UNIVERSITY OF LODZ
DEPARTMENT OF LOGIC

BULLETIN
OF THE SECTION OF LOGIC

VOLUME 52, NUMBER 4

ŁÓDŹ, DECEMBER 2023

C O P E
Member since 2018
JM13707

Layout
Michał Zawidzki

Initiating Editor
Katarzyna Smyczek

Printed directly from camera-ready materials provided
to the Lodz University Press

Published by Lodz University Press

First edition. W.10996.23.0.C

Printing sheets 11.25

Lodz University Press
90-237 Łódź, 34A Jana Matejki St.
www.wydawnictwo.uni.lodz.pl
e-mail: ksiegarnia@uni.lodz.pl
+48 42 635 55 77

The **Bulletin of the Section of Logic** (*BSL*) is a quarterly peer-reviewed journal published with the support from the University of Lodz. Its aim is to act as a forum for a wide and timely dissemination of new and significant results in logic through rapid publication of relevant research papers. *BSL* publishes contributions on topics dealing directly with logical calculi, their methodology, and algebraic interpretation.

Papers may be submitted through the *BSL* online editorial platform at https://czasopisma.uni.lodz.pl/bulletin. While preparing the munuscripts for publication please consult the Submission Guidelines.

* * *

Editorial Office: Department of Logic, University of Lodz
ul. Lindleya 3/5, 90-131 Łódź, Poland
e-mail: bulletin@uni.lodz.pl

Homepage: https://czasopisma.uni.lodz.pl/bulletin

TABLE OF CONTENTS

Bulletin of the **S**ection of **L**ogic
Volume 52/4 (2023), pp. 411–440
https://doi.org/10.18778/0138-0680.2023.20

Mohammad Hamidi

EXTENDED BCK-IDEAL BASED ON SINGLE-VALUED NEUTROSOPHIC HYPER BCK-IDEALS

Abstract

This paper introduces the concept of single-valued neutrosophic hyper BCK-subalgebras as a generalization and alternative of hyper BCK-algebras and on any given nonempty set constructs at least one single-valued neutrosophic hyper BCK-subalgebra and one a single-valued neutrosophic hyper BCK-ideal. In this study level subsets play the main role in the connection between single-valued neutrosophic hyper BCK-subalgebras and hyper BCK-subalgebras and the connection between single-valued neutrosophic hyper BCK-ideals and hyper BCK-ideals. The congruence and (strongly) regular equivalence relations are the important tools for connecting hyperstructures and structures, so the major contribution of this study is to apply and introduce a (strongly) regular relation on hyper BCK-algebras and to investigate their categorical properties (quasi commutative diagram) via single-valued neutrosophic hyper BCK-ideals. Indeed, by using the single-valued neutrosophic hyper BCK-ideals, we define a congruence relation on (weak commutative) hyper BCK-algebras that under some conditions is strongly regular and the quotient of any (single-valued neutrosophic)hyper BCK-(sub)algebra via this relation is a (single-valued neutrosophic)(hyper BCK-subalgebra) BCK-(sub)algebra.

Keywords: single-valued neutrosophic (hyper)BCK-subalgebra, quasi commutative diagram, extendable single-valued neutrosophic (hyper)BCK-ideal.

2020 Mathematical Subject Classification: 03B47 , 06F35 , 03B52.

Presented by: Janusz Czelakowski
Received: January 21, 2021
Published online: August 10, 2023

1. Introduction

Theory of neutrosophic set as an extension of classical set, (intuitionistic) fuzzy set [21] and interval-valued (intuitionistic) fuzzy set is introduced by Smarandache for the first time in 2005 [18] and novel concept of neutrosophy theory titled neutro-(hyper)algebra as the development of classical (hyper)algebra and partial-(hyper)algebra [19]. This concept handles problems involving ambiguous, hesitancy, and conflicting data and describes the main tool in modeling unsure hypernetworks in all sciences, see in more detail, accessible single-valued neutrosophic graphs [3], derivable single-valued neutrosophic graphs based on KM-single-valued neutrosophic metric [5] and single-valued neutrosophic directed (hyper)graphs and applications in networks [4], single-valued neutrosophic general machine [17] and a novel similarity measure of single-valued neutrosophic sets based on modified manhattan distance and its applications [22]. Today, in the scope of logical (hyper)algebras, (hyper)BCK-algebras and their generalization such as fuzzy hyper BCK-subalgebras and single-valued neutrosophic hyper BCK-subalgebras are investigated and applied in related interdisciplinary sciences such as inf-hesitant fuzzy ideals in BCK/BCI-algebras [10], length neutrosophic subalgebras of BCK=BCI-algebras [9], fuzzy soft positive implicative hyper BCK-ideals of several types [13], implicative neutrosophic quadruple BCK-Algebras and ideals [15], construction of an HV-K-algebra from a BCK-algebra based on ends lemma [16], and implicative ideals of BCK-algebras based on MBJ-neutrosophic sets [20]. The fundamental relations make an important role in the connection between hyper BCK-subalgebras and BCK-subalgebras and some research is published in this scopes such as on fuzzy quotient, BCK-algebras [2], (semi)topological quotient BCK-algebras [14] and extended fuzzy BCK-subalgebras [23].

Recently in the scope of neutro logical (hyper) algebra Hamidi, et al. introduced the concept of neutro BCK-subalgebras [6] and single-valued neutro hyper BCK-subalgebras [7] as a generalization of BCK-algebras and hyper BCK-subalgebras, respectively and presented the main results in this regard.

Regarding these points, we try to develop the notation of fuzzy hyper BCK-subalgebras to the concept of single-valued neutrosophic hyper BCK-subalgebras and so we want to seek the connection between single-valued neutrosophic BCK-algebras and single-valued neutrosophic hyper BCK-algebras. In this paper, we consider single-valued neutrosophic hyper BCK-ideals and describe the relationship between (BCK-ideals) hyper BCK-ideals and single-valued neutrosophic hyper BCK-ideals. The connection between of category of logical algebras and the category of logical hyperalgebras (as quasi commutative diagram) is based on fundamental relation and this problem is made a motivation to introduce some relation on hyper BCK-subalgebras via the single-valued neutrosophic hyper BCK-subalgebras and single-valued neutrosophic hyper BCK-ideals, it is the main and major contribution of this study. We apply a fundamental relation to any given hyper BCK-algebras and discuss the quotient of single-valued neutrosophic hyper BCK-algebras to the convert of single-valued neutrosophic BCK-algebras and discuss the quotient of single-valued neutrosophic hyper BCK-ideals to the convert of single-valued neutrosophic BCK-ideals. Moreover, applying the concept of single-valued neutrosophic hyper BCK-ideals, we get a congruence relation on (weak commutative) hyper BCK-algebras that the quotient of any given hyper BCK-algebra via this relation is a (hyper BCK-algebra) BCK-algebra. An isomorphism theorem of single-valued neutrosophic hyper BCK-ideals is obtained using the special single-valued neutrosophic hyper BCK-ideals. In the section 3, we investigated on single-valued neutrosophic hyper BCK-subalgebras, especially we converted any given nonempty set to hyper BCK-subalgebra and obtained a family of single-valued neutrosophic hyper BCK-subalgebras. In the section 4, it is presented the concepts of single-valued neutrosophic hyper BCK-ideals, especially any given nonempty set extended to a hyper BCK-algebra with at least a single-valued neutrosophic hyper BCK-ideal.

2. Preliminaries

In this section, we recall some concepts that need to our work.

DEFINITION 2.1. [8] Let $X \neq \emptyset$. Then a universal algebra $(X, \vartheta, 0)$ of type $(2, 0)$ is called a *BCK-algebra*, if $\forall\ x, y, z \in X$:
(BCI-1) $((x\vartheta\ y)\vartheta\ (x\vartheta\ z))\vartheta\ (z\vartheta\ y) = 0$,
(BCI-2) $(x\vartheta\ (x\vartheta\ y))\vartheta\ y = 0$,
(BCI-3) $x\vartheta\ x = 0$,
(BCI-4) $x\vartheta\ y = 0$ and $y\vartheta\ x = 0$ imply $x = y$,
(BCK-5) $0\vartheta\ x = 0$,

where $\vartheta(x, y)$ is denoted by $x\vartheta\ y$.

Definition 2.2. [1, 11] Let $X \neq \emptyset$ and $P^*(X) = \{Y \mid \emptyset \neq Y \subseteq X\}$. Then for a map $\varrho : X^2 \to P^*(X)$ a hyperalgebraic system $(X, \varrho, 0)$ is called a *hyper BCK-algebra*, if $\forall\ x, y, z \in X$:
$(H1)$ $(x\ \varrho\ z)\ \varrho\ (y\ \varrho\ z) \ll x\ \varrho\ y$,
$(H2)$ $(x\ \varrho\ y)\ \varrho\ z = (x\ \varrho\ z)\ \varrho\ y$,
$(H3)$ $x\ \varrho\ X \ll x$,
$(H4)$ $x \ll y$ and $y \ll x$ imply $x = y$,
where $x \ll y$ is defined by $0 \in x\ \varrho\ y$, $\forall\ W, Z \subseteq X$, $W \ll Z \Leftrightarrow\ \forall\ a \in W\ \exists\ b \in Z$ s.t $a \ll b$, $(W\ \varrho\ Z) = \bigcup\limits_{a\in W, b\in Z} (a\ \varrho\ b)$ and $\varrho(x, y)$ is denoted by $x\varrho\ y$.

We will call X is a *weak commutative* hyper BCK-algebra if, $\forall\ x, y \in X, (x\ \varrho\ (x\ \varrho\ y)) \cap (y\ \varrho\ (y\ \varrho\ x)) \neq \emptyset$.

Theorem 2.3. [11] *Let $(X, \varrho, 0)$ be a hyper BCK-algebra. Then $\forall\ x, y, z \in X$ and $W, Z \subseteq X$,*

(i) $(0\ \varrho\ 0) = 0, 0 \ll x, (0\ \varrho\ x) = 0$, $x \in (x\ \varrho\ 0)$ *and* $(W \ll 0 \Rightarrow W = 0)$,

(ii) $x \ll x$, $x\ \varrho\ y \ll x$ *and* $(y \ll z \Rightarrow x\ \varrho\ z \ll x\ \varrho\ y)$,

(iii) $W\ \varrho\ Z \ll W$, $W \ll W$ *and* $(W \subseteq Z \Rightarrow W \ll Z)$.

Definition 2.4. [18] Let V be a universal set. A neutrosophic subset (NS) X of V is an object having the following form $X = \{(x, T_X(x), I_X(x), F_X(x)) \mid x \in V\}$, or $X : V \to [0, 1] \times [0, 1] \times [0, 1]$ which is characterized by a truth-membership function T_X, an indeterminacy-membership function I_X and a falsity-membership function F_X. There is no restriction on the sum of $T_X(x), I_X(x)$ and $F_X(x)$.

From now on, $\forall\ x, y \in [0, 1]$, consider $T_{min}(x, y) = \min\{x, y\}$ and $S_{max}(x, y) = \max\{x, y\}$ as triangular norm and triangular conorm, respectively.

Definition 2.5. [12] Let $(X, \varrho, 0)$ be a hyper BCK-algebra. A single-valued neutrosophic subset $A = (T_A, I_A, F_A)$ of X is called a single-valued neutrosophic hyper BCK-ideal, if $\forall\ x, y \in X$ it satisfies the following properties:
$(FH1)$ $x \ll y \Rightarrow T_A(x) \geq T_A(y), I_A(x) \geq I_A(y)$ and $F_A(x) \leq F_A(y)$,

$(FH2)$ $T_A(x) \geq T_{min}\{T_A(y), \bigwedge(T_A(x\varrho\, y))\}, I_A(x) \geq T_{min}\{I_A(y), \bigwedge(I_A(x \varrho\, y))\}$ and
$F_A(x) \leq S_{max}\{F_A(y), \bigvee(F_A(x\varrho\, y))\}$.

3. Single-valued neutrosophic hyper BCK-subalgebras

In this section, we make the concept of single-valued neutrosophic hyper BCK-subalgebras as an extension of fuzzy hyper BCK-subalgebras and seek some of their properties.

From now on, consider (X, ϱ) as a hyper BCK-subalgebra.

Definition 3.1. A single-valued neutrosophic subset $A = (T_A, I_A, F_A)$ of (X, ϱ) is called a single-valued neutrosophic hyper BCK-subalgebra of $(X, \varrho, 0)$, if

(*i*) $\bigwedge(T_A(x \;\varrho\; y)) \geq T_{min}(T_A(x), T_A(y))$;

(*ii*) $\bigvee(I_A(x \;\varrho\; y)) \leq S_{max}(I_A(x), I_A(y))$;

(*iii*) $\bigvee(F_A(x \;\varrho\; y)) \leq S_{max}(F_A(x), F_A(y))$.

Theorem 3.2. *Let $A = (T_A, I_A, F_A)$ be a single-valued neutrosophic hyper BCK-subalgebra of $(X, \varrho, 0)$. Then*

(*i*) $T_A(0) \geq T_A(x)$;

(*ii*) $\bigwedge(T_A(x \;\varrho\; 0)) = T_A(x)$;

(*iii*) $\bigwedge(T_A(0 \;\varrho\; x)) = T_A(0)$;

Proof: (*i*) Let $x \in X$. Since $0 \in x \;\varrho\; x$, we get that $T_A(0) \geq \bigwedge(T_A(x \;\varrho\; x)) \geq T_{min}(T_A(x), T_A(x)) = T_A(x)$.

(*ii*) Let $x \in X$. Since $x \in x \;\varrho\; 0$, we get that $T_A(x) \geq \bigwedge(T_A(x \;\varrho\; 0)) \geq T_{min}(T_A(x), T_A(0)) = T_A(x)$. So $\bigwedge(T_A(x \;\varrho\; 0)) = T_A(x)$.

(*iii*) Immediate by Theorem 2.3. □

Theorem 3.3. *Let $A=(T_A,I_A,F_A)$ be a single-valued neutrosophic hyper BCK-subalgebra of X. Then*

(i) $I_A(0)\leq I_A(x)$;

(ii) $\bigvee(I_A(x\ \varrho\ 0))=I_A(x)$;

(iii) $\bigvee(I_A(0\ \varrho\ x))=I_A(0)$;

Proof: (i) Let $x\in X$. Since $0\in x\ \varrho\ x$, we get that $I_A(0)\leq\bigvee(I_A(x\ \varrho\ x))\leq S_{max}(I_A(x),I_A(x))=I_A(x)$.

(ii) Let $x\in X$. Since $x\in x\ \varrho\ 0$, we get that $I_A(x)\leq\bigvee(I_A(x\ \varrho\ 0))\leq S_{max}(I_A(x),I_A(0))=I_A(x)$. So $\bigvee(I_A(x\ \varrho\ 0))=I_A(x)$.

(iii) Immediate by Theorem 2.3. □

Corollary 3.4. Let $A=(T_A,I_A,F_A)$ be a single-valued neutrosophic hyper BCK-subalgebra of $(X,\varrho,0)$. Then

(i) $F_A(0)\leq F_A(x)$;

(ii) $\bigvee(F_A(x\ \varrho\ 0))=F_A(x)$;

(iii) $\bigvee(F_A(0\ \varrho\ x))=F_A(0)$;

(iv) $T_{min}(T_A(x),I_A(0),F_A(0))\leq T_{min}(T_A(0),I_A(x),F_A(x))$.

Theorem 3.5. *Let $0\notin X\neq\emptyset$. Then X converted to a hyper BCK-algebra $(X',\varrho,0)(X'=X\cup\{0\})$ with at least a single-valued neutrosophic hyper BCK-subalgebra.*

Proof: Let $x,y\in X'$. Define " ϱ " on X' by $0\ \varrho\ y=0, x\ \varrho\ x=\{0,x\}(x\neq 0)$, else $x\ \varrho\ y=x$. Clearly $(X',\ \varrho,0)$ is a hyper BCK-algebra. Now, it is easy to see that every single-valued neutrosophic set $A=(T_A,I_A,F_A)$ that $T_A(0)=1, I_A(0)=F_A(0)=0$, is a single-valued neutrosophic hyper BCK-subalgebra of X'. □

Let $\mathcal{SVN}h=\{A=(T_A,I_A,F_A)\mid A\}$, whence X is a hyper BCK-algebra, A is a single-valued neutrosophic hyper BCK-subalgebra of X and $|X|\geq 1$.

COROLLARY 3.6. Let $X \neq \emptyset$. Then X can be extended to a hyper BCK-algebra that $|\mathcal{SVN}h| = |\mathbb{R}|$.

PROOF: Let $|X| = 1$. Then (X, ϱ, x) is a hyper BCK-algebra such that $x \varrho x = X$. Then for a single-valued neutrosophic set $A = (T_A, I_A, F_A)$ by $T_A(x) = I_A(x) = F_A(x) = \alpha$ is a single-valued neutrosophic hyper BCK-subalgebra of X where $\alpha \in [0, 1]$. If $|X| \geq 2$, then by Theorem 3.5, define $A = (T_{A_\alpha}, I_{A_\alpha}, F_{A_\alpha})$ by

$$T_{A_\alpha}(x) = \begin{cases} 1, & \text{if } x = 0, \\ \alpha, & \text{if } x \neq 0 \end{cases}, I_{A_\alpha}(x) = \begin{cases} 0, & \text{if } x = 0, \\ \alpha, & \text{if } x \neq 0 \end{cases}$$

and $F_{A_\alpha}(x) = \begin{cases} 0, & \text{if } x = 0, \\ \alpha, & \text{if } x \neq 0, \end{cases}$ Obviously, $A = (T_{A_\alpha}, I_{A_\alpha}, F_{A_\alpha})$ a single-valued neutrosophic hyper BCK-subalgebra of X and so $|\mathcal{SVN}h| = |[0, 1]|$. □

Let X be a hyper BCK-algebra, $A = (T_A, I_A, F_A)$ a single-valued neutrosophic hyper BCK-subalgebra of X and $\alpha, \beta, \gamma \in [0,\ 1]$. Define $T_A^\alpha = \{x \in X \mid T_A(x) \geq \alpha\}, I_A^\beta = \{x \in X \mid I_A(x) \leq \beta\}$, $F_A^\gamma = \{x \in X \mid F_A(x) \leq \gamma\}$ and $A^{(\alpha,\beta,\gamma)} = \{x \in X \mid T_A(x) \geq \alpha, I_A(x) \leq \beta, F_A(x) \leq \gamma\}$.

THEOREM 3.7. *Let $A = (T_A, I_A, F_A)$ be a single-valued neutrosophic hyper BCK-subalgebra of X. Then*

(*i*) $0 \in A^{(\alpha,\beta,\gamma)} = T_A^\alpha \cap I_A^\beta \cap F_A^\gamma$,

(*ii*) *$A^{(\alpha,\beta,\gamma)}$ is a hyper BCK-subalgebra of X,*

(*iv*) *if $0 \leq \alpha \leq \alpha' \leq 1$, then $T_A^{\alpha'} \subseteq T_A^\alpha, I_A^{\alpha'} \supseteq I_A^\alpha$ and $F_A^{\alpha'} \supseteq F_A^\alpha$.*

PROOF: (*i*) Clearly $A^{(\alpha,\beta,\gamma)} = A^\alpha \cap A^\beta \cap A^\gamma$ and by Theorems 3.2, 3.3, and Corollary 3.4, we get that $0 \in A^{(\alpha,\beta,\gamma)}$.

(*ii*) Let $x, y \in T_A^\alpha$. Then $T_{min}(T_A(x), T_A(y)) \geq \alpha$. Now, for any $z \in x \varrho y, T_A(z) \geq T_{min}(T_A(x \varrho y)) \geq T_{min}(T_A(x), T_A(y)) \geq \alpha$. Hence $z \in T_A^\alpha$ and so $x \varrho y \subseteq T_A^\alpha$. In similar a way $x, y \in I_A^\beta \cap F_A^\gamma$, implies that $x \varrho y \subseteq (I_A^\beta \cap F_A^\gamma)$. Then $A^{(\alpha,\beta,\gamma)}$ is a hyper BCK-subalgebra of X.

(*iii*) Immediate. □

Corollary 3.8. Let $A = (T_A, I_A, F_A)$ be a single-valued neutrosophic hyper BCK-subalgebra of X. If $0 \leq \alpha \leq \alpha' \leq 1$, then $A^{(\alpha',\alpha,\alpha)}$ is a hyper BCK-subalgebra of $A^{(\alpha,\alpha',\alpha')}$.

Let X be a hyper BCK-algebra, S be a hyper BCK-subalgebra of X and $\alpha, \alpha', \beta, \beta', \gamma, \gamma' \in [0,1]$. Define

$$T_A^{[\alpha,\alpha']}(x) = \begin{cases} \alpha', & \text{if } x \in S, \\ \alpha, & \text{if } x \notin S, \end{cases}, \quad I_A^{[\beta,\beta']}(x) = \begin{cases} \beta', & \text{if } x \in S, \\ \beta, & \text{if } x \notin S, \end{cases}, \textit{and}$$

$$F_A^{[\gamma,\gamma']}(x) = \begin{cases} \gamma', & \text{if } x \in S, \\ \gamma, & \text{if } x \notin S, \end{cases}$$. Thus we have the following theorem.

Theorem 3.9. *Let X be a hyper BCK-algebra and S be a hyper BCK-subalgebra of X. Then*

(i) $T_A^{[\alpha,\alpha']}$ *is a fuzzy hyper BCK-subalgebra of X.*

(ii) $I_A^{[\beta,\beta']}$ *is a fuzzy hyper BCK-subalgebra of X.*

(iii) $F_A^{[\gamma,\gamma']}$ *is a fuzzy hyper BCK-subalgebra of X.*

(iv) $A = (T_A^{[\alpha,\alpha']}, I_A^{[\beta,\beta']}, F_A^{[\gamma,\gamma']})$ *is a single-valued neutrosophic hyper BCK-subalgebra of X.*

Proof: (i) Let $x, y \in X$. If $x, y \in S$, since S is a hyper subalgebra of X, we get that $x \varrho y \subseteq S$ and so

$$\bigwedge T_A^{[\alpha,\alpha']}(x \varrho y) \geq \bigwedge T_A^{[\alpha,\alpha']}(S) = \alpha' \geq T_{\min}(T_A^{[\alpha,\alpha']}(x), T_A^{[\alpha,\alpha']}(y)).$$

If ($x \in S$ and $y \notin S$) or ($x \notin S$ and $y \in S$) or ($x \notin S$ and $y \notin S$) then $\bigwedge T_A^{[\alpha,\alpha']}(x \varrho y)) \in \{\alpha, \alpha'\}$. Thus $\bigwedge T_A^{[\alpha,\alpha']}(x \varrho y)) \geq T_{\min}(T_A^{[\alpha,\alpha']}(x), T_A^{[\alpha,\alpha']}(y))$ and so $T_A^{[\alpha,\alpha']}$ is a fuzzy hyper BCK-subalgebra of X.

(ii), (iii) Are similar to (i).

(iv) Let $x, y \in X$. If $x, y \in S$, since S is a hyper BCK-subalgebra of X, we get that $x \varrho y \subseteq S$ and so $\bigvee I_A^{[\beta,\beta']}(x \varrho y)) \leq \bigvee I_A^{[\beta,\beta']}(S) = \alpha' \leq S_{\max}(I_A^{[\beta,\beta']}(x), I_A^{[\beta,\beta']}(y))$. If ($x \in S$ and $y \notin S$) or ($x \notin S$ and $y \in S$) or ($x \notin S$ and $y \notin S$) then $\bigvee I_A^{[\beta,\beta']}(x \varrho y)) \in \{\beta, \beta'\}$. Thus

$\bigvee T_A^{[\beta,\beta']}(x \ \varrho \ y)) \leq S_{\max}(I_A^{[\beta,\beta']}(x), I_A^{[\beta,\beta']}(y))$. In similar a way, can see that $\bigvee F_A^{[\gamma,\gamma']}(x \ \varrho \ y)) \leq S_{\max}(F_A^{[\gamma,\gamma']}(x), F_A^{[\gamma,\gamma']}(y))$ an by item (i), $A = (T_A^{[\alpha,\alpha']}, I_A^{[\beta,\beta']}, F_A^{[\gamma,\gamma']})$ is a single-valued neutrosophic hyper BCK-subalgebra of X. □

4. Single-valued neutrosophic hyper BCK-ideals of hyper BCK-algebras

In this section, we extended any given nonempty set to a hyper BCK-algebra with at least a single-valued neutrosophic hyper BCK-ideal and investigate their properties. Also, single-valued neutrosophic hyper BCK-ideals are converted to hyper BCK-ideal via valued cuts. The homomorphisms play the main role in the extension of single-valued neutrosophic hyper BCK-ideals and consequently in the extension of hyper BCK-ideals. A fundamental relation is applied to generate single-valued neutrosophic BCK-ideals from single-valued neutrosophic hyper BCK-ideal and so it is considered their properties of via related diagrams. We consider the (weak commutative) hyper BCK-algebras and define a regular equivalence relation on any given hyper BCK-algebras via single-valued neutrosophic hyper BCK-ideals and prove some isomorphism theorems in this regard, that is the major contribution of this section.

Throughout this work, we denote hyper BCK-algebra $(X, \varrho, 0)$ by X.

PROPOSITION 4.1. Let $(X, \varrho, 0)$ be a hyper BCK-algebra and $A = (T_A, I_A, F_A)$ be a single-valued neutrosophic hyper BCK-ideal. Then

$$S_{max}(T_A(0), I_A(0), F_A(x)) \geq S_{max}(T_A(x), I_A(x), F_A(0)).$$

PROOF: Immediate by definition. □

THEOREM 4.2. *Let $0 \in X$ be an arbitrary set. Then X extended to a hyper BCK-algebra $(X, \varrho, 0)$ with at least a single-valued neutrosophic hyper BCK-ideal.*

PROOF: Let $x, y \in X$. Define " ϱ "on X by Theorem 3.5. Clearly, $(X, \varrho, 0)$ is a hyper BCK-algebra. Let $A = (T_A, I_A, F_A)$ be a single-valued neutrosophic set, where $A(0) = (1, 1, 0)$ and $x, y \in X$, then $F_A(0) = 0 \leq F_A(y)$. If $x \neq y$, then

$$F_A(x) \leq S_{max}(F_A(y), F_A(x)) = S_{max}(F_A(y), \bigvee(F_A(x\varrho\, y))).$$

If $0 \neq x = y$, then

$$F_A(x) \leq S_{max}(F_A(y), F_A(x)) = S_{max}(F_A(y), \bigvee(F_A(x\varrho\, y))).$$

In similar a way,
$\forall\, x, y \in X, T_A(x) \geq T_{min}(T_A(y), T_A(x)) = T_{min}(T_A(y), \bigwedge(T_A(x\varrho\, y)))$ and $I_A(x) \geq T_{min}(I_A(y), I_A(x)) = T_{min}(I_A(y), \bigwedge(I_A(x\varrho\, y)))$. Therefore, A is a single-valued neutrosophic hyper BCK-ideal. □

Let $(X, \varrho, 0)$ be a hyper BCK-algebra which is defined in Theorem 4.2 and

$$\mathcal{SVN}hi = \{\mu \mid \mu \text{is a single-valued neutrosophic hyper BCK-ideal on}(X, \varrho, 0)\},$$

then we have the following result.

Corollary 4.3. Let $(X, \varrho, 0)$ be a hyper BCK-algebra. If $|X| \geq 1$, then $|\mathcal{SVN}hi| = |\mathbb{R}|$.

Example 4.4. Let $X = \{-1, -2, -3, -4, -5\} \subseteq \mathbb{Z}$. Then $(X, \varrho\ , -1)$ is a hyper BCK-algebra as follows:

ϱ	-1	-2	-3	-4	-5
-1	$\{-1\}$	$\{-1\}$	$\{-1\}$	$\{-1\}$	$\{-1\}$
-2	$\{-2\}$	$\{-1, -2\}$	$\{-2\}$	$\{-2\}$	$\{-2\}$
-3	$\{-3\}$	$\{-3\}$	$\{-1, -3\}$	$\{-3\}$	$\{-3\}$
-4	$\{-4\}$	$\{-4\}$	$\{-4\}$	$\{-1, -4\}$	$\{-4\}$
-5	$\{-5\}$	$\{-5\}$	$\{-5\}$	$\{-5\}$	$\{-1, -5\}$

Define $A : X \to [0, 1]^3$ by $T_A(x) = I_A(x) = \dfrac{1}{-x}$ and $F_A(x) = \dfrac{1}{x}$. It is easy to see that $A = (T_A, I_A, F_A)$ is a single-valued neutrosophic hyper BCK-ideal.

Theorem 4.5. *Let $(X, \varrho, 0)$ be a hyper BCK-algebra and $A = (T_A, I_A, F_A)$ be a single-valued neutrosophic hyper BCK-ideal of X. Then $\forall\, x, y \in X$ and $Y, Z \subset X$:*

(i) *if* $Y \ll Z$, *then* $\exists\, z \in Z$ *such that* $T_{min}(\bigvee(T_A(Y)), \bigvee(I_A(Y))) \geq T_{min}(T_A(z), I_A(z))$ *and* $\bigwedge(F_A(Y)) \leq F_A(z)$;

(ii) *if* $Y \ll Z$, *then* $T_{min}(\bigvee(T_A(Y)), \bigvee(I_A(Y))) \geq T_{min}(\bigwedge(T_A(Z)), \bigwedge(I_A(Z)))$ *and* $\bigvee(F_A(Z)) \geq \bigwedge(F_A(Y))$;

(iii) $T_{min}(T_A(x), I_A(x)) \leq T_{min}(\bigvee(T_A(x\varrho\, y), \bigvee(I_A(x\varrho\, y))$ *and* $F_A(x) \geq \bigwedge(F_A(x\varrho\, y))$.

(iv) $T_{min}(T_A(x), I_A(x)) \leq T_{min}(\bigwedge(T_A(x\varrho\, y), \bigwedge(I_A(x\varrho\, y))$ *and* $F_A(x) \geq \bigvee(F_A(x\varrho\, y))$.

PROOF: (i) Since $Y \ll Z$, $\forall\, y \in Y$, $\exists\, z \in Z$ such that $y \ll z$. Hence $\bigvee(T_A(Y)) \geq T_A(y) \geq T_A(z)$. In similar a way, $\bigvee(I_A(Y)) \geq I_A(y) \geq I_A(z)$ and so $T_{min}(\bigvee(T_A(Y)), \bigvee(I_A(Y))) \geq T_{min}(T_A(z), I_A(z))$. In addition, $\forall\, y \in Y$, $\exists\, z \in Z$ such that $\bigwedge(F_A(Y)) \leq F_A(y) \leq F_A(z)$.

(ii) Let $Y \ll Z$. Then $\forall\, y \in Y$, $\exists\, z \in Z$ such that $y \ll z$, so $T_A(y) \geq T_A(z), I_A(y) \geq I_A(z)$ and $F_A(y) \leq F_A(z)$. It follows that $\bigvee(T_A(Y)) \geq T_A(y) \geq T_A(z) \geq \bigwedge(T_A(Z))$, $\bigvee(I_A(Y)) \geq I_A(y) \geq I_A(z) \geq \bigwedge(I_A(Z))$ and $\bigvee(F_A(Z)) \geq F_A(z) \geq F_A(y) \geq \bigwedge(F_A(Y))$. Hence $T_{min}(\bigvee(T_A(Y)), \bigvee(I_A(Y))) \geq T_{min}(\bigwedge(T_A(Z)), \bigwedge(I_A(Z)))$ and $\bigvee(F_A(Z)) \geq \bigwedge(F_A(Y))$.

(iii) By Theorem 2.3, $x\varrho\, y \ll x$. Then by (ii), we get that $T_A(x) \leq \bigvee T_A(x\varrho\, y), I_A(x) \leq \bigvee(I_A(x\varrho\, y))$ and $F_A(x) \geq \bigwedge(F_A(x\varrho\, y))$.

(iv) By Theorem 2.3, $x\varrho\, y \ll x$. Then $\forall\, t \in (x\varrho\, y), t \ll x$, we get that $T_A(t) \geq T_A(x)$, so $\bigwedge T_A(x\varrho\, y) \geq T_A(x)$ and similar a way $\bigwedge I_A(x\varrho\, y) \geq I_A(x)$ is obtained. Also $x\varrho\, y \ll x$ implies that $\forall\, t \in (x\varrho\, y), t \ll x$ so $F_A(t) \leq F_A(x)$. Thus $\bigvee(F_A(x\varrho\, y)) \leq F_A(x)$. □

COROLLARY 4.6. Let $(X, \varrho, 0)$ be a hyper BCK-algebra and A be a single-valued neutrosophic hyper BCK-ideal of X. Then $\forall\, x, y \in X$ and $Y, Z \subset X$, get $T_{min}(\bigvee(T_A(Y\ \varrho\ Z)), \bigvee(I_A(Y\ \varrho\ Z))) \geq T_{min}(\bigwedge(T_A(Y)), \bigwedge(I_A(Y)))$ and $\bigvee(F_A(Y)) \geq \bigwedge(F_A(Y\ \varrho\ Z))$.

Let $\alpha, \beta, \gamma \in [0,\ 1]$ and $A = (T_A, I_A, F_A)$ be a single-valued neutrosophic hyper BCK-ideal of X. Define $A^{\lfloor \alpha,\beta,\gamma \rfloor} = T^{(\alpha)} \cap I^{(\beta)} \cap F^{(\gamma)}$, where $T^{(\alpha)} = \{x \in X \mid T_A(x) \geq \alpha\}, I^{(\beta)} = \{x \in X \mid I_A(x) \geq \beta\}$ and $F^{(\gamma)} = \{x \in X \mid F_A(x) \leq \gamma\}$.

THEOREM 4.7. *neutrosophic hyper BCK-ideal is a single-valued neutrosophic hyper BCK-ideal. Let $(X, \varrho, 0)$ be a hyper BCK-algebra and $A = (T_A, I_A, F_A)$ be a single-valued neutrosophic hyper BCK-ideal of X such that $T^{(\alpha)}, I^{(\beta)}, F^{(\gamma)} \neq \emptyset$. Then $\forall\ x, y, z \in X$:*

(i) $0 \in A^{\lfloor \alpha,\beta,\gamma \rfloor}$;

(ii) *if $y \in A^{\lfloor \alpha,\beta,\gamma \rfloor}$ and $x \ll y$, then $x \in A^{\lfloor \alpha,\beta,\gamma \rfloor}$;*

(iii) *$(y\varrho\ z) \ll x$ implies that $T_A(y) \geq T_{min}(T_A(z), T_A(x)), I_A(y) \geq T_{min}(I_A(z), I_A(x)), F_A(y) \leq S_{max}(F_A(z), F_A(x))$;*

(iv) *$A^{\lfloor \alpha,\beta,\gamma \rfloor}$ is a hyper BCK-ideal of X.*

PROOF: (i) There exists $x \in A^{\lfloor \alpha,\beta,\gamma \rfloor}$ such that $T_A(x) \geq \alpha, I_A(x) \geq \beta$ and $F_A(x) \leq \gamma$. By Corollary 4.1, $T_A(0) \geq T_A(x), I_A(0) \geq I_A(x), F_A(0) \leq F_A(x)$, conclude that $0 \in A^{\lfloor \alpha,\beta,\gamma \rfloor}$.

(ii) Since $x \ll y$, by definition, we get that $T_A(x) \geq T_A(y), I_A(x) \geq I_A(y)$ and $F_A(x) \leq F_A(y)$. Now, $y \in A^{\lfloor \alpha,\beta,\gamma \rfloor}$ implies that $x \in A^{\lfloor \alpha,\beta,\gamma \rfloor}$.

(iii) $(y\varrho\ z) \ll x$ implies that $0 \in (y\varrho\ z)\varrho\ x$, then by Theorem 4.5, we get that $T_A(x) \leq \bigwedge(T_A(y\varrho\ z))$, $I_A(x) \leq \bigwedge(I_A(y\varrho\ z))$ and $F_A(x) \geq \bigvee(F_A(y\varrho\ z))$. Now, A is a single-valued neutrosophic hyper BCK-ideal so

$$T_A(y) \geq T_{min}(T_A(z), \bigwedge(T_A(y\varrho\ z))) \geq T_{min}(T_A(z), T_A(x))$$
$$I_A(y) \geq T_{min}(I_A(z), \bigwedge(I_A(y\varrho\ z))) \geq T_{min}(I_A(z), I_A(x))$$
$$F_A(y) \leq S_{max}(F_A(z), \bigvee(F_A(y\varrho\ z))) \leq S_{max}(F_A(z), F_A(x)).$$

(iv) Let $x, y \in X, x\varrho\ y \ll A^{\lfloor \alpha,\beta,\gamma \rfloor}$ and $y \in A^{(\alpha,\beta,\gamma)}$. Then $T_A(y) \geq \alpha, I_A(y) \geq \beta, F_A(y) \leq \gamma$ and by Theorem 4.5,
$\bigwedge(T_A(x\varrho\ y)) \geq \alpha, \bigwedge(I_A(x\varrho\ y)) \geq \beta$ and $\bigvee(F_A(x\varrho\ y)) \leq \gamma$. Hence

$$T_A(x) \geq T_{min}(T_A(y), \bigwedge(T_A(x\varrho\ y))) \geq T_{min}(\alpha, \alpha) = \alpha$$
$$I_A(x) \geq T_{min}(I_A(y), \bigwedge(I_A(x\varrho\ y))) \geq T_{min}(\beta, \beta) = \beta$$
$$F_A(x) \leq S_{max}(F_A(y), \bigvee(F_A(x\varrho\ y))) \geq S_{max}(\gamma, \gamma) = \gamma.$$

Therefore, $x \in A^{\lfloor \alpha,\beta,\gamma \rfloor}$ and so $A^{\lfloor \alpha,\beta,\gamma \rfloor}$ is a hyper BCK-ideal. □

Let $(X, \varrho, 0)$ be a hyper BCK-algebra. A map $f : X \to X$ is called a homomorphism, if $f(0) = 0$ and $\forall\ x, y \in X, f(x\varrho y) = f(x)\varrho f(y)$. If f be an onto homomorphism and $A = (T_A, I_A, F_A)$ a single-valued neutrosophic subset of X. Define $A_f = (T_{A_f}, I_{A_f}, F_{A_f})$ by

$$A_f(x) = (T_A(f(x)), I_A(f(x)), F_A(f(x)).$$

Thus, have the following theorem.

THEOREM 4.8. *Let $(X, \varrho, 0)$ be a hyper BCK-algebra. Then the single-valued neutrosophic set $A = (T_A, I_A, F_A)$, is a single-valued neutrosophic hyper BCK-ideal of X if and only if $A_f = (T_{A_f}, I_{A_f}, F_{A_f})$ is a single-valued neutrosophic hyper BCK-ideal of X.*

PROOF: Let $A = (T_A, I_A, F_A)$ be a single-valued neutrosophic hyper BCK-ideal of X and $x \in X$. Then

$$T_{A_f}(0) = T_A(f(0)) = T_A(0) \geq T_A(f(x)) = T_{A_f}(x)$$
$$I_{A_f}(0) = I_A(f(0)) = I_A(0) \geq I_A(f(x)) = I_{A_f}(x)$$
$$F_{A_f}(0) = F_A(f(0)) = F_A(0) \leq F_A(f(x)) = F_{A_f}(x)$$

and $\forall\ x, y \in X$,

$$\begin{aligned} T_{A_f}(y) = T_A(f(y)) &\geq T_{min}(T_A(f(x)), \bigwedge(T_A(f(y)\varrho\ f(x)))) \\ &= T_{min}(T_A(f(x)), \bigwedge(T_A(f(y\varrho\ x))) \\ &= T_{min}(T_{A_f}(x), \bigwedge(T_{A_f}(y\varrho\ x))). \end{aligned}$$

In similar a way, $I_{A_f}(y) \geq T_{min}(I_{A_f}(x), \bigwedge(I_{A_f}(y\varrho\ x)))$ and $F_{A_f}(y) \leq S_{miax}(F_{A_f}(x), \bigvee(F_{A_f}(y\varrho\ x)))$ are obtained. Hence $A_f = (T_{A_f}, I_{A_f}, F_{A_f})$ is a single-valued neutrosophic hyper BCK-ideal of X.

Conversely, assume that $A_f = (T_{A_f}, I_{A_f}, F_{A_f})$ is a single-valued neutrosophic hyper BCK-ideal of X and $y \in X$. Since f is onto, $\exists\, x \in X$ such that $f(x) = y$. Then

$$\begin{aligned} T_A(0) = T_A(f(0)) = T_{A_f}(0) \geq T_{A_f}(x) = T_A(y) \\ I_A(0) = I_A(f(0)) = I_{A_f}(0) \geq I_{A_f}(x) = I_A(y) \\ F_A(0) = F_A(f(0)) = F_{A_f}(0) \leq F_{A_f}(x) = F_A(y). \end{aligned}$$

Let $x, y \in X$. Then there exists $a, b \in X$ such that $f(a) = x$ and $f(b) = y$. Hence we get that

$$\begin{aligned} T_A(y) = T_A(f(b)) &= T_{A_f}(b) \\ &\geq T_{min}(T_{A_f}(a), \bigwedge(T_{A_f}(b\varrho\ a))) \\ &= T_{min}(T_A(f(a)), \bigwedge(T_A(f(b\varrho\ a)))) \\ &= T_{min}(T_A(f(a)), \bigwedge(T_A(f(b)\varrho\ f(a))) \\ &= T_{min}(T_A(x), \bigwedge(T_A(y\varrho\ x)). \end{aligned}$$

In similar a way, can see that $I_A(y) \geq T_{min}(I_A(x), \bigwedge(I_A(y\varrho\ x))$ and $F_A(y) \leq S_{max}(F_A(x), \bigvee(F_A(y\varrho\ x))$. Therefore $A = (T_A, I_A, F_A)$ is a single-valued neutrosophic hyper BCK-ideal of X. □

THEOREM 4.9. *Let $(X, \varrho, 0)$ be a hyper BCK-algebra, $A = (T_A, I_A, F_A)$ be a single-valued neutrosophic hyper BCK-ideal of X and $f : X \to X$ be a homomorphism,*

- (i) *if $x \in ker(f)$, then $\forall\ y \in X, T_{min}(T_{A_f}(x), I_{A_f}(x)) \geq T_{min}(T_A(y), I_A(y))$ and $F_{A_f}(x) \leq F_A(y)$.*
- (ii) *if at least one of T_A or I_A or F_A is one to one, then $ker(f)$ is a hyper BCK-ideal.*
- (iii) *if $\exists\, x \in X$ such that $A(x) = (1, 1, 0)$, then $A_{(1,0)} = \{x \in X \mid T_A(x) = I_A(x) = 1, F_A(x) = 0\}$ is a hyper BCK-ideal in X.*
- (iv) *$A_{(0,0)}$ is a single-valued neutrosophic hyper BCK-ideal in X.*

PROOF: (i) Let $x \in ker(f)$. Then, $T_{A_f}(x) = T_A(f(x)) = T_A(0)$, $I_{A_f}(x) = I_A(f(x)) = I_A(0)$ and $F_{A_f}(x) = F_A(f(x)) = F_A(0)$. Thus $\forall\ y \in X$, $T_{A_f}(x) \geq T_A(y), I_{A_f}(x) \geq I_A(y)$ and $F_{A_f}(x) \leq F_A(y)$.

(ii) Clearly $0 \in ker(f)$. Let $y \in ker(f)$ and $x\varrho\, y \ll ker(f)$, where $x, y \in X$. Then $T_{A_f}(y) = T_A(0), I_{A_f}(y) = I_A(0), F_{A_f}(y) = F_A(0)$, $\bigwedge(T_{A_f}(x\varrho\, y)) = T_A(0), \bigwedge(I_{A_f}(x\varrho\, y)) = I_A(0)$ and $\bigvee(F_{A_f}(x\varrho\, y)) = F_A(0)$ So

$$T_{A_f}(x) \geq T_{min}(T_{A_f}(y), \bigwedge(T_{A_f}(x\varrho\, y))) = T_{min}(T_A(0), T_A(0)) = T_A(0)$$

$$I_{A_f}(x) \geq T_{min}(I_{A_f}(y), \bigwedge(I_{A_f}(x\varrho\, y))) = T_{min}(I_A(0), I_A(0)) = I_A(0)$$

$$F_{A_f}(x) \leq S_{max}(F_{A_f}(y), \bigvee(F_{A_f}(x\varrho\, y))) = S_{max}(F_A(0), F_A(0)) = F_A(0).$$

Hence $T_{A_f}(x) = T_A(0), I_{A_f}(x) = I_A(0)$ and $F_{A_f}(x) = F_A(0)$. If if at least one of T_A or I_A or F_A is a one to one map, then $x \in ker(f)$.

(iii) Since there exists $x \in X$ such that $A(x) = (1, 1, 0)$, we get that $1 = T_A(x) \leq T_A(0), 1 = I_A(x) \leq I_A(0)$ and $0 = F_A(x) \geq F_A(0)$. Hence $T_A(0) = I_A(0) = 1, F_A(0) = 0$ and so $0 \in A_{(1,0)}$. Now, let $y \in A_{(1,0)}$ and $x\varrho\, y \ll A_{(1,0)}$, where $x, y \in X$. Then, $T_A(y) = I_A(y) = 1, F_A(y) = 0$, $\bigwedge(T_A(x\varrho\, y)) = \bigwedge(I_A(x\varrho\, y)) = 1$ and $\bigvee(F_A(x\varrho\, y)) = 0$. So

$$T_A(x) \geq T_{min}(T_A(y), \bigwedge(T_A(x\varrho\, y))) = T_{min}(1, 1) = 1$$

$$I_A(x) \geq T_{min}(I_A(y), \bigwedge(I_A(x\varrho\, y))) = T_{min}(1, 1) = 1$$

$$F_A(x) \leq S_{max}(F_A(y), \bigvee(F_A(x\varrho\, y))) = S_{max}(0, 0) = 0.$$

Hence $T_A(x) = I_A(x) = 1, F_A(x) = 0$ and so $x \in A_{(1,0)}$.

(iv) Since $A_{(0,0)} = X$, then the proof is clear. □

THEOREM 4.10. *Let* $(X, \varrho, 0)$ *be a hyper* BCK*-algebra,* I *be a hyper* BCK*-ideal and* $A = (T_A, I_A, F_A), A' = (T_{A'}, I_{A'}, F_{A'})$ *be single-valued neutrosophic hyper* BCK*-ideals of* X*. Then*

(i) $X_A = \{x \in X \mid T_A(x) = T_A(0), I_A(x) = I_A(0), F_A(x) = F_A(0)\}$ *is a hyper* BCK*-ideal of* X*;*

(ii) *if* $A'(0) = A(0)$*, then* $X_{A'}\varrho\, X_A = \bigcup_{\substack{a' \in X_{A'} \\ a \in X_A}} (a'\varrho\, a)$*, is a hyper* BCK*-ideal;*

(*iii*) *X_A is a hyper BCK-ideal of $X_A \varrho\, X_{A'}$;*

(*iv*) *if A is restricted to I, then A is a single-valued neutrosophic hyper BCK-ideal of I.*

PROOF: (*i*) Let $x, y \in X$ such that $x\varrho\, y \ll X_A$ and $y \in X_A$. Then $T_A(y) = T_A(0), I_A(y) = I_A(0), F_A(y) = F_A(0), \bigwedge(T_A(x\varrho\, y)) = T_A(0)$, $\bigwedge(I_A(x\varrho\, y)) = I_A(0)$ and $\bigvee(F_A(x\varrho\, y)) = F_A(0)$, So $T_A(x) \geq T_{min}\{T_A(y)$, $\bigwedge(T_A(x\varrho\, y))\} = T_A(0), I_A(x) \geq T_{min}\{I_A(y), \bigwedge(I_A(x\varrho\, y))\} = I_A(0)$ and $F_A(x) \leq S_{max}\{F_A(y), \bigvee(F_A(x\varrho\, y))\} = F_A(0)$. So $T_A(x) = T_A(0), I_A(x) = I_A(0), F_A(x) = F_A(0)$, hence $x \in X_A$ and X_A is a hyper BCK-ideal.

(*ii*) Clearly $0 \in X_{A'}\varrho\, X_A$. Let $t, t' \in X$ such that $t'\varrho\, t \ll X_{A'}\varrho\, X_A$ and $t \in X_{A'}\varrho\, X_A$. Then there exist $a' \in X_{A'}$ and $a \in X_A$ such that $t \in a'\varrho\, a$ so by Theorem 4.5,

$$\begin{aligned} & T'_A(t) \geq \bigwedge(T'_A(a'\varrho\, a)) \geq T'_A(a') = T'_A(0), I'_A(t) \geq \bigwedge(I'_A(a'\varrho\, a)) \\ \geq\ & I'_A(a') = I'_A(0)F'_A(t) \leq \bigvee(F'_A(a'\varrho\, a)) \leq F'_A(a') = F'_A(0) \end{aligned}$$

and so

$$\begin{aligned} T'_A(t') &\geq T_{min}(T'_A(t), \bigwedge(T'_A(t'\varrho\, t))) \geq T_{min}(T'_A(t), T'_A(0)) \\ I'_A(t') &\geq T_{min}(I'_A(t), \bigwedge(I'_A(t'\varrho\, t))) \geq T_{min}(I'_A(t), I'_A(0)) \\ F'_A(t') &\leq S_{max}(F'_A(t), \bigwedge(F'_A(t'\varrho\, t))) \geq S_{max}(F'_A(t), F'_A(0)). \end{aligned}$$

Hence $t' \in X_{A'}$ and so $t' \in t'\varrho\, 0 \subseteq X_{A'}\varrho\, X_A$. Therefore $X_{A'}\varrho\, X_A$ is a hyper BCK-ideal in X.

(*iii*) Let $x \in X_A$. Since $x \in x\varrho\, 0$, we get that $x \in X_A \subseteq X_A\varrho\, X_{A'}$ and by (*i*), X_A is a hyper BCK-ideal of $X_A\varrho\, X_{A'}$.

(*iv*) The proof is clear. □

Let X be a hyper BCK-algebra and $x, y \in X$. Then $x\beta y \Leftrightarrow \exists n \in \mathbb{N}, (a_1, \ldots, a_n) \in X^n$ and $\exists u \in \varrho(a_1, \ldots, a_n)$ such that $\{x, y\} \subseteq u$. The relation β is a reflexive and symmetric relation, but not transitive relation. Let $C(\beta)$ be the *transitive closure* of β (the smallest transitive relation such that contains β). Hamidi, et.al in [1], proved that for any given weak commutative hyper BCK-algebra X, $C(\beta)$ is a strongly regular relation on X and $(X/C(\beta), \vartheta, \overline{0})$ is a BCK-algebra, where $C(\beta)(x)\vartheta\ C(\beta)(y) = C(\beta)(x\ \varrho\ y)$ and $\overline{0} = C(\beta)(0)$.

Theorem 4.11. *Let $(X, \varrho, 0)$ be a hyper BCK-algebra. If $A = (T_A, I_A, F_A)$ is a single-valued neutrosophic hyper BCK-ideal of X, then there exists a single-valued neutrosophic hyper BCK-ideal $\overline{A} = (\overline{T_A}, \overline{I_A}, \overline{F_A})$ of $(X/C(\beta), \vartheta, \overline{0})$ such that $\forall\ x, y \in X$,*

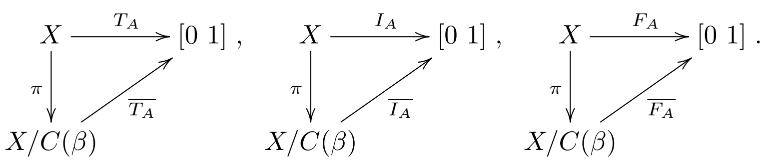
(i) $\overline{A}(\overline{0}) \geq \overline{A}(C(\beta)(x))$;

(ii) $\overline{T_A}(C(\beta)(y)) \geq T_{min}(\overline{T_A}(C(\beta)(x), \bigwedge \big(\overline{T_A}(\vartheta(C(\beta)(y), C(\beta)(x)))\big)$,

(iii) $\overline{I_A}(C(\beta)(y)) \geq T_{min}(\overline{I_A}(C(\beta)(x), \bigwedge \big(\overline{I_A}(\vartheta(C(\beta)(y), C(\beta)(x)))\big)$,

(iv) $\overline{F_A}(C(\beta)(y)) \leq S_{max}(\overline{F_A}(C(\beta)(x), \bigwedge \big(\overline{F_A}(\vartheta(C(\beta)(y), C(\beta)(x)))\big)$.

Proof: *(i)* We define $\overline{A} : X/C(\beta) \to [0,\ 1]^3$ by $(\overline{T_A}(C(\beta)(t)), \overline{I_A}(C(\beta)(t)), \overline{F_A}(C(\beta)(t))) = (\bigvee\limits_{t\ C(\beta)\ x} T_A(x), \bigvee\limits_{t\ C(\beta)\ x} I_A(x), \bigwedge\limits_{t\ C(\beta)\ x} F_A(x))$, where $x, t \in X$. Consider the following diagram:



Firstly we show that $\overline{A}$ is well-defined. Let $t, t', x \in X$ and $C(\beta)(t) = C(\beta)(t')$. Then $t\ C(\beta)\ t'$ and

$$\overline{T_A}(C(\beta)(t)) = \bigvee_{x\ C(\beta)\ t} T_A(x) = \bigvee_{x\ C(\beta)\ t'} T_A(x) = \overline{T_A}(C(\beta)(t'))$$

$$\overline{I_A}(C(\beta)(t)) = \bigvee_{x\ C(\beta)\ t} I_A(x) = \bigvee_{x\ C(\beta)\ t'} I_A(x) = \overline{I_A}(C(\beta)(t'))$$

$$\overline{F_A}(C(\beta)(t)) = \bigwedge_{x\ C(\beta)\ t} F_A(x) = \bigwedge_{x\ C(\beta)\ t'} F_A(x) = \overline{F_A}(C(\beta)(t')).$$

In addition, $\forall\ x, t \in X$, we get that

$$\overline{T_A}(C(\beta)(0)) = \bigvee_{t\ C(\beta)\ 0} T_A(t) = T_A(0) \geq \bigvee_{t\ C(\beta)\ x} T_A(t) = \overline{T_A}(C(\beta)(x))$$

$$\overline{I_A}(C(\beta)(0)) = \bigvee_{t\ C(\beta)\ 0} T_A(t) = I_A(0) \geq \bigvee_{t\ C(\beta)\ x} I_A(t) = \overline{I_A}(C(\beta)(x))$$

$$\overline{F_A}(C(\beta)(0)) = \bigwedge_{t\ C(\beta)\ 0} F_A(t) = F_A(0) \leq \bigwedge_{t\ C(\beta)\ x} F_A(t) = \overline{F_A}(C(\beta)(x)).$$

(ii) Let $x, y \in X$. Since $\forall\ t \in C(\beta)(y)$ and $\forall\ t' \in C(\beta)(x)$,

$$\bigvee_{t\ C(\beta)\ y} T_A(t) \geq T_A(t) \geq T_{min}(T_A(t'), \bigwedge(T_A(t\varrho\ t')))$$

$$\bigvee_{t\ C(\beta)\ y} I_A(t) \geq I_A(t) \geq T_{min}(I_A(t'), \bigwedge(I_A(t\varrho\ t')))$$

$$\bigwedge_{t\ C(\beta)\ y} F_A(t) \leq F_A(t) \leq S_{max}(F_A(t'), \bigwedge(F_A(t\varrho\ t')))$$

we get that

$$\begin{aligned}
\overline{T_A}(C(\beta)(y)) &= \bigvee_{t\ C(\beta)\ y} T_A(t) \\
&\geq \bigvee_{\substack{t'\in C(\beta)(x) \\ t\ C(\beta)\ y}} (T_{min}(T_A(t'), \bigwedge(T_A(t\varrho\ t'))) \\
&\geq T_{min}(\bigvee_{t'\in C(\beta)(x)} T_A(t'), \bigvee_{t'\in C(\beta)(x)} \bigwedge_{t\in C(\beta)(y)} (T_A(t\varrho\ t'))) \\
&\geq T_{min}(\bigvee_{t'\in C(\beta)(x)} T_A(t'), \bigwedge_{m\in\vartheta(C(\beta)(y), C(\beta)(x))} \bigvee(T_A(m)) \\
&\geq T_{min}(\overline{T_A}(C(\beta)(x), \bigwedge(\overline{T_A}(\vartheta(C(\beta)(y), C(\beta)(x)))).
\end{aligned}$$

(iii, iv) Similar to item (ii), can see that
$\overline{I_A}(C(\beta)(y)) \geq T_{min}(\overline{I_A}(C(\beta)(x), \bigwedge(\overline{I_A}(\vartheta(C(\beta)(y), C(\beta)(x)))))$ and
$\overline{F_A}(C(\beta)(y)) \leq S_{max}(\overline{F_A}(C(\beta)(x), \bigvee(\overline{F_A}(\vartheta(C(\beta)(y), C(\beta)(x))))$. □

Let $(Y, \vartheta, 0, \preceq)$ be a BCK-algebra and $B = (T_B, I_B, F_B)$ a single-valued neutrosophic subset of Y. Then $B = (T_B, I_B, F_B)$ is called a single-valued neutrosophic BCK-ideal of Y, if (1);$\forall\ x, y \in Y, x \preceq y \Rightarrow T_A(x) \geq T_A(y), I_A(x) \geq I_A(y)$ and $F_A(x) \leq F_A(y)$,

(2); $T_A(x) \geq T_{min}\{T_A(y), T_A(x\vartheta\ y)\}, I_A(x) \geq T_{min}\{I_A(y), I_A(x\vartheta\ y)\}$ and $F_A(x) \leq S_{max}\{F_A(y), F_A(x\vartheta\ y)\}$.

Corollary 4.12. Let $(X, \varrho, 0)$ be a weak commutative hyper BCK-algebra. If $A = (T_A, I_A, F_A)$ is a single-valued neutrosophic hyper BCK-ideal of X, then there exists a single-valued neutrosophic BCK-ideal $B = (T_B, I_B, F_B)$ of BCK-algebra $(X/C(\beta), \vartheta, \overline{0})$, such that $T_B \circ \pi \geq T_A, I_B \circ \pi \geq I_A$, and $F_B \circ \pi \leq F_A$.

Proof: By Theorem 4.11, consider $B = \overline{T_A}$. For any $x \in X$, since $xC(\beta)x$, we get that $(T_B \circ \pi)(x) = T_B(C(\beta)(x)) = \bigvee_{t\ C(\beta)\ x} T_A(t) \geq T_A(x), (I_B \circ \pi)(x) = I_B(C(\beta)(x)) = \bigvee_{t\ C(\beta)\ x} I_A(t) \geq I_A(x)$ and

$$(F_B \circ \pi)(x) = F_B(C(\beta)(x)) = \bigwedge_{t\ C(\beta)\ x} F_A(t) \leq F_A(x). \qquad \square$$

Example 4.13. Let $X = \{0, b, c, d\}$. Then $A = (T_A, I_A, F_A)$ is a single-valued neutrosophic hyper BCK-subalgebra of hyper BCK-algebra $(X, \varrho, 0)$ as follows:

ϱ	0	b	c	d
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{0\}$	$\{0\}$	$\{0\}$
c	$\{c\}$	$\{c\}$	$\{0\}$	$\{0\}$
d	$\{d\}$	$\{d\}$	$\{c\}$	$\{0, c\}$

and

	0	b	c	d
T_A	1	0.9	0.3	0.3
I_A	1	0.9	0.3	0.3
F_A	0.1	0.15	0.25	0.25

.

Clearly (X, ϱ, A) is not weak commutative and T is a single-valued neutrosophic hyper BCK-ideal. Now we get that $X/C(\beta) = \{C(\beta)(0) = \{0, c\}, C(\beta)(b) = \{b\}, C(\beta)(d) = \{d\}\}$,

ϑ	$C(\beta)(0)$	$C(\beta)(b)$	$C(\beta)(d)$
$C(\beta)(0)$	$C(\beta)(0)$	$C(\beta)(0)$	$C(\beta)(0)$
$C(\beta)(b)$	$C(\beta)(b)$	$C(\beta)(0)$	$C(\beta)(0)$
$C(\beta)(d)$	$C(\beta)(d)$	$C(\beta)(d)$	$C(\beta)(0)$

and

	$C(\beta)(0)$	$C(\beta)(b)$	$C(\beta)(d)$
$\overline{T_A}$	1	0.9	0.3
$\overline{I_A}$	1	0.9	0.3
$\overline{F_A}$	0.1	0.25	0.25

It is easy to see that $(X/C(\beta), \vartheta, C(\beta)(0), \overline{A})$ is a hyper BCK-algebra.

DEFINITION 4.14. Let $(X, \varrho, 0)$ be a hyper BCK-algebra and $A = (T_A, I_A, F_A)$ be a single-valued neutrosophic hyper BCK-ideal of X. For any $x, y \in X$, define binary relations $R^{T_A}, R^{I_A}, R^{F_A}$ on X as follows:

$$xR^{T_A}y \Leftrightarrow T_A(x) \leq T_A(y) \text{ and } \bigwedge(T_A(\varrho(x,y))) \geq T_A(y)$$
$$xR^{I_A}y \Leftrightarrow \ I_A(x) \leq I_A(y) \text{ and } \bigwedge(I_A(\varrho(x,y))) \geq I_A(y)$$
$$xR^{F_A}y \Leftrightarrow \ F_A(x) \geq F_A(y)$$
$$\text{and } \bigvee(F_A(\varrho(x,y))) \leq F_A(y) \text{ and } R = R^{T_A} \cap R^{I_A} \cap R^{F_A}.$$

THEOREM 4.15. *Let $(X, \varrho, 0)$ be a hyper BCK-algebra, $A = (T_A, I_A, F_A)$ be a single-valued neutrosophic hyper BCK-ideal of X and $x, y \in X$.*

(*i*) *R is an equivalence relation on X.*

(*ii*) *if A is one to one and xRy, then $\forall\ z \in X$ we have $(x\varrho\ z)R(y\varrho\ z)$ and $(z\varrho\ x)R(z\varrho\ y)$.*

(*iii*) *if A is one to one, xRy and uRw then $(x\varrho\ u)R(y\varrho\ w)\ \forall\ u, w \in X$.*

PROOF: (*i*) By Theorem 4.5, $T_A(x) \leq \bigwedge(T_A(x\varrho\, x)), I_A(x) \leq \bigvee(I_A(x\varrho\, x))$, $F_A(x) \geq \bigwedge(F_A(x\varrho\ x))$ and so R is a reflexive relation. Let $x, y \in X$ such that xRy. Then $T_A(x) \leq T_A(y), I_A(x) \leq I_A(y)$, $F_A(x) \geq F_A(y)$, $\bigwedge(T_A(\varrho(x,y))) \geq T_A(y), \bigwedge(I_A(\varrho(x,y))) \geq I_A(y)$ and $\bigvee(F_A(\varrho(x,y))) \leq F_A(y)$. Since

$$T_A(x) \geq T_{min}(T_A(y), \bigwedge(T_A(x\varrho\ y))) \geq T_{min}(T_A(y), T_A(y)) = T_A(y)$$
$$I_A(x) \geq T_{min}(I_A(y), \bigwedge(I_A(x\varrho\ y))) \geq T_{min}(I_A(y), I_A(y)) = I_A(y)$$
$$F_A(x) \leq S_{max}(F_A(y), \bigvee(F_A(x\varrho\ y))) \leq S_{max}(F_A(y), F_A(y)) = F_A(y)$$

we get that $T_A(x) = T_A(y), I_A(x) = I_A(y), F_A(x) = F_A(y)$. Using Theorem 4.5, $\bigwedge(T_A(y\varrho\ x)) \geq T_A(y) = T_A(x), \bigwedge(I_A(y\varrho\ x)) \geq I_A(y) = I_A(x)$ and $\bigvee(F_A(y\varrho\ x)) \leq F_A(y) = F_A(x)$ so R is a symmetric relation. Let xRy and yRz. Then $T_A(x) = T_A(y) = T_A(z), I_A(x) = I_A(y) = I_A(z), F_A(x) = F_A(y) = F_A(z)$ and clearly R is a transitive relation.

(ii) Let xRy and $z \in X$. Then by (i), $T_A(x) = T_A(y), I_A(x) = I_A(y), F_A(x) = F_A(y)$ and since A is a one to one map, we have $x = y$. Hence there exists $a \in x\varrho\ z$ and $y \in y\varrho\ z$ such that $T_A(a) \leq T_A(b)$, $\bigwedge(T_A(a\varrho\ b)) \geq T_A(b), I_A(a) \leq I_A(b), \bigwedge(I_A(a\varrho\ b)) \geq I_A(b)$ and $F_A(a) \geq F_A(b), \bigvee(F_A(a\varrho\ b)) \leq F_A(b)$. Therefore $(x\varrho\ z)R(y\varrho\ z)$ and in a similar way get that $(z\varrho\ x)R(z\varrho\ y)$.

(iii) Let xRy and uRw. Then by (ii), $(x\varrho\ u)R(y\varrho\ u)$ and $(y\varrho\ u)R(y\varrho\ w)$. Using the transitivity of R, we get that $(x\varrho\ u)R(y\varrho\ w)$. □

Corollary 4.16. Let $(X, \varrho, 0)$ be a hyper BCK-algebra and $A = (T_A, I_A, F_A)$ be a single-valued neutrosophic hyper BCK-ideal of X and $x, y \in X$.

(i) if A is one to one, then R is a congruence relation on X;

(ii) $R(0) = X_A$ and if A is one to one, then $R(0) = \{0\}$;

(iii) if A is one to one, then R is a strongly regular relation on X.

Proof: (i) Immediate by Theorem 4.15.

(ii) Let $x \in R(0)$. Then by Theorem 4.15, $T_A(x) = T_A(0), I_A(x) = I_A(0), F_A(x) = F_A(0)$ and so $R(0) = X_A$. Since A is one to one, we get that $X_A = \{x \mid T_A(x) = T_A(0), I_A(x) = I_A(0), F_A(x) = F_A(0)\} = \{0\}$.

(iii) Let $x, y, z \in X$ and xRy. Then $x = y$ and so $x\varrho\ z = y\varrho\ z$. Therefore $(x\varrho\ z)\overline{\overline{R}}(y\varrho\ z), (z\varrho\ x)\overline{\overline{R}}(z\varrho\ y)$ and so R is a strongly regular relation. □

Theorem 4.17. *Let $(X, \varrho, 0)$ be a (weak commutative) hyper BCK-algebra and $A = (T_A, I_A, F_A)$ be a one to one single-valued neutrosophic hyper BCK-ideal of X. Then, $(X/R, \varrho', R(0))$ is a (BCK-algebra) hyper BCK-algebra such that $\forall\ x, y \in X, R(x)\varrho' R(y) = R(x\varrho\ y)$.*

Proof: By Corollary 4.16, ϱ' is well-defined and the proof is straightforward. □

Theorem 4.18. *Let $(X, \varrho_1, 0)$ and $(Y, \varrho_2, 0')$ be (weak commutative) hyper BCK-algebras and $A = (T_A, I_A, F_A)$ be a one to one single-valued neutrosophic hyper BCK-ideal of Y. If $f : X \to Y$ is an epimorphism, then* (*i*) A_f *is a single-valued neutrosophic hyper* BCK*-ideal of* X*;*

(*ii*) $X/R_f \cong Y/R$ *such that* $xR_fx' \Leftrightarrow T_A(f(x)) \leq T_A(f(x')), I_A(f(x)) \leq I_A(f(x')), F_A(f(x)) \geq F_A(f(x')), \bigwedge(T_A(f(x\varrho\ x'))) \geq T_A(f(x')),$ $\bigwedge(I_A(f(x\varrho\ x'))) \geq I_A(f(x'))$ *and* $\bigvee(F_A(f(x\varrho\ x'))) \leq F_A(f(x')),$ *where* $x, x' \in X$*.*

Proof: (*i*) Clearly for all $x \in X$, $T_{A_f}(0) = T_A(f(0)) = T_A(0') \geq T_A(f(x)) = T_{A_f}(x), I_{A_f}(0) = I_A(f(0)) = I_A(0') \geq I_A(f(x)) = I_{A_f}(x)$ and $F_{A_f}(0) = F_A(f(0)) = F_A(0') \leq F_A(f(x)) = F_{A_f}(x)$. Let $x, x' \in X$. Since A is a single-valued neutrosophic hyper BCK-ideal of Y, we get that

$$\begin{aligned} T_{A_f}(x) = T_A(f(x)) &\geq T_{min}\{T_A(f(x')), \bigwedge(T_A(f(x)\varrho_2 T_A(f(x')))\} \\ &= T_{min}\{T_A(f(x'), \bigwedge(T_A(f(x\varrho_1 x')))\} \\ &= T_{min}\{T_{A_f}(x'), \bigwedge(T_{A_f}(x\varrho_1 x'))\}. \end{aligned}$$

In similar a way, can see that $I_{A_f}(x) \geq T_{min}\{I_{A_f}(x'), \bigwedge(I_{A_f}(x\varrho_1 x'))\}$ and $I_{A_f}(x) \leq S_{max}\{F_{A_f}(x'), \bigvee$
$(F_{A_f}(x\varrho_1 x'))\}$.

(*ii*) Since T_A and T_{A_f} are single-valued neutrosophic hyper BCK-ideals of Y, X, respectively, then by Theorem 4.17, $(X/R_f, \varrho\,', R_f(0))$ and $(Y/R, \varrho\,', R(0'))$ are (BCK-algebras) hyper BCK-algebras. Now, define a map $\varphi : X/R_f \to Y/R$ by $\varphi(R_f(x)) = R(f(x))$. Let $x, x' \in X$. Then

$$\begin{aligned} \varphi(R_f^{T_A}(x)) &= \varphi(R_f^{T_A}(x')) \Leftrightarrow f(x)R^{T_A}f(x') \\ &\Leftrightarrow T_A(f(x)) \leq T_A(f(x')), \bigwedge(T_A(f(x)\varrho\,_2 f(x'))) \geq T_A(f(x')) \\ &\Leftrightarrow T_{A_f}(x) \leq T_{A_f}(x') \text{ and } \bigwedge(T_A(f(x\varrho_1 x'))) \geq T_A(f(x')) \\ &\Leftrightarrow T_{A_f}(x) \leq T_{A_f}(x') \text{ and } \bigwedge(T_{A_f}(x\varrho_1 x')) \geq T_{A_f}(x') \\ &\Leftrightarrow R_f^{T_A}(x) = R_f^{T_A}(x'). \end{aligned}$$

In similar a way, $\varphi(R_f^{I_A}(x)) = \varphi(R_f^{I_A}(x')) \Leftrightarrow R_f^{I_A}(x) = R_f^{I_A}(x')$ and $\varphi(R_f^{F_A}(x)) = \varphi(R_f^{F_A}(x')) \Leftrightarrow R_f^{F_A}(x) = R_f^{F_A}(x')$. It follows that $\varphi(R_f(x)) = \varphi(R_f(x')) \Leftrightarrow R_f(x) = R_f(x')$ and hence φ is a well-defined and one to one map. Clearly φ is an epimorphism, and so it is an isomorphism. □

COROLLARY 4.19. (Isomorphism Theorem) Let $(X, \varrho, 0)$ be a hyper BCK-algebra and $A = (T_A, I_A, F_A), A' = (T'_A, I'_A, F'_A)$ be one to one single-valued neutrosophic hyper BCK-ideals of X such that $A(0) = A'(0)$. Then
(i) $A' \cap A$ is a single-valued neutrosophic hyper BCK-ideal of X;
(ii) $(X_A \varrho X_{A'})/R_A \cong X_A/R_{A' \cap A}$.

PROOF: (i) Let $x \in X$. Then

$$(T'_A \cap T_A)(0) = T_{min}(T'_A(0), T_A(0)) \geq T_{min}(T'_A(x), T_A(x)) = (T'_A \cap T_A)(x), (I'_A \cap I_A)(0) = T_{min}(I'_A(0), I_A(0)) \geq T_{min}(I'_A(x), I_A(x)) = (I'_A \cap I_A)(x), (F'_A \cap F_A)(0) = S_{max}(F'_A(0), F_A(0)) \leq S_{max}(F'_A(x), F_A(x)) = (F'_A \cap F_A)(x).$$

Let $x, y \in X$. Then

$$\begin{aligned} & (T'_A \cap T_A)(x) = T_{min}(T'_A(x), T_A(x)) \\ \geq \; & T_{min}[T_{min}[T'_A(y), \bigwedge(T'_A(x\varrho\, y))], T_{min}[T_A(y), \bigwedge(T_A(x\varrho\, y))]] \\ = \; & T_{min}[T_{min}[T'_A(y), T_A(y)], T_{min}[\bigwedge(T'_A(x\varrho\, y)), \bigwedge(T_A(x\varrho\, y))]] \\ = \; & T_{min}[(T'_A \cap T_A)(y), \bigwedge((T'_A \cap T_A)(x\varrho\, y))] \end{aligned}$$

In similar a way can see that $(I'_A \cap I_A)(x) \geq T_{min}[(I'_A \cap I_A)(y), \bigwedge((I'_A \cap I_A)$ and $(F'_A \cap F_A)(x) \leq S_{max}[(F'_A \cap F_A)(y), \bigvee((F'_A \cap F_A)$.

(ii) By Theorem 4.10, $A' \cap A$ is a single-valued neutrosophic hyper BCK-ideal of X_A, then we define $\varphi : X_A/R_{A' \cap A} \to (X_A \varrho\; X_{A'})/R_A$ by $\varphi(R_{A' \cap A}(x)) = R_A(x)$. Let $x, x' \in X_A$ and $R_{A' \cap A}(x) = R_{A' \cap A}(x')$. Then $(A' \cap A)(x) = (A' \cap A)(x')$ and since $A' \cap A$ is one to one, we get that $x = x'$. Hence $R_A(x) = R_A(x')$. Moreover, $\varphi(R_{A' \cap A}(x)\varrho\;' R_{A' \cap A}(x')) = \varphi(R_{A' \cap A}(x\varrho\; x')) = R_A(x\varrho\; x') = R_A(x)\varrho\;' R_A(x')$ and so φ is a homomorphism. Clearly φ is bijection and so is an isomorphism. □

Example 4.20. Let $X = \{0, 1, 2, 3, 4, 5\}$. Then $A = (T_A, I_A, F_A)$ is a single-valued neutrosophic hyper BCK-subalgebra of X hyper BCK-algebra $(X, \varrho, 0)$ as follows:

ϱ	0	1	2	3	4	5
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
2	$\{2\}$	$\{2\}$	$\{0,2\}$	$\{2\}$	$\{2\}$	$\{2\}$
3	$\{3\}$	$\{3\}$	$\{3\}$	$\{0,3\}$	$\{3\}$	$\{3\}$
4	$\{4\}$	$\{4\}$	$\{4\}$	$\{4\}$	$\{0,4\}$	$\{0\}$
5	$\{5\}$	$\{5\}$	$\{5\}$	$\{5\}$	$\{5\}$	$\{0,5\}$

and

	0	1	2	3	4	5
T_A	0.72	0.61	0.54	0.34	0.27	0.19
I_A	0.19	0.8	0.2	0.21	0.26	0.25
F_A	0.15	0.28	0.34	0.39	0.48	0.61

(i) If $\alpha = 0.5, \beta = 0.7, \gamma = 0.4$, then $T^{\alpha} = \{0,1,2\}, I^{\beta} = \{0,2,3,4,5\}$, $F^{\gamma} = \{0,1,2,3\}$ and so $A^{(\alpha,\beta,\gamma)} = \{0,2\}$, which is a hyper BCK-subalgebra of $(X,\varrho,0)$.

(ii) Consider $S = \{0,5\}$, $\alpha = 0.5, \alpha' = 0.7, \beta = 0.6, \beta' = 0.8, \gamma = 0.85$ and $\gamma' = 0.9$. Then

$$T_A^{[\alpha,\alpha']} = \{(0,0.5),(1,0.7),(2,0.7),(3,0.7),(4,0.7),(5,0.5)\}$$
$$I_A^{[\beta,\beta']} = \{(0,0.6),(1,0.8),(2,0.8),(3,0.8),(4,0.8),(5,0.6)\}$$
$$F_A^{[\gamma,\gamma\beta']} = \{(0,0.85),(1,0.9),(2,0.9),(3,0.9),(4,0.9),(5,0.85)\}$$

are fuzzy hyper BCK-subalgebras of X and $A = (T_A^{[\alpha,\alpha']}, I_A^{[\beta,\beta']}, F_A^{[\gamma,\gamma']})$ is a single-valued neutrosophic hyper BCK-subalgebra of X.

(iii) Let $\alpha = 0.3, \beta = 0.1$ and $\gamma = 0.5$. Then $A^{\lfloor\alpha,\beta,\gamma\rfloor} = T^{(\alpha)} \cap I^{(\beta)} \cap F^{(\gamma)} = \{0,1,2,3\} \cap \{0,1,2,3,4,5\} \cap \{0,1,2,3,4\} = \{0,1,2,3\}$. Clearly $2 \in A^{\lfloor\alpha,\beta,\gamma\rfloor}$ and $0 \ll 2$, then $0 \in A^{\lfloor\alpha,\beta,\gamma\rfloor}$.

Example 4.21. Let $X = \{0,1,2,3\}$ and $Y = \{0',a,b,c\}$. Then $A = (T_A, I_A, F_A)$ is a single-valued neutrosophic hyper BCK-ideal of hyper BCK-algebra $(X,\varrho,0)$ as follows:

ϱ	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$	$\{1\}$
2	$\{2\}$	$\{2\}$	$\{0,2\}$	$\{2\}$
3	$\{3\}$	$\{3\}$	$\{3\}$	$\{0,3\}$

ϱ'	$0'$	a	b	c
$0'$	$\{0'\}$	$\{0'\}$	$\{0'\}$	$\{0'\}$
a	$\{a\}$	$\{0',a\}$	$\{a\}$	$\{a\}$
b	$\{b\}$	$\{b\}$	$\{0',b\}$	$\{b\}$
c	$\{c\}$	$\{c\}$	$\{c\}$	$\{0',c\}$

and

	0	1	2	3
T_A	0.93	0.73	0.13	0.13
I_A	0.87	0.67	0.1	0.05
F_A	0.13	0.23	0.33	0.4

(i) Define $f : Y \to X$ by $f = \{(0', 0), (c, 1), (b, 2), (a, 3)\}$, clearly f is a homomorphism. Hence A_f is a single-valued neutrosophic hyper BCK-ideal of hyper BCK-algebra $(Y, \varrho', 0')$ that is obtained as follows:

	$0'$	a	b	c
T_{A_f}	0.93	0.13	0.13	0.73
I_{A_f}	0.87	0.05	0.1	0.67
F_{A_f}	0.13	0.4	0.33	0.23

(ii) Computations show that $R^{T_A} = \{(x, x), (2, 3), (3, 2) \mid x \in X\}, R^{I_A} = \{(x, x) \mid x \in X\}, R^{F_A} = \{(x, x) \mid x \in X\}$ and so $R = \{(x, x) \mid x \in X\}$ that is a congruence relation. It follows that $R_f = \{(x, x) \mid x \in Y\}$ and so $X/R \cong X \cong Y \cong Y/R_f$.

(iii) Clearly $X_A = R(0) = \{0\}$ and $ker(f) = \{0\}$ that is a trivial (hyper) BCK-ideal. Also for all $x \in ker(f)$ and for all $y \in X, T_{min}(T_{A_f}(x), I_{A_f}(x)) \geq T_{min}(T_A(y), I_A(y))$.

Example 4.22. Let $X = \{0, 1, 2, 3\}$. Then $A = (T_A, I_A, F_A)$ and $A' = (T_{A'}, I_{A'}, F_{A'})$ are single-valued neutrosophic hyper BCK-ideals of hyper BCK-algebra $(X, \varrho, 0)$ as follows:

ϱ	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$	$\{0\}$
2	$\{2\}$	$\{2\}$	$\{0, 2\}$	$\{0\}$
3	$\{3\}$	$\{3\}$	$\{3\}$	$\{0\}$

	0	1	2	3
T_A	0.95	0.85	0.25	0.25
I_A	0.9	0.8	0.2	0.1
F_A	0.15	0.2	0.3	0.35

and

	0	1	2	3
$T_{A'}$	0.95	0.75	0.15	0.15
$I_{A'}$	0.9	0.7	0.1	0.05
$F_{A'}$	0.15	0.25	0.35	0.4

Then

$$A \cap A' = A', X_A = X_{A'} = \{0\}, R^{T_A} = R^{T_{A'}} = \{(x,x),(2,3),(3,2)|x \in X\},$$
$$R^{I_A} = R^{I_{A'}} = \{(x,x),(2,3),(3,2) \mid x \in X\}, R^{F_A} = R^{F_{A'}} =$$
$$\{(x,x) \mid x \in X\} \text{ and so } R_A = R^{T_A} \cap R^{I_A} \cap R^{F_A} = R_{A'} =$$
$$R^{T_{A'}} \cap R^{I_{A'}} \cap R^{F_{A'}} = \{(x,x) \mid x \in X\}.$$

It follows that $(X_A \varrho X_{A'}) = \{0\}$ and so $(X_A \varrho X_{A'})/R_A = \{0\}/R_A \cong \{0\}/R_{A'} \cong \{0\}/R_{A' \cap A} = X_A/R_{A' \cap A}$.

5. Conclusion

In some problems in the real world, there are many uncertainties (such as fuzziness, incompatibilities, and randomness), in an expert system, belief system, and information fusion, especially in some scopes of computer sciences such as artificial intelligence. Thus we need to deal with uncertain information and logic establishes the foundations for it, because computer sciences are based on classical logic. The concept of BCK-algebra is one of the important logical algebras that are applied in computer sciences and other networking sciences. In addition, defects in classical algebras that can not work in groups and have limitations can be eliminated with the help of logical hyperalgebra. Thus the concept of hyper BCK-algebra is an important logical hyperalgebra that is applied in the computer sciences and other hypernetworking sciences that some groups of elements must be operated together and have been proposed for semantical hypersystems of logical hypersystems. In addition in some applications such as expert systems, belief systems, and information fusion, we should consider not only the truth membership supported by the evidence but also the falsity-membership against the evidence, which is beyond the scope of fuzzy subsets. Thus the concept of a neutrosophic subset is a powerful general formal framework that generalizes the concept of the classic set and the fuzzy subset is characterized by a truth-membership function, an indeterminacy-membership function, and a falsity-membership function. This assumption is very important in a lot of situations such as information fusion when we try to combine the data from different sensors. In this paper, we consider the collectivity of logical (hyper)BCK-algebras and single-valued neutrosophic hyper BCK-subalgebras to solve some complex real problems dealing with the principles of logical hyperalgebra(one or more groups based on these principles must be combined) and have uncertain information such as complex intelligent hypernetworks and related other sciences. Thus the non-classical mathematics together with the concept of neutrosophic subset, therefore, has nowadays become a useful tool in applications mathematics and complex hypernetworks. Moreover, we can refer to some academic contributions of single-valued neutrosophic subsets such as single-valued neutrosophic directed (hyper)graphs and applications in networks [4], application of single-valued neutrosophic in lifetime in wireless sensor (hyper)network [4], an application of single-valued neutrosophic subsets in social (hyper)networking [4], application of single-valued neutrosophic sets in medical diagnosis, application of neutro hyper BCK-algebras and single-valued neutrosophic hyper BCK-subalgebras in economic hypernetwork [7], and application of neutro hyper BCK-algebras and single-valued neutrosophic hyper BCK-subalgebras in data (hyper)networks [7]. To conclude, we considered the notion of single-valued neutrosophic hyper BCK-ideals and investigated some of their new useful properties. We considered that for any $\alpha \in [0,\ 1]$ there is an algebraic relation between of a single-valued neutrosophic subset hyper BCK-subalgebra, $A = (T_A, I_A, F_A)$ and $A = ({T_A}^{\alpha}, {I_A}^{\alpha}, {F_A}^{\alpha})$. In addition, with respect to the concept of hyper BCK-ideals of given hyper BCK-algebra, is constructed quotient BCK-algebra structures. On any nonempty set, is constructed an extendable single-valued neutrosophic BCK-(ideal)subalgebra and isomorphism theorem of single-valued neutrosophic hyper BCK-ideals is obtained. One of the advantages of this study is the conversion of complex hypernetworks to complex networks in such a way that all the details of the complex hypernetworks are preserved and transferred to the complex networks, but there are some limitations in this work. Although neutrosophic subsets are more flexible and useful as compared to all fuzzy theories, there are some limitations whence we need more than three functions in designing and modeling the real problem with complexity and high dimension. Also, the computations of single-valued neutrosophic hyper BCK-ideals for any given hyper BCK-algebras with large cardinal is hard and so the related mathematical tools such as congruence and strongly relations, nontrivial homomorphisms are complicated. Hence these problems prevent us from having a definite and simple algorithm for our computations.

We wish this research is important for the next studies in logical hyperalgebras. In our future studies, we hope to obtain more results regarding single-valued neutrosophic (hyper)BCK-subalgebras and their applications in handing information regarding various aspects of uncertainty, non-classical mathematics (fuzzy mathematics or great extension and development of classical mathematics) that are considered to be a more powerful technique than classical mathematics.

Acknowledgements. We thank the anonymous referee for the useful comments and suggestions, which helped us to improve the overall quality of the paper.

References

[1] R. A. Borzooei, R. Ameri, M. Hamidi, *Fundamental relation on hyper BCK-algebras*, **Analele Universitatii din Oradea-Fascicola Matematica**, vol. 21(1) (2014), pp. 123–136.

[2] S. S. Goraghani, *On Fuzzy Quotient BCK-algebras*, **TWMS Journal of Applied and Engineering Mathematics**, vol. 10(1) (2020), pp. 59–68, URL: https://jaem.isikun.edu.tr/web/index.php/archive/104-vol10no1/491-on-fuzzy-quotient-bck-algebras.

[3] M. Hamidi, A. B. Saied, *Accessible single-valued neutrosophic graphs*, **Journal of Computational and Applied Mathematics**, vol. 57 (2018), pp. 121–146, DOI: https://doi.org/10.1007/s12190-017-1098-z.

[4] M. Hamidi, F. Smarandache, *Single-valued neutrosophic directed (Hyper)graphs and applications in networks*, **Journal of Intelligent & Fuzzy Systems**, vol. 37(2) (2019), pp. 2869–2885, DOI: https://doi.org/10.3233/JIFS-190036.

[5] M. Hamidi, F. Smarandache, *Derivable Single Valued Neutrosophic Graphs Based on KM-single-valued neutrosophic Metric*, **IEEE Access**, vol. 8 (2020), pp. 131076–131087, DOI: https://doi.org/10.1109/ACCESS.2020.3006164.

[6] M. Hamidi, F. Smarandache, *Neutro-BCK-Algebra*, **International Journal of Neutrosophic Science**, vol. 8 (2020), pp. 110–117, URL: https://digitalrepository.unm.edu/math_fsp.

[7] M. Hamidi, F. Smarandache, *Single-Valued Neutro Hyper BCK-Subalgebras*, **Journal of Mathematics**, vol. 2021 (2021), pp. 1–11, DOI: https://doi.org/10.1155/2021/6656739.

[8] Y. Imai, K. Iseki, *Proceedings of the Japan Academy, Series A, Mathematical Sciences*, **Bulletin of the Section of Logic**, vol. 42 (1966), pp. 19–22, DOI: https://doi.org/10.3792/pja/1195522169.

[9] Y. B. Jun, M. Khan, F. Smarandache, S. Z. Song, *Length Neutrosophic Subalgebras of BCK/BCI-Algebras*, **Bulletin of the Section of Logic**, vol. 49(4) (2020), pp. 377–400, DOI: https://doi.org/10.18778/0138-0680.2020.21.

[10] Y. B. Jun, S. Z. Song, *Inf-Hesitant Fuzzy Ideals in BCK/BCI-Algebras*, **Bulletin of the Section of Logic**, vol. 49(1) (2020), pp. 53–78, DOI: https://doi.org/10.18778/0138-0680.2020.03.

[11] Y. B. Jun, M. M. Zahedi, X. L. Xin, R. A. Borzooei, *On hyper BCK-algebras*, **Italian Journal of Pure and Applied Mathematics**, vol. 10 (2000), pp. 127–136.

[12] S. Khademan, M. M. Zahedi, R. A. Borzooei, Y. B. Jun, *Neutrosophic Hyper BCK-Ideals*, **Neutrosophic Sets and Systems**, vol. 27 (2019), pp. 201–217, URL: http://fs.unm.edu/NSS2/index.php/111/article/view/626.

[13] S. Khademan, M. M. Zahedi, R. A. Borzooei, Y. B. Jun, *Fuzzy Soft Positive Implicative Hyper BCK-ideals Of Several Types*, **Miskolc Mathematical Notes**, vol. 22(1) (2021), pp. 299–315, DOI: https://doi.org/10.18514/MMN.2021.2855.

[14] N. Kouhestani, S. Mehrshad, *(Semi)topological quotient BCK-algebras*, **Afrika Matematika**, vol. 28 (2017), pp. 1235–1251, DOI: https://doi.org/10.1007/s13370-017-0513-9.

[15] G. Muhiuddin, A. N. Al-Kenani, E. H. Roh, Y. B. Jun, *Implicative Neutrosophic Quadruple BCK-Algebras and Ideals*, **Symmetry**, vol. 11(2) (2019), p. 277, DOI: https://doi.org/10.3390/sym11020277.

[16] R. Naghibi, S. M. Anvariyeh, *Construction of an HV-K-algebra from a BCK-algebra based on Ends Lemma*, **Journal of Discrete Mathematical Sciences and Cryptography**, vol. 25(2) (2022), pp. 405–425, DOI: https://doi.org/10.1080/09720529.2019.1689606.

[17] M. Shamsizadeh, *Single Valued Neutrosophic General Machine*, **Neutrosophic Sets and Systems**, vol. 490 (2022), pp. 509–530, URL: http://fs.unm.edu/NSS2/index.php/111/article/view/2502.

[18] F. Smarandache, *Neutrosophic Set, a generalisation of the intuitionistic single-valued neutrosophic sets*, **International Journal of Pure**

and Applied Mathematics, vol. 24 (2005), pp. 287–297, URL: https://digitalrepository.unm.edu/math_fsp.

[19] F. Smarandache, *Generalizations and Alternatives of Classical Algebraic Structures to Neutroalgebraic Structures and Antialgebraic Structures*, **Journal of Fuzzy Extension and Applications**, vol. 1(2) (2020), pp. 85–87, DOI: https://doi.org/10.22105/jfea.2020.248816.1008.

[20] M. M. Takallo, R. A. Borzooei, S. Z. Song, Y. B. Jun, *Implicative ideals of BCK-algebras based on MBJ-neutrosophic sets*, **AIMS Mathematics**, vol. 6(10) (1965), pp. 11029–11045, DOI: https://doi.org/10.3934/math.2021640.

[21] L. A. Zadeh, *Fuzzy sets*, **Information and Control**, vol. 8 (2021), pp. 338–353, DOI: https://doi.org/10.1016/S0019-9958(65)90241-X.

[22] Y. Zeng, H. Ren, T. Yang, S. Xiao, N. Xiong, *A Novel Similarity Measure of Single-Valued Neutrosophic Sets Based on Modified Manhattan Distance and Its Applications*, **Electronics**, vol. 11(6) (2022), p. 941, DOI: https://doi.org/10.3390/electronics11060941.

[23] J. Zhan, M. Hamidi, A. B. Saeid, *Extended Fuzzy BCK-Subalgebras*, **Iranian Journal of Fuzzy Systems**, vol. 13(4) (2016), pp. 125–144, DOI: https://doi.org/10.22111/IJFS.2016.2600.

Mohammad Hamidi
University of Payame Noor
Department of Mathematics
19395-4697
Tehran, IRAN
e-mail: m.hamidi@pnu.ac.ir

Bulletin of the **S**ection of **L**ogic
Volume 52/4 (2023), pp. 441–458
https://doi.org/10.18778/0138-0680.2023.10

Farzad Iranmanesh
Mansoor Ghadiri
Arsham Borumand Saeid

FUNDAMENTAL RELATION ON H_vBE-ALGEBRAS

Abstract

In this paper, we are going to introduce a fundamental relation on H_vBE-algebra and investigate some of properties, also construct new $(H_v)BE$-algebras via this relation. We show that quotient of any H_vBE-algebra via a regular regulation is an H_vBE-algebra and this quotient, via any strongly relation is a BE-algebra. Furthermore, we investigate that under what conditions some relations on H_vBE-algebra are transitive relations.

Keywords: $(H_v, Hyper)BE$-algebra, fundamental relation, quotient.

2020 Mathematical Subject Classification: 06F35, 03G25.

1. Introduction and preliminaries

Hyperstructures represent a natural extension of classical structures and they were introduced by French Mathematician F. Marty in 1934 [10]. A hyperstructure is a nonempty set H, together with a function $\circ : H \times H \longrightarrow P^*(H)$ called hyper operation, where $P^*(H)$ denotes the set of all nonempty subsets of H. Marty introduced hypergroups as a generalization of groups [4, 3]. Hyperstructures have many applications to several sectors of both pure and applied sciences as geometry, graphs and hypergraphs, fuzzy sets and rough sets, automata, cryptography, codes, relation algebras, C-algebras, artificial intelligence, probabilities, chemistry, physics, especially in atomic physic and in harmonic analysis [2, 7].

Presented by: Janusz Ciuciura
Received: April 10, 2022
Published online: August 9, 2023

H. S. Kim and Y. H. Kim introduced the notation of BE-algebra as a generalization of dual BCK algebra [9]. R. A. Borzooei et al. defined the notation of a hyper K-algebra, bounded hyper K-algebra and considered the zero condition in hyper K-algebras. They showed that every hyper K-algebra with the zero condition can be extended to a bounded hyper K-algebra [1].

A. Radfar et al. combined BE-algebra with hyperstructures and defined the notation of hyper BE-algebra. Also, they focused on some types of hyper BE-algebras and show that every dual hyper K-algebra is a hyper BE-algebra [11].

We know that the class of the H_v-structures, introduced by Vougiouklis in 1990 [13, 14], is the largest class of hyperstructures. In the classical hyperstructures, in any axiom where the equality is used, if we replace the equality by the nonempty intersection, then we obtain a corresponding H_v structures.

F. Iranmanesh et al. present the notation of the H_vBE-algebra as generalization of hyper BE-algebra [8]. They defined new H_v-structures and considered some of their useful properties. Also discuss H_v-filters and homomorphism on this structure. Furthermore, they got more results in H_vBE-algebras [8]. Fundamental relations are one of the main tools in algebraic hyperstructures theory.

Algebraic hyperstructures are extension of algebraic structures and for better understanding their properties we want some connections between algebraic hyperstructures and algebraic structures, a fundamental relation is an interesting concept in algebraic hyperstructures that makes this connection. In this paper, for obtain a relationship between BE, hyper BE and $(H_v)BE$-algebra, we define a fundamental relation on $(H_v)BE$-algebra that is called "δ" also, we study "δ^*" as a transitive closure of "δ" in such away that is the smallest equivalence relation that contains "δ". Finally, a BE-algebras which is quotient of H_vBE-algebra via "δ^*" is obtained, therefore we find a connection between algebraic structures and (H_v)hyper algebraic structures.

Definition 1.1 ([9]). Let X be a nonempty set, "$*$" be a binary operation on X and a constant $0 \in X$. Then $(X, *, 0)$ is called a BCK-algebra if for all $x, y, z \in X$ it satisfies the following conditions:

(BCI-1) $((x * y) * (x * z)) * (z * y) = 0$,

(BCI-2) $(x * (x * y)) * y = 0$,

(BCI-3) $x * x = 0$,

(BCI-4) $x * y = 0$ and $y * x = 0$, imply $x = y$,

(BCI-5) $0 * x = 0$.

We define a binary relation "$\leq$" on X by $x \leq y$ if and only if $x * y = 0$.

DEFINITION 1.2 ([9]). Let X be a nonempty set, "$*$" be a binary operation on X and $1 \in X$. An algebra $(X, *, 1)$ of type $(2, 0)$ is called a BE-algebra if the following axioms hold:

($BE1$) $x * x = 1$,

($BE2$) $x * 1 = 1$,

($BE3$) $1 * x = x$,

($BE4$) $x * (y * z) = y * (x * z)$, for all $x, y, z \in X$.

We introduce the relation "$\leq$" on X by $x \leq y$ if and only if $x * y = 1$.
The BE-algebra $(X, *, 1)$ is said to be commutative, if for all $x, y \in X$, $(x * y) * y = (y * x) * x$.

PROPOSITION 1.3 ([9]). Let X be a BE-algebra. Then

(i) $x * (y * x) = 1$.

(ii) $y * ((y * x) * x) = 1$, for all $x, y \in X$.

Example 1.4 ([12]). Let $X = \{1, 2, ...\}$. Define the operation "$*$" as follows:

$$x * y = \begin{cases} 1 & \text{if } y \leq x \\ y & \text{otherwise.} \end{cases}$$

then $(X, *, 1)$ is a BE-algebra.

Definition 1.5 ([4]). Let H be a nonempty set and $\circ : H \times H \longrightarrow P^*(H)$ be a hyperoperation. Then $(H, \circ)$ is called an H_v-group if it satisfies the following axioms:

$(H1)$ $x \circ (y \circ z) \bigcap (x \circ y) \circ z \neq \phi$,

$(H2)$ $a \circ H = H \circ a = H$, for all $x, y, z, a \in H$,

where $a \circ H = \bigcup\limits_{h \in H} a \circ h$, $H \circ a = \bigcup\limits_{h \in H} h \circ a$.

Definition 1.6 ([11]). Let H be a nonempty set and $\circ : H \times H \longrightarrow P^*(H)$ be a hyperoperation. Then $(H, \circ, 0)$ is called a hyper K-algebra if satisfies the following axioms:

$(HK1)$ $(x \circ z) \circ (y \circ z) < x \circ y$,

$(HK2)$ $(x \circ y) \circ z = (x \circ z) \circ y$,

$(HK3)$ $x < x$,

$(HK4)$ $x < y$ and $y < x$ implies $x = y$,

$(HK5)$ $0 < x$, for all $x, y, z \in H$,

where the relation " $<$ " is defined by $x < y \iff 0 \in x \circ y$. For every $A, B \subseteq H$, $A < B$ if and only if there exist $a \in A$ and $b \in B$ such that $a < b$. Note that if $A, B \subseteq H$, then by $A \circ B$ we mean the subset $\bigcup\limits_{a \in A, b \in B} a \circ b$ of H.

Definition 1.7 ([11]). Let H be a nonempty set and $\circ : H \times H \longrightarrow P^*(H)$ be a hyperoperation. Then $(H, \circ, 1)$ is called a hyper $BE-$algebra if satisfies the following axioms:

$(HBE1)$ $x < 1$ and $x < x$,

$(HBE2)$ $x \circ (y \circ z) = y \circ (x \circ z)$,

$(HBE3)$ $x \in 1 \circ x$,

$(HBE4)$ $1 < x$ implies $x = 1$, for all $x, y, z \in H$.

$(H, \circ, 1)$ is called a dual hyper K-algebra if it satisfies $(HBE1), (HBE2)$ and the following axioms:

$(DHK1)$ $x \circ y < (y \circ z) \circ (x \circ z)$,

$(DHK4)$ $x < y$ and $y < x$ imply that $x = y$,

where the relation " $<$ " is defined by $x < y \iff 1 \in x \circ y$.

Definition 1.8 ([8]). Let H be a nonempty set and $\circ : H \times H \longrightarrow P^*(H)$ be a hyperoperation. Then $(H, \circ, 1)$ is called an H_vBE-algebra if satisfies the following axioms:

(H_vBE1) $x < 1$ *and* $x < x$,

(H_vBE2) $x \circ (y \circ z) \bigcap y \circ (x \circ z) \neq \phi$,

(H_vBE3) (H_vBE3) $x \in 1 \circ x$,

(H_vBE4) $1 < x$ implies $x = 1$, for all $x, y, z \in H$,

where the relation "$<$" is defined by $x < y \iff 1 \in x \circ y$.

Also $A < B$ if and only if there exist $a \in A$ and $b \in B$ such that $a < b$.

Proposition 1.9 ([6]). Every dual hyper K-algebra is a hyper BE-algebra.

2. On H_vBE-algebras and some results

In this section, we consider H_vBE-structure with some results on its.

Example 2.1.

(i) Let $(H, *, 1)$ be a BE-algebra and we know that $\circ : H \times H \longrightarrow P^*(H)$ with $x \circ y = \{x * y\}$ is a hyperoperation. Then $(H, \circ, 1)$ is a trivial hyper BE-algebra and an H_vBE-algebra.

(ii) Let $H = \{1,a,b\}$. Define hyperoperation " $\circ$ " as follows:

$\circ$	1	a	b
1	{1}	{a,b}	{b}
a	{1}	{1,a}	{1,b}
b	{1}	{1,a,b}	{1}.

Then $(H, \circ, 1)$ is an H_vBE-algebra.

(iii) Define the hyperoperation " $\circ$ " on $\mathbb{R}$ as follows:

$$x \circ y = \begin{cases} \{y\} & \text{if } x = 1 \\ \mathbb{R} & \text{otherwise,} \end{cases}$$

then $(\mathbb{R}, \circ, 1)$ is an $H_v BE$-algebra.

PROPOSITION 2.2 ([8]). Any hyper BE-algebra is an $H_v BE$-algebra.

Example 2.3 shows that the converse of Proposition 2.2 does not hold in general.

Example 2.3. Define a hyperoperation " $\circ$ " on the set H = {1,a,b} as follows:

$\circ$	1	a	b
1	{1}	{a}	{b}
a	$\{1, b\}$	{1}	{1,a,b}
b	{1}	{1,b}	{1,b}.

Then $(H, \circ, 1)$ is an $H_v BE$-algebra. And we have that:

$$a \circ (b \circ b) = a \circ (\{1, b\}) = \{1, a, b\} \neq \{1, b\} = b \circ (\{1, a, b\}) = b \circ (a \circ b).$$

So $(H, \circ, 1)$ does not satisfy $(HBE2)$, therefore $(H, \circ, 1)$ is not a hyper BE-algebra.

THEOREM 2.4. *Let* $(H, \circ, 1)$ *be an* $H_v BE$*-algebra. Then*

(i) $A \circ (B \circ C) \bigcap B \circ (A \circ C) \neq \phi$ *for every* $A, B, C \in P^*(H)$,

(ii) $A < A$,

(iii) $1 < A$ *implies* $1 \in A$,

(iv) $1 \in x \circ (x \circ x)$,

(v) $x < x \circ x$.

PROOF: (i) Let $a \in A, b \in B$ and $c \in C$. We have $a \circ (b \circ c) \subseteq A \circ (B \circ C)$, $b \circ (a \circ c) \subseteq B \circ (A \circ C)$, Then by $(H_v BE2)$, $a \circ (b \circ c) \bigcap b \circ (a \circ c) \neq \phi$, therefore $A \circ (B \circ C) \bigcap B \circ (A \circ C) \neq \phi$.

Other cases are similar. □

DEFINITION 2.5 ([6]). Let F be a nonempty subset of H_vBE-algebra H and $1 \in F$. Then F is called

(i) a weak H_v-filter of H if $x \circ y \subseteq F$ and $x \in F$ imply $y \in F$, for all $x, y \in H$.

(ii) an H_v-filter of H if $x \circ y \approx F(i.e\ \phi \neq (x \circ y) \bigcap F)$ and $x \in F$ imply $y \in F$, for all $x, y \in H$.

Example 2.6. Let $H = \{1,a,b\}$. Define the hyperoperation "$\circ_1$" and "$\circ_2$" as follows:

$\circ_1$	1	a	b
1	$\{1\}$	$\{a, b\}$	$\{b\}$
a	$\{1\}$	$\{1, a\}$	$\{1, b\}$
b	$\{1\}$	$\{1, a, b\}$	$\{1\}$

$\circ_2$	1	a	b
1	$\{1\}$	$\{a, b\}$	$\{b\}$
a	$\{1\}$	$\{1, a, b\}$	$\{b\}$
b	$\{1, b\}$	$\{1, a, b\}$	$\{1, a, b\}$.

We see that $(H, \circ_1, 1)$ is an H_vBE-algebra and $F_1 = \{1, a\}$ is a weak H_v-filter of H. Also $(H, \circ_2, 1)$ is an H_vBE-algebra and $F_2 = \{1, a\}$ is an H_v-filter of H.

In Example 2.6, F_1 is not an H_v-filter, because $a \circ_1 b \approx F_1$ and $a \in F_1$, but $b \notin F_1$.

THEOREM 2.7. *Every H_v-filter is a weak H_v-filter.*

Notation. By Example 2.6, we can see that the notion of weak H_v-filter and H_v-filter are different in H_vBE-algebra.

THEOREM 2.8 ([8]). *Let F be a subset of an H_vBE-algebra H and $1 \in F$. For all $x, y \in H$, if $x \circ y < F$ and $x \in F$ implies $y \in F$, then F=H.*

3. Relations on H_vBE-algebras

In this section, let $(H, \circ, 1)$ be a H_vBE-algebra and presents in summary with H. We show that there exists a connection between hyper algebraic structures and algebraic structures by strongly regular relations.

Definition 3.1. Let $(H, \circ, 1)$ be an H_vBE-algebra and R be an equivalence relation on H. If A, B are nonempty subsets of H, then

(i) $A\ \bar{R}\ B$ means that, for all $a \in A$, there exists $b \in B$ in such away that aRb and for all $b' \in B$, there exists $a' \in A$ in such away that $b'Ra'$,

(ii) $A\ \bar{\bar{R}}\ B$ means that, for all $a \in A$ and $b \in B$, we have aRb,

(iii) R is called right regular(left regular), if for all $x \in H$, from aRb, it follows that $(a \circ x)\bar{R}(b \circ x)((x \circ a)\bar{R}(x \circ b))$.

(iv) R is called strongly right regular(strongly left regular), if for all $x \in H$, from aRb, it follows that $(a \circ x)\bar{\bar{R}}(b \circ x)((x \circ a)\bar{\bar{R}}(x \circ b))$.

(v) R is called (strongly) regular, if it is (strongly) right regular and (strongly) left regular,

(vi) R is called good, if $(a \circ b)\ R\ 1$ and $(b \circ a)\ R\ 1$ imply aRb, for all $a, b \in H$.

It is clear that $(a \circ b)\ R\ 1$ means that there exists $x \in a \circ b$ in such a way that $xR1$.

Example 3.2. Let $H = \{1, a, b\}$. Define the hyperoperation "$\circ$" as follows:

$\circ$	1	a	b
1	$\{1\}$	$\{a, b\}$	$\{b\}$
a	$\{1\}$	$\{1, a, b\}$	$\{b\}$
b	$\{1, b\}$	$\{1, a, b\}$	$\{1, a, b\}$

Then $(H, \circ, 1)$ is an H_vBE-algebra. It is easy to see that

$$R = \{(1,1), (a,a), (b,b), (a,b), (b,a), (1,b), (b,1), (a,1), (1,a)\}$$

is a good strongly regular relation on H and for any $A, B \in P^*(H)$, $A\ \bar{\bar{R}}\ B$.

Example 3.3. Let $H = \{1, d, b, c\}$. Define the hyperoperation "$\circ$" as follows:

$\circ$	1	b	c	d
1	$\{1\}$	$\{b\}$	$\{c\}$	$\{d\}$
b	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
c	$\{1\}$	$\{b\}$	$\{1\}$	$\{d\}$
d	$\{1\}$	$\{b\}$	$\{1, c\}$	$\{1\}$

Then $(H, \circ, 1)$ is an H_vBE-algebra. It is easy to see that

$$R = \{(1,1),(d,d),(b,b),(c,c),(c,b),(b,c),(d,c),(c,d)\}$$

is not regular and strongly regular relation on H.

Notation. Let R be regular relation on H. We denote the set of all equivalence classes of R by H/R. Hence $H/R = \{\bar{x} : x \in H\}$. For any $\bar{x}, \bar{y} \in H/R$, define a hyperoperation "$*$" on H/R by

$$\bar{x} * \bar{y} = \{\bar{z} : z \in x \circ y\}$$

and a binary relation " $<$ " on H/R by

$$\text{“}\bar{x} < \bar{y}\text{”} \Longleftrightarrow \bar{1} \in \bar{x} * \bar{y}.$$

Lemma 3.4. *Let R be a regular relation on H. Then $(H/R; *)$, is a hypergroupoid.*

Proof: We must show that $*$ be well defined. Let $\bar{x_1}, \bar{x_2}, \bar{y_1}, \bar{y_2} \in H/R$ such that $\bar{x_1} = \bar{x_2}$, $\bar{y_1} = \bar{y_2}$. Then $x_1\ R\ x_2$ and $y_1\ R\ y_2$. Since R is a regular relation, we have $(x_1 \circ y_1)\bar{R}(x_2 \circ y_2)$ [5]. Let $\bar{r} \in \bar{x_1} * \bar{y_1}$. Then there exists $z \in x_1 \circ y_1$ in such a way that $\bar{r} = \bar{z}$. Now $z \in x_1 \circ y_1$ and $(x_1 \circ y_1)\bar{R}(x_2 \circ y_2)$, then there exists $u \in (x_2 \circ y_2)$ such that zRu then $\bar{z} = \bar{u}$ and $\bar{r} = \bar{u}$, thus $\bar{x_1} * \bar{y_1} \subseteq \bar{x_2} * \bar{y_2}$ and in a similar way we get $\bar{x_2} * \bar{y_2} \subseteq \bar{x_1} * \bar{y_1}$, i.e $\bar{x_1} * \bar{y_1} = \bar{x_2} * \bar{y_2}$ therefore $*$ is well defined and $(H/R; *)$ is a hypergroupoid. □

Theorem 3.5. *If R is a regular relation on H then $(H/R; *; \bar{1})$ is a H_vBE-algebra.*

Proof: Let R be a regular relation on H. If $x \in H$ then $\bar{x} \circ \bar{1} = \{\bar{t} : t \in x \circ 1\}$. Since H is an H_vBE- algebra by (H_vBE1) we conclude that $1 \in x \circ 1$ and so $\bar{1} \in \bar{x} * \bar{1}$. Therefore $\bar{x} < \bar{1}$. Also $1 \in x \circ x$ and $\bar{x} \circ \bar{x} = \{\bar{t} : t \in x \circ x\}$, then $\bar{1} \in \bar{x} * \bar{x}$ and $\bar{x} < \bar{x}$.

(H_vBE2) Let $x, y, z \in H$. Since $(H, \circ, 1)$ is an H_vBE- algebra, then $x \circ (y \circ z) \bigcap y \circ (x \circ z) \neq \phi$. If $t \in x \circ (y \circ z) \bigcap y \circ (x \circ z)$, then there exists $s_1 \in y \circ z$ in such away that $t \in x \circ s_1$ by a similar way there exists $s_2 \in x \circ z$ in such away that $t \in y \circ s_2$. We get the $\bar{t} \in \bar{x} * \bar{s}_1 \subseteq \bar{x} * (\bar{y} * \bar{z})$ and $\bar{t} \in \bar{y} * \bar{s}_2 \subseteq \bar{y} * (\bar{x} * \bar{z})$. Therefore $\bar{x} * (\bar{y} * \bar{z}) \bigcap \bar{y} * (\bar{x} * \bar{z}) \neq \phi$.

(H_vBE3) if $x \in H$ then $\bar{1} \circ \bar{x} = \{\bar{t} : t \in 1 \circ x\}$. Since H is a H_vBE-algebra, we have $x \in 1 \circ x$ and $\bar{x} \in \bar{1} * \bar{x}$.

(H_vBE4) $x \in H$ and $\bar{1} < \bar{x}$ then $\bar{1} \in \bar{1} * \bar{x}$. Hence $1 \in 1 \circ x$ and $1 < x$. Since H is a H_vBE- algebra, we have $x = 1$ and so $\bar{x} = \bar{1}$. □

Corollary 3.6. Let $(H, \circ, 1)$ be a dual hyper K-algebra and R be an equivalence relation on H. If R is a regular relation on H, then $(H/R; *; \bar{1})$ is an H_vBE-algebra.

Theorem 3.7. *If R is strongly regular relation on H, then $(H/R; *; \bar{1})$ is a BE-algebra.*

Proof: If $\bar{z_1}, \bar{z_2} \in \bar{x} * \bar{y}$, for any $\bar{x}, \bar{y} \in H/R$, then $z_1, z_2 \in x \circ y$. Since R is strongly regular, for all $x, y \in H$, yRy then $(x \circ y)\bar{\bar{R}}(x \circ y)$ and $z_1, z_2 \in x \circ y$, we have $z_1 \ R \ z_2$, therefore $\bar{z_1} = \bar{z_2}$ and $|\bar{x} * \bar{y}| = 1$ and so by Theorem 3.5, $(H/R; *; \bar{1})$ is a BE-algebra. □

Example 3.8. Let $H = \{1, a, b, c, d, e\}$. Define the hyperoperation " $\circ$ " as follows:

$\circ$	1	a	b	c	d	e
1	$\{1, c\}$	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$	$\{e\}$
a	$\{1, c\}$	$\{1, c\}$	$\{a\}$	$\{1, c\}$	$\{c\}$	$\{d\}$
b	$\{1, c\}$	$\{1, c\}$	$\{1, c\}$	$\{1, c\}$	$\{c\}$	$\{c\}$
c	$\{1, c\}$	$\{a\}$	$\{b\}$	$\{1, c\}$	$\{a\}$	$\{b\}$
d	$\{1, c\}$	$\{1, c\}$	$\{a\}$	$\{1, c\}$	$\{1, c\}$	$\{a\}$
e	$\{1, c\}$	$\{1, c\}$	$\{1, c\}$	$\{1, c\}$	$\{1, c\}$	$\{1, c\}$

Then $(H, \circ, 1)$ is an H_vBE-algebra. It is easy to see that R=$\{(1, 1)$, (a, a), (b, b), (c, c), (d, d), (e, e), $(1, c)$, $(c, 1)$, (e, b), $(b.e)$, (a, d), $(d, a)\}$ is a good strongly regular relation on H and

$$H/R = \{\{1, c\}, \{a, d\}, \{e, b\}\} = \{R(1), R(a), R(b)\}.$$

Now we have:

$*$	$R(1)$	$R(a)$	$R(b)$
$R(1)$	$R(1)$	$R(a)$	$R(b)$
$R(a)$	$R(1)$	$R(1)$	$R(a)$
$R(b)$	$R(1)$	$R(1)$	$R(1)$

Clearly, $(H/R; *; R(1))$ is a BE-algebra.

In this place, we present some results and examples about dual hyper K-algebras and hyper BE-algebras that are useful.

LEMMA 3.9 ([6]). *Let $(X;\circ,1)$ be a dual hyper K-algebra and R be a regular relation on X. Then for any $\bar{x},\bar{y},\bar{z} \in X/R$, $(\bar{x} * \bar{y}) < (\bar{y} * \bar{z}) * (\bar{x} * \bar{z})$.*

THEOREM 3.10 ([6]). *Let $(X,\circ,1)$ be a dual hyper K-algebra and R be a regular relation on X. If R is a good relation, then $(X/R;*,\bar{1})$ is a dual hyper K-algebra.*

THEOREM 3.11 ([6]). *Let $(X,\circ,1)$ be a dual hyper K-algebra and R be a strongly regular relation on X. If R is a good relation, then $(X/R;*,\bar{1})$ is a dual BCK-algebra.*

Example 3.12. Let $X = \{1,a,b,c,d,e\}$. Define the hyperoperation "$\circ$" as follows:

$\circ$	1	a	b	c	d	e
1	$\{1,e\}$	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$	$\{e\}$
a	$\{1,e\}$	$\{1,e\}$	$\{b\}$	$\{c\}$	$\{d\}$	$\{e\}$
b	$\{1,e\}$	$\{a\}$	$\{1,e\}$	$\{c\}$	$\{d\}$	$\{e\}$
c	$\{1,e\}$	$\{a\}$	$\{b\}$	$\{1,e\}$	$\{d\}$	$\{e\}$
d	$\{1,e\}$	$\{a\}$	$\{b\}$	$\{c\}$	$\{1,e\}$	$\{e\}$
e	$\{1,e\}$	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$	$\{1,e\}$

Then $(X,\circ,1)$ is a dual hyper K-algebra (H_vBE-algebra). It is easy to see that R={(1,1),(a,a),(b,b),(c,c),(d,d),(e,e),(1,c),(c,1),(e,c),(c,e)} is a good strongly regular relation on X and

$$X/R = \{\{1,e\},\{a\},\{b\},\{c\},\{d\}\} = \{R(1),R(a),R(b),R(c),R(d)\}.$$

Now we have:

$*$	$R(1)$	$R(a)$	$R(b)$	$R(c)$	$R(d)$
$R(1)$	$R(1)$	$R(a)$	$R(b)$	$R(c)$	$R(d)$
$R(a)$	$R(1)$	$R(1)$	$R(b)$	$R(c)$	$R(d)$
$R(b)$	$R(1)$	$R(a)$	$R(1)$	$R(c)$	$R(d)$
$R(c)$	$R(1)$	$R(a)$	$R(b)$	$R(1)$	$R(d)$
$R(d)$	$R(1)$	$R(a)$	$R(b)$	$R(c)$	$R(1)$

Clearly $(X/R;*,R(1))$ is a dual BCK-algebra.

4. δ- relation on H_vBE-algebra

Let $(H;\circ,1)$ be a H_vBE-algebra and A be a subset of H. The set of all finite combinations of A with hyperoperation $\circ$ and $\bigodot_{i=1}^{n} a_i = a_1 \circ a_2 \circ ...a_n$, is denoted by L(A) [5].

Definition 4.1. Let $(H;\circ,1)$ be a H_vBE-algebra. Consider:

$$\delta_1 = \{(x,x) : x \in H\}$$

and for every natural number $n \geq 1$, δ_n is defined as follows:

$x\delta_n y \Longleftrightarrow \exists(a_1,a_2,...,a_n) \in H^n, \exists u \in L(a_1,a_2,...,a_n)$ such that $\{x,y\} \subseteq u$.

Obviously for every $n \geq 1$ the relations $\delta\ _n$ are symmetric and no reflexive and transitive, but the relation $\delta = \bigcup_{n\geq 1} \delta_n$ is a reflexive and symmetric relation. Let δ^* be *transitive closure* of δ (the smallest transitive relation such that contains δ).

In the following theorem we show that δ^* is a strongly regular relation.S

Example 4.2. Let $H = \{1,a,b\}$. Define the hyperoperation "$\circ$" as follows:

$\circ$	1	a	b
1	$\{1\}$	$\{a,b\}$	$\{b\}$
a	$\{1\}$	$\{1,a,b\}$	$\{b\}$
b	$\{1,b\}$	$\{1,a,b\}$	$\{1,a,b\}$

Then $(H,\circ,1)$ is an H_vBE-algebra. $\delta\ _1 = \{(x,x) : x \in H\} = \{(1,1),(a,a),(b,b)\}$.

Since $\{1,a\},\{1,b\},\{a,b\} \subseteq b \circ a$ then $1\delta_2$a, $1\delta_2$b, $a\delta_2$b. Also, we know that $\{1,a\} \subseteq (1 \circ a) \circ b = \bigcup_{x \in 1\circ a} (x \circ b)$ therefore $1\delta_3$a.

Similarly, $1\delta_3$b, $a\delta_3$b. Obviously, $1\delta_n$a, $1\delta_n$b and $a\delta_n$b, since $\delta = \bigcup_{n\geq 1} \delta_n$, then $1\delta a$, $1\delta b$ and $a\delta b$.

THEOREM 4.3. *Let $(H, \circ, 1)$ be a H_vBE-algebra. Then δ^* is a strongly regular relation on H.*

PROOF: Let $x, y \in H$ and $x\ \delta^*\ y$. Then we show that for any $s \in H$:

$$(x \circ s)\overline{\overline{\delta^*}}(y \circ s).$$

Since $\delta = \bigcup_{n \geq 1} \delta\ _n$ and δ^* is the smallest transitive relation such that contains δ, then there exist $a_0, a_1, ..., a_n \in H$ such that $a_0 = x, a_n = y$ and there exist $q_1, q_2, ..., q_n \in \mathbb{N}$ such that

$$x = a_0\delta_{q_1}a_1\delta_{q_2}a_2 \ldots a_{n-1}\delta_{q_n}a_n = y,$$

where $n \in \mathbb{N}$. Since for any $1 \leq i \leq n$, $a_{i-1}\ \delta_{q_i}\ a_n$, then there exists $z_t^j \in H$ such that

$$\{a_i, a_{i+1}\} \subseteq z_1^{i+1} \circ z_2^{i+1} \circ \ldots \circ z_{q_{i+1}}^{i+1},$$

where for $1 \leq m \leq n-1$, we have $1 \leq t \leq q_m$, and $1 \leq j \leq n-1$. Now, since $s \in H$, then for all $0 \leq i \leq n-1$,

$$a_i \circ s \subseteq z_1^{i+1} \circ z_2^{i+1} \circ ... \circ z_{q_{i+1}}^{i+1} \circ s.$$

In a similar way, we get that

$$a_{i+1} \circ s \subseteq z_1^{i+1} \circ z_2^{i+1} \circ ... \circ z_{q_{i+1}}^{i+1} \circ s.$$

Then for all $1 \leq i \leq n$, and for all $u \in a_i \circ s, v \in a_{i+1} \circ s$, We have $\{u, v\} \subseteq z_1^{i+1} \circ z_2^{i+1} \circ ... \circ z_{q_{i+1}}^{i+1} \circ s$. Therefore $u\ \delta_{q_{i+1}}\ v$, and so for all $z \in a_0 \circ s = x \circ s, w \in a_n \circ s = y \circ s$, We have $z\ \delta^*\ w$. Then δ^* is a strongly right regular and similarly is a strongly left regular relation, therefore δ^* is a strongly regular relation on H. □

COROLLARY 4.4. Let $(H, \circ, 1)$ be a hyper BE-algebra. Then δ^* is a strongly regular relation on H.

THEOREM 4.5. *Let $(H, \circ, 1)$ be a H_vBE-algebra. $(H/\delta^*; *, \bar{1})$ is a BE algebra.*

PROOF: By Theorem 3.7 and 4.3, the proof is obvious. □

Example 4.6. Let $H = \{1, x, y, z, t\}$. Define hyperoperation "$\circ$" as follows:

$\circ$	1	x	y	z	t
1	$\{1, t\}$	$\{x\}$	$\{y\}$	$\{z\}$	$\{t\}$
x	$\{1, t\}$	$\{1, t\}$	$\{1, t\}$	$\{1, t\}$	$\{1, t\}$
y	$\{1, t\}$	$\{1, t\}$	$\{1, t\}$	$\{1, t\}$	$\{1, t\}$
z	$\{1, t\}$	$\{1, t\}$	$\{1, t\}$	$\{1, t\}$	$\{1, t\}$
t	$\{1, t\}$	$\{1, t\}$	$\{1, t\}$	$\{1, t\}$	$\{1, t\}$

Then $(H, \circ, 1)$ is a H_vBE-algebra. We have $(x \circ y) \circ x = \{1, x, t\}$, $(x \circ y) \circ y = \{1, y, t\}$, $(x \circ y) \circ t = \{1, t\}$, $(x \circ y) \circ z = \{1, z, t\}$. Then for any $u \in H$, $1\ \delta^* u$ and so $\delta^*\ (1) = \{u \in H : 1\ \delta^*\ \text{u}\} = H = \delta^*(u)$. Therefore H/δ^*=$\{\delta^*(1)\}$ and we see that $(H/\delta^*; *, \delta^*(1))$ is a trivial BE-algebra.

Example 4.7. Let $H = \{1, x, y, z\}$. Define hyperoperation "$\circ$" as follows:

$\circ$	1	x	y	z
1	$\{1\}$	$\{x\}$	$\{y\}$	$\{z\}$
x	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
y	$\{1\}$	$\{x\}$	$\{1\}$	$\{z\}$
z	$\{1\}$	$\{x\}$	$\{1, y\}$	$\{1\}$

Then $(H, \circ, 1)$ is a H_vBE-algebra. We conclude that $H/\delta^* = \{\{1, y\}, \{x\}, \{z\}\} = \{\delta^*(1), \delta^*(x), \delta^*(z)\}$ and then:

$*$	$\delta^*(1)$	$\delta^*(x)$	$\delta^*(z)$
$\delta^*(1)$	$\delta^*(1)$	$\delta^*(x)$	$\delta^*(z)$
$\delta^*(x)$	$\delta^*(1)$	$\delta^*(1)$	$\delta^*(1)$
$\delta^*(z)$	$\delta^*(1)$	$\delta^*(x)$	$\delta^*(x)$

Now, by Theorem 4.3, $(H/\delta^*; *, \delta(1))$ is a BE-algebra.

Notation. We know that δ is reflexive and symmetric but is not transitive on H. If R is an equivalence relation on H, then H/R is defined and we have the following theorem;

THEOREM 4.8 ([6]). *Let $(H, \circ, 1)$ be a hyper BE-algebra and R be an equivalence relation on H. Then, R is a regular relation on H if and only if $(H/R; *, \bar{1})$ is a hyper BE algebra.*

DEFINITION 4.9. Let M be a nonempty subset of H. M is called δ-part if for any $n \in \mathbb{N}$, $a_i \in H$, and $L(a_1, a_2, \dots, a_n) \cap M \neq \emptyset$, then $L(a_1, a_2, \dots, a_n) \subseteq M$.

Example 4.10. Let $H = \{1, x, y, z\}$. Define hyperoperation "$\circ$" as follows:

$\circ$	1	x	y	z
1	$\{1, x\}$	$\{1, x\}$	$\{y\}$	$\{z\}$
x	$\{1, x\}$	$\{1, x\}$	$\{y\}$	$\{z\}$
y	$\{1, x\}$	$\{1, x\}$	$\{1, x\}$	$\{z\}$
z	$\{1, x\}$	$\{1, x\}$	$\{1, x\}$	$\{1, x\}$

Then $(H, \circ, 1)$ is a H_vBE-algebra. It is easy to verify that for any $M \subseteq H$ that $M \neq \{1\}$ and $M \neq \{a\}$, M is a δ-part.

COROLLARY 4.11. Let $(H, \circ, 1)$ be a H_vBE-algebra and M, N are δ-part of H. Then $M \cap N$ is a δ-part of H.

PROOF: For any $n \in \mathbb{N}$, $a_i \in H$, if $L(a_1, a_2, ..., a_n) \cap (M \cap N) \neq \emptyset$, then $L(a_1, a_2, ..., a_n) \cap M \neq \emptyset, L(a_1, a_2, ..., a_n) \cap N \neq \emptyset$. Since M, N are δ-part, we have $L(a_1, a_2, ..., a_n) \subseteq M, L(a_1, a_2, ..., a_n) \subseteq N$ and then $L(a_1, a_2, ..., a_n) \subseteq M \cap N$. Therefore $M \cap N$ is a δ-part of H. □

LEMMA 4.12 ([6]). *Let M be a non-empty subset of a dual hyper K-algebra H. Then the following conditions are equivalent:*

(i) *M is a δ-part of H,*

(ii) *$x \in M$, $x\ \delta\ y$ imply $y \in M$,*

(iii) *$x \in M$, $x\ \delta^*\ y$ imply $y \in M$.*

THEOREM 4.13. *Let $(H, \circ, 1)$ be a H_vBE-algebra. If H be a dual hyper K-algebra and for any $x \in H$, $\delta^*(x)$ is a δ-part, then δ is transitive relation.*

PROOF: Let $x\ \delta\ y$ and $y\ \delta\ z$. Then there exist $m, n \in \mathbb{N}$, $a_i, b_j \in H$ such that $\{x, y\} \subseteq (\bigodot_{i=1}^{n} a_i)$ and $\{y, z\} \subseteq (\bigodot_{j=1}^{m} b_j)$. Now, $\delta^*(x)$ is a δ-part, $x \in \delta^*(x) \cap (\bigodot_{i=1}^{n} a_i)$ and $y \in (\bigodot_{i=1}^{n} a_i) \cap (\bigodot_{j=1}^{m} b_j)$. Since $\delta^*(x)$ is a δ-part, then $(\bigodot_{i=1}^{n} a_i) \subseteq \delta^*(x)$ therefore $y \in \delta^*(x) \cap (\bigodot_{j=1}^{m} b_j)$. Since $\delta^*(x)$ is a δ-part, then $(\bigodot_{j=1}^{m} b_j) \subseteq \delta^*(x)$ therefore $z \in \delta^*(x)$. But $z \in \delta(z)$ by above $z\ \delta^*\ x$, set $M = \delta(z)$ and know that $\delta^*(x) = \delta(z)$ then by Lemma 4.12 , $x\ \delta\ z$, therefore δ is transitive relation. □

Open problem: Under what conditions converse of above theorem is true?

5. Conclusion

In the present paper, we have introduced new H_vBE-algebras and BE-algebras based on equivalence relations.

This work focused on fundamental relations on H_vBE-algebras and we investigated some of their properties. The relations δ^* and δ are constructed and studied, they are one of the most main tools for better understanding the algebraic hyperstructures. In future, we try to find an answer to above open problem.

Acknowledgements. We wish to thank the reviewers for excellent suggestions that have been incorporated into the paper.

References

[1] R. A. Borzoei, A. Hasankhani, M. M. Zahedi, Y. B. Jun, *On hyper K-algebras*, **Mathematica japonicae**, vol. 52(1) (2000), pp. 113–121.

[2] J. Chvalina, Š. Hošková-Mayerová, A. D. Nezhad, *General actions of hyperstructures and some applications*, **Analele ştiinţifice ale Universităţii" Ovidius" Constanţa. Seria Matematică**, vol. 21(1) (2013), pp. 59–82, DOI: https://doi.org/10.2478/auom-2013-0004.

[3] P. Corsini, **Prolegmena of hypergroup theory**, Aviani, Tricesimo (1993).

[4] P. Corsini, V. Leoreanu, **Applications of hyperstructure theory**, vol. 5, Springer Science & Business Media, New York (2013), DOI: https://doi.org/10.1007/978-1-4757-3714-1.

[5] B. Davvaz, V. Leoreanu-Fotea, **Hyperring theory and applications**, vol. 347, International Academic Press, Palm Harbor (2007).

[6] M. Hamidi, A. Rezaei, A. Borumand Saeid, *δ-relation on dual hyper K-algebras*, **Journal of Intelligent & Fuzzy Systems**, vol. 29(5) (2015), pp. 1889–1900, DOI: https://doi.org/10.3233/IFS-151667.

[7] Š. Hošková-Mayerová, *Topological hypergroupoids*, **Computers & Mathematics with Applications**, vol. 64(9) (2012), pp. 2845–2849, DOI: https://doi.org/10.1016/j.camwa.2012.04.017.

[8] F. Iranmanesh, M. Ghadiri, A. B. Saeid, *On $H_v BE$-algebras*, **Miskolc Mathematical Notes**, vol. 21(2) (2020), pp. 897–909, DOI: https://doi.org/10.18514/MMN.2020.2836.

[9] H. S. Kim, Y. H. Kim, *On BE-algebras*, **Scientiae Mathematicae Japonicae**, vol. 66(1) (2007), pp. 113–116, DOI: https://doi.org/10.32219/isms.66.1_113.

[10] F. Marty, *Sur une generalization de la notion de groups*, [in:] **8th Scandinavian Congress of Mathematicians, Stockholm**, Stockholm (1934).

[11] A. Radfar, A. Rezaei, A. B. Saeid, *Hyper BE-algebras*, **Novi Sad Journal of Math**, vol. 44(2) (2014), pp. 137–147.

[12] A. Rezaei, A. B. Saeid, *On fuzzy subalgebras of BE-algebras*, **Afrika Matematika**, vol. 22 (2011), pp. 115–127, DOI: https://doi.org/10.1007/s13370-011-0011-4.

[13] T. Vougiouklis, *The fundamental relation in hyperrings. The general hyperfield*, [in:] **Proceedings of the Fourth International Congress on Algebraic Hyperstructures and Applications (AHA 1990)**, World Scientific (1991), pp. 203–211, DOI: https://doi.org/10.1142/9789814539555.

[14] T. Vougiouklis, *From rings to minimal H_v-fields*, **Journal of Algebraic Hyperstructures and Logical Algebras**, vol. 1(3) (2020), pp. 1–14, DOI: https://doi.org/10.29252/hatef.jahla.1.3.1.

Farzad Iranmanesh
University of Yazd
Department of Mathematics
8915818411, University Blvd
Yazd, Iran
e-mail: farzad.iranmanesh67@gmail.com

Mansoor Ghadiri
University of Yazd
Department of Mathematics
8915818411, University Blvd
Yazd, Iran
e-mail: mghadiri@yazd.ac.ir

Arsham Borumand Saeid
Shahid Bahonar University of Kerman
Department of Pure Mathematics
Faculty of Mathematics and Computer
7616913439, Pazhouhesh Sq.
Kerman, Iran
e-mail: arsham@uk.ac.ir

Bulletin of the **S**ection of **L**ogic
Volume 52/4 (2023), pp. 459–495
https://doi.org/10.18778/0138-0680.2023.22

Simone Martini*
Andrea Masini
Margherita Zorzi**

CUT ELIMINATION FOR EXTENDED SEQUENT CALCULI

Abstract

We present a syntactical cut-elimination proof for an extended sequent calculus covering the classical modal logics in the K, D, T, K4, D4 and S4 spectrum. We design the systems uniformly since they all share the same set of rules. Different logics are obtained by "tuning" a single parameter, namely a constraint on the applicability of the cut rule and on the (left and right, respectively) rules for $\Box$ and $\Diamond$. Starting points for this research are 2-sequents and indexed-based calculi (sequents and tableaux). By extending and modifying existing proposals, we show how to achieve a syntactical proof of the cut-elimination theorem that is as close as possible to the one for first-order classical logic. In doing this, we implicitly show how small is the proof-theoretical distance between classical logic and the systems under consideration.

Keywords: proof theory, sequent calculus, cut elimination, modal logic, 2-sequents.

2020 Mathematical Subject Classification: 03B22, 03B45, 03F05.

*Partially supported by project SERICS (PE00000014) under the Italian MUR National Recovery and Resilience Plan funded by the European Union – NextGenerationEU; and by INdAM GNSAGA.

**Partially supported by INdAM GNSAGA, project "Computazione, Infinito e Reversibilità".

Presented by: Norihiro Kamide
Received: April 15, 2022
Published online: September 25, 2023

1. Introduction

In the thirties, Gentzen introduced the sequent calculus (and natural deduction) to prove Hilbert's consistency assertion for pure logic and Peano Arithmetic. Gentzen's work marked the beginning of structural proof theory, by sanctioning its role to understand the structure of mathematical proofs and isolate and solve methodological problems in the foundations of mathematics. Proof theory is a wide research area that provides tools, methodologies, and solutions also to computer science and philosophical logic. It still offers interesting open problems, especially if we move away from classical and intuitionistic logic. Proof theory of *modal logic*, in particular, is subtle, since a uniform, technically elegant treatment of modalities ($\Box$, $\Diamond$) is generally difficult.

During the last decades, many modal systems have been introduced. Among these, some of the most interesting ones are the labeled systems [24, 26, 23], which extend ordinary calculi by explicitly mirroring in the deductive apparatus the accessibility relation of Kripke models. While such labeled frameworks provide a smart solution to represent structural properties, a more implicit representation of the semantical structure is sometimes preferable, especially if one wants to reduce the formal *iatus* between classical proof theory and the modal one.

In this regards a number of calculi have been introduced, e.g. [23] [4, 5, 7, 9, 15, 13, 18, 17, 16, 23, 21, 19, 22] (see section 5 for a detailed comparison between our proposal and some related ones).

These systems have been defined by taking into account some basic principles: *analyticity* (e.g., the subformula property), *modularity* (to be able to capture an entire family of logics instead of only one), and, if possible, an explicit *syntactical cut elimination procedure.*

Despite the number of calculi introduced and studied, *syntactical cut elimination* remains a "precious" property—many papers claim its validity but they do not exhibit detailed syntactical proofs (or do not prove it at all).

Cut elimination is often obtained either by semantical methods or by translation from other cut-free systems [14, 3]. We believe, on the contrary, that an explicit cut elimination procedure—in the spirit of Gentzen's original ideas—is still an important asset for modal proof theory.

For this reason, the present paper focuses on a syntactical cut elimination theorem, proposing a *modular* system based on *extended sequents* (in the following, simply *e–sequents*)—which allow for a uniform cut elimination argument for all the modal logics in the spectrum K, D, T, K4, D4, S4.

This paper is the natural companion of our [18], where we study a natural deduction calculus for the same family of modal logics and we prove a normalization theorems by a syntactical argument.

We pursue a strong form of modularity, since *all systems share the same set of rules*. Differently from other proposals, to obtain a specific system we do not add or drop characteristic axioms on a "kernel" calculus: we simply set some constraints on the left rule for $\Box$, on the right rule for $\Diamond$, and on the (eliminable) cut rules.

The main idea behind extended sequents is to equip formulas with a *position*—a sequence of uninterpreted tokens—which adds a *spatial dimension*. Positions allow us to interpret sequents *geometrically*, thus permitting a proof theoretical treatment of modalities as close as possible to that of first-order quantifiers.

Here are the main features of our system:

- there is exactly one left and one right rule for each connective;
- the right rule for $\Box$, and its dual left rule for $\Diamond$, are formulated using constraints on positions, with a strong analogy with the constraints on the eigenvariable of the right $\forall$ rule (and $\exists$ left rule, respectively) of standard first-order calculus;
- no direct formalization of the accessibility relation appears;
- only modal operators can change the spatial positions of formulas;
- all the logics share the same set of rules—different systems can be obtained by "tuning" some constraints on the applicability of the cut rule and on the (left and right, respectively) rules for $\Box$ and $\Diamond$.

The result is a parametric system, which we show proves the same theorems of the standard (Hilbert-style) systems for the same logics.

In Section 2 we present extended sequents (*e–sequents*); Section 3 is devoted to the syntactical proof of cut elimination; in Section 4 we show that e–sequent calculi are equivalent (they prove the same theorems) to the standard systems for the same modal logics. Comparison with related proposals and review of the state of the art are in Section 5.

2. Extended Sequent calculi

In this section we introduce *extended sequents* (briefly: *e–sequents*), an extensions of the 2–Sequents originally introduced in [20, 19] and then developed in [17, 16] (see section 5.1 for more on this approach).

To treat uniformly all the logics in the K, D, T, K4, D4 and S4 spectrum, we introduce *positions*—sequences of uninterpreted *tokens*. We start with basic notations and operations.

DEFINITION 2.1. Given a set X, X^* is the set of ordered finite sequences on X. With $\langle x_1, ..., x_n\rangle$ we denote a finite non-empty sequence such that $x_1, \ldots, x_n \in X$; $\langle\,\rangle$ is the empty sequence.

The (associative) concatenation of sequences $\circ : X^* \times X^* \to X^*$ is defined as

- $\langle x_1, ..., x_n\rangle \circ \langle z_1, ..., z_m\rangle = \langle x_1, ..., x_n, z_1, ..., z_m\rangle$,
- $s \circ \langle\,\rangle = \langle\,\rangle \circ s = s$.

For $s \in X^*$ and $x \in X$, we sometimes write $s \circ x$ for $s \circ \langle x\rangle$; and $x \in s$ as a shorthand for $\exists t, u \in X^*.\ s = t \circ \langle x\rangle \circ u$. On X^* we define the successor relation $s \triangleleft_X t \Leftrightarrow \exists x \in X.\ t = s \circ \langle x\rangle$. In the following:

- $\triangleleft^0_X$ denotes the reflexive closure of $\triangleleft_X$;
- $\sqsubset_X$ denotes the transitive closure of $\triangleleft_X$;
- $\sqsubseteq_X$ denotes the reflexive and transitive closure of $\triangleleft_X$.

Given three sequences $s, u, v \in X^*$, the *prefix replacement* $s[u \Rsh v]$ is so defined

$$s[u \Rsh v] = \begin{cases} v \circ t & \text{if } s = u \circ t \\ s & \text{otherwise.} \end{cases}$$

When u and v have the same length, the replacement is called *renaming* of u with v.

2.1. A class of normal modal systems

We introduce a class of systems for normal (i.e. extensions of system K) modal logics.

We first define the propositional modal language $\mathcal{L}$ which contains countably infinite *proposition symbols*, $p_0, p_1, \ldots$; the *propositional connectives* $\vee$, $\wedge$, $\rightarrow$, $\neg$; the *modal operators* $\Box$, $\Diamond$; the parenthesis as *auxiliary symbols*.

Definition 2.2. The set $\mathfrak{mf}$ of propositional *modal formulas* of $\mathcal{L}$ is the least set that contains the propositional symbols and is closed under application of the propositional connectives $\rightarrow, \wedge, \vee$ (binary), $\neg$ (unary), and the modal operators $\Box$, $\Diamond$ (unary).

In the following $\mathcal{T}$ denotes a denumerable set of *tokens*, ranged by meta-variables x, y, z, possibly indexed. Let $\mathcal{T}^*$ be the sequences on $\mathcal{T}$, called *positions*; meta-variables α, β, γ, possibly indexed, range over $\mathcal{T}^*$.

Now, *extended–sequents* are tuples of finite sequences of *position-formulas*, i.e. formulas labeled with positions.

Definition 2.3.

1. A *position-formula* (briefly: *p-formula*) is an expression of the form A^α, where A is a modal formula and $\alpha \in \mathcal{T}^*$; $\mathfrak{pf}$ is the set of position formulas.
2. An *extended sequent* (briefly: *e–sequent*) is an expression of the form $\Gamma \vdash \Delta$, where Γ and Δ are finite sequences of p–formulas.

Remark 2.4. An e–sequent is a *linear notation* for the so-called *tree sequents*, or with more modern terminology, *nested sequents*. All this will be clarified in section 5.

Given a sequence Γ of p-formulas, with $\mathfrak{Init}[\Gamma]$ we mean the set $\{\beta : \exists A^\alpha \in \Gamma.\ \beta \sqsubseteq \alpha\}$.

We briefly recall the axiomatic ("Hilbert-style") presentation of normal modal systems. Let Z be a set of formulas. The normal modal logic $\mathfrak{M}[Z]$ is defined as the smallest set X of formulas verifying the following properties:

(i) $Z \subseteq X$

(ii) X contains all instances of the following schemas:

- **1.** $A \rightarrow (B \rightarrow A)$
- **2.** $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$

Axiom schema	Logic
D $\Box A \to \Diamond A$	$\mathsf{K} = \mathfrak{M}[\varnothing]$
T $\Box A \to A$	$\mathsf{D} = \mathfrak{M}[\mathbf{D}]$
4 $\Box A \to \Box\Box A$	$\mathsf{T} = \mathfrak{M}[\mathbf{T}]$
	$\mathsf{K4} = \mathfrak{M}[\mathbf{4}]$
	$\mathsf{S4} = \mathfrak{M}[\mathbf{T}, \mathbf{4}]$
	$\mathsf{D4} = \mathfrak{M}[\mathbf{D}, \mathbf{4}]$

Figure 1. Axioms for systems K, D, T, K4, D4 and S4

3. $((\neg B \to \neg A) \to ((\neg B \to A) \to B))$

K. $\Box(A \to B) \to (\Box A \to \Box B)$

MP If $A, A \to B \in X$ then $B \in X$;

NEC If $A \in X$ then $\Box A \in X$.

We write $\vdash_{\mathfrak{M}[Z]} A$ for $A \in \mathfrak{M}[Z]$. If $N_1, .., N_k$ are names of schemas, the sequence $N_1 \dots N_k$ denotes the set $[N_1] \cup ... \cup [N_1]$, where $[N_i] = \{A : A$ is an instance of the schema $N_i\}$. Figure 1 lists the standard axioms for the well-known modal systems K, D, T, K4, S4. We use $\mathbb{M}$ as generic name for one of these systems.

2.2. The sequent calculi $\mathsf{E}_\mathsf{K}, \mathsf{E}_\mathsf{D}, \mathsf{E}_\mathsf{T}, \mathsf{E}_\mathsf{K4}, \mathsf{E}_\mathsf{S4}, \mathsf{E}_{\mathsf{D}4}$

We introduce a class of e–sequent calculi for the logics K, D, T, K4, and S4. The system is presented only once (Figure 2) for S4: the other calculi are obtained by imposing some constraints on the modal rules and the cut (see Figure 3).

Observe that, as usual in sequent calculi presentations, sequences of formulas (Γ, Δ), or positions (α, β) *may be empty*, except when explicitly forbidden. The constraint on necessitation (rule $\vdash \Box$, and its dual $\Diamond \vdash$) is formulated as a constraint on position occurrences in the context, *analogously to the usual constraint on variable occurrences for* $\forall$*-introduction* ($\exists$-elimination, respectively).

Systems for other logics are obtained by restricting the application of some rules, exployting positions. In particular, rules $\Box \vdash$ and $\vdash \Diamond$ are constrained for all the systems but E_S4. Moreover, for E_K4 and E_K also the cut rule is restricted. Figure 3 lists such constraints.

Axiom and cut

$$A^\alpha \vdash A^\alpha \quad Ax \qquad\qquad \frac{\Gamma_1 \vdash A^\alpha, \Delta_1 \quad \Gamma_2, A^\alpha, \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ Cut$$

Structural rules

$$\frac{\Gamma \vdash \Delta}{\Gamma, A^\alpha \vdash \Delta}\ W\vdash \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A^\alpha, \Delta}\ \vdash W$$

$$\frac{\Gamma, A^\alpha, A^\alpha \vdash \Delta}{\Gamma, A^\alpha \vdash \Delta}\ C\vdash \qquad \frac{\Gamma \vdash A^\alpha, A^\alpha, \Delta}{\Gamma \vdash A^\alpha, \Delta}\ \vdash C$$

$$\frac{\Gamma_1, A^\alpha, B^\beta, \Gamma_2 \vdash \Delta}{\Gamma_1, B^\beta, A^\alpha, \Gamma_2 \vdash \Delta}\ Exc\vdash \qquad \frac{\Gamma \vdash \Delta_1, A^\alpha, B^\beta, \Delta_2}{\Gamma \vdash \Delta_1, B^\beta, A^\alpha, \Delta_2}\ \vdash Exc$$

Propositional rules

$$\frac{\Gamma \vdash A^\alpha, \Delta}{\Gamma, \neg A^\alpha \vdash \Delta}\ \neg\vdash \qquad \frac{\Gamma, A^\alpha \vdash \Delta}{\Gamma \vdash \neg A^\alpha, \Delta}\ \vdash\neg$$

$$\frac{\Gamma, A^\alpha \vdash \Delta}{\Gamma, A \wedge B^\alpha \vdash \Delta}\ \wedge_1\vdash \qquad \frac{\Gamma, B^\alpha \vdash \Delta}{\Gamma, A \wedge B^\alpha \vdash \Delta}\ \wedge_2\vdash$$

$$\frac{\Gamma_1 \vdash A^\alpha, \Delta_1 \quad \Gamma_2 \vdash B^\alpha, \Delta_2}{\Gamma_1, \Gamma_2 \vdash A \wedge B^\alpha, \Delta_1, \Delta_2}\ \vdash\wedge \qquad \frac{\Gamma_1, A^\alpha \vdash \Delta_1 \quad \Gamma_2, B^\alpha \vdash \Delta_2}{\Gamma_1, \Gamma_2, A \vee B^\alpha \vdash \Delta_1, \Delta_2}\ \vee\vdash$$

$$\frac{\Gamma \vdash A^\alpha, \Delta}{\Gamma \vdash A \vee B^\alpha, \Delta}\ \vdash\vee_1 \qquad \frac{\Gamma \vdash B^\alpha, \Delta}{\Gamma \vdash A \vee B^\alpha, \Delta}\ \vdash\vee_2$$

$$\frac{\Gamma_1, B^\alpha \vdash \Delta_1 \quad \Gamma_2 \vdash A^\alpha, \Delta_2}{\Gamma_1, \Gamma_2, A \to B^\alpha \vdash \Delta_1, \Delta_2}\ \to\vdash \qquad \frac{\Gamma, A^\alpha \vdash B^\alpha, \Delta}{\Gamma \vdash A \to B^\alpha, \Delta}\ \vdash\to$$

Figure 2. Rules for the System E_{S4}

Modal rules

$$\frac{\Gamma, A^{\alpha\circ\beta} \vdash \Delta}{\Gamma, \Box A^{\alpha} \vdash \Delta}\ \Box\vdash \qquad \frac{\Gamma \vdash A^{\alpha\circ x}, \Delta}{\Gamma \vdash \Box A^{\alpha}, \Delta}\ \vdash\Box$$

$$\frac{\Gamma, A^{\alpha\circ x} \vdash \Delta}{\Gamma, \Diamond A^{\alpha} \vdash \Delta}\ \Diamond\vdash \qquad \frac{\Gamma \vdash A^{\alpha\circ\beta}, \Delta}{\Gamma \vdash \Diamond A^{\alpha}, \Delta}\ \vdash\Diamond$$

Constraints:
In rules $\vdash \Box$ and $\Diamond \vdash$, no position in Γ, Δ may start with $\alpha \circ x$; that is, $\alpha \circ x \notin \mathfrak{Init}[\Gamma, \Delta]$.

Figure 2 (cont.). Rules for the System E_{S4}

Note that both E_{K4} and E_{K}, in addition to the constraint on the main position β, have also constraints on the context: in the modal rules $\Box \vdash$ and $\vdash \Diamond$ *there must be another formula occurrence* $B^{\alpha\circ\beta\circ\eta}$ *in either* Γ *or* Δ (of course, α and/or η may be empty). This prevents the derivation of $\Box A \to \Diamond A^{\gamma}$ (the p-formula representing axiom **D**).

The notions of proof, provable sequent and height $h(\Pi)$ of a proof Π are standard.

Notation 2.5. In order to simplify the graphical representation of proofs, we will use a double deduction line to indicate application of a rule preceded or followed by a sequence of structural rules. So we will write

$$\frac{\Gamma \vdash \Delta}{\overline{\Sigma \vdash \Theta}}\ r$$

when the e–sequent $\Sigma \vdash \Theta$ has been obtained from the e–sequent $\Gamma \vdash \Delta$ by means of an application of rule r and of a finite number of structural rules.

Remark 2.6 (On the cut rule for E_{K}, E_{K4}).
The constraint is necessary for E_{K4} and E_{K}, since it prevents the derivation of the unsound schema $\Diamond(A \to A)^{\langle\rangle}$ (remember that K and $\mathsf{K4}$ do not validate $\Diamond$**true**). Indeed, without the constraint we could construct the proof-tree:

Calculus	**Constraints on the rules $\Box\vdash$ and $\vdash\Diamond$**
E_{S4}	no constraints
E_{T}	$\beta = \langle\ \rangle$, or β is a singleton sequence $\langle z\rangle$
E_{D}	β is a singleton sequence $\langle z\rangle$
E_{D4}	β is a non-empty sequence
E_{K4}	β is a non-empty sequence; there is at least a formula $B^{\alpha\circ\beta\circ\eta}$ in either Γ or Δ
E_{K}	β is a singleton sequence $\langle z\rangle$; there is at least a formula $B^{\alpha\circ\beta\circ\eta}$ in either Γ or Δ

	Constraints on the cut rule
E_{D}, E_{T}, E_{S4} E_{D4}	no constraints
E_{K}, E_{K4}	$\alpha \in \mathfrak{Init}[\Gamma_1, \Delta_1 - A^\alpha]$ or $\alpha \in \mathfrak{Init}[\Gamma_2 - A^\alpha, \Delta_2]$

Figure 3. Constraints

$$\dfrac{\dfrac{A^{\langle x\rangle} \vdash A^{\langle x\rangle}}{\vdash A \to A^{\langle x\rangle}} \qquad \dfrac{A \to A^{\langle x\rangle} \vdash A \to A^{\langle x\rangle}}{A \to A^{\langle x\rangle} \vdash \Diamond(A \to A)^{\langle\rangle}}}{\vdash \Diamond(A \to A)^{\langle\rangle}}$$

Using the terminology we will introduce shortly in Definition 2.8, we will say that, in order to be sound for E_{K} or E_{K4}, cut formulas must have a *sentinel*. It is easy to see that *modus ponens* (from $\vdash A \to B^\alpha$ and $\vdash A^\alpha$, obtain a derivation of $\vdash B^\alpha$) remains derivable also in presence of this constraint.

Characteristic axioms of normal modal systems are easily derivable, as shown in Section 4.

We introduce now some definitions. The position $\alpha \circ x$ in the rules $\vdash\Box$ and $\Diamond\vdash$ is the *eigenposition* of the rule, by analogy to first-order sequent calculus. It is well known that in first order sequent calculus eigenvariables should be considered as bound variables. In particular, any eigenvariable in a derivation may always be substituted with a *fresh* one (that is, a variable which does not occur in any other place in that derivation), without affecting the provable end sequent (up to renaming of its bound variables). Indeed, one may guarantee that each eigenvariable in a derivation is the eigenvariable of exactly one right $\forall$ or left $\exists$ rule (and, moreover, that variable occurs in the derivation only above the rule of which it is eigenvariable, and it never occurs as a bound variable.) We will show analogous properties for the eigenpositions of e–sequents, in order to define in a sound way a notion of *prefix replacement for proofs* (that we defined at the end of Section 2 for positions). We denote with $\Gamma[\alpha \Rsh \beta]$ the obvious extension of prefix replacement to a sequence Γ of p-formulas.

FACT 2.7. Let $\alpha \circ z$ be an eigenposition. It is always possible to rename the eigenposition as $\alpha \circ z_0$, where z_0 fresh token w.r.t the whole derivation in which the eigenposition occurs.
This assumption ensures that, after a renaming, we cannot have e–sequents $\Gamma \vdash \Delta$ containing both a formula $A^{\alpha \circ z_0}$ with $\alpha \circ z_0$ as eigenposition and other formulas of the shape B^β with $\beta \in \mathfrak{Init}[\alpha \circ z_0]$.

DEFINITION 2.8. An occurrence of a formula A^α in an e–sequent $\Gamma \vdash \Delta$ is said *guarded* if there exists in $\Gamma \vdash \Delta$ an occurrence of a formula $B^{\alpha\circ\delta}$ (δ possibly empty) different from A^α. The formula $B^{\alpha\circ\delta}$ is the *sentinel* of A^α.

PROPOSITION 2.9. Let $\Gamma \vdash \Delta$ be an e–sequent. If a formula A^α is guarded in $\Gamma \vdash \Delta$, then, for any substitution $[\delta \circ z \Rsh \delta \circ \tau]$, the formula $A^{\alpha[\delta\circ z \Rsh \delta\circ\tau]}$ is guarded in $\Gamma[\delta \circ z \Rsh \delta \circ \tau] \vdash \Delta[\delta \circ z \Rsh \delta \circ \tau]$.

PROOF: Let B^β a sentinel formula of A^α (so $\beta = \alpha \circ \gamma$). We distinguish some cases:

1. $A^{\alpha[\delta\circ z \Rsh \delta\circ\tau]} = A^\alpha$ (the prefix of α is different from $\delta \circ z$) and then $B^{\alpha\circ\gamma[\delta\circ z \Rsh \delta\circ\tau]} \equiv B^{\alpha\circ\gamma}$. The are two subcases:

   (a) if $\delta \circ z \notin \mathfrak{Init}[\alpha \circ \gamma]$ then $B^{\alpha\circ\gamma[\delta\circ z \Rsh \delta\circ\tau]} \equiv B^{\alpha\circ\gamma}$.

   (b) if $\delta \circ z \in \mathfrak{Init}[\alpha \circ \gamma]$, since $\alpha[\delta \circ z \Rsh \delta \circ \tau] \equiv \alpha$, then $\alpha \circ \gamma = \alpha\alpha' z\gamma'$ where $\alpha\alpha' = \delta$. Then $\alpha \circ \gamma[\delta \circ z \Rsh \delta \circ \tau] = \alpha\alpha'\tau\gamma'$ and $B^{\alpha\alpha'\tau\gamma'}$ is a supervisor of A^α.

2. $\alpha = \delta \circ z \circ \mu$. In this case we have $A^{\alpha[\delta\circ z \Rsh \delta\circ\tau]} = A^{\delta\circ\tau\circ\mu}$ and $B^{\alpha\circ\gamma[\delta\circ z \Rsh \delta\circ\tau]} = B^{\alpha\circ\tau\circ\mu\circ\gamma}$ and then $A^{\delta\circ\tau\circ\mu}$ is still guarded. $\square$

We now extend the notion of prefix replacement to proofs. The lemmas are valid for all the systems (that is, in presence of the constraints) of Figure 3.

Lemma 2.10. *Let* Π *be an e–sequent proof with conclusion* $\Gamma \vdash \Delta$*. Let* $\delta \circ z$ *be a position, and let* b *be a fresh token (that is, not occurring in either* Π *or* $\delta \circ z$*). Then we may define the* prefix replacement $\Pi[\delta \circ z \Rsh \delta \circ b]$*, a proof with conclusion* $\Gamma[\delta \circ z \Rsh \delta \circ b] \vdash \Delta[\delta \circ z \Rsh \delta \circ b]$*.*

Proof: If Π is an axiom $A^\alpha \vdash A^\alpha$, than $\Pi[\delta \circ z \Rsh \delta \circ b]$ is $A^{\alpha[\delta \circ z \Rsh \delta \circ b]} \vdash A^{\alpha[\delta \circ z \Rsh \delta \circ b]}$.
All inductive cases are trivial, except the modal rules.

If the last rule of Π is

$$\frac{\Gamma \vdash A^{\alpha \circ x}, \Delta}{\Gamma \vdash \Box A^\alpha, \Delta} \quad \vdash \Box$$

let Π' be the subproof rooted at this rule. We have two cases, depending on whether the position $\delta \circ z$ is the eigenposition of the rule. (i) If $\alpha \circ x = \delta \circ z$, obtain by induction the proof $\Pi'[\alpha \circ x \Rsh \alpha \circ b]$ with conclusion $\Gamma \vdash A^{\alpha \circ b}, \Delta$ (remember that $\alpha \circ x \notin \mathfrak{Init}[\Gamma, \Delta]$). Then $\Pi[\delta \circ z \Rsh \delta \circ b]$ is obtained from $\Pi'[\alpha \circ x \Rsh \alpha \circ b]$ by an application of $\vdash \Box$. (ii) If $\alpha \circ x \neq \delta \circ z$, obtain by induction the proof $\Pi'[\delta \circ z \Rsh \delta \circ b]$ with conclusion $\Gamma[\delta \circ z \Rsh \delta \circ b] \vdash A^{\alpha[\delta \circ z \Rsh \delta \circ b] \circ x}, \Delta[\delta \circ z \Rsh \delta \circ b]$. Observe now that $\alpha[\delta \circ z \Rsh \delta \circ b] \circ x$ cannot be an initial segment of a formula in $\Gamma[\delta \circ z \Rsh \delta \circ b], \Delta[\delta \circ z \Rsh \delta \circ b]$. Indeed, if for some B^γ in Γ, Δ we had $\alpha[\delta \circ z \Rsh \delta \circ b] \circ x \sqsubseteq \gamma[\delta \circ z \Rsh \delta \circ b]$, since b is fresh, this could only result from $\alpha \circ x$ being a prefix of γ, which is impossible. Therefore, we may conclude with an application of $\vdash \Box$, since its side-condition is satisfied.

If the last rule of Π is

$$\frac{\Gamma \vdash A^{\alpha \circ \beta}, \Delta}{\Gamma \vdash \Diamond A^\alpha, \Delta} \quad \vdash \Diamond$$

let, as before, Π' be the subproof rooted at this rule and construct by induction the proof $\Pi'[\delta \circ z \Rsh \delta \circ b]$ with conclusion $\Gamma[\delta \circ z \Rsh \delta \circ b] \vdash A^{\alpha \circ \beta[\delta \circ z \Rsh \delta \circ b]}, \Delta[\delta \circ z \Rsh \delta \circ b]$. It is easy to verify that any side condition of the $\vdash \Diamond$ rule (which depends on the specific system, according to the table above), is still verified after the prefix replacement. We may then conclude with a $\vdash \Diamond$ rule.

The left modal rules are analogous. $\Box$

By repeatedly using the previous lemma, we obtain the following.

Proposition 2.11 (Eigenposition renaming). Given a proof Π of an e–sequent $\Gamma \vdash \Delta$, we may always find a proof Π' ending with $\Gamma \vdash \Delta$ where all eigenpositions are distinct from one another.

Π' differs from Π only for the names of positions. In practice we will freely use such a renaming all the times it is necessary (or, in other words, proofs are de facto equivalence classes modulo renaming of eigenpositions). In a similar way to the previous lemmas we may obtain the following, which allows the prefix replacement of arbitrary positions (once eigenpositions are considered as bound variables, and renamed so that any confusion is avoided). When we use prefix replacement for proofs we will always assume that the premises of the following lemma are satisfied, implicitly calling for eigenposition renaming if this is not the case.

Lemma 2.12 (Sequents Prefix Replacement). *Let $\mathbb{M}$ be one of the modal systems* K, D, T, K4, D4, S4*, and let β a position taken according to the constraint for β of figure 3. Let $\delta \circ z$ be a position, and let Π be an $\mathsf{E}_{\mathbb{M}}$ proof of $\Gamma \vdash \Delta$, where all eigenpositions are distinct from one another, and are different from $\delta \circ z$. Then we may define the* prefix replacement *$\Pi[\delta \circ z \Rsh \delta \circ \beta]$, an $\mathsf{E}_{\mathbb{M}}$ proof with conclusion $\Gamma[\delta \circ z \Rsh \delta \circ \beta] \vdash \Delta[\delta \circ z \Rsh \delta \circ \beta]$.*

Proof: The proof proceeds by induction on the length of the proof and by cases on the last rule. Propositional cases are trivial. We focus on modal rules and in particular on the non-serial cases E_{K} and E_{K4}.

System E_{K}: in this case β is a single token.

1. The last rule is $\vdash \Box$. We have two cases:
   (a) The proof has the structure

$$\dfrac{\begin{matrix}\Pi_1\\ \Gamma \vdash A^{\alpha \circ z}, \Delta\end{matrix}}{\Gamma \vdash \Box A^{\alpha}, \Delta}$$

   We can exclude this case by eigenposition renaming.

   (b) The proof has the structure

$$\dfrac{\begin{matrix}\Pi_1\\ \Gamma \vdash A^{\alpha \circ y}, \Delta\end{matrix}}{\Gamma \vdash \Box A^{\alpha}, \Delta}$$

By inductive hypothesis, we have a proof

$$\frac{\Pi_1[\delta \circ z \Rsh \delta \circ \beta]}{\Gamma[\delta \circ z \Rsh \delta \circ \beta] \vdash A^{\alpha \circ z[\delta \circ z \Rsh \delta \circ \beta]}, \Delta[\delta \circ z \Rsh \delta \circ \beta]}$$

Since $\alpha \circ y \notin \mathfrak{Init}[\Gamma, \Delta]$ and we can assume that the token $y \notin \mathfrak{Init}[\Gamma, \Delta, \beta]$, we have $\Gamma[\delta \circ z \Rsh \delta \circ \beta] = \Gamma$, $\Delta[\delta \circ z \Rsh \delta \circ \beta] = \Delta$. If not, by renaming we can replace $\alpha \circ y$ with $\alpha \circ y_0$ with y_0 fresh. Therefore we have that $\alpha \circ y \notin \mathfrak{Init}[\Gamma[\delta \circ z \Rsh \delta \circ \beta], \Delta[\delta \circ z \Rsh \delta \circ \beta]]$ ($\alpha \circ y$ can not appear in the substitution). We can conclude by applying the $\vdash \Box$ rule:

$$\cfrac{\cfrac{\Pi_1[\delta \circ z \Rsh \delta \circ \beta]}{\Gamma[\delta \circ z \Rsh \delta \circ \beta] \vdash A^{\alpha \circ y[\delta \circ z \Rsh \delta \circ \beta]}, \Delta[\delta \circ z \Rsh \delta \circ \beta]}}{\Gamma[\delta \circ z \Rsh \delta \circ \beta] \vdash \Box A^{\alpha[\delta \circ z \Rsh \delta \circ \beta]}, \Delta[\delta \circ z \Rsh \delta \circ \beta]} \vdash \Box$$

There are no additional constraints to satisfy, so this case is clearly sound for $\mathsf{E_K}$

2. The last rule is $\Diamond \vdash$: symmetric to the previous case.

3. The last rule is $\vdash \Diamond$, so we have a proof

$$\cfrac{\cfrac{\Pi_1}{\Gamma \vdash A^{\alpha \circ c}, \Delta}}{\Gamma \vdash \Diamond A^{\alpha}, \Delta}$$

where c is a token and there exists at least a formula $B^{\alpha \circ c \circ \mu}$ in either Γ or Δ. Notice that, since we are in $\mathsf{E_K}$ the $\vdash \Diamond$ rule always modifies the position of the main formula $A^{\alpha \circ c}$ (i.e. in the conclusion we have the formula A^{α}).

By i.h., we apply the prefix replacement on the subproof Π_1:

$$\frac{\Pi_1[\delta \circ z \Rsh \delta \circ \beta]}{\Gamma[\delta \circ z \Rsh \delta \circ \beta] \vdash A^{\alpha \circ c[\delta \circ z \Rsh \delta \circ \beta]}, \Delta[\delta \circ z \Rsh \delta \circ \beta]}$$

We have some cases:

(a) $\delta \circ z \notin \mathfrak{Init}[\alpha \circ c]$, so $\alpha \circ c[\delta \circ z \Rsh \delta \circ \beta] = \alpha \circ c$.
We apply the substitution:

$$\dfrac{\begin{array}{c}\Pi_1[\delta \circ z \Rsh \delta \circ \beta] \\ \Gamma[\delta \circ z \Rsh \delta \circ \beta] \vdash A^{\alpha \circ c[\delta \circ z \Rsh \delta \circ \beta]}, \Delta[\delta \circ z \Rsh \delta \circ \beta]\end{array}}{\Gamma[\delta \circ z \Rsh \delta \circ \beta] \vdash \Diamond A^{\alpha}, \Delta[\delta \circ z \Rsh \delta \circ \beta]}$$

By Proposition 2.9, derivation is sound for K.

(b) $\delta \circ z \in \mathfrak{Init}[\alpha \circ c]$ and $\delta \circ z \in \mathfrak{Init}[\alpha]$, so $\alpha \circ c[\delta \circ z \Rsh \delta \circ \beta] = \alpha[\delta \circ z \Rsh \delta \circ \beta] \circ c$.
Then

$$\dfrac{\begin{array}{c}\Pi_1[\delta \circ z \Rsh \delta \circ \beta] \\ \Gamma[\delta \circ z \Rsh \delta \circ \beta] \vdash A^{\alpha[\delta \circ z \Rsh \delta \circ \beta] \circ c}, \Delta[\delta \circ z \Rsh \delta \circ \beta]\end{array}}{\Gamma[\delta \circ z \Rsh \delta \circ \beta] \vdash \Diamond A^{\alpha[\delta \circ z \Rsh \delta \circ \beta]}, \Delta[\delta \circ z \Rsh \delta \circ \beta]}$$

The proof is sound for K thanks to Proposition 2.9.

(c) $\delta \circ z \in \mathfrak{Init}[\alpha \circ c]$ and $\delta \circ z \notin \mathfrak{Init}[\alpha]$. We have $\alpha \circ c = \delta \circ z$ and $\alpha = \delta$ and $\beta = \langle c \rangle$.
We apply the inductive hypothesis, and we obtain the following proof:

$$\dfrac{\begin{array}{c}\Pi_1[\delta \circ z \Rsh \delta \circ \beta] \\ \Gamma[\delta \circ z \Rsh \delta \circ \beta] \vdash A^{\alpha \circ c[\delta \circ z \Rsh \delta \circ \beta]}, \Delta[\delta \circ z \Rsh \delta \circ \beta]\end{array}}{\Gamma[\delta \circ z \Rsh \delta \circ \beta] \vdash \Diamond A^{\alpha[\delta \circ z \Rsh \delta \circ \beta]}, \Delta[\delta \circ z \Rsh \delta \circ \beta]}$$

since $\Diamond A^{\delta} = \Diamond A^{\alpha[\delta \circ z \Rsh \delta \circ \beta]} = \Diamond A^{\alpha}$
The last step of the derivation is sound for K, by means of Proposition 2.9.

4. The last rule is $\Box \vdash$: symmetric to the previous case.

5. The last rule is a cut. Let A^{α} be the cut-formula. In $\mathsf{E_K}$ we have the constraint $\alpha \in \mathfrak{Init}[\Gamma_1, \Delta_1 - A^{\alpha}]$ or $\alpha \in \mathfrak{Init}[\Gamma_2 - A^{\alpha}, \Delta_2]$.

By i.h., we obtain the proofs

$$\dfrac{\Pi_1[\delta \circ z \Rsh \delta \circ \beta]}{\Gamma_1[\delta \circ z \Rsh \delta \circ \beta] \vdash A^{\alpha[\delta \circ z \Rsh \delta \circ \beta]}, \Delta_1[\delta \circ z \Rsh \delta \circ \beta]}$$

and

$$\dfrac{\Pi_2[\delta \circ z \Rsh \delta \circ \beta]}{\Gamma_2[\delta \circ z \Rsh \delta \circ \beta], A^{\alpha[\delta \circ z \Rsh \delta \circ \beta]} \vdash \Delta_2[\delta \circ z \Rsh \delta \circ \beta]}$$

And therefore we can conclude with a cut

$$\dfrac{\dfrac{\Pi_1[\delta \circ z \Rsh \delta \circ \beta]}{S_1} \qquad \dfrac{\Pi_2[\delta \circ z \Rsh \delta \circ \beta]}{S_2}}{\Gamma \vdash \Delta} \; Cut$$

where:
$S_1 = \Gamma_1[\delta \circ z \Rsh \delta \circ \beta] \vdash A^{\alpha[\delta \circ z \Rsh \delta \circ \beta]}, \Delta_1[\delta \circ z \Rsh \delta \circ \beta]$ and $S_2 = \Gamma_2[\delta \circ z \Rsh \delta \circ \beta], A^{\alpha[\delta \circ z \Rsh \delta \circ \beta]} \vdash \Delta_2[\delta \circ z \Rsh \delta \circ \beta]$.
Notice that the proof above is sound: the constraint on the cut rule ensures that there is at least a formula $B^{\alpha \circ \gamma}$ which is a sentinel for A^{α}. This still holds after the replacement $[\delta \circ z \Rsh \delta \circ \beta]$ by means of Proposition 2.9.

System E_{K4}: in this case β is an arbitrary non void position.

1. The last rule is $\vdash \Box$: as for E_{K}, case 1.
2. The last rule is $\Diamond \vdash$: as for E_{K}, case 2.
3. The last rule is $\vdash \Diamond$, so we have a proof

$$\dfrac{\dfrac{\Pi_1}{\Gamma \vdash A^{\alpha \circ \tau}, \Delta}}{\Gamma \vdash \Diamond A^{\alpha}, \Delta}$$

where τ is a non-empty sequence and there exists at least a formula $B^{\alpha \circ \tau \circ \mu}$ in either Γ or Δ. Since we are in E_{K4}, as in the previous case the $\vdash \Diamond$ rule always modifies the position of the main formula $A^{\alpha \circ \tau}$ (i.e. in the conclusion we have the formula A^{α}).

By i.h., we apply the prefix replacement on the subproof Π_1:

$$\Pi_1[\delta \circ z \Rsh \delta \circ \beta]$$
$$\Gamma[\delta \circ z \Rsh \delta \circ \beta] \vdash A^{\alpha \circ \tau[\delta \circ z \Rsh \delta \circ \beta]}, \Delta[\delta \circ z \Rsh \delta \circ \beta]$$

We have some cases:

(a) $\delta \circ z \notin \mathfrak{Init}[\alpha \circ \tau]$, so $\alpha \circ \tau[\delta \circ z \Rsh \delta \circ \beta] = \alpha \circ \tau$. As for system $\mathsf{E_K}$ case 3a.

(b) $\delta \circ z \in \mathfrak{Init}[\alpha \circ \tau]$ and $\delta \circ z \in \mathfrak{Init}[\alpha]$, so $\alpha \circ \tau[\delta \circ z \Rsh \delta \circ \beta] = \alpha[\delta \circ z \Rsh \delta \circ \beta] \circ \tau$. As for system $\mathsf{E_K}$ case 3b.

(c) $\delta \circ z \in \mathfrak{Init}[\alpha \circ \tau]$ and $\delta \circ z \notin \mathfrak{Init}[\alpha]$.
The position $\alpha \circ \tau$ has the shape $\alpha \circ \tau = \alpha \circ \tau_1 \circ z \circ \tau_2$ (so $\alpha \circ \tau_1 = \delta$) and $\alpha \circ \tau[\delta \circ z \Rsh \delta \circ \beta] = \alpha \circ \tau_1 \circ \beta \circ \tau_2$.
We apply the inductive hypothesis, and we obtain the following proof:

$$\Pi_1[\delta \circ z \Rsh \delta \circ \beta]$$
$$\frac{\Gamma[\delta \circ z \Rsh \delta \circ \beta] \vdash A^{\alpha \tau_1 \circ \beta \circ \tau_2}, \Delta[\delta \circ z \Rsh \delta \circ \beta]}{\Gamma[\delta \circ z \Rsh \delta \circ \beta] \vdash \Diamond A^{\alpha}, \Delta[\delta \circ z \Rsh \delta \circ \beta]}$$

We know that there exists at least a formula $B^{\alpha \circ \tau \circ \mu}$ in either Γ or Δ and thanks to Proposition 2.9 the formula $A^{\alpha \tau_1 \circ \beta \circ \tau_2}$ is still guarded in $\Gamma[\delta \circ z \Rsh \delta \circ \beta]$ or $\Delta[\delta \circ z \Rsh \delta \circ \beta]$ by some $B^{\alpha \circ \tau \circ \mu}[\delta \circ z \Rsh \delta \circ \beta]$. Then the proof is sound.

4. The last rule is $\Box \vdash$: as for system $\mathsf{E_K}$, case 4.
5. The last rule is a cut formula: as for system $\mathsf{E_K}$, case 5 □

3. The cut elimination theorem

We prove the cut-elimination theorem for the e–sequent systems, by adapting the standard techniques for the classical predicate calculus [11]. In particular the reader could appreciate the strong similarity, in the proofs of the mix lemmas, between positions in e–systems and first-order terms in classical logic.

Thanks to the modularity of our proposal, we can prove the mix lemmas *only twice*, once for serial systems and once for non-serial ones.

We start with the usual notions of *subformula* and *degree*. Observe that, as the set of first-order (Gentzen) subformulas of $\forall x A(x)$ contain all the term-instances of $A(x)$, here the set of (position, modal) subformulas of $\Box A^\alpha$ contain all the extensions of the position α in A^α.

DEFINITION 3.1 (subformula). The set $Sub(A^\alpha)$ of *subformulas* of a formula A^α is recursively defined as follows:

$Sub(p^\alpha) = \{p^\alpha\}$ if p is a proposition symbol;

$Sub(\neg A^\alpha) = \{\neg A^\alpha\} \cup Sub(A^\alpha)$;

$Sub(A\#B^\alpha) = \{A\#B^\alpha\} \cup Sub(A^\alpha) \cup Sub(B^\alpha)$, when $\# \in \{\rightarrow, \vee, \wedge\}$;

$Sub(\#A^\alpha) = \{\#A^\alpha\} \cup \{Sub(A^{\alpha\circ\beta}) : \beta \in P\}$, when $\# \in \{\Box, \Diamond\}$.

DEFINITION 3.2 (degree). The *degree* of modal formulas, p-formulas, and e–sequent proofs are defined as follows.

1. The *degree of a modal formula* A, $\mathsf{dg}(A)$, is recursively defined as:
   (a) $\mathsf{dg}(p) = 0$ if p is a proposition symbol;
   (b) $\mathsf{dg}(\neg A) = \mathsf{dg}(\Box A) = \mathsf{dg}(\Diamond A) = \mathsf{dg}(A) + 1$;
   (c) $\mathsf{dg}(A \wedge B) = \mathsf{dg}(A \vee B) = \mathsf{dg}(A \rightarrow B) = \max\{\mathsf{dg}(A), \mathsf{dg}(B)\} + 1$.
2. The *degree of a p-formula* A^α, $\mathsf{dg}(A^\alpha)$, is just $\mathsf{dg}(A)$.
3. The *degree of a proof* Π, $\delta[\Pi]$, is the natural number defined as follows:

$$\delta[\Pi] = \begin{cases} 0 & \text{if } \Pi \text{ is cut-free;} \\ \sup\{\mathsf{dg}(A^\alpha) + 1 : A^\alpha \text{ is a cut formula in } \Pi\} & \text{otherwise.} \end{cases}$$

Let Γ be a sequence of formulas. We denote by $\Gamma - A^\alpha$ the sequence obtained by removing all occurrences of A^α in Γ. When writing $\Gamma, \Gamma' - A^\alpha$ we actually mean $\Gamma, (\Gamma' - A^\alpha)$. In the sequel, ordered pairs of natural numbers are intended to be lexicographically ordered. Hence one can make proofs by induction on pairs of numbers. The height $h(\Pi)$ of a proof Π is defined in the usual way.

We will prove two different "mix lemmas", to take into account that the cut-rule for the systems $\mathsf{E_K}$ and $\mathsf{E_{K4}}$ have special constraints, which are mirrored into the hypothesis of the lemma.

LEMMA 3.3 (Mix Lemma for $\mathsf{E_D}$, $\mathsf{E_T}$, $\mathsf{E_{S4}}$). *Let $\mathcal{S}$ be one of the systems $\mathsf{E_D}$, $\mathsf{E_T}$, $\mathsf{E_{S4}}$. Let $n \in \mathbb{N}$ and let A^α be a formula of degree n. Let now Π, Π' be proofs of the e–sequents $\Gamma \vdash \Delta$ and $\Gamma' \vdash \Delta'$, respectively, satisfying the property $\delta[\Pi], \delta[\Pi'] \leq n$. Then one can obtain in an effective way from Π and Π' a proof $\mathsf{Mix}(\Pi, \Pi')$ of the e–sequent $\Gamma, \Gamma' - A^\alpha \vdash \Delta - A^\alpha, \Delta'$ satisfying the property $\delta[\mathsf{Mix}(\Pi, \Pi')] \leq n$.*

PROOF: The proof proceeds in a standard way, by induction on the pair $\langle h(\Pi), h(\Pi')\rangle$. We highlight only the main points. Let Π and Π' be

$$\frac{\left\{\begin{array}{c}\Pi_i\\ \Gamma_i \vdash \Delta_i\end{array}\right\}_{i\in I}}{\Gamma \vdash \Delta}\, r \qquad \text{and} \qquad \frac{\left\{\begin{array}{c}\Pi'_j\\ \Gamma'_j \vdash \Delta'_j\end{array}\right\}_{j\in I'}}{\Gamma' \vdash \Delta'}\, r'$$

respectively, where I and I' are $\varnothing$ (in case of an axiom), $\{1\}$ or $\{1, 2\}$. We proceed by cases.

1. r is Ax.

   If $\Gamma \vdash \Delta$ is $A^\alpha \vdash A^\alpha$, then one gets $\mathsf{Mix}(\Pi, \Pi')$ from Π' by means of a suitable sequence of structural rules.

   If $\Gamma \vdash \Delta$ is $B^\beta \vdash B^\beta$, for $B \neq A$ or $\beta \neq \alpha$, then one gets $\mathsf{Mix}(\Pi, \Pi')$ from Π by a suitable sequence of structural rules.

2. r' is Ax. This case is symmetric to case 1.

3. r is a structural rule. Apply the induction hypothesis to the pair $\langle \Pi_1, \Pi'\rangle$, then apply a suitable sequence of structural rules to get the conclusion.

4. r' is a structural rule. This case is symmetric to 3.

5. r is a cut or a logical rule not introducing A^α to the right.
   Apply the induction hypothesis to each pair $\langle \Pi_i, \Pi'\rangle$, so obtaining the proof $\mathsf{Mix}(\Pi_i, \Pi')$, for $i \in I$. The proof $\mathsf{Mix}(\Pi, \Pi')$ is then

$$\frac{\left\{\begin{array}{c}\mathsf{Mix}(\Pi_i, \Pi')\\ \Gamma_i, \Gamma' - A^\alpha \vdash \Delta_i - A^\alpha, \Delta'\end{array}\right\}_{i\in I}}{\Gamma, \Gamma' - A^\alpha \vdash \Delta - A^\alpha, \Delta'}\, r$$

6. r' is a cut or a logical rule not introducing A^α to the left. This case is symmetric to 5.

7. r is a logical rule introducing A^α to the right and r' is a logical rule introducing A^α to the left.

   (a) r is a propositional rule. This subcase is treated as in the first order case (see, for instance, [11] or [25]).

   (b) A is $\Box B$. Let Π and Π' be

$$\dfrac{\begin{array}{c}\Pi_1\\ \Gamma \vdash B^{\alpha\circ x}, \Delta_1\end{array}}{\Gamma \vdash A^\alpha, \Delta_1} \qquad \text{and} \qquad \dfrac{\begin{array}{c}\Pi'_1\\ \Gamma'_1, B^{\alpha\circ\beta} \vdash \Delta'\end{array}}{\Gamma'_1, A^\alpha \vdash \Delta'}$$

   respectively. Apply the induction hypothesis to the pairs of proofs $\langle \Pi_1[\alpha\circ x \Rsh \alpha\circ\beta], \Pi' \rangle$ and $\langle \Pi, \Pi'_1 \rangle$, obtaining $\mathsf{Mix}(\Pi_1[\alpha\circ x \Rsh \alpha\circ\beta], \Pi')$ and $\mathsf{Mix}(\Pi, \Pi'_1)$, respectively (both of degree less or equal n). The proof $\mathsf{Mix}(\Pi, \Pi')$ is then

$$\dfrac{\dfrac{\begin{array}{c}\mathsf{Mix}(\Pi_1[\alpha\circ x \Rsh \alpha\circ\beta], \Pi')\\ \Gamma, \Gamma'_1 - A^\alpha \vdash B^{\alpha\circ\beta}, \Delta_1 - A^\alpha, \Delta'\end{array} \qquad \begin{array}{c}\mathsf{Mix}(\Pi, \Pi'_1)\\ \Gamma, \Gamma'_1 - A^\alpha, B^{\alpha\circ\beta} \vdash \Delta_1 - A^\alpha, \Delta'\end{array}}{\Gamma, \Gamma'_1 - A^\alpha, \Gamma, \Gamma'_1 - A^\alpha \vdash \Delta_1 - A^\alpha, \Delta', \Delta_1 - A^\alpha, \Delta'}\,Cut}{\Gamma, \Gamma'_1 - A^\alpha \vdash \Delta_1 - A^\alpha, \Delta'}$$

   (c) A is $\Diamond B$. This subcase is symmetric to the case $\Box B$.

In all cases involving new cuts, since the additional cuts are performed on strict subformulas of A^α with degree less than n, we immediately get $\delta[\mathsf{Mix}(\Pi, \Pi')] \leq n$. $\Box$

The above proof does not go through for the systems $\mathsf{E_K}$ and $\mathsf{E_{K4}}$, because of the constraint on the context for the rules $\Box\vdash$ and $\vdash\Diamond$. Indeed, the case (5) of the proof would fail, as shown by the following two proof fragments. Let $\alpha = \beta \circ x$ be the position of the statement of the lemma,

$$\dfrac{\begin{array}{c}\Pi_1\\ \vdash B^{\beta\circ x}, A^{\beta\circ x}\end{array}}{\vdash \Diamond B^\beta, A^{\beta\circ x}} \qquad \text{and} \qquad \begin{array}{c}\Pi'\\ A^{\beta\circ x} \vdash C^\beta\end{array}$$

If we apply the induction hypothesis to the pair $\langle \Pi_1, \Pi' \rangle$ we obtain

$$\begin{array}{c} \mathsf{Mix}(\Pi,\Pi') \\ \vdash B^{\beta \circ x}, C^{\beta} \end{array}$$

and now it is impossible to conclude with the $\vdash \Diamond$ rule, because via the induction hypothesis we deleted the only formula essential to validate the $\vdash \Diamond$ rule. To fix the problem, we need a stronger statement of the lemma, which mirrors the constraint of the cut rule of $\mathsf{E_K}$ and $\mathsf{E_{K4}}$.

Lemma 3.4 (Mix Lemma for $\mathsf{E_K}$, $\mathsf{E_{K4}}$). *Let $\mathcal{S}$ be one of the systems $\mathsf{E_K}$ or $\mathsf{E_{K4}}$. Let $n \in \mathbb{N}$ and let A^α be a formula of degree n. Let now Π, Π' be proofs of the e–sequents $\Gamma \vdash \Delta$ and $\Gamma' \vdash \Delta'$, respectively, satisfying the properties:*

- $\delta[\Pi], \delta[\Pi'] \leq n$;
- $\alpha \in \mathfrak{Init}[\Gamma, \Delta - A^\alpha]$, *or* $\alpha \in \mathfrak{Init}[\Gamma' - A^\alpha, \Delta']$

Then one can obtain in an effective way from Π and Π' a proof $\mathsf{Mix}(\Pi, \Pi')$ of the e–sequent $\Gamma, \Gamma' - A^\alpha \vdash \Delta - A^\alpha, \Delta'$ satisfying the property $\delta[\mathsf{Mix}(\Pi, \Pi')] \leq n$.

Proof: The proof proceeds as for the previous lemma, with special care for cases (5) and (7).

1.–4. As in Lemma 3.3

5. r is a cut or a logical rule not introducing A^α to the right.
   Apply the induction hypothesis to each pair $\langle \Pi_i, \Pi' \rangle$, so obtaining the proof $\mathsf{Mix}(\Pi_i, \Pi')$, for $i \in I$. The proof $\mathsf{Mix}(\Pi, \Pi')$ is then

$$\dfrac{\left\{ \begin{array}{c} \mathsf{Mix}(\Pi_i, \Pi') \\ \Gamma_i, \Gamma' - A^\alpha \vdash \Delta_i - A^\alpha, \Delta' \end{array} \right\}_{i \in I}}{\Gamma, \Gamma' - A^\alpha \vdash \Delta - A^\alpha, \Delta'} \, r$$

   Notice that in the case of r is a cut rule one has the further constrains from Figure 3: $\alpha \in \mathfrak{Init}[\Gamma_1, \Delta_1 - A^\alpha]$ or $\alpha \in \mathfrak{Init}[\Gamma_2 - A^\alpha, \Delta_2]$.

6. r' is a cut or a logical rule not introducing A^α to the left.
   This case is symmetric to 3.

7. r is a logical rule introducing A^α to the right and r' is a logical rule introducing A^α to the left.

   (a) r is a propositional rule.
   This subcase is treated as in the classical logic case (see, for instance, [11] or [25]). Here we show only the case when A is of the form $B \to C$.

   Let Π and Π' be respectively

$$\dfrac{\begin{array}{c}\Pi_1\\ \Gamma, B^\alpha \vdash C^\alpha, \Delta\end{array}}{\Gamma \vdash B \to C^\alpha, \Delta} \quad \text{and} \quad \dfrac{\begin{array}{c}\Pi'_1\\ \Gamma'_1, C^\alpha \vdash \Delta'_1\end{array} \qquad \begin{array}{c}\Pi'_2\\ \Gamma'_2 \vdash B^\alpha, \Delta'_2\end{array}}{\Gamma'_1, \Gamma'_2, B \to C^\alpha \vdash \Delta'_1, \Delta'_2}$$

   Apply the induction hypothesis to the pairs of proofs $\langle \Pi, \Pi'_2 \rangle$, $\langle \Pi_1, \Pi' \rangle$ and $\langle \Pi, \Pi'_1 \rangle$, obtaining the following proofs:

   - $\mathsf{Mix}(\Pi, \Pi'_2)$ of the sequent $\Gamma, \Gamma'_2 - A^\alpha \vdash B^\alpha, \Delta - A^\alpha, \Delta'_2$ with constraints $\alpha \in \mathfrak{Init}[\Gamma, \Delta - A^\alpha]$ or $\alpha \in \mathfrak{Init}[\Gamma'_2 - A^\alpha, \Delta'_2, B^\alpha]$ (B^α acts as a sentinel formula for A^α).
   - $\mathsf{Mix}(\Pi_1, \Pi')$ of the sequent $\Gamma, \Gamma'_1 - A^\alpha, \Gamma'_2 - A^\alpha B^\alpha \vdash C^\alpha, \Delta - A^\alpha, \Delta'_1, \Delta'_2$ with constraints $\alpha \in \mathfrak{Init}[\Gamma, B^\alpha, \Delta - A^\alpha, C^\alpha]$ or $\alpha \in \mathfrak{Init}[\Gamma'_1 - A^\alpha, \Gamma'_2 - A^\alpha, \Delta'_1, \Delta'_2]$ (both B^α and C^α act as sentinel formulas for A^α).
   - $\mathsf{Mix}(\Pi, \Pi'_1)$ of the sequent $\Gamma, \Gamma'_1 - A^\alpha, C^\alpha \vdash \Delta - A^\alpha, \Delta'_1$ with constraints $\alpha \in \mathfrak{Init}[\Gamma, \Delta - A^\alpha]$ or $\alpha \in \mathfrak{Init}[\Gamma'_1 - A^\alpha, C^\alpha, \Delta'_1]$ (C^α acts as a sentinel formula for A^α).

   The proof $\mathsf{Mix}(\Pi, \Pi')$ is then obtained as follows. Cut first $\mathsf{Mix}(\Pi, \Pi'_2)$ against $\mathsf{Mix}(\Pi_1, \Pi')$ to obtain the following proof Υ:

$$\dfrac{\begin{array}{c}\mathsf{Mix}(\Pi, \Pi'_2)\\ \Gamma, \Gamma'_2 - A^\alpha \vdash B^\alpha, \Delta - A^\alpha, \Delta'_2\end{array} \qquad \begin{array}{c}\mathsf{Mix}(\Pi_1, \Pi')\\ \Gamma, \Gamma'_1 - A^\alpha, \Gamma'_2 - A^\alpha, B^\alpha \vdash C^\alpha, \Delta - A^\alpha, \Delta'_1, \Delta'_2\end{array}}{\Gamma, \Gamma'_1 - A^\alpha, \Gamma'_2 - A^\alpha \vdash C^\alpha, \Delta - A^\alpha, \Delta'_1, \Delta'_2}\ Cut$$

Cut now Υ against $\mathsf{Mix}(\Pi,\Pi'_1)$, obtaining the final proof

$$\dfrac{\begin{matrix}\Upsilon \\ \Gamma,\Gamma'_1-A^\alpha,\Gamma'_2-A^\alpha\vdash C^\alpha,\Delta-A^\alpha,\Delta'_1,\Delta'_2\end{matrix}\qquad\begin{matrix}\mathsf{Mix}(\Pi,\Pi'_1)+\text{right wekenings of }\Delta'_2 \\ \Gamma,\Gamma'_1-A^\alpha,C^\alpha\vdash\Delta-A^\alpha,\Delta'_1,\Delta'_2\end{matrix}}{\Gamma,\Gamma'_1-A^\alpha,\Gamma'_2-A^\alpha\vdash\Delta-A^\alpha,\Delta'_1,\Delta'_2}\,Cut$$

The cut in Υ is soundly applied, since at least C^α acts as a sentinel for the rule, so the constraints are verified. As for the last cut, let as check that in all possible subcases there exists a sentinel formula for the cut formula C^α. In building $\mathsf{Mix}(\Pi,\Pi')$ we know there is a sentinel formula for A^α, of shape $D^{\alpha\circ\mu}$, somewhere in the contexts: either $D^{\alpha\circ\mu}$ is in Γ, or is in $\Delta-A^\alpha$, or is in Γ'_1-A^α, or is in Γ'_2-A^α, or is in Δ'_1, or finally is in Δ'_2.

In these cases:

i. if $D^{\alpha\circ\mu}$ is in Γ, or is in Γ'_1-A^α, or is in Γ'_2-A^α, then $D^{\alpha\circ\mu}$ is a sentinel for C^α, because it appears on the left of $\vdash$ in the first premise of the cut;

ii. if $D^{\alpha\circ\mu}$ is in $\Delta-A^\alpha$, or is in Δ'_1, then $D^{\alpha\circ\mu}$ is a sentinel for C^α, because it appears on the right of $\vdash$ in the second premise of the cut;

iii. if $D^{\alpha\circ\mu}$ is in Δ'_2, note that we have added Δ'_2 (with suitable right weakenings) to the conclusion of $\mathsf{Mix}(\Pi,\Pi'_1)$, so that $D^{\alpha\circ\mu}$ could be a sentinel for the cut formula (observe that the conclusion of the whole proof does not change, since Δ'_2 is already present there.)

(b) A is $\Box B$.

Let Π, r, Π' and r' be respectively

$$\dfrac{\begin{matrix}\Pi_1\\ \Gamma\vdash B^{\alpha\circ x},\Delta\end{matrix}}{\Gamma\vdash A^\alpha,\Delta}\vdash\Box \qquad\text{and}\qquad \dfrac{\begin{matrix}\Pi'_1\\ \Gamma',B^{\alpha\circ\beta}\vdash\Delta'\end{matrix}}{\Gamma',A^\alpha\vdash\Delta'}\Box\vdash$$

Recall that for r we have the constraint for the $\vdash \Box$ rule $\alpha \circ x \notin \mathfrak{Init}[\Gamma, \Delta]$ and, since we are in K or in $\mathsf{K4}$, we have also the constraints on $r' = \Box\vdash$, namely there must be a sentinel for $B^{\alpha\circ\beta}$.

Let us suppose to be in $\mathsf{E_K}$.
In this case β is a singleton, i.e. $\beta = \langle z \rangle$ and one also requires that there exists at least a formula $D^{\alpha\circ z\circ\mu}$ in Γ' or in Δ'. Apply the induction hypothesis to the pairs of proofs $\langle \Pi_1[\alpha \circ x \Rsh \alpha \circ z], \Pi' \rangle$ and $\langle \Pi, \Pi'_1 \rangle$, obtaining a proof $\mathsf{Mix}(\Pi_1[\alpha \circ x \Rsh \alpha \circ z], \Pi')$ of $\Gamma[\alpha \circ x \Rsh \alpha \circ z], \Gamma' - A^\alpha \vdash B^{\alpha\circ z}, \Delta[\alpha \circ x \Rsh \alpha \circ z] - A^\alpha, \Delta'$ and a proof $\mathsf{Mix}(\Pi, \Pi'_1)$ of $\Gamma, \Gamma' - A^\alpha, B^{\alpha\circ z} \vdash, \Delta - A^\alpha, \Delta'$.

Thanks to the constraints on the $\vdash \Box$ rule and Proposition 2.11 (eigenposition renaming), it holds that $\Gamma[\alpha \circ x \Rsh \alpha \circ z] = \Gamma$ and $\Delta[\alpha \circ x \Rsh \alpha \circ z] = \Delta$, so we can drop the substitution from the contexts Γ and Δ.

Notice that we soundly applied the induction hypothesis. To check this, it enough to verify the constraint $\alpha \in \mathfrak{Init}[\Gamma, \Delta - A^\alpha]$ or $\alpha \in \mathfrak{Init}[\Gamma' - A^\alpha, \Delta']$ is satisfied both by $\mathsf{Mix}(\Pi_1[\alpha \circ x \Rsh \alpha \circ z], \Pi')$ and $\mathsf{Mix}(\Pi, \Pi'_1)$.

For the proof $\mathsf{Mix}(\Pi_1[\alpha \circ x \Rsh \alpha \circ z], \Pi')$ of the sequent $\Gamma, \Gamma' - A^\alpha \vdash B^{\alpha\circ z}, \Delta - A^\alpha, \Delta'$ we know there is at least a formula $D^{\alpha\circ z\circ\mu}$ in Γ' or in Δ', and so the constraint is verified. This holds also for for the proof $\mathsf{Mix}(\Pi, \Pi'_1)$ of the sequent $\Gamma, \Gamma' - A^\alpha, B^{\alpha\circ z} \vdash \Delta - A^\alpha, \Delta'$, thanks to the presence of $B^{\alpha\circ z}$.

The proof $\mathsf{Mix}(\Pi, \Pi')$ is then

$$\dfrac{\dfrac{\dfrac{\mathsf{Mix}(\Pi_1[\alpha \circ x \Rsh \alpha \circ z], \Pi')}{\Gamma, \Gamma' - A^\alpha \vdash B^{\alpha\circ z}, \Delta - A^\alpha, \Delta'} \qquad \dfrac{\mathsf{Mix}(\Pi, \Pi'_1)}{\Gamma, \Gamma' - A^\alpha, B^{\alpha\circ z} \vdash \Delta - A^\alpha, \Delta'}}{\Gamma, \Gamma' - A^\alpha, \Gamma, \Gamma' - A^\alpha \vdash \Delta - A^\alpha, \Delta', \Delta - A^\alpha, \Delta'}\ Cut}{\Gamma, \Gamma' - A^\alpha \vdash \Delta - A^\alpha, \Delta'}$$

Notice that the application of the cut rule is sound, i.e. the following constraints is satisfied: $\alpha \circ z \in \mathfrak{Init}[\Gamma, \Gamma' - A^{\alpha}, (\Delta - A^{\alpha}, \Delta') - B^{\alpha \circ z}]$ or $\alpha \circ z \in \mathfrak{Init}[(\Gamma, \Gamma' - A^{\alpha}) - B^{\alpha \circ z}, \Delta - A^{\alpha}, \Delta']$. We have two cases: if there exists in Δ' a formula $C^{\alpha \circ z \circ \mu}$ (constraint of $\mathsf{K}$to the $\Box \vdash \Gamma'$) we are done; if there is a formula $C^{\alpha \circ z}$ that belongs to Γ', it also belongs to $\Gamma' - A^{\alpha}$ and we can conclude.

If we are in $\mathsf{E_K}4$, then $\beta = \delta$ with $\delta \neq \langle\rangle$ and we proceed exactly as for $\mathsf{E_K}$.

(c) A is $\Diamond B$. This subcase is symmetric to the previous one.

In all cases involving new cuts, since the additional cuts are performed on strict subformulas of A^{α}, with degree less than n we immediately get $\delta[\mathsf{Mix}(\Pi, \Pi')] \leq n$. □

Theorem 3.5 (Cut elimination for $\mathsf{E_D}$, $\mathsf{E_T}$, $\mathsf{E_{D4}}$ $\mathsf{E_{S4}}$). *Let $\mathbb{M}$ be one of the modal systems $\mathsf{E_D}$, $\mathsf{E_T}$, $\mathsf{E_{D4}}$ and $\mathsf{E_{S4}}$. If Π is a $\mathsf{E}_{\mathbb{M}}$–proof of $\Gamma \vdash \Delta$, then there exists a cut-free $\mathsf{E}_{\mathbb{M}}$–proof Π^* of $\Gamma \vdash \Delta$.*

Proof: By induction on the pair $\langle \delta[\Pi], h(\Pi) \rangle$. Suppose Π is not cut-free and let r be the last rule applied in Π. We distinguish two cases:

1. r is not a cut. Let Π be

$$\frac{\left\{ \begin{array}{c} \Pi_i \\ \Gamma_i \vdash \Delta_i \end{array} \right\}_{i \in I}}{\Gamma \vdash \Delta} \, r,$$

where I is one of $\{1\}$, $\{1,2\}$ Apply the induction hypothesis to each Π_i, obtaining cut-free proofs Π_i^*, for $i \in I$. A cut-free proof Π^* of $\Gamma \vdash \Delta$ is then

$$\frac{\left\{ \begin{array}{c} \Pi_i^* \\ \Gamma_i \vdash \Delta_i \end{array} \right\}_{i \in I}}{\Gamma \vdash \Delta} \, r$$

2. r is a cut. Let Π be

$$\dfrac{\begin{array}{c}\Pi_1\\ \Gamma_1 \vdash A^\alpha, \Delta_1\end{array} \qquad \begin{array}{c}\Pi_2\\ \Gamma_2, A^\alpha \vdash \Delta_2\end{array}}{\Gamma \vdash \Delta}\ Cut$$

We apply the induction hypothesis to Π_1 and Π_2 in order to obtain cut-free proofs Π_1^* and Π_2^* of $\Gamma_1 \vdash A^\alpha, \Delta_1$ and $\Gamma_2, A^\alpha \vdash \Delta_2$ respectively.

Applying Lemma 3.3 to the pair $\langle \Pi_1^*, \Pi_2^* \rangle$, one gets a proof Π_0 of sequent $\Gamma_1, \Gamma_2 - A^\alpha \vdash \Delta_1 - A^\alpha, \Delta_2$ such that $\delta[\Pi_0] \leq \mathsf{dg}(A^\alpha) < \delta[\Pi]$.

Finally one gets a cut-free proof of $\Gamma_1, \Gamma_2 - A^\alpha \vdash \Delta_1 - A^\alpha, \Delta_2$ from Π_0 by induction hypothesis and, from it, a cut-free proof of $\Gamma \vdash \Delta$ by application of a suitable sequence of structural rules. $\square$

THEOREM 3.6 (Cut elimination for $\mathsf{E_K}$, $\mathsf{E_{K4}}$). *Let $\mathbb{M}$ be one of the modal systems $\mathsf{E_K}$ and $\mathsf{E_{K4}}$. If Π is a $\mathsf{E_\mathbb{M}}$–proof of $\Gamma \vdash \Delta$, then there exists a cut-free $\mathsf{E_\mathbb{M}}$–proof Π^* of $\Gamma \vdash \Delta$.*

PROOF: By induction on the pair $\langle \delta[\Pi], h(\Pi) \rangle$. Suppose Π is not cut-free and let r be the last rule applied in Π. We distinguish two cases:

1. r is not a cut. Let Π be

$$\dfrac{\left\{\begin{array}{c}\Pi_i\\ \Gamma_i \vdash \Delta_i\end{array}\right\}_{i \in I}}{\Gamma \vdash \Delta}\ r,$$

where I is one of $\{1\}$, $\{1, 2\}$ Apply the induction hypothesis to each Π_i, obtaining cut-free proofs Π_i^*, for $i \in I$. A cut-free proof Π^* of $\Gamma \vdash \Delta$ is then

$$\dfrac{\left\{\begin{array}{c}\Pi_i^*\\ \Gamma_i \vdash \Delta_i\end{array}\right\}_{i \in I}}{\Gamma \vdash \Delta}\ r$$

2. r is a cut. Let Π be

$$\dfrac{\begin{array}{c}\Pi_1 \\ \Gamma_1 \vdash A^\alpha, \Delta_1\end{array} \qquad \begin{array}{c}\Pi_2 \\ \Gamma_2, A^\alpha \vdash \Delta_2\end{array}}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ Cut$$

where we know that $\alpha \in \mathfrak{Init}[\Gamma_1, \Delta_1 - A^\alpha]$ or $\alpha \in \mathfrak{Init}[\Gamma_2 - A^\alpha, \Delta_2]$.

Apply the induction hypothesis to Π_1 and Π_2 to obtain cut-free proofs Π_1^* and Π_2^* of $\Gamma_1 \vdash A^\alpha, \Delta_1$ and $\Gamma_2, A^\alpha \vdash \Delta_2$ respectively. Notice that $\delta[\Pi_1^*], \delta[\Pi_2^*] \le \delta[A^\alpha] = n$.

Applying Lemma 3.4 to the pair $\langle \Pi_1^*, \Pi_2^* \rangle$, one gets a proof Π_0 of sequent $\Gamma_1, \Gamma_2 - A^\alpha \vdash \Delta_1 - A^\alpha, \Delta_2$ such that $\delta[\Pi_0] \le \delta[A^\alpha] < \delta[\Pi]$ and $\alpha \in \mathfrak{Init}[\Gamma_1, \Delta_1 - A^\alpha]$, or $\alpha \in \mathfrak{Init}[\Gamma_2 - A^\alpha, \Delta_2]$. Notice that this is the same as we had for the last rule of Π.

Finally one gets a cut-free proof of $\Gamma_1, \Gamma_2 - A^\alpha \vdash \Delta_1 - A^\alpha, \Delta_2$ from Π_0 by induction hypothesis and, from it, a cut-free proof of $\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2$ via a suitable sequence of structural rules. □

Let $\mathbb{M}$ be one of the systems $\mathsf{K}, \mathsf{D}, \mathsf{T}, \mathsf{K4}, \mathsf{D4}, \mathsf{S4}$. Subformula Property and Consistency follows as immediate corollaries of cut-elimination.

Corollary 3.7 (Subformula Property). Each formula occurring in a cut-free $\mathsf{E}_\mathbb{M}$-proof Π is a subformula of some formula occurring in the conclusion of Π.

Corollary 3.8 (Consistency). $\mathsf{E}_\mathbb{M}$ is consistent, namely there is no $\mathsf{E}_\mathbb{M}$-proof of the empty sequent $\vdash$.

4. E–sequent calculi are equivalent to standard calculi

The systems introduced in the previous sections prove the same theorems of the Hilbert-style presentation of the corresponding logics. Let $\mathbb{M}$ be one of the logics K, $\mathsf{K4}$, D, $\mathsf{D4}$ and $\mathsf{S4}$. We start with a proof that, if $\mathbb{M}$ proves A, then $\mathsf{E}_\mathbb{M}$ proves $\vdash A^{\langle\rangle}$. We show the derivations for the modal axioms. Observe that the proof of each axiom satisfies the constraints on $\Box\vdash$ and $\vdash\Diamond$ of the corresponding e–system.

Axiom K

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B^{\langle x\rangle} \vdash B^{\langle x\rangle} \quad A^{\langle x\rangle} \vdash A^{\langle x\rangle}}{A^{\langle x\rangle}, A \to B^{\langle x\rangle} \vdash B^{\langle x\rangle}}\to\vdash}{A^{\langle x\rangle}, \Box(A \to B)^{\langle\ \rangle} \vdash B^{\langle x\rangle}}\Box\vdash}{\Box A^{\langle\ \rangle}, \Box(A \to B)^{\langle\ \rangle} \vdash B^{\langle x\rangle}}\Box\vdash}{\Box A^{\langle\ \rangle}, \Box(A \to B)^{\langle\ \rangle} \vdash \Box B^{\langle\ \rangle}}\vdash\Box}{\Box(A \to B)^{\langle\ \rangle} \vdash \Box A \to \Box B^{\langle\ \rangle}}\vdash\to}{\vdash \Box(A \to B) \to (\Box A \to \Box B)^{\langle\ \rangle}}\vdash\to$$

Axiom D

$$\dfrac{\dfrac{\dfrac{A^{\langle x\rangle} \vdash A^{\langle x\rangle}}{\Box A^{\langle\ \rangle} \vdash A^{\langle x\rangle}}\Box\vdash}{\Box A^{\langle\ \rangle} \vdash \Diamond A^{\langle\ \rangle}}\vdash\Diamond}{\vdash \Box A \to \Diamond A^{\langle\ \rangle}}\vdash\to$$

Axiom T

$$\dfrac{\dfrac{A^{\langle\ \rangle} \vdash A^{\langle\ \rangle}}{\Box A^{\langle\ \rangle} \vdash A^{\langle\ \rangle}}\Box\vdash}{\Box A \to A^{\langle\ \rangle}}\vdash\to$$

Axiom 4

$$\dfrac{\dfrac{\dfrac{\dfrac{A^{\langle y,x\rangle} \vdash A^{\langle y,x\rangle}}{\Box A^{\langle\ \rangle} \vdash A^{\langle y,x\rangle}}\Box\vdash}{\Box A^{\langle\ \rangle} \vdash \Box A^{\langle y\rangle}}\vdash\Box}{\Box A^{\langle\ \rangle} \vdash \Box\Box A^{\langle\ \rangle}}\vdash\Box}{\vdash \Box A \to \Box\Box A^{\langle\ \rangle}}\vdash\to$$

Closure under **MP** is trivially obtained by means of the cut rule. We provide a similar construction in [18] where we study a natural deduction formulations of e–systems.

Finally, closure under **NEC** is obtained by showing that all positions in a provable sequent may be "lifted" by any prefix. Observe first that, for $\Gamma = A_1^{\gamma_1}, \ldots, A_n^{\gamma_n}$, we have $\Gamma[\langle\rangle \Rsh \beta] = A_1^{\beta\circ\gamma_1}, \ldots, A_n^{\beta\circ\gamma_n}$.

PROPOSITION 4.1 (lift). Let $\mathbb{M}$ be one of the modal systems K, D, T, $\mathsf{K4}$, $\mathsf{D4}$, $\mathsf{S4}$, and let β be a position. If $\Gamma \vdash \Delta$ is provable in $\mathsf{E}_{\mathbb{M}}$, so is the e–sequent $\Gamma[\langle\rangle \Rsh \beta] \vdash \Delta[\langle\rangle \Rsh \beta]$.

PROOF: Like Lemma 2.12: Standard induction on derivations (with a suitable renaming of eigenpositions). It is easily verified that the constraints on the modal rules remain satisfied. □

COROLLARY 4.2 (closure under **NEC**). Let $\mathbb{M}$ be one of the modal systems K, D, T, $\mathsf{K4}$, $\mathsf{D4}$, $\mathsf{S4}$. If $\vdash A^{\langle\,\rangle}$ is provable in $\mathsf{E}_{\mathbb{M}}$ so is the e–sequent $\vdash \Box A^{\langle\,\rangle}$.

We can finally state the first direction of the equivalence result.

THEOREM 4.3. *Let $\mathbb{M}$ be one of the modal systems K, D, T, $\mathsf{K4}$, $\mathsf{D4}$, $\mathsf{S4}$. If $\vdash_{\mathbb{M}} A$, the e–sequent $\vdash A^{\langle\,\rangle}$ is provable in $\mathsf{E}_{\mathbb{M}}$.*

As for the other direction, Fitting introduced tableaux systems for a large class of modal logics (see also Section 5) and proved their equivalence to the corresponding Hilbert style systems [8, pages 398–400]. Labels in Fitting's tableaux play the same role as our positions, and the semantics he proposes works for our systems. In particular, his proof of soundness also readily gives the proof we need. We simply state the result:

THEOREM 4.4. *Let $\mathbb{M}$ be one of the modal systems K, D, T, $\mathsf{K4}$, $\mathsf{D4}$, $\mathsf{S4}$. If the e–sequent $\vdash A^{\langle\,\rangle}$ is provable in $\mathsf{E}_{\mathbb{M}}$, then $\vdash_{\mathbb{M}} A$.*

Alternatively, a direct proof can be found in [18], although formulated in an equivalent natural deduction presentation of our e–sequents.

5. Related work

We discuss in this section some alternative proposals of modal systems, related to our extended sequents.

We start from the 2–sequents/linear nested sequents tradition. We then analyse the work of Fitting [9, 7, 8], Mints [22] and Cerrato [4, 5]. We also make a quick comparison with the so-called *Labeled Deductive Systems*, which represent an important field of studies. A more in-depth comparison, in the setting of natural deduction systems, may be found in [18].

5.1. Starting point: 2–sequents and linear nested sequents

The systems we studied in this paper are, in their current formulation, strongly similar to the ones proposed by Fitting [9, 7, 8] and by Mints [22]. However, our research started from other grounds, that of 2–Sequents [19, 27], especially as presented in [17]. In that paper the first and second authors propose a natural deduction system for the negative fragment ($\to$, $\wedge$, $\Box$) of modal logic, towards a proof theory for the normal modal logics D,T,D4 and S4. If we rephrase in a sequent calculus setting the natural deduction rules of that paper, we obtain the following (intuitionistic) rules for $\Box$, where each formula is decorated with a natural number, representing its *level*:

$$\frac{\Gamma, A^{n+k} \vdash B}{\Gamma, \Box A^{n} \vdash B}\ \Box\vdash \qquad\qquad \frac{\Gamma \vdash A^{n+1}}{\Gamma \vdash \Box A^{n}}\ \vdash\Box$$

In the rule $\vdash\Box$ one requires that, for each formula C^k in Γ, $k \leq n$. Different modal systems are obtained by suitable restrictions of the $\Box\vdash$ rule. For example, if $k = 1$ we have D; if $k < 0$ we have D4, and so on.

The idea works fine for the negative $\bot$-free fragments of the modal logics D, T, D4 and S4, and for the corresponding MELL (Multiplicative Exponential Linear Logic) subsystems [12].

Unfortunately, at that time we could not extend this formulation of 2–Sequents to the full classical modal logics considered in this paper, since the notion of level of a formula is too simple and does not interact well with a standard cut elimination procedure.

Few years ago Lellmann et al. reinterpret 2–Sequents as *Linear Nested Sequents* (LNS) [15], a restricted form of Nested Sequents (in their turn a generalization of relational Hypersequents, see Section 5.5 for some references) where the tree-structure is restricted to a no-branching (linear) structure. Lellmann's reformulation allows to extends 2–Sequents to a large class of logics, also avoiding the complexity of nested sequent calculi. In [15]

Lellmann et al. state a cut elimination theorem through an indirect argument. They prove cut elimination for the standard formulation (no levels, no nested sequents) of the considered modal logics and then obtain a cut elimination statement for LNS by means of a translation into the standard cut free formalisms.

The extended sequents of the present paper result from the realization that to obtain a direct syntactical proof of cut elimination, we must enrich the notion of level in 2–Sequents/LNS, moving to a set of uninterpreted names. Instead of indexed formula A^n, at level n we should have (position) formulas of the shape A^α, with *positions* α of length n (namely $A^{\langle x_1,\dots,x_n\rangle}$). The constraints on levels of [17]—the key point of the system design—can be naturally translated (and extended) in constraints on positions.

Non-surprisingly we obtain a system with interesting similarities with those of Fitting [9, 7, 8], Mints [22] and Cerrato [4, 5]. We focus now on these authors, starting with Fitting's research. Even if Cerrato's system is antecedent to Mints' one, to simplify the presentation we introduce first Mints' tableaux and then we discuss Cerrato's by analogy.

5.2. Fitting's indexed tableaux

From now on, we use the standard notion of tableau for classical logic as given in Bell and Machover's textbook [2].

In Fitting's prefixed tableaux, formulas are labeled with a prefix α. Intuitively, one could think of $\alpha.A$ as saying A is true at the world named by α; and that a prefix $\alpha.\beta$ is a naming for a world that is accessible from the world that is named by α. Despite the semantical meaning and intent of prefixes, we can identify the notion of prefix with that of position—a prefixed formula $\alpha.A$ can be viewed[1] as a position formulas A^α. Fitting introduces two kinds of rules, that he calls π and ν rules that model the behaviour of modal operators and of their negations. Here we use directly the symbols $\Box$ and $\Diamond$ and we take into account only the rules for $\Box$ in the basic normal system K (obviously the rules for $\Diamond$ are symmetric). To facilitate the reading, we adopt our notation:

[1] In Fitting's [8] tokens are natural numbers, but this is simply syntactic sugar.

$\Box$–rule: $\Pi = \left\{ \begin{array}{c} \vdots \\ \Box A^{\alpha} \\ \vdots \end{array} \right. \quad \begin{array}{c} | \\ A^{\alpha \circ \beta} \end{array}$ **$\neg\Box$–rule:** $\Pi = \left\{ \begin{array}{c} \vdots \\ \neg\Box A^{\alpha} \\ \vdots \end{array} \right. \quad \begin{array}{c} | \\ \neg A^{\alpha \circ x} \end{array}$

where Π is the branch that is extended by means of the rule and the following constraints hold: (i) in the $\Box$–rule the prefix $\alpha \circ \beta$ is not new in the branch of the tableaux; (ii) in the $\neg\Box$–rule the prefix $\alpha \circ x$ has to be fresh.

The constraints above match the constraints we introduced in e–sequents for K. In particular, for $\Box \vdash$ we require that β is a singleton, and that the main formula has at least a sentinel in the context (notice the analogy with Fitting's $\Box$–rule); for $\vdash \Box$ we force the usual constraint on the quantification, by requiring that $\alpha \circ x$ does not appear as a prefix in any formula of the contexts. Also in this latter case we can observe the analogy with the $\neg\Box$–rule.

As shown in [9], indexed tableau can be easily "overturned" to obtain prefixed sequent and translated to obtain nested sequents, and conversely. Fitting's tableaux (as well as their reformulation and translations) enjoy a form of modularity. Prefixed tableaux can be easily reconfigured to move from K to D: if one no longer requires that on the rule $\vdash \Box$ above the prefix $\alpha \circ \beta$ is not new in the branch, it is possible to derive the D axiom. This happens very similarly in our framework: constraints on $\Box \vdash$ for K (and $\mathsf{K4}$) prevents the derivation of the p-formula $(\Box A \to \Diamond A)^{\gamma}$. Instead, for the other normal logics, specific rules (modeling the characteristic axioms) must be added to those for K.

5.3. Mints' sequents

In [22] G. Mints introduces a calculus for a family of modal logic inspired by Kripke's semantic tableaux. Even if the paper uses the term "tableau", the framework is technically a sequent calculus. To facilitate the comparison, we show how to reformulate the calculus of Mints in terms of our e–sequents. To avoid misunderstanding, from now on we call *sequents* the standard (i.e. non indexed) sequents and *indexed sequents* (in Mints' terminology) expressions of the kind $\alpha(\Gamma \vdash \Delta)$, where α is a position and

$\Gamma \vdash \Delta$ is a sequent (in particular a sequent may be seen as an indexed sequent with $\langle\rangle$ as index.)

The objects of Mints' calculus finite are multisets of indexed sequents called *tableaux*. A tableau is therefore a multiset of the shape

$$\Gamma_0 \vdash \Delta_0; \alpha_1(\Gamma_1 \vdash \Delta_1); \cdots ; \alpha_n(\Gamma_n \vdash \Delta_n)$$

where each $(\Gamma_i \vdash \Delta_i)$ is a sequent. In spite of the name, a tableaux is then an e–sequent, under the following translation (up to exchange rules):

$$\Gamma_0 \vdash \Delta_0; \alpha_1(\Gamma_1 \vdash \Delta_1); \cdots ; \alpha_n(\Gamma_n \vdash \Delta_n) \rightsquigarrow \Gamma_0, \Gamma_1^{\alpha_1}, \ldots \Gamma_n^{\alpha_n} \vdash \Delta_0, \Delta_1^{\alpha_1}, \ldots \Delta_n^{\alpha_n}$$

Under this interpretation, we can now compare Mints' rules with those of e–sequents.

The rules for K are the same as our rules, although modularity is obtained differently than in e–sequents. Mints defines a relation r between positions that at first glance seems similar to our notion of $\mathfrak{Init}[\cdot]$. This is not the case. For example, it is true that $\alpha \; r \; \alpha \circ z$, but $(\alpha, \alpha \circ \beta) \notin r$ if the length of β is greater than 1. In particular Mints forces that r is not transitive. The way r is defined and used to formulate the different logics do not allow to use it to handle directly the transitivity.

Transitivity (axiom $\mathsf{4}$) is obtained by "adding" to the basic rule of K the following one (expressed in our notation):

$$\frac{\Gamma, \Box A^{\alpha \circ z} \vdash \Delta}{\Gamma, \Box A^{\alpha} \vdash \Delta}$$

Also the system for KT is quite different. In fact, Mints introduces two rules for $\Box \vdash$. The basic rule for K (with the constraint that there must be a sentinel in the sequent) plus a new one:

$$\frac{\Gamma, A^{\alpha} \vdash \Delta}{\Gamma, \Box A^{\alpha} \vdash \Delta}$$

The two rules cannot be merged, since the basic rule for $\Box \vdash$ has to satisfy a suitable constraint.

Mints' formulation of the rule allows to prove a cut elimination theorem, at the price of having a proof that does not follow the standard steps of cut elimination for classical first order logic.

5.4. Cerrato's modal tree sequents

In [4, 5] Cerrato proposes *modal tree-sequents* as a formalism for a family of normal modal logics, from K to S5. A modal tree-sequent is indeed a tree of sequents. In spite of a heavy graphical formalism, tree sequents correspond (modulo a direct simple translation) to Mints' tableaux, and, more interestingly, with the same rules. All systems share the same right $\Box$ rule, while dedicated left $\Box$ rules allow the derivation of the characteristic axioms of the different logics. Moreover, both Mints and Cerrato manage transitivity in the same way. If $\mathcal{F}$ is the set of formulas, one can define a function $\mathsf{index} : \mathcal{P}(\mathcal{F}) \to X^*$ that returns the position of the node in the tree. Starting from a "labeled version" of Cerrato's tree sequents, one can easily define a translation into Mints-style multiset of sequents. Therefore what we said for Mints also applies to Cerrato's tree sequents. In particularly, differently from Cerrato, we insist that we obtain a syntactical proof of cut elimination via the same standard argument which is used for first order logic, by leaning on a Mix Lemma (see [25]).

5.5. Other systems

In the previous subsections, we focused on systems strongly similar to our proposal. In the literature, of course, there are many other proof-theoretical approaches to modal logics. Among these, *display calculi* [28], (relational) *hypersequents* [1, 23, 6], and *labelled deductive systems* (LDS) [10, 24, 26, 23], on which we conclude our review.

At a first glance, our system (or those of Fitting-Cerrato-Mints) seems just a syntactical variant (a "rephrasing") of LDS. One can define a translation (objectively, quite cumbersome) of our extended sequents into the formalism of LDS. We present a detailed comparison (formulated in a natural deduction version of the present system) in our [18]. That one system could be translated into another one does not mean that the two are the same, or that one of them is uninteresting (think, for example, about natural deduction and the calculus of sequents).

The basic idea of the translation is to associate a new label a_i to each position and then define suitable relational formulas: each position $\langle x_1, \ldots, x_n \rangle$ is translated into a set of formulas $\{a_0 R a_1, \ldots, a_{n-1} R a_n\}$[2]. These relational formulas are treated in LDS with explicit logical rules,

[2] The simpler $\{x_1 R x_2, \ldots, x_{n-1} R x_n\}$ would not work.

whereas in our e–systems positions are treated in the same way as the terms of the first-order logic, thus with no need for additional special machinery.

For example, seriality and transitivity are handled in LDS through the following rules (for details, see e.g. [23]):

$$\frac{\Gamma, aRb \vdash c : A, \Delta}{\Gamma, \vdash c : A, \Delta}\, seriality \qquad \frac{\Gamma, aRb, bRc \vdash c : A, \Delta}{\Gamma, aRc \vdash c : A, \Delta}\, trans$$

Dispensing from ad hoc rules like these is the very purpose of e–systems, see [18] for more details.

References

[1] A. Avron, *The method of hypersequents in the proof theory of propositional non-classical logics*, [in:] W. Hodges, M. Hyland, C. Steinhorn, J. Truss (eds.), **Logic: From Foundations to Applications**, Oxford Science Publications, Clarendon Press, New York (1996), pp. 1–32.

[2] J. L. Bell, M. Machover, **A course in mathematical logic**, North Holland, Amsterdam (1977).

[3] S. Burns, R. Zach, *Cut-free completeness for modular hypersequent calculi for modal Logics K, T, and D*, **Review of Symbolic Logic**, vol. 14(4) (2021), pp. 910–929, DOI: https://doi.org/10.1017/s1755020320000180.

[4] C. Cerrato, *Cut-free modal sequents for normal modal logics*, **Notre Dame Journal of Formal Logic**, vol. 34(4) (1993), pp. 564–582, DOI: https://doi.org/10.1305/ndjfl/1093633906.

[5] C. Cerrato, *Modal tree-sequents*, **Mathematical Logic Quarterly**, vol. 42(2) (1996), pp. 197–210, DOI: https://doi.org/10.1002/malq.19960420117.

[6] A. Ciabattoni, R. Ramanayake, H. Wansing, *Hypersequent and display calculi—a unified perspective*, **Studia Logica**, vol. 102(6) (2014), pp. 1245–1294, DOI: https://doi.org/10.1007/s11225-014-9566-z.

[7] M. Fitting, *Tableau methods of proof for modal logics*, **Notre Dame Journal of Formal Logic**, vol. 13(2) (1972), pp. 237–247, DOI: https://doi.org/10.1305/ndjfl/1093894722.

[8] M. Fitting, **Proof methods for modal and intuitionistic logics**, Springer–Verlag (1983).

[9] M. Fitting, *Prefixed tableaus and nested sequents*, **Annals of Pure and Applied Logic**, vol. 163(3) (2012), pp. 291–313, DOI: https://doi.org/10.1016/j.apal.2011.09.004.

[10] D. M. Gabbay, R. J. G. B. de Queiroz, *Extending the Curry-Howard interpretation to linear, relevant and other resource logics*, **Journal of Symbolic Logic**, vol. 57(4) (1992), pp. 1319–1365, DOI: https://doi.org/10.2307/2275370.

[11] J.-Y. Girard, **Proof theory and logical complexity**, vol. 1, Bibliopolis, Naples (1987).

[12] S. Guerrini, S. Martini, A. Masini, *An analysis of (linear) exponentials based on extended sequents*, **Logic Journal of the IGPL**, vol. 6(5) (1998), pp. 735–753, DOI: https://doi.org/10.1093/jigpal/6.5.735.

[13] B. Lellmann, *Linear nested sequents, 2-sequents and hypersequents*, [in:] H. de Nivelle (ed.), **Automated Reasoning with Analytic Tableaux and Related Methods, TABLEAUX 2015**, vol. 9323 of Lecture Notes in Computer Science, Springer (2015), pp. 135–150, DOI: https://doi.org/10.1007/978-3-319-24312-2_10.

[14] B. Lellmann, D. Pattinson, *Constructing cut free sequent systems with context restrictions based on classical or intuitionistic logic*, [in:] K. Lodaya (ed.), **Logic and Its Applications, 5th Indian Conference, ICLA 2013**, vol. 7750 of Lecture Notes in Computer Science, Springer (2013), pp. 148–160, DOI: https://doi.org/10.1007/978-3-642-36039-8_14.

[15] B. Lellmann, E. Pimentel, *Modularisation of sequent calculi for normal and non-normal modalities*, **ACM Transactions on Compututational Logic**, vol. 20(2) (2019), pp. 7:1–7:46, DOI: https://doi.org/10.1145/3288757.

[16] S. Martini, A. Masini, *On the fine structure of the exponential rule*, [in:] J.-Y. Girard, Y. Lafont, L. Regnier (eds.), **Advances in Linear Logic**, Cambridge University Press (1993), pp. 197–210.

[17] S. Martini, A. Masini, *A Computational Interpretation of Modal Proofs*, [in:] H. Wansing (ed.), **Proof Theory of Modal Logics**, Kluwer (1994), pp. 213–241.

[18] S. Martini, A. Masini, M. Zorzi, *From 2-sequents and linear nested sequents to natural eduction for normal modal logics*, **ACM Transactions on Compututational Logic**, vol. 22(3) (2021), pp. 19:1–19:29, DOI: https://doi.org/10.1145/3461661.

[19] A. Masini, *2-Sequent calculus: A proof theory of modalities*, **Annals of Pure and Applied Logic**, vol. 58(3) (1992), pp. 229–246, DOI: https://doi.org/10.1016/0168-0072(92)90029-Y.

[20] A. Masini, *2-Sequent calculus: intuitionism and natural deduction*, **Journal of Logic and Computation**, vol. 3(5) (1993), pp. 533–562, DOI: https://doi.org/10.1093/logcom/3.5.533.

[21] A. Masini, L. Viganò, M. Volpe, *Labelled natural deduction for a bundled branching temporal logic*, **Journal of Logic and Computation**, vol. 21(6) (2011), pp. 1093–1163, DOI: https://doi.org/10.1093/logcom/exq028.

[22] G. Mints, *Indexed systems of sequents and cut-elimination*, **Journal of Philosophical Logic**, vol. 26(6) (1997), pp. 671–696, DOI: https://doi.org/10.1023/A:1017948105274.

[23] S. Negri, *Proof theory for modal logic*, **Philosophy Compass**, vol. 6(8) (2011), pp. 523–538, DOI: https://doi.org/10.1111/j.1747-9991.2011.00418.x.

[24] A. Simpson, **The proof theory and semantics of intuitionistic modal logic**, Ph.D. thesis, University of Edinburgh, UK (1993).

[25] G. Takeuti, **Proof theory**, 2nd ed., vol. 81 of Studies in Logic and the Foundations of Mathematics, North-Holland Publishing Co., Amsterdam (1987).

[26] L. Viganò, **Labelled non-classical logics**, Kluwer Academic Publishers, Dordrecht (2000).

[27] H. Wansing, *Sequent calculi for normal modal propositional logics*, **Journal of Logic and Computation**, vol. 4(2) (1994), pp. 125–142, DOI: https://doi.org/10.1093/logcom/4.2.125.

[28] H. Wansing, *Predicate logics on display*, **Studia Logica**, vol. 62(1) (1999), pp. 49–75, DOI: https://doi.org/10.1023/A:1005125717300.

Simone Martini
Università di Bologna
Dipartimento di Informatica – Scienza e Ingegneria
Mura Anteo Zamboni, 7
40126 Bologna, Italy
e-mail: simone.martini@unibo.it

Andrea Masini
Università di Verona
Dipartimento di Informatica
Strada le Grazie, 15
37134 Verona, Italy
e-mail: andrea.masini@univr.it

Margherita Zorzi
Università di Verona
Dipartimento di Informatica
Strada le Grazie, 15
37134 Verona, Italy
e-mail: margherita.zorzi@univr.it

Bulletin of the Section of Logic
Volume 52/4 (2023), pp. 497–515
https://doi.org/10.18778/0138-0680.2023.21

Andrzej Walendziak

ON IMPLICATIVE AND POSITIVE IMPLICATIVE GE ALGEBRAS

Abstract

GE algebras (generalized exchange algebras), transitive GE algebras (tGE algebras, for short) and aGE algebras (that is, GE algebras verifying the antisymmetry) are a generalization of Hilbert algebras. Here some properties and characterizations of these algebras are investigated. Connections between GE algebras and other classes of algebras of logic are studied. The implicative and positive implicative properties are discussed. It is shown that the class of positive implicative GE algebras (resp. the class of implicative aGE algebras) coincides with the class of generalized Tarski algebras (resp. the class of Tarski algebras). It is proved that for any aGE algebra the property of implicativity is equivalent to the commutative property. Moreover, several examples to illustrate the results are given. Finally, the interrelationships between some classes of implicative and positive implicative algebras are presented.

Keywords: GE algebra, tGE algebra, BCK algebra, Hilbert algebra, (positive) implicativity.

2020 Mathematical Subject Classification: 03G25, 06A06, 06F35.

1. Introduction

L. Henkin [6] introduced the notion of "implicative model", as a model of positive implicative propositional calculus. In 1960, A. Monteiro [16] has given the name "Hilbert algebras" to the dual algebras of Henkin's implicative models. In 1966, K. Iséki [9] introduced a new notion called a BCK

Presented by: Janusz Czelakowski
Received: July 20, 2022
Published online: September 25, 2023

algebra. It is an algebraic formulation of the BCK-propositional calculus system of C. A. Meredith [15], and generalize the concept of implicative algebras (see [1]). In 2021, P. Cintula and C. Noguera [4] presented of the most important logics that one can find in the literature. In particular, they considered the $\mathcal{BCK}$ logic and its many extensions. To solve some problems on BCK algebras, Y. Komori [14] introduced BCC algebras. These algebras (also called BIK$^+$-algebras) are an algebraic model of $\mathcal{BIK}^+$ logic. In [12], as a generalization of BCK algebras, H. S. Kim and Y. H. Kim defined BE algebras. In 2008, A. Walendziak [18] defined commutative BE algebras and proved that they are BCK algebras. Later on, in 2010, D. Buşneag and S. Rudeanu [3] introduced the notion of pre-BCK algebra. A BCK algebra is just a pre-BCK algebra satisfying also the antisymmetry. In 2016, A. Iorgulescu [7] introduced new generalizations of BCK and Hilbert algebras (RML, aBE, pi-BE, pimpl-RML algebras and many others). Recently, R. Bandaru et al. [2] introduced the concepts of GE algebra (generalized exchange algebra) and transitive GE algebra (tGE algebra for short). These algebras are a generalization of Hilbert algebras.

In 1978, K. Iséki and S. Tanaka [10] introduced the concepts of implicativity and positive implicativity in the theory of BCK algebras. The present paper is a continuation of the author's paper [19], where the property of implicativity for various generalizations of BCK algebras was studied. Implicative BE algebras were presented in [21] (see also [23]).

Here we consider RML, BE, GE, tGE, pre-BCC and pre-BCK algebras and investigate the implicative and positive implicative properties for these algebras. We obtain some characterizations of GE and transitive GE algebras. We study connections between GE algebras and other classes of algebras of logic. We show that the class of positive implicative GE algebras (resp. the class of implicative GE algebras satisfying the property of antisymmetry) coincides with the class of generalized Tarski algebras (resp. the class of Tarski algebras). We prove that for any GE algebra with the antisymmetry the property of implicativity is equivalent to the commutative property. Moreover, we give several examples to illustrate the results. Finally, we present the interrelationships between the classes of implicative and positive implicative algebras considered here.

2. Preliminaries

Let $\mathcal{A} = (A, \rightarrow, 1)$ be an algebra of type $(2, 0)$. We consider the following list of properties ([7]) that can be satisfied by $\mathcal{A}$ (the properties in the list are the most important properties satisfied by a BCK algebra):

(An) (Antisymmetry) $x \rightarrow y = 1 = y \rightarrow x \Longrightarrow x = y$,

(B) $(y \rightarrow z) \rightarrow [(x \rightarrow y) \rightarrow (x \rightarrow z)] = 1$,

(BB) $(y \rightarrow z) \rightarrow [(z \rightarrow x) \rightarrow (y \rightarrow x)] = 1$,

(C) $[x \rightarrow (y \rightarrow z)] \rightarrow [y \rightarrow (x \rightarrow z)] = 1$,

(D) $y \rightarrow [(y \rightarrow x) \rightarrow x] = 1$,

(Ex) (Exchange) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,

(K) $x \rightarrow (y \rightarrow x) = 1$,

(L) (Last element) $x \rightarrow 1 = 1$,

(M) $1 \rightarrow x = x$,

(Re) (Reflexivity) $x \rightarrow x = 1$,

(Tr) (Transitivity) $x \rightarrow y = 1 = y \rightarrow z \Longrightarrow x \rightarrow z = 1$,

(*) $y \rightarrow z = 1 \Longrightarrow (x \rightarrow y) \rightarrow (x \rightarrow z) = 1$,

(**) $y \rightarrow z = 1 \Longrightarrow (z \rightarrow x) \rightarrow (y \rightarrow x) = 1$.

The following lemma will be used many times throughout the rest of this paper.

LEMMA 2.1 ([7], Proposition 2.1 and Theorem 2.7). *Let* $\mathcal{A} = (A, \rightarrow, 1)$ *be an algebra of type* $(2, 0)$. *Then the following hold:*

(i) *(M) + (B) imply (Re), (*) and (**),*

(ii) *(M) + (*) imply (Tr),*

(iii) *(M) + (**) imply (Tr),*

(iv) *(An) + (C) imply (Ex),*

(v) *(M) + (BB) imply (B).*

Definition 2.2 ([7]).

1. A *RML algebra* is an algebra $\mathcal{A} = (A, \to, 1)$ of type $(2, 0)$ verifying (Re), (M), (L).
2. A *BE algebra* is a RML algebra verifying (Ex).
3. An *aBE algebra* is a BE algebra verifying (An).
4. A *pre-BCC algebra* is a RML algebra verifying (B).
5. A *pre-BCK algebra* is a pre-BCC algebra verifying (Ex).
6. A *BCC algebra* is a pre-BCC algebra verifying (An).
7. A *BCK algebra* is a pre-BCK algebra verifying (An).

Denote by **RML**, **BE**, **aBE**, **pre-BCC**, **pre-BCK**, **BCC** and **BCK** the classes of RML, BE, aBE, pre-BCC, pre-BCK, BCC and BCK algebras, respectively.

Let $\mathcal{A} = (A, \to, 1)$ be an algebra of type $(2, 0)$. We define the binary relation $\leq$ by: for all $x, y \in A$,

$$x \leq y \Longleftrightarrow x \to y = 1.$$

It is known that $\leq$ is an order relation in BCC and BCK algebras. By definition, in RML and BE algebras, $\leq$ is a reflexive relation; in aBE algebras, $\leq$ is reflexive and antisymmetric. By Lemma 2.1 (i) and (ii), in pre-BCC and pre-BCK algebras, $\leq$ is reflexive and transitive (i.e., it is a pre-order relation).

Definition 2.3 ([2]). A *GE algebra* (*generalized exchange algebra*) is an algebra $\mathcal{A} = (A, \to, 1)$ of type $(2, 0)$ verifying (Re), (M) and

(GE) $x \to (y \to z) = x \to [y \to (x \to z)]$.

Following [2],

- a *transitive GE algebra* (*tGE algebra*, for short) is a GE algebra verifying (B),
- an *aGE algebra* is a GE algebra verifying (An).

Denote by **GE**, **tGE** and **aGE** the classes of all GE algebras, transitive GE algebras and aGE algebras, respectively.

Proposition 2.4. Any GE algebra satisfies the folowing property

(pi) $x \to y = x \to (x \to y)$.

Proof: Let $\mathcal{A}$ be a GE algebra and $x, y \in A$. We have $x \to y \overset{(M)}{=} x \to (1 \to y) \overset{(GE)}{=} x \to [1 \to (x \to y)] \overset{(M)}{=} x \to (x \to y)$, that is, (pi) holds in $\mathcal{A}$. □

Example 2.5. Consider the set $A = \{a, b, c, d, e, 1\}$ and the operation $\to$ given by the following table:

$\to$	a	b	c	d	e	1
a	1	1	c	c	1	1
b	a	1	d	d	1	1
c	a	1	1	1	1	1
d	a	1	1	1	1	1
e	a	1	1	1	1	1
1	a	b	c	d	e	1

.

We can observe that the properties (Re), (M), (L), (GE) (hence (pi)) are satisfied. Therefore, $(A, \to, 1)$ is a GE algebra. It does not satisfy (An) for $(x, y) = (c, d)$; (Ex) for $(x, y, z) = (a, b, c)$; (Tr) and (B) for $(x, y, z) = (a, e, c)$. Then, $\mathcal{A}$ is not transitive.

Example 2.6. Let $A = \{a, b, c, d, 1\}$ and $\to$ be defined as follows:

$\to$	a	b	c	d	1
a	1	1	c	c	1
b	1	1	d	d	1
c	a	a	1	1	1
d	b	b	1	1	1
1	a	b	c	d	1

.

The algebra $\mathcal{A} = (A, \to, 1)$ verifies (Re), (M), (L), (GE), (B). It does not verify (An) for $x = a, y = b$; (Ex) for $x = a, y = b, z = c$. Thus $\mathcal{A}$ is a tGE algebra which is not a pre-BCK algebra.

Following [7], a *pi-RML algebra* (respectively: *pi-BE, pi-aBE, pi-pre-BCC, pi-pre-BCK, pi-BCC, pi-BCK algebra*) is a RML algebra (respectively: BE, aBE, pre-BCC, pre-BCK, BCC, BCK algebra) verifying (pi).

Denote by **pi-RML**, **pi-BE**, **pi-aBE**, **pi-pre-BCC**, **pi-pre-BCK**, **pi-BCC**, **pi-BCK** the classes of pi-RML, pi-BE, pi-aBE, pi-pre-BCC, pi-pre-BCK, pi-BCC, pi-BCK algebras, respectively.

PROPOSITION 2.7. *Let $\mathcal{A} = (A, \to, 1)$ be an algebra of type $(2, 0)$. Then the following hold:*

(i) (Re) + (pi) *imply* (L),

(ii) (Ex) + (pi) *imply* (GE).

PROOF: (i) It follows immediately from Proposition 6.4 (ii) of [7].

(ii) Let $x, y, z \in A$. We obtain

$$x \to (y \to z) \overset{\text{(pi)}}{=} x \to [x \to (y \to z)] \overset{\text{(Ex)}}{=} x \to [y \to (x \to z)].$$

Thus (GE) holds. □

By Propositions 2.4 and 2.7 (i), we have

COROLLARY 2.8. *Any GE algebra is a pi-RML algebra.*

By Proposition 2.7 (ii), we get

COROLLARY 2.9. *Any pi-BE algebra is a GE algebra.*

Remark 2.10. By Corollaries 2.8 and 2.9, **pi-BE** $\subset$ **GE** $\subset$ **pi-RML**. Observe that these inclusions are proper. Indeed, the algebra given in Example 2.5 is a GE algebra not satisfying (Ex). The algebra from Example 10.1 of [8] is a pi-RML algebra that is not a GE algebra.

The interrelationships between the classes of algebras mentioned before are visualized in Figure 1. (An arrow indicates proper inclusion, that is, if **X** and **Y** are classes of algebras, then $\mathbf{X} \longrightarrow \mathbf{Y}$ means $\mathbf{X} \subset \mathbf{Y}$.)

In [17], S. Tanaka introduced the notion of commutativity in the theory of BCK algebras. A BCK algebra $\mathcal{A} = (A, \to, 1)$ is called *commutative* if, for all $x, y \in A$,

(Com) $(x \to y) \to y = (y \to x) \to x$.

H. Yutani [22] proved that the class of commutative BCK algebras is equationally definable. A. Walendziak [18] showed that any commutative

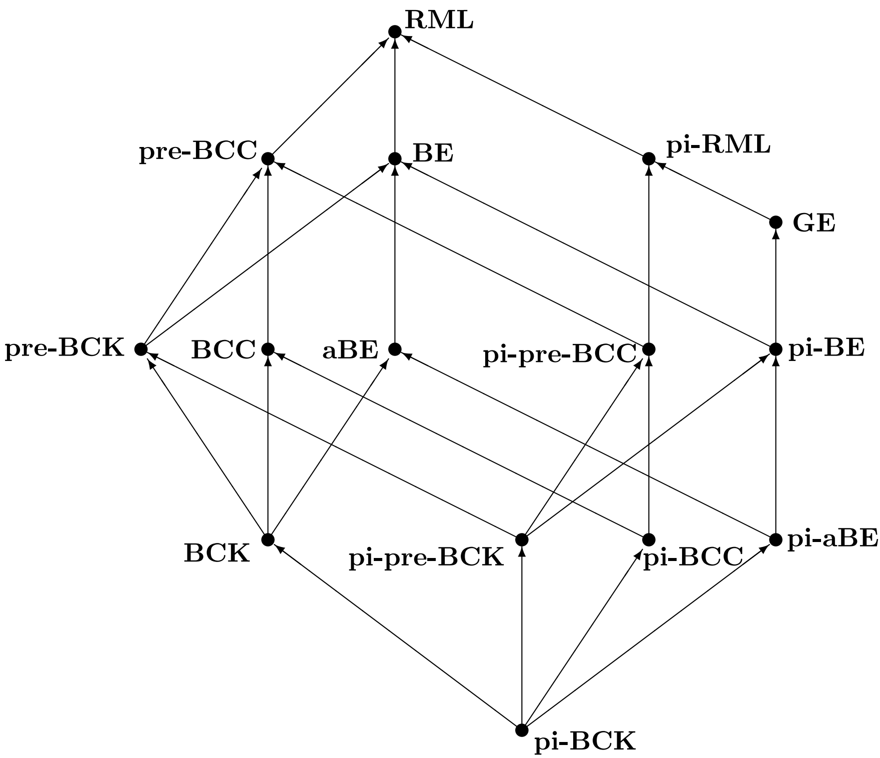


Figure 1.

BE algebra is a BCK algebra. The property of commutativity for other generalizations of BCK algebras was investigated in [20].

As in the case of BCK algebras, we define:

Definition 2.11. A RML algebra $\mathcal{A} = (A, \to, 1)$ is called *commutative* if it satisfies (Com).

Denote by **com-RML** the class of commutative RML algebras. Similarly, if **X** is a subclass of the class **RML**, then **com–X** denotes the class of all commutative algebras belonging to **X**.

Remark 2.12. Since every commutative BE algebra is a BCK algebra, we have **com-BE** = **com-BCK**. Moreover, following [20], we obtain

com-BE = **com-pre-BCC** = **com-pre-BCK** = **com-BCC** = **com-BCK**.

As a preparation for the next results we need the following

LEMMA 2.13. ([20], Proposition 3.3) *Let* $\mathcal{A} = (A, \to, 1)$ *be an algebra of type* $(2, 0)$ *verifying* (M) *and* (Com). *Then* $\mathcal{A}$ *verifies* (An).

Remark 2.14. Note that commutative GE algebras were introduced and studied in [2].

3. On GE and transitive GE algebras

First we present the following

PROPOSITION 3.1. Let $\mathcal{A} = (A, \to, 1)$ be an algebra of type $(2, 0)$. Then the following hold:

(i) (K) + (GE) *imply* (C),

(ii) (Re) + (GE) + (L) *imply* (D) *and* (K),

(iii) (GE) + (K) + (An) *imply* (Ex),

(iv) (C) + (D) + (M) + (Tr) *imply* (**).

PROOF: (i) Let $x, y, z \in A$. We have $[x \to (y \to z)] \to [y \to (x \to z)] \overset{(GE)}{=} [x \to (y \to z)] \to [y \to (x \to (y \to z))] \overset{(K)}{=} 1$, that is, (C) holds in $\mathcal{A}$.

(ii) Let $x, y \in A$. We obtain

$$y \to [(y \to x) \to x] \overset{(GE)}{=} y \to [(y \to x) \to (y \to x)] \overset{(Re)}{=} y \to 1 \overset{(L)}{=} 1,$$

that is, (D) holds in $\mathcal{A}$.

Now, applying (GE), (Re) and (L), we get $x \to (y \to x) = x \to [y \to (x \to x)] = 1$, that is, (K) holds in $\mathcal{A}$.

(iii) It follows from above (i) and Lemma 2.1 (iv).

(iv) Let $x, y, z \in A$ and $y \leq z$. By (D), $z \leq (z \to x) \to x$. Applying (Tr), we get $y \leq (z \to x) \to x$. From (C) it follows that

$$1 = y \to [(z \to x) \to x] \leq (z \to x) \to (y \to x).$$

Hence, by (M), $(z \to x) \to (y \to x) = 1$. Therefore $z \to x \leq y \to x$, thus (**) holds in $\mathcal{A}$. □

From Propositions 2.4, 2.7 (i) and 3.1 (i), (ii) we have

Corollary 3.2. *Any GE algebra satisfies (pi), (L), (C), (D) and (K).*

Corollary 3.3. *In GE algebras, we have*

$$(Tr) \Longleftrightarrow (^{**}).$$

Proof: Let $\mathcal{A}$ be a GE algebra verifying (Tr). By Proposition 3.1 (iv), $\mathcal{A}$ verifies (**). The converse follows from Lemma 2.1 (iii). □

Remark 3.4. Applying Proposition 3.1 (iii), we have **aGE** $\subseteq$ **pi-aBE**. Since **pi-BE** $\subset$ **GE**, see Remark 2.9, we get **pi-aBE** $\subseteq$ **aGE**. Consequently, **pi-aBE** $=$ **aGE**.

Since (M) + (B) imply (Re), see Lemma 2.1 (i), we obtain

Proposition 3.5. *An algebra $\mathcal{A} = (A, \to, 1)$ of type $(2, 0)$ is a transitive GE algebra if and only if it satisfies (M), (GE), (B).*

Now we consider the following properties; they are the most important properties satisfied by a Hilbert algebra:

(p-1) $x \to (y \to z) \le (x \to y) \to (x \to z)$,

(p-2) $(x \to y) \to (x \to z) \le x \to (y \to z)$,

(pimpl) $x \to (y \to z) = (x \to y) \to (x \to z)$.

Remark 3.6. It is easy to see that (p-1) + (p-2) + (An) imply (pimpl).

Proposition 3.7. *Let $\mathcal{A} = (A, \to, 1)$ be an algebra of type $(2, 0)$. Then the following hold:*

(i) (pi) + (pimpl) *imply* (GE),

(ii) (Re) + (M) + (pimpl) *imply* (pi),

(iii) (Re) + (M) + (pimpl) *imply* (GE),

(iv) (M) + (C) + (B) +(pi) *imply* (p-1),

(v) (M) + (K) + (C) + (**) *imply* (p-2),

(vi) (K) + (Tr) + (p-1) *imply* (B),

(vii) (M) + (L) + (p-1) *imply* (*).

Proof: (i) Let $x, y, z \in A$. We obtain

$$\begin{aligned} x \to (y \to z) &\overset{\text{(pimpl)}}{=} (x \to y) \to (x \to z) \\ &\overset{\text{(pi)}}{=} (x \to y) \to [x \to (x \to z)] \\ &\overset{\text{(pimpl)}}{=} x \to [y \to (x \to z)]. \end{aligned}$$

Thus (GE) holds.

(ii) Follows from Proposition 6.4 (iii) of [7].

(iii) Follows from above (i) and (ii).

(iv) By Lemma 2.1 (i) and (ii), $\mathcal{A}$ satisfies (Tr). Let $x, y, z \in A$. From (C) it follows $x \to (y \to z) \leq y \to (x \to z)$. Applying (B) and (pi), we get $y \to (x \to z) \leq (x \to y) \to [x \to (x \to z)] = (x \to y) \to (x \to z)$. By (Tr), $x \to (y \to z) \leq (x \to y) \to (x \to z)$, that is, (p-1) holds.

(v) Let $x, y, z \in A$. By (K), $y \leq x \to y$, and hence, using (**), we obtain

$$(x \to y) \to (x \to z) \leq y \to (x \to z). \tag{3.1}$$

By (C),

$$y \to (x \to z) \leq x \to (y \to z). \tag{3.2}$$

Since $\mathcal{A}$ satisfies (M) and (**), from Lemma 2.1 (iii) we see that (Tr) holds in $\mathcal{A}$. Therefore, applying (3.1) and (3.2), we get $(x \to y) \to (x \to z) \leq x \to (y \to z)$, that is, (p-2) holds.

(vi) Let $x, y, z \in A$. By (K) and (p-1), $y \to z \leq x \to (y \to z)$ and $x \to (y \to z) \leq (x \to y) \to (x \to z)$. Then, from (Tr) we have $y \to z \leq (x \to y) \to (x \to z)$. Thus (B) holds.

(vii) Let $x, y, z \in A$ and $y \to z = 1$. Using (L) and (p-1), we get $1 = x \to (y \to z) \leq (x \to y) \to (x \to z)$. By (M), $(x \to y) \to (x \to z) = 1$. Therefore (*) holds. □

Since (M) + (B) imply (**), see Lemma 2.1 (i), from Prposition 3.7 (iv), (v) we obtain

Corollary 3.8. *Any transitive GE algebra verifies properties (p-1) and (p-2).*

PROPOSITION 3.9. In GE algebras, *(p-1)* $\Longrightarrow$ *(p-2)*.

PROOF: Let $\mathcal{A}$ be a GE algebra verifying (p-1). By Proposition 3.7 (vii), $\mathcal{A}$ verifies (*). Therefore, (Tr) also holds, and hence $\mathcal{A}$ verifies (**), by Proposition 3.1 (iv). Applying Proposition 3.7 (v), we get (p-2). □

THEOREM 3.10. *In GE algebras, we have*

$$\text{(BB)} \Longleftrightarrow \text{(B)} \Longleftrightarrow \text{(p-1)} \Longleftrightarrow (*).$$

PROOF: By Lemma 2.1 (v), (BB) $\Longrightarrow$ (B), and, by Proposition 3.7 (iv), (vii), we conclude that (B) $\Longrightarrow$ (p-1) and (p-1) $\Longrightarrow$ (*). Let $\mathcal{A}$ be a GE algebra with (*). Let $x, y, z \in A$. From (C) we see that $x \to [(y \to z) \to z] \leq (y \to z) \to (x \to z)$, and hence

$$(x \to y) \to [x \to ((y \to z) \to z)] \leq (x \to y) \to [(y \to z) \to (x \to z)]$$

by (*). Observe that

$$(x \to y) \to [x \to ((y \to z) \to z)] = 1. \tag{3.3}$$

Indeed, from (D) we conclude that $y \leq (y \to z) \to z$. Applying (*), we obtain (3.3). Therefore, $x \to y \leq (y \to z) \to (x \to z)$, that is, (BB) holds. □

COROLLARY 3.11. An algebra $\mathcal{A} = (A, \to, 1)$ *of type* $(2, 0)$ is a transitive GE algebra if and only if $\mathcal{A}$ verifies *(Re), (M), (GE)* and *(p-1)*.

COROLLARY 3.12. Any transitive GE algebra verifies *(B), (BB), (*), (**), (Tr), (p-1), (p-2)*.

4. Implicative and positive implicative GE algebras

The well-known implicative and positive implicative BCK algebras were introduced by K. Iséki and S. Tanaka [10].

Let $\mathcal{A} = (A, \to, 1)$ be an algebra of type $(2, 0)$. We first consider the following property:

(im) $(x \to y) \to x = x$.

PROPOSITION 4.1. Let $\mathcal{A} = (A, \to, 1)$ be an algebra of type $(2, 0)$. Then:

(i) (Re) + (im) *imply* (M),

(ii) (M) + (im) *imply* (L),

(iii) (im) *implies* (pi),

(iv) (Re) + (pimpl) *imply* (L) *and* (B),

(v) (Re) + (M) + (pimpl) + (An) *imply* (Ex).

PROOF: (i)–(iii) follow from Proposition 3.5 of [19].

(iv) and (v) follow from Propositions 6.4, 6.9 and Theorem 6.16 of [7]. □

Similarly as in the case of BCK algebras, we say that a RML algebra (in particular, a GE algebra) $\mathcal{A} = (A, \to, 1)$ is *implicative* if it satisfies (im).

A *positive implicative RML algebra* ([7]), or a *pimpl-RML algebra* for short, is a RML algebra verifying (pimpl).

Remark 4.2. Note that from Theorem 8 of [10] it follows that for BCK algebras, (pimpl) and (pi) are equivalent. By Theorem 9 of [10], a commutative BCK algebra is implicative if and only if it is positive implicative.

Denote by **im-RML** and **pimpl-RML** the classes of implicative and positive implicative RML algebras, respectively; similarly for subclasses of the class of all RML algebras.

It is easy to check that the algebra from Example 2.6 is an implicative tGE algebra. However, the algebra given in Example 2.5 is not implicative, since $(b \to a) \to b = 1 \neq b$.

Example 4.3. Consider the set $A = \{a, b, c, d, 1\}$ with the following table of $\to$:

$\to$	a	b	c	d	1
a	1	b	b	1	1
b	a	1	1	a	1
c	a	1	1	a	1
d	1	c	c	1	1
1	a	b	c	d	1

The algebra $\mathcal{A} = (A, \to, 1)$ verifies (Re), (M), (L), (GE) (hence (C), (D), (K), (pi)), (B) (hence (*), (**), (Tr)) and (pimpl). It does not verify (An) for b, c; (Ex) for a, d, b; (im) for c, a. Thus $\mathcal{A}$ is a pimpl-tGE algebra which is not implicative.

Remark 4.4. Any implicative RML and pimpl-RML algebra is a pi-RML algebra by Propositions 4.1 (iii) and 3.7 (ii).

We recall the following definitions:

DEFINITION 4.5. Let $\mathcal{A} = (A, \rightarrow, 1)$ be an algebra of type $(2, 0)$.

8. $\mathcal{A}$ is a *Hilbert algebra* ([5]) if it verifies (An), (K) and (p-1).
9. $\mathcal{A}$ is a generalized *Hilbert algebra* (*GH-algebra* for short) if it verifies (Re), (M), (Ex) and (pimpl).
10. $\mathcal{A}$ is a *Tarski algebra* ([11])if it verifies (Re), (M), (pimpl) and (Com).
11. $\mathcal{A}$ is a *generalized Tarski algebra* (*GT-algebra* for short) if it verifies (Re), (M) and (pimpl).

Denote by **H**, **GH**, **T** and **GT** the classes of Hilbert algebras, GH-algebras, Tarski algebras and GT-algebras, respectively.

Remark 4.6. Hilbert algebras were introduced in 1950, in a dual form, by L. Henkin [6], under the name "implicative model". A. Monteiro has given the name "Hilbert algebras" to the dual algebras of Henkin's implicative models (see [6, 2]). In [5], A. Diego proved that the class of all Hilbert algebras is a variety. From Remarks 6.18 and 6.19 of [7] and Remark 3.4 we conclude that

$$\begin{aligned}\textbf{pimpl-BCC} &= \textbf{pimpl-BCK}\\ &= \textbf{pi-BCK}\\ &= \textbf{pimpl-aBE}\\ &= \textbf{pimpl-aGE}\\ &= \textbf{H}.\end{aligned}$$

PROPOSITION 4.7 ([7], Corollary 6.17). Any algebra $(A, \rightarrow, 1)$ verifying (Re), (M), (An) and (pimpl) is a Hilbert algebra.

Remark 4.8. By definition, generalized Hilbert algebras coincide with positive implicative BE algebras, that is, **pimpl-BE** $=$ **GH**. Note that a self-distributive BE algebra (see [12]) is in fact our pimpl-BE algebra. By Remark 6.19 of [7], **pimpl-BE** $=$ **pimpl-pre-BCK** ($=$ **GH**).

Remark 4.9. Note that GT-algebras were introduced and studied in [13]. Since (Re) + (pimpl) imply (L) and (B), see Proposition 4.1 (iv), we have $\mathbf{GT} = \textbf{pimpl-RML} = \textbf{pimpl-pre-BCC}$. By Proposition 3.7 (iii), $\mathbf{GT} = \textbf{pimpl-GE} = \textbf{pimpl-tGE}$.

Remark 4.10. A Tarski algebra is in fact a commutative GT-algebra. By Lemma 2.13, a Tarski algebra verifies (An), hence, by Proposition 4.7, it is a Hilbert algebra. Therefore, Tarski algebras coincide with commutative Hilbert algebras, and with commutative GE algebras by Theorem 3.9 of [2]. Thus $\mathbf{T} = \textbf{com-GT} = \textbf{com-GE} = \textbf{com-H}$, where **com-GT**, **com-GE** and **com-H** denote commutative GT, commutative GE and commutative Hilbert algebras, respectively.

By above remarks, we obtain that

$$\mathbf{T} = \textbf{com-H} \overset{a)}{\subset} \mathbf{H} = \textbf{pimpl-aBE} \overset{b)}{\subset} \mathbf{GH} = \\ \textbf{pimpl-BE} \overset{c)}{\subset} \mathbf{GT} = \textbf{pimpl-RML} = \textbf{pimpl-tGE} \overset{d)}{\subset} \mathbf{GE}.$$

These inclusions are proper; see Examples 3.10 [2], for a), 10.8 [8], for b); 4.3, for c); and finally, Example 2.5, for d).

By definition, we have

$$\textbf{im-BCK} \subset \textbf{im-pre-BCK} \subset \textbf{im-tGE} \subset \\ \textbf{im-pre-BCC} \subset \textbf{im-RML} \subset \textbf{pi-RML}.$$

These inclusions are proper; see Examples 4.11, 2.6, 4.12, 4.13 and Example 10.1 of [8].

Example 4.11. Let $A = \{a, b, c, d, e, 1\}$ and $\to$ be defined as follows:

$\to$	a	b	c	d	e	1
a	1	1	e	d	e	1
b	1	1	d	d	d	1
c	1	1	1	1	1	1
d	a	b	b	1	1	1
e	a	a	a	1	1	1
1	a	b	c	d	e	1

It is easy to see that the properties (Re), (M), (L), (Ex), (B), (im) (hence (pi)) are satisfied; (An) is not satisfied for $(x, y) = (a, b)$, (pimpl) is not satisfied for $(x, y, z) = (a, b, c)$, Therefore, $(A, \rightarrow, 1)$ is an implicative pre-BCK algebra that is not positive implicative.

Example 4.12. Consider the set $A = \{a, b, c, d, e, 1\}$ and the operation $\rightarrow$ given by the following table:

$\rightarrow$	a	b	c	d	e	1
a	1	1	e	d	e	1
b	1	1	c	d	d	1
c	b	b	1	1	1	1
d	a	b	1	1	1	1
e	a	a	1	1	1	1
1	a	b	c	d	e	1

We can observe that the properties (Re), (M), (L), (B) and (im) are satisfied. Hence, $(A, \rightarrow, 1)$ is an implicative pre-BCC algebra. It does not satisfy (An) for $(x, y) = (a, b)$; (Ex) and (GE) for $(x, y, z) = (a, b, c)$; (pimpl) for $(x, y, z) = (a, b, e)$.

Example 4.13. ([19], Example 3.24) Let $A = \{a, b, c, d, 1\}$ and $\rightarrow$ be defined as follows:

$\rightarrow$	a	b	c	d	1
a	1	b	b	d	1
b	a	1	a	a	1
c	1	1	1	1	1
d	a	1	1	1	1
1	a	b	c	d	1

It is easy to see that the properties (Re), (M), (L) and (im) (hence (pi)) are satisfied; (An) is not satisfied for $(x, y) = (c, d)$, (GE), (Ex) and (pimpl) are not satisfied for $(x, y, z) = (b, a, d)$, (Tr) is not satisfied for $(x, y, z) = (d, c, a)$. Therefore, $(A, \rightarrow, 1)$ is an implicative RML algebra (hence also a pi-RML algebra) that is not a pre-BCC algebra.

PROPOSITION 4.14 ([19], Proposition 3.14). Let $\mathcal{A} = (A, \rightarrow, 1)$ be an algebra verifying (Re), (D), (**) and (im). Then

$$y \leq x \Longrightarrow (x \rightarrow y) \rightarrow y \leqslant x \tag{4.1}$$

for all $x, y \in A$.

Theorem 4.15. *If $\mathcal{A} = (A, \to, 1)$ is an implicative GE algebra with* (Tr), *then $\mathcal{A}$ satisfies the following condition:*

(wCom) $(x \to y) \to y \leq (y \to x) \to x$.

Proof: Let $\mathcal{A}$ be an implicative GE algebra verifying (Tr). By Proposition 4.14, $\mathcal{A}$ satisfies (4.1). Let $x, y \in A$. From (K) we have $x \leq (y \to x) \to x$. Applying (**) twice, we obtain

$$(x \to y) \to y \leq (((y \to x) \to x) \to y) \to y. \tag{4.2}$$

By (D), $y \leq (y \to x) \to x$, and hence, using (4.1), we get

$$(((y \to x) \to x) \to y) \to y \leq (y \to x) \to x. \tag{4.3}$$

Since $\mathcal{A}$ satisfies (Tr), from inequalities (4.2) and (4.3) we have (wCom). □

Proposition 4.16 ([23]). Implicative aBE algebras satisfy (Tr).

Proposition 4.17. Implicative aGE algebras concide with implicative aBE algebras.

Proof: From Remark 3.4 it follows that **pi-aBE** = **aGE**. Since (im) implies (pi), we have **im-aBE** = **im-aGE**. □

Proposition 4.18. In GE algebras, we have

$$\text{(im)} + \text{(An)} \iff \text{(Com)}.$$

Proof: Let $\mathcal{A} = (A, \to, 1)$ be a GE algebra. Assume that (im) and (An) hold in $\mathcal{A}$. By Propositions 4.16 and 4.17, $\mathcal{A}$ satisfies (Tr). From Theorem 4.15 we conclude that $\mathcal{A}$ is commutative.

Conversely, suppose that $\mathcal{A}$ satisfies (Com). By Lemma 2.13, (An) is satisfied. To prove (im), let $x, y \in A$. We have $((x \to y) \to x) \to x \overset{\text{(Com)}}{=} (x \to (x \to y)) \to (x \to y) \overset{\text{(pi)}}{=} (x \to y) \to (x \to y) \overset{\text{(Re)}}{=} 1$. Then $(x \to y) \to x \leq x$. Applying Proposition 3.1 (ii), we see that $\mathcal{A}$ satisfies (K). Therefore, $x \leq (x \to y) \to x$. Then, using (An), we obtain $x = (x \to y) \to x$, that is, (im) holds in $\mathcal{A}$. □

Corollary 4.19. Let $\mathcal{A}$ be a GE algebra satisfying (An). Then the property of implicativity is equivalent to the commutative property.

From Corollary 4.19 it follows that **com-GE** = **im-aGE**. Since **T** = **com-GE** (see Remark 4.10), we have **T** = **im-aGE**. Hence we obtain

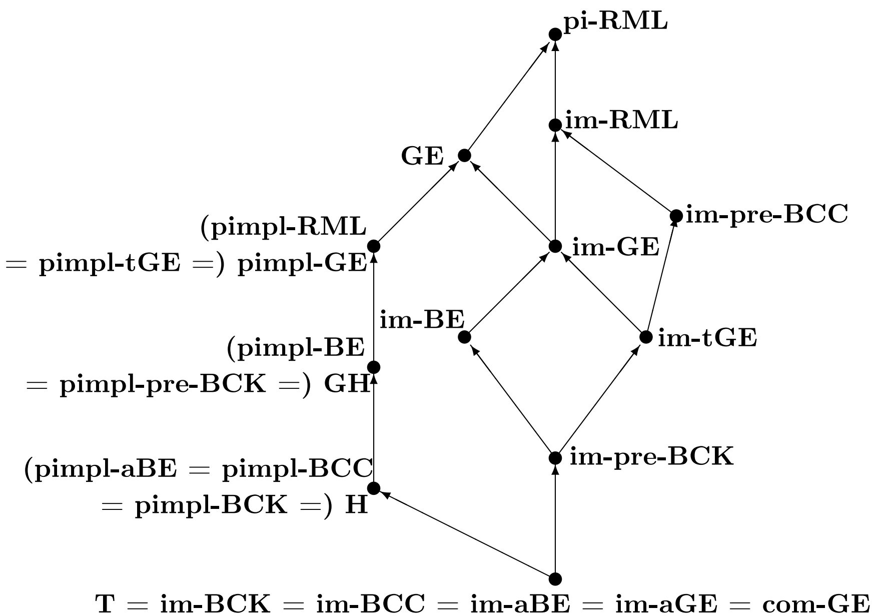


Figure 2.

COROLLARY 4.20. Any implicative aGE algebra is a Tarski algebra.

We draw now the interrelationships between some classes of implicative and positive implicative algebras mentioned before (see Figure 2).

References

[1] J. C. Abbott, *Semi-boolean algebras*, **Matematički Vesnik**, vol. 4(19) (1967), pp. 177–198.

[2] R. Bandaru, A. B. Saeid, Y. B. Jun, *On GE-algebras*, **Bulletin of the Section of Logic**, vol. 50(1) (2021), pp. 81–96, DOI: https://doi.org/10.18778/0138-0680.2020.20.

[3] D. Buşneag, S. Rudeanu, *A glimpse of deductive systems in algebra*, **Central European Journal of Mathematics**, vol. 8(4) (2010), pp. 688–705, DOI: https://doi.org/10.2478/S11533-010-0041-4.

[4] P. Cintula, C. Noguera, **Logic and Implication: An Introduction to the General Algebraic Study of Non-Classical Logics**, vol. 57 of Trends in Logic, Springer, Berlin (2010), DOI: https://doi.org/10.1007/978-3-030-85675-5.

[5] A. Diego, **Sur les algébras de Hilbert**, vol. 21 of Collection de Logigue Mathématique, Serie A, Gauthier-Villars, Paris (1966).

[6] L. Henkin, *An algebraic characterization of quantifilers*, **Fundamenta Mathematicae**, vol. 37 (1950), pp. 63–74, URL: http://eudml.org/doc/213228.

[7] A. Iorgulescu, *New generalizations of BCI, BCK and Hilbert algebras – Part I*, **Journal of Multiple-Valued Logic and Soft Computing**, vol. 27(4) (2016), pp. 353–406.

[8] A. Iorgulescu, *New generalizations of BCI, BCK and Hilbert algebras – Part II*, **Journal of Multiple-Valued Logic and Soft Computing**, vol. 27(4) (2016), pp. 407–456.

[9] K. Iséki, *An algebra related with a propositional calculus*, **Proceedings of the Japan Academy**, vol. 42 (1966), pp. 26–29, DOI: https://doi.org/10.3792/pja/1195522171.

[10] K. Iséki, S. Tanaka, *An introduction to the theory of BCK-algebras*, **Mathematica Japonica**, vol. 23 (1978), pp. 1–26.

[11] Y. B. Jun, M. S. Kang, *Fuzzifications of generalized Tarski filters in Tarski algebras*, **Computers and Mathematics with Applications**, vol. 61 (2011), pp. 1–7, DOI: https://doi.org/10.1016/j.camwa.2010.10.024.

[12] H. S. Kim, Y. H. Kim, *On BE-algebras*, **Scientiae Mathematicae Japonicae**, vol. 66(1) (2007), pp. 113–116, DOI: https://doi.org/10.32219/isms.66.1_113.

[13] J. Kim, Y. Kim, E. H. Roh, *A note on GT-algebras*, **The Pure and Applied Mathematics**, vol. 16(1) (2009), pp. 59–69, URL: https://koreascience.kr/article/JAKO200910335351650.page.

[14] Y. Komori, *The class of BCC-algebras is not a variety*, **Mathematica Japonica**, vol. 29(3) (1984), pp. 391–394.

[15] C. A. Meredith, **Formal Logics**, 2nd ed., Clarendon Press, Oxford (1962).

[16] A. Monteiro, **Lectures on Hilbert and Tarski algebras**, 2nd ed., Insitituto de Mathemática, Universuidad Nacional del Sur, Bahía Blanca, Argentina (1960).

[17] S. Tanaka, *A new class of algebras*, **Mathematics Seminar Notes**, vol. 3 (1975), pp. 37–43.

[18] A. Walendziak, *On commutative BE-algebras*, **Scientiae Mathematicae Japonicae**, vol. 69(2) (2009), pp. 281–284, DOI: https://doi.org/10.32219/ISMS.69.2_281.

[19] A. Walendziak, *The implicative property for some generalizations of BCK algebras*, **Journal of Multiple-Valued Logic and Soft Computing**, vol. 31 (2018), pp. 591–611.

[20] A. Walendziak, *The property of commutativity for some generalizations of BCK algebras*, **Soft Computing**, vol. 23 (2019), pp. 7505–7511, DOI: https://doi.org/10.1007/s00500-018-03691-9.

[21] A. Walendziak, *On implicative BE algebras*, **Annales Universitatis Mariae Curie-Skłodowska, Sectio A**, vol. 76 (2022), pp. 45–54, DOI: https://doi.org/10.17951/a.2022.76.2.45-54.

[22] H. Yutani, *On a system of axioms of commutative BCK-algebras*, **Mathematics Seminar Notes**, vol. 5 (1977), pp. 255–256.

[23] D. Zelent, *Transitivity of implicative aBE algebras*, **Annales Universitatis Mariae Curie-Skłodowska, Sectio A**, vol. 76 (2022), pp. 55–58, DOI: https://doi.org/10.17951/a.2022.76.2.55-58.

Andrzej Walendziak
Siedlce University of Natural Sciences and Humanities
Faculty of Exact and Natural Sciences
Institute of Mathematics
3 Maja 54
08-110 Siedlce, Poland
e-mail: walent@interia.pl

Bulletin of the **S**ection of **L**ogic
Volume 52/4 (2023), pp. 517–550
https://doi.org/10.18778/0138-0680.2023.23

Marcelo Esteban Coniglio
Guilherme Vicentin de Toledo

A CATEGORY OF ORDERED ALGEBRAS EQUIVALENT TO THE CATEGORY OF MULTIALGEBRAS

Abstract

It is well known that there is a correspondence between sets and complete, atomic Boolean algebras (*CABA*s) taking a set to its power-set and, conversely, a complete, atomic Boolean algebra to its set of atomic elements. Of course, such a correspondence induces an equivalence between the opposite category of **Set** and the category of *CABA*s.

We modify this result by taking multialgebras over a signature Σ, specifically those whose non-deterministic operations cannot return the empty-set, to *CABA*s with their zero element removed (which we call a *bottomless Boolean algebra*) equipped with a structure of Σ-algebra compatible with its order (that we call *ord-algebras*). Conversely, an ord-algebra over Σ is taken to its set of atomic elements equipped with a structure of multialgebra over Σ. This leads to an equivalence between the category of Σ-multialgebras and the category of ord-algebras over Σ.

The intuition, here, is that if one wishes to do so, non-determinism may be replaced by a sufficiently rich ordering of the underlying structures.

Keywords: multialgebras, ordered algebras, non-deterministic semantics.

Introduction

It is a seminal result (see [24] for a proof) that a correlation between sets and complete, atomic Boolean algebras (*CABA*s) exists: a set is taken to

Presented by: Janusz Ciuciura
Received: September 25, 2022
Published online: August 16, 2023

its power-set, while a *CABA* is taken to its set of atomic elements. These two assignments can be made into functors, giving rise to an equivalence of $\mathbf{Set}^{op}$ and **CABA**, the category with *CABA*s as objects.

This is part of a broader area of study, known as Stone dualities, which studies relationships between posets and topological spaces and was established by Stone ([22]) and his representation theorem, which states that every Boolean algebra is isomorphic to a field of sets, specifically the algebra of clopen sets of its Stone space (a topological space where points are ultrafilters of the original Boolean algebra). Of course, this corresponds to an equivalence between the category **BA** of Boolean algebras and that of Stone spaces.

In the search of further such equivalences, we focus on a more concrete one, associated to the one between $\mathbf{Set}^{op}$ and **CABA** in the sense that: we look at an enriched category of sets on one side, namely a category of multialgebras (multialgebras having been originally introduced by Marty in [15] through the notion of hypergroups) over a signature Σ, obtained by adding multioperations to **Set**; and on the other side, at a category attained by equipping the objects of **CABA** with Σ-operations compatible with their orders. And reaching such an equivalence using the aforementioned most general definition of multialgebras on one side, and *CABA*s on the other, is possible: indeed, we do so briefly on Section 5 as a corollary of other of our results. But we choose to focus most of our efforts instead on slightly distinct categories: we are most interested in non-partial multialgebras, where the result of an operation never returns the empty set. Consequently, we exchange *CABA*s for posets corresponding to power-sets with the empty-set removed (that is, *CABA*s without minimum elements, that we call *bottomless Boolean algebras*). This way, a multialgebra, with universe A, is taken to an algebra over the set of non-empty subsets of A, with order given by inclusion and operations given by "accumulating" the operations of the multialgebra, while conversely, a bottomless Boolean algebras is taken to its set of atomic elements, transformed into a multialgebra.

In the area of research of non-deterministic semantics ([2]), specially paraconsistent logics ([7]), this offers an alternative: many logicians are reluctant to appeal to multialgebras in order to characterize a given logic, and the equivalence we here present shows one can, if one chooses to, replace such non-deterministic structures with more classically-behaved algebras, with an added underlying order. Furthermore, using bottomless Boolean algebras follows a trend: in logic, we are used to considering ordered algebraic structures without bottoms; for instance, implicative lattices, which are bottomless Heyting algebras. The use of bottomless Boolean algebras feels then justified because the definition of the functor between the categories is much simpler and seems to better correspond to the intuition found in using non-deterministic semantics.

This paper is organized as follows: In the first section, we give the definition of multialgebras we will use and introduce a brief characterization of power-sets without the empty-set. In the second section, we introduce a naive approach to what we would like to accomplish, and show why it fails. In the third section, we introduce the categories for which our desired result actually holds and the functors that will establish an equivalence between them, which we detail in section four. The final section is reserved for related results.

Preliminary versions of this paper can be found in the PhD thesis [23] and in the preprint [11].

1. Preliminary notions

A signature is a collection $\Sigma = \{\Sigma_n\}_{n\in\mathbb{N}}$ of possibly empty, disjoint sets indexed by the natural numbers; when there is no risk of confusion, the union $\bigcup_{n\in\mathbb{N}} \Sigma_n$ will also be denoted by Σ.

A $\Sigma-$multialgebra (also known as multialgebra) is a pair $\mathcal{A} = (A, \{\sigma_{\mathcal{A}}\}_{\sigma\in\Sigma})$ such that: A is a non-empty set and, for $\sigma \in \Sigma_n$, $\sigma_{\mathcal{A}}$ is a function of the form

$$\sigma_{\mathcal{A}} : A^n \to \mathcal{P}(A) \setminus \{\emptyset\},$$

where $\mathcal{P}(A)$ denotes the power-set of A. If $\sigma_{\mathcal{A}}(\vec{a})$ is a singleton for every $\sigma_{\mathcal{A}}$ and $\vec{a} \in A^n$, then $\mathcal{A}$ is said to be deterministic, and can be identified with a standard algebra.

A homomorphism between $\Sigma-$multialgebras $\mathcal{A} = (A, \{\sigma_{\mathcal{A}}\}_{\sigma\in\Sigma})$ and $\mathcal{B} = (B, \{\sigma_{\mathcal{B}}\}_{\sigma\in\Sigma})$ is a function $h : A \to B$ satisfying, for any $n \in \mathbb{N}$, $\sigma \in \Sigma_n$ and $a_1, \dots, a_n \in A$,

$$\{h(a) : a \in \sigma_{\mathcal{A}}(a_1, \dots, a_n)\} \subseteq \sigma_{\mathcal{B}}(h(a_1), \dots, h(a_n)).$$

If the inclusion, in the previous equation, were to be replaced by an equality, the resulting h would be a full homomorphism; and a bijective full homomorphism is called an isomorphism. Whenever h is a homomorphism from $\mathcal{A}$ to $\mathcal{B}$, we write $h : \mathcal{A} \to \mathcal{B}$. If both $\mathcal{A}$ and $\mathcal{B}$ are deterministic, then h can be identified with a standard homomorphism between algebras.

1.1. Related approaches

Stone-like dualities in particular, and related categorial equivalences in general, are a fertile ground for new results since their conception. Given multialgebras, at least when conceived as relational structures, have permeated mathematics for a long time, it is natural that both concepts have in a way or another interacted in the past.

The simple idea of taking a multialgebra (or a relational structure if one wishes to be more general) and, from that multialgebra, constructing an ordered algebra goes back at least to Dresher and Ore [13]: the constructed algebra is usually referred to as a power algebra, complex, or global algebra. An interesting analysis of this construction is found in Brink [5], which however deals mostly with the broader notion of a relational algebra instead of multialgebras *per se*; a, less studied, alternative to ordered algebras, which attempt to capture inclusion, are the ε-algebras of [8], that in turn try to codify those properties of membership.

One can, however, also find examples of the procedure from which one obtains the power algebra from a multialgebra being applied to the multialgebras we are more interested in, the non-partial ones. Pickett [20], Bruck [6], Walicki and Meldal [25], and Breaz and Pelea [19] do precisely that, although Pickett and Bruck appear to be more preoccupied with the applications of this notion to multigroups and, respectively, multiquasigroups and multigrupoids.

And, despite the fact we use the same construction, not only none of the aforementioned papers delve in the study of Stone-dualities or any related categorical equivalences, they use markedly different definitions of homomorphisms (except Walicki and Meldal, that use no definition of homomorphism as their work is mostly devoted to generalizing identities to the context of multialgebras): more specifically, they use what some logicians call in today's literature full homomorphisms (see [7]); that is not unexpected, since that definition is very appropriate when dealing with the theories of multigroups and hyperrings, but not when dealing with Nmatrices.

Bošnjak and Madarász [4] also use power algebras of non-partial multialgebras, as long as one considers the obvious connection between the latter and multigraphs, see [9] for a definition of this generalization of graphs (and its applications to logic), and [10] for a few constructions of multialgebras from multigraphs.

In one of the first papers dealing with multialgebras and duality, Hansoul achieves in [14] results very similar to those we wish to obtain, but for different multialgebras: specifically, he uses what some call nowadays partial multialgebras, which are full relational structures; in other words, the result of a multioperation may equal the empty set, and that makes his analysis necessarily very different from ours, non-partial multialgebras being preferred as semantics for non-classical logics (see [2] for the first discussion on the use of non-partial multialgebras as semantic objects, and [3] for the first discussion on the use of partial multialgebras). Nolan, in his thesis ([18]), obtains more general results than Hansoul, taking into his analysis both ordered algebras and Boolean algebras with operators, but also restricts himself to partial multialgebras, giving again preference to full homomorphisms.

To summarize, although the construction leading to power algebras, and multialgebras have interacted before, this was done for the sake of these algebraic structures themselves, or for applications within the realm of pure mathematics, such as in the theory of hypergroups. The difference here lies in the very basic structures we aim to deal with: we are using non-partial multialgebras (that we take the liberty of addressing simply by multialgebras from here forward), together with a weak notion of homomorphism that, although not specially useful for the study of, e.g., hypergroups, is the standard when applying multialgebras to logics. And these profound differences in the category to be analyzed lead to equally profound differences on how the method of producing power algebras behaves, suggesting the use of what we chose to call bottomless Boolean algebras, as we now set out to define.

1.2. Bottomless Boolean algebras

Here, we will understand Boolean algebras mostly as partially-ordered sets (*poset*). A pair $(A, \leq)$ is a Boolean algebra if: $\leq$ is reflexive, anti-symmetric and transitive; there are a maximum (denoted by 1) and a minimum (0), which we shall assume distinct; for every pair of elements $(a, b) \in A^2$, the set $\{a, b\}$ has a supremum, denoted by $\sup\{a, b\}$ or $a \vee b$, and an infimum, denoted by $\inf\{a, b\}$ or $a \wedge b$; and every element a has a complement b which satisfies

$$b = \min\{c \in A : \sup\{a, c\} = 1\}$$

and

$$b = \max\{c \in A : \inf\{a, c\} = 0\}.$$

A Boolean algebra $(A, \leq)$ is said to be complete if every $S \subseteq A$ has a supremum.

LEMMA 1.1. (i) *Every Boolean algebra* $(A, \leq)$ *is distributive, meaning*

$$\sup\{a, \inf\{b, c\}\} = \inf\{\sup\{a, b\}, \sup\{a, c\}\}$$

and

$$\inf\{a, \sup\{b, c\}\} = \sup\{\inf\{a, b\}, \inf\{a, c\}\},$$

for any $a, b, c \in A$;

(ii) *every complete Boolean algebra* $(A, \leq)$ *is infinite distributive, meaning that for any* $S \cup \{a\} \subseteq A$, $\sup\{\inf\{a, s\} : s \in S\} = \inf\{a, \sup S\}$ *and* $\inf\{\sup\{a, s\} : s \in S\} = \sup\{a, \inf S\}$.

An element a of a Boolean algebra $(A, \leq)$ is said to be an atom if it is minimal in $A \setminus \{0\}$ according to $\leq$, which means that if $b \leq a$, then either $b = 0$ or $b = a$. The set of all the atoms d such that $d \leq a$ will be denoted by A_a. Finally, a Boolean algebra is said to be atomic if, for every one of its elements a, $a = \sup A_a$.

Notice that complete, atomic Boolean algebras are power-sets, at least up to an equivalence (of categories). In one direction, this equivalence takes a Boolean algebra $\mathcal{A} = (A, \leq)$ to the power-set $\mathcal{P}(A_1)$ of the set A_1 of all its atoms, an element $a \in A \setminus \{0\}$ being mapped (by the associated natural isomorphisms of the equivalence) to A_a, and 0 to $\emptyset$. Conversely, a set X its taken by this equivalence to the complete, atomic Boolean algebra that is the power-set of X, $\mathcal{P}(X)$. For more information, look at Theorem 2.4 of [24].

We would like to work with Boolean algebras that are, simultaneously, complete, atomic and bottomless, meaning they lack a bottom element: this seems a contradiction, given we assume Boolean algebras to have bottom elements, but it can be adequately formalized.

Definition 1.2. Given a non-empty partially ordered set $\mathcal{A} = (A, \leq_{\mathcal{A}})$, we define $\mathcal{A}_0$ as the partially ordered set

$$(A \cup \{0\}, \leq_{\mathcal{A}_0}),$$

where we assume $0 \notin A$, such that $a \leq_{\mathcal{A}_0} b$ if and only if:

(i) either $a \leq_{\mathcal{A}} b$;

(ii) or $a = 0$.

Definition 1.3. A non-empty partially ordered set $\mathcal{A}$ is a *bottomless Boolean algebra* whenever $\mathcal{A}_0$ is a Boolean algebra; if $\mathcal{A}_0$ is a complete or atomic, $\mathcal{A}$ is also said to be complete or atomic, respectively.

Notice that, since $\mathcal{P}(\emptyset)$ only has $\emptyset$ as element, for any complete, atomic and bottomless Boolean algebra $\mathcal{A}$ we cannot have $\mathcal{A}_0 = \mathcal{P}(\emptyset)$, given $\mathcal{A}$ has at least one element and therefore $\mathcal{A}_0$ must have at least two. This means complete, atomic and bottomless Boolean algebras correspond to the power-set of non-empty sets with $\emptyset$ removed.

Definition 1.4. A partially ordered set $(A, \leq)$ with maximum 1 is called semi-complemented when for every $a \in A \setminus \{1\}$ there exists $b \in A$, named a complement of a, such that

$$b = \min\{c \in A : \sup\{a, c\} = 1\}$$

and

$$b = \max\{c \in A : \inf\{a, c\} \text{ does not exist}\}.$$

Theorem 1.5. *Consider the following properties a partially ordered set* $\mathcal{A} = (A, \leq)$ *may have.*

(i) *It has a maximum element* 1.

(ii) $(A, \leq)$ *is semi-complemented.*

(iii) *All subsets with two elements* $\{a, b\}$ *of* A *have a supremum.*

(iii)* *All non-empty subsets* S *of* A *have a supremum.*

(iv) *Denoting by* A_a *the set of minimal elements smaller than* a, $a = \sup A_a$.

If $\mathcal{A}$ *satisfies* (i), (ii) *and* (iii), *it is a bottomless Boolean algebra; if* $\mathcal{A}$ *satisfies* (i), (ii) *and* (iii)*, *it is a complete bottomless Boolean algebra; and if it satisfies* (i), (ii), (iii)* *and* (iv), *it is an atomic, complete Bottomless Boolean algebra.*

Proof: Suppose that $\mathcal{A} = (A, \leq_{\mathcal{A}})$ is a partially ordered set satisfying (i), (ii) and (iii). Since $\mathcal{A}$ is a partially ordered set, so is $\mathcal{A}_0$ from Definition 1.2. The maximum 1 of $\mathcal{A}$ remains a maximum in $\mathcal{A}_0$, while 0 becomes a minimum. For non-zero elements a and b, the supremum in $\mathcal{A}_0$ of $\{a, b\}$ remains the same as in $\mathcal{A}$, while if $a = 0$ or $b = 0$ the supremum is simply the largest of the two.

If a or b are equal to 0, the infimum is 0, while if $a, b \in A$ there are two cases to consider: if $\inf\{a, b\}$ was defined in $\mathcal{A}$, it remains the same in $\mathcal{A}_0$. If the infimum was not defined in $\mathcal{A}$, it must be 0 in $\mathcal{A}_0$: indeed, $\inf\{a, b\}$ certainly exists in $\mathcal{A}_0$, given that is a Boolean algebra; and if it is not 0, it is in $\mathcal{A}$, being an infimum in this poset as well.

Every element $a \in A\backslash\{1\}$ already has a complement b in $\mathcal{A}$ such that $b = \min\{c \in A : \sup\{a, c\} = 1\}$ and $b = \max\{c \in A : \inf\{a, c\}$ does not exist$\}$. Of course the first equality keeps on holding in $\mathcal{A}_0$, while the second becomes, remembering that the non-defined infima in $\mathcal{A}$ become 0 in $\mathcal{A}_0$,

$$b = \max\{c \in A : \inf\{a, c\} = 0\};$$

the complement of 1 is clearly 0 and vice-versa. This proves that $\mathcal{A}_0$ is a Boolean algebra, and so $\mathcal{A}$ is a bottomless Boolean algebra.

Suppose now $\mathcal{A}$ satisfies instead (i), (ii) and (iii)*: since (iii)* implies (iii), it is clear that $\mathcal{A}$ is a bottomless Boolean algebra; and, since $\mathcal{A}$ is now closed under the suprema of any non-empty sets, and $\sup\emptyset = 0$ in $\mathcal{A}_0$, it is clear that $\mathcal{A}_0$ is closed under any suprema.

Finally, if $\mathcal{A}$ satisfies (i), (ii), (iii)* and (iv), it is to begin with a complete bottomless Boolean algebra; furthermore, clearly $\mathcal{A}_0$ remains atomic, since $\mathcal{A}$ is atomic, what completes the proof that the previous list of conditions imply $\mathcal{A}$ is a complete, atomic and bottomless Boolean algebra. □

Theorem 1.6. *The converses of Theorem 1.5 hold, meaning that: a bottomless Boolean algebra satisfies conditions* (i), (ii) *and* (iii)*; a complete bottomless Boolean algebra satisfies conditions* (i), (ii) *and* (iii)**; and an atomic, complete bottomless Boolean algebra satisfies conditions* (i), (ii), (iii)* *and* (iv).

Proof: Given a partially ordered set $\mathcal{A}$, suppose $\mathcal{A}_0$ is a Boolean algebra.

(i) The maximum 1 of $\mathcal{A}_0$ is still a maximum in $\mathcal{A}$.

(ii) Given any element $a \neq 1$, its complement b in $\mathcal{A}_0$ ends up being also its complement in $\mathcal{A}$. Clearly

$$b = \min\{c \in A : \sup\{a, c\} = 1\}.$$

Now, $\inf\{a, c\}$ does not exist in $\mathcal{A}$ if, and only if, $\inf\{a, c\} = 0$ in $\mathcal{A}_0$: we already proved that if $\inf\{a, c\}$ does not exist in $\mathcal{A}$ then $\inf\{a, c\} = 0$ in $\mathcal{A}_0$, remaining to show the converse; if the infimum of a and c existed in $\mathcal{A}$, it would equal 0 in $\mathcal{A}_0$ given the unicity of the infimum, contradicting that 0 is not in $\mathcal{A}$. This way, we find that in $\mathcal{A}$

$$b = \max\{c \in A : \inf\{a, c\} \text{ does not exist}\},$$

as required.

(iii) The supremum of any set $\{a, b\}$ of cardinality 2 in $\mathcal{A}$ is just its supremum in $\mathcal{A}_0$.

Suppose now $\mathcal{A}_0$ is a complete Boolean algebra.

(iii)* Then the supremum of any non-empty set in $\mathcal{A}$ is its supremum in $\mathcal{A}_0$.

Finally, let $\mathcal{A}_0$ now be an atomic, complete Boolean algebra.

(iv) Clearly $\mathcal{A}_0$ being atomic implies $\mathcal{A}$ being atomic. □

PROPOSITION 1.7. If $(A, \leq_{\mathcal{A}})$ is a complete, atomic and bottomless Boolean algebra, for any $S \subseteq A$, if

$$S^a = \{s \in S : \inf\{a, s\} \text{ exists}\} \neq \emptyset,$$

then

$$\sup\{\inf\{a, s\} : s \in S^a\} = \inf\{a, \sup S\};$$

if $S^a = \emptyset$, $\inf\{a, \sup S\}$ also does not exist.

PROOF: If $S^a = \emptyset$, this means that $\inf\{a, s\} = 0$ for every $s \in S$ in $\mathcal{A}_0$, and therefore $\inf\{a, \sup S\} = 0$, so that the same infimum no longer exists in $\mathcal{A}$.

If $S^a \neq \emptyset$, all infima and suprema in $\sup\{\inf\{a, s\} : s \in S^a\}$ and $\inf\{a, \sup S\}$ exist in $\mathcal{A}$ and are therefore equal to their counterparts in $\mathcal{A}_0$; given $\sup\{\inf\{a, s\} : s \in S^a\} = \sup\{\inf\{a, s\} : s \in S\}$ in $\mathcal{A}_0$, since $s \in S \setminus S^a$ implies $\inf\{a, s\} = 0$, by the infinite-distributivity of $\mathcal{A}_0$ one proves the desired result. □

The lesson to be taken from this short exposition is that a complete, atomic and bottomless Boolean algebra is a power-set (of a non-empty set) with the empty-set removed. This will be important to us given our multialgebras cannot return the empty-set as the result of an operation.

1.3. Tarski algebras and classical implicative lattices

Now, the use of bottomless Boolean algebras may seem an odd choice of structures to take into consideration, given their proximity to Boolean algebras, but there are two important reasons for that choice. First of all, they are very natural when considering the multialgebras, as well as the homomorphisms, typically found when studying non-deterministic semantics. Second, this choice is not as odd as it may appear at first when considering the vast diversity of algebraic structures that are required when dealing with algebraic logic.

We then make a brief comparison of bottomless Boolean algebras with two varieties of algebras, Tarski algebras and classical implicative lattices, both designed to capture the behavior of some negation-free fragment of classical logic: this likeness follows from the fact that, by ignoring the empty-set, we are also, in a sense, considering a positive fragment of something when defining bottomless Boolean algebras. We start with Tarski algebras. In the 1960s, J. Abbott [1] and A. Monteiro [16], with the aim of capturing the implicational fragment of classical logic, independently introduced and studied a class of implication algebras related to Boolean algebras. The latter called them *Tarski algebras* in lectures delivered at Universidad Nacional del Sur (cf. [17]), while the former called them *implicational algebras*. These algebraic structures have only a binary connective $\to$ and satisfy the following axioms (we use infix notation).

(i) $(x \to y) \to x = x$;
(ii) $x \to (y \to z) = y \to (x \to z)$;
(iii) $(x \to y) \to y = (y \to x) \to x$.

Considering our previous commentary, the following result is perhaps not entirely surprising.

THEOREM 1.8. *Given a bottomless Boolean algebra $\mathcal{A} = (A, \leq)$, define*

$$a \to b = \begin{cases} \sup\{c, b\} & \text{if } a \neq 1, \text{ where } c \text{ is the complement of } a; \\ b & \text{if } a = 1. \end{cases}$$

Then A, equipped with $\to$, is a Tarski algebra.

PROOF: Although this can be brute-forced, it is easier to see that $a \to b$ is just the implication of the Boolean algebra $\mathcal{A}_0$, restricted to its non-zero elements: indeed, if $a \neq 1$, and c is the complement of a, $c = \neg a$ and $\sup\{c, b\} = \neg c \vee b$; if $a = 1$, $\neg a = 0$, and $b = 0 \vee b = \neg 1 \vee b$. Notice that $\neg a \vee b$ can never be 0, if $b \neq 0$.

As every Boolean algebra is a Taski algebra, we are done. □

Example 1.9. Consider $A = \{a, b, c, 1\}$ and the following table for an implication on A that gives us a structure $\mathcal{A}$.

$\to$	a	b	c	1
a	1	b	c	1
b	a	1	c	1
c	a	b	1	1
1	a	b	c	1

This structure can be shown to be a Tarski algebra, but is not a bottomless Boolean algebra: if it were, $\mathcal{A}_0$ would be a Boolean algebra with 5 elements, what is impossible.

We can show even more: the structure in Example 1.9 is not a classical implicative lattice either. To better explain what that means, a classical implicative lattice, introduced in [12], is an algebra on the signature with symbols $\vee$, $\wedge$ and $\to$ of arity 2, and 1 of arity 0, such that $\vee$, $\wedge$ and 1 make of the structure a lattice with top element 1 (and the usual order, where $x \leq y$ iff $x \vee y = y$), and the following axioms are satisfied:

(i) $x \wedge y \leq z$ iff $y \leq x \to z$;

(ii) $(x \to y) \to x = x$.

As Tarski algebras attempt to capture the implicative fragment of classical propositional logic, classical implicative lattices attempt to capture the positive fragment of the same logic. It is relatively easy to see ([21] being a possibility) that all classical implicative lattices are Tarski algebras. Furthermore, as one can, in a finite classical implicative lattice, obtain a bottom by taking the infimum of all elements, it is also true that all non-trivial finite classical implicative lattices are Boolean algebras.[1]

[1] By non-trivial we mean with cardinality bigger than one: the only element of a one-element classical implicative lattice is both a top and a bottom, what makes of the one-element classical implicative lattice not a Boolean algebra; the situation changes in a foreseeable way if one wishes to entertain the possibility of a one-element Boolean algebra.

Notice that the structure from Example 1.9 is not a classical implicative lattice because if $\to$ were the implication of a Boolean algebra, $0 \to x$ would equal 1 for every x in $\mathcal{A}$, what clearly is not true whether 0 equals a, b or c.

Now, it is obvious that any Boolean algebra, whether finite or infinite, is a classical implicative lattice. As shown below, the reciprocal is not true.

Example 1.10. Take an infinite set X (say $\mathbb{N}$), and define $A(X)$ as the set of subsets a of X such that a^c is finite, where a^c is the complement $X \setminus a$ of a: these are called the cofinite sets of X.

$A(X)$ has an obvious order, such that $a \leq b$ iff $a \subseteq b$, and a maximal element under this order, X itself. Then $a \vee b = a \cup b$ is the supremum of a and b (and is cofinite since $|(a \cup b)^c| = |a^c \cap b^c| \leq |a^c|$, which is finite), and $a \wedge b = a \cap b$ is the infimum of a and b (and is cofinite since $|(a \cap b)^c| = |a^c \cup b^c| \leq |a^c| + |b^c|$, which is finite since $|a^c|$ and $|b^c|$ are finite).

We then define $a \to b$ as $a^c \cup b$: this is cofinite since $|(a^c \cup b)^c| = |a \cap b^c| \leq |b^c|$, which is finite given that b is cofinite.

(i) if $a \wedge b \leq c$ then $a \cap b \subseteq c$ and so

$$b \subseteq b \cup a^c = X \cap (b \cup a^c) = (a \cup a^c) \cap (b \cup a^c) = (a \cap b) \cup a^c \subseteq c \cup a^c,$$

that is, $b \leq a \to c$. Conversely, $b \leq a \to c$ implies that $b \subseteq c \cup a^c$, hence

$$a \cap b \subseteq a \cap (c \cup a^c) = (a \cap c) \cup (a \cap a^c) = a \cap c \subseteq c,$$

that is, $a \wedge b \leq c$.

(ii) $(a \to b) \to a$ equals

$$(a^c \cup b)^c \cup a = (a \cap b^c) \cup a = (a \cup a) \cap (b^c \cup a) = a \cap (b^c \cup a) = a.$$

So, we have proven $A(X)$ is a classical implicative lattice. But it cannot be a Boolean algebra: no element a of $A(X)$ can be a bottom, since removing a single element of X from a gives an element of $A(X)$ strictly smaller than a itself.

There is, however, a more involved, although also more natural, example of a classical implicative lattice that is not a Boolean algebra: the Lindenbaum-Tarski algebra of the positive fragment of classical logic in an infinite number of variables. That is, in fact, the very reason why classical implicative lattices were defined, to model the properties of these fragments.

Finally, to completely characterize the relationship between Tarski, Boolean and Bottomless Boolean algebras, and classical implicative lattices, we just need to prove that no non-trivial Bottomless Boolean algebra is a classical implicative lattice: of course, the one-element bottomless Boolean algebra is also a classical implicative lattice. Suppose, then, that the poset $\mathcal{A} = (A, \leq)$ is a bottomless Boolean algebra, a classical implicative lattice, and has more than one element in its domain, so $\mathcal{A}_0$ is a Boolean algebra with at least four elements. There is, therefore, an element a in $\mathcal{A}_0$ that is neither 0 nor 1, and so is $\neg a$: both a and $\neg a$ must then be in $\mathcal{A}$, and so must $0 = a \wedge \neg a$ since $\mathcal{A}$ is a classical implicative lattice. This is a contradiction, given that $\mathcal{A}$, as a bottomless Boolean algebra with at least two elements, cannot have a bottom.

We therefore reach the characterization shown in Figure 1.

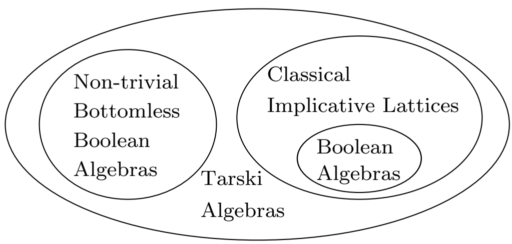


Figure 1. Several classes of Boolean-like algebras

2. A naive approach to an equivalence of categories

Consider the categories $\mathbf{Alg}(\Sigma)$ of Σ-algebras, with homomorphisms between Σ-algebras as morphisms, and $\mathbf{MAlg}(\Sigma)$ of Σ-multialgebras, with homomorphisms between Σ-multialgebras as morphisms, and with composition of morphisms and identity morphisms as in $\mathbf{Set}$.

For simplicity, denote the set of non-empty subsets of A, $\mathcal{P}(A) \setminus \{\emptyset\}$, by $\mathcal{P}^*(A)$. For a Σ-multialgebra $\mathcal{A} = (A, \{\sigma_{\mathcal{A}}\}_{\sigma \in \Sigma})$, consider the Σ-algebra $\mathsf{P}(\mathcal{A}) = (\mathcal{P}^*(A), \{\sigma_{\mathsf{P}(\mathcal{A})}\}_{\sigma \in \Sigma})$ where, for a $\sigma \in \Sigma_n$ and nonempty $A_1, \dots, A_n \subseteq A$,

$$\sigma_{\mathsf{P}(\mathcal{A})}(A_1, \dots, A_n) = \bigcup_{(a_1,\dots,a_n) \in A_1 \times \cdots \times A_n} \sigma_{\mathcal{A}}(a_1, \dots, a_n).$$

Again, for simplicity, we may write the previous equation as $\sigma_{\mathsf{P}(\mathcal{A})}(A_1, \ldots, A_n) = \bigcup\{\sigma_{\mathcal{A}}(a_1, \ldots, a_n) : a_i \in A_i\}$. We also define, for $\mathcal{A}$ and $\mathcal{B}$ two Σ-multialgebras, and a homomorphism $h : \mathcal{A} \to \mathcal{B}$, the function $\mathsf{P}(h) : \mathsf{P}(A) \to \mathsf{P}(B)$ such that, for every $\emptyset \neq A' \subseteq A$,

$$\mathsf{P}(h)(A') = \{h(a) : a \in A'\}.$$

One could hope that $\mathsf{P}(h)$ is actually a Σ-homomorphism, perhaps making of P a functor from $\mathbf{MAlg}(\Sigma)$ to $\mathbf{Alg}(\Sigma)$, but the following result shows this is usually not the case.

Lemma 2.1. *For $\mathcal{A}$ and $\mathcal{B}$ two Σ-multialgebras and $h : \mathcal{A} \to \mathcal{B}$ a homomorphism, $\mathsf{P}(h)$ satisfies*

$$\mathsf{P}(h)(\sigma_{\mathsf{P(A)}}(A_1, \ldots, A_n)) \subseteq \sigma_{\mathsf{P(B)}}(\mathsf{P}(h)(A_1), \ldots, \mathsf{P}(h)(A_n))$$

for all $\sigma \in \Sigma$ and nonempty $A_1, \ldots, A_n \subseteq A$. If h is a full homomorphism, $\mathsf{P}(h)$ is a homomorphism.

Proof: Given $\sigma \in \Sigma_n$ and nonempty $A_1, \ldots, A_n \subseteq A$, we have that

$$\sigma_{\mathsf{P}(\mathcal{B})}(\mathsf{P}(h)(A_1), \ldots, \mathsf{P}(h)(A_n)) = \bigcup\{\sigma_{\mathcal{B}}(b_1, \ldots, b_n) : b_i \in \mathsf{P}(h)(A_i)\} =$$

$$\bigcup\{\sigma_{\mathcal{B}}(b_1, \ldots, b_n) : b_i \in \{h(a) : a \in A_i\}\} =$$

$$\bigcup\{\sigma_{\mathcal{B}}(h(a_1), \ldots, h(a_n)) : a_i \in A_i\} \supseteq$$

$$\bigcup\{\{h(a) : a \in \sigma_{\mathcal{A}}(a_1, \ldots, a_n)\} : a_i \in A_i\} =$$

$$\{h(a) : a \in \bigcup\{\sigma_{\mathcal{A}}(a_1, \ldots, a_n) : a_i \in A_i\}\} =$$

$$\{h(a) : a \in \sigma_{\mathsf{P}(\mathcal{A})}(A_1, \ldots, A_n)\} = \mathsf{P}(h)(\sigma_{\mathsf{P}(\mathcal{A})}(A_1, \ldots, A_n)),$$

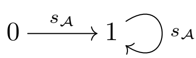
so that $\mathsf{P}(h)$ satisfies the required property.

If h is a full homomorphism, $\sigma_{\mathcal{B}}(h(a_1), \ldots, h(a_n)) = \{h(a) : a \in \sigma_{\mathcal{A}}(a_1, \ldots, a_n)\}$, and the inclusions above become identities. □

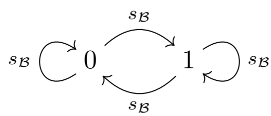
So, let us restrict P for a moment to the category $\mathbf{MAlg}_{=}(\Sigma)$, of Σ-multialgebras with only full homomorphisms between them as morphisms, and let us call this new transformation $\mathsf{P}_{=} : \mathbf{MAlg}_{=}(\Sigma) \to \mathbf{Alg}(\Sigma)$.

PROPOSITION 2.2. $\mathsf{P}_=$ is a functor from $\mathbf{MAlg}_=(\Sigma)$ to $\mathbf{Alg}(\Sigma)$.

Unfortunately, $\mathsf{P}_=$ is not injective on objects: take the signature Σ_s with a single unary operator s, and consider the Σ_s-multialgebras $\mathcal{A} = (\{0,1\},\{s_\mathcal{A}\})$ and $\mathcal{B} = (\{0,1\},\{s_\mathcal{B}\})$ such that: $s_\mathcal{A}(0) = s_\mathcal{A}(1) = \{1\}$ and $s_\mathcal{B}(0) = s_\mathcal{B}(1) = \{0,1\}$.



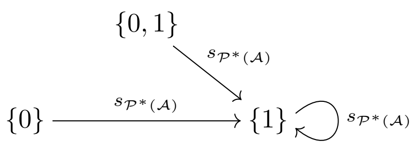
The Σ_s-multialgebra $\mathcal{A}$



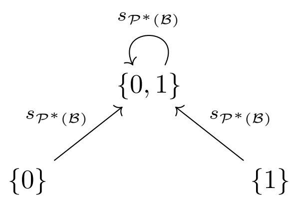
The Σ_s-multialgebra $\mathcal{B}$

Clearly the two of them are not isomorphic, given that the result of an operation in $\mathcal{A}$ always has cardinality 1 and in $\mathcal{B}$ alway has cardinality 2.

However, we have that $s_{\mathsf{P}_=(\mathcal{A})}(\{0\}) = s_{\mathsf{P}_=(\mathcal{A})}(\{1\}) = s_{\mathsf{P}_=(\mathcal{A})}(\{0,1\}) = \{1\}$, while $s_{\mathsf{P}_=(\mathcal{B})}(\{0\}) = s_{\mathsf{P}_=(\mathcal{A})}(\{1\}) = s_{\mathsf{P}_=(\mathcal{A})}(\{0,1\}) = \{0,1\}$.



The Σ_s-algebra $\mathsf{P}_=(\mathcal{A})$



The Σ_s-algebra $\mathsf{P}_=(\mathcal{B})$

Taking the function $h : \mathcal{P}^*(A) \to \mathcal{P}^*(B)$ such that $h(\{0\}) = \{0\}$, $h(\{1\}) = \{0,1\}$, and $h(\{0,1\}) = \{1\}$, we see that it is a bijection and a homomorphism, and therefore $h : \mathsf{P}_=(\mathcal{A}) \to \mathsf{P}_=(\mathcal{B})$ is an isomorphism.

3. A solution: ord-algebras

The problem with our definition of $\mathsf{P}_=$ is that it disregards the structure of the universe of $\mathcal{P}(\mathcal{A})$. So, we change our target category to reflect this structure.

3.1. The category $\mathbf{OAlg}(\Sigma)$, and the functor $\mathbb{P}$

DEFINITION 3.1. Given a signature Σ, an ord-algebra over Σ is a triple $\mathcal{A} = (A, \{\sigma_{\mathcal{A}}\}_{\sigma\in\Sigma}, \leq_{\mathcal{A}})$ such that:

(i) $(A, \{\sigma_{\mathcal{A}}\}_{\sigma\in\Sigma})$ is a Σ-algebra;

(ii) $(A, \leq_{\mathcal{A}})$ is a complete, atomic and bottomless Boolean algebra;

(iii) if A_a is the set of minimal elements of $(A, \leq_{\mathcal{A}})$ (atoms) less than or equal to a, for all $\sigma \in \Sigma_n$ and $a_1, \ldots, a_n$ we have that

$$\sigma_{\mathcal{A}}(a_1, \ldots, a_n) = \sup\{\sigma_{\mathcal{A}}(b_1, \ldots, b_n) : (b_1, \ldots, b_n) \in A_{a_1} \times \cdots \times A_{a_n}\}.$$

Here, it should be clear that ord-algebras are a class of ordered algebras that aim to capture precisely those properties of the power algebras of those multialgebras which interest us: the non-partial ones. This shall be formalized further ahead.

PROPOSITION 3.2. Let $\mathcal{A}$ be an ord-algebra over Σ, $\sigma \in \Sigma_n$ and a_1 through a_n, and $b_1$through b_n in A such that $a_1 \leq_{\mathcal{A}} b_1$, $\ldots$, $a_n \leq_{\mathcal{A}} b_n$. Then, $\sigma_{\mathcal{A}}(a_1, \ldots, a_n) \leq_{\mathcal{A}} \sigma_{\mathcal{A}}(b_1, \ldots, b_n)$.

PROOF: Since, for every $i \in \{1, \ldots, n\}$, $a_i \leq_{\mathcal{A}} b_i$, we have that $A_{a_i} \subseteq A_{b_i}$, one concludes that $A_{a_1} \times \cdots \times A_{a_n} \subseteq A_{b_1} \times \cdots \times A_{b_n}$; this way,

$$\sigma_{\mathcal{A}}(a_1, \ldots, a_n) = \sup\{\sigma_{\mathcal{A}}(c_1, \ldots, c_n) : c_i \in A_{a_i}\} \leq_{\mathcal{A}}$$
$$\sup\{\sigma_{\mathcal{A}}(c_1, \ldots, c_n) : c_i \in A_{b_i}\} = \sigma_{\mathcal{A}}(b_1, \ldots, b_n). \qquad \square$$

For a Σ-multialgebra $\mathcal{A} = (A, \{\sigma_{\mathcal{A}}\}_{\sigma\in\sigma})$, we define $\mathbb{P}(\mathcal{A})$ as the ord-algebra

$$(\mathcal{P}^*(A), \{\sigma_{\mathbb{P}(\mathcal{A})}\}_{\sigma\in\Sigma}, \leq_{\mathbb{P}(\mathcal{A})})$$

over Σ such that $(\mathcal{P}^*(A), \{\sigma_{\mathbb{P}(\mathcal{A})}\}_{\sigma\in\Sigma})$ is exactly the Σ-algebra $\mathsf{P}(\mathcal{A})$ defined at the beginning of Section 2 and, for nonempty subsets A_1 and A_2 of A, $A_1 \leq_{\mathbb{P}(\mathcal{A})} A_2$ if and only if $A_1 \subseteq A_2$. Since:

(i) $\mathsf{P}(\mathcal{A})$ is a Σ-algebra;

(ii) $(\mathcal{P}^*(A), \leq_{\mathbb{P}(\mathcal{A})})$ is a complete, atomic and bottomless Boolean algebra, given that $\mathcal{P}(A)$ is a complete, atomic Boolean algebra with at least two elements;

(iii) and, for $\sigma \in \Sigma_n$ and $\emptyset \neq A_1, \ldots, A_n \subseteq A$, since the atoms below A_i are exactly $A_{A_i} = \{\{a\} : a \in A_i\}$,

$$\sigma_{\mathbb{P}(\mathcal{A})}(A_1, \ldots, A_n) = \bigcup\{\sigma_{\mathcal{A}}(a_1, \ldots, a_n) : a_i \in A_i\} =$$

$$\bigcup\{\sigma_{\mathbb{P}(\mathcal{A})}(\{a_1\}, \ldots, \{a_n\}) : \{a_i\} \in A_{A_i}\};$$

we indeed have that $\mathbb{P}(\mathcal{A})$ is an ord-algebra over Σ.

Definition 3.3. Let $\mathcal{A} = (A, \{\sigma_{\mathcal{A}}\}_{\sigma\in\Sigma}, \leq_{\mathcal{A}})$ and $\mathcal{B} = (B, \{\sigma_{\mathcal{B}}\}_{\sigma\in\Sigma}, \leq_{\mathcal{B}})$ be two ord-algebras over Σ. A function $h : A \to B$ is said to be a homomorphism of ord-algebras, in which case we write $h : \mathcal{A} \to \mathcal{B}$, when:

(i) for all $\sigma \in \Sigma_n$ and $a_1, \ldots, a_n \in A$ we have that

$$h(\sigma_{\mathcal{A}}(a_1, \ldots, a_n)) \leq_{\mathcal{B}} \sigma_{\mathcal{B}}(h(a_1), \ldots, h(a_n));$$

(ii) h is continuous, meaning that, for every non-empty subset $A' \subseteq A$,

$$h(\sup A') = \sup\{h(a) : a \in A'\};$$

(iii) h maps minimal elements of $(A, \leq_{\mathcal{A}})$ to minimal elements of $(B, \leq_{\mathcal{B}})$.

Notice that a homomorphism of ord-algebras is essentially an "almost Σ-homomorphism" which is also continuous and minimal-elements-preserving. Notice also that a homomorphism of ord-algebras is order preserving: if $a \leq_{\mathcal{A}} b$, then $b = \sup\{a, b\}$, and therefore $h(b) = \sup\{h(a), h(b)\}$, meaning that $h(a) \leq_{\mathcal{B}} h(b)$.

Lemma 3.4. *The composition of homomorphisms of ord-algebras returns a homomorphism of ord-algebras.*

Proof: Take ord-algebras $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ over Σ, and homomorphisms of ord-algebras $h : \mathcal{A} \to \mathcal{B}$ and $h' : \mathcal{B} \to \mathcal{C}$.

(i) $h' \circ h$ obviously is a function from A to C, so let $\sigma \in \Sigma_n$ and $a_1, \ldots, a_n \in A$: we have that, since both h' and h are homomorphisms of ord-algebras,

$$h' \circ h(\sigma_{\mathcal{A}}(a_1, \ldots, a_n)) = h'(h(\sigma_{\mathcal{A}}(a_1, \ldots, a_n))) \leq_{\mathcal{C}} h'(\sigma_{\mathcal{B}}(h(a_1), \ldots, h(a_n))),$$

because h' is order-preserving and

$$h(\sigma_{\mathcal{A}}(a_1, \ldots, a_n)) \leq_{\mathcal{B}} \sigma_{\mathcal{B}}(h(a_1), \ldots, h(a_n));$$

and

$$h'(\sigma_{\mathcal{B}}(h(a_1),\ldots,h(a_n))) \leq_{\mathcal{C}} \sigma_{\mathcal{C}}(h'(h(a_1)),\ldots,h'(h(a_n))) =$$
$$\sigma_{\mathcal{C}}(h'\circ h(a_1),\ldots,h'\circ h(a_n))$$

since h' is an "almost homomorphism".

(ii) Given a non-empty $A' \subseteq A$, we have that $h(\sup A') = \sup\{h(a) : a \in A'\}$ and, denoting $\{h(a) : a \in A'\}$ as B', we have that $h'(\sup B') = \sup\{h'(b) : b \in B'\}$; since $\sup B' = h(\sup A')$, we obtain

$$h'\circ h(\sup A') = \sup\{h'(b) : b \in B'\} = \sup\{h'\circ h(a) : a \in A'\},$$

which means that $h'\circ h$ is continuous.

(iii) Finally, if $a \in A$ is a minimal element, $h(a) \in B$ is a minimal element, since h preserves minimal elements, and for the same reason $h'\circ h(a) = h'(h(a)) \in C$ remains a minimal element still, and from all of the above $h'\circ h$ is a homomorphism of ord-algebras. □

Proposition 3.5. When we take as objects all ord-algebras over Σ and as morphisms all the homomorphisms of ord-algebras between them, with composition of morphisms and identity morphisms as in **Set**, the resulting structure is a category, denoted by $\mathbf{OAlg}(\Sigma)$.

Theorem 3.6. *The transformation taking a Σ-multialgebra $\mathcal{A}$ to $\mathbb{P}(\mathcal{A})$, and a homomorphism $h : \mathcal{A} \to \mathcal{B}$ to the homomorphism $\mathbb{P}(h) : \mathbb{P}(\mathcal{A}) \to \mathbb{P}(\mathcal{B})$ of ord-algebras such that, for every $\emptyset \neq A' \subseteq A$,*

$$\mathbb{P}(h)(A') = \{h(a) : a \in A'\},$$

is a functor, of the form $\mathbb{P} : \boldsymbol{MAlg}(\Sigma) \to \boldsymbol{OAlg}(\Sigma)$.

Proof: First we must show that $\mathbb{P}(h)$ is, in fact, a homomorphism of ord-algebras: given Lemma 2.1 and the fact that $\mathsf{P}(h) = \mathbb{P}(h)$, we have that $\mathbb{P}(h)$ satisfies the first condition for being a homomorphism of ord-algebras; and, if $\emptyset \neq A''$ is a subset of $\mathcal{P}(A)$, we have that

$$\mathbb{P}(h)(\sup A'') = \{h(a) : a \in \sup A''\} = \{h(a) : a \in \bigcup A''\} =$$
$$\bigcup\{\{h(a) : a \in A'\} : A' \in A''\} = \bigcup\{\mathbb{P}(h)(A') : A' \in A''\} =$$
$$\sup\{\mathbb{P}(h)(A') : A' \in A''\},$$

what proves the satisfaction of the second condition; for the third condition, we remember that the minimal elements of $(\mathcal{P}^*(A), \subseteq)$ are the singletons, that is, sets of the form $\{a\}$ with $a \in A$, and since $\mathbb{P}(h)(\{a\}) = \{h(a)\}$, $\mathbb{P}(h)$ preserves minimal elements.

If $h : \mathcal{A} \to \mathcal{A}$ is the identity $id_{\mathcal{A}}$ of $\mathcal{A}$, then

$$\mathbb{P}(id_{\mathcal{A}})(A') = \{id_{\mathcal{A}}(a) : a \in A'\} = \{a : a \in A'\} = A',$$

meaning $\mathbb{P}(id_{\mathcal{A}})$ is again the identity. Finally, if $h : \mathcal{A} \to \mathcal{B}$ and $h' : \mathcal{B} \to \mathcal{C}$ are homomorphisms of multialgebras,

$$\mathbb{P}(h' \circ h)(A') = \{h' \circ h(a) : a \in A'\} =$$

$$\{h'(b) : b \in \mathbb{P}(h)(A')\} = \mathbb{P}(h') \circ \mathbb{P}(h)(A'),$$

and thus $\mathbb{P}$ is indeed a functor. □

Here, we start to better understand the role played by power algebras: if $\mathcal{A}$ is a multialgebra, $\mathbb{P}(\mathcal{A})$ is a certain power algebra of $\mathcal{A}$; specifically, the one presented conveniently as a bottomless Boolean algebra.

3.2. $\mathbb{P}$ may be seen as part of a monad

As is the case with the power-set functor, from **Set** to itself, we may see $\mathbb{P}$, or even P and $\mathsf{P}_=$, as being part of a monad, although some minor modifications are necessary. So, consider the endofunctor $\tilde{\mathbb{P}} : \mathbf{MAlg}(\Sigma) \to \mathbf{MAlg}(\Sigma)$ such that, for a Σ-multialgebra $\mathcal{A} = (A, \{\sigma_{\mathcal{A}}\}_{\sigma\in\Sigma})$, $\tilde{\mathbb{P}}\mathcal{A}$ is the Σ-multialgebra with universe $\mathcal{P}^*(A)$ and operations given by

$$\sigma_{\tilde{\mathbb{P}}\mathcal{A}}(A_1, \dots, A_n) = \{\{a\} \in \mathcal{P}^*(A) : a \in \bigcup\{\sigma_{\mathcal{A}}(a_1, \dots, a_n) : a_i \in A_i\}\},$$

for σ an n-ary symbol and A_1 through A_n non-empty subsets of A; and for Σ-multialgebras $\mathcal{A}$ and $\mathcal{B}$, a homomorphism $h : \mathcal{A} \to \mathcal{B}$ and a non-empty $A' \subseteq A$, $\tilde{\mathbb{P}}h : \tilde{\mathbb{P}}\mathcal{A} \to \tilde{\mathbb{P}}\mathcal{B}$ satisfying $\tilde{\mathbb{P}}h(A') = \{h(a) : a \in A'\}$ (we omit a pair of parenthesis in this expression only for this section, given it is heavy in compositions of functors). Notice that $\tilde{\mathbb{P}}\mathcal{A}$ is almost the same as $\mathsf{P}(\mathcal{A})$, with the difference that in the latter, operations return subsets of A, while in the former they return sets of singletons of A, whose union is exactly the result of the operation as performed in $\mathsf{P}(\mathcal{A})$.

For the natural transformations to form a monad together with $\tilde{\mathbb{P}}$, we chose the obvious candidates: $\eta : 1_{\mathbf{MAlg}(\Sigma)} \to \tilde{\mathbb{P}}$ and $\epsilon : \tilde{\mathbb{P}} \circ \tilde{\mathbb{P}} \to \tilde{\mathbb{P}}$ given by, for a Σ-multialgebra $\mathcal{A}$, an element a of $\mathcal{A}$ and a non-empty collection $\{A_i\}_{i\in I}$ of non-empty subsets of A, $\eta_{\mathcal{A}}(a) = \{a\}$ and $\epsilon_{\mathcal{A}}(\{A_i\}_{i\in I}) = \bigcup\{A_i : i \in I\}$.

Proposition 3.7. For any Σ-multialgebra $\mathcal{A}$, $\eta_{\mathcal{A}}$ and $\epsilon_{\mathcal{A}}$ are homomorphisms.

Proposition 3.8. For any Σ-multialgebras $\mathcal{A}$ and $\mathcal{B}$, and homomorphism $h : \mathcal{A} \to \mathcal{B}$, the identities $\tilde{\mathbb{P}}h \circ \eta_{\mathcal{A}} = \eta_{\mathcal{B}} \circ h$ and $\tilde{\mathbb{P}}h \circ \epsilon_{\mathcal{A}} = \epsilon_{\mathcal{B}} \circ \tilde{\mathbb{P}}\tilde{\mathbb{P}}h$ are satisfied, meaning that η and ϵ are natural transformations.

Proof: Let a be an element of $\mathcal{A}$. We have that $\tilde{\mathbb{P}}h \circ \eta_{\mathcal{A}}(a) = \tilde{\mathbb{P}}h(\eta_{\mathcal{A}}(a))$, and since $\eta_{\mathcal{A}}(a) = \{a\}$, we have that $\tilde{\mathbb{P}}h \circ \eta_{\mathcal{A}}(a) = \{h(a)\}$. Meanwhile, $\eta_{\mathcal{B}} \circ h(a) = \eta_{\mathcal{A}}(h(a)) = \{h(a)\}$, and as stated both expressions coincide.

Now, let $\{A_i\}_{i\in I}$ be an element of $\tilde{\mathbb{P}}\tilde{\mathbb{P}}\mathcal{A}$, meaning it is a non-empty set of non-empty subsets of $\mathcal{A}$: $\tilde{\mathbb{P}}h \circ \epsilon_{\mathcal{A}}(\{A_i\}_{i\in I}) = \tilde{\mathbb{P}}h(\epsilon_{\mathcal{A}}(\{A_i\}_{i\in I}))$, and since $\epsilon_{\mathcal{A}}(\{A_i\}_{i\in I}) = \bigcup\{A_i : i \in I\}$, the whole expression simplifies to $\{h(a) : a \in \bigcup\{A_i : i \in I\}\}$. In turn,

$$\epsilon_{\mathcal{B}} \circ \tilde{\mathbb{P}}\tilde{\mathbb{P}}h(\{A_i\}_{i\in I}) = \epsilon_{\mathcal{B}}(\{\{h(a) \ : \ a \in A_i\} \ : \ i \in I\}),$$

which is equal to

$$\bigcup\{\{h(a) \ : \ a \in A_i\} i \in I\} = \{h(a) \ : \ a \in \bigcup\{A_i : i \in I\}\},$$

giving us the desired equality. □

Theorem 3.9. *The triple of $\tilde{\mathbb{P}}$, η and ϵ forms a monad.*

Proof: Let $\mathcal{A}$ be a Σ-multialgebra. We first must prove $\epsilon \circ \tilde{\mathbb{P}}\epsilon = \epsilon \circ \epsilon\tilde{\mathbb{P}}$, what amounts to $\epsilon_{\mathcal{A}} \circ \tilde{\mathbb{P}}\epsilon_{\mathcal{A}} = \epsilon_{\mathcal{A}} \circ \epsilon_{\tilde{\mathbb{P}}\mathcal{A}}$, as homomorphisms from $\tilde{\mathbb{P}}^3\mathcal{A}$ to $\tilde{\mathbb{P}}\mathcal{A}$. So, let $\{\{A_i^j\}_{i\in I}\}_{j\in J}$ be an element of $\tilde{\mathbb{P}}^3\mathcal{A}$, where I and J are non-empty sets of indexes and all A_i^j are non-empty subsets of A:

$$\epsilon_{\mathcal{A}} \circ \tilde{\mathbb{P}}\epsilon_{\mathcal{A}}(\{\{A_i^j\}_{i\in I}\}_{j\in J}) = \epsilon_{\mathcal{A}}(\{\epsilon_{\mathcal{A}}(\{A_i^j \ : \ i \in I\}) \ : \ j \in J\}) =$$

$$\epsilon_{\mathcal{A}}(\{\bigcup\{A_i^j : i \in I\} \ : \ j \in J\}),$$

what equals $\bigcup\{\bigcup\{A_i^j : i \in I\} : j \in J\}$, while $\epsilon_{\mathcal{A}} \circ \epsilon_{\tilde{\mathbb{P}}\mathcal{A}}(\{\{A_i^j\}_{i\in I}\}_{j\in J}) = \epsilon_{\mathcal{A}}(\bigcup\{\{A_i^j : j \in J\}\}_{i\in I}) = \bigcup\{\bigcup\{A_i^j : j \in J\} : i \in I\}$, and it is clear that both sets are the same.

It remains to be proven $\epsilon \circ \tilde{\mathbb{P}}\eta = \epsilon \circ \eta\tilde{\mathbb{P}} = 1_{\tilde{\mathbb{P}}}$, meaning that $\epsilon_{\mathcal{A}} \circ \eta_{\tilde{\mathbb{P}}\mathcal{A}} = \epsilon_{\mathcal{A}} \circ \tilde{\mathbb{P}}\eta_{\mathcal{A}}$, as homomorphisms from $\tilde{\mathbb{P}}\mathcal{A}$ to $\tilde{\mathbb{P}}\mathcal{A}$, and this coincides with the identity homomorphism on this multialgebra as well. So, we take a non-empty subset A' of A, and we have that $\epsilon_{\mathcal{A}} \circ \eta_{\tilde{\mathbb{P}}\mathcal{A}}(A') = \epsilon_{\mathcal{A}}(\{A'\}) = A'$, while for the other expression one derives

$$\epsilon_{\mathcal{A}} \circ P\eta_{\mathcal{A}}(A') = \epsilon_{\mathcal{A}}(\{\eta_{\mathcal{A}}(a) \ : \ a \in A'\}) = \epsilon_{\mathcal{A}}(\{\{a\} \ : \ a \in A'\}) =$$
$$\bigcup\{\{a\} : a \in A'\} = A',$$

what completes the proof. □

3.3. Multialgebras of atoms

Given an ord-algebra $\mathcal{A}$ over Σ, take the set $\mathbb{A}((A, \leq_{\mathcal{A}}))$ of atoms of $(A, \leq_{\mathcal{A}})$, that is, the set of minimal elements of this partially ordered set (equal to A_1 as well). For a $\sigma \in \Sigma_n$ and atoms $a_1, \ldots, a_n \in \mathbb{A}((A, \leq_{\mathcal{A}}))$, we define

$$\sigma_{\mathbb{A}(\mathcal{A})}(a_1, \ldots, a_n) = \{a \in \mathbb{A}((A, \leq_{\mathcal{A}})) : a \leq_{\mathcal{A}} \sigma_{\mathcal{A}}(a_1, \ldots, a_n)\} = A_{\sigma_{\mathcal{A}}(a_1,\ldots,a_n)}.$$

This way, $(\mathbb{A}((A, \leq_{\mathcal{A}})), \{\sigma_{\mathbb{A}(\mathcal{A})}\}_{\sigma\in\Sigma})$ becomes a Σ-multialgebra, that we will denote by $\mathbb{A}(\mathcal{A})$ and call the multialgebra of atoms of $\mathcal{A}$. Given ord-algebras $\mathcal{A}$ and $\mathcal{B}$ over Σ, and a homomorphism of ord-algebras $h : \mathcal{A} \to \mathcal{B}$, we also define

$$\mathbb{A}(h) : \mathbb{A}((A, \leq_{\mathcal{A}})) \to \mathbb{A}((B, \leq_{\mathcal{B}}))$$

as the restriction of h to $\mathbb{A}((A, \leq_{\mathcal{A}})) \subseteq A$. It is well-defined since every homomorphism of ord-algebras preserves minimal elements, that is, atoms.

For $\sigma \in \Sigma_n$ and atoms $a_1, \ldots, a_n \in \mathbb{A}((A, \leq_{\mathcal{A}}))$ we have that

$$\{\mathbb{A}(h)(a) : a \in \sigma_{\mathbb{A}(\mathcal{A})}(a_1, \ldots, a_n)\} = \{h(a) : a \in \sigma_{\mathbb{A}(\mathcal{A})}(a_1, \ldots, a_n)\} =$$
$$\{h(a) \in \mathbb{A}((B, \leq_{\mathcal{B}})) : a \leq_{\mathcal{A}} \sigma_{\mathcal{A}}(a_1, \ldots, a_n)\}$$

and, since $a \leq_{\mathcal{A}} \sigma_{\mathcal{A}}(a_1, \ldots, a_n)$ implies $h(a) \leq_{\mathcal{B}} h(\sigma_{\mathcal{A}}(a_1, \ldots, a_n))$ given that h is order preserving, which in turn implies that $h(a) \leq_{\mathcal{B}}$

$\sigma_{\mathcal{B}}(h(a_1), \ldots, h(a_n))$ since h is an "almost homomorphism", we get that

$$\{h(a) \in \mathbb{A}((B, \leq_{\mathcal{B}})) : a \leq_{\mathcal{A}} \sigma_{\mathcal{A}}(a_1, \ldots, a_n)\} \subseteq$$

$$\{b \in \mathbb{A}((B, \leq_{\mathcal{B}})) : b \leq_{\mathcal{B}} \sigma_{\mathcal{B}}(h(a_1), \ldots, h(a_n)\} =$$

$$\sigma_{\mathbb{A}(\mathcal{B})}(h(a_1), \ldots, h(a_n)) = \sigma_{\mathbb{A}(\mathcal{B})}(\mathbb{A}(h)(a_1), \ldots, \mathbb{A}(h)(a_n)),$$

what proves that $\mathbb{A}(h)$ is a homomorphism between Σ-multialgebras, and we may write $\mathbb{A}(h) : \mathbb{A}(\mathcal{A}) \to \mathbb{A}(\mathcal{B})$.

The natural question is if $\mathbb{A} : \mathbf{OAlg}(\Sigma) \to \mathbf{MAlg}(\Sigma)$ is a functor, to which the answer is yes: it is easy to see that it distributes over the composition of morphisms and preserves the identity ones.

4. $\mathbf{OAlg}(\Sigma)$ and $\mathbf{MAlg}(\Sigma)$ are equivalent

Now, we aim to prove that $\mathbf{OAlg}(\Sigma)$ and $\mathbf{MAlg}(\Sigma)$ are actually equivalent categories, the equivalence being given by the functors $\mathbb{P}$ and $\mathbb{A}$. In order to prove that $\mathbb{P}$ and $\mathbb{A}$ form an equivalence of categories it is enough to prove that both are full and faithful and $\mathbb{A}$ is a right adjoint of $\mathbb{P}$.

4.1. $\mathbb{P}$ and $\mathbb{A}$ are full and faithful

Proposition 4.1. $\mathbb{P}$ is faithful.

Proof: Given Σ-multialgebras $\mathcal{A}$ and $\mathcal{B}$, and homomorphisms $h, h' : \mathcal{A} \to \mathcal{B}$, if $\mathbb{P}(h) = \mathbb{P}(h')$, we have that, for every $a \in A$,

$$\{h(a)\} = \mathbb{P}(h)(\{a\}) = \mathbb{P}(h')(\{a\}) = \{h'(a)\},$$

and therefore $h = h'$. □

Proposition 4.2. $\mathbb{A}$ is faithful.

Proof: Given ord-algebras $\mathcal{A}$ and $\mathcal{B}$ over Σ, and homomorphisms of ord-algebras $h, h' : \mathcal{A} \to \mathcal{B}$, suppose that $\mathbb{A}(h) = \mathbb{A}(h')$. Then, for every $a \in A$, we can write $a = \sup A_a$, since $(A, \leq_{\mathcal{A}})$ is atomic.

Since h and h' are continuous, $h(a) = \sup\{h(a') : a' \in A_a\}$ and $h'(a) = \sup\{h'(a') : a' \in A_a\}$. But, since $\mathbb{A}(h) = \mathbb{A}(h')$, h and h' are the same when restricted to atoms, and therefore $\{h(a') : a' \in A_a\} = \{h'(a') : a' \in A_a\}$. This means that $h(a) = h'(a)$ and, since a is arbitrary, $h = h'$. □

THEOREM 4.3. $\mathbb{P}$ *is full.*

PROOF: Given Σ-multialgebras $\mathcal{A}$ and $\mathcal{B}$, and a homomorphism of ord-algebras $h : \mathbb{P}(\mathcal{A}) \to \mathbb{P}(\mathcal{B})$, to prove that $\mathbb{P}$ is full we must find a homomorphism $h' : \mathcal{A} \to \mathcal{B}$ such that $\mathbb{P}(h') = h$.

For every $a \in A$, $\{a\}$ is an atom and, since h preserves atoms, $h(\{a\})$ is an atom of $\mathbb{P}(B)$, and therefore of the form $\{b_a\}$ for some $b_a \in B$. We define $h' : \mathcal{A} \to \mathcal{B}$ by $h'(a) = b_a$. First of all, we must show that h' is in fact a homomorphism, which is quite analogous to the proof of the same fact for $\mathbb{A}(h)$. Given $\sigma \in \Sigma_n$ and $a_1, \ldots, a_n \in A$,

$$\{h'(a) : a \in \sigma_{\mathcal{A}}(a_1, \ldots, a_n)\} = \{b_a : a \in \sigma_{\mathcal{A}}(a_1, \ldots, a_n)\} =$$

$$\sup\{\{b_a\} : a \in \sigma_{\mathcal{A}}(a_1, \ldots, a_n)\} = \sup\{h(\{a\}) : a \in \sigma_{\mathcal{A}}(a_1, \ldots, a_n)\} =$$

$$h(\sup\{\{a\} : a \in \sigma_{\mathcal{A}}(a_1, \ldots, a_n)\}),$$

given that h is continuous. Since it is an "almost homomorphism", this equals

$$\begin{aligned} h(\sigma_{\mathcal{A}}(a_1, \ldots, a_n)) &= h(\sigma_{\mathbb{P}(\mathcal{A})}(\{a_1\}, \ldots, \{a_n\})) \\ &\subseteq \sigma_{\mathbb{P}(\mathcal{B})}(h(\{a_1\}), \ldots, h(\{a_n\})) \\ &= \sigma_{\mathbb{P}(\mathcal{B})}(\{b_{a_1}\}, \ldots, \{b_{a_n}\}) \\ &= \sigma_{\mathcal{B}}(b_{a_1}, \ldots, b_{a_n}) \\ &= \sigma_{\mathcal{B}}(h'(a_1), \ldots, h'(a_n)). \end{aligned}$$

Now, when we consider $\mathbb{P}(h')$, we see that $\mathbb{P}(h')(\{a\}) = \{b_a\} = h(\{a\})$ for every atom $\{a\}$ of $\mathbb{P}(\mathcal{A})$, and so the restrictions of h and $\mathbb{P}(h')$ to atoms are the same, and therefore $\mathbb{A}(h) = \mathbb{A}(\mathbb{P}(h'))$. Since $\mathbb{A}$ is faithful, we discover that $h = \mathbb{P}(h')$ and, therefore, $\mathbb{P}$ is full. □

Now it remains to be shown that $\mathbb{A}$ is also full. Given ord-algebras $\mathcal{A}$ and $\mathcal{B}$ over Σ, and a homomorphism $h : \mathbb{A}(\mathcal{A}) \to \mathbb{A}(\mathcal{B})$, we then define $h' : \mathcal{A} \to \mathcal{B}$ by

$$h'(a) = \sup\{h(c) : c \in A_a\}.$$

First of all, we must prove that h' is a homomorphism of ord-algebras, for which we shall need a few lemmas.

Lemma 4.4. *In a complete, atomic and bottomless Boolean algebra $\mathcal{A}$, take a non-empty family of indexes I and, for every $i \in I$, $X_i \subseteq A$. Suppose we have $x_i = \sup X_i$, for $i \in I$, and $X = \bigcup\{X_i : i \in I\}$. Then, $\sup\{x_i : i \in I\} = \sup X$.*

Proof: We define $a = \sup\{x_i : i \in I\}$ and $b = \sup X$: first, we show that a is an upper bound for X, so that $a \geq_{\mathcal{A}} b$. For every $x \in X$, we have that, since $X = \bigcup\{X_i : i \in I\}$, there exists $j \in I$ such that $x \in X_j$, and therefore $x_j \geq_{\mathcal{A}} x$. Since $a = \sup\{x_i : i \in I\}$, we have that $a \geq_{\mathcal{A}} x_j$, and by transitivity $a \geq_{\mathcal{A}} x$, and therefore a is indeed an upper bound for X.

Now we show that b is an upper bound for $\{x_i : i \in I\}$, and so $b \geq_{\mathcal{A}} a$ (and $a = b$). For every $i \in I$, we have that b is an upper bound for X_i, since $X_i \subseteq X$ and b is an upper bound for X, and therefore $b \geq_{\mathcal{A}} x_i$, since x_i is the smallest upper bound for X_i. It follows that b is indeed an upper bound for $\{x_i : i \in I\}$, what completes the proof. □

Lemma 4.5. *In a complete, atomic and bottomless Boolean algebra $\mathcal{A}$, for a non-empty $C \subseteq A$ one has that $\bigcup\{A_c : c \in C\} = A_{\sup C}$.*

Proof: If $d \in A_c$ for a $c \in C$, d is an atom such that $d \leq_{\mathcal{A}} c$. Since $c \leq_{\mathcal{A}} \sup C$, $d \leq_{\mathcal{A}} \sup C$, and therefore d belongs to $A_{\sup C}$. Thus $\bigcup\{A_c : c \in C\} \subseteq A_{\sup C}$.

Conversely, suppose that $d \in A_{\sup C}$. Then, d is an atom such that $d \leq_{\mathcal{A}} \sup C$, and therefore $\inf\{d, \sup C\} = d$. It follows that the subset $C^d \subseteq C$, of $c \in C$ such that $\inf\{d, c\}$ exists, is not empty, by Proposition 1.7. But if $c \in C^d$, $\inf\{d, c\}$ exists, and since d is an atom, we have that $d \in A_c \subseteq \bigcup\{A_c : c \in C\}$, and from that $\bigcup\{A_c : c \in C\} = A_{\sup C}$. □

Since $\sigma_{\mathcal{A}}(a_1, \ldots, a_n)$ is equal to the supremum of $\{\sigma_{\mathcal{A}}(c_1, \ldots, c_n) : (c_1, \ldots, c_n) \in A_{a_1} \times \cdots \times A_{a_n}\}$, from Lemma 4.5 we have that $A_{\sigma_{\mathcal{A}}(a_1,\ldots,a_n)}$ is equal to

$$\bigcup\{A_{\sigma_{\mathcal{A}}(c_1,\ldots,c_n)} : c_1 \in A_{a_1}, \ldots, c_n \in A_{a_n}\},$$

that is, we have the following lemma.

Lemma 4.6. *For $\sigma \in \Sigma_n$ and $a_1, \ldots, a_n \in A$,*

$$A_{\sigma_{\mathcal{A}}(a_1,\ldots,a_n)} = \bigcup\{A_{\sigma_{\mathcal{A}}(c_1,\ldots,c_n)} : c_i \in A_{a_i}\}.$$

Theorem 4.7. *$\mathbb{A}$ is full.*

Proof: First of all, we prove that h' is a homomorphism of ord-algebras.

(i) First, it is clear that h' maps atoms into atoms: if a is an atom, $A_a = \{a\}$ and

$$h'(a) = \sup\{h(c) : c \in A_a\} = \sup\{h(a)\} = h(a),$$

which is an atom since h is a map between $\mathbb{A}(\mathcal{A})$ and $\mathbb{A}(\mathcal{B})$.

(ii) h' is continuous: for any non-empty set $C \subseteq A$, we remember that $h'(c) = \sup\{h(d) : d \in A_c\}$, and from Lemmas 4.4 and 4.5 we get that

$$\begin{aligned}\sup\{h'(c) : c \in C\} &= \sup\{\sup\{h(d) : d \in A_c\} : c \in C\}\\ &= \sup \bigcup \{\{h(d) : d \in A_c\} : c \in C\}\\ &= \sup\{h(d) : d \in A_{\sup C}\}\\ &= h'(\sup C).\end{aligned}$$

(iii) Since $\{h(a) : a \in \sigma_{\mathbb{A}(\mathcal{A})}(a_1, \ldots, a_n)\} \subseteq \sigma_{\mathbb{A}(\mathcal{B})}(h(a_1), \ldots, h(a_n))$, given that h is a homomorphism of multialgebras, it follows from Lemma 4.6 that

$$\begin{aligned}h'(\sigma_{\mathcal{A}}(a_1, \ldots, a_n)) &= \sup\{h(c) : c \in \bigcup\{A_{\sigma_{\mathcal{A}}(c_1, \ldots, c_n)} : c_i \in A_{a_i}\}\}\\ &= \sup \bigcup \{\{h(c) : c \in \sigma_{\mathbb{A}(\mathcal{A})}(c_1, \ldots, c_n)\} : c_i \in A_{a_i}\}\\ &\leq_{\mathcal{B}} \sup \bigcup \{\sigma_{\mathbb{A}(\mathcal{B})}(h(c_1), \ldots, h(c_n)) : c_i \in A_{a_i}\},\end{aligned}$$

where we have used that, for atoms $c_1, \ldots, c_n$ of $\mathcal{A}$, $\sigma_{\mathbb{A}(\mathcal{A})}(c_1, \ldots, c_n) = A_{\sigma_{\mathcal{A}}(c_1, \ldots, c_n)}$. Since, for atoms $d_1, \ldots, d_n$ of $\mathcal{B}$, we also have that $\sigma_{\mathbb{A}(\mathcal{B})}(d_1, \ldots, d_n) = A_{\sigma_{\mathcal{B}}(d_1, \ldots, d_n)}$, this is equal to

$$\sup \bigcup \{A_{\sigma_{\mathcal{B}}(h(c_1), \ldots, h(c_n))} : c_i \in A_{a_i}\} = \sup \bigcup \{A_{\sigma_{\mathcal{B}}(h'(c_1), \ldots, h'(c_n))} : c_i \in A_{a_i}\}.$$

Since h' is continuous, $c_i \leq_{\mathcal{A}} a_i$, for every $i \in \{1, \ldots, n\}$, implies $h'(c_i) \leq_{\mathcal{B}} h'(a_i)$, and therefore $\sigma_{\mathcal{B}}(h'(c_1), \ldots, h'(c_n)) \leq_{\mathcal{B}} \sigma_{\mathcal{B}}(h'(a_1), \ldots, h'(a_n))$ for $(c_1, \ldots, c_n) \in A_{a_1} \times \cdots \times A_{a_n}$. It follows that the union, for $(c_1, \ldots, c_n)$ in $A_{a_1} \times \cdots \times A_{a_n}$, of $A_{\sigma_{\mathcal{B}}(h'(c_1), \ldots, h'(c_n))}$, is contained on $A_{\sigma_{\mathcal{B}}(h'(a_1), \ldots, h'(a_n))}$, and therefore

$$\begin{aligned}\sup \bigcup \{A_{\sigma_{\mathcal{B}}(h'(c_1), \ldots, h'(c_n))} : c_i \in A_{a_i}\} &\leq_{\mathcal{B}} \sup A_{\sigma_{\mathcal{B}}(h'(a_1), \ldots, h'(a_n))}\\ &= \sigma_{\mathcal{B}}(h'(a_1), \ldots, h'(a_n)).\end{aligned}$$

Now, for every atom a of $\mathcal{A}$, we have that $h'(a) = h(a)$, and therefore the restriction of h' to atoms coincides with h, that is, $\mathbb{A}(h') = h$, and since h was taken arbitrarily, $\mathbb{A}$ is full. □

4.2. $\mathbb{P}$ and $\mathbb{A}$ are adjoint

It remains to be shown that $\mathbb{P}$ and $\mathbb{A}$ are adjoint. To this end, consider the bijections

$$\Phi_{\mathcal{B},\mathcal{A}} : Hom_{\mathbf{MAlg}(\Sigma)}(\mathbb{A}(\mathcal{B}), \mathcal{A}) \to Hom_{\mathbf{OAlg}(\Sigma)}(\mathcal{B}, \mathbb{P}(\mathcal{A})),$$

for $\mathcal{A}$ a Σ-multialgebra and $\mathcal{B}$ an ord-algebra over Σ, given by, for $h : \mathbb{A}(\mathcal{B}) \to \mathcal{A}$ a homomorphism and b an element of $\mathcal{B}$,

$$\Phi_{\mathcal{B},\mathcal{A}}(h)(b) = \{h(c) : c \in A_b\}.$$

Lemma 4.8. *$\Phi_{\mathcal{B},\mathcal{A}}(h)$ is a homomorphism of ord-algebras.*

Proof: (i) If b is an atom, $A_b = \{b\}$, and therefore $\Phi_{\mathcal{B},\mathcal{A}}(h)(b) = \{h(c) : c \in A_b\} = \{h(b)\}$, which is a singleton and therefore an atom of $\mathbb{P}(\mathcal{A})$.

(ii) Let D be a non-empty subset of $\mathcal{B}$. We have that

$$\begin{aligned}\Phi_{\mathcal{B},\mathcal{A}}(h)(\sup D) &= \{h(c) : c \in A_{\sup D}\}\\ &= \{h(c) : c \in \bigcup\{A_d : d \in D\}\}\\ &= \bigcup\{\{h(c) : c \in A_d\} : d \in D\}\\ &= \bigcup\{\Phi_{\mathcal{B},\mathcal{A}}(h)(d) : d \in D\}\\ &= \sup\{\Phi_{\mathcal{B},\mathcal{A}}(h)(d) : d \in D\},\end{aligned}$$

since $A_{\sup D} = \bigcup_{d \in D} A_d$ and the supremum in $\mathbb{P}(\mathcal{A})$ is simply the union.

(iii) For $\sigma \in \Sigma_n$ and $b_1, \ldots, b_n$ elements of $\mathcal{B}$, we have that

$$\begin{aligned}\Phi_{\mathcal{B},\mathcal{A}}(h)(\sigma_{\mathcal{B}}(b_1, \ldots, b_n)) &= \{h(c) : c \in A_{\sigma_{\mathcal{B}}(b_1,\ldots,b_n)}\}\\ &= \{h(c) : c \in \bigcup\{A_{\sigma_{\mathcal{B}}(c_1,\ldots,c_n)} : c_i \in A_{b_i}\}\}\\ &= \bigcup\{\{h(c) : c \in A_{\sigma_{\mathcal{B}}(c_1,\ldots,c_n)}\} : c_i \in A_{b_i}\},\end{aligned}$$

and, since $c_1, \ldots, c_n$ are atoms, this is equal to

$$\begin{aligned}
&\bigcup\{\{h(c) : c \in \sigma_{\mathbb{A}(\mathcal{B})}(c_1, \ldots, c_n)\} : c_i \in A_{b_i}\}\\
\subseteq&\bigcup\{\sigma_{\mathcal{A}}(h(c_1), \ldots, h(c_n)) : c_i \in A_{b_i}\}\\
=&\bigcup\{\sigma_{\mathcal{A}}(a_1, \ldots, a_n) : a_i \in \{h(c) : c \in A_{b_i}\}\}\\
=&\bigcup\{\sigma_{\mathcal{A}}(a_1, \ldots, a_n) : a_i \in \Phi_{\mathcal{B},\mathcal{A}}(h)(b_i)\}\\
=&\sigma_{\mathbb{P}(\mathcal{A})}(\Phi_{\mathcal{B},\mathcal{A}}(h)(b_1), \ldots, \Phi_{\mathcal{B},\mathcal{A}}(h)(b_n)).
\end{aligned}$$

□

LEMMA 4.9. *The function $\Phi_{\mathcal{B},\mathcal{A}}$ is a bijection between $Hom_{\boldsymbol{MAlg}(\Sigma)}(\mathbb{A}(\mathcal{B}), \mathcal{A})$ and $Hom_{\boldsymbol{OAlg}(\Sigma)}(\mathcal{B}, \mathbb{P}(\mathcal{A}))$.*

PROOF: The functions $\Phi_{\mathcal{B},\mathcal{A}}$ are certainly injective: if $\Phi_{\mathcal{B},\mathcal{A}}(h) = \Phi_{\mathcal{B},\mathcal{A}}(h')$, for every atom b we have that

$$\{h(b)\} = \Phi_{\mathcal{B},\mathcal{A}}(h)(b) = \Phi_{\mathcal{B},\mathcal{A}}(h')(b) = \{h'(b)\},$$

and therefore $h = h'$.

For the surjectivity, take a homomorphism of ord-algebras $h : \mathcal{B} \to \mathbb{P}(\mathcal{A})$. We then define $h' : \mathbb{A}(\mathcal{B}) \to \mathcal{A}$ by $h'(b) = a$ for an atom b in $\mathcal{B}$, where $h(b) = \{a\}$. It is well-defined since a homomorphism of ord-algebras takes atoms to atoms, and the atoms of $\mathbb{P}(\mathcal{A})$ are exactly the singletons.

We must show that h' is indeed a homomorphism. For $\sigma \in \Sigma_n$ and atoms $b_1, \ldots, b_n$ in $\mathbb{A}(\mathcal{B})$ such that $h(b_i) = \{a_i\}$ for every $i \in \{1, \ldots, n\}$, we have that $h(\sigma_{\mathcal{B}}(b_1, \ldots, b_n)) \subseteq \sigma_{\mathbb{P}(\mathcal{A})}(h(b_1), \ldots, h(b_n))$, since h is a homomorphism of ord-algebras, and therefore

$$\begin{aligned}
\{h'(b) : b \in \sigma_{\mathbb{A}(\mathcal{B})}(b_1, \ldots, b_n)\} &= \{h'(b) : b \in A_{\sigma_{\mathcal{B}}(b_1,\ldots,b_n)}\}\\
&= \bigcup\{h(b) : b \in A_{\sigma_{\mathcal{B}}(b_1,\ldots,b_n)}\}\\
&= h(\sup A_{\sigma_{\mathcal{B}}(b_1,\ldots,b_n)})\\
&= h(\sigma_{\mathcal{B}}(b_1, \ldots, b_n))\\
&\subseteq \sigma_{\mathbb{P}(\mathcal{A})}(h(b_1), \ldots, h(b_n))\\
&= \sigma_{\mathbb{P}(\mathcal{A})}(\{a_1\}, \ldots, \{a_n\})\\
&= \sigma_{\mathcal{A}}(a_1, \ldots, a_n)\\
&= \sigma_{\mathcal{A}}(h'(b_1), \ldots, h'(b_n)).
\end{aligned}$$

Finally, we state that $\Phi_{\mathcal{B},\mathcal{A}}(h') = h$ since, for any element b in $\mathcal{B}$, we have that

$$\Phi_{\mathcal{B},\mathcal{A}}(h')(b) = \{h'(c) : c \in A_b\} = \bigcup\{h(c) : c \in A_b\} = h(\sup A_b) = h(b),$$

and therefore each $\Phi_{\mathcal{B},\mathcal{A}}$ is, indeed, bijective. □

THEOREM 4.10. *$\mathbb{P}$ and $\mathbb{A}$ are adjoint.*

PROOF: Given $\mathcal{A}$ and $\mathcal{C}$ two Σ-multialgebras, $\mathcal{B}$ and $\mathcal{D}$ two ord-algebras over Σ, $h : \mathcal{A} \to \mathcal{C}$ a homomorphism and $h' : \mathcal{D} \to \mathcal{B}$ a homomorphism of ord-algebras, we must now only prove that the following diagram commutes.

$$\begin{array}{ccc} Hom_{\mathbf{MAlg}(\Sigma)}(\mathbb{A}(\mathcal{B}), \mathcal{A}) & \xrightarrow{\Phi_{\mathcal{B},\mathcal{A}}} & Hom_{\mathbf{OAlg}(\Sigma)}(\mathcal{B}, \mathbb{P}(\mathcal{A})) \\ \downarrow{\scriptstyle Hom(\mathbb{A}(h'),h)} & & \downarrow{\scriptstyle Hom(h',\mathbb{P}(h))} \\ Hom_{\mathbf{MAlg}(\Sigma)}(\mathbb{A}(\mathcal{D}), \mathcal{C}) & \xrightarrow{\Phi_{\mathcal{D},\mathcal{C}}} & Hom_{\mathbf{OAlg}(\Sigma)}(\mathcal{D}, \mathbb{P}(\mathcal{C})) \end{array}$$

So, we take a homomorphism $g : \mathbb{A}(\mathcal{B}) \to \mathcal{A}$ and an element d of $\mathcal{D}$. We have that

$$Hom(h', \mathbb{P}(h))(\Phi_{\mathcal{B},\mathcal{A}}(g)) = \mathbb{P}(h) \circ \Phi_{\mathcal{B},\mathcal{A}}(g) \circ h',$$

and therefore the right side of the diagram gives us

$$\mathbb{P}(h) \circ \Phi_{\mathcal{B},\mathcal{A}}(g) \circ h'(d) = \mathbb{P}(h)(\{g(b) : b \in A_{h'(d)}\}) = \{h \circ g(b) : b \in A_{h'(d)}\}.$$

The left side gives us

$$\Phi_{\mathcal{D},\mathcal{C}}(h \circ g \circ \mathbb{A}(h'))(d) = \{h \circ g \circ \mathbb{A}(h')(e) : e \in A_d\} = \{h \circ g \circ h'(e) : e \in A_d\}.$$

If d is an atom, the right side becomes the singleton containing only $h \circ g \circ h'(d)$, since in this case $A_d = \{d\}$ and, given that h' preserves atoms, $A_{h'(d)} = \{h'(d)\}$. The left side becomes also the singleton formed by $h \circ g \circ h'(d)$, because again $A_d = \{d\}$. As a homomorphism of ord-algebras is determined by its action on atoms, we find that the left and right sides of the diagram are equal, and therefore the diagram commutes. As observed before, this proves that $\mathbb{A}$ and $\mathbb{P}$ are adjoint. □

Corollary 4.11. The categories **MAlg**(Σ) and **OAlg**(Σ) are equivalent.

Proof: Follows from the fact that $\mathbb{P}$ and $\mathbb{A}$ are an equivalence between the two categories, proven in Theorem 4.10. □

5. Some consequences and related results

The result that **MAlg**(Σ) and **OAlg**(Σ) are equivalent has a few consequences, and related results, we would like to stress. First of all, we start by taking the empty signature: in that case, given that all multialgebras are non-empty, **MAlg**(Σ) becomes the category of non-empty sets $\mathbf{Set}^*$, with functions between them as morphisms.

Meanwhile, **OAlg**(Σ) becomes the category with complete, atomic and bottomless Boolean algebras as objects (given we simply drop the operations from an ord-algebra over Σ), with continuous, atoms-preserving functions between them as morphisms. Notice this is very closely related to the equivalence between **CABA** and $\mathbf{Set}^{op}$: the morphisms on the former are merely continuous functions, so the only extra requirement to the morphisms we are making is that they should preserve atoms. This, of course, allows one to exchange the opposite category of **Set** by **Set** itself (or rather $\mathbf{Set}^*$).

A generalization of our result is to partial multialgebras. That is, pairs $\mathcal{A} = (A, \{\sigma_{\mathcal{A}}\}_{\sigma\in\Sigma})$ such that, if $\sigma \in \Sigma_n$, $\sigma_{\mathcal{A}}$ is a function from A^n to $\mathcal{P}(A)$ (no longer $\mathcal{P}(A) \setminus \{\emptyset\}$). In other words, a partial multialgebra is a multialgebra where operations may return the empty-set. Given partial Σ-multialgebras $\mathcal{A}$ and $\mathcal{B}$, a homomorphism between them is a function $h : A \to B$ such that, as is the case for homomorphisms for multialgebras,

$$\{h(a) : a \in \sigma_{\mathcal{A}}(a_1, \dots, a_n)\} \subseteq \sigma_{\mathcal{B}}(h(a_1), \dots, h(a_n)),$$

for $\sigma \in \Sigma_n$ and $a_1, \dots, a_n \in A$. The class of all partial Σ-multialgebras, with these homomorphisms between them as morphisms, becomes a category, which we shall denote by **PMAlg**(Σ).

It is easy to find an equivalence, much alike the one between **MAlg**(Σ) and **OAlg**(Σ), between **PMAlg**(Σ) and a category related to **OAlg**(Σ): it is sufficient to replace the requirement that, in an ord-algebra, $(A, \leq_{\mathcal{A}})$ is a complete, atomic and bottomless Boolean algebra by the requisite that it is actually a complete, atomic Boolean algebra, and accordingly, change the morphisms in the correspondent category by requiring they preserve the supremum of any sets, not necessarily non-empty.

Finally, let us slightly modify the notion of homomorphism between Σ-multialgebras: a multihomomorphism h from $\mathcal{A}$ to $\mathcal{B}$, what may be written as $h : \mathcal{A} \to \mathcal{B}$ for simplicity, is a function $h : A \to \mathcal{P}(B) \setminus \{\emptyset\}$ that satisfies

$$\bigcup_{a\in\sigma_{\mathcal{A}}(a_1,\ldots,a_n)} h(a) \subseteq \bigcup_{(b_1,\ldots,b_n)\in h(a_1)\times\cdots\times h(a_n)} \sigma_{\mathcal{B}}(b_1,\ldots,b_n).$$

The category with: Σ-multialgebras as objects; multihomomorphisms as morphisms; and the composition of morphisms $h : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{B} \to \mathcal{C}$ given by, for an element a of $\mathcal{A}$, $g \circ h(a) = \bigcup_{b\in h(a)} g(b)$, will be denoted by $\mathbf{MMAlg}(\Sigma)$. If, in the category $\mathbf{OAlg}(\Sigma)$, we change morphisms by not longer demanding that they map atoms into atoms, it is easy to adapt the proof given in Section 4 to show that the resulting category is equivalent to $\mathbf{MMAlg}(\Sigma)$.

Conclusion and future work

As we explained before, the main results here presented involve an equivalence similar to the one between the categories of complete, atomic Boolean algebras and $\mathbf{Set}^{op}$: the first is modified by addition of operations to said Boolean algebras, that are required to furthermore be compatible with the algebra's order; meanwhile, to the second we attach non-deterministic (yet still non-partial) operations, leading us to a category of multialgebras.

Although not specially complicated, this result is useful as it allows to treat non-deterministic matrices (Nmatrices) as, not precisely algebraic semantics, but mixed methods that combine both an algebraic component and one relative to its order. This may seem to increase the complexity of decision methods, but this sacrifice is made precisely to avoid non-determinism; and it is made, not because we distrust the use of multialgebras as semantics for non-classical logics, but as an alternative to those logicians that have philosophical objections against that very use.

More pragmatically, we feel encouraged to develop a further study of the categories of multialgebras, now from the viewpoint of categories of partially ordered sets, far better understood than the former ones. Moreover, we can now recast several non-deterministic characterizations of logics found in the literature in the terms here presented. Specifically, there are several paraconsistent logics uncharacterizable by finite matrices, but characterized by finite Nmatrices, which can now have semantics presented only in classical terms of algebras and orders.

Acknowledgements. We thank the suggestions of the two anonymous referees, which helped us to improve the overall quality of this paper. We also thank Martin Figallo for drawing our attention to Tarski algebras, and to Rodolfo Ertola for improving our intuitions on implicative lattices. The first author acknowledges financial support from the National Council for Scientific and Technological Development (CNPq), Brazil, under research grant 306530/2019-8. The second author was initially supported by a doctoral scholarship from the *Coordenação de Aperfeiçoamento de Pessoal de Nível Superior – Brasil (CAPES)* – Finance Code 001 (Brazil), and later on by a postdoctoral fellowship by NSF-BSF, grant number 2020704.

References

[1] J. C. Abbott, *Implicational algebras*, **Bulletin mathématique de la Société des Sciences Mathématiques de la République Socialiste de Roumanie**, vol. 11(59)(1) (1967), pp. 3–23, URL: http://www.jstor.org/stable/43679502.

[2] A. Avron, I. Lev, *Canonical Propositional Gentzen-type Systems*, [in:] R. Gore, A. Leitsch, T. Nipkow (eds.), **Proceedings of the 1st International Joint Conference on Automated Reasoning (IJCAR 2001)**, vol. 2083 of LNAI, Springer Verlag (2001), pp. 529–544, DOI: https://doi.org/10.1007/3-540-45744-5_45.

[3] M. Baaz, O. Lahav, A. Zamansky, *A Finite-valued Semantics for Canonical Labelled Calculi*, **Journal of Automated Reasoning**, vol. 51 (2013), pp. 401–430, DOI: https://doi.org/10.1007/s10817-013-9273-x.

[4] I. Bošnjak, R. Madarász, *On power structures*, **Algebra and Discrete Mathematics**, vol. 2003(2) (2003), pp. 14–35.

[5] C. Brink, *Power structures*, **Algebra Universalis**, vol. 30(2) (1993), pp. 177–216, DOI: https://doi.org/10.1007/BF01196091.

[6] R. H. Bruck, **A Survey of Binary Systems**, Springer Berlin Heidelberg (1971), DOI: https://doi.org/10.1007/978-3-662-43119-1.

[7] W. A. Carnielli, M. E. Coniglio, **Paraconsistent Logic: Consistency, Contradiction and Negation**, vol. 40 of Logic, Epistemology, and the Unity of Science, Springer International Publishing, Cham, Switzerland (2016), DOI: https://doi.org/10.1007/978-3-319-33205-5.

[8] J. Cīrulis, *A first-order logic for multi-algebras*, **Novi Sad Journal of Mathematics**, vol. 34(2) (2004), pp. 27–36.

[9] M. E. Coniglio, A. Sernadas, C. Sernadas, J. Rasga, *A graph-theoretic account of logics*, **Journal of Logic and Computation**, vol. 19 (2009), pp. 1281–1320, DOI: https://doi.org/10.1093/logcom/exp023.

[10] M. E. Coniglio, G. V. Toledo, *Weakly Free Multialgebras*, **Bulletin of the Section of Logic**, vol. 51(1) (2021), pp. 109–141, URL: https://czasopisma.uni.lodz.pl/bulletin/article/view/5680.

[11] M. E. Coniglio, G. V. Toledo, *A Category of Ordered Algebras Equivalent to the Category of Multialgebras*, arXiv 2209.08158 [math.CT] (2022), URL: https://arxiv.org/abs/2209.08158.

[12] H. B. Curry, **Foundations of mathematical logic**, 2nd ed., Dover Books on Mathematics, Dover Publications, Mineola, NY (1977).

[13] M. Dresher, O. Ore, *Theory of Multigroups*, **American Journal of Mathematics**, vol. 60(3) (1938), pp. 705–733, DOI: https://doi.org/https://doi.org/10.2307/2371606.

[14] G. Hansoul, *A duality for Boolean algebras with operators*, **Algebra Universalis**, vol. 17(1) (1983), pp. 34–49, DOI: https://doi.org/10.1007/BF01194512.

[15] F. Marty, *Sur une generalization de la notion de groupe*, [in:] **Comptes rendus du huitième Congrès des mathématiciens scandinaves tenu à Stockholm 14–18 août 1934** (1935), pp. 45–49.

[16] A. Monteiro, *Cours sur les algebrés de Hilbert et de Tarski*, Instituto de Matemática, Universidad Nacional del Sur, Bahía Blanca, Argentina (1960).

[17] A. Monteiro, L. Iturrioz, *Les algebrés de Tarski avec un nombre fini de générateurs libres*, 1965. A. Monteiro: Unpublished papers I. Notas de Lógica Matemática, 40, Universidad Nacional del Sur, Instituto de Matemática, Bahía Blanca, Argentina (1996).

[18] F. M. Nolan, **Multi algebras & related structures**, Ph.D. thesis, University of Canterbury, Christchurch, New Zealand (1979), DOI: https://doi.org/10.26021/2302.

[19] C. Pelea, S. Breaz, *Multialgebras and term functions over the algebra of their nonvoid subsets*, **Mathematica**, vol. 43(2) (2001), pp. 143–149.

[20] H. E. Pickett, *Homomorphisms and subalgebras of multialgebras*, **Pacific Journal of Mathematics**, vol. 21 (1967), pp. 327–342, DOI: https://doi.org/10.2140/pjm.1967.21.327.

[21] U. Rivieccio, *Implicative twist-structures*, **Algebra Universalis**, vol. 71(2) (2014), pp. 155–186, DOI: https://doi.org/10.1007/s00012-014-0272-5.

[22] M. H. Stone, *The Theory of Representations for Boolean Algebras*, **Transactions of the American Mathematical Society**, vol. 40 (1936), pp. 37–111, DOI: https://doi.org/10.2307/1989664.

[23] G. V. Toledo, **Multialgebras and non-deterministic semantics applied to paraconsistent logics**, Ph.D. thesis, University of Campinas, Campinas, SP, Brazil (2022), URL: https://repositorio.unicamp.br/acervo/detalhe/1244055.

[24] J. van Oosten, *Basic Category Theory*, Basic Research in Computer Science. BRICS Lecture Series LS-95-1. Ultrecht University, Netherlands (1995), URL: https://www.brics.dk/LS/95/1/BRICS-LS-95-1.ps.gz.

[25] M. Walicki, S. Meldal, *Multialgebras, power algebras and complete calculi of identities and inclusions*, [in:] E. Astesiano, G. Reggio, A. Tarlecki (eds.), **Recent Trends in Data Type Specification**, Springer Berlin Heidelberg, Berlin, Heidelberg (1995), pp. 453–468, DOI: https://doi.org/https://doi.org/10.1007/BFb0014444.

Marcelo Esteban Coniglio
University of Campinas (Unicamp)
Institute of Philosophy and the Humanities (IFCH) and Centre for Logic, Epistemology and the History of Science (CLE)
13083-859, Rua Sérgio Buarque de Holanda, 251
Campinas, SP, Brazil
e-mail: coniglio@unicamp.br

Guilherme Vicentin de Toledo
Bar Ilan University
Department of Computer Science
The Spiegel Mathematics & Computer Center
Ramat Gan, Tel Aviv, Israel
e-mail: guivtoledo@gmail.com

Bulletin of the **S**ection of **L**ogic
Volume 52/4 (2023), pp. 551–581
https://doi.org/10.18778/0138-0680.2023.25

Tin Adlešić*
Vedran Čačić

THE CARDINAL SQUARING PRINCIPLE AND AN ALTERNATIVE AXIOMATIZATION OF NFU

Abstract

In this paper, we rigorously prove the existence of type-level ordered pairs in Quine's New Foundations with atoms, augmented by the axiom of infinity and the axiom of choice (NFU + Inf + AC). The proof uses the cardinal squaring principle; more precisely, its instance for the (infinite) universe (VCSP), which is a theorem of NFU + Inf + AC. Therefore, we have a justification for proposing a new axiomatic extension of NFU, in order to obtain type-level ordered pairs almost from the beginning. This axiomatic extension is NFU + Inf + AC + VCSP, which is equivalent to NFU + Inf + AC, but easier to reason about.

Keywords: Quine's New Foundations, cardinal multiplication, axiomatization.

2020 Mathematical Subject Classification: Primary 03E20; Secondary 03E30, 03E25.

Introduction

Quine's New Foundations (NF) [12] can be viewed as an improved and simplified version of *Principia mathematica*. However, its (relative) consistency has not been proved for thirty years, until Jensen proved that a slight modification of NF admits a consistency proof [11]. Jensen weakened the axiom of extensionality and allowed the atoms to exist in the theory,

*This author was supported by the Croatian Science Foundation under project UIP-2017-05-9219 (FORMALS).

Presented by: Andrzej Indrzejczak
Received: October 29, 2022
Published online: September 28, 2023

named *New Foundations with atoms* or urelements (NFU). By Jensen's results, NFU is relatively consistent with the axiom of infinity (Inf) and the axiom of choice (AC). So we can work in NFU + Inf + AC, which seems to us a good alternative theory to ZF for the foundation of mathematics. As far as NF (without atoms) is concerned, the search for its consistency proof is still in progress, most prominently by Holmes [8] and Gabbay [4].

Since NFU embodies a kind of type theory, it is important to keep track of (relative) types during syntactic manipulations, the procedure which can sometimes be arduous. The notion that makes this particularly cumbersome is that of ordered pairs. It is impossible to define type-level ordered pairs in NFU alone since their existence implies the axiom of infinity [7], which is independent of NFU [11]. "Type-level" means that an ordered pair has the same type as its components. There are essentially three approaches to deal with type-level ordered pairs "problem" in NFU.

First one can be called *the way of resignation*. In this approach, one simply rejects the necessity of type-level ordered pairs and works with Kuratowski's ones. Although that is a valid approach, Kuratowski's ordered pairs are difficult to work with because the type of an ordered pair is two higher than its components, so "type explosion" happens very soon, and that can be a liability in theory development. For instance, cardinal arithmetic is defined in an unnatural way, and the proofs about it are very cumbersome (see section 4) if one is using Kuratowski's ordered pairs. For that reason, we would prefer to avoid the resignation way.

The second approach is due to Holmes [7] and its main feature is the introduction of a new *axiom of ordered pairs* (which we denote OP) to NFU. This axiom introduces ordered pairs as a primitive notion and in that way enables the existence of type-level ordered pairs of any two entities (sets or atoms). This axiom solves most of our problems and is a good option for theory development. It does have a justification, for it can be proved that inside a model of NFU + Inf one can construct a model of NFU + OP [5]. Nonetheless, its motivation is entirely pragmatical, since it is solely envisioned to solve one technical problem, and introducing ordered pairs as primitives requires extending our language by some (at least one, usually three) function symbols, without intrinsic rules for type assignment. Such an extension changes the notion of atomic formulas, complicating the notion of stratification. Moreover, even though OP is solving a purely technical problem, its ontological commitment is enormous, for it implies the existence of infinitely many arbitrary objects [7] in a non-obvious way.

The third approach is the new one we are proposing: an alternative axiomatic extension of NFU. We have already mentioned that in every model of NFU + Inf one can obtain a model of NFU + OP, but that is only possible by interpreting OP in the signature specifically extended for OP to have its intended meaning. On the contrary, NFU + Inf + AC does not need any artificial signature extension since it can prove VCSP from which type-level ordered pairs can be easily obtained. Therefore, we have a justification to introduce a new axiomatization NFU + Inf + AC + VCSP, which is an axiomatic extension of NFU+Inf+AC. The main advantage of this approach is that ordered pairs are available almost from the start, it is well-motivated, the language does not need to be extended (thus the notion of stratification remains the same), and it does not have any ontological commitment since it is a conservative axiomatic extension. However, there is a need for some Kuratowski's ordered pair-dependent theory, in order to be able to state the new axiom, VCSP. Fortunately, the theory needed for that is rather small (which will be seen soon enough). It is important to note that theory NFU + Inf + VCSP does not prove AC (see theorem 5.3), which means that it cannot be used as a satisfactory theory for development. Therefore, we find NFU + Inf + AC + VCSP to be the best approach proposed so far for the development of set theory in Quine's style.

In order to show that the third approach is a viable option, we need to prove in NFU+Inf+AC (with Kuratowski's ordered pairs) that the cardinal squaring principle holds. From there, we can easily prove the existence of type-level ordered pairs, completing our justification of alternative extension NFU + Inf + AC + VCSP. The cardinal squaring principle has not yet been rigorously proved in our setting, although it seems to be a well-known fact. The main motivation for this paper is a remark about the cardinal squaring principle in [10]. The same remark is also stated in [6] and [5].

Our proof is based on the one in [2]—but it is not the same since we need to take the peculiarities of NFU into consideration. Moreover, our proof that every infinite set has a countable subset, using Kuratowski's ordered pairs, is correct, unlike the ones in contemporary literature. The proof in [7] is using Zorn's lemma on a set of functions from arbitrary subsets of natural numbers to an infinite set, which does not work, for example, for a set of all even natural numbers as a starting infinite set. Moreover, the proof in [9] is using Zorn's lemma on an empty set, if we use any uncountable set as a starting infinite set.

Overview

In the first section, we introduce the necessary syntax and notation. This is done in more detail in [1].

In the second section, we introduce the axioms of NFU, along with some well-known facts needed for the proof of the cardinal squaring principle. Axiom of choice is also introduced in this section, as well as Kuratowski's ordered pairs.

In the third section, we introduce natural and cardinal numbers and state the axiom of infinity. The main result of this section is the proof that every infinite set has a countable subset.

In the fourth section, we introduce the sum and product of cardinal numbers using Kuratowski's ordered pairs. After a few theorems of preparation, we finally prove that the cardinal squaring principle holds in our setting, and then we show how the existence of type-level ordered pairs can be proved.

In the fifth section, we present the resulting axiomatic extension of NFU, as well as some results regarding the mutual provability of various claims we have introduced.

1. Syntax

In this section, we introduce the syntax of NFU as well as some other necessary notions. Most results are stated without proof; the proofs can be found in [1].

An **alphabet** is a collection of:

- (individual) variables $v_0, v_1, \ldots$
- logical symbols (connectives and quantifiers) $\neg$, $\rightarrow$, $\exists$
- non-logical (relation) symbols $\in$, $=$, *set*
- auxiliary symbols (brackets) (,)

Relation symbols $\in$ and $=$ have the usual interpretation, and *set* is a unary relation symbol expressing that an entity is a set. All the other usual logical symbols ($\vee$, $\wedge$, $\leftrightarrow$, $\forall$, $\exists!$) can be defined in terms of the existing ones in the standard way. Formulas are defined in the following way:

$$\varphi ::= x \in y \mid x = y \mid set(x) \mid (\varphi_1 \rightarrow \varphi_2) \mid \neg\varphi_1 \mid \exists x\varphi,$$

where x and y denote variables, while φ, φ_1 and φ_2 denote formulas. We will denote with $\varphi(x_1, \ldots, x_n)$ that $x_1, \ldots, x_n$ are all free variables occuring in φ. We will usually write $(\exists x \in y)\varphi$, $(\forall x \in y)\varphi$, instead of $\exists x(x \in y \wedge \varphi)$ and $\forall x(x \in y \rightarrow \varphi)$ respectively. We write $(\forall x, y \in t)\varphi$ as a shorthand for $(\forall x \in t)(\forall y \in t)\varphi$.

A formula φ is **stratified** if there exists a mapping $type_\varphi$ from the variables of φ to the positive natural numbers such that: for every subformula of φ of the form $x = y$, we have $type_\varphi(x) = type_\varphi(y)$, and for every subformula of φ of the form $x \in y$, we have $type_\varphi(y) = type_\varphi(x) + 1$. The number $type_\varphi(x)$ is called the **type** of the variable x in the formula φ. Conditions imposed on the mapping $type_\varphi$ are called **stratification conditions**. We will call mappings satisfying stratification conditions, **type mappings**. Types of variables will be written in superscript.

Definition 1.1. Let $\varphi(x, x_1, \ldots, x_n)$ be a stratified formula. An expression of the form $\{z \mid \varphi(z, x_1, \ldots, x_n)\}$ is called an **abstraction term**. We extend the notion of (atomic) formulas by allowing them to contain abstraction terms in addition to variables. Formulas containing abstraction terms we call **formulas of the extended language**. Abstraction terms that appear in atomic formulas are eliminated in the following way:

1. $x \in \{z \mid \varphi(z, x_1, \ldots, x_n)\} :\Leftrightarrow \varphi(x, x_1, \ldots, x_n)$
2. $x = \{z \mid \varphi(z, x_1, \ldots, x_n)\} :\Leftrightarrow$ $set(x) \wedge \forall y\big(y \in x \leftrightarrow y \in \{z \mid \varphi(z, x_1, \ldots, x_n)\}\big)$
3. $\{z \mid \varphi(z, x_1, \ldots, x_n)\} \in x \;:\Leftrightarrow (\exists y \in x)\big(y = \{z \mid \varphi(z, x_1, \ldots, x_n)\}\big)$
4. $set(\{z \mid \varphi(z, x_1, \ldots, x_n)\}) :\Leftrightarrow \exists y\big(y = \{z \mid \varphi(z, x_1, \ldots, x_n)\}\big)$

Definition 1.2. Let $\varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ and $\psi(w, x_1, \ldots, x_n)$ be formulas. Nested abstraction terms are eliminated in the following way:

$$\big\{\{w \mid \psi(w, x_1, \ldots, x_n)\} \mid \varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)\big\} := \big\{z \;\big|\; set(z) \wedge \exists x_1 \cdots \exists x_n\big(\varphi(x_1, \ldots, x_n, y_1, \ldots, y_m) \wedge z = \{w \mid \psi(w, x_1, \ldots, x_n)\}\big)\big\}.$$

It is important to be able to assign types to abstraction terms. If $\varphi(z, z_1, \ldots, z_n)$ is a stratified formula and $\mathsf{t} = \{z \mid \varphi(z, z_1, \ldots, z_n)\}$ is an abstraction term, then the type of t in some stratified formula is determined by the type of a variable t in a formula $z \in t \leftrightarrow \varphi(z, z_1, \ldots, z_n)$. All this has been done more formally and precisely in [1].

When defining sets, we will usually not check whether the defining formulas are stratified. That can be easily done using type assignments in the extended language (again, details about this procedure can be found in [1]). Explicit checking will only be done in some proofs.

2. NFU set theory

Axiom of extensionality:

$$\forall x \forall y \big(set(x) \land set(y) \land \forall z (z \in x \leftrightarrow z \in y) \to x = y\big).$$

Axiom of sethood:

$$\forall x\, (\forall y \in x)\, set(x).$$

Axiom schema of stratified comprehension: if $\varphi(z, x_1, \ldots, x_n)$ is stratified, then

$$\forall x_1 \cdots \forall x_n \exists y \big(set(y) \land \forall z \big(z \in y \leftrightarrow \varphi(z, x_1, \ldots, x_n)\big)\big).$$

We say that x is a **subset** of y if $set(x) \land set(y) \land (\forall z \in x)(z \in y)$, which is written $x \subseteq y$. We write $x \subset y$ if $x \subseteq y$ and $x \neq y$. Some simple sets and operations in NFU are:

$$\begin{gathered}
\emptyset := \{x \mid x \neq x\}, \qquad V := \{x \mid x = x\}, \qquad SET := \{x \mid set(x)\}, \\
x \cup y := \{z \mid z \in x \lor z \in y\}, \qquad x \cap y := \{z \mid z \in x \land z \in y\}, \\
\bigcup x := \{z \mid (\exists t \in x)(z \in t)\}, \qquad \bigcap x := \{z \mid (\forall t \in x)(z \in t)\}, \\
x \setminus y := \{z \mid z \in x \land z \notin y\}, \qquad x^c := \{z \mid z \notin x\}, \\
\{x\} := \{z \mid z = x\}, \qquad \mathscr{P}_1(x) := \big\{\{t\} \bigm| t \in x\big\}
\end{gathered}$$

It is easy to check the following equivalence:

$$\forall x \big(set(x) \leftrightarrow x = \emptyset \lor (\exists y \in x)\big).$$

Definition 2.1. For $x, y \in V$ we define their **ordered pair** $(x, y) := \big\{\{x\}, \{x, y\}\big\}$ (where $\{a, b\} := \{a\} \cup \{b\}$).

These ordered pairs are the usual Kuratowski's ordered pairs and they have the unfortunate property of not being type-level. More precisely, if x and y have type n in some stratified formula, then (x, y) has type $n+2$ in that same formula.

Definition 2.2. For sets X and Y we define their **Cartesian product** $X \times Y := \{(x, y) \mid x \in X \wedge y \in Y\}$.

Definition 2.3. Let X and Y be sets. We say that R is a **(binary) relation** between X and Y if $R \subseteq X \times Y$, which we write $\mathit{rel}(R, X, Y)$. We say that R is a **relation** if $\mathit{rel}(R, V, V)$.

Let R be a relation. Instead of $(x, y) \in R$ we will write $x\ R\ y$. In addition, if $(x, y) \notin R$, we will write $x \not R\ y$.

Definition 2.4. Let R be a relation. We define its **domain** $\mathit{dom}(R) := \{x \mid \exists y(x\ R\ y)\}$ and **range** $\mathit{rng}(R) := \{y \mid \exists x(x\ R\ y)\}$.

Definition 2.5.

1. We define **identity** on a set X as $\mathit{id}_X := \{(x, x) \bigm| x \in X\}$.
2. For relations R and R' we define their **composition** as $R' \circ R := \{(x, z) \bigm| \exists y(x\ R\ y\ R'\ z)\}$.
3. For a relation R, we define its **inverse** as $R^{-1} := \big\{(y, x) \bigm| x\ R\ y\big\}$.

Definition 2.6. Let X and Y be sets. We say that f is a **function** from X to Y if $\mathit{rel}(f, X, Y) \wedge (\forall x \in X)(\exists! y \in Y)(x\ f\ y)$, which we write $\mathit{func}(f, X, Y)$.

For a function f and $x \in \mathit{dom}(f)$, we introduce the standard notation $f(x)$ for the unique y such that $x\ f\ y$.

Definition 2.7. Let X and Y be sets.

1. f is an **injection** from X to Y if $\mathit{func}(f, X, Y) \wedge (\forall x_1, x_2 \in X)$ $\big(f(x_1) = f(x_2) \to x_1 = x_2\big)$, which we write $\mathit{inj}(f, X, Y)$.
2. f is a **bijection** between X and Y if $\mathit{inj}(f, X, Y) \wedge \mathit{rng}(f) = Y$, which we write $\mathit{bij}(f, X, Y)$.

It is easy to see that every relation R is a relation on $dom(R) \cup rng(R)$. We say that R is a **reflexive** relation if $x \mathrel{R} x$ for every $x \in dom(R)$.

Definition 2.8. We say that a relation $\preceq$ is a **partial order** if it is reflexive, antisymmetric and transitive; symbolically, if

$$rel(\preceq, V, V) \wedge \big(\forall x \in dom(\preceq)\big)(x \preceq x) \wedge$$
$$\wedge\, \forall x \forall y (x \preceq y \preceq x \to x = y) \wedge \forall x \forall y \forall z (x \preceq y \preceq z \to x \preceq z),$$

which is written $Po(\preceq)$. We will say that $\preceq$ is a partial order on a set X if $Po(\preceq) \wedge dom(\preceq) = X$, which is written $Po(\preceq, X)$.

Definition 2.9. We say that a relation R on X is an **equivalence relation** if it is reflexive, symmetric, and transitive. For every $x \in X$ we define its **equivalence class** $[x]_R := \{y \mid y \mathrel{R} x\}$.

Definition 2.10. For a set X we define its **quotient set** by an equivalence relation R on X as $X/R := \{[x]_R \mid x \in X\}$.

Definition 2.11. Let $\preceq$ be a partial order, $Y \subseteq dom(\preceq)$ and $y_0 \in dom(\preceq)$. We say that y_0 is

1. a **maximal** element of Y if $y_0 \in Y \wedge (\forall y \in Y)(y_0 \preceq y \to y_0 = y)$;
2. an **upper bound** of Y if $(\forall y \in Y)(y \preceq y_0)$;
3. the **greatest** element of Y if $y_0 \in Y$ and y_0 is an upper bound of Y;
4. the **least** element of Y if y_0 is the greatest element of Y with respect to the relation $\succeq := \preceq^{-1}$.

Definition 2.12. Let $\preceq$ be a partial order.
We say that $L \subseteq dom(\preceq)$ is a **chain** in $\preceq$ if $(\forall x, y \in L)(x \preceq y \vee y \preceq x)$.

Lemma 2.13. *Let X be a set of functions ordered by inclusion, and let $C \subseteq X$ be a chain. Then*

1. $\big(\bigcup C\big)^{-1} = \bigcup\{f^{-1} \mid f \in C\}$.
2. $\bigcup C$ *is a function.*
3. $dom\big(\bigcup C\big) = \bigcup\big\{dom(f) \mid f \in C\big\}$
4. $rng\big(\bigcup C\big) = \bigcup\big\{rng(f) \mid f \in C\big\}$
5. *If every function $f \in C$ is an injection, then $\bigcup C$ is an injection.*

PROOF:

1. Let $z \in \left(\bigcup C\right)^{-1}$, then $z = (x, y)$ for some x, y such that $(y, x) \in \bigcup C$. Then there exists $f \in C$ such that $(y, x) \in f$, which implies $(x, y) \in f^{-1} \subseteq \bigcup\{f^{-1} \mid f \in C\}$. The other direction is analogous.

2. Let $x, y \in dom\left(\bigcup C\right)$. Then there exist $a, b \in rng\left(\bigcup C\right)$ such that $(x, a), (y, b) \in \bigcup C$. It means there exist functions $f_1, f_2 \in C$ such that $(x, a) \in f_1$ and $(y, b) \in f_2$. Because C is a chain, without the loss of generality, we can assume $f_1 \subseteq f_2$. From that we get $(x, a), (y, b) \in f_2 \subseteq \bigcup C$. If $x = y$, since f_2 is a function, we get $a = b$. Therefore, $\bigcup C$ is a function.

3. Let $z \in dom\left(\bigcup C\right)$. Then there exists $y \in rng\left(\bigcup C\right)$ such that $(z, y) \in \bigcup C$. Then there exists a function $f \in C$ such that $(z, y) \in f$, that is, $z \in dom(f)$. From that, we get $z \in \bigcup\left\{dom(f) \mid f \in C\right\}$. If $z \in \bigcup\left\{dom(f) \mid f \in C\right\}$, then there exists $f \in C$ such that $z \in dom(f)$. That means there exists $y \in rng(f)$ such that $(z, y) \in f \subseteq \bigcup C$, which implies $z \in dom\left(\bigcup C\right)$.

4. Follows from (1) and (3).

5. Follows from (1) and (2). □

We say that a relation on X is a **well-order** if it is a partial order and every nonempty subset of X has the least element in that order.

AXIOM OF CHOICE: $\forall x\big(set(x) \land \emptyset \notin x \land (\forall y, z \in x)(y \neq z \rightarrow y \cap z = \emptyset) \rightarrow \exists u(\forall w \in x)\exists! v(v \in w \cap u)\big)$.

***Zorn's lemma*:** Let $\preceq$ be a partial order. If every chain C in $\preceq$ has an upper bound, then $dom(\preceq)$ has a maximal element.

***Zermelo's theorem*:** Every set can be well-ordered.

THEOREM 2.14. *Axiom of choice* $\Leftrightarrow$ *Zorn's lemma* $\Leftrightarrow$ *Zermelo's theorem.*

The equivalence proof resembles the usual one (from ZF) and can be found in [7]. It is worth noting that in Zorn's lemma, we can assume that the chain is nonempty provided we prove first that $\preceq$ is nonempty (we can always use any element of $dom(\preceq)$ as an upper bound for the empty chain).

3. Cardinal numbers

DEFINITION 3.1.

1. The set $0 := \{\emptyset\}$ is **zero**.
2. For a set x we define its **successor** $succ(x) := \{y \mid (\exists z \in y)(y \setminus \{z\} \in x)\}$.
3. The set $\mathbb{N} := \bigcap\{x \mid 0 \in x \wedge (\forall y \in x)(succ(y) \in x)\}$ is called the set of **natural numbers**.
4. The set $FIN := \bigcup \mathbb{N}$ is the set of **finite sets**.

It can be proved that $succ(x) = \{y \cup \{z\} \mid y \in x \wedge z \notin y\}$.
We define $1 := succ(0) = \mathscr{P}_1(V)$ and $2 := succ(1)$.
We say that a set x is **finite** if $x \in FIN$, otherwise it is **infinite**.

AXIOM OF INFINITY: $V \notin FIN$.

Peano's axioms are the following:

1. $0 \in \mathbb{N}$.
2. $(\forall n \in \mathbb{N})(succ(n) \in \mathbb{N})$.
3. $(\forall n \in \mathbb{N})(0 \neq succ(n))$.
4. $(\forall n \in \mathbb{N})(\forall m \in \mathbb{N})(succ(n) = succ(m) \to n = m)$.
5. If $\varphi(x)$ is a stratified formula, then

$$\varphi(0) \wedge (\forall n \in \mathbb{N})(\varphi(n) \to \varphi(succ(n))) \to (\forall n \in \mathbb{N})\varphi(n).$$

The first three Peano axioms can be easily proved in NFU. However, the axiom schema of mathematical induction must be restricted to stratified formulas only. Lastly, the fourth Peano axiom is equivalent to the axiom of infinity. This proof can be found in [14], but we will provide one too.

LEMMA 3.2. *Every natural number n is 0 or a successor of some natural number.*

PROOF: We need to prove $(\forall n^1 \in \mathbb{N}^2)(n^1 = 0^1 \vee (\exists m^1 \in \mathbb{N}^2)(n^1 = succ(m^1)^1))$, which is stratified, so we can prove it by induction on n.

The claim trivially holds for $n = 0$. Assume the claim holds for some $n \in \mathbb{N}$ and prove it for $succ(n)$. However, $succ(n)$ is the successor of n, and n is a natural number. □

Definition 3.3. We define the **equipotence** relation between sets as $(\sim) := \{(x, y) \mid set(x) \wedge set(y) \wedge \exists f\ bij(f, x, y)\}$.

Definition 3.4. We define the set of **cardinal numbers** $Card := SET/(\sim) = \{[x]_\sim \mid set(x)\}$.

We will denote $[x]_\sim$ with $|x|$ and call it the **cardinal number of set** x.

Definition 3.5. We define the relation $\leq$ on $Card$ with $\kappa \leq \lambda :\Longleftrightarrow (\forall X \in \kappa)(\forall Y \in \lambda)\exists f\ inj(f, X, Y)$.

It is easy to show that it is enough, in definition 3.5, to require just the existence of X and/or Y (the existence of an injection between two sets is invariant with respect to equipotence).

Theorem 3.6. *The relation $\leq$ is a well-order.*

Proof (Sketch of proof:): Reflexivity and transitivity are easy.

The antisymmetricity is actually Cantor–Bernstein's theorem which can be found in [7].

The fact that every nonempty set of cardinals has the least element is proved using Zermelo's theorem. See [7, p. 123–124]. □

We write $\kappa < \lambda$ for $\kappa \leq \lambda \wedge \kappa \neq \lambda$.

Definition 3.7. We say that a set x is **Dedekind-infinite**, if there exists $y \subset x$ such that $x \sim y$.

For now, we can only prove that Dedekind-infinity implies infinity.

Theorem 3.8. *If a set is Dedekind-infinite, then it is infinite.*

Proof: We will prove the contrapositive of the claim: if a set is finite (that is, it is an element of some natural number), then there is no bijection between it and its any proper subset.

Formula $(\forall n^2 \in \mathbb{N}^3)(\forall x^1 \in n^2)\forall y^1(y^1 \subset x^1 \to x^1 \not\sim^4 y^1)$ is stratified, so we can prove it by induction on n.

Let $n = 0$ and $x \in n$ be arbitrary. Then $x = \emptyset$, so the statement holds vacuously since x does not have any nonempty proper subsets. Assume that for a natural number n the statement holds. Let us prove the statement for $succ(n)$. Let $x \in succ(n)$ be arbitrary. Then $x = y \cup \{z\}$, for some $y \in n$ and $z \notin y$. Assume that there exists a $u \subset x$ and a bijection $f\colon x \to u$.

First case is when $z \notin u$. Then we have $u \subseteq y$, and so $u \setminus \{f(z)\} \subset y$, but we have $bij(f \setminus \{(z, f(z))\}, y, u \setminus \{f(z)\})$, which is a contradiction.

Second case is when $z \in u$. Then there exists $w \in x$ such that $f(w) = z$. Define $h := id_{x\setminus\{w,z\}} \cup \{(z, w), (w, z)\}$. It is obvious that $bij(h, x, x)$, so $bij(f \circ h, x, u)$. The same argument holds if $z = w$. Since $(f \circ h)(z) = f(w) = z$, we have $bij(f \setminus \{(z, z)\}, y, u \setminus \{z\})$. However, we have $u \setminus \{z\} \subset y$, which is a contradiction. □

Lemma 3.9. *The following holds:* $(\forall n \in \mathbb{N})(\forall y \in n)\forall z(z \in n \leftrightarrow y \sim z)$.

Proof: Formula $(\forall n^2 \in \mathbb{N}^3)(\forall y^1 \in n^2)\forall z^1(z^1 \in x^2 \leftrightarrow z^1 \sim^4 y^1)$ is stratified, so we can prove it by induction on x.

Let $n = 0$, let $y \in n$ and z be arbitrary. From $y \in n$ we get $y = \emptyset$. If $z \in n$, we get $z = \emptyset = y$. Obviously, $bij(\emptyset, \emptyset, \emptyset)$. If $y \sim z$, then there exists a bijection f between y and z. Because $y = \emptyset$, f is a bijection between $\emptyset$ and z, so we have $z = rng(f) = rng(\emptyset) = \emptyset \in 0 = n$.

Assume that the claim holds for a natural number n. Let us prove it for $succ(n)$. If $succ(n) = \emptyset$, the claim trivially holds. Let $y \in succ(n)$ and z be arbitrary. By the definition of successor we have $y = a \cup \{b\}$ such that $a \in n$ and $b \notin a$. If $z \in succ(n)$, then $z = u \cup \{v\}$, where $u \in n$ and $v \notin u$. By the assumption, we have $u \sim a$, so there exists a bijection $f\colon a \to u$ and the function $g := f \cup \{(b, v)\}$ is obviously a bijection between y and z. If $y \sim z$, then there exists a bijection $f\colon y \to z$. We define $x := \{f(t) \mid t \in a\}$ and $y := f(b)$. Obviously, $y \notin x$ and since $x \sim a \in n$, we have $z = x \cup \{y\} \in succ(n)$. □

Theorem 3.10. $V \notin FIN \iff \emptyset \notin \mathbb{N} \iff \mathbb{N} \subseteq Card \iff$ (P4).

Proof: We will first prove $V \notin FIN \implies \emptyset \notin \mathbb{N}$. Assume $V \notin FIN$. Formula $(\forall n^1 \in \mathbb{N}^2)(n^1 \neq \emptyset^1)$ is stratified, so we can prove it by induction on n. For $n = 0$ we have $0 = \{\emptyset\} \neq \emptyset$. Let us assume that the claim holds for an $n \in \mathbb{N}$ and prove it for $succ(n)$. Assume that $succ(n) = \emptyset$. We know $succ(n) = \big\{y \cup \{z\} \bigm| y \in n \wedge z \notin y\big\} = \big\{t \mid (\exists y \in n)(\exists z \notin y)(t = y \cup \{z\})\big\}$. Now from $succ(n) = \emptyset$ follows $(\forall y \in n)\forall z(z \in y)$, and from the axiom of extensionality and the fact that V is the universal set we get $\forall y\big(y \in n \to$

$V = y\big)$. From that we get $n = \emptyset$ or $n = \{V\}$. It is impossible to have $n = \emptyset$ by the induction hypothesis, and from $n = \{V\}$ we get $V \in FIN$, which contradicts the assumption.

The claim $\emptyset \notin \mathbb{N} \implies \mathbb{N} \subseteq \mathit{Card}$ follows from lemma 3.9. Assume $\emptyset \notin \mathbb{N}$. Formula $(\forall n^1 \in \mathbb{N}^2)(n^1 \in \mathit{Card}^2)$ is stratified, so we can prove it by induction on n. For $n = 0$ we have $0 = |\emptyset| \in \mathit{Card}$. Assume that for an $n \in \mathbb{N}$ there exists x such that $n = |x|$, and prove the claim for $\mathit{succ}(n)$. By the assumption, we have $\mathit{succ}(n) \neq \emptyset$, so there exists a $y \in \mathit{succ}(n)$. By lemma 3.9 we have $z \in \mathit{succ}(n) \leftrightarrow z \in |y|$. Therefore, $\mathit{succ}(n) = |y|$.

Let us prove $\mathbb{N} \subseteq \mathit{Card} \implies$ (P4). Assume $\mathbb{N} \subseteq \mathit{Card}$. Let $n, m \in \mathbb{N} \subseteq \mathit{Card}$ and assume $\mathit{succ}(n) = \mathit{succ}(m) = |z|$ for some z. By definitions of successor we have $z = a \cup \{b\} = c \cup \{d\}$, where $a \in n$, $c \in m$, $b \notin a$ and $d \notin c$. If $b = d$, then we have $a = c$, for if $w \in a \subseteq z = c \cup \{d\}$, then $w \in c \cup \{d\}$, and since $w \neq d$, then $w \in c$. The converse when $w \in c$ is proved analogously. If $b \neq d$, then $d \in a$ and $b \in c$. Then $g := id_{z \setminus \{b,d\}} \cup \{(b,d)\}$ is obviously a bijection between a and c. In both cases, we have $a \sim c$, meaning $n = m$.

It remains to prove (P4) $\implies V \notin FIN$. Assume $V \in FIN$. Then there exists an $n \in \mathbb{N}$ such that $V \in n$. We claim $n = \{V\}$. Assume $x \in n$ such that $x \neq V$. Then we have $x \subset V$ and because $x, V \in n$, by lemma 3.9 we get $x \sim V$. Now from theorem 3.8 we get that V is an infinite set, that is, $V \notin FIN$, which is a contradiction. Therefore, $n = \{V\}$. Now by the definition of successor we have $\mathit{succ}(n) = \{y \cup \{z\} \mid y \in n \land z \notin y\} = \{y \cup \{z\} \mid z \notin V\} = \emptyset \in \mathbb{N}$ and $\mathit{succ}(\emptyset) = \emptyset$. Then we have $\mathit{succ}(n) = \mathit{succ}(\emptyset)$, but clearly $n \neq \emptyset$. Therefore, fourth Peano axiom does not hold. □

We will say that a cardinal number is an **infinite cardinal number** if it is not a natural number (that is, if it is $|X|$ for an infinite X).

THEOREM 3.11. *Let X be a set, $x_0 \in X$, and $f\colon X \to X$ be a function. Then there exists a unique function $g\colon \mathbb{N} \to X$ such that $g(0) = x_0$ and $g\big(\mathit{succ}(n)\big) = f\big(g(n)\big)$ for every $n \in \mathbb{N}$.*

PROOF: Let X, x_0 and f be as stated. Formula

$$\varphi(t) := \big((0^1, x_0^1)^3 \in t^4 \land (\forall n^1 \in \mathbb{N}^2)(\forall y^1 \in X^2)$$
$$\big((n^1, y^1)^3 \in t^4 \to (\mathit{succ}(n^1)^1, f^4(y^1)^1)^3 \in t^4\big)\big)$$

is stratified, so we can define the set $S := \{t \mid \varphi(t)\}$. Obviously $\mathbb{N} \times X \in S$, so $g := \bigcap S \subseteq \mathbb{N} \times X$. In other words, we have $\mathit{rel}(g, \mathbb{N}, X)$, so we can use the usual infix notation for g.

First, we prove $\varphi(g)$. For every $t \in S$ we have $\varphi(t)$, so $0\ t\ x_0$, therefore also $0\ g\ x_0$. In the same manner, if $n \in \mathbb{N}$, $y \in X$ and $n\ g\ y$, then for every $t \in S$ we have $n\ t\ y$. By $\varphi(t)$, we have $\mathit{succ}(n)\ t\ f(y)$, and therefore also $\mathit{succ}(n)\ g\ f(y)$. So, we can conclude $g \in S$.

Formula $\psi(n) := (\exists!\, y^1 \in X^2)(n^1 g^4 y^1)$ is stratified, so we can prove $(\forall n \in \mathbb{N})\psi(n)$ by induction (fifth Peano axiom). For $n = 0$ we have $0\ g\ x_0$ by $\varphi(g)$, and for any $x' \in X \setminus \{x_0\}$ we can see that $g' := g \setminus \{(0, x')\}$ also satisfies φ. Namely, $(0, x')$ is different from any ordered pair forced into t by φ, since $x' \neq x_0$, and $0 \neq \mathit{succ}(n)$ for any n by third Peano axiom. Therefore we have $g' \in S$, from which $g \subseteq g' = g \setminus \{(0, x')\}$, hence $0 \not g\ x'$.

In much the same manner, suppose that (for a particular $k \in \mathbb{N}$) there is a unique $y \in X$ such that $k\ g\ y$. By $\varphi(g)$ we have $\mathit{succ}(k)\ g\ f(y)$, so existence holds. To prove uniqueness, suppose $\mathit{succ}(k)\ g\ u$ for some $u \in X \setminus \{f(y)\}$. Now we prove that $g'' := g \setminus \{(\mathit{succ}(k), u)\}$ satisfies φ: it obviously satisfies the first conjunct since $0 \neq \mathit{succ}(k)$ by third Peano axiom. For the second, let $m \in \mathbb{N}$ and $z \in X$ be such that $m\ g''\ z$. Then $m\ g\ z$ since $g'' \subseteq g$, and therefore $\mathit{succ}(m)\ g\ f(z)$. But if $\big(\mathit{succ}(m), f(z)\big) = \big(\mathit{succ}(k), u\big)$, then $m = k$ by fourth Peano axiom, and also $u = f(z)$. However, $u \neq f(y)$ means $y \neq z$, and that contradicts uniqueness for k.

Therefore we have $\mathit{func}(g, \mathbb{N}, X)$, so we can use the usual function notation for g. We have already proved $g(0) = x_0$. For any $n \in \mathbb{N}$, we have $n\ g\ g(n)$ (since $\mathit{dom}(g) = \mathbb{N}$), and from $\varphi(g)$ we also have $\mathit{succ}(n)\ g\ f\big(g(n)\big)$, which in function notation is exactly $g\big(\mathit{succ}(n)\big) = f\big(g(n)\big)$.

It remains to prove that such g is unique. Assume the opposite, that there is $h\colon \mathbb{N} \to X$ such that $h \neq g$, $h(0) = x_0$, and for all $n \in \mathbb{N}$, $h\big(\mathit{succ}(n)\big) = f\big(h(n)\big)$. Formula $(\forall n^1 \in \mathbb{N}^2)\big(h^4(n^1)^1 = g^4(n^1)^1\big)$ is stratified, so we can prove it by induction on n. For $n = 0$ we have $h(0) = x_0 = g(0)$. Assume that the claim holds for some $n \in \mathbb{N}$, and prove it for $\mathit{succ}(n)$. We have $h\big(\mathit{succ}(n)\big) = f\big(h(n)\big) = f\big(g(n)\big) = g\big(\mathit{succ}(n)\big)$. Therefore, $h = g$, which is a contradiction. □

It is easy to see that, in theorem 3.11, if x_0 has type k, X has type $k+1$, and f has type $k+3$, then g has type $k+3$.

Theorem 3.12. *Let* $\mathfrak{t}$ *be a term with variable* x *free, which is type-level—that is, in every stratified formula in the extended language where* $\mathfrak{t}$ *appears,* $type(\mathfrak{t}) = type(x)$*. Then for every set* A*, there is a function* f *such that*

$$f(x) = \mathfrak{t}, \qquad \text{for every } x \in A. \tag{3.1}$$

Furthermore, f *is unique if we require additionally that* $dom(f) = A$*.*

That is, we can *define* a function by expression (3.1). We write shorthand "$f(x) := \mathfrak{t},\ x \in A$" and call it the *definition of a function from a domain and a type-level term.* We treat A as a constant—it can also be a variable, but it must not appear in $\mathfrak{t}$ then.

Proof: We define $f := \{(x, \mathfrak{t}) \mid x \in A\} = \{p \mid (\exists x^1 \in A^2)(p^3 = (x^1, \mathfrak{t}^1)^3)\}$. Since $\mathfrak{t}$ is type-level, this is well-defined, and the existence (and sethood) of f follows from the axiom of stratified comprehension. The uniqueness follows from extensionality. □

In a completely analogous way, we can prove the following.

Corollary 3.13. Let $\mathfrak{t}$ be a term with variables x and y free, such that in every stratified formula in the extended language where $\mathfrak{t}$ appears, $type(\mathfrak{t}) = type((x, y))$. Then for every two sets A and B, there is a function f such that

$$f(x, y) = \mathfrak{t}, \qquad \text{for every } x \in A \text{ and } y \in B. \tag{3.2}$$

Furthermore, f is unique if we require additionally that $dom(f) = A \times B$.

Besides the infinitude of V, stated in the axiom of infinity, we can now prove the infinitude of another set.

Theorem 3.14. *The set* $\mathbb{N}$ *is Dedekind-infinite.*

Proof: Since $succ(n)$ is type-level in n, by theorem 3.12 there is a function s such that $s(n) := succ(n),\ n \in \mathbb{N}$. By second Peano axiom s is a function from $\mathbb{N}$ to $\mathbb{N}$, and it is an injection because of fourth Peano axiom. Therefore, $bij\big(s, \mathbb{N}, rng(s)\big)$, and from third and first Peano axiom, $rng(s) \subseteq \mathbb{N} \setminus \{0\} \subset \mathbb{N}$. □

The fact that $\mathbb{N}$ is infinite is an easy consequence of theorems 3.8 and 3.14. The cardinal number of natural numbers is $\aleph_0 := |\mathbb{N}|$. We say that a set X is **countable** if $|X| = \aleph_0$.

From theorems 3.10 and 3.11 we can define the sum and product of natural numbers. Note that *succ* can be viewed as a function on the set of natural numbers, which we denote by s as in the proof of theorem 3.14. For an arbitrary $m \in \mathbb{N}$ by theorem 3.11 there exists a function s_m such that $s_m(0) = m$ and $s_m\big(succ(n)\big) = s\big(s_m(n)\big)$ for every $n \in \mathbb{N}$. Since s_m has the same type as s, it is easy to see that if m has a type k, then s_m has a type $k+3$, because m plays the role of x_0 from theorem 3.11. We now define the **sum of natural numbers** as a function term $add(m,n) := s_m(n)$. Obviously, if m and n have type k, then $s_m(n)$ has type k. We use the usual notation for summation $m + n := add(m,n)$.

The product of natural numbers is defined in the same way. For arbitrary $m \in \mathbb{N}$ by theorem 3.11 there exists a function p_m such that $p_m(0) = 0$ and $p_m\big(succ(n)\big) = s_m\big(p(n)\big)$. We now define the **product of natural numbers** $n, m \in \mathbb{N}$ as a function term $mul(m,n) := p_m(n)$. Similarly (since s_m plays the role of f from theorem 3.11), if m and n have type k, then $mul(m,n)$ has a type k. We use the usual notation for multiplication $m \cdot n := p_m(n)$. It is easy to prove by induction the usual properties of addition and multiplication.

Lemma 3.15. *If X is a finite set, and $Y \subseteq X$, then Y is finite.*

Proof: Let X be an arbitrary finite set, that is, $X \in FIN$. That means there exists $n \in \mathbb{N}$ such that $X \in n$. It is enough to prove the formula $(\forall n^2 \in \mathbb{N}^3)(\forall X^1 \in n^2)\forall Y^1(Y^1 \subseteq X^1 \rightarrow Y^1 \in FIN^2)$, which is stratified, so we can prove it by induction on n.

If $n = 0$, we have $X \in 0$. That implies $X = \emptyset$. Now for $Y \subseteq \emptyset$ we have $Y = \emptyset \in 0 \subseteq FIN$. Let us assume the claim holds for some n and prove it for $succ(n)$.

Let $X \in succ(n)$ and $Y \subseteq X$. That means $X = x \cup \{z\}$, for some $x \in n$ and $z \notin x$. If $z \notin Y$, then $Y \subseteq x \in n$, and from the induction assumption we have $Y \in FIN$. If $z \in Y$, then $Y \setminus \{z\} \subseteq x$, so from the induction assumption we have $Y \setminus \{z\} \in FIN$, that is, there exists some $k \in \mathbb{N}$ such that $Y \setminus \{z\} \in k$. Because $z \notin Y \setminus \{z\}$, we have $Y = Y \setminus \{z\} \cup \{z\} \in succ(k) \subseteq FIN$ by definition of successor. □

Lemma 3.16. *The following statements hold:*

1. *For all $n \in \mathbb{N}$, if $x \in succ(n)$ and $y \in x$, then $x \setminus \{y\} \in n$.*
2. *There is no natural number n such that $n < 0$.*

3. *For all* $n \in \mathbb{N}$ *we have* $n < succ(n)$.
4. *For every* $n, m \in \mathbb{N}$, $m \leq n$ *if and only if* $m < succ(n)$.
5. *Every nonempty partial ordered finite set has a maximal element.*
6. *For all* $n \in \mathbb{N}$ *we have* $n < \aleph_0$.
7. *For* $n, m \in \mathbb{N}$, *if* $n < m$, *then* $succ(n) < succ(m)$.
8. *For* $n, m \in \mathbb{N}$, *if* $n < m$, *then for every* $x \in \mathbb{N}$, $n + x < m + x$.
9. *For* $n, m \in \mathbb{N}$, *if* $n < m$, *then for every* $x \in \mathbb{N}$, $n \cdot x \leq m \cdot x$.

PROOF:

1. Let $n \in \mathbb{N}$ be arbitrary, $x \in succ(n)$ and $y \in x$. Because $x \in succ(n)$, there exists $z \in x$ such that $x \setminus \{z\} \in n$. Then one bijection between $x \setminus \{y\}$ and $x \setminus \{z\}$ is given by $id_{x \setminus \{y\}}$ if $y = z$, and by $id_{x \setminus \{y,z\}} \cup \{(z, y)\}$ otherwise.

2. Assume the contrary, that there exists $n \in \mathbb{N}$ such that $n \leq 0 \wedge n \neq 0$. Let $A \in n$ and $B \in 0$ be arbitrary. By definition of relation $\leq$, there exists an injection from A to B. However, $B \in 0$ means $B = \emptyset$, therefore, that injection must be empty, hence $A = \emptyset$. That is impossible because $n \neq 0$.

3. Let $A \in n \in \mathbb{N}$ be arbitrary. Then $A \neq V$, so there exists $x \in V$ such that $x \notin A$. Now by the characterization of a successor, we have $A \cup \{x\} \in succ(n)$. Obviously $inj(id_A, A, A \cup \{x\})$, so $n \leq succ(n)$. Assume that $n = succ(n)$. Then there exists a bijection $f \colon A \to A \cup \{x\}$. Obviously $A \subset A \cup \{x\}$, and by theorem 3.8 we have that $A \cup \{x\}$ is infinite, therefore $A \cup \{x\} \notin FIN$. But we also have $A \cup \{x\} \in succ(n) \subseteq FIN$, which means that the assumption is wrong, therefore $n < succ(n)$.

4. Let n and m be arbitrary. Assume $m \leq n$. From 3.16(3) we have $n \leq succ(n)$ and $n \neq succ(n)$. From transitivity of relation $\leq$ we get $m \leq succ(n)$. Assume $m = succ(n)$. Then we have $succ(n) \leq n$ and $n \leq succ(n)$, which gives $n = succ(n)$. That is a contradiction with $n \neq succ(n)$, so $m \leq succ(n)$ and $m \neq succ(n)$, that is, $m < succ(n)$.

   Assume $m < succ(n)$. Let $A \in m$ and $B \in succ(n)$ be arbitrary. By the assumption, there exists an injection $f \colon A \to B$, which is not a bijection. That means there exists $b \in B$ such that $b \notin \mathit{rng}(f)$. That implies f is also an injection from A to $B \setminus \{b\}$, and because lemma 3.16(1) implies $B \setminus \{b\} \in n$, we get $m \leq n$.

5. We need to prove

$$(\forall X \in \mathit{FIN} \setminus \{\emptyset\}) \forall R \big(\mathit{Po}(R, X) \rightarrow \\ \rightarrow (\exists x_0 \in X)(\forall y \in X \setminus \{x_0\})(x_0 \not R\, y)\big),$$

which follows from the stratified formula

$$(\forall n^3 \in \mathbb{N}^4 \setminus \{0^3\}^4)(\forall X^2 \in n^3) \forall R^4 \big(\mathit{Po}(R^4, X^2) \rightarrow \\ \rightarrow (\exists x_0^1 \in X^2)(\forall y^1 \in X^2 \setminus \{x_0^1\}^2)(x_0^1 \not R^4 y^1)\big).$$

We will prove it by induction. Let $n = 1$, $X \in 1$ be arbitrary and R be a partial order on X. From $X \in 1 = \mathit{succ}(0)$ we get that there exists z_0 such that $X = \{z_0\}$, and then $R = \{(z_0, z_0)\}$, so z_0 is a maximal element of X under relation R. Assume the claim holds for a natural number $n \geq 1$, and prove it for $\mathit{succ}(n)$.

Let $X \in \mathit{succ}(n)$ be arbitrary, and R be a partial order on X. From characterization of successor, we have $X = x \cup \{y\}$, where $x \in n$ and $y \notin x$. Since R is a partial order, $R' := R \cap (x \times x)$ is partial order on x. From the induction hypothesis, we have that there exists a maximal element z_0 of x under relation R'. If $z_0\ R\ y$, then y must be a maximal element of X under R. For if there existed some $w_0 \neq y$ such that $y\ R\ w_0$, then from $z_0\ R\ y$ and transitivity of R, we would have $z_0\ R\ w_0$. Since $w_0 \neq y$, we have $w_0 \in x$, which is a contradiction with maximality of z_0 in x. If $y\ R\ z_0$, or z_0 and y are not comparable, then z_0 is a maximal element of X under R.

6. Let $n \in \mathbb{N}$ be arbitrary. Assume $n \geq \aleph_0$. By definition of $\leq$, there exists and injection $f \colon \mathbb{N} \rightarrow A$, where $A \in n$. That means $\mathbb{N} \sim \mathit{rng}(f)$, and also $\mathit{rng}(f) \subseteq A$. Because A is finite, by theorem 3.15 we have $\mathit{rng}(f)$ finite, which implies that $\mathbb{N}$ is finite, which is a contradiction. Therefore, $n < \aleph_0$.

7. Take $n, m \in \mathbb{N}$ such that $n < m$. Assume $\mathit{succ}(m) \leq \mathit{succ}(n)$. By (3) we have $m < \mathit{succ}(m) \leq \mathit{succ}(n)$, and from (4) we get $m \leq n$, which is a contradiction.

8. Take arbitrary $n, m \in \mathbb{N}$. Formula $(\forall x^1 \in \mathbb{N}^2)(n^1 <^4 m^1 \to (m^1 + x^1)^1 <^4 (n^1 + x^1)^1)$ is stratified, so we can prove it by induction on x. If $x = 0$, then from $n < m$ we get $n + x = n + 0 = n < m = m + x$. Assume that the claim holds for some $x \in \mathbb{N}$, and let us prove it for $succ(x)$. If $n < m$, then from the associativity we get $n + succ(x) = succ(n + x)$, and by the induction assumption, we have $n + x < m + x$. Now from (7) we get $succ(n + x) < succ(m + x)$. Therefore, $n + succ(x) \leq succ(m + x) = m + succ(x)$.

9. Take arbitrary $n, m \in \mathbb{N}$. Formula $(\forall x^1 \in \mathbb{N}^2)\big(n^1 <^4 m^1 \to (n^1 \cdot x^1)^1 \leq^4 (m^1 \cdot x^1)^1\big)$ is stratified, so we can prove it by induction on x. If $x = 0$, then the claim trivially holds. Let us assume the claim for some $x \in \mathbb{N}$, and prove it for $succ(x)$. If $n < m$, then from the induction assumption, (8) and commutativity of addition, we get $n \cdot succ(x) = n \cdot x + n \leq m \cdot x + m = m \cdot succ(x)$. □

Remark 3.17. It is useful to note that if the partial order in lemma 3.16(5) is a well-order, then a maximal element is also the greatest element.

Definition 3.18. For every $n \in \mathbb{N}$ we define its initial segment as

$$A_n := \{m \in \mathbb{N} \mid m < n\}.$$

Note that if n has type s, then A_n has type $s + 1$.

Lemma 3.19.

1. *For every $n \in \mathbb{N}$, the set A_n is finite.*

2. *If X is a set of initial segments of natural numbers and $\bigcup X \subset \mathbb{N}$, then $\bigcup X$ is an initial segment of natural numbers.*

Proof:

1. Formula $(\forall n^1 \in \mathbb{N}^2)(A^2_{n^1} \in FIN^3)$ is stratified, so we can perform mathematical induction on n. For $n = 0$ we have $A_0 = \{m \in \mathbb{N} \mid m < 0\} = \emptyset \in 0 \subseteq FIN$. Assume that for some $n \in \mathbb{N}$, set A_n is finite, and let us prove the statement for $A_{succ(n)}$. Because $A_n \in FIN$, there exists $k \in \mathbb{N}$ such that $A_n \in k$ and there exists $x \notin A_n$ such that $A_n \cup \{x\} \in succ(k)$. From lemma 3.16(4) we have $A_{succ(n)} = A_n \cup \{n\}$,

therefore $id_{A_n} \cup \{(x, n)\}$ is a bijection between $A_n \cup \{x\}$ and $A_n \cup \{n\}$, which means $A_n \cup \{n\} = A_{n+1} \in succ(k) \subseteq FIN$.

2. Assume that X is a set of initial segments of natural numbers such that $\bigcup X \subset \mathbb{N}$. If $X = \{A_0\}$, then obviously $\bigcup X = A_0$, so assume $X \neq \{A_0\}$. The set $\bigcup X$ is a proper subset of $\mathbb{N}$, so there is an $m \in \mathbb{N}$ such that $m \notin \bigcup X$. Then $\bigcup X \subseteq A_m$, so by (1) and lemma 3.15, $\bigcup X$ is a nonempty finite subset of $\mathbb{N}$. Therefore, by remark 3.17, it has the greatest element r. Then for every $x \in \bigcup X$, $x \leq r$, therefore by lemma 3.16(4) $x < succ(r)$. So, $\bigcup X \subseteq A_{succ(r)}$. For the opposite inclusion, suppose $x \in A_{succ(r)}$. Since $r \in \bigcup X$, there is $A_i \in X$ such that $r \in A_i$. Then $x \leq r < i$ implies $x \in A_i \subseteq \bigcup X$. □

The following theorem is very important for accomplishing our goal.

THEOREM 3.20. *Every infinite set has a countable subset.*

PROOF: Let X be an infinite set. We will prove that there is an injection from $\mathbb{N}$ to X.

Formula $(\exists n^1 \in \mathbb{N}^2) inj(f^4, A^2_{n^1}, X^2) \vee inj(f^4, \mathbb{N}^2, X^2)$ is stratified, so we can define a set $K := \{f \mid (\exists n \in \mathbb{N}) inj(f, A_n, X) \vee inj(f, \mathbb{N}, X)\}$. Set K is nonempty because for $n = 0$ we have $A_0 = \emptyset$ by lemma 3.16(2), which means $inj(\emptyset, A_0, X)$. We order K by inclusion and prove that it satisfies the remaining condition of Zorn's lemma.

Let $C \subseteq K$ be an arbitrary nonempty chain. We need to prove $\bigcup C \in K$. From lemma 2.13 we get that $\bigcup C$ is an injection, $rng(\bigcup C) \subseteq X$, and $dom(\bigcup C) \subseteq \mathbb{N}$. If $dom(\bigcup C) \neq \mathbb{N}$, then since the domain of every element of C is an initial segment of natural numbers, from lemma 2.13 and lemma 3.19(2) there exists some $n_0 \in \mathbb{N}$ such that $dom(\bigcup C) = A_{n_0}$, which implies $\bigcup C \in K$. If $dom(\bigcup C) = \mathbb{N}$, then obviously $\bigcup C \in K$. Now from Zorn's lemma, there exists a maximal element of K, which we denote by f_0.

If $dom(f_0) \neq \mathbb{N}$, then there exists $n \in \mathbb{N}$ such that $dom(f_0) = A_n$. If $rng(f_0) = X$, we have $bij(f_0, A_n, X)$, which is a contradiction because A_n is finite by lemma 3.19(1) and X is infinite by assumption. If $rng(f_0) \neq X$, then there exists $x \in X \setminus rng(f_0)$. Define the function $F := f_0 \cup \{(n, x)\}$. Obviously $f_0 \subset F \in K$, which is a contradiction with the maximality of f_0. Therefore, $dom(f_0) = \mathbb{N}$. So, we have $inj(f_0, \mathbb{N}, X)$, and $rng(f) \sim \mathbb{N}$ is a desired countable subset of X. □

THEOREM 3.21. *If $X \subseteq \mathbb{N}$ is an infinite set, then X is countable.*

PROOF: From $X \subseteq \mathbb{N}$ we have $|X| \leq \aleph_0$. On the other hand, X is an infinite set so by theorem 3.20 there exists $X_0 \subseteq X$ such that $|X_0| = \aleph_0$. But now we have $\aleph_0 = |X_0| \leq |X|$, and because $\leq$ is antisymmetric, that means $|X| = \aleph_0$. □

It is now easy to prove that infinity implies Dedekind-infinity, but this result is not needed for our purposes.

4. The cardinal squaring principle

DEFINITION 4.1. Let κ and λ be cardinal numbers. We define their **level sum** as $\kappa +_L \lambda := \{z \mid (\exists x \in \kappa)(\exists y \in \lambda)(x \times \{0\} \cup y \times \{1\} \sim z \times \{2\})\}$ and their **level product** as $\kappa \cdot_L \lambda := \{z \mid (\exists x \in \kappa)(\exists y \in \lambda)(x \times y \sim z \times \{2\})\}$.

These two operations are defined in such a way as to assure that their types are the same as the types of their operands.

Remark 4.2. **However**, they do not necessarily capture what we expect of the sum and product of cardinal numbers. More precisely, their results don't have to be cardinal numbers. In order for $\kappa \cdot_L \lambda$ to be a cardinal number, it must be nonempty, therefore there must exist $x \in \kappa$, $y \in \lambda$ and z such that $x \times y \sim z \times \{2\}$. But if in particular $\kappa := \lambda := |V|$, then we must have

$$V \times V \sim x \times y \sim z \times \{2\} \subseteq V \times \{2\} \subseteq V \times V,$$

and therefore by Cantor–Bernstein's theorem $V \times V \sim V \times \{2\}$, which is equivalent to VCSP. The other direction is even easier: if VCSP holds, then for every two cardinals κ and λ, for every $x \in \kappa$ and $y \in \lambda$, we can restrict the bijection between $V \times V$ and $\mathscr{P}_1^2(V)$ to $x \times y$, and its image obviously must be of the form $\mathscr{P}_1^2(z)$ for some z. The same bijection can also be restricted to $x \times \{0\} \cup y \times \{1\}$, giving us the nonemptiness of $\kappa + \lambda$.

So, definition 4.1 really defines binary operations on *Card* if and only if VCSP holds. While itself a good motivation for the inclusion of VCSP as an axiom, this argument shows that we must define the cardinal sum and product differently in order to be able to prove VCSP. We will define the aforementioned operations in the usual way, but some claims will then be stated with type raising operation T.

DEFINITION 4.3. For every $\kappa = |x| \in \mathit{Card}$, we define $T(\kappa) := |\mathscr{P}_1(x)|$.

It is important to note that $T(\kappa)$ for a cardinal number κ does not depend on the representative $x \in \kappa$. It is immediate from the definition that if κ has type n, then $T(\kappa)$ has type $n+1$. We also define $T^0(\kappa) := \kappa$ and $T^{k+1}(\kappa) = T\big(T^k(\kappa)\big)$. It easily follows from the definition that if $T(\kappa) = T(\lambda)$ for some cardinal numbers κ and λ, then $\kappa = \lambda$.

In addition, we introduce the symbol for singleton ι with $\iota^0(x) := x$ and $\iota^{k+1}(x) := \{\iota^k(x)\}$. Obviously, if x has type n, then $\iota^k(x)$ has type $n+k$.

DEFINITION 4.4. For cardinal numbers κ and λ, we define their **outer sum** and **outer product** as

$$\kappa \oplus \lambda := \big\{z \;\big|\; (\exists x \in \kappa)(\exists y \in \lambda)(z \sim x \times \{0\} \cup y \times \{1\})\big\},$$
$$\kappa \odot \lambda := \{z \mid (\exists x \in \kappa)(\exists y \in \lambda)(z \sim x \times y)\}.$$

If κ and λ have type n, then $\kappa \oplus \lambda$ and $\kappa \odot \lambda$ have type $n+2$.

It is easy to see that the outer sum and the outer product are commutative. However, for $n, m \in \mathbb{N} \subseteq \mathit{Card}$, $n+m$ and $n \cdot m$ are generally not the same objects as $n \oplus m$ and $n \odot m$ respectively.

THEOREM 4.5. *Let X be a set and $A \subseteq X$. Then $|X \setminus A| \oplus |A| = T^2(|X|)$.*

PROOF: We need to prove $(X \setminus A) \times \{0\} \cup A \times \{1\} \sim \mathscr{P}_1^2(X)$. Since $\iota^2(x)$ is type-level with $(x,0)$ and $\iota^2(y)$ is type-level with $(y,1)$, by corollary 3.13 we can define functions $h_1(x,0) := \iota^2(x), x \in X \setminus A$ and $h_2(y,1) := \iota^2(y), y \in A$. Then obviously $\mathit{bij}\big(h_1 \cup h_2, (X \setminus A) \times \{0\} \cup A \times \{1\}, \mathscr{P}_1^2(X)\big)$. $\square$

THEOREM 4.6. *For every infinite cardinal number κ and for every natural number n we have $\kappa \oplus n = T^2(\kappa)$.*

PROOF: Let $\kappa = |X|$ and $n = |A|$. We need to prove $X \times \{0\} \cup A \times \{1\} \sim \mathscr{P}_1^2(X)$. If $n = 0$, then $A = \emptyset = A \times \{1\}$. Therefore, we need to prove $X \times \{0\} \sim \mathscr{P}_1^2(X)$, and one bijection is $(x,0) \mapsto \iota^2(x)$.

Let $n \neq 0$. Then from theorem 3.20 there exists an injection $f\colon \mathbb{N} \to X$ and by theorem 3.16(6) there exists an injection $g\colon A \to \mathbb{N}$. By assumption, A is a finite set, therefore $\mathit{rng}(g)$ is finite (and nonempty). From theorem 3.16(5) it follows that $\mathit{rng}(g)$ has the greatest element a_0.

By application of corollary 3.13, we can define the following functions:

$$h_1(x,0) := \iota^2\big(f\big(f^{-1}(x) + a_0 + 1\big)\big),\ x \in rng(f)$$
$$h_2(x,0) := \iota^2(x),\ x \in X \setminus rng(f)$$
$$h_3(a,1) := \iota^2\big(f(g(a))\big),\ a \in A.$$

Then $bij(h_1 \cup h_2 \cup h_3, X \times \{0\} \cup A \times \{1\}, \mathscr{P}_1^2(X))$ can be proved by cases, and that means we have $\kappa \oplus n = T^2(\kappa)$. □

THEOREM 4.7. $\aleph_0 \oplus \aleph_0 = T^2(\aleph_0)$.

PROOF: We need to prove $\mathbb{N} \times \{0\} \cup \mathbb{N} \times \{1\} \sim \mathscr{P}_1^2(\mathbb{N})$. By corollary 3.13 we can define functions $f_1(n,0) := \iota^2(2 \cdot n), n \in \mathbb{N}$ and $f_2(n,1) := \iota^2(2 \cdot n + 1), n \in \mathbb{N}$. Obviously, $bij\big(f_1 \cup f_2, \mathbb{N} \times \{0\} \cup \mathbb{N} \times \{1\}, \mathscr{P}_1^2(\mathbb{N})\big)$. □

THEOREM 4.8. *For every infinite cardinal* κ *we have* $\kappa \oplus \kappa = T^2(\kappa)$.

PROOF: Let $\kappa = |X|$. We need to prove $X \times \{0\} \cup X \times \{1\} \sim \mathscr{P}_1^2(X)$. Formula $\exists Y^2\big(Y^2 \subseteq X^2 \wedge Y^2 \notin FIN^3 \wedge bij(f^6, (Y^2 \times \{0^1\}^2)^4 \cup (Y^2 \times \{1^1\}^2)^4, \mathscr{P}_1^2(Y^2)^4)\big)$ is stratified so we define the set $K := \big\{f \mid \exists Y\big(Y \subseteq X \wedge Y \notin FIN \wedge bij(f, Y \times \{0\} \cup Y \times \{1\}, \mathscr{P}_1^2(Y))\big)\big\}$, which we order by inclusion. Because X is infinite, by theorem 3.20 there exists a countable $X_0 \subseteq X$. Now from theorem 4.7 we get $X_0 \times \{0\} \cup X_0 \times \{1\} \sim \mathscr{P}_1^2(X_0)$, so there exists a bijection $f_0 \colon X_0 \times \{0\} \cup X_0 \times \mathscr{P}_1^2(X_0)$, which means $f_0 \in K$, so K is nonempty. Let C be an arbitrary nonempty chain in K. By lemma 2.13 we get that $\bigcup C$ is an injection. We need to prove $\bigcup C \in K$.

Formula $(\exists f^5 \in C^6)\big((\bigcup\bigcup rng(f^5)^3)^1 = z^1\big)$ is stratified, so we define the set $S := \big\{z \mid (\exists f \in C)\big(\bigcup\bigcup rng(f) = z\big)\big\}$. Application of the double union on the set $rng(f)$ is necessary to get rid of double $\mathscr{P}_1$. We claim $\bigcup S \subseteq X$, $\bigcup S$ is infinite, and $rng\big(\bigcup C\big) = \mathscr{P}_1^2\big(\bigcup S\big)$.

Let us prove $\bigcup S \subseteq X$. Let $z \in \bigcup S$. Then there exists $t \in S$ such that $z \in t$. There exists $f \in C$ such that $t = \bigcup\bigcup rng(f)$ and $z \in t$. From $rng(f) \subseteq \mathscr{P}_1^2(X)$, we have $rng(f) = \mathscr{P}_1^2(A)$ for some infinite $A \subseteq X$. But then $\mathscr{P}_1^2(t) = \mathscr{P}_1^2\big(\bigcup\bigcup rng(f)\big) = \mathscr{P}_1^2\big(\bigcup\bigcup \mathscr{P}_1^2(A)\big) = \mathscr{P}_1^2(A) = rng(f)$. Therefore, $t \subseteq X$, which implies $z \in X$.

Let us prove that $\bigcup S$ is infinite. Assume the contrary, that it is finite. Let us fix $f \in C$. Then $rng(f) = \mathscr{P}_1^2(A)$ for some infinite $A \subseteq X$. Now we have $A = \bigcup\bigcup \mathscr{P}_1^2(A) = \bigcup\bigcup rng(f) \subseteq \bigcup S$. From lemma 3.15 we get that A is finite, which is a contradiction.

Let us prove $rng\big(\bigcup C\big) = \mathscr{P}_1^2\big(\bigcup S\big)$. If $z \in rng\big(\bigcup C\big)$, then by lemma 2.13 there exists $f \in C$ such that $z \in rng(f)$. That means that there exists infinite $A \subseteq X$ such that $z \in rng(f) = \mathscr{P}_1^2(A)$. Then there exists $a \in A$ such that $z = \iota^2(a)$. We have $\bigcup\bigcup \iota^2(a) = a \in A = \bigcup\bigcup \mathscr{P}_1^2(A) = \bigcup\bigcup rng(f)$, so $A \in S$, and from that we get $a \in \bigcup S$. Now we have $z = \iota^2(a) \in \mathscr{P}_1^2\big(\bigcup S\big)$.

If $z \in \mathscr{P}_1^2\big(\bigcup S\big)$, then there exists $b \in \bigcup S$ such that $z = \iota^2(b)$. That means there exists $B \in S$ such that $b \in B$. That implies there exists $f \in C$ such that $\bigcup\bigcup rng(f) = B$ and $b \in B$. We know that $rng(f) = \mathscr{P}_1^2(A)$ for some infinite $A \subseteq X$. From that we get $z = \iota^2(b) \in \mathscr{P}_1^2(B) = \mathscr{P}_1^2\big(\bigcup\bigcup rng(f)\big) = \mathscr{P}_1^2\big(\bigcup\bigcup \mathscr{P}_1^2(A)\big) = \mathscr{P}_1^2(A) = rng(f) \subseteq rng\big(\bigcup C\big)$. So, there exists infinite $Z := \bigcup S \subseteq X$ such that $rng\big(\bigcup C\big) = \mathscr{P}_1^2(Z)$.

It remains to prove $dom\big(\bigcup C\big) = Z \times \{0\} \cup Z \times \{1\}$. Let $z \in dom\big(\bigcup C\big)$, then by lemma 2.13 there exists $f \in C$ such that $z \in dom(f) = T \times \{0\} \cup T \times \{1\}$, for some infinite $T \subseteq X$. Because $f \subseteq \bigcup C$, we have $\mathscr{P}_1^2(T) = rng(f) \subseteq rng\big(\bigcup C\big) = \mathscr{P}_1^2(Z)$, which implies $T \subseteq Z$, that is, $T \times \{0\} \cup T \times \{1\} \subseteq Z \times \{0\} \cup Z \times \{1\}$. Therefore, $z \in Z \times \{0\} \cup Z \times \{1\}$, that is, $dom\big(\bigcup C\big) \subseteq Z \times \{0\} \cup Z \times \{1\}$.

If $z \in Z \times \{0\} \cup Z \times \{1\}$, then $z = (a, s)$, where $a \in Z$ and $s \in \{0, 1\}$. Then $\iota^2(a) \in \mathscr{P}_1^2(Z) = rng\big(\bigcup C\big)$, which means there exists $f \in C$ such that $\iota^2(a) \in rng(f) = \mathscr{P}_1^2(U)$, for infinite $U \subseteq X$. Then we have $a \in U$, which implies $z = (a, s) \in U \times \{0\} \cup U \times \{1\} = dom(f)$, so $z \in dom\big(\bigcup C\big)$.

Finally, we can conclude $\bigcup C \in K$, and then by Zorn's lemma, there exists a maximal element of K. Denote it by $f_0 \colon A_0 \times \{0\} \cup A_0 \times \{1\} \to \mathscr{P}_1^2(A_0)$, where $A_0 \subseteq X$ is infinite. We want to prove $|X| = |A_0|$.

By theorem 4.5 we have $T^2(|X|) = |X \setminus A_0| \oplus |A_0|$. We claim that $X \setminus A_0$ is finite. Assume the contrary, that it is infinite. Then there exists a countable set $B \subseteq X \setminus A_0$, which implies $A_0 \subsetneq A_0 \cup B \subseteq X$. Because B is countable, by theorem 4.7 we get $B \times \{0\} \cup B \times \{1\} \sim \mathscr{P}_1^2(B)$, so there exists a bijection $g_0 \colon B \times \{0\} \cup B \times \{1\} \to \mathscr{P}_1^2(B)$. Obviously, $bij\big(f_0 \cup g_0, (A_0 \cup B) \times \{0\} \cup (A_0 \cup B) \times \{1\}, \mathscr{P}_1^2(A_0 \cup B)\big)$. But now we have $f_0 \subset f_0 \cup g_0$, which is a contradiction with the maximality of f_0. Therefore, $X \setminus A_0$ is finite. Now from theorem 4.6 we get $T^2(|X|) = |X \setminus A_0| \oplus |A_0| = T^2(|A_0|)$, that is, $|X| = |A_0|$. $\square$

THEOREM 4.9. *Let κ be an infinite cardinal and $\lambda \leq \kappa$. Then $\kappa \oplus \lambda = \lambda \oplus \kappa = T^2(\kappa)$.*

PROOF: Let $A \in \kappa$ and $B \in \lambda$. We need to prove $A \times \{0\} \cup B \times \{1\} \sim \mathscr{P}_1^2(A)$. From $\lambda \leq \kappa$ we have an injection $f\colon B \to A$. Denote $X := rng(f)$, which is obviously equipotent with B. By theorem 4.8 we have $\mathscr{P}_1^2(X) \sim X \times \{0\} \cup X \times \{1\} \sim X \times \{0\} \cup B \times \{1\}$. Then $A \times \{0\} \cup B \times \{1\} = (A \setminus X \cup X) \times \{0\} \cup B \times \{1\} = (A \setminus X) \times \{0\} \cup (X \times \{0\} \cup B \times \{1\}) \sim \mathscr{P}_1^2(A \setminus X) \cup \mathscr{P}_1^2(X) \sim \mathscr{P}_1^2(A \setminus X \cup X) = \mathscr{P}_1^2(A)$. □

LEMMA 4.10. *For any family of finitely many equipotent infinite sets, their union is also equipotent with each of them.*

PROOF: Denote the number of sets with n. The claim is trivial for $n = 0$ and $n = 1$. It is enough to prove the claim for $n = 2$; then the claim for $n \geq 3$ follows by induction.

Let $A \sim B$ be arbitrary sets and define $C := A \setminus B$. Then $C \subseteq A$, which means $inj(id_C, C, A)$, so $|C| \leq |A|$. By theorem 4.9, $|A| \oplus |C| = T^2(|A|)$. On the other hand, for $U := A \cup B = C \cup B$ we have $U \setminus C = B$, so by theorem 4.5, we have $|A| \oplus |C| = |B| \oplus |C| = |U \setminus C| \oplus |C| = T^2(|U|)$. From these two facts, $|A| = |U|$ follows. □

THEOREM 4.11. $\aleph_0 \odot \aleph_0 = T^2(\aleph_0)$.

PROOF: Formula $(\exists n^1 \in \mathbb{N}^2)\big(a^3 = \iota^2(n^1)^3 \wedge b^3 = (n^1, n^1)^3 \wedge t^5 = (a^3, b^3)^5\big)$ is stratified, so we can define a relation $g := \big\{\big(\iota^2(n), (n, n)\big) \;\big|\; n \in \mathbb{N}\big\}$. Then $inj\big(g, \mathscr{P}_1^2(\mathbb{N}), \mathbb{N} \times \mathbb{N}\big)$, which implies $T^2(\aleph_0) \leq \aleph_0 \odot \aleph_0$.

By corollary 3.13 we can define a function $f(m, n) := \iota^2\big((m+n) \cdot (m+n) + m\big)$ for every $m, n \in \mathbb{N}$. We need to prove that f is an injection. Let $n, m, a, b \in \mathbb{N}$ be such that $(m, n) \neq (a, b)$.

The first case is when $m + n \neq a + b$, without the loss of generality $m + n < a + b$. Then $m + n + 1 = succ(m + n) \leq a + b$. So we have

$$\begin{aligned}(m+n)\cdot(m+n)+m &\leq (m+n)\cdot(m+n)+m+n+m+n < \\ &< succ\big((m+n)\cdot(m+n+2)\big) = (m+n+1)\cdot(m+n+1) \leq \\ &\leq (a+b)\cdot(m+n+1) \leq (a+b)\cdot(a+b) \leq (a+b)\cdot(a+b)+a,\end{aligned}$$

which implies $f(m, n) \neq f(a, b)$.

The second case is when $m+n = a+b$, and then obviously $m \neq a$, without the loss of generality $m < a$. Then we have $(m+n)\cdot(m+n)+m = (a+b)\cdot(a+b)+m < (a+b)+a$, and also $f(m,n) \neq f(a,b)$.

Therefore, $f\colon \mathbb{N}\times\mathbb{N} \to \mathscr{P}_1^2(\mathbb{N})$ is an injection, so we have $\aleph_0 \times \aleph_0 \leq T^2(\aleph_0)$. By Cantor–Bernstein's theorem we get $\aleph_0 \times \aleph_0 = T^2(\aleph_0)$. $\square$

THEOREM 4.12. *For every infinite cardinal κ we have $\kappa \odot \kappa = T^2(\kappa)$.*

PROOF: Let $\kappa = |X|$. We need to prove $X \times X \sim \mathscr{P}_1^2(X)$. Formula $\exists Y^1\big(Y^1 \subseteq X^1 \wedge Y^1 \notin \mathit{FIN}^2 \wedge \mathit{bij}(f^5, (Y\times Y)^3), \mathscr{P}_1^2(Y)^3\big)$ is stratified, so we can define the set $K := \big\{ f \bigm| \exists Y\big(Y \subseteq X \wedge Y \notin \mathit{FIN} \wedge \mathit{bij}\big(f, Y\times Y, \mathscr{P}_1^2(Y)\big)\big)\big\}$, which we order by inclusion. By theorem 3.20 there exists a countable $X_0 \subseteq X$ and by theorem 4.11 we have $X_0 \times X_0 \sim \mathscr{P}_1^2(X_0)$, so there exists a bijection $f_0\colon X_0 \times X_0 \to \mathscr{P}_1^2(X_0)$, which means $f_0 \in K$, so K is nonempty. Let C be an arbitrary nonempty chain in K. By lemma 2.13 we get that $\bigcup C$ is an injection. We need to prove $\bigcup C \in K$.

We can prove analogously as in the proof of theorem 4.8 that there exists an infinite $Z \subseteq X$ such that $\mathit{rng}\big(\bigcup C\big) = \mathscr{P}_1^2(Z)$.

It remains to prove $\mathit{dom}\big(\bigcup C\big) = Z \times Z$. Let $z \in \mathit{dom}\big(\bigcup C\big)$, then by lemma 2.13 there exists $f \in C$ such that $z \in \mathit{dom}(f) = T \times T$, for some infinite $T \subseteq X$. Because $f \subseteq \bigcup C$, we have $\mathscr{P}_1^2(T) = \mathit{rng}(f) \subseteq \mathit{rng}\big(\bigcup C\big) = \mathscr{P}_1^2(Z)$, which implies $T \subseteq Z$, that is, $T\times T \subseteq Z \times Z$. Therefore, $z \in Z\times Z$ and then we have $\mathit{dom}\big(\bigcup C\big) \subseteq Z \times Z$.

If $z \in Z \times Z$, then $z = (u,w)$, where $u, w \in Z$. Then $\iota^2(u), \iota^2(w) \in \mathscr{P}_1^2(Z) = \mathit{rng}\big(\bigcup C\big)$, which means there exist $f_1, f_2 \in C$ such that $\iota^2(u) \in \mathit{rng}(f_1)$ and $\iota^2(w) \in \mathit{rng}(f_2)$. Because C is a chain, without the loss of generality we can assume $f_1 \subseteq f_2$, therefore, $\iota^2(u), \iota^2(w) \in \mathit{rng}(f_2) = \mathscr{P}_1^2(U)$ for some infinite $U \subseteq X$. Then we have $u, w \in U$, which implies $z = (u,w) \in U \times U = \mathit{dom}(f_2) \subseteq \mathit{dom}\big(\bigcup C\big)$, so $z \in \mathit{dom}\big(\bigcup C\big)$.

We can conclude $\bigcup C \in K$, and then by Zorn's lemma, there exists a maximal element of K. Denote it by $f_0\colon A_0 \times A_0 \to \mathscr{P}_1^2(A_0)$, where $A_0 \subseteq X$ is infinite. Then f_0 shows $\lambda \odot \lambda = T^2(\lambda)$, where $\lambda := |A_0|$. It remains to prove $|A_0| = |X|$.

From $A_0 \subseteq X$, we get $|A_0| \leq |X|$. Assume $|A_0| < |X|$. Because $\leq$ is well-order, either $|X \setminus A_0| \leq |A_0|$ or $|A_0| < |X \setminus A_0|$. If $|X \setminus A_0| \leq |A_0|$, by theorems 4.5 and 4.9 we have $T^2(|X|) = |X \setminus A_0| \oplus |A_0| = T^2(|A_0|)$, so we get $|X| = |A_0|$, a contradiction. Therefore, $|A_0| < |X \setminus A_0|$, so there exists an injection from A_0 to $X \setminus A_0$, which is not a bijection; hence there exists $Z \subset X \setminus A_0$ such that $|Z| = |A_0| = \lambda$.

By distributivity we have $(A_0 \cup Z) \times (A_0 \cup Z) = (A_0 \times A_0) \cup (A_0 \times Z) \cup (Z \times A_0) \cup (Z \times Z)$. Now from $A_0 \sim Z$ we get $A_0 \times Z \sim Z \times A_0 \sim Z \times Z$, and then from lemma 4.10

$$(A_0 \times Z) \cup (Z \times A_0) \cup (Z \times Z) \sim Z \times Z \sim \mathscr{P}_1^2(Z).$$

Therefore, there exists a bijection $g\colon (A_0 \times Z) \cup (Z \times A_0) \cup (Z \times Z) \to \mathscr{P}_1^2(Z)$.

Define $h := (f_0 \cup g) : (A_0 \cup Z) \times (A_0 \cup Z) \to \mathscr{P}_1^2(A_0 \cup Z)$. Since $A_0 \cap Z = \emptyset$, h is a bijection such that $f_0 \subseteq h$, and $h \in K$, because $A_0 \cup Z$ is an infinite subset of X.

We also have $f_0 \neq h$ because for any $z \in Z \neq \emptyset$, $\big((z, z), \iota^2(z)\big) \in h \setminus f_0$, since $Z \subseteq X \setminus A_0$. Now we have $f_0 \subset h \in K$, a contradiction with the maximality of f_0.

Therefore, the assumption $|A_0| < |X|$ was wrong, which implies $|X| \leq |A_0|$, so $\lambda = |A_0| = |X| = \kappa$. Now $\kappa \odot \kappa = \lambda \odot \lambda = T^2(\lambda) = T^2(\kappa)$. □

Remark 4.13. The proofs of theorems about cardinal arithmetic are good examples of why working with Kuratowski's ordered pair (or any other pairs that are not type-level) is tedious. Even the statements of theorems must be modified in order to accommodate this. Using type-level ordered pairs greatly reduces the complexity of said proofs.

Theorem 4.14. *In* $\mathsf{NFU} + \mathsf{Inf} + \mathsf{AC}$ *there exist type-level ordered pairs.*

Proof: Denote the cardinal number of the universe as $|V| =: \kappa$. We know from the axiom of infinity that V is an infinite set, so κ is an infinite cardinal number. From theorem 4.12 we have $\kappa \odot \kappa = T^2(\kappa)$, which means there is a bijection $F\colon V \times V \to \mathscr{P}_1^2(V)$.

Formula $F^6\big((x^1, y^1)^3\big)^3 = \iota^2(w^1)^3$ is stratified, so we can define the set $S_{xy} := \{w \mid F\big((x, y)\big) = \iota^2(w)\}$. Note that S_{xy} is a singleton: for if $z_1, z_2 \in S_{xy}$, then $\iota^2(z_1) = F\big((x, y)\big) = \iota^2(z_2)$, which implies $z_1 = z_2$.

For $x, y \in V$ we define new ordered pair $\langle x, y\rangle := \bigcup S_{xy}$. Let us prove that it satisfies the usual property of ordered pairs and that it is type-level.

Let us first prove the usual property. Let $x, y, a, b \in V$ be such that $\langle x, y\rangle = \langle a, b\rangle$. By definition, we have $F\big((x, y)\big) = \iota^2(\langle x, y\rangle)$ and $F\big((a, b)\big) = \iota^2(\langle a, b\rangle)$, so $F\big((x, y)\big) = F\big((a, b)\big)$. Since F is an injection, we have $(x, y) = (a, b)$, which implies $x = a$ and $y = b$. If $x = a$ and $y = b$, then $\iota^2(\langle x, y\rangle) = F\big((x, y)\big) = F\big((a, b)\big) = \iota^2(\langle a, b\rangle)$, which implies $\langle x, y\rangle = \langle a, b\rangle$.

Let us prove that they are type-level. Let $x, y \in V$ be arbitrary. We have

$$z^1 \in \langle x, y\rangle^2 \leftrightarrow \exists w^2\big(F^7\big((x^2, y^2)^4\big)^4 = \iota^2(w^2)^4 \wedge z^1 \in w^2\big).$$

That proves that if x and y have type n, then $\langle x, y\rangle$ has type n. Therefore, we have type-level ordered pairs. □

Remark 4.15. Since here we're primarily concerned with set theory and not with logic, we are somewhat sloppy with respect to proving existence versus "pinpointing" some mathematical object. However, in the interest of completeness, it is important to note that using the logical principle of *existential instantiation* [3] we can in fact, having proved $\exists F\, bij\big(F, V \times V, \mathscr{P}_1^2(V)\big)$, expand the signature of our theory by a new constant symbol F and an axiom $bij\big(F, V \times V, \mathscr{P}_1^2(V)\big)$, and it will be a conservative extension. Then the new constant symbol can be used in other ways, for instance, to define a two-place function term for the new ordered pair $\langle \_, \_\rangle$.

5. Axiomatic extension

We will briefly show how to use the third approach from the introduction. We start by introducing axioms of NFU (the axiom of extensionality, the axiom of sethood, and the axiom of stratified comprehension). Next, we need a few basic notions independent of the usage of ordered pairs.

We are then able to introduce the axiom of choice. The next step is to introduce natural numbers or, more precisely, the notion of finite sets. Then we are able to introduce the axiom of infinity.

The only thing left is the introduction of the notion of (Kuratowski's) bijection and then we can state the *universe cardinal squaring principle*.

DEFINITION 5.1.

1. For $x, y \in V$ we define their **Kuratowski's ordered pair** $(x, y)_K := \big\{\{x\}, \{x, y\}\big\}$.
2. For $X, Y \in SET$ we define their **Kuratowski's product** $X \times_K Y := \{(x, y)_K \mid x \in X \wedge y \in Y\}$.
3. For X and Y we define the notion of **Kuratowski's bijection** between them as $bij_K(f, X, Y) :\Leftrightarrow f \subseteq X \times_K Y \wedge \forall x \exists! y\big((x, y)_K \in f\big) \wedge$ $\wedge\, \forall y \exists! x\big((x, y)_K \in f\big)$.

Universe cardinal squaring principle:

$$V \notin FIN \to \exists f\ bij_K\big(f, V \times_K V, \mathscr{P}_1^2(V)\big).$$

The *universe cardinal squaring principle* can be interpreted as a claim that there exists a (Kuratowski's) bijection between $V \times_K V$ and $\mathscr{P}_1^2(V)$.

The finishing touch is theorem 4.14, and via it, we can define (see remark 4.15) type-level ordered pairs. We can now develop the theory in any way needed. Notions independent of ordered pairs will stay the same, and few should be redefined, replacing Kuratowski's definitions with type-level ones.

One important notion that should also be redefined is the notion of applying the function to an argument, since now the type of f must be only one higher than the type of x, in order for $f(x)$ to be meaningful and have a type (equal to the type of x).

The next two results were given to us by an anonymous reviewer.

Theorem 5.2. $\mathsf{NFU} + \mathsf{OP}$ *proves* VCSP.

Proof: Denote with $(x, y)_K$ Kuratowski's ordered pairs and with (x, y) type-level ordered pairs. Assume $V \notin FIN$. Since $(x, y)_K$ and $\iota^2\big((x, y)\big)$ have the same type, by corollary 3.13 we can define a function $f\big((x, y)_K\big) = \iota^2\big((x, y)\big)$ for every $x, y \in V$. Function f is obviously an injection from $V \times_K V$ to $\mathscr{P}_1^2(V)$. On the other hand, function $\iota^2(x) \mapsto (x, x)_K$ is obviously an injection from $\mathscr{P}_1^2(V) \to V \times_K V$. Now Cantor–Bernstein's theorem implies that there exists a bijection between $V \times_K V$ and $\mathscr{P}_1^2(V)$. Therefore, the universe cardinal squaring principle holds. □

Theorem 5.3. $\mathsf{NFU} + \mathsf{Inf} + \mathsf{VCSP}$ *does not prove* AC.

Proof (Sketch of proof:): First, we know that $\mathsf{NFU} + \mathsf{Inf}$ interprets $\mathsf{NFU} + \mathsf{OP}$: within any model M of $\mathsf{NFU} + \mathsf{Inf}$ we can find a smaller model M' of $\mathsf{NFU} + \mathsf{OP}$. More precisely, M' is obtained as a doubly iterated partitive set of V in M [5]. Therefore, the truth of Zermelo's theorem (and also of AC) is the same in both M and M'.

We also know that $\mathsf{NFU}+\mathsf{Inf}$ does not prove AC [11]: there is a model M of $\mathsf{NFU} + \mathsf{Inf}$ which does not validate AC. If we carry out the construction from the previous paragraph, we obtain M' which validates NFU and also Inf [9], while proving VCSP by theorem 5.2 and not validating AC (since if AC were to hold in M', it would also hold in M by the previous paragraph, which is a contradiction). □

Conclusion

It is apparent that non-type-level ordered pairs are causing many difficulties. By proving the cardinal squaring principle using Kuratowski's ordered pairs we are able to justify $\mathsf{NFU}+\mathsf{Inf}+\mathsf{AC}+\mathsf{VCSP}$. Not only that, we have presented the development of NFU with Kuratowski's ordered pairs that can be used for further reference, without the need to go through it again every time type-level ordered pairs are needed.

It is worth emphasizing that everything in this article is done without the appeal to Rosser's axiom of counting, which is prominently used in Rosser's [13] and Holmes' book [7]. In our opinion, this shows that the usage of the axiom of counting, although sometimes making proofs simpler, is not essential to our approach.

Acknowledgements. We thank the anonymous reviewer for their remarks, suggestions, and perspectives, which have improved our paper. We are also thankful to the same reviewer for pointing out and providing us with the proofs of theorems 5.2 and 5.3.

We would like to express our gratitude to Tin Perkov for reading our paper and for the comments provided, and to Marko Doko for proofreading it.

References

[1] T. Adlešić, V. Čačić, *A Modern Rigorous Approach to Stratification in NF/NFU*, **Logica Universalis**, vol. 16(3) (2022), pp. 451–468, DOI: https://doi.org/10.1007/s11787-022-00310-y.

[2] H. B. Enderton, **Elements of set theory**, Academic press (1977), DOI: https://doi.org/https://doi.org/10.1016/C2009-0-22079-4.

[3] H. B. Enderton, **A Mathematical Introduction to Logic**, Academic Press (2001).

[4] M. J. Gabbay, *Consistency of Quine's New Foundations* (2014), DOI: https://doi.org/10.48550/ARXIV.1406.4060.

[5] M. R. Holmes, *Systems of combinatory logic related to Quine's 'New Foundations'*, **Annals of Pure and Applied Logic**, vol. 53(2) (1991), pp. 103–133, DOI: https://doi.org/10.1016/0168-0072(91)90052-N.

[6] M. R. Holmes, *The set-theoretical program of Quine succeeded, but nobody noticed*, **Modern Logic**, (1994).

[7] M. R. Holmes, *Elementary set theory with a universal set*, https://randall-holmes.github.io/head.pdf (1998).

[8] M. R. Holmes, *A new pass at the NF consistency proof*, https://randall-holmes.github.io/Nfproof/newattempt.pdf (2020).

[9] M. R. Holmes, *Proof, Sets, and Logic*, https://randall-holmes.github.io/proofsetslogic.pdf (2021).

[10] M. R. Holmes, T. E. Forster, T. Libert, *Alternative Set Theories*, **Sets and extensions in the twentieth century**, vol. 6 (2012), pp. 559–632.

[11] R. B. Jensen, *On the consistency of a slight (?) modification of Quine's New Foundations*, [in:] J. Hintikka (ed.), **Words and objections: Essays on the Work of W. V. Quine**, Springer (1969), pp. 278–291.

[12] W. V. Quine, *New foundations for mathematical logic*, **The American mathematical monthly**, (1937), DOI: https://doi.org/10.2307/2267377.

[13] J. B. Rosser, **Logic for mathematicians**, Dover Publications (2008), DOI: https://doi.org/10.2307/2273431.

[14] G. Wagemakers, **New Foundations—A survey of Quine's set theory**, Master's thesis, Instituut voor Tall, Logica en Informatie Publication Series, X-89-02 (1989).

Tin Adlešić
University of Zagreb
Faculty of Teacher education
10000, Savska cesta 77
Zagreb, Croatia
e-mail: tin.adlesic@ufzg.hr

Vedran Čačić
University of Zagreb
Faculty of Science
10000, Bijenička cesta 30
Zagreb, Croatia
e-mail: veky@math.hr

Submission Guidelines

Manuscripts Papers submitted to the *BSL* should be formatted using the `BSLstyle` LaTeX class with the manuscript option loaded, which can be downloaded at https://czasopisma.uni.lodz.pl/bulletin/libraryFiles/downloadPublic/603. All prospective authors should read the "Instructions for authors" file included in the style files folder and follow the guidelines included there. Abstract and keywords are compulsory parts of each submission as they will be used in the *BSL* online search tools. Mind that an abstract should contain no references and the list of keywords should consist of at least 3 items. It is also recommended that each author having an ORCID number provides it in the `.tex` source file. Authors who are unable to comply with these requirements should contact the Editorial Office in advance.

Paper Length There is no fixed limit imposed on the length of submitted papers, however one can expect that for shorter papers, up to 18 pages long, the Editorial Board will be able to reduce the time needed for the reviewing process.

Footnotes should be avoided as much as possible, however it is not disallowed to use them if necessary.

Bibliography should be formatted using BibTeX and the `BSLbibstyle` bibliography style (to be found in the style files folder). It is essential that to each bibliography item a plain DOI number (i.e., not a full link) is attached whenever applicable. If a submitted paper is accepted for publication, the author(s) should provide the bibliography file in the `.bib` format among other source files. For more details on bibliography processing the authors are referred to the "Instructions for authors". Authors unfamiliar with BibTeX are advised to familiarize themselves with this short tutorial (https://www.overleaf.com/blog/532-creating-and-managing-bibliographies-with-bibtex-on-overleaf) or video tutorial (https://www.overleaf.com/learn/latex/Questions/How_to_include_a_bibliography_using_bibtex) on managing bibliographies with BibTeX.

Affiliation and mailing addresses of all the authors should be included in the `\Affiliation` and `\AuthorEmail` fields, respectively, in the source `.tex` file.

Submission When the manuscript is ready, it should be submitted through our editorial platform, using the the «Make a Submission» button. If the paper is meant to be included in a special issue, the appropriate section name should be selected before submitting it. If the paper is regular, the authors can indicate the editor they would like to supervise the editorial process or leave this decision to the Editorial Office by leaving the "Comments for the Editor" section blank. For the duration of the whole editorial process of the manuscript it must not be submitted for review to any other venue.

Publication Once the manuscript has been accepted for publication and the galley proof has been revised by the authors, the article is given a DOI number and published in the *Early View* section, where articles accepted for publication and awaiting assignment to an issue are made available to the public. The authors will be notified when their article is assigned to an issue.

Copyright permission It is the authors' responsibility to obtain the necessary copyright permission from the copyright owner(s) of the submitted paper or extended abstract to publish the submitted material in the *BSL*.