5\}$ or $\lambda(x \wedge y) < \min\{\lambda(x), \lambda(y), 0.5\}$ for some $x, y \in X$. Since $|Im(\lambda)| = 2$, it follows that $\lambda(x \sim y) = 0$ or $\lambda(x \wedge y) = 0$, and $\min\{\lambda(x), \lambda(y), 0.5\} = t$. Since $t < 0.5$, we have $\lambda(x) = t = \lambda(y)$ and so $x, y \in S$. Then $x \sim y \in S$ and $x \wedge y \in S$, which imply that $\lambda(x \sim y) = t$ and $\lambda(x \wedge y) = t$, which is a contradiction, and so $\lambda(x \sim y) \geq \min\{\lambda(x), \lambda(y), 0.5\}$ and $\lambda(x \wedge y) \geq \min\{\lambda(x), \lambda(y), 0.5\}$. Hence, by Theorem 4.4, we know that λ is an $(\in, \in \vee q)$-fuzzy sub-equality algebra of X. □

For any fuzzy set λ in X and $t \in [0, 1]$, we consider the following sets and we call then *q-level set* and *$\in \vee q$-level set*, respectively.

$$\lambda_q^t := \{x \in X \mid x_t \, q \, \lambda\} \text{ and } \lambda_{\in \vee q}^t := \{x \in X \mid x_t \in \vee \, q \lambda\}$$

Clearly, $\lambda_{\in \vee q}^t = \lambda_{\in}^t \cup \lambda_q^t$.

THEOREM 4.12. *A fuzzy set λ in X is an $(\in, \in \vee q)$-fuzzy sub-equality algebra of X if and only if $\lambda_{\in \vee q}^t$ is a sub-equality algebra of X for all $t \in [0, 1]$.*

PROOF: Assume that λ is an $(\in, \in \vee q)$-fuzzy sub-equality algebra of X. Let $x, y \in \lambda_{\in \vee q}^t$ for $t \in [0, 1]$. Then $x_t \in \vee \, q \lambda$ and $y_t \in \vee \, q \lambda$, i.e., $\lambda(x) \geq t$ or $\lambda(x) + t > 1$, and $\lambda(y) \geq t$ or $\lambda(y) + t > 1$. It follows from (4.2) that $\lambda(x \sim y) \geq \min\{t, 0.5\}$ and $\lambda(x \wedge y) \geq \min\{t, 0.5\}$. In fact, if $\lambda(x \sim y) < \min\{t, 0.5\}$ or $\lambda(x \wedge y) < \min\{t, 0.5\}$, then $x_t \, \overline{\in \vee q} \, \lambda$ or $y_t \, \overline{\in \vee q} \, \lambda$, a contradiction. If $t \leq 0.5$, then $\lambda(x \sim y) \geq \min\{t, 0.5\} = t$ and $\lambda(x \wedge y) \geq$

$\min\{t, 0.5\} = t$. Hence $x \sim y \in U(\lambda; t) \subseteq \lambda^t_{\in\vee q}$ and $x \wedge y \in U(\lambda; t) \subseteq \lambda^t_{\in\vee q}$. If $t > 0.5$, then $\lambda(x \sim y) \geq \min\{t, 0.5\} = 0.5$ and $\lambda(x \wedge y) \geq \min\{t, 0.5\} = 0.5$. Hence $\lambda(x \sim y) + t > 0.5 + 0.5 = 1$ and $\lambda(x \wedge y) + t > 0.5 + 0.5 = 1$, that is, $(x \sim y)_t\, q\, \lambda$ and $(x \wedge y)_t\, q\, \lambda$. It follows that $x \sim y \in \lambda^t_q \subseteq \lambda^t_{\in\vee q}$ and $x \wedge y \in \lambda^t_q \subseteq \lambda^t_{\in\vee q}$. Therefore $\lambda^t_{\in\vee q}$ is a sub-equality algebra of X for all $t \in (0, 1]$.

Conversely, let λ be a fuzzy set in X and $t \in [0, 1]$ such that $\lambda^t_{\in\vee q}$ is a sub-equality algebra of X. Suppose that $\lambda(x \sim y) < \min\{\lambda(x), \lambda(y), 0.5\}$ or $\lambda(x \wedge y) < \min\{\lambda(x), \lambda(y), 0.5\}$ for some $x, y \in X$. Then $\lambda(x \sim y) < k < \min\{\lambda(x), \lambda(y), 0.5\}$ or $\lambda(x \wedge y) < k < \min\{\lambda(x), \lambda(y), 0.5\}$ for some $k \in (0, 0.5)$. Hence $x, y \in U(\lambda; k) \subseteq \lambda^k_{\in\vee q}$, and so $x \sim y \in \lambda^k_{\in\vee q}$ and $x \wedge y \in \lambda^k_{\in\vee q}$. Thus $\lambda(x \sim y) \geq k$ or $\lambda(x \sim y) + k > 1$, and $\lambda(x \wedge y) \geq k$ or $\lambda(x \wedge y) + k > 1$. This is a contradiction, and therefore $\lambda(x \sim y) \geq \min\{\lambda(x), \lambda(y), 0.5\}$ and $\lambda(x \wedge y) \geq \min\{\lambda(x), \lambda(y), 0.5\}$ for all $x, y \in X$. Consequently, λ is an $(\in, \in\vee q)$-fuzzy sub-equality algebra of X by Theorem 4.4. □

THEOREM 4.13. *If λ is an $(\in, \in\vee q)$-fuzzy sub-equality algebra of X, then the q-set λ^t_q is a sub-equality algebra of X for all $t \in (0.5, 1]$.*

PROOF: Let $x, y \in \lambda^t_q$ for $t \in (0.5, 1]$. Then $\lambda(x) + t > 1$ and $\lambda(y) + t > 1$, and so $\lambda(x) > 1 - t$, and $\lambda(y) > 1 - t$. By assumption, we have $(x \sim y)_{1-t} \in\vee q\lambda$ and $(x \wedge y)_{1-t} \in\vee q\lambda$. Thus, by Theorem 4.4 that

$$\lambda(x \sim y) \geq \min\{\lambda(x), \lambda(y), 0.5\} > \min\{1 - t, 0.5\},$$

since $t \in (0.5, 1]$, we have $1 - t \in [0, 0.5)$ and so $1 - t < 0.5$. Thus, $\lambda(x \sim y) \geq \min\{1 - t, 0.5\} = 1 - t$ and so $\lambda(x \sim y) + t > 1$. Hence $x \sim y \in \lambda^t_q$. Similarly, we have $x \wedge y \in \lambda^t_q$. □

THEOREM 4.14. *Let $f : X \to Y$ be a homomorphism of equality algebras. If λ and μ are $(\in, \in\vee q)$-fuzzy sub-equality algebras of X and Y, respectively, then*

(1) *$f^{-1}(\mu)$ is an $(\in, \in\vee q)$-fuzzy sub-equality algebra of X.*

(2) *If f is onto and λ satisfies the condition*

$$(\forall T \subseteq X)(\exists x_0 \in T) \left(\lambda(x_0) = \sup_{x \in T} \lambda(x)\right), \tag{4.6}$$

then $f(\lambda)$ is an $(\in, \in\vee q)$-fuzzy sub-equality algebra of Y.

PROOF: (1) Let $x, y \in X$ and $t, k \in [0,1]$ be such that $x_t \in f^{-1}(\mu)$ and $y_k \in f^{-1}(\mu)$. Then $(f(x))_t \in \mu$ and $(f(y))_k \in \mu$. Since μ is an $(\in, \in\vee q)$-fuzzy sub-equality algebra of Y, we have

$$(f(x \sim y))_{\min\{t,k\}} = (f(x) \sim f(y))_{\min\{t,k\}} \in\vee q\, \mu$$

and

$$(f(x \wedge y))_{\min\{t,k\}} = (f(x) \wedge f(y))_{\min\{t,k\}} \in\vee q\, \mu.$$

Hence $(x \sim y)_{\min\{t,k\}} \in\vee q\, f^{-1}(\mu)$ and $(x \wedge y)_{\min\{t,k\}} \in\vee q\, f^{-1}(\mu)$. Therefore, $f^{-1}(\mu)$ is an $(\in, \in\vee q)$-fuzzy sub-equality algebra of X.

(2) Let $a, b \in Y$ and $t, k \in [0,1]$ be such that $a_t \in f(\lambda)$ and $b_k \in f(\lambda)$. Then $(f(\lambda))(a) \geq t$ and $(f(\lambda))(b) \geq k$. Using the condition (4.6), there exist $x \in f^{-1}(a)$ and $y \in f^{-1}(b)$ such that

$$\lambda(x) = \sup_{z \in f^{-1}(a)} \lambda(z) \text{ and } \lambda(y) = \sup_{w \in f^{-1}(b)} \lambda(w).$$

Then $x_t \in \lambda$ and $y_k \in \lambda$, which imply that $(x \sim y)_{\min\{t,k\}} \in\vee q\, \lambda$ and $(x \wedge y)_{\min\{t,k\}} \in\vee q\, \lambda$, since λ is an $(\in, \in\vee q)$-fuzzy sub-equality algebra of X. Now, $x \sim y \in f^{-1}(a \sim b)$ and $x \wedge y \in f^{-1}(a \wedge b)$, and so $(f(\lambda))(a \sim b) \geq \lambda(x \sim y)$ and $(f(\lambda))(a \wedge b) \geq \lambda(x \wedge y)$. Hence,

$$(f(\lambda))(a \sim b) \geq \min\{t,k\} \text{ or } (f(\lambda))(a \sim b) + \min\{t,k\} > 1$$

and

$$(f(\lambda))(a \wedge b) \geq \min\{t,k\} \text{ or } (f(\lambda))(a \wedge b) + \min\{t,k\} > 1,$$

that is, $(a \sim b)_{\min\{t,k\}} \in\vee q\, f(\lambda)$ and $(a \wedge b)_{\min\{t,k\}} \in\vee q\, f(\lambda)$. Therefore, $f(\lambda)$ is an $(\in, \in\vee q)$-fuzzy sub-equality algebra of Y. □

5. Conclusion

Our aim was to define the concepts of an $(\in, \in)$-fuzzy sub-equality algebra, an $(\in, \in\vee q)$-fuzzy sub-equality algebra and we discussed some properties and found some equivalent definitions of them. Then, we discussed characterizations of an $(\in, \in)$-fuzzy sub-equality algebra and an $(\in, \in\vee q)$-fuzzy sub-equality algebra. Also, we found relations between an $(\in, \in)$-fuzzy sub-equality algebra and an $(\in, \in\vee q)$-fuzzy sub-equality algebra.

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A SYNTACTIC PROOF OF THE DECIDABILITY OF FIRST-ORDER MONADIC LOGIC

Abstract

Decidability of monadic first-order classical logic was established by Löwenheim in 1915. The proof made use of a semantic argument and a purely syntactic proof has never been provided. In the present paper we introduce a syntactic proof of decidability of monadic first-order logic in innex normal form which exploits **G3**-style sequent calculi. In particular, we introduce a cut- and contraction-free calculus having a (complexity-optimal) terminating proof-search procedure. We also show that this logic can be faithfully embedded in the modal logic **T**.

Keywords: proof theory, classical logic, decidability, Herbrand theorem.

1. Introduction

A cornerstone result in the field of classical logic is the undecidability of first-order logic (FOL) [3]. Indeed, the set of first-order (FO) logical truths is recursively enumerable and so semidecidable, but essentially undecidable. Even before the discovery of this crucial feature, some decidable fragments have been isolated and investigated.

One of the most representative ones is the monadic fragment obtained by restricting the language to one-place predicates, thus excluding relations therefrom. A first proof of the decidability of monadic classical FOL (MFOL) was given by Löwenheim [6]. The proof employed semantic arguments (in particular, a form of finite model property) and it can thus be

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regarded as partially satisfactory, as it uses a semantic method to establish a syntactic result.[1]

Other proofs were provided by Quine [7] and, later, by Boolos [1]. A key ingredient in these arguments is the reduction of formulas of MFOL to a kind of normal form, which pushes quantifiers inside formulas. Hence, validity of the formulas thus obtained—to be called *innex formulas*—is checked via semantic arguments. However, a purely syntactic and proof-theoretic version of decidability has not been presented yet. In the present paper, we aim at filling this gap.

The design of a terminating sequent calculus for monadic logic is not a trivial task. Indeed, we need to observe that the rule of contraction cannot be *a priori* dispensed with. An example is the formula $\exists x(P(x) \supset \forall y P(y))$ which is a monadic valid formula that is not provable without a (possibly implicit in the rule) step of contraction.

Therefore we focus on a specific fragment of MFOL, i.e. the innex one, and we show that we can give a terminating sequent calculus in which every rule is height-preserving invertible without the need for any form of contraction. The calculus **G3INT** is obtained by combining a form of focusing—i.e., a specific ordering in the application of the rules [5]—with a new rule for the existential quantifier.

These aspects contribute to complicating the structural analysis of the system which has some peculiar traits. Furthermore, we offer an extremely simple syntactic proof of cut-elimination which is based on a single inductive parameter, the degree of the cut formula, instead of two parameters—e.g., the degree and the height of the cut—as in calculi for FOL [8].

Finally, we offer another perspective on the decidability of the innex fragment of monadic logic. In particular, we show that it can be soundly and faithfully embedded in the modal system **T** enhanced with a first-order language (but without quantifiers). This reduction highlights some specific characteristics of the fragment by identifying $\forall$ and $\exists$ with the modal operators $\Box$ and $\Diamond$, respectively.

The plan of the paper is as follows. Section 2 introduces innex normal form and a preliminary calculus for MFOL. Section 3 is devoted to the calculus **G3INT** whose properties are thoroughly investigated in Section 4.

[1] As observed by a reviewer, under the completeness of monadic logic, the decidability result might be considered a semantic as well as syntactic problem. In our opinion, the problem of whether a logic is decidable concerns derivability in a formal system and thus it has a more intrinsically proof-theoretic content.

Soundness and completeness are discussed in Section 5 and Section 6 deals with the modal interpretation of the system. Finally, Section 7 adds some concluding remarks and sketches some themes which may be an object of future research.

2. MFOL and innex normal form

Let us fix a signature $\mathcal{S}$ containing a countable and non-empty set of monadic predicates. Given a denumerable set of variables $\mathcal{V}$, the language of MFOL (in negative normal form) is defined by the following grammar:

$$A ::= P(x) \mid \overline{P}(x) \mid A \wedge A \mid A \vee A \mid \exists x A \mid \forall x A$$

where $P \in \mathcal{S}$ and $x \in \mathcal{V}$.

Parentheses are used as usual and negation is defined via (De Morgan's) dualities and double-negation elimination—e.g., $\neg\overline{P}(x) \equiv \overline{\overline{P}} \equiv P(x)$ and $\neg\forall x A \equiv \exists x \neg A$. A *literal* is a formula of shape $P(x)$ or $\overline{P}(x)$. We use the following metavariables: x, y, z for variables, P, R, S for literals, and A, B, C for arbitrary formulas. $A[y/x]$ stands for the formula obtained by replacing in A each free occurrence of x with an occurrence of y, provided that y is free for x in A. When convenient, we use $B(y)$ for the formula obtained from QxB by removing the quantifier Qx and substituting y for x.

In (classical) FOL it is often preferred to work with formulas which have a precise shape. In this sense, a normal form for FOL is the so-called *prenex normal form*. A formula is in *prenex normal form* whenever it is of the form: $Q_{x_1}...Q_{x_n}A$, where $Q_{x_1}...Q_{x_n}$ is a finite string of quantifiers and A is a quantifier-free formula—i.e., A contains only propositional connectives.

Proposition 2.1. Each first-order formula A is logically equivalent to a formula A' in prenex normal form: A and A' are satisfied by the same FO-models.

Proof: A standard induction on the structure of A making use of De Morgan's dualities, of the distributivity of $\vee$ over $\wedge$ and of the following FO-validities:[2]

[2] Without loss of generality, we are assuming that each quantifier binds a different variable, no variable has both free and bound occurrences in a formula, and $x \notin B$.

- $\neg\exists x A \supset\subset \forall x \neg A$
- $\neg\forall x A \supset\subset \exists x \neg A$
- $\exists x A \vee \exists y C \supset\subset \exists x(A \vee C[x/y])$
- $\forall x A \wedge \forall y C \supset\subset \forall x(A \wedge C[x/y])$
- $\exists x A \vee B \supset\subset \exists x(A \vee B)$
- $\forall x A \vee B \supset\subset \forall x(A \vee B)$
- $\exists x A \wedge B \supset\subset \exists x(A \wedge B)$
- $\forall x A \wedge B \supset\subset \forall x(A \wedge B)$ □

This is a property which is specific of classical FOL which does not usually extend to non-classical logics or modal logics. In particular, neither FO-intuitionistic nor FO-modal logics do validate the prenexation laws.

In this paper we are actually interested in a sort of converse transformation which pushes quantifiers inside the formulas.

DEFINITION 2.2. A first-order formula is in *innex normal form* (INF) if it is a boolean combination of formulas A and QxB, where A is a quantifier-free formula and QxB is a formula of the form $\exists x(P_1(x) \wedge ... \wedge P_n(x))$ or $\forall x(P_1(x) \vee ... \vee P_n(x))$ where P_i is a literal.

In general, FO-formulas are not equivalent to formulas in INF, but this holds if we consider the *monadic fragment* of the language—i.e, a FO-language containing only unary predicates.

PROPOSITION 2.3. Each formula A of the monadic fragment of the FO-language is logically equivalent to a formula A' in INF.

PROOF: Analogous to the proof of Prop. 2.1, applying the same equivalences in reverse direction, cf. [1, Lemma 21.12]. □

MFOL has been shown to be decidable already in [6] by means of a semantic argument. In particular, we have the following results:

THEOREM 2.4.

1. *If a monadic sentence containg k predicates is satisfiable, it has a model of size no greater than 2^k [1, Lemma 21.8];*
2. *The satisfiability problem for monadic FO-logic is* NEXP*-complete [4].*

In this paper we are interested in giving a proof-theoretic proof of decidability of MFOL. A key ingredient for this result will be Proposition

Initial Sequents: $\dfrac{}{\Gamma, P, \overline{P}}\, Ax$

Rules: $\dfrac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \wedge B}\, \wedge \qquad \dfrac{\Gamma, A, B}{\Gamma, A \vee B}\, \vee \qquad \dfrac{\Gamma, A[y/x]}{\Gamma, \forall x A}\, \forall, y \text{ fresh} \qquad \dfrac{\Gamma, \exists x A, A[z/x]}{\Gamma, \exists x A}\, \exists$

Figure 1. The sequent calculus **G3S**

2.3: the possibility of defining an innex normal form for monadic formulas is crucial in order to develop a terminating calculus. Indeed, innex formulas remove the nesting of quantifiers and allow for a full elimination of contraction which is harmful for proof search. Interestingly, also Quine [7] has given a proof of decidability for MFOL exploiting Prop. 2.3. However, his method uses truth tables which are arguably less immediate than the method of terminating sequent calculi adopted in this paper.

2.1. Decidability of MFOL

The one-sided sequent calculus **G3S** for MFOL is given in Figure 1, we refer the reader to [8] for its properties and to Section 3 for some basic definition. The difficulty in directly establishing a decidability proof of MFOL within **G3S** is due to the formulation of the rule $\exists$ in which the principal formula is repeated in the premise of the rule. This design choice is necessary in order to make the rule invertible, but it has a hidden contraction. In principle, there is no bound on the possible number of applications of the rule $\exists$.

In order to prove the decidability result we need to show a weak version of Herbrand's theorem which will be essential in order to obtain the proof.

Lemma 2.5. *For every finite multisets of quantifier-free formulas Γ and all quantifier-free formulas B_i, if $\Gamma, \exists x_1 B_1, ..., \exists x_\ell B_\ell$ is* **G3S***-derivable, then, for some m and n in $\mathbb{N}$, there is a derivation of the same height of*

$$\Gamma, \{B_1[y_i/x_1] : i \leq m\}, ..., \{B_\ell[y_j/x_\ell] : j \leq n\}$$

Proof: We argue by induction on the height h of the derivation. If $h = 0$, the proof is immediate, because quantified formulas cannot be principal.

If $h > 0$, then we distinguish cases according to the last rule applied. If it is a propositional rule, we apply the induction hypothesis to the premise(s) and then the rule again. If it is a quantifier rule, then it can only be the rule $\exists$. It is enough to apply the induction hypothesis. □

THEOREM 2.6. *For every sequent* Γ*, where* Γ *is a finite multiset of formulas of MFOL in INF, there is a procedure outputting either a* **G3S***-proof or a finite failed attempt to it.*

PROOF: The decision procedure consists in applying the invertibility of every propositional rule. This will imply that the derivability of the sequent Γ is equivalent to that of the sequents Γ_1,...,Γ_n, where, for each $i \in \{1, ..., n\}$, Γ_i is of the form (for Γ'_i, D_j, and B_ℓ quantifier-free):

$$\Gamma'_i, \forall x_1 D_1, \ldots, \forall x_m D_m, \exists z_1 B_1, \ldots, \exists z_k B_k$$

for some $m \geq 0$ and $k \geq 0$.

We now apply the invertibility of the rule $\forall$ to get:

$$\Gamma'_i, D_1[y_1/x_1], ..., D_m[y_m/x_r], \exists z_1 B_1, \ldots, \exists z_k B_k$$

Each sequent thus obtained satisfies the hypotheses of Lemma 2.5 and therefore we can reduce its derivability to that of a sequent Γ^*_i which does not contain any quantified formula. The derivability of each of these sequents is decidable. □

Theorem 2.6 shows that if we restrict our attention to formulas of MFOL that are in INF then we can bound the number of contractions hidden inside of the rule $\exists$ so as to obtain a decision procedure for MFOL. Observe that Theorem 2.6 does not mean that **G3S** is a terminating calculus for MFOL. Even if we have a sequent (whose formulas are) in INF, proof-search is non-terminating because of the contraction hidden in the repetition of the principal formula in the premise of the rule $\exists$. More precisely, we have defined a strategy to halt the search for a derivation or a countermodel, but the decidability is not intrinsic to the calculus **G3S**.

The system **G3S** represents a bridge towards a terminating calculus for MFOL. In order to obtain it, we will impose that the rule for the existential quantifier can be applied only when we already know all variables over which it can be instantiated, so that it can be instantiated over each of them at the same time. As it will be shown in Section 4.1, this terminating calculus has all the rules invertible without having to resort to any (hidden or explicit) instance of contraction and, hence, it will allow for a decision procedure for MFOL that is optimal complexity-wise.

Initial Sequents $\dfrac{}{\Gamma, P, \overline{P}}\ Ax,\ \Gamma \text{ innex}$

Logical Rules $\dfrac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \wedge B}\ R\wedge$ $\qquad \dfrac{\Gamma, A, B}{\Gamma, A \vee B}\ R\vee$

$$\dfrac{\Gamma, P_1[y/x] \vee \ldots \vee P_n[y/x]}{\Gamma, \forall x(P_1 \vee \ldots \vee P_n)}\ R\forall,\ y \text{ fresh} \qquad \dfrac{\Gamma, \{P_1[z_i/x] \wedge \ldots \wedge P_n[z_i/x] : z_i \in Var(\Gamma)\}}{\Gamma, \exists x(P_1 \wedge \ldots \wedge P_n)}\ R\exists,\ \Gamma \text{ reduced}$$

Figure 2. The sequent calculus **G3INT**.

3. The calculus G3INT

To define a contraction-free calculus, we shall introduce another sequent calculus for MFOL in INF. In particular, we will use a **G3**-style calculus to obtain the result. The reason of the choice lies in the fact that **G3**-style calculi have good structural properties and they are suitable for backward reasoning, due to the invertibility of every rule.

The rules of the (one-sided) calculus **G3INT** are given in Figure 2. In particular, initial sequents have the side condition that Γ is a multiset of (monadic) formulas in INF. By $Var(\Gamma)$ we denote the set of free variables occurring in the multiset Γ, if any, otherwise the singleton containing some fixed variable y. Rule $R\exists$ has the side condition that Γ is a multiset of reduced formulas, where the notion of *reduced multiset* is defined as follows:

Definition 3.1. A multiset Γ is *reduced* whenever it does not contain universal quantifiers.

A *derivation* is a finite rooted tree where the leaves are initial sequents and every node is constructed by applications of the rules. The *height* of a derivation is the number of nodes in a branch of maximal length in the derivation minus one. The *degree* of a formula is the number of logical symbols occurring in the formula. A rule is *(height-preserving) admissible* if, whenever each premise of the rule is derivable (with a derivation of height $\leq n$), so is the conclusion (with a derivation of height $\leq n$). Without loss of generality, we always assume to be working up to renaming of bound variables, i.e. *modulo* α-conversion.

We briefly recall some properties of **G3INT**. We start by proving that the rules are such that the property of being in innex normal form propagates from the leaves to each node of a derivation. This allows us to restrict attention to (finite multisets of) formulas in innex normal form.

Lemma 3.2. *If* Γ *is derivable in* **G3INT**, *then (each formula occurring in) it is in innex normal form.*

Proof: The proof is by induction on the height of the derivation. If Γ is an initial sequent, the proof is trivial. Otherwise we distinguish cases according to the last rule applied. In each case it is enough to apply the induction hypothesis to (each of) the premise(s) of the rule and then observing that the rules preserve the innex normal form. □

Lemma 3.3. *The rules* $R\wedge, R\vee$ *and* $R\exists$ *are height-preserving invertible.*

Proof: The proof runs by induction on the height of the derivation. We discuss the case of $R\exists$ (the other cases are as for **G3S**). If $\Gamma, \exists x(P_1 \wedge \dots P_n)$ is an initial sequent, so is $\Gamma, \{P_1[z_i/x] \wedge \dots \wedge P_n[z_i/x] : z_i \in Var(\Gamma)\}$. If the last rule applied is $R\exists$ and $\exists x(P_1 \wedge \dots \wedge P_n)$ is principal, the proof is immediate. Otherwise, the last rule applied cannot be R$\forall$ since $\Gamma, \exists x(P_1 \wedge \dots \wedge P_n)$ is reduced. Therefore, we simply apply the induction hypothesis to each of the premises and then the rule again. □

Lemma 3.4. *The sequent* $\Gamma, A, \overline{A}$ *is provable in* **G3INT**.

Proof: We argue by induction on the degree of A. If A is a literal then there is nothing to prove. If A (or $\overline{A}$) is of the shape $B \vee C$, the proof is immediate. If it is of the shape $\forall x B$, we first apply root-first the rules to obtain sequents with a reduced context Γ' and then we proceed as follows:[3]

$$\dfrac{\dfrac{\overline{\overline{\Gamma', A[y/x], \overline{A}[z_1/x], \dots, \overline{A}[z_n/x], \overline{A}[y/x]}}\,\text{IH}}{\Gamma', A[y/x], \exists x\overline{A}}\,R\exists}{\Gamma', \forall x A, \exists x\overline{A}}\,R\forall$$

Where $z_1, \dots, z_n, y$ are all variables occurring free in $\Gamma', A[y/x]$. □

[3]The doubleline derivation symbol marks a step that is admissible.

4. Structural analysis of G3INT

LEMMA 4.1. *The rule:*

$$\frac{\Gamma, \Delta, \Delta}{\Gamma, \Delta}\ \textit{RedC}$$

where Γ, Δ, Δ *is reduced, is height-preserving admissible.*

PROOF: We proceed by induction on the height of the derivation of the sequent Γ, Δ, Δ. If it is an initial sequent, then so is Γ, Δ.

If no formula in Δ is principal, we apply the induction hypothesis and then the rule again.

If a formula A is principal in Δ, we distinguish cases according to the last rule applied. The strategy consists in applying Lemma 3.3, the induction hypothesis and then the rule again. We consider the case of the rule $R\exists$.

$$\frac{\Gamma, A[z_1/x], ..., A[z_n/x], \exists x A, \Delta', \Delta'}{\Gamma, \exists x A, \exists x A, \Delta', \Delta'}\ R\exists$$

where A is a finite conjunction of atomic formulas. We construct the following derivation:

$$\cfrac{\cfrac{\cfrac{\Gamma, A[z_1/x], ..., A[z_n/x], \exists x A, \Delta', \Delta'}{\Gamma, A[z_1/x], ..., A[z_n/x], A[z_1/x], ..., A[z_n/x], \Delta', \Delta'}\ \text{Lemma 3.3}}{\Gamma, A[z_1/x], ..., A[z_n/x], \Delta'}\ IH}{\Gamma, \exists x A, \Delta'}\ R\exists$$

□

Proceeding in a slightly unusual order, we now prove the admissibility of substitution. As usual, we extend substitutions to multisets of formulas.

LEMMA 4.2 (Substitution). *The rule:*

$$\frac{\Gamma}{\Gamma[y/x]}\ \textit{Sub}$$

is height-preserving admissible

PROOF: By induction on the height of the derivation $\mathcal{D}$ of the premise Γ.

If $\mathtt{h}(\mathcal{D}) = 0$ then the lemma obviously holds. If $\mathtt{h}(\mathcal{D}) = n + 1$ then we have cases according to the last rule applied in $\mathcal{D}$. If the last rule is an instance of rule $R\wedge$ or $R\vee$, the proof follows from the induction hypothesis.

If the last step in $\mathcal{D}$ is the following instance of rule $R\exists$:

$$\frac{\Gamma, A[z_1/z], \ldots, A[z_n/z]}{\Gamma, \exists z A} R\exists$$

where, w.l.o.g. $z \notin \{x, y\}$, then we transform the derivation as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, A[z_1/z], \ldots, A[z_n/z]}{\Gamma[y/x], (A[z_1/z])[y/x], \ldots, (A[z_n/z])[y/x]} \text{IH}}{\Gamma[y/x], (A[y/x])[(z_1[y/x])/z], \ldots, (A[y/x])[(z_n[y/x])/z]} \star}{\Gamma[y/x], \exists z(A[y/x])} R\exists}{\Gamma[y/x], (\exists z A)[y/x]} \star\star$$

where the steps marked with $\star$ and $\star\star$ are syntactic rewritings that do not increase the height of the derivation. The application of the rule $R\exists$ is justified as the set of terms occurring in $\Gamma[y/x]$ is a subset of the set of terms occurring in $(A[y/x])[(z_1[y/x])/z], \ldots, (A[y/x])[(z_n[y/x])/z]$.

Furthermore, note that if z is free in $A[y/x]$ and, for some $j, k \leq n$, $x \equiv z_j$ and $y \equiv z_k$, then $(A[y/x])[(z_j[y/x])/z] \equiv (A[y/x])[(z_k[y/x])/z]$. This is not a problem since by the design of the rules the sequent

$$\Gamma[y/x], (A[y/x])[(z_1[y/x])/z], \ldots, (A[y/x])[(z_n[y/x])/z]$$

is reduced and so we can safely apply Lemma 4.1.

Finally, suppose the last step in $\mathcal{D}$ is the following instance of rule:

$$\frac{\Gamma, A[y_2/y_1]}{\Gamma, \forall y_1 A} R\forall;\ y_2!$$

where neither y nor x is y_1. We apply the inductive hypothesis (IH) twice to the derivation of the premise, the first time to replace y_2 with a variable y_3 that is new to the premise and the second time to replace x with y. By applying an instance of rule $R\forall$ we conclude $(\Gamma, \forall y_1 A)[y/x]$. □

Lemma 4.3 (Invertibility). *All rules of* **G3INT** *are height-preserving invertible.*

Proof: The case of rules $R\vee$, $R\wedge$ and $R\exists$ has been proved in Lemma 3.3. We show, by induction on the height of the derivation $\mathcal{D}$, that rule $R\forall$ is height-preserving invertible.

If $\mathtt{h}(\mathcal{D}) = 0$, or if the formula we are inverting is principal in the last step of $\mathcal{D}$, then the proof is trivial.

If $\mathcal{D}$ is

$$\frac{\Delta', \forall y_1 A \qquad (\Delta'', \forall y_1 A)}{\Delta, \forall y_1 A}\,\mathrm{R}$$

we know that R is not an instance of rule $R\exists$. Once again we apply IH to the premise(s) (possibly with a height-preserving admissible step of substitution to avoid clashes of variables) and then an instance of R to conclude $\Delta, A[y_2/y_1]$. □

THEOREM 4.4. *The rules of contraction are height-preserving admissible in* **G3INT**.

PROOF: The proof is by induction on the height of the derivation $\mathcal{D}$. If Γ, A, A is an initial sequent, so is Γ, A. If A is not principal in the last rule applied, we apply the induction hypothesis to each of the premises of the rule and then the rule again, e.g., if $\mathcal{D}$ is

$$\frac{\Delta', A, A \qquad (\Delta'', A, A)}{\Delta, A, A}\,\mathrm{R}$$

We construct the following derivation:

$$\dfrac{\dfrac{\Delta', A, A}{\Delta', A}\,\mathrm{IH} \qquad \dfrac{(\Delta'', A, A)}{(\Delta'', A)}\,\mathrm{IH}}{\Delta, A}\,\mathrm{R}$$

If, instead, A is principal, we distinguish cases according to its shape. The strategy consists in applying the invertibility lemma followed by the induction hypothesis. We focus on the case of the existential quantifier. We suppose that the set of variables free in Γ is not empty and we have:

$$\frac{\Gamma, \exists x(P_1 \wedge \ldots P_n), \{P_1[z_i/x] \wedge \ldots \wedge P_n[z_i/x] : z_i \in Var(\Gamma)\}}{\Gamma, \exists x(P_1 \wedge \ldots \wedge P_n), \exists x(P_1 \wedge \ldots \wedge P_n)}\,\mathrm{R}\exists$$

We proceed as follows:

$$\dfrac{\dfrac{\dfrac{\Gamma, \exists x(P_1 \wedge \ldots P_n), \{P_1[z_i/x] \wedge \ldots \wedge P_n[x_i/x] : z_i \in Var(\Gamma)\}}{\Gamma, \{P_1[z_i/x] \wedge \ldots \wedge P_n[z_i/x] : z_i \in Var(\Gamma)\}, \{P_1[z_i/x] \wedge \ldots \wedge P_n[z_i/x] : z_i \in Var(\Gamma)\}}\,\mathrm{inv}\ \exists}{\Gamma, \{P_1[z_i/x] \wedge \ldots \wedge P_n[z_i/x] : z_i \in Var(\Gamma)\}}\,\mathrm{IH}}{\Gamma, \exists x(P_1 \wedge \ldots P_n)}\,\mathrm{R}\exists$$

Where IH stands for possibly multiple applications of the inductive hypothesis. □

Finally, we can prove the admissibility of the rule of weakening. Contrarily to usual **G3**-style systems, weakening is admissible without preservation of height. We start by discussing a specific case, i.e. weakening for reduced sequents.

Lemma 4.5. *The rule:*

$$\frac{\Gamma}{\Gamma, \Delta}\ \mathit{Weak}_{Red}\ ,$$

where Δ *is (innex and) reduced, is height-preserving admissible in* **G3INT***.*

Proof: The proof is by induction on the height of the derivation. If Γ is an initial sequent, so is Γ, Δ. If the last rule applied is $\wedge$ or $\vee$, the proof follows from the application of the induction hypothesis and the rule again. As an example, we detail the case of $\vee$:

$$\frac{\Gamma, A, B}{\Gamma, A \vee B}\ \vee \quad \rightsquigarrow \quad \cfrac{\cfrac{\Gamma, A, B}{\Gamma, A, B, \Delta}\ \mathrm{IH}}{\Gamma, A \vee B, \Delta}\ \vee$$

If the last rule applied is $R\exists$ and Δ is reduced, we proceed as follows:

$$\frac{\Gamma', B(z_1), \ldots, B(z_m)}{\Gamma', \exists x B}\ R\exists \quad \rightsquigarrow \quad \cfrac{\cfrac{\Gamma', B(z_1), \ldots, B(z_m)}{\Gamma', B(z_1), \ldots, B(z_m), B(z_{m+1}), \ldots, B(z_{m+n}), \Delta}\ IH}{\Gamma', \exists x B, \Delta}\ R\exists$$

where $z_{m+1}, \ldots, z_{m+n}$ are the variables occurring free in Δ but not Γ. □

Lemma 4.6. *The rule:*

$$\frac{\Gamma}{\Gamma, \Delta}\ \mathit{Weak},\ \Delta\ \mathit{innex}$$

is admissible in **G3INT***.*

Proof: The proof runs by induction on the height of the derivation. We detail—for the sake of readability—the case in which Δ consists of a single formula A.

If Γ is an initial sequent, then Γ, A is an initial sequent too. In the remaining cases except for $R\exists$, we apply the induction hypothesis (and possibly height-preserving admissibility of substitution in order to avoid clashes of variables) and then the rule again.

If the last rule applied is $R\exists$, we have:

$$\frac{\Gamma, B(z_1), \ldots, B(z_m)}{\Gamma, \exists x B} \; R\exists$$

By the induction hypothesis we get a derivation of $\Gamma, B(z_1), \ldots, B(z_m), A$. We decompose it into reduced sequents via height-preserving invertibility of the rules $R\forall, R\wedge$ and $R\vee$ to get sequents of the shape:

$$\Gamma, B(z_1), \ldots, B(z_m), A_1, \ldots, A_n$$

where A_i is reduced for $i \in \{1, \ldots, n\}$. Next, we proceed as follows:

$$\dfrac{\dfrac{\Gamma, B(z_1), \ldots, B(z_m), A_1, \ldots, A_n}{\Gamma, B(z_1), \ldots, B(z_m), B(z_{m+1}), \ldots, B(z_{m+l}), A_1, \ldots, A_n} \; Weak_{Red}}{\Gamma, \exists x B, A_1, \ldots, A_n} \; R\exists$$

The formulas $B(z_{m+1}), \ldots, B(z_{m+l})$ are instantiations of B over terms occurring in $A_1, \ldots, A_n$ (introduced by the analysis of A). The application $Weak_{Red}$ is justified by the previous lemma.

The desired conclusion is obtained from $\Gamma, \exists x B, A_1, \ldots, A_n$ via the application of the rules used to decompose A in the reverse order. □

We are now in the position to state and prove the admissibility of the cut rule. In this case, we shall argue by induction on a single parameter, the degree of the cut formula.

Theorem 4.7. *The cut rule is admissible in* **G3INT**.

Proof: The proof is by induction on the degree of the cut formula. We consider an upper-most instance of a context-sharing cut:

$$\frac{\Gamma, A \qquad \Gamma, \overline{A}}{\Gamma} \; \text{Cut}$$

The admissibility of a context-free cut follows by the admissibility of weakening and contraction.

If the cut formula is atomic, it is of the shape P and $\overline{P}$ and we have:

$$\dfrac{\begin{array}{c}\vdots\mathcal{D}\\ \Gamma, P\end{array} \qquad \Gamma, \overline{P}}{\Gamma}\ \text{Cut}$$

We consider the topmost sequents of the derivation $\mathcal{D}$. They will be the sequents Γ_i, P, for $1 \leq i \leq n$. We substitute P with Γ. We claim the resulting sequent is derivable. Indeed, if P is not principal in the initial sequent Γ_i, P, then also Γ_i, Γ is an initial sequent. Else, P is principal in Γ_i, P and $\Gamma_i \equiv \Gamma_i', \overline{P}$. The sequent $\Gamma_i', \overline{P}, \Gamma$ is cut-free derivable by applying an admissible instance of weakening to the right premise of the cut rule. We can, thus replace each premise Γ_i, P of $\mathcal{D}$ with Γ_i, Γ. We have the following cut-free derivation of Γ:

$$\dfrac{\begin{array}{c}\Gamma_1, P \quad \cdots \quad \Gamma_n, P\\ \vdots\mathcal{D}\\ \Gamma, P\end{array} \qquad \Gamma, \overline{P}}{\Gamma}\ \text{Cut} \quad \rightsquigarrow \quad \begin{array}{c}\dfrac{\dfrac{\Gamma_1, \Gamma}{\Gamma_1, \Gamma_1}\ \text{Inv}}{\Gamma_1}\ \text{Ctr} \quad \cdots \quad \dfrac{\dfrac{\Gamma_n, \Gamma}{\Gamma_n, \Gamma_n}\ \text{Inv}}{\Gamma_n}\ \text{Ctr}\\ \vdots\mathcal{D}\\ \Gamma\end{array}$$

In the cases in which the formula is compound, but not quantified, we exploit invertibility and then cuts on formulas of lesser degrees. In particular, we have:

$$\dfrac{\Gamma, A \wedge B \qquad \Gamma, \overline{A} \vee \overline{B}}{\Gamma}\ \text{Cut}$$

We transform the derivation as follows:

$$\dfrac{\dfrac{\Gamma, A \wedge B}{\Gamma, B}\ \text{Inv} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\Gamma, A \wedge B}{\Gamma, A}\ \text{Inv}}{\Gamma, A, \overline{B}}\ \text{Weak} \qquad \dfrac{\Gamma, \overline{A} \vee \overline{B}}{\Gamma, \overline{A}, \overline{B}}\ \text{Inv}}{\Gamma, \overline{B}}\ \text{Cut}}{}}{\Gamma}\ \text{Cut}$$

The cuts are removed by induction on the degree of the cut formula.

If the formula is quantified, we have:

$$\frac{\Gamma, \exists x(P_1 \wedge \ldots \wedge P_n) \qquad \Gamma, \forall x(\overline{P}_1 \vee \ldots \vee \overline{P}_n)}{\Gamma}$$

In this case we first apply height-preserving invertibility to both premises of the cut in order to reach reduced sequents in Γ. In particular, this yields two sets of sequents:

$$\mathbf{A} = \{\Gamma_i, \exists x(P_1 \wedge \ldots \wedge P_n) \,|\, 1 \leq i \leq k\} \text{ and}$$
$$\mathbf{B} = \{\Gamma_i, \forall x(\overline{P}_1 \vee \ldots \vee \overline{P}_n) \,|\, 1 \leq i \leq k\}$$

By applying height-preserving invertibility of the rules for the existential quantifier, we get the set $\mathbf{A}'$:

$$\mathbf{A}' = \{\Gamma_i, \{P_1(z_j) \wedge \ldots \wedge P_n(z_j) \,:\, z_j \in VAR(\Gamma_i)\} \,:\, 1 \leq i \leq k\}$$

By invertibility of the rule $R\forall$, we get derivations of: $\Gamma_i, \overline{P}_1(z_j) \vee \ldots \vee \overline{P}_n(z_j)$ for each i and each z_j. For every i we proceed as follows:

$$\cfrac{\cfrac{\cfrac{\Gamma_i, \{P_1(z_j) \wedge \ldots \wedge P_n(z_j) \,:\, 1 \leq j \leq \ell\} \qquad \Gamma_i, \overline{P}_1(z_1) \vee \ldots \vee \overline{P}_n(z_1)}{\Gamma_i, \{P_1(z_j) \wedge \ldots \wedge P_n(z_j) \,:\, 2 \leq j \leq \ell\}}\,\text{Cut}}{\vdots}}{\cfrac{\Gamma_i, P_1(z_\ell) \wedge \ldots \wedge P_n(z_\ell) \qquad \Gamma_i, \overline{P}_1(z_\ell) \vee \ldots \vee \overline{P}_n(z_\ell)}{\Gamma_i}\,\text{Cut}}$$

All the cuts are eliminated invoking the induction hypothesis on the degree of the cut formula. Finally, we apply the rules in the reverse order to get a derivation of Γ. □

4.1. Termination and bounds on cut-free proofs

In this subsection, we establish the termination of the proof search and we define bounds on the height of cut-free derivations. It is easy to see that each bottom-up application of a rule either decreases the number of quantifiers or the number of connectives occurring in the endsequent.

Proposition 4.8. The calculus **G3INT** is terminating.

Proof: Given a sequent Γ we argue by induction on lexicographically ordered pairs (m, n), where m is the number of quantifiers occurring in the endsequent and n is the number of connectives occurring in Γ.

It is immediate to see that the rules $R\forall$ and $R\exists$ decrease the number of quantifiers. The latter potentially increases the number of connectives, but this is not problematic, because it is the second inductive parameter.

The rules $R\wedge$ and $R\vee$ do not increase the number of quantifiers, but strictly decrease the number of connectives in the endsequent. Therefore we can infer that the proof search terminates. □

Next, we would like to compute explicit bounds on the height of cut-free derivations. As it is well-known, in classical (and intuitionistic) FOL the elimination of cuts can lead to an hyperexponential increase of the height of the proofs. In the case of the innex fragment of classical MFOL, we can show that the bounds on cut-free proofs is way lower than for FOL. Indeed, since the proof search terminates for every sequent, we can define a maximal height of any derivation.

Definition 4.9. Given a sequent Γ, we define a measure of complexity for every formula A occurring in it, in symbols $\sigma_\Gamma(A)$. If A is a literal, then $\sigma_\Gamma(A) = 0$. If A is $B\#C$, with $\# \in \{\wedge, \vee\}$, then $\sigma_\Gamma(A) = \sigma_\Gamma(B)+\sigma_\Gamma(C)+1$. If A is $\forall xB$, then $\sigma_\Gamma(\forall xB) = \sigma_\Gamma(B) + 1$ and if A is $\exists xB$, then $\sigma_\Gamma(\exists xB) = \sigma_\Gamma(B)\cdot sw(n(\forall)_\Gamma+n(VAR)_\Gamma)+1$, where $sw(k) = 1$ if $k = 0$ and k otherwise, $n(\forall)_\Gamma$ is the number of universal quantifiers occurring in Γ and $n(VAR)_\Gamma$ is the number of variables having free occurrences in Γ. The complexity of a sequent $\sigma(\Gamma)$ is $\Sigma_{A\in\Gamma}\sigma_\Gamma(A)$.

Proposition 4.10. Given a derivable sequent Γ, the maximal height of a cut-free derivation is $\sigma(\Gamma)$.

Proof: The proof is straightforward by observing that the maximal number of rules which are bottom-up applicable to Γ is precisely $\sigma(\Gamma)$. □

This gives us a decision procedure for the derivablity problem in **G3INT** whose complexity is in co-NP. The procedure is shown in Table 1; where universal choice handles the branching caused by rule $R\wedge$ and Lemma 4.3 allows us to freely choose which rule to apply at each step.

Proposition 4.11. The algorithm in Table 1 runs in co-NP.

Proof: The procedure is in the form of a non-deterministic Turing machine with universal choice whose computations are bounded by $\sigma(\Gamma)$. □

Observe that Prop. 4.11 entails that the satisfiability problem for monadic formulas in INF is in NP. However, this result does not clash with the

Table 1. Decision procedure for **G3INT**-derivability.

Input:	A sequent Γ in innex normal form.
Output:	If Γ is derivable then 'yes', else a sequent.

If for some A, both A and $\overline{A}$ are in Γ **then** return 'yes' and halt;
else if some rule is applicable **then**
| pick the first rule instance applicable;
| universally choose one premise Γ' of this rule instance;
| check recursively the derivability of Γ', output the answer and halt;
else return Γ and halt;
end.

NEXP-hardness of the satisfiability problem for monadic FO-logic [4] since the conversion of an arbitrary monadic formula into an innex one can lead to an exponential explosion of $\sigma(\Gamma)$.

5. Characterisation

THEOREM 5.1 (soundness). *If* Γ *is* **G3INT**-*derivable then* $\bigvee \Gamma$ *is valid in classical FO-logic.*

PROOF: An easy induction on the height of the derivation of Γ. □

In order to prove completeness, we show that all rules of **G3INT** are semantically invertible:

LEMMA 5.2. *If there is a countermodel for all formulas in one premise of an instance of a rule of* **G3INT** *then there is a countermodel for its conclusion.*

PROOF: The case of rules $R\wedge$, $R\vee$, and $R\forall$ are immediate. For rule $R\exists$ we assume $\mathcal{M} = \langle \mathcal{D}, \mathcal{V} \rangle$ is a model and μ an assignment defined over $\mathcal{D}$ such that $\mathcal{M}, \mu$ falsifies all formulas in

$$\Gamma, P_1(z_1) \wedge \cdots \wedge P_n(z_1), \ldots, P_1(z_\ell) \wedge \cdots \wedge P_n(z_\ell) \qquad (\Delta)$$

We construct a countermodel for all formulas in $\Gamma, \exists x(P_1(x) \wedge \cdots \wedge P_n(x))$.

Initial Sequents: $\dfrac{}{\Gamma, P, \overline{P}}\ Ax$

Logical Rules: $\dfrac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \wedge B}\ R\wedge$ $\quad \dfrac{\Gamma, A, B}{\Gamma, A \vee B}\ R\vee$ $\quad \dfrac{\Sigma, A}{\Gamma, \Diamond\Sigma, \Box A}\ R\Box$ $\quad \dfrac{\Gamma, \Diamond A, A}{\Gamma, \Diamond A}\ R\Diamond$

Figure 3. The sequent calculus **G3T**

Given that Δ, being reduced, contains no instance of $\forall$, we can apply Lemma 4.3 to it until it becomes a multiset Δ' of literals such that $X = \{z_1, \dots, z_\ell\}$ is the finite set of all variables occurring (free) in Δ. It is easy to see that that Δ' is falsified by $\mathcal{M}^\Delta = \langle \mathcal{D}^\Delta, \mathcal{V}^\Delta \rangle, \mu^\Delta$, where $\mathcal{D}^\Delta = \mathcal{D} \cap \mu(X)$, $\mathcal{V}^\Delta(P) = \mathcal{V}(P) \cap \mu(X)$, and μ^Δ behave like μ for all variables occurring free in Δ' and maps all other variables to $\mu(z_1)$. $\mathcal{M}^\Delta, \mu^\Delta$ falsifies also $\exists x(P_1(x) \wedge \cdots \wedge P_n(x))$ since each conjunct in $P_1(x) \wedge \cdots \wedge P_n(x)$ is false of some object in $\mathcal{D}^\Delta$. □

Theorem 5.3 (Completeness). *If* $\bigvee \Gamma$ *is valid then* Γ *is* **G3INT**-*derivable.*

Proof: By Prop. 2.3 we can assume Γ is in INF. If **G3INT** $\nvdash \Gamma$ then there is a finite proof-search tree for Γ having at least one leaf Δ that is not an initial sequent. We can easily define a countermodel for Δ from that leaf and, by Lemma 5.2, we conclude that $\bigvee \Gamma$ has a countermodel. □

6. Modal interpretation

It is well known that there is a sound and faithful interpretation of the propositional modal logic **S5** into MFOL [2]. We show in this section that the innex fragment of MFOL can be soundly and faithfully interpreted in the quantifier-free monadic fragment of the FO-modal logic **T**. This will be done by using the sequent calculus for **T** given in Figure 3, cf. [8].

Let $\mathcal{L}^\Box$ be the language obtained from the language of MFOL (cf. Section 2) by replacing $\forall$ and $\exists$ with $\Box$ and $\Diamond$, respectively. We define inductively a pair of translations τ_1, τ_2 from the language of MFOL to $\mathcal{L}^\Box$ ($\tau = \tau_2 \circ \tau_1$). Formally, given an innex sequent Γ, we have:

- $(P(y))^{\tau_1} = P(y)$
- $(\overline{P}(y))^{\tau_1} = \overline{P}(y)$
- $(A \# B)^{\tau_1} = A^\tau \# B^\tau$, with $\# \in \{\wedge, \vee\}$

- $(\forall x A)^{\tau_1} = \Box A[y/x]$, where y does not occur in Γ
- $(\exists x A)^{\tau_2} = \Diamond(A[z_1/x] \vee \ldots \vee A[z_n/x])$, where $z_1, \ldots, z_n$ are all the variables free in $(\Gamma)^{\tau_1}$.

We start by showing a preliminary lemma concerning derivability in **G3INT**.

LEMMA 6.1. *Let* Γ*,* Π *and* Σ *be multisets of quantifier-free, universal and existential formulas in innex normal form, respectively. If* Γ, Π, Σ*, is derivable, then* Γ, Σ *or* Σ, A*, where* $A \in \Pi$*, is derivable with at most the same height.*

PROOF: The proof runs by induction on the height of the derivation. Every case is trivial with the exception of the case in which the last rule applied is $R\forall$. In the latter case we have:

$$\frac{\Gamma, \Pi', \Sigma, P_1(y) \vee \ldots \vee P_n(y)}{\Gamma, \Pi', \Sigma, \forall x(P_1(x) \vee \ldots \vee P_n(x))} R\forall$$

The induction hypothesis yields the derivability of $\Gamma, \Sigma, P_1(y) \vee \ldots \vee P_n(y)$ or of A, Σ for some A in Π'. In the second case, we already have obtained the desired conclusion. In the first one, due to the eigenvariable condition, we observe that $\Sigma, P_1(y) \vee \ldots \vee P_n(y)$ is derivable or Γ, Σ is derivable. In the first case we get the desired conclusion via an application of the rule $R\forall$, the other case is trivial. $\Box$

The previous lemma allows us to prove the soundness of the embedding.

THEOREM 6.2. *If* **G3INT** *proves* Δ*, then* **G3T** *proves* $(\Delta)^{\tau}$*.*

PROOF: The proof is by induction on the height of the derivation. We detail the case of the quantifiers. Let Γ, Π and Σ be multisets of quantifier-free, universal and existential formulas in innex normal form, respectively. If the last rule applied is $R\forall$, we have:

$$\frac{\Gamma, \Sigma, \Pi, P_1(y) \vee \ldots \vee P_n(y)]}{\Gamma, \Sigma, \Pi, \forall x(P_1(x) \vee \ldots \vee P_n(x))} R\forall$$

Since $\Gamma, \Sigma, \Pi, P_1(y) \vee \ldots \vee P_n(y)$ is derivable, then Lemma 6.1 entails that $\Gamma, \Sigma, P_1(y) \vee \ldots \vee P_n(y)$ is derivable or Σ, C is derivable, where C is a formula in Π. The latter case is trivial and the conclusion follows from the induction hypothesis and an application of weakening. In the first case, due to the eigenvariable condition either $\Sigma, P_1(y) \vee \ldots \vee P_n(y)$ is derivable or Γ, Σ is derivable. Once again, in the latter subcase the conclusion can be obtained by the induction hypothesis and weakening. In the first subcase, we first apply the height-preserving invertibility of the rule $R\exists$ to get $A_1[y/x_1], \ldots, A_m[y/x_m], P_1(y) \vee \ldots \vee P_n(y)$, where $\Sigma = \exists x_1 A_1, \ldots, \exists x_m A_m$. Next, we have the following **G3T**-derivation:

$$\dfrac{\dfrac{\dfrac{A_1[y/x_1], \ldots, A_m[y/x_m], P_1(y) \vee \ldots \vee P_n(y)}{A_1^\tau[y/x], \ldots, A_m^\tau[y/x], P_1(y) \vee \ldots \vee P_n(y)}\ \text{IH}}{\Diamond\Sigma^\tau, P_1(y) \vee \ldots \vee P_n(y)}\ \text{several Weak and } R\Diamond}{(\Gamma, \Pi)^\tau, \Diamond\Sigma^\tau, \Box(P_1(y) \vee \ldots \vee P_n(y))}\ R\Box$$

If the last rule applied is $R\exists$, we proceed as follows:

$$\dfrac{\Gamma, \Sigma, A[z_1/x], \ldots, A[z_n/x]}{\Gamma, \Sigma, \exists x A}\ R\exists \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\Gamma, \Sigma, A[z_1/x], \ldots, A[z_n/x]}{\Gamma^\tau, \Sigma^\tau, A^\tau[z_1/x], \ldots, A^\tau[z_n/x]}\ \text{IH}}{\Gamma^\tau, \Sigma^\tau, A^\tau[z_1/x] \vee \ldots \vee A^\tau[z_n/x]}\ R\vee}{\Gamma^\tau, \Sigma^\tau, \Diamond(A^\tau[z_1/x] \vee \ldots \vee A^\tau[z_n/x])}\ \text{Weak and } R\Diamond$$

Where $A^\tau \equiv A$ since A is conjunction of literals. □

We can also prove the faithfulness of the embedding.

THEOREM 6.3. *Given a sequent Δ of monadic formulas in innex normal form, if $(\Delta)^\tau$ is derivable in* **G3T***, then Δ is derivable in* **G3INT***.*

PROOF: If the sequent is initial, the proof is immediate. If it is the conclusion of $R\wedge$, $R\vee$, the proof is straightforward by the induction hypothesis. If the last rule applied is $R\Box$, we have:

$$\dfrac{\Sigma^\tau, A^\tau[y/x]}{\Gamma^\tau, \Diamond\Sigma^\tau, \Box A^\tau}\ \Box$$

We apply the induction hypothesis, the rules $R\exists$ and $R\forall$ and weakening.

If the last rule applied is $R\Diamond$, we proceed as follows (where A is $B[z_1/x] \vee \cdots \vee B[z_n/x]$ and $\{z_1, \ldots z_n\}$ are all variables free in Γ):

$$\dfrac{A, \Diamond A^\tau, \Gamma^\tau}{\Diamond A^\tau, \Gamma^\tau} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{A^\tau, \Diamond A^\tau, \Gamma^\tau}{B[z_1/x], \ldots, B[z_n/x], \exists x B, \Gamma}\ \text{IH+ } \vee\text{-inv}}{\exists x B, \exists x B, \Gamma}\ R\exists}{\exists x B, \Gamma}\ \text{Ctr}$$

□

Let us observe that the structural properties established for **G3INT**—including cut elimination—can now be proved indirectly via the embedding in the modal system.

7. Concluding remarks and future work

We have introduced a terminating sequent calculus for a fragment of MFOL. This, combined with a normal form theorem, gives a fully syntactic decision procedure for monadic classical first-order logic.

It is natural to ask whether it is possible to design a sequent calculus for the full language of monadic logic. One way to do so is to define rules which directly convert formulas in innex normal form and then to proceed as for **G3INT**. We leave this theme for future investigations.

Finally, we would like to generalize the cut-elimination strategy to other classes of logics, showing how to eliminate the cuts by induction on the degree of the cut formula. Particularly promising would be to spell out sufficient conditions for cut-elimination.

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SEQUENT SYSTEMS FOR CONSEQUENCE RELATIONS OF CYCLIC LINEAR LOGICS

Abstract

Linear Logic is a versatile framework with diverse applications in computer science and mathematics. One intriguing fragment of Linear Logic is Multiplicative-Additive Linear Logic (MALL), which forms the exponential-free component of the larger framework. Modifying MALL, researchers have explored weaker logics such as Noncommutative MALL (Bilinear Logic, BL) and Cyclic MALL (CyMALL) to investigate variations in commutativity. In this paper, we focus on Cyclic Nonassociative Bilinear Logic (CyNBL), a variant that combines noncommutativity and nonassociativity. We introduce a sequent system for CyNBL, which includes an auxiliary system for incorporating nonlogical axioms. Notably, we establish the cut elimination property for CyNBL. Moreover, we establish the strong conservativeness of CyNBL over Full Nonassociative Lambek Calculus (FNL) without additive constants. The paper highlights that all proofs are constructed using syntactic methods, ensuring their constructive nature. We provide insights into constructing cut-free proofs and establishing a logical relationship between CyNBL and FNL.

Keywords: linear logic, Lambek calculus, nonassociative logics, noncommutative logics, substructural logics, consequence relation, nonlogical axioms, conservativeness.

1. Introduction and preliminaries

Linear Logic (PLL), introduced by Girard [7], is a powerful framework widely applied in computer science and mathematics. It offers a rich set of

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tools for reasoning about resources and provides a foundation for various formal systems. One intriguing fragment of PLL is Multiplicative-Additive Linear Logic (MALL), which focuses on the exponential-free aspects of PLL. In MALL, we encounter four binary connectives: $\otimes$ (product; multiplicative conjuntion), $\parr$ (par; multiplicative disjunction), $\wedge$ (additive conjuntion) and $\vee$ (additive disjunction). Additionally, MALL includes one unary connective: $^{\sim}$ (linear negation), and four constants: 1, 0, $\bot$, and $\top$. It is worth noting that MALL exhibits associativity and commutativity, as defined by the algebraic interpretation of the $\parr$ connective, further enhancing its expressive capabilities.

Abrusci [1] investigates Noncommutative MALL, a variant of the logic where the $\otimes$ connective is not required to be commutative. This exploration of noncommutativity adds an intriguing dimension to MALL and offers new avenues for reasoning about resources and implications. In Noncommutative MALL, we encounter two negations, $^{\sim}$ and $^{-}$, which exhibit an interesting algebraic property: for all a, the following equivalences hold: $a^{\sim -} = a = a^{-\sim}$. This property highlights the interplay between the two negations and underscores the expressive power of Noncommutative MALL. It is worth noting that this variant is also known as Bilinear Logic (BL), as named by Lambek [8].

Yetter [12] introduces Cyclic MALL (CyMALL) as a compromise between MALL and BL. While CyMALL maintains the noncommutative nature of BL, it distinguishes itself by adopting only one negation that satisfies the double negation law. This unique choice of negation adds a distinct flavor to the reasoning capabilities of CyMALL. Additionally, CyMALL allows for the relaxation of associativity, further differentiating it from traditional Bilinear Logic. Nonassociative Bilinear Logic (NBL) is another intriguing logic that explores the implications of nonassociativity. In this paper, we specifically focus on Cyclic NBL (CyNBL), a variant of NBL that inherits the noncommutative property from CyMALL while also incorporating nonassociativity.

In this paper, we present the sequent system for CyNBL in Section 2. Additionally, we introduce an auxiliary sequent system which allows for the inclusion of nonlogical axioms by treating them as special cases of the cut rules. As a result, we obtain an equivalent system that incorporates a form of the cut elimination property. Specifically, the cut elimination property applies to the pure logic, while the cut rules are restricted to handling assumptions only.

The proof of cut elimination and the development of the sequent systems in this paper draw inspiration from prior research. Specifically, Buszkowski [5] establishes the cut elimination property for constant-free MLL, which corresponds to the multiplicative fragment of MALL. Furthermore, Płaczek [10] extends this method to prove cut elimination for NBL. Notably, in the author's doctoral dissertation [11], there are remarks regarding the potential approach for proving cut elimination in CyNBL. It is worth mentioning that none of these previous results involve assumptions, as they focus primarily on the cut elimination property within the given logical frameworks.

Lin [9] has previously explored sequent systems for specific extensions of NL with assumptions. In these systems, confined to intuitionistic sequents of the form $\Gamma \Rightarrow A$, the assumption $A \Rightarrow B$ is replaced by a specific instance of the cut rule: from $\Gamma[B] \Rightarrow C$ and $\Delta => A$, we derive $\Gamma[\Delta] => C$. In this study, we adapt this concept to systems employing classical sequents.

Building upon these foundations, our paper further explores the cut elimination property within the context of CyNBL, considering both the pure logic aspects and the incorporation of nonlogical axioms.

In the third section, we prove the strong conservativeness of CyNBL over Full Nonassociative Lambek Calculus (FNL). This result highlights the relationship between CyNBL and FNL, shedding light on the expressive power and logical properties of CyNBL within the context of nonassociative Lambek calculus. Abrusci [2] has previously demonstrated that CyMALL, the commutative variant of CyNBL, is not a conservative extension of Full Lambek Calculus (FL) when considering the inclusion of additive constants such as $\perp$ and $\top$. However, when the additive constants are omitted, CyMALL exhibits conservativeness. V. M. Abrusci's work presents a sequent that is provable in CyMALL but not in FL with additive constants. A similar example can be provided for the nonassociative version.

In this paper, we establish that CyNBL without additive constants serves as a strongly conservative extension of FNL without additive constants, highlighting the logical relationship between the two systems. Additionally, a similar result can be obtained for CyMALL, as discussed in Section 4. Notably, this outcome has been previously demonstrated in Płaczek's work [11] and may also be inferred from other algebraic findings.

The crucial contribution of this paper lies in the application of syntactic methods, ensuring that all proofs are constructive in nature. We present a systematic approach to construct cut-free proofs based on existing theorems in CyNBL. Furthermore, we demonstrate how to construct proofs in FNL based on the proofs available in CyNBL. By showing these constructive methods, we provide valuable insights into the practical aspects of reasoning within CyNBL and its relationship with FNL, establishing a foundation for future research and application.

As a consequence of the results of this paper we can tell more about complexity of these logics. CyNBL has undecidable consequence relation, since it is a strongly conservative extension of FNL; see [6]. Also CyMALL has undecidable consequence relation, because it is a strongly conservative extension of FL; see [3]. The finitary consequence relation of multiplicative part of CyNBL is decidable in *PTIME*; see [4].

The other consequence is that NBL is also a strongly conservative extension of FNL. The open problem in this matter remains the decidability (and complexity) of the finitary consequence relation for multiplicative fragment of NBL.

1.1. Algebras

We will briefly introduce certain algebras that serve as models for the logics examined in this paper.

Let $(P, \leq)$ be a poset and let $^\sim$ be a unary operation on P such that for all $a, b \in P$: (i) if $a \leq b$, then $b^\sim \leq a^\sim$; (ii) $a^{\sim\sim} = a$. Such an operation $^\sim$ is called a *De Morgan negation*.

Definition 1.1. Let $\mathbf{M} = (M, \otimes, \wedge, \vee, {}^\sim, 1, \bot, \leq)$ be a structure such that $\otimes, \wedge, \vee$ are binary operations, $^\sim$ is a unary operation, 1 and $\bot$ are constants and $\leq$ is a partial order on M. We say that $\mathbf{M}$ is a *bounded CyNBL–algebra*, if the following conditions hold:

(i) $^\sim$ is a De Morgan negation;

(ii) $(M, \wedge, \vee, \leq)$ is a lattice;

(iii) $a \otimes b \leq c$ iff $b \otimes c^\sim \leq a^\sim$ iff $c^\sim \otimes a \leq b^\sim$ for all $a, b, c \in M$;

(iv) $1 \otimes a = a = a \otimes 1$ for all $a \in M$;

(v) $\bot \leq a$ for all $a \in M$.

The analogous structre without constant $\bot$ and (iv) is called an *unbounded CyNBL–algebra.* One defines $a \parr b = (b^{\sim} \otimes a^{\sim})^{\sim}$ and $0 = 1^{\sim}$ and $\top = \bot^{\sim}$.

Bounded CyNBL–algebras serve as models of CyNBL, while unbounded CyNBL–algebras model CyNBL without additive constants.

Definition 1.2. Let $\mathbf{M} = (M, \otimes, \multimap, \multimapinv, \wedge, \vee, 1, \leq)$ be a structure such that $\otimes, \multimap, \multimapinv, \wedge, \vee$ are binary operations, $1 \in M$ and $\leq$ is a partial order on M. We say that $\mathbf{M}$ is an *FNL–algebra*, if the following conditions hold:

(i) $(M, \wedge, \vee, \leq)$ is a lattice;

(ii) $a \otimes b \leq c$ iff $a \leq c \multimapinv b$ iff $b \leq a \multimap c$ for all $a, b, c \in M$;

(iii) $1 \otimes a = a = a \otimes 1$ for all $a \in M$.

FNL–algebras serve as models of FNL. It is possible to extend FNL by introducing additive constants $\bot$ and $\top$, or solely $\bot$ (given that $\top$ can be defined), resulting in bounded FNL–algebras. However, it's important to note that our paper does not explore FNL with additive constants.

It can be proved that every CyNBL–algebra, whether bounded or unbounded, is also an FNL–algebra. We define $a \multimap b = a^{\sim} \parr b$ and $a \multimapinv b = a \parr b^{\sim}$. One checks that the condition (ii) holds.

2. Sequent systems

Let $\mathcal{V}$ be an arbitrary, countable set of variables. We define the set of atoms $\mathcal{V}'$ as follows: if $p \in \mathcal{V}$, then both p and $p^{\sim}$ are elements of $\mathcal{V}'$. Variables in this set are referred to as *positive* atoms, while their negations ($p^{\sim}$) are termed *negative* atoms. We construct the set of *CyNBL–formulas* from $\mathcal{V}'$ by the binary connectives: $\otimes$, $\parr$, $\wedge$ and $\vee$ and the constants 1, 0, $\top$ and $\bot$.

It's worth noting that we do not treat negation as a connective. The systems we introduce adhere to the negation normal form, meaning that negations only appear in the form of atoms.

We define the metalanguage negation ${}^{\sim}$:

$$(p)^{\sim} = p^{\sim} \qquad (p^{\sim})^{\sim} = p$$

$$1^{\sim} = 0 \qquad 0^{\sim} = 1$$

$$\top^{\sim} = \bot \qquad \bot^{\sim} = \top$$

$$(A \otimes B)^{\sim} = B^{\sim} \parr A^{\sim} \qquad (A \parr B)^{\sim} = B^{\sim} \otimes A^{\sim}$$

$$(A \wedge B)^{\sim} = A^{\sim} \vee B^{\sim} \qquad (A \vee B)^{\sim} = A^{\sim} \wedge B^{\sim}$$

One notices $A^{\sim\sim} = A$ for all formulas A.

We define *CyNBL–bunches*. A CyNBL–bunch is an element of the free unital groupoid generated by the set of all CyNBL–formulas. The neutral element of this unital groupoid is referred to as an empty bunch and is denoted by ϵ. A *CyNBL–sequent* is defined as any nonempty bunch, and we represent bunches using capital Greek letters.

An *anonymous variable* is a unique formula represented as $\_$, serving as a placeholder for substitution. It's important to note that if a bunch contains multiple anonymous variables, they are considered distinct, even if they share the same symbol. A *CyNBL–context* is a bunch with an anonymous variable. Contexts are denoted by $\Gamma[\_]$, and when we perform the substitution of Δ in place of $\_$, we represent it as $\Gamma[\Delta]$.

The axioms of CyNBL are:

$$\text{(a-id)}\ \frac{}{p, p^{\sim}} \qquad \text{(a-0)}\ \frac{}{0}$$

$$\text{(a-}\bot\text{)}\ \frac{}{\Gamma[\bot]}$$

The introduction rules (rules introducing connectives and constants) are:

$$\text{(r-}\otimes\text{)}\ \frac{\Gamma[(A, B)]}{\Gamma[A \otimes B]} \qquad \text{(r-1)}\ \frac{\Gamma[\Delta]}{\Gamma[(1, \Delta)]} \quad \frac{\Gamma[\Delta]}{\Gamma[(\Delta, 1)]}$$

$$\text{(r-}\parr\text{1)}\ \frac{\Gamma[B] \qquad \Delta, A}{\Gamma[(\Delta, A \parr B)]} \qquad \text{(r-}\parr\text{2)}\ \frac{\Gamma[A] \qquad B, \Delta}{\Gamma[(A \parr B, \Delta)]}$$

$$\text{(r-}\wedge\text{)}\ \frac{\Gamma[A]}{\Gamma[A \wedge B]} \quad \frac{\Gamma[A]}{\Gamma[B \wedge A]} \qquad \text{(r-}\vee\text{)}\ \frac{\Gamma[A] \qquad \Gamma[B]}{\Gamma[A \vee B]}$$

In (r-1) we assume $\Delta \neq \epsilon$.

The structural rules and the cut rule are:

$$\text{(r-shift)}\frac{(\Gamma, \Delta), \Theta}{\Gamma, (\Delta, \Theta)} \qquad \text{(r-cyc)}\frac{\Gamma, \Delta}{\Delta, \Gamma}$$

$$\text{(r-cut)}\frac{\Gamma[A] \qquad \Delta, A^{\sim}}{\Gamma[\Delta]}$$

The rules (r-shift) and (r-cyc) are reversible. For (r-cyc) it is obvious. To obtain reversed (r-shift) we apply consequtively (r-cyc), (r-shift), (r-cyc), (r-shift) and again (r-cyc). The reversibility of these rules is an important fact we use later. For the simplicity of proofs, we do not assume this fact in the definition of the system.

The models of CyNBL are bounded CyNBL–algebras. A valuation is a homomorphism μ of a free algebra of CyNBL–formulas to a bounded CyNBL–algebra extended by the following properties: $\mu((\Gamma, \Delta)) = \mu(\Gamma) \otimes \mu(\Delta)$ and $\mu(\epsilon) = 1$. A sequent Γ is satisfied by a valuation μ, if $\mu(\Gamma) \leq 0$.

CyNBL is strongly complete with respect to bounded CyNBL–algebras. The (r-shift) rule express the condition (iii) from definition 1.1. The rule (r-cyc) express the fact, that we have a De Morgan negation. One proves strong completeness in a usual way, using Lindenbaum–Tarski algebras.

2.1. Auxilary system

Let Φ be a set of sequents of the form $C, D^{\sim}$. We define the system $\mathbf{S}_{\Phi}$. The system $\mathbf{S}_{\Phi}$ has all axioms and introduction rules of CyNBL. We add the following rules and axioms:

$$\text{(a-id2)}\frac{}{p^{\sim}, p}$$

$$\text{(r-}⅋\text{3)}\frac{A, \Gamma \qquad B, \Delta}{A ⅋ B, (\Delta, \Gamma)} \qquad \text{(r-}⅋\text{4)}\frac{\Gamma, A \qquad \Delta, B}{(\Delta, \Gamma), A ⅋ B}$$

In (r-⅋3) and (r-⅋4) we assume Γ, Δ are nonempty; otherwise they are special cases of (r-⅋2) and (r-⅋1).

For every $(C, D^{\sim}) \in \Phi$ we add the *assumption rules*:

$$\text{(r-assm1)}\frac{D, \Gamma \qquad \Delta, C^{\sim}}{\Delta, \Gamma} \qquad \text{(r-assm2)}\frac{D, \Gamma \qquad \Delta, C^{\sim}}{\Gamma, \Delta}$$

$$\text{(r-assm3)}\frac{D, \Gamma_1 \qquad (\Gamma_2, \Gamma_3), C^{\sim}}{\Gamma_2, (\Gamma_3, \Gamma_1)} \qquad \text{(r-assm4)}\frac{D, (\Gamma_1, \Gamma_2) \quad \Gamma_3, C^{\sim}}{\Gamma_1, (\Gamma_2, \Gamma_3)}$$

$$\text{(r-assm5)}\frac{D, \Gamma_1 \qquad (\Gamma_2, \Gamma_3), C^{\sim}}{(\Gamma_3, \Gamma_1), \Gamma_2} \qquad \text{(r-assm6)}\frac{D, (\Gamma_1, \Gamma_2) \quad \Gamma_3, C^{\sim}}{(\Gamma_2, \Gamma_3), \Gamma_1}$$

$$\text{(r-assm7)}\frac{D, \Gamma_1 \qquad (\Gamma_2, \Gamma_3), C^{\sim}}{\Gamma_3, (\Gamma_1, \Gamma_2)} \qquad \text{(r-assm8)}\frac{D, (\Gamma_1, \Gamma_2) \quad \Gamma_3, C^{\sim}}{\Gamma_2, (\Gamma_3, \Gamma_1)}$$

$$\text{(r-assm9)}\frac{D, \Gamma_1 \qquad (\Gamma_2, \Gamma_3), C^{\sim}}{(\Gamma_1, \Gamma_2), \Gamma_3} \qquad \text{(r-assm10)}\frac{D, (\Gamma_1, \Gamma_2) \quad \Gamma_3, C^{\sim}}{(\Gamma_3, \Gamma_1), \Gamma_2}$$

We assume none of $\Gamma_1, \Gamma_2, \Gamma_3$ is empty. From now on we denote $\vdash_{\mathbf{S}_\Phi} \Gamma$ the provability of Γ in $\mathbf{S}_\Phi$.

We define inductively a function f:

$$f(A) = (A), \text{ for all CyNBL–formulas } A$$
$$f((\Gamma, \Delta)) = f(\Gamma) \otimes f(\Delta)$$
$$f(\epsilon) = 1$$

One proves that $\vdash_{\mathbf{S}_\Phi} \Gamma$ iff $\vdash_{\mathbf{S}_\Phi} f(\Gamma)$. Let Γ be a CyNBL–sequent. We represent Γ in the form $C, D^{\sim}$. If $\Gamma = (\Gamma_1, \Gamma_2)$, then $C = f(\Gamma_1)$ and $D = f(\Gamma_2)^{\sim}$. If $\Gamma = A$, then $C = A, D = 0$. Hence, every CyNBL–sequent may be represented by the sequent of the form $C, D^{\sim}$.

We define the relation $\Gamma \sim \Delta$, which holds for the CyNBL–bunches Γ and Δ, if Δ can be derived from Γ by finitely many applications of (r-cyc) and (r-shift).

Since both (r-cyc) and (r-shift) are reversible, this relation is an equivalence relation. The following lemma is a modification of lemma from Buszkowski [4] or Płaczek [11].

Lemma 2.1. *Let* $\Gamma[\_]$ *be an CyNBL–context. Then, there exists the unique CyNBL–bunch* Δ *such that* $\Gamma[\_] \sim (\Delta, \_)$.

Proof: We provide an algorithm which reduces $\Gamma[\_]$ to some sequent $(\Delta, \_)$. The reduction rules are as follows:

$$\text{(R1)} \qquad (\Psi[\_], \Phi) \to (\Phi, \Psi[\_])$$

$$\text{(R2)} \quad (\Phi, (\Psi, \Xi[\_])) \to ((\Phi, \Psi), \Xi[\_])$$

$$\text{(R3)} \quad (\Phi, (\Psi[\_], \Xi)) \to ((\Xi, \Phi), \Psi[\_])$$

(R1) is an application of (r-cyc), (R2) is an application of (r-cyc), (r-shift), (r-cyc), (r-shift), consequtively (i.e. reversed (r-shift) and (R3) is an application of (r-cyc), (r-shift) and (r-cyc), consequtively. The algorithm is deterministic and hence, after finitely many steps, terminates and yields $(\Delta, \_)$.

The rest of the proof is similar to Buszkowski [4] and Płaczek [11]. □

COROLLARY 2.2. Let $\Gamma[\_] \sim (\Delta, \_)$ and let Θ be a substructure of $\Gamma[\_]$, which does not contain this occurence of $\_$ (but it can contain occurences of other anonymous variables). Then, the reduction preserves Θ.

As a consequence, the relation $\sim$ is closed under substitution.

PROPOSITION 2.3. Let $\Gamma \sim \Delta$. Then Γ is provable in $\mathbf{S}_\Phi$ iff Δ is provable.

PROOF: We use the outer induction on the number of (r-shift) and (r-cyc) used to obtain Δ from Γ and the inner induction on the proof of Γ.

1° Assume Δ arises from Γ by one application of (r-cyc) or (r-shift); we denote: $\Gamma \sim_1 \Delta$. We run the inner induction. Let Γ be an axiom. Then Δ is an axiom, too.

Now we assume Γ is the conclusion of a rule.

1.1° We consider (r-$\otimes$). We have $\Gamma = \Theta[A \otimes B]$.

Let $\Theta[\_] \sim_1 \Delta'[\_]$ and $\Delta = \Delta'[A \otimes B]$. The premise of (r-$\otimes$) is $\Theta[(A, B)]$. By the inner induction hypothesis and corollary 2.2, $\vdash_{\mathbf{S}_\Phi} \Delta'[(A, B)]$, so we apply (r-$\otimes$) and obtain $\vdash_{\mathbf{S}_\Phi} \Delta'[A \otimes B]$.

1.2° The cases for (r-$\vee$), (r-$\wedge$) and (r-1) are similar to (r-$\otimes$).

1.3° We consider (r-⅋1). We have

$$\frac{\Theta[B] \qquad \Xi, A}{\Theta[(\Xi, A \parr B)]}$$

and $\Gamma = \Theta[(\Xi, A \parr B)]$.

Assume Δ arises from Γ by an application of (r-cyc). We consider cases: (1) $\Theta[B] = B$, (2) $\Theta[B] \neq B$.

In the first case $\Gamma = (\Xi, A \parr B)$ and $\Delta = (A \parr B, \Xi)$. By the inner induction hypothesis, $\vdash_{\mathbf{S}_\Phi} A, \Xi$. So we apply (r-⅋2) to A, Ξ and B and obtain $\vdash_{\mathbf{S}_\Phi} A \parr B, \Xi$.

In the second case, we apply (r-cyc) to the premise $\Theta[B]$ and obtain $\Theta'[B]$. By the inner induction hypothesis, $\vdash_{\mathbf{S}_\Phi} \Theta'[B]$. We use (r-⅋1) with the premises $\Theta'[B]$ and Ξ, A.

Assume Δ arises from Γ by an application of (r-shift). We consider cases.

1.3°(i) We assume $\Theta[B] = B$. The derivation is as follows:

$$\frac{B \qquad (\Xi_1, \Xi_2), A}{(\Xi_1, \Xi_2), A \parr B}$$

Then $\Gamma = \big((\Xi_1, \Xi_2), A \parr B\big)$ and $\Delta = \big(\Xi_1, (\Xi_2, A \parr B)\big)$. By the inner induction hypothesis, $\vdash_{\mathbf{S}_\Phi} \Xi_1, (\Xi_2, A)$. We apply (r-⅋2).

1.3°(ii) We assume $\Theta[B] \neq B$. If $\Theta[B]$ consists of two formulas, then (r-shift) is not applicable to the conclusion. Otherwise we apply (r-shift) the first premise and use the same rule.

1.4° We consider (r-⅋2). We have

$$\frac{\Theta[A] \qquad B, \Xi}{\Theta[(A \parr B, \Xi)]}$$

The case when Δ arises by one application of (r-cyc) from Γ is similar to the previous one. The more interesting case is Δ arising by one application of (r-shift). The only possible case is when $\Theta[A] = \Theta'[A], \Psi$; otherwise (r-shift) is not applicable to the conclusion. In such a case, we apply (r-shift) to the first premise and then we use the same rule.

1.5° We consider (r-⅋3). We have:

$$\frac{A, \Theta \qquad B, \Xi}{A ⅋ B, (\Theta, \Xi)}$$

Δ must arise by an application of (r-cyc). Then $\Delta = ((\Theta, \Xi), A⅋B)$. We apply (r-cyc) to the premises. By the inner induction hypothesis, $\vdash_{\mathbf{S}_\Phi} \Theta, A$ and $\vdash_{\mathbf{S}_\Phi} \Xi, B$. We apply (r-⅋4) with these premises.

1.6° The case for (r-⅋4) is analogous to the previous cases.

1.7° We consider the assumption rules. Assume Δ arises by an application of (r-cyc). We apply other rule (as described in the table below) with the same premises:

original rule	new rule	original rule	new rule
(r-assm1)	(r-assm2)	(r-assm2)	(r-assm1)
(r-assm3)	(r-assm5)	(r-assm5)	(r-assm3)
(r-assm4)	(r-assm6)	(r-assm6)	(r-assm4)
(r-assm7)	(r-assm9)	(r-assm9)	(r-assm7)
(r-assm8)	(r-assm10)	(r-assm10)	(r-assm8)

Analogously, if Δ arises by (r-shift), we apply the table below:

original rule	new rule	original rule	new rule
(r-assm1)	(r-assm3)	(r-assm6)	(r-assm8)
(r-assm2)	(r-assm4)	(r-assm9)	(r-assm2)
(r-assm5)	(r-assm7)	(r-assm10)	(r-assm1)

2° Assume Δ arises from Γ by $n+1$ applications of (r-cyc) or (r-shift). Then there exists Γ' such that Γ' arises from Γ by n applications and

Δ arises from Γ' by one application. By the outer induction hypothesis, $\vdash_{\mathbf{S}_\Phi} \Gamma'$. We have again only one application, so we proceed as above. □

Remark 2.4. The transformation provided in the proof above does not change the length of the proof.

Corollary 2.5. The rules (r-shift) and (r-cyc) are admissible in $\mathbf{S}_\Phi$.

2.2. Cut elimination

In this section we prove that the cut rule (r-cut) is admissible in $\mathbf{S}_\Phi$ for every Φ. As a consequence, we obtain the cut–elimination property for CyNBL, since $\mathbf{S}_\emptyset$ is equivalent to CyNBL (they have the same theorems). The cut elimination for CyNBL can be proved simpler, since it is a property for pure logic (i.e. without assumptions), but our proof shows us not only cut elimination for pure logic, but also something like partial cut elimination for logics with nonlogical axioms (assumptions). This result will be useful later.

Lemma 2.6. *Let* $(C, D^{\sim}) \in \Phi$.

(1) *if* $\vdash_{\mathbf{S}_\Phi} D, \Gamma$ *and* $\vdash_{\mathbf{S}_\Phi} \Delta[C^{\sim}]$, *then* $\vdash_{\mathbf{S}_\Phi} \Delta[\Gamma]$,

(2) *if* $\vdash_{\mathbf{S}_\Phi} \Gamma[D]$ *and* $\vdash_{\mathbf{S}_\Phi} \Delta, C^{\sim}$, *then* $\vdash_{\mathbf{S}_\Phi} \Gamma[\Delta]$

Proof: We consider (1). We assume $\vdash_{\mathbf{S}_\Phi} D, \Gamma$ and $\vdash_{\mathbf{S}_\Phi} \Delta[C^{\sim}]$. Then, by proposition 2.3, $\vdash_{\mathbf{S}_\Phi} \Delta', C^{\sim}$ for some Δ' such that $\Delta[\_] \sim (\Delta', \_)$. Then, $\Delta[\Gamma] \sim \Delta', \Gamma$. We apply (r-assm1) to D, Γ and $\Delta', C^{\sim}$ and obtain $\vdash_{\mathbf{S}_\Phi} \Delta', \Gamma$. By proposition 2.3, $\vdash_{\mathbf{S}_\Phi} \Delta[\Gamma]$.

We consider (2). We assume $\vdash_{\mathbf{S}_\Phi} \Gamma[D]$ and $\vdash_{\mathbf{S}_\Phi} \Delta, C^{\sim}$. Let Γ' be such that $\Gamma[\_] \sim (\Gamma', \_)$. Then $\vdash_{\mathbf{S}_\Phi} D, \Gamma'$ by proposition 2.3, since $(D, \Gamma') \sim (\Gamma', D)$. We apply (r-assm2) and obtain $\vdash_{\mathbf{S}_\Phi} \Gamma', \Delta$. Hence, $\vdash_{\mathbf{S}_\Phi} \Gamma[\Delta]$, by proposition 2.3. □

Theorem 2.7. *The rule (r-cut) is admissible in* $\mathbf{S}_\Phi$, *i.e. if* $\vdash_{\mathbf{S}_\Phi} \Gamma[A]$ *and* $\vdash_{\mathbf{S}_\Phi} \Delta, A^{\sim}$, *then* $\vdash_{\mathbf{S}_\Phi} \Gamma[\Delta]$.

Proof: We assume $\vdash_{\mathbf{S}_\Phi} \Theta[C]$ and $\vdash_{\mathbf{S}_\Phi} \Psi, C^{\sim}$. We show $\vdash_{\mathbf{S}_\Phi} \Theta[\Psi]$.

The proof proceeds by the outer induction on the number of connectives in C, the intermediate induction on the length of the proof of $\Theta[C]$ and the inner induction on the length of the proof of $\Psi, C^{\sim}$.

We run the outer induction.

1° $C = p$. Then $C^{\sim} = p^{\sim}$. We run the intermediate induction.

1.1° Let $\Theta[C]$ be an axiom (a-id) or (a-id2). We have two possibilities: $p, p^{\sim}$ and $p^{\sim}, p$. We run the inner induction.

If $\Psi, C^{\sim}$ is an axiom, then $\Psi = p = C$ or $\Psi = \Xi[\bot]$ and $\Theta[\Psi]$ is an instance of (a-$\bot$). Now let $\Psi, C^{\sim}$ be the conclusion of an introduction rule. $C^{\sim}$ cannot be the active formula of any rule. We apply the inner induction hypothesis to the premise(s) with $C^{\sim}$ and use the same rule.

We consider the following special case:

$$\frac{A \qquad B, C^{\sim}}{A \parr B, C^{\sim}},$$

with $\Psi = A \parr B$. This may be obtained by (r-$\parr$1) or (r-$\parr$2). We apply the inner induction hypothesis to the premise $B, C^{\sim}$ and use (r-$\parr$1).

Now let $\Psi, C^{\sim}$ be the conclusion of an assumption rule. We have the following possibilities:

$$(1)\ \frac{F, \Psi \qquad C^{\sim}, E^{\sim}}{\Psi, C^{\sim}} \qquad (2)\ \frac{F, C^{\sim} \qquad \Psi, E^{\sim}}{\Psi, C^{\sim}}$$

$$(3)\ \frac{F, \Psi_2 \quad (C^{\sim}, \Psi_1), E^{\sim}}{\Psi, C^{\sim}} \qquad (4)\ \frac{F, (C^{\sim}, \Psi_1) \quad \Psi_2, E^{\sim}}{\Psi, C^{\sim}}$$

$$(5)\ \frac{F, \Psi_1 \quad (\Psi_2, C^{\sim}), E^{\sim}}{\Psi, C^{\sim}} \qquad (6)\ \frac{F, (\Psi_2, C^{\sim}) \quad \Psi_1, E^{\sim}}{\Psi, C^{\sim}}$$

where $(\Psi_1, \Psi_2) = \Psi$.

(1) By proposition 2.3 we have $\vdash_{\mathbf{S}_\Phi} E^{\sim}, C^{\sim}$ and the length of the proof of this sequent is the same as the length of the proof of $C^{\sim}, E^{\sim}$. We apply the inner induction hypothesis to $E^{\sim}, C^{\sim}$ and $\Theta[C]$ and obtain $\Theta[E^{\sim}]$. By lemma 2.6, $\vdash_{\mathbf{S}_\Phi} \Theta[\Psi]$.

(2) We apply the inner induction hypothesis to $F, C^{\sim}$ and $\Theta[C]$ and obtain $\Theta[F]$. Then, by lemma 2.6, $\vdash_{\mathbf{S}_\Phi} \Theta[\Psi]$.

(3) By proposition 2.3, $\vdash_{\mathbf{S}_\Phi} (\Psi_1, E^\sim), C^\sim$ and it has the proof of the same length as $(C^\sim, \Psi), E^\sim$. We apply the inner induction hypothesis to $(\Psi_1, E^\sim), C^\sim$ and $\Theta[C]$ and obtain $\vdash_{\mathbf{S}_\Phi} \Theta[(\Psi_1, E^\sim)]$. By lemma 2.6 we obtain $\vdash_{\mathbf{S}_\Phi} \Theta[\Psi]$.

(4)–(6) are similar to (1)–(3).

1.2° Let $\Theta[C]$ be an axiom (a-$\perp$). Then $\Theta[C] = \Theta'[\perp][C]$ and $\Theta'[\perp][\Psi]$ is another instance of (a-$\perp$).

1.3° We assume that $\Theta[C]$ is not an axiom, hence it is obtained by a rule. C cannot be the active formula of any rule. Hence it occurs in at least one premise, so we apply the intermediate induction hypothesis to the premise(s) with C and use the same rule.

2° The case for $C = p^\sim$ is similar to the previous one.

3° $C = 0$. Then $C^\sim = 1$. We run the intermediate induction.

Let $\Theta[0]$ be an axiom (a-0), then $\Theta[C] = C = 0$ and $\Theta[\Psi] = \Psi$. We run the inner induction. If $\Psi, 1$ is an axiom, then $\Psi = \Xi[\perp]$ and $\Theta[\Psi]$ is an instance of (a-$\perp$). Let $\Psi, 1$ be obtained by a rule. If $C^\sim = 1$ is not the active formula of a rule, we proceed as for $C = p$. If 1 is the active formula, then the rule is (r-1) of the form:

$$\frac{\Psi}{\Psi, 1}.$$

The premise is $\Psi = \Theta[\Psi]$.

Let $\Theta[C]$ be an axiom (a-$\perp$). Then $\Theta[C] = \Theta'[\perp][C]$ and $\Theta'[\perp][\Psi]$ is another instance of (a-$\perp$).

Now let $\Theta[C]$ be the conclusion of a rule. $C = 0$ cannot be the active formula of any rule. We apply the intermediate induction hypothesis to the premise(s) with $C = 0$ and use the same rule.

4° $C = 1$. Then $C^\sim = 0$. We run the intermediate induction.

Let $\Theta[C]$ be an axiom. Then $\Theta[C] = \Theta'[\perp][C]$ and $\Theta'[\perp][\Psi]$ is another instance of (a-$\perp$).

We assume $\Theta[1]$ is obtained by a rule. If $C = 1$ is the active formula, then $\Theta[1]$ is obtained by (r-1) admitting $\Delta = \epsilon$ in $\Theta[\Delta]$ as the premise. We run the inner induction. If $\Psi, 0$ is an axiom, then $\Psi = \epsilon$ and

$\Theta[\Psi] = \Theta[\epsilon]$ or $\Psi = \Xi[\bot]$ and $\Theta[\Psi]$ is an instance of (a-$\bot$). Let $\Psi, 0$ be obtained by a rule. $C^{\sim} = 0$ cannot be the active formula of any rule, so we proceed as for $C = p$.

If $\Theta[1]$ is obtained by a rule and $C = 1$ is not the active formula, then we proceed as above.

5° $C = \bot$. Then $C^{\sim} = \top$. We run the intermediate induction. Let $\Theta[C]$ be an axiom, we run the inner induction. We assume $\Psi, C^{\sim}$ is an axiom. Then $\Psi = \Xi[\bot]$ and $\Theta[\Psi]$ is another instance of (a-$\bot$). We assume $\Psi, C^{\sim}$ is the conclusion of a rule. Since $\top$ cannot be the active formula, then we apply the inner induction hypothesis to the premise(s) with $\top$ and use te same rule.

We assume $\Theta[C]$ is the conclusion of a rule. Since $\bot$ cannot be the active formula, then we proceed as above.

6° $C = \top$. Then $C^{\sim} = \bot$. We run the intermediate induction. Let $\Theta[C]$ be an axiom, then $\Theta[C] = \Theta'[\bot][\top]$ and $\Theta'[\bot][\Psi]$ is another instance of (a-$\bot$). We assume $\Theta[C]$ is the conclusion of a rule. Since $\top$ cannot be the active formula, then we proceed as above.

7° C is not an atomic formula. We run the intermediate induction.

Since C is not atomic, $\Theta[C]$ cannot be an instance of axiom (a-id), (a-id2) or (a-0). Let $\Theta[C]$ be an axiom (a-$\bot$). Then $\Theta[C] = \Theta'[\bot][C]$ and $\Theta'[\bot][\Psi]$ is another instance of (a-$\bot$). Let $\Theta[C]$ be the conclusion of a rule. If C is not the active formula, we apply the intermediate induction hypothesis to the premise(s) with C and use the same rule. We assume that C is the active formula.

7.1° $C = A \otimes B$. So $C^{\sim} = B^{\sim} \parr A^{\sim}$ and $\Theta[C]$ arises by (r-$\otimes$):

$$\frac{\Theta[(A, B)]}{\Theta[A \otimes B]}$$

We run the inner induction. Let $\Psi, C^{\sim}$ be an axiom, then $\Psi = \Xi[\bot]$ and $\Theta[\Psi]$ is another instance of (a-$\bot$). We assume $\Psi, C^{\sim}$ is the conclusion of a rule.

In the cases when $C^{\sim}$ does not occur in the active bunch, we apply the inner induction hypothesis to $\Theta[C]$ and the premise(s) with $C^{\sim}$, and use the same rule.

For example:

$$\frac{\Gamma[(D,E)], C^{\sim}}{\Gamma[D \otimes E], C^{\sim}}$$

changes into:

$$\frac{\Theta[\Gamma[(D,E)]]}{\Theta[\Gamma[D \otimes E]]},$$

where $\Psi = \Gamma[D \otimes E]$.

We consider cases when $C^{\sim}$ occurs in the active bunch, but is not the active formula.

$$\frac{D \qquad E, C^{\sim}}{D \parr E, C^{\sim}} \qquad \frac{D, C^{\sim} \qquad E}{D \parr E, C^{\sim}}$$

We apply the inner induction hypothesis to the premise with $C^{\sim}$ and use (r-$\parr$1).

Let $C^{\sim}$ be the active formula:

$$\frac{\Psi, A^{\sim} \qquad B^{\sim}}{\Psi, C^{\sim}} \qquad \frac{\Psi, B^{\sim} \qquad A^{\sim}}{\Psi, C^{\sim}} \qquad \frac{\Psi_2, B^{\sim} \quad \Psi_1, A^{\sim}}{(\Psi_1, \Psi_2), C^{\sim}}$$

The first case is obtained by (r-$\parr$1). We apply the outer induction hypothesis to $\Theta[(A,B)]$ and $\Psi, A^{\sim}$ and then to $\Theta[(\Psi, B)]$ and $B^{\sim}$, obtaining $\Theta[\Psi]$. The second one is obtained by (r-$\parr$1) or (r-$\parr$2). We proceed as above: we apply twice the outer induction hypothesis to both premises. The third case is obtained by (r-$\parr$4), where $\Psi = (\Psi_1, \Psi_2)$. We apply the outer induction hypothesis twice, obtaining $\Theta[(\Psi_1, \Psi_2)] = \Theta[\Psi]$.

7.2° $C = A \parr B$, then $C^{\sim} = B^{\sim} \otimes A^{\sim}$. We have to consider four cases, one for each (r-$\parr$i).

(1) $$\frac{\Gamma[B] \qquad \Delta, A}{\Gamma[(\Delta, A \parr B)]}$$

We run the inner induction. Let $\Psi, C^{\sim}$ be an axiom, then $\Psi = \Xi[\bot]$ and $\Theta[\Psi]$ is another instance of (a-$\bot$). We assume $\Psi, C^{\sim}$ is the conclusion of a rule. We skip cases when $C^{\sim}$ is not the active formula of a rule (in these cases we proceed as above). We consider (r-$\otimes$) as the only possibility:

$$\frac{\Psi, (B^{\sim}, A^{\sim})}{\Psi, C^{\sim}}$$

We apply proposition 2.3 to $\Psi, (B^{\sim}, A^{\sim})$, then we apply the outer induction hypothesis to Δ, A and $(\Psi, B^{\sim}), A^{\sim}$ and obtain: $\Delta, (\Psi, B^{\sim})$. By proposition 2.3 and the outer induction hypothesis applied to $\Theta[B]$ and $(\Delta, \Psi), B^{\sim}$ we obtain $\Gamma[(\Delta, \Psi)] = \Theta[\Psi]$.

(2) $\dfrac{\Gamma[A] \qquad B, \Delta}{\Gamma[(A \parr B, \Delta)]}$

We run the inner induction and consider the same instance as above. We apply proposition 2.3 to $\Psi, (B^{\sim}, A^{\sim})$, obtaining $(\Psi, B^{\sim}), A^{\sim}$. By proposition 2.3 we get $A^{\sim}, (\Psi, B^{\sim})$. We use proposition 2.3 and apply the outer induction hypothesis to $(A^{\sim}, \Psi), B^{\sim}$ and B, Δ. We obtain $(A^{\sim}, \Psi), \Delta$ and apply proposition 2.3 and proposition 2.3. We use the outer induction hypothesis with $(\Psi, \Delta), A^{\sim}$ and $\Gamma[A]$, obtaining $\Gamma[(\Psi, \Delta)] = \Theta[\Psi]$.

(3) $\dfrac{A, \Gamma \qquad B, \Delta}{A \parr B, (\Delta, \Gamma)}$

We run the inner induction and consider the same instance as above. We apply proposition 2.3 to $\Psi, (B^{\sim}, A^{\sim})$ and obtain $(\Psi, B^{\sim}), A^{\sim}$. We apply proposition 2.3 and get $A^{\sim}, (\Psi, B^{\sim})$. We use proposition 2.3 and apply the outer induction hypothesis to $(A^{\sim}, \Psi), B^{\sim}$ and B, Δ. We have $(A^{\sim}, \Psi), \Delta$. We apply proposition 2.3 and proposition 2.3. We use the outer induction hypothesis to $(\Psi, \Delta), A^{\sim}$ and A, Γ, obtaining $(\Psi, \Delta), \Gamma$. We use proposition 2.3.

(4) $\dfrac{\Gamma, A \qquad \Delta, B}{(\Delta, \Gamma), A \parr B}$

We run the inner induction and consider the same instance as above. We apply proposition 2.3 to $\Psi, (B^{\sim}, A^{\sim})$, obtaining $(\Psi, B^{\sim}), A^{\sim}$. We apply the outer induction hypothesis to $(\Psi, B^{\sim}), A^{\sim}$ and Γ, A. We get $\Gamma, (\Psi, B^{\sim})$. We use proposition 2.3 and apply the outer induction hypothesis to $(\Gamma, \Psi), B^{\sim}$ and Δ, B. We obtain $\Delta, (\Gamma, \Psi)$ and use proposition 2.3.

7.3° $C = A \wedge B$. So $C^{\sim} = A^{\sim} \vee B^{\sim}$. We have the following instances:

$$\frac{\Theta[A]}{\Theta[C]} \qquad \frac{\Theta[B]}{\Theta[C]}$$

We run the inner induction. Let $\Psi, C^{\sim}$ be an axiom, then $\Psi = \Xi[\bot]$ and $\Theta[\Psi]$ is another instance of (a-$\bot$). We assume $\Psi, C^{\sim}$ is the conclusion of a rule. We skip the cases with $C^{\sim}$ not being the active formula. We have only one possibility:

$$\frac{\Psi, A^{\sim} \qquad \Psi, B^{\sim}}{\Psi, C^{\sim}}$$

We apply the outer induction hypothesis to $\Theta[A]$ and $\Psi, A^{\sim}$ or to $\Theta[B]$ and $\Psi, B^{\sim}$, depending on the proof of $\Theta[C]$. In both cases we obtain $\Theta[\Psi]$.

7.4° $C = A \vee B$. So $C^{\sim} = A^{\sim} \wedge B^{\sim}$. We have the following case:

$$\frac{\Theta[A] \qquad \Theta[B]}{\Theta[C]}$$

We run the inner induction. Let $\Psi, C^{\sim}$ be an axiom, then $\Psi = \Xi[\bot]$ and $\Theta[\Psi]$ is another instance of (a-$\bot$). We assume Ψ, $C^{\sim}$ is the conclusion of a rule. We consider only the cases with $C^{\sim}$ as the active formula:

$$\frac{\Psi, A^{\sim}}{\Psi, C^{\sim}} \qquad \frac{\Psi, B^{\sim}}{\Psi, C^{\sim}}$$

In the first case we apply the outer induction hypothesis to $\Theta[A]$ and $\Psi, A^{\sim}$ and in the second case to $\Theta[B]$ and $\Psi, B^{\sim}$. □

Lemma 2.8. *Let A be any CyNBL–formula. Then $A, A^{\sim}$ and $A^{\sim}, A$ are provable in $\mathbf{S}_{\Phi}$.*

Proof: The proof proceeds by the induction on the complexity of the formula A. Let $A = p$. Then $(A, A^{\sim}) = (p, p^{\sim})$ and $(A^{\sim}, A) = (p^{\sim}, p)$ are axioms. Analogously for $A = p^{\sim}$.

Let $A = 1$, then $(A, A^{\sim}) = (1, 0)$. We have $\vdash 0$. We apply (r-1) and obtain $\vdash 1, 0$. Analogously, $\vdash (0, 1)$. Similarly for $A = 0$.

Now let $A = A_1 \otimes A_2$. Then $(A, A^{\sim}) = (A_1 \otimes A_2, A_2^{\sim} \parr A_1^{\sim})$ and $(A^{\sim}, A) = (A_2^{\sim} \parr A_1^{\sim}, A_1 \otimes A_2)$. By the induction hypothesis: $\vdash A_1, A_1^{\sim}$, $\vdash A_1^{\sim}, A_1$, $\vdash A_2, A_2^{\sim}$ and $\vdash A_2^{\sim}, A_2$. We have the following derivations:

$$\dfrac{\dfrac{\dfrac{\dots}{A_1, A_1^{\sim}} \qquad \dfrac{\dots}{A_2, A_2^{\sim}}}{(A_1, A_2), A_2^{\sim} \parr A_1^{\sim}}}{A_1 \otimes A_2, A_2^{\sim} \parr A_1^{\sim}}\ \text{(r-}\parr\text{4)} \qquad \dfrac{\dfrac{\dfrac{\dots}{A_1^{\sim}, A_1} \qquad \dfrac{\dots}{A_2^{\sim}, A_2}}{A_2^{\sim} \parr A_1^{\sim}, (A_1, A_2)}}{A_2^{\sim} \parr A_1^{\sim}, A_1 \otimes A_2}\ \text{(r-}\parr\text{3)}$$

Let $A = A_1 \parr A_2$. Then $(A, A^{\sim}) = (A_1 \parr A_2, A_2^{\sim} \otimes A_1^{\sim})$ and $(A^{\sim}, A) = (A_2^{\sim} \otimes A_1^{\sim}, A_1 \parr A_2)$. The proof is analogous to the case for $\otimes$, but we use in the first case (r-$\parr$3) instead of (r-$\parr$4) and (r-$\parr$4) instead of (r-$\parr$3).

Let $A = A_1 \wedge A_2$. Then $(A, A^{\sim}) = (A_1 \wedge A_2, A_1^{\sim} \vee A_2^{\sim})$. By the induction hypothesis: $\vdash A_1, A_1^{\sim}$ and $\vdash A_2, A_2^{\sim}$. We use (r-$\wedge$) and obtain $\vdash A_1 \wedge A_2, A_1^{\sim}$ and $\vdash A_1 \wedge A_2, A_2^{\sim}$. We apply (r-$\vee$) and obtain $\vdash A_1 \wedge A_2, A_1^{\sim} \vee A_2^{\sim}$. The second part is proved in the similar way.

The case $A = A_1 \vee A_2$ is similar to the previous one. □

Proposition 2.9 (Phi=SPhi). $\Phi \vdash \Gamma$ iff $\vdash_{\mathbf{S}_\Phi} \Gamma$.

Proof: Let $\Phi \vdash \Gamma$. We show $\vdash_{\mathbf{S}_\Phi} \Gamma$. All rules of CyNBL are admissible in $\mathbf{S}_\Phi$. We show that every sequent $(C, D^{\sim}) \in \Phi$ is provable in $\mathbf{S}_\Phi$. By lemma 2.8, $\vdash D, D^{\sim}$ and $\vdash C, C^{\sim}$. We apply (r-assm1) and obtain $\vdash_{\mathbf{S}_\Phi} C, D^{\sim}$. Hence, $\vdash_{\mathbf{S}_\Phi} \Gamma$.

Now we assume $\vdash_{\mathbf{S}_\Phi} \Gamma$. We show $\Phi \vdash \Gamma$. We take the proof of Γ in $\mathbf{S}_\Phi$ and replace all applications of the assumption rules as follows:

$$\dfrac{\dfrac{\dots}{D, \Gamma} \qquad \dfrac{\dots}{\Delta, C^{\sim}}}{\Delta, \Gamma} \quad \to \quad \dfrac{\dfrac{\dfrac{\dots}{D, \Gamma} \qquad C, D^{\sim}}{C, \Gamma} \qquad \dfrac{\dots}{\Delta, C^{\sim}}}{\Delta, \Gamma}\ \text{(r-cut)}$$

$$\dfrac{\dfrac{\dots}{D, \Gamma} \qquad \dfrac{\dots}{\Delta, C^{\sim}}}{\Gamma, \Delta} \quad \to \quad \dfrac{\dfrac{\dfrac{\dfrac{\dots}{D, \Gamma} \qquad C, D^{\sim}}{C, \Gamma} \qquad \dfrac{\dots}{\Delta, C^{\sim}}}{\Delta, \Gamma}\ \text{(r-cut)}}{\Gamma, \Delta}\ \text{(r-cut)}$$

$$\dfrac{\dfrac{\dots}{D, \Gamma_1} \qquad \dfrac{\dots}{(\Gamma_2, \Gamma_3), C^{\sim}}}{\Gamma_2, (\Gamma_3, \Gamma_1)} \quad \to \quad \dfrac{\dfrac{\dfrac{\dfrac{\dots}{D, \Gamma_1} \qquad C, D^{\sim}}{C, \Gamma_1} \qquad \dfrac{\dots}{(\Gamma_2, \Gamma_3), C^{\sim}}}{(\Gamma_2, \Gamma_3), \Gamma_1}\ \text{(r-cut)}}{\Gamma_2, (\Gamma_3, \Gamma_1)}\ \text{(r-cut)}$$

And analogously with other rules. The rules (r-$\parr$3) and (r-$\parr$4) are admissible in CyNBL. □

Provability in pure CyNBL is equivalent to provability in $\mathbf{S}_{\emptyset}$. Hence, CyNBL admits the cut elimination.

Let T be an arbitrary set of CyNBL–formulas. By a *T–sequent* we mean a CyNBL–sequent containing only formulas from T. By a *T–proof* we mean a proof consisting of only T–sequents.

Proposition 2.10 (subformula property). Let Φ be a set of sequents of the form $C, D^{\sim}$. Let Γ be a CyNBL–sequent and let T be a subformula–closed set such that every formula in Γ occurs in T and for every $(C, D^{\sim}) \in \Phi$ we have $C, D, C^{\sim}, D^{\sim} \in T$. Then, Γ is provable in $\mathbf{S}_{\Phi}$ iff it has a T–proof in $\mathbf{S}_{\Phi}$.

3. Strong conservativeness

In this section we define Full Nonassociative Lambek Calculus (FNL) without $\bot$. We know, that CyNBL is not a conservative extension of FNL with $\bot$. We prove that CyNBL is strongly conservative extension of FNL. This result may be easily proved for CyNBL without additive constants, using the subformula property.

Let $\mathcal{V}$ be an arbitrary, countable set of variables. We construct the set of *FNL*–formulas from $\mathcal{V}$ by the binary connectives $\otimes, \multimap, \multimapinv, \wedge, \vee$ and the constant 1.

An *FNL–bunch* is an element of free unital groupoid generated by the set of all FNL–formulas. The neutral element of this unital groupoid is called an empty bunch and denoted ϵ. We define an *FNL–context* analogously as a CyNBL–context. An *FNL–sequent* is a pair $\Gamma \Rightarrow A$, where Γ is an FNL–bunch and A is an FNL–formula.

The axioms and the rules of FNL are as follows:

$$\text{(id)}\ \frac{}{A \Rightarrow A} \qquad \text{(cut)}\ \frac{\Gamma \Rightarrow A \qquad \Delta[A] \Rightarrow C}{\Delta[\Gamma] \Rightarrow C}$$

$$(\otimes \Rightarrow)\ \frac{\Gamma[(A, B)] \Rightarrow C}{\Gamma[A \otimes B] \Rightarrow C} \qquad (\Rightarrow \otimes)\ \frac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A \otimes B}$$

$$(\multimap\Rightarrow)\ \frac{\Gamma[B] \Rightarrow C \qquad \Theta \Rightarrow A}{\Gamma[(\Theta, A \multimap B)] \Rightarrow C} \qquad (\Rightarrow\multimap)\ \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \multimap B}$$

$$(\multimapinv\Rightarrow)\ \frac{\Gamma[A] \Rightarrow C \qquad \Theta \Rightarrow B}{\Gamma[(A \multimapinv B, \Theta)] \Rightarrow C} \qquad (\Rightarrow\multimapinv)\ \frac{\Gamma, B \Rightarrow A}{\Gamma \Rightarrow A \multimapinv B}$$

$$(1 \Rightarrow)\ \frac{\Gamma[\Delta] \Rightarrow C}{\Gamma[(1,\Delta)] \Rightarrow C} \quad \frac{\Gamma[\Delta] \Rightarrow C}{\Gamma[(\Delta,1)] \Rightarrow C} \qquad (\Rightarrow 1)\ \frac{}{\epsilon \Rightarrow 1}$$

$$(\vee \Rightarrow)\ \frac{\Gamma[A] \Rightarrow C \qquad \Gamma[B] \Rightarrow C}{\Gamma[A \vee B] \Rightarrow C} \qquad (\Rightarrow \vee)\ \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2}(i=1,2)$$

$$(\wedge \Rightarrow)\ \frac{\Gamma[A] \Rightarrow C}{\Gamma[A \wedge B] \Rightarrow C} \quad \frac{\Gamma[B] \Rightarrow C}{\Gamma[A \wedge B] \Rightarrow C} \quad (\Rightarrow \wedge)\ \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}$$

FNL is strongly complete with respect to FNL–algebras. One proves that fact in a standard way, using Lindenbaum–Tarski algebras. Since every CyNBL–algebra is an FNL–algebra, then CyNBL is an extension of FNL.

DEFINITION 3.1. We define two sets of CyNBL–formulas: $\mathcal{F}_1$ and $\mathcal{F}_2$. The former is the set of FNL–forumlas translated into CyNBL and the latter is the set of negated translated FNL–formulas.

(i) For every $p \in \mathcal{V}$, $p \in \mathcal{F}_1$ and $p^\sim \in \mathcal{F}_2$.

(ii) $1 \in \mathcal{F}_1$ and $0 \in \mathcal{F}_2$.

(iii) If $A, B \in \mathcal{F}_1$, then $A \otimes B, A \wedge B, A \vee B \in \mathcal{F}_1$.

(iv) If $A, B \in \mathcal{F}_2$, then $A \parr B, A \wedge B, A \vee B \in \mathcal{F}_2$.

(v) If $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$, then $A \parr B, B \parr A \in \mathcal{F}_1$ and $A \otimes B, B \otimes A \in \mathcal{F}_2$.

(vi) No other formula belongs to $\mathcal{F}_1$ nor $\mathcal{F}_2$.

Notice that for every $A \in \mathcal{F}_1$ its metalanguage negation $A^\sim \in \mathcal{F}_2$ and conversely, if $A \in \mathcal{F}_2$, then $A^\sim \in \mathcal{F}_1$. Moreover, $\mathcal{F}_1 \cap \mathcal{F}_2 = \emptyset$. We define $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$. We see that $\mathcal{F}$ is not the set of all CyNBL–formulas; e.g. $p \vee p^\sim \notin \mathcal{F}$.

CyNBL is an extension of FNL. We translate $A \multimap B$ into $A^\sim \parr B$ and $A \multimapinv B$ into $A \parr B^\sim$. We translate the FNL–sequent $\Gamma \Rightarrow A$ to the CyNBL–sequent $\Gamma, A^\sim$. One notices that every FNL–bunch is a CyNBL–bunch (if we replace $\multimap$ and $\multimapinv$ with $\parr$). Also, every translated FNL–sequent is an $\mathcal{F}$–sequent.

Lemma 3.2. *Let Φ be a set of sequents of the form $C, D^{\sim}$, where C, D are FNL–formulas. If an $\mathcal{F}$–sequent contains some formulas $A, B \in \mathcal{F}_2$, then this sequent is unprovable in $\mathbf{S}_\Phi$.*

Proof: Since $\mathcal{F}$ is closed under subformulas and contains all formulas $C, D, C^{\sim}$ and $D^{\sim}$ for every sequent in Φ, then every provable (in $\mathbf{S}_\Phi$) $\mathcal{F}$–sequent has an $\mathcal{F}$–proof, by the subformula property.

We observe that none of the axioms of $\mathbf{S}_\Phi$ has more than one formula from $\mathcal{F}_2$. We show, that if the premises of a rule are $\mathcal{F}$–sequents with at most one formula from $\mathcal{F}_2$, then the conclusion also has at most one $\mathcal{F}_2$–formula or is not an $\mathcal{F}$–sequent.

We consider (r-$\otimes$). The premise is of the form $\Gamma[(A, B)]$. The conclusion is of the form $\Gamma[A \otimes B]$. We consider two cases: (1) A or B belongs to $\mathcal{F}_2$, (2) Neither A, B belongs to $\mathcal{F}_2$. In the first case, $A \otimes B \in \mathcal{F}_2$, by definition 3.1(v). There is no other $\mathcal{F}_2$–formula, since in premise there is only one. In the second case, if $A, B \in \mathcal{F}_1$, then $A \otimes B \in \mathcal{F}_1$, by definition 3.1(iii). Hence, it is impossible to be more $\mathcal{F}_2$–formulas in the conclusion than in the premise.

We consider (r-1). Since $1 \in \mathcal{F}_1$, then there cannot be two negated formulas in the conclusion.

We consider (r-$\wedge$). Let the premise be $\Gamma[A]$. If $A \in \mathcal{F}_1$, then $A \wedge B \in \mathcal{F}_1$ or $A \wedge B \notin \mathcal{F}$. If $A \in \mathcal{F}_2$, then $A \wedge B \in \mathcal{F}_2$ or $A \wedge B \notin \mathcal{F}$.

We consider (r-$\vee$). Let the premises be $\Gamma[A]$ and $\Gamma[B]$. Then the conclusion is $\Gamma[A \vee B]$. The formula $A \vee B$ belongs to $\mathcal{F}$ iff both $A, B \in \mathcal{F}_1$ or both $A, B \in \mathcal{F}_2$.

We consider (r-$\parr$1). Let the premises be $\Gamma[B]$ and Δ, A. If $A, B \in \mathcal{F}_1$, then $A \parr B \notin \mathcal{F}$ and the conclusion is not an $\mathcal{F}$–sequent. If $A, B \in \mathcal{F}_2$, then there are only $\mathcal{F}_1$–formulas in $\Gamma[\_]$ and Δ. If one of A, B is in $\mathcal{F}_1$ and the other in $\mathcal{F}_2$, then $A \parr B \in \mathcal{F}_1$, by definition 3.1(v). So if in the conclusion were two $\mathcal{F}_2$–formulas, then one of the premises would also have two $\mathcal{F}_2$–formulas, which is impossible by assumption.

Cases for (r-$\parr$2), (r-$\parr$3), (r-$\parr$4) are similar.

We consider the assumption rules. All of them have premises of the form D, Γ and $\Delta, C^{\sim}$. Since C, D are FNL–formulas, then $D \in \mathcal{F}_1$ and $C^{\sim} \in \mathcal{F}_2$. Thus, Γ contains at most one formula from $\mathcal{F}_2$ and Δ does not contain any formula of $\mathcal{F}_2$. For every assumption rule the conclusion is built from the formulas of Γ and Δ, so the conclusion can have at most one $\mathcal{F}_2$–formula.

Now, assume that some $\mathcal{F}$–sequent with two formulas from $\mathcal{F}_2$ is provable in $\mathbf{S}_\Phi$. Since it is provable, it has an $\mathcal{F}$–proof. But none of the axioms and none of the conclusions of the rules has two formulas form $\mathcal{F}_2$, unless they are not $\mathcal{F}$–sequents. Hence, there is no $\mathcal{F}$–proof and the sequent is unprovable. □

One notices that the lemma above does not hold if we admit $\bot$ and $\top$ in FNL. In such a case, $\mathcal{F}_1 \cap \mathcal{F}_2 \neq \emptyset$. We see, that $\bot, (A, B)$ is an axiom for all formulas A, B, even those belonging to $\mathcal{F}_2$.

Theorem 3.3. *Let* $\Gamma[A^\sim]$ *be an* $\mathcal{F}$*–sequent where* $A \in \mathcal{F}_1$ *and* $\Gamma[A^\sim] \sim \Delta, A^\sim$ *for some* Δ*. Then* $\vdash_{\mathbf{S}_\Phi} \Gamma[A^\sim]$*, if and only if* $\Phi \vdash_{FNL} \Delta \Rightarrow A$*.*

Proof: The if-part immediately follows from **??** and the fact that CyNBL is an extension of FNL. We prove only the if-part.

Let $\Theta[C^\sim]$ be an $\mathcal{F}$–sequent provable in $\mathbf{S}_\Phi$ and $\Theta[\_] \sim (\Psi, \_)$. We show the construction of the proof in FNL. We proceed by the outer induction on the number of connectives of C and the inner induction on the proof of $\Theta[C^\sim]$. Notice that $\Theta[\_]$ has only $\mathcal{F}_1$–formulas.

1° Let $C = p$. We run the inner induction. If $\Theta[p^\sim]$ is an axiom, then $\Theta[p^\sim] = (p, p^\sim)$ or $\Theta[p^\sim] = (p^\sim, p)$. In both cases, $(\Psi, C^\sim) = (p, p^\sim)$ and $p \Rightarrow p$ is an axiom in FNL.

Now we assume $\Theta[p^\sim]$ is the conclusion of a rule. We observe $p^\sim$ cannot be the active formula of a rule.

We consider (r-$\otimes$). The premise is $\Gamma[(A, B)][p^\sim]$. Then, by proposition 2.3, $\vdash_{\mathbf{S}_\Phi} \Delta[(A, B)], p^\sim$ for $(\Psi, p^\sim) = (\Delta[A \otimes B], p^\sim)$. By the inner induction hypothesis, $\Delta[(A, B)] \Rightarrow p$ is provable in FNL from Φ. We apply $(\otimes \Rightarrow)$ and obtain $\Delta[A \otimes B] \Rightarrow p$.

The cases for (r-$\vee$), (r-$\wedge$) and (r-1) are analogous.

We consider (r-$⅋$1). We recall that $A ⅋ B = A^\sim \multimap B = A \multimapinv B^\sim$. We consider the following cases:

$$\frac{\Gamma[B][p^\sim] \qquad \Delta, A^\sim}{\Gamma[(\Delta, A \multimap B)][p^\sim]} \qquad \frac{\Gamma[B^\sim] \qquad \Delta[p^\sim], A}{\Gamma[(\Delta[p^\sim], A \multimapinv B)]}$$

$$\frac{\Gamma[B^\sim][p^\sim] \qquad \Delta, A}{\Gamma[(\Delta, A \multimapinv B)][p^\sim]} \qquad \frac{\Gamma[B] \qquad \Delta[p^\sim], A^\sim}{\Gamma[(\Delta[p^\sim], A \multimap B)]}$$

In the first case we have $\Gamma[B][p^{\sim}] \sim \Gamma'[B], p^{\sim}$. By the inner induction hypothesis, $\Phi \vdash_{\mathrm{FNL}} \Gamma'[B] \Rightarrow p$ and $\Phi \vdash_{\mathrm{FNL}} \Delta \Rightarrow A$. We apply $(\multimap\Rightarrow)$ and obtain $\Phi \vdash_{\mathrm{FNL}} \Gamma'[(\Delta, A \multimap B)] \Rightarrow p$. Also, $(\Psi, p^{\sim}) = (\Gamma'[(\Delta, A \multimap B)], p^{\sim})$ by corollary 2.2.

In the second case we know that $\Gamma[B^{\sim}]$ reduces to $\Gamma', B^{\sim}$ and, by the inner induction hypothesis, $\Phi \vdash_{\mathrm{FNL}} \Gamma' \Rightarrow B$. Also, $\Delta[p^{\sim}], A$ reduces to $\Delta'[A], p^{\sim}$. So, $\Phi \vdash_{\mathrm{FNL}} \Delta'[A] \Rightarrow p$. We apply $(\multimapinv\Rightarrow)$ and obtain $\Phi \vdash_{\mathrm{FNL}} \Delta'[(A \multimapinv B, \Gamma')] \Rightarrow p$. One easily checks that $\Gamma[(\Delta[p^{\sim}], A \multimapinv B)] \sim (\Delta'[(A \multimapinv B, \Gamma')], p^{\sim})$.

The last two cases have premises with two negated FNL–formulas. By lemma 3.2, it is impossible. The cases for (r-$\parr$2), (r-$\parr$3) and (r-$\parr$4) are similar.

We consider (r-assm1). We have two possibilities.

$$(1)\ \frac{F, \Gamma[p^{\sim}] \qquad \Delta, E^{\sim}}{\Delta, \Gamma[p^{\sim}]},$$

where $(\Delta, \Gamma[p^{\sim}]) = \Theta[p^{\sim}]$ and $(\Delta, \Gamma[p^{\sim}]) \sim \Gamma'[\Delta]$ and $\Gamma'[\Delta] = \Psi$.

By the inner induction hypothesis, $\Phi \vdash_{\mathrm{FNL}} \Delta \Rightarrow E$ and $\Phi \vdash_{\mathrm{FNL}} \Gamma'[F] \Rightarrow p$. Since $E \Rightarrow F \in \Phi$, then $\Phi \vdash_{\mathrm{FNL}} E \Rightarrow F$. We apply (cut) to $\Gamma'[F] \Rightarrow p$ and $E \Rightarrow F$ and obtain $\Phi \vdash_{\mathrm{FNL}} \Gamma'[E] \Rightarrow p$. We apply (cut) to this and $\Delta \Rightarrow E$ and obtain $\Phi \vdash_{\mathrm{FNL}} \Gamma'[\Delta] \Rightarrow p$.

$$(2)\ \frac{D, \Gamma \qquad C^{\sim}, \Delta[p^{\sim}]}{\Delta[p^{\sim}], \Gamma},$$

where $(\Delta[p^{\sim}], \Gamma) = \Theta[p^{\sim}]$ and $\Delta'[\Gamma] = \Psi$. This case is impossible, since, by lemma 3.2, the second premise is unprovable.

The cases for other assumption rules are similar.

2° Let $C = 1$. We run the inner induction. $\Theta[0]$ may be an axiom (a-0). So $\Theta[\_] = \_$ and $\Psi = \epsilon$. Hence, $(\Psi, C^{\sim}) = 1^{\sim}$ and $\Phi \vdash_{\mathrm{FNL}} \epsilon \Rightarrow 1$ by $(\Rightarrow 1)$.

We assume $\Theta[0]$ is not an axiom. Then, 0 is not the active formula of any rule. We proceed as above.

3° Let $C = A \otimes B$. Then $C^{\sim} = B^{\sim} \parr A^{\sim}$. We run the inner induction. We know that $\Theta[C^{\sim}]$ cannot be an axiom. So it has to be the conclusion of a rule. If $C^{\sim}$ is not the active formula of a rule, we proceed as above. We assume $C^{\sim}$ is the active formula, so we have the following possibilities:

$$(1)\,\frac{\Gamma[A^{\sim}] \qquad \Delta, B^{\sim}}{\Gamma[(\Delta, B^{\sim} \parr A^{\sim})]} \qquad (2)\,\frac{\Gamma[B^{\sim}] \qquad A^{\sim}, \Delta}{\Gamma[(B^{\sim} \parr A^{\sim}, \Delta)]}$$

$$(3)\,\frac{B^{\sim}, \Gamma \qquad A^{\sim}, \Delta}{B^{\sim} \parr A^{\sim}, (\Delta, \Gamma)} \qquad (4)\,\frac{\Gamma, B^{\sim} \qquad \Delta, A^{\sim}}{(\Delta, \Gamma), B^{\sim} \parr A^{\sim}}$$

In (1) we have $\Gamma[\_] \sim (\Gamma', \_)$, so, $\Gamma[(\Delta, B^{\sim} \parr A^{\sim})] \sim ((\Gamma', \Delta), B^{\sim} \parr A^{\sim})$. By the inner induction hypothesis, $\Phi \vdash_{\mathrm{FNL}} \Gamma' \Rightarrow A$ and $\Phi \vdash_{\mathrm{FNL}} \Delta \Rightarrow B$. We apply $(\Rightarrow \otimes)$ and obtain $\Phi \vdash_{\mathrm{FNL}} \Gamma', \Delta \Rightarrow A \otimes B$. In (2), (3) and (4) we proceed similarly.

4° Let $C = A \multimap B$. Then $C = A^{\sim} \parr B$ and $C^{\sim} = B^{\sim} \otimes A$. We run the inner induction. $\Theta[C^{\sim}]$ cannot be an axiom. Hence it has to be the conclusion of a rule. If $C^{\sim}$ is not the active formula of a rule, we proceed as above. We assume $C^{\sim}$ is the active formula, so we have only one possibility:

$$\frac{\Gamma[(B^{\sim}, A)]}{\Gamma[B^{\sim} \otimes A]}$$

Then $\Gamma[\_] \sim (\Gamma', \_)$ and $\Gamma[(B^{\sim}, A)] \sim ((A, \Gamma'), B^{\sim})$. By the induction hypothesis, $\Phi \vdash_{\mathrm{FNL}} A, \Gamma' \Rightarrow B$. We apply $(\Rightarrow \multimap)$ and obtain $\Phi \vdash_{\mathrm{FNL}} \Gamma' \Rightarrow A \multimap B$.

The case when $C = A \multimapinv B$ is analogous.

5° Let $C = A \vee B$, then $C^{\sim} = A^{\sim} \wedge B^{\sim}$. We run the inner induction. $\Theta[C^{\sim}]$ cannot be an axiom. Hence it has to be the conclusion of a rule. If $C^{\sim}$ is not the active formula of a rule, we proceed as above. We assume $C^{\sim}$ is the active formula, so we have two possibilities:

$$\frac{\Theta[A^{\sim}]}{\Theta[A^{\sim} \wedge B^{\sim}]} \qquad \frac{\Theta[B^{\sim}]}{\Theta[A^{\sim} \wedge B^{\sim}]}$$

In both cases we apply the inner induction hypothesis to the premise and then we apply $(\Rightarrow \vee)$.

6° Let $C = A \wedge B$, then $C^{\sim} = A^{\sim} \vee B^{\sim}$. We run the inner induction. $\Theta[C^{\sim}]$ cannot be an axiom. Hence it has to be the conclusion of a rule. If $C^{\sim}$ is not the active formula of a rule, we proceed as above. We assume $C^{\sim}$ is the active formula, so we have one case:

$$\frac{\Theta[A^{\sim}] \qquad \Theta[B^{\sim}]}{\Theta[A^{\sim} \vee B^{\sim}]}$$

We apply the inner induction hypothesis to both premises and then we apply $(\Rightarrow \wedge)$. □

Corollary 3.4. CyNBL is a strongly conservative extension of FNL.

4. Application to similar logics

The results of this paper may be adapted to other logics similar to CyNBL. In this section we provide sequent systems for these logics and some remarks about the results of this paper.

4.1. Cyclic Multiplicative–Additive Linear Logic

Cyclic Multiplicative-Additive Linear Logic (CyMALL) serves as the associative counterpart to CyNBL. Key distinctions in the sequent systems include the use of finite sequences of formulas rather than bunches and the absence of the (r-shift) rule.

CyMALL employs the same formulas as CyNBL, and metalanguage negation is defined in a similar manner. CyMALL–sequents consist of nonempty finite sequences of CyMALL–formulas. Notably, an empty sequence, denoted by ϵ, is not considered a sequent.

The axioms of CyMALL are:

(a-id) $\dfrac{}{p, p^{\sim}}$ (a-0) $\dfrac{}{0}$

(a-$\bot$) $\dfrac{}{\Gamma, \bot, \Delta}$

The introduction rules are:

$$(\text{r-}\otimes)\ \frac{\Gamma, A, B, \Delta}{\Gamma, A \otimes B, \Delta} \qquad (\text{r-}1)\ \frac{\Gamma, \Delta}{\Gamma, 1, \Delta}$$

$$(\text{r-}⅋1)\ \frac{\Gamma_1, B, \Gamma_2 \qquad \Delta, A}{\Gamma_1, \Delta, A ⅋ B, \Gamma_2} \qquad (\text{r-}⅋2)\ \frac{\Gamma_1, A, \Gamma_2 \qquad B, \Delta}{\Gamma_1, A ⅋ B, \Delta, \Gamma_2}$$

$$(\text{r-}\wedge)\ \frac{\Gamma, A, \Delta}{\Gamma, A \wedge B, \Delta} \quad \frac{\Gamma, A, \Delta}{\Gamma, B \wedge A, \Delta} \qquad (\text{r-}\vee)\ \frac{\Gamma, A, \Delta \qquad \Gamma, B, \Delta}{\Gamma, A \vee B, \Delta}$$

The structural rule and the cut rule are:

$$(\text{r-cyc})\ \frac{\Gamma, \Delta}{\Delta, \Gamma} \qquad (\text{r-cut})\ \frac{\Gamma_1, A, \Gamma_2 \qquad \Delta, A^{\sim}}{\Gamma_1, \Delta, \Gamma_2}$$

Let Φ be a set of sequents of the form $C, D^{\sim}$. We define the system $\mathbf{S}_{\Phi}$. The system $\mathbf{S}_{\Phi}$ has all axioms and introduction rules of CyMALL. We add the following axioms:

$$(\text{a-id2})\ p^{\sim}, p$$

For every $(C, D^{\sim}) \in \Phi$ we add the assumption rules:

$$(\text{r-assm1})\ \frac{D, \Gamma \qquad \Delta, C^{\sim}}{\Delta, \Gamma} \qquad (\text{r-assm2})\ \frac{D, \Gamma \qquad \Delta, C^{\sim}}{\Gamma, \Delta}$$

As one may notice, the system for associative version is much less complex. The proofs are then also simpler. For example, the analogue for lemma 2.1 is the following:

Lemma. *Let* Γ_1, A, Γ_2 *be an CyMALL–sequent. Then, there exists unique* Γ *such that* $\Gamma_1, A, \Gamma_2 \sim \Gamma, A$.

Two sequences of formulas are related by $\sim$ if one can be obtained from the other through finitely many applications of (r-cyc). This signifies that one sequence is a cyclic permutation of the other. It's worth noting that this can always be achieved in a single application of (r-cyc). Furthermore, the counterpart of (r-shift) in the associative system is a trivial rule.

We consider CyMALL as an extension of FL, i.e. associative version of FNL. The FL–formulas are the same as for FNL. Instead of bunches we use finite sequences of formulas. The axioms and rules of FL are as follows:

$$\text{(id)} \quad A \Rightarrow A$$

$$\text{(cut)} \ \frac{\Gamma \Rightarrow A \qquad \Delta_1, A, \Delta_2 \Rightarrow C}{\Delta_1, \Gamma, \Delta_2 \Rightarrow C}$$

$$(\otimes \Rightarrow) \frac{\Gamma, A, B, \Delta \Rightarrow C}{\Gamma, A \otimes B, \Delta \Rightarrow C} \qquad (\Rightarrow \otimes) \frac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A \otimes B}$$

$$(\multimap\Rightarrow) \frac{\Gamma, B, \Delta \Rightarrow C \qquad \Theta \Rightarrow A}{\Gamma, \Theta, A \multimap B, \Delta \Rightarrow C} \qquad (\Rightarrow\multimap) \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \multimap B}$$

$$(\multimapinv\Rightarrow) \frac{\Gamma, A, \Delta \Rightarrow C \qquad \Theta \Rightarrow B}{\Gamma, A \multimapinv B, \Theta, \Delta \Rightarrow C} \qquad (\Rightarrow\multimapinv) \frac{\Gamma, B \Rightarrow A}{\Gamma \Rightarrow A \multimapinv B}$$

$$(1 \Rightarrow) \frac{\Gamma, \Delta \Rightarrow C}{\Gamma, 1, \Delta \Rightarrow C} \qquad (\Rightarrow 1)\ \epsilon \Rightarrow 1$$

$$(\vee \Rightarrow) \frac{\Gamma, A, \Delta \Rightarrow C \qquad \Gamma, B, \Delta \Rightarrow C}{\Gamma, A \vee B, \Delta \Rightarrow C} \qquad (\Rightarrow \vee) \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2} (i = 1, 2)$$

$$(\wedge \Rightarrow) \frac{\Gamma, A_i, \Delta \Rightarrow C}{\Gamma, A_1 \wedge A_2, \Delta \Rightarrow C} (i = 1, 2) \qquad (\Rightarrow \wedge) \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}$$

One may easily adjust the proofs of all the results in this paper, since the rules are similar.

4.2. Logics without multiplicative constants

We can explore variations of CyNBL and CyMALL by removing the multiplicative constants 1 and 0. These versions are denoted as CyNBL$^+$ and CyMALL$^+$. We eliminate all rules and axioms involving 1 and 0 and adjust the definitions of sequents.

For CyNBL$^+$, we define bunches as the element of the free groupoid generated by the set of all CyNBL$^+$–formulas. A CyNBL$^+$–sequent is every bunch which has at least two formulas.

CyNBL$^+$ is not an extension of FNL, but of FNL$^+$. FNL$^+$ is derived from FNL by removing constant 1, all axioms and rules associated with 1, and imposing a restriction on sequents to have a nonempty antecedent (known as the Lambek restriction).

Since CyMALL$^+$ is an associative variant of CyNBL$^+$, we define a CyMALL$^+$–sequent as a sequence of CyMALL$^+$–formulas, consising of at least two formulas. All other modifications are similar to those for CyNBL$^+$.

All the results proved in this paper remain true, since we do not use 1 in any important way.

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