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Hamzeh Mohammadi

LINEAR ABELIAN MODAL LOGIC

Abstract

A many-valued modal logic, called linear abelian modal logic $\mathbf{LK}(\mathbf{A})$ is introduced as an extension of the abelian modal logic $\mathbf{K}(\mathbf{A})$. Abelian modal logic $\mathbf{K}(\mathbf{A})$ is the minimal modal extension of the logic of lattice-ordered abelian groups. The logic $\mathbf{LK}(\mathbf{A})$ is axiomatized by extending $\mathbf{K}(\mathbf{A})$ with the modal axiom schemas $\Box(\varphi \vee \psi) \to (\Box\varphi \vee \Box\psi)$ and $(\Box\varphi \wedge \Box\psi) \to \Box(\varphi \wedge \psi)$. Completeness theorem with respect to algebraic semantics and a hypersequent calculus admitting cut-elimination are established. Finally, the correspondence between hypersequent calculi and axiomatization is investigated.

Keywords: many-valued logic, modal logic, abelian logic, hypersequent calculus, cut-elimination.

2020 Mathematical Subject Classification: 03B45, 03B50, 03F03, 03F05.

1. Introduction

Many-valued modal logics combine the Kripke frame semantics of classical modal logic with a many-valued semantics at each world. As in the classical setting, they provide a compromise between the good computational properties (decidability and low complexity) of propositional logics and the expressivity of first-order logics. Such logics have been used to model modal notions such as fuzzy similarity measures [14], fuzzy modal logic for belief functions (see, e.g., [13, 11]), probabilistic logics (see, e.g., [12, 21]), many-valued tense logics (see, e.g., [9, 16]), Łukasiewicz μ-calculus [22], continuous propositional modal logic [3], and serve as a basis for defining

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fuzzy description logics (see, e.g., [2, 15, 24]), dealing with fuzzy concepts and ontologies.

Several many-valued modal logics with propositional connectives interpreted in the ordered additive group of real numbers have been studied. These logics make use of basic operations on the real numbers and have been studied in a wide range of different contexts.

Recently, monadic logic of ordered abelian groups [19] and abelian modal logic $\mathbf{K}(\mathbf{A})$ [10] are introduced by G. Metcalfe and co-authors. Monadic logic of ordered abelian groups serves as a modal counterpart of the one-variable fragment of a (monadic) first-order real-valued logic. Propositional connectives are interpreted as the usual lattice and group operations over the real numbers in abelian modal logic $\mathbf{K}(\mathbf{A})$.

Abelian modal logic $\mathbf{K}(\mathbf{A})$ is the minimal modal extension of the abelian logic $\mathbf{A}$. Abelian logic $\mathbf{A}$ is the logic of lattice-ordered abelian groups, introduced independently by Meyer and Slaney [20] as a relevance logic, and Casari [4] as a comparative logic. In both settings, $\mathbf{A}$ was defined via axiom systems that are complete with respect to validity in the variety of lattice-ordered abelian groups.

As mentioned in [19], there are several advantages to focusing on modal extensions of Abelian logic, including that the language is rich enough to interpret other logics (e.g., modal extensions of Lukasiewicz logic), the semantics are based directly on structures studied in algebra and computer science, and the logics are naturally separated into the group and lattice fragments.

In [17], two embeddings of Łukasiewicz logic into Meyer and Slaney's Abelian logic and analytic proof systems for abelian logic are presented. In [10], a tableau calculus for the full logic $\mathbf{K}(\mathbf{A})$ and a sequent calculus for the modal-multiplicative fragment of $\mathbf{K}(\mathbf{A})$ as first steps towards addressing the corresponding (much more challenging) problems for the full logic, and complexity result are obtained.

The first main contribution of this work is to provide an axiomatization and algebraic semantics for the full logic $\mathbf{K}(\mathbf{A})$, which is addressed as an open question in the concluding remarks of [10]. The second aim is to develop a hypersequent calculus for the full logic $\mathbf{K}(\mathbf{A})$.

A real-valued modal logic, called linear abelian modal logic $\mathbf{LK}(\mathbf{A})$, as an extension of the minimal normal modal logic $\mathbf{K}(\mathbf{A})$ is introduced. An axiom system and also algebraic semantics for $\mathbf{LK}(\mathbf{A})$ are presented. Indeed, $\mathbf{LK}(\mathbf{A})$ is an extension of $\mathbf{K}(\mathbf{A})$ with the modal axiom schemas:

$\Box(\varphi \vee \psi) \to (\Box\varphi \vee \Box\psi)$ and $(\Box\varphi \wedge \Box\psi) \to \Box(\varphi \wedge \psi)$. The converse of the these axioms, i.e., $(\Box\varphi \vee \Box\psi) \to \Box(\varphi \vee \psi)$ and $(\Box\varphi \wedge \Box\psi) \to \Box(\varphi \wedge \psi)$ is derivable in $\mathbf{LK}(\mathbf{A})$. Thus, the modal operator $\Box$ distributes over the both operators $\vee$ and $\wedge$ with an equivalence. It is well known that usually, necessity doesn't distribute over disjunction with an equivalence in the modal logic. So, it may be interesting to study logics like $\mathbf{LK}(\mathbf{A})$ in which necessity distributes over disjunction with an equivalence.

Moreover, completeness of the axiom system with respect to both corresponding appropriate algebras and linearly ordered algebras with a lattice-ordered abelian groups reduct, using methods of abstract algebraic logic is investigated. A hypersequent calculus called **HLK(A)** for $\mathbf{LK}(\mathbf{A})$, extending the sequent calculus for the modal-multiplicative fragment of $\mathbf{K}(\mathbf{A})$ (introduced in [10]) is presented. Finally, the cut-elimination theorem and the correspondence between the hypersequent calculus and the axiomatization are established.

The paper is structured as follows. In the next section, syntax and semantics of Linear Abelian Modal Logic are introduced. Then, in Section 3 the completeness theorem with respect to both appropriate algebras and linearly ordered algebras is proved. The cut-elimination theorem as well as the correspondence between the hypersequent calculus and the axiomatization are investigated in Section 4. Finally, Section 5 concludes the paper.

2. Linear abelian modal logic

In this section, we introduce a many-valued modal logic, namely linear abelian modal logic $\mathbf{LK}(\mathbf{A})$ as an extension of the minimal normal modal logic $\mathbf{K}(\mathbf{A})$ extending Abelian logic $\mathbf{A}$, the logic of lattice-ordered abelian groups. We provide an axiom system and also algebraic and Kripke semantics for $\mathbf{LK}(\mathbf{A})$. Finally, we establish a connection between algebraic and Kripke semantics.

2.1. Axiomatizations

The language $\mathcal{L}_A^{\Box}$ of linear abelian modal logic $\mathbf{LK}(\mathbf{A})$ is consisting of the binary connective $\wedge, \vee, \to$ and unary connective $\Box$. The formula of $\mathbf{LK}(\mathbf{A})$ is defined inductively by

$$\varphi := p \,|\, \varphi \wedge \psi \,|\, \varphi \vee \psi \,|\, \varphi \to \psi \,|\, \Box\varphi,$$

where p is a propositional variable. To define further connectives, let

$$\bar{0} := p \to p, \quad \neg\varphi := \varphi \to \bar{0}, \quad \varphi + \psi := \neg\varphi \to \psi, \quad \Diamond\varphi := \neg\Box\neg\varphi,$$

and $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$. We also define $0\varphi := \bar{0}$ and $(n+1)\varphi := \varphi + (n\varphi)$ for each $n \in \mathbb{N}$. Let us also denote by Fm the set of formulas of $\mathcal{L}_A^\Box$ over a countably infinite set of variables. An axiomatization of the minimal normal modal logic $\mathbf{K}(\mathbf{A})$ is presented in Table 1. An axiom system of

Table 1. An Axiom System for Abelian Modal Logic $\mathbf{K}(\mathbf{A})$

(B)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(I)	$\varphi \to \varphi$
(C)	$(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$
(A)	$((\varphi \to \psi) \to \psi) \to \varphi$
(+1)	$\varphi \to (\psi \to \varphi + \psi)$
(+2)	$(\varphi \to (\psi \to \chi)) \to ((\varphi + \psi) \to \chi)$
($\bar{0}$1)	$\bar{0}$
($\bar{0}$2)	$\varphi \to (\bar{0} \to \varphi)$
($\wedge$1)	$(\varphi \wedge \psi) \to \varphi$
($\wedge$2)	$(\varphi \wedge \psi) \to \psi$
($\wedge$3)	$((\varphi \to \psi) \wedge (\varphi \to \chi)) \to (\varphi \to (\psi \wedge \chi))$
($\vee$1)	$\varphi \to (\varphi \vee \psi)$
($\vee$2)	$\psi \to (\varphi \vee \psi)$
($\vee$3)	$((\varphi \to \chi) \wedge (\psi \to \chi)) \to ((\varphi \vee \psi) \to \chi)$
(K)	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
($\mathrm{D_n}$)	$\Box(n\varphi) \to n\Box\varphi \quad (n \geq 2)$
	$\dfrac{\varphi \qquad \varphi \to \psi}{\psi}$ (mp) $\qquad \dfrac{\varphi}{\Box\varphi}$ (nec) $\qquad \dfrac{\varphi \qquad \psi}{\varphi \wedge \psi}$ (adj)

linear abelian modal logic $\mathbf{LK}(\mathbf{A})$ is defined over $\mathcal{L}_A^\Box$ by extending $\mathbf{K}(\mathbf{A})$ with the following modal axiom schemas:

$$(\vee_\Box) \quad \Box(\varphi \vee \psi) \to (\Box\varphi \vee \Box\psi),$$
$$(\wedge_\Box) \quad (\Box\varphi \wedge \Box\psi) \to \Box(\varphi \wedge \psi).$$

For a formula $\varphi \in \mathrm{Fm}$, we write $\vdash_{\mathbf{LK(A)}} \varphi$ if there exists a $\mathbf{LK}(\mathbf{A})$-derivation of φ, defined as usual as a finite sequence of $\mathcal{L}_A^\Box$-formulas that ends with φ and is constructed inductively using the axioms and rules of $\mathbf{LK}(\mathbf{A})$.

Proposition 2.1. For any $\varphi, \psi \in \mathrm{Fm}$,

(i) $\vdash_{\mathbf{LK(A)}} (\Box\varphi \vee \Box\psi) \to \Box(\varphi \vee \psi)$,

(ii) $\vdash_{\mathbf{LK(A)}} \Box(\varphi \wedge \psi) \to (\Box\varphi \wedge \Box\psi)$,

(iii) $\vdash_{\mathbf{LK(A)}} n\Box\varphi \to \Box(n\varphi) \quad (n \geq 2)$.

Proof: Derivation for (i) is obtained, using the axiom schemas $(\vee 1), (\mathrm{K})$, and $(\vee 3)$, and also rules $(\mathrm{nec}), (\mathrm{mp})$ and (adj) as follows:

1. $\vdash_{\mathbf{LK(A)}} \varphi \to (\varphi \vee \psi) \qquad (\vee 1)$
2. $\vdash_{\mathbf{LK(A)}} \Box(\varphi \to (\varphi \vee \psi)) \qquad (\mathrm{nec})$
3. $\vdash_{\mathbf{LK(A)}} \Box(\varphi \to (\varphi \vee \psi)) \to (\Box\varphi \to \Box(\varphi \vee \psi)) \qquad (\mathrm{K})$
4. $\vdash_{\mathbf{LK(A)}} \Box\varphi \to \Box(\varphi \vee \psi) \qquad (2, 3 \text{ and } (mp))$
5. $\vdash_{\mathbf{LK(A)}} \Box\psi \to \Box(\varphi \vee \psi) \qquad (\text{similarly})$
6. $\vdash_{\mathbf{LK(A)}} (\Box\varphi \to \Box(\varphi \vee \psi)) \wedge (\Box\psi \to \Box(\varphi \vee \psi)) \qquad (4, 5 \text{ and } (\mathrm{adj}))$
7. $\vdash_{\mathbf{LK(A)}} (\Box\varphi \to \Box(\varphi \vee \psi)) \wedge (\Box\psi \to \Box(\varphi \vee \psi)) \to ((\Box\varphi \vee \Box\psi) \to (\Box(\varphi \vee \psi))) \qquad (\vee 3)$
8. $\vdash_{\mathbf{LK(A)}} (\Box\varphi \vee \Box\psi) \to (\Box(\varphi \vee \psi)) \qquad (6, 7 \text{ and } (\mathrm{mp})$

Derivation for (ii) is obtained, similar to the derivation of (i), using the axiom schemas $(\wedge 1), (\mathrm{K})$ and $(\wedge 3)$, and also rules $(\mathrm{nec}), (\mathrm{mp})$ and (adj), and is omitted here. For derivation of (iii), observe that $n\Box\varphi \to \Box(n\varphi)$ is derivable in $\mathbf{LK(A)}$ for $n \geq 2$ using (nec) and (mp) together with the axioms of $\mathbf{LK(A)}$. For instance, $(\Box\varphi + \Box\varphi) \to \Box(\varphi + \varphi)$ is derivable as follows:

1. $\vdash_{\mathbf{LK(A)}} \varphi \to (\varphi \to (\varphi + \varphi)) \qquad (+1)$
2. $\vdash_{\mathbf{LK(A)}} \Box(\varphi \to (\varphi \to (\varphi + \varphi))) \qquad (\mathrm{nec})$
3. $\vdash_{\mathbf{LK(A)}} \Box(\varphi \to (\varphi \to (\varphi + \varphi))) \to (\Box\varphi \to \Box(\varphi \to (\varphi + \varphi))) \qquad (\mathrm{K})$

4. $\vdash_{\mathbf{LK(A)}} \Box\varphi \to \Box(\varphi \to (\varphi + \varphi))$ (2, 3 and (mp))

5. $\vdash_{\mathbf{LK(A)}} \Box(\varphi \to (\varphi + \varphi)) \to (\Box\varphi \to \Box(\varphi + \varphi))$ (K)

6. $\vdash_{\mathbf{LK(A)}} \Box\varphi \to (\Box\varphi \to \Box(\varphi + \varphi))$ ((B), 4, 5 and (mp))

7. $\vdash_{\mathbf{LK(A)}} (\Box\varphi \to (\Box\varphi \to \Box(\varphi+\varphi))) \to (\Box\varphi+\Box\varphi \to \Box(\varphi+\varphi))$ (+2)

8. $\vdash_{\mathbf{LK(A)}} \Box\varphi + \Box\varphi \to \Box(\varphi + \varphi)$ (6, 7 and (mp)) $\Box$

2.2. Semantics

In this subsection, algebraic semantics for $\mathbf{LK}(\mathbf{A})$ are presented. Appropriate class of algebras for **LK(A)** is defined over lattice-ordered abelian groups.

Definition 2.2. A lattice-ordered abelian group (abelian ℓ-group for short) is an algebraic structure $(A, \wedge, \vee, +, \neg, \bar{0})$ such that $(A, +, \neg, \bar{0})$ is an abelian group, $(A, \wedge, \vee)$ is a lattice, and $a+(b\vee c) = (a+b)\vee(a+c)$ for all $a, b, c \in A$. In addition, we define $a \to b = \neg a + b$, and $a \leq b$ iff $a \vee b = b$.

Well-known examples of abelian ℓ-groups are

$$\begin{aligned}
&\text{the integers } \mathcal{Z} = (\mathbb{Z}, \min, \max, +, -, 0),\\
&\text{the rationals } \mathcal{Q} = (\mathbb{Q}, \min, \max, +, -, 0),\\
&\text{and the reals } \mathcal{R} = (\mathbb{R}, \min, \max, +, -, 0).
\end{aligned}$$

In fact, any of them generates the variety of Abelian ℓ-groups (see [18] for more details).

Below we introduce algebras for the logic defined in the previous section, the idea being to consider particular classes of residuated lattices where the modal operator is interpreted by a special unary operator I on the corresponding algebras.

Definition 2.3 (**LK(A)**-algebra). An **LK(A)**-algebra is an algebra $\mathcal{A} = (A, \wedge, \vee, +, \neg, \bar{0}, I)$, where the reduct $(A, \wedge, \vee, +, \neg, \bar{0})$ is an abelian ℓ-group and I is an unary operation satisfying:

1. $I(x \to y) \leq I(x) \to I(y)$,
2. $I(x \vee y) = I(x) \vee I(y)$,
3. $I(x \wedge y) = I(x) \wedge I(y)$,

4. $I(x + x) = I(x) + I(x)$,
5. $I(\bar{0}) = \bar{0}$.

An $\mathcal{A}$-valuation is a function $V : \mathrm{Fm} \to A$ satisfying $V(\varphi \star \psi) = V(\varphi) \star V(\psi)$ for $\star \in \{\wedge, \vee, \to, +\}$, and $V(\Box\varphi) = I(V(\varphi))$. Formula φ is $\mathcal{A}$-valid if $V(\varphi) \geq \bar{0}$ for each $\mathcal{A}$-valuation V. We write $\models_{\mathbf{LK(A)}} \varphi$ iff φ is valid in all $\mathbf{LK(A)}$-algebras.

Example 2.4. Consider the real number structure $\mathcal{R} = (\mathbb{R}, \min, \max, +, -, 0, I)$, where I is defined as follows:

$$I : \mathbb{R} \longrightarrow \mathbb{R}$$
$$I(x) = \min\{x, 0\},$$

One can easily prove that this structure is an $\mathbf{LK(A)}$-algebra. Note that $\min\{x + y, 0\} \neq \min\{x, 0\} + \min\{y, 0\}$ (consider, for example $x = 1$ and $y = -1$), i.e., $I(x + y) \neq I(x) + I(y)$. While, $\min\{x + x, 0\} = \min\{x, 0\} + \min\{x, 0\}$, i.e., $I(x + x) = I(x) + I(x)$.

3. Completeness

In this section, we will establish the completeness theorem with respect to the corresponding algebraic semantics proceeding in the standard way (see e.g [18, 5, 8]). Given $T \subseteq \mathrm{Fm}$, the Lindenbaum algebra is defined in the usual way as $\mathcal{A}_T = (A_T, \wedge_T, \vee_T, +_T, \neg_T, \bar{0}_T, I_T)$ where $A_T = \{[\varphi]_T : \varphi \in \mathrm{Fm}\}$, $[\varphi]_T = \{\psi \in \mathrm{Fm} : T \vdash_{\mathbf{LK(A)}} \varphi \leftrightarrow \psi\}$, $[\varphi]_T \star_T [\psi]_T = [\varphi \star \psi]_T$ for $\star \in \{+, \vee, \wedge\}$, $\neg_T[\varphi] = [\neg\varphi]_T$, $\bar{0}_T = [\bar{0}]_T$, and $I_T[\varphi]_T = [\Box\varphi]_T$. The next Lemma follows from various provabilities in $\mathbf{LK}(\mathbf{A})$ and the axioms.

Lemma 3.1. *$\mathcal{A}_T$ is an $\mathbf{LK(A)}$-algebra.*

To show that $\mathcal{A}_T$-validity corresponds to $\mathbf{LK}(\mathbf{A})$-derivability from T, we make use of a specially defined valuation for this algebra that maps each formula to its corresponding equivalence class.

Lemma 3.2. *For any $T \subseteq \mathrm{Fm}$ and $\varphi \in \mathrm{Fm}$:*

$$T \vdash_{\mathbf{LK(A)}} \varphi \quad \text{iff} \quad \bar{0} \leq V_T(\varphi),$$

where V_T is the $\mathcal{M}_T$-valuation defined by $V_T(p) = [p]_T$ for each propositional variable p.

Proof: We first prove that $V_T(\varphi) = [\varphi]_T$ for all formulas φ, by induction on the complexity of φ. The case where φ is a variable follows by definition. For the other cases, just note that for any connective $\star \in \{+, \vee, \wedge\}$ (using the induction hypothesis):

$$\begin{aligned} V_T(\varphi \star \psi) &= V_T(\varphi) \star V_T(\psi) \\ &= [\varphi]_T \star [\psi]_T \\ &= [\varphi \star \psi]_T \end{aligned}$$

For unary connective $\Box$, we have: $V_T(\Box\varphi) = I_T(V_T(\varphi)) = I_T([\varphi]_T) = [\Box\varphi]_T$. The result then follows because $[\bar{0}]_T \leq [\varphi]_T$ iff $T \vdash_{\mathbf{LK(A)}} \bar{0} \to \varphi$ iff $T \vdash_{\mathbf{LK(A)}} \varphi$. □

Theorem 3.3 (Completeness). *$T \models_{\mathbf{LK(A)}} \varphi$ iff $T \vdash_{\mathbf{LK(A)}} \varphi$.*

Proof: Soundness proceeds as usual by an induction on the height of a derivation of φ in $\mathbf{LK}(\mathbf{A})$, showing that each axiom is valid and each rule sound in all $\mathbf{LK(A)}$-algebras. For the reverse direction, assume that $T \nvdash_{\mathbf{LK(A)}} \varphi$. By the previous lemma, $V_T(\psi) \geq \bar{0}$ for each $\psi \in T$ where $V_T(\varphi) \ngeq \bar{0}$. So $T \nvDash \varphi$. □

We now turn our attention, following [6, 7, 8], next to completeness with respect to linearly ordered algebras. First, let us say that a congruence filter of an $\mathbf{LK(A)}$-algebra $\mathcal{A}$ is a set $F = \{x \in A : \exists y \leq x \ (y\theta\bar{0})\}$, for some congruence θ on $\mathcal{A}$. The next Lemma follows from the fact that the reduct of an $\mathbf{LK(A)}$-algebra is an abelian ℓ-group.

Lemma 3.4. *Let $\mathcal{A} = (A, \wedge, \vee, +, \neg, \bar{0}, I)$ be an $\mathbf{LK(A)}$-algebra and $a, b, c, d \in A$. If $a \leq b$ and $c \leq d$, then $a + c \leq b + d$.*

Corollary 3.5. Let $\mathcal{A} = (A, \wedge, \vee, +, \neg, \bar{0}, I)$ be an $\mathbf{LK(A)}$-algebra and $a, b \in A$. If $a, b \leq \bar{0}$, then $(a + b) \leq (a \vee b)$.

Proof: Let $a, b \leq \bar{0}$, then $a \vee b \leq \bar{0}$ and so, by Lemma 3.4, $(a \vee b) + (a \vee b) \leq a \vee b$ since $(a \vee b) \leq (a \vee b)$. Now, $a \leq a \vee b$ and $b \leq a \vee b$ follows that $a + b \leq (a \vee b) + (a \vee b) \leq a \vee b$. □

Lemma 3.6. *F is a congruence filter of an $\mathbf{LK(A)}$-algebra $\mathcal{A}$ iff (i) $\bar{0} \in F$ (ii) if $a \in F$ and $a \to b \in F$, then $b \in F$ (iii) if $a \in F$ then $I(a) \in F$.*

Proof: That a congruence filter must satisfy (i), is almost immediate. We check (ii) and (iii). If $a \in F$ and $a \to b \in F$, then there are $u, v \in A$

such that $u \leq a$, $v \leq a \to b$ and $u\theta a$ and $v\theta(a \to b)$. So, by Lemma 3.4, $u + v \leq a + (a \to b)$ i.e., $u + v \leq a + (\neg a + b)$. Therefore, by equations $\bar{0} + a = a$ and $\neg a + a = \bar{0}$ of the definition of abelian ℓ-group and the compatibility of congruence with $(+)$, we have $(u + v) \leq b$ and $(u + v)\theta b$. Thus, $b \in F$. If $a \in F$, then there is $u \in A$ such that $u \leq a$ and $u\theta a$. It follows that $I(u) \leq I(a)$ and $I(u)\theta I(a)$, since $u \leq a$ i.e., $u \vee a = a$ follows that $I(u \vee a) = I(a)$ so $I(u) \vee I(a) = I(a)$ i.e., $I(u) \leq I(a)$, and hence $I(a) \in F$. Conversely, let F be a subset of A that satisfies the conditions, and let θ be defined by $a\theta b$ iff $a \to b \in F$ and $b \to a \in F$. One can easily show that θ is a equivalence relation. Thus, we may define equivalence classes $[a]_F = \{b \mid a\theta b\}$. We prove that θ is compatible with the operations of **LK(A)**-algebras.

- θ is compatible with $(+)$: If $a\theta b$ and $c\theta d$, then $a \to b, b \to a \in F$ and $c \to d, d \to c \in F$, therefore $(a \to b)+(c \to d), (b \to a)+(d \to c) \in F$, as F is closed under $(+)$. It follows that $(\neg a + b) + (\neg c + d), (\neg b + a) + (\neg d + c) \in F$, and so $\neg(a + c) + (b + d), \neg(b + d) + (a + c) \in F$ i.e., $(a + c) \to (b + d), (b + d) \to (a + c) \in F$. Thus, $(a + c)\theta(b + d)$.

- θ is compatible with $(\vee)$: Since θ is an equivalence relation, we define equivalence classes $[a]_\theta = \{b \mid a\theta b\}$. Let A/θ_F be the set of all equivalence classes. One verifies that $(A/\theta_F, \cap, \cup, +_F, \neg_F, 0_F, I_F)$, where $\cap, \cup, +_F, \neg_F, 0_F, I_F$ are defined component-wise from the ones of $\mathcal{A}$, is an **LK(A)**-algebra. If $a\theta b$ and $c\theta d$, then $[a]_\theta = [b]_\theta$ and $[c]_\theta = [d]_\theta$. It follows that $[a]_\theta \cup [c]_\theta = [b]_\theta \cup [d]_\theta$, and so $[a \vee c]_\theta = [b \vee d]_\theta$. Therefore, $(a \vee c)\theta(b \vee d)$. The compatibility of θ with $(\wedge)$ is treated similarly.

- θ is compatible with $(\neg)$: If $a\theta b$, then $a \to b, b \to a \in F$, i.e., $\neg a + b, \neg b + a \in F$. Therefore, $\neg b + \neg(\neg a), \neg a + \neg(\neg b) \in F$, i.e., $\neg a \to \neg b, \neg b \to \neg a \in F$. Thus $\neg a\theta\neg b$.

- θ is compatible with (I): If $a\theta b$, then $a \to b, b \to a \in F$. Therefore $I(a \to b), I(b \to a) \in F$, as F is closed under I. It follows that $I(a) \to I(b), I(b) \to I(a) \in F$. Thus, $I(a)\theta I(b)$. □

Now, by imitating [6], we define $\mathrm{Fg}(a)$ be the smallest congruence filter containing a, and define inductively: $I_0(a) = a$ and $I_{n+1}(a) = I(I_n(a)) \wedge I_n(a)$ for an **LK(A)**-algebra $\mathcal{A}$ and $a \in A$. Note that $I_{n+1}(a) \leq I_n(a)$, thus, by induction, $I_n(a) \leq I_m(a)$ for $m \leq n$.

Lemma 3.7. *Let $\mathcal{A} = (A, \wedge, \vee, +, \neg, \bar{0}, I)$ be an* **LK(A)***-algebra and $a, b \in A$. If $a \leq b$, then $I_n(a) \leq I_n(b)$ for all $n \in \mathbb{N}$.*

Proof: We first observe that $a \leq b$ if and only if $I(a) \leq I(b)$:

$$a \leq b \,\text{iff}\, a \vee b = b \,\text{iff}\, I(a \vee b) = I(b) \,\text{iff}\, I(a) \vee I(b) = I(b) \,\text{iff}\, I(a) \leq I(b).$$

Let $a \leq b$, by induction on n we can easily prove $I_n(a) \leq I_n(b)$. For $n = 0$, obviously $I_0(a) \leq I_0(b)$. Suppose $I_n(a) \leq I_n(b)$, then $I(I_n(a)) \leq I(I_n(b))$. It follows that $I(I_n(a)) \wedge I_n(a) \leq I(I_n(b)) \wedge I_n(b)$ i.e., $I_{n+1}(a) \leq I_{n+1}(b)$. □

Lemma 3.8. *Let $\mathcal{A} = (A, \wedge, \vee, +, \neg, \bar{0}, I)$ be an* **LK(A)***-algebra and $a, b \in A$. Then $I_n(a \vee b) = I_n(a) \vee I_n(b)$ for all $n \in \mathbb{N}$.*

Proof: First observe that by induction on n we can easily prove $I_n(a) \leq a$ for all $n \in \mathbb{N}$: For $n = 0$, $I_0(a) = a \leq a$. Suppose $I_n(a) \leq a$, then $I_{n+1}(a) = I(I_n(a)) \wedge I_n(a) \leq I_n(a) \leq a$. Suppose now that $I_n(a \vee b) = I_n(a) \vee I_n(b)$, then $I(I_n(a \vee b)) = I(I_n(a) \vee I_n(b))$, so $I(I_n(a \vee b)) = I(I_n(a)) \vee I(I_n(b))$. It follows that $I(I_n(a \vee b)) \wedge I_n(a \vee b) = (I(I_n(a)) \wedge I_n(a)) \vee (I(I_n(b)) \wedge I_n(b))$ i.e., $I_{n+1}(a \vee b) = I_{n+1}(a) \vee I_{n+1}(b)$. □

Lemma 3.9. *Let $\mathcal{A} = (A, \wedge, \vee, +, \neg, \bar{0}, I)$ be an* **LK(A)***-algebra and $a \in A$. Then*

$$\mathrm{Fg}(a) = \{x \in A \,|\, \exists n, m \in \mathbb{N}\, (mI_n(a) \leq x)\},$$

where $1I_n(a) = I_n(a)$ and $(n + 1)I_n(a) = I_n(a) + nI_n(a)$.

Proof: Let $G = \{x \in A \,|\, \exists n, m \in \mathbb{N}\, (mI_n(a) \leq x)\}$. We show that $G \subseteq \mathrm{Fg}(a)$; suppose $x \in G$, then there is $n, m \in \mathbb{N}$ such that $mI_n(a) \leq x$. It follows that $x \in \mathrm{Fg}(a)$ because $a \in \mathrm{Fg}(a)$ and $\mathrm{Fg}(a)$ is closed upwards and closed under $I, +$, and $\wedge$. For the opposite direction, since $a \in G$, it suffices to prove that G is a congruence filter. It is trivial that $\bar{0} \in G$. If $x, x \to y \in G$, then there are $m_1, n_1, m_2, n_2 \in \mathbb{N}$ such that $m_1(I_{n_1}(a)) \leq x$ and $m_2(I_{n_2}(a)) \leq x \to y$. But then easily $(m_1 + m_2)(I_{n_1+n_2}(a)) \leq x + (x \to y) = x + (\neg x + y) = y$, and hence $y \in G$. Finally, G is closed under I. If $x \in G$, then there are an m, n such that $m(I_n(a)) \leq x$. It follows that $mI_{n+1}(a) \leq mI(I_n(a)) = I(mI_n(a)) \leq I(x)$, and $I(x) \in G$. Thus, by Lemma 3.6, G is a filter and $a \in G$. It follows that $\mathrm{Fg}(a) \subseteq G$. □

Theorem 3.10. *Every subdirectly irreducible* **LK(A)***-algebra $\mathcal{A}$ is linearly ordered.*

Proof: Assume for a contradiction that $\mathcal{A}$ is a subdirectly irreducible **LK(A)**-algebra with minimum non-trivial filter F and elements a, b such that $a \not\leq b$ and $b \not\leq a$. Then, both $\mathrm{Fg}(a \to b)$ and $\mathrm{Fg}(b \to a)$ are non-trivial filters; hence they both contain F. Let $c \in F$ with $c < \bar{0}$. Then, there are $m_1, n_1, m_2, n_2 \in \mathbb{N}$ such that $I_{n_1}(m_1(a \to b)) = m_1 I_{n_1}(a \to b) \leq c < \bar{0}$ and $I_{n_2}(m_2(b \to a)) = m_2 I_{n_2}(b \to a) \leq c < \bar{0}$. It follows, by Lemma 3.7, that $m_1(a \to b) < \bar{0}$ and $m_2(b \to a) < \bar{0}$. Let $m = \max\{m_1, m_2\}$, then $m(a \to b) < \bar{0}$ and $m(b \to a) < \bar{0}$. Therefore, by Lemma 3.5, $m(a \to b) + m(b \to a) \leq m(a \to b) \vee m(b \to a)$. Then, again by Lemma 3.7, $I_n(m(a \to b) + m(b \to a)) \leq I_n(m(a \to b) \vee m(b \to a))$ for all n. Now, letting $n = \max\{n_1, n_2\}$, we have the following contradiction:

$$\begin{aligned}
\bar{0} = I_n(\bar{0}) &= I_n((m(\neg a) + mb) + (m(\neg b) + ma)) \\
&= I_n(m(a \to b) + m(b \to a)) \\
&\leq I_n(m(a \to b) \vee m(b \to a)) \\
&= I_n(m(a \to b)) \vee I_n(m(b \to a)) \\
&= mI_n(a \to b) \vee mI_n(b \to a) \\
&\leq m_1 I_n(a \to b) \vee m_2 I_n(b \to a) \\
&\leq m_1 I_{n_1}(a \to b) \vee m_2 I_{n_2}(b \to a) \\
&\leq c \vee c = c < \bar{0}.
\end{aligned}$$

□

Hence, making use of Birkhoff's subdirect representation theorem, we have the following Corollary.

Corollary 3.11. Every **LK(A)**-algebra is isomorphic to a subdirect product of a family of linearly ordered **LK(A)**-algebras.

4. A hypersequent calculus for LK(A)

In this section, a proof system for **LK(A)**, called **HLK(A)** in the framework of hypersequent, is presented. Hypersequent is a generalization of sequents introduced independently by Avron [1] and Pottinger [23]. **HLK(A)** extends the sequent calculus for the modal multiplicative fragment of **K**(**A**) [10]. Then, the cut elimination theorem is established and finally it is shown that the axiomatic and hypersequent presentations really characterize the same logics.

Since in this section we will often be dealing with quite complicated structures, let us recall some notational conveniences:

- φ, ψ, χ and $\Gamma, \Delta, \Pi, \Sigma$ (sometimes with primes or numerical subscripts) denote arbitrary formulas and finite multisets of formulas, respectively. The multiset union $\Gamma \uplus \Delta$ is often denoted by Γ, Δ. In addition, $n\Gamma$ or sometimes Γ^n is used for $\Gamma, \ldots, \Gamma$ (n times), and $\Box\Gamma$ for $\{\Box\varphi : \varphi \in \Gamma\}$.

- a sequent is an ordered pair of finite multisets of formulas Γ and Δ, written $\Gamma \Rightarrow \Delta$. A hypersequent is a finite multiset of ordinary sequents, written $\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$.

- $G, H, \mathcal{G}, \mathcal{H}$ (possibly with primes) denote hypersequents, $[\mathcal{G}_i]_{i=1}^n$ denotes the hypersequent $\mathcal{G}_1 \mid \ldots \mid \mathcal{G}_n$, and also $\{\mathcal{G}_i\}_{i=1}^n$ denotes a set of hypersequents $\mathcal{G}_1, \ldots, \mathcal{G}_n$ (perhaps the premises of some rule application).

The intended interpretation of the hypersequent $H = \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$ is defined as follows:

$$\mathcal{I}(H) = (\sum \Gamma_1 \to \sum \Delta_1) \vee \cdots \vee (\sum \Gamma_n \to \sum \Delta_n),$$

where $\Sigma\{\varphi_1, \ldots, \varphi_m\} := \varphi_1 + \ldots + \varphi_m$ and $\Sigma\emptyset = \bar{0}$. Axioms and rules of hypersequent calculus **HLK(A)** is presented in Table 2. For a hypersequent H, we write $\vdash_{\mathbf{HLK(A)}} H$ if there is a **HLK(A)**-derivation of H.
The following rules for other connectives are **HLK(A)**-derivable:

$$\frac{\Gamma, \varphi, \psi \Rightarrow \Delta \mid H}{\Gamma, \varphi + \psi \Rightarrow \Delta \mid H}\,(L+) \qquad \frac{\Gamma \Rightarrow \varphi, \psi, \Delta \mid H}{\Gamma \Rightarrow \varphi + \psi, \Delta \mid H}\,(R+)$$

$$\frac{\Gamma \Rightarrow \varphi, \Delta \mid H}{\Gamma, \neg\varphi \Rightarrow \Delta \mid H}\,(L\neg) \qquad \frac{\Gamma, \varphi \Rightarrow \Delta \mid H}{\Gamma \Rightarrow \neg\varphi, \Delta \mid H}\,(R\neg)$$

$$\frac{\Gamma \Rightarrow \Delta \mid H}{\Gamma, \bar{0} \Rightarrow \Delta \mid H}\,(L\bar{0}) \qquad \frac{\Gamma \Rightarrow \Delta \mid H}{\Gamma \Rightarrow \bar{0}, \Delta \mid H}\,(R\bar{0})$$

Example 4.1. Below we provide an example of a **HLK(A)**-derivation to get more familiar with this calculus.

Table 2. Hypersequent Calculus **HLK(A)**

Axiom:

$$\dfrac{}{\Gamma \Rightarrow \Gamma \mid H}\ (\mathrm{AX})$$

Logical rules:

$$\dfrac{\Gamma, \psi \Rightarrow \varphi, \Delta \mid H}{\Gamma, \varphi \to \psi \Rightarrow \Delta \mid H}\ (L \to) \qquad \dfrac{\Gamma, \varphi \Rightarrow \psi, \Delta \mid H}{\Gamma \Rightarrow \Delta, \varphi \to \psi \mid H}\ (R \to)$$

$$\dfrac{\Gamma, \varphi \Rightarrow \Delta \mid \Gamma, \psi \Rightarrow \Delta \mid H}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta \mid H}\ (L\wedge) \qquad \dfrac{\Gamma \Rightarrow \varphi, \Delta \mid H \qquad \Gamma \Rightarrow \psi, \Delta \mid H}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi \mid H}\ (R\wedge)$$

$$\dfrac{\Gamma, \varphi \Rightarrow \Delta \mid H \qquad \Gamma, \psi \Rightarrow \Delta \mid H}{\Gamma, \varphi \vee \psi \Rightarrow \Delta \mid H}\ (L\vee) \qquad \dfrac{\Gamma \Rightarrow \varphi, \Delta \mid \Gamma \Rightarrow \psi, \Delta \mid H}{\Gamma \Rightarrow \Delta, \varphi \vee \psi \mid H}\ (R\vee)$$

Modal rule:

$$\dfrac{\Gamma \Rightarrow n\varphi \mid H}{\Box\Gamma \Rightarrow n\Box\varphi \mid H}\ (\Box_n)$$

Structural rules:

$$\dfrac{\Gamma, \varphi \Rightarrow \Delta \mid H \qquad \Pi \Rightarrow \varphi, \Sigma \mid H}{\Gamma, \Pi \Rightarrow \Sigma, \Delta \mid H}\ (\mathrm{Cut}) \qquad \dfrac{\Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta \mid H}{\Gamma \Rightarrow \Delta \mid H}\ (\mathrm{EC})$$

$$\dfrac{\Gamma \Rightarrow \Delta \mid H \qquad \Pi \Rightarrow \Sigma \mid H}{\Gamma, \Pi \Rightarrow \Delta, \Sigma \mid H}\ (\mathrm{Mix}) \qquad \dfrac{\Gamma, \Pi \Rightarrow \Sigma, \Delta \mid H}{\Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma \mid H}\ (\mathrm{Split})$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \varphi \mid \psi \Rightarrow \varphi \wedge \psi}\ (\mathrm{AX}) \qquad \dfrac{\dfrac{\dfrac{}{\varphi, \psi \Rightarrow \varphi, \psi}\ (\mathrm{AX})}{\varphi \Rightarrow \psi \mid \psi \Rightarrow \varphi}\ (\mathrm{Split}) \qquad \dfrac{}{\varphi \Rightarrow \psi \mid \psi \Rightarrow \psi}\ (\mathrm{AX})}{\varphi \Rightarrow \psi \mid \psi \Rightarrow \varphi \wedge \psi}\ (R\wedge)}{\varphi \Rightarrow \varphi \wedge \psi \mid \psi \Rightarrow \varphi \wedge \psi}\ (R\wedge)}{\varphi \Rightarrow \varphi \wedge \psi \mid \Box\psi \Rightarrow \Box(\varphi \wedge \psi)}\ (\Box_1)}{\Box\varphi \Rightarrow \Box(\varphi \wedge \psi) \mid \Box\psi \Rightarrow \Box(\varphi \wedge \psi)}\ (\Box_1)}{\Box\varphi \wedge \Box\psi \Rightarrow \Box(\varphi \wedge \psi)}\ (L\wedge)}{\Rightarrow \Box\varphi \wedge \Box\psi \to \Box(\varphi \wedge \psi)}\ (R \to)$$

We now consider a more complicated family of rules, indexed by $k \in \mathbb{N} \setminus \{0\}$ and $n \in \mathbb{N}$, that is inspired by Denisa Diaconescu et al [10] and will be very useful in subsequent cut-elimination and completeness proofs:

$$\frac{\Gamma_0 \Rightarrow \ \mid H \quad \Gamma_1 \Rightarrow k\varphi_1 \mid H \quad \cdots \quad \Gamma_n \Rightarrow k\varphi_n \mid H}{\Delta, \Box\Gamma \Rightarrow \Box\varphi_1, \ldots, \Box\varphi_n, \Delta \mid H}\,(\Box_{k,n}) \quad \text{where } k\Gamma{=}\Gamma_0,\ldots,\Gamma_n$$

Critically for our later considerations, $\Box_{k,n}$ is **HLK(A)**-derivable for all $k \in \mathbb{N} \setminus \{0\}, n \in \mathbb{N}$ (for $k = 1$, omitting the applications of (EC) and (Split)):

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(AX)}
\UnaryInfC{$\Delta \Rightarrow \Delta \mid H$}
\AxiomC{$\Gamma_0 \Rightarrow \ \mid H$}
\RightLabel{$(\Box_0)$}
\UnaryInfC{$\Box\Gamma_0 \Rightarrow \ \mid H$}
\AxiomC{$\Gamma_1 \Rightarrow k\varphi_1 \mid H$}
\RightLabel{$(\Box_k)$}
\UnaryInfC{$\Box\Gamma_1 \Rightarrow k\Box\varphi_1 \mid H$}
\AxiomC{$\Gamma_n \Rightarrow k\varphi_n \mid H$}
\RightLabel{$(\Box_k)$}
\UnaryInfC{$\Box\Gamma_n \Rightarrow k\Box\varphi_n \mid H$}
\RightLabel{(Mix)}
\UnaryInfC{$\vdots$}
\RightLabel{(Mix)}
\BinaryInfC{$\Box(\Gamma_1 \ldots, \Gamma_n) \Rightarrow k\Box\varphi_1, \ldots, k\Box\varphi_n \mid H$}
\RightLabel{(Mix)}
\BinaryInfC{$\Box(\Gamma_0, \Gamma_1 \ldots, \Gamma_n) \Rightarrow k\Box\varphi_1, \ldots, k\Box\varphi_n \mid H$}
\RightLabel{(Split), (EC)}
\UnaryInfC{$\Box\Gamma \Rightarrow \Box\varphi_1, \ldots, \Box\varphi_n \mid H$}
\RightLabel{(Mix)}
\BinaryInfC{$\Delta, \Box\Gamma \Rightarrow \Box\varphi_1, \ldots, \Box\varphi_n, \Delta \mid H$}
\end{prooftree}
$$

In order to prove the cut elimination theorem, we begin by showing that every cut-free **HLK(A)**-derivation can be transformed into a derivation in a restricted calculus $\mathbf{HLK}(\mathbf{A})^{\mathrm{r}}$ consisting only of the rules (AX), logical rules, $(\Box_{k,n})(k \in \mathbb{N} \setminus \{0\}, n \in \mathbb{N})$, (Split) and (EC).

LEMMA 4.2. *The following rules are height-preserving* $\mathbf{HLK}(\mathbf{A})^{\mathrm{r}}$*-admissible.*

$$\frac{H}{\Gamma \Rightarrow \Delta \mid H}\,(\mathrm{EW}) \qquad \frac{\Gamma \Rightarrow \Delta \mid H}{\Gamma, \Pi \Rightarrow \Delta, \Pi \mid H}\,(\mathrm{IW})$$

PROOF: By induction on the height of the premises. □

LEMMA 4.3. *All logical rules are* $\mathbf{HLK}(\mathbf{A})^{\mathrm{r}}$*-invertible.*

PROOF: To cope with multiple occurrences of formulas, we will need to show the invertibility of more general rules. To show that $(L \to)$ is $\mathbf{HLK}(\mathbf{A})^{\mathrm{r}}$-invertible, we prove that the following rule is admissible in $\mathbf{HLK}(\mathbf{A})^{\mathrm{r}}$

$$\frac{[\Gamma_i, [\varphi \to \psi]^{\lambda_i} \Rightarrow \Delta_i]_{i=1}^{n} \mid H}{[\Gamma_i, [\psi]^{\lambda_i} \Rightarrow [\varphi]^{\lambda_i}, \Delta_i]_{i=1}^{n} \mid H}$$

proceeding by induction on the height of a $\mathbf{HLK(A)}^{\mathrm{r}}$-derivation of $[\Gamma_i, [\varphi \to \psi]^{\lambda_i} \Rightarrow \Delta_i]_{i=1}^n \,|\, H$. If $\lambda_1 = \dots = \lambda_n = 0$, then the result follows immediately, so let us assume without loss of generality that $\lambda_1 \geq 1$. Then for the base case, $\Delta_j = \Gamma_j \uplus [\varphi \to \psi]^{\lambda_j}$ for $j \in \{1, \dots, n\}$, and it suffices to observe that $\vdash_{\mathbf{HLK(A)}^{\mathrm{r}}} \Gamma_j, [\psi]^{\lambda_j} \Rightarrow [\varphi]^{\lambda_j}, \Gamma_j, [\varphi \to \psi]^{\lambda_j} \,|\, \mathcal{H}$. For the inductive step, we observe that when the last rule applied is not $(\Box_{k,n})$, the claim follows immediately by applying the induction hypothesis, where necessary twice, and the relevant rule. Suppose now that the last rule applied is $(\Box_{k,n})$, so $[\varphi \to \psi]^{\lambda_j}$ must occur also on the right of the sequent as follows:

$$\frac{\Gamma'_0 \Rightarrow \,|\, \mathcal{H} \quad \Gamma'_1 \Rightarrow k[\chi_1] \,|\, \mathcal{H} \quad \cdots \quad \Gamma'_n \Rightarrow k[\chi_n] \,|\, \mathcal{H}}{\Omega_j, [\varphi \to \psi]^{\lambda_j}, \Box\Gamma' \Rightarrow \Box\chi_1, \dots, \Box\chi_n, [\varphi \to \psi]^{\lambda_j}, \Omega_j \,|\, \mathcal{H}} \; (\Box_{k,n})$$

where $\Gamma_j = \Omega_j \uplus [\varphi \to \psi]^{\lambda_j} \uplus \Box\Gamma'$ and $\Delta_j = \Box\chi_1 \uplus \dots \uplus \Box\chi_n \uplus [\varphi \to \psi]^{\lambda_j} \uplus \Omega_j$, and also $k\Gamma' = \Gamma'_0 \uplus \Gamma'_1 \uplus \dots \uplus \Gamma'_n$. Then the claim follows by first applying the induction hypothesis and then applying the rule $(\Box_{k,n})$ and $(R \to)$ (λ_j times) as follows: where $\mathcal{G}$ is obtained from $\mathcal{H}$ by applying induction hypothesis.

$$\dfrac{\dfrac{\Gamma'_0 \Rightarrow \,|\, \mathcal{G} \quad \Gamma'_1 \Rightarrow k\chi_1 \,|\, \mathcal{G} \quad \cdots \quad \Gamma'_n \Rightarrow k\chi_n \,|\, \mathcal{G}}{\Omega_j, [\varphi]^{\lambda_j}, [\psi]^{\lambda_j}, \Box\Gamma' \Rightarrow \Box\chi_1, \dots, \Box\chi_n, [\varphi]^{\lambda_j}, [\psi]^{\lambda_j}, \Omega_j \,|\, \mathcal{G}} \; (\Box_{k,n})}{\Omega_j, [\psi]^{\lambda_j}, \Box\Gamma' \Rightarrow \Box\chi_1, \dots, \Box\chi_n, [\varphi]^{\lambda_j}, [\varphi \to \psi]^{\lambda_j}, \Omega_j \,|\, \mathcal{G}} \; (R \to)(\lambda_j \text{ times})$$

The proof of $\mathbf{HLK(A)}^{\mathrm{r}}$-invertibility of the rule $(R \to)$ is very similar. To show that $(L\wedge)$ is $\mathbf{HLK(A)}^{\mathrm{r}}$-invertible, we prove, more generally, that the following rule is admissible in $\mathbf{HLK(A)}^{\mathrm{r}}$

$$\frac{[\Gamma_i, [\varphi \wedge \psi]^{\lambda_i} \Rightarrow \Delta_i]_{i=1}^n \,|\, H}{[\Gamma_i, [\varphi]^{\lambda_i} \Rightarrow \Delta_i \,|\, \Gamma_i, [\psi]^{\lambda_i} \Rightarrow \Delta_i]_{i=1}^n \,|\, H}$$

proceeding by induction on the height of a $\mathbf{HLK(A)}^{\mathrm{r}}$-derivation of $[\Gamma_i, [\varphi \wedge \psi]^{\lambda_i} \Rightarrow \Delta_i]_{i=1}^n \,|\, H$. If $\lambda_1 = \dots = \lambda_n = 0$, then the result follows immediately using (EC), so let us assume without loss of generality that $\lambda_1 \geq 1$. For the base case, $\Delta_j = \Gamma_j \uplus [\varphi \wedge \psi]^{\lambda_j}$ for $j \in \{1, \dots, n\}$ and it suffices to observe that $\Gamma_j, [\varphi]^{\lambda_j} \Rightarrow \Gamma_j, [\varphi \wedge \psi]^{\lambda_j} \mid \Gamma_j, [\psi]^{\lambda_j} \Rightarrow \Gamma_j, [\varphi \wedge \psi]^{\lambda_j} \mid \mathcal{G}$ is derivable. For example, suppose $\lambda_j = 1$ for $j = 1$, first we have the following derivation:

$$\dfrac{\dfrac{\dfrac{}{\Gamma, \Gamma, \varphi, \psi \Rightarrow \Gamma, \Gamma, \psi, \varphi \mid \mathcal{G}}\,(\mathrm{AX})}{\Gamma, \varphi \Rightarrow \Gamma, \psi \mid \Gamma, \psi \Rightarrow \Gamma, \varphi \mid \mathcal{G}}\,(\mathrm{Split}) \quad \dfrac{}{\Gamma, \varphi \Rightarrow \Gamma, \psi \mid \Gamma, \psi \Rightarrow \Gamma, \psi \mid \mathcal{G}}\,(\mathrm{AX})}{\Gamma, \varphi \Rightarrow \Gamma, \psi \mid \Gamma, \psi \Rightarrow \Gamma, \varphi \wedge \psi \mid \mathcal{G}.}\,(R\wedge)$$

Then, the conclusion is derived as follows:

$$\dfrac{\dfrac{}{\Gamma, \varphi \Rightarrow \Gamma, \varphi \mid \Gamma, \psi \Rightarrow \Gamma, \varphi \wedge \psi \mid \mathcal{G}}\,(\mathrm{AX}) \quad \Gamma, \varphi \Rightarrow \Gamma, \psi \mid \Gamma, \psi \Rightarrow \Gamma, \varphi \wedge \psi \mid \mathcal{G}}{\Gamma, \varphi \Rightarrow \Gamma, \varphi \wedge \psi \mid \Gamma, \psi \Rightarrow \Gamma, \varphi \wedge \psi \mid \mathcal{G}}\,(R\wedge)$$

For the inductive step, we observe that when the last rule applied is not $(\Box_{k,n})$, the claim follows immediately by applying the induction hypothesis, where necessary twice, and the relevant rule (see e.g. [18] Lemma 5.18 for more details). Suppose now that the last rule applied is $(\Box_{k,n})$, so $[\varphi \wedge \psi]^{\lambda_j}$ must occur also on the right of the sequent as follows:

$$\dfrac{\Gamma'_0 \Rightarrow \ \mid \mathcal{H} \quad \Gamma'_1 \Rightarrow k\chi_1 \mid \mathcal{H} \quad \cdots \quad \Gamma'_n \Rightarrow k\chi_n \mid \mathcal{H}}{\Omega, \Box\Gamma', [\varphi \wedge \psi]^{\lambda_j} \Rightarrow \Box\chi_1, \ldots, \Box\chi_n, \Omega, [\varphi \wedge \psi]^{\lambda_j} \mid \mathcal{H}}\,(\Box_{k,n})$$

where $\Gamma_j = \Omega \uplus \Box\Gamma'$, and $\Delta_j = \Omega \uplus \Box\chi_1 \uplus \cdots \uplus \Box\chi_n \uplus [\varphi \wedge \psi]^{\lambda_j}$ and also $k\Gamma' = \Gamma'_0 \uplus \ldots \uplus \Gamma'_n$. Then the conclusion is obtained by first applying the induction hypothesis to the premises and then applying $(\Box_{k,n})$, (EW), (Split) and $(R\wedge)$ as required. For example suppose that $\lambda_j = 1$, the claim is derived as follows:

$$\begin{array}{c}\mathcal{D}_1 \\ \dfrac{\dfrac{\Gamma'_0 \Rightarrow \ \mid \mathcal{G} \quad \Gamma'_1 \Rightarrow k\chi_1 \mid \mathcal{G} \quad \cdots \quad \Gamma'_n \Rightarrow k\chi_n \mid \mathcal{G}}{\Omega, \Box\Gamma', \varphi \Rightarrow \Box\chi_1, \ldots, \Box\chi_n, \Omega, \varphi \mid \mathcal{G}}\,(\Box_{k,n})}{\Omega, \Box\Gamma', \varphi \Rightarrow \Box\chi_1, \ldots, \Box\chi_n, \Omega, \varphi \mid \Omega, \Box\Gamma', \psi \Rightarrow \Box\chi_1, \ldots, \Box\chi_n, \Omega, \varphi \mid \mathcal{G}}\,(\mathrm{EW})\end{array}$$

where $\mathcal{G}$ is obtained from $\mathcal{H} = [\Gamma_i, [\varphi \wedge \psi]^{\lambda_i} \Rightarrow \Delta_i]_{i=2}^n \mid H$ by applying induction hypothesis. Similarly, we have

$$\begin{array}{c}\mathcal{D}_2 \\ \Omega, \Box\Gamma', \varphi \Rightarrow \Box\chi_1, \ldots, \Box\chi_n, \Omega, \varphi \mid \Omega, \Box\Gamma', \psi \Rightarrow \Box\chi_1, \ldots, \Box\chi_n, \Omega, \psi \mid \mathcal{G}.\end{array}$$

Then, by applying $(R\wedge)$ we have:

$$\dfrac{\mathcal{D}_1 \qquad \mathcal{D}_2}{\Omega, \Box\Gamma', \varphi \Rightarrow \Box\chi_1, \ldots, \Box\chi_n, \Omega, \varphi \mid \Omega, \Box\Gamma', \psi \Rightarrow \Box\chi_1, \ldots, \Box\chi_n, \Omega, \varphi \wedge \psi \mid \mathcal{G}}\,(R\wedge).$$

Now, by a similar argument, we have:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1' \\ 2\Gamma_0' \Rightarrow \ \mid \mathcal{G} \quad 2\Gamma_1' \Rightarrow 2k\chi_1 \mid \mathcal{G} \quad \cdots \quad 2\Gamma_n' \Rightarrow 2k\chi_n \mid \mathcal{G}\end{array}}{2\Omega, 2\Box\Gamma', \varphi, \psi \Rightarrow 2\Box\chi_1, \ldots, 2\Box\chi_n, 2\Omega, \varphi, \psi \mid \mathcal{G}}\,(\Box_{2k,2n})}{\Omega, \Box\Gamma', \varphi \Rightarrow \Box\chi_1, \ldots, \Box\chi_n, \Omega, \psi \mid \Omega, \Box\Gamma', \psi \Rightarrow \Box\chi_1, \ldots, \Box\chi_n, \Omega, \varphi \mid \mathcal{G}}\,(\text{Split}).$$

And, similar to the derivations $\mathcal{D}_1$ and $\mathcal{D}_2$, we have:

$$\begin{array}{c}\mathcal{D}_2' \\ \Omega, \Box\Gamma', \varphi \Rightarrow \Box\chi_1, \ldots, \Box\chi_n, \Omega, \psi \mid \Omega, \Box\Gamma', \psi \Rightarrow \Box\chi_1, \ldots, \Box\chi_n, \Omega, \psi \mid \mathcal{G}.\end{array}$$

Now, by applying $(R\wedge)$ we have:

$$\dfrac{\mathcal{D}_1' \qquad \mathcal{D}_2'}{\Omega, \Box\Gamma', \varphi \Rightarrow \Box\chi_1, \ldots, \Box\chi_n, \Omega, \psi \mid \Omega, \Box\Gamma', \psi \Rightarrow \Box\chi_1, \ldots, \Box\chi_n, \Omega, \varphi \wedge \psi \mid \mathcal{G}.}\,(R\wedge).$$

Finally, again by applying $(R\wedge)$ the claim is obtained. The proof of $\mathbf{HLK(A)}^{\mathrm{r}}$-invertibility of the rule $(R\vee)$ is very similar. To show that $(L\vee)$ is $\mathbf{HLK(A)}^{\mathrm{r}}$-invertible, we prove that the following rules are admissible in $\mathbf{HLK(A)}^{\mathrm{r}}$

$$\dfrac{[\Gamma_i, [\varphi \vee \psi]^{\lambda_i} \Rightarrow \Delta_i]_{i=1}^n \mid H}{[\Gamma_i, [\varphi]^{\lambda_i} \Rightarrow \Delta_i]_{i=1}^n \mid H} \qquad \dfrac{[\Gamma_i, [\varphi \vee \psi]^{\lambda_i} \Rightarrow \Delta_i]_{i=1}^n \mid H}{[\Gamma_i, [\psi]^{\lambda_i} \Rightarrow \Delta_i]_{i=1}^n \mid H}$$

proceeding by induction on the height of the derivations of the premises. We only consider the case that the last rule applied in the derivation of the premise is $(\Box_{k,n})$; the other cases are treated easily. Suppose that the last rule applied is $(\Box_{k,n})$, so $[\varphi \vee \psi]^{\lambda_j}$ must occur also on the right of the sequent as follows:

$$\dfrac{\Gamma_0' \Rightarrow \ \mid \mathcal{H} \quad \Gamma_1' \Rightarrow k\chi_1 \mid \mathcal{H} \quad \cdots \quad \Gamma_n' \Rightarrow k\chi_n \mid \mathcal{H}}{\Omega, \Box\Gamma', [\varphi \vee \psi]^{\lambda_j} \Rightarrow \Box\chi_1, \ldots, \Box\chi_n, \Omega, [\varphi \vee \psi]^{\lambda_j} \mid \mathcal{H}}\,(\Box_{k,n}),$$

where $\Gamma_j = \Omega \uplus \Box\Gamma'$, and $\Delta_j = \Omega \uplus \Box\chi_1 \uplus \cdots \uplus \Box\chi_n \uplus [\varphi \wedge \psi]^{\lambda_j}$ and also $k\Gamma' = \Gamma_0' \uplus \ldots \uplus \Gamma_n'$. Then, for $\lambda_j = 1$, the conclusion is obtained as follows:

$$\dfrac{\dfrac{\dfrac{\Gamma_0' \Rightarrow \ \mid \mathcal{G} \quad \Gamma_1' \Rightarrow k\chi_1 \mid \mathcal{G} \quad \cdots \quad \Gamma_n' \Rightarrow k\chi_n \mid \mathcal{G}}{\Omega, \Box\Gamma', \varphi \Rightarrow \Box\chi_1, \ldots, \Box\chi_n, \Omega, \varphi \mid \mathcal{G}}\,(\Box_{k,n})}{\Omega, \Box\Gamma', \varphi \Rightarrow \Box\chi_1, \ldots, \Box\chi_n, \Omega, \varphi \mid \Omega, \Box\Gamma', \varphi \Rightarrow \Box\chi_1, \ldots, \Box\chi_n, \Omega, \psi \mid \mathcal{G}}\,(\text{EW})}{\Omega, \Box\Gamma', \varphi \Rightarrow \Box\chi_1, \ldots, \Box\chi_n, \Omega, \varphi \vee \psi \mid \mathcal{G}}\,R\vee$$

where $\mathcal{G}$ is obtained from $\mathcal{H} = [\Gamma_i, [\varphi \vee \psi]^{\lambda_i} \Rightarrow \Delta_i]_{i=2}^n \mid H$ by applying induction hypothesis. The $\mathbf{HLK(A)}^{\mathrm{r}}$-invertibility of the rule $(R\wedge)$ is proved similarly. $\square$

Lemma 4.4. *The rule* (Mix) *is* $\mathbf{HLK(A)}^{\mathrm{r}}$*-admissible.*

Proof: To show the $\mathbf{HLK(A)}^{\mathrm{r}}$-admissibility of (Mix), we prove, more generally, that the following rule is admissible

$$\frac{[\Gamma_i \Rightarrow \Delta_i]_{i=1}^n \mid H \qquad [\Pi_j \Rightarrow \Sigma_j]_{j=1}^m \mid H}{[r_{i_1}\Gamma_i, s_{i_1}\Pi_1 \Rightarrow r_{i_1}\Delta_i, s_{i_1}\Sigma_1]_{i=1}^n \mid \cdots \mid [r_{i_m}\Gamma_i, s_{i_m}\Pi_m \Rightarrow r_{i_m}\Delta_i, s_{i_m}\Sigma_m]_{i=1}^n \mid H}$$

for all $r_{i_j}, s_{i_j} \in \mathbb{N} \cup \{0\}$. Proceeding by induction on the lexicographically ordered pair consisting of the sum of the modal depth of the formulas in the premises and the sum of the height of $\mathbf{HLK(A)}^{\mathrm{r}}$-derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ of $[\Gamma_i \Rightarrow \Delta_i]_{i=1}^n \mid H$ and $[\Pi_j \Rightarrow \Sigma_j]_{j=1}^m \mid H$, respectively. If $\mathcal{D}_1$ and $\mathcal{D}_2$ have height 0, then $[\Gamma_i \Rightarrow \Delta_i]_{i=1}^n \mid H$ and $[\Pi_j \Rightarrow \Sigma_j]_{j=1}^m \mid H$ are instances of (AX). i.e., $\Gamma_i = \Delta_i$ for some $1 \leqslant i \leqslant n$, and $\Pi_j = \Sigma_j$ for some $1 \leqslant j \leqslant m$, (in particular if Γ_i, Δ_i, Π_j, and Σ_j contain only variables), then $r_{i_j}\Gamma_i \uplus s_{i_j}\Pi_j = r_{i_j}\Delta_i \uplus s_{i_j}\Sigma_j$ and so $[r_{i_1}\Gamma_i, s_{i_1}\Pi_1 \Rightarrow r_{i_1}\Delta_i, s_{i_1}\Sigma_1]_{i=1}^n \mid \cdots \mid [r_{i_m}\Gamma_i, s_{i_m}\Pi_m \Rightarrow r_{i_m}\Delta_i, s_{i_m}\Sigma_m]_{i=1}^n \mid H$ is an instance of (AX). If the last application of rules in $\mathcal{D}_1$ and $\mathcal{D}_2$ are not $(\Box_{k,n})$ then the result follows easily by one (or two) applications of the induction hypothesis and further applications of the rule. For example, suppose $\mathcal{D}_2$ ends with

$$\frac{\Pi', \varphi \Rightarrow \Sigma_1 \mid \Pi', \psi \Rightarrow \Sigma_1 \mid [\Pi_j \Rightarrow \Sigma_j]_{j=2}^m \mid H}{\Pi', \varphi \wedge \psi \Rightarrow \Sigma_1 \mid [\Pi_j \Rightarrow \Sigma_j]_{j=2}^m \mid H}\ (L\wedge),$$

where $\Pi_1 = \Pi' \uplus [\varphi \wedge \psi]$. An application of the induction hypothesis to the $\mathbf{HLK(A)}^{\mathrm{r}}$-derivation of the premise $[\Gamma_i \Rightarrow \Delta_i]_{i=1}^n$ together with a $\mathbf{HLK(A)}^{\mathrm{r}}$-derivation of

$$\Pi', \varphi \Rightarrow \Sigma_1 \mid \Pi', \psi \Rightarrow \Sigma_1 \mid [\Pi_j \Rightarrow \Sigma_j]_{j=2}^m \mid H$$

yields

$$[r_{i_1}\Gamma_i, s_{i_1}\Pi', s_{i_1}\varphi \Rightarrow r_{i_1}\Delta_i, s_{i_1}\Sigma_1]_{i=1}^n \mid [r_{i_1}\Gamma_i, s_{i_1}\Pi', s_{i_1}\psi \Rightarrow r_{i_1}\Delta_i, s_{i_1}\Sigma_1]_{i=1}^n \mid$$
$$\cdots \mid [r_{i_m}\Gamma_i, s_{i_m}\Pi_m \Rightarrow r_{i_m}\Delta_i, s_{i_m}\Sigma_m]_{i=1}^n \mid H.$$

It follows then that the following hypersequent is $\mathbf{HLK}(\mathbf{A})^{\mathrm{r}}$-derivable using $\sum_{i=1}^{n} s_{i_1}$ times applications of the rule $(L\wedge)$:

$$[r_{i_1}\Gamma_i, s_{i_1}\Pi', s_{i_1}(\varphi\wedge\psi) \Rightarrow r_{i_1}\Delta_i, s_{i_1}\Sigma_1]_{i=1}^{n} \mid \cdots \mid [r_{i_m}\Gamma_i, s_{i_m}\Pi_m \Rightarrow r_{i_m}\Delta_i, s_{i_m}\Sigma_m]_{i=1}^{n} \mid H.$$

The case where $\mathcal{D}_2$ ends with $(R\vee)$, $(L\rightarrow)$, $(R\rightarrow)$, (EC) or (Split) is treated by a similar argument. If $\mathcal{D}_2$ ends with

$$\frac{\Pi', \varphi \Rightarrow \Sigma_1 \mid [\Pi_j \Rightarrow \Sigma_j]_{j=2}^{m} \mid H \qquad \Pi', \psi \Rightarrow \Sigma_1 \mid [\Pi_j \Rightarrow \Sigma_j]_{j=2}^{m} \mid H}{\Pi', \varphi \vee \psi \Rightarrow \Sigma_1 \mid [\Pi_j \Rightarrow \Sigma_j]_{j=2}^{m} \mid H} \ (L\vee),$$

where $\Pi_1 = \Pi' \uplus [\varphi \vee \psi]$. Then, by the induction hypothesis,

$$\vdash_{\mathbf{HLK}(\mathbf{A})} [r_{i_1}\Gamma_i, s_{i_1}\Pi', s_{i_1}\varphi \Rightarrow r_{i_1}\Delta_i, s_{i_1}\Sigma_1]_{i=1}^{n} \mid \cdots \mid [r_{i_m}\Gamma_i, s_{i_m}\Pi_m \Rightarrow r_{i_m}\Delta_i, s_{i_m}\Sigma_m]_{i=1}^{n} \mid H$$

$$\vdash_{\mathbf{HLK}(\mathbf{A})} [r_{i_1}\Gamma_i, s_{i_1}\Pi', s_{i_1}\psi \Rightarrow r_{i_1}\Delta_i, s_{i_1}\Sigma_1]_{i=1}^{n} \mid \cdots \mid [r_{i_m}\Gamma_i, s_{i_m}\Pi_m \Rightarrow r_{i_m}\Delta_i, s_{i_m}\Sigma_m]_{i=1}^{n} \mid H$$

So, the conclusion is derived by $\sum_{i=1}^{n} s_{i_1}$ times applications of $(L\vee)$. The case where $\mathcal{D}_2$ ends with $(R\wedge)$ is treated by a similar argument. Finally, let us consider the case where $\mathcal{D}_1$ ends with an application of $(\Box_{k,p})$ as follows:

$$\frac{\Gamma_0 \Rightarrow \ \mid \mathcal{G} \quad \Gamma_1 \Rightarrow k\varphi_1 \mid \mathcal{H} \cdots \Gamma_p \Rightarrow k\varphi_p \mid \mathcal{H}}{\Omega, \Box\Gamma' \Rightarrow \Box\varphi_1, \ldots, \Box\varphi_p, \Omega \mid \mathcal{H}} \ (\Box_{k,p}),$$

where $\Gamma_1 = \Omega \uplus \Box\Gamma'$ and $\Delta_1 = [\Box\varphi_1] \uplus \ldots \uplus [\Box\varphi_p] \uplus \Omega$, in addition $k\Gamma' = \Gamma_0 \uplus \ldots \uplus \Gamma_p$ and $\mathcal{H} = [\Gamma_i \Rightarrow \Delta_i]_{i=2}^{n} \mid H$, and suppose $\mathcal{D}_2$ ends with

$$\frac{\Pi_0 \Rightarrow \ \mid \mathcal{H} \quad \Pi_1 \Rightarrow l\psi_1 \mid \mathcal{H} \cdots \Pi_q \Rightarrow l\psi_q \mid \mathcal{H}}{\Theta, \Box\Pi' \Rightarrow \Box\psi_1, \ldots, \Box\psi_q, \Theta \mid \mathcal{H}} \ (\Box_{l,q}),$$

where $\Pi_1 = \Theta \uplus \Box\Pi'$ and $\Sigma_1 = [\Box\psi_1] \uplus \ldots \uplus [\Box\psi_p] \uplus \Theta$, in addition $l\Pi' = \Pi_0 \uplus \ldots \uplus \Pi_q$ and $\mathcal{H} = [\Pi_j \Rightarrow \Sigma_j]_{j=2}^{m} \mid H$. Then, applying the rule $(\Box_{kl, r_{1_1}p + s_{1_1}q})$, we obtain the required $\mathbf{HLK}(\mathbf{A})^{\mathrm{r}}$-derivation

$$\frac{r_{1_1} l\Gamma_0, s_{1_1} k\Pi_0 \Rightarrow \ \mid \mathcal{G} \qquad \{l\Gamma_i \Rightarrow kl\varphi_i \mid \mathcal{G}\}_{i=1}^{r_{1_1}p} \cdots \{k\Pi_j \Rightarrow kl\psi_j \mid \mathcal{G}\}_{j=1}^{s_{1_1}q}}{\begin{array}{l} r_{1_1}\Omega, s_{1_1}\Theta, r_{1_1}\Box\Gamma', s_{1_1}\Box\Pi' \Rightarrow \\ r_{1_1}\Box\varphi_1, \ldots, r_{1_1}\Box\varphi_p, s_{1_1}\Box\psi_1, \ldots, s_{1_1}\Box\psi_q, r_{1_1}\Omega, s_{1_1}\Theta \mid \mathcal{G} \end{array}}$$

where,

$$\mathcal{G} = [r_{i_1}\Gamma_i, s_{i_1}\Pi_1 \Rightarrow r_{i_1}\Delta_i, s_{i_1}\Sigma_1]_{i=2}^n | \cdots \\ |[r_{i_m}\Gamma_i, s_{i_m}\Pi_m \Rightarrow r_{i_m}\Delta_i, s_{i_m}\Sigma_m]_{i=2}^n \,|\, H$$

and the premises are all $\mathbf{HLK}(\mathbf{A})^{\mathrm{r}}$-derivable using the induction hypothesis. For example,

$$r_{1_1} l\Gamma_0, s_{1_1} k\Pi_0 \Rightarrow \;|[r_{i_1}\Gamma_i, s_{i_1}\Pi_1 \Rightarrow r_{i_1}\Delta_i, s_{i_1}\Sigma_1]_{i=2}^n | \cdots \\ |[r_{i_m}\Gamma_i, s_{i_m}\Pi_m \Rightarrow r_{i_m}\Delta_i, s_{i_m}\Sigma_m]_{i=2}^n | \,H$$

is derived as follows using the induction hypothesis (note that $r_{i_j}, s_{i_j} \in \mathbb{N} \cup \{0\}$):

$$\frac{\Gamma_0 \Rightarrow \;|\, [\Gamma_i \Rightarrow \Delta_i]_{i=2}^n \,|\, H \qquad \Pi_0 \Rightarrow \;|\, [\Pi_j \Rightarrow \Sigma_j]_{j=2}^m \,|\, H}{\begin{array}{l} r_{1_1} l\Gamma_0, s_{1_1} k\Pi_0 \Rightarrow \;|[r_{i_1}\Gamma_i, s_{i_1}\Pi_1 \Rightarrow r_{i_1}\Delta_i, s_{i_1}\Sigma_1]_{i=2}^n | \cdots \\ |[r_{i_m}\Gamma_i, s_{i_m}\Pi_m \Rightarrow r_{i_m}\Delta_i, s_{i_m}\Sigma_m]_{i=2}^n | \,H \end{array}}$$

□

Theorem 4.5. **HLK(A)** *admits cut-elimination.*

Proof: To establish cut-elimination for $\mathbf{HLK}(\mathbf{A})$, it suffices to prove that an uppermost application of (Cut) in a $\mathbf{HLK}(\mathbf{A})$-derivation can be eliminated; that is, we show that cutfree $\mathbf{HLK}(\mathbf{A})$-derivations of the premises of an instance of (Cut) can be transformed into a cut-free $\mathbf{HLK}(\mathbf{A})$-derivation of the conclusion. Observe first that the rule $(\Box_n)$ is $\mathbf{HLK}(\mathbf{A})^{\mathrm{r}}$-derivable using $(\Box_{k,n})$ with $k = n, \varphi_1 = \cdots = \varphi_n = \varphi$ and $\Gamma_1 = \ldots = \Gamma_n = \Gamma$. Hence, the proof of Lemma 4.4 shows that any cut-free $\mathbf{HLK}(\mathbf{A})$-derivation can be transformed algorithmically into a $\mathbf{HLK}(\mathbf{A})^{\mathrm{r}}$-derivation. We prove (constructively) that the following rule called "cancellation" rule is $\mathbf{HLK}(\mathbf{A})^{\mathrm{r}}$-admissible:

$$\frac{[\Gamma_i, \varphi_i \Rightarrow \varphi_i, \Delta_i]_{i=1}^n \,|\, H}{[\Gamma_i \Rightarrow \Delta_i]_{i=1}^n \,|\, H}\ (\mathrm{CAN}).$$

Suppose then that there are cut-free $\mathbf{HLK}(\mathbf{A})$-derivations of the premises $\Gamma, \varphi \Rightarrow \Delta \,|\, H$ and $\Pi \Rightarrow \varphi, \Sigma \,|\, H$ of an uppermost application of (Cut). By (Mix), we obtain a cut-free $\mathbf{HLK}(\mathbf{A})$- derivation of $\Gamma, \Pi, \varphi \Rightarrow \varphi, \Delta, \Sigma \,|\, H$ and hence a $\mathbf{HLK}(\mathbf{A})^{\mathrm{r}}$-derivation of this sequent. By cancellation rule, we obtain a $\mathbf{HLK}(\mathbf{A})^{\mathrm{r}}$-derivation of $\Gamma, \Pi \Rightarrow \Delta, \Sigma \,|\, H$, which also gives the desired cut-free $\mathbf{HLK}(\mathbf{A})$-derivation. We prove the admissibility of the cancellation rule by induction on the lexicographically ordered triple consisting of the sum of the modal depth of the formulas φ_i, $1 \le i \le n$, sum of the complexities of the formulas φ_i, $1 \leq i \leq n$, and the height of the derivation of the premise. For the base case, suppose that the formulas φ_i for all $1 \leq i \leq n$ are variables. If the premise is an instance of (AX), then $\Gamma_i, \varphi_i = \varphi_i, \Delta_i$ for some $1 \leq i \leq n$, i.e., $\Gamma_i = \Delta_i$ and so $[\Gamma_i \Rightarrow \Delta_i]_{i=1}^n$ is an instance of (AX). We observe that when the last rule applied is not $(\Box_{k,n})$, the claim follows immediately by applying the induction hypothesis and, where necessary, the relevant rule. Let us consider some cases; suppose that the last rule applied is $(L \to)$ as follows:

$$\frac{\Gamma_1', \varphi_1, \chi \Rightarrow \psi, \varphi_1, \Delta_1 \mid [\Gamma_i, \varphi_i \Rightarrow \varphi_i, \Delta_i]_{i=2}^n \mid H}{\psi \to \chi, \Gamma_1', \varphi_1 \Rightarrow \varphi_1, \Delta_1 \mid [\Gamma_i, \varphi_i \Rightarrow \varphi_i, \Delta_i]_{i=2}^n \mid H} \ (L \to),$$

where $\Gamma_1 = \psi \to \chi, \Gamma_1'$. Then, the height of the premise is reduced and so by applying the induction hypothesis the conclusion is obtained as follows:

$$\cfrac{\cfrac{\Gamma_1', \varphi_1, \chi \Rightarrow \psi, \varphi_1, \Delta_1 \mid [\Gamma_i, \varphi_i \Rightarrow \varphi_i, \Delta_i]_{i=2}^n \mid H}{\Gamma_1', \chi \Rightarrow \psi, \Delta_1 \mid [\Gamma_i \Rightarrow \Delta_i]_{i=2}^n \mid H} \ (\text{IH})}{\psi \to \chi, \Gamma_1' \Rightarrow \Delta_1 \mid [\Gamma_i \Rightarrow \Delta_i]_{i=2}^n \mid H} \ (L \to).$$

The cases where the last rule applied is $(R \to)$ or (Split) are very similar. Suppose that the last rule applied is $(L\wedge)$ as follows:

$$\frac{\psi, \Gamma_1', \varphi_1 \Rightarrow \varphi_1, \Delta_1 \mid \chi, \Gamma_1', \varphi_1 \Rightarrow \varphi_1, \Delta_1 \mid [\Gamma_i, \varphi_i \Rightarrow \varphi_i, \Delta_i]_{i=2}^n \mid H}{\psi \wedge \chi, \Gamma_1', \varphi_1 \Rightarrow \varphi_1, \Delta_1 \mid [\Gamma_i, \varphi_i \Rightarrow \varphi_i, \Delta_i]_{i=2}^n \mid H} \ (L\wedge),$$

where $\Gamma_1 = \psi \wedge \chi, \Gamma_1'$. Then, the height of the premise is reduced and so by applying the induction hypothesis we have:

$$\frac{\psi, \Gamma_1', \varphi_1 \Rightarrow \varphi_1, \Delta_1 \mid \chi, \Gamma_1', \varphi_1 \Rightarrow \varphi_1, \Delta_1 \mid [\Gamma_i, \varphi_i \Rightarrow \varphi_i, \Delta_i]_{i=2}^n \mid H}{\psi, \Gamma_1', \varphi_1 \Rightarrow \varphi_1, \Delta_1 \mid \chi, \Gamma_1', \Rightarrow \Delta_1 \mid [\Gamma_i \Rightarrow \Delta_i]_{i=2}^n \mid H} \ (\text{IH})$$

Therefore, the sum of the complexities of the formulas φ_i is reduced, again by applying the induction hypothesis the conclusion is obtained as follows:

$$\cfrac{\cfrac{\psi, \Gamma_1', \varphi_1 \Rightarrow \varphi_1, \Delta_1 \mid \chi, \Gamma_1', \Rightarrow \Delta_1 \mid [\Gamma_i \Rightarrow \Delta_i]_{i=2}^n \mid H}{\psi, \Gamma_1' \Rightarrow \Delta_1 \mid \chi, \Gamma_1', \Rightarrow \Delta_1 \mid [\Gamma_i \Rightarrow \Delta_i]_{i=2}^n \mid H} \ (\text{IH})}{\psi \wedge \chi, \Gamma_1' \Rightarrow \Delta_1 \mid [\Gamma_i \Rightarrow \Delta_i]_{i=2}^n \mid H} \ (L\wedge).$$

The cases where the last rule applied is $(R\vee)$ or (EC) are very similar. Suppose now that the last rules applied is $(\Box_{k,m})$ as follows:

$$\frac{\Pi_0 \Rightarrow \;|\, \mathcal{H} \quad \Pi_1 \Rightarrow k\psi_1 \,|\, \mathcal{H} \quad \cdots \quad \Pi_m \Rightarrow k\psi_m \,|\, \mathcal{H}}{\Sigma, \Box\Pi, \varphi_1 \Rightarrow \varphi_1, \Box\psi_1, \ldots, \Box\psi_m, \Sigma \,|\, \mathcal{H}} \;(\Box_{k,m}),$$

where $\mathcal{H} = [\Gamma_i, \varphi_i \Rightarrow \varphi_i, \Delta_i]_{i=2}^n \,|\, H$, and $k\Pi = \Pi_0 \uplus \Pi_1 \uplus \ldots \uplus \Pi_m$, in addition $k = k_0 + k_1 + \ldots + k_m$. Thus, the sum of the complexities of the formulas φ_i is reduced, by applying the induction hypothesis we have $\mathbf{HLK}(\mathbf{A})^{\mathrm{r}}$-derivations of

$$\begin{aligned}
&\Pi_0 \Rightarrow \;|\, [\Gamma_i \Rightarrow \Delta_i]_{i=2}^n \,|\, H \\
&\Pi_1 \Rightarrow k\psi_1 \,|\, [\Gamma_i \Rightarrow \Delta_i]_{i=2}^n \,|\, H \\
&\vdots \\
&\Pi_m \Rightarrow k\psi_m \,|\, [\Gamma_i \Rightarrow \Delta_i]_{i=2}^n \,|\, H
\end{aligned}$$

Then, by applying the rule $(\Box_{k,m})$, we have a $\mathbf{HLK}(\mathbf{A})^{\mathrm{r}}$-derivation of

$$\Sigma, \Box\Pi, \Rightarrow \Box\psi_1, \ldots, \Box\psi_m, \Sigma \,|\, [\Gamma_i \Rightarrow \Delta_i]_{i=2}^n \,|\, H$$

For the inductive step, suppose that $\varphi_i = \psi \to \chi$ for some $1 \leq i \leq n$, then we use the invertibility of $(L \to)$ and $(R \to)$ and apply the induction hypothesis twice. If φ_i has the form $\psi \wedge \chi$ for some $1 \leq i \leq n$, then the conclusion is obtained as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma_1, \psi \wedge \chi \Rightarrow \psi \wedge \chi, \Delta_1 \,|\, [\Gamma_i, \varphi_i \Rightarrow \varphi_i, \Delta_i]_{i=2}^n \,|\, H}{\Gamma_1, \psi \Rightarrow \psi \wedge \chi, \Delta_1 \,|\, \Gamma_1, \chi \Rightarrow \psi \wedge \chi, \Delta_1 \,|\, [\Gamma_i, \varphi_i \Rightarrow \varphi_i, \Delta_i]_{i=2}^n \,|\, H} \;(L\wedge^{-1})}{\Gamma_1, \psi \Rightarrow \psi, \Delta_1 \,|\, \Gamma_1, \chi \Rightarrow \chi, \Delta_1 \,|\, [\Gamma_i, \varphi_i \Rightarrow \varphi_i, \Delta_i]_{i=2}^n \,|\, H} \;(R\wedge^{-1}) \text{ twice}}{\Gamma_1 \Rightarrow \Delta_1 \,|\, \Gamma_1 \Rightarrow \Delta_1 \,|\, [\Gamma_i \Rightarrow \Delta_i]_{i=2}^n \,|\, H} \;(\mathrm{IH}) \text{ twice}}{\Gamma_1 \Rightarrow \Delta_1 \,|\, [\Gamma_i \Rightarrow \Delta_i]_{i=2}^n \,|\, H} \;(\mathrm{EC})$$

Note that by applying the invertibility of the logical rules the height and sum of the complexities of the formulas in the premise can increase, but the sum of the complexities of the formulas φ_i is reduced. The cases where φ_i for some $1 \leq i \leq n$ has the form $\psi \vee \chi$ are very similar. Lastly, suppose that $\varphi_i = \Box\chi$ for some $1 \leq i \leq n$, and the derivation ends with

$$\frac{\Pi_0, k_0\chi \Rightarrow \;|\, \mathcal{H} \quad \Pi_1, k_1\chi \Rightarrow k\chi \,|\, \mathcal{H} \quad \{\Pi_i, k_i\chi \Rightarrow k\psi_i \,|\, \mathcal{H}\}_{i=2}^p}{\Sigma, \Box\Pi, \Box\chi \Rightarrow \Box\chi, \Box\psi_2, \ldots, \Box\psi_n, \Sigma \,|\, \mathcal{H}} \;(\Box_{k,p}),$$

where $\mathcal{H} = [\Gamma_i, \varphi_i \Rightarrow \varphi_i, \Delta_i]_{i=2}^n \,|\, H$, and $k\Pi = \Pi_0 \uplus \Pi_1 \uplus \ldots \uplus \Pi_p$, in addition $k = k_0 + k_1 + \ldots + k_p$. In this case, the sum of the modal depth of the formulas φ_i is reduced. By the induction hypothesis, $\vdash_{\mathbf{HLK(A)}^{\mathrm{r}}} \Pi_1 \Rightarrow (k - k_1)\chi \,|\, \mathcal{H}$. By the $\mathbf{HLK(A)}^{\mathrm{r}}$-admissibility of the rule (mix), we have $\mathbf{HLK(A)}^{\mathrm{r}}$-derivations of

$$k_0\Pi_1, (k - k_1)\Pi_0, (k - k_1)k_0\chi \Rightarrow (k - k_1)k_0\chi \,|\, \mathcal{H}$$

$$k_i\Pi_1, (k-k_1)\Pi_i, (k-k_1)k_i\chi \Rightarrow (k-k_1)k_i\chi, (k-k_1)k\psi_i \,|\, \mathcal{H} \text{ for } i \in \{2, \ldots, p\}.$$

So, by the induction hypothesis, we have $\mathbf{HLK(A)}^{\mathrm{r}}$-derivations of

$$k_0\Pi_1, (k - k_1)\Pi_0 \Rightarrow \,|\, \mathcal{G}$$

$$k_i\Pi_1, (k - k_1)\Pi_i \Rightarrow (k - k_1)k\psi_i \,|\, \mathcal{G} \text{ for } i \in \{2, \ldots, p\},$$

where $\mathcal{G} = [\Gamma_i \Rightarrow \Delta_i]_{i=2}^n \,|\, H$. Now by an application of $(\Box_{((k-k_1)k,n-1)})$, we have a $\mathbf{HLK(A)}^{\mathrm{r}}$-derivation ending with

$$\frac{k_0\Pi_1, (k - k_1)\Pi_0 \Rightarrow \,|\, \mathcal{G} \qquad \{k_i\Pi_1, (k - k_1)\Pi_i \Rightarrow (k - k_1)k\psi_i \,|\, \mathcal{G}\}_{i=2}^p}{\Sigma, \Box\Pi \Rightarrow \Box\psi_2, \ldots, \Box\psi_n, \Sigma \,|\, \mathcal{G}}$$

where $(k - k_1)k\Pi = (k_0 + k_2 + \cdots + k_p)(\Pi_0 \uplus \Pi_1 \uplus \ldots \uplus \Pi_p)$. ☐

We now turn our attention to showing that the axiomatic and hypersequent presentations really characterize the same logics, writing $+\{\varphi_1, \ldots, \varphi_n\}$ as shorthand for $\varphi_1 + \ldots + \varphi_n$.

Lemma 4.6.

(i) *If* $\vdash_{\mathbf{HLK(A)}} \Gamma, \varphi + \psi \Rightarrow \Delta \,|\, H$, *then* $\vdash_{\mathbf{HLK(A)}} \Gamma, \varphi, \psi \Rightarrow \Delta \,|\, H$.

(ii) *If* $\vdash_{\mathbf{HLK(A)}} \Gamma \Rightarrow \Delta, \varphi + \psi \,|\, H$, *then* $\vdash_{\mathbf{HLK(A)}} \Gamma \Rightarrow \Delta, \varphi, \psi \,|\, H$.

Proof: For (i), since $\vdash_{\mathbf{HLK(A)}} \varphi, \psi \Rightarrow \varphi + \psi \,|\, H$, if $\vdash_{\mathbf{HLK(A)}} \Gamma, \varphi + \psi \Rightarrow \Delta \,|\, H$, then by (Cut), $\vdash_{\mathbf{HLK(A)}} \Gamma, \varphi, \psi \Rightarrow \Delta \,|\, H$. The case (ii) is similar. ☐

Lemma 4.7. *If* $\vdash_{\mathbf{HLK(A)}} \Rightarrow \mathcal{I}(H)$, *then* $\vdash_{\mathbf{HLK(A)}} H$.

Proof: Let $H = \Gamma_1 \Rightarrow \Delta_1 \,|\, \ldots \,|\, \Gamma_n \Rightarrow \Delta_n$. If

$$\vdash_{\mathbf{HLK(A)}} (\sum \Gamma_1 \to \sum \Delta_1) \vee \cdots \vee (\sum \Gamma_n \to \sum \Delta_n),$$

then by invertibility of the rules $(R\vee)$ and $(R\to)$,

$$\vdash_{\mathbf{HLK(A)}} (\sum \Gamma_1 \Rightarrow \sum \Delta_1) \mid \cdots \mid (\sum \Gamma_n \Rightarrow \sum \Delta_n).$$

Hence, by Lemma 4.6, $\vdash_{\mathbf{HLK(A)}} H$. $\square$

THEOREM 4.8. *$\vdash_{\mathbf{HLK(A)}} H$ iff $\vdash_{\mathbf{LK(A)}} \mathcal{I}(H)$.*

PROOF: For the left-to-right direction we proceed by induction on the height of the derivation of H in $\mathbf{HLK}(\mathbf{A})$. If H is an instance of an axiom of $\mathbf{HLK}(\mathbf{A})$, then it is easy to check that $\vdash_{\mathbf{LK(A)}} \mathcal{I}(H)$. For the inductive step, suppose that H follows by some rule of $\mathbf{HLK}(\mathbf{A})$ from $H_1, \ldots, H_n$. By the induction hypothesis n times, we have $\vdash_{\mathbf{LK(A)}} \mathcal{I}(H_1), \ldots, \vdash_{\mathbf{LK(A)}} \mathcal{I}(H_n)$. For the non-modal rules of $\mathbf{HLK}(\mathbf{A})$ (see e.g. [18] for details), it is easy to check that

$$\vdash_{\mathbf{LK(A)}} \mathcal{I}(H_1) \to (\mathcal{I}(H_2) \to (\cdots \to (\mathcal{I}(H_n) \to \mathcal{I}(H))) \cdots)$$

and that hence, by (mp) n times, $\vdash_{\mathbf{LK(A)}} \mathcal{I}(H)$. For the modal rule, suppose that $\vdash_{\mathbf{LK(A)}} \mathcal{I}(H) \vee \mathcal{I}(\Gamma \Rightarrow n\varphi)$. By Theorem 3.3, it is sufficient to show that $\mathcal{I}(\Box\Gamma \Rightarrow n\Box\varphi \mid H)$ is valid in every $\mathbf{LK(A)}$-algebra. Consider a valuation v for such an algebra. Either $v(\mathcal{I}(H)) \geq \bar{0}$ and hence $v(\mathcal{I}(H) \vee \mathcal{I}(\Box\Gamma \Rightarrow n\Box\varphi) \geq \bar{0}$ or $v(\mathcal{I}(\Gamma \Rightarrow n\varphi)) \geq \bar{0}$. If the latter, then $I(v(\mathcal{I}(\Gamma \Rightarrow n\varphi)) \geq I(\bar{0}))$. But $I(v(\mathcal{I}(\Gamma \Rightarrow n\varphi))) = v(\mathcal{I}(\Box\Gamma \Rightarrow n\Box\varphi))$ so we are done.
For the right-to-left direction, we have (an easy exercise) that the axioms of $\mathbf{LK}(\mathbf{A})$ are derivable in $\mathbf{HLK}(\mathbf{A})$. Moreover, (nec) corresponds to $(\Box)$, (adj) corresponds to (R$\wedge$), and (mp) can be derived from $\vdash_{\mathbf{HLK(A)}} \Rightarrow \varphi$ and $\vdash_{\mathbf{HLK(A)}} \Rightarrow \varphi \to \psi$, by using (Cut) twice with $\vdash_{\mathbf{HLK(A)}} \varphi, \varphi \to \psi \Rightarrow \psi$. Hence, if $\vdash_{\mathbf{LK(A)}} \mathcal{I}(H)$, then $\vdash_{\mathbf{HLK(A)}} \Rightarrow \mathcal{I}(H)$, and so by Lemma 4.7, $\vdash_{\mathbf{HLK(A)}} \Rightarrow H$. $\square$

5. Concluding remarks

The paper is devoted to a proof-theoretic account of continuous modal logics: many-valued modal logics with connectives interpreted locally by continuous functions over sets of real numbers [10]. I have introduced linear abelian modal logic $\mathbf{LK}(\mathbf{A})$, which is an extension of the abelian modal logic $\mathbf{K}(\mathbf{A})$, where propositional connectives are interpreted using lattice ordered group operations over the real numbers. I have provided a hypersequent calculus admitting cut-elimination for $\mathbf{LK}(\mathbf{A})$. Moreover, the correspondence between this calculus and the complete axiomatization with respect to both appropriate algebras and linearly ordered algebras is established.

I have only focused in this work on the extension of the sequent calculus for the modal multiplicative fragment of $\mathbf{K}(\mathbf{A})$ to a hypersequent calculus for the full logic. Clearly, there are many open questions still to be addressed. The most pressing issue is to provide a suitable Kripke model for $\mathbf{LK}(\mathbf{A})$ and prove the completeness theorem with respect to it. It seems that adapting the Kripke semantics and prove completeness with respect to the Kripke semantics is more tricky. Since the distributivity of box over the operator "+", i.e., $\Box(\varphi + \psi) \to \Box\varphi + \Box\psi$ is not derivable in the provided hypersequent calculus, this formula should not be valid in Kripke models. Therefore, it seems that we need some conditions on the accessibility relation in the Kripke models in which the formula $\Box(\varphi \vee \psi) \to \Box\varphi \vee \Box\psi$ is valid, while the formula $\Box(\varphi + \psi) \to \Box\varphi + \Box\psi$ is not valid.

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References

[1] A. Avron, *Hypersequents, logical consequence and intermediate logics for concurrency*, **Annals of Mathematics and Artificial Intelligence**, vol. 4(3) (1991), pp. 225–248, DOI: https://doi.org/10.1007/BF01531058.

[2] F. Baader, S. Borgwardt, R. Penaloza, *Decidability and complexity of fuzzy description logics*, **KI-Künstliche Intelligenz**, vol. 31(1) (2017), pp. 85–90, DOI: https://doi.org/10.1007/s13218-016-0459-3.

[3] S. Baratella, *Continuous propositional modal logic*, **Journal of Applied Non-Classical Logics**, vol. 28(4) (2018), pp. 297–312, DOI: https://doi.org/10.1080/11663081.2018.1468677.

[4] E. Casari, *Comparative logics and Abelian l-groups*, [in:] **Studies in Logic and the Foundations of Mathematics**, vol. 127, Elsevier (1989), pp. 161–190, DOI: https://doi.org/https://doi.org/10.1016/S0049-237X(08)70269-6.

[5] A. Ciabattoni, G. Metcalfe, F. Montagna, *Adding modalities to MTL and its extensions*, [in:] **Proceedings of the Linz Symposium**, vol. 2005, Citeseer (2005).

[6] A. Ciabattoni, G. Metcalfe, F. Montagna, *Algebraic and proof-theoretic characterizations of truth stressers for MTL and its extensions*, **Fuzzy Sets and Systems**, vol. 161(3) (2010), pp. 369–389, DOI: https://doi.org/10.1016/j.fss.2009.09.001.

[7] P. Cintula, *Weakly implicative (fuzzy) logics I: Basic properties*, **Archive for Mathematical Logic**, vol. 45(6) (2006), pp. 673–704, DOI: https://doi.org/10.1007/s00153-006-0011-5.

[8] P. Cintula, C. Noguera, *A general framework for mathematical fuzzy logic*, [in:] P. Cintula, P. Hajek, C. Noguera (eds.), **Handbook of Mathematical Fuzzy Logic**, vol. 1, College publications, London (2011), pp. 103–207.

[9] D. Diaconescu, G. Georgescu, *Tense operators on MV-algebras and Łukasiewicz-Moisil algebras*, **Fundamenta Informaticae**, vol. 81(4) (2007), pp. 379–408.

[10] D. Diaconescu, G. Metcalfe, L. Schnüriger, *A Real-Valued Modal Logic*, **Logical Methods in Computer Science**, vol. 14(1) (2018), DOI: https://doi.org/10.23638/LMCS-14(1:10)2018.

[11] T. Flaminio, L. Godo, E. Marchioni, *Logics for belief functions on MV-algebras*, **International Journal of Approximate Reasoning**, vol. 54(4) (2013), pp. 491–512, DOI: https://doi.org/10.1016/j.ijar.2012.08.006.

[12] T. Flaminio, F. Montagna, *MV-algebras with internal states and probabilistic fuzzy logics*, **International Journal of Approximate Reasoning**, vol. 50(1) (2009), pp. 138–152, DOI: https://doi.org/10.1016/j.ijar.2008.07.006.

[13] L. Godo, P. Hájek, F. Esteva, *A fuzzy modal logic for belief functions*, **Fundamenta Informaticae**, vol. 57(2–4) (2003), pp. 127–146.

[14] L. Godo, R. O. Rodríguez, *A fuzzy modal logic for similarity reasoning*, [in:] G. Chen, M. Ying, K. Cai (eds.), **Fuzzy Logic and Soft Computing. The International Series on Asian Studies in Computer and Information Science**, Springer, Boston, MA (1999), pp. 33–48, DOI: https://doi.org/10.1007/978-1-4615-5261-1_3.

[15] P. Hájek, *Making fuzzy description logic more general*, **Fuzzy Sets and Systems**, vol. 154(1) (2005), pp. 1–15, DOI: https://doi.org/10.1016/j.fss.2005.03.005.

[16] P. Hájek, D. Harmancová, R. Verbrugge, *A qualitative fuzzy possibilistic logic*, **International Journal of Approximate Reasoning**, vol. 12(1) (1995), pp. 1–19, DOI: https://doi.org/10.1016/0888-613X(94)00011-Q.

[17] G. Metcalfe, N. Olivetti, D. Gabbay, *Sequent and hypersequent calculi for abelian and Łukasiewicz logics*, **ACM Transactions on Computational Logic (TOCL)**, vol. 6(3) (2005), pp. 578–613, DOI: https://doi.org/10.1145/1071596.1071600.

[18] G. Metcalfe, N. Olivetti, D. M. Gabbay, **Proof theory for fuzzy logics**, vol. 36 of Applied Logic Series, Springer, Dordrecht (2008), DOI: https://doi.org/10.1007/978-1-4020-9409-5.

[19] G. Metcalfe, O. Tuyt, *A Monadic Logic of Ordered Abelian Groups*, [in:] N. Olivetti, R. Verbrugge, S. Negri, G. Sandu (eds.), **Advances in Modal Logic 13**, College Publications, London (2020), pp. 441–457.

[20] R. K. Meyer, J. K. Slaney, *Abelian Logic (From A to Z)*, [in:] G. Priest, R. Routley, J. Norman (eds.), **Paracoconsistent Logic**, Philosophia Verlag, Munich (1989).

[21] M. Mio, R. Mardare, R. Furber, *Probabilistic logics based on Riesz spaces*, **Logical Methods in Computer Science**, vol. 16 (2020), DOI: https://doi.org/10.23638/LMCS-16(1:6)2020.

[22] M. Mio, A. Simpson, *Łukasiewicz μ-calculus*, **Fundamenta Informaticae**, vol. 150(3–4) (2017), pp. 317–346, DOI: https://doi.org/10.3233/FI-2017-1472.

[23] G. Pottinger, *Uniform, cut-free formulations of T, S4 and S5*, **Journal of Symbolic Logic**, vol. 48(3) (1983), p. 900, DOI: https://doi.org/10.2307/2273495.

[24] U. Straccia, *Reasoning within fuzzy description logics*, **Journal of Artificial Intelligence Research**, vol. 14 (2001), pp. 137–166, DOI: https://doi.org//10.1613/jair.813.

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ON PARACOMPLETE VERSIONS OF JAŚKOWSKI'S DISCUSSIVE LOGIC

Abstract

Jaśkowski's discussive (discursive) logic $\mathbf{D_2}$ is historically one of the first paraconsistent logics, i.e., logics that 'tolerate' contradictions. Following Jaśkowski's idea to define his discussive logic by means of the modal logic **S5** *via* special translation functions between discussive and modal languages and supporting at the same time the tradition of paracomplete logics being the counterpart of paraconsistent ones, we present a paracomplete discussive logic $\mathbf{D_2^p}$.

Keywords: discussive logic, discursive logic, modal logic, paracomplete logic, paraconsistent logic.

2020 Mathematical Subject Classification: 03B53, 03B60, 03B45.

1. Introduction

The idea to set up *paracomplete* versions of Jaśkowski's discussive[1] logic $\mathbf{D_2}$ [23, 22], commonly accepted to be among the pioneering *paraconsis-*

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[1] *Discursive* is another dubbing for this logic. Without any additional reason we choose the former dubbing through the paper.

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tent logics, may appear to the reader as a contradiction in terms.[2] S/He might think that the present paper is, at best, of insignificant and technical importance or is, at worst, about an unconventional and deliberately provocative interpretation of $\mathbf{D_2}$ which has now become a classic. What we would like to avoid in this paper, foremost, is giving the reader the impression that $\mathbf{D_2}$ might be argued not to be paraconsistent. We would not like the reader to get the impression that setting up a system, where $p \rightarrow_{\mathrm{d}} (\neg p \rightarrow_{\mathrm{d}} q)$ is a theorem, is in line with Jaśkowski's original ideas on discussive implication $\rightarrow_{\mathrm{d}}$.

On the contrary, we would like to stress that while presenting his view on the Duns Scotus law (see [23]), Jaśkowski points out that, since antiquity, Aristotle's view that two contradictory statements are not both true has been a subject of criticism. Jaśkowski emphasizes that in the nineteenth and twentieth centuries, these views revived, under which it was pointed out that there are convincing arguments that lead to contradictory conclusions. So, he aimed to construct a system in which the implicational law of overfilling $p \rightarrow (\neg p \rightarrow q)$ is not valid. The idea behind the construction of such a system was as follows: first, with regard to inconsistent sets of statements, such a system does not always lead to overfilling of the set of conclusions; second, it is supposed to be so rich as to enable practical inference; and third, it should have an intuitive justification.

Due to the form of the posed problem, the choice of the implication plays a crucial role in building such a system. Originally, using the modal operator of possibility, Jaśkowski introduced discussive implication and, on its basis, also the discussive equivalence. The system of modal sentences that results from enriching the modal logic **S5** with the relevant definitions of discussive connectives is denoted as $\mathbf{M_2}$. On the basis of $\mathbf{M_2}$ Jaśkowski is defining *the system of the two-valued discussive sentential calculus*. This logic is quite rich and allows for the rejection of the implicational law of overfilling.

[2] Following Jaśkowski's approach [23, 22], we call a logic $\mathbf{L}$ paraconsistent iff there are formulas A and B such that $A \rightarrow (\neg A \rightarrow B)$ is not valid in $\mathbf{L}$. Following Akama and da Costa [3], we call a logic $\mathbf{L}$ paracomplete iff there is a formula A such that $A \vee \neg A$ is not valid in $\mathbf{L}$. This definition has already been used by Sette and Carnieli [47], although they preferred the term (nowadays, rarely used) weakly-intuitionistic logic. For the other definitions of paralogics (logics that are paraconsistent or paracomplete), the reader is addressed to [3, 40].

Interestingly, due to the use of classical conjunction in Jaśkowski's first paper [23] on discussive logic, one might suppose that discussive logic is non-adjunctive. And indeed, while presenting examples of formulas that are not theses of discussive logic, Jaśkowski considers $p \to_{\mathrm{d}} (q \to_{\mathrm{d}} (p \wedge q))$. He gives intuitions accompanying this: due to the fact that a certain thesis $\mathfrak{P}$ and another thesis $\mathfrak{Q}$ were put forward in the discussion, it does not follow that the thesis $\mathfrak{P} \wedge \mathfrak{Q}$ was also put forward, as it may happen that theses $\mathfrak{P}$ and $\mathfrak{Q}$ were sustained by different participants in the discussion. The intuitive explanation goes along with the formal justification. Of course, the use of classical conjunction leads to $(p \wedge \neg p) \to_{\mathrm{d}} q$, a version of the overfilling law that is a thesis of such a variant of the discussive system.[3] And only thanks to the non-adjunctive character of it, the implicational version of the overfilling law does not become a thesis.

However, the history of discussive logic does not end with [23]. What is nowadays treated as proper discussive logic is its variant with discussive conjunction. Discussive conjunction completing the language of $\mathbf{D_2}$ is given in a short paper [22]. Only there discussive conjunction $\wedge_{\mathrm{d}}$ is introduced and in this way, the formula $p \to_{\mathrm{d}} (q \to_{\mathrm{d}} (p \wedge_{\mathrm{d}} q))$ is becoming a thesis of $\mathbf{D_2}$. On the other hand, $(p \wedge_{\mathrm{d}} \neg p) \to_{\mathrm{d}} q$ is not a thesis of this final version of $\mathbf{D_2}$. At least to some point, the second paper on discussive logic was much less known than the first one. This was due to the fact that the paper on discussive conjunction was written in Polish and much later translated into English.[4]

On the other hand, we do not mean here Jaśkowski's discussive logic could not tolerate non-standard approaches. See, for example, [32], where an extended version of Jaśkowski's model of discussion with debaters employing modal operators explicitly is presented.

The motivation for this paper stems from conventional sources. First, it correlates to the fact that paraconsistent and paracomplete logics co-exist harmonically, with the latter being a junior counterpart of the former. (See also endnote 21.) Moreover, we argue below that the logic we present here is not the first paracomplete discussive logic in the literature. The second conventional source is to employ modal logic (generally, **S5**) in defining

[3] Already, this fact could be used as a factor showing that Jaśkowski model of discussion can be applied to explore a non-paraconsistent domain as well.

[4] A detailed discussion of the subject may be found in [13].

discussive connectives. Hence, we explore the possibility of changing some of the usual definitions to achieve the clear-cut effect that paracomplete logic results in.[5] Let us detail these sources.

Despite the fact that in [2] Akama, Abe, and Nakamatsu do not dub their *constructive discursive logic with strong negation*, **CDLSN**, paracomplete, there is no counterargument for not doing it. In Section 6 below, we provide a comparative analysis of **CDLSN** and our logic. Here we would like to stress the fact that the Jaśkowskian ideas, which led to one of the first paraconsistent logics ever, permit a discussive paracomplete logic such as **CDLSN** and a non-discussive paracomplete logic such as ours. This view reveals the pair of paraconsistency–paracompleteness as a harmonious tandem rather than a strictly opposed dichotomy. Indeed, the dichotomy in question reveals itself strikingly in the case of many-valued logics.[6] In contradistinction to paraconsistent logic, which is sometimes dubbed *logic with truth-value gluts* because a formula might be true and false simultaneously, paracomplete logic[7] is sometimes dubbed *logic with truth-value gaps* because a formula might be neither true nor false simultaneously. Let us confine ourselves to the case of three-valued logics for this approach to paracomplete logic seems to be the most popular in the literature: A formula of this kind is assigned the third truth-value which is not a designated one. Hence, the law of excluded middle and certain inference rules related to it fail (the *italics* are not ours): “A *paracomplete logic* is a logic, in which the *principle of excluded middle*, i.e., $A \vee \neg A$ is not a theorem of that

[5] Notice that the first formal system, consciously conceived as a logic invalidating Duns Scotus law, was developed by Stanisław Jaśkowski in 1948 [23], while ideas that can be regarded as the basis of paracomplete logics were explored in the 1960s (for example, [50]), with formal investigations in [29].

[6] Note that each logic that is thoroughly discussed in this paper is bivalent. Many-valuedness is needed to clarify the argument. For this aim, we use the terms ‘logic with truth-value gluts’ and ‘logic with truth-value gaps’, which come from many-valued logic. At that, we warn against the identification of ‘logic with truth-value gluts’ with paraconsistent logic and ‘logic with truth-value gaps’ with paracomplete logic. First, not all paraconsistent and paracomplete logics are many-valued. Second, logics with truth-value gluts and gaps are not always paraconsistent and paracomplete ones. As was shown in [48], many-valued logics (with gaps and gluts) satisfying Rosser and Turquette’s standard conditions [45, p. 26] have classical consequence relation. However, in many-valued semantics, gluts usually lead to paraconsistency, and gaps usually lead to paracompleteness.

[7] Among the first papers where this term appears are [29, 30].

logic" [3, p. 8]. Note that $A \to (\neg A \to B)$, one of the Jaśkowskian stimuli, might be a paracomplete theorem, and the situation is upside-down in the case of paraconsistency ($A \vee \neg A$ is a $\mathbf{D_2}$-theorem).

To conclude the detalization of the second source, we find it proper to briefly enlist some of the most famous paracomplete logics (apart from intuitionistic logic and Kolmogorov's system [25]): Kleene's strong three-valued logic $\mathbf{K_3}$ [24], whose absence of theoremhood led to setting up three-valued paracomplete logics with theoremhoods and robust implications (the ones validating modus ponens, etc.). Obviously, its implication extensions are paracomplete, among them Łukasiewicz's three-valued logic $\mathbf{Ł_3}$ [31] and the three-valued logic **PComp** (Słupecki, Bryll, and Prucnal are likely to be its authors [49]; the author of the name **PComp** is Popov [42]). There are other well-known three-valued paracomplete logics: Kleene's weak three-valued logic $\mathbf{K_3^w}$ [24], Bochvar's $\mathbf{B_3}$ [10], Heyting-Gödel-Jaśkowski's $\mathbf{G_3}$ [20, 18, 21], and Sette and Carnielli's $\mathbf{I^1}$ [47]. As for four-valued paracomplete logics, Pietz and Riveccio's [41] **ETL** deserves attention. Most fuzzy logics are paracomplete (and almost none of them is paraconsistent; see [7, 15] for some rare examples of paraconsistent fuzzy logics). At last, let us mention *paradefinite* [6] or *paranormal* [9, 43] logics, i.e., logics which are both paracomplete and paraconsistent. The most influential logic among them is Anderson-Belnap's **FDE** [5].

The harmonious tandem discussed above seems to be more striking if the reader pays attention to the fact that the Jaśkowskian ideas in question were not axiomatized by himself but later. (Let us, again, refer the reader to *status quo* in [38].) As a result, Jaśkowski's followers give his semi-formal intuitions about connecting robust discussive reasoning with **S5**[8] different (even mutually contradicting) formal insights. Roughly, the key points of those techniques are the same, though: two translation functions from a discussive language into a modal language, and *vice versa* together with the notion of the 'M-counterpart of **S5**' introduced by Perzanowski [39], where 'M' stands for the modal possibility operator. Our approach, in fact, follows the *spirit* of the techniques in question in spite of mirror-like transformations of their key points. The notion of the 'L-counterpart of **S5**', where 'L' stands for the modal necessity operator, is employed because the transformed translation functions employ the necessity operator rather

[8] Furmanowski proves that **S4** is enough to establish this connection [16]. This result was strengthened by Perzanowski [39], and Nasieniewski and Pietruszczak [33, 34].

than the possibility one. As one can see, it is not essential for **S5**, but it would be if the whole strategy were applied, for example, to non-normal modal logics. The notion of 'L-counterpart of **S5**' was also introduced by Perzanowski, and it naturally corresponds to the set **LS5** of all those **S5**-theses at the beginning of which there is the necessity operator that was used, in particular, by Kotas in giving his axiomatization of $\mathbf{D_2}$. Now the ray highlights discussive left-, right disjunction rather than discussive left-, right conjunction, as it has been done in the literature before. Hence, the difference between paraconsistency and paracompleteness in the discussive setting is shifted from the standard criterion that certain formulae are (not) theorems of the logic in question to the non-standard criterion that certain connectives are to be especially treated within certain discussive logics. On this path, we believe, some alternatives to Jaśkowski's, Akama-Abe-Nakamatsu's, and our approaches to discussive logic might be discovered.

2. On Jaśkowski's discussive logic $\mathbf{D_2}$

Following Omori and Alama [38], we distinguish between the three languages, $\mathscr{L}$, $\mathscr{L}_{\mathrm{r}}$, and $\mathscr{L}_{\mathrm{l}}$, over which $\mathbf{D_2}$ can be built. Note that such a possibility is also used in a question asked by João Marcos and considered in [35]. The former has the alphabet $\{\mathcal{P}, \neg, \vee, \rightarrow_{\mathrm{d}}, \wedge, (,)\}$, where $\mathcal{P} = \{p, q, r, s, p_1, \ldots\}$ is the set of propositional variables. The languages $\mathscr{L}_{\mathrm{r}}$ and $\mathscr{L}_{\mathrm{l}}$ have right and left discussive conjunctions $\wedge^{\mathrm{r}}_{\mathrm{d}}$ and $\wedge^{\mathrm{l}}_{\mathrm{d}}$, respectively, instead of $\wedge$. The sets of all $\mathscr{L}$-, $\mathscr{L}_{\mathrm{r}}$-, and $\mathscr{L}_{\mathrm{l}}$-formulas are defined in the standard way and denoted *via* $\mathscr{F}$, $\mathscr{F}_{\mathrm{r}}$, and $\mathscr{F}_{\mathrm{l}}$, respectively. We denote a propositional variable (in the metalanguage) by P, Q, etc., a discussive formula by A, B, C, etc., a modal formula by φ, ψ, γ, etc., and a set of discussive formulas by X. The language $\mathscr{L}_{\mathrm{m}}$ of the modal logic **S5** has the alphabet $\langle \mathcal{P}, \{\neg, \vee, \rightarrow, \wedge, \Box, \Diamond, (,)\}\rangle$. The set of all $\mathscr{L}_{\mathrm{m}}$-formulas is defined in the standard way and denoted *via* $\mathscr{F}_{\mathrm{m}}$. We write $\varphi \leftrightarrow \psi$ for $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$.

Following Jaśkowski, we give a translation function τ from $\mathscr{F} \cup \mathscr{F}_{\mathrm{r}} \cup \mathscr{F}_{\mathrm{l}}$ into $\mathscr{L}_{\mathrm{m}}$.

- $\tau(P) = P$, for any $P \in \mathcal{P}$,
- $\tau(\neg A) = \neg\tau(A)$,
- $\tau(A \vee B) = \tau(A) \vee \tau(B)$,
- $\tau(A \rightarrow_{\mathrm{d}} B) = \Diamond\tau(A) \rightarrow \tau(B)$,
- $\tau(A \wedge B) = \tau(A) \wedge \tau(B)$,
- $\tau(A \wedge^{\mathrm{r}}_{\mathrm{d}} B) = \tau(A) \wedge \Diamond\tau(B)$,
- $\tau(A \wedge^{\mathrm{l}}_{\mathrm{d}} B) = \Diamond\tau(A) \wedge \tau(B)$.

Jaśkowski [23] originally formulated $\mathbf{D_2}$ in $\mathscr{L}$. However, in his next paper [22] he switched to $\mathscr{L}_\mathrm{r}$.[9] The language $\mathscr{L}_\mathrm{l}$ was used by da Costa, Dubikajtis, Kotas, Achtelik and others [27, 14, 1] as well as by Vasyukov [51].

As noted in [38, Proposition 1], $\tau(A \wedge^\mathrm{r}_\mathrm{d} B) \leftrightarrow \tau(\neg(B \rightarrow_\mathrm{d} \neg A)) \in$ **S5** and $\tau(A \wedge^\mathrm{l}_\mathrm{d} B) \leftrightarrow \tau(\neg(A \rightarrow_\mathrm{d} \neg B)) \in$ **S5**. Obviously, in all three languages $\wedge$ can be expressed *via* $\neg$ and $\vee$. One may think about the fourth conjunction, $\wedge^\mathrm{b}_\mathrm{d}$ (we write 'b' for 'both', $\tau(A \wedge^\mathrm{b}_\mathrm{d} B) \leftrightarrow \Diamond\tau(A) \wedge \Diamond\tau(B)$). However, as noted by Ciuciura [12, p. 85], $(A \wedge^\mathrm{b}_\mathrm{d} B) \rightarrow_\mathrm{d} (\neg(A \wedge^\mathrm{b}_\mathrm{d} B) \rightarrow_\mathrm{d} C)$ will be valid then.

We will denote the formulations of $\mathbf{D_2}$ in $\mathscr{L}_\mathrm{r}$ and $\mathscr{L}_\mathrm{l}$, respectively, *via* $\mathbf{D_2}_\mathrm{r}$ and $\mathbf{D_2}_\mathrm{l}$. The set $\mathbf{D_2}$-tautologies is $\{A \in \mathscr{F} \mid \Diamond\tau(A) \in \mathbf{S5}\}$. Similarly, the sets of $\mathbf{D_2}_\mathrm{r}$- and $\mathbf{D_2}_\mathrm{l}$-tautologies are $\{A \in \mathscr{F}_\mathrm{r} \mid \Diamond\tau(A) \in \mathbf{S5}\}$ and $\{A \in \mathscr{F}_\mathrm{l} \mid \Diamond\tau(A) \in \mathbf{S5}\}$, respectively. Nowadays, $\mathbf{D_2}$ is usually referred to as the corrected version from [22] rather than from [23]: $\mathbf{D_2} = \{\mathbf{A} \in \mathscr{F}_\mathrm{r} \mid \Diamond\tau(\mathbf{A}) \in \mathbf{S5}\}$. It is this version of $\mathbf{D_2}$ with right discussive conjunction that we employ in this paper.

3. Paracomplete versions of $\mathbf{D_2}$

We fix three languages, $\mathscr{L}^*_\mathrm{r}$, $\mathscr{L}^*_\mathrm{l}$, and $\mathscr{L}^*_\mathrm{b}$, over which a variant of $\mathbf{D_2}$ can be built. The former has the alphabet $\{\mathcal{P}, \neg, \vee^\mathrm{r}_\mathrm{d}, \rightarrow^\mathrm{w}_\mathrm{d}, \wedge, (,)\}$, where $\vee^\mathrm{r}_\mathrm{d}$ is right discussive *disjunction*. The language $\mathscr{L}^*_\mathrm{l}$ has left discussive *disjunction* $\vee^\mathrm{l}_\mathrm{d}$ instead of $\vee^\mathrm{r}_\mathrm{d}$.[10] The language $\mathscr{L}^*_\mathrm{b}$ has $\vee^\mathrm{l}_\mathrm{b}$[11] instead of $\vee^\mathrm{r}_\mathrm{d}$. The sets of all $\mathscr{L}^*_\mathrm{r}$-, $\mathscr{L}^*_\mathrm{l}$-, and $\mathscr{L}^*_\mathrm{b}$-formulas are defined in the standard way and denoted *via* $\mathscr{F}^*_\mathrm{r}$, $\mathscr{F}^*_\mathrm{l}$, and $\mathscr{F}^*_\mathrm{b}$, respectively. We define a translation function σ from one of the languages $\mathscr{L}^*_\mathrm{r}$ or $\mathscr{L}^*_\mathrm{l}$ or $\mathscr{L}^*_\mathrm{b}$ to $\mathscr{L}_\mathrm{m}$:

- $\sigma(P) = P$, for any $P \in \mathcal{P}$,
- $\sigma(\neg A) = \neg\sigma(A)$,
- $\sigma(A \wedge B) = \sigma(A) \wedge \sigma(B)$,

[9] Strictly speaking, it was not explicitly said whether the new conjunction was meant to extend the original language or to replace the classical conjunction.

[10] As in the case of Jaśkowski's original model, we also refer to a model of discussion and try to articulate the respective translations in terms of the possible strategy of debaters that could be applied by them while formulating their own statements. That is why we let ourselves to treat the connectives of implication and disjunction as discussive.

[11] Again, 'b' stands for 'both'.

- $\sigma(A \to^{\mathrm{w}}_{\mathrm{d}} B) = \Box\sigma(A) \to \sigma(B)$,
- $\sigma(A \vee^{\mathrm{r}}_{\mathrm{d}} B) = \sigma(A) \vee \Box\sigma(B)$,
- $\sigma(A \vee^{\mathrm{l}}_{\mathrm{d}} B) = \Box\sigma(A) \vee \sigma(B)$,
- $\sigma(A \vee^{b}_{d} B) = \Box\sigma(A) \vee \Box\sigma(B)$.

We define the following logics (we associate a logic with its set of tautologies), where $i \in \{\mathrm{l}, \mathrm{r}, \mathrm{b}\}$:

- $\mathbf{D}^{\mathsf{p}}_{\mathbf{2}i} = \{A \in \mathscr{F}^{*}_{i} \mid \Box\sigma(A) \in \mathbf{S5}\}$.

In what follows, we write $\mathbf{D}^{\mathsf{p}}_{\mathbf{2}}$ for $\mathbf{D}^{\mathsf{p}}_{\mathbf{2}\mathrm{l}}$.

Let us emphasize that, as follows from this definition, $\mathbf{D}^{\mathsf{p}}_{\mathbf{2}}$ is embeddable into $\mathbf{S5}$ the translation σ. It is not excluded that one can map $\mathbf{S5}$ in $\mathbf{D}^{\mathsf{p}}_{\mathbf{2}}$, but this issue requires further research.

Let us say a few words about the intuitions that led to such an understanding of discussive connectives. In the original formulation, we have: "If anyone states that p, then q", so from the point of view of a given participant, there is not too much needed to say q. Here we have a much more mistrustful or misgiving position: to say q, it is needed that all participants state p.

However, the reader is not supposed to consider that such a debating model is unrealistic. In a sense, each debating model with some debaters having the power of veto is of this kind. To put it differently, participants are equal in the Jaśkowskian paraconsistent debating model and are not in the paracomplete one in quite the same way debaters are equal in expressing their views on international policy in Łazienki Królewskie, but in order to reach consensus in the United Nations Security Council, the power of veto that its five permanent members have is to be overcome. Generally, to some extent, at least some part of scientific knowledge and juridical process is built in this way: it is only when all the sources (witnesses, observers, experiments, participants in an experiment) jointly say some thesis that a specific conclusion can be added to the current state of knowledge.

One can similarly understand the case of disjunction (say, $\vee^{\mathrm{r}}_{\mathrm{d}}$). Either I am saying p or everyone is stating q, so this would express a kind of dilemma where we have an opposition of my own statements against statements expressed by all the other debaters. Note that the Jaśkowskian disjunction is classical; hence, $p \vee \neg p$ is valid there to the effect that "Everyone is stating that $p \vee \neg p$" expresses the capacity of any participant to state this particular logical law in the course of a debate independently of statements expressed by the other debaters. Paracomplete disjunction here is not classical; hence, $p \vee^{\mathrm{r}}_{\mathrm{d}} \neg p$ is invalid here to the effect that the underlying dilemma "Either I am saying p or everyone is stating that $\neg p$" is, obviously, not characteristic of any debate. To be sure, the paracomplete model of discussion allows any participant to state logical laws, too, but $p \vee^{\mathrm{r}}_{\mathrm{d}} \neg p$ is not among them.

Interestingly, discussive conjunction and disjunction are, in a sense, interdefinable. More strictly, if we would like to compare standard discussive logics and paracomplete discussive ones, we can use some additional translations that show that particular logics are interdefinable. This could be proved inductively by restricting languages to the $\{\neg, \wedge^{\mathrm{l}}_{\mathrm{d}}\}$-part on the discussive side and to the $\{\neg, \vee^{\mathrm{l}}_{\mathrm{d}}\}$-part on the paracomplete side. Of course, this could also be extended to the full languages. So, using inductive hypotheses, we would obtain:

$\tau(\neg(\neg A \wedge^{\mathrm{l}}_{\mathrm{d}} \neg B)) = \neg(\Diamond\neg\tau(A) \wedge \neg\tau(B)) \leftrightarrow \neg\Diamond\neg\tau(A) \vee \neg\neg\tau(B) \leftrightarrow \Box\tau(A) \vee \tau(B) \leftrightarrow_{\textsf{by ind}} \Box\sigma(A) \vee \sigma(B) = \sigma(A \vee^{\mathrm{l}}_{\mathrm{d}} B)$,

$\sigma(\neg(\neg A \vee^{\mathrm{l}}_{\mathrm{d}} \neg B)) = \neg(\Box\neg\sigma(A) \vee \neg\sigma(B)) \leftrightarrow \neg\Box\neg\sigma(A) \wedge \neg\neg\sigma(B) \leftrightarrow \Diamond\sigma(A) \wedge \sigma(B) \leftrightarrow_{\textsf{by ind}} \Diamond\tau(A) \wedge \tau(B) = \tau(A \wedge^{\mathrm{l}}_{\mathrm{d}} B)$.

The logic $\mathbf{D}^{\mathsf{p}}_{\mathbf{2}}$ has the following axioms (where $\bot$ denotes $p \wedge \neg p$).

Ax$_1$ $A \to^{\mathrm{w}}_{\mathrm{d}} (B \to^{\mathrm{w}}_{\mathrm{d}} A)$

Ax$_2$ $(A \to^{\mathrm{w}}_{\mathrm{d}} (B \to^{\mathrm{w}}_{\mathrm{d}} C)) \to^{\mathrm{w}}_{\mathrm{d}} ((A \to^{\mathrm{w}}_{\mathrm{d}} B) \to^{\mathrm{w}}_{\mathrm{d}} (A \to^{\mathrm{w}}_{\mathrm{d}} C))$

Ax$_3$ $\neg\big(A \wedge \neg(A \wedge A)\big)$

Ax$_4$ $\neg\big((A \wedge B) \wedge \neg A\big)$

Ax$_5$ $\neg\big(\neg(A \wedge \neg B) \wedge \neg\neg(\neg(B \wedge C) \wedge \neg\neg(C \wedge A))\big)$

Ax$_6$ $\neg\neg A \to^{\mathrm{w}}_{\mathrm{d}} A$

Ax$_7$ $\neg(\neg(A \to^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg A)$

Ax$_8$ $\neg(\neg\neg((A \to^{\mathrm{w}}_{\mathrm{d}} \bot) \to^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg A)$

Ax$_9$ $\neg\big(\neg(\neg(A \wedge \neg B) \to^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg\neg(\neg(A \to^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg\neg(B \to^{\mathrm{w}}_{\mathrm{d}} \bot))\big)$

Ax$_{10}$ $\neg\big(A \wedge \neg\neg(A \to^{\mathrm{w}}_{\mathrm{d}} \bot)\big)$

Ax$_{11}$ $\neg\big(\neg\big(\neg(A \to^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg B\big) \wedge \neg(A \to^{\mathrm{w}}_{\mathrm{d}} B)\big)$.

Ax$_{12}$ $\neg\big((A \to^{\mathrm{w}}_{\mathrm{d}} B) \wedge \neg\neg(\neg(A \to^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg B)\big)$

Ax_{13} $\neg\big((A \vee^{\mathrm{l}}_{\mathrm{d}} B) \wedge \neg\neg(\neg\neg(A \to^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg B)\big)$

Ax_{14} $\neg\big(\neg(\neg\neg(A \to^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg B) \wedge \neg(A \vee^{\mathrm{l}}_{\mathrm{d}} B)\big)$

Ax_{15} $A \to^{\mathrm{w}}_{\mathrm{d}} \big((\neg\neg\big(\neg(A \wedge \neg B) \to^{\mathrm{w}}_{\mathrm{d}} \bot\big) \to^{\mathrm{w}}_{\mathrm{d}} \bot\big) \to^{\mathrm{w}}_{\mathrm{d}} B\big)$

Ax_{16} $A \to^{\mathrm{w}}_{\mathrm{d}} \neg(A \to^{\mathrm{w}}_{\mathrm{d}} \bot)$

Ax_{17} $\neg(A \wedge \neg B) \to^{\mathrm{w}}_{\mathrm{d}} (A \to^{\mathrm{w}}_{\mathrm{d}} B)$

$$\frac{A \quad A \to^{\mathrm{w}}_{\mathrm{d}} B}{B} \tag{$\mathrm{MP}^{\mathrm{w}}_{\mathrm{d}}$}$$

Lemma 3.1 (Deduction theorem). *$X, A \vdash_{\mathbf{D}_2^{\mathsf{p}}} B$ iff $X \vdash_{\mathbf{D}_2^{\mathsf{p}}} A \to^{\mathrm{w}}_{\mathrm{d}} B$.*

Proof: The proof is textbookian in the presence of Ax_1, Ax_2, and ($\mathrm{MP}^{\mathrm{w}}_{\mathrm{d}}$). □

One can see that any proof given on the basis of classical logic expressed in the language with $\neg$ and $\wedge$ by means of Ax_3–Ax_5 and the respective form of modus ponens can be transferred into a $\mathbf{D}_2^{\mathsf{p}}$-proof by Ax_{17}.

Fact 3.2. For any thesis A of classical logic in the language with $\neg$ and $\wedge$, $\vdash_{\mathbf{D}_2^{\mathsf{p}}} A$.

3.1. Lewis's intensional implication and disjunction

Intensional implication and disjunction introduced by Lewis in the systems **S1-S5** had a deep influence on modern modal logic (especially the former, which is mostly dubbed *strict implication*). It is the well-knownness of intensional implication and disjunction that allows us to skip details (Lewis's motivation to introduce it, analyzing its pros and cons, etc.) and address those properties of them that concern the purpose of our study only.[12] We begin with strict implication, which we do not denote with the Lewisian fishhook but with $\to_L$, so that its traditional definition looks as follows: $\varphi \to_L \psi =_{\mathrm{df}} \Box(\varphi \to \psi)$. We are interested in two arguments: the one by Jaśkowski who rejects $\to_L$ in the quality of discussive implication, and the one by Lewis, who rejects the known classical equivalence between implication and disjunction.

[12] An accurate introduction to Lewis's ideas and their impact on modern modal logic is in [8].

Let us remind the reader that Jaśkowski devotes a passage to the Lewisian implication while describing the known solutions to the problem of formulating the logic of inconsistent systems [23, p. 40]. In more detail, he rejects it due to its weakness: "But the set of the theses which include strict implication only, and do not include material implication, is very limited" [23, p. 40].[13] Our implication is stronger than $\rightarrow_L$ because $\Box(\varphi \rightarrow \psi) \models_{\mathbf{S5}} \Box(\Box\varphi \rightarrow \psi)$, and the opposite is false, though.[14] Moreover, in contrast to $\rightarrow_L$, our implication is not paraconsistent because $\neg A \rightarrow^{\mathrm{w}}_{\mathrm{d}} (A \rightarrow^{\mathrm{w}}_{\mathrm{d}} B)$ is valid. On the other hand, one of Lewis's motivations is to avoid *paradoxes of material implication* which are valid in our system.[15] Our implication is similar to $\rightarrow_L$ with respect to the classical one, viz., it is stronger than it because $\varphi \rightarrow \psi \models_{\mathbf{S5}} \Box\varphi \rightarrow \psi$, and the opposite is false, though. This fact means that Quine's critique on $\rightarrow_L$, which roughly bases on the fact that even for **S1**, if $\varphi \rightarrow \psi$ is a theorem, then $\varphi \rightarrow_L \psi$ is a theorem, either, holds true for our implication [8].

Another suggestion by Lewis is about the classical equivalence between implication and disjunction: $A \rightarrow B =_{\mathrm{df}} \neg A \vee B$. In our logic, it fails: $\neg A \vee^{\mathrm{l}}_{\mathrm{d}} B \models A \rightarrow^{\mathrm{w}}_{\mathrm{d}} B$, and the opposite is false, though, because $\Box(\Box\neg\varphi \vee \psi) \models_{\mathbf{S5}} \Box(\Box\varphi \rightarrow \psi)$ with the opposite being false. This is in sharp contrast to Lewis, who bases upon MacColl's ideas that the failure of the above-mentioned equivalence is caused by $\neg A \vee_L B \not\models A \rightarrow_L B$, where $\vee_L$ stands for the Lewisian disjunction, while the opposite is true: "Lewis infers that disjunction too must be given a new intensional sense, according to which $(p \vee q)$ holds just in case if p were not the case it would have to be the case that q. Considerations of this sort, based on the distinction between extensional and intensional readings of the connectives, were not original to Lewis. Already [...] MacColl [...] claimed that $(p \rightarrow q)$ and $(\neg p \vee q)$ are not equivalent: $(\neg p \vee q)$ follows from $(p \rightarrow q)$, but not *vice versa*" [8].

[13] Perzanowski, who is the editor of the contemporary translation of both Jaśkowski's papers, notes: "Observe that the present criticism in comparison with the previous one, is rather weak. Some calculi of the strict implication can thereby be treated as paraconsistent ones" [23, p. 56].

[14] Recall that the same analysis shows that our implication is stronger than the Jaśkowskian one.

[15] Note that avoiding those paradoxes is not our motivation whatsoever.

4. Soundness and completeness

4.1. L-counterpart of S5

First of all, let us recall the axiomatics of the modal logic **S5**. It can be axiomatized by following axioms and rules[16]:

$$\text{All the axiom schemes of } \mathbf{CPL} \tag{CPL}$$

$$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \quad \text{(K)} \qquad \Box\varphi \to \Box\Box\varphi \quad \text{(4)}$$

$$\Box\varphi \to \varphi \quad \text{(T)} \qquad \Diamond\Box\varphi \to \varphi \quad (\mathsf{B^d})$$

$$\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi \tag{Def$_\Diamond$}$$

$$\frac{\varphi \quad \varphi \to \psi}{\psi} \quad \text{(MP)} \qquad \frac{\varphi}{\Box\varphi} \quad \text{(Nec)}$$

Instead of ($\mathsf{B^d}$), the formula (B) $\varphi \to \Box\Diamond\varphi$ is usually used. As it is known (see, e. g., [17, p. 44–45]), these formulas are replaceable in all normal logics.

Da Costa and Dubikajtis [14] as well as Omori and Alama [38] used the notion of "M-counterpart of **S5**" denoted as M(**S5**) (following Perzanowski's terminology [39]), where $\mathrm{M}(\mathbf{S5}) = \{\varphi \in \mathscr{F}_{\mathrm{m}} | \vdash_{\mathbf{S5}} \Diamond\varphi\}$. While changing $\Diamond$ to $\Box$ in the definitions of the discussive connectives, we also incline towards an application of the same point of view when formally explicating the point of view of an external observer in Jaśkowski's model of discussion. In the presented variant, the external observer would be more careful by accepting a given discussive thesis only when its translated modal version is necessarily accepted. That is why we use the notion of "L-counterpart of **S5**" denoted as L(**S5**) (following Perzanowski's terminology again, where $\mathrm{L}(\mathbf{S5}) = \{\varphi \in \mathscr{F}_{\mathrm{m}} | \vdash_{\mathbf{S5}} \Box\varphi\}$), in the definition of the proposed variant of discussive logic. However, observe the below Fact that follows from [39, (3.6)].

FACT 4.1. L(**S5**) = **S5**.

Let us give an axiomatization of L(**S5**) (taking into account the given above Fact 4.1, it is also an axiomatization of **S5**) corresponding to the

[16] '**CPL**' is for classical propositional logic, of course. We can consider any fixed axiomatization of **CPL** or just take all theses of **CPL**.

recalled above axiomatization of **S5** and useful for our next considerations. Thus, we present the system **JL**. It corresponds also in a way to Kotas's axiomatization of **LS5** — the set of all theses of **S5** having $\Box$ at the beginning [26]. The relation $\vdash_{\mathbf{JL}}$ is determined by the following axioms and rules of inference.

AJL$_1$ $\Diamond\Box\varphi$, where φ is an axiom scheme of a fixed axiomatization of **CPL**,

AJL$_2$ $\Diamond\Box(\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi))$

AJL$_3$ $\Diamond\Box(\Box\varphi \to \varphi)$

AJL$_4$ $\Diamond\Box(\varphi \to \Diamond\varphi)$[17]

AJL$_5$ $\Diamond\Box(\Box\varphi \to \Box\Box\varphi)$

AJL$_6$ $\Diamond\Box(\Diamond\Box\varphi \to \varphi)$

AJL$_7$ $\Diamond\Box(\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi)$

RJL$_1$ $\dfrac{\varphi \quad \Diamond\Box(\varphi \to \psi)}{\psi}$

RJL$_2$ $\dfrac{\varphi}{\Box\varphi}$

Lemma 4.2. *If* $\vdash_{\mathbf{JL}} \varphi$, *then* $\vdash_{\mathbf{S5}} \Box\varphi$.

Proof: Induction on the length of a derivation of φ in **JL**. Suppose that $\vdash_{\mathbf{JL}} \varphi$. Then φ is an axiom or is obtained by RJL$_1$ or RJL$_2$.

Let φ be an axiom. It is standard that if $\varphi \in$ **CPL** or it is an instance of (K), (T), (T$^{\mathsf{d}}$), (4), (B$^{\mathsf{d}}$) or (Def$_\Diamond$), then $\Box\Diamond\Box\varphi \in$ **S5**. Thus, $\vdash_{\mathbf{S5}} \Box\Diamond\Box\varphi$, i.e., $\vdash_{\mathbf{S5}} \Box\varphi$.

[17] Of course, (T$^{\mathsf{d}}$): $\varphi \to \Diamond\varphi$ is derivable on basis of the given axiomatization of **S5**. Note also that the axiom AJL$_4$ is needed to rebuild $\Diamond\Box$ before formulae obtained by (MP) (see Lemma 4.3 below). The need for AJL$_4$, whose derivability on the basis of the rest of the system $\vdash_{\mathbf{JL}}$ goes beyond the scope of the paper, is connected with saving 1–1 correspondence between L(**S5**) and **JL**.

Let φ be obtained by RJL$_1$ from some formulas ψ and $\Diamond\Box(\psi \to \varphi)$. By the inductive hypothesis, $\vdash_{\mathbf{S5}} \Box\psi$ and $\vdash_{\mathbf{S5}} \Box\Diamond\Box(\psi \to \varphi)$. Since $\vdash_{\mathbf{S5}} \Box\Diamond\Box F \leftrightarrow \Box F$, for any formula F (see, e. g. [17, p. 43]), we have $\vdash_{\mathbf{S5}} \Box(\psi \to \varphi)$. Since $\vdash_{\mathbf{S5}} \Box(\psi \to \varphi) \to (\Box\psi \to \Box\varphi)$, we have $\vdash_{\mathbf{S5}} \Box\varphi$.

Let φ be obtained by RJL$_2$ from some formula ψ. Then φ has the form of $\Box\psi$. By the inductive hypothesis, $\vdash_{\mathbf{S5}} \Box\psi$. By Gödel's rule, we get $\vdash_{\mathbf{S5}} \Box\Box\psi$, i.e., $\vdash_{\mathbf{S5}} \Box\varphi$. □

Lemma 4.3. *If* $\vdash_{\mathbf{S5}} \varphi$, *then* $\vdash_{\mathbf{JL}} \Diamond\Box\varphi$.

Proof: Induction on the length of a derivation of φ in **S5**. Suppose that $\vdash_{\mathbf{S5}} \varphi$. Then φ is an axiom or is obtained by modus ponens (MP) or by Gödel's rule.

Let φ be an axiom. Then $\Diamond\Box\varphi$ is an axiom of **JL**.

Let φ be obtained by (MP) from some formulas γ and $\gamma \to \varphi$. By the inductive hypothesis, $\vdash_{\mathbf{JL}} \Diamond\Box\gamma$ and $\vdash_{\mathbf{JL}} \Diamond\Box(\gamma \to \varphi)$. Using AJL$_6$ (i.e., $\Diamond\Box(\Diamond\Box\gamma \to \gamma)$) and RJL$_1$, we get $\vdash_{\mathbf{JL}} \gamma$. Then, by RJL$_1$, $\vdash_{\mathbf{JL}} \varphi$. By RJL$_2$, $\vdash_{\mathbf{JL}} \Box\varphi$. By AJL$_4$, $\vdash_{\mathbf{JL}} \Diamond\Box(\Box\varphi \to \Diamond\Box\varphi)$. Hence, by RJL$_1$, $\vdash_{\mathbf{JL}} \Diamond\Box\varphi$.

Let φ be obtained by Gödel's rule from some formula γ. Then φ has the form of $\Box\gamma$. By the inductive hypothesis, $\vdash_{\mathbf{JL}} \Diamond\Box\gamma$. Using AJL$_6$ and RJL$_1$, we get $\vdash_{\mathbf{JL}} \gamma$. Applying RJL$_2$ twice, we obtain $\vdash_{\mathbf{JL}} \Box\Box\gamma$. By AJL$_6$ and RJL$_1$, $\vdash_{\mathbf{JL}} \Diamond\Box\Box\gamma$, i.e., $\vdash_{\mathbf{JL}} \Diamond\Box\varphi$. □

Lemma 4.4. $\varphi \in \mathrm{L}(\mathbf{S5})$ *iff* $\vdash_{\mathbf{JL}} \varphi$.

Proof: Suppose that $\varphi \in \mathrm{L}(\mathbf{S5})$. Then $\vdash_{\mathbf{S5}} \Box\varphi$, by the definition of $\mathrm{L}(\mathbf{S5})$. By Lemma 4.3, $\vdash_{\mathbf{JL}} \Diamond\Box\Box\varphi$. By AJL$_6$, $\vdash_{\mathbf{JL}} \Diamond\Box(\Diamond\Box\Box\varphi \to \Box\varphi)$. By RJL$_1$, $\vdash_{\mathbf{JL}} \Box\varphi$. By AJL$_3$, $\vdash_{\mathbf{JL}} \Diamond\Box(\Box\varphi \to \varphi)$, hence, using RJL$_1$, we infer $\vdash_{\mathbf{JL}} \varphi$.

Suppose that $\vdash_{\mathbf{JL}} \varphi$. Then $\vdash_{\mathbf{S5}} \Box\varphi$, by Lemma 4.2. By the definition of $\mathrm{L}(\mathbf{S5})$, $\varphi \in \mathrm{L}(\mathbf{S5})$. □

4.2. L-counterpart of S5 and paracomplete discussive logic

The system **JL** introduced above employs a modal rather than a discussive language.

Let us introduce a translation function π from $\mathscr{L}_{\mathrm{m}}$ to $\mathscr{L}_{\mathrm{r}}^*$, $\mathscr{L}_{\mathrm{l}}^*$, and $\mathscr{L}_{\mathrm{b}}^*$ (where $\perp$ denotes $p \wedge \neg p$):

- $\pi(P) = P$, for any $P \in \mathcal{P}$,
- $\pi(\neg\varphi) = \neg\pi(\varphi)$,
- $\pi(\Diamond\varphi) = \neg\pi(\varphi) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot$,
- $\pi(\Box\varphi) = \neg(\pi(\varphi) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot)$,
- $\pi(\varphi \wedge \psi) = \pi(\varphi) \wedge \pi(\psi)$.
- $\pi(\varphi \vee \psi) = \neg(\neg\pi(\varphi) \wedge \neg\pi(\psi))$,
- $\pi(\varphi \rightarrow \psi) = \neg(\pi(\varphi) \wedge \neg\pi(\psi))$.

We are going to use the following two axiomatizations of **S5** in the $\{\neg, \wedge, \Box\}$-language for the reasons connected with the translation π given above. We use Rosser's [44, p. 55] axiomatization of **CPL** in the $\{\neg, \wedge\}$-language.[18]

The given below consequence relation meant for **S5** is denoted by $\vdash_{\mathbf{S5}^{\neg\wedge}}$:

Ax$_1$ $\neg\big(\varphi \wedge \neg(\varphi \wedge \varphi)\big)$

Ax$_2$ $\neg\big((\varphi \wedge \psi) \wedge \neg\varphi\big)$

Ax$_3$ $\neg\big(\neg(\varphi \wedge \neg\psi) \wedge \neg\neg(\neg(\psi \wedge \gamma) \wedge \neg\neg(\gamma \wedge \varphi))\big)$

Ax$_4$ $\neg\big(\Box\neg(\varphi \wedge \neg\psi) \wedge \neg\neg(\Box\varphi \wedge \neg\Box\psi)\big)$

Ax$_5$ $\neg(\Diamond\varphi \wedge \neg\neg\Box\neg\varphi) \wedge \neg(\neg\Box\neg\varphi \wedge \neg\Diamond\varphi)$

Ax$_6$ $\neg(\Box\varphi \wedge \neg\varphi)$

Ax$_7$ $\neg(\Box\varphi \wedge \neg\Box\Box\varphi)$

Ax$_8$ $\neg(\Diamond\Box\varphi \wedge \neg\varphi)$

RS5$_1$ $\dfrac{\varphi \quad \neg(\varphi \wedge \neg\psi)}{\psi}$ RS5$_2$ $\dfrac{\varphi}{\Box\varphi}$

Let us denote the function from $\mathscr{L}_{\mathrm{m}}$ to $\mathscr{L}_{\mathrm{m}}$ by δ that operates as π for $\neg$, $\wedge$, $\vee$, and $\rightarrow$, while for modal operators we assume that $\delta(\Diamond\varphi) = \Diamond\delta(\varphi)$ and $\delta(\Box\varphi) = \Box\delta(\varphi)$. We have an easy-to-see:

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[18]The original Rosserian axioms look as follows: (1) $P \supset PP$, (2) $PQ \supset P$, (3) $P \supset Q. \supset .{\sim}(QR) \supset {\sim}(RP)$. Note that due to the invalidity of $(A \rightarrow^{\mathrm{w}}_{\mathrm{d}} B) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \neg(A \wedge \neg B)$ on the basis of $\mathbf{D}^{\mathsf{p}}_{\mathbf{2}}$, one cannot interpret implication in the Rosserian axiomatization as $\rightarrow^{\mathrm{w}}_{\mathrm{d}}$.

FACT 4.5. For any formula $\varphi \in \mathscr{F}_{\mathrm{m}}$

1. $\varphi \in \mathbf{S5}$ iff $\vdash_{\mathbf{S5}^{\neg\wedge}} \delta(\varphi)$,
2. if φ is expressed in the language with $\{\Diamond, \Box, \neg, \wedge\}$ and $\vdash_{\mathbf{S5}^{\neg\wedge}} \varphi$, then $\varphi \in \mathbf{S5}$.

We will use another consequence relation denoted as $\vdash_{\mathbf{JL}^{\neg\wedge}}$ that corresponds to $\vdash_{\mathbf{JL}}$.

AJL$_1$ $\Diamond\Box\neg\big(\varphi \wedge \neg(\varphi \wedge \varphi)\big)$

AJL$_2$ $\Diamond\Box\neg\big((\varphi \wedge \psi) \wedge \neg\varphi\big)$

AJL$_3$ $\Diamond\Box\neg\big(\neg(\varphi \wedge \neg\psi) \wedge \neg\neg(\neg(\psi \wedge \gamma) \wedge \neg\neg(\gamma \wedge \varphi))\big)$

AJL$_4$ $\Diamond\Box\neg\big(\Box\neg(\varphi \wedge \neg\psi) \wedge \neg\neg(\Box\varphi \wedge \neg\Box\psi)\big)$

AJL$_5$ $\Diamond\Box(\neg(\Diamond\varphi \wedge \neg\neg\Box\neg\varphi) \wedge \neg(\neg\Box\neg\varphi \wedge \neg\Diamond\varphi))$

AJL$_6$ $\Diamond\Box\neg(\Box\varphi \wedge \neg\varphi)$

AJL$_7$ $\Diamond\Box\neg(\varphi \wedge \neg\Diamond\varphi)$

AJL$_8$ $\Diamond\Box\neg(\Box\varphi \wedge \neg\Box\Box\varphi)$

AJL$_9$ $\Diamond\Box\neg(\Diamond\Box\varphi \wedge \neg\varphi)$

RJL$_1$ $\dfrac{\varphi \quad \Diamond\Box\neg(\varphi \wedge \neg\psi)}{\psi}$ RJL$_2$ $\dfrac{\varphi}{\Box\varphi}$

FACT 4.6. For any formula φ in the language with $\Diamond, \Box, \neg, \wedge$:

$$\vdash_{\mathbf{JL}} \varphi \text{ iff } \vdash_{\mathbf{JL}^{\neg\wedge}} \varphi$$

LEMMA 4.7. *The following rule is inferable on the basis of* $\mathbf{D}_\mathbf{2}^{\mathsf{p}}$*:*

$$\frac{D}{\neg\neg(D \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot} \qquad (\Diamond\Box^{\pi})$$

PROOF:

1. D assumption
2. $\neg\neg(D \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot)$ premiss
3. $\neg\neg(D \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot) \rightarrow^{\mathrm{w}}_{\mathrm{d}} (D \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot)$ Ax_6
4. $D \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot$ $(\mathrm{MP}^{\mathrm{w}}_{\mathrm{d}})$: 2, 3
5. $\bot$ $(\mathrm{MP}^{\mathrm{w}}_{\mathrm{d}})$: 1, 4
6. $D \vdash_{\mathbf{D_2^p}} \neg\neg(D \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot$ Lemma 3.1: 1–5

$\square$

LEMMA 4.8. *For any axiom Ax of $\vdash_{\mathbf{JL}^{\neg\wedge}}$, $\vdash_{\mathbf{D_2^p}} \pi(Ax)$.*

PROOF:

- The case of AJL_1: $\Diamond\Box\neg\big(\varphi \wedge \neg(\varphi \wedge \varphi)\big)$.

  Since $\pi(\Diamond\Box\neg\big(\varphi\wedge\neg(\varphi\wedge\varphi)\big)) = \neg\neg\Big(\neg\big(\pi(\varphi)\wedge\neg(\pi(\varphi)\wedge\pi(\varphi))\big) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot\Big) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot$, we apply $(\Diamond\Box^{\pi})$ for $D = \neg\big(\pi(\varphi) \wedge \neg(\pi(\varphi) \wedge \pi(\varphi))\big)$—an instance of Ax_3.

- The case of AJL_2: $\Diamond\Box\neg\big((\varphi \wedge \psi) \wedge \neg\varphi\big)$.

  Since $\pi\big(\Diamond\Box\neg((\varphi\wedge\psi)\wedge\neg\varphi)\big) = \neg\neg\Big(\neg\big((\pi(\varphi)\wedge\pi(\psi))\wedge\neg\pi(\varphi)\big) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot\Big) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot$, we apply $(\Diamond\Box^{\pi})$ for $D = \neg\big((\pi(\varphi) \wedge \pi(\psi)) \wedge \neg\pi(\varphi)\big)$—an instance of Ax_4.

- The case of AJL_3: $\Diamond\Box\neg\big(\neg(\varphi \wedge \neg\psi) \wedge \neg\neg(\neg(\psi \wedge \gamma) \wedge \neg\neg(\gamma \wedge \varphi))\big)$.

  Since $\pi\big(\Diamond\Box\neg(\neg(\varphi\wedge\neg\psi)\wedge\neg\neg(\neg(\psi\wedge\gamma)\wedge\neg\neg(\gamma\wedge\varphi)))\big) = \neg\neg\Big(\neg\big(\neg(\pi(\varphi) \wedge \neg\pi(\psi)) \wedge \neg\neg(\neg(\pi(\psi) \wedge \pi(\gamma)) \wedge \neg\neg(\pi(\gamma) \wedge \pi(\varphi)))\big) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot\Big) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot$, we apply $(\Diamond\Box^{\pi})$ for $D = \neg\big(\neg(\pi(\varphi) \wedge \neg\pi(\psi)) \wedge \neg\neg(\neg(\pi(\psi) \wedge \pi(\gamma)) \wedge \neg\neg(\pi(\gamma) \wedge \pi(\varphi)))\big)$—an instance of Ax_5.

- The case of AJL_4: $\Diamond\Box\neg\big(\neg\Box(\varphi \wedge \neg\psi) \wedge \neg\neg(\Box\varphi \wedge \neg\Box\psi)\big)$.

  Since $\pi\big(\Diamond\Box\neg(\neg\Box(\varphi \wedge \neg\psi) \wedge \neg\neg(\Box\varphi \wedge \neg\Box\psi))\big) = \neg\neg\Big(\neg\big(\neg\neg((\pi(\varphi) \wedge \neg\pi(\psi)) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg\neg(\neg(\pi(\varphi) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg\neg(\pi(\psi) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot))\big) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot\Big) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot$, we apply ($\Diamond\Box^{\pi}$) for $D = \neg\big(\neg\neg((\pi(\varphi) \wedge \neg\pi(\psi)) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg\neg(\neg(\pi(\varphi) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg\neg(\pi(\psi) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot))\big)$—an instance of Ax_9.

- The case of AJL_5: $\Diamond\Box(\neg(\Diamond\varphi \wedge \neg\neg\Box\neg\varphi) \wedge \neg(\neg\Box\neg\varphi \wedge \neg\Diamond\varphi))$.

  Since $\pi(\Diamond\Box(\neg(\Diamond\varphi \wedge \neg\neg\Box\neg\varphi) \wedge \neg(\neg\Box\neg\varphi \wedge \neg\Diamond\varphi))) = \neg\neg\Big(\big(\neg((\neg\pi(\varphi) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg\neg\neg(\neg\pi(\varphi) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot)) \wedge \neg(\neg\neg(\neg\pi(\varphi) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg(\neg\pi(\varphi) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot))\big) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot\Big) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot$. We apply ($\Diamond\Box^{\pi}$) for $D = \big(\neg((\neg\pi(\varphi) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg\neg\neg(\neg\pi(\varphi) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot)) \wedge \neg(\neg\neg(\neg\pi(\varphi) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg(\neg\pi(\varphi) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot))\big)$—an instance of classical thesis that is inferable by Fact 3.2.

- The case of AJL_6: $\Diamond\Box\neg(\Box\varphi \wedge \neg\varphi)$.

  Since $\pi(\Diamond\Box\neg(\Box\varphi \wedge \neg\varphi)) = \neg\neg\big(\neg(\neg(\pi(\varphi) \rightarrow \bot) \wedge \neg\pi(\varphi)) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot\big) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot$, we apply ($\Diamond\Box^{\pi}$) for $D = \neg(\neg(\pi(\varphi) \rightarrow \bot) \wedge \neg\pi(\varphi))$—an instance of Ax_7.

- The case of AJL_7: $\Diamond\Box\neg(\varphi \wedge \neg\Diamond\varphi)$

  Since $\pi(\Diamond\Box\neg(\varphi \wedge \neg\Diamond\varphi)) = \neg\neg\big(\neg(\pi(\varphi) \wedge \neg(\neg\pi(\varphi) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot)) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot\big) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot$, we apply ($\Diamond\Box^{\pi}$) for $D = \neg(\pi(\varphi) \wedge \neg(\neg\pi(\varphi) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot))$—following from Ax_7 by Fact 3.2.

- The case of AJL_8: $\Diamond\Box\neg(\Box\varphi \wedge \neg\Box\Box\varphi)$.

  Since $\pi(\Diamond\Box\neg(\Box\varphi \wedge \neg\Box\Box\varphi)) = \neg\neg\Big(\neg\big(\neg(\pi(\varphi) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg\neg(\neg(\pi(\varphi) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot)\big) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot\Big) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot$, we apply ($\Diamond\Box^{\pi}$) for $D = \neg\big(\neg(\pi(\varphi) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg\neg(\neg(\pi(\varphi) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot)\big)$—an instance of Ax_{10}.

- The case of AJL_9: $\Diamond\Box\neg(\Diamond\Box\varphi \wedge \neg\varphi)$.

  Since $\Diamond\Box\neg(\Diamond\Box\varphi \wedge \neg\varphi) = \neg\neg\Big(\neg\big((\neg\neg(\pi(\varphi) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg\pi(\varphi)\big) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot\Big) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot$, we apply ($\Diamond\Box^{\pi}$) for $D = \neg\big((\neg\neg(\pi(\varphi) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg\pi(\varphi)\big)$—an instance of Ax_8. $\Box$

Now we want to show that in $\mathbf{D_2^p}$, the specific discussive connectives $\to_\mathrm{d}^\mathrm{w}$ and $\vee_\mathrm{d}^\mathrm{l}$ are properly characterized. Such characterizations are in the form of discussive implications in both directions.

LEMMA 4.9. *For any* $C \in \mathscr{L}_\mathrm{l}^*$, *it holds* $\vdash_{\mathbf{D_2^p}} \pi(\sigma(C)) \to_\mathrm{d}^\mathrm{w} C$.

PROOF: First, we show that $\vdash_{\mathbf{D_2^p}} \neg(\pi(\sigma(A)) \wedge \neg A)$. To obtain that for any A, $\vdash_{\mathbf{D_2^p}} \neg(\pi(\sigma(A)) \wedge \neg A)$, we can prove $\neg(\pi(\sigma(A)) \wedge \neg A)$ and additionally $\neg(A \wedge \neg\pi(\sigma(A)))$ using simultaneous induction on the construction of A.

The case when A is a variable is trivial due to the fact that (Ax$_3$)–(Ax$_5$) with (MP) expressed for the language $\{\wedge, \neg\}$ constitute the complete axiomatization of classical logic. Similarly, due to the fact that $\pi(\sigma(\neg B)) = \neg\pi(\sigma(B))$ and $\pi(\sigma(B \wedge C)) = \pi(\sigma(B)) \wedge \pi(\sigma(C))$, the cases of $\wedge$ and $\neg$ follow by inductive hypotheses for B and C, and extensionality for classical logic expressed in $\{\wedge, \neg\}$.

Case $A = B \to_\mathrm{d}^\mathrm{w} C$.

By definitions $\neg\big(\pi(\sigma(B \to_\mathrm{d}^\mathrm{w} C)) \wedge \neg(B \to_\mathrm{d}^\mathrm{w} C)\big) = \neg\big(\pi(\Box\sigma(B) \to \sigma(C)) \wedge \neg(B \to_\mathrm{d}^\mathrm{w} C)\big) = \neg\big(\neg(\neg(\pi(\sigma(B)) \to_\mathrm{d}^\mathrm{w} \bot) \wedge \neg\pi(\sigma(C))) \wedge \neg(B \to_\mathrm{d}^\mathrm{w} C)\big)$ and $\neg\big((B \to_\mathrm{d}^\mathrm{w} C) \wedge \neg\pi(\sigma(B \to_\mathrm{d}^\mathrm{w} C))\big) = \neg\big((B \to_\mathrm{d}^\mathrm{w} C) \wedge \neg\pi(\Box\sigma(B) \to \sigma(C))\big) = \neg\big((B \to_\mathrm{d}^\mathrm{w} C) \wedge \neg\neg(\neg(\pi(\sigma(B)) \to_\mathrm{d}^\mathrm{w} \bot) \wedge \neg\pi(\sigma(C)))\big)$.

Consider the following inference.

1. $\neg(B \wedge \neg\pi(\sigma(B)))$ — inductive hypothesis
2. $\neg(B \wedge \neg\pi(\sigma(B))) \to_\mathrm{d}^\mathrm{w} \neg(\neg(B \wedge \neg\pi(\sigma(B))) \to_\mathrm{d}^\mathrm{w} \bot)$ — Ax$_{16}$
3. $\neg(\neg(B \wedge \neg\pi(\sigma(B))) \to_\mathrm{d}^\mathrm{w} \bot)$ — 1, 2 and (MP$_\mathrm{d}^\mathrm{w}$)
4. $\neg\big(\neg(\neg(B \wedge \neg\pi(\sigma(B))) \to_\mathrm{d}^\mathrm{w} \bot) \wedge \neg\neg(\neg(B \to_\mathrm{d}^\mathrm{w} \bot) \wedge \neg\neg(\pi(\sigma(B)) \to_\mathrm{d}^\mathrm{w} \bot))\big)$ — Ax$_9$
5. $\neg(\neg(B \wedge \neg\pi(\sigma(B))) \to_\mathrm{d}^\mathrm{w} \bot) \to_\mathrm{d}^\mathrm{w} \neg(\neg(B \to_\mathrm{d}^\mathrm{w} \bot) \wedge \neg\neg(\pi(\sigma(B)) \to_\mathrm{d}^\mathrm{w} \bot))$ — 4, Ax$_{17}$ and (MP$_\mathrm{d}^\mathrm{w}$)
6. $\neg(\neg(B \to_\mathrm{d}^\mathrm{w} \bot) \wedge \neg\neg(\pi(\sigma(B)) \to_\mathrm{d}^\mathrm{w} \bot))$ — 3, 5, and (MP$_\mathrm{d}^\mathrm{w}$)

Next, applying the above inferred formula, an instance of the axiom Ax$_{11}$: $\neg\big(\neg\big(\neg(B \to_\mathrm{d}^\mathrm{w} \bot) \wedge \neg C\big) \wedge \neg(B \to_\mathrm{d}^\mathrm{w} C)\big)$, the inductive hypothesis for C: $\neg(\pi(\sigma(C)) \wedge \neg C)$, and classical logic expressed in $\{\wedge, \neg\}$ (due to Fact 3.2) we obtain the required thesis $\neg\big(\pi(\sigma(B \to_\mathrm{d}^\mathrm{w} C)) \wedge \neg(B \to_\mathrm{d}^\mathrm{w} C)\big)$.

For the case of $\neg\big((B \to^{\mathrm{w}}_{\mathrm{d}} C) \wedge \neg\pi(\sigma(B \to^{\mathrm{w}}_{\mathrm{d}} C))\big)$, consider the following inference.

1. $\neg(\pi(\sigma(B)) \wedge \neg B)$ inductive hypothesis
2. $\neg(\pi(\sigma(B)) \wedge \neg B) \to^{\mathrm{w}}_{\mathrm{d}} \neg(\neg(\pi(\sigma(B)) \wedge \neg B) \to^{\mathrm{w}}_{\mathrm{d}} \bot)$ Ax_{16}
3. $\neg(\neg(\pi(\sigma(B)) \wedge \neg B) \to^{\mathrm{w}}_{\mathrm{d}} \bot)$ 1, 2 and ($\mathrm{MP}^{\mathrm{w}}_{\mathrm{d}}$)
4. $\neg\big(\neg(\neg(\pi(\sigma(B)) \wedge \neg B) \to^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg\neg(\neg(\pi(\sigma(B)) \to^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg\neg(B \to^{\mathrm{w}}_{\mathrm{d}} \bot))\big)$ Ax_{9}
5. $\neg(\neg(\pi(\sigma(B))\wedge\neg B) \to^{\mathrm{w}}_{\mathrm{d}} \bot) \to^{\mathrm{w}}_{\mathrm{d}} \neg(\neg(\pi(\sigma(B)) \to^{\mathrm{w}}_{\mathrm{d}} \bot)\wedge\neg\neg(B \to^{\mathrm{w}}_{\mathrm{d}} \bot))$ 4, Ax_{17} and ($\mathrm{MP}^{\mathrm{w}}_{\mathrm{d}}$)
6. $\neg(\neg(\pi(\sigma(B)) \to^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg\neg(B \to^{\mathrm{w}}_{\mathrm{d}} \bot))$ 3, 5, and ($\mathrm{MP}^{\mathrm{w}}_{\mathrm{d}}$)

Similarly, using the obtained formula, Ax_{12}, the inductive hypothesis for C: $\neg(C \wedge \neg\pi(\sigma(C)))$ and classical logic expressed in $\{\wedge, \neg\}$ we obtain the required formula.

Case $A = B \vee^{\mathrm{l}}_{\mathrm{d}} C$.

By definitions $\neg\big(\pi(\sigma(B \vee^{\mathrm{l}}_{\mathrm{d}} C)) \wedge \neg(B \vee^{\mathrm{l}}_{\mathrm{d}} C)\big) = \neg\big(\pi(\Box\sigma(B) \vee \sigma(C)) \wedge \neg(B \vee^{\mathrm{l}}_{\mathrm{d}} C)\big) = \neg\big(\neg(\neg\neg(\pi(\sigma(B)) \to^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg\pi(\sigma(C))) \wedge \neg(B \vee^{\mathrm{l}}_{\mathrm{d}} C)\big)$.

Again, taking into account $\neg(\neg(\pi(\sigma(B)) \to^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg\neg(B \to^{\mathrm{w}}_{\mathrm{d}} \bot))$ and $\neg(\neg(B \to^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg\neg(\pi(\sigma(B)) \to^{\mathrm{w}}_{\mathrm{d}} \bot))$, applying an instance of the axiom Ax_{14} $\neg\big(\neg(\neg\neg(B \to^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg C) \wedge \neg(B \vee^{\mathrm{l}}_{\mathrm{d}} C)\big)$, the inductive hypothesis for C, and extensionality for classical logic expressed in $\{\wedge, \neg\}$ (due to Fact 3.2) we obtain the required thesis.

The case of $\neg\big((B \vee^{\mathrm{l}}_{\mathrm{d}} C) \wedge \neg\pi(\sigma(B \vee^{\mathrm{l}}_{\mathrm{d}} C))\big)$ is being treated analogously with the help of Ax_{13}.

Having proved $\vdash_{\mathbf{D}_2^{\mathrm{p}}} \neg(\pi(\sigma(A)) \wedge \neg A)$, the required thesis follows by Ax_{17}. □

We need an additional, easy-to-see fact.

Fact 4.10. For any $\varphi \in \mathscr{F}_{\mathrm{m}}$, it holds $\pi(\varphi) = \pi(\delta(\varphi))$.

Theorem 4.11. *For any formula A of the discussive language:*

$$A \in \mathbf{D}_2^{\mathrm{p}} \textit{ iff } \vdash_{\mathbf{D}_2^{\mathrm{p}}} A$$

Proof: Assume that $A \in \mathbf{D}^{\mathsf{p}}_{\mathbf{2}}$. By definitions, $\Box\sigma(A) \in \mathbf{S5}$, while by Fact 4.5 and definition of δ, $\vdash_{\mathbf{S5}^{\neg\wedge}} \Box\delta(\sigma(A))$, and again by Fact 4.5, $\Box\delta(\sigma(A)) \in \mathbf{S5}$. Hence, by Lemma 4.4 we have $\vdash_{\mathbf{JL}} \delta(\sigma(A))$. By Fact 4.6, $\vdash_{\mathbf{JL}^{\neg\wedge}} \delta(\sigma(A))$. Consider a respective proof $\varphi_1, \ldots, \varphi_n = \delta(\sigma(A))$, on the basis of $\vdash_{\mathbf{JL}^{\neg\wedge}}$. Now let us consider $C_1 = \pi(\varphi_1), \ldots, C_n = \pi(\varphi_n) = \pi(\delta(\sigma(A)))$, $C_{n+1} = \pi(\delta(\sigma(A))) \rightarrow^{\mathrm{w}}_{\mathrm{d}} A$, $C_{n+2} = A$. Using Fact 4.10, we see that $C_n = \pi(\sigma(A))$, $C_{n+1} = \pi(\delta(\sigma(A))) \rightarrow^{\mathrm{w}}_{\mathrm{d}} A = \pi(\sigma(A)) \rightarrow^{\mathrm{w}}_{\mathrm{d}} A$. By induction on the length of the proof, we show that for each $1 \leqslant i \leqslant n+2$, $\vdash_{\mathbf{D}^{\mathsf{p}}_{\mathbf{2}}} C_i$. The case of axioms follows by Lemma 4.8.

Consider the cases of rules. Assume that φ_i, where $1 \leqslant i \leqslant n$ results from an application of RJL$_1$, that is, there are $1 < j, k < i$ such that $\varphi_k = \Diamond\Box\neg(\varphi_j \wedge \neg\varphi_i)$. By inductive hypothesis $\vdash_{\mathbf{D}^{\mathsf{p}}_{\mathbf{2}}} \pi(\varphi_j)$ and $\vdash_{\mathbf{D}^{\mathsf{p}}_{\mathbf{2}}} \pi(\Diamond\Box\neg(\varphi_j \wedge \neg\varphi_i))$, i.e., $\vdash_{\mathbf{D}^{\mathsf{p}}_{\mathbf{2}}} \neg\neg(\neg(\pi(\varphi_j) \wedge \neg\pi(\varphi_i)) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot$. Consider the following sequence:

1. $\pi(\varphi_j)$ — by inductive hypothesis
2. $\neg\neg(\neg(\pi(\varphi_j) \wedge \neg\pi(\varphi_i)) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot$ — by inductive hypothesis
3. $\pi(\varphi_j) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \Big((\neg\neg(\neg(\pi(\varphi_j) \wedge \neg\pi(\varphi_i)) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \pi(\varphi_i)\Big)$ — Ax$_{15}$
4. $\pi(\varphi_i)$ — $2 \times$ (MP$^{\mathrm{w}}_{\mathrm{d}}$): 1, 2, 3

Assume that φ_i, where $1 \leqslant i \leqslant n$ results from an application of RJL$_2$, that is, there is $1 < k < i$ such that $\varphi_i = \Box\varphi_j$. We have to show that $\vdash_{\mathbf{D}^{\mathsf{p}}_{\mathbf{2}}} \pi(\Box\varphi_j)$, i.e., $\vdash_{\mathbf{D}^{\mathsf{p}}_{\mathbf{2}}} \neg(\pi(\varphi_j) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot)$. By the inductive hypothesis $\vdash_{\mathbf{D}^{\mathsf{p}}_{\mathbf{2}}} \pi(\varphi_j)$. Consider the following sequence:

1. $\pi(\varphi_j)$ — by inductive hypothesis
2. $\pi(\varphi_j) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \neg(\pi(\varphi_j) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot)$ — Ax$_{16}$
3. $\neg(\pi(\varphi_j) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot)$ — (MP$^{\mathrm{w}}_{\mathrm{d}}$): 1, 2

For the last two elements in $C_1, \ldots, C_n, C_{n+1}, C_{n+2} = A$ we use Lemma 4.9 and (MP$^{\mathrm{w}}_{\mathrm{d}}$).

The reverse implication is obtained by routine checking. $\Box$

5. A modification of axiomatization

Now let us consider only the $\{\neg, \to^{\mathrm{w}}_{\mathrm{d}}, \wedge\}$-part of $\mathbf{D}_{\mathbf{2}}^{\mathsf{p}}$.[19] As the only rule of inference, let $\Vdash_{\mathbf{D}_{\mathbf{2}}^{\mathsf{p}}}$ denote the consequence relation as determined by ($\mathrm{MP}^{\mathrm{w}}_{\mathrm{d}}$) together with the axiom schemes listed below: Ax_1–Ax_5, Ax_{11}, Ax_{16}–Ax_{17}, and

$$\neg((A \to^{\mathrm{w}}_{\mathrm{d}} B) \wedge (\neg(A \to^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg B)) \qquad (\mathrm{Imp}^{\mathsf{cl}})$$

$$\neg(((A \to^{\mathrm{w}}_{\mathrm{d}} \bot) \to^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg A) \qquad (\mathrm{B}^{\mathsf{d}})$$

One can easily see that by Ax_{16}, Ax_{17}, positive logic expressed with $\to^{\mathrm{w}}_{\mathrm{d}}$, and classical logic in $\neg$ and $\wedge$, and ($\mathrm{Imp}^{\mathsf{cl}}$) we have:

Fact 5.1.

$$\Vdash_{\mathbf{D}_{\mathbf{2}}^{\mathsf{p}}} \neg(A \wedge \neg B) \to^{\mathrm{w}}_{\mathrm{d}} (A \to^{\mathrm{w}}_{\mathrm{d}} \neg(B \to^{\mathrm{w}}_{\mathrm{d}} \bot)) \qquad (\mathrm{K})$$

$$\Vdash_{\mathbf{D}_{\mathbf{2}}^{\mathsf{p}}} (A \to^{\mathrm{w}}_{\mathrm{d}} B) \to^{\mathrm{w}}_{\mathrm{d}} \neg(\neg(A \to^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg B) \qquad (\mathrm{Imp})$$

Lemma 5.2.

$$\Vdash_{\mathbf{D}_{\mathbf{2}}^{\mathsf{p}}} \mathrm{Ax}_9$$

$$\Vdash_{\mathbf{D}_{\mathbf{2}}^{\mathsf{p}}} \mathrm{Ax}_{12}$$

Proof: By (K), (Imp), positive logic for $\to^{\mathrm{w}}_{\mathrm{d}}$ we have:

- $\neg(A \wedge \neg B) \to^{\mathrm{w}}_{\mathrm{d}} (A \to^{\mathrm{w}}_{\mathrm{d}} \neg(B \to^{\mathrm{w}}_{\mathrm{d}} \bot)$
- $(A \to^{\mathrm{w}}_{\mathrm{d}} \neg(B \to^{\mathrm{w}}_{\mathrm{d}} \bot) \to^{\mathrm{w}}_{\mathrm{d}} \neg(\neg(A \to^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg\neg(B \to^{\mathrm{w}}_{\mathrm{d}} \bot))$
- $\neg(A \wedge \neg B) \to^{\mathrm{w}}_{\mathrm{d}} \neg(\neg(A \to^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg\neg(B \to^{\mathrm{w}}_{\mathrm{d}} \bot))$
- $\neg\big(\neg(\neg(A \wedge \neg B) \to^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg\neg(\neg(A \to^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg\neg(B \to^{\mathrm{w}}_{\mathrm{d}} \bot))\big)$,

The case of Ax_{12} is obvious from ($\mathrm{Imp}^{\mathsf{cl}}$) and classical logic expressed in the language with $\neg$ and $\wedge$. □

[19] Since $\vee^{\mathrm{l}}_{\mathrm{d}}$ is definable in the considered language, for the language with $\vee^{\mathrm{l}}_{\mathrm{d}}$, one could just add two axioms: $\neg((A \vee^{\mathrm{l}}_{\mathrm{d}} B) \wedge ((A \to^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg B))$, $\neg(\neg(A \vee^{\mathrm{l}}_{\mathrm{d}} B) \wedge \neg((A \to^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg B))$.

LEMMA 5.3. *For any formula C in the language with $\{\neg, \rightarrow^{\mathrm{w}}_{\mathrm{d}}, \wedge\}$, it holds* $\Vdash_{\mathbf{D}^{\mathsf{p}}_{\mathbf{2}}} \pi(\sigma(C)) \rightarrow^{\mathrm{w}}_{\mathrm{d}} C$.

PROOF: First, we show that $\Vdash_{\mathbf{D}^{\mathsf{p}}_{\mathbf{2}}} \neg(\pi(\sigma(A)) \wedge \neg A)$. To obtain that for any A, $\Vdash_{\mathbf{D}^{\mathsf{p}}_{\mathbf{2}}} \neg(\pi(\sigma(A)) \wedge \neg A)$, we can prove $\neg(\pi(\sigma(A)) \wedge \neg A)$ and additionally $\neg(A \wedge \neg\pi(\sigma(A)))$ using simultaneous induction on the construction of A.

The cases of a variable, $\neg$ and $\wedge$ are being handled in the same way as in the proof of Lemma 4.9.

Using the axioms Ax_{11}, Ax_{16}, and Ax_{17}, as well as Ax_{9} and Ax_{12} (inferable by Lemma 5.2), we can repeat the proof of Lemma 4.9 in its part for the case of $\rightarrow^{\mathrm{w}}_{\mathrm{d}}$.

Having proved $\vdash_{\mathbf{D}^{\mathsf{p}}_{\mathbf{2}}} \neg(\pi(\sigma(A)) \wedge \neg A)$, the required thesis follows by Ax_{17}. □

THEOREM 5.4. *For any formula A of the discussive language:*

$$A \in \mathbf{D}^{\mathsf{p}}_{\mathbf{2}} \text{ iff } \Vdash_{\mathbf{D}^{\mathsf{p}}_{\mathbf{2}}} A$$

PROOF: Assume that $A \in \mathbf{D}^{\mathsf{p}}_{\mathbf{2}}$. By definition, $\Box\sigma(A) \in \mathbf{S5}$, so also $\sigma(A) \in \mathbf{S5}$. By Fact 4.5 $\vdash_{\mathbf{S5}^{\neg\wedge}} \delta(\sigma(A))$. There is a proof $\varphi_1, \ldots, \varphi_n = \delta(\sigma(A))$ on the basis of the relation $\vdash_{\mathbf{S5}^{\neg\wedge}}$. Now we consider a sequence $C_1 = \pi(\varphi_1)$, $\ldots$, $C_n = \pi(\varphi_n) = \pi(\delta(\sigma(A)))$, $C_{n+1} = \pi(\delta(\sigma(A))) \rightarrow^{\mathrm{w}}_{\mathrm{d}} A$, $C_{n+2} = A$. By Fact 4.10, we see that $C_n = \pi(\sigma(A))$ and $C_{n+1} = \pi(\sigma(A)) \rightarrow^{\mathrm{w}}_{\mathrm{d}} A$.

By induction on the length of the proof, we show that for each $1 \leqslant i \leqslant n+2$, $\Vdash_{\mathbf{D}^{\mathsf{p}}_{\mathbf{2}}} C_i$. For the case of an axiom scheme $\mathrm{Ax} \in \{\mathrm{Ax}_1, \mathrm{Ax}_2, \mathrm{Ax}_3\}$, we have $\pi(\mathrm{Ax})$ is an instance of an axiom scheme of $\Vdash_{\mathbf{D}^{\mathsf{p}}_{\mathbf{2}}}$. For the case of Ax_4, we see that $\pi\big(\neg\big(\Box\neg(\varphi\wedge\neg\psi)\wedge\neg\neg(\Box\varphi\wedge\neg\Box\psi)\big)\big) = \neg\big(\neg(\neg(\pi(\varphi)\wedge\neg\pi(\psi)) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg\neg(\neg(\pi(\varphi) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg\neg(\pi(\psi) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot))\big)$. Thus, by Lemma 5.2, the required thesis is an instance of a formula provable on the basis of $\Vdash_{\mathbf{D}^{\mathsf{p}}_{\mathbf{2}}}$.

For the case of Ax_6, we see that $\pi(\neg(\Box\varphi \wedge \neg\varphi)) = \neg(\neg(\pi(\varphi) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg\pi(\varphi))$ But this follows from the thesis $A \rightarrow^{\mathrm{w}}_{\mathrm{d}} A$ and (Imp).

For the case of Ax_7, we have $\pi(\neg(\Box\varphi \wedge \neg\Box\Box\varphi)) = \neg\big(\neg(\pi(\varphi) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg\neg(\neg(\pi(\varphi) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot)\big)$. By Ax_{16} we have $A \rightarrow^{\mathrm{w}}_{\mathrm{d}} \neg(A \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot)$ and $\neg(A \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \neg(\neg(A \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot)$, hence $A \rightarrow^{\mathrm{w}}_{\mathrm{d}} \neg(\neg(A \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot)$, so the required formula follows by (Imp).

For the case of Ax_8, we have $\pi(\neg(\varphi \wedge \neg\Box\Diamond\varphi)) = \neg\big(\pi(\varphi) \wedge \neg\neg((\neg\pi(\varphi) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot)\big)$. However, by ($\mathrm{B}^{\mathrm{d}}$) we have $\neg(((\neg A \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg\neg A)$, hence by classical logic expressed in $\{\neg, \wedge\}$, Ax_{17}, by $\mathrm{MP}^{\mathrm{w}}_{\mathrm{d}}$ we obtain $\neg\big(A \wedge$

$\neg\neg((\neg A \to_{\mathrm{d}}^{\mathrm{w}} \bot) \to_{\mathrm{d}}^{\mathrm{w}} \bot)\big)$, so the required scheme is an instance of the last scheme.

Consider the rule cases. Assume that φ_i, where $1 \leqslant i \leqslant n$ results from application of RS5$_1$, that is, there are $1 < j, k < i$ such that $\varphi_k = \neg(\varphi_j \wedge \neg\varphi_i)$. By inductive hypothesis $\Vdash_{\mathbf{D}_2^{\mathsf{p}}} \pi(\varphi_j)$ and $\Vdash_{\mathbf{D}_2^{\mathsf{p}}} \pi(\neg(\varphi_j \wedge \neg\varphi_i))$, i.e., $\Vdash_{\mathbf{D}_2^{\mathsf{p}}} (\neg(\pi(\varphi_j) \wedge \neg\pi(\varphi_i)))$, but by Ax$_{17}$ $\Vdash_{\mathbf{D}_2^{\mathsf{p}}} \pi(\varphi_j) \to_{\mathrm{d}}^{\mathrm{w}} \pi(\varphi_i)$, so the required formula follows by ($\mathrm{MP}_{\mathrm{d}}^{\mathrm{w}}$). The case of RS5$_2$ is a direct consequence of the application of Ax$_{16}$ and ($\mathrm{MP}_{\mathrm{d}}^{\mathrm{w}}$).

For the formula C_{n+1} we use Lemma 5.3, while C_{n+2} is obtained by the application of ($\mathrm{MP}_{\mathrm{d}}^{\mathrm{w}}$).

The fact that if $\Vdash_{\mathbf{D}_2^{\mathsf{p}}} A$, then $A \in \mathbf{D}_2^{\mathsf{p}}$ follows by routine checking. □

6. Related work

Arguably, Akama, Abe, and Nakamatsu's discursive logic is the first paracomplete discussive logic [2] that Jaśkowski's discussive logic inspires.[20] Being based on Nelson's constructive logic with a strong negation **N4** [4, 36] (the name **N4** is due to Wansing [52]), Akama et al. propose **CDLSN**, *constructive discursive logic with strong negation*, where "discursive negation is defined similar to intuitionistic negation and discursive implication is defined as material implication using discursive negation [2, p. 395] [...] **CDLSN** can be defined in two ways. One is to extend **N4** with discursive negation $\neg_{\mathrm{d}}$. The other is to weaken intuitionistic negation in **N4**. We adopt the first approach [...] Intuitionistic negation is not a discursive negation" [2, p. 398]. Below, we highlight some (dis)similarities between this and our approaches.

First, Akama et al.'s approach is not standard because it unemploys a classically-based modal logic: "Most works on discursive logic utilize classical logic and **S5** as a basis. However, we do not think that these are essential. For instance, an intuitionist hopes to have a discursive system in a constructive setting" [2, p. 397]. However, they argue that **CDLSN** is a discussive logic (see [2, pp. 406–407]). Our motivation is not to set up a discussive logic by any means. Rather, we would like to show that *non-discussive* logics are obtainable if one sticks to the standard approach for setting up discussive logic on the basis of a classically-based modal

[20]The below-mentioned exposition of both **CDLSN** itself and the ideas beyond it does not claim completeness. The reader is consulted to address [2] for details.

logic and at the same time employs a certain non-standard interpretation of discussive connectives. In particular, the non-discussive logics in this paper are obtained *via* non-standard interpretations of discussive disjunction. Alternatively and quite analogously, the non-standard interpretation of discussive negation as $\neg_{\mathrm{d}} A =_{\mathrm{df}} \Box\neg A$, which we briefly outline in Section 3 above might be employed to set up non-discussive logic. We do not get into details here, for it deserves a separate paper.

Second, both logics are similar because no version of *the law of excluded middle* is valid. Hence, both of them are paracomplete.[21] However, the invalidity of these versions stems from different reasons. The intuitionistic-like version, $A \vee {\sim}A$ ($\sim$ is the original notation in [2] for the strong negation), as well as the discussive one, $A \vee \neg_{\mathrm{d}} A$, have intuitionistic disjunction and are **CDLSN**-invalid due to the well-known properties of the given intuitionistic(-like) negations and disjunction. Our version, $A \vee_{\mathrm{d}} \neg A$, is quite opposite in a sense that it contains classical negation and discussive disjunction. And its invalidity is due only to the interpretation of discussive disjunction as $A \vee_{\mathrm{d}} \neg A =_{\mathrm{df}} \Box A \vee \neg A$, where $\Box A \vee \neg A$ is **S5**-invalid.[22]

Third, with regard to discussive implications in both logics, one of ours proves each formula from the classical implicative fragment, which is obviously not in line with Akama et al.'s intuitionistic-like motivation. Hence, $((A \to^{\mathrm{w}}_{\mathrm{d}} B) \to^{\mathrm{w}}_{\mathrm{d}} A) \to^{\mathrm{w}}_{\mathrm{d}} A$ is a theorem in our logic only.[23] Moreover, being a theorem in our logic, $A \to^{\mathrm{w}}_{\mathrm{d}} A$ is not a **CDLSN**-theorem. Well-known intuitionistically invalid formulae with (both strong and discussive) negations

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[21] Their paracompleteness is in line with the history of logic, where paraconsistency and paracompleteness often go hand-in-hand. As J.-Y. Béziau puts it: "Paraconsistent logic and paracomplete logic appear therefore like husband and wife" [9, p. 12].

[22] The alternative approach which we sketch in Section 3 above is to interpret discussive negation in a non-standard way as $\neg_{\mathrm{d}} A =_{\mathrm{df}} \Box\neg A$. It gives us an **S5**-invalid formula $A \vee \Box\neg A$.

[23] Let us notice that in general, paracompleteness (when referring to the invalidity of the law of excluded middle) has not to entail that the implicational-negative part cannot behave classically (it can be easily justified by considering a similar translation to ours, where in the case of implication no modality is added). On the other hand, in $\mathbf{D}^{\mathsf{p}}_{\mathbf{2}}$ for example, the formula $(\neg p \to^{\mathrm{w}}_{\mathrm{d}} \neg q) \to^{\mathrm{w}}_{\mathrm{d}} (q \to^{\mathrm{w}}_{\mathrm{d}} p)$ belonging to classical logic expressed in the implicational-negative language, is not a thesis of $\mathbf{D}^{\mathsf{p}}_{\mathbf{2}}$. It can be invalidated by using our translations. Indeed, consider the formulas obtained *via* the translation σ given in Section 3 and equivalent on the basis of **S5** to each of the following formulas: $\Box(\Box\neg p \to \neg q) \to (\Box q \to p)$; $\Box(\Diamond p \vee \neg q) \to (\Box q \to p)$; $(\Diamond p \vee \Box\neg q) \to (\Box q \to p)$; $((\Diamond p \wedge \Box q) \vee (\Box\neg q \wedge \Box q)) \to p$. One can easily see that the last formula is not a thesis of **S5**, so, $(\neg p \to^{\mathrm{w}}_{\mathrm{d}} \neg q) \to^{\mathrm{w}}_{\mathrm{d}} (q \to^{\mathrm{w}}_{\mathrm{d}} p)$ is indeed not a thesis of $\mathbf{D}^{\mathsf{p}}_{\mathbf{2}}$.

predictably fail in **CDLSN**, say, $\neg_{\rm d} \neg_{\rm d} A \rightarrow^{\rm w}_{\rm d} A$, while their analogues with the classical negation, say, $\neg\neg A \rightarrow^{\rm w}_{\rm d} A$, are theorems in our logic. On the other hand, say, $A \rightarrow^{\rm w}_{\rm d} \neg_{\rm d} \neg_{\rm d} A$ as well as its analogue with the classical negation $A \rightarrow^{\rm w}_{\rm d} \neg\neg A$ are valid in **CDLSN** and in our logic, respectively. At last, in [2], the authors define discursive implication as material implication using discursive negation, i.e., $A \rightarrow^{\rm w}_{\rm d} B =_{\rm df} \neg_{\rm d} A \vee B$. Our analogue of this definition, $A \rightarrow^{\rm w}_{\rm d} B =_{\rm df} \neg A \vee_{\rm d} B$, as well as $\neg(A \wedge B) \rightarrow^{\rm w}_{\rm d} (\neg A \vee_{\rm d} \neg B)$ do not hold. Let us recall to the reader that our logic does not employ any discussive conjunction, for the motivation is not focused on it but on discussive disjunction.

7. Conclusion

With regard to future topics to study, let us point out two directions. The former deals with developing the target logic. Following Perzanowski's idea (which he introduced in a comment on his translation of Jaśkowski's paper [22, p. 59]), Ciuciura [11] considers a quasi-discursive system $\mathbf{ND_2^+}$ which has a discursive negation defined as follows:

- $\tau(\neg_d A) = \Diamond\neg\tau(A)$.

One may consider a paracomplete version of $\mathbf{ND_2^+}$ with the following negation:

- $\sigma(\neg_d A) = \Box\neg\sigma(A)$.

As the reviewer kindly drew our attention, it should be clear by looking at [37, Definition 11] that the three-valued logic $\mathbf{I^1}$ [47] is characterized in a similar manner by considering the translation as follows, where $\sim$ and $\rightarrow^I$ are negation and implication of $\mathbf{I^1}$:

- $\sigma({\sim}A) = \Box\neg\sigma(A)$,
- $\sigma(A \rightarrow^I B) = \Box\sigma(A) \rightarrow \Box\sigma(B)$.

The paper [37] also makes use of the 'diamond' type implication and, similarly to [11], the 'diamond-not' type negation in capturing $\mathbf{P^1}$ [46]:

- $\sigma({\sim}A) = \Diamond\neg\sigma(A)$,
- $\sigma(A \rightarrow^I B) = \Diamond\sigma(A) \rightarrow \Diamond\sigma(B)$.

Yet another similar translation can be found in Kovač's paper [28]:

- $\sigma(A \wedge B) = \Diamond\sigma(A) \wedge \Diamond\sigma(B)$,
- $\sigma(A \vee B) = \Box\sigma(A) \vee \Box\sigma(B)$,
- $\sigma(A \to B) = \Diamond\sigma(A) \to \Box\sigma(B)$.

Jaśkowski is known to have rejected a many-valued tabular approach to $\mathbf{D_2}$ with providing (arguably, quite weak) arguments in favour of the modal approach that has been fruitfully developing for more than six decades already (and the present paper is another one evidence of it). Nevertheless, as in the case of $\mathbf{D_2}$, it would be interesting to develop the tabular many-valued approach to $\mathbf{D_2^{\mathsf{p}}}$: in particular, to find out $\mathbf{D_2^{\mathsf{p}}}$-provabiity of (some of) the formulae that are characteristic of paracomplete reasoning.

The latter direction of future research deals with applications of $\mathbf{D_2^{\mathsf{p}}}$. Generally, the reader's brief look at the axioms of $\mathbf{D_2^{\mathsf{p}}}$ acknowledges their awkwardness: $\mathbf{D_2^{\mathsf{p}}}$ inherits this property from $\mathbf{D_2}$. As a result, it is extremely difficult to implement the current Hilbert-style axiomatization of $\mathbf{D_2^{\mathsf{p}}}$ in practice, which implies the problem with proof searching. A possible solution to this problem would be to axiomatize $\mathbf{D_2^{\mathsf{p}}}$ as a Gentzen-style (sequent-style) or a natural deduction calculus. To the best of the authors' knowledge, no such calculi have yet been set up in the literature, not even for $\mathbf{D_2}$. We believe that on this path there will be found a solution for the notoriously difficult problem of independence of the axioms of discussive logics that is still open even for $\mathbf{D_2}$.

On the other hand, let us remind the reader about the passage on page 36 about $\mathbf{D_2^{\mathsf{p}}}$ modeling a discussion whose debaters are not equal in the sense in which they are equal in a discussion modeled with $\mathbf{D_2}$. Such modeling, which is an application of the target logic to argumentation theory, would also stimulate setting up axiomatizations of $\mathbf{D_2^{\mathsf{p}}}$ mentioned above.

Last, but not least, the present approach could be generalized by using other (weaker) modal logics as a basis for corresponding systems, similar to how the minimal variant $\mathbf{D_0}$ of $\mathbf{D_2}$ is axiomatized with the help of the deontic normal logic $\mathbf{D}$ in [19].

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References

[1] G. Achtelik, L. Dubikajtis, E. Dudek, J. Konior, *On independence of axioms of Jaśkowski's discussive propositional calculus*, **Reports on Mathematical Logic**, vol. 11 (1981), pp. 3–11.

[2] S. Akama, J. M. Abe, K. Nakamatsu, *Constructive discursive logic with strong negation*, **Logique et Analyse. Nouvelle Série**, vol. 54(215) (2011), pp. 395–408, URL: https://www.jstor.org/stable/44085016.

[3] S. Akama, N. C. A. da Costa, *Why paraconsistent logics?*, [in:] S. Akama (ed.), **Towards paraconsistent engineering**, vol. 110 of Intelligent Systems Reference Library, Springer, Cham (2016), pp. 7–24, DOI: https://doi.org/10.1007/978-3-319-40418-9_2.

[4] A. Almukdad, D. Nelson, *Constructible falsity and inexact predicates*, **The Journal of Symbolic Logic**, vol. 49(1) (1984), pp. 231–233, DOI: https://doi.org/10.2307/2274105.

[5] A. R. Anderson, N. D. Belnap, Jr., **Entailment: The Logic of Relevance and Necessity, Volume 1**, Princeton University Press, Princeton, N.J.–London (1975).

[6] O. Arieli, A. Avron, *Four-valued paradefinite logics*, **Studia Logica**, vol. 105(6) (2017), pp. 1087–1122, DOI: https://doi.org/10.1007/s11225-017-9721-4.

[7] A. Avron, *Paraconsistent fuzzy logic preserving non-falsity*, **Fuzzy Sets and Systems**, vol. 292 (2016), pp. 75–84, DOI: https://doi.org/10.1016/j.fss.2014.07.001.

[8] R. Ballarin, *Modern Origins of Modal Logic*, [in:] E. N. Zalta, U. Nodelman (eds.), **The Stanford Encyclopedia of Philosophy**, Fall 2023 ed., Metaphysics Research Lab, Stanford University (2023), URL: https://plato.stanford.edu/archives/fall2023/entries/logic-modal-origins/.

[9] J.-Y. Béziau, *The Future of Paraconsistent Logics*, **Logical Studies**, vol. 2 (1999), pp. 1–28, URL: http://wwwa.unine.ch/unilog/jyb/future-pl.pdf.

[10] D. A. Bochvar, *On a three-valued logical calculus and its application to the analysis of the paradoxes of the classical extended functional calculus*, **History and Philosophy of Logic**, vol. 2 (1981), pp. 87–112, DOI:

https://doi.org/10.1080/01445348108837023, translated from the Russian by Merrie Bergmann.

[11] J. Ciuciura, *A quasi-discursive system* ND_2^+, **Notre Dame Journal of Formal Logic**, vol. 47(3) (2006), pp. 371–384, DOI: https://doi.org/10.1305/ndjfl/1163775444.

[12] J. Ciuciura, *Frontiers of the discursive logic*, **Bulletin of the Section of Logic**, vol. 37(2) (2008), pp. 81–92.

[13] J. Ciuciura, *Non-adjunctive discursive logic*, **Bulletin of the Section of Logic**, vol. 42(3-4) (2013), pp. 169–181, DOI: https://doi.org/10.1007/bf03322857.

[14] N. da Costa, L. Dubikajtis, *On Jaśkowski's Discussive Logic*, [in:] A. I. Arruda, N. C. A. da Costa, R. Chuaqui (eds.), **Non-Classical Logics, Model Theory, And Computability**, vol. 89 of Studies in Logic and the Foundations of Mathematics, Elsevier (1977), pp. 37–56, DOI: https://doi.org/10.1016/S0049-237X(08)70644-X.

[15] R. Ertola, F. Esteva, T. Flaminio, L. Godo, C. Noguera, *Paraconsistency properties in degree-preserving fuzzy logics*, **Soft Computing**, vol. 19(3) (2015), pp. 531–546, DOI: https://doi.org/10.1007/s00500-014-1489-0.

[16] T. Furmanowski, *Remarks on discussive propositional calculus*, **Studia Logica**, vol. 34(1) (1975), pp. 39–43, DOI: https://doi.org/10.1007/BF02314422.

[17] J. W. Garson, **Modal logic for philosophers**, Cambridge University Press, Cambridge (2006), DOI: https://doi.org/10.1017/CBO9780511617737.

[18] K. Gödel, *Zum Intuitionistischen Aussagenkalkül*, **Anzeiger der Akademie der Wissenschaften in Wien**, vol. 69 (1932), pp. 65–66.

[19] O. Grigoriev, M. Nasieniewski, K. Mruczek-Nasieniewska, Y. Petrukhin, V. Shangin, *Axiomatizing a minimal discussive logic*, **Studia Logica**, vol. 111(5) (2023), pp. 855–895, DOI: https://doi.org/10.1007/s11225-023-10042-9.

[20] A. Heyting, **Die formalen Regeln der intuitionistischen Logik**, Sitzungsberichte der Preussischen Akademie der Wissenschaften zu Berlin (1930).

[21] S. Jaśkowski, *Recherches sur le système de la logique intuitioniste*, [in:] **Actes du Congrès International de Philosophie Scientifique,**

VI Philosophie des mathématiques, Hermann & Cie, Paris (1936), pp. 58–61, (in: *Actualités scientifiques et industrielles*, vol. 393).

[22] S. Jaśkowski, *On the discussive conjunction in the propositional calculus for inconsistent deductive systems*, **Logic and Logical Philosophy**, vol. 7 (1999), pp. 57–59, DOI: https://doi.org/10.12775/llp.1999.004, translated by Jerzy Perzanowski from the Polish *O koniunkcji dyskusyjnej w rachunku zdań dla systemów dedukcyjnych sprzecznych*, Studia Societatis Scientiarum Torunensis. Sectio A, Mathematica-Physica, vol. 1(8) (1949), no. 8, pp. 171–172.

[23] S. Jaśkowski, *A propositional calculus for inconsistent deductive systems*, **Logic and Logical Philosophy**, vol. 7 (1999), pp. 35–56, DOI: https://doi.org/10.12775/llp.1999.003, translated by Jerzy Perzanowski from the Polish *Rachunek zdań dla systemow dedukcyjnych sprzecznych*, Studia Societatis Scientiarum Torunensis. Sectio A, Mathematica-Physica, vol. 1(5) (1948), pp. 57–77; another English translation, by Olgierd Wojtasiewicz: *Propositional calculus for contradictory deductive systems*, Studia Logica, vol. 24 (1969), pp. 143–157.

[24] S. C. Kleene, *An addendum*, **The Journal of Symbolic Logic**, vol. 28 (1963), pp. 154–156, DOI: https://doi.org/10.2307/2271596.

[25] A. Kolmogoroff, *Zur Deutung der intuitionistischen Logik*, **Mathematische Zeitschrift**, vol. 35(1) (1932), pp. 58–65, DOI: https://doi.org/10.1007/BF01186549.

[26] J. Kotas, *The axiomatization of S. Jaśkowski's discussive system*, **Studia Logica**, vol. 33 (1974), pp. 195–200, DOI: https://doi.org/10.1007/BF02120494.

[27] J. Kotas, N. C. A. da Costa, *On some Modal Logical Systems Defined in Connexion with Jaśakowski's Problem*, [in:] A. I. Arruda, N. C. A. da Costa, R. Chuaqui (eds.), **Non-Classical Logics, Model Theory, And Computability**, vol. 89 of Studies in Logic and the Foundations of Mathematics, Elsevier (1977), pp. 57–73, DOI: https://doi.org/10.1016/S0049-237X(08)70645-1.

[28] S. Kovač, *In what sense is Kantian principle of contradiction non-classical?*, **Logic and Logical Philosophy**, vol. 17(3) (2008), pp. 251–274, DOI: https://doi.org/10.12775/LLP.2008.013.

[29] A. Loparić, N. C. A. da Costa, *Paraconsistency, paracompleteness, and valuations*, **Logique et Analyse. Nouvelle Série**, vol. 27(106) (1984), pp. 119–131, URL: https://www.jstor.org/stable/44084079.

[30] A. Loparić, N. C. A. da Costa, *Paraconsistency, paracompleteness, and induction*, **Logique et Analyse. Nouvelle Série**, vol. 29(113) (1986), pp. 73–80.

[31] J. Łukasiewicz, *On three-valued logic (in Polish)*, **Ruch Filozoficzny**, vol. 5 (1920), pp. 170–171, English translation: J. Łukasiewicz: Selected Works, L. Borkowski (ed.), North-Holland Publishing Company, Amsterdam (1970), pp. 87–88.

[32] K. Mruczek-Nasieniewska, M. Nasieniewski, A. Pietruszczak, *A modal extension of Jaśkowski's discussive logic* $\mathbf{D_2}$, **Logic Journal of the IGPL**, vol. 27(4) (2019), pp. 451–477, DOI: https://doi.org/10.1093/jigpal/jzz014.

[33] M. Nasieniewski, A. Pietruszczak, *A method of generating modal logics defining Jaśkowski's discussive logic* D_2, **Studia Logica**, vol. 97(1) (2011), pp. 161–182, DOI: https://doi.org/10.1007/s11225-010-9302-2.

[34] M. Nasieniewski, A. Pietruszczak, *On the weakest modal logics defining Jaśkowski's logic* D_2 *and the* D_2*-consequence*, **Bulletin of the Section of Logic**, vol. 41(3–4) (2012), pp. 215–232.

[35] M. Nasieniewski, A. Pietruszczak, *Axiomatisations of minimal modal logics defining Jaśkowski-like discussive logics*, [in:] A. Indrzejczak, J. Kaczmarek, M. Zawidzki (eds.), **Trends in Logic XIII**, Wydawnictwo Uniwersytetu Łódzkiego, Łódź (2014), pp. 149–163.

[36] D. Nelson, *Constructible falsity*, **The Journal of Symbolic Logic**, vol. 14 (1949), pp. 16–26, DOI: https://doi.org/10.2307/2268973.

[37] H. Omori, *Sette's logics, revisited*, [in:] **Logic, rationality, and interaction**, vol. 10455 of Lecture Notes in Computer Science, Springer, Berlin (2017), pp. 451–465, DOI: https://doi.org/10.1007/978-3-662-55665-8.

[38] H. Omori, J. Alama, *Axiomatizing Jaśkowski's discussive logic* $\mathbf{D_2}$, **Studia Logica**, vol. 106(6) (2018), pp. 1163–1180, DOI: https://doi.org/10.1007/s11225-017-9780-6.

[39] J. Perzanowski, *On M-fragments and L-fragments of normal modal propositional logics*, **Reports on Mathematical Logic**, vol. 5 (1975), pp. 63–72.

[40] Y. Petrukhin, *Generalized correspondence analysis for three-valued logics*, **Logica Universalis**, vol. 12(3–4) (2018), pp. 423–460, DOI: https://doi.org/10.1007/s11787-018-0212-9.

[41] A. Pietz, U. Rivieccio, *Nothing but the truth*, **Journal of Philosophical Logic**, vol. 42(1) (2013), pp. 125–135, DOI: https://doi.org/10.1007/s10992-011-9215-1.

[42] V. M. Popov, *Between the logic Par and the set of all formulae*, [in:] **The Proceeding of the 6th Smirnov Readings in Logic, Contemporary Notebooks**, Moscow (2009), pp. 93–95, (in Russian).

[43] V. M. Popov, *Between* $\mathrm{Int}_{\langle\omega,\omega\rangle}$ *and intuitionistic propositional logic*, **Logical investigations**, vol. 19 (2013), pp. 197–199, URL: https://iphras.ru/uplfile/logic/log19/LI19_Popov.pdf.

[44] J. B. Rosser, **Logic for mathematicians**, 2nd ed., Chelsea Publishing Co., New York (1978).

[45] J. B. Rosser, A. R. Turquette, **Many-valued logics**, Studies in Logic and the Foundations of Mathematics, North-Holland Publishing Co., Amsterdam (1951).

[46] A. M. Sette, *On the propositional calculus* P^1, **Mathematica Japonica**, vol. 18 (1973), pp. 173–180.

[47] A. M. Sette, W. A. Carnielli, *Maximal weakly-intuitionistic logics*, **Studia Logica**, vol. 55(1) (1995), pp. 181–203, DOI: https://doi.org/10.1007/BF01053037.

[48] D. J. Shoesmith, T. J. Smiley, **Multiple-conclusion logic**, Cambridge University Press, Cambridge–New York (1978), DOI: https://doi.org/10.1017/CBO9780511565687.

[49] J. Słupecki, G. Bryll, T. Prucnal, *Some remarks on three-valued logic of J. Łukasiewicz*, **Studia Logica**, vol. 21 (1967), pp. 45–70, DOI: https://doi.org/10.1007/BF02123418.

[50] B. C. van Fraassen, *Singular Terms, Truth-Value Gaps, and Free Logic*, **The Journal of Philosophy**, vol. 63(17) (1966), pp. 481–495, URL: http://www.jstor.org/stable/2024549.

[51] V. L. Vasyukov, *A new axiomatization of Jaśkowski's discussive logic*, **Logic and Logical Philosophy**, vol. 9 (2001), pp. 35–46, DOI: https://doi.org/10.12775/llp.2001.003.

[52] H. Wansing, *Negation*, [in:] L. Goble (ed.), **The Blackwell Guide to Philosophical Logic**, Blackwell Philosophy Guides, Blackwell, Oxford (2001), pp. 415–436, DOI: https://doi.org/10.1002/9781405164801.ch18.

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MATHEMATICAL METHODS IN REGION-BASED THEORIES OF SPACE: THE CASE OF WHITEHEAD POINTS

Abstract

Regions-based theories of space aim—among others—to define *points* in a geometrically appealing way. The most famous definition of this kind is probably due to Whitehead. However, to conclude that the objects defined are points indeed, one should show that they are points of a geometrical or a topological space constructed in a specific way. This paper intends to show how the development of mathematical tools allows showing that Whitehead's method of extensive abstraction provides a construction of objects that are fundamental building blocks of specific topological spaces.

Keywords: Boolean contact algebras, region-based theories of space, point-free theories of space, points, spatial reasoning, Grzegorczyk, Whitehead, extensive abstraction.

2020 Mathematical Subject Classification: 00A30, 03G05, 06E25.

1. Introduction

Imagine yourself trying to read out space's structure from the flux of data that reach your senses. After Russell [38] we might say that you are submerged in the perspective space—the space of private experience, a small

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fragment of the world. Yet your ambitions go way beyond that. You aim at a general mathematical theory that will reflect the essential, structural properties of a large fragment of what we know as the universe. You know it is feasible. Faithful and efficient systems of geometry are exactly such theories, and they have been with us since antiquity. Developed at the outset as tools to handle practical problems of relatively small communities, they turned into theories describing universal properties of larger fragments of space, including the properties of the universe as such after the emergence of non-Euclidean geometries. The rise of topology has been driven by the search for space's most general features, as well as for the solution of real-world problems, Euler's *Königsberg bridges puzzle tour* to be one of them. Purely mathematical enterprise at the beginning, topology flourished as a branch of mathematics with applications in macro- and micro-scale. All those achievements were obtained by experiencing fragments of our world only but turned out to be so powerful as to describe its most general properties.

Put yourself into the shoes of an admirer of geometry and topology who, at the same time, finds one thing to be a bit troubling—the fundamental constituents of geometrical and topological spaces are points, highly idealized, dimensionless objects that cannot be found in the space of private experience. Thus you ask yourself the question: *could points be mathematically satisfactory explained employing the objects from the perspective space?*

One of the very first endeavors toward a positive answer to the question was due to Alfred N. Whitehead [46, 47, 48]. He presented various constructions of points out of which the one from *Process and reality* was best developed and gained the attention of the community of logicians, mathematicians, and philosophers.[1] However, having defined points, the English mathematician never bothered himself to show that the entities constructed are building blocks of any space.

This paper's goals are very modest, as we aim to show how the development of formal methods from the XXth century lets us carry out Whitehead's construction in a rigorous mathematical manner and formulate a partial positive solution to the problem of existence of non-trivial

[1] To tell the truth, the constructions of *points* from [46, 47] were wrong, as observed by de Laguna [7]. The reason was that initially, Whitehead worked with *part of* relation only, and de Laguna suggested—and rightly so—going beyond it and adopting the notion of *containing* (the modern *non-tangential inclusion*) as one of the primitives.

topologies based on Whitehead points. In light of this, the paper does not provide any new groundbreaking results in the field of region-based topology but rather shows how various results obtained within it allow us to draw a positive conclusion concerning Whitehead points: not only do exist structures with Whitehead points, but these points are also building blocks of topological spaces that were constructed in the area of representation theory for Boolean algebras and their extensions.

2. The informal construction

Observe that the data you receive through your senses, concerning the spatial entities, contain various objects that we may collectively call *regions*. Both the laptop on your desk and the courtyard you see from the window of your office are regions, chunks of space. Those chunks are related to each other in various ways, of which two seem to be the most general: (a) one region may be part of another, as the screen is part of the laptop, (b) two regions may touch each other, as in the case of the laptop and the surface of the desk, or can be separated, like the pen in your backpack and the cup of coffee standing next to your left hand. Next to these, we have the idea of points as precise locations in space. These can be represented as collections of shrinking regions in space, tapering down to the precise locations. One of the main driving forces of region-based theories is to capture this vague idea through parthood and contact.

One way is to write down axioms that could be justified by how we seem to experience regions and their relations. We may engage both *parthood* and *contact*, as many authors did, or only just one of them, as was done originally by Whitehead [48]. Let '$\sqsubseteq$' and 'C' be the two symbols that denote, respectively, *parthood* and (binary) *contact*. We read '$x \sqsubseteq y$' as x *is part of* y and '$x \mathrel{\mathsf{C}} y$, as x *is contact with* y or x *touches* y. The most reasonable axioms for the former are probably those for one of the possible systems of mereology[2] that is a faithful representation of spatio-temporal properties between regions and their parts. For contact, the standard axioms to be assumed are the following: every region is in contact with itself: $x \mathrel{\mathsf{C}} x$, the contact is symmetrical: if $x \mathrel{\mathsf{C}} y$, then $y \mathrel{\mathsf{C}} x$; if $x \mathrel{\mathsf{C}} y$ and y is part of z, then $x \mathrel{\mathsf{C}} z$, which intuitively means that if x touches y, then every region of which y is part must also touch x. This is the axiomatic basis.

[2] See e.g. [31, 32] and [44] for expositions of various mereological theories.

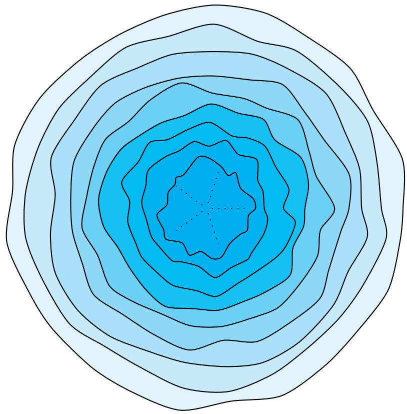

Figure 1. Point as a limit of shrinking system of regions

Other axioms may be introduced, and we will get back to this in the sequel. The way of proceeding directly from the sense data to axioms of a theory can be named, after Pratt-Hartmann [34], the *empiricist* approach.

Whitehead's [48] characterization of contact and definition of point extends over six pages of *Process and reality* and is preceded by 24 assumptions and 15 other definitions, a solid overkill, to say the least. Let's get straight to the bottom of Whitehead's points as easy as it gets without delving into his philosophical motivations. For these, we refer the interested reader to the excellent exposition of Varzi [45].

The English mathematician [48] follows the idea of the point from Figure 1. To do this properly, one must first say what it means for one region to be *a non-tangential part* (we will often use the phrase 'well-inside' as a synonym of 'non-tangential inclusion') of another: it is the case when the former is not in contact with the complement of the latter[3] or, as we will often say, *is separated from* the complement (see Figure 2).

[3] If we are working in the classical mereology we have to be careful what we mean by the *complement* as the zero region is absent. See [31] for details. In the case the main theory does not assume a region that is the largest region, the notion of the complement may have no sense at all, and we have to define the situation from Figure 2 in a different way. This can be done, e.g. by requiring that y does not touch any region outside x. We refer the reader again to the paper by Varzi [45].

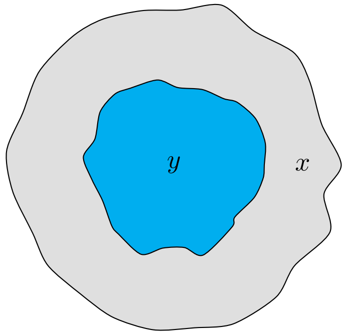

Figure 2. Region y is a non-tangential part of region x.

On the Whiteheadian road to points, we begin with the definition of *abstractive* sets of regions, i.e., those sets that:

(a) do not have any minimum with respect to *part of* relation, that is every region constituting an abstractive set has a proper part that is also in the set,

(b) their any two distinct elements are comparable with respect to *non-tangential part* relation.

The idea is that abstractive sets represent objects such as two-dimensional figures, planes, one-dimensional lines or segments (see Figure 3), and—last but not least—points, as the readers will convince themselves looking at Figure 1 again. The question is *how* to identify these abstractive sets that represent points? To this end, we define the *covering* relation between abstractive sets as follows: A *covers* B (in symbols: $A \succeq B$) iff for every region x in A there is a region y in B such that y is part of x. Now, if A covers B and vice versa, both sets represent the same object, and we can say those sets are equivalent. It is routine to verify that the equivalence of abstractive sets is indeed an equivalence relation: reflexive, symmetrical, and transitive. An equivalence class, say $[A]$, represents a unique object and therefore deserves to be called *a geometrical object*. Still, it does not have to satisfy our intuition of *point* as dimensionless, «infinitely» small entity. How to identify these geometrical objects that do? A way out is via comparing geometrical objects in the following manner: $[A] \trianglerighteq [B]$ if and only if $A \succeq B$. The relation $\trianglerighteq$ is a partial order, and if the partially ordered set of all geometrical elements happens to have minimal elements, then these elements truly deserve the name of *points*.

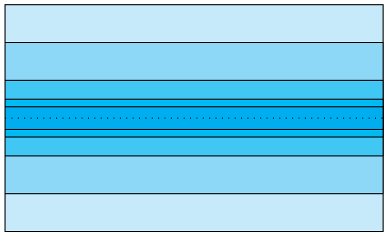

Figure 3. A small fragment of an abstractive set of two-dimensional rectangles representing a one-dimensional segment, marked above by dots.

Yet do they? How can we be sure that these are good candidates for points? After all, we have nothing to support this claim except for our intuition: when we *think* about regions as extended objects of the spatio-temporal continuum, ordered by the aforementioned armory of relational concepts, then what we defined as points are abstract objects that are, in a way, so «tiny» that they must be good representations of what we may ever want to declare points. So far, so good, the problem is that the intuition may fail, and the best way to avoid failure is to put it to strict mathematical tests. To do this, we need proper *formal* machinery, and thus we have to leave empiricism behind and take the path of *rationalism*, as characterized by Pratt-Hartmann [34].

3. The cornerstone

How can we test objects for «pointhood»? The best method we have is that of the representation theory known from universal algebra, which allows us to show that given objects from some abstract or concrete algebraic structure are indeed points. The idea of *representation* is a formal embodiment of reducing the *unfamiliar* and *abstruse* to *familiar* and *comprehensible*. Or turning *abstract* into *concrete*.

To get the feeling of the mechanism of representation, let us divide the class of Boolean algebras into two subclasses of (a) *abstract* and (b) *concrete* Boolean algebras, respectively. Abstract BAs are defined as specific structures with a distinguished domain whose elements are to satisfy certain conditions (axioms), usually for the binary operations of meet and join, the unary operation of complement, and the two individual constants: zero and one (unity). In the case of concrete BAs, we have a fixed set X, and take as the domain of the algebra a family $\mathcal{S}$ of its subsets that contains X and $\emptyset$, and is closed for the set-theoretical operations of intersection, union, and complement. Such a family is called a *field of sets*. More precisely, a concrete BA may be identified with a pair $\langle X, \mathcal{S}\rangle$ such that $\mathcal{S}$ is a field of sets over X (see [43]). It is evident that every concrete BA is an abstract one. It is also true that every abstract algebra is isomorphic to a concrete algebra, although this statement is far from obvious. It was proved by Marshall Stone [42], who created the representation method relevant from this paper's point of view.

Stone's work's motivations were purely mathematical—he aimed to understand what Boolean algebras are and how they relate to other mathematical entities. The first step towards understanding was to show that given any (abstract) Boolean algebra B we can construct (in a canonical way) a concrete algebra $\langle X, \mathcal{S}\rangle$ that is isomorphic to B.

With an algebra B at hand, everything we have at our disposal is this algebra (plus various mathematical tools that are normally used). The situation is analogous to a construction of a term model of a first-order theory by means of the Henkin method—we start with syntactical data, and we turn it into a model of the theory. To tell the truth, Henkin's construction may be viewed as a special case of the Stone theorem (see e.g., Exercises 4, 5, and 6 in pages 37–38 of [26])

With every—either abstract or concrete—algebra, there is associated the notion of a *filter*, a non-empty subset $\mathscr{F}$ of (the domain of) B that does not contain the zero element, is upward closed (in the sense of the standard Boolean order), and is closed for the binary meet operation.

The special place in the representation theory is occupied by *ultrafilters*, i.e., filters that are maximal in the family of all filters (in the sense of set-theoretical inclusion), or equivalently, filters $\mathscr{F}$ that satisfy the following condition: for any $x \in B$, either x is in $\mathscr{F}$ or its Boolean complement $-x$ is in $\mathscr{F}$. Given an algebra B, we will denote the family of all its ultrafilters by $\mathbf{Ult}(B)$, and we are going to use the letter '$\mathscr{U}$' as ranging over $\mathbf{Ult}(B)$.

Applying set-theoretical machinery, we can prove that every non-zero object $x \in B$ is an element of an ultrafilter. Moreover, we can show that if x and y are distinct, then there is an ultrafilter $\mathscr{U}$ that contains exactly one of these. Therefore, every object in B can be unequivocally represented by all these ultrafilters to which it belongs. More formally, with every Boolean algebra B we may associate an injective operation $\mathscr{U}\colon B \to \mathcal{P}(\mathbf{Ult}B)$ (from the domain to the power set of the family of all ultrafilters of B) such that $\mathscr{U}(x) := \{\mathscr{U} \in \mathbf{Ult}(B) \mid x \in \mathscr{U}\}$ (*the Stone mapping*). It is routine to verify that the image of this operation:

$$\mathscr{U}[B] = \{\mathscr{U}(x) \mid x \in B\}$$

is a field of sets. Indeed, $\mathbf{Ult}(B) = \mathscr{U}(\mathsf{1}) \in \mathscr{U}[B]$ and $\emptyset = \mathscr{U}(\mathsf{0}) \in \mathscr{U}[B]$. $\mathscr{U}[B]$ is closed for intersections and unions since:

$$\mathscr{U}(x) \cap \mathscr{U}(y) = \mathscr{U}(x \cdot y) \quad \text{and} \quad \mathscr{U}(x) \cup \mathscr{U}(y) = \mathscr{U}(x + y)\,,$$

where $\cdot$ and $+$ are the Boolean operations of meet and join, respectively; and the closure for set-theoretical complementation stems from the following equivalence:

$$\mathscr{U} \notin \mathscr{U}(x) \longleftrightarrow x \notin \mathscr{U} \longleftrightarrow -x \in \mathscr{U} \longleftrightarrow \mathscr{U} \in \mathscr{U}(-x)\,. \tag{3.1}$$

To conclude, to an abstract Boolean algebra B we can always associate a concrete isomorphic algebra $\langle \mathbf{Ult}(B), \mathcal{S}\rangle$ (with $\mathcal{S} := \mathscr{U}[B]$), that is isomorphic with B, its canonical representation. This is the content of the set-theoretical version of the Stone representation theorem.

However, the construction may be carried on to a topological representation. The main advantage of this is that it allows using spatial intuitions to draw consequences about the algebraic properties of Boolean algebras. In the case of set-theoretical representation above, the algebra B is shown to be isomorphic to a field of sets. In the case of the topological representation, it is proven that the field consists of distinguished—in one way or another—subsets of a topological space.

With respect to these, two crucial observations are that (a) we may treat ultrafilters as points—building blocks of point-based topologies, (b) with the topological structure induced by sets $\mathscr{U}(x)$ taken as basic open sets. The fact that $\mathscr{U}[B]$ satisfies the conditions of a basis stems from earlier observations for this family: every ultrafilter is in $\mathscr{U}(\mathsf{1})$, and $\mathscr{U}(x) \cap \mathscr{U}(y) =$

$\mathscr{U}(x \cdot y)$. Let $\mathscr{S}$ be the topology on $\mathbf{Ult}B$ with $\mathscr{U}[B]$ as a basis[4], the *Stone topology*. The pair $\langle \mathbf{Ult}(B), \mathscr{S} \rangle$ bears the name of the *Stone space* for the algebra B.

Let us have a look at the basic features of Stone spaces. Firstly, observe that given any open basic set $\mathscr{U}(x)$, it is a straightforward consequence of (3.1) that its complement is open too. This means that the basis for $\mathscr{S}$ is built out of sets that are both closed and open (and are called *clopen* for this reason). Such spaces are called *zero-dimensional*, and they are not very intuitive from the point of view of properties of the perspective space. If we take, e.g., the three-dimensional Cartesian space that serves as the standard model of the (static) world around us, then we only find two clopen sets: the whole space $\mathbb{R}^3$ and the empty set. For the other

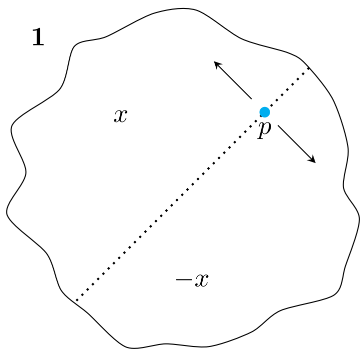

Figure 4. In Stone spaces, points cannot be located on boundaries between regions, as there are no boundaries. The point p is either a point of x or a point of the complement of x.

crucial property of Stone spaces, look at Figure 4. The intuition from the perspective space is that when we divide a region into two parts, there is such a thing as the boundary between the parts, and there are points that are located on the boundary. However, this is impossible in Stone spaces. The point p from the figure is an ultrafilter. Therefore either x is in p, or

[4]Recall that a basis for a topology on the set X is a family $\mathscr{B}$ of subsets of X such that $X = \bigcup \mathscr{B}$ and for every $B_1, B_2 \in \mathscr{B}$ and every $x \in B_1 \cap B_2$ there is $B_3 \in \mathscr{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

the Boolean complement of x is in p. In topological parlance, we say that the space is disconnected. For Stone spaces, the discontinuity phenomenon takes an extreme form: the only connected components of those spaces are singletons of points, i.e., the spaces are *totally disconnected*. Again, this is not a very intuitive property from the point of view of the perspective space. Actually, for the class of compact and Hausdorff spaces (a larger class than that of Stone space), the two properties are equivalent, in the sense that every compact Hausdorff space is zero-dimensional iff it is totally disconnected (see [26, Theorem 7.5, p. 97]).

The aforementioned compactness is—in a way—a topological version of finiteness: a space X is *compact* if for every family of open sets that covers the whole space, there is its finite subfamily that covers X either. As every open set is the sum of some family of basic open sets, we may replace 'open' with 'basic open' in the definition. In the case of Stone spaces, compactness is a consequence of the Ultrafilter Theorem, which says that every set F of elements of a BA such that F has the *finite intersection property* is contained in an ultrafilter, where F has the finite intersection property iff any finite subcollection of F has the non-zero meet: if $x_1, \dots x_n \in F$, then $x_1 \cdot \ldots \cdot x_n \neq \mathsf{0}$. Using this, it is relatively easy to show that the Stone space $\mathbf{Ult}(B)$ is compact.

Another key feature of Stone spaces is the Hausdorff separation axiom: any two distinct points x and y can be separated by open sets, in the sense that there are disjoint open set U and V around x and y, respectively. If ultrafilters $\mathscr{U}_1$ and $\mathscr{U}_2$ are distinct, there must be an x which is in only one of them, say $\mathscr{U}_1$. But then $-x$ must be in $\mathscr{U}_2$, and thus $\mathscr{U}(x)$ and $\mathscr{U}(-x)$ are disjoint (basic) open sets around the two ultrafilters, i.e., points of the Stone space.

To conclude, with every Boolean algebra B, we can associate a topological space, the Stone space of B, which is Hausdorff, compact, and zero-dimensional.[5] Moreover, the algebra B is isomorphic to the family $\mathrm{CO}(\mathbf{Ult}(B)))$ of clopen sets of this space. Thusly, there is a way from Boolean algebras to topological Stone spaces, i.e., structures with certain spatial data.

[5]Topological spaces that have these three properties are often called *Boolean spaces*, and the name is used with the intention to treat such spaces somewhat independently from the Stone spaces of ultrafilters. However, as we will see, every Boolean space X is a Stone space, in the sense that we can associate with X a Boolean algebra B whose Stone space $\mathbf{Ult}(B)$ is an exact copy of X.

However, there is also a way in the other direction. Any topological space X carries a Boolean algebra $\mathrm{CO}(X)$ of all its clopen subsets. In the case of Euclidean spaces $\mathbb{R}^n$, this algebra will have only two elements: the whole space $\mathbb{R}^n$ and the empty set. More generally, every connected space will carry the two-element Boolean algebra of its clopen subsets. Things get interesting if we limit our attention to Stone spaces only. In such a case, we obtain a deep dependence between the class **Stone** of all Hausdorff, compact, and zero-dimensional spaces, and the class **BA** of all Boolean algebras.

Let us start with a Boolean algebra B. As we have seen, there is a topological space that can be naturally associated with B, the Stone space $\mathbf{Ult}(B)$. This space, on the other hand, carries a Boolean algebra of its clopen subsets $\mathrm{CO}(\mathbf{Ult}(B))$, that is isomorphic to B, i.e., B and $\mathrm{CO}(\mathbf{Ult}(B))$ cannot be structurally distinguished.

$$
\begin{tikzcd}
\mathsf{BA} \ni B \arrow[r, dashed] \arrow[dr, shift right, "i"'] & \mathbf{Ult}(B) \in \mathsf{Stone} \arrow[d, dashed] \\
& \mathrm{CO}(\mathbf{Ult}(B)) \in \mathsf{BA} \arrow[ul, shift right, "i^{-1}"']
\end{tikzcd}
$$

Figure 5. Any Boolean algebra B is indistinguishable from the Boolean of clopen sets of the Stone space of B.

On the other hand, if we start from a Stone space X, then $\mathrm{CO}(X)$ is a Boolean algebra, and $\mathbf{Ult}(\mathrm{CO}(X))$ is its Stone space, that is, as the reader could expect, indistinguishable (*homeomorphic* is the technical jargon) from X.

$$
\begin{tikzcd}
\mathsf{Stone} \ni X \arrow[r, dashed] \arrow[dr, shift right, "h"'] & \mathrm{CO}(X) \in \mathsf{BA} \arrow[d, dashed] \\
& \mathbf{Ult}(\mathrm{CO}(B)) \in \mathsf{Stone} \arrow[ul, shift right, "h^{-1}"']
\end{tikzcd}
$$

Figure 6. Any Stone space B is indistinguishable from the Stone space of the Boolean algebra $\mathrm{CO}(X)$ of the clopen sets of X.

Thus, Stone [42] demonstrated that there is a kinship between the world of algebraic structures and the world of topological spaces. In particular, when we focus on Boolean algebras and Stones spaces, the bondage is so strong that we can say they are two sides of the same coin or two aspects of the same abstract phenomenon.[6]

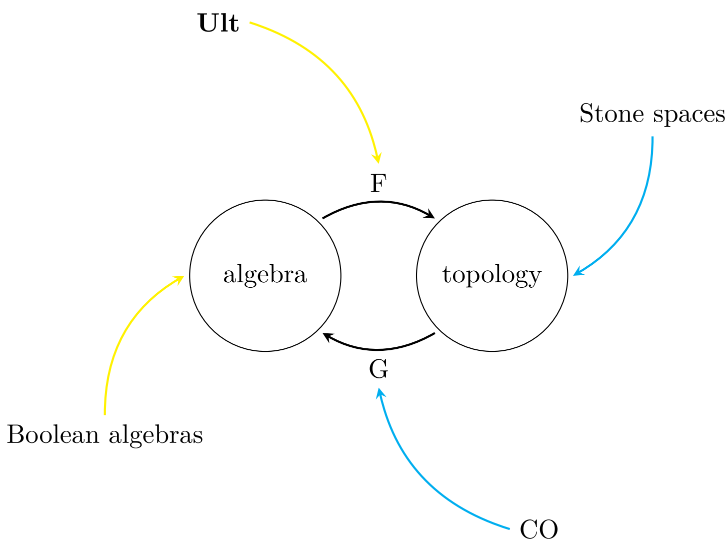

Figure 7. Boolean algebras and Stone topological spaces are very closely related.

[6] The kinship also extends to homomorphisms between algebras and continuous mappings between the spaces, in the sense that to every homomorphism between BAs corresponds a continuous mapping between their Stone mapping, and vice versa—with every continuous mapping between Stone spaces, there is associated a homomorphism between the algebras of their clopen sets. It is, roughly, the content of the famous Stone duality between the categories of Boolean algebras with homomorphisms, and Stone spaces with continuous mappings. For details, see, e.g., [24].

4. The extension: De Vries algebras

If we look at the diagram in Figure 7 we see that there are two very natural ways towards extending Stone results: we may encompass a larger (or a different) class of topological spaces, but we may also tinker with algebras taken into account. One of such tinkerings may, in particular, involve extending the signature (the non-logical language).

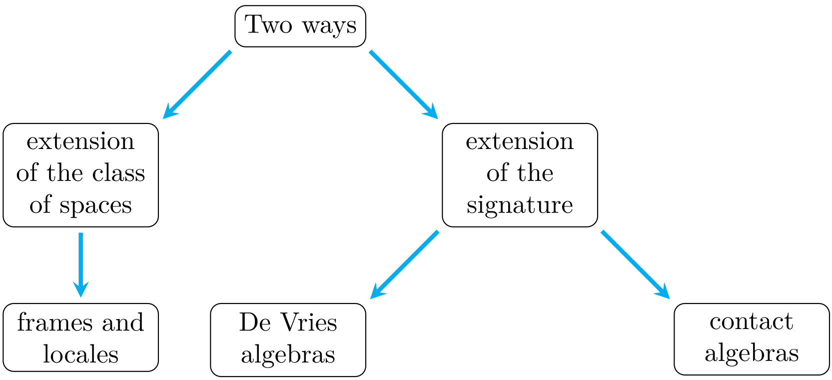

Figure 8. Possible extensions of the Stone duality

The extension of the class of topological spaces leads to a fruitful and fascinating theory of frames and locales (see, e.g., [24], [29, 30]), which in a nutshell can be described as a region-based theory of space in which the notion of *open set* is taken as basic. It is probably most developed among all region-based approaches. Yet, its objectives and main motivations (for the exposition of these see, e.g., [25]) are not, at least directly, connected to the leading topic of this paper.[7] This is mainly due to the fact that we want to

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[7] Mormann [28] presents a solution of what he calls *a Whitehead's problem* in the framework of Heyting algebras and continuous lattices, structures that are of particular importance in the theory of frames and locales. However, his paper does not mention Whitehead points and instead constructs topological spaces whose points are *Dedekind ideals* (in the terminology adopted by us further in the paper, these could be called *round* ideals). This is because Mormann defines the Whitehead problem as constructing spaces of points from regions of a uniform dimension that sets of points can faithfully represent. If the reader wishes, they may think about our paper as presenting a solution to the same problem yet utilizing the specific technique of conjuring up points put forward in *Process and reality*.

narrow down the notion of *region* to those interpretations that are faithful models of fragments of the perspective space, while the notion of *open set* is probably the most encompassing among primitive concepts of point-free theories.

Thus, we will follow the way of the signature extensions. The reasons to do this may vary, and our primary motivation is that the language of Boolean algebras does not differentiate between situations in which regions are incompatible (in the sense that their Boolean product is zero) and separated, and those where regions are incompatible but touch each other (see Figure 9). Equivalently, Boolean algebras cannot discern the difference between the situation in which x is part of y but does not touch the complement of y, and the one in which x is part of y and touches the complement of y, i.e., from the point of view of Boolean algebras there is no difference between the two scenarios in Figure 10.

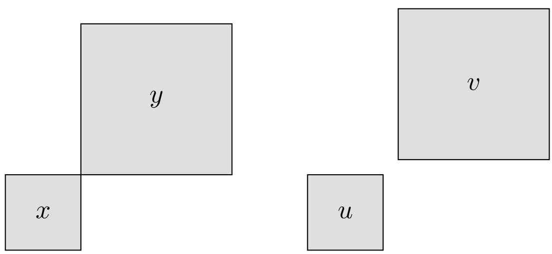

Figure 9. The regions x and y are incompatible and touch one other, while u and v are incompatible and separated.

Again, there may be different reasons to ponder Boolean algebras' extensions, either with the touching relation or well-inside relation. As we already saw in Section 2, proper (whatever it means now) construction of points may require it. Nevertheless, the reasons may be less philosophical and more practical as in the case of de Vries's work [8], which will serve as our starting point towards the justification of Whitehead's construction.

De Vries's aim was mathematical at heart: algebraization of the notion of *compactness* of a topological space. De Vries's algebras are just (complete) Boolean algebras extended with a binary relation $\ll$ whose intended

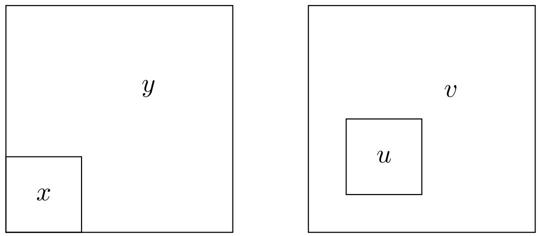

Figure 10. The region x is part of y but touches the complement of y, while u is well-inside v.

interpretation is non-tangential inclusion (well-inside part). The axioms concerning $\ll$ are the following:

$$\mathsf{1} \ll \mathsf{1}\,, \tag{DV1}$$

$$x \ll y \longrightarrow x \leq y\,, \tag{DV2}$$

$$x \leq y \wedge y \ll z \wedge z \leq w \longrightarrow x \ll w\,, \tag{DV3}$$

$$x \ll y \wedge x \ll z \longrightarrow x \ll y \cdot z\,, \tag{DV4}$$

$$x \ll y \longrightarrow -y \ll -x\,, \tag{DV5}$$

$$x \ll y \longrightarrow \exists z\,(x \ll z \wedge z \ll y)\,, \tag{DV6}$$

$$(\forall x \neq \mathsf{0})(\exists y \neq \mathsf{0})\, y \ll x\,. \tag{DV7}$$

These may not be self-evident at first sight, so let us explain them in a proper setting. The concrete De Vries algebras can be obtained from regular open algebras of κ-normal topological spaces.[8] A subset x of a topological space is *regular open* if x is equal to the interior of its closure: $x = \operatorname{Int}\operatorname{Cl} x$.[9] From a geometrical point of view, regular open sets of $\mathbb{R}^n$ are those open sets that do not have «surprises» in the form of cracks, holes,

[8] A space X is *κ-normal* (or *weakly normal*) iff any pair of its disjoint regular closed sets can be separated by open sets (see [40]).

[9] Alternatively, regular open sets can be characterized as regular elements in the lattice $\Omega(X)$ of all open sets of X. Such a lattice is a Heyting algebra and thus may have elements that are not regular, in the sense that if x^* is a relative complement of x, then $x^{**} \not\leq x$ (the reverse inclusion is always true). Thus x is regular open if $x = x^{**}$.

punctures, or snags. For this reason, they are sometimes considered good candidates for mathematical models of regions of the perspective space.[10]

As is well known, given a topological space X, the family of $\mathrm{RO}(X)$ of its regular opens is a complete Boolean algebra with the operations defined as follows:

$$U \cdot V := U \cap V \qquad U + V := \operatorname{Int}\operatorname{Cl}(U \cup V) \qquad -U := \operatorname{Int}(X \setminus U)$$
$$\bigvee_{i \in I} U_i := \operatorname{Int}\operatorname{Cl}\bigcup_{i \in I} U_i \qquad \bigwedge_{i \in I} U_i := \operatorname{Int}\bigcap_{i \in I} U_i \,.$$

If we interpret the non-tangential inclusion in the standard way as:

$$U \ll V \;:\longleftrightarrow\; \operatorname{Cl} U \subseteq V \,,$$

and assume that space X is κ-normal, then we will see that $\mathrm{RO}(X)$ is a De Vries algebra, with (DV6) (the so-called *interpolation axiom*) corresponding to the κ-normality of the space, and (DV7) to its weak version of regularity according to which every non-empty open set V has a nonempty set U whose closure is a subset of V. Since there are κ-normal spaces, there are De Vries algebras.

To pin down points in a De Vries algebra, the Stone-like technique of treating points as sets of regions is applied. To this end, the family of *round* filters is distinguished, i.e., filters $\mathscr{F}$ that have the following property:

$$(\forall x \in \mathscr{F})(\exists y \in \mathscr{F})\, y \ll x \,.$$

It is easy to see that every De Vries algebra must have a round filter: $\{\mathsf{1}\}$, trivial as it is. A less trivial example may be obtained if there is a non-zero region distinct from the unity, say x. Then, by (DV7) and the Axiom of Dependent Choices, we can come up with a sequence of non-zero elements:

$$\ldots x_2 \ll x_1 \ll x_0 = x \,,$$

and the filter generated by the sequence: $\mathscr{F} := \{y \mid (\exists n \in \omega)\ x_n \ll y\}$ must be round. An easy application of the Kuratowski-Zorn lemma

[10] Nowadays the class of all regular open sets of $\mathbb{R}^n$ is usually considered too large to model regions of the surrounding world. Various authors put forward different limitations on it, see, e.g., [9, 10, 27, 33, 39].

shows that there exist *maximal* round filters, and they are meant to be points of spaces.[11]

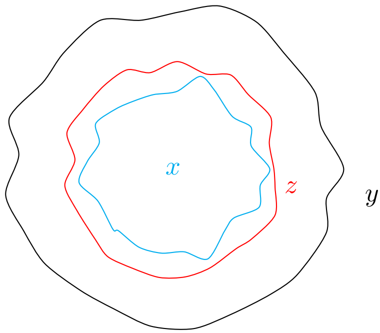

Figure 11. A geometrical interpretation of (DV6) axiom: between any two regions x and y such that x is well-inside y we can squeeze in a third region well-above x and well-inside y.

Let us have a look at two concrete examples. Take the real line $\mathbb{R}$, which is a normal (and the more so κ-normal) space. Consider the family of intervals $\{(-1-{}^1\!/\!_n, 1+{}^1\!/\!_n) \mid n \in \omega^+\}$ whose elements are regular open in $\mathbb{R}$. The filter $\mathscr{F}$ generated by this family is round. However, it is not maximal in the family of round filters. We can extend $\{(-1-{}^1\!/\!_n, 1+{}^1\!/\!_n) \mid n \in \omega^+\}$ with some regular open sets well-inside $(-1,1)$ which will result in a proper extension of $\mathscr{F}$. Thus $\mathscr{F}$ does not represent a point, which is good, since what all the regions in $\mathscr{F}$ have in common is the interval $(-1,1)$, a continuum of points. For a positive example, take any point $x \in \mathbb{R}$ and let $\mathrm{RO}(\mathscr{O}_x)$ be the family of all regular opens around x. $\mathrm{RO}(\mathscr{O}_x)$ is obviously a filter, and since the real line is regular, it is round. But it must also be maximal. For suppose $\mathscr{F}$ is a round filter extending $\mathrm{RO}(\mathscr{O}_x)$, and let V be an element of $\mathscr{F}$ but not of $\mathrm{RO}(\mathscr{O}_x)$. Therefore $x \notin V$. $\mathscr{F}$ is round, so there is $M \in \mathscr{F}$ with $\mathrm{Cl}\, M \subseteq V$. Thus, regularity entails existence of a regular open set R around x that is disjoint from M. But both R and M

[11] The original de Vries [8] terminology was different: he called *concordant* and *maximal concordant filters* round filters and maximal round filters, respectively. With other authors, the reader may also encounter terms *contracting* and *maximal contracting* filters. The latter are often called *ends* in the framework of proximity approach the mereotopology. We have decided to use 'round' as it is currently the most established practice among researchers within the field.

are elements of $\mathscr{F}$, so $\mathscr{F}$ is an improper filter (i.e. $\mathscr{F} = \mathrm{RO}(X)$). Thus $\mathrm{RO}(\mathscr{O}_x)$ is maximal round filter. This is good since the family of all regular open sets around x should uniquely determine x.

Due to the latter example, we may be tempted to think that De Vries might have had a geometrical intuition of point similar to Whitehead's. However, a certain example shows that the ideas of compactness and compactification were the leading ones for the Dutch mathematician, and it's a point of discrepancy between his and Whitehead's approach. Consider the following chain of regions of $\mathrm{RO}(\mathbb{R})$: $\{(n, +\infty) \mid n \in \omega\}$. The filter $\mathscr{F}$ that it generates is round and thus is contained in a maximally round filter $\mathscr{F}'$, a point. This filter represents a point at infinity in $\mathbb{R}$, since it cannot be $\mathrm{RO}(\mathscr{O}_x)$ for any real number x. See also Figure 12 for a geometrical intuition in the case of two-dimensional space.

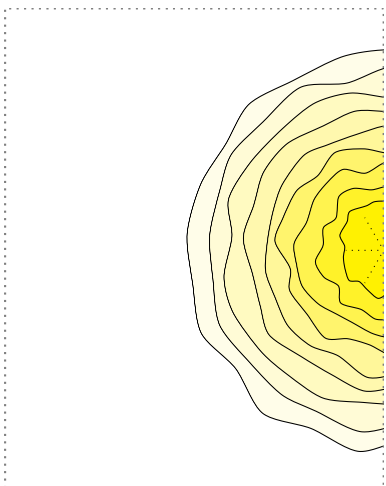

Figure 12. De Vries points involve points at infinity.

Why do we maintain that this example shows that Whitehead and De Vries had different objectives? The thing is that if we are to treat points as unique locations in the perspective space, points at infinity do not fit into this. Figuratively speaking, they are too far from our experience to enter the domain of points. At the very end of Section 7 we will demonstrate that the above constructed maximal contracting filter is not a point in the sense of Whitehead.

Maximal round filters are exactly those round filters that satisfy the following condition:

$$x \ll y \longrightarrow -x \in \mathscr{F} \vee y \in \mathscr{F}\,, \tag{†}$$

or, as the readers may easily convince themselves:

$$x \not\mathsf{C}\, y \longrightarrow -x \in \mathscr{F} \vee -y \in \mathscr{F}\,, \tag{‡}$$

where contact is defined as:

$$x \mathrel{\mathsf{C}} y \;:\!\longleftrightarrow\; \neg(x \ll -y)\,.$$

On the other hand, for the whole class of filters of a Boolean algebra we have that $\mathscr{F}$ (not necessarily round) is an ultrafilter if and only if:

$$x \cdot y = \mathsf{0} \longrightarrow -x \in \mathscr{F} \vee -y \in \mathscr{F}\,.$$

Therefore, if we have additional information that $\ll$ coincides with $\leq$ (or, equivalently, contact is overlap), the family of maximal round filters is exactly the family of ultrafilters since every region is incompatible with its complement. However, in general, we cannot exclude existence of points living on the borders of regions and their complements, as we did in the case of spaces of ultrafilters (see figures 4 and 13). Even more can be said: if x is in contact with $-x$, then there is a maximally round filter $\mathscr{E}$ such that $x \notin \mathscr{E}$ and $-x \notin \mathscr{E}$. This leads to an interesting conclusion: if every non-zero region is in contact with its complement, then the space of maximal round filters should be connected (if only there are such spaces).

There are, of course. The standard Stone-like assignment $\mathscr{E}\colon B \to \mathcal{P}(\mathbf{MRF}(B))$, where B is a De Vries algebra and $\mathbf{MRF}(B)$ is the set of all its maximal round filters leads to the family $\mathscr{B} := \{\mathscr{E}(x) \mid x \in B\}$ which satisfies the standard properties of a basis. The spaces $\langle \mathbf{MRF}(B), \mathscr{O}\rangle$ thus constructed are Hausdorff, since if $\mathscr{E}_1 \neq \mathscr{E}_2$, and there is a region x in, say $\mathscr{E}_1 \setminus \mathscr{E}_2$, then there is a region $y \in \mathscr{E}_1$ well-inside x. By (†) either $-y \in \mathscr{E}_2$ or $x \in \mathscr{E}_2$, and since the second disjunct does not hold, the first is true. But $\mathscr{E}(y) \cap \mathscr{E}(-y) = \emptyset$, and $\mathscr{E}_1 \in \mathscr{E}(y)$ and $\mathscr{E}_2 \in \mathscr{E}(-y)$.

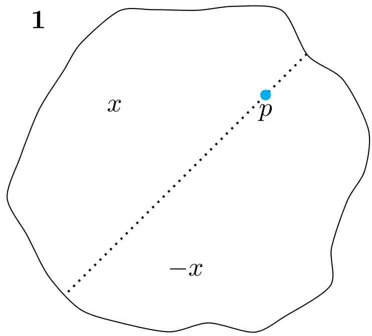

Figure 13. In spaces of maximally round filters, points may inhabit the boundaries of regions and their complements, if the regions and their complements are in contact, in the sense that $\neg(x \ll x)$.

Every $\mathbf{MRF}(B)$ must also be compact. The proof is slightly more complicated than the one for Stone spaces, and we skip it not to mar the gist of the paper with unnecessary technicalities. The important thing is that to every De Vries algebra corresponds a certain topological space that is Hausdorff compact. Similarly to the situation for Boolean algebras and Stone spaces, given a topological space X that is Hausdorff compact, its family of $\mathrm{RO}(X)$ with $\ll$ interpreted as the topological well-inside inclusion must be a De Vries algebra. Again, if we start with B, go to $\mathbf{MRF}(B)$ and to $\mathrm{RO}(\mathbf{MRF}(B))$, then we have that either B can be densely embedded in $\mathrm{RO}(\mathbf{MRF}(B))$, or is isomorphic with $\mathrm{RO}(\mathbf{MRF}(B))$, if complete. Since we changed the class of algebras from Boolean to De Vries we need an appropriate notion of isomorphism that remains essentially the same as the standard one, with an extra condition stipulating that $\ll$ is preserved in the following sense:

$$x \ll y \longleftrightarrow h(x) \ll h(y)\,.$$

We will call the mapping h *De Vries isomorphism*. To repeat, if the initial algebra B is not complete, then the mapping $\mathscr{E}\colon B \to \mathrm{RO}(\mathbf{MRF}(B))$ is a dense De Vries embedding[12] of B, and in the case B is complete, the same mapping is a De Vries isomorphism.

$$\begin{tikzcd} \textsf{\textbf{DV}} \ni B \arrow[r, dashed] \arrow[dr, hook, "\mathscr{E}"'] & \mathbf{MRF}(B) \ \in \ \textsf{\textbf{KHaus}} \arrow[d, dashed] \\ & \mathrm{RO}(\mathbf{MRF}(B)) \ \in \ \textsf{\textbf{DV}}^c \end{tikzcd}$$

Figure 14. Let **DV** be the class of De Vries algebras, and **KHaus** the class of compact topological spaces. Any De Vries algebra embeds densely into the algebra of regular open sets of the compact Hausdorff space for B.

$$\begin{tikzcd} \textsf{\textbf{DV}}^c \ni B \arrow[r, dashed] \arrow[dr, shift right, "\mathscr{E}"'] & \mathbf{MRF}(B) \ \in \ \textsf{\textbf{KHaus}} \arrow[d, dashed] \\ & \mathrm{RO}(\mathbf{MRF}(B)) \ \in \ \textsf{\textbf{DV}}^c \arrow[ul, shift right, "\mathscr{E}^{-1}"'] \end{tikzcd}$$

Figure 15. Let $\textsf{\textbf{DV}}^c$ be the class of complete De Vries algebras. Any its element B is indistinguishable from the De Vries algebra of regular open sets of the compact Hausdorff space for B.

To conclude, De Vries, through pursuing his algebraic objectives, showed a way to represent structures with a version of a point-free topological nearness as fully-fledged topological spaces. In the next section, we will see how it helps to understand another classical point-free topology by a Polish logician Andrzej Grzegorczyk, which on the other hand, will let us show that Whitehead points (or at least some of them) are indeed points of a certain class of topological spaces.

[12] The embedding $\mathscr{E}$ is dense in quite a strong sense, that is if $x, y \in \mathrm{RO}(\mathbf{MRF}(B))$ are such that $x \ll y$, then there is $z \in B$ for which $x \ll \mathscr{E}(z) \ll y$.

$$\begin{tikzcd}
\mathsf{KHaus} \ni X \arrow[r, dashed] \arrow[dr, shift right, "h"'] & \mathrm{RO}(X) \quad \in \quad \mathsf{DV}^c \arrow[d, dashed] \\
& \mathbf{MRF}(\mathrm{RO}(X)) \in \mathsf{KHaus} \arrow[ul, shift right, "h^{-1}"']
\end{tikzcd}$$

Figure 16. Any compact Hausdorff topological space X is homeomorphic to the compact Hausdorff space of the complete De Vries algebra of regular open sets of X.

5. The criterion of points

From the above, we can see that we have a general method of constructing spaces from algebraic data *via* mimicking Stone's technique to treat points as subsets of the domain. So what would it mean to achieve the Whitehead's goal?, i.e., explain points in a geometrically appealing way. On the intuitive level, Whitehead's points are collections of regions related to each other via spatially motivated relations. The intuition may be turned into a precise notion in two steps: firstly, by imposing an algebraic structure on regions to reflect the most general properties of the perspective space (i.e., extend the signature); secondly, by showing that the Whitehead's minimal geometrical objects reconstructed within such a structure as higher-order objects are indeed points of a certain space.

More precisely—and more generally—suppose $\langle A, R_1, \ldots, R_n\rangle$ is an algebraic structure with relations $R_1, \ldots, R_n$, all these together modelling the universe of regions. Suppose $\mathscr{P}$ is the set of higher-order objects defined within this structure. The main problem is now to find a topological space with $\mathscr{P}$ as the underlying set of points (similarly as ultrafilters are taken as points of Stone spaces, and maximally round filters as points of compact Hausdorff spaces) that naturally models regions (elements of the domain) and relations R_i. That is, $\mathfrak{A} := \langle A, R_1, \ldots, R_n\rangle$ is captured within $\mathscr{P}$ as $\mathfrak{A}' = \langle A', R_1', \ldots, R_n'\rangle$ in a similar way as any Boolean algebra B is captured as the algebra of clopen sets of its Stone space $\mathbf{Ult}(B)$. This, in particular, means that A' is a family of subsets of $\mathscr{P}$, and that the set $\mathscr{P}$ of points may be given an appropriate topology in which every R_i can be modeled in such a way that R_i holds among regions iff R_i' holds among their point-based counterparts. Usually, subsets of regular open (or regular closed) sets of $\mathscr{P}$ are taken as models of regions (see footnote 10). If this has been achieved, then we may say we have the solution to the problem of points, as we have the representation of the original structure in the structure built from higher-order elements of $\mathscr{P}$ that now deserve the name of *points*. In this manner, every BA is represented in the space of ultrafilters, and every De Vries algebra in the space of maximally round filters. This justifies naming both ultrafilters and maximally round filters as points. The idea is now to repeat the above steps with a proper algebraic structure in place of $\mathfrak{A}$, and a set of Whitehead points in place of $\mathscr{P}$.

In light of the theorems of Stone's and De Vries's, one could naturally ask could either ultrafilters of maximally round filters serve as Whitehead points? Why do they fall short? In the case of ultrafilters, the main problem is hidden in the fact that if they are points, the contact relation collapses to overlap. Indeed, suppose we have a mapping f that represents regions of a Boolean algebra in $\mathcal{P}(\mathbf{Ult}(B))$, and that f is the standard Stone-like function, that is, for every region x its points are all these ultrafilters $\mathscr{U}$ that has x as an element. But then, as we observed earlier (see page 71), there are no points on the boundaries of the regions, so the contact can only be the overlap, i.e., we cannot model the situations in which objects are external to each other and touch each other at the same time. Moreover, in [16] it was shown that if contact and overlap coincide, in complete algebras, there are no Whitehead points, so in general, ultrafilters cannot serve as them.

This does not mean that ultrafilters are always bad candidates for building blocks of spaces of points in which the contact relation is to be modeled. For example, Peter Roeper [36] starts with them, yet his points are not ultrafilters themselves, but equivalence classes of ultrafilters that, in the end, can be shown to be maximal round filters (see [15]).

As for De Vries points, we have shown above that their class is too large for the class of Whitehead points, in the sense that Whitehead points may only form a proper subset of the set of all maximal round filters. We'll get back to this problem in Section 7.

For the completeness of the presentation, it should be emphasized that higher-order constructions are not the only method of explaining points, and some scholars either defined points in terms of regions (elements of the domain) or distinguished a subset of the domain as the collection of points. The most important examples—in either topological or geometrical setting—are [12, 13, 14, 21, 22, 35, 39].

6. Grzegorczyk points

This time we start with the contact relation. The difference is irrelevant from a logical point of view, as with enough axioms the two approaches, either via contact or via non-tangential inclusion, are definitionally equivalent. However, the proper terminology well-chosen at the outset will equip us with a user-friendly language. The purpose is to expose the point-free topology of Grzegorczyk's from [20], who by the way chose the third way and based his system on the notion of *separation*, yet this is again an equivalent approach to those used in this paper.[13]

Before we begin a proper, mathematical exposition of Grzegorczyk's construction and before we draw an analogy between this and Whitehead's, let us remind that it was Clarke [4, 5] who was the first scholar to undertake the task of developing Whitehead's meretopological ideas. He based his system on the binary relation of *connection*, and the definition of a point different from the original proposal of the English logician. However, as it was later demonstrated by Biacino and Gerla [2], Clarke's contact relation collapses to overlap, and his axioms characterize the atomless complete Boolean algebras. In consequence, Clarke's points as defined in [5] are nothing but ultrafilters. Thus, his approach falls short.

So, let us turn to contact and Boolean contact algebras as a unifying framework. By a *Boolean contact algebra*[14] we mean a Boolean algebra extended with a binary relation C of *contact* that satisfies the following constraints:

---

[13] Strictly speaking, Grzegorczyk did not work with Boolean algebras, but with mereology, which is closely related to the former, see, e.g., [31]. The differences are mainly hidden in technical intricacies, as mereologies generally do not have zero elements and are thus semi-lattices.

[14] For an exposition of Boolean contact algebras see [1, 41].

$$0 \not\mathrel{\mathsf{C}} x \tag{C0}$$

$$x \leq y \wedge x \neq 0 \longrightarrow x \mathrel{\mathsf{C}} y, \tag{C1}$$

$$x \mathrel{\mathsf{C}} y \longrightarrow y \mathrel{\mathsf{C}} x, \tag{C2}$$

$$x \leq y \longrightarrow \forall z \in B(z \mathrel{\mathsf{C}} x \longrightarrow z \mathrel{\mathsf{C}} y)\,, \tag{C3}$$

$$x \mathrel{\mathsf{C}} y + z \longrightarrow x \mathrel{\mathsf{C}} y \vee x \mathrel{\mathsf{C}} z\,. \tag{C4}$$

We extend the inventory of relations by introducing non-tangential inclusion via the expected definition:

$$x \ll y \;:\longleftrightarrow\; x \not\mathrel{\mathsf{C}} -y\,.$$

The reader will check easily that so defined $\ll$ has properties (DV1)–(DV5). The remaining two De Vries axioms need additional assumptions about C.

Grzegorczyk's idea to introduce points was somewhat similar to those of Whitehead and De Vries.[15] Take a region and shrink it till you «squeeze» a point out of it. However, what distinguishes his definition from the other two is that he demanded that every set of regions that is a candidate for a point satisfy the following (geometrical in spirit) property: if x and y are regions such that each one overlaps all regions in a point candidate, then x must touch y (see Figure 17). This requirement singles out Grzegorczyk points among De Vries points, as we will see in a moment.

Formally, a *Grzegorczyk representative of a point* (*G-representative* for short)[16] in a Boolean contact algebra is a non-empty set Q of regions such that:

$$0 \notin Q\,, \tag{r0}$$

$$\forall u, v \in Q(u = v \vee u \ll v \vee v \ll u)\,, \tag{r1}$$

$$(\forall u \in Q)(\exists v \in Q)\; v \ll u\,, \tag{r2}$$

$$(\forall x, y \in R)(\forall u \in Q)\big((u \bigcirc x \wedge u \bigcirc y) \longrightarrow x \mathrel{\mathsf{C}} y\big)\,, \tag{r3}$$

where:

$$x \bigcirc y \;:\longleftrightarrow\; x \cdot y \neq 0\,.$$

[15] Historically, Grzegorczyk precedes De Vries, yet it is virtually impossible that the two scholars were aware of each other's work.

[16] Both the term and its abbreviation adopted from [3].

It is not hard to see that if we take $\mathrm{RO}(\mathbb{R})$ and a point r, then the family $Q := \{(r - {}^1\!/\!_n, r + {}^1\!/\!_n) \mid n \in \omega^+\}$ is a G-representative. Of course, different sets may represent the same point, as for example $Q_e := \{(r - {}^1\!/\!_n, r + {}^1\!/\!_n) \mid n \in \mathbb{E}^+\}$, $Q_o := \{(r - {}^1\!/\!_n, r + {}^1\!/\!_n) \mid n \in \mathbb{O}^+\}$ and Q do ($\mathbb{E}^+$ and $\mathbb{O}^+$ are, respectively, the sets of all positive even integers and of all positive odd integers). It is easy to see that in the case of r (and any other real number) there are uncountably many G-representatives. More generally, if Q is a G-representative in a Boolean contact algebra, and $x \in Q$, then the set $\{y \in Q \mid y \leq x\}$ is also a G-representative. To circumvent the problem of a unique point identification we declare Grzegorczyk points (G-points) of a Boolean contact algebra B to be filters generated by G-representatives (whose set is denoted by $\mathbf{Q}(B)$):

$$\mathscr{G} \in \mathbf{Grz}(B) \longleftrightarrow (\exists Q \in \mathbf{Q}(B))\, \mathscr{G} = \{y \in B \mid \exists x \in Q\, y \leq x\}\,.$$

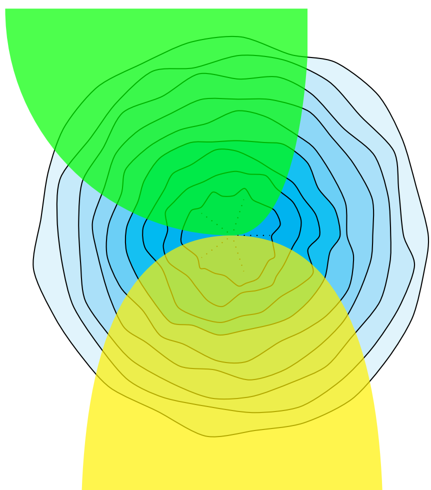

Figure 17. A representative of a point in the sense of Grzegorczyk: if two regions overlap all elements of the representative, then they must be in contact.

Let $x \infty \mathscr{F}$ hold iff region x is in contact with every region in a filter $\mathscr{F}$: $(\forall y \in \mathscr{F})\, y \mathsf{C}\, x$. Accordingly, $x \not\infty \mathscr{F}$ iff there is a region in $\mathscr{F}$ that is separated from x. The reader will easily check that if $\mathscr{F}$ is round, then:

$$x \in \mathscr{F} \longleftrightarrow -x \not\infty \mathscr{F} .$$

Interestingly, every Grzegorczyk point is a maximal round filter. Firstly, every G-point $\mathscr{G}$ is a round filter, for if $x \in \mathscr{G}$ and Q generates $\mathscr{G}$, then in Q there is a region $y \leq x$. But in Q there is z well-inside y, so z is well-inside x either. Secondly, in [15] it was proven that a round filter $\mathscr{F}$ satisfies the following condition:

$$(\forall x, y \in B)\, (x \infty \mathscr{F} \infty y \longrightarrow x \mathsf{C}\, y) \tag{✠}$$

iff the condition (‡) is also true about $\mathscr{F}$. Indeed, if x is separated from y, applying (✠) in the contraposed form we obtain that either $x \not\infty \mathscr{F}$ or $y \not\infty \mathscr{F}$, and so either $-x$ is an element of $\mathscr{F}$ or $-y$ is, as required. The reverse implication is proven analogously. As the property (‡) uniquely identifies maximal round filters, so does (✠). At the same time, the condition (r3) for G-representatives, together with the definition of G-points, entail that every Grzegorczyk point must satisfy (✠). So $\mathbf{Grz}(B) \subseteq \mathbf{MRF}(B)$. Does the other inclusion hold? In general, no. If we look back at Figure 12 we can see a fragment of a point at infinity that, in general, does not have to be a G-point. To see this, imagine that we color the regions of the point with two alternating colors, as in Figure 18. After coloring, we choose only blue stripes, number them with natural numbers, and divide them into two sets: of stripes tagged with even and of stripes tagged with odd numbers, respectively. We can now take the suprema of the first set and the second set to obtain regions that are apart yet overlap every region in the chain we began with. The chain is included in a maximal round filter $\mathscr{E}$, yet $\mathscr{E}$ cannot be generated by any G-representative. Precisely because any such a G-representative would have to be covered by the chain, thus failing to satisfy the condition (r3). We must be careful here as the situation is subtle, so let us repeat: the maximal round filter $\mathscr{E}$ must satisfy (✠), since every maximal round filter satisfies the condition; it is only that no G-representative can give rise to $\mathscr{E}$.

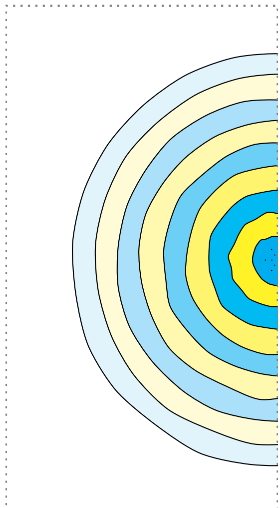

Figure 18. A construction towards showing that not every maximal round filter is a Grzegorczyk point.

Let us show that G-points satisfy the criteria of the method from page 84. The first step towards demonstrating this was done by Grzegorczyk himself, and more elaborate constructions were delivered in [15], and [17, 18]. Grzegorczyk demonstrated that his points, together with the Stone-like mapping, form a topological Hausdorff space, in which his mereology-based separation structures can be represented. Grzegorczyk also maintained that the spaces of his points have the following property: for every point p, there exists an infinite strictly decreasing family of open sets such that the intersection of the family is $\{p\}$. Yet this is not true, as there are finite structures that are models of Grzegorczyk axioms, which was proven in the papers by Gruszczyński and Pietruszczak. In those papers, a class of the so-called *concentric* topological spaces was singled out, which are T_1 spaces additionally satisfying the condition (R1) displayed below on page 91. Later in [19] it was proven that this class forms a subclass of the so-called *lob-spaces* – topological spaces with linearly ordered basis at every point (see [6]). The subclass contains only regular spaces; that is, concentric spaces are those lob-spaces that are T_1 and regular. Further, both authors proved that every Grzegorczyk structure can be represented as a subalgebra of the regular open algebra of a concentric space of Grzegorczyk points. Moreover, it was also proven that there is a one-to-one correspondence between Grzegorczyk structures that satisfy the countable chain condition and concentric spaces that satisfy the topological version of the condition. As a result, abstract Grzegorczyk structures obtained concrete representation, and their existence was also established. The latter follows from the fact that, e.g., the real line with the standard Euclidean topology is a concentric space.

In the BCA setting, a *Grzegorczyk contact algebra* may be defined as a Boolean contact algebra that satisfies two additional second-order Grzegorczyk's axioms. The first of them says that every region has a G-representative (and consequently, a G-point):

$$(\forall x \in B)(\exists Q \in \mathbf{Q})\, x \in Q\,. \tag{G1}$$

According to the second, G-representatives (and so G-points either) exist in those locations of space (understood as the unity of the algebra) where regions touch each other:

$$x \mathrel{\mathsf{C}} y \longrightarrow (\exists Q \in \mathbf{Q})(\forall u \in Q)\,(u \bigcirc x \wedge u \bigcirc y)\,. \tag{G2}$$

More precisely, the class of Grzegorczyk contact algebras is determined by axioms (C0)–(C3), (G1), (G2), as (C4) is their consequence.

It is provable that the set of all values of the Stone-like mapping $\mathscr{G}\colon B \to \mathcal{P}(\mathbf{Grz}(B))$ such that $\mathscr{G}(x) := \{\mathscr{G} \in \mathbf{Grz}(B) \mid x \in \mathscr{G}\}$ is a basis, and thus gives rise to a topological space $\langle \mathbf{Grz}(B), \mathscr{O}\rangle$. As we wrote above, the key notion to understanding this space is the concept of a *concentric* space, which is formally defined as a T_1 space in which every point p has a local basis $\mathscr{B}_p$ of regular open sets such that:

$$(\forall U, V \in \mathscr{B}_p)\,(U = V \vee \operatorname{Cl} U \subseteq V \vee \operatorname{Cl} V \subseteq U)\,. \tag{R1}$$

The reader will notice that the condition is a point-based counterpart of (r1) from page 87. Every concentric space is a regular space, yet generally, the converse is not true. For example, the uncountable product of the discrete space $\{0, 1\}$ is regular but not concentric [37].

If B is a Grzegorczyk contact algebra, then the space $\mathbf{Grz}(B)$ must be a concentric space. Given any G-point $\mathscr{G}$ we know that it has been generated by some G-representative Q, and thusly, the family $\mathscr{B}_{\mathscr{G}} := \{\mathscr{G}(x) \mid x \in Q\}$ is a local basis at the point $\mathscr{G}$ that satisfies the condition (R1). The fact that $\mathbf{Grz}(B)$ is T_1 is routinely verified, since if $\mathscr{G}_1 \neq \mathscr{G}_2$, then $\mathscr{G}_1 \nsubseteq \mathscr{G}_2$ and $\mathscr{G}_2 \nsubseteq \mathscr{G}_1$ (for G-points are maximal objects). Therefore there is a region x in $\mathscr{G}_1$ but not in $\mathscr{G}_2$, so $\mathscr{G}_1(x)$ is an open set around the point $\mathscr{G}_1$ but not around $\mathscr{G}_2$.

On the other hand, given a concentric space X, its algebra $\mathrm{RO}(X)$ is a (complete) Grzegorczyk contact algebra.[17]

In [15] it was shown that every Grzegorczyk contact algebra B embeds into a Grzegorczyk contact algebra of a concentric topological space, and the embedding is an isomorphism in the case of completeness of B.

$$\begin{tikzcd}
\mathsf{GCA} \ni B \arrow[r, dashed] \arrow[dr, hook, "\mathscr{G}"'] & \mathbf{Grz}(B) \in \mathsf{Conc} \arrow[d, dashed] \\
& \mathrm{RO}(\mathbf{Grz}(B)) \in \mathsf{GCA}^c
\end{tikzcd}$$

Figure 19. Let **GCA** be the class of Grzegorczyk contact algebras, and **Conc** the class of concentric topological spaces. Any Grzegorczyk algebra B embeds densely into the algebra of regular open sets of the concentric space for B.

The path from the concentric topological spaces to Grzegorczyk algebras is a bit more complicated, and it was only proven for Grzegorczyk contact algebras and concentric spaces that satisfy, respectively, algebraical and topological versions of the countable chain condition, which has not been circumvented so far. By an antichain of a Boolean algebra we mean, standardly, a subset of its regions that are pairwise incompatible. In the case of topological spaces, an antichain is a family of open sets whose intersections are pairwise empty. The countable chain condition is satisfied either by an algebra or a topological space if any antichain is at most countable.

[17] The proof of this fact can be found in [15] and [18].

$$\begin{tikzcd} \mathsf{GCA}^c \ni B \arrow[r, dashed] \arrow[dr, shift right, "\mathscr{G}"'] & \mathbf{Grz}(B) \in \mathsf{Conc} \arrow[d, dashed] \\ & \mathrm{RO}(\mathbf{Grz}(B)) \in \mathsf{GCA}^c \arrow[ul, shift right, "\mathscr{G}^{-1}"'] \end{tikzcd}$$

Figure 20. Any complete Grzegorczyk contact algebra B is indistinguishable from the Grzegorczyk algebra of regular open sets of the concentric space for B.

Firstly, if the condition holds for a Grzegorczyk algebra B, then its space $\mathbf{Grz}(B)$ satisfies the topological version of the condition, and the algebraical version transfers to $\mathrm{RO}(\mathbf{Grz}(B))$. The first dependence stems from the fact that if every antichain of regions is at most countable and the family of all sets of the form $\mathscr{G}(x)$ is a basis for $\mathbf{Grz}(B)$, then the space must also satisfy the condition. If it did not, for an uncountable antichain of its open sets we would find an uncountable antichain of sets of the form $\mathscr{G}(x)$, and since:

$$x \perp y \longleftrightarrow \mathscr{G}(x) \cap \mathscr{G}(y) = \emptyset$$

the pre-images of $\mathscr{G}(x)$s would form an uncountable chain of regions in B.

Secondly, it is evident that if a topological space satisfies the countable chain condition, then its algebra of regular open sets must also satisfy it.

In light of these, it is easily seen that the situations from figures 19 and 20 transfer immediately to those structures that satisfy the condition. Moreover, we can extend the representation to the one from Figure 21. For complete Grzegorczyk algebras, we have then a one-to-one correspondence between these that satisfy ccc, and concentric structures that have ccc.

To conclude, the results presented let us affirmatively respond to the question: are Grzegorczyk points really points?[18] As it turns out, thanks to the results for G-points, we can positively answer the main problem of this paper: are there any spaces of *Whitehead points* (in the sense of the method from page 5)?

[18] Technical details of all constructions can be found in [15] and [17, 18].

$$\begin{tikzcd}
\mathsf{Conc}_{\mathrm{ccc}} \ni X \arrow[r, dashed] \arrow[dr, shift right, "h"'] & \mathrm{RO}(X) \in \mathsf{GCA}^{c}_{\mathrm{ccc}} \arrow[d, dashed] \\
& \mathbf{Grz}(\mathrm{RO}(X)) \in \mathsf{Conc}_{\mathrm{ccc}} \arrow[ul, shift right, "h^{-1}"']
\end{tikzcd}$$

Figure 21. Any concentric space X satisfying the countable chain condition is homeomorphic to the concentric space of the complete Grzegorczyk algebra of regular open sets of X that satisfies the condition either.

7. Spaces of Whitehead points

In [3] we find the proof that, under some reasonable constraints, the classes of Grzegorczyk points and Whitehead points for a certain connection structures (mereological structures with the contact relation) coincide. In this section, we rephrase the results of Biacino and Gerla in the framework of contact algebras in order to apply their result (together with the results from earlier sections) to the problem of representation theorem for Whitehead points.

As we saw, Grzegorczyk points may be defined as filters, but they can also be characterized as quotients with respect to the covering relation from section 2. In the case of G-representatives we have that if Q_1 covers Q_2, then Q_2 covers Q_1. This is a consequence of two facts: (a) if Q_1 does not cover Q_2, then there are regions $x \in Q_1$ and $y \in Q_2$ that are separated from each other, and (b) if Q_1 covers Q_2, then for all $x \in Q_1$ and $y \in Q_2$, x and y are compatible.

Since covering is also transitive and reflexive, it must be an equivalence relation (in the family of G-representatives, but not generally in the family of all abstractive sets), and thus we can say that G-representatives Q_1 and Q_2 *represent the same location* (in symbols: $Q_1 \sim Q_2$) if and only if Q_1 covers Q_2 (and Q_2 covers Q_1).

The relation $\sim$ may be recovered from the set of G-points via the following equivalence:

$$Q_1 \sim Q_2 \longleftrightarrow (\exists \mathscr{G} \in \mathbf{Grz})\, Q_1 \cup Q_2 \subseteq \mathscr{G}\,.$$

The family of all equivalence classes of the relation $\sim$ in the set of Grzegorczyk representatives:

$$\mathbf{Eq} := \mathbf{Q}/_{\sim}$$

may now be treated as the set of points, as there is a bijective correspondence between elements of $\mathbf{Eq}$ and G-points given by function $f\colon \mathbf{Eq} \to \mathbf{Grz}$ such that $f([Q]) := \mathscr{G}_Q$, where $\mathscr{G}_Q$ is the G-point generated by Q. Thus, Grzegorczyk points can be characterized by the Whiteheadian covering relation.

Let us observe that Whitehead's abstractive sets are sets of regions that satisfy Grzegorczyk conditions (r0), (r1), plus non-minimality constraint:

$$\neg(\exists x \in B)(\forall y \in A)\, x \leq y\,. \tag{A}$$

Thus, it is immediate that if the Boolean contact algebra in focus is atomless, then its Grzegorczyk representatives must be abstractive sets. A less obvious conclusion is that in every atomless contact algebra, every G-representative must be a Whitehead representative of a point either. To see this, let us couch—after Biacino and Gerla—a mathematically satisfactory definition of a Whitehead representative and a Whitehead point.

Unlike the covering relation on Grzegorczyk representatives, covering on abstractive sets does not have to be an equivalence relation since it is not generally symmetric. However, it is reflexive and transitive, so it gives rise to the following equivalence relation[19]:

$$A_1 \sim A_2 \;:\longleftrightarrow\; A_1 \succeq A_2 \wedge A_2 \succeq A_1\,.$$

In the case $A_1 \sim A_2$, we say that the objects A_1 and A_2 are *similar*. The intended meaning of *similarity* is a representation of the same geometrical figure in space. Of course, unlike G-representatives, abstractive sets do not have to represent the same precise *location*, and the idea is to identify those that do. As $\sim$ is an equivalence, we can define—in Whitehead's spirit—geometrical objects as equivalence classes of abstractive sets with respect to similarity, i.e., as elements of the family $\mathbf{A}/_{\sim}$. This family equipped with the following binary relation:

[19]Recall that $A_1 \succeq A_2$ means A_1 *covers* A_2.

$$[A_1] \trianglerighteq [A_2] \;:\longleftrightarrow\; A_1 \succeq A_2$$

is a partially ordered set (i.e., $\trianglerighteq$ is reflexive, anti-symmetrical, and transitive).

We can now define Whitehead points and Whitehead representatives. $[A] \in \mathbf{A}/_\sim$ is a *Whitehead point* (*W-point*) iff $[A]$ is maximal with respect to $\trianglerighteq$: for every $[A'] \in \mathbf{A}/_\sim$, if $[A] \trianglerighteq [A']$, then $[A] = [A']$. $A \in \mathbf{A}$ is a *Whitehead representative* of a point (a *W-representative*) iff $[A]$ is a Whitehead point. The set of all Whitehead points and of all Whitehead representatives will be denoted by, respectively, '$\mathbf{W}$' and '$\mathbf{Q}_W$'.

Observe that we can also characterize as W-representatives those abstractive sets that satisfy the following equivalence:

$$A \in \mathbf{Q}_W \longleftrightarrow (\forall X \in \mathbf{A})\,(A \succeq X \longrightarrow X \succeq A)\,.$$

As it was demonstrated in [16], the notion of the Whitehead point is consistent, i.e., there are contact algebras with Whitehead points. However, we can still ask: can we prove that there are topological spaces based on Whitehead points obtained in the way described on page 84?, and can we find any form of representation theorems for such spaces? Both questions may be answered affirmatively in an indirect way using the result of Biacino and Gerla: under additional assumptions, the set of Whitehead points of a given contact algebra coincides with the set of Grzegorczyk points.

To prove that every G-point is a W-point it is enough to show that every G-representative is a W-representative. This part is relatively easy, and the result from [3] can actually be strengthened to the following (for details, see [16])

THEOREM 7.1. *If* $\mathfrak{B}$ *is a Boolean contact algebra that satisfies* (DV7) *then*: $\mathfrak{B}$ *is atomless iff in* $\mathfrak{B}$ *every G-representative is a W-representative.*

Proving that every W-representative is a G-representative is a bit harder, and the original demonstration of [3] calls for a small modification. In the class of all abstractive sets of a given Boolean algebra B we distinguish those that countable abstractive sets can represent. By an ω-abstractive set, we understand an abstractive set A for which there is a countable abstractive set A' such that A both covers A' and is covered by A'. Accordingly, W^ω-representatives will be those Whitehead representatives that are ω-abstractive sets. Let $\mathbf{Q}_W^\omega$ be the set of all W^ω-representatives of a given Boolean contact algebra. We have:

THEOREM 7.2. *If $\mathfrak{B}$ is a Boolean contact algebra that satisfies* (DV6) *and:*

$$x \notin \{0,1\} \longrightarrow x \,\mathsf{C}\, {-x}\,, \tag{C6}$$

then every W^ω-representative is a G-representative.

The small modification we mentioned is the inclusion of (C6) in the premises of the theorem. Here, (C6) is a region-based version of connectedness, i.e., it says that every non-zero and non-unity region touches its Boolean complement. For a more detailed analysis of this, we again refer the reader to [16].

In light of the above and the earlier results, we may conclude that the set $\mathbf{Q}_G^\omega$ of those Grzegorczyk representatives that countable sets can faithfully represent, we have the equality: $\mathbf{Q}_G^\omega = \mathbf{Q}_W^\omega$, and in consequence, $\mathbf{Grz}^\omega = \mathbf{W}^\omega$, where the former set is the set of Grzegorczyk points obtained from the elements of $\mathbf{Q}_G^\omega$ and the latter the set of Whitehead points obtained from the elements of $\mathbf{Q}_W^\omega$.

We thus have reached a point at which we can formulate the following theorem:

THEOREM 7.3. *Let B be an atomless Boolean contact algebra that satisfies the interpolation axiom* (DV6) *and the connectedness axiom* (C6)*. Suppose we introduce both definitions of points—by Grzegorczyk and by Whitehead—and extend the axioms with Grzegorczyk postulates* (G1) *and* (G2)*. Suppose $\mathbf{Grz}^\omega \neq \emptyset$. Let $\langle \mathbf{Grz}, \mathscr{O}\rangle$ be the concentric topological space for B. Then its subspace $\langle \mathbf{Grz}^\omega, \mathscr{O}^\omega\rangle$ (where $\mathscr{O}^\omega := \{\mathbf{Grz}^\omega \cap V \mid V \in \mathscr{O}\}$) is a topological space whose points are W-points.*

We can also conclude that there are spaces in which both sets of points coincide on the whole space, not only its subspace. To this end, observe that in the case of abstractive sets *covering* is anything but a form of *cofinality* for $\geq$-relation: an abstractive set A covers an abstractive set B iff B is cofinal with A. Putting the dual $\geq$ of *part of* relation in focus, and assuming Axiom of Choice, every chain C in any Boolean contact algebra has a cofinal well-ordered subchain C' with respect to $\geq$, where we refer to the dual notion of the well-ordered set by requiring the existence of the maximal element for $\geq$ in every non-empty subset of C'. On the other hand, the countable chain condition entails that every infinite well-ordered set of regions must be countable. Therefore:

THEOREM 7.4. *Let B be an atomless Boolean contact algebra that satisfies the interpolation axiom* (DV6)*, the connectedness axiom* (C6)*, and the countable chain condition. Suppose we introduce both definitions of points—by Grzegorczyk and by Whitehead—and we extend the axioms with Grzegorczyk postulates* (G1) *and* (G2)*. The concentric topological space $\langle \mathbf{Grz}, \mathscr{O} \rangle$ for B is a topological space in which $\mathbf{Grz} = \mathbf{Grz}^{\omega}$, so it is a space whose points are W-points.*

Thanks to the above theorem, we can see that Grzegorczyk and Whitehead points coincide in a large subclass of regular spaces: concentric spaces that satisfy countable chain condition.[20]

Since the algebra $\mathrm{RO}(\mathbb{R}^n)$ of regular open subsets of the n-dimensional Euclidean space has all the properties from the premises of the theorem above, we can conclude that:

COROLLARY 7.5. There are spaces of Whitehead points satisfying the requirements of the method from page 84.

Let us conclude this section with a strict justification of the difference between De Vries's and Whitehead points mentioned on page 80. We know there are structures in which Whitehead points are exactly Grzegorczyk points. Yet on page 6 we have demonstrated how to construct a maximally round filter that is not a G-point. This construction can be carried out in $\mathbb{R}^2$, which is a space that satisfies all premises of Theorem 7.4. Thus, in $\mathrm{RO}(\mathbb{R}^2)$ there is a De Vries point that is not a Whitehead point.

8. Summary

From the intuitions about the perspective space, we have come a long way through the topological representation theorems for Boolean algebras and De Vries algebras, Grzegorczyk contact algebras, to spaces of Whitehead points. Because there are spaces of Grzegorczyk points and Grzegorczyk

[20] The result concerning the relationship between Grzegorczyk and Whitehead points can be generalized by eliminating the countability assumption. This, however, calls for a stronger, second-order version of (DV6). Details, again, can be found in [16].

contact algebras whose G-points coincide with Whitehead points, we concluded that there are topological spaces constructed in the Stone-like manner whose fundamental objects are the English logician's points.

One might say that this is a roundabout way to show that there are topological spaces built out of Whitehead points. However, to our knowledge, no better way has been found so far. The earlier analyses only presented the way to points via extensive abstraction or compared them to other similar constructions. Yet, none of them pointed out that there are indeed topological spaces of Whitehead points obtained via methods of representation theorems.

The natural questions at this point are: can we generalize the result?, can we drop the reference to Grzegorczyk points and build any representation (or, even better, duality) for Whitehead points directly? With positive answers to these, we may try extending the scrutiny of both Grzegorczyk and Whitehead constructions to algebraic structures weaker than Boolean contact algebras, e.g., (extended) distributive contact lattices [11, 23], or Stonian p-ortholattices [49], to name few.

These, in our opinion, are problems concerning the classical Whitehead construction that has been neglected for too long. The path to understanding what Whitehead points are leads through the realms of logic and mathematics.

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References

[1] B. Bennett, I. Düntsch, *Axioms, algebras and topology*, [in:] M. Aiello, I. Pratt-Hartmann, V. Benthem (eds.), **Handbook of Spatial Logics**, chap. 3, Springer, Dordrecht (2007), pp. 99–159, DOI: https://doi.org/10.1007/978-1-4020-5587-4_3.

[2] L. Biacino, G. Gerla, *Connection structures*, **Notre Dame Journal of Formal Logic**, vol. 32(2) (1991), pp. 242–247, DOI: https://doi.org/10.1305/ndjfl/1093635748.

[3] L. Biacino, G. Gerla, *Connection structures: Grzegorczyk's and Whitehead's definitions of point*, **Notre Dame Journal of Formal Logic**,

vol. 37(3) (1996), pp. 431–439, DOI: https://doi.org/10.1305/ndjfl/1039886519.

[4] B. L. Clarke, *A calculus of individuals based on 'connection'*, **Notre Dame Journal of Formal Logic**, vol. 22(3) (1981), pp. 204–217, DOI: https://doi.org/10.1305/ndjfl/1093883455.

[5] B. L. Clarke, *Individuals and points*, **Notre Dame Journal of Formal Logic**, vol. 26(1) (1985), pp. 61–67, DOI: https://doi.org/10.1305/ndjfl/1093870761.

[6] S. W. Davis, *Spaces with linearly ordered local bases*, **Topology Proceedings**, vol. 3 (1978), pp. 37–51.

[7] T. de Laguna, *Extensive abstraction: A suggestion*, **The Philosophical Review**, vol. 30(2) (1921), pp. 216–218.

[8] H. de Vries, **Compact spaces and compactifications**, Van Gorcum and Comp. N.V., Amsterdam (1962).

[9] G. Del Piero, *A class of fit regions and a universe of shapes for continuum mechanics*, **Journal of Elasticity**, vol. 70 (2003), pp. 175–195, DOI: https://doi.org/10.1023/B:ELAS.0000005551.92536.c6.

[10] G. Del Piero, *A new class of fit regions*, **Note di Matematica**, vol. 27(2) (2007), pp. 55–67.

[11] I. Düntsch, W. MacCaull, D. Vakarelov, M. Winter, *Distributive contact lattices: Topological representations*, **The Journal of Logic and Algebraic Programming**, vol. 76(1) (2008), pp. 18–34, DOI: https://doi.org/10.1016/j.jlap.2007.10.002.

[12] C. Eschenbach, *A Mereotopological Definition of 'Point'*, [in:] C. Eschenbach, C. Habel, B. Smith (eds.), **Topological Foundations of Cognitive Science**, Graduiertenkolleg Kognitionswissenschaft, Hamburg (1994), pp. 63–80, DOI: https://doi.org/10.1007/978-1-4020-5587-4_3.

[13] A. Galton, *The mereotopology of discrete space*, [in:] C. Freksa, D. M. Mark (eds.), **COSIT '99: Proceedings of the International Conference on Spatial Information Theory: Cognitive and Computational Foundations of Geographic Information Science**, Springer, Berlin–Heidelberg (1999), pp. 251–266, DOI: https://doi.org/10.1007/3-540-48384-5_17.

[14] A. Galton, *Multidimensional mereotopology*, [in:] D. Dubois, C. Welty, M.-A. Williams (eds.), **Principles of Knowledge Representation and**

Reasoning: Proceedings of the Ninth International Conference, AAAI Press (2004), pp. 45–54.

[15] R. Gruszczyński, **Niestandardowe teorie przestrzeni**, Wydawnictwo Naukowe Uniwersytetu Mikołaja Kopernika, Toruń (2016).

[16] R. Gruszczyński, S. J. Martinez, *Grzegorczyk and Whitehead points: the story continues* (2023), arxiv:2303.08664.

[17] R. Gruszczyński, A. Pietruszczak, *A study in Grzegorczyk point-free topology. Part I: Separation and Grzegorczyk structures*, **Studia Logica**, vol. 106 (2018), pp. 1197–1238, DOI: https://doi.org/10.1007/s11225-018-9786-8.

[18] R. Gruszczyński, A. Pietruszczak, *A study in Grzegorczyk point-free topology. Part II: Spaces of points*, **Studia Logica**, vol. 107 (2019), pp. 809–843, DOI: https://doi.org/10.1007/s11225-018-9822-8.

[19] R. Gruszczyński, A. Pietruszczak, *Grzegorczyk points and filters in Boolean contact algebras*, **The Review of Symbolic Logic**, vol. 16(2) (2023), pp. 509–528, DOI: https://doi.org/10.1017/S1755020321000459.

[20] A. Grzegorczyk, *Axiomatizability of geometry without points*, **Synthese**, vol. 12(2–3) (1960), pp. 228–235, DOI: https://doi.org/10.1007/BF00485101.

[21] T. Hahmann, *CODI: A Multidimensional Theory of Mereotopology with Closure Operations*, **Applied Ontology**, vol. 15(3) (2020), pp. 1–61, DOI: https://doi.org/10.3233/AO-200233.

[22] E. Huntington, *A set of postulates for abstract geometry, expressed in terms of the simple relation of inclusion*, **Mathematische Annale**, vol. 73 (1913), pp. 522–559, DOI: https://doi.org/10.1007/BF01455955.

[23] T. Ivanova, D. Vakarelov, *Distributive mereotopology: extended distributive contact lattices*, **Annals of Mathematics and Artificial Intelligence**, vol. 77 (2016), pp. 3–41, DOI: https://doi.org/10.1007/s10472-016-9499-5.

[24] P. T. Johnstone, **Stone spaces**, vol. 3 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge (1982).

[25] P. T. Johnstone, *The point of pointless topology*, **Bulletin (New Series) of the American Mathematical Society**, vol. 8(1) (1983), pp. 41–53, URL: https://projecteuclid.org/journals/bulletin-of-the-

american-mathematical-society-new-series/volume-8/issue-1/The-point-of-pointless-topology/bams/1183550014.full.

[26] S. Koppelberg, **Handbook of Boolean Algebras**, vol. 1, Elsevier, Amsterdam (1989).

[27] T. Lando, D. Scott, *A calculus of regions respecting both measure and topology*, **Journal of Philosophical Logic**, vol. 48(5) (2019), pp. 825–850, DOI: https://doi.org/10.1007/s10992-018-9496-8.

[28] T. Mormann, *Continuous lattices and Whiteheadian theory of space*, **Logic and Logical Philosophy**, vol. 6(6) (1998), pp. 35–54, DOI: https://doi.org/10.12775/LLP.1998.002.

[29] J. Picado, A. Pultr, **Frames and Locales**, Frontiers in Mathematics, Birkhäuser, Basel (2012), DOI: https://doi.org/10.1007/978-3-0348-0154-6.

[30] J. Picado, A. Pultr, **Separation in Point-Free Topology**, Birkhäuser, Basel (2021), DOI: https://doi.org/10.1007/978-3-030-53479-0.

[31] A. Pietruszczak, **Metamereology**, Nicolaus Copernicus University Publishing House, Toruń (2018).

[32] A. Pietruszczak, **Foundations of the theory of parthood**, Springer, Toruń (2020), DOI: https://doi.org/10.1007/978-3-030-36533-2.

[33] I. Pratt, O. Lemon, *Ontologies for plane, polygonal mereotopology*, **Notre Dame Journal of Formal Logic**, vol. 38(2) (1997), pp. 225–245, DOI: https://doi.org/10.1305/ndjfl/1039724888.

[34] I. Pratt-Hartmann, *Empiricism and rationalism in region-based theories of space*, **Fundamenta Informaticae**, vol. 46(1–2) (2001), pp. 159–186.

[35] K. Robering, *"The whole is greater than the part." Mereology in Euclid's Elements*, **Logic and Logical Philosophy**, vol. 25(3) (2016), pp. 371–409, DOI: https://doi.org/10.12775/LLP.2016.011.

[36] P. Roeper, *Region-based topology*, **Journal of Philosophical Logic**, vol. 26(3) (1997), pp. 251–309, DOI: https://doi.org/10.1023/A:1017904631349.

[37] H. Rüping, *An example of a regular but not linear-based topological space* (2016), URL: https://mathoverflow.net/q/240345, mathOverflow, version: 2016-06-03.

[38] B. Russell, **Our knowledge of the external world**, George Allen and Unwin LTD, London (1914).

[39] D. J. Schoop, *Points in point-free mereotopology*, **Fundamenta Informaticae**, vol. 46(1–2) (2001), pp. 129–143.

[40] E. V. Shchepin, *Real functions and near-normal spaces*, **Fundamenta Informaticae**, vol. 13 (1972), pp. 820–830, DOI: https://doi.org/10.1007/BF00968394.

[41] J. Stell, *Boolean connection algebras: A new approach to the Region-Connection Calculus*, **Artificial Intelligence**, vol. 122(1) (2000), pp. 111–136, DOI: https://doi.org/10.1016/S0004-3702(00)00045-X.

[42] M. H. Stone, *The theory of representations for Boolean algebras*, **Transactions of the American Mathematical Society**, vol. 40(1) (1936), pp. 37–111, DOI: https://doi.org/10.1090/S0002-9947-1936-1501865-8.

[43] T. Tao, *245b notes 4: The Stone and Loomis-Sikorski representation theorems (optional)* (2009), URL: https://terrytao.wordpress.com/2009/01/12/245b-notes-1-the-stone-and-loomis-sikorski-representation-theorems-optional/, accessed: August 01, 2010.

[44] A. C. Varzi, *Mereology*, [in:] E. N. Zalta (ed.), **The Stanford Encyclopedia of Philosophy** (2016), URL: https://plato.stanford.edu/entries/mereology/, spring 2016 edition.

[45] A. C. Varzi, *Points as higher-order constructs*, [in:] G. Hellman, S. Shapiro (eds.), **The History of Continua: Philosophical and Mathematical Perspectives**, Oxford University Press, Oxford (2020), pp. 347–378, DOI: https://doi.org/10.1093/oso/9780198809647.003.0015.

[46] A. N. Whitehead, **Enquiry Concerning the Principles of Human Knowledge**, Cambridge University Press, Cambridge (1919).

[47] A. N. Whitehead, **The Concept of Nature**, Cambridge University Press, Cambridge (1920).

[48] A. N. Whitehead, **Process and Reality**, MacMillan, New York (1929).

[49] M. Winter, T. Hahmann, M. Grüninger, *On the algebra of regular sets*, **Annals of Mathematics and Artificial Intelligence**, vol. 65 (2012), pp. 25–60, DOI: https://doi.org/10.1007/s10472-012-9301-2.

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STABILIZERS ON L-ALGEBRAS

Abstract

The main goal of this paper is to introduce the notion of stabilizers in L-algebras and develop stabilizer theory in L-algebras. In this paper, we introduced the notions of left and right stabilizers and investigated some related properties of them. Then, we discussed the relations among stabilizers, ideal and co-annihilators. Also, we obtained that the set of all ideals of a CKL-algebra forms a relative pseudo-complemented lattice. In addition, we proved that right stabilizers in CKL-algebra are ideals. Then by using the right stabilizers we produced a basis for a topology on L-algebra. We showed that the generated topology by this basis is Baire, connected, locally connected and separable and we investigated the other properties of this topology.

Keywords: L-algebra, stabilizer, ideal, co-annihilators, Baire space, topological space.

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1. Introduction

L-algebras, which are related to algebraic logic and quantum structures, were introduced by Rump [8]. Many examples shown that L-algebras are very useful. Yang and Rump [10], characterized pseudo-MV-algebras and

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Bosbach's non-commutative bricks as L-algebras. Wu and Yang [13] proved that orthomodular lattices form a special class of L-algebras in different ways. It was shown that every lattice-ordered effect algebra has an underlying L-algebra structure in Wu et al. [12]. Also, they proved that a basic algebra which satisfies

$$(z \oplus \neg x) \oplus \neg(y \oplus \neg x) = (z \oplus \neg y) \oplus \neg(x \oplus \neg y),$$

can be converted into an L-algebra. Conversely, if an L-algebra with 0 and some conditions such that it is an involutive bounded lattice can be organized into a basic algebra, it must be a lattice-ordered effect algebra. In addition, Aaly in [1], and Ciung in [5] studied the relationship between logical algebraic structures and basic algebras with L-algebras, such as $BCK/\ BCI$-algebras, hoop, residuated lattice, equality and EQ-algebras.

A stabilizer is a part of a monoid acting on a set. Specifically, let $\mathbb{X}$ be a monoid operating on a set $\mathbb{X}$ and let $\mathbb{H}$ be a subset of $\mathbb{X}$. The stabilizer of $\mathbb{H}$, sometimes denoted $St(\mathbb{H})$ is the set of elements as a of $\mathbb{X}$ for which $a(\mathbb{H}) \subseteq \mathbb{H}$. The strict stabilizer is the set of $a \in \mathbb{X}$ for which $a(\mathbb{H}) = \mathbb{H}$. In the other words, the stabilizer of $\mathbb{H}$ is the transporter of $\mathbb{H}$ to itself. In recent years, many mathematicians have studied and investigated the characteristics of stabilizers in logical algebraic structures. Also, some of them have used a special type of stabilizers called co-annihilators and have obtained interesting results in this field, and this concept has been investigated on different structures, such as BL-algebra, EQ-algebra, hoop and etc. For more information in this field, we refer the readers to the references [3, 4, 6, 7, 11].

The main goal of this paper is to introduce the notion of stabilizers in L-algebras and develop stabilizer theory in L-algebras. In this paper, we introduce the notions of left and right stabilizers and investigate some related properties of them. Then, we discuss the relations among stabilizers, ideal and co-annihilators. Also, we obtain that the set of all ideals in a CKL-algebra forms a relative pseudo-complemented lattice. In addition, we prove that right stabilizers in CKL-algebra are ideals. Then by using the right stabilizers produce a basis for a topology on L-algebra. We show that the generated topology by this basis is Baire, connected, locally connected and separable and we investigate the other properties of this topology.

2. Preliminaries

In this section, we gather some basic notions relevent to L-algebras which will need in the next sections.

DEFINITION 2.1 ([8]). An *L-algebra* is an algebraic structure $(\mathbb{L}; \to, 1)$ of type $(2, 0)$ satisfying

$(L1)$ $x \to x = x \to 1 = 1$ and $1 \to x = x$,

$(L2)$ $(x \to y) \to (x \to z) = (y \to x) \to (y \to z)$,

$(L3)$ if $x \to y = y \to x = 1$, then $x = y$,

for any $x, y, z \in \mathbb{L}$. Condition $(L1)$ states that 1 is a logical unit, while $(L2)$ is related to the quantum Yang-Baxter equation. Note that a logical unit is always unique. In addition, we can define the relation

$$x \lesssim y \text{ if and only if } x \to y = 1,$$

on $\mathbb{L}$. By (L_1) and (L_2), clearly this relation is reflexive and transitive, respectively and by $(L3)$, untisymmetric is proved. So, $(\mathbb{L}, \lesssim)$ is a poset. If $\mathbb{L}$ admits a smallest element 0, then it is called a *bounded L-algebra*.

Let $\mathbb{L}$ be bounded. We define a binary operation "$'$" on $\mathbb{L}$ by $x' = x \to 0$, for all $x \in \mathbb{L}$. If for any $x \in \mathbb{L}$, $x'' = x$, then the bounded L-algebra $\mathbb{L}$ is called to have the double negation properties.

PROPOSITION 2.2 ([10]). Let $\mathbb{L}$ be an L-algebra. Then $x \lesssim y$ implies $z \to x \lesssim z \to y$, for any $x, y, z \in \mathbb{L}$.

PROPOSITION 2.3 ([10]). For an L-algebra $\mathbb{L}$, the following are equivalent:

(i) $x \lesssim y \to x$,

(ii) if $x \lesssim z$, then $z \to y \lesssim x \to y$,

(iii) $((x \to y) \to z) \to z \lesssim ((x \to y) \to z) \to ((y \to x) \to z)$,

for any $x, y, z \in \mathbb{L}$.

DEFINITION 2.4 ([9]). An L-algebra $\mathbb{L}$ which satisfies

$$x \to (y \to x) = 1, \qquad (K)$$

for any $x, y \in \mathbb{L}$ is called a *KL-algebra*.

A *CKL-algebra* is an L-algebra which satisfies

$$x \to (y \to z) = y \to (x \to z), \qquad (C)$$

for any $x, y, z \in \mathbb{L}$ (see [9]).

Clearly, every CKL-algebra is a KL-algebra, since for any $x, y \in \mathbb{L}$, we have

$$x \to (y \to x) = y \to (x \to x) = y \to 1 = 1.$$

PROPOSITION 2.5 ([2]). Assume $(\mathbb{L}, \to, 1)$ is a CKL-algebra. Then for any $x, y, z \in \mathbb{L}$, the following properties hold:

(i) if $x \lesssim y$, then $z \to x \lesssim z \to y$,

(ii) $x \to (y \to x) = 1$, i.e., $x \lesssim y \to x$,

(iii) $x \lesssim (x \to y) \to y$,

(iv) $x \lesssim y \to z$ if and only if $y \lesssim x \to z$,

(v) if $x \lesssim y$, then $y \to z \lesssim x \to z$,

(vi) $((x \to y) \to z) \to z \lesssim ((x \to y) \to z) \to ((y \to x) \to z)$,

(vii) $z \to y \lesssim (y \to x) \to (z \to x)$,

$(viii)$ $z \to y \lesssim (x \to z) \to (x \to y)$,

If $\mathbb{L}$ has a least element 0, then

(ix) if $x \lesssim y$, then $y' \lesssim x'$, where $x' = x \to 0$,

(x) $x \lesssim x''$, and $x' = x'''$,

(xi) $x' \lesssim x \to y$,

(xii) $((x \to y) \to y) \to y = x \to y$,

$(xiii)$ If $\mathbb{L}$ has double negation, then $x \to y = y' \to x'$.

DEFINITION 2.6 ([8]). An L-algebra $\mathbb{L}$ is said to be a **semi-regular** if the equation

$$((x \to y) \to z) \to ((y \to x) \to z) = ((x \to y) \to z) \to z,$$

holds in $\mathbb{L}$. Also, $\mathbb{L}$ is called a **regular L-algebra** if in addition, for any pair element $x \lesssim y$ in $\mathbb{L}$, there is an element $z \gtrsim x$ in $\mathbb{L}$ such that $z \to x = y$.

For a bounded L-algebra with negation, we set

$$x \curlywedge y = ((x \to y) \to x')', \qquad x \curlyvee y = (x' \to y') \to x. \tag{2.1}$$

PROPOSITION 2.7 ([10]). Let $\mathbb{L}$ be a semi-regular L-algebra with negation. Then the equations

$$x \to (y \curlywedge z) = (x \to y) \curlywedge (x \to z)$$
$$(x \curlyvee y) \to z = (x \to z) \curlywedge (y \to z)$$

hold for any $x, y, z \in \mathbb{L}$.

DEFINITION 2.8 ([8]). A subset $\mathbb{I}$ of an L-algebra $\mathbb{L}$ is called an *ideal of* $\mathbb{L}$ if it satisfies the following conditions for all $x, y \in \mathbb{L}$,

(I_1) $1 \in \mathbb{I}$,

(I_2) if $x \in \mathbb{I}$ and $x \to y \in \mathbb{I}$, then $y \in \mathbb{I}$,

(I_3) if $x \in \mathbb{I}$, then $(x \to y) \to y \in \mathbb{I}$,

(I_4) if $x \in \mathbb{I}$, then $y \to x \in \mathbb{I}$ and $y \to (x \to y) \in \mathbb{I}$.

The set of all ideals of $\mathbb{L}$ is denoted by $\mathcal{I}d(\mathbb{L})$.

PROPOSITION 2.9 ([2]). Every ideal of $\mathbb{L}$ is upset.

If we consider the ideal of CKL-algebra, the conditions (I_3) and (I_4) can be dropped. In fact, for any $x \in \mathbb{I}$, by (C) and (I_1) we have

$$x \to ((x \to y) \to y) = (x \to y) \to (x \to y) = 1 \in \mathbb{I},$$

for any $y \in \mathbb{L}$. It follows by (I_2) that $(x \to y) \to y \in \mathbb{I}$. Thus ($I_3$) holds. Furthermore, if $x \in \mathbb{I}$, then for any $y \in \mathbb{L}$, by (K) we have $x \to (y \to x) = 1 \in \mathbb{I}$ and by (I_2), $y \to x \in \mathbb{I}$.

For an L-algebra such as $\mathbb{L}$, a binary relation $\sim$ is a *congruence relation* [8] on $\mathbb{L}$ if it is an equivalence relation such that for any $x, y, z \in \mathbb{L}$,

$$x \sim y \Leftrightarrow \ (z \to x) \sim (z \to y) \ \text{ and } \ (x \to z) \sim (y \to z).$$

THEOREM 2.10 ([8]). *Let* $(\mathbb{L}, \to, 1)$ *be an* L*-algebra. Then every ideal* $\mathbb{I}$ *of* $\mathbb{L}$ *defines a congruence relation on* $\mathbb{L}$*, for any* $x, y \in \mathbb{L}$*, where*

$$x \sim_{\mathbb{I}} y \ \Leftrightarrow \quad x \to y, \ y \to x \in \mathbb{I}.$$

Conversely, every congruence relation $\sim$ *defines an ideal*

$$\mathbb{I} = \{x \in \mathbb{L} \mid x \sim 1\}.$$

Definition 2.11 ([8]). Let $\mathbb{L}$ and $\mathbb{H}$ be two L-algebras. Then a map $f : \mathbb{L} \to \mathbb{H}$ is called an *L-homomorphism* if for any $x, y \in \mathbb{L}$ we have

$$f(x \to_{\mathbb{L}} y) = f(x) \to_{\mathbb{H}} f(y).$$

Obviously, $f(1_{\mathbb{L}}) = 1_{\mathbb{H}}$.

Note. From now on, we let $(\mathbb{L}, \to, 1)$ or $\mathbb{L}$, for short, be an L-algebra and $\mathbb{X}$ be a non-empty subset of $\mathbb{L}$.

3. Main results

3.1. Stabilizers on L-algebras

In this section, we introduce the notions of left and right stabilizers on L-algebras and investigate some properties of them.

Definition 3.1. A **left stabilizer and a right stabilizer of** $\mathbb{X}$ are defined as follows:

$$\mathcal{S}_r(\mathbb{X}) = \{a \in \mathbb{L} \mid \text{for any } x \in \mathbb{X},\ a \to x = x\}.$$
$$\mathcal{S}_l(\mathbb{X}) = \{a \in \mathbb{L} \mid \text{for any } x \in \mathbb{X},\ x \to a = a\}.$$

Example 3.2. (i) Assume $(\mathbb{L} = \{a, b, c, 1\}, \lesssim)$ is a chain where $a < b < c < 1$. Then $(\mathbb{L}, \to, 1)$ is an L-algebra such that

$\to$	a	b	c	1
a	1	1	1	1
b	c	1	1	1
c	b	c	1	1
1	a	b	c	1

Clearly, $\mathcal{S}_r(\{b\}) = \mathcal{S}_l(\{b\}) = \{1\}$.
(ii) Suppose $(\mathbb{L} = \{a, b, c, 1\}, \lesssim)$ is a chain where $a < b < c < 1$. Then $(\mathbb{L}, \to, 1)$ is an L-algebra such that

$\to$	a	b	c	1
a	1	1	1	1
b	b	1	1	1
c	a	b	1	1
1	a	b	c	1

If $\mathbb{X}_1 = \{c\}$ and $\mathbb{X}_2 = \{a, b, 1\}$, then $\mathcal{S}_l(\mathbb{X}_1) = \{a, b, 1\}$ and $\mathcal{S}_r(\mathbb{X}_2) = \{c, 1\}$.

Note. $\mathcal{S}_r(\mathcal{S}_l(\mathbb{X}))$ is called a **right-left stabilizer of** $\mathbb{X}$ and we denote it by $(\mathcal{S}_l(\mathbb{X}))_r$, for short. Similarly, $(\mathcal{S}_r(\mathbb{X}))_l$ is a **left-right stabilizer of** $\mathbb{X}$.

Proposition 3.3. For all $x, y \in \mathbb{L}$ and $\emptyset \neq \mathbb{X}, \mathbb{Y} \subseteq \mathbb{L}$, the following statements hold:

(*i*) $1 \in \mathcal{S}_r(\mathbb{X}) \cap \mathcal{S}_l(\mathbb{X})$.

(*ii*) If $\mathbb{X} \subseteq \mathbb{Y}$, then $\mathcal{S}_r(\mathbb{Y}) \subseteq \mathcal{S}_r(\mathbb{X})$ and $\mathcal{S}_l(\mathbb{Y}) \subseteq \mathcal{S}_l(\mathbb{X})$.

(*iii*) $\mathbb{X} \subseteq (\mathcal{S}_r(\mathbb{X}))_l \cap (\mathcal{S}_l(\mathbb{X}))_r$.

(*iv*) $\mathcal{S}_r(\mathbb{X}) = (((\mathcal{S}_r(\mathbb{X}))_l)_r$ and $\mathcal{S}_l(\mathbb{X}) = (((\mathcal{S}_l(\mathbb{X}))_r)_l$.

(*v*) If $\{\mathbb{X}_i\}_{i\in I}$ is a family of non-empty subsets of $\mathbb{L}$, then $\mathcal{S}_r(\bigcup_{i\in I} \mathbb{X}_i) = \bigcap_{i\in I} \mathcal{S}_r(\mathbb{X}_i)$ and $\mathcal{S}_l(\bigcup_{i\in I} \mathbb{X}_i) = \bigcap_{i\in I} \mathcal{S}_l(\mathbb{X}_i)$.

(*vi*) $\mathcal{S}_r(\mathbb{L}) = \mathcal{S}_l(\mathbb{L}) = \{1\}$.

(*vii*) $\mathcal{S}_r(\{1\}) = \mathcal{S}_l(\{1\}) = \mathbb{L}$.

(*viii*) If $x \in \mathcal{S}_r(\{x\}) \cap \mathcal{S}_l(\{x\})$, then $x = 1$.

(*ix*) If $h : \mathbb{L} \to \mathbb{L}$ is a homomorphism and $x \in \mathbb{L}$, then $h(\mathcal{S}_r(\{x\})) \subseteq \mathcal{S}_r(\{h(x)\})$ and $h(\mathcal{S}_l(\{x\})) \subseteq \mathcal{S}_l(\{h(x)\})$.

(*x*) If $\mathbb{L}$ is a bounded L-algebra with DNP, then $\mathcal{S}_r(\{0\}) = \{1\}$.

(*xi*) If $\mathbb{L}$ is a bounded L-algebra, then $\mathcal{S}_r(\{0\}) = \{1\}$ if and only if for any $x, y \in \mathbb{L}$, $x \to y, y \to x \in \mathcal{S}_r(\{0\})$ implies $x = y$.

Proof: (*i*) Clearly, by ($L1$), since for any $x \in \mathbb{X}$, $1 \to x = x$, we get $1 \in \mathcal{S}_r(\mathbb{X})$. In addition, by ($L1$), $x \to 1 = 1$, and so $1 \in \mathcal{S}_l(\mathbb{X})$. Hence, $1 \in \mathcal{S}_r(\mathbb{X}) \cap \mathcal{S}_l(\mathbb{X})$.

(ii) Assume $a \in \mathcal{S}_r(\mathbb{Y})$. Then for any $y \in \mathbb{Y}$, $a \to y = y$. Since $\mathbb{X} \subseteq \mathbb{Y}$, clearly, for any $x \in \mathbb{X}$, $a \to x = x$ and so $a \in \mathcal{S}_r(\mathbb{X})$. Hence, $\mathcal{S}_r(\mathbb{Y}) \subseteq \mathcal{S}_r(\mathbb{X})$. The proof of the other case is similar.

(iii) Suppose $a \in \mathbb{X}$. Then for any $y \in \mathcal{S}_r(\mathbb{X})$, $y \to a = a$, and so $a \in (\mathcal{S}_r(\mathbb{X}))_l$. In addition, for any $y \in \mathcal{S}_l(\mathbb{X})$, $a \to y = y$ and so $a \in (\mathcal{S}_l(\mathbb{X}))_r$. Hence, $a \in (\mathcal{S}_r(\mathbb{X}))_l \cap (\mathcal{S}_l(\mathbb{X}))_r$. Therefore, $\mathbb{X} \subseteq (\mathcal{S}_r(\mathbb{X}))_l \cap (\mathcal{S}_l(\mathbb{X}))_r$.

(iv) By (iii), we have $\mathbb{X} \subseteq (\mathcal{S}_r(\mathbb{X}))_l$ and by (ii), we get $((\mathcal{S}_r(\mathbb{X}))_l)_r \subseteq \mathcal{S}_r(\mathbb{X})$. Also, by (iii), $\mathbb{Y} \subseteq (\mathcal{S}_l(\mathbb{Y}))_r$. Consider $\mathbb{Y} = \mathcal{S}_r(\mathbb{X})$. Then $\mathcal{S}_r(\mathbb{X}) \subseteq ((\mathcal{S}_r(\mathbb{X}))_l)_r$. Hence, $\mathcal{S}_r(\mathbb{X}) = (((\mathcal{S}_r(\mathbb{X}))_l)_r$. The proof of the other case is similar.

(v) Since $\mathbb{X}_i \subseteq \bigcup\limits_{i\in I} \mathbb{X}_i$, by (ii), $\mathcal{S}_r(\bigcup\limits_{i\in I} \mathbb{X}_i) \subseteq \mathcal{S}_r(\mathbb{X}_i)$, and so $\mathcal{S}_r(\bigcup\limits_{i\in I} \mathbb{X}_i) \subseteq \bigcap\limits_{i\in I} \mathcal{S}_r(\mathbb{X}_i)$. Conversely, assume $a \in \bigcap\limits_{i\in I} \mathcal{S}_r(\mathbb{X}_i)$, then for any $i \in I$, $a \in \mathcal{S}_r(\mathbb{X}_i)$, and so for any $x_i \in \mathbb{X}_i$, $a \to x_i = x_i$. Thus for any $x \in \bigcup\limits_{i\in I} \mathbb{X}_i$, there exists $i \in I$ such that $x \in \mathbb{X}_i$, and so $a \to x = x$. So, $a \in \mathcal{S}_r(\bigcup\limits_{i\in I} \mathbb{X}_i)$. Therefore, $\mathcal{S}_r(\bigcup\limits_{i\in I} \mathbb{X}_i) = \bigcap\limits_{i\in I} \mathcal{S}_r(\mathbb{X}_i)$.

(vi) Clearly, by (i), $\{1\} \subseteq \mathcal{S}_r(\mathbb{L})$. Assume $1 \neq a \in \mathcal{S}_r(\mathbb{L})$. Then for any $x \in \mathbb{L}$, $a \to x = x$. Let $x = a$. Then $1 = a \to a = a$, and so $a = 1$, which is a contradiction. Hence, $\mathcal{S}_r(\mathbb{L}) = \{1\}$. The proof of the other case is similar.

(vii) Obviously, $\mathcal{S}_r(\{1\}), \mathcal{S}_l(\{1\}) \subseteq \mathbb{L}$. Suppose $a \in \mathbb{L}$. Then by $(L1)$, $a \to 1 = 1$ and $1 \to a = a$. Thus $a \in \mathcal{S}_r(\{1\}) \cap \mathcal{S}_l(\{1\})$. Hence $\mathcal{S}_r(\{1\}) = \mathcal{S}_l(\{1\}) = \mathbb{L}$.

$(viii)$ Straightforward.

(ix) Assume $y \in h(\mathcal{S}_l(\{x\}))$. Then there exists $a \in \mathcal{S}_l(\{x\})$ such that $y = h(a)$. Since $x \to a = a$ and h is a homomorphism on $\mathbb{L}$, we have

$$y = h(a) = h(x \to a) = h(x) \to h(a) = h(x) \to y.$$

Thus $y \in \mathcal{S}_l(\{h(x)\})$. Hence, $h(\mathcal{S}_l(\{x\})) \subseteq \mathcal{S}_l(h(x))$. The proof of the other case is similar.

(x) Assume $a \in \mathcal{S}_r(\{0\})$. Then $a \to 0 = 0$, and so $a' = 0$. By hypothesis, $a'' = 0' = 1$ and so $a = 1$. Thus $\mathcal{S}_r(\{0\}) = \{1\}$.

(xi) If $\mathcal{S}_r(\{0\}) = \{1\}$ and $x \to y, y \to x \in \mathcal{S}_r(\{0\})$, then $x \to y = y \to x = 1$, and by $(L3)$, we have $x = y$. Conversely, by (i), $\{1\} \subseteq \mathcal{S}_r(\{0\})$. Consider

$a \in \mathcal{S}_r(\{0\})$. Since $1 \to a = a \in \mathcal{S}_r(\{0\})$ and $a \to 1 = 1 \in \mathcal{S}_r(\{0\})$, we get $1 \to a$, $a \to 1 \in \mathcal{S}_r(\{0\})$. So, by assumption, we have $a = 1$. Hence, $\mathcal{S}_r(\{0\}) = \{1\}$. $\square$

In the following example we show that for any non-empty subset $\mathbb{X}$ of $\mathbb{L}$, $\mathcal{S}_r(\mathbb{X})$ and $\mathcal{S}_l(\mathbb{X})$ are not ideals of $\mathbb{L}$, in general.

Example 3.4. (*i*) According to Example 3.2(i), $\mathcal{S}_r(\{b\}) = \mathcal{S}_l(\{b\}) = \{1\}$. So, both are ideals of $\mathbb{L}$.

(*ii*) According to Example 3.2(ii), $\mathcal{S}_r(\{a, b, 1\}) = \{c, 1\}$ is an ideal of $\mathbb{L}$ but $\mathcal{S}_l(\{c\}) = \{a, b, 1\}$ is not an ideal of $\mathbb{L}$ since $b \to c = 1 \in \{a, b, 1\}$ and $b \in \{a, b, 1\}$ but $c \notin \{a, b, 1\}$.

(*iii*) Suppose $(\mathbb{L} = \{a, b, c, 1\}, \lesssim)$ is a poset where $a, c < b < 1$. Then $(\mathbb{L}, \to, 1)$ is an L-algebra such that

$\to$	a	b	c	1
a	1	1	a	1
b	a	1	c	1
c	b	1	1	1
1	a	b	c	1

If $\mathbb{X} = \{b\}$, then $\mathcal{S}_l(\mathbb{X}) = \{a, c, 1\}$ is not an ideal of $\mathbb{L}$, because $a \to b = 1 \in \mathcal{S}_l(\mathbb{X})$ and $a \in \mathcal{S}_l(\mathbb{X})$, but $b \notin \mathcal{S}_l(\mathbb{X})$.

(*iv*) Suppose $(\mathbb{L} = \{a, b, c, 1\}, \lesssim)$ is a poset where $a < b, c < 1$. Then $(\mathbb{L}, \to, 1)$ is an L-algebra such that

$\to$	a	b	c	1
a	1	1	1	1
b	c	1	c	1
c	b	b	1	1
1	a	b	c	1

Assume $\mathbb{X} = \{b\}$. Then $\mathcal{S}_r(\mathbb{X}) = \mathcal{S}_l(\mathbb{X}) = \{c, 1\}$ are ideals of $\mathbb{L}$.

Proposition 3.5. If $\mathbb{L}$ is a KL-algebra and for any $x, y \in \mathbb{L}$, $(x \to y) \to y = (y \to x) \to x$, then for any $\mathbb{X} \subseteq \mathbb{L}$, $\mathcal{S}_r(\mathbb{X}) = \mathcal{S}_l(\mathbb{X})$.

Proof: Let $a \in \mathcal{S}_r(\mathbb{X})$. Then for any $x \in \mathbb{X}$, $a \to x = x$. Since $\mathbb{L}$ is a KL-algebra, by Proposition 2.3, $a \lesssim x \to a$. By assumption,

$$(x \to a) \to a = (a \to x) \to x = x \to x = 1.$$

Thus, $x \to a \lesssim a$, and so by $(L3)$, we have $x \to a = a$. Hence, $a \in \mathcal{S}_l(\mathbb{X})$ and so $\mathcal{S}_r(\mathbb{X}) \subseteq \mathcal{S}_l(\mathbb{X})$. The proof of the other case is similar. □

Example 3.6. Assume $\mathbb{L}$ is an L-algebra as in Example 3.4(iv). This example demonstrates Proposition 3.5.

In the following example we show that the condition $(x \to y) \to y = (y \to x) \to x$ in Proposition 3.5 is necessary.

Example 3.7. Let $\mathbb{L}$ be an L-algebra as in Example 3.2(ii). Clearly, $\mathbb{L}$ is a KL-algebra but

$$(a \to c) \to c = 1 \to c = c \neq 1 = a \to a = (c \to a) \to a.$$

As we see in this example, if $\mathbb{X} = \{c\}$, then $\mathcal{S}_r(\mathbb{X}) = \{1\} \neq \{a, b, 1\} = \mathcal{S}_l(\mathbb{X})$.

Theorem 3.8. *If $\mathbb{L}$ is a CKL-algebra, $\mathcal{S}_r(\mathbb{X})$ is an ideal of $\mathbb{L}$, for any non-empty subset $\mathbb{X}$ of $\mathbb{L}$.*

Proof: By Proposition 3.3(i), $1 \in \mathcal{S}_r(\mathbb{X})$. Assume $a, a \to b \in \mathcal{S}_r(\mathbb{X})$, for any $a, b \in \mathbb{L}$. Then for any $x \in \mathbb{X}$, $a \to x = x$ and $(a \to b) \to x = x$. Thus,

$$\begin{aligned} b \to x &\lesssim (a \to b) \to (a \to x) && \text{by Proposition 2.5(viii)} \\ &= a \to ((a \to b) \to x) && \text{by (C)} \\ &= a \to x && \text{since } a \to b \in \mathcal{S}_r(\mathbb{X}) \\ &= x. && \text{since } a \in \mathcal{S}_r(\mathbb{X}) \end{aligned}$$

Thus, $b \to x \lesssim x$. By Proposition 2.5(ii), $x \lesssim b \to x$. Hence, $b \to x = x$, and so $b \in \mathcal{S}_r(\mathbb{X})$. Therefore, $\mathcal{S}_r(\mathbb{X})$ is an ideal of $\mathbb{L}$. □

The next example shows that the condition CKL-algebra in Theorem 3.8 is necessary.

Example 3.9. Suppose $(\mathbb{L} = \{a, b, c, 1\}, \lesssim)$ is a poset where $a, c < b < 1$. Then $(\mathbb{L}, \to, 1)$ is an L-algebra such that

$\to$	a	b	c	1
a	1	1	a	1
b	c	1	a	1
c	a	1	1	1
1	a	b	c	1

But $\mathbb{L}$ is not a CKL-algebra, since

$$a \to (b \to c) = a \to a = 1 \neq c = b \to a = b \to (a \to c).$$

If $\mathbb{X} = \{a\}$, then $\mathcal{S}_r(\mathbb{X}) = \{c, 1\}$, which is not an ideal of $\mathbb{L}$, because since $c \in \mathcal{S}_r(\mathbb{X})$ by (I_3), we have to have $b \to c \in \mathcal{S}_r(\mathbb{X})$, but $b \to c = a \notin \mathcal{S}_r(\mathbb{X})$.

By the following example we show that in any CKL-algebra, $\mathcal{S}_l(\mathbb{X})$ is not an ideal of $\mathbb{L}$.

Example 3.10. Suppose $(\mathbb{L} = \{a, b, c, 1\}, \lesssim)$ is a chain where $a < b < c < 1$. Then $(\mathbb{L}, \to, 1)$ is a CKL-algebra such that

$\to$	a	b	c	1
a	1	1	1	1
b	a	1	1	1
c	a	b	1	1
1	a	b	c	1

Assume $\mathbb{X} = \{b\}$. Then $\mathcal{S}_l(\mathbb{X}) = \{a, 1\}$ which is not an ideal of $\mathbb{L}$, since $a \to b = 1 \in \mathcal{S}_l(\mathbb{X})$ and $a \in \mathcal{S}_l(\mathbb{X})$ but $b \notin \mathcal{S}_l(\mathbb{X})$.

Remark 3.11. If $\mathbb{L}$ is a CKL-algebra with DNP, then by Proposition 2.5(xiii) and (2.1), we have

$$x \curlyvee y = (x' \to y') \to x = (y \to x) \to x.$$

Proposition 3.12. Assume $\mathbb{L}$ is a CKL-algebra with DNP and $a \in \mathbb{L}$. If $a \curlyvee x = 1$, then for any $x \in \mathbb{X}$, $a \in \mathcal{S}_r(\mathbb{X}) \cap \mathcal{S}_l(\mathbb{X})$.

Proof: Since $\mathbb{L}$ is a CKL-algebra, by Proposition 2.5(ii), $a \lesssim x \to a$, for any $x \in \mathbb{X}$. By assumption and Remark 3.11, we have

$$1 = a \curlyvee x = (a' \to x') \to a = (x \to a) \to a.$$

So, $x \to a \lesssim a$. Thus by Proposition 2.5(ii), we get $x \to a = a$, and so $a \in \mathcal{S}_l(\mathbb{X})$. By similar discussion, we can prove $a \in \mathcal{S}_r(\mathbb{X})$. Hence, $a \in \mathcal{S}_r(\mathbb{X}) \cap \mathcal{S}_l(\mathbb{X})$. $\square$

Theorem 3.13. *Let $\mathbb{L}$ be a semi-regular L-algebra with negation. If $x \curlyvee y \in \mathcal{S}_r(\mathbb{X})$, then $x \in \mathcal{S}_r(\mathbb{X})$ or $y \in \mathcal{S}_r(\mathbb{X})$.*

Proof: Suppose $x \curlyvee y \in \mathcal{S}_r(\mathbb{X})$ such that $x \notin \mathcal{S}_r(\mathbb{X})$ and $y \notin \mathcal{S}_r(\mathbb{X})$. Then for any $a \in \mathbb{X}$, $(x \curlyvee y) \to a = a$, $x \to a \neq a$ and $y \to a \neq a$. By Proposition 2.3, $a < x \to a$ and $a < y \to a$. Thus by Proposition 2.7 we have

$$a < (x \to a) \curlywedge (y \to a) = (x \curlyvee y) \to a = a,$$

which is a contradiction. Hence, $x \in \mathcal{S}_r(\mathbb{X})$ or $y \in \mathcal{S}_r(\mathbb{X})$. $\square$

Theorem 3.14. *Let $\mathbb{L}$ be a semi-regular L-algebra with negation. If $\mathbb{I}_1, \mathbb{I}_2 \in \mathcal{I}d(\mathbb{X})$ such that $\mathcal{S}_r(\mathbb{X}) = \mathbb{I}_1 \cap \mathbb{I}_2$, then $\mathcal{S}_r(\mathbb{X}) = \mathbb{I}_1$ or $\mathcal{S}_r(\mathbb{X}) = \mathbb{I}_2$.*

Proof: By the assumption, $\mathcal{S}_r(\mathbb{X}) = \mathbb{I}_1 \cap \mathbb{I}_2$. So, clearly, $\mathcal{S}_r(\mathbb{X}) \subseteq \mathbb{I}_1$ and $\mathcal{S}_r(\mathbb{X}) \subseteq \mathbb{I}_2$. Now, suppose $\mathbb{I}_1, \mathbb{I}_2 \nsubseteq \mathcal{S}_r(\mathbb{X})$. Then there exist $a \in \mathbb{I}_1 \setminus \mathcal{S}_r(\mathbb{X})$ and $b \in \mathbb{I}_2 \setminus \mathcal{S}_r(\mathbb{X})$. Since $\mathbb{I}_1, \mathbb{I}_2 \in \mathcal{I}d(\mathbb{X})$, by Proposition 2.9, $\mathbb{I}_1$ and $\mathbb{I}_2$ are upset. So, $a \lesssim a \curlyvee b$ and $b \lesssim a \curlyvee b$, we get $a \curlyvee b \in \mathbb{I}_1 \cap \mathbb{I}_2$. Thus $a \curlyvee b \in \mathcal{S}_r(\mathbb{X})$. By Theorem 3.13, we obtain $a \in \mathcal{S}_r(\mathbb{X})$ or $b \in \mathcal{S}_r(\mathbb{X})$, which is a contradiction. Hence, $\mathbb{I}_1 \subseteq \mathcal{S}_r(\mathbb{X})$ or $\mathbb{I}_2 \subseteq \mathcal{S}_r(\mathbb{X})$. Therefore, $\mathcal{S}_r(\mathbb{X}) = \mathbb{I}_1$ or $\mathcal{S}_r(\mathbb{X}) = \mathbb{I}_2$. $\square$

Note. The set ${}^{\perp}\mathbb{X} = \{a \in \mathbb{L} \mid a \curlyvee x = 1, \text{ for all } x \in \mathbb{X}\}$, if $x \curlyvee a$ exists, is called a *co-annihilator* of $\mathbb{X}$. In the following theorem we investigate the condition showing ${}^{\perp}\mathbb{X} = \mathcal{S}_r(\mathbb{X}) \cap \mathcal{S}_l(\mathbb{X})$.

Theorem 3.15. *Consider $\mathbb{L}$ be a CKL-algebra with DNP. Then ${}^{\perp}\mathbb{X} = \mathcal{S}_r(\mathbb{X}) \cap \mathcal{S}_l(\mathbb{X})$.*

Proof: Assume $a \in^{\perp} \mathbb{X}$. Then for all $x \in \mathbb{X}$, $a \curlyvee x = 1$, and by Remark 3.11 and since $x \curlyvee a = a \curlyvee x$ we have

$$1 = a \curlyvee x = (a \to x) \to x = (x \to a) \to a.$$

Thus, $(a \to x) \to x = 1$ and $(x \to a) \to a = 1$. Hence, $a \to x \lesssim x$ and $x \to a \lesssim a$. Also, by Proposition 2.5(ii), we have $x \lesssim a \to x$ and $a \lesssim x \to a$. Then $a \to x = x$ and $x \to a = a$. Therefore, $a \in \mathcal{S}_r(\mathbb{X}) \cap \mathcal{S}_l(\mathbb{X})$. Conversely, suppose $a \in \mathcal{S}_r(\mathbb{X}) \cap \mathcal{S}_l(\mathbb{X})$. Then $a \in \mathcal{S}_r(\mathbb{X})$ and $a \in \mathcal{S}_l(\mathbb{X})$. Thus, for any $x \in \mathbb{X}$, $a \to x = x$ and $x \to a = a$. So $(a \to x) \to x = 1$ and $(x \to a) \to a = 1$. By Remark 3.11, we have

$$(a \to x) \to x = (x \to a) \to a = a \curlyvee x = 1.$$

Hence, $a \in^{\perp} \mathbb{X}$. Therefore, ${}^{\perp}\mathbb{X} = \mathcal{S}_r(\mathbb{X}) \cap \mathcal{S}_l(\mathbb{X})$. $\square$

Stabilizer topology

In this section, we use of the right and left stabilizers of an L-algebra and produce a basis for a topology on it. Then we show that the generated topology by this basis is Baire, connected, locally connected and separable and investigate the other properties of this topology.

DEFINITION 3.16. A map $C : \mathcal{P}(\mathbb{L}) \to \mathcal{P}(\mathbb{L})$ is a **closure operator** if for any $\mathbb{X}, \mathbb{Y} \in \mathcal{P}(\mathbb{L})$ we have

(C_1) $\mathbb{X} \subseteq C(\mathbb{X})$,

(C_2) If $\mathbb{X} \subseteq \mathbb{Y}$, then $C(\mathbb{X}) \subseteq C(\mathbb{Y})$,

(C_3) $C(C(\mathbb{X})) = C(\mathbb{X})$.

THEOREM 3.17. *Define* $\omega : \mathcal{P}(\mathbb{L}) \to \mathcal{P}(\mathbb{L})$ *such that* $\omega(\mathbb{X}) = (\mathcal{S}_l(\mathbb{X}))_r$, *for all* $\mathbb{X} \in \mathcal{P}(\mathbb{L})$. *Then*

(*i*) ω *is a closure map.*

(*ii*) $\mathbb{X} \subseteq \omega(\mathbb{Y})$ *if and only if* $\omega(\mathbb{X}) \subseteq \omega(\mathbb{Y})$, *for all* $\mathbb{Y} \subseteq \mathbb{L}$.

(*iii*) $\gamma_\omega = \{\mathbb{X} \in \mathcal{P}(\mathbb{L}) \mid \omega(\mathbb{X}) = \mathbb{X}\}$ *is a basis for a topology on* $\mathbb{L}$.

PROOF: (*i*) By Proposition 3.3(ii), (iii) and (iv) the proof is clear.

(*ii*) By (i) is clear.

(*iii*) Let $\gamma_\omega = \{\mathbb{X} \in \mathcal{P}(\mathbb{L}) \mid \omega(\mathbb{X}) = \mathbb{X}\}$. Obviously, $\emptyset \in \gamma_\omega$. Also, by Proposition 3.3(vi) and (vii), $\omega(\mathbb{L}) = (\mathcal{S}_l(\mathbb{L}))_r = \mathcal{S}_r(\{1\}) = \mathbb{L}$. Thus, $\omega(\mathbb{L}) = \mathbb{L}$, and so $\mathbb{L} \in \gamma_\omega$. Now, suppose $\mathbb{X}, \mathbb{Y} \in \gamma_\omega$. Then $\omega(\mathbb{X}) = \mathbb{X}$ and $\omega(\mathbb{Y}) = \mathbb{Y}$. We show $\mathbb{X} \cap \mathbb{Y} \in \gamma_\omega$. Since $\mathbb{X} \cap \mathbb{Y} \subseteq \mathbb{X}, \mathbb{Y}$, by (i), $\omega(\mathbb{X} \cap \mathbb{Y}) \subseteq \omega(\mathbb{X})$ and $\omega(\mathbb{Y})$. Thus, $\omega(\mathbb{X} \cap \mathbb{Y}) \subseteq \omega(\mathbb{X}) \cap \omega(\mathbb{Y})$. In addition, from $\mathbb{X}, \mathbb{Y} \in \gamma_\omega$, we have $\omega(\mathbb{X} \cap \mathbb{Y}) \subseteq \mathbb{X} \cap \mathbb{Y}$. Moreover, by Proposition 3.3(iii), $\mathbb{X} \cap \mathbb{Y} \subseteq \omega(\mathbb{X} \cap \mathbb{Y})$. Then $\omega(\mathbb{X} \cap \mathbb{Y}) = \mathbb{X} \cap \mathbb{Y}$, and so $\mathbb{X} \cap \mathbb{Y} \in \gamma_\omega$. Therefore, γ_ω is a basis. □

Note. (*i*) According to the definition γ_ω, clearly, $(St_l(\mathbb{L}))_r = \mathbb{L}$ and $(St_l(\emptyset))_r = \emptyset$, so $\emptyset, \mathbb{L} \in \gamma_\omega$ and by Proposition 3.3(i), for any $\emptyset \neq \mathbb{X} \subseteq \mathbb{L}$, $1 \in (St_l(\mathbb{X}))_r$, so for any $\mathbb{X} \in \gamma_\omega$, $1 \in \mathbb{X}$. We have to notice that in general form, $\mathbb{X} \in \gamma_\omega$ is not an ideal of $\mathbb{L}$.
(*ii*) Since $(St_l(\emptyset))_r = \emptyset$, by Proposition 3.3(*vi*) and (*vii*), $\{\emptyset, \{1\}, \mathbb{L}\} \subseteq \gamma_\omega$.

DEFINITION 3.18. According to Theorem 3.17, the topological space, $(\mathbb{L}, \tau_\omega)$ is called a *stabilizer topology*.

Note. Since in any CKL-algebra, $\mathcal{S}_r(\mathbb{X}) \in \mathcal{I}d(\mathbb{L})$, for any $\mathbb{X} \subseteq \mathbb{L}$, every element of γ_ω is an ideal of $\mathbb{L}$.

Example 3.19.

(i) In Example 3.2(i), $(\mathbb{L}, \rightarrow, 1)$ is an L-algebra. By Proposition 3.3(i) and (vii), $\{1\} \in \omega(\mathbb{X})$, for all $\emptyset \neq \mathbb{X} \subseteq \mathbb{L}$. So, if $1 \notin \mathbb{X}$, then $\mathbb{X} \notin \gamma_\omega$. By some manipulations, we get $\gamma_\omega = \{\emptyset, \mathbb{L}, \{1\}\}$. Thus, $\tau_\omega = \{\emptyset, \mathbb{L}, \{1\}\}$. In addition, $\{1, b\} \notin \gamma_\omega$, because $\mathcal{S}_l(\{1, b\}) = \{1\}$ and by Proposition 3.3(vii), $\mathcal{S}_r(\{1\}) = \mathbb{L}$, then $\omega(\{1, b\}) = \mathbb{L}$, and so $\omega(\{1, b\}) \neq \{1, b\}$.

(ii) Assume $\mathbb{L}$ is an L-algebra as in Examples 3.2(ii) and 3.10. Then $\gamma_\omega = \{\emptyset, \{1\}, \{c, 1\}, \mathbb{L}\}$.

(iii) Consider an L-algebra as in Example 3.4(iii). Then $\gamma_\omega = \{\emptyset, \{1\}, \{b, 1\}, \mathbb{L}\}$.

(iv) According to Example 3.4(iv), $\gamma_\omega = \{\emptyset, \{1\}, \{b, 1\}, \{c, 1\}, \mathbb{L}\}$.

THEOREM 3.20. *The stabilizer topology* $(\mathbb{L}, \tau_\omega)$ *is*

(i) *connected.*

(ii) *locally connected.*

(iii) *Hausdorff space if and only if* $\mathbb{L} = \{1\}$.

THEOREM 3.21. *Let* $(\mathbb{L}, \tau_\omega)$ *be a stabilizer topology. If* $\emptyset \neq \mathbb{X} \subseteq \mathbb{L}$ *such that* $1 \in \mathbb{X}$, *then* $\overline{\mathbb{X}} = \mathbb{L}$.

PROOF: Suppose $\emptyset \neq \mathbb{X} \subseteq \mathbb{L}$ such that $1 \in \mathbb{X}$. Consider $x \in \mathbb{L}$. If $x = 1$, then $x \in \mathbb{X}$. Hence, $\overline{\mathbb{X}} = \mathbb{L}$. Now, suppose $1 \neq x \in \mathbb{L}$. Then there exists an open subset $\mathcal{U} \in \gamma_\omega$ such that $x \in \mathcal{U}$. Since $1 \in \mathcal{U}$, we have $\mathcal{U} \cap (\mathbb{X} \backslash \{x\}) \neq \emptyset$. Hence, $x \in \overline{\mathbb{X}}$, and so $\overline{\mathbb{X}} = \mathbb{L}$. □

Note. A topological space is called *separable* if it contains a countable dense subset.

COROLLARY 3.22. $(\mathbb{L}, \tau_\omega)$ is separable.

PROOF: Since $\{1\} \in \gamma_\omega$, by Theorem 3.21, $\overline{\{1\}} = \mathbb{L}$. Hence, $(\mathbb{L}, \tau_\omega)$ is separable. □

THEOREM 3.23. *$(\mathbb{L}, \tau_\omega)$ is Baire space, where $\mathbb{L}$ is a CKL-algebra.*

PROOF: Let $\mathcal{U} \in \tau_\omega$. Since $\mathbb{L}$ is a CKL-algebra, by Theorem 3.8, $\mathcal{U} \in \mathcal{I}d(\mathbb{L})$ and so $1 \in \mathcal{U}$. Since $1 \in \mathcal{U}$ by Theorem 3.21, $\overline{\mathcal{U}} = \mathbb{L}$. Thus, every open set of $(\mathbb{L}, \tau_\omega)$ is dense. On the other side, for each collection of open set $\mathcal{U}_n$, $1 \in \bigcap\limits_{n \in \mathbb{N}} \mathcal{U}_n$. Thus, by Theorem 3.21, $\overline{\bigcap\limits_{n \in \mathbb{N}} \mathcal{U}_n} = \mathbb{L}$, and so $\bigcap\limits_{n \in \mathbb{N}} \mathcal{U}_n$ is dense. Therefore, $(\mathbb{L}, \tau_\omega)$ is Baire space. □

In the following example, we show that $(\mathbb{L}, \tau_\omega)$ is not a T_0-space or T_1-space.

Example 3.24. In Example 3.2(i), $\gamma_\omega = \{\emptyset, \mathbb{L}, \{1\}\}$. Since $b \neq c$, for $b, c \in \mathbb{L}$, there is not $\mathcal{U} \in \gamma_\omega$ such that $b \in \mathcal{U}$ and $c \notin \mathcal{U}$. Therefore, $(\mathbb{L}, \tau_\omega)$ is not a T_0-space. Obviously, $(\mathbb{L}, \tau_\omega)$ is not a T_1-space.

THEOREM 3.25. *Let $\mathbb{L}$ be a bounded CKL-algebra. If $\mathbb{L}$ has a cover of $\mathcal{U}_i \in \gamma_\omega$, for $i \in I$, then there exists $i \in I$ such that $\mathcal{U}_i = \mathbb{L}$. Particularly, $\mathbb{L}$ is compact.*

PROOF: Let $\mathbb{L}$ be bounded and $\{\mathcal{U}_i\}_{i \in I}$ be a cover of $\mathbb{L}$ such that, for all $i \in I$, $\mathcal{U}_i \in \gamma_\omega$ and $\mathbb{L} \subseteq \bigcup\limits_{i \in I} \mathcal{U}_i$. Since, for all $i \in I$, $\mathcal{U}_i \in \gamma_\omega$, by Theorem 3.8, we have $\mathcal{U}_i \in \mathcal{I}d(\mathbb{L})$. On the other side, $\mathbb{L}$ is bounded, then $0 \in \mathbb{L}$, and so $0 \in \bigcup\limits_{i \in I} \mathcal{U}_i$. Thus, there exists $i \in I$ such that $0 \in \mathcal{U}_i$. Since $\mathcal{U}_i \in \mathcal{I}d(\mathbb{L})$ and $0 \in \mathcal{U}_i$, by Proposition 2.9, $\mathcal{U}_i = \mathbb{L}$. Hence, there exists a finite family of $\{\mathcal{U}_i\}_{i \in I}$ such that $\mathbb{L} \subseteq \bigcup_{i=1}^{n} \mathcal{U}_i$. □

3.2. Generalization of stabilizers on L-algebras

In this section, we introduce the generalization of stabilizers on L-algebra and investigate their properties and relation of them with stabilizers.

DEFINITION 3.26. Let $\mathbb{X}, \mathbb{Y}$ be two non-empty subsets of $\mathbb{L}$. Then a right(left) stabilizer of $\mathbb{X}$ with respect to $\mathbb{Y}$ are defined by

$$\mathcal{S}t_r(\mathbb{X}, \mathbb{Y}) = \{a \in \mathbb{L} \mid \text{for all } x \in \mathbb{X}, (a \to x) \to x \in \mathbb{Y}\},$$
$$\mathcal{S}t_l(\mathbb{X}, \mathbb{Y}) = \{a \in \mathbb{L} \mid \text{for all } x \in \mathbb{X}, (x \to a) \to a \in \mathbb{Y}\}.$$

Example 3.27. According to Example 3.2(i), let $\mathbb{X} = \{a, b\}$ and $\mathbb{Y} = \{b, c, 1\}$. Then $\mathcal{S}t_r(\mathbb{X}, \mathbb{Y}) = \{b, c, 1\}$ and $\mathcal{S}t_l(\mathbb{X}, \mathbb{Y}) = \{b, c, 1\}$.

PROPOSITION 3.28. Let $\mathbb{X}, \mathbb{Y}, \mathbb{X}_i, \mathbb{Y}_i$ be non-empty subsets of $\mathbb{L}$ and $\mathbb{I} \in \mathcal{I}d(\mathbb{L})$. Then the following statements hold:

(*i*) If $\mathcal{S}t_r(\mathbb{X}, \mathbb{Y}) = \mathbb{L}$ or $\mathcal{S}t_l(\mathbb{X}, \mathbb{Y}) = \mathbb{L}$, then $\mathbb{X} \subseteq \mathbb{Y}$.

(*ii*) If $\mathbb{L}$ is a CKL-algebra and $\mathbb{I} \subseteq \mathbb{Y}$, then $\mathcal{S}t_r(\mathbb{I}, \mathbb{Y}) = \mathbb{L}$ and $\mathcal{S}t_l(\mathbb{I}, \mathbb{Y}) = \mathbb{L}$.

(*iii*) If $\mathbb{L}$ is a CKL-algebra, then $\mathcal{S}t_r(\mathbb{I}, \mathbb{I}) = \mathbb{L}$ and $\mathcal{S}t_l(\mathbb{I}, \mathbb{I}) = \mathbb{L}$.

(*iv*) $\mathcal{S}_r(\mathbb{X}) \subseteq \mathcal{S}t_r(\mathbb{X}, \mathbb{I})$ and $\mathcal{S}_l(\mathbb{X}) \subseteq \mathcal{S}t_l(\mathbb{X}, \mathbb{I})$.

(*v*) If $\mathbb{L}$ is a KL-algebra, then $\mathcal{S}_r(\mathbb{X}, \{1\}) = \mathcal{S}t_r(\mathbb{X})$ and $\mathcal{S}_l(\mathbb{X}, \{1\}) = \mathcal{S}t_l(\mathbb{X})$.

(*vi*) If $\mathbb{X}_i \subseteq \mathbb{Y}_i$ and $\mathbb{X}_j \subseteq \mathbb{Y}_j$, then $\mathcal{S}_r(\mathbb{Y}_i, \mathbb{X}_j) \subseteq \mathcal{S}t_r(\mathbb{X}_i, \mathbb{Y}_j)$ and $\mathcal{S}_l(\mathbb{Y}_i, \mathbb{X}_j) \subseteq \mathcal{S}t_l(\mathbb{X}_i, \mathbb{Y}_j)$.

(*vii*) $\mathcal{S}t_r(\mathbb{X}, \bigcap_{i\in I} \mathbb{Y}_i) = \bigcap_{i\in I} \mathcal{S}t_r(\mathbb{X}, \mathbb{Y}_i)$ and $\mathcal{S}t_l(\mathbb{X}, \bigcap_{i\in I} \mathbb{Y}_i) = \bigcap_{i\in I} \mathcal{S}t_l(\mathbb{X}, \mathbb{Y}_i)$.

PROOF: (*i*) Assume $x \in \mathbb{X}$. Since $\mathbb{X} \subseteq \mathbb{L}$, we get $x \in \mathbb{L}$, and so $x \in \mathcal{S}t_r(\mathbb{X}, \mathbb{Y})$. Thus, for any $a \in \mathbb{X}$, $(x \to a) \to a \in \mathbb{Y}$. Consider $a = x$. Then by $(L1)$ we have

$$x = 1 \to x = (x \to x) \to x \in \mathbb{Y}.$$

Hence, $\mathbb{X} \subseteq \mathbb{Y}$. The proof of the other case is similar.

(*ii*) Clearly, $\mathcal{S}t_r(\mathbb{I}, \mathbb{Y}) \subseteq \mathbb{L}$. Assume $x \in \mathbb{L}$. Then for any $a \in \mathbb{I}$, by Proposition 2.5(iii), $a \lesssim (x \to a) \to a$. Thus by Proposition 2.9, $(x \to a) \to a \in \mathbb{I}$, and so $(x \to a) \to a \in \mathbb{Y}$. Hence, $x \in \mathcal{S}t_r(\mathbb{I}, \mathbb{Y})$, thus $\mathbb{L} \subseteq \mathcal{S}t_r(\mathbb{I}, \mathbb{Y})$. Therefore, $\mathbb{L} = \mathcal{S}t_r(\mathbb{I}, \mathbb{Y})$. The proof of the other case is similar.

(*iii*) By (ii) the proof is clear.

(*iv*) Let $a \in \mathcal{S}_r(\mathbb{X})$. Then for any $x \in \mathbb{X}$, $a \to x = x$ and clearly, $(a \to x) \to x = 1$. Since $\mathbb{I} \in \mathcal{I}d(\mathbb{L})$, by (I_1), $1 \in \mathbb{I}$ and so $(a \to x) \to x \in \mathbb{I}$. Thus, $a \in \mathcal{S}t_r(\mathbb{X}, \mathbb{I})$. Hence, $\mathcal{S}_r(\mathbb{X}) \subseteq \mathcal{S}t_r(\mathbb{X}, \mathbb{I})$. The proof of the other case is similar.

(*v*) Since $\{1\} \in \mathcal{I}d(\mathbb{L})$, by (iv), we have $\mathcal{S}_r(\mathbb{X}) \subseteq \mathcal{S}t_r(\mathbb{X}, \{1\})$. Assume $a \in \mathcal{S}t_r(\mathbb{X}, \{1\})$. Then for any $x \in \mathbb{X}$, $(a \to x) \to x \in \{1\}$, and so $a \to x \lesssim x$. By hypothesis and Proposition 2.3, $x \lesssim a \to x$, and so $a \to x = x$. Hence, $x \in \mathcal{S}_r(\mathbb{X})$. Therefore, $\mathcal{S}_r(\mathbb{X}, \{1\}) = \mathcal{S}t_r(\mathbb{X})$. The proof of the other case is similar.

(vi) Assume $a \in \mathcal{S}_r(\mathbb{Y}_i, \mathbb{X}_j)$. Then for any $x \in \mathbb{Y}_i$, $(a \to x) \to x \in \mathbb{X}_j$. By assumption, $\mathbb{X}_i \subseteq \mathbb{Y}_i$, thus for $x \in \mathbb{X}_i$, we get $(a \to x) \to x \in \mathbb{X}_j$. In addition, $\mathbb{X}_j \subseteq \mathbb{Y}_j$, so $(a \to x) \to x \in \mathbb{Y}_j$. Hence, $a \in \mathcal{S}t_r(\mathbb{X}_i, \mathbb{Y}_j)$. Therefore, $\mathcal{S}_r(\mathbb{Y}_i, \mathbb{X}_j) \subseteq \mathcal{S}t_r(\mathbb{X}_i, \mathbb{Y}_j)$. The proof of the other case is similar.
(vii) Consider $a \in \mathcal{S}t_r(\mathbb{X}, \bigcap\limits_{i\in I} \mathbb{Y}_i)$. Then for any $x \in \mathbb{X}$, we have $(a \to x) \to x \in \bigcap\limits_{i\in I} \mathbb{Y}_i$. Thus, for all $i \in I$, we have $(a \to x) \to x \in \mathbb{Y}_i$. So, $a \in \mathcal{S}t_r(\mathbb{X}, \mathbb{Y}_i)$. Hence, $a \in \bigcap\limits_{i\in I} \mathcal{S}t_r(\mathbb{X}, \mathbb{Y}_i)$. Therefore, $\mathcal{S}t_r(\mathbb{X}, \bigcap\limits_{i\in I} \mathbb{Y}_i) \subseteq \bigcap\limits_{i\in I} \mathcal{S}t_r(\mathbb{X}, \mathbb{Y}_i)$. The proof of other side is similar. $\square$

In the following example we show that the condition CKL-algebra in Proposition 3.28(ii) is necessary.

Example 3.29. According to Example 3.4(iii), $\mathbb{L}$ is not a CKL-algebra, since

$$b \to (c \to a) = b \to b = 1 \neq b = c \to a = c \to (b \to a).$$

Consider $\mathbb{I} = \{1\}$, $\mathbb{Y} = \{c, 1\}$ and $\mathbb{X} = \{a\}$. Then $\mathcal{S}t_r(\mathbb{X}, \mathbb{Y}) = \{b, 1\} \neq \mathbb{L}$.

Proposition 3.30. Consider $\emptyset \neq \mathbb{X}, \mathbb{Y} \subseteq \mathbb{L}$. If for any $x, y \in \mathbb{L}$, $(x \to y) \to y = (y \to x) \to x$, then $\mathcal{S}t_r(\mathbb{X}, \mathbb{Y}) = \mathcal{S}t_l(\mathbb{X}, \mathbb{Y})$.

Proof: Let $a \in \mathcal{S}t_r(\mathbb{X}, \mathbb{Y})$. Then for any $x \in \mathbb{X}$, $(a \to x) \to x \in \mathbb{Y}$. By assumption, $(x \to a) \to a \in \mathbb{Y}$, and so $a \in \mathcal{S}t_l(\mathbb{X}, \mathbb{Y})$. By the similar way, $\mathcal{S}t_l(\mathbb{X}, \mathbb{Y}) \subseteq \mathcal{S}t_r(\mathbb{X}, \mathbb{Y})$. Hence, $\mathcal{S}t_r(\mathbb{X}, \mathbb{Y}) = \mathcal{S}t_l(\mathbb{X}, \mathbb{Y})$. $\square$

Proposition 3.31. Consider $\mathbb{L}$ be a CKL-algebra and $\mathbb{I}, \mathbb{J} \in \mathcal{I}d(\mathbb{L})$. Then $\mathcal{S}t_r(\mathbb{I}, \mathbb{J}) \in \mathcal{I}d(\mathbb{L})$.

Proof: By $(L1)$, since for any $a \in \mathbb{I}$, $(1 \to a) \to a = 1 \in \mathbb{J}$, we get $1 \in \mathcal{S}t_r(\mathbb{I}, \mathbb{J})$. Assume $a, a \to b \in \mathcal{S}t_r(\mathbb{I}, \mathbb{J})$. Then for any $x \in \mathbb{I}$, $(a \to x) \to x \in \mathbb{J}$ and $((a \to b) \to x) \to x \in \mathbb{J}$. Since $x \in \mathbb{I}$, by assumption and Proposition 2.5(ii), $x \lesssim a \to x$, and by Proposition 2.9 we get $a \to x \in \mathbb{I}$. So, $((a \to b) \to (a \to x)) \to (a \to x) \in \mathbb{J}$. In addition, by Proposition 2.5(viii) we have $b \to x \lesssim (a \to b) \to (a \to x)$, and by Proposition 2.5(vii), we have

$$((a \to b) \to (a \to x)) \to (a \to x) \lesssim (b \to x) \to (a \to x),$$

Since $((a \to b) \to (a \to x)) \to (a \to x) \in \mathbb{J}$ and $\mathbb{J} \in \mathcal{I}d(\mathbb{L})$, by Proposition 2.9, $(b \to x) \to (a \to x) \in \mathbb{J}$. Moreover, by Proposition 2.5(xii), $a \to x = ((a \to x) \to x) \to x$. Thus

$$\begin{aligned}((a \to x) \to x) \to ((b \to x) \to x) &= (b \to x) \to (((a \to x) \to x) \to x) \\ &= (b \to x) \to (a \to x) \in \mathbb{J}.\end{aligned}$$

From $\mathbb{J} \in \mathcal{I}d(\mathbb{L})$ and $(a \to x) \to x \in \mathbb{J}$, by (I_2), we have $(b \to x) \to x \in \mathbb{J}$. Hence, $b \in \mathcal{S}t_r(\mathbb{I}, \mathbb{J})$. Therefore, $\mathcal{S}t_r(\mathbb{I}, \mathbb{J}) \in \mathcal{I}d(\mathbb{L})$. □

Theorem 3.32.

(i) *For any* $\mathbb{I}, \mathbb{J} \in \mathcal{I}d(\mathbb{L})$, $\mathcal{S}t_r(\mathbb{I}, \mathbb{J}) \cap \mathbb{I} \subseteq \mathbb{J}$.

(ii) *If* $\mathbb{L}$ *is a* CKL*-algebra, then* $\mathcal{S}t_r(\mathbb{I}, \mathbb{J})$ *is the greatest ideal of* $\mathbb{L}$ *such that* $\mathcal{S}t_r(\mathbb{I}, \mathbb{J}) \cap \mathbb{I} \subseteq \mathbb{J}$.

Proof: (i) Let $a \in \mathcal{S}t_r(\mathbb{I}, \mathbb{J}) \cap \mathbb{I}$. Then $a \in \mathbb{I}$ and $a \in \mathcal{S}t_r(\mathbb{I}, \mathbb{J})$. Thus for any $x \in \mathbb{I}$, $(a \to x) \to x \in \mathbb{J}$. Consider $x = a$, so by $(L1)$, $(a \to a) \to a = a \in \mathbb{J}$. Thus, $\mathcal{S}t_r(\mathbb{I}, \mathbb{J}) \cap \mathbb{I} \subseteq \mathbb{J}$.

(ii) By (i), obviously, $\mathcal{S}t_r(\mathbb{I}, \mathbb{J}) \cap \mathbb{I} \subseteq \mathbb{J}$. Suppose there exists $\mathbb{K} \in \mathcal{I}d(\mathbb{L})$, where $\mathbb{K} \cap \mathbb{I} \subseteq \mathbb{J}$. We show that $\mathbb{K} \subseteq \mathcal{S}t_r(\mathbb{I}, \mathbb{J})$. For this, assume $a \in \mathbb{K}$ and $x \in \mathbb{I}$. Thus by Proposition 2.5(ii) and (iii), $a, x \lesssim (a \to x) \to x$. Since $\mathbb{I}, \mathbb{K} \in \mathcal{I}d(\mathbb{L})$, by Proposition 2.9, we get $(a \to x) \to x \in \mathbb{K} \cap \mathbb{I}$, and so $(a \to x) \to x \in \mathbb{J}$. Thus, $a \in \mathcal{S}t_r(\mathbb{I}, \mathbb{J})$, and so $\mathbb{K} \subseteq \mathcal{S}t_r(\mathbb{I}, \mathbb{J})$. Therefore, $\mathcal{S}t_r(\mathbb{I}, \mathbb{J})$ is the greatest ideal of $\mathbb{L}$ such that $\mathcal{S}t_r(\mathbb{I}, \mathbb{J}) \cap \mathbb{I} \subseteq \mathbb{J}$. □

Corollary 3.33. Assume $\mathbb{L}$ is a CKL-algebra. Then $\langle \mathcal{I}d(\mathbb{L}), \sqcap, \sqcup, \{1\}, \mathbb{L} \rangle$ is a relative pseudo-complement lattice where $\mathcal{S}t_r(\mathbb{I}, \mathbb{J})$ is the relative pseudo-complement of $\mathbb{I}$ with respect to $\mathbb{J}$ in $\mathcal{I}d(\mathbb{L})$ such that $\mathbb{I} \sqcap \mathbb{J} = \mathbb{I} \cap \mathbb{J}$ and $\mathbb{I} \sqcup \mathbb{J}$ is a generated ideal of $\mathbb{L}$ contains $\mathbb{I} \cup \mathbb{J}$.

Proof: By Theorem 3.32 and [5, Definition 3.5 and Proposition 3.6], the proof is straightforward. □

4. Conclusion

The aim of this paper is to introduce the notion of stabilizers in L-algebras and develop stabilizer theory in L-algebras. In this paper, the notions of left and right stabilizers are introduced and some properties related to them has been investigated. Then, the relations among stabilizers, ideals and co-annihilators are discussed. Also, it was shown that the set of all ideals in a CKL-algebra forms a relative pseudo-complemented lattice. Also, it was proved that all right stabilizers in CKL-algebra are ideals. Then by using the right stabilizers, a basis for a topology on L-algebra was produced. Finally, it was proved that the generated topology by this basis is Baire, connected, locally connected and separable and the other properties of this topology are investigated.

In future, we can introduce the notions of fuzzy left and right stabilizers and investigate their related properties and discuss the relations among fuzzy stabilizers, fuzzy ideals and fuzzy co-annihilators.

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References

[1] M. Aaly Kologani, *Relations between L-algebras and other logical algebras*, **Journal of Algebraic Hyperstructures and Logical Algebras**, vol. 4(1) (2023), pp. 27–46, DOI: https://doi.org/10.52547/HATEF.JAHLA.4.1.3.

[2] M. Aaly Kologani, *Some results on L-algebars*, **Soft Computing**, vol. 27 (2023), pp. 13765–13777, DOI: https://doi.org/10.1007/s00500-023-08965-5.

[3] R. A. Borzooei, M. Aaly Kologani, *Stabilizers topology of hoops*, **Algebraic Structures and Their Applications**, vol. 1(1) (2014), pp. 35–48, DOI: https://doi.org/10.1090/S0002-9904-1935-06166-X.

[4] X. Y. Cheng, M. Wang, W. Wang, J. T. Wang, *Stabilizers in EQ-algebras*, **Open Mathematics**, vol. 17 (2019), pp. 998–1013, DOI: https://doi.org/10.1515/math-2019-0078.

[5] L. C. Ciungu, *Results in L-algebras*, **Algebra Univers**, vol. 87 (2021), p. 7, DOI: https://doi.org/10.1007/s00012-020-00695-1.

[6] M. Haveshki, M. Mohamadhasani, *Stabilizer in BL-algebras and its properties*, **International Mathematical Forum**, vol. 5(57) (2010), pp. 2809–2816, DOI: https://doi.org/10.1090/S0002-9904-1935-06166-X.

[7] B. Meng, X. L. Xin, *Generalized co-annihilator of BL-algebras*, **Open Mathematics**, vol. 13(1) (2015), pp. 639–654, DOI: https://doi.org/10.1515/math-2015-0060.

[8] W. Rump, *L-algebras, self-similarity, and ℓ-groups*, **Journal of Algebra**, vol. 320 (2008), pp. 2328–2348, DOI: https://doi.org/10.1016/j.jalgebra.2008.05.033.

[9] W. Rump, *A general Glivenko theorem*, **Algebra Universalis**, vol. 61 (2009), pp. 455–473, DOI: https://doi.org/10.1007/s00012-009-0018-y.

[10] W. Rump, Y. Yang, *Interval in ℓ-groups as L-algebras*, **Algebra Universalis**, vol. 67(2) (2012), pp. 121–130, DOI: https://doi.org/10.1007/s00012-012-0172-5.

[11] W. J. Tao, A. B. Saeid, P. F. He, *Stabilizers in MTL-algebras*, **Journal of Intelligent and Fuzzy Systems**, vol. 35 (2018), pp. 717–727, DOI: https://doi.org/10.3233/JIFS-171105.

[12] Y. L. Wu, J. Wang, Y. C. Yang, *Lattice-ordered effect algebras and L-algebras*, **Fuzzy Sets and Systems**, vol. 369 (2019), pp. 103–113, DOI: https://doi.org/10.1016/j.fss.2018.08.013.

[13] Y. L. Wu, Y. C. Yang, *Orthomodular lattices as L-algebras*, **Soft Computing**, vol. 24 (2020), pp. 14391–14400, DOI: https://doi.org/10.1007/s00500-020-05242-7.

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L-MODULES

Abstract

In this paper, considering L-algebras, which include a significant number of other algebraic structures, we present a definition of modules on L-algebras (L-*modules*). Then we provide some examples and obtain some results on L-modules. Also, we present definitions of *prime ideals* of L-algebras and L-*submodules* (*prime* L-*submodules*) of L-modules, and investigate the relationship between them. Finally, by proving a number of theorems, we provide some conditions for having prime L-submodules.

Keywords: L-algebra, L-module, L-submodule, prime L-submodule.

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1. Introduction

In the study of set-theoretical solutions of the Yang-Baxter equation, the cycloid equation, $(x \cdot y) \cdot (x \cdot z) = (y \cdot x) \cdot (y \cdot z)$, plays a fundamental role, see for example [6, 15]. Finding a solution to the Young-Baxter equation is a research topic for many authors. Rump's research in order to find a solution for that equation led to the introduction of L-algebras [16]. L-algebras are related to algebraic logic and quantum structures. They are closely related to non-classical logical algebras and quantum Yang-Baxter equation solutions. It was shown that many non-classical logical algebras can be unified into L-algebras. For instance, the pseudo MV-algebras can be characterized as semiregular L-algebras with negation [21]; Orthomodular

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lattices can be characterized as L-algebras [20], and every lattice-ordered effect algebra gives rise to an L-algebra [19]. Also, Rump showed that an L-algebra can be represented as an interval in a lattice ordered group if and only if it is semiregular with an smallest element and bijective negation [18]. In short, there are effective relationships between L-algebras and other algebraic structures. For example, we can consider them as Hilbert algebras, locales, hoops, pseudo MV-algebras, etc. Other recent results on the structure of the category of L-algebras can be found in [8].
Discussions about modular structures on algebraic structures have long been of interest to scientists. For instance, the notion of BCK-module was introduced in 1994 as an action of a BCK-algebra over a commutative group [2], and it was extended in 2014 [3]; The notion of MV-modules was introduced as an action of a PMV-algebra over an MV-algebra in 2003 [1]; Also, the notion of MV-semimodules was introduced in 2013 [14], and the new definition of MV-semimodules was presented in 2021 [13]. As mentioned, there are effective connections between most algebraic structures. These connections show a relationship between the modular structures associated with these algebras. L- Algebras under conditions can be equivalent to other algebras such as BCK-algebras, MV-algebras, etc. Considering that we have spent a relatively large amount of time studying modular structures (for instance, see [3, 4, 9, 10, 11, 12, 13]), in order to complete and consolidate our study in this field, we have decided to define L-modules as an action of an L-algebra over an Abelian group. We hope that this definition can help us to clarify the structure of L-algebras.

2. Preliminaries

In this section, we review the material that we will use in the paper.

Definition 2.1 ([7]). An *L-algebra* is an algebra $(L; \to, 1)$ of type $(2, 0)$ satisfying

$(L1)$ $x \to x = x \to 1 = 1$, $1 \to x = x$;

$(L2)$ $(x \to y) \to (x \to z) = (y \to x) \to (y \to z)$;

$(L3)$ $x \to y = y \to x = 1$ implies $x = y$, for all $x, y, z \in L$.

The relation $x \leq y$ if and only if $x \to y = 1$, defines a partial order for any L-algebra L. If L admits a smallest element 0, then it is called a bounded L-algebra.

Moreover, in the bounded L-algebra L, if the map $' : L \longrightarrow L$ defined, by $x \longrightarrow x' = x \to 0$ for every $x \in L$, is bijective, then we say that L has negation.

Definition 2.2 ([17]). A *KL-algebra* is an L-algebra $(L, \to, 1)$ such that

$$x \to (y \to x) = 1 \qquad (K)$$

for every $x, y \in L$.
A *CL-algebra* is an L-algebra $(L, \to, 1)$ such that

$$(x \to (y \to z)) \to (y \to (x \to z)) = 1 \quad (C)$$

for every $x, y, z \in L$.

Definition 2.3 ([16]). Let $(L; \to, 1)$ be an L-algebra. Then a subset K of L is called an *L-subalgebra* if $x \to y, y \to x \in K$, for all $x, y \in K$.
A subset I of L is called an *ideal* if the following hold for all $x, y \in L$:

($I1$) $1 \in I$,

($I2$) $x, x \to y \in I$ implies $y \in I$,

($I3$) $x \in I$ implies $(x \to y) \to y \in I$,

($I4$) $x \in I$ implies $y \to x$, $y \to (x \to y) \in I$. Denote by $\mathcal{ID}(L)$ the set of all ideals of L.

If L satisfies condition (K), then (I_4) can be omitted. Also, if L satisfies condition (C), then , (I_3) and (I_4) can be omitted.

Definition 2.4 ([5]). For every subset $Y \subseteq L$, the smallest ideal of L containing Y (i.e. the intersection of all ideals $I \in \mathcal{ID}(L)$ such that $Y \subseteq I$) is called the ideal generated by Y and it will be denoted by $[Y)$. If $Y = \{x\}$ we write $[x)$ instead of $[\{x\})$. In this case $[x)$ is called a principal ideal of L.

3. L-modules

In this section, we present our definition of L-modules, and obtain some results on them. Then we introduce the concepts of L-submodules and prime L-submodules in L-modules. Finally, we investigate some conditions for having a prime L-submodule.

Note. If L is an L-algebra, then we denote $(l \to u) \to u$ by $l \uparrow u$, for every $l, u \in L$.

Definition 3.1. Let $L = (L; \to, 0, 1)$ be a bounded L-algebra, and $M = (M, +)$ be an Abelian group. Then M is called an *L-module*, if there is an operation $\cdot : L \times M \longrightarrow M$ by $(l, m) \longmapsto l \cdot m$ such that for every $l, u \in L$ and $m, n \in M$, we have:

($LM1$) $1 \cdot m = m$;

($LM2$) $l \cdot (m + n) = l \cdot m + l \cdot n$;

($LM3$) $(l \to u) \cdot m = l' \cdot m + u \cdot m$, for all pairs (l, u) with $u \neq 1$. Moreover, if we have

($LM4$) $(l \uparrow u) \cdot m = l \cdot (u \cdot m)$, for all pairs (l, u) with $l \neq 0$,

then M is called an *Extended L-module* (or briefly EL-module).

Example 3.2. (i) Let $L = \{0, 1\}$ and define an operation " $\to$ " on L by

$\to$	0	1
0	1	1
1	0	1

Then $L = (L; \to, 0, 1)$ is a bounded L-algebra. The map $' : L \longrightarrow L$ by $0' = 1$ and $1' = 0$ is bijective. Consider the operation $\cdot : L \times \mathbb{Z} \longrightarrow \mathbb{Z}$ by $0 \cdot n = 0$ and $1 \cdot n = n$, for every $n \in \mathbb{Z}$. Then ($L\mathbb{Z}1$) and ($L\mathbb{Z}2$) are clear. ($L\mathbb{Z}3$) We have $(0 \to 0).n = 0'.n + 0.n$, $(1 \to 1).n = 1'.n + 1.n$ and $(1 \to 0).n = 1'.n + 0.n$, for every $n \in \mathbb{Z}$. Then $\mathbb{Z}$ is an L-module. Moreover, ($L\mathbb{Z}4$) We have $(0 \uparrow 0).n = 0.(0.n)$ and $(1 \uparrow 1).n = 1.(1.n)$, for every $n \in \mathbb{Z}$. Therefore, $\mathbb{Z}$ is an EL-module.
(ii) Let A be a non-empty set. Then it is routine to see that $(\rho(A); \to, \emptyset, A)$ is a bounded L-algebra, where $X \to Y = X' \cup Y$, for every $X, Y \in \rho(A)$. Since $\emptyset \to \emptyset = \emptyset \to A = A \to A = A$ and $A \to \emptyset = \emptyset$, we get $L = \{\emptyset, A\}$ is an L-subalgebra of $\rho(A)$ and so it is an L-algebra. Consider $M = (\rho(A), \Delta)$, where $X \Delta Y = X \cup Y \setminus X \cap Y$, for every $X, Y \in \rho(A)$. It is easy to see that M is an abelian group. Now, let the operation $\cdot : L \times M \to M$ be defined by $T \cdot Y = T \cap Y$, for any $T \in L$ and $Y \in M$. Then

($LM1$) $A \cdot Y = A \cap Y = Y$, for every $Y \in M$;

($LM2$) It is routine to see that

$$T \cdot (X+Y) = T \cap (X \Delta Y) = (T \cap X)\Delta(T \cap Y) = (T \cdot X) + (T \cdot Y),$$

for every $T \in L$ and $X, Y \in M$;

$(LM3)$ We have

$$\begin{aligned}(A \to A).X &= (A \cup A') \cap X = X = X \cap A = X \cap (A'\Delta A)\\ &= (A' \cap X)\Delta(A \cap X) = A' \cdot X + A \cdot X,\end{aligned}$$

for every $X \in M$. By the similar way, we have

$(\emptyset \to \emptyset) \cdot X = \emptyset' \cdot X + \emptyset \cdot X$ and $(A \to \emptyset) \cdot X = A' \cdot X + \emptyset \cdot X$, for every $X \in M$.

Hence, M is an L-module. Moreover,

$(LM4)$ Since

$$A \uparrow A = (A \to A) \to A = (A' \cup A) \to A = (A \cap A') \cup A = A,$$

we have $(A \uparrow A) \cdot X = A \cdot (A \cdot X)$, for every $X \in M$. By the similar way, we have $(\emptyset \uparrow \emptyset) \cdot X = \emptyset \cdot (\emptyset \cdot X)$, for every $X \in M$. Therefore, M is an EL-module.

Note. From now on, in this paper, we let $L = (L; \to, 1)$ be an L-algebra.

Definition 3.3. If $l \uparrow u = u \uparrow l$, for every $l, u \in L$, then we say that L is L-commutative.

Example 3.4. (i) Let $L = \{0, l, u, 1\}$ and define an operation " $\to$ " on L by

$\to$	0	l	u	1
0	1	1	1	1
l	u	1	u	1
u	l	l	1	1
1	0	l	u	1

Then $(L; \to, 1)$ is an L-algebra. Moreover, L is L-commutative.
(ii) According to Example 3.2 (i), L is L-commutative.

(iii) Let $L = \{0, l, u, t, 1\}$ and define operation " $\rightarrow$ " on L by

$\rightarrow$	0	l	u	t	1
0	1	1	1	1	1
l	0	1	l	t	1
u	0	l	1	t	1
t	t	1	1	1	1
1	0	l	u	t	1

Then $(L; \rightarrow, 1)$ is an L-algebra. Since $l \uparrow t = (l \rightarrow t) \rightarrow t = 1 \neq l = (t \rightarrow l) \rightarrow l = t \uparrow l$, L is not L-commutative.

In the following, we present a general example of L-module.

PROPOSITION 3.5. Let $L = (L; \rightarrow, 0, 1)$ be bounded and L-commutative. Then $(L, +)$ is an Abelian group, where

$$l + u = (l \rightarrow u)' \uparrow (u \rightarrow l)', \text{ for every } l, u \in L.$$

PROOF: At first, we show that $0 + l = l + 0 = l$, for every $l \in L$. We have

$$l + 0 = (l \rightarrow 0)' \uparrow (0 \rightarrow l)' = (l')' \uparrow 1' = l \uparrow 0 = (l \rightarrow 0) \rightarrow 0 = (l')' = l.$$

By the similar way, we have $0 + l = l$ and so $0 + l = l + 0 = l$, for every $l \in L$. Also, since

$$l + l = (l \rightarrow l)' \uparrow (l \rightarrow l)' = 1' \uparrow 1' = 0 \uparrow 0 = (0 \rightarrow 0) \rightarrow 0 = 1 \rightarrow 0 = 0,$$

we conclude that every member of L has a counterpart in L. Now, with a long and routine method, it can be seen

$$l + (u + t) = (l + u) + t, \text{ for every } l, u, t \in L.$$

Finally, since L is L-commutative, we have

$$l + u = (l \rightarrow u)' \uparrow (u \rightarrow l)' = (u \rightarrow l)' \uparrow (l \rightarrow u)' u + l, \text{ for every } l, u \in L.$$

Therefore, $(L, +)$ is an Abelian group. $\square$

PROPOSITION 3.6. Let $L = (L; \wedge, \vee, ', 0, 1)$ be a Boolean-algebra. Then L is a bounded L-algebra. Moreover, L is L-commutative.

Proof: We define $l \to u = l' \vee u$, for every $l, u \in L$. Then

($L1$) It is clear that $l \to l = l \to 1 = 1$ and $1 \to l = l$, for every $l \in L$.

($L2$) For every $l, u \in L$, we have

$$\begin{aligned}(l \to u) \to (l \to t) &= (l' \vee u) \to (l' \vee t) = (l' \vee u)' \vee (l' \vee t)\\ &= (l \wedge u') \vee (l' \vee t) = ((l \wedge u') \vee l') \vee t\\ &= ((l \vee l') \wedge (u' \vee l')) \vee t = (1 \wedge (u' \vee l')) \vee t\\ &= (u' \vee l') \vee t.\end{aligned}$$

On the other hand, by the similar way, we have $(u \to l) \to (u \to t) = (u' \vee l') \vee t$. Hence

$$(l \to u) \to (l \to t) = (u \to l) \to (u \to t), \text{ for every } l, u \in L.$$

($L3$) Let $l \to u = u \to l = 1$, for any $l, u \in L$. Then $l' \vee u = u' \vee l = 1$ and so

$$l \wedge u = (l \wedge l') \vee (l \wedge u) = l \wedge (l' \vee u) = l \wedge 1 = l.$$

This means that $l \leq u$. By the similar way, we have $u \leq l$ and so $u = l$. Thus, $(L, \to, 1)$ is an L-algebra. Note that $0 \to l = 0' \vee l = 1 \vee l = 1$. So $0 \leq l$, for every $l \in L$ and so L is bounded. Moreover, we have

$$\begin{aligned}l \uparrow u &= (l \to u) \to u = (l' \vee u)' \vee u = (l \wedge u') \vee u = (l \vee u) \wedge (u \vee u')\\ &= l \vee u = (l \vee u) \wedge (l \vee l') = l \vee (u \wedge l') = l \vee (u' \vee l)' = l \vee (u \to l)'\\ &= (u \to l) \to l = u \uparrow l, \text{ for every } u, l \in L.\end{aligned}$$

Therefore, L is L-commutative. □

Example 3.7. Let $L = (L; \wedge, \vee, ', 0, 1)$ be a Boolean-algebra. If $l \to u \neq 1$ implies $u \leq l$, for every $u, l \in L$, then L is an L-module.

Proof: By Proposition 3.6, L is bounded and L-commutative, and by Proposition 3.5, $M = (L, +)$ is an Abelian group, where $l + u = (l \to u)' \uparrow (u \to l)'$, for every $l, u \in L$. We define the operation $\cdot : L \times M \longrightarrow M$ by $l.m = l \wedge m$, for every $l \in L$ and $m \in M$. Then

($LM1$) $1 \cdot m = 1 \wedge m$, for every $m \in M$;

($LM2$) Since for every $m, n \in M$,

$$\begin{aligned}
m+n &= (m \to n)' \uparrow (n \to m)' = ((m \to n)' \to (n \to m)') \to (n \to m)' \\
&= ((m' \vee n)' \to (n' \vee m)') \to (n' \vee m)' \\
&= ((m' \vee n) \vee (n \wedge m'))' \vee (n \wedge m') \\
&= ((m \wedge n') \wedge (n' \vee m)) \vee (n \wedge m') \\
&= ((m \wedge n') \vee (n \wedge m')) \wedge ((n \wedge m') \vee (n' \vee m)) \\
&= ((m \wedge n') \vee n) \wedge ((m \wedge n') \vee m) \wedge ((n \vee m \vee n) \wedge (n' \vee m' \vee m')) \\
&= ((n \vee m) \wedge (n \vee n')) \wedge ((m \vee m') \wedge (m' \vee n')) \wedge (m \wedge m) \\
&= (n \vee m) \wedge (m' \vee n') = ((n \vee m) \wedge m') \vee ((n \vee m) \wedge n') \\
&= ((n \wedge m') \vee (m \wedge m')) \vee ((n \wedge n') \vee (m \wedge n')) \\
&= (n \wedge m') \vee (m \wedge n'),
\end{aligned}$$

we have

$$\begin{aligned}
l \cdot (m+n) &= l \wedge ((n \wedge m') \vee (m \wedge n')) = (l \wedge n \wedge m') \vee (l \wedge m \wedge n') \\
&= ((l \wedge m) \wedge (l \wedge n)') \vee ((l \wedge m)' \wedge (l \wedge n)) \\
&= (l \wedge m) + (l \wedge n) = l \cdot m + l \cdot n,
\end{aligned}$$

for every $l \in L$ and $m, n \in M$.

$(LM3)$ Let $l \to u \neq 1$ or $l = u$, for any $l, u \in L$. Then $u \leq l$ and so $u \vee l = l$ and $u \wedge l = u$. Thus, for every $m \in M$,

$$\begin{aligned}
l'.m + u.m &= (l' \wedge m) + (u \wedge m) \\
&= ((l' \wedge m)' \wedge (u \wedge m)) \vee ((l' \wedge m) \wedge (u \wedge m)') \\
&= ((l \vee m') \wedge (u \wedge m)) \vee ((l' \wedge m) \wedge (u' \vee m')) \\
&= ((u \wedge m \wedge l) \vee (u \wedge m \wedge m') \vee (l' \wedge m \wedge u') \vee (l' \wedge m \wedge m')) \\
&= (u \wedge m \wedge l) \vee (l' \wedge m \wedge u') = m \wedge ((u \wedge l) \vee (l' \wedge u')) \\
&= ((l \vee u) \to (l \wedge u)).m = (l \to u).m.
\end{aligned}$$

Note that if $l \to u = 1$, then $l \leq u$. So by the similar way, we have $(l \to u).m = l'.m + u.m$. Hence,

$$(l \to u) \cdot m = l' \cdot m + u \cdot m, \text{ for all pairs (l,u) with } u \neq 1.$$

Therefore, L is an L-module. $\square$

Proposition 3.8. Let $L = (L; \rightarrow, 0, 1)$ be bounded and L-commutative, I be an ideal of L and L be an L-module. Then $\frac{L}{I}$ is an L-module. Moreover, if L is an EL-module, then $\frac{L}{I}$ is an EL-module.

Proof: Since $(L, +)$ is an Abelian group, it is easy to see that $(\frac{L}{I}, \oplus)$ is An abelian group, where $[l] \oplus [u] = [l + u]$, for every $l, u \in L$. We define the operation $\bullet : L \times \frac{L}{I} \longrightarrow \frac{L}{I}$ by $l \bullet [m] = [l \cdot m]$, for every $l \in L$ and $[m] \in \frac{L}{I}$. Then

$(L\frac{L}{I}1)$ By $(LL1)$, we have $1 \bullet [m] = [m]$, for every $[m] \in \frac{L}{I}$;

$(L\frac{L}{I}2)$ By $(LL2)$, for every $l \in L$ and $[m], [n] \in \frac{L}{I}$, we have

$$1 \bullet ([m] \oplus [n]) = l \bullet [m+n] = [l \cdot (m+n)] = [l \cdot m + l \cdot n] = [l \cdot m] \oplus [l \cdot n] = l \bullet [m] \oplus l \bullet [n];$$

$(L\frac{L}{I}3)$ By $(LL3)$, for every $[m] \in \frac{L}{I}$ and for all pairs (l, u) with $u \neq 1$, we have

$$(l \rightarrow u) \bullet [m] = [(l \rightarrow u) \cdot m] = [l' \cdot m + u \cdot m] = [l' \cdot m] \oplus [u \cdot m] = l' \bullet [m] \oplus u \bullet [m].$$

Then $\frac{L}{I}$ is an L-module. Moreover,

$(L\frac{L}{I}4)$ By $(LL4)$, for every $[m] \in \frac{L}{I}$ and for all pairs (l, u) with $l \neq 0$, we have

$$(l \uparrow u) \bullet [m] = [(l \uparrow u) \cdot m] = [l \cdot (u \cdot m)] = l \bullet [u \cdot m] = l \bullet (u \bullet [m]).$$

Therefore, $\frac{L}{I}$ is an EL-module. $\square$

Note. From now on, in this paper, we let M be an Abelian group.

Let $I \in \mathcal{ID}(L)$. The relation $\sim$ on L is defined by

$$u \sim l \Leftrightarrow u \rightarrow l, l \rightarrow u \in I, \text{ for every } u, l \in L.$$

It was proved that $\sim$ is a congruence on L. Then $(\frac{L}{I}; \to, [1])$ is an L-algebra, where $[u] \to [l] = [u \to l]$, for every $u, l \in L$ (see [16]).

Theorem 3.9. *Let M be an L-module, and I be an ideal of L such that $I \subseteq Ann_L(M)$, where $Ann_L(M) = \{l \in L : l \cdot m = 0,$ for every $m \in M\}$. Then M is an $\frac{L}{I}$-module. Moreover, if M is an EL-module, then M is an $E\frac{L}{I}$-module.*

Proof: Consider $' : \frac{L}{I} \longrightarrow \frac{L}{I}$ by $([l])' = [l']$, for every $l \in L$ which is a bijective mapping. Define the operation $\bullet : \frac{L}{I} \times M \longrightarrow M$ by $[l] \bullet m = l \cdot m$, for every $[l] \in \frac{L}{I}$ and $m \in M$. Let $[l] = [u]$ and $m = n$, for every $[l], [u] \in \frac{L}{I}$ and $m, n \in M$. Then $l \to u, u \to l \in I \subseteq Ann_L(M)$ and so $(l \to u) \cdot m = (u \to l) \cdot m = 0$, for every $m \in M$. It results that $l' \cdot m + u \cdot m = u' \cdot m + l \cdot m = 0$ and so $l \cdot m - u \cdot m = l' \cdot m - u' \cdot m$ and $l \cdot m = -u' \cdot m$. Hence $l \cdot m - u \cdot m = l' \cdot m + l \cdot m = (l \to l) \cdot m = 1 \cdot m$ and so $l \cdot m - u \cdot m = 1 \cdot m$. By the similar way, we have $u \cdot m - l \cdot m = 1 \cdot m$. It results that $l \cdot m - u \cdot m = u \cdot m - l \cdot m$ and so $l \cdot m = u \cdot m$. It means that $\bullet$ is well defined. Now, we have:

$(\frac{L}{I}M1)$ By $(LM1)$, it is clear that $[1] \bullet m = m$, for every $m \in M$;

$(\frac{L}{I}M2)$ By $(LM2)$, we have

$$[l] \bullet (m + n) = l \cdot (m + n) = l \cdot m + l \cdot n = [l] \bullet m + l \bullet n,$$

for every $[l] \in \frac{L}{I}$ and $m, n \in M$;

$(\frac{L}{I}M3)$ By $(LM3)$, for every $m \in M$ and for all pairs $([l], [u])$ with $[u] \neq [1]$, we have

$$([l] \to [u]) \bullet m = [l \to u] \bullet m = (l \to u) \cdot m = l' \cdot m + u \cdot m = [l]' \bullet m + [u] \bullet m.$$

Note that $l \neq 1$ implies $[l] \neq [1]$. Hence, M is an $\dfrac{L}{I}$-module. Moreover,

$(\dfrac{L}{I}M4)$ by $(LM4)$, for every $m \in M$ and for all pairs $([l],[u])$ with $[l] \neq [0]$, we have

$$([l] \uparrow [u]) \bullet m = [l \uparrow u] \bullet m = (l \uparrow u) \cdot m = l \cdot (u \cdot m) = [l] \bullet (u \cdot m) = [l] \bullet ([u] \bullet m).$$

Note that $l = 0$ implies $[l] = [0]$. Therefore, M is an $E\dfrac{L}{I}$-module. $\square$

DEFINITION 3.10. Let M be an L-module, and S be a subgroup of M. If S satisfies

$$l \cdot s \in S, \text{ for every } l \in L \text{ and } s \in S,$$

then it is called an *L-submodule* of M.

Example 3.11. (i) By Example 3.2 (i), $2\mathbb{Z}$ is an L-submodule of M.
(ii) According to Example 3.2 (ii), consider $A = \{a, b\}$. Then $S_1 = \{\emptyset, \{a\}\}$ and $S_2 = \{\emptyset, \{b\}\}$ are L-submodules of M.

Let M be an L-module, and S be an L-submodule of M. Since $(M, +)$ is an Abelian group and S is a subgroup of M, we can apply the module theory to present quotient L-module. So it is clear that $(\dfrac{M}{S}, \oplus)$ is an Abelian group, where $(m+S) \oplus (n+S) = (m+n) \oplus S$, for every $m, n \in M$.

PROPOSITION 3.12. Let M be an L-module, and S be an L-submodule of M. Then $\dfrac{M}{S}$ is an L-module. Moreover, if M is an EL-module, then $\dfrac{M}{S}$ is an EL-module.

PROOF: We define the operation $\bullet : L \times \dfrac{M}{S} \longrightarrow \dfrac{M}{S}$ by $l \bullet (m+S) = l \cdot m + S$, for every $l \in L$ and $m + S \in \dfrac{M}{S}$. It is routine to see that $\bullet$ is well defined. By $(LM1)$ and $(LM2)$, the proofs of $(L\dfrac{M}{S}1)$ and $(L\dfrac{M}{S}2)$ are routine.
$(L\dfrac{M}{S}3)$ By $(LM3)$, for all pairs (l, u) with $u \neq 1$, we have

$$\begin{aligned}(l \to u) \bullet (m+S) &= (l \to u) \cdot m + S = (l' \cdot m + u \cdot m) + S \\ &= (l' \cdot m + S) \oplus (u \cdot m + S) \\ &= l' \bullet (m+S) \oplus u \bullet (m+S),\end{aligned}$$

for every $m + S \in \dfrac{M}{S}$. Then $\dfrac{M}{S}$ is an L-module. Moreover,
$(L\dfrac{M}{S}4)$ by $(LM4)$, for all pairs (l, u) with $l \neq 0$, we have

$$\begin{aligned}(l \uparrow u) \bullet (m+S) &= (l \uparrow u) \cdot m + S = l \cdot (u \cdot m) + S \\ &= l \bullet (u \cdot m + S) = l \bullet (u \bullet (m+S)),\end{aligned}$$

for every $m + S \in \dfrac{M}{S}$. Therefore, $\dfrac{M}{S}$ is an EL-module. □

LEMMA 3.13. *Let M be an EL-module, and I be an ideal of L. Then*

$$I_L(M) = \{\Sigma_{i=1}^{n} t_i \cdot m_i : 0 \neq t_i \in I, m_i \in M, n \in \mathbb{N}\}$$

is an L-submodule of M.

PROOF: It is clear that $I_L(M)$ is a subgroup of M. Now, for every $l \in L$ and $\Sigma_{i=1}^{n} t_i \cdot m_i \in I_L(S)$, by $(LM2)$, we have

$$l \cdot \Sigma_{i=1}^{n} t_i \cdot m_i = l \cdot (t_1 \cdot m_1) + l \cdot (t_2 \cdot m_2) + \cdots + l \cdot (t_n \cdot m_n)$$

and so by $(LM4)$,

$$l \cdot \Sigma_{i=1}^{n} t_i \cdot m_i = (l \uparrow t_1) \cdot m_1 + (l \uparrow t_2) \cdot m_2 + \cdots + (l \uparrow t_n) \cdot m_n.$$

Since by (I_3), $t_i \cdot m_i \in I$, for every $1 \leq i \leq n$, we get $l \cdot \Sigma_{i=1}^{n} t_i \cdot m_i \in I_L(M)$. Therefore, $I_L(M)$ is an L-submodule of M. □

DEFINITION 3.14. Let I be a proper ideal of L. Then I is called a *prime ideal* of L, if $l \uparrow u \in I$ implies $l \in I$ or $u \in I$, where $l, u \in L$.

Example 3.15. According to Example 3.4 (i), it is easy to see that $I_1 = \{1, l\}$ and $I_2 = \{1, u\}$ are prime ideals of L.

THEOREM 3.16. *Let M be an EL-module, S be an L-submodule of M and P be a prime ideal of L. Then*

$$S_{N,P} = \{m \in M : c \cdot m \in P_L(M) + S,\ \exists\, 0 \neq c \in (L \setminus P) \cup \{1\}\}$$

is an L-submodule of M and $P_L(M) + S \subseteq S_{N,P}$.

PROOF: Let $m, n \in S_{N,P}$. Then there are $c_1, c_2 \in (L \setminus P) \cup \{1\}$ such that $0 \neq c_1$, $0 \neq c_2$ and $c_1 \cdot m,\ c_2 \cdot n \in P \cdot M + S$. Consider $c = c_1 \uparrow c_2$. It is clear that $c \in (L \setminus P) \cup \{1\}$. Then by $(LM4)$, we have

$$\begin{aligned} c \cdot (m - n) &= (c_1 \uparrow c_2) \cdot (m - n) = c_1 \cdot (c_2 \cdot (m - n)) \\ &= c_1 \cdot (c_2 \cdot m - c_2 \cdot n) = c_1 \cdot (c_2 \cdot m) - c_1 \cdot (c_2 \cdot n) \end{aligned}$$

and so by Lemma 3.13, $c \cdot (m - n) \in P_L(M) + S$. Now, for every $l \in L$ and $m \in S_{N,P}$, we show that $l \cdot m \in S_{N,P}$. Since $m \in S_{N,P}$, there is $0 \neq c \in (L \setminus P) \cup \{1\}$ such that $c \cdot m \in P_L(M)$. Then by Lemma 3.13 and $(LM4)$,

$$c \cdot (l \cdot m) = (c \uparrow l) \cdot m = (l \uparrow c) \cdot m = l \cdot (c \cdot m) \in P_L(M).$$

Hence, $S_{N,P}$ is an L-submodule of M. Finally, let $t \cdot m \in P_L(M)$. Then we have $1 \cdot (t \cdot m) \in P_L(M) + S$, where $c = 1 \in (L \setminus P) \cup \{1\}$. Therefore, $t \cdot m \in S_{N,P}$ and so $P_L(M) \subseteq S_{N,P}$. □

THEOREM 3.17. *Let I be an ideal of L, and M be an EL-module. Then $\dfrac{M}{I_L(M)}$ is an $E\dfrac{L}{I}$-module. Moreover, if M is an EL-module, then $\dfrac{M}{I_L(M)}$ is an $E\dfrac{L}{I}$-module.*

PROOF: The module $\dfrac{M}{I_L(M)}$ can be defined by Lemma 3.13. Then we define the operation
$\bullet : \dfrac{L}{I} \times \dfrac{M}{I_L(M)} \longrightarrow \dfrac{M}{I_L(M)}$ by $[l] \bullet (m + I_L(M)) = l \cdot m + I_L(M)$, for every $[l] \in \dfrac{L}{I}$ and $m + I_L(M) \in \dfrac{M}{I_L(M)}$. Since

$$\begin{aligned} I \bullet \frac{M}{I_L(M)} &= \{l \bullet (m + I_L(M)) : l \in L, m \in M\} \\ &= \{l \cdot m + I_L(M) : l \in L, m \in M\} = I_L(M), \end{aligned}$$

we have $I \subseteq Ann_L(\dfrac{M}{I_L(M)})$ and so with a proof similar to the proof of Theorem 3.9, $\bullet$ is well defined.

$(\dfrac{L}{I}\dfrac{M}{I_L(M)}1)$ By $(LM1)$, $[1]\bullet(m+I_L(M)) = 1\cdot m+I_L(M) = m+I_L(M)$, for every $m \in M$;

$(\dfrac{L}{I}\dfrac{M}{I_L(M)}2)$ By $(LM2)$, we have

$$\begin{aligned}
[l]\bullet((m+I_L(M))\oplus(n+I_L(M))) &= [l]\bullet(m+n+I_L(M)) \\
&= l\cdot(m+n)+I_L(M) \\
&= l\cdot m+l\cdot n+I_L(M) \\
&= (l\cdot m+I_L(M))\oplus(l\cdot n+I_L(M)) \\
&= [l]\bullet(m+I_L(M))\oplus[l]\bullet(n+I_L(M)),
\end{aligned}$$

for every $[l] \in \dfrac{L}{I}$ and $(m+I_L(M)),(n+I_L(M)) \in \dfrac{M}{I_L(M)}$;

$(\dfrac{L}{I}\dfrac{M}{I_L(M)}3)$ By $(LM3)$, for every $m+I_L(M) \in \dfrac{M}{I_L(M)}$ and for all pairs $([l],[u])$ with $[u] \neq [1]$, we have

$$\begin{aligned}
([l]\rightarrow[u])\bullet(m+I_L(M)) &= [l\rightarrow u]\bullet(m+I_L(M)) \\
&= (l\rightarrow u)\cdot m+I_L(M) \\
&= (l'\cdot m+u\cdot m)+I_L(M) \\
&= (l'\cdot m+I_L(M))\oplus(u\cdot m+I_L(M)) \\
&= [l]'\bullet(m+I_L(M))\oplus[u]\bullet(m+I_L(M));
\end{aligned}$$

Hence, M is an $\dfrac{L}{I}$-module. Moreover,

$(\dfrac{L}{I}\dfrac{M}{I_L(M)}4)$ by $(LM4)$, for every $m+I_L(M) \in \dfrac{M}{I_L(M)}$ and for all pairs $([l],[u])$ with $[l] \neq [0]$, we have

$$\begin{aligned}
([l]\uparrow[u])\bullet(m+I_L(M)) &= [l\uparrow u]\bullet(m+I_L(M)) = (l\uparrow u)\cdot m+I_L(M) \\
&= l\cdot(u\cdot m)+I_L(M) = [l]\bullet(u\cdot m+I_L(M)) \\
&= [l]\bullet([u]\bullet(m+I_L(M)).
\end{aligned}$$

Therefore, $\dfrac{M}{I_L(M)}$ is an $E\dfrac{L}{I}$-module. $\square$

DEFINITION 3.18. Let M be an L-module and S be a proper L-submodule of M. Then S is called a *prime L-submodule* of M, if by $l \cdot m \in S$, we have $m \in S$ or $l \in (S : M) = \{l \in L : l \cdot M \subseteq S\}$.

Example 3.19. By Example 3.2(i), $2\mathbb{Z}$ is a prime L-submodule of $\mathbb{Z}$.

Note. Let M be an L-module, $I \subseteq L$ and $D \subseteq M$. Then we set $ID = \{i \cdot d : i \in I \text{ and } d \in D\}$, and $I_t = \{\alpha \in L : t \to \alpha = 1\}$, for every $t \in L$. It is clear that $1, t \in I_t$ and so $I_t \neq \emptyset$.

THEOREM 3.20. *Let L be bounded and L-commutative, M be an L-module and S be a proper L-submodule of M. Then S is a prime L-submodule of M if and only if $I_t D \subseteq S$ implies $D \subseteq S$ or $I_t \subseteq (S : M)$, for any L-submodule D of M and $t \in L$.*

PROOF: ($\Rightarrow$) Let S be a prime L-submodule of M and $I_t D \subseteq S$, where D is an L-submodule of M and $t \in L$. We show that $D \subseteq S$ or $I_t \subseteq (S : M)$. Let $I_t \nsubseteq (S : M)$ and $D \nsubseteq S$. Then there are $x \in I_t$ and $d \in D$ such that $x \cdot M \nsubseteq S$ and $d \notin S$. Since $ID \subseteq S$, we have $x \cdot d \in S$ and so by $d \notin S$, we get $x \in (S : M)$, which is a contradiction.
($\Leftarrow$) Let by $I_t D \subseteq S$, we have $D \subseteq S$ or $I_t \subseteq (S : M)$, for any L-submodule D of M and $t \in L$. Suppose $x \cdot m \in S$ and $m \notin S$, for any $x \in L$ and $m \in M$. For every $\alpha \in I_x$, we have

$$\begin{aligned} \alpha \cdot m &= (1 \to \alpha) \cdot m = ((x \to \alpha) \to \alpha) \cdot m = (x \uparrow \alpha) \cdot m = (\alpha \uparrow x) \cdot m \\ &= \alpha \cdot (x \cdot m) \in S. \end{aligned}$$

Now, consider $D = \prec m \succ = \{y \cdot m : y \in L\}$. Then

$$I_x D = \{\alpha \cdot (y \cdot m) : \alpha, y \in L\} = \{y \cdot (\alpha \cdot m) : \alpha, y \in L\} \subseteq S$$

and so $I_x \subseteq (S : M)$ or $D \subseteq S$. Since $m \notin S$, we have $I_x \subseteq (S : M)$ and so $x \in (S : M)$. Therefore, S is a prime L-submodule of M. $\square$

PROPOSITION 3.21. For every $x, y \in L$,

(i) $x' \to (x \to y) = 1$;

(ii) $(x \to y) \to x' = (y \to x) \to y'$.

Proof: (i) By $(L2)$, we have
$x' \to (x \to y) = (x \to 0) \to (x \to y) = (0 \to x) \to (0 \to y) = 1 \to 1 = 1$, for every $x, y \in L$. (ii) By $(L2)$, we have
$(x \to y) \to x' = (x \to y) \to (x \to 0) = (y \to x) \to (y \to 0) = (y \to x) \to y'$, for every $x, y \in L$. □

Lemma 3.22. *Let L be a bounded KL-algebra, M be an EL-module and S be a proper L-submodule of M. Then $P_S = (S : M) \cup \{1\}$ is an ideal of L.*

Proof: $(I1)$ It is clear that $1 \in P_S$.
$(I2)$ Let $x, x \to y \in P_S$. Because of the nature of the definition of P_S, we need to consider three cases:

(1) If $x = 1$, then $y = 1 \to y = x \to y \in P_S$.

(2) Let $x \to y = 1$. Then for $y = 1$, the problem is solved. Thus, let $y \neq 1$. In this case, if $x = 0$, then by $(LM3)$, $m = 1 \cdot m = (0 \to y) \cdot m = 1 \cdot m + y \cdot m = m + y \cdot m$ and so $y \cdot m = 0$, for every $m \in M$. It means that $y \in (S : M)$ and so $y \in P_S$. Hence, suppose $x \neq 0$ and $y \neq 1$. Since $y = 1 \to y = (x \to y) \to y = x \uparrow y$, by $(LM4)$, we have

$$y \cdot m = (x \uparrow y) \cdot m = (y \uparrow x) \cdot m = y \cdot (x \cdot m) \in S, \text{ for every } m \in M.$$

Thus, $y \in (S : M)$ and so $y \in P_S$.

(3) Let $x \neq 1$ and $x \to y \neq 1$. Then $x \cdot m, (x \to y) \cdot m \in S$, for every $m \in M$. It results that $x \cdot m + (x \to y) \cdot m \in S$, for every $m \in M$. Now, by Proposition 3.21(i) and $(LM3)$, for every $m \in M$, we have

$$m = 1 \cdot m = (x' \to (x \to y)) \cdot m = x \cdot m + (x \to y) \cdot m \in S,$$

which is a contradiction.

Therefore, $P_S = (S : M) \cup \{1\}$ is an ideal of L. □

Definition 3.23. Let L be bounded and M be an L-module. Then M is called a *torsion free* L-module, if $l \cdot m = 0$ implies $l = 0$ or $m = 0$, for every $l \in L$ and $m \in M$.

Example 3.24. By Example 3.2(ii), M is a torsion free L-module.

THEOREM 3.25. *Let L be a bounded KL-algebra, M be an EL-module and S be a proper L-submodule of M. Then S is a prime L-submodule of M if and only if $P_S = (S : M) \cup \{1\}$ is a prime ideal of L and $\dfrac{M}{S}$ is a torsion free $\dfrac{L}{P_S}$-module.*

PROOF: ($\Rightarrow$) Let S is a prime L-submodule of M. By Lemma 3.22, P_S is an ideal of L. At first, we show that P_S is a prime ideal of L. Let $x \uparrow y \in P_S$, for any $x, y \in P_S$. We consider three cases:

(1) If $x = 1$ or $y = 1$, then $x \in P_S$ or $y \in P_S$.

(2) If $x \uparrow y \neq 1$, $x \neq 1$ and $y \neq 1$, then by $(LM4)$, we have $x \cdot (y \cdot m) = (x \uparrow y) \cdot m \in S$, for every $m \in S$. Hence, $x \in (S : M)$ or $y \cdot m \in S$, for every $m \in M$. It results that $x \in P_S$ or $y \in P_S$.

(3) Let $x \uparrow y = 1$, $x \neq 1$ and $y \neq 1$. Then $(x \to y) \to y = x \uparrow y = 1$ and so $x \to y \leq y$. Since $y \leq x \to y$, we have $x \to y = y$ and so by $(LM3)$,

$$(x \to y) \cdot m = x' \cdot m + y \cdot m = y \cdot m, \text{ for every, } m \in M.$$

Then $x' \cdot m = 0 \in S$ and so $x' \in (S : M)$ or $m \in S$, for every $m \in M$. If $m \in S$, for every $m \in M$, then $M = S$, which is a contradiction. Thus, $x' \in (S : M) \subseteq P_S$ and so by $(I3)$, we have $y = x \to y = y' \to x' \in P_S$. Hence, P_S is a prime ideal of L.

Now, we define the operation $\bullet : \dfrac{L}{P_S} \times \dfrac{M}{S} \longrightarrow \dfrac{M}{S}$ by $[l] \bullet (m+S) = l \cdot m + S$, for every $[l] \in \dfrac{L}{P_S}$ and $m + S \in \dfrac{M}{S}$. By the similar way to the proof of Theorem 3.17, $\dfrac{M}{S}$ is an $\dfrac{L}{P_S}$-module. Finally, let $[l] \bullet (m + S) = S$, for any $[l] \in \dfrac{L}{P_S}$ and $m + S \in \dfrac{M}{S}$. Then $l \cdot m + S = S$ and so $l \cdot m \in S$. It results that $l \in (S : M) \subseteq P_S$ or $m \in S$ and so $[l] = P_S$ or $m + S = S$. Therefore, $\dfrac{M}{S}$ is a torsion free $\dfrac{L}{P_S}$-module.

($\Leftarrow$) Let $P_S = (S : M) \cup \{1\}$ be a prime ideal of L and $\dfrac{M}{S}$ be a torsion free $\dfrac{L}{P_S}$-module. If $l \cdot m \in S$, for any $l \in L$ and $m \in S$, then $[l] \bullet (m + S) = l \cdot m + S = S$ and so $[l] = P_S = [1]$ or $m + S = S$. It means that $l = 1 \to l \in P_S$. Therefore, S is a prime L-submodule of M. $\square$

4. Conclusions and future works

In this paper, we have presented the definitions of L-modules, L-submodules and prime L-submodules, and some results about prime L-submodules. We intend to study L-modules in specific cases, too. For examples, free L-modules, projective(injective) L-modules, and so on. Because L-algebras cover a number of algebraic structures (such as BCK-algebras, etc.), the results of this paper can be generalized to those algebraic structures. We hope that we have taken an effective step in this regard.

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References

[1] P. F. A. Di Nola, I. Leustean, *MV-modules*, **Journal of Algebra**, vol. 267 (2003), pp. 21–40, DOI: https://doi.org/10.1016/S0021-8693(03)00332-6.

[2] H. A. S. Abojabal, M. Aslam, A. B. Thaheem, *On actions of BCK-algebras on groups*, **Pan American Mathematical Journal**, vol. 4 (1994), pp. 727–735.

[3] R. A. Borzooei, S. S. Goraghani, *Prime submodules in extended BCK-module*, **Italian Journal of Pure and Applied Mathematics**, vol. 33 (2014), pp. 433–444.

[4] R. A. Borzooei, S. S. Goraghani, *Free MV-modules*, **Journal of Intelligent and Fuzzy System**, vol. 3(1) (2016), pp. 151–161, DOI: https://doi.org/10.3233/IFS-162128.

[5] L. C. Ciungu, *Results in L-algebras*, **Algebra Universalis**, vol. 82(7) (2021), pp. 1–18, DOI: https://doi.org/10.1007/s00012-020-00695-1.

[6] P. Dehornoy, *Set-theoretic solutions of the Yang-Baxter equation, RC-calculus, and Garside germs*, **Advances in Mathematics**, vol. 282 (2015), pp. 93–127, DOI: https://doi.org/10.1016/j.aim.2015.05.008.

[7] V. G. Drinfeld, *On some unsolved problems in quantum group theory*, [in:] P. P. Kulish (ed.), **Proceedings of Workshops held in the Euler International Mathematical Institute, Leningrad, Fall 1990**, vol. 1510 of Lecture Notes in Mathematics, Springer, Berlin–Heidelberg (1992), pp. 1–8, DOI: https://doi.org/10.1007/BFB0101175.

[8] A. Facchini, M. Gran, M. Pompini, *Ideals and congruences in L-algebra and PRE-L-algebras* (2023), arXiv:2305.19042 [math.CT].

[9] S. S. Goraghani, R. A. Borzooei, *Most results on A-ideals in MV-moduls*, **Journal of Algebraic Systems**, vol. 5(1) (2017), pp. 1–13, DOI: https://doi.org/10.22044/jas.2017.994.

[10] S. S. Goraghani, R. A. Borzooei, *Results on prime ideals in PMV-algebras and MV-modules*, **Italian Journal of Pure and Applied Mathematics**, vol. 37 (2017), pp. 183–196.

[11] S. S. Goraghani, R. A. Borzooei, *Decomposition of A-ideals in MV-modules*, **Annals of the University of Craiova-mathematics and Computer Science Series**, vol. 45(1) (2018), pp. 66–77, DOI: https://doi.org/10.52846/ami.v45i1.872.

[12] S. S. Goraghani, R. A. Borzooei, *On injective MV-modules*, **Bulletin of the Section of Logic**, vol. 47(4) (2018), pp. 283–298, DOI: https://doi.org/10.18778/0138-0680.47.4.04.

[13] S. S. Goraghani, R. A. Borzooei, S. S. Ann, *New definition of MV-semimoduls*, **Journal of Intelligent and Fuzzy Systems**, vol. 41(2) (2021), pp. 3715–3726, DOI: https://doi.org/10.3233/JIFS-211130.

[14] A. D. Nola, C. Russo, *Semiring and semimodules issues in MV-algebras*, **Communications in Algebra**, vol. 41(3) (2013), pp. 1017–1048, DOI: https://doi.org/10.1080/00927872.2011.610074.

[15] W. Rump, *A decomposition theorem for square-free unitary solutions of the quantum YangBaxter equation*, **Advances in Mathematics**, vol. 193(1) (2005), pp. 40–55, DOI: https://doi.org/10.1016/j.aim.2004.03.019.

[16] W. Rump, *L-algebra, self-similarity and l-groups*, **Journal of Algebra**, vol. 320 (2008), pp. 2328–2348, DOI: https://doi.org/10.1016/j.jalgebra.2008.05.033.

[17] W. Rump, *A general Glivenko theorem*, **Algebra Universal**, vol. 61 (2009), pp. 455–473, DOI: https://doi.org/10.1007/s00012-009-0018-y.

[18] W. Rump, Y. Yang, *Intervals in l-groups as L-algebras*, **Algebra Universal**, vol. 6(7) (2012), pp. 121–130, DOI: https://doi.org/10.1007/s00012-012-0172-5.

[19] Y. Wu, Y. C. Y. J. Wang, *Lattice-ordered effect algebras and L-algebras*, **Fuzzy Sets and Systems**, vol. 369 (2019), pp. 103–113, DOI: https://doi.org/10.1016/j.fss.2018.08.013.

[20] Y. Wu, Y. Yang, *Orthomodular lattices as L-algebras*, **Soft Computing**, vol. 24 (2020), pp. 14391–14400, DOI: https://doi.org/10.1007/s00500-020-05242-7.

[21] Y. Yang, W. Rump, *Pseudo-MV algebras as L-algebras*, **Journal of Multiple-Valued Logic and Soft Computing**, vol. 19 (2012), pp. 621–632.

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