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Bilateralism and Proof-Theoretic Semantics (Part II)

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Sara AYHAN, Germany

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Michał Zawidzki

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Katarzyna Smyczek

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Editorial Office: Department of Logic, University of Lodz
ul. Lindleya 3/5, 90-131 Łódź, Poland
e-mail: bulletin@uni.lodz.pl

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INTRODUCTION: BILATERALISM AND PROOF-THEORETIC SEMANTICS (PART II)

Most of the papers contained in this special issue[1] are results from contributions at a conference on this topic, which took place at the Ruhr University Bochum in March 2022. Since the topic of proof-theoretic semantics (PTS) can by now be considered as well-established in the logic community and has been exclusively dealt with at several conferences and in many publications[2], this introduction's focus will be on the part of logical bilateralism. Before summarizing the content of this special issue, a brief overview of the development in the field will be given, though this is not meant and does not aim to be an exhaustive account of the existing literature.[3]

There are rather different approaches branded as *bilateralism* in the literature, whose differences are mostly not made explicit, though. Although the origin of bilateralism is Rumfitt's [20] seminal paper in the sense that the concrete term and idea are introduced therein and spelled out thoroughly, there are some predecessors to the general idea that are frequently cited, like [12], [22], and [8].[4] The most frequent characterization that is

[1] For editorial reasons it was decided to have actually two issues on this topic, which is why this introduction will appear in both parts and only differ in the presentation of the papers contained in the respective issue.

[2] See, e.g., [21, 4, 9, 11].

[3] Parts of the following paragraphs can also be found in a joint paper by Heinrich Wansing and myself on the topic of multilateralism [27]. In its introductory part we give an overview of the literature on bilateralism as well as of the existing but scarce literature extending this concept to multilateralism.

[4] A paper which is *not* often mentioned in this context, probably due to the fact that it was written in German, but which deserves recognition in this context is [24]. Von Kutschera is concerned with the relation between the notions of proof and refutation

used for bilateralism is that it is a theory of meaning displaying a symmetry between certain notions (or often rather: conditions governing these notions), which have not been considered being on a par by 'conventional' theories of meaning. The relevant notions are most often *assertion* and *denial*, or *assertibility* and *deniability*, sometimes also *acceptance* and *rejection*.[5] While the former are usually taken to describe speech acts, the latter are usually – though not always (see [19] for a thorough distinction) – considered to describe the corresponding internal cognitive states or attitudes. 'Assertibility' and 'deniability', on the other hand, are of a third kind, since they can be seen to describe something like properties of propositions. The symmetry between these respective concepts is often described with expressions like "both being primitive", "not reducible to each other", "being on a par", and "of equal importance". Another point to characterize bilateralism, which is often mentioned, though not as frequent or central as the former point,[6] is that in a bilateral approach the denial of A is not interpreted in terms of, or *as* the assertion of the negation of A but that it is the other way around: In bilateralism rejection and/or denial are usually considered as conceptually prior to negation.

Ripley [18, 19] distinguishes two camps of bilateral theories of meaning in terms of "what kinds of condition on assertion and denial they appeal to" [19, p. 50]: a warrant-based approach and a coherence-based approach, for the latter of which he himself argues [17] and which was firstly devised by Restall [13, 14].[7] As references for the first camp, which Ripley calls the 'orthodox' bilateralism, [12], [22], and [20] are given. Warrant-based bilateralism takes the relevant conditions to be the ones under which propositions can be *warrantedly* asserted or denied. Coherence-based bilateralism,

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and claims, e.g., that it is not necessary to define the latter in terms of the former but that it could just as well be done the other way around, or, although in the paper he does differently, that both could be seen as primitive. Thus, it seems that he voices quite bilateralist ideas.

[5] To give some examples of references using a characterization of essentially this flavor: [5, 7, 10, 16, 20, 26].

[6] The following use this as an additional characterization (while also using the essential characterization that the references in fn. 4 use): [2, 3, 17, 23]. This is not to say that this point does not occur in other works on bilateralism but that it is not used as a *characterizing feature* of bilateralism there.

[7] In [19] this one is called the "bounds-based bilateralism". Interestingly, Restall does not use the expression "bilateralism" at all in the cited works, only later does this term become part of his terminology, e.g., in [15].

on the other hand, takes the relevant conditions to be the conditions under which *collections* of propositions can be *coherently* asserted and/or denied together.

What the two approaches have in common is that they were both meant, as they were originally devised, to motivate a PTS approach using *classical* instead of intuitionistic logic. What they tend to differ in, though, is their design and interpretations of proof systems. Rumfitt [20] uses a natural deduction system with signed formulas for assertion and denial, i.e., rules do not apply to propositions but to speech acts. He argues that the shortcomings that a classical natural deduction calculus has from a PTS point of view are overcome once we consider a calculus containing introduction and elimination rules determining not only the assertion conditions for formulas containing the connective in question but also the denial conditions. Thus, he means to give a motivation how the rules of classical logic lay down the meaning of the connectives.[8]

Restall [13], opting for the coherence-based approach, does the same but comes from another direction in suggesting a bilateral reading of classical sequent calculus (i.e., with multiple conclusions) incorporating the speech acts of assertion and denial. In a nutshell, he proposes that having the derivation of a sequent $\Gamma \vdash \Delta$, means that the position of asserting each of the members of Γ while simultaneously denying each of the members of Δ would be 'out of bounds'. In a recent paper, though, Restall [15] seems convinced by Steinberger's [23] criticism of multiple-conclusion systems as not adhering to our natural inferential practice and he considers an approach using a natural deduction system instead, which does not employ signed formulas but rather uses different positions for certain commitments from which the inference is drawn to the conclusion.[9]

What Ripley [19] mentions in a footnote is that there are also other kinds of bilateralism, which do not fit into either camp because they do not consider speech acts (i.e., assertion and denial) as the primary notions to act upon in the context of PTS but rather notions being on a par with proof, provability, or verification, i.e., refutation, refutability, or falsification, respectively. The point of interest is, thus, to implement different

[8] For critical assessments of that paper, see, e.g., [6, 2, 10, 5].

[9] The motivation is still to make a case for classical logic being usable in a PTS framework, although Restall does not seem too dogmatic about anything being 'the best' logic. He also wants to show how such a system can be used for substructural logics.

derivability relations in a proof-theoretic framework expressing a duality between different inferential relationships, which has been devised, e.g., in [25, 26].

These different varieties of bilateralism depicted above are actually very well represented in this special issue. It is even the majority of the contributions dealing with what can be called – in one way or another – 'unorthodox' bilateralism.

In the paper "Fractional-valued modal logic and soft bilateralism" Mario Piazza, Gabriele Pulcini and Matteo Tesi outline yet another, unorthodox variety of bilateralism, which they call *soft bilateralism* to demarcate their approach from more traditional conceptions. It is 'bilateral' because the rules in the calculi they introduce are meant to deal with both derivability and underivability. It is only 'softly' bilateral due to their conception of the speech act of denial, namely as rejection in the sense of proving the unprovability of a formula rather than in the sense of the stronger notion of directly refuting that formula. Based on this approach they argue for considering *fractional semantics* – a semantics whose values are the rational numbers in the closed interval [0,1] – for a family of modal logics and investigate and prove certain properties for these systems.

There are also papers, though, which deal with issues of 'orthodox' bilateralism. Nils Kürbis' paper "Supposition: A problem for bilateralism" spells out an important objection that can be raised against a system of natural deduction with signed formulas to be interpreted as speech acts in Rumfitt-style. The argument against such a system is as simple as it is compelling: Natural deduction systems work with assumptions. Making an assumption is also to be considered as a kind of speech act. Embedding speech acts within other speech acts is – as it is widely agreed upon – not possible. Thus, we cannot make sense of the use of assumptions in a proof system which implements bilateralism in such a way.

Leonardo Ceragioli's paper "Bilateral rules as complex rules" deals with the same kind of proof system and more specifically, two objections raised in [5] about issues caused by the so-called *coordination principles*, which are needed in such a bilateralist system besides the operational rules. The first objection is that in a bilateralist framework the notorious connective `tonk` cannot be ruled out by the criterion of harmony as it can be usually done in a unilateralist framework and that thus, there can be (at least on a certain understanding of the term) a reduction procedure for `tonk`, which indeed would be highly undesirable from the viewpoint of PTS. The second objection concerns a bilateralist version of rules for a paradoxical zero-ary connective, which Gabbay [5] presents and which he claims to be in harmony. However, together with the coordination principles they would trivialize the system, i.e., they should not be admitted, although they seem fine by the harmony criterion. Ceragioli's proposed solution to these two issues is based on reinterpreting bilateral systems as systems with *complex rules* and applies the results existing on such systems to the special case of bilateralism.

Last but not least, the paper by Pedro del Valle-Inclan "Harmony and normalisation in bilateral logic" builds upon former work by the author and co-author Julian Schlöder [1] in which they propose a specific notion of proof-theoretic harmony for bilateralist contexts. In the present paper del Valle-Inclan argues that this notion also leads to a special notion of normal form. Based on this, he goes on to prove normalization results for two (Rumfitt-style) bilateralist calculi for classical logic, which are subsequently compared to other existing results in this area.

Acknowledgements. I would like to thank the speakers and participants of the conference "Bilateralism and Proof-Theoretic Semantics" for the lively discussion, the vital input on the presented works and for contributing to this issue. I am also very grateful to the editors of this journal for their kind help with administrative and technical questions, to all the reviewers for their constructive feedback on the contributions and last but not least, to Heinrich Wansing for helping out with all of the above and much more.

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Sara Ayhan
Ruhr University Bochum
Institute of Philosophy
Universitätsstraße 150
44780 Bochum, Germany
e-mail: sara.ayhan@rub.de

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Mario Piazza
Gabriele Pulcini
Matteo Tesi

FRACTIONAL-VALUED MODAL LOGIC AND SOFT BILATERALISM

Abstract

In a recent paper, under the auspices of an unorthodox variety of bilateralism, we introduced a new kind of proof-theoretic semantics for the base modal logic **K**, whose values lie in the closed interval $[0, 1]$ of rational numbers [14]. In this paper, after clarifying our conception of bilateralism – dubbed "soft bilateralism" – we generalize the fractional method to encompass extensions and weakenings of **K**. Specifically, we introduce well-behaved hypersequent calculi for the deontic logic **D** and the non-normal modal logics **E** and **M** and thoroughly investigate their structural properties.

Keywords: modal logic, general proof theory (including proof-theoretic semantics), many-valued logics.

2010 Mathematical Subject Classification: 03B45, 03F03, 03B50.

1. Introduction

From a general perspective, the distinctive aspect of bilateralism is that it recognizes and isolates two different dimensions of logic which are placed on a par: *assertion* and *denial*. Although often neglected in the history of logic, denial can be seen as a perfectly sensible logical notion which follows its own specific inferential trajectories [6, 17]. Since the notion of logical denial admits several consistent meanings, the proper logical realm

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of bilateralism is still a matter of philosophical controversy. Therefore, over the last few decades, various proposals concerning the possibility of a bilateral reading of logic have flourished [19, 4, 22, 17].

On the one hand, Rumfitt has argued that the natural theoretical backdrop against which bilateralism takes place is classical logic; and in effect, bilateralism has traditionally been adopted to give a coherent proof-theoretic account of classical logic. On the other hand, more recently, this view has been challenged by Kürbis, who claims that a bilateral account of intuitionistic logic is also possible [8, 9]. This stance seems perfectly sensible, as the acts of assertion and denial can also be rephrased in proper intuitionistic terms.

In what follows, we propose a particular conception of bilateralism, which can accommodate non-classical logics or extensions of classical logic, such as substructural logics and modal logic. As it is well known, the notion of denial in bilateralism is primitive and cannot be reduced to the assertion of a negation. Our proposal is based on interpreting the act of denial by means of the logically "soft" notion of *rejection*. A formula A can be considered as rejected just in case it does not admit a proof within the reference system. For example, in classical propositional logic contradictions and truth-functional contingencies all qualify as rejectable formulas [18]. This is why we label this type of bilateralism as "soft" to distinguish it from other narrower interpretations, whereby denial is logically analyzed as refutation, i.e. in terms of a derivation of grounds for the denial of the proposition.

In this paper, we introduce calculi for a family of modal logics that operate within a soft bilateral framework by combining rules for handling derivable as well as underivable sequents.[1] This hybrid approach to inference rules is both technically useful, as it allows for a more comprehensive understanding of the logic without reducing it to the set of its theorems, and conceptually profound, as it is closely linked to the venerable notion of analyticity, which is essential for manipulating information about underivability in a well-behaved proof-theoretic setting.

Mainstream proof-theoretic semantics embraces the meaning-as-use paradigm, which entails shifting the focus from analyzing truth-conditions to understanding the inference patterns that govern the recursive construc-

[1] Proof-systems combining together rules for dealing with valid and invalid syntactic expressions are sometimes called 'hybrid' in the literature on rejection systems [20, 6].

tion of proofs [21, 15, 5]. In proof-theoretic semantics, the meaning of connectives is primarily conveyed through the top-down reading of their respective introduction rules.

As standard bilateralism is conceptually linked to proof-theoretic semantics, our account of bilateralism also yields its peculiar semantics in terms of proofs, which we call *fractional semantics*. While proof-theoretic semantics is mainly concerned with intuitionistic logic, we have recently shown how a fractional semantics can be provided for a wide class of logics, including classical logic [12], the minimal normal modal logic K [14], and the multiplicative-additive fragment of linear logic MALL [13].

The term "fractional" is used to describe semantics in which formulas are interpreted as values in the closed interval $[0, 1]$ of rational numbers. In the fractional setting, a reference proof system is used as an algorithm to decompose a formula A into a set of clauses $\mathcal{C}(A)$, which are ordinary sequents in the case of classical logic, and hypersequents when K and MALL are being analyzed. The interpretation of A, denoted by $[\![A]\!]$, is obtained by calculating the ratio of true clauses in $\mathcal{C}(A)$ to the total number of clauses produced by the decomposition. This interpretation function measures the degree to which A is satisfied, or the "quantity of truth" in A[2]. Needless to say, we must be able to carry out such a decomposition for *any* formula A in the language, including the case in which A is neither provable nor refutable. Therefore, a "soft" variety of bilateralism is necessary to ensure that this is possible.

Methodologically, the proof-theoretic platform on which the fractional evaluation is built needs to meet the following requirements:

- *Invertibility:* for each logical rule in the calculus, the derivability of the conclusion always implies the derivability of (each of) the premise(s).
- *Stability*: any complete decomposition of the endsequent (end-hypersequent) always returns the *same* set of top-sequents (top-hypersequents).

[2] In interpreting the formulas of classical logic, we use Kleene's system $\mathsf{G4}$ enriched with a 'complementary' axiom introducing whatever clause $\Gamma \vdash \Delta$ such that $\Gamma \cap \Delta = \varnothing$ [12]. Consider for instance the formula $A \equiv p \to (p \wedge q)$. The enriched system decomposes it into the set of clauses $\{p \vdash p; p \vdash q\}$, so that $[\![A]\!] = {}^1/_2 = 0.5$. Actually, this formula can be rewritten as $(p \to p) \wedge (p \to q)$ and this form clearly displays that fact that A is formed by two components of which only one displays an identity.

- *Termination of the proof search*: any decomposition of a given end-sequent (end-hypersequent) always terminates yielding either a proof or a rejection.

On one hand, invertibility and termination guarantee the possibility of turning any set of clauses $\mathcal{C}(A)$ into some sort of canonical form for A (its conjunctive normal form, in classical logic). On the other hand, stability is what allows us to call the described fractional evaluation a 'semantics', making the value $[\![A]\!]$ a derivation-invariant.

The technical aim of this paper is to extend the fractional approach proposed for modal logic to other systems beyond **K**. After reviewing the main proof-theoretic ingredients, the paper shows how to apply the fractional approach to basic deontic logic **D** as well as non-normal modal logics **E** and **M**. **E** is the minimal non-normal modal logic characterized by neighborhood semantics. **M** extends **E** by introducing the axiom of distributivity of $\Box$ over conjunction. The paper investigates the structural properties of these systems and establishes the admissibility of the rules of weakening, contraction, and cut using purely finitary and constructive methods.

2. The systems

2.1. Separating modality and classicality

As we have remarked above, in order to apply the fractional method to modal logic, we need to design a calculus which meets some proof-theoretic *desiderata*. In particular, stability, finiteness of the proof-search space and invertibility.

Achieving finiteness of the proof-search space is perhaps the most delicate item when dealing with non-classical logics or extensions of classical logic. In fact, if we stick to a standard sequent calculus setting, we often lose invertibility. On the other hand, if we supplement the structure of sequents, we can obtain invertible rules, but often at the cost of losing finiteness of the proof-search space.

To meet all of these requirements, we find it natural to switch to a hypersequent formulation of the modal logics we are considering. The use of hypersequents proves to be well-suited as it maintains a strong version of the formula interpretation, meaning that any syntactic object can be interpreted as a formula in the language. Furthermore, hypersequents provide a way to disentangle the classical content of a sequent from its modal residual elements, which is a key step in obtaining finiteness of the proof-search space.

2.2. The calculus $\overline{\overline{\mathsf{HK}}}$

We shall be mainly working with *hypersequents*, introduced under a different name by Mints in the early seventies of the last century [11, 10] and independently by Pottinger [16], then further elaborated (and so named) by Avron [1, 2, 3]. Hypersequents come as a generalization of the standard notion of sequent in the style of Gentzen. A *sequent* is a syntactic expression of the form $\Gamma \Rightarrow \Delta$, where Γ, Δ are finite multisets of modal formulas from the set $\mathscr{F}$ recursively defined by the grammar:

$$\mathscr{F} ::= AT \,|\, \neg\mathscr{F} \,|\, \mathscr{F} \to \mathscr{F} \,|\, \mathscr{F} \wedge \mathscr{F} \,|\, \mathscr{F} \vee \mathscr{F} \,|\, \Box\mathscr{F}$$

with AT collecting the atomic sentences. As usual, $\Diamond A$ is taken to abridge the formula $\neg\Box\neg A$. If $\Gamma = [A_1, A_2, \ldots, A_n]$, then $\bigwedge\Gamma$ and $\bigvee\Gamma$ are the two formulas $A_1 \wedge A_2 \wedge \cdots \wedge A_n$ and $A_1 \vee A_2 \vee \cdots \vee A_n$, respectively. If $\Gamma = \varnothing$, then we set $\bigwedge\Gamma = \top$ and $\bigvee\Gamma = \bot$, where $\top$ and $\bot$ stand for an arbitrarily selected tautology and contradiction, respectively. With $\Box\Gamma$ we mean the multiset $[\Box A_1, \Box A_2, \ldots, \Box A_n]$. For any formula A we denote with A^n the multiset containing exactly n occurrences of A.

In general, if M and N are two multisets, we indicate with $M \uplus N$ and $\#M$ their multiset union and M's cardinality, respectively. A *hypersequent*, denoted by $\mathcal{G}, \mathcal{H}, \ldots$, is defined as a finite (possibly empty) multiset of sequents written as follows:

$$\Gamma_1 \Rightarrow \Delta_1 \,|\, \Gamma_2 \Rightarrow \Delta_2 \,|\, \cdots \,|\, \Gamma_n \Rightarrow \Delta_n.$$

We shall keep calling 'sequents' those hypersequents listing exactly one sequent. The set collecting hypersequents is here indicated with $\mathscr{H}$. Practically speaking, a hypersequent $\mathcal{G}$ turns out to be *valid* whenever at least one of the sequents listed in $\mathcal{G}$ is valid. Here the meaning of the term 'valid' has to be specified in progress, depending on the logical context.

The following definition introduces the notion of *hyperclause* which extends that of clause for standard sequents of classical logic.

AXIOMS

$$\frac{}{\mid\!\!\frac{1}{}\; \Box\Pi_1, \Gamma_1, p \Rightarrow \Delta_1, p \mid \cdots \mid \Box\Pi_n, \Gamma_n \Rightarrow \Delta_n}\; ax$$

$$\frac{}{\mid\!\!\frac{0}{}\; \Box\Pi_1, \Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Box\Pi_n, \Gamma_n \Rightarrow \Delta_n}\; \overline{ax} \qquad \Gamma_i \cap \Delta_i = \varnothing \text{ for } 1 \leqslant i \leqslant n$$

LOGICAL RULES

$$\frac{\mid\!\!\frac{i}{}\; \mathcal{G} \mid \Gamma \Rightarrow \Delta, A}{\mid\!\!\frac{i}{}\; \mathcal{G} \mid \Gamma, \neg A \Rightarrow \Delta}\; \neg\Rightarrow \qquad \frac{\mid\!\!\frac{i}{}\; \mathcal{G} \mid A, \Gamma \Rightarrow \Delta}{\mid\!\!\frac{i}{}\; \mathcal{G} \mid \Gamma \Rightarrow \Delta, \neg A}\; \Rightarrow\neg$$

$$\frac{\mid\!\!\frac{i}{}\; \mathcal{G} \mid \Gamma, A, B \Rightarrow \Delta}{\mid\!\!\frac{i}{}\; \mathcal{G} \mid \Gamma, A \wedge B \Rightarrow \Delta}\; \wedge\Rightarrow \qquad \frac{\mid\!\!\frac{i}{}\; \mathcal{G} \mid \Gamma \Rightarrow \Delta, A \qquad \mid\!\!\frac{j}{}\; \mathcal{G} \mid \Gamma \Rightarrow \Delta, B}{\mid\!\!\frac{i\cdot j}{}\; \mathcal{G} \mid \Gamma \Rightarrow \Delta, A \wedge B}\; \Rightarrow\wedge$$

$$\frac{\mid\!\!\frac{i}{}\; \mathcal{G} \mid \Gamma, A \Rightarrow \Delta \qquad \mid\!\!\frac{j}{}\; \mathcal{G} \mid \Gamma, B \Rightarrow \Delta}{\mid\!\!\frac{i\cdot j}{}\; \mathcal{G} \mid \Gamma, A \vee B \Rightarrow \Delta}\; \vee\Rightarrow \qquad \frac{\mid\!\!\frac{i}{}\; \mathcal{G} \mid \Gamma \Rightarrow \Delta, A, B}{\mid\!\!\frac{i}{}\; \mathcal{G} \mid \Gamma \Rightarrow \Delta, A \vee B}\; \Rightarrow\vee$$

$$\frac{\mid\!\!\frac{i}{}\; \mathcal{G} \mid \Gamma \Rightarrow \Delta, A \qquad \mid\!\!\frac{j}{}\; \mathcal{G} \mid \Gamma, B \Rightarrow \Delta}{\mid\!\!\frac{i\cdot j}{}\; \mathcal{G} \mid \Gamma, A \to B \Rightarrow \Delta}\; \to\Rightarrow \qquad \frac{\mid\!\!\frac{i}{}\; \mathcal{G} \mid \Gamma, A \Rightarrow \Delta, B}{\mid\!\!\frac{i}{}\; \mathcal{G} \mid \Gamma \Rightarrow \Delta, A \to B}\; \Rightarrow\to$$

MODAL OPERATOR RULE

$$\frac{\mid\!\!\frac{i}{}\; \mathcal{G} \mid \Gamma \Rightarrow A \mid \Box\Gamma, \Gamma' \Rightarrow \Box\Delta, \Delta'}{\mid\!\!\frac{i}{}\; \mathcal{G} \mid \Box\Gamma, \Gamma' \Rightarrow \Box A, \Box\Delta, \Delta'}\; \Box, \quad \text{where } \Gamma' \uplus \Delta' \subseteq AT$$

Figure 1. The $\overline{\overline{\mathsf{HK}}}$ sequent calculus (read $\mid\!\!\frac{1}{}$ as $\vdash$ and $\mid\!\!\frac{0}{}$ as $\dashv$).

Definition 2.1 (Hyperclauses). A hyperclause is a hypersequent

$$\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$$

such that no rule of the calculus can be upwardly applied to it. An *identity* hyperclause is such that, for some i, $\Gamma_i \uplus \Delta_i \neq \varnothing$; otherwise, it is *complementary.*

Example 2.2. An identity hyperclause and a complementary hyperclause, respectively:

$$p \Rightarrow p \mid \Box(\Box p \to p) \Rightarrow \qquad\qquad \Rightarrow p \mid \; \Rightarrow p \mid \Box(\Box p \to p) \Rightarrow$$

Figure 1 presents the ‘softly’ bilateral hypersequent calculus $\overline{\overline{\mathsf{HK}}}$. The rules of $\overline{\overline{\mathsf{HK}}}$ operate on hypersequents prefixed by the symbols ‘$\vdash$’ and ‘$\dashv$’: we write $\vdash \mathcal{G}$ and $\dashv \mathcal{G}$ to assert the validity and *in*validity of $\mathcal{G}$, respec-

$$
\dfrac{
\dfrac{\dfrac{}{\dashv \Rightarrow p \mid\; \Rightarrow p \mid \Box(\Box p \to p) \Rightarrow}\,\overline{ax}}{\dashv \Rightarrow \Box p, p \mid \Box(\Box p \to p) \Rightarrow}\,\Box
\qquad
\dfrac{}{\vdash p \Rightarrow p \mid \Box(\Box p \to p) \Rightarrow}\,ax.
}{
\dfrac{\dfrac{\dashv \Box p \to p \Rightarrow p \mid \Box(\Box p \to p) \Rightarrow}{\dashv \Box(\Box p \to p) \Rightarrow \Box p}\,\Box}{\dashv \Rightarrow \Box(\Box p \to p) \to \Box p}\,{\Rightarrow}{\to}
}\,{\to}{\Rightarrow}
$$

Figure 2. An example of $\overline{\overline{\mathsf{HK}}}$ proof

tively. For the sake of a more compact notation, in Figure 1 the $\overline{\overline{\mathsf{HK}}}$ rules are expressed by writing $\vdash^{1}$ and $\vdash^{0}$ to indicate the two signs '$\vdash$' and '$\dashv$', respectively. The calculus is equipped with two axiom rules: the ordinary ax-rule introduces any identity hyperclause, whilst the $\overline{ax}$-rule specifically introduces complementary hyperclauses.

From now on, we will indicate derivations with small Greek letters $\pi, \rho, \ldots$. We recall that the height $h(\pi)$ of a derivation π is given by the number of hypersequents figuring in one of its longest branches. Moreover, we indicate with $\mathsf{top}(\pi)$ the multiset of π's top-hypersequents.

Example 2.3. Figure 2 displays a $\overline{\overline{\mathsf{HK}}}$-derivation ending in $\dashv \Rightarrow \Box(\Box p \to p) \to \Box p$, that is a formal *rejection* for the sequent $\Rightarrow \Box(\Box p \to p) \to \Box p$.

Remark 2.4. The $\Box$-rule is the only inference schema in which the hypersequent structure comes effectively into play. Intuitively speaking, a $\Box$-application in its bottom-up reading allows us to decompose a sequent-component in a hypersequent by splitting its classical part from modal residues. In fact, each time the rule is applied, a new hypersequent component is added, thus starting a parallel derivation.

Furthermore, notice that the side condition on the $\Box$-rule about contexts Γ' and Δ' is crucial to avoid pathological situations like the one indicated below, in which $\overline{\overline{\mathsf{HK}}}$ proves both $\vdash \mathcal{G}$ and $\dashv \mathcal{G}$.

$$
\dfrac{
\dfrac{\dfrac{}{\vdash t \mid p \Rightarrow p}\,ax}{\vdash p \Rightarrow p, \Box t}\,\Box
\quad
\dfrac{\dfrac{}{\vdash t \Rightarrow t \mid \Box t \Rightarrow p}\,ax}{\vdash p, \Box t \Rightarrow \Box t}\,\Box
}{\vdash p, p \to \Box t \Rightarrow \Box t}\,{\to}{\Rightarrow}
\qquad
\dfrac{
\dfrac{
\dfrac{}{\vdash \Rightarrow t \mid p \Rightarrow p}\,ax
\quad
\dfrac{}{\dashv \Rightarrow t \mid p, \Box t \Rightarrow}\,\overline{ax}
}{\dashv \Rightarrow t \mid p, p \to \Box t \Rightarrow}\,{\to}{\Rightarrow}
}{\dashv p, p \to \Box t \Rightarrow \Box t}\,\Box
$$

The other modal systems are obtained by adjusting the system $\overline{\overline{\mathsf{HK}}}$ as indicated below.

- $\overline{\overline{\mathsf{HD}}}$ is obtained by adding to $\overline{\overline{\mathsf{HK}}}$ the rule:

$$\dfrac{\models^{i} \mathcal{G} \,|\, \Pi \Rightarrow \Sigma \,|\, \Gamma \Rightarrow}{\models^{i} \mathcal{G} \,|\, \Box\Gamma, \Pi \Rightarrow \Sigma}\ \mathsf{d} \quad \text{where } \Pi, \Sigma \subset \mathsf{AT}$$

and by revising the $\overline{ax}$-rule as follows:

$$\dfrac{}{\dashv \Gamma_1 \Rightarrow \Delta_1 \,|\, \dots \,|\, \Gamma_n \Rightarrow \Delta_n}\ \text{where } \Gamma_i, \Delta_i \subset AT$$

- $\overline{\overline{\mathsf{HM}}}$ is obtained by substituting the $\Box$-rule in $\overline{\overline{\mathsf{HK}}}$ with the following inference pattern:

$$\dfrac{\models^{i} \mathcal{G} \,|\, A_1 \Rightarrow B \,|\, \dots \,|\, A_n \Rightarrow B \,|\, \Box A_1, \dots, \Box A_n, \Pi \Rightarrow \Box\Delta, \Sigma}{\models^{i} \mathcal{G} \,|\, \Box A_1, \dots, \Box A_n, \Pi \Rightarrow \Box\Delta, \Box B, \Sigma}\ \mathsf{m}$$

where Π, Σ are multisets of atomic formulas, $i \in \{1, ..., m\}$, and $j \in \{1, ..., n\}$. We also need to replace the $\overline{ax}$-rule with the following version:

$$\dfrac{}{\dashv \Box\Pi_1, \Gamma_1 \Rightarrow \Delta_1 \,|\, \dots \,|\, \Gamma_n \Rightarrow \Delta_n, \Box\Sigma_n}\ \text{where } \Gamma_i, \Delta_i \subset AT$$

- The system $\overline{\overline{\mathsf{HE}}}$ is obtained from $\overline{\overline{\mathsf{HK}}}$ by replacing the $\Box$-rule with the following inference schema:

$$\dfrac{\models^{i} \mathcal{G} \,|\, [\Rightarrow A_i \leftrightarrow B_j] \,|\, \Gamma \Rightarrow \Delta}{\models^{i} \mathcal{G} \,|\, \Box A_1, ..., \Box A_m, \Gamma \Rightarrow \Delta, \Box B_1, ..., \Box B_n}\ \mathsf{e}$$

where Γ, Δ are multisets of atomic formulas and $i \in \{1, ..., m\}$ and $j \in \{1, ..., n\}$. We also need to replace the $\overline{ax}$-rule with the following version:

$$\dfrac{}{\dashv \Box\Pi_1, \Gamma_1 \Rightarrow \Delta_1 \,|\, \dots \,|\, \Gamma_n \Rightarrow \Delta_n, \Box\Sigma_n}\ \text{where } \Gamma_i, \Delta_i \subset AT$$

$$\frac{\vdash \mathcal{G} \,|\, \Gamma \Rightarrow \Delta}{\vdash \mathcal{G} \,|\, A, \Gamma \Rightarrow \Delta}\ LW \qquad \frac{\vdash \mathcal{G}}{\vdash \mathcal{G} \,|\, \mathcal{H}}\ EW \qquad \frac{\vdash \mathcal{G} \,|\, A, A, \Gamma \Rightarrow \Delta}{\vdash \mathcal{G} \,|\, A, \Gamma \Rightarrow \Delta}\ LC$$

$$\frac{\vdash \mathcal{G} \,|\, \Gamma \Rightarrow \Delta}{\vdash \mathcal{G} \,|\, \Gamma \Rightarrow \Delta, A}\ RW \qquad \frac{\vdash \mathcal{G} \,|\, \Gamma \Rightarrow \Delta \,|\, \Gamma \Rightarrow \Delta}{\vdash \mathcal{G} \,|\, \Gamma \Rightarrow \Delta}\ EC \qquad \frac{\vdash \mathcal{G} \,|\, \Gamma \Rightarrow \Delta, A, A}{\vdash \mathcal{G} \,|\, \Gamma \Rightarrow \Delta, A}\ RC$$

$$\frac{\vdash \mathcal{G} \,|\, \Gamma \Rightarrow \Delta, A \qquad \vdash \mathcal{H} \,|\, A, \Pi \Rightarrow \Sigma}{\vdash \mathcal{G} \,|\, \mathcal{H} \,|\, \Gamma, \Pi \Rightarrow \Delta, \Sigma}\ Cut$$

Figure 3. Admissible structural rules

3. Structural analysis

In this section we spell out the details of a purely syntactical cut-elimination procedure for these systems. In a previous work [13], cut-elimination was established in the form of closure under cut due to soundness and completeness of the system. We shall now give a purely syntactic proof thereof.

We recall the standard proof-theoretic definitions and measures. In particular, the *degree* of a formula is defined as the number of occurrences of connectives in it.

We also recall that a rule is *height-preserving admissible* when (i) the derivability of the premises entails the derivability of the conclusion and (ii) the height of the conclusion's derivation does not exceed that of the derivations of the premises. Additionally, we need the following notation: given a calculus $\overline{\overline{\mathbf{HX}}}$, we denote by **HX** the calculus obtained by removing its complementary axiom.

LEMMA 3.1. *The rules of the calculus* **HK** *are height-preserving invertible.*

PROOF: The proof is by induction on the height of the derivation of the conclusion of the rule. We consider only the case of the modal operator, the other ones are routine. Given a hypersequent shaped as

$$\vdash \mathcal{G} \,|\, \Box\Gamma, \Gamma' \Rightarrow \Box A, \Box\Delta, \Delta',$$

by inspection of the rules of the system, it can only come as a conclusion of the $\Box$-rule. On the other hand, if $\Box A$ is the principal formula, then the premise is the desired conclusion. If the principal formula is a formula in $\Box\Delta$, say $\Box B$, then we have:

$$\frac{\vdash \mathcal{G} \,|\, \Gamma \Rightarrow B \,|\, \Box\Gamma, \Gamma' \Rightarrow \Box A, \Box\Delta'', \Delta'}{\vdash \mathcal{G} \,|\, \Box\Gamma, \Gamma' \Rightarrow \Box A, \Box\Delta'', \Box B, \Delta'} \Box$$

Since the height gets decreased, we can apply the induction hypothesis which yields a derivation ending in $\vdash \mathcal{G} \,|\, \Gamma \Rightarrow B \,|\, \Box\Gamma, \Gamma' \Rightarrow \Box A, \Box\Delta'', \Delta'$. The desired conclusion then follows by a final application of the $\Box$-rule. $\Box$

LEMMA 3.2. *The weakening rules* (EW)*,* (LW) *and* (RW) *are both admissible.*

PROOF: Admissibility of the rule of external weakening (EW) follows from a straightforward induction on the height of derivations. On the contrary, to establish the admissibility of the weakening rules (LW) and (RW) we need to argue by double induction, with the main induction hypothesis on the degree of the formula to be added and the secondary induction hypothesis on the height of the derivation under consideration. In particular:

If $n = 0$, then if the hypersequent $\vdash \mathcal{G} \,|\, \Box\Gamma, \Gamma' \Rightarrow \Delta$ is derivable, so are both $\vdash \mathcal{G} \,|\, A, \Box\Gamma, \Gamma' \Rightarrow \Delta$ and $\vdash \mathcal{G} \,|\, \Box\Gamma, \Gamma' \Rightarrow \Delta, A$.

If $n > 0$ and the last rule is not a $\Box$-application, then we apply the secondary induction hypothesis to the premise(s) and then the rule again. Otherwise, if the last rule applied is a $\Box$-application, we distiguish three subcases.

- If A is an atomic formula, then we apply the secondary induction hypothesis and then the rule again.

- If A is a modal formula $\Box B$ we have:

$$\frac{\vdash \mathcal{G} \,|\, \Gamma \Rightarrow C \,|\, \Box\Gamma, \Gamma' \Rightarrow \Box\Delta, \Delta'}{\vdash \mathcal{G} \,|\, \Box\Gamma, \Gamma' \Rightarrow \Box C, \Box\Delta, \Delta'} \Box$$

  If we want to add $\Box B$ to the succedent we can simply apply the secondary induction hypothesis and then the rule again. Otherwise, we get the following configuration:

$$\dfrac{\dfrac{\dfrac{\vdash \mathcal{G} \,|\, \Gamma \Rightarrow C \,|\, \Box\Gamma, \Gamma' \Rightarrow \Box\Delta, \Delta'}{\vdash \mathcal{G} \,|\, \Gamma \Rightarrow C \,|\, \Box\Gamma, \Box B, \Gamma' \Rightarrow \Box\Delta, \Delta'} LW}{\vdash \mathcal{G} \,|\, \Gamma, B \Rightarrow C \,|\, \Box\Gamma, \Box B, \Gamma' \Rightarrow \Box\Delta, \Delta'} LW}{\vdash \mathcal{G} \,|\, \Box\Gamma, \Gamma', \Box B, \Rightarrow \Box C, \Box\Delta, \Delta'} \Box$$

The first application of LW is removed by secondary induction hypothesis, while the second by the primary induction hypothesis.

- It remains to consider the case in which A is a formula whose principal connective is one among $\wedge$, $\vee$, and $\to$. In these case, we decompose the formula A by applying invertibility of the rules for the classical connectives, then we add the formulas as described in the preceding subcases. $\square$

LEMMA 3.3. *The rules of contraction* (LC) *and* (RC) *and external contraction* (EC) *are all height-preserving admissible.*

PROOF: By simultaneous induction on the height of derivations. External contraction follows by a straightforward induction on the height of the derivation under analysis by applying height-preserving invertibility of the logical rules.

Internal contraction is slightly more delicate to handle. The critical situation is the one in which we have a hypersequent $\vdash \mathcal{G} \,|\, \Box\Gamma, \Gamma' \Rightarrow \Box A, \Box A, \Box\Delta, \Delta'$ and the formula $\Box A$ is principal in the last rule applied. In this case, we consider the premise

$$\vdash \mathcal{G} \,|\, \Gamma \Rightarrow A \,|\, \Box\Gamma, \Gamma' \Rightarrow \Box A, \Box\Delta, \Delta'$$

and we proceed in the following way

$$\cfrac{\cfrac{\cfrac{\vdash \mathcal{G} \,|\, \Gamma \Rightarrow A \,|\, \Box\Gamma, \Gamma' \Rightarrow \Box A, \Box\Delta, \Delta'}{\vdash \mathcal{G} \,|\, \Gamma \Rightarrow A \,|\, \Gamma \Rightarrow A \,|\, \Box\Gamma, \Gamma' \Rightarrow \Box\Delta, \Delta'}\,\mathit{Inv}\text{-}\Box}{\vdash \mathcal{G} \,|\, \Gamma \Rightarrow A \,|\, \Box\Gamma, \Gamma' \Rightarrow \Box\Delta, \Delta'}\,\mathrm{EC}}{\vdash \mathcal{G} \,|\, \Box\Gamma, \Gamma' \Rightarrow \Box A, \Box\Delta, \Delta'}\,\Box$$

$\square$

THEOREM 3.4. *The cut-rule is admissible.*

PROOF: The proof is by double induction with main induction hypothesis on the degree of the cut-formula and the secondary induction hypothesis on the sum of the height of the derivation of the premises of the cut.

We distinguish the following cases. If the right premise of the cut is an initial sequent, then, when the cut formula is not active, we remove it. Otherwise, the conclusion follows by weakening.

If the right premise of the cut is the conclusion of a logical rule different from $\Box$, we distinguish two subcases according to whether the cut-formula is principal or not. In the former case, we apply the invertibility of the corresponding rule and we replace the cut-application under consideration with cuts on formulas of smaller degree. In the latter case we permute the cut upwards.

If the last inference step is a $\Box$-application, then the cut-formula is either atomic or a modal formula. In both cases, we argue by induction on the left premise of the cut. The relevant case is the one in which the last rule applied is $\Box$. We have:

$$\dfrac{\dfrac{\vdash \mathcal{G} \,|\, \Gamma \Rightarrow A \,|\, \Box\Gamma, \Gamma' \Rightarrow \Box\Delta, \Delta'}{\vdash \mathcal{G} \,|\, \Box\Gamma, \Gamma' \Rightarrow \Box\Delta, \Box A, \Delta'}\,\Box \qquad \dfrac{\vdash \mathcal{H} \,|\, A, \Pi \Rightarrow B \,|\, \Box A, \Box\Pi, \Pi' \Rightarrow \Box\Sigma, \Sigma'}{\vdash \mathcal{H} \,|\, \Box A, \Box\Pi, \Pi' \Rightarrow \Box\Sigma, \Box B, \Sigma'}\,\Box}{\vdash \mathcal{G} \,|\, \mathcal{H} \,|\, \Box\Gamma, \Box\Pi, \Gamma', \Pi' \Rightarrow \Box\Delta, \Box\Sigma, \Box B, \Delta', \Sigma'}\,Cut$$

The cut is removed as follows (we avoid writing the contexts for better readability). First, we apply a cross-cut:

$$\dfrac{\vdash \Box\Gamma, \Gamma' \Rightarrow \Box\Delta, \Box A, \Delta' \qquad \vdash A, \Pi \Rightarrow B \,|\, \Box A, \Box\Pi, \Pi' \Rightarrow \Box\Sigma, \Sigma'}{\vdash A, \Pi \Rightarrow B \,|\, \Box\Gamma, \Box\Pi, \Gamma', \Pi' \Rightarrow \Box\Delta, \Box\Sigma, \Delta', \Sigma'}\,Cut$$

The cut is removed by applying the secondary induction hypothesis. The reduction is then completed as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash \Gamma \Rightarrow A \,|\, \Box\Gamma, \Gamma' \Rightarrow \Box\Delta, \Delta' \qquad \vdash A, \Pi \Rightarrow B \,|\, \Box\Gamma, \Box\Pi, \Gamma', \Pi' \Rightarrow \Box\Delta, \Box\Sigma, \Delta', \Sigma'}{\vdash \Gamma, \Pi \Rightarrow B \,|\, \Box\Gamma, \Gamma' \Rightarrow \Box\Delta, \Delta' \,|\, \Box\Gamma, \Box\Pi, \Gamma', \Pi' \Rightarrow \Box\Delta, \Box\Sigma, \Delta', \Sigma'}\,Cut}{\vdash \Gamma, \Pi \Rightarrow B \,|\, (\Box\Gamma, \Box\Pi, \Gamma', \Pi' \Rightarrow \Box\Delta, \Box\Sigma, \Delta', \Sigma')^2}\,LW, RW}{\vdash \Gamma, \Pi \Rightarrow B \,|\, \Box\Gamma, \Box\Pi, \Gamma', \Pi' \Rightarrow \Box\Delta, \Box\Sigma, \Delta', \Sigma'}\,\mathrm{EC}}{\vdash \Box\Gamma, \Box\Pi, \Gamma', \Pi' \Rightarrow \Box\Delta, \Box B, \Box\Sigma, \Delta', \Sigma'}\,\Box$$

where the cut-rule is removed by primary induction hypothesis on the degree of the cut-formula. $\Box$

We consider now the system **HD**. In this case the analysis proceeds analogously. Of course, the admissibility of the structural rules needs to be established once again.

LEMMA 3.5. *Every rule is height-preserving invertible in* **HD**.

PROOF: The only new case to be detailed is the one involving the rule d. In this case the proof is immediate, as the only applicable rule is *d* which acts on all the formulas in the antecedents. $\Box$

LEMMA 3.6. *The weakening rules* (EW), (LW) *and* (RW) *are admissible.*

PROOF: External weakening is established by a straightforward induction on the height of the derivation. Proving the admissibility of W requires a double induction, with main induction hypothesis on the degree of the formula and secondary induction hypothesis on the height of derivations.

The only new case to detail is the one involving rule d. As usual, we need to proceed by cases. If the formula to be added is an atomic formula, then we simply apply the secondary induction hypothesis and then the rule again. If it is a boxed formula to be added in the antecedent, then we apply the primary induction hypothesis on the degree of the formula and then the rule again.

In the remaining cases we first decompose the formulaand we then obtain some hypersequents which contain only boxed formulas in the antecedents of the components and atomic formulas. Hence we apply the primary induction hypothesis and then we apply the rules in the reverse order. $\square$

LEMMA 3.7. *The rules of contraction are height-preserving admissible.*

PROOF: The proof is by induction on the height of the derivation. The only new case to discuss is the one involving the rule d. We have:

$$\frac{\vdash \mathcal{G} \mid A, A, \Gamma \Rightarrow \ \mid \Pi \Rightarrow \Sigma}{\vdash \mathcal{G} \mid \Box A, \Box A, \Box\Gamma, \Pi \Rightarrow \Sigma}\ \mathsf{d}$$

We proceed as follows:

$$\dfrac{\dfrac{\vdash \mathcal{G} \mid A, A, \Gamma \Rightarrow \ \mid \Pi \Rightarrow \Sigma}{\vdash \mathcal{G} \mid A, \Gamma \Rightarrow \ \mid \Pi \Rightarrow \Sigma}\ LC}{\vdash \mathcal{G} \mid \Box A, \Box\Gamma, \Pi \Rightarrow \Sigma}\ \mathsf{d}$$

The application of LC is removed by the induction hypothesis on the height of the derivation. $\square$

THEOREM 3.8. *The cut rule is admissible in* **HD**.

PROOF: By double induction. We discuss only the new interesting case.

$$\dfrac{\dfrac{\vdash \mathcal{G} \mid \Gamma \Rightarrow A \mid \Box\Gamma, \Gamma' \Rightarrow \Box\Delta, \Delta'}{\vdash \mathcal{G} \mid \Box\Gamma, \Gamma' \Rightarrow \Box\Delta, \Box A, \Delta'}\ \Box \qquad \dfrac{\vdash \mathcal{H} \mid A, \Pi \Rightarrow \ \mid \Theta \Rightarrow \Sigma}{\vdash \mathcal{H} \mid \Box A, \Box\Pi, \Theta \Rightarrow \Sigma}\ \mathsf{d}}{\vdash \mathcal{G} \mid \mathcal{H} \mid \Box\Gamma, \Box\Pi, \Gamma', \Theta \Rightarrow \Box\Delta, \Sigma, \Delta'}\ Cut$$

We proceed as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash \mathcal{G} \,|\, \Gamma \Rightarrow A \,|\, \Box\Gamma, \Gamma' \Rightarrow \Box\Delta, \Delta' \qquad \vdash \mathcal{H} \,|\, A, \Pi \Rightarrow \;|\, \Theta \Rightarrow \Sigma}{\vdash \mathcal{G} \,|\, \mathcal{H} \,|\, \Gamma, \Pi \Rightarrow \;|\, \Theta \Rightarrow \Sigma \,|\, \Box\Gamma, \Gamma' \Rightarrow \Box\Delta, \Delta'}\; Cut}{\vdash \mathcal{G} \,|\, \mathcal{H} \,|\, \Box\Gamma, \Box\Pi, \Theta \Rightarrow \Sigma \,|\, \Box\Gamma, \Gamma' \Rightarrow \Box\Delta, \Delta'}\; \mathsf{d}}{\vdash \mathcal{G} \,|\, \mathcal{H} \,|\, (\Box\Gamma, \Box\Pi, \Gamma', \Theta \Rightarrow \Box\Delta, \Sigma, \Delta')^2}\; LW, RW}{\vdash \mathcal{G} \,|\, \mathcal{H} \,|\, \Box\Gamma, \Box\Pi, \Gamma', \Theta \Rightarrow \Box\Delta, \Sigma, \Delta'}\; EC$$

The cut is replaced by a cut on a formula of smaller degree and the conclusion is obtained applying the rule d followed by weakening and contraction. □

We now consider the case of **HM**. Since by now the reader should be acquainted with the strategies employed to establish the structural properties of this kind of calculi we shall not get into the details.

LEMMA 3.9. *Every rule is height-preserving invertible.*

PROOF: We deal with m. If $\vdash \mathcal{G} \,|\, \Box A_1, \ldots, \Box A_n, \Pi \Rightarrow \Box\Delta, \Box B, \Box C, \Sigma$ is an initial sequent, so is $\vdash \mathcal{G} \,|\, A_1 \Rightarrow C \,|\, \ldots \,|\, A_n \Rightarrow C \,|\, \Box A_1, \ldots, \Box A_n, \Pi \Rightarrow \Box\Delta, \Box B, \Sigma$. If it is the conclusion of a rule, we apply the induction hypothesis to each of the premises and then the rule again. For example, we have:

$$\dfrac{\vdash \mathcal{G} \,|\, A_1 \Rightarrow B \,|\, \ldots \,|\, A_n \Rightarrow B \,|\, \Box A_1, \ldots, \Box A_n, \Pi \Rightarrow \Box\Delta, \Box C, \Sigma}{\vdash \mathcal{G} \,|\, \Box A_1, \ldots, \Box A_n, \Pi \Rightarrow \Box\Delta, \Box B, \Box C, \Sigma}\; \mathsf{m}$$

We proceed as follows:

$$\dfrac{\dfrac{\vdash \mathcal{G} \,|\, A_1 \Rightarrow B \,|\, \ldots \,|\, A_n \Rightarrow B \,|\, \Box A_1, \ldots, \Box A_n, \Pi \Rightarrow \Box\Delta, \Box C, \Sigma}{\vdash \mathcal{G} \,|\, A_1 \Rightarrow B \,|\, \ldots \,|\, A_n \Rightarrow B \,|\, A_1 \Rightarrow C \,|\, \ldots \,|\, A_n \Rightarrow C \,|\, \Box A_1, \ldots, \Box A_n, \Pi \Rightarrow \Box\Delta, \Sigma}\; \mathrm{IH}}{\vdash \mathcal{G} \,|\, A_1 \Rightarrow C \,|\, \ldots \,|\, A_n \Rightarrow C \,|\, \Box A_1, \ldots, \Box A_n, \Pi \Rightarrow \Box\Delta, \Box B, \Sigma}\; \mathsf{m}$$

□

LEMMA 3.10. *The rules $(EW), (LW)$ and (RW) are admissible.*

PROOF: EW. Straightforward by induction on the height of the derivation. With respect to W we argue by double induction as above with minor changes. □

LEMMA 3.11. *The rules $(EC), (LC)$ and (RC) are height-preserving admissible.*

PROOF: By induction on the height of the derivation. We deal with the only relevant cases.

$$\dfrac{\vdash \mathcal{G} \,|\, A_1 \Rightarrow B \,|\, \ldots \,|\, A_n \Rightarrow B \,|\, \Box A_1, \ldots, \Box A_n, \Pi \Rightarrow \Box\Delta, \Box B, \Sigma}{\vdash \mathcal{G} \,|\, \Box A_1, \ldots, \Box A_n, \Pi \Rightarrow \Box\Delta, \Box B, \Box B, \Sigma}\ \mathsf{m}$$

We proceed as follows:

$$\dfrac{\dfrac{\dfrac{\vdash \mathcal{G} \,|\, A_1 \Rightarrow B \,|\, \ldots \,|\, A_n \Rightarrow B \,|\, \Box A_1, \ldots, \Box A_n, \Pi \Rightarrow \Box\Delta, \Box B, \Sigma}{\vdash \mathcal{G} \,|\, (A_1 \Rightarrow B)^2 \,|\, \ldots \,|\, (A_n \Rightarrow B)^2 \,|\, \Box A_1, \ldots, \Box A_n, \Pi \Rightarrow \Box\Delta, \Sigma}\ \mathit{Inv}\text{-}\mathsf{m}}{\vdash \mathcal{G} \,|\, A_1 \Rightarrow B \,|\, \ldots \,|\, A_n \Rightarrow B \,|\, \Box A_1, \ldots, \Box A_n, \Pi \Rightarrow \Box\Delta, \Sigma}\ \mathit{EC}}{\vdash \mathcal{G} \,|\, \Box A_1, \ldots, \Box A_n, \Pi \Rightarrow \Box\Delta, \Box B, \Sigma}\ \mathsf{m}$$

If the formula to contract is in the antecedent, we proceed analogously, possibly exploiting external contraction and the induction hypothesis on the height of the derivation. □

The last step is the cut-elimination theorem.

THEOREM 3.12. *The cut rule is admissible in* **HM**.

PROOF: By double induction on the degree of the cut formula and the sum of the height of the derivations of the premises of the cut. We discuss the case in which the cut formula is principal in both the premises in an application of the rule m.

$$\dfrac{\dfrac{\mathcal{G} \,|\, A_1 \Rightarrow C_1 \,|\, \ldots \,|\, A_n \Rightarrow C_1 \,|\, \Box A_1, \ldots, \Box A_n, \Gamma \Rightarrow \Box\Delta, \Delta'}{\mathcal{G} \,|\, \Box A_1, \ldots, \Box A_n, \Gamma \Rightarrow \Box\Delta, \Box C_1, \Delta'}\ \mathsf{m} \qquad \dfrac{\mathcal{H} \,|\, C_1 \Rightarrow D \,|\, \ldots \,|\, C_n \Rightarrow D \,|\, \Box C_1, \ldots, \Box C_n, \Pi \Rightarrow \Box\Sigma, \Sigma'}{\mathcal{H} \,|\, \Box C_1, \ldots, \Box C_n, \Pi \Rightarrow \Box\Sigma, \Box D, \Sigma'}\ \mathsf{m}}{\mathcal{G} \,|\, \mathcal{H} \,|\, \Box A_1, \ldots, \Box A_n, \Gamma, \Box C_2, \ldots, \Box C_n, \Pi \Rightarrow \Box\Sigma, \Box D, \Sigma', \Box\Delta, \Delta'}\ \mathit{Cut}$$

We construct the following derivation (we omit the contexts for better readability):

$$\dfrac{\vdash \Box A_1, \ldots, \Box A_m, \Gamma \Rightarrow \Box\Delta, \Box C_1, \Delta' \qquad \vdash C_1 \Rightarrow D \,|\, \ldots \,|\, C_n \Rightarrow D \,|\, \Box C_1, \ldots, \Box C_n, \Pi \Rightarrow \Box\Sigma, \Sigma'}{\vdash C_1 \Rightarrow D \,|\, \ldots \,|\, C_n \Rightarrow D \,|\, \Box A_1, \ldots, \Box A_m, \Gamma, \Box C_2, \ldots, \Box C_n, \Pi \Rightarrow \Box\Sigma, \Sigma', \Box\Delta, \Delta'}\ \mathit{Cut}$$

The cut is removed by secondary induction hypothesis. Next, we cut on C_1. We write $\mathcal{S}$ as an abbreviation for $\vdash \Box A_1, \ldots, \Box A_m, \Gamma, \Box C_2, \ldots, \Box C_n, \Pi \Rightarrow \Box\Sigma, \Sigma', \Box\Delta, \Delta'$. We have:

$$\dfrac{\dfrac{\vdash A_1 \Rightarrow C_1 \,|\, \ldots \,|\, A_m \Rightarrow C_1 \,|\, \Box A_1, \ldots, \Box A_m, \Gamma \Rightarrow \Box\Delta, \Delta' \qquad \vdash C_1 \Rightarrow D \,|\, \ldots \,|\, C_n \Rightarrow D \,|\, \mathcal{S}}{\vdash A_1 \Rightarrow D \,|\, \ldots \,|\, A_m \Rightarrow C_1 \,|\, \ldots \,|\, C_n \Rightarrow D \,|\, \Box A_1, \ldots, \Box A_m, \Gamma \Rightarrow \Box\Delta, \Delta' \,|\, \mathcal{S}}\ \mathit{Cut}}{\vdash A_1 \Rightarrow D \,|\, \ldots \,|\, A_m \Rightarrow C_1 \,|\, \ldots \,|\, C_n \Rightarrow D \,|\, \mathcal{S}}\ \mathit{LW}, \mathit{RW}, \mathit{EC}$$

We now apply again a cut on C_1 between $\vdash A_1 \Rightarrow D \,|\, \dots \,|\, A_m \Rightarrow C_1 \,|\, \dots \,|\, C_n \Rightarrow D \,|\, \mathcal{S}$ and $\vdash C_1 \Rightarrow D \,|\, \dots \,|\, C_n \Rightarrow D \,|\, \mathcal{S}$ which yields (modulo contraction)

$$\vdash A_1 \Rightarrow D \,|\, A_2 \Rightarrow D \,|\, \dots \,|\, A_m \Rightarrow C_1 \,|\, \dots \,|\, C_n \Rightarrow D \,|\, \mathcal{S}$$

By repeating this procedure (formalizable by induction on m), we get:

$$\vdash A_1 \Rightarrow D \,|\, A_2 \Rightarrow D \,|\, \dots \,|\, A_m \Rightarrow D \,|\, \dots \,|\, C_n \Rightarrow D \,|\, \mathcal{S}$$

An application of the rule m gives the desired conclusion. $\square$

The last system that we analyze is **HE**. We state the preliminary structural properties omitting the proofs which can be obtained along the same lines as the previously discussed systems.

PROPOSITION 3.13. The rule of weakening is admissible. Every rule of the calculus is height-preserving invertible. The rule of contraction is height-preserving admissible.

To conclude the section we discuss cut-elimination for the case of **HE**. Instead of lingering on abstract technicalities, we give a concrete example of reduction and we leave to the reader the generalization of the argument.

$$\dfrac{\dfrac{\vdash \mathcal{G} \,|\, \Rightarrow A \leftrightarrow C \,|\, \Rightarrow B \leftrightarrow C \,|\, \Gamma \Rightarrow \Delta}{\vdash \mathcal{G} \,|\, \Box A, \Box B, \Gamma \Rightarrow \Delta, \Box C}\,\mathsf{e} \qquad \dfrac{\vdash \mathcal{G}' \,|\, \Rightarrow C \leftrightarrow D \,|\, \Rightarrow C \leftrightarrow E \,|\, \Pi \Rightarrow \Sigma}{\vdash \mathcal{G}' \,|\, \Box C, \Pi \Rightarrow \Sigma, \Box D, \Box E}\,\mathsf{e}}{\vdash \mathcal{G} \,|\, \mathcal{G}' \,|\, \Box A, \Box B, \Gamma, \Pi \Rightarrow \Delta, \Sigma, \Box D, \Box E}\,Cut$$

We first observe that the rule:

$$\dfrac{\vdash \mathcal{G} \,|\, \Rightarrow A \leftrightarrow B \qquad \vdash \mathcal{G}' \,|\, \Rightarrow B \leftrightarrow C}{\vdash \mathcal{G} \,|\, \mathcal{G}' \,|\, \Rightarrow A \leftrightarrow C}\,Eq$$

is admissible via cuts on formulas of lower size. Hence we propose the following reduction containing applications of Eq (we omit the contexts and the turnstiles and the applications of the rule EC for reasons of space):

$$\dfrac{\dfrac{\Rightarrow A \leftrightarrow C \,|\, \Rightarrow B \leftrightarrow C \qquad \dfrac{\dfrac{\Rightarrow A \leftrightarrow C \,|\, \Rightarrow B \leftrightarrow C \qquad \Rightarrow C \leftrightarrow D \,|\, \Rightarrow C \leftrightarrow E}{\Rightarrow A \leftrightarrow D \,|\, \Rightarrow B \leftrightarrow C \,|\, \Rightarrow C \leftrightarrow E} \qquad \Rightarrow C \leftrightarrow D \,|\, \Rightarrow C \leftrightarrow E}{\Rightarrow A \leftrightarrow D \,|\, \Rightarrow B \leftrightarrow D \,|\, \Rightarrow C \leftrightarrow E}}{\Rightarrow A \leftrightarrow D \,|\, \Rightarrow B \leftrightarrow D \,|\, \Rightarrow A \leftrightarrow E \,|\, \Rightarrow B \leftrightarrow C} \qquad \Rightarrow A \leftrightarrow D \,|\, \Rightarrow B \leftrightarrow D \,|\, \Rightarrow C \leftrightarrow E}{\Rightarrow A \leftrightarrow D \,|\, \Rightarrow B \leftrightarrow D \,|\, \Rightarrow A \leftrightarrow E \,|\, \Rightarrow B \leftrightarrow E}$$

All the cuts are removed by primary induction hypothesis on the degree of the cut formula.

THEOREM 3.14. *The cut rule is admissible in* **HE**.

As a matter of fact, proofs in the hypersequent calculi here proposed amount to the decomposition of the endsequent into non further analyzable top-hypersequents. The calculi enjoy invertibility of every rule with preservation of the height. In addition, as it will be shown in the next section, the decomposition is unique or, which is equivalent, the calculus enjoys the stability property.

4. Development of fractional semantics

4.1. Conservativity over the base logic

Conservativity stems from the soundness and the completeness of the calculus. Soundness is established with respect to structures which interpret modal logics.

DEFINITION 4.1. An **E**-*neighborhood model* is a triple $\langle \mathcal{W}, \mathcal{I}, \mathcal{V} \rangle$, where $\mathcal{W}$ is a non-empty set, $\mathcal{I} : \mathcal{W} \to \mathcal{P}(\mathcal{P}(\mathcal{W}))$ and $\mathcal{V} : AT \to \mathcal{P}(\mathcal{W})$. Truth conditions for a formula A in a world x in a model are inductively defined as follows:

- $x \Vdash p$ if and only if $x \in \mathcal{V}(P)$.
- $x \Vdash B \wedge C$ if and only if $x \Vdash B$ and $x \Vdash C$.
- $x \Vdash B \vee C$ if and only if $x \Vdash B$ or $x \Vdash C$.
- $x \Vdash \neg B$ if and only if $x \nVdash B$.
- $x \Vdash \Box B$ if and only if $\{y \mid y \Vdash B\} \in \mathcal{I}(x)$.

An **M**-*neighborhood model* is an **E**-*neighborhood model* with the additional condition: if $a \in \mathcal{I}(x)$ and $a \subseteq b$ then $b \in \mathcal{I}(x)$. A **K**-*neighborhood model* is an **M**-*neighborhood model* in which, if $a \in \mathcal{I}(x)$ and $b \in \mathcal{I}(x)$ then we get both $a \cap b \in \mathcal{I}(x)$ and $\mathcal{I}(x) \neq \emptyset$, for every x. A **D**-*neighborhood model* is a **K**-*neighborhood model* satisfying the following additional condition: $a \in \mathcal{I}(x) \Rightarrow a^c \notin \mathcal{I}(x)$.

The definition of validity for a hypersequent in this setting is as follows: $\mathcal{G}$ is valid if one of its components it valid.

Proposition 4.2. If **HX** proves $\vdash\Rightarrow A$, then A is valid.

Proof: The proof is by induction on the height of the derivation in the corresponding hypersequent calculus. We discuss the case of **HE** as an example. Suppose the hypersequent $\vdash \mathcal{G} \,|\, [\Rightarrow A_i \leftrightarrow B_j] \,|\, \Gamma \Rightarrow \Delta$ is valid, hence one of the components is valid. If any component in $\mathcal{G}$ or $\Gamma \Rightarrow \Delta$ is valid, then so is the conclusion, trivially. If for some i, j $A_i \leftrightarrow B_j$ is valid, then this implies that $\Box A_i \leftrightarrow \Box B_j$ is valid and therefore the validity of the conclusion follows. □

As regards completeness, it suffices to establish that whenever we have a derivation of the Hilbert style calculus for a given modal logic, the corresponding sequent is derivable in our calculus too.

We recall here the modular presentation of the Hilbert style systems for the logics considered here.

- The system **E** is axiomatized by adding to a Hilbert-style calculus for classical propositional logic the rule:

$$\frac{\vdash A \leftrightarrow B}{\vdash \Box A \leftrightarrow \Box B}\ \mathrm{E}$$

- The system **M** is axiomatized by adding to **E** the rule:

$$\frac{\vdash A \rightarrow B}{\vdash \Box A \rightarrow \Box B}\ \mathrm{M}$$

- The system **K** is axiomatized by adding to a Hilbert-style calculus for classical propositional logic the axiom $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$ and the rule:

$$\frac{\vdash A}{\vdash \Box A}\ \mathrm{RN}$$

- The system **D** is axiomatized by adding to **K** the axiom $\Box A \rightarrow \Diamond A$.

Theorem 4.3. *If* **X** *proves* $\vdash A$*, then* $\overline{\overline{\mathbf{HX}}} \vdash\Rightarrow A$ *for* $\mathbf{X} \in \{\mathbf{K}, \mathbf{M}, \mathbf{D}\}$.

Proof: The proof is by induction on the height of the derivation in the system **X**. We give an example of the derivation of the axiom **D** in **HD**:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash A \Rightarrow A}{\vdash A, \neg A \Rightarrow}\,\mathrm{L}\neg}{\vdash \Box A, \Box \neg A \Rightarrow}\,\mathsf{d}}{\vdash \Box A \Rightarrow \neg \Box \neg B}\,\mathrm{R}\!\to}{\vdash \Rightarrow \Box A \to \neg \Box \neg A}\,\mathrm{R}\!\to$$

With respect to the rules of the calculus, we show the admissibility of the rule M in the calculus **HM**:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash \Rightarrow A \to B}{\vdash A \Rightarrow B}\,\mathrm{Inv}\!\to}{\vdash A \Rightarrow B \mid \Box A \Rightarrow}\,\mathrm{EW}}{\vdash \Box A \Rightarrow \Box B}\,\mathsf{m}}{\vdash \Rightarrow \Box A \to \Box B}\,\mathrm{R}\!\to$$

of modus ponens:

$$\dfrac{\vdash \Rightarrow A \qquad \dfrac{\vdash \Rightarrow A \to B}{\vdash A \Rightarrow B}\,\mathrm{Inv}\!\to}{\vdash \Rightarrow B}\,\mathrm{Cut}$$

and of the E rule in **HE**:

$$\dfrac{\dfrac{\dfrac{\vdash \Rightarrow A \leftrightarrow B}{\vdash \Box B \Rightarrow \Box A}\,\mathsf{e}}{\Rightarrow \Box B \to \Box A}\,\mathrm{R}\!\to \qquad \dfrac{\dfrac{\vdash \Rightarrow B \leftrightarrow A}{\vdash \Box A \Rightarrow \Box B}\,\mathsf{e}}{\vdash \Rightarrow \Box A \to \Box B}\,\mathrm{R}\!\to}{\vdash \Rightarrow \Box A \leftrightarrow \Box B}\,\mathrm{R}\wedge$$

□

As a corollary of the embedding we get the completeness of the resulting system. Soundness is obtained as usual through a straightforward induction on the height of the derivation of the system and thus we omit the details.

Corollary 4.4. The systems $\overline{\overline{\overline{\mathbf{HX}}}}$ are sound and complete with respect to the logics **X**.

Proof: If A is valid, then it is derivable in the corresponding axiomatic calculus and so in $\overline{\overline{\overline{\mathbf{HX}}}}$. □

4.2. Fractional valued non-normal modal logics

In order to develop a fractional interpretation of non-normal modal logics, we need to show that the assignment of values to formula does not depend on the specific shape of the derivations.

THEOREM 4.5 (Stability). *If π and ρ are two $\overline{\overline{\overline{\mathbf{HX}}}}$-derivations ending with the same hypersequent, then* $\mathsf{top}(\pi) = \mathsf{top}(\rho)$.

PROOF: The proof is standardly led by induction on the height n of the derivation of π. If $n = 0$, then the claim comes straightforwardly. Otherwise we distinguish cases according to the last rule applied. We consider the case in which the last inference is an application of a unary rule, that is:

$$\dfrac{\begin{array}{c}\pi' \\ \vdots \\ \overset{i}{\vdash} \mathcal{G}'\end{array}}{\overset{i}{\vdash} \mathcal{G}}\;\mathsf{r}$$

We apply the invertibility of the rule r to get a derivation ρ' of $\mathcal{G}'$. Since the height of π' is strictly lower than that of π, we can apply the induction hypothesis to get $\mathsf{top}(\pi') = \mathsf{top}(\rho')$, which immediately yields the desired conclusion. □

Due to the stability property, we can now consider the multiset of top-hypersequents associated with a given formula as a derivation-invariant notion. That is, the multiset decomposition remains stable through different derivations of the same hypersequent.

DEFINITION 4.6. Given a formula A, $\mathsf{top}_{\mathbf{X}}(A)$ is the multiset of the top-hyperclauses in any of the $\overline{\overline{\overline{\mathbf{HX}}}}$-derivation ending in ($\vdash$ or $\dashv$) $\Rightarrow A$. The multiset $\mathsf{top}_{\mathbf{X}}(A)$ is partitioned into the two multisets $\mathsf{top}^1_{\mathbf{X}}(A)$ and $\mathsf{top}^0_{\mathbf{X}}(A)$ collecting all the hyperclauses signed by '$\vdash$' and the hyperclauses signed by '$\dashv$', respectively.

DEFINITION 4.7 (Fractional evaluation function). Let $\mathbb{Q}^* = [0, 1] \cap \mathbb{Q}$, i.e., $\mathbb{Q}^*$ is the set of the rational numbers in the closed interval $[0, 1]$. For each system $\mathbf{X}$, the evaluation function $\llbracket \cdot \rrbracket_{\mathbf{X}} : \mathscr{F} \mapsto \mathbb{Q}^*$ is defined as follows: for any logical formula A,

$$\llbracket A \rrbracket_{\mathbf{X}} = \frac{\#\mathsf{top}^1_{\mathbf{X}}(A)}{\#\mathsf{top}_{\mathbf{X}}(A)}$$

Let us emphasize some basic features about the evaluation function defined above. First, as already noticed, the Stability property makes the fractional evaluation of formulas a derivation-invariant, therefore the fractional method can be regarded as a semantics to all intents and purposes. Second, invertibility of of the rules of the calculus ensures that the relevant information stored in the conclusion is entirely preserved through the decomposition procedure. Third, the assignment is conservative over the base logic, as valid formulas are mapped to the maximum fractional value. The next theorem establishes the latter point.

Theorem 4.8 (Conservativity). *The formula A is $\mathbf{X}$-valid just in case $\llbracket A \rrbracket_{\mathbf{X}} = 1$.*

Proof: ($\Leftarrow$) If $\llbracket A \rrbracket_{\mathbf{X}} = 1$, then there is a $\mathbf{HX}$ ending in $\vdash\Rightarrow A$. By applying the soundness theorem we can infer the $\mathbf{X}$-validity of A.

($\Rightarrow$) If A is $\mathbf{X}$-valid, then by completeness there is a $\mathbf{HX}$ derivation ending in $\vdash\Rightarrow A$, so every initial top-hypersequent expresses an identity and therefore we get

$$\llbracket A \rrbracket_{\mathbf{X}} = \frac{\#\mathsf{top}^1_{\mathbf{X}}(A)}{\#\mathsf{top}_{\mathbf{X}}(A)} = \frac{\#\mathsf{top}^1_{\mathbf{X}}(A)}{\#\mathsf{top}^1_{\mathbf{X}}(A)} = 1$$

$\square$

Let $\mathscr{F}^c$ be the language of classical propositional logic. The next theorem establishes the surjectivity of the interpretation function $\llbracket \cdot \rrbracket$. In particular, we have:

Theorem 4.9. *For any $q \in \mathbb{Q}^*$: (i) there is a formula $A \in \mathscr{F}^c$ s.t. $\llbracket A \rrbracket_{\mathbf{X}} = q$, and ($ii$) there is a formula $B \in \mathscr{F} - \mathscr{F}^c$ s.t. $\llbracket B \rrbracket_{\mathbf{X}} = q$.*

Proof: Let $q = {}^m\!/_n$, where $m, n \in \mathbb{N}^+$ and $m \leqslant n$. (i) Consider the formula $\bigwedge(p \vee \neg p)^m \wedge \bigwedge p^{n-m}$. It is immediate to see that $\llbracket \bigwedge(p \vee \neg p)^m \wedge \bigwedge p^{n-m} \rrbracket_{\mathbf{X}} = {}^m\!/_n = q$.

(ii) We provide details for the modal logic $\mathbf{E}$, other systems can be handled analogously. We consider now the modal formula $\bigwedge(\Box p \to \Box p)^m \wedge$

$\bigwedge(\Box p)^{2n-m}$ in $\mathscr{F} - \mathscr{F}^c$. It turns out, similarly, that $\llbracket \bigwedge(\Box p \to \Box p)^m \wedge \bigwedge(\Box p)^{2n-m}\rrbracket_{\mathbf{X}} = {}^{2m}/_{2n} = {}^{m}/_{n} = q$. $\Box$

Remark 4.10. By combining Theorem 4.9 and the density of $\mathbb{Q}^*$, it is easy to verify that for, any modal system $\mathbf{X}$ and any pair of modal formulas A, B with $\llbracket A\rrbracket_{\mathbf{X}} < \llbracket B\rrbracket_{\mathbf{X}}$, we can always find a third formula $C \in \mathscr{F}^c$ such that $\llbracket A\rrbracket_{\mathbf{X}} < \llbracket C\rrbracket_{\mathbf{X}} < \llbracket B\rrbracket_{\mathbf{X}}$.

The previous theorem extends the result that has already been established for the modal logic **K** and serves as a bridge between classical and modal propositional logic. Specifically, for any modal formula, it is possible to provide a classical formula that has the same identity content as the modal one, as determined by the fractional interpretation. To illustrate this qualitative analysis, consider the modal formula $\Box(\Box p \to p) \to \Box p$ such that $\llbracket\Box(\Box p \to p) \to \Box p\rrbracket_{\mathbf{M}} = 0.5$. The decomposition algorithm ejects the modal component and returns the classical formula $(p \vee \neg p) \wedge p$ whose fractional interpretation is $\llbracket(p \vee \neg p) \wedge p\rrbracket_{\mathbf{M}} = 0.5$. In fact, the decomposition of the formula leads to two initial sequents: a tautological one and a complementary one.

5. Concluding remarks

We have developed new logical calculi for modal logic **D**, as well as the non-normal modal logics **M** and **E**. These systems are able to combine some of the most important proof-theoretic features: the subformula property (as a consequence of the cut-elimination theorem), finiteness of the proof-search space, and invertibility of the logical rules. By fine-tuning a variety of bilateralism based on the notion of rejection as underivability, we showed how to articulate a proof-based interpretation of the modal logics under focus.

We acknowledge that there are differences between canonical proof-theoretic semantics and fractional semantics, to the extent that a semantics in terms of proofs does not necessarily qualify as proof-theoretic. In particular, the fractional technique results in a multi-valued interpretation of the formulas in the language, whereas proof-theoretic semantics is completely disengaged from any "quantitative" form of evaluation. This fact deserves special consideration as it suggests that, when decidable systems are under consideration, the syntax/semantics dichotomy can be overcome by means of a proof-based interpretation, which nonetheless entails a quantitative evaluation of the formulas in the language.

To conclude, we would like to say something about the problem of devising a proof-theoretic semantics for the modal operator of necessity. According to Kürbis, a proof-theoretic semantics should be seriously regarded as defective without a proper account of the $\Box$-modality [7]. The technical achievements in this paper show that modal formulas can be maximally analyzed by means of a set of logical rules which have the effect of progressively detecting the modal components as residual elements. That is, the "quantity of identity" present in a modal formula can be measured in essentially the same way as in classical logic, provided that the classical content has been properly isolated. The lesson to be learned is that, if we consider the fractional method as a legitimate variant of proof-theoretic semantics, the issue raised by Kürbis can be circumvented inasmuch as modal formulas can be evaluated without taking the meaning of the $\Box$-modality directly into account. In this sense, we believe that our work is a step towards a proof-theoretic semantics for modal logics Nonetheless, the problem of providing a fully satisfactory proof-theoretic account of the $\Box$-modality remains an open and challenging task, which requires further investigation and research.

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Mario Piazza
Scuola Normale Superiore
Classe di Lettere e Filosofia
Piazza dei Cavalieri 7
56126 Pisa, Italy
e-mail: mario.piazza@sns.it

Gabriele Pulcini
University of Rome "Tor Vergata"
Department of Literary, Philosophical and Art History Studies
Via Columbia 1
00133 Roma, Italy
e-mail: gabriele.pulcini@uniroma2.it

Matteo Tesi
Scuola Normale Superiore
Classe di Lettere e Filosofia
Piazza dei Cavalieri 7
56126 Pisa, Italy
e-mail: matteo.tesi@sns.it

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Nils Kürbis*

SUPPOSITION: A PROBLEM FOR BILATERALISM

Abstract

In bilateral logic formulas are signed by + and −, indicating the speech acts assertion and denial. I argue that making an assumption is also speech act. Speech acts cannot be embedded within other speech acts. Hence we cannot make sense of the notion of making an assumption in bilateral logic. Attempts to solve this problem are considered and rejected.

Keywords: Assertion, denial, negation, supposition, assumption, speech acts.

1. Introduction

According to bilateralist inferentialist semantics for the logical constants, their meanings are determined, not merely by rules of inference specifying their use in deductive arguments, but by rules specifying their use in deductive arguments that appeal to two primitive speech acts of assertion and denial. It is part of a wider position in the theory of meaning, proposed by Price, which, quite generally, 'takes the fundamental notion for a recursive theory of sense to be not assertion conditions alone, but these

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in conjunction with rejection, or *denial* conditions' [30, 162]. As Rumfitt puts it, 'mastering the sense of an atomic sentence A will involve learning methods whose deployment might entitle one either to affirm it or to reject it' [34, 797]. Accordingly, rules of inference in bilateral logic do not merely specify which conclusions follow from which premises, but they do so in a way that construes premises and conclusions as assertions or denials.

The most prominent system of bilateral logic has been proposed by Rumfitt [34], building on work by Smiley [37]. Humberstone [20] proposed a similar system around the same time as Rumfitt. Their formalism has been taken up by various writers with an interest in inferentialism or proof-theoretic semantics, such as Restall [32] and Francez [4].[1]

In this paper I shall point out a fundamental problem for the framework of bilateral logic. In bilateral logic, all formulas are supposed to be asserted or denied. Logical inference involves making and discharging assumptions, as witnessed also by the rules of bilateral logic. Assertion and denial are speech acts. Making an assumption is also a speech act. Hence bilateral logic demands that assertions and denials may be assumed and discharged. But this cannot be done, as speech acts cannot be iterated. Bilateral logic as it stands is thus incoherent.[2]

The final section considers two attempts to solve this problem by incorporating speech acts for supposition within bilateral logic or interpreting deductions in bilateral logic as conditional assertions and denials. I conclude that neither approach is successful.

2. Bilateral logic

The rules of bilateral logic are applied to asserted or denied formulas. It builds on the claim that there is 'a readily comprehensible variety of actual deductive practice in which the components of arguments express

[1] Rumfitt's system was devised with an eye on a formalisation of classical logic that respects Dummettian considerations on harmony. For a different and striking account of a natural proof system for classical logic see Restall [33].

[2] It has been observed before that treating assumptions like assertions or denials may pose a problem for bilateralist logic, e.g. by Incurvati and Smith [22, 230], Hjortland [19, 464, footnote 23] and myself ([26], [27, 221]), but as far as I am aware the present paper contains the first sustained discussion of the issue. Although I hope there to be some agreement, the other authors' remarks are too brief for it to be possible to assess whether they would accept the analysis of the source and the precise nature of the problem put forward here. The present paper keeps a promise to expound the details of my objection.

the assignation of affirmative or negative force to propositional contents' [34, 798]. Smiley and Rumfitt motivate this by examples of how questions and answers may figure in arguments.

Frege suggested that we can represent the content of a sentence, which we may also call a *proposition* or a *thought*, by a 'propositional question'[3], a question that asks for the answer 'Yes' or 'No'.[4] An assertion can then be effected by answering 'Yes' to such a question. This is as far as Frege went, who did not afford the answers 'Yes' and 'No' the same status, but preferred to keep only 'Yes' as primitive and to treat 'No' as analysed in terms 'Yes' and sentential negation. With some justice bilateralists observe that *prima facie* the answers 'Yes' and 'No' are on a par. Bilateralists hold that, just as an assertion may be effected by the answer 'Yes', a denial may be effected by answering 'No'. According to Smiley, 'a mechanism for rejection is there for anyone who wishes to use it, in the shape of an answer to a yes-or-no question. Questioner and answerer are usually different people, but if one puts the question to oneself, one comes up with the forms "P? Yes" and "P? No". I suggest that "...? Yes?" is a very passable realization of Frege's assertion-sign, the "judgment stroke" in his turnstile notation, and that "...? No" is an equally passable realization of a rejection-sign' [37, 1]. Notice that there are not two things, answering a propositional question with 'Yes' or 'No' and asserting or denying the corresponding declarative sentence: answering 'Yes' to a propositional question just is to assert the thought expressed; answering 'No' just is to deny it.

Rumfitt adapts an example of Smiley's, itself inspired by one of Frege's, to illustrate how propositional questions and their answers, and accordingly assertions and denials, may be used in deductive arguments:[5]

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[3]This is Geach's translation of Frege's *Satzfrage* [14, 143].

[4]Frege's view is slightly more nuanced, as he also acknowledges that the propositional question contains more than just the thought, namely the request that the question be answered. This nuance is of no consequence for present purposes. See [10, 62] and [11, 145].

[5]It is questionable whether Rumfitt's is a good example to motivate the bilateral cause. Weiss [43, 98] observes that if 'No' in the first premise is taken to reject the entire conditional, and that conditional is material, as Smiley and Rumfitt agree it is, then, by bilateral logic, the first premise already entails the conclusion and the second premise is superfluous. For the example to work as one that illustrates a two premise argument with a conclusion, the first premise must be understood as an assertion of the conditional 'If the accused was in Berlin at the time of the murder, he could not have committed the crime', i.e. as the answer 'Yes' to the corresponding propositional question.

If the accused was in Berlin at the time of the murder, could he have committed it? No.
Was the accused in Berlin at the time of the murder? Yes.
So: Could he have committed the murder? No.

Smiley's example, where $*$ indicates rejection and no star assertion, is:

If the accused was not in Berlin at the time of the murder, he did not commit the murder.
$*$The accused was in Berlin at the time of the murder.
So: $*$ The accused committed the murder.

Transposing it into question and answer format, the result is:

If the accused was not in Berlin, he did not commit the murder? Yes.
Was the accused in Berlin at the time of the murder? No.
So: Did the accused commit the murder? No.

Humberstone observes that Smiley's notation is confusing and does not capture the supposedly equal status of assertion and denial [20, 345]. It is preferable to introduce two symbols, one for assertion and one for denial. Humberstone and Rumfitt use + and −: 'Where A is a declarative sentence (or formula), let us introduce the signed sentences (or formulae) $+A$ and $-A$ to abbreviate Smiley's amalgams of questions with answers ⌜Is it the case that A? Yes⌝ and ⌜Is it the case that A? No⌝.' [34, 800] Result:

$+$ If the accused was not in Berlin, he did not commit the murder.
$-$ The accused was in Berlin at the time of the murder.
So: $-$ The accused committed the murder.

Thus, bilateralists argue, assertions and denials can be premises and conclusions in deductive arguments.

To specify the meanings of the logical constants in a bilateral inferential semantics, rules of inference must be formulated that specify the conditions under which formulas with the constants as main operators may be asserted and denied.

Rumfitt and Humberstone call the premises and conclusions of the rules of their bilateral logics *signed formulas*, i.e. signed by + and − representing the speech acts of assertion and denial. Lower case Greek letters range over signed formulas. α^* designates the *conjugate* of α, the result of reversing its sign from + to − and conversely. For each connective **c**, there are assertive rules specifying the grounds for and consequences of asserting a formula with **c** as main operator and rejective rules specifying the grounds for and consequences of denying such a formula [34, 800ff]. For purposes of illustration it suffices to give only some rules of some of the connectives of Rumfitt's system. The system below is, however, complete in the sense that the missing assertive and rejective rules for $\neg$ and $\supset$ as well as those for the other logical constants, defined as usual in terms of $\neg$ and $\supset$, are derivable:[6]

$$+\!\supset\! I : \dfrac{\begin{array}{c}\overline{+\ A}^{\,i}\\ \Pi \\ +\ B\end{array}}{+\ A \supset B}\,i \qquad\qquad +\!\supset\! E : \dfrac{+\ A \supset B \qquad +\ A}{+\ B}$$

$$-\neg I : \dfrac{+\ A}{-\ \neg A} \qquad\qquad -\neg E : \dfrac{-\ \neg A}{+\ A}$$

$$\text{Reductio:} \quad \dfrac{\begin{array}{c}\overline{\alpha}^{\,i}\\ \Pi \\ \bot\end{array}}{\alpha *}\,i \qquad\qquad \text{Non-Contradiction:} \quad \dfrac{\alpha \qquad \alpha *}{\bot}$$

Reductio and Non-Contradiction are bilateral versions of common principles, but here they have the character of structural rules governing the framework in which deductions are carried out rather than that of operational rules for logical constants. They codify relations between assertions and denials.

[6]This claim assumes that Rumfitt's requirement that the rule of Non-Contradiction be restricted to atomic premises is not imposed. This restriction is not relevant to what is at issue in the present paper. Kürbis's normalisation proof for Rumfitt's system [28] appeals to the unrestricted version, which may speak against imposing it. The proof contains a deplorable oversight, noted with a sketch of a correction in [29]

3. Force and content

Frege distinguishes the *content* of a sentence from the *force* with which it is put forward. Following the widely accepted treatment of this distinction by Hare, Searle and others, the same content can be asserted to be true, it can be asked whether it is true, it can be commanded that it be made true, it can be wished that it were true, etc..[7] Asserting, asking, commanding, wishing are activities speakers engage in: they are speech acts. Different such acts can have the same content. A speech act, being an activity, is not a proposition, and so it is not the kind of thing that can be used as a component in constructing larger propositions by sentential operators. Actions cannot be embedded into contexts that require propositions.[8]

A typical account of why the speech act of assertion cannot form part of propositions is found in Reichenbach's *Elements of Symbolic Logic*:

> Assertion is used in three different meanings: it denotes, first, the act of asserting; second, the result of this act, i.e., an expression of the form '$\vdash p$'; third, a statement which is asserted, i.e. a statement 'p' occurring within an expression '$\vdash p$'. It should be noticed that it is not possible to define the verb 'assert' in terms of the assertion sign. One might suppose that such a definition could be constructed by regarding the sentence '"p" is asserted' as having the same meaning as the expression '$\vdash p$'. But the coordination is not possible because '$\vdash p$' is not a sentence. [31, 346]

[7] See, e.g., Hare [18, Sec 2.1], Searle [36, 22f, 29ff], Stenius [38, 1f]. Frege's view is once more more nuanced than the received view, as was pointed out to me by Mark Textor. In 'On Sense and Reference', Frege expresses the view that imperatives and optatives do not express thoughts: 'A subordinate clause with "that" after "command," "ask," "forbid," would appear in direct speech as an imperative. Such a clause has no referent but only a sense. A command, a request, are indeed not thoughts, yet they stand on the same level as thoughts. Hence in subordinate clauses depending upon "command," "ask," etc., words have their indirect referents. The referent of such a clause is therefore not a truth value but a command, a request, and so forth.' [7, 38f] (Black's translation [14, 68].) This is a curious passage and of great interest, but I set it aside. The received view surely has much to be said for it, and Frege's point is orthogonal to present issues in as far as imperatives and optatives are adduced only for heuristic purposes and the focus of this paper lies elsewhere. I set this nuance aside, too.

[8] I set aside the question whether there are mental acts corresponding to speech acts that do not involve linguistic items. The discussion of Frege below mentions judgements, but for present purposes these can be assimilated to assertions.

Reichenbach then refers to his earlier analysis of the assertion sign as an example of a pragmatic sign. 'Expressions including a pragmatic sign are not propositions. They are not true or false, as is shown by the fact that they cannot be negated. [...] Since assertive expressions are not propositions, they cannot be combined by propositional operations.' [31, 337] This position is widely accepted, in particular by bilateralists.[9]

Speech acts can be described or reported by sentences in the third person[10] such as 'He asked whether p' and 'She asserted that q'. This differs from the performance of the speech act. If I report that she asserted that q, no such speech act with content expressed by 'q' need have been performed: my report may be mistaken. By contrast, if she asserts that q, a speech act with content expressed by 'q' has been performed, no matter whether q is true or false. Sentences describing or reporting speech acts are true or false. Speech acts are performed or not.

In bilateral logic, A represents the content of a speech act, + and − the forces assertion and denial. It makes no sense to put an action into the antecedent of a conditional, for instance: hence the sequence of symbols $(+\ A) \supset B$ is meaningless. It is crucial that that which is represented by + and − in bilateral logic cannot be embedded:

> It would be a confusion to construe the sign of rejection "−" as a notational variant for the negation operator "¬". Whether in a formal or a natural language, a sign of negation is a freely iterating sentence-forming operator on sentences: A, $\ulcorner\neg A\urcorner$, $\ulcorner\neg\neg A\urcorner$, etc. are all well-formed formulae. The sign of rejection, by contrast, was explained as the formal correlate of the operation of forming an interrogative sentence from a declarative sentence and appending the answer "No", and this operation cannot be iterated. "Is it the case that two is not a prime number? No" makes perfectly good sense, but "Is it the case that is it the case that two is a prime number? No? No" is gibberish. The sign "−", then, does not contribute to propositional content, but indicates the force with which that content is promulgated.

[9] Of course, now that the option has been mentioned, it may only be a matter of time before someone appears who rejects it.

[10] Sentences in the first person, such as 'I assert that q', by contrast, may achieve both, describe the utterer as performing a speech act and performing it, as pointed out to me by Mark Textor.

> Just as one asserts the entire content expressed by A by inscribing $\ulcorner + A\urcorner$, so one expressly rejects that same content by inscribing $\ulcorner - A\urcorner$. The symbol "+", in a word, is a Fregean assertion sign or Urtheilsstrich; and the symbol "−" is a cognate rejection sign or Verneinungsstrich. [34, 802f]

If expressions such as $(+\ A) \supset B$ or $-\ -\ A$ were legitimate, − and + would be mere notational variants of negation and the truth operator, expressing the trivial truth function mapping True to True and False to False, rather than indicators of the speech acts assertion and denial.

Bilateralists accept what Geach calls the *The Frege Point*: 'A thought may have just the same content whether you assent to its truth or not; a proposition may occur in discourse now asserted, now unasserted, and yet be recognizably the same proposition.' [13, 254f] According to Geach, a phrase cannot carry the assertoric force of an utterance or inscription if a sentence containing that phrase can be embedded into larger sentences, in particular if it can form the antecedent of a conditional: in such a context, the sentence is not asserted, hence the phrase that supposedly carried assertoric force cannot, after all, have done so [13, 262f]. The Frege Point provides a test for whether an expression carries the force of a speech act: if a sentence containing the expression can be embedded into a larger sentence so that the speech act is not performed by an utterance of the latter, then the expression cannot carry the force of the speech act.[11]

Geach went so far as to conclude that in ordinary language 'there is no naturally used sign of assertion [...]. That is why Frege had to devise a special sign.' [13, 262f] Bilateralists disagree with Geach's verdict: 'Yes' is such a sign. To argue this point, Rumfitt turns the test provided by the Frege Point into one for signs for speech acts: if a sentence containing a

[11] Geach observes that expressions such as 'the fact that' carry assertoric force even when occurring in embedded sentences. Geach analyses 'Jim is aware of the fact that his wife is unfaithful' as a 'double barrelled assertion' 'equivalent to the pair of assertions "Jim is convinced that his wife is unfaithful" and "Jim's wife is unfaithful".' [13, 259] The occurrence of the phrase 'the fact that' is not, however, a sign carrying the assertoric force of the sentence as a whole, but only of the clause following 'that'. In asserting 'If Jim is aware of the fact that his wife is unfaithful, then he is not showing it', an example I owe to Mark Textor, I do not assert that Jim is aware of the fact that his wife is unfaithful, but only that his wife is unfaithful. Standing alone the phrase 'the fact that' cannot be used to indicate the assertoric force of a sentence, as it forms a noun phrase from a sentence, not a sentence. 'The fact that p obtains' is again not the sign of assertoric force, as in asserting 'If the fact that p obtains, then q', I am not asserting that the fact that p obtains.

certain expression, or just some individual expression, cannot be embedded, this indicates that the sentence contains, or that the expression is, a sign of a speech act.[12] Other expressions that indicate speech acts are those for greetings, such as 'Hullo', or for valedictions, such as 'Adieu', or for the expression of gratitude, such as 'Thank you': none of these can be embedded.

'It is assertible that' or 'It is deniable that' are not correct renderings of the bilateralist's + and −: sentences beginning with them can be embedded in larger sentence, such as 'If it is assertible that p, then p', 'If it is deniable that p, then $\neg p$', 'It is deniable that it is assertible that p' or 'It is assertible that it is deniable that p'. 'It is assertible that' and 'It is deniable that' are sentential operators. The + and − of bilateral logic are signs that convey the forces of speech acts, in the same category as Frege's judgement stroke.

4. Supposition

One reason why Frege held that the distinction between sense and force is necessary is that it is possible to assume a proposition without asserting it, or, as he put it, without judging it to be true: 'This separation of the judgement from that which is judged appears to be unavoidable, as otherwise a mere assumption, the positing of a case without judging whether it arises, could not be expressed.' [6, 21f] Consequently, Frege explains, he introduces the judgement stroke, a vertical line, to indicate that a proposition is judged or asserted to be true. It is to be put to the left of 'the horizontal', in his early work called 'the content stroke' [5, § 2]. In a footnote, Frege continues: 'The judgement stroke cannot be used in the formation of a functional expression, because in combination with other symbols it does not serve to designate an object. "$\vdash 2+3=5$" does not designate anything, it asserts something.' [6, 22] In another footnote Frege writes: 'To judge is not merely to grasp a thought, but to acknowledge its truth.' [7, 34] We can assume propositions 'for the sake of the argument' and derive logical consequences from them without thereby having to take a stance on whether they are true or not. In assuming that there is a

[12] This may need qualification, if there are expressions that prevent a sentence containing them from being embeddable or that cannot be embedded, but that are not signs of speech acts.

set containing all and only those sets that do not contain themselves for purposes of *reductio*, I am not asserting that proposition.

It is true that, despite his acknowledgement of the need for distinguishing assertion from assumption, Frege did not apply it as one might expect: all propositions of *Begriffsschrift* and *Grundgesetze* are marked with the judgement stroke and thus asserted. Propositions that could form assumptions in the process of reasoning appear in the antecedents of asserted conditionals.[13] It took later developments until systems of logic were formalised that deploy Frege's insight. According to Gentzen, the main difference between his systems of natural deduction and the 'logistic' calculi, as he called them, is that in his systems deductions begin with formulas that are assumed, rather than with axioms that are asserted:

> The essential difference between NJ-derivations [i.e. in natural deduction for intuitionist logic] and derivations in the systems of Russell, Hilbert and Heyting is the following: In the latter, correct formulas are derived from a number of 'logically basic formulas' [i.e. axioms] by means of few rules of inference; natural deduction, however, does not in general start from logically basic propositions, but from a s s u m p t i o n s [...], which are followed by logical inferences. A later inference then makes the result again independent of the assumption. [15, 184][14]

[13]Notice, however, how Frege proceeds in the appendix to the second volume of *Grundgesetze*: deriving Russell's contradiction in Begriffsschrift, Frege informs us that he will 'leave out the judgement stroke because truth is in doubt' [8, 256]. Curiously, this passage is omitted by Geach and Black in their translation of the appendix. Frege here draws logical inferences from propositions that are not judged; his practice betrays his doctrine that from mere assumptions nothing can be inferred ([9, 387], [12, 47]). There would, hence, be a way of expressing mere assumptions in Frege's logical practice, namely, by refraining from applying the judgement stroke. But this is not a method Frege uses in his official development of logic. At the beginning of *Grundgesetze* and elsewhere, a formula without a judgement stroke attached is taken to be the name of a truth value. A proposition can only ever name a truth value, be it the True or the False. To judge is to take the step from the sense of a sentence, the thought, to its reference, its truth value [7, 35]. In judging, we proceed from a thought to its truth value, or rather from the thought to the True. According to Frege's official doctrine, inference requires that process to have been made. See Textor's reconstruction of Frege's theory of judgement, where he explains: 'Judgement and inference are "level-crossing" mental acts. In them the judger advances from a thought to its truth-value.' [40, 639]

[14]This differs slightly from Szabo's translation [16, 75].

Roughly around the same time Jaśkowski makes virtually the same observation:

> In 1926 Prof. J. Łukasiewicz called attention to the fact that mathematicians in their proofs do not appeal to the theses of the theory of deduction, but make use of other methods of reasoning. The chief means employed in their method is that of an arbitrary supposition. The problem raised by Mr. Łukasiewicz was to put these methods under the form of structural rules and to analyze their relation to the theory of deduction. [23, 5]

Jaśkowski solves Łukasiewicz's problem by formalising a system of natural deduction. Like Gentzen, Jaśkowski continues to point out that assumptions are made to be discharged, that an implication derived from a conclusion derived under a supposition does not depend on the supposition: 'It would remain true even in case the suppositions used [in its derivation] should be false.' [23, 6] Both Gentzen and Jaśkowski underline that making and discharging assumptions is essential to the process of logical inference as captured by natural deduction.

Making an assumption is not often listed amongst examples of speech acts. It is, however, quite clear that to make an assumption is to perform a speech act. It is to do something with the content of a sentence, with a proposition or a thought, and to engage in a linguistic activity. An assumption can have the same content as an assertion, a question, a command or a wish and is distinguished from them by what is done with the content. Dummett concurs that 'in supposition, a thought is expressed but not asserted: "Suppose ..." must be taken as a sign of the *force* [...] with which the sentence is uttered.' [2, 309] Although this observation is virtually immediate once the distinction between force and content is drawn, it is possible to give more evidence and argument for it. Doing so contributes to an analysis of this speech act. I shall follow Jaśkowski and call speech act of the making of an assumption *supposition*.[15]

Supposition shares features with other speech acts. It is similar to requests and commands in that suppositions are often expressed using the

[15] For a detailed analysis of the norms governing suppositions and how supposition differs from other speech acts, see [17]. Green also remarks, as I will below, on the fact that there are conventional ways of marking supposition in natural deduction, showing that supposition is a speech act.

imperative: 'Let a be an F', 'Assume p', 'Suppose q'. If the use of the imperative indicates a speech act, this suggests that its use in supposition does so, too.

Supposition has a specific purpose: it marks the first step in argumentation or logical deduction. Supposition comes with the intention to draw an inference. Gentzen and Jaśkowski go even further: the intention is to produce a chain of inferences with the aim of discharging the assumption made. Thus, like commands, requests and questions, suppositions prompt further actions, the former answers and the carrying out of the command or request, the latter further steps in an argument or deduction. If in the former cases, this is due to the fact that speech acts have been performed, this suggests that supposition is also a speech act.[16]

There are conventions marking assertions, question and commands. Often these are not sure-fire indications, but in general fair enough to determine which speech act has been performed. If a sentence ends with a full stop, this is a rough and ready indication that it is an assertion; if it ends in a question mark, this is a rough and ready indication that it is a question; if it ends in an exclamation mark and is in the imperative mood, this is a rough and ready indication that it is a command, if issued by a person with the relevant authority. There are also conventions marking when something has been assumed. In the four most popular systems of natural deduction, these are quite precise rather than rough and ready:

(1) By writing formulas at the top nodes of a proof tree with no line on top (Gentzen);
(2) By writing an S at the beginning of the formula and a numeral to the left, with a prefix if the assumption is in the scope of other assumptions (Jaśkowski);
(3) By writing 'hyp' to the right of the formula and $|\_$ to its left, with further lines $|$ to the left if the assumption is in the scope of other assumptions (Fitch);
(4) By writing an assumption number to the left of the formula and an 'A' to its right (Lemmon).

[16] When I presented an early version of this material at a work in progress seminar in London, Mark Textor asked whether one can't just suppose without drawing inferences. Keith Hossack responded that this is not supposition, but entertaining a thought.

Thus, just as there are conventions marking the speech acts assertion, question, command and request, there are conventions marking supposition. This feature, too, puts supposition into the realm of speech acts. Notice that there is no conventional mark indicating that a thought is expressed: the sentence expressing the thought suffices.

The test provided by the Frege Point provides further reasons for counting supposition amongst the speech acts. If the conventional signs of supposition are such that they cannot be embedded, this is an indication that it is a speech act. And this is indeed the case. Conventional signs for supposition in ordinary English, such as 'Suppose A', 'Let a be an F' and 'Assume p' cannot be embedded. 'If suppose A, then B', 'Suppose assume A', 'It is not the case that let a be an F' are gibberish. Similarly for the conventional signs of supposition in formal systems of natural deduction. Expressions such as '$(Sp) \supset q$', '(1. $|_{-}$ p hyp) $\supset q$' and (1 p A)$\supset q$ are illformed, and although the formula that occupies a top node of a proof tree can occur in the antecedent of a conditional, it makes no sense to put the top nodes of proof trees into that position. In Jaśkowski's and Fitch's system, assumptions can be made in the scope of other assumptions, but this is not the same as embedding an assumption in another. To make an assumption in the scope of another is to perform two speech acts one after the other. It is not to embed one speech act in another. No provision has been made for strings of symbols such as '1. 2. $SSpq$' or '$|_{-}$ $|_{-}$ p hyp q hyp', or similar strings with q omitted. But as the systems of Gentzen and Lemmon show, the notion of the scope of an assumption is not essential. Be that as it may, no provision has been made in Lemmon's system for strings of symbols such as '1 2 p A q A' either, and in Gentzen's system, where supposition is effected by writing a formula on top of a line indicating an inference with nothing above it, there isn't even anything that might count as an attempt to embed one supposition within another.

Finally, although we can describe or report that an assumption has been made by a sentence such as 'It is assumed that p', this is not the same as supposition. It does not have the same effect. We cannot render what is being done when an assumption is made by the phrase 'It is assumed that'. The inference 'It is assumed that p, it is assumed that $p \to q$, therefore q' is invalid: It may be true that it is assumed that p and that it is assumed that $p \to q$, while it is false that q, because it is possible to assume falsehoods. Nonetheless, from the supposition that $p \to q$ and the supposition that p, q follows logically. 'It is assumed that' is a sentential operator, not an indicator of a speech act: it can be used to describe or report which assumptions have been made, but 'It is assumed that p' cannot take the place of performing the speech act of assuming that p. The description or report may be true or false; the assumption is made or not. Compare with Frege's assertion sign: it indicates the assertoric force of an inscription without asserting that the inscription is asserted; the latter is done by means of the sentential operator 'It is asserted that'. Like assertion, supposition is not something that is part of the proposition assumed. It is something that is done with a proposition.[17]

Supposition is different from merely grasping or expressing a thought. Thoughts can be grasped without being assumed, e.g. when I grasp the components p and q in complex sentences such as $\neg p$ and $p \to q$. We can test whether someone has grasped a thought expressed by 'p' by asking 'Do you understand this sentence?'. Even when the answer is 'Yes', this need not be the preparation for a chain of reasoning. Grasping or expressing a thought need not be followed by inferences.

As pointed out by Frege, supposition is evidently something other than assertion. For further illustration, consider Descartes at the end of his first meditation: 'I will suppose [...] that there is an evil spirit who is supremely powerful and intelligent, and does his utmost to deceive me.' [1, 65] Descartes assumes this, but does not assert it: he assumes it to see what follows in his quest for a rational reconstruction of his beliefs on firm foundations. It is also an assumption to be discharged. Descartes aims to draw conclusions that do not depend on this assumption. Anything can be assumed, at least in formal logic, and maybe even in philosophy, but assertion is governed by stricter norms and not anything can be (felicitously, sincerely) asserted. Not many people have ever been in a position where they would assert 'I am being deceived by an evil spirit'.[18] The assertion

[17] As Frege says, to judge is something utterly peculiar and incomparable. [7, 35] Nothing other than a judgement has the effects of a judgement; in particular, a description that a judgement that p has been made (by someone or other) need not involve a judgement that p. Van der Schaar gives an account of the difference between judging and describing a judgement, of the first person perspective and the third person perspective on judgements [42]. Similar remarks apply to supposition.

[18] Saints Ignatius of Loyola and Teresa of Avila came close. Both report in their autobiographies the realisation that some of the thoughts and feelings that arose during their meditations were temptations and effectively assert that they were being deceived by evil demons. But even they do not quite report having asserted 'I am being deceived by an evil demon' in the present tense. Note the difference: For Descartes, the thought 'I

that p does, the supposition that p does not, commit to the truth of p. Speakers use assertions to express their beliefs; they do not use suppositions for that purpose. Although supposition has features in common with command, question or request, it is a speech act that differs from them, too. As Dummett observes, a command can be followed up by a question 'Have you done it yet?', but 'Let a be an F' or 'Suppose p', can't be followed up by such a question. [2, 309] For a similar reason, suppositions are not requests either. Supposition is also different from asking a question: I can assume a proposition without wondering whether it is true or not. A question is a challenge to provide an answer; a supposition can be made without any view on settling the question whether it is true or not – indeed, once an assumption is discharged, its truth value is irrelevant to the truth of the conclusion.

5. Supposition as a problem for bilateralism

Formulas of bilateral logic are prefixed by + or −, representing the speech acts of assertion and denial. Being a speech act, supposition requires a propositional content as that which is supposed. An assertion or a denial is not a propositional content. Thus it is not possible to assume an expression such as $+\,A$ or $-\,A$. Every formula of bilateral logic is already put forward with assertive or rejective force. There is therefore no sense to assuming such a formula. Speech acts cannot be embedded. 'Assume $+\,A$' and 'Assume $-\,A$' are therefore meaningless.

Nonetheless, formulas of the form $+\,A$ and $-\,A$ are supposed to feature as assumptions in deductions in bilateral logic. The rules $+\supset I$ and Reductio show as much. They permit the discharge of signed formulas. The conclusion of an application of these rules no longer depends on the signed formulas discharged. That which is discharged is an assumption.

As pointed out by Jaśkowski and Gentzen, supposition is an essential feature of inference. But we cannot make sense of the notion of making an assumption in bilateral logic, where every formula is prefixed with a sign for assertion or denial. Bilateral logic demands we do something that

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am being deceived by an evil demon' occurs within disinterested philosophical reflection. For Teresa and Ignatius, it is the cause of extreme distress. The difference between the supposition and the assertion couldn't be more dramatic.

cannot be done: to embed the speech acts of assertion and denial within the speech act of supposition. Bilateral logic as it stands is thus incoherent.

According to bilateralists, $+ A$ and $- A$ can be rendered as propositional questions and their answers. Doing so starkly presents the predicament. 'Suppose was the accused in Berlin at the time of the murder? Yes' makes no sense. 'Suppose is it the case that A? No' and 'Assume is it the case that A? Yes' are of the same kind of gibberish as Rumfitt's example to illustrate that $+$ and $-$ cannot be embedded (p. 307), and so is 'Let is a an F? Yes'.[19]

We can assume that something has been asserted or that something is assertible. 'Suppose it is asserted that A' or 'Assume that A is assertible' make sense. But they are different from assuming that A. To assume a proposition is not to assume that anyone asserted it. To assume that A is assertible is different from assuming that A. If B follows from the assumption that A, then I can infer 'If A, then B'. If B follows from the assumption that A is assertible, I can infer that 'If it is assertible that A, then B'. These are not the same. To take an example of Dummett's, let A be 'You will go into that room' and B 'You will die before nightfall', so that in 'If you go into that room, you will die before nightfall', 'the event stated in the consequent is predicted on condition of the truth of the antecedent (construed as in the future tense proper [i.e., not the future tense expressing present tendencies]), not of its justifiability.' [3, 193] Suppose that the present tendencies are that you will go into that room, but you later change your mind, don't go and don't die before nightfall. Then the conditional 'If it is assertible that you go into that room, you will die before nightfall' is false, as the antecedent is true and the consequent is false, while the conditional 'If you go into that room, you will die before nightfall' is true, if the room is one in which everyone is killed who enters before nightfall. The distinction between assuming that a proposition is assertible and assuming the proposition is pertinent for bilateralists like Price and Rumfitt, for whom a crucial aspect of the motivation for adopting the bilateral approach to meaning is their claim that it enables them to draw the distinction between truth and assertibility. (See [30, 167] and [35].) Besides, 'Suppose it is asserted that A' or 'Assume that it is assertible that A' cannot render correctly the bilateralists' attempts at assuming $+ A$ and $- A$: $+ A$ and $- A$ represent speech acts of assertion and denial, not

[19] How about 'P? Suppose Yes' or 'Let a be an F? Yes'? See Section 6.

reports that any such speech acts have been performed or assertions that they could be performed.[20] 'It is asserted that' and 'It is assertible that' are sentential operators, not indicators of speech acts.[21]

Maybe bilateralists could respond that their logic is one that works without supposition: the rules specify how to proceed to further assertions and denials from assertions and denials that have in fact been made.[22] Compare with Frege's view that only from true premises can something be concluded. We can, however, discount this option: the framework of natural deduction chosen by bilateral logicians betrays that this cannot be the intention, as it does not fit the Fregean account of inference. Seeking a way out along the Fregean route and providing an axiomatic system of logic in which some axioms are asserted, others denied is also not conducive to the expressed aim of providing an inferential semantics for the logical constants: it is to give up on the project of specifying the meanings of the connectives in terms of rules of inference.

The notion of the discharge of assumption merits further consideration. It is the second essential aspect of inference pointed out by Gentzen and Jaśkowski. In unilateral systems of logic, if a rule permitting discharge of assumptions is applied, the conclusion no longer depends on their truth. What could it mean to discharge a speech act of assertion or denial? Dis-

[20]To assume that an assertion has been made or a question answered is irrelevant to logic, or at least it does not cover all the cases logic is concerned with: Descartes need not have asserted that he is deceived by an evil demon, nonetheless he and we can proceed from that assumption and see what follows, draw consequences and potentially reject the assumption, if we reach a contradiction. And rejecting an assumption here means: to derive its negation, which we are then entitled to assert, if we assert also all the other premises used in the argument. Rejecting an assumption in the sense relevant to logic is not like rejecting an assertion (as in metalinguistic negation): it is the step after deriving a contradiction (or otherwise unpalatable proposition) from it (and other assumptions or asserted propositions), that is, it is to derive and assert its negation (on the basis of other assumptions).

[21]The items to which bilateralist logic is applied can hardly be possible assertions. I doubt that mainstream bilateralists are happy to admit that there are possible assertions, so the only way to make sense of the claim that A is a possible assertion is to say that A is assertible. Maybe bilateralists could reject the Frege Point and adopt a view that aims to imbue propositions with an intrinsic assertoric or rejective force, a force that is canceled if they are embedded into other speech acts, such as supposition? Jesperson argues forcefully against such a view [24]. The view also goes against the evidence provided by Rumfitt that bilateralists accept the Frege Point and do not think speech acts can be embedded.

[22]As Dorothy Edgington and Mark Textor wondered.

charging an assumption is not like retracting an assertion. To see this, it suffices to compare Frege's retraction of Basic Law V and Descartes's discharge of the assumption that he is being deceived by an evil spirit. But a few more words may be in order. If an assertion has been made, or a question raised and answered, a speech act has been performed. And even though I can retract an assertion or change my mind what the answer to a question is, the assertion or question and answer cannot be made undone: they are events that have happened, and we cannot, as it were, remove them from the universe by a process such as applying implication introduction or *reductio ad absurdum*. It is possible to cancel the commitment to the correctness of an assertion previously made, but that is not like discharging an assumption: cancelling a previous assertion is not done by an application of a rule of inference. Cancelling a commitment to a previous assertion is not analogous to the process of inferring further propositions that have the content of the assertion as a component; indeed, no such process would appear to make sense, as it would appear to require having the assertion as a component. Discharging an assumption is also a notion bilateralists cannot make sense of. There is no process that does to the speech act of assertion (or denial, for that matter) what discharge does to assumptions.

That it makes no sense to discharge an assertion is further evidence that we cannot assume assertions either: the possibility of its discharge is an essential feature of making an assumption.

What about Smiley's and Rumfitt's 'readily comprehensible variety of actual deductive practice' that uses propositional questions and their answers? There is no need to appeal to the bilateralist machinery to make sense of such inferences. One may, instead, appeal to Textor's account, who argues that 'Yes' and 'No' used in answering questions are not force indicators, but prosentences [41]. Thus any such aspect of deductive practice can be reconstructed without appeal to speech acts of assertion and denial.

6. Attempts to solve the problem

One might object that the difficulty pointed out in the last section is less than a problem and more of an omission: bilateral logic is incomplete and needs to recognise further speech acts besides those marked by + and −;

in particular, it needs to recognise also the speech act of supposition.[23] Could we not answer ‘Suppose yes’ to questions such as ‘Was the accused in Berlin?’ and mark the speech act of supposition thereby? The first thing to note here is that this is not what Smiley, Rumfitt and Humberstone are doing, according to whom $+$ and $-$ are to be read as assertion and denial, not supposition.[24] ‘Yes’ and ‘No’ are not ‘Suppose yes’ and ‘Suppose no’, and if the former are represented by $+$ and $-$, the latter are not represented by them, and hence we should have to add further signs for supposition.

Such an approach has been followed up by Kearns [25]. Kearns is a bilateralist at heart: he accepts that there are primitive speech acts of assertion and rejection, which he represents by $\vdash$ and $\dashv$. Correspondingly, there are two kinds of supposition, supposing as true, which he represents by $\llcorner$, and supposing as false, represented by $\neg$ [25, 335]. Every formula in a deduction is signed by one of these four symbols. To keep the system simple, Kearns only considers rules for $\vdash$ and $\llcorner$, and he only explicitly states some of the rules for conjunction, disjunction and negation. Even so, the system brings with it certain complications, as it needs to be settled what to do with conclusions that are derived from a mixture of asserted and supposed premises. This leads to a large number of rules: conjunction introduction, for instance, has three forms. Kearns has a principle for deciding whether the conclusion of an inference is asserted or supposed: if the only suppositions on which the premises of a rule depend are those to be discharged by its application, then the conclusion is asserted; otherwise it is supposed. Only suppositions can be discharged.[25]

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[23] Mark Textor suggested I put my point like this. The following also owes to discussions with Greg Restall.

[24] Evidently, we can’t read $+$ and $-$ as ‘Suppose the accused as in Berlin etc.? Yes’ as that, if we admit it at all, asserts that it is supposed that the accused was in Berlin, and this is different from supposing that the accused was in Berlin. Cf. pp. 313 and 316.

[25] There are unexplained question marks in the rules for disjunction and negation elimination [25, 336]. I interpret them as meaning that these formulas may either be asserted or supposed. There is a typo in negation elimination, a version of classical *reductio ad absurdum*: negations are missing from the discharged suppositions. Vacuous discharge appears to be forbidden in disjunction elimination, which is why its minor premises can only be supposed, while in negation elimination, it is permitted above supposed premises. Kearns says versions of *ex contradictione quodlibet* (vacuous discharge above both premises in negation elimination) are valid, as long as at least one premise and the conclusion are supposed; if both premises are asserted, it is not permitted to proceed to the assertion (and presumably the supposition) of an arbitrary formula; rather, ‘once a person finds herself [in such a position], she must abandon some of her beliefs’

There is, however, no need to consider all these variations. A rule of inference of Kearns's logic is that from an assertion of A, its supposition follows: from $\vdash A$ infer $\llcorner A$. It would therefore suffice to formulate only rules for when all premises are supposed and leave the cases with asserted premises as derived rules of inference. What is more, as a consequence of Kearns's principle for deciding whether a conclusion is asserted or supposed, it suffices to give rules that conclude with suppositions, and then add the global condition that the conclusion of a deduction is asserted if all the premises it depends on are asserted, supposed otherwise. It is clear, however, that prefixing $\llcorner$ to all premises and conclusions is superfluous. The situation is thus exactly as in a system such as Gentzen's, where assumptions are not marked in any special way, and a deduction of A from formulas Γ entitles us to assert A if we assert all formulas in Γ.[26]

The fact that Kearns permits suppositions to be conclusions of inferences presents a more general problem. Kearns's notion of supposition is prised apart from the notion of discharge. For instance, if $\llcorner A \wedge B$ is concluded from $\llcorner A$ and $\vdash B$, it cannot be discharged further down in the proof. Kearns considers supposition to be something weaker than assertion. In some sense or other that may be true: a supposition does not commit in the way an assertion does. But no such sense is pertinent for logic. Supposition is not a weaker kind of assertion, but something different altogether. Assumptions stand at the beginning of deductions and are not the result of inference. And as Gentzen and Jaśkowski observe, assumptions are made to be discharged: it is of the essence of an assumption that it may be discharged by an application of a rule of inference further down in the de-

[25, 337]. Abandoning a belief is then not discharging it and deriving the assertion (or supposition) of its negation.

[26] In Jaśkowski's system, suppositions are marked by S, while asserted formulas are not marked by anything. 'The above conventions [of how to construct deductions in his system of natural deduction] lead us to some new expressions [those beginning with 'S'] which must be considered as significant ones. [...] We shall retain for the term "proposition" the meaning already given, namely the significant propositions of the usual theory of deduction' [23, 7], i.e. an axiomatisation of the propositional calculus by Łukasiewicz in which all formulas are asserted. There would be no need for a separate symbol indicating supposition, as suppositions are already marked by their position in the deduction, standing, as they do, to the right of prefixes composed of numerals indicating scope, each supposition with its unique prefix. Propositions, whether concluded by the discharge of suppositions or used as premises, do not get prefixes. It is worth noting that formulas concluded under suppositions are not marked by anything other than the prefix of the suppositions under which they stand.

duction. Consequently, in their systems of natural deduction, assumptions *only* stand at the beginning of deductions and introduce the formulas from which the deduction is going to take its course. I conclude that Kearns's $\llcorner$ does not represent the speech act of supposition, as supposition plays a different role from that played by $\llcorner$ in Kearns's system.[27]

The forgoing considerations also shows Kearns's rule 'from $\vdash A$ infer $\llcorner A$' to be absurd. A conclusion drawn on the basis of a deduction in which all assumptions are discharged is asserted outright and there is no sense in which it is supposed. Kearns, however, must say that it is, as according to him, from the assertion of the conclusion of this deduction, its assumption follows.

These problems are not just problems that mar Kearns's approach. Any approach that insists on adding speech acts of supposition to those of assertion and denial would need to answer the questions Kearns has aimed addressed, and if all formulas in a deduction are supposed to be marked by signs for speech act, the question remains what speech act is supposed to follow from supposed formulas.

Maybe the most reasonable thing would be not to sign conclusions of inferences by markers for speech acts at all. Evidently, even marking conclusions only by + and − if they are concluded exclusively from asserted and denied formulas would open up the problem of Kearns's approach again: what should we conclude if one premise of a rule is signed and the other isn't. And so we are back to Jaśkowski.

One might try to solve the problem posed by supposition for bilateralism by observing that even if speech acts cannot be embedded, there are speech acts that are conditional: there are conditional commands, requests and bets, for instance, such as 'If you go to the shops, get some beers' or 'I bet a tenner it'll rain if I don't take an umbrella'. The request and the bet are made on condition of other things taking place. If you don't go to the shops, the request is void. If I take an umbrella the bet is off. It is plausible to add conditional assertion to the list of conditional speech acts. Indeed, it is plausible that if a conclusion is drawn on from assumptions, it is asserted conditionally on those assumptions. An expression like 'therefore'

[27]The act of drawing a conclusion is often marked by 'therefore': 'Suppose A and suppose B, therefore suppose $A \wedge B$' is once more gibberish, to use Rumfitt's word, and hence, as according to Kearns concluding $\llcorner A$ is meaningful, $\llcorner$ cannot be a sign for the speech act of supposition that is pertinent to logic. One will observe that putting a question and answer after 'therefore' does not fare much better.

that signals an inference also bears the marks of a speech act. Consider this example: Let a be an F. But no F is a G. Therefore, a is not a G. Here we have three speech acts: a supposition, an assertion and the announcement of a conclusion. The result is a conditional assertion: The conclusion that a is not a G is asserted conditionally upon a's being an F. There is also a practice of recording the conditional assertions resultant upon a deduction as $\Gamma \vdash A$ and of calculi for deriving the further commitments incurred by conditional assertions, i.e. single conclusion sequent calculus. A deduction, then, consists in a series of speech acts: it begins with suppositions or propositions the truths of which are accepted, continues with announcements of propositions inferred, and results in a conditional assertion of its conclusion, if any assumptions remain undischarged, or its outright assertion, if not.[28]

This is an attractive way of understanding the result of a unilateral deduction. The question is how to apply it to bilateral logic.

If there are conditional assertions, the bilateralist can add conditional denials. The product of a deduction is a conditional assertion or a conditional denial. Undischarged formulas $+ A$ and $- A$ should then represent the conditions on on which the conclusion of the deduction is asserted or denied. But this does not get the conditions right. A speech act conditional upon the assertion or denial of a proposition is different from a speech act conditional upon its truth or falsity. A conditional request, command or bet is conditional upon the truth of the proposition expressing the condition, not the performance of a speech act with that proposition as its content. Similarly, the condition of a conditional assertion should be expressed by A or $\neg A$, as it is done in a unilateral system, not by $+ A$ or $- A$, as the bilateralist would have to insist. It is crucial to the bilateralist that $+$ and $-$ are speech acts: the bilateralist needs to ensure that $+$ and $-$ are not merely notational variants of the trivial truth function and negation, and the way to do this is to insist on their status as speech acts. A must be different from $+ A$, $\neg A$ from $- A$. But if $+ A$ and $- A$ mark the conditions of the speech act, they are no different from A and $\neg A$.

[28]Eliot Michaelson, Michael Potter and Bernhard Weiss pressed me on this issue, and it is to them that I owe the objection. In our four systems of natural deduction, it is not marked whether any of the formulas from which the deduction proceeds are accepted as true unless they are axioms: they are all treated as assumptions. But it would be straightforward to add a convention for indicating such formulas, the most immediate one being to treat them the way axioms are already treated.

This is also seen by looking at the conditional. A conditional assertion of the kind that is at issue here, one put forward on the grounds of a deduction of a conclusion from assumptions, is equivalent to the assertion of a conditional. This follows from the bilateral rules for $\supset$.[29] This shows that the condition of the assertion is A, not $+\ A$: $+\ A$ cannot go into the antecedent of the conditional. Only A can go there.

7. Conclusion

The core of the argument of this paper is the following. Supposition is a speech act. In bilateral logic, the premises and conclusions of inferences are asserted or denied. Speech acts cannot be iterated. Thus there cannot be any assumptions in bilateral logic. But this is absurd: assumptions and their discharge are essential to logic.

One might protest: something in this argument must be wrong, for if that were so, then what is it that bilateral logicians are doing when they assume and discharge formulas of the form $+\ A$ and $-\ A$. My best diagnosis is that the practice of bilateral logician shows that their $+$ and $-$ are non-embeddable truth and negation operators. The description of $-$ and $+$ as speech acts does not match their use.

All this may just point to an incompleteness in the bilateral account of deduction: other speech acts need to be acknowledged besides those marked by $+$ and $-$. However, an attempt of doing just that has been shown to be inadequate. It is also not adequate to read deductions in bilateral logic, analogously to a plausible way of reading deductions in unilateral logic,

[29] This marks a difference between conditional assertion and conditional requests and commands. There is a difference between a conditional bet and a bet on a conditional; a conditional request and a request of a conditional. If bet on the conditional, that if I don't take an umbrella, then it will rain, I've won if I take an umbrella. If I request the conditional that if you go to the shop, then you buy beer, you've complied if you don't go to the shop. By contrast, there seems to be no such distinction in the case of assertion: a conditional assertion and the assertion of a conditional amount to the same thing, at least in the system we are considering, where the deduction theorem holds. If the condition of a conditional assertion reached by deductive inference is not fulfilled, although I am not committed to the assertion of the conclusion, I am still committed to the conditional that follows by implication introduction. In conditional assertion, I am committed to asserting the conclusion if the condition holds; in the assertion of a conditional, I am committed to the consequence drawn by *modus ponens*, if the condition holds. In this respect conditional assertion is thus different from conditional bets and requests.

as conditional assertions and denials. The burden of proof lies on the bilateralist to indicate how the account is to be amended.

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Nils Kürbis
University of Lodz
Department of Logic and Methodology of Science
Williama H. Lindleya 3/5
90-131 Lodz, Poland

University of Bochum
Department of Philosophy I
44801, Universitätsstraße 150
Bochum, Germany

University College London
Department of Philosophy
WC1E 6BT, Gowerstreet
London, UK

e-mail: nils.kuerbis@ruhr-uni-bochum.de

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Leonardo Ceragioli

BILATERAL RULES AS COMPLEX RULES

Abstract

Proof-theoretic semantics is an inferentialist theory of meaning originally developed in a unilateral framework. Its extension to bilateral systems opens both opportunities and problems. The problems are caused especially by Coordination Principles (a kind of rule that is not present in unilateral systems) and mismatches between rules for assertion and rules for rejection. In this paper, a solution is proposed for two major issues: the availability of a reduction procedure for *tonk* and the existence of harmonious rules for the paradoxical zero-ary connective $\bullet$. The solution is based on a reinterpretation of bilateral rules as complex rules, that is, rules that introduce or eliminate connectives in a subordinate position. Looking at bilateral rules from this perspective, the problems faced by bilateralism can be seen as special cases of general problems of complex systems, which have been already analyzed in the literature. In the end, a comparison with other proposed solutions underlines the need for further investigation in order to complete the picture of bilateral proof-theoretic semantics.

Keywords: bilateralism, separability, harmony.

1. Introduction

The aim of this paper is to solve some problems faced by a specific flavour of proof-theoretic semantics when applied to bilateral systems. A complete reconstruction of the state of the art of this field of study or its history is far beyond the limits of this contribution, but some of its key aspects have to be reminded. The same holds for bilateralism: we do not intend to give

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the full picture regarding this vast topic, but we will remind some aspects that will be relevant to the problems here at issue.

Proof-theoretic semantics is an approach toward meaning (especially for the logical language), which – as opposed to model-theoretic semantics – assigns meaning without referring to things external to the language and the linguistic practices, such as models or structures. It is a vast and heterogeneous field of study, with different ramifications, tied together by the adoption of proof – as opposed to truth – as the key ingredient of semantics. In its original formulation due to Dummett and Prawitz, proof-theoretic semantics focuses primarily (if not only) on natural deduction systems, and so on systems containing only Operational Rules.[1] For these rules, some *criteria* of acceptability are given:

- For I-rules, something like a complexity condition is usually imposed, with the clause that in all its applications, the conclusion should be more complex than both the premises and the discharged assumptions;[2]

- For E-rules, a criterion called *harmony* guarantees that they are consequences of, and so justified by, the corresponding I-rules.

While there is no consensus about which shape the criterion of *harmony* should take, it is usually agreed that Inversion Principle should be at least one of its ingredients or presuppositions:[3]

> "Let α be an application of an elimination rule that has B as consequence. Then, deductions that satisfy the sufficient condition $[\cdots]$ for deriving the major premiss of α, when combined with deductions of the minor premisses of α (if any), already "contain" a deduction of B; the deduction of B is thus obtainable directly from the given deductions without the addition of α."

In practice, a pair of rules for a logical constant suits this principle iff there are some *reduction steps* that: take every derivation in which the

[1] See [28] (general proof theory) and [6].

[2] See [6] p. 258. This criterion has been criticized in [8], where the author proposes a new criterion.

[3] [27], p. 33. For a historical account of the development of this principle, see [24].

major premise of an E-rule is derived using an I-rule as input; return a derivation of the conclusion of the E-rule which is constructed by combining the derivations of the premises of the I-rule and the (eventual) derivations of the minor premises of the E-rule.

To make this intuition more precise, Prawitz introduces the notion of *maximal formulae*:

Definition 1.1 (Maximal Formulae). Given a derivation $\mathfrak{D}$, a maximal formula in it is a formula that is the conclusion of an I-rule and the major premise of an E-rule.

With this definition, we can observe that Inversion Principle asks that for each *maximal formula* generated applying a pair of I and E-rules, there is a reduction step that removes it. As an example, the maximal formula in the derivation on the left is removed in that on the right:

$$\supset\mathrm{E}\,\dfrac{\begin{array}{c}\mathfrak{D}_1\ \vdots \\ A\end{array} \qquad \supset\mathrm{I}\,\dfrac{\begin{array}{c}[A]\\ \mathfrak{D}_2\ \vdots\\ B\end{array}}{A \supset B}}{B} \qquad \rightsquigarrow \qquad \begin{array}{c}\mathfrak{D}_1\ \vdots\\ A\\ \mathfrak{D}_2\ \vdots\\ B\end{array}$$

It should be clear that Inversion Principle does not entail that *maximal formulae* can be avoided in general, since a reduction step can generate new *maximal formulae*. As an example, if in the previous example the derivation $\mathfrak{D}_1$ of A ends with an application of an I-rule and the derivation $\mathfrak{D}_2$ of B from A starts with an application of an E-rule of which A is its major premise, the reduction gives birth to a new *maximal formula*, that is A. The generation of new *maximal formulae* poses the problem of circular reductions and in general of the eliminability of *all maximal formulae*.[4] So, defining as in *normal form* a derivation in which there are no *maximal formulae*, there are two properties eligible for harmony:

existence of normal form Given a derivation of C from Γ, there is a derivation in *normal form* of the same conclusion from at most the same assumptions;

[4] As opposed to the eliminability of *each maximal formula*, for which Inversion Principle is enough.

normalization Given a derivation of C from Γ, there is an effective procedure leading from it to a derivation in *normal form* of the same conclusion from at most the same assumptions.

In the most traditional versions of proof-theoretic semantics, when harmony is not equated with Inversion Principle *tout court*,[5] it usually entails both this principle and the request for normalization.[6] In our discussion about Rumfitt's bilateral system, we will follow him and consider normalization as the key ingredient of harmony. Our aim will be to remain as adherent as possible to this traditional conception of proof-theoretic semantics, while endorsing bilateralism and solving the issues pointed out against Rumfitt's system.

Of course, this overview of proof-theoretic semantics is far from complete, and covers just the orthodox developments of this discipline that follow Dummett and Prawitz in favouring single-conclusion natural deduction and harmony *criteria* based on normalization. Admittedly, there are generalizations and different approaches departing from this traditional flavour of proof-theoretic semantics. As an example, there are some attempts to generalize this kind of investigation in the direction of sequent calculus.[7] Nonetheless, Rumfitt's bilateral system is a development of this early tradition, and the author is explicitly skeptical about meaning-theoretical usages of sequent calculus.[8] Moreover, none of the criticisms that we will consider about this system crosses the limits of this traditional approach. Hence, later alternative approaches to proof-theoretic semantics can be overlooked in what follows.

Even though there are some issues with *ex falso quodlibet*, we can consider part of the received wisdom that traditional unilateral proof-theoretic semantics leads to the justification of intuitionistic logic.[9] On the contrary,

[5] As suggested in [16].

[6] To be honest, the situation is far more complex than that. For a recent analysis of the precise relation between normalization and validity in proof-theoretic semantics, see [38].

[7] *Inter alia*, see [14] for inferentialism and proof-theoretic semantics, and [34] for a bilateralist analysis of these calculi.

[8] [36], p. 795.

[9] See [17] for the problems that proof-theoretic semantics has in defining the meaning of $\perp$, and [1] for the problems encountered in trying to prove that *ex falso quodlibet* suits Dummett and Prawitz's definitions of proof-theoretic validity.

$$\wedge\mathrm{I}^{+}\ \frac{+A \qquad +B}{+(A \wedge B)} \qquad \wedge\mathrm{E}^{+}\ \frac{+(A \wedge B)}{+A} \qquad \wedge\mathrm{E}^{+}\ \frac{+(A \wedge B)}{+B}$$

$$\wedge\mathrm{E}^{-}\ \frac{-(A \wedge B) \qquad \begin{array}{c}[-A]\\ \vdots\\ C\end{array} \qquad \begin{array}{c}[-B]\\ \vdots\\ C\end{array}}{C} \qquad \begin{array}{c}\wedge\mathrm{I}^{-}\ \dfrac{-A}{-(A \wedge B)}\\[2ex] \wedge\mathrm{I}^{-}\ \dfrac{-B}{-(A \wedge B)}\end{array}$$

$$\vee\mathrm{E}^{+}\ \frac{+(A \vee B) \qquad \begin{array}{c}[+A]\\ \vdots\\ C\end{array} \qquad \begin{array}{c}[+B]\\ \vdots\\ C\end{array}}{C} \qquad \begin{array}{c}\vee\mathrm{I}^{+}\ \dfrac{+A}{+(A \wedge B)}\\[2ex] \vee\mathrm{I}^{+}\ \dfrac{+B}{+(A \wedge B)}\end{array}$$

$$\vee\mathrm{I}^{-}\ \frac{-A \qquad -B}{-(A \vee B)} \qquad \vee\mathrm{E}^{-}\ \frac{-(A \vee B)}{-A} \qquad \vee\mathrm{E}^{-}\ \frac{-(A \vee B)}{-B}$$

$$\supset\mathrm{I}^{+}\ \frac{\begin{array}{c}[+A]\\ \vdots\\ +B\end{array}}{+(A \supset B)} \qquad \supset\mathrm{I}^{-}\ \frac{+A \qquad -B}{-(A \supset B)} \qquad \supset\mathrm{E}^{+}\ \frac{+(A \supset B) \qquad +A}{+B}$$

$$\supset\mathrm{E}^{-}\ \frac{-(A \supset B)}{+A} \qquad \neg\mathrm{I}^{-}\ \frac{+A}{-(\neg A)} \qquad \supset\mathrm{E}^{-}\ \frac{-(A \supset B)}{-B}$$

$$\neg\mathrm{I}^{+}\ \frac{-A}{+(\neg A)} \qquad \neg\mathrm{E}^{-}\ \frac{-(\neg A)}{+A} \qquad \neg\mathrm{E}^{+}\ \frac{+(\neg A)}{-A}$$

Figure 1. Operational rules

while there are some attempts in this direction, there is no clear and uncontended justification of classical logic inside such a unilateral perspective.[10]

In [36], Rumfitt investigates the possibility of justifying classical logic by focusing on a bilateral reformulation of natural deduction. He uses $+A$ to mean that A is asserted and $-A$ to mean that A is rejected, and proposes a system consisting of two kinds of rules:

Operational Rules: rules governing the introduction or elimination of connectives inside propositions prefixed by a sing $+$ or by a sign $-$;

Coordination Principles: principles dealing with propositions prefixed by a sing $+$ or by a sign $-$ regardless of their logical structure.

In Figure 1 the Operational Rules endorsed by Rumfitt are displayed.[11] In Figure 2 two alternative sets of Coordination Principles for a bilateral classical system are displayed: the three on the top (the two rules of Reductio and the rule of Non-Contradiction) or the two on the bottom (the two rules of Smiley).[12] We will work mostly with the system composed of the Operational Rules together with the two Smiley, but sometimes we will consider Reductio and Non-Contradiction as well.

Rumfitt explicitly endorses a *criterion* of harmony based on normalization for the acceptability of the Operational Rules, but has to provide new *criteria* for the Coordination Principles. Indeed, being developed in a unilateral framework, proof-theoretic semantics deals traditionally only with Operational Rules and gives *criteria* only for the acceptability of such rules. Rumfitt proposes different *criteria* for the Coordination Principles, which nonetheless have been proved to be untenable.[13] At the beginning

[10] An anonymous referee suggests that Sandqvist's semantics for classical logic in [37] counts as such an uncontended justification. I thank them for this suggestion. Sandqvist's result is surely thought-provoking for proof-theoretic semantics and uncontested as a formal result, leaving aside some formal issues regarding disjunction and existential quantifier. Anyway, it develops a notion of validity that is very different from the one based on harmony, as remarked also in [26]. What I mean here is that there are no uncontroversial justifications of classical logic inside the specific flavour of proof-theoretic semantics that relies on harmony and normalization, and to which Rumfitt's work belongs, even though there are some attempts in this direction: [22], [32] and [25] *inter alia*.

[11] [36], pp. 800–802.

[12] [36], p. 804.

[13] [9] and [18], p. 635.

of our investigation, it will just be enough to focus on harmony for Operational Rules and leave open the issue of Coordination Principles. Later on, the problem of providing a working *criterion* for Coordination Principles will become central and we will see that a common *criterion* for both Operational Rules and Coordination Principles can be found.

Before moving to more technical topics, let us discuss a conceptual difficulty about bilateral systems, since (apart from its intrinsic interest) it will become relevant later on. Kürbis has shown (see the contribution to this volume) that the interpretation of $+$ and $-$ as speech acts is untenable, and has proposed a proof-and-refutation route to bilateralism, as opposed to an assertion-and-rejection one. In a nutshell, his argument is the following: since asserting, denying, and making an assumption are all speech acts, and speech acts cannot be iterated (as an example, in Rumfitt's vocabulary the expression $++p$ is forbidden), then Rumfitt cannot adopt assumptions in his system.[14] While I find his objection well-defended and convincing, in this paper I will focus on what seems to me an orthogonal issue. I will just treat $+$ and $-$ as two modalities, without discussing their nature, and try to address some well-known problems of this system.[15] What will come out are considerations coherent with Kürbis' objection, but independent from it.[16]

The structure of the article is the following. In section 2 we will deal with the first objection regarding bilateral systems in proof-theoretic semantics. In particular, in subsection 2.1 we will display the problem and argue the need for a formal solution, in subsection 2.2 we will propose our

[14] An early exposition of this argument can be found in [19]. An anonymous referee asks whether Hjortland's bilateral sequent system in [15] escapes Kürbis' objection. Even though it is an interesting observation, I have some reservations about such a solution. Indeed, while sequent *calculi* are formalized without assumptions, their inferential interpretation considers *formulae* in the antecedent as open assumptions, and this speaks against the referee's proposal. Moreover, see note 7 for inferentialism and sequent *calculi*.

[15] These modalities are reminiscent of Wittgenstein's reading of negation (see [31], pp. 178–182 and [21] pp. 60–61), even though in Rumfitt's system they are endorsed not in place of negation, but alongside it.

[16] An anonymous referee objected that modalities can be nested, while this is forbidden for $+$ and $-$. I agree that this restriction is quite peculiar. Anyway, when I say that $+$ and $-$ should be treated as modalities, I mean that they should undergo the same scrutiny of the rest of the language and, first of all, be considered for harmony and separability. The restriction on their occurrence only as outermost terms can maybe undermine their reading as simple modalities, but not this point.

$$\text{Reductio}\ \dfrac{\begin{matrix}[+A]\\ \vdots\\ \bot\end{matrix}}{-A} \qquad \text{Non-Contradiction}\ \dfrac{+A \quad -A}{\bot} \qquad \text{Reductio}\ \dfrac{\begin{matrix}[-A]\\ \vdots\\ \bot\end{matrix}}{+A}$$

$$\text{Smiley}\ \dfrac{\begin{matrix}[+A]\\ \vdots\\ +B\end{matrix} \qquad \begin{matrix}[+A]\\ \vdots\\ -B\end{matrix}}{-A} \qquad \text{Smiley}\ \dfrac{\begin{matrix}[-A]\\ \vdots\\ +B\end{matrix} \qquad \begin{matrix}[-A]\\ \vdots\\ -B\end{matrix}}{+A}$$

Figure 2. Coordination principles

solution, and in subsection 2.3 we will discuss some consequences regarding separability. In section 3 we will deal with the second objection. In particular, in subsection 3.1 we will display the problem and evaluate a formal solution present in the recent literature, and in subsection 3.2 we will extend the proposal developed in subsection 2.2 so to cover this objection as well. In section 4 we will develop a comparison between our proposal and the other alternatives present in the literature. In section 5 we will sum up and conclude.

2. *Tonk* in bilateral systems

2.1. Gabbay's reduction for *tonk*

Prior's connective *tonk*

$$tonk\mathrm{I}\ \dfrac{A}{A\ tonk\ B} \qquad tonk\mathrm{E}\ \dfrac{A\ tonk\ B}{B}$$

is the most famous example of pathological connective in proof-theoretic semantics.[17] It was presented as an objection to an early version of inferentialism proposed by Popper and Kneale, which adopted a completely descriptive approach toward rules: no restrictions were imposed for the rules to attach meaning to the logical constants. One of the features of normative approaches to inferentialism like proof-theoretic semantics, which on

[17] [30].

the countrary impose *criteria* for the acceptability of rules, should be that they exclude pathological connectives like *tonk*. Indeed, it can be observed that unilateral proof-theoretic semantics excludes *tonk*, on the ground that it leads to non-reducible *maximal formulae*: the only derivation of B from A pass through an application of *tonk*I followed immediately by an application of *tonk*E.

The first objection to bilateralism that we will consider focuses precisely on the behavior of *tonk* inside this framework. Michael Gabbay points out that, as opposed to standard unilateral systems, in Rumfitt's bilateral system the rules for *tonk* cannot be excluded by the usual criterion of harmony.[18] Of course, in order to comply with bilateralism, *tonk*-rules have to be modified to work with assertions and rejections, but this step does not pose any problem: we just add + to both the premise and the conclusion of each *tonk*-rule. The problem emerges when we observe that the usual *maximal formula* obtained for *tonk* can now be "reduced" by inserting some applications of Coordination Principles between the conclusion of the I-rule and the assumption of the E-rule.

$$\textit{tonk}\mathrm{E}\,\dfrac{\textit{tonk}\mathrm{I}\,\dfrac{+A}{+(A\,\textit{tonk}\,B)}}{+B} \quad\rightsquigarrow\quad \text{Smiley,2}\,\dfrac{\textit{tonk}\mathrm{I}\,\dfrac{+A}{+(A\,\textit{tonk}\,B)} \qquad \text{Smiley,1}\,\dfrac{\textit{tonk}\mathrm{E}\,\dfrac{[+(A\,\textit{tonk}\,B)]^1}{+B} \qquad [-B]^2}{-(A\,\textit{tonk}\,B)}}{+B}$$

Even though this "reduction" manages to derive B from A without passing through *maximal formulae*, Francez argues that the second derivation does not qualify as a real reduction of the first, because it does not avoid the need to introduce and then to eliminate a *tonk*-formula, but just spreads it out in the derivation.[19] In this way, the detour is still there, although it is not in plain view.

While I agree with Francez in his evaluation that this should not count as a proper reduction, I believe that he misinterpreted Gabbay's intentions. Indeed Gabbay never claims that what he proposes is a valid reduction, but just points out that there is no formal *criterion* that detects a *maximal formula* in the derivation on the right. In other words, since according to Francez the derivation on the right does not qualify as in normal form, it should qualify as containing a *maximal formula*. Nonetheless, the standard

[18] [13].

[19] [12], section 5.

definition of such a notion is useless for this purpose, and the observation that the detour on the left is just "spread out" in the derivation on the right is just an intuitive observation, which cannot do the work of a formal *criterion*. As a consequence, in the bilateral systems we should take care also of these hidden detours (or fake normal derivations) by providing a formal *criterion* for them. When this is done, the reduction procedure can be evaluated and, if necessary, updated. I am not sure whether this is the original objection planned by Gabbay or a reformulation of it, but what is important is that there seems to be no answer to it in Francez's paper.

It can be seen that something similar happens when we have disjunction in a natural deduction system and we are forced to consider *maximal sequences* in addition to *maximal formulae* in defining *normal form*. If we do not define *maximal sequences*, then we could eventually use $\lor$-rules to "reduce" some *maximal formulae*, by moving an application of $\lor$E between the I and the E-rule in the following way:[20]

$$
\lor\mathrm{E}\,\dfrac{A \lor B \qquad \oplus\mathrm{E}\,\dfrac{\oplus\mathrm{I}\,\dfrac{\begin{matrix}[A]\\ \vdots\\ C\end{matrix}}{D}}{E} \qquad \oplus\mathrm{E}\,\dfrac{\oplus\mathrm{I}\,\dfrac{\begin{matrix}[B]\\ \vdots\\ C\end{matrix}}{D}}{E}}{E}
\quad\rightsquigarrow\quad
\oplus\mathrm{E}\,\dfrac{\lor\mathrm{E}\,\dfrac{A \lor B \qquad \oplus\mathrm{I}\,\dfrac{\begin{matrix}[A]\\ \vdots\\ C\end{matrix}}{D} \qquad \oplus\mathrm{I}\,\dfrac{\begin{matrix}[B]\\ \vdots\\ C\end{matrix}}{D}}{D}}{E}
$$

This move is available both in "normal" unilateral systems and in bilateral ones, before the definition of *maximal sequences*, and mirrors Gabbay's objection that *tonk-formulae* are reducible in bilateral systems.

Admittedly, this fake reduction procedure cannot be applied to every *maximal formula*, since it asks for a very specific position of the *maximal formula* in relation to an application of $\lor$E. As a consequence, it cannot be used to argue for the harmony of the *tonk*-rules, or in general of rules that generate irreducible *maximal formulae*. Indeed, if in general a pair of rules gives rise to non-reducible *maximal formulae*, even accepting this fake reduction, the vast majority of *maximal formulae* would remain without reduction. Nonetheless, when we have rules with particularly restrictive

[20] We will use $\oplus$ and $\ominus$ for generic logical constant.

side conditions, this "reduction" could be sufficient to argue for their harmony. As an example, let us consider what happens if we add to *tonk*I the following conditions of applicability:

- the conclusion of *tonk*I or the conclusion of the rule that is applied immediately after *tonk*I must be one of the minor premises of $\vee$E;
- an identical application of *tonk*I and, eventually, of the other rule occurring immediately after must conclude also the sub-derivation of the other premise of $\vee$E.[21]

These odd clauses entail that the only acceptable applications of *tonk*-rules that generate *maximal formulae* have the form

$$\vee\mathrm{E}\,\dfrac{A \vee B \qquad \mathit{tonk}\mathrm{E}\,\dfrac{\mathit{tonk}\mathrm{I}\,\dfrac{\begin{matrix}[A]\\ \vdots \\ C\end{matrix}}{C\ \mathit{tonk}\ D}}{D} \qquad \mathit{tonk}\mathrm{E}\,\dfrac{\mathit{tonk}\mathrm{I}\,\dfrac{\begin{matrix}[B]\\ \vdots \\ C\end{matrix}}{C\ \mathit{tonk}\ D}}{D}}{D}$$

But this is precisely the maximality that *can* be reduced if we do not consider *maximal sequences* alongside *maximal formulae*. So, the exclusion of this weakened version of *tonk* requires *maximal sequences*.

The situation here clearly resembles Gabbay's objection because, without a generalization of maximality that includes *maximal sequences*, we are forced to conclude that all *maximal formulae* generated by these weakened *tonk*-rules can be reduced according to the pattern that we already saw: by moving $\vee$E between *tonk*I and *tonk*E. Hence, without the definition of *maximal sequences* it seems that we need to accept this weakened reformulation of the *tonk*-rules. However, some of these "reducible" derivations prove blatantly unacceptable consequences, such as:

[21] The extra clause that the major premise of $\vee$E is an assumption (open or closed) may be added in order to prevent problems for the reduction of *maximal formulae* of disjunctive form.

$$\vee\mathrm{E}\,\dfrac{A \vee A \qquad \textit{tonk}\mathrm{E}\,\dfrac{\textit{tonk}\mathrm{I}\,\dfrac{[A]}{A\,\textit{tonk}\,B}}{B} \qquad \textit{tonk}\mathrm{E}\,\dfrac{\textit{tonk}\mathrm{I}\,\dfrac{[A]}{A\,\textit{tonk}\,B}}{B}}{B}$$

Of course, the solution here is just to extend the characterization of maximality so as to cover *maximal sequences* as well. This leads to rejecting the alleged reduction, because the derivation on the right is not in normal form and it even contains a *maximal sequence* longer than the one contained in the derivation on the left.[22] In the same way, the rejection of Gabbay's proposed reduction should be grounded on an extension of *maximality.* Unfortunately, such an explicit generalization is missing in Francez's paper, so we can not see his answer as satisfactory. In the following sections, we will search for a generalization of maximality that deals with Gabbay's objection.

2.2. Complex maximality

My proposal is to apply to Gabbay's provocatory reduction a generalization of maximality developed by Milne in order to justify classical logic in traditional unilateral proof-theoretic semantics.

Dummett defined the following notions regarding the structure of an I-rule:[23]

Purity Only one logical constant figures in each rule;

Simplicity Every logical constant which occurs in a rule, occurs as principal operator;

Directness Discharged assumptions are completely general, rules do not specify some connectives that must occur in them.

The I-rules of Gentzen's system **NJ** suit all these properties, apart from $\neg$I, which is not pure since $\perp$ occurs in it. Moreover, Dummett proposes a pure, even though oblique (that is, non-direct), rule for negation, so that

[22] Of course, this is not the whole story about *maximal sequences*, which must be considered to prove normalization, regardless of these fake reductions. See [27].

[23] [6], p. 257.

at least purity and simplicity can be obtained for the complete unilateral system for intuitionistic logic.[24]

Nevertheless, while all these properties are clearly *desiderata* for an I-rule, not least for feasibility reasons, it is far from clear that they should be required as necessary conditions. Indeed, as we have just seen, in each system for intuitionistic logic at least one of them fails and even Dummett pointed out that they together constituted an "exorbitant" demand.[25] Despise this early and authoritative declaration, there are quite few attempts to generalize proof-theoretic semantics by allowing for impure and complex (non-simple) rules.[26] The first real attempt of including complex rules in proof-theoretic semantics is [22], in which the author gives a harmonious and, to some extent, separable unilateral system for classical logic.[27]

Milne proposes the following impure and complex rules for classical conditional and classical negation:[28]

$$\supset\mathrm{I}_{Mln}\ \dfrac{\begin{matrix}[A]\\ \vdots\\ B\{\vee D\}\end{matrix}}{(A\supset B)\{\vee D\}} \qquad \neg\mathrm{I}_{Mln}\ \dfrac{\begin{matrix}[A]\\ \vdots\\ D\end{matrix}}{\neg A\vee D}$$

Here, the meaning of curly brackets is that the formula between them may either be or not be present, the rule remaining valid anyway. In other words, Milne's rules can introduce $\supset$ both as the principal connective of a formula and inside a disjunction, depending on the premise.

[24] The rule (displayed in [7], p. 89) is $\dfrac{\begin{matrix}[A]\\ \vdots\\ B\end{matrix}\quad\begin{matrix}[A]\\ \vdots\\ \neg B\end{matrix}}{\neg A}$. Read shows why obliquity, in this case, is not a problem by pointing out that all derivations containing applications of this rule can be modified so as to ensure that, for each of these applications, the discharged hypotheses are always less complex than the conclusion (and so the rule follows Dummett's complexity condition). See [33] for a complete analysis.

[25] [6], p. 257.

[26] Having worked on this subject, I strongly suspect that the main reasons are not ideological, but rather practical: trying to prove something about or in a system with complex rules can be very frustrating!

[27] I have pushed further some of Milne's intuitions in [2], which nonetheless lacks much of the elegance of Milne's work.

[28] [22], p. 514.

The adoption of these complex rules comes with the need to revise the definition of *maximal formula*. Indeed, when $\supset$ and $\neg$ are introduced inside a disjunction, they cannot be directly eliminated. There need to be an application of $\vee$E that enables such an elimination. As a consequence, in the following derivation $(A \supset B) \vee C$ counts as a *maximal formula*:

$$\dfrac{\dfrac{\begin{matrix}[A]^1\\ \vdots\\ \mathfrak{D}_1\quad B\vee C\end{matrix}}{(A\supset B)\vee C}\supset\mathrm{I},\,1 \qquad \begin{matrix}\dfrac{[A\supset B]^2 \qquad A}{B}\supset\mathrm{E}\\ \vdots\\ \mathfrak{D}_2\quad D\end{matrix} \qquad \begin{matrix}[C]^2\\ \vdots \\ \mathfrak{D}_3 \quad D\end{matrix}}{D}\vee\mathrm{E},\,2$$

The reduction of such a derivation should remove both I and E-rule for $\supset$, possibly maintaining $\vee$E. Milne proposes the following reduction step:

$$\dfrac{\begin{matrix}A\\ \vdots\\ \mathfrak{D}_1\quad B\vee C\end{matrix} \qquad \begin{matrix}[B]^2\\ \vdots\\ \mathfrak{D}_2\quad D\end{matrix} \qquad \begin{matrix}[C]^2\\ \vdots\\ \mathfrak{D}_3\quad D\end{matrix}}{D}\vee\mathrm{E},\,2$$

The need for such a revision of maximality can be shown by considering the following weakened rule for *tonk*.

$$tonk\mathrm{I}\ \dfrac{A \vee C}{(A\, tonk\, B) \vee C} \qquad tonk\mathrm{E}\ \dfrac{A\, tonk\, B}{B}$$

Since there are no curly brackets, as opposed to Milne's rule for $\supset$, the introduction of *tonk* inside a disjunction is not optional but explicitly required by the rule. In other words, the premise of *tonk*I *must* be a disjunction and *tonk* cannot be introduced as the principal connective of the conclusion, but only inside the disjunction itself.

The complex structure of *tonk*I does not allow for the construction of a traditional *maximal formula*, since it cannot be paired with an immediate application of *tonk*E. Moreover, *tonk*I should not count as an I-rule for disjunction, so that an immediate application of $\vee$E to its conclusion does not generate any maximality by itself.[29] As a consequence, in order to reject this complex reformulation of *tonk* we need an extension in

[29] For some objections to this conclusion, see [39] and [40], p. 345.

the definition of *maximal formula* which singles out the maximality in the following derivation of B from $A \vee B$.

$$\dfrac{\dfrac{A \vee B}{(A\,tonk\,B) \vee B}\,tonk\mathrm{I} \qquad \dfrac{[A\,tonk\,B]^1}{B}\,tonk\mathrm{E} \qquad [B]^1}{B}\,\vee\mathrm{E},\,1$$

There seems to be some obvious similarities between this derivation and the alleged reduction proposed by Gabbay for the bilateral version of *tonk*. Indeed, in both cases there is an application of *tonk*I that is followed (indirectly) by an application of *tonk*E, with some applications of extra rules between them. These extra rules are in both cases rules for the more external logical constants in the conclusions of the I-rules for *tonk*: in Milne's unilateral case, they are $\vee$-rules; in Gabbay's bilateral case, they are Coordination Principles, that is rules for $+$ and $-$. Moreover, also the meaning-theoretical justification of the application of such extra rules could rely on the same basis in both cases: the dependence of meaning of classical conditional and negation (and of the complex reformulation of *tonk*, of course) upon disjunction in Milne's system, and the dependence of meaning of all the connectives (*tonk* included) upon $+$ and $-$ in Rumfitt's bilateral system. Of course, this is just an intuitive analysis of the analogy between Milne's complex maximality and Gabbay's reduction for *tonk*. To check whether complex maximality can solve Gabbay's objection, we need to take into consideration the formal developments of Milne's ideas.

Unfortunately, Milne does not propose any formal *criterion* for his generalization of maximality, even though he discusses informally the cases with negation and implication and provides an interesting semantic analysis for them. However, in a previous work, I have provided a general definition of *maximal formulae* in unilateral systems with complex rules, and I have proved that there are harmonious complex systems for both intuitionistic and classical logic.[30] In the rest of this section, we will see how this definition can be adapted to bilateral systems, and what follows from its application in this framework.

[30]See [2]. To solve an issue about circularity of meaning, in this previous paper I have worked with single-assumption (and single-conclusion) systems, but I do not want to pose the same restriction here. Also because circularity of meaning seems to be ineliminable for bilateral systems, as concluded in section 2.3.

First of all, the formal definition of dependence of meaning is the expected one:[31]

Definition 2.1 (Dependence of Meaning). For every pair of logical terms $\ominus$ and $\oplus$, the meaning of $\oplus$ depends on the meaning of $\ominus$ ($\ominus \prec \oplus$) iff there is a sequence of logical terms $\circ_1, \dots, \circ_n$ such that $\circ_1 = \ominus$, $\circ_n = \oplus$ and for every $1 \leq i < n$, $\circ_i$ occurs in the premisses or in the discharged assumptions of an I-rule for $\circ_{i+1}$.

For example, since $\bot$ occurs in $\neg$I in **NJ** the meaning of $\neg$ depends on that of $\bot$, and since $\vee$ occurs in the premise of $\supset\mathrm{I}_{Mln}$ in Milne's system the meaning of $\supset$ depends on that of $\vee$. As for Rumfitt's bilateral system, the meaning of the connectives depends on that of $+$ and $-$, since these terms occur in their I-rules. Moreover, the Coordination Principles characterize the meaning of $+$ and $-$, pointing out that each of them depends circularly on the other one.[32] Indeed, we will treat each Coordination Principle as contemporarily an I-rule for the modality in the conclusion and an E-rule for the modalities in the premises or in the discharged assumptions.[33] In particular, Reductio introduces one of the modalities in the conclusion, eliminating the other one, and for this reason it counts as an element of $\oplus\mathrm{I}^+$ or $\oplus\mathrm{I}^-$, depending on the modality of its conclusion, and an element of $\oplus\mathrm{E}^+$ or $\oplus\mathrm{E}^-$, depending on the modality of its discharged assumption. Moreover, since Non-Contradiction eliminates both modalities, it belongs to both $\oplus\mathrm{E}^+$ and $\oplus\mathrm{E}^-$. Hence, since meaning dependence is transitive by definition, each connective depends on both $+$ and $-$.

As for Smiley, it surely works as an E-rule for the modality that is not in the conclusion, and as an I-rule for the modality that is in it. However, the occurrence of the introduced modality in one of the premises of Smiley makes inaccurate to characterize it as just an I-rule for it: it rather seems both an I and an E-rule for the modality in the conclusion. For this reason, it seems less confusing to use Reductio and Non-Contradiction when dealing with separability and harmony.

[31] Notice that meaning-dependence between logical terms is defined on the base of their occurrence in the rules, not in their applications (that is, in inferences).

[32] For now, let us take for granted that this is not a problem. See end of section 2.3.

[33] This interpretation of the logical terms in the discharged assumptions as eliminated is not uncontroversial. As an example, [23] treats them as introduced connectives, and so Peirce's rule and Classical *reductio ad absurdum* as I-rules.

To be more precise, the use of Reductio and Non-Contradiction, in place of Smiley, seems needed to prove that $+$ and $-$ can be characterized with harmonious rules, that is, to prove that Coordination Principles are in harmony with each other. However, given the circular dependence between $+$ and $-$ and the fact that each Coordination Principle eliminates at least one of these modalities, the choice between Reductio and Non-Contradiction, or Smiley is irrelevant to prove harmony and separability of the Operational Rules, regardless of complex maximality. These aspects will become clearer after the display of the formal *criteria* for harmony and separability, in this and in the following section. In particular, in the proof of Theorem 2.6, we will use Smiley until we take into account *maximal formulae* that are the conclusion of Reductio.

The definition of *maximal formulae* for complex systems rests on the notions of elimination path and active logical term in an inference:

Definition 2.2 (Elimination Path (E-path)). Given a derivation $\mathfrak{D}$, a list of formulae $A_1, \ldots, A_n$ is an E-path iff for every m such that $1 \leq m \leq n$, A_m is:

1. the *major* premise of an E-rule, A_{m+1} is one of its discharged assumption, and A_m does not depend on A_{m+1} before the discharge;[34] or
2. the *major* premise of an E-rule that does not discharge assumptions, and A_{m+1} is its conclusion.[35]

Definition 2.3 (Active Logical Term in an Inference). An occurrence of a logical term in a formula A is active in an inference iff the inference is an application of a rule in which, in the formula exemplified by A, the term already has the same occurrence.

The first definition is quite simple. An E-path is a list of *major* premises of E-rules such that, the next element after such a premise is one of the discharged assumptions of the rule, if there is one, or its conclusion otherwise. Sometimes, for brevity we will speak of E-rules of an E-path, to refer

[34] The last clause about the dependence of A_m on A_{m+1} excludes E-paths that go from the major premise of an E-rule A_m to the discharged assumption that is above A_m. This clause is not needed in [2], because the systems there presented do not contain E-rules that discharge open assumptions above their major premises.

[35] We will apply this second clause only for Non-Contradiction, as we will see in the proof of Theorem 2.6.

to the E-rules that have formulae of the E-path as major premises. The second notion is a little tricky, but an example will clarify its definition. In the derivation

$$\dfrac{\supset\mathrm{I}^+\ \dfrac{[+B]^1}{+(A \wedge C) \supset B} \qquad -(A \wedge C) \supset B}{-B}\ \text{Smiley, 1}$$

$\supset$ and $+$ in $+(A \wedge C) \supset B$ are active in the instantiation of $\supset\mathrm{I}^+$, since their occurrence are already present in the schema of $\supset\mathrm{I}^+$. On the contrary, the occurrence of $\wedge$ in the same formula is not active, since conjunction does not occur in the schema of $\supset\mathrm{I}^+$.

With the previous notions at hand, we can give the following definition:

DEFINITION 2.4 (Maximal Formulae). Given a derivation $\mathfrak{D}$, a formula A that occurs in it is a *maximal formula* iff it is the conclusion of an application of $\oplus\mathrm{I}^+$ ($\oplus\mathrm{I}^-$) and the first formula of an E-path such that:

1. the last rule of the E-path is $\oplus\mathrm{E}^+$ ($\oplus\mathrm{E}^-$), and the last formula of the E-path is identical to A;[36]
2. each rule in the E-path eliminates occurrences of logical terms that are active in the conclusion of the application of $\oplus\mathrm{I}$, or logical terms on which those depend.

If the E-path contains only one formula, we say that the *maximal formula* is *simple*. Otherwise, we say that it is *complex*. Notice that, since Coordination Principles are considered as I and/or E-rules for the modalities, a formula that is conclusion of Reductio (which counts as $\oplus\mathrm{I}^+$ or $\oplus\mathrm{I}^-$) and premise of Non-Contradiction (which counts as $\oplus\mathrm{E}^+$ or $\oplus\mathrm{E}^-$) is maximal according to this definition. Moreover, notice that Reductio can introduce only *simple maximal formulae*, not complex ones, since only one logical term occurs actively in its conclusion, and the E-path can go on only with an application of Non-Contradiction, which ends the E-path.

Regarding Operational Rules, in each application of an I-rule of Rumfitt's bilateral system, the only logical terms active in the conclusion are the connective introduced and the modality in which it is introduced (+

[36] In the unilateral systems presented in [2], the restriction on the form of the last formula of the E-path is not needed because of the specific form of its E-rules.

or $-$). Moreover, the only dependences of meaning are the circular one between $+$ and $-$, and the dependence of every connective on at least one of the modalities (and so indirectly on both of them, for transitivity). Apart from this, there is no other occurrence of connectives in the I-rules that establish dependences of meaning. As a consequence, complex *maximal formulae* can only be composed of an I-rule for a connective $\oplus$ inside a modality, followed by an E-path of Coordination Principles that ends with $\oplus$E for the same modality. Simple *maximal formulae* remain the standard ones individuated by Rumfitt in which an I-rule is immediately followed by an E-rule for the same connective and the same modality. Let us see some examples of *maximal formulae* and some of their properties.

First of all, it is quite clear that according to Definition 2.4 we have a *simple maximal formula* for *tonk* in

$$\dfrac{\dfrac{+A}{+(A\,tonk\,B)}\,\textit{tonk}\mathrm{I}}{+B}\,\textit{tonk}\mathrm{E}$$

and a *complex maximal formula* in

$$\dfrac{\dfrac{+A}{+(A\,tonk\,B)}\,\textit{tonk}\mathrm{I} \qquad \dfrac{\dfrac{[+(A\,tonk\,B)]^1}{+B}\,\textit{tonk}\mathrm{E} \qquad [-B]^2}{-(A\,tonk\,B)}\,\text{Smiley, 1}}{+B}\,\text{Smiley, 2}$$

Indeed, the conclusion of *tonk*I starts an E-path, since it is the major premise of a Smiley. The E-path continues with the occurrence of $-B$ discharged by this Smiley (see the first clause of the Definition 2.2), and then with the occurrence of $+(A\,tonk\,B)$ discharged by the Smiley that has $-B$ as premise. This occurrence of $+(A\,tonk\,B)$ then ends the E-path, being the premise of *tonk*E.

So, our formal *criterion* of maximality confirms the previous intuition about the non-normality of Gabbay's reduction. There is however something more to say about complex maximality in bilateral systems. What is peculiar in Rumfitt's system is that there are complex *maximal formulae* for every pair of I and E-rules, such as:

$$\text{Smiley, 2}\ \dfrac{\wedge\mathrm{I}^+\ \dfrac{+A \qquad +B}{+A \wedge B} \qquad \text{Smiley, 1}\ \dfrac{\wedge\mathrm{E}^+\ \dfrac{[+A \wedge B]^1}{+A} \qquad [-A]^2}{-A \wedge B}}{+A}$$

Moreover, they are all quite easily reducible to traditional simple *maximal formulae*. In this case, to:

$$\wedge\mathrm{E}^+\ \dfrac{\wedge\mathrm{I}^+\ \dfrac{+A \qquad +B}{+A \wedge B}}{+A}$$

It should not come as a surprise that equivalent *maximal formulae* can be individuated when Reductio and Non-Contradiction are used in place of Smiley, such as:

$$\text{Reductio, 2}\ \dfrac{\text{Non-Contradiction}\ \dfrac{\wedge\mathrm{I}^+\ \dfrac{+A \qquad +B}{+A \wedge B} \qquad \text{Reductio, 1}\ \dfrac{\text{Non-Contradiction}\ \dfrac{\wedge\mathrm{E}^+\ \dfrac{[+A \wedge B]^1}{+A} \qquad [-A]^2}{\bot}}{-A \wedge B}}{\bot}}{+A}$$

Here, the E-path originated with the conclusion of $\wedge\mathrm{I}^+$ continues with the $\bot$ that is concluded by Non-Contradiction (an E-rule that does not discharge assumptions, see the second clause of Definition 2.2), then with $-A$ that is discharged by Reductio, then $\bot$ and $+A \wedge B$, which ends the E-path. Moreover, also this complex *maximal formula* reduces to its simple counterpart.

This general reducibility of the complex maximality to the simple one is particularly interesting. Indeed, in Rumfitt's system the generalization of the definition of *maximal formula* to include complex cases seems ineffective – even though justified from a meaning-theoretic point of view –, and poses just minor problems for normalizability, as we will see. Nonetheless, it is the key ingredient to reject Gabbay's alleged reduction for *tonk*. On the contrary, in Milne's system *complex maximality* properly extends *simple maximality.*[37] Indeed, *maximal formulae* obtained using his complex version of I-rules for negation and implication are *not* reducible to *maximal formulae* obtained using their simple counterparts, and this is the reason why they must be addressed independently, as Milne himself does.

In Rumfitt's system, the reducibility of complex *maximal formulae* to simple *maximal formulae* holds for *tonk*-rules as well. The difference is that, while with well-behaving connectives the reduction can then go on and lead to a normal derivation, simple *maximal formulae* for *tonk* are not reducible, so the reduction has to stop there. Hence, Gabbay is wrong not only because his alleged reduction for *tonk* is not in normal form, but also because the reduction procedure goes in the opposite direction of what he claims: what he is proposing is not a reduction to a normal form, but a step backward from a non-normal derivation with a simple *maximal formula* to one with a complex *maximal formula.*

Let us now substantiate the previous intuitive analysis with a formal treatment of normalization for Rumfitt's system that deals with complex maximality. First of all, we will need a formal definition of *maximal sequence*:

DEFINITION 2.5 (Maximal Sequences). Given a derivation $\mathfrak{D}$, a list of identical formulae that occur in it $A_1, \ldots, A_n$ is a *maximal sequence* iff A_1 is the conclusion of an application of $\oplus\mathrm{I}^+$ ($\oplus\mathrm{I}^-$) and the first formula of an E-path such that:

1. each rule in the E-path eliminates occurrences of logical terms that are active in the conclusion of the application of $\oplus$I, or logical terms on which those depend;

2. if the E-path contains more than one formula, then the last formula of the E-path is the second formula of *the maximal sequence* A_2, and, for each $1 < i < n$, A_i is a minor premise of an application of $\vee\mathrm{E}^+$ or $\wedge\mathrm{E}^-$;

[37] This holds for the systems in [2] as well.

3. if the E-path contains just one formula, then it is the first formula of *the maximal sequence* A_1, and, for each $1 \leq i < n$, A_i is a minor premise of an application of $\vee$E$^+$ or $\wedge$E$^-$;

4. A_n is the major premise of $\oplus$E$^+$ ($\oplus$E$^-$).

In summary, the structure of a *maximal sequence* is the following. It starts with the conclusion of an I-rule $\oplus$I. If it is the major premise of the E-rule $\oplus$E or the minor premise of $\vee$E$^+$ or $\wedge$E$^-$, we have a simple *maximal sequence* like the usual ones considered by Prawitz. Otherwise, the conclusion of $\oplus$I is the first formula of an E-path. In this case, the last formula of the E-path is the second formula of the *maximal sequence* and can be a minor premise of $\vee$E$^+$ or $\wedge$E$^-$, or the major premise of the E-rule $\oplus$E. In the last case, we have a complex *maximal formula*, which is a specific case of complex *maximal sequence*. As usual, a derivation in which there are no *maximal sequences* is called in *normal form*.

Finally, let us prove normalization for this complex reformulation of maximality:

Theorem 2.6 (Normalization). *For every derivation $\mathfrak{D}$ of the system consisting of Rumfitt's Operational Rules in table 1 together with both the rules of Smiley in table 2, or together with both the rules of Reductio and the rule of Non-Contradiction in the same table, there is a reduction procedure that leads from $\mathfrak{D}$ to a derivation $\mathfrak{D}'$ in normal form with the same conclusion of $\mathfrak{D}$ and the same or less open assumptions.*

Proof: The structure of the proof is the following. In the first part, we will prove the result for the system constructed with the two Smiley as the only Coordination Principles. Then, we will extend the result to the system with Reductio and Non-Contradiction in place of Smiley.

First of all, let us see the reduction steps for the complex *maximal sequences*. For reasons of space, we will not show the reduction steps for all of them. In particular, we will focus on the case of positive sequences and we will assume that the Smiley that starts the E-path discharges positive formulae. Nonetheless, all the other cases are trivial variations of those here displayed.

Let us consider the complex *maximal sequence*:

$$
\dfrac{\dfrac{\begin{matrix}[+A]^1\\ \vdots\ \mathfrak{D}_1\\ +B\end{matrix}}{+C}\oplus\mathrm{I}^+ \qquad \begin{matrix}\dfrac{[+A]^1 \qquad \begin{matrix}\dfrac{+C\vee F \qquad [+C]^n \qquad \begin{matrix}[+F]^{n+1}\\ \vdots\ \mathfrak{D}_2\\ +C\end{matrix}}{+C}\vee\mathrm{E}^+,\ \mathrm{n{+}1}\\ \vee\mathrm{E}^+ \text{ or } \wedge\mathrm{E}^-,\ \mathrm{n{+}2,\ldots,m}\\ \vdots\ \mathfrak{D}_3\\ \dfrac{+C}{+E}\oplus\mathrm{E}^+\\ \text{Smiley} \times n-2,\ 3,\ldots,\mathrm{n}\\ \vdots\ \mathfrak{D}_4\\ -A\end{matrix}}{-D}\text{Smiley, 2}\\ \vdots\ \mathfrak{D}_5\\ -C\end{matrix}}{-A}\text{Smiley, 1}
$$

It can be reduced to a complex *maximal formula* in just one step:

$$
\text{Smiley, 1}\ \dfrac{\oplus\mathrm{I}^+\dfrac{\begin{matrix}[+A]^1\\ \vdots\ \mathfrak{D}_1\\ +B\end{matrix}}{+C} \qquad \begin{matrix}\text{Smiley, 2}\ \dfrac{[+A]^1 \qquad \begin{matrix}\oplus\mathrm{E}^+\dfrac{[+C]^n}{+E}\\ \text{Smiley} \times n-2,\ 3,\ldots,\mathrm{n}\\ \vdots\ \mathfrak{D}_4\\ -A\end{matrix}}{-D}\\ \vdots\ \mathfrak{D}_5\\ -C\end{matrix}}{-A}
$$

This in turn can be reduced to a simple *maximal formula* in the following way:

$$
\text{Smiley, 1}\ \dfrac{[+A]^1 \qquad \begin{array}{c} [+A]^1 \\ \mathfrak{D}_1\ \vdots \\ \dfrac{\dfrac{+B}{+C}\,\oplus\mathrm{I}^+}{+E}\,\oplus\mathrm{E}^+ \\ \text{Smiley} \times n-2,\ 3,\ldots,\text{n} \\ \mathfrak{D}_4\ \vdots \\ -A \end{array}}{-A}
$$

Notice that no new *maximal sequences* can be generated. For simple ones this is obvious. For complex ones, $\mathfrak{D}_1$ is composed *above* $\mathfrak{D}_2$, so from the last point of Definition 2.2 it follows that no new E-path for complex *maximal sequences* is generated. When A is not an open assumption of the derivation $\mathfrak{D}_1$, we can drop the last application of Smiley, as done in the examples already shown for *maximal formulae* introduced by $\wedge\mathrm{I}^+$. Of course, the simple *maximal formulae* and *maximal sequences* are reducible, as already noticed by Rumfitt.[38] Hence, for *each maximal sequence* a reduction step is available. Let us now prove normalization *stricto sensu*, that is that reduction steps can be composed to reduce *all maximal sequences*, and so lead to a derivation in normal form.

Our proof of normalization is a development of Prawitz's normalization for NJ.[39] The definitions of degree and length of a *maximal sequence* are as usual. The proof is by induction on $\langle d, l, e\rangle$, where d is the highest degree of a *maximal sequence* in $\mathfrak{D}$, l is the sum of the lengths of *maximal sequences* in $\mathfrak{D}$ of degree d, and e is the sum of the lengths of the E-paths of complex *maximal sequences* in $\mathfrak{D}$ of degree d. We assume that $\langle d', l', e'\rangle < \langle d, l, e\rangle$ iff $d' < d$, or $d' = d$ and $l' < l$, or $d' = d$, $l' = l$ and $e' < e$.

We prove that, given a derivation $\mathfrak{D}$ not in normal form with induction value $v = \langle d, l, e\rangle$, we can find a derivation $\mathfrak{D}'$ with an induction value less than v of the same conclusion, from the same or fewer assumptions. Let us choose a *maximal sequence* α of degree d with the following properties:

[38] In general, the availability of reduction steps for simple *maximal sequences* in Rumfitt's bilateral system is a well-established starting point of the discussion about bilateralism. The problem is how to extend this result to obtain at least coherence, and how to treat Coordination Principles.

[39] [27], pp. 50–51.

1. there are no *maximal sequences* of degree d above α; and

2. if α is a simple *maximal sequence*, then there are no simple *maximal sequences* of degree d that are above or contain the other premises of the inference of which the last formula of α is premise.[40]

If the *maximal sequence* is simple, we apply the standard reduction steps for Rumfitt's bilateral system. They clearly reduce d or l, if it is a simple *maximal formula*, and l, if it is a simple *maximal sequence* longer than one. In both cases, we obtain a derivation $\mathfrak{D}'$ with a value of $\langle d, l, e \rangle$ that is less than v. The reduction steps do not generate any new *maximal sequence* of degree d. This is well known for simple *maximal sequences* and easy to see for complex *maximal sequences* as well. As an example, let us focus on the case of *permutative conversions* that reduce the length of simple *maximal sequences*. If $+B \vee C$ (or $-B \wedge C$) is major premise of $\vee\mathrm{E}^+$ (respectively $\wedge\mathrm{E}^-$), it does not begin an E-path, sice $\vee$ (respectively $\wedge$) is active only in applications of $\vee\mathrm{I}^+$ (respectively $\wedge\mathrm{I}^-$). Hence, even though there is a change in the derivation of the minor premises of $\vee\mathrm{E}^+$ ($\wedge\mathrm{E}^-$), it cannot generate any new complex *maximal sequence*.

If the *maximal sequence* is complex, we apply the reduction steps displayed at the beginning of this proof. If the *maximal sequence* is longer than two, then the reduction step reduces the value of l. Otherwise, the reduction step reduces the value of e. In both cases, the value of $\langle d, l, e \rangle$ is decreased. As we have already argued in the first part of the proof, no new *maximal sequence* is generated by these reduction steps. Hence, the result follows by induction.

So far, we have considered a system with Smiley as Coordination Principle. Let us show that the use of Reductio and Non-Contradiction in its place does not constitute any problem. First of all, Smiley is clearly derivable from the other two rules: just derive $\perp$ from $+B$ and $-B$ using Non-Contradiction, and then discharge the open assumption $+A$ (or $-A$) to derive $-A$ (or $+A$) using Reductio. Moreover, as we have seen, the cyclic dependence of meaning between $+$ and $-$ entails that we can use

[40] This second clause is required by Prawitz to make *permutative conversions* effective in reducing the inductive value of the derivation. Since the reduction procedure for complex *maximal sequences* is very different and does not use *permutative conversions*, this clause can be dropped for them.

both Reductio and Non-Contradiction in the E-path of a *maximal sequence*. Hence, the occurrences of Smiley in the reductions can be substituted with occurrences of Reductio and Non-Contradiction without affecting the normalization process. This is the reason why we claimed that for the harmony of the Operational Rules (and for separability as well), the adoption of Reductio and Non-Contradiction, or Smiley makes no difference.

However, while using Smiley it is not clear how we should address simple maximality regarding Coordination Principles, that is regarding $+$ and $-$, with Reductio and Non-Contradiction we clearly have *maximal formulae*, when the conclusion of an application of Reductio is premise of an application of Non-Contradiction. Fortunately, in these cases we can easily find a reduction, such as:

$$\text{Non-Contradiction}\ \dfrac{+A \qquad \text{Reductio, 1}\ \dfrac{\begin{matrix}[+A]^1\\ \vdots\\ \bot\end{matrix}}{-A}}{\bot} \qquad\rightsquigarrow\qquad \begin{matrix}+A\\ \vdots\\ \bot\end{matrix}$$

Moreover, given a derivation $\mathfrak{D}$ and chosen a *maximal sequence* according to the instruction seen previously, this reduction step clearly reduces the value of d or l.

The conclusion of an application of Reductio can also be the first formula of a simple *maximal sequence* that ends with an application of Non-Contradiction. In this case, a *permutative conversion* can be provided. As an example, the derivation

$$\text{Non-Contradiction}\ \dfrac{\vee\text{E}^+,\, 2\ \dfrac{+B \vee C \qquad \text{Reductio, 1}\ \dfrac{\begin{matrix}[+A]^1 \quad [+B]^2\\ \mathfrak{D}_1 \quad \vdots\\ \bot\end{matrix}}{-A} \qquad \begin{matrix}[+C]^2\\ \mathfrak{D}_2 \quad \vdots\\ -A\end{matrix}}{-A} \qquad +A}{\bot}$$

can be reduced to the derivation

$$\dfrac{+B \vee C \qquad \dfrac{\dfrac{\begin{matrix}[+A]^1 \quad [+B]^2\\ \vdots\ \mathfrak{D}_1 \\ \bot\end{matrix}}{-A}\text{Reductio, 1} \qquad +A}{\bot}\text{Non-Con} \qquad \dfrac{\begin{matrix}[+C]^2\\ \vdots\ \mathfrak{D}_2\\ -A\end{matrix} \qquad +A}{\bot}\text{Non-Con}}{\bot}\vee\mathrm{E}^+, 2$$

Of course, a similar reduction can be provided also when the *maximal sequence* applies $\wedge\mathrm{E}^-$ instead of $\vee\mathrm{E}^+$. In this case as well, given a derivation $\mathfrak{D}$ and chosen a *maximal sequence* according to the instructions seen previously, the reduction step reduces the value of l.[41]

As for complex *maximal sequences*, let me remind the reader that the conclusion of Reductio cannot be the first formula of an E-path, since there is only one active logical term in its conclusion. Hence, complex maximality is not an issue in this case. In conclusion, normalization holds also for the system composed with Reductio and Non-Contradiction in place of Smiley. □

Given Theorem 2.6, it follows that the generalization of the notion of *maximal formula* does not exclude the well-behaved system proposed by Rumfitt. In other words, the complex *maximal formulae* and *maximal sequences* constructed using Rumfitt's Operational Rules and Coordination Principles are all reducible. Moreover, in [2] I have shown that also in unilateral systems this generalization does not exclude well-behaved systems, but on the contrary extends the class of acceptable systems. As we have claimed in the introduction, normalization is usually the key ingredient of any proof-theoretic *criterion* of harmony that decides the acceptability for a set of rules, at least in the flavour of proof-theoretic semantics that is based on Dummett's and Prawitz's works – and to which Rumfitt's bilateralism belongs. Hence, since the adoption of complex maximality results in a proof of normalization for a classical bilateral system, it is compatible with the standard approach to validity endorsed in proof-theoretic semantics, and at most extends the class of its valid systems. The only systems ruled

[41] Notice that the second clause imposed for the selection of the *maximal sequence* to be reduced is relevant here, since if there is a *maximal sequence* of degree d in the derivation of the right minor premise of $\vee\mathrm{E}^+$ the value of l remains unchanged at the end of the reduction.

out by this extension of maximality are pathological systems that use complex rules to introduce paradoxical connectives avoiding simple *maximal formulae* (such as the introduction of *tonk* in the scope of the disjunction), or to fake a reduction process (such as in Gabbay's case).

2.3. Weak separability

Complex rules, like all impure rules, impose a dependence of meaning between logical constants. As an example, it is part of the received wisdom that the meaning of intuitionistic negation depends upon that of $\bot$. This is the semantic counterpart of the occurrence of this constant in the I-rule for negation, that is of the impurity of $\neg$I. In the same way, the occurrence of disjunction in Milne's rule for the introduction of classical negation and conditional entails that the meaning of both $\supset$ and $\neg$ depends upon that of $\vee$.

For separability, the fact that the meaning of $\neg$ depends upon that of $\bot$ means that in order to prove that C follows logically from Γ the $\bot$-rules (that is *ex falso quodlibet*) could be needed together with those for $\neg$, even if $\neg$ occurs in $\Gamma \cup \{C\}$ but $\bot$ does not.[42] A clear example of this phenomenon is that any derivation of $A \wedge \neg A \vdash C$, with C fully general, requires *ex falso*. Traditionally, instead of considering the meaning of intuitionistic negation as depending on that of absurdity, $\neg A$ has been sometimes considered just a shortening for $A \supset \bot$. With complex rules, this solution is not viable and meaning-dependence becomes an incontestable phenomenon of the system.[43] This leads to revising the traditional definition of separability in the following way:

Definition 2.7 (Weak Separability). To prove a logical consequence $\Gamma \vdash C$ we only need to use the rules for the logical constants that occur in Γ or C, together with the rules for the constants on which those depend. That is, in order to prove a logical consequence $\Gamma \vdash C$, it is enough to use the rules for the constants $\circ_1, \dots, \circ_n$ such that for every $1 \leq i \leq n$:

- $\circ_i$ occurs in Γ or C; or
- for some $j \neq i$ such that $1 \leq j \leq n$, $\circ_j$ occurs in Γ or C and $\circ_i \prec \circ_j$.

[42] We will use $\Gamma \vdash C$ to indicate that C is a logical consequence of Γ.

[43] Cozzo investigated something similar, even though for non-logical terms: see [3], pp. 246–250, [4], pp. 32–34 and [5], p. 305. Also Prawitz developed a similar idea for logical terms, apparently independently of Milne: see [29].

To see a clear example of why such a weakening of separability is needed, let us just consider the following proof of the purely implicational classical theorem usually called *Peirce's law*:[44]

$$\vee\mathrm{E},2,3\ \dfrac{\supset\mathrm{I},1\ \dfrac{\vee\mathrm{I}\ \dfrac{[A]^1}{A\vee B}}{A\vee(A\supset B)} \qquad \supset\mathrm{E}\ \dfrac{[(A\supset B)\supset A]^4 \qquad [A\supset B]^2}{A} \qquad [A]^3}{\supset\mathrm{I},4\ \dfrac{A}{((A\supset B)\supset A)\supset A}}$$

Since disjunction does not occur in the conclusion, Peirce's law is not intuitionistically valid and the only difference between Milne's classical $\supset$I and intuitionistic $\supset$I is the possibility of introducing $\supset$ inside a disjunction, the usage of $\vee$-rules is obviously needed in the previous derivation. Hence, Milne's system does not suit separability, even though it can be shown to suit *weak* separability.

Rumfitt claims that his system is separable, but he considers only Operational Rules to show this result. It is far from obvious that separability holds when we consider $+$ and $-$ as well, asking for example that in order to prove purely assertive consequences (that is, consequences that have only $+$-*formulae* both between the assumptions and as the conclusion) only rules for assertion are needed. On the contrary, any derivation of the purely assertive consequence $+\neg\neg A \vdash +A$ seems to require an application of rules for the rejection of $\neg$-*formulae*, such as:

$$\neg\mathrm{E}^-\ \dfrac{\neg\mathrm{E}^+\ \dfrac{+(\neg\neg A)}{-(\neg A)}}{+A}$$

The generalization of maximality seen in the previous paragraph gives ground for considering $+$ and $-$ too for separability, since they contribute to the meaning of the connectives and Coordination Principles can be used to construct complex maximality. Moreover, Kürbis' observations about the problems of interpreting these signs as standing for speech acts suggests that they should be treated more like modalities and so, arguably, considered for separability as well.[45] Other circumstantial pieces of evidence that

[44] [22], p. 527.

[45] We have seen briefly Kürbis' observations at the end of section 1.

$+$ and $-$ should be considered for separability can be provided: changing Coordination Principles entails a change in the logic,[46] some *criteria* are needed to balance $+$ and $-$ rules for the same connective.[47] Nonetheless, considering these signs for separability also means adopting a weakened version of this requirement. Indeed, assertion and rejection are used to give meaning to all the connectives, and if the Coordination Principles define the meaning of $+$ and $-$, it seems obvious that there can only be a cyclic dependence of meaning of each of them upon the other. Hence, weak separability for Rumfitt's bilateral system asks that all logical consequences are provable using only rules for the connectives that occur in the premises or in the conclusion of the consequence, signed with the signs that occur in the consequence, together with Coordination Principles.[48] As an example, if both premises and conclusion of a consequence are signed $+$ and $\oplus$ occurs in its premises or in its conclusion, we can use $\oplus^+$-rules and Coordination Principles, while we should not use $\oplus^-$-rules. With this clarification, we obtain:

Theorem 2.8 (Weak Separability). *Weak separability holds for the system consisting of Rumfitt's Operational Rules in table 1 together with both the rules of Smiley in table 2, or together with both the rules of Reductio and the rule of Non-Contradiction in the same table.*

Proof: The part about the connectives is given by Rumfitt: in order to prove $\Gamma \vdash C$ we need to use only rules for connectives that occur in $\Gamma \cup \{C\}$.[49] About $+$ and $-$, showing that for every connective $\oplus$, $\oplus^+$-rules ($\oplus^-$-rules) are derivable from $\oplus^-$-rules ($\oplus^+$-rules) together with Coordination Principles is sufficient to establish the result. Indeed, from this derivability it follows that any application of a $\oplus^-$-rule can be substituted with an application of the $\oplus^+$-rules and of Coordination Principles. Hence, it cannot be necessary to use $\oplus^-$-rules to prove that C follows from Γ if $-$ does not occur neither in the premises nor in the conclusion. The same

[46] [18]

[47] We will deal with this issue in section 3.1.

[48] In his [20], the author asks for the normalization of Reductio followed by E-rules or Non-Contradiction. In concrete, this means considering Coordination Principles (and sometimes even Operational Rules, as we will see) as meaning conferring rules for $+$ and $-$.

[49] See [36].

argument proves that $\oplus^+$-rules are not needed to derive consequences that regard only rejections.

We will not give a complete proof of the derivability of all the rules of assertion (rejection) for a connective from the rules of rejection (assertion) for the same connective together with Coordination Principles, since they are quite easy. The example of implication will be sufficient to illustrate the procedure:

- The rule $\supset^+$I is derived by

$$\dfrac{\begin{array}{c}\dfrac{[-A \supset B]^1}{+A}\,{\scriptstyle \supset E^-} \\ \vdots \\ +B\end{array} \qquad \dfrac{[-A \supset B]^1}{-B}\,{\scriptstyle \supset E^-}}{+A \supset B}\,\text{Smiley, 1}$$

- The rule $\supset^+$E is derived by

$$\dfrac{+A \supset B \qquad \dfrac{+A \qquad [-B]^1}{-A \supset B}\,{\scriptstyle \supset I^-}}{+B}\,\text{Smiley, 1}$$

- The rule $\supset^-$I is derived by

$$\dfrac{\dfrac{+A \qquad [+A \supset B]^1}{+B}\,{\scriptstyle \supset E^+} \qquad -B}{-A \supset B}\,\text{Smiley, 1}$$

- The rule $\supset^{-}\mathrm{E}_1$ is derived by

$$\dfrac{-A \supset B \qquad \dfrac{\dfrac{[-A]^2 \qquad [+A]^1}{+B}\,\text{Smiley}}{+A \supset B}\,{\supset}\mathrm{I}^{+}, 1}{+A}\,\text{Smiley, } 2$$

- The rule $\supset^{-}\mathrm{E}_2$ is derived by

$$\dfrac{-A \supset B \qquad \dfrac{[+B]^1}{+A \supset B}\,{\supset}\mathrm{I}^{+}}{-B}\,\text{Smiley, } 1$$

□

Whether cyclic dependencies of meaning like the one between + and − are acceptable in logic is at least a controversial issue.[50] Here I will not discuss this issue, which would need a further article in itself, but I will be satisfied with having pointed at what seems to me the deepest problem of the bilateral systems.

That bilateral systems have problems with *complexity criterion* and non-circularity of meaning-dependence should not come as a surprise. Indeed, Milne already considered the possibility of reading

$$\dfrac{\begin{array}{c}[\neg A]\\ \vdots \\ \bot\end{array}}{A}\,\text{Classical Reductio}$$

as an I-rule for the atomic formula in the conclusion, and of course the main obstacle for such a reading is the *complexity criterion*.[51] The same *criterion* is still violated if we substitute − to ¬ and add + to the conclusion, so obtaining the bilateral rule of Reductio. Moreover, it can be observed that in the bilateral framework the *complexity condition* is violated less heavily, since the conclusion of Reductio is not less complex but has the same complexity of its discharged assumption, but only at the cost of a violation of the circularity of meaning-dependence between + and −.

---

[50] Dummett clearly rejects such a possibility in [6], p. 257. Nonetheless, the same criticism that has been raised against his complexity *criterion* could maybe be used against this non-cyclic requirement.

[51] [21], p. 59.

3. Bullet and the balance between assertion and rejection

3.1. Gabbay's reappraisal of Read's •

In section 2 we have seen one of the objections raised by Gabbay against bilateral systems and we have evaluated a possible solution that rests on a generalization of *maximality*. Here we will consider another objection raised in the same paper and evaluate whether a common solution to both these problems can be found.

While the first objection was a reinterpretation of *tonk* inside the bilateral framework, the second one is a reinterpretation in the same framework of Read's •, an inferentialist version of liar's paradox.[52] Gabbay presents the following set of rules for this zero-ary connective:

$$\bullet\mathrm{I}^+\ \frac{+A \qquad -A}{+\bullet} \qquad\qquad \bullet\mathrm{I}^-\ \frac{+A \qquad -A}{-\bullet}$$

$$\bullet\mathrm{E}^+\ \frac{+\bullet}{+A} \qquad \bullet\mathrm{E}^+\ \frac{+\bullet}{-A} \qquad \bullet\mathrm{E}^-\ \frac{-\bullet}{+A} \qquad \bullet\mathrm{E}^-\ \frac{-\bullet}{-A}$$

It can be shown that they are harmonious in bilateral systems, since any *maximal formula* obtained by pairing an I-rule (for assertion or for rejection) with the corresponding E-rule can be reduced. They are nonetheless unacceptable, since they lead any system that is equipped with the standard Coordination Principles to trivialism:

$$\bullet\mathrm{E}^-\ \dfrac{\text{Smiley, 1}\ \dfrac{\bullet\mathrm{E}^+\ \dfrac{[+\bullet]^1}{+B} \qquad \bullet\mathrm{E}^+\ \dfrac{[+\bullet]^1}{-B}}{-\bullet}}{+A} \qquad\qquad \bullet\mathrm{E}^-\ \dfrac{\text{Smiley, 1}\ \dfrac{\bullet\mathrm{E}^+\ \dfrac{[+\bullet]^1}{+B} \qquad \bullet\mathrm{E}^+\ \dfrac{[+\bullet]^1}{-B}}{-\bullet}}{-A}$$

Francez proposes a diagnosis of what is wrong with these rules and a formal *criterion* to exclude them.[53] He claims that the problem is not a disharmony between I and E-rules, but a lack of balance between rules for assertion and rules for rejection. According to Francez, bilateralism should

[52] [32], p. 141. Prawitz's inferentialist interpretation of Russell's paradox displays some similarities with Read's rules; see [27], p. 95.

[53] His answer to Gabbay is in [12], but the principle he employs in his reply was already formulated in his [11].

endorse a *principle of coherence* that prohibits the assertion and the rejection of the same formula. Technically, in order to assure this principle, he asks that the rules of rejection be a function of the rules of assertion, labeling *Horizontal Balance* this condition. The formal details of this functionality are very complex, and there is no need to go into the details here. What is important is that, technically speaking, his principle works fine for this objection (even though not for the one regarding *tonk*, still raised by Gabbay).[54]

There are, however, some more conceptual perplexities that could be raised. Francez's requirement of coherence is explicitly inspired by a similar requirement imposed by Restall, who nonetheless works in a completely different framework. Restall proposes a meaning-theoretical and inferentialist reading of sequent calculus, and his principle of coherence is just a reading of the uncontroversial axiom $A \Rightarrow A$, which works as starting point of every sequent calculus. On the contrary, in Rumfitt's bilateralism, to endorse coherence means both to exclude the rules of Incoherence

$$\text{Incoherence}\ \frac{+A}{-A} \qquad \text{Incoherence}\ \frac{-A}{+A}$$

from the set of Coordination Principles, and to ask that they are not derivable rules of the system. So in this case there is a positive restriction to be imposed, which moreover raises both conceptual and formal issues.

The first formal problem is that Horizontal Balance applies only to Operational Rules and leads to Coherence *only if* Coordination Principles behave well. Hence, since Francez poses no restriction on Coordination Principles, we can only be sure that the rules of Incoherence are not derivable using Operational Rules, as would be the case with Gabbay's $\bullet$-rules, but we cannot prevent their adoption as Coordination Principles or exclude that they are derivable because of the Coordination Principles. The reason why Francez seems not to consider this as an issue is that he works with a predetermined set of Coordination Principles, but this solution seems *ad hoc*: it would be clearly preferable to have a *criterion* for Coordination Principles as we have for Operational Rules.

The second formal problem is that rejecting the rules of Incoherence by decree seems to be unjustified, since these rules have nothing wrong *per se*. Indeed they lead to trivialism only together with standard Coordination Principles, while by themselves they "just" establish the interderivability

[54] The formal details are developed in [10] and in section 4.4.1.7 of [11].

of $+A$ and $-A$, leading to maybe unpalatable consequences but neither to trivialism nor to proof of $\bot$. Arguably, this outcome prevents endorsing Incoherence if we want to preserve the reading of $+$ and $-$ as assertion and rejection, but the acceptability of a rule in itself, due to inferentialist reasons, should not be mixed with its adequacy to its standard interpretation. As an example, the issue of inferentialist acceptability of a set of rules for classical conditional should not be confused with the issue of whether the conditional we use in everyday arguments has classical properties or not.

In summary, from a formal point of view both issues suggest that, without a criterion that deals with Coordination Principles as well as Operational Rules, the problems raised by $\bullet$ can only be moved, not solved. Moreover, also from a purely conceptual point of view the adoption of a principle of coherence is far from obvious from the perspective of proof-theoretic semantics. Indeed, Restall's inferentialism, which inspires Francez's solution, explicitly departs from the standard Dummettian antirealist theory of meaning, which is at the core of proof-theoretic semantics, and relies on a theory of meaning based on Brandom's works.[55]

3.2. The search for a solution

Having dismissed Francez's analysis of Gabbay's rules for $\bullet$, we are in need of a solution that explains what is wrong with them. It seems to me that, following the interpretation of bilateralism outlined in section 2 (and especially in subsection 2.2), we can retort a criticism pointed out by Gabbay himself against Read's $\bullet$ to Gabbay's bilateral reformulation of this zero-ary connective.

Read proposed the following rules for $\bullet$ in a standard unilateral framework

$$\bullet\mathrm{I}\ \frac{\neg\bullet}{\bullet} \qquad\qquad \bullet\mathrm{E}\ \frac{\bullet \qquad \begin{matrix}[\neg\bullet]\\ \vdots \\ C\end{matrix}}{C}$$

arguing that they are in harmony with each other, since *each maximal formula* obtained from an application of $\bullet$I immediately followed by $\bullet$E

[55] See [34] and [35] *inter alia*. For a comparison between these two approaches to inferentialism, see [42] and [41].

can be removed. Nonetheless, together with standard rules for negation they supply a closed proof of $\bot$:

$$\dfrac{\neg\text{I},2\,\dfrac{\neg\text{E}\,\dfrac{[\bullet]^2 \qquad \bullet\text{E},1\,\dfrac{[\bullet]^2 \qquad [\neg\bullet]^1}{\neg\bullet}}{\bot}}{\neg\bullet} \qquad \bullet\text{I}\,\dfrac{\neg\text{I},2\,\dfrac{\neg\text{E}\,\dfrac{[\bullet]^2 \qquad \bullet\text{E},1\,\dfrac{[\bullet]^2 \qquad [\neg\bullet]^1}{\neg\bullet}}{\bot}}{\neg\bullet}}{\bullet}}{\bot}\,\neg\text{E}$$

A first standard objection to the acceptability of these rules is that they are not really harmonious, but just suit Inversion Principle. Indeed, even though *each maximal formula* can be removed, not for every derivation this procedure leads to a derivation without *maximal formulae*, that is to a normalization of the derivation. As an example, the closed derivation of $\bot$ just seen cannot be given in normal form.[56] In the original formulation of proof-theoretic semantics, this objection leads to rejecting the set of rules that causes the non-normalizability. This is indeed the choice made by Prawitz when the same issue arises regarding his inferentialist version of Russell's paradox.[57] Nonetheless, Read rejects this objection, accepting Inversion Principle as the only criterion for harmony. A discussion of Read's position regarding harmony and Inversion Principle exceeds the scope of this article. Nonetheless, it should be noticed that the lack of normalizability cannot be used as an objection against Gabbay's bilateral version of $\bullet$. Indeed, first of all, only the E-rules of bilateral $\bullet$ are used in the proof of $\bot$ that we have just displayed. Moreover, the reduction step for a *maximal formula* obtained via introduction and elimination rules for $\bullet$ completely erases the occurrence of $\bullet$ itself, as opposed to the reduction step for the unilateral version of these rules proposed by Read. So, even taking for granted Prawitz's requirement of normalizability, it can at most exclude Read's $\bullet$, but not Gabbay's.

There is anyway another objection to Read's $\bullet$ that, taken together with the interpretation of $+$ and $-$ presented in section 2, could be extended to cover Gabbay's bilateral version as well. Ironically enough, this objection is raised by Gabbay himself in the same paper in which he proposes his

[56] Tranchini shows that the reduction procedure goes in circle: [43], p. 413.

[57] [27], p. 95.

bilateral $\bullet$. Gabbay observes that, even though Read considers it only as an introduction rule, $\bullet$I is also an elimination rule for $\neg$.[58] As a consequence, $\bullet$I should suit Inversion Principle also *qua* E-rule for negation. But this is clearly not the case, since there is no reduction procedure for the *maximal formula* $\neg\bullet$ in the derivation:[59]

$$\dfrac{\dfrac{\begin{matrix}[\bullet]\\ \vdots \\ \bot\end{matrix}}{\neg\bullet}\,\neg\mathrm{I}}{\bullet}\,\bullet\mathrm{I}$$

Of course, since Inversion Principle is at least a necessary requirement for harmony, lacking it $\bullet$-rules cannot be harmonious.

In the previous section, we have seen that to solve Gabbay's puzzle regarding the reduction of *tonk*-formulae we have to consider bilateral Operational Rules as if they were complex rules introducing or eliminating their connectives in the scope of $+$ and $-$, and to extend the notion of *maximal formulae* accordingly. We have also seen that according to this interpretation the meaning of the logical constants depends on those of $+$ and $-$, and that this dependence of meaning requires both a weakening of separability and considering Coordination Principles as meaning-determining rules for $+$ and $-$.

Given this reinterpretation of what goes on in bilateralism, Gabbay's objection against Read's $\bullet$-rules can be retorted against his own $\bullet$-rules. Indeed, following his argument, $\bullet\mathrm{I}^+$ and $\bullet\mathrm{E}^-$ count also as E-rules for $-$, $\bullet\mathrm{I}^-$ and $\bullet\mathrm{E}^+$ also as E-rules for $+$, $\bullet\mathrm{E}^+$ also as an I-rule for $-$, and $\bullet\mathrm{E}^-$ also as an I-rule for $+$. This entails that Gabbay's rules are not really harmonious if considered together with standard Coordination Principles. Indeed, for example, the derivation

$$\dfrac{\dfrac{+\bullet}{-A}\,\bullet\mathrm{E}^+ \qquad +A}{\bot}\,-\mathrm{E}$$

cannot be reduced, as $\bullet\mathrm{E}^+$ intended *qua* I-rule for $-$ would on the contrary require.

[58] [13], p. S113.

[59] Peter Milne claimed that $\bullet$-rules do not suit Inversion Principle already in his [23], but gives only an indirect argument for such a conclusion.

This line of reasoning makes the unacceptability of $\bullet$ rely on the availability of the standard Coordination Principles. Indeed, their disharmony holds only in relation to this or other sets of Coordination Principles, and not in itself. Far from being a problem, it seems to me that, since Coordination Principles are needed to obtain $\perp$ from the rules for $\bullet$, it is plausible that they should be excluded by our *criterion* only when these Principles are present. Moreover, this feature could work as a solution to the problems that we saw at the end of subsection 3.1 regarding the rules of Incoherence and Francez's proposal of a Horizontal Balance between rules for assertion and rules for rejection. Our main reservations against Francez's *criterion* and the exclusion of the rules of Incoherence were that:

1. the rules of Incoherence lead to triviality only if endorsed together with some sets of Coordination Principles;

2. Horizontal Balance cannot be applied to Coordination Principles;

3. when applied to Operational Rules, it does not take into consideration the Coordination Principles, which are nonetheless needed to obtain triviality of the system.

Let us now check how our proposal could deal with them, and whether the involvement of Coordination Principles in our solution could be of some use.

About the first point, such as bilateral rules for $\bullet$ are evaluated in combination with Coordination Principles, the same can be done for the rules of Incoherence. Also in this case, an example of irreducible maximality can be displayed. Indeed, in the derivation

$$\dfrac{\dfrac{+A}{-A}\,\text{Incoherence} \qquad +A}{\perp}\,\text{Non-Contradiction}$$

Incoherence works as an I-rule for $-$ and Non-Contradiction as an E-rule for the same term, and so $-A$ is a *maximal formula* for which no reduction is available. As a consequence, Incoherence is excluded by our *criterion*, but not for itself; only on the background assumption of the standard Coordination Principles.[60] Since Incoherence is a Coordination Principle and it is not excluded by *ad hoc* decisions but because of a general *criterion*, this observation answers both point 1 and point 2. As for the third point, it clearly does not hold for our *criterion*, since the evaluation of the Operational Rules for $\bullet$ leads to a rejection explicitly because of the Coordination Principles, so our proposal seems to solve all the open issues seen in the previous section.

Nevertheless, someone could object that we are posing a too strong restriction on the Operational Rules by asking that they cohere also with Coordination Principles when considered as I or E-rules for $+$ and $-$. Indeed, if $\neg\mathrm{I}^+$ is also an I-rule for $+$ and $\supset\mathrm{E}^-$ is also an E-rule for $+$, they form together the following maximality, which is irreducible:

$$\dfrac{\dfrac{-A}{+(\neg A)}{+\mathrm{I}} \qquad -B}{-(\neg A \supset B)}{+\mathrm{E}}$$

So, we are in danger of proving that Rumfitt's system is not in harmony and throwing away the baby with the dirty water.

Nonetheless, we should not be too hasty in abandoning our alleged solution. Indeed, it should be taken into account that Gabbay's rules for $\bullet$ extend the logical consequences regarding only $+$ and $-$, since they make provable $+A \dashv\vdash -A$, while Rumfitt's rules for negation and implication do not. Of course they extend the provable results regarding $\neg$, $\supset$, $+$ and $-$, but not the consequences regarding only these last two terms. The rules for negation and those for implication constitute, in more formal terms, a conservative extension of the system composed of only the Coordination Principles. As a consequence, there is ground to claim that while Gabbay's rules for $\bullet$ work also as I and E-rules for $+$ and $-$, this is not true for the rules of negation and implication. The intuitive principle that we are

[60] An anonymous referee asks whether there are some *formal* reasons to drop Incoherence instead of Non-Contradiction. It seems to me that the only reasons for this choice regard the intended meaning of $+$ and $-$, and that there are no formal reasons to dismiss Incoherence in a context in which the other Coordination Principles are absent. The situation here resembles the one evaluated by Dummett about harmonious rules that are nonetheless unacceptable, if proposed to capture the meaning of counterfactual conditional. In other words, harmony can be enough for the formal acceptability of a rule, but not for its adequacy with respect to actual usage. See [6], p. 206. [33] as well proposes harmony as just a precondition for validity.

applying here is the same that we implicitly relied on when we, *contra* Steinberger, rejected to Milne's I-rules for classical negation and conditional the *status* of I-rules for disjunction.[61] In conclusion, even though some technicalities are needed, our *criterion* can exclude bilateral $\bullet$, bilateral *tonk* and the rule of Incoherence, without at the same time excluding the well-behaving part of Rumfitt's system.[62]

4. Comparison with other proposals

We have already seen that our proposal of treating bilateral rules as complex has some advantages over Francez's proposal of a principle of Horizontal Balance. Indeed, using this approach we have provided both a solution to Gabbay's puzzle of the 'reduction' for *tonk*-rules, and a *criterion* for Coordination Principles. Nonetheless, it must be admitted that Francez's criterion is much more elegant than mine, since the adoption of complex rules entails a great complication both in constructing the proofs inside the system, and in proving metatheorems about the system itself.

There is nonetheless another, more recent proposal that we have considered only in passing, and that we should compare with our analysis. In a recent paper, Kürbis has proposed a normalization procedure for a variation of Rumfitt's system.[63] What is peculiar about his work is that he has taken into consideration Coordination Principles as well, asking for the reduction of *maximal formulae* obtained: with only Operational Rules, with only Coordination Principles, and with both Operational Rules and Coordination Principles.

[61] See section 2.2. The same principle is applied also in my work about complex rules in unilateral systems; see [2], p. 1043.

[62] A referee wonders whether $\neg\mathrm{I}^+$ could be considered an I-rule for $\neg$ and $+$ together, instead of an I-rule for both of them taken separately. I thank them for this suggestion. I share their feeling about this interpretation of $\neg\mathrm{I}^+$. Anyway, there are some difficulties, and this reinterpretation cannot be seen as a general solution to the apparent maximality between Operation Rules and Coordination Principles. Indeed, in order to keep our rejection of Gabbay's $\bullet$, we need to interpret $\bullet\mathrm{E}^+$ as an I-rule for $-$. Moreover, even rejecting to $\neg\mathrm{I}^+$ its status of I-rule for $+$, on the basis that it introduces positive formulae only if their most external connective is $\neg$, it is hard to do the same for $\supset\mathrm{E}^-$. Indeed, just like for $\bullet\mathrm{E}^+$, the only logical term occurring in the conclusion of $\supset\mathrm{E}^-$ is $+$. So, in order to reject Gabbay's $\bullet$ but keep the ordinary rules for the logical connectives, a *criterion* relying on *conservative extension* seems needed anyway.

[63] [20]; a strengthening of Kürbis' result has been proposed by Pedro del Valle-Inclan in his contribution to this conference.

While there are some similarities between Kürbis' approach and mine, they should not be confused with each other. Confronting the requirements of these two *criteria* in general would take too long, so we will instead analyze how they behave in some relevant situations, focusing on a variation of Gabbay's rules for $\bullet$ presented by Kürbis himself: two binary connectives that he calls *conk* and *honk* and that, like $\bullet$-rules, are harmonious but leads to triviality.[64] Of course, Kürbis' revision of *harmony* is expressly designed to exclude these connectives, so the issue is whether our *criterion* can do the same.

Let us start by considering the rules for *honk*:

$$honk\mathrm{I}^{+}\ \frac{-A \qquad +B}{+A\ honk\ B} \qquad honk\mathrm{E}_1^{+}\ \frac{+A\ honk\ B}{-A} \qquad honk\mathrm{E}_2^{+}\ \frac{+A\ honk\ B}{+B}$$

$$honk\mathrm{I}^{-}\ \frac{+A \qquad -B}{-A\ honk\ B} \qquad honk\mathrm{E}_1^{-}\ \frac{-A\ honk\ B}{+A} \qquad honk\mathrm{E}_2^{-}\ \frac{-A\ honk\ B}{-B}$$

Our *criterion* excludes *honk* in the same way in which it excludes $\bullet$, that is $honk\mathrm{E}_1^{+}$ can be read also as an I-rule for $-$ and $honk\mathrm{E}_1^{-}$ also as an I-rule for $+$. Interpreted in this way, they do not suit harmony with respect to the other Coordination Principles. The other *honk*-rules are harmonious and so we could propose an amended version of *honk* composed of solely *honk*I, $honk\mathrm{E}_2^{+}$ and $honk\mathrm{E}_2^{-}$, which indeed does not lead to triviality. So, at least for this first connective, our *criterion* can be used in place of Kürbis' one.

The rules for *conk* are similar to those for *honk*, but with a relevant difference: all rules have the same modality both in the premise and in the conclusion.

$$conk\mathrm{I}^{+}\ \frac{+A \qquad +B}{+A\ conk\ B} \qquad conk\mathrm{E}^{+}\ \frac{+A\ conk\ B}{+A} \qquad conk\mathrm{E}^{+}\ \frac{+A\ conk\ B}{+B}$$

$$conk\mathrm{I}^{-}\ \frac{-A \qquad -B}{-A\ conk\ B} \qquad conk\mathrm{E}^{-}\ \frac{-A\ conk\ B}{-A} \qquad conk\mathrm{E}^{-}\ \frac{-A\ conk\ B}{-B}$$

This being the case, it is plain that we cannot exclude *conk* by applying the same strategy seen for $\bullet$. Indeed, none of *conk*-rules can be interpreted as introducing or eliminating $+$ or $-$. On the contrary, Kürbis' *criterion* excludes this connective as well, and so we have to admit the incompleteness of our *criterion*.

[64] [20], pp. 537–538.

Moreover, even though Kürbis never explicitly states this, his *criterion* excludes also Gabbay's alleged reduction for *tonk*, which we have seen in section 2. Indeed, in Gabbay's reduction of a *tonk*-maximality

$$
\text{Reductio}_2\,\dfrac{\text{Non-Contradiction}\,\dfrac{\textit{tonk}\text{I}\,\dfrac{+A}{+(A\,\textit{tonk}\,B)} \qquad \text{Reductio}_1\,\dfrac{\text{Non-Contradiction}\,\dfrac{\textit{tonk}\text{E}\,\dfrac{[+(A\,\textit{tonk}\,B)]^1}{+B} \qquad [-B]^2}{\bot}}{-(A\,\textit{tonk}\,B)}}{\bot}}{+B}
$$

the occurrence of $-AtonkB$ immediately after $\bot$ is a *maximal formula* according to the definition given by Kürbis, since it is the conclusion of an application of Reductio and a premise of an application of Non-Contradiction.[65] Hence, Kürbis' extension of *maximality* can be used in place of our proposal in order to solve this puzzle.

We can nonetheless strike a blow for our *criterion*. Indeed, even though our proposal has problems to exclude *conk*, it seems an open issue whether Kürbis' *criterion* can exclude Gabbay's rules for $\bullet$, which are on the contrary excluded by our *criterion*.[66] Surely, it is possible to proof both $+\bullet$ and $-\bullet$ using derivations that are normal according to Kürbis definition, as displayed in:

$$
\text{Red}_1\,\dfrac{\text{Non-Contr}\,\dfrac{\bullet\text{E}^+\,\dfrac{[+\bullet]^1}{+B} \qquad \bullet\text{E}^+\,\dfrac{[+\bullet]^1}{-B}}{\bot}}{-\bullet}
\qquad
\text{Red}_1\,\dfrac{\text{Non-Contr}\,\dfrac{\bullet\text{E}^+\,\dfrac{[-\bullet]^1}{+B} \qquad \bullet\text{E}^+\,\dfrac{[-\bullet]^1}{-B}}{\bot}}{+\bullet}
$$

Admittedly, in order to go a step further and obtain triviality, we need an application of an E-rule for $\bullet$, which would lead to maximality according to Kürbis' definition. Nonetheless, the availability of normal closed proofs for both $+\bullet$ and $-\bullet$ suggests at least that some more careful reflection is needed regarding these requirements.

[65] The occurrence of $+(AtonkB)$ on the left branch should not be *maximal*, since the other premise of Non-Contradiction is not a conclusion of an I-rule. Moreover, notice that according to our Definition 2.4, $-(A\,tonk\,B)$ is a simple *maximal formula* like for Kürbis, and $+(AtonkB)$ is a complex *maximal formula*.

[66] Kürbis seems to be aware of this lack, see [20], p. 539 note 5.

In conclusion, both *criteria* seem unable to exclude at least one clearly pathological set of rules, with no clear solution on the horizon: my proposal being unable to exclude *conk* and Kürbis' proposal having at least some trouble to exclude bilateral $\bullet$. Even though the objections that could be raised against my *criterion* are more serious than the ones raised against Kürbis', there seems to be enough ground to argue that neither of them is the full story. Maybe a more in-depth comparison between the proposals could lead to a deeper understanding of what is still lacking in the picture, but such a comparison would require a further paper of its own.

5. Conclusion

We have opened this article with a brief introduction about the development of bilateral systems in proof-theoretic semantics, focusing on the difficulty of individuating a clear *criterion* of acceptability for Coordination Principles. Then we have moved to two major objections raised by Gabbay against bilateral systems: an alleged reduction procedure for *tonk* and the availability of paradoxical but harmonious rules for a bilateral reformulation of Read's $\bullet$.

First of all, in Section 2 we have focused on *tonk*, arguing that an extension in the definition of *maximal formulae* is needed in order to reject Gabbay's reduction. We have found such an extension by applying to bilateralism some ideas taken from Milne's work on complex rules. Nonetheless, this solution has forced us to consider $+$ and $-$ as well for separability and so to skip to a weakened version of this notion, a change that is in line with Milne's work. In passing, we have observed that the main problems regarding bilateralism seem to remain: the circular interdependence of meaning of $+$ and $-$ and the violation of the *complexity condition* by Coordination Principles.

Then, in Section 3 we have moved to $\bullet$, analyzing the solution proposed by Francez and finding it conceptually unsatisfactory (even though formally unquestionable), the main problem being the lack of a *criterion* for Coordination Principles. Hence, we have claimed that some Operational Rules should be considered as I and E-rules for $+$ and $-$, together with Coordination Principles. This gives ground for a common *criterion* for both Coordination Principles and Operational Rules, which suffices to exclude $\bullet$ and accept the standard Coordination Principles. The worry that this *criterion* excludes well-behaving Operational Rules as well is dealt with through a solution that is still in line with Milne's work.

In the end, in Section 4 we displayed a comparison with other solutions to the problems of bilateralism present in the literature. In particular, the comparison with Kürbis' *criterion* seems to show that both proposals are to some extent incomplete, even though Kürbis' one in a less serious way than mine, and so that further investigations seem desirable.

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Leonardo Ceragioli
University of Pisa
Department of Civilizations and Forms of Knowledge
Via Pasquale Paoli, #15
56126 Pisa, Italy
e-mail: leo.crg.uni@gmail.com

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Pedro del Valle-Inclan

HARMONY AND NORMALISATION IN BILATERAL LOGIC[1]

Abstract

In a recent paper del Valle-Inclan and Schlöder argue that bilateral calculi call for their own notion of proof-theoretic harmony, distinct from the usual (or 'unilateral') ones. They then put forward a specifically bilateral criterion of harmony, and present a harmonious bilateral calculus for classical logic.

In this paper, I show how del Valle-Inclan and Schlöder's criterion of harmony suggests a notion of normal form for bilateral systems, and prove normalisation for two (harmonious) bilateral calculi for classical logic, $\mathbf{HB}_1$ and $\mathbf{HB}_2$. The resulting normal derivations have the usual desirable features, like the separation and subformula properties. $\mathbf{HB}_1$-normal form turns out to be strictly stronger that the notion of normal form proposed by Nils Kürbis, and $\mathbf{HB}_2$-normal form is neither stronger nor weaker than a similar proposal by Marcello D'Agostino, Dov Gabbay, and Sanjay Modgyl.

Keywords: bilateralism, normalisation, harmony.

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1. Introduction

According to inferentialists the meaning of logical vocabulary is given by the rules governing its use in inferences. There is nothing more to the meaning of, for example, disjunction, than the rules governing when to infer, and what to infer from, certain sentences containing 'or'. It follows that one can define a connective by laying down rules that govern it, like the introduction and elimination rules of natural deduction systems.

Inferentialism faces an objection first posed by Arthur Prior [8]. Consider the binary operator 'tonk' defined by the following rules:

$$\frac{A}{A \text{ tonk } B}\,(\text{tonk I}) \qquad\qquad \frac{A \text{ tonk } B}{B}\,(\text{tonk E})$$

By chaining an application of (tonk I) with an application of (tonk E) one can deduce two arbitrary sentences from each other. According to Prior, inferentialists have to conclude that any sentence follows from any other. Inferentialists, on their part, typically reject the assumption that *any* set of rules adequately defines a connective. They hold that there is something wrong with the rules for 'tonk', something that makes it an illegitimate piece of vocabulary. This has given rise to the search for a criterion to determine which rules are acceptable definitions, a project which has come to be known, following Dummett [3], as the search for a criterion of *proof-theoretic harmony*.

The most common approach to harmony appeals to an intuitive notion of 'balance'. A set of introduction and elimination rules is balanced if the elimination rules are neither too strong nor too weak with respect to the introductions (and vice versa). The elimination rule for tonk, for instance, is held to be too strong with respect to its introduction rule. The idea is that (tonk E) allows one to derive 'too much' from 'A tonk B', given what (tonk I) requires in order to derive such a sentence. Over the years a host of non-equivalent explications of this intuitive notion of balance have been put forward (see Steinberger [10] for a brief overview). By and large they all have something in common: the usual formalisations of classical logic come out disharmonious. Thus, or so it is argued, inferentialism is incompatible with classical logic.

Ian Rumfitt [9] has argued that bilateralism can solve this incompatibility. According to Rumfitt's bilateralism the speech acts of assertion and rejection should be taken as primitive, rather than analysed in terms of each other. Furthermore, he argues, the meaning of classical connectives must be given bilaterally, by means of rules governing the assertion *and* the rejection of sentences containing them. By stipulating assertive and rejective rules for each connective, he is able to provide a calculus for classical logic that satisfies the usual requirements of harmony.

Rumfitt's position has been recently challenged by del Valle-Inclan and Schlöder [2]. They argue that bilateral calculi require their own notion of harmony, distinct from the standard (or 'unilateral') ones. Thus, although Rumfitt's system is harmonious *according to criteria fit for unilateral systems*, this is not enough to vindicate classical logic from an inferentialist point of view. They propose a bilateral criterion of harmony and show, using a result by Fernando Ferreira [4], that Rumfitt's system is not harmonious according to it. To solve the problem, they put forward a new Rumfitt-style formalisation of classical logic.

The aim of this paper is to explore the relation between del Valle-Inclan and Schlöder's criterion of harmony, on the one hand, and normalisation on the other. I will first show how their harmony criterion suggests a natural notion of normal form for bilateral calculi. Then, I will show that their calculus, as well as a closely related one, normalise. Derivations in normal form have the usual desirable features; the subformula and separation properties, in particular, can be obtained as corollaries of normalisation. Finally, I will briefly compare the present notion of normal form with proposals by Nils Kürbis [6] and Marcello D'Agostino, Dov Gabbay, and Sanjay Modgyl [1].

The paper is structured as follows: Section 2 recaps Rumfitt's position and del Valle-Inclan and Schlöder's criticism. Sections 3 and 4 prove normalisation and corollaries for two (harmonious) bilateral calculi for classical logic. Section 5 compares the present normalisation results with previous ones, and Section 6 concludes.

2. Rumfitt, bilateralism and harmony

There are two core tenets to Rumfitt's bilateralism. The first is that assertion and rejection are distinct, primitive speech acts that serve to express different attitudes towards propositional content (assent and dissent, respectively). The second is that both assertion and rejection play a role in our inferential practice. From an inferentialist point of view, it follows that to specify the meaning of a connective one must give rules that govern both the assertion and rejection of sentences containing it. Rumfitt does this by means of a natural deduction calculus for signed formulae, that is, standard formulae preceded by force indicators '+' and '−'. If A is a propositional formula, $+A$ is to be interpreted as the assertion of A, and $-A$ as its rejection; force indicators cannot be iterated or embedded. Rumfitt proposes the following operational rules for the classical connectives:[2]

Conjunction:

$$\frac{+A_1 \qquad +A_2}{+A_1 \wedge A_2}\ (+\wedge\mathrm{I}) \qquad\qquad \frac{-A_i}{-A_1 \wedge A_2}\ (-\wedge\mathrm{I})$$

$$\frac{+A_1 \wedge A_2}{+A_i}\ (+\wedge\mathrm{E}) \qquad\qquad \frac{-A_1 \wedge A_2 \qquad \begin{matrix}[-A_1]^1 \\ \mathcal{D}_1 \\ \varphi\end{matrix} \qquad \begin{matrix}[-A_2]^1 \\ \mathcal{D}_2 \\ \varphi\end{matrix}}{\varphi}\ (-\wedge\mathrm{E})^1$$

Disjunction:

$$\frac{+A_i}{+A_1 \vee A_2}\ (+\vee\mathrm{I}) \qquad\qquad \frac{-A_1 \qquad -A_2}{-A_1 \vee A_2}\ (-\vee\mathrm{I})$$

$$\frac{+A_1 \vee A_2 \qquad \begin{matrix}[+A_1]^1 \\ \mathcal{D}_1 \\ \varphi\end{matrix} \qquad \begin{matrix}[+A_2]^1 \\ \mathcal{D}_2 \\ \varphi\end{matrix}}{\varphi}\ (+\vee\mathrm{E})^1 \qquad\qquad \frac{-A_1 \vee A_2}{-A_i}\ (-\vee\mathrm{E})$$

Implication:

$$\frac{\begin{matrix}[+A_1]^1 \\ \mathcal{D} \\ +A_2\end{matrix}}{+A_1 \rightarrow A_2}\ (+\rightarrow\mathrm{I})^1 \qquad\qquad \frac{+A_1 \rightarrow A_2 \qquad +A_1}{A_2}\ (+\rightarrow\mathrm{E})$$

[2] A note about notation: roman letters range over unsigned formulae, greek letters over signed formulae, brackets indicate discharged assumptions, and both vacuous and multiple discharges are allowed. When there are two formulae separated by '/' below the horizontal line, as in rule $(-\rightarrow\mathrm{E})$, an application of the rule in question can conclude *either* formulae, not both simultaneously (all rules are single-conclusion).

$$\frac{+A_1 \qquad -A_2}{-A_1 \to A_2}\,(-\to\mathrm{I}) \qquad\qquad \frac{-A_1 \to A_2}{+A_1 / -A_2}\,(-\to\mathrm{E})$$

Negation:

$$\frac{-A}{+\neg A}\,(+\neg\mathrm{I}) \qquad\qquad \frac{+A}{-\neg A}\,(-\neg\mathrm{I})$$

$$\frac{+\neg A}{-A}\,(+\neg\mathrm{E}) \qquad\qquad \frac{-\neg A}{+A}\,(-\neg\mathrm{E})$$

In addition to operational rules Rumfitt's calculus contains *coordination principles*. These are rules that govern the interaction between '+' and '−', rather than specific connectives. They are meant to capture our conventions regarding the assertion and rejection of the same content. Rumfitt's coordination principles are (Rejection) and *Smilean reductio*:

$$\frac{+A \qquad -A}{\bot}\,(\text{Rejection}) \qquad \frac{\begin{matrix}[+A]^1\\ \mathcal{D}\\ \bot\end{matrix}}{-A}\,(\mathrm{SR}_1)^1 \qquad \frac{\begin{matrix}[-A]^1\\ \mathcal{D}\\ \bot\end{matrix}}{+A}\,(\mathrm{SR}_2)^1$$

The principle of (Rejection) states that the assertion and rejection of the same content are incompatible.[3] The two halves of *Smilean reductio* state (respectively) that if the assertion of a formula leads to absurdity one may reject it, and if the rejection of a formula leads to absurdity, then one may assert it. On top of this, *Smilean reductio* also encodes a form of explosion, through the vacuous discharge of assumptions. A more perspicuous representation of Rumfitt's commitments about the interplay between assertion and rejection can be given by the following coordination principles of *Explosion* and *Bilateral Excluded Middle*:

$$\frac{+A \qquad -A}{\varphi}\,(\mathrm{ex}) \qquad \frac{\begin{matrix}[+A]^1\\ \mathcal{D}_1\\ \varphi\end{matrix} \qquad \begin{matrix}[-A]^1\\ \mathcal{D}_2\\ \varphi\end{matrix}}{\varphi}\,(\mathrm{bem})^1$$

[3] Note that Rumfitt, following Tennant [11], takes '⊥' as a *punctuation sign* indicating a logical dead end. It is not a sentence, and therefore cannot be signed, embedded in formulae, or appear as a topmost node in derivations.

It is routine to check that:

Remark 2.1. (Rejection), (SR1) and (SR_2) are derivable from (ex) and (bem) and vice versa.

Rumfitt's operational rules satisfy all the standard criteria of harmony, and it is intuitively clear why this should be so. Take the rules for negation: in order to apply ($+\neg$ I) one needs to derive a sentence of the form $-A$, which is exactly what one gets from an application of ($+\neg$ E). Similarly, in order to apply ($-\neg$ I) one needs to derive a sentence of the form $+A$, which is precisely what an application of ($+\neg$ E) yields. In other words, its operational rules of the same sign are inverses of each other. Something similar, of course, applies to the other connectives.

Del Valle-Inclan and Schlöder [2] argue that Rumfitt-style bilateral calculi call for a more demanding notion of harmony. In unilateral natural deduction what one can do with a connective is determined by operational rules alone; the relation between operational rules, then, is all that unilateral harmony needs to take into account. In bilateral calculi, however, coordination principles permit further inferential steps. And crucially, this means that connectives whose operational rules are balanced according to all the usual standards can become tonk-like when they interact with coordination principles. They give the following connective 'bink' as an example:

$$\frac{+A \qquad -A}{+\,\text{bink}\,A}\,(+\,\text{bink I}) \qquad \frac{+\,\text{bink}\,A}{+A}\,(+\,\text{bink E}_1) \qquad \frac{+\,\text{bink}\,A}{-A}\,(+\,\text{bink E}_2)$$

$$\frac{-A}{-\,\text{bink}\,A}\,(-\,\text{bink I}) \qquad \frac{-\,\text{bink}\,A}{-A}\,(-\,\text{bink E})$$

The assertive introduction and elimination rules of bink are inverses of each other, and so are its rejective rules. Indeed, bink is harmonious according to all the usual (unilateral) standards of harmony. If bink interacts with *Smilean reductio*, however, it trivialises the calculus it is part of. The following derivation, for instance, shows that there is a proof of $-A$ for any A:

$$\dfrac{\dfrac{\dfrac{\dfrac{[+\,\text{bink}\,A]^1}{+A}\,(+\,\text{bink}\,\mathrm{E}_1) \qquad \dfrac{[+\,\text{bink}\,A]^1}{-A}\,(+\,\text{bink}\,\mathrm{E}_2)}{\bot}}{-\,\text{bink}\,A}\,(\mathrm{SR}_1)^1}{-A}\,(-\,\text{bink}\,\mathrm{E})$$

Examples like this show that a bilateral criterion of harmony must take into consideration *both* the relation between introduction and elimination rules, on the one hand, and the relation between operational rules and coordination principles on the other. Del Valle-Inclan and Schlöder propose the following criterion of bilateral harmony:

> **Bilateral harmony:** A connective **c** is bilaterally harmonious iff (i) (+**c**I) and (+**c**E) are unilaterally harmonious; (ii) (−**c**I) and (−**c**E) are unilaterally harmonious; (iii) all coordination principles are preserved by the rules for **c** (i.e. when all coordination principles are restricted to atomic sentences, all their instances for sentences containing **c** as their main operator are derivable).

To put it simply: whatever unilateral harmony may be, bilateral harmony is *that* plus preservation of all coordination principles. For further examples, and a more thorough defence of the criterion, the reader is referred to [2].

Fernando Ferreira [4] has shown that Rumfitt's operational rules do not preserve *Smilean reductio*. Rumfitt's system, therefore, is not harmonious. To solve the problem del Valle Inclan and Schlöder propose slight modifications to the rejective rules for conjunction and the assertive rules for disjunction. Their rules for rejected conjunctions are:[4]

$$\dfrac{\begin{array}{c}[+A]^1\\ \mathcal{D}\\ -B\end{array}}{-A\wedge B}\,(-\wedge\mathrm{I})^1 \qquad \dfrac{-A\wedge B \qquad +A}{-B}\,(-\wedge\mathrm{E})$$

[4]These rules for conjunction are also independently discussed in Nils Kürbis' [5]

Their rules for asserted disjunctions are:

$$\dfrac{\begin{array}{c}[-A]^1\\ \mathcal{D}\\ +B\end{array}}{+A \vee B}\,(+\vee\mathrm{I})^1 \qquad \dfrac{+A \vee B \qquad -A}{+B}\,(+\vee\mathrm{E})$$

These rules are analogous to the usual rules for the material conditional, and it is easy to check that they are harmonious according the usual unilateral criteria. In addition, they preserve all the coordination principles we have considered. More generally, call the set of Rumfitt's operational rules with the present modifications **B**. Then:

Remark 2.2. All the rules in **B** preserve *Smilean reductio*, (Rejection), (ex) and (bem).

Remark 2.2. follows trivially from the normalisation results to be proved below.

In what follows I will refer to the calculus consisting of **B**, (SR_1), (SR_2) and (Rejection) as $\mathbf{HB}_1$, and the calculus consisting of **B**, (ex) and (bem) as $\mathbf{HB}_2$. It follows from the observation that the modified operational rules are unilaterally harmonious, combined with Remark 2.2, that $\mathbf{HB}_1$ and $\mathbf{HB}_2$ are harmonious in del Valle-Inclan and Schlöder's sense. It is also routine to check that:

Remark 2.3. $\mathbf{HB}_1$ and $\mathbf{HB}_2$ are equivalent to Rumfitt's calculus (i.e. φ is derivable from Γ in Rumfitt's calculus iff it is derivable from Γ in $\mathbf{HB}_1$ and $\mathbf{HB}_2$).

$\mathbf{HB}_1$ and $\mathbf{HB}_2$, then, are bilaterally harmonious formalisations of classical logic. We can finally examine the relation between harmony and normalisation. I will discuss $\mathbf{HB}_1$ first, and deal with $\mathbf{HB}_2$ in Section 4.

3. Harmony and normalisation: $\mathbf{HB}_1$

All derivations in normal form, according to Dag Prawitz's original result [7], share a central feature: no formula occurrence in them is simultaneously the consequence of an introduction rule and the major premise of an elimination. This is usually thought to be related to harmony. The idea is that if the operational rules of a connective are 'balanced', one should gain nothing by first introducing and then immediately eliminating a connective. Therefore, it should be possible to eliminate all such steps within a derivation.[5] This is the core principle behind normalisation, and so for normal derivations in $\mathbf{HB}_1$ we also require that:

> **(i)** No conclusion of an I-rule is a major premise of an E-rule.

The notion of bilateral harmony proposed by del Valle-Inclan and Schlöder suggests a similar principle, this time regarding the interaction between operational rules and coordination principles. Their idea is that a connective is 'balanced' with respect to a coordination principle if one can lay down the coordination principle for atoms and prove it for complex sentences. To reflect this at the level of derivations we should, as before, require that applications of coordination principles to complex formulae should be eliminable. In other words, that for normal derivations in $\mathbf{HB}_1$:

> **(ii)** Coordination principles are applied only to atoms.

Clauses **(i)** and **(ii)** are enough to ensure the separation property for normal derivations. They are not, however, enough to secure the stronger subformula property, as the following derivation shows:

$$\dfrac{\dfrac{\dfrac{\dfrac{+p \qquad -p}{\bot}}{+q}\,(\mathrm{SR}_2)^0 \qquad \dfrac{\dfrac{+p \qquad -p}{\bot}}{-q}\,(\mathrm{SR}_1)^0}{\bot}}{+r}\,(\mathrm{SR}_2)^0$$

The derivation satisfies **(i)** and **(ii)**, but contains signed formula occurrences $+q$ and $-q$ which are subformulae of neither the assumptions nor the conclusion. This is due to the fact that the form of explosion encoded by *Smilean reductio* is used twice, consecutively. To avoid this kind of configuration in normal derivations, and thus ensure the subformula property, we need only require that:

> **(iii)** No conclusion of *Smilean reductio* is a premise of (Rejection).

[5]This is not the whole story. In the presence of, for example, the usual rules for disjunction, one can introduce a connective and eliminate it a few steps below in the derivation, rather than immediately after the introduction. Because of the modified operational rules of $\mathbf{HB}_1$, however, this cannot happen in normal derivations, so we need not worry about it.

Putting everything together, we have the following definition.

DEFINITION 3.1. (Normal form)
A derivation in $\mathbf{HB}_1$ is in normal form if in it: **(i)** No conclusion of an I-rule is a major premise of an E-rule. **(ii)** Coordination principles are applied only to atoms. **(iii)** No conclusion of *Smilean reductio* is a premise of (Rejection).

Formula occurrences that infringe clauses **(i)**, **(ii)** and **(iii)** are called **maximal operational formulae**, **maximal coordination formulae** and **ancillary maximal formulae**, respectively.

The rest of this section proves normalisation and corollaries for $\mathbf{HB}_1$. The first step is providing appropriate reduction procedures. Since the rules for disjunction and conjunction are analogous, I will not explicitly provide reduction steps involving the latter. It may be useful to keep the overall normalisation strategy in mind when examining the reduction steps. Reductions for maximal coordination formulae may create maximal operational formulae of the same complexity. Reduction steps for maximal operational formulae, on the other hand, may create new ancillary maximal formulae only. Finally, reduction steps for *atomic* ancillary maximal formulae create no new maximal formulae of any kind. Therefore, the normalisation process reduces maximal coordination formulae first, followed by maximal operational formulae, and then ancillary maximal formulae.

3.1. Operational reductions

Negation:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ -A\end{matrix}}{+\neg A}\,(+\neg\ \mathrm{I})}{-A}\,(+\neg\ \mathrm{E}) \qquad \rightsquigarrow \qquad \begin{matrix}\mathcal{D}_1\\ -A\end{matrix}$$

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ +A\end{matrix}}{-\neg A}\,(-\neg\ \mathrm{I})}{+A}\,(-\neg\ \mathrm{E}) \qquad \rightsquigarrow \qquad \begin{matrix}\mathcal{D}_1\\ +A\end{matrix}$$

Implication:

$$\dfrac{\dfrac{\begin{array}{c}[+A]^1\\ \mathcal{D}_1\\ +B\end{array}}{+A \to B}\,(+\to \mathrm{I})^1 \qquad \begin{array}{c}\mathcal{D}_2\\ +A\end{array}}{+B}\,(+\to \mathrm{E}) \quad \rightsquigarrow \quad \begin{array}{c}\mathcal{D}_2\\ +A\\ \mathcal{D}_1\\ +B\end{array}$$

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ +A\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ -B\end{array}}{-A \to B}\,(-\to \mathrm{I})}{+A/-B}\,(-\to \mathrm{E}) \quad \rightsquigarrow \quad \begin{array}{c}\mathcal{D}_{1/2}\\ +A/-B\end{array}$$

Disjunction:

$$\dfrac{\dfrac{\begin{array}{c}[-A]^1\\ \mathcal{D}_1\\ +B\end{array}}{+A \vee B}\,(+\vee\ \mathrm{I})^1 \qquad \begin{array}{c}\mathcal{D}_2\\ -A\end{array}}{+B}\,(+\vee\ \mathrm{E}) \quad \rightsquigarrow \quad \begin{array}{c}\mathcal{D}_2\\ -A\\ \mathcal{D}_1\\ +B\end{array}$$

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ -A\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ -B\end{array}}{-A \vee B}\,(-\vee\ \mathrm{I})}{-A/-B}\,(-\vee\ \mathrm{E}) \quad \rightsquigarrow \quad \begin{array}{c}\mathcal{D}_{1/2}\\ -A/-B\end{array}$$

3.2. Reducing (rejection) to atomic applications

Negation:

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ +\neg A\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ -\neg A\end{array}}{\bot}\,(\mathrm{Rej}) \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ +\neg A\end{array}}{-A} \qquad \dfrac{\begin{array}{c}\mathcal{D}_2\\ -\neg A\end{array}}{+A}}{\bot}\,(\mathrm{Rej})$$

Implication:

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ +A\to B\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ -A\to B\end{array}}{\bot}\,(\mathrm{Rej}) \quad\rightsquigarrow\quad \dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ +A\to B\end{array} \qquad \dfrac{\begin{array}{c}\mathcal{D}_2\\ -A\to B\end{array}}{+A}}{+B} \qquad \dfrac{\begin{array}{c}\mathcal{D}_2\\ -A\to B\end{array}}{-B}}{\bot}\,(\mathrm{Rej})$$

Disjunction:

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ +A\vee B\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ -A\vee B\end{array}}{\bot}\,(\mathrm{Rej}) \quad\rightsquigarrow\quad \dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ +A\vee B\end{array} \qquad \dfrac{\begin{array}{c}\mathcal{D}_2\\ -A\vee B\end{array}}{-A}}{+B} \qquad \dfrac{\begin{array}{c}\mathcal{D}_2\\ -A\vee B\end{array}}{-B}}{\bot}\,(\mathrm{Rej})$$

3.3. Reducing *Smilean reductio* to atomic applications

Negation: (the other case is analogous)

$$\begin{array}{c}[+\neg A]^1\\ \mathcal{D}\\ \dfrac{\bot}{-\neg A}\,(\mathrm{SR}_1)^1\end{array} \qquad\rightsquigarrow\qquad \begin{array}{c}\dfrac{[-A]^1}{+\neg A}\\ \mathcal{D}\\ \dfrac{\dfrac{\bot}{+A}\,(\mathrm{SR}_2)^1}{-\neg A}\end{array}$$

Implication:

$$\begin{array}{c}[+A\to B]^1\\ \mathcal{D}\\ \dfrac{\bot}{-A\to B}\,(\mathrm{SR}_1)^1\end{array} \quad\rightsquigarrow\quad \dfrac{\begin{array}{c}\dfrac{\dfrac{\dfrac{[+A]^1 \qquad [-A]^2}{\bot}}{+B}\,(\mathrm{SR}_2)^0}{+A\to B}\,(+\to\mathrm{I})^1\\ \mathcal{D}\\ \dfrac{\bot}{+A}\,(\mathrm{SR}_2)^2\end{array} \qquad \begin{array}{c}\dfrac{[+B]^3}{+A\to B}\\ \mathcal{D}\\ \dfrac{\bot}{-B}\,(\mathrm{SR}_1)^3\end{array}}{-A\to B}$$

$$
\dfrac{\begin{matrix}[-A \to B]^1 \\ \mathcal{D} \\ \bot\end{matrix}}{+A \to B}\,(\mathrm{SR}_2)^1
\quad\rightsquigarrow\quad
\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\dfrac{[+A]^1 \qquad [-B]^2}{-A \to B} \\ \mathcal{D} \\ \bot\end{matrix}}{-A}\,(\mathrm{SR}_1)^1 \qquad [+A]^3}{\bot}}{+B}\,(\mathrm{SR}_2)^2}{+A \to B}\,(+\to\mathrm{I})^3
$$

Disjunction:

$$
\dfrac{\begin{matrix}[+A \lor B]^1 \\ \mathcal{D} \\ \bot\end{matrix}}{-A \lor B}\,(\mathrm{SR}_1)^1
\quad\rightsquigarrow\quad
\dfrac{\dfrac{\begin{matrix}\dfrac{\dfrac{\dfrac{[-A]^1 \qquad [+A]^2}{\bot}}{+B}\,(\mathrm{SR}_2)^0}{+A \lor B}\,(+\lor\mathrm{I})^1 \\ \mathcal{D} \\ \bot\end{matrix}}{-A}\,(\mathrm{SR}_2)^2 \qquad \dfrac{\begin{matrix}\dfrac{[+B]^3}{+A \lor B} \\ \mathcal{D} \\ \bot\end{matrix}}{-B}\,(\mathrm{SR}_1)^3}{-A \lor B}
$$

$$
\dfrac{\begin{matrix}[-A \lor B]^1 \\ \mathcal{D} \\ \bot\end{matrix}}{+A \lor B}\,(\mathrm{SR}_2)^1
\quad\rightsquigarrow\quad
\dfrac{\dfrac{\begin{matrix}\dfrac{[-A]^2 \qquad [-B]^1}{-A \lor B} \\ \mathcal{D} \\ \bot\end{matrix}}{+B}\,(\mathrm{SR}_1)^1}{+A \lor B}\,(-\lor\mathrm{I})^2
$$

3.4. Ancillary reductions

Ancillary reductions eliminate formulae that are consequences of *Smilean reductio* and premises of (Rejection). Because of the way the normalisation process takes place, we need only give them for atomic formulae. In what follows, then, α ranges over arbitrary atoms, and $\overline{\alpha}$ denotes the conjugate of α.[6] There are three cases to consider:

[6]The conjugate of a signed formula $+A$ is $-A$ and vice versa.

Case 1: One of the premises of (Rejection) is not the conclusion of *Smilean Reductio*. Suppose, without loss of generality, that it is the left one:

$$\dfrac{\begin{matrix}\mathcal{D}_1\\ \alpha\end{matrix} \qquad \dfrac{\begin{matrix}[\alpha]^1\\ \mathcal{D}_2\\ \bot\end{matrix}}{\overline{\alpha}}\,(\mathrm{SR})^1}{\bot}\,(\mathrm{Rejection}) \qquad \rightsquigarrow \qquad \begin{matrix}\mathcal{D}_1\\ \alpha\\ \mathcal{D}_2\\ \bot\end{matrix}$$

Note that since α is an atom, this eliminates the ancillary maximal formula in question whilst introducing no further maximal formulae of any kind.

Case 2: Both premises of (Rejection) are the conclusion of *Smilean Reductio*, and at least one of the applications of *Smilean Reductio* discharges no premises of (Rejection). Suppose, without loss of generality, that the rightmost application is of this type:

$$\dfrac{\begin{matrix}\mathcal{D}_1\\ \alpha\end{matrix} \qquad \dfrac{\begin{matrix}[\alpha]^1\\ \mathcal{D}_2\\ \bot\end{matrix}}{\overline{\alpha}}\,(\mathrm{SR})^1}{\bot}\,(\mathrm{Rejection}) \qquad \rightsquigarrow \qquad \begin{matrix}\mathcal{D}_1\\ \alpha\\ \mathcal{D}_2\\ \bot\end{matrix}$$

Note that again this reduces the number of ancillary maximal formulae and gives rise to no maximal formulae of any other type.

Case 3: Both premises of (Rejection) are conclusions of *Smilean reductio* and discharge some premise of (Rejection).

$$\dfrac{\dfrac{\begin{matrix}\dfrac{[\alpha]^1 \qquad \begin{matrix}\mathcal{D}_0\\ \overline{\alpha}\end{matrix}}{\bot}\\ \mathcal{D}_1\\ \bot\end{matrix}}{\overline{\alpha}}\,(\mathrm{SR})^1 \qquad \begin{matrix}\mathcal{D}_2\\ \alpha\end{matrix}}{\bot}\,(\mathrm{Rejection}) \qquad \rightsquigarrow \qquad \begin{matrix}\dfrac{\begin{matrix}\mathcal{D}_2\\ \alpha\end{matrix} \qquad \begin{matrix}\mathcal{D}_0\\ \overline{\alpha}\end{matrix}}{\bot}\,(\mathrm{Rejection})\\ \mathcal{D}_1\\ \bot\end{matrix}$$

Suppose we apply this reduction to an ancillary maximal formula such that there are no ancillary maximal formulae above it or above a formula side connected with it. In the original derivation there may be further occurrences of α with discharge label 1 besides the one explicitly represented above. We also replace them with a copy of $\mathcal{D}_2$ ending in α. Those occurrences that were not premises of (Rejection) are unproblematic. Those that were, on the other hand, become new ancillary maximal formulae of the same complexity. By assumption, though, they are ancillary maximal formulae of the type covered in **Case 1**. We eliminate them as part of the current reduction step, and as a result the number of ancillary maximal formulae decreases, and we give rise to no maximal formulae of other kinds.

3.5. Normalisation and corollaries

THEOREM 3.2 (Normalisation). *If there is a derivation $\mathcal{D}$ of φ from Γ then there is a normal derivation $\mathcal{D}'$ of φ from $\Gamma' \subseteq \Gamma$.*

PROOF: To each derivation $\mathcal{D}$ we assign a coordination rank $(n, m) \in \mathbb{N}\times\mathbb{N}$, where n is the highest complexity of a maximal coordination formula, and m the number of maximal coordination formulae of maximal complexity. A derivation without maximal coordination formulae has rank $(0, 0)$, and coordination ranks are ordered lexicographically. We also assign it an operational rank $(j, k) \in \mathbb{N} \times \mathbb{N}$ defined analogously but with respect to maximal operational formulae, and order operational ranks with their own lexicographical order. The following is an effective procedure to normalise derivations:

1. Take a maximal coordination formula of the highest complexity such that there are no coordination formulae of the highest complexity above it or above a formula side-connected with it. Apply the appropriate reduction from Sections 3.2 and 3.3. The coordination rank strictly decreases. Thus, after a finite number of steps, our derivation has coordination rank (0,0).

2. Take a maximal operational formula of the highest complexity such that there are no maximal operational formulae of the highest complexity above it or above a formula side-connected with it. Apply the appropriate reduction from Section 3.1. The operational rank

strictly decreases, and the coordination rank stays $(0, 0)$. Thus, after a finite number of steps, our derivation has coordination and operational ranks (0,0).

3. Take an ancillary maximal formula (note that it must be atomic) such that that there are no ancillary maximal formulae above it or above a formula side-connected with it. Apply the appropriate reduction from Section 3.4. The number of ancillary maximal formulae goes down, and the coordination and operational ranks stay the same. After a finite number of steps, the derivation is in normal form. □

Definition 3.3. (Branch)
A branch π in a derivation $\mathcal{D}$ is a sequence $\varphi_1, ..., \varphi_n$ of occurrences of formulae or of $\perp$ such that: **(i)** φ_1 is a leaf (an assumption), discharged or not. **(ii)** φ_{i+1} stands immediately below φ_i. **(iii)** φ_n is either the conclusion of $\mathcal{D}$ or the first formula occurrence in the sequence that is the minor premise of $(+\to \text{E})$, $(+\vee \text{E})$ or $(-\wedge\text{E})$.

Lemma 3.4. *Every formula in a derivation belongs to some branch.*

Proof: By induction on derivations. □

The following theorem characterises the shape of normal derivations (see also Remark 3.6 for a comparison with Prawitz's normal form).

Theorem 3.5 (Shape of normal derivations). *Let $\mathcal{D}$ be a normal derivation, $\pi = \varphi_1, ..., \varphi_n$ a branch in it. Then there is a minimum formula φ_i dividing π into two (possibly empty) parts, the E part and the I-part, such that:*
(i)*Each φ_j in the E-part (i.e. $j < i$) is the major premise of an E-rule.*
(ii) *If $i \neq n$ then φ_i is a premise of (Rejection) or an I-rule.*
(iii) *Each φ_k in the I-part (i.e. $i < k$) is a premise of an I-rule, except φ_{i+1}, which may be a premise $\perp$ of Smilean reductio.*

Proof: Let $\pi = \varphi_1, ..., \varphi_n$ be a branch in a normal derivation $\mathcal{D}$. Then in π there are a) no applications of an E-rule after an I-rule, b) no applications of *Smilean reductio* after an I-rule, c) no applications of (Rejection) after an I-rule and d) no applications of an E-rule after *Smilean reductio*. I will prove a) as an example; b)-d) are proved analogously.

No applications of an E-rule after an I-rule: suppose for a contradiction that there are, let φ_k be the first consequence of an E-rule applied after an I-rule, consider φ_{k-1}. Since the derivation is normal, φ_{k-1} is not the consequence of an I-rule. It cannot be the consequence of *Smilean reductio* or (Rejection) either, as then we couldn't obtain φ_k from it through an E-rule. Thus, φ_{k-1} must be the consequence of an E-rule, contradicting the assumption that φ_k was first.

The remainder of the theorem is easy to prove: consider the last rule applied in π: if it is an E-rule, let $\varphi_i = \varphi_n$. If it is *Smilean reductio*, let $\varphi_i = \varphi_{n-2}$. If it is (Rejection), let $\varphi_i = \varphi_1$. Finally, if the last rule is an I-rule, let φ_i be the only formula occurrence in π that is a premise of (Rejection) – if there is one –, or else let φ_i be the first premise of an I-rule.

□

Remark 3.6. An alternative way of phrasing Theorem 3.5 is to say that a branch in a normal derivation consists of three (possibly empty) parts: an E-part, where every formula occurrence is the major premise of an E-rule, a C-part, where every formula occurrence is atomic and a premise of a coordination principle, and an I-part, where every formula occurrence is a premise of an I-rule. Branches in Prawitz's classical normal derivations (see [7]) consist of an E-part and an I-part, joined together by a (possibly empty) part where classical reductio is applied to an atom.

We can now obtain the subformula and separation properties as corollaries.

Definition 3.7. (Subformula)
Signed formula ψ is a subformula of signed formula φ if the unsigned part of ψ is a subformula (in the standard sense) of the unsigned part of φ. Thus, for example, all of $+p$, $-p$, $+q$, $-q$ are signed subformulae of $+p \to q$. Note that $\perp$ is not a formula but a punctuation sign.

Definition 3.8 (Order of a branch). A branch $\pi = \varphi_1, ..., \varphi_n$ in a derivation $\mathcal{D}$ is of order 0 if φ_n is the conclusion of $\mathcal{D}$, and of order $k+1$ if it ends on the minor premise of an E-rule the major premise of which belongs to a branch π' order k.

Corollary 3.9 (Subformula property). All the formulae that occur in a normal derivation of φ from Γ are subformulae of some $\gamma \in \Gamma$ or of φ.

Proof: By induction on the order of branches. Let $\pi = \varphi_1, ..., \varphi_n$ be a branch of order k and assume the result for branches of order $j < k$. We will think of the E, C and I-parts of a branch as defined in Remark 3.6. The result is obvious for the I-part: if $k = 0$ all formulae in it are subformulae of $\varphi_n = \varphi$. Similarly, if $k > 0$ then all formulae in the I-part are subformulae of φ_n, which is in its turn a subformula of the major premise ψ of an elimination rule that belongs to a branch of lower order. By inductive hypothesis ψ is itself subformula of some $\gamma \in \Gamma$ or of φ, and therefore so are all the formulae in the I-part.

It remains to show the result for the E and C-parts. Note that all remaining formulae are subformulae of φ_1, the first formula of the branch. Now, if φ_1 is an undischarged assumption the result follows trivially. If φ_1 is a discharged assumption, there are two cases to consider:

Case 1: If φ_1 is discharged by *Smilean reductio* then φ_1 must be an atom, and so the E-part of our branch π is empty. Moreover, the application of *Smilean reductio* in question concludes $\overline{\varphi}_1$, the conjugate of φ_1. Note that φ_1 is a subformula of $\overline{\varphi}_1$, and that $\overline{\varphi}_1$ must be a subformula of φ_n, the last formula of the branch. Thus, φ_1 is a subformula of φ_n. If the branch π is of order 0 this means that φ_1 is a subformula of the conclusion, and if π is of order > 0 then the result follows by inductive hypothesis.

Case 2: If φ_1 is discharged by an I-rule, then it is a subformula of the consequence φ_k of that application, and φ_k is in its turn a subformula of φ_n, the last formula in the branch. Once again, if the branch π is of order 0 this means that φ_1 is a subformula of the conclusion, and if π is of order > 0 then the result follows by inductive hypothesis. □

Corollary 3.10 (Separation property). In a normal derivation of φ from Γ only operational rules for connectives in φ and Γ (and perhaps coordination principles) are used.

Proof: Follows immediately from Corollary 3.9. □

4. Harmony and normalisation: $\mathbf{HB}_2$

The first two clauses of the definition of normal form for $\mathbf{HB}_2$ are identical to those of $\mathbf{HB}_1$. In other words, we require that for all normal derivations of $\mathbf{HB}_2$:

- **(i)** No conclusion of an I-rule is a major premise of an E-rule.
- **(ii)** Coordination principles are applied only to atoms.

The motivation behind them remains the same: **(i)** is taken from Prawitz, and **(ii)** is its analogue for bilateral systems, suggested by del Valle-Inclan and Schlöder's notion of harmony. This is, as before, enough to ensure that normal derivations satisfy the separation property. Once again, however, it is not enough to obtain the subformula property, as the following derivation shows:

$$\dfrac{\dfrac{+p \qquad -p}{+q}\,(\mathrm{ex}) \qquad \dfrac{+p \qquad -p}{-q}\,(\mathrm{ex})}{+r}\,(\mathrm{ex})$$

In order to secure the subformula property we follow the same strategy as before: imposing constraints on the way coordination principles interact *with each other*. These constraints are given by clause **(iii)** of Definition 4.1.

Definition 4.1. (Normal form)
A derivation in $\mathbf{HB}_2$ is in normal form if in it: **(i)** No conclusion of an I-rule is a major premise of an E-rule. **(ii)** Coordination principles are applied only to atoms. **(iii)** (a) No conclusion of (ex) is a premise of (ex), (b) no application of (ex) has both premises discharged by (bem) and, (c) no conclusion of (bem) is a premise of (ex).

Formula occurrences that infringe clauses **(i)** and **(ii)** are called maximal operational formulae and maximal coordination formulae, respectively. Formula occurrences that infringe clause **(iii)** are called ancillary maximal formulae.

$\mathbf{HB}_1$ and $\mathbf{HB}_2$ share the same operational rules, so the reduction steps for operational maximal formulae are identical. The obvious similarity between the rules (Rejection) and (ex) means that the reduction steps to restrict (ex) to atomic premises are analogous to the steps restricting (Rejection) to atomic premises; I will omit this type of reduction as well, for reasons of space. The remaining reduction steps are as follows.

4.1. Reducing (ex) to atomic conclusions

Negation:

$$\dfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2\\ +A & -A\end{array}}{+\neg B}\,(\text{ex}) \quad\leadsto\quad \dfrac{\dfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2\\ +A & -A\end{array}}{-B}\,(\text{ex})}{+\neg B}\,(-\neg\ \text{I})$$

$$\dfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2\\ +A & -A\end{array}}{-\neg B}\,(\text{ex}) \quad\leadsto\quad \dfrac{\dfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2\\ +A & -A\end{array}}{+B}\,(\text{ex})}{-\neg B}\,(-\neg\ \text{I})$$

Implication:

$$\dfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2\\ +A & -A\end{array}}{+B \to C}\,(\text{ex}) \quad\leadsto\quad \dfrac{\dfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2\\ +A & -A\end{array}}{+C}\,(\text{ex})}{+B \to C}\,(+\to\ \text{I})^0$$

$$\dfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2\\ +A & -A\end{array}}{-B \to C}\,(\text{ex}) \quad\leadsto\quad \dfrac{\dfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2\\ +A & -A\end{array}}{+B}\,(\text{ex}) \qquad \dfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2\\ +A & -A\end{array}}{-C}\,(\text{ex})}{-B \to C}\,(-\to\ \text{I})$$

Disjunction:

$$\dfrac{\begin{matrix}\mathcal{D}_1 \\ +A\end{matrix} \qquad \begin{matrix}\mathcal{D}_2 \\ -A\end{matrix}}{+B \vee C}\,(\text{ex}) \quad\rightsquigarrow\quad \dfrac{\dfrac{\begin{matrix}\mathcal{D}_1 \\ +A\end{matrix} \qquad \begin{matrix}\mathcal{D}_2 \\ -A\end{matrix}}{+C}\,(\text{ex})}{+B \vee C}\,(+\vee\ \text{I})^0$$

$$\dfrac{\begin{matrix}\mathcal{D}_1 \\ +A\end{matrix} \qquad \begin{matrix}\mathcal{D}_2 \\ -A\end{matrix}}{-B \vee C}\,(\text{ex}) \quad\rightsquigarrow\quad \dfrac{\dfrac{\begin{matrix}\mathcal{D}_1 \\ +A\end{matrix} \qquad \begin{matrix}\mathcal{D}_2 \\ -A\end{matrix}}{-B}\,(\text{ex}) \qquad \dfrac{\begin{matrix}\mathcal{D}_1 \\ +A\end{matrix} \qquad \begin{matrix}\mathcal{D}_2 \\ -A\end{matrix}}{-C}\,(\text{ex})}{-B \vee C}\,(-\vee\ \text{I})$$

4.2. Reducing assumptions to atoms in (bem)

Negation:

$$\dfrac{\begin{matrix}[+\neg A]^1 \\ \mathcal{D}_1 \\ \varphi\end{matrix} \qquad \begin{matrix}[-\neg A]^1 \\ \mathcal{D}_2 \\ \varphi\end{matrix}}{\varphi}\,(\text{bem})^1 \quad\rightsquigarrow\quad \dfrac{\begin{matrix}\dfrac{[-A]^1}{+\neg A} \\ \mathcal{D}_1 \\ \varphi\end{matrix} \qquad \begin{matrix}\dfrac{[+A]^1}{-\neg A} \\ \mathcal{D}_2 \\ \varphi\end{matrix}}{\varphi}\,(\text{bem})^1$$

Implication:

$$\dfrac{\begin{matrix}[+A \to B]^1 \\ \mathcal{D}_1 \\ \varphi\end{matrix} \qquad \begin{matrix}[-A \to B]^1 \\ \mathcal{D}_2 \\ \varphi\end{matrix}}{\varphi}\,(\text{bem})^1 \quad\rightsquigarrow$$

$$\dfrac{\begin{matrix}\dfrac{[+B]^3}{+A \to B} \\ \mathcal{D}_1 \\ \varphi\end{matrix} \qquad \dfrac{\begin{matrix}\dfrac{[-B]^3 \qquad [+A]^2}{-A \to B} \\ \mathcal{D}_2 \\ \varphi\end{matrix} \qquad \begin{matrix}\dfrac{\dfrac{[-A]^2 \qquad [+A]^1}{+B}\,(\text{ex})}{+A \to B}\,(+\to\ \text{I})^1 \\ \mathcal{D}_1 \\ \varphi\end{matrix}}{\varphi}\,(\text{bem})^2}{\varphi}\,(\text{bem})^3$$

Disjunction:

$$\dfrac{\begin{matrix}[+A \vee B]^1\\ \mathcal{D}_1\\ \varphi\end{matrix} \qquad \begin{matrix}[-A \vee B]^1\\ \mathcal{D}_2\\ \varphi\end{matrix}}{\varphi}\,(\mathrm{bem})^1 \qquad \rightsquigarrow$$

$$\dfrac{\begin{matrix}\dfrac{[+B]^3}{+A \vee B}\\ \mathcal{D}_1\\ \varphi\end{matrix} \qquad \dfrac{\begin{matrix}\dfrac{[-B]^3 \quad [-A]^2}{-A \vee B}\\ \mathcal{D}_2\\ \varphi\end{matrix} \qquad \begin{matrix}\dfrac{\dfrac{[-A]^1 \quad [+A]^2}{+B}\,(\mathrm{ex})}{+A \vee B}\,(+\vee\ \mathrm{I})^1\\ \mathcal{D}_1\\ \varphi\end{matrix}}{\varphi}\,(\mathrm{bem})^2}{\varphi}\,(\mathrm{bem})^3$$

4.3. Reducing conclusions to atoms in (bem)

Negation:

$$\dfrac{\begin{matrix}[+A]^1\\ \mathcal{D}_1\\ +\neg B\end{matrix} \qquad \begin{matrix}[-A]^1\\ \mathcal{D}_2\\ +\neg B\end{matrix}}{+\neg B}\,(\mathrm{bem})^1 \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\dfrac{\begin{matrix}[+A]^1\\ \mathcal{D}_1\\ +\neg B\end{matrix}}{-B}\,(+\neg\ \mathrm{E}) \qquad \dfrac{\begin{matrix}[-A]^1\\ \mathcal{D}_2\\ +\neg B\end{matrix}}{-B}\,(+\neg\ \mathrm{E})}{-B}\,(\mathrm{bem})^1}{+\neg B}\,(+\neg\ \mathrm{I})$$

The case where $\varphi = -\neg B$ is analogous.

Implication:

$$\dfrac{\begin{matrix}[+A]^1\\ \mathcal{D}_1\\ +B\to C\end{matrix} \qquad \begin{matrix}[-A]^1\\ \mathcal{D}_2\\ +B\to C\end{matrix}}{+B\to C}\,(\mathrm{bem})^1 \qquad\qquad \rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\begin{matrix}[+A]^1\\ \mathcal{D}_1\\ +B\to C\end{matrix}\qquad [+B]^2}{+C}\qquad \dfrac{\begin{matrix}[-A]^1\\ \mathcal{D}_2\\ +B\to C\end{matrix}\qquad [+B]^2}{+C}}{+C}\,(\mathrm{bem})^1}{+B\to C}\,(+\to\mathrm{I})^2$$

$$\dfrac{\begin{matrix}[+A]^1\\ \mathcal{D}_1\\ -B\to C\end{matrix} \qquad \begin{matrix}[-A]^1\\ \mathcal{D}_2\\ -B\to C\end{matrix}}{-B\to C}\,(\mathrm{bem})^1 \qquad\qquad \rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\begin{matrix}[+A]^1\\ \mathcal{D}_1\\ -B\to C\end{matrix}}{+B}\,(-\to\mathrm{E}) \qquad \dfrac{\begin{matrix}[-A]^1\\ \mathcal{D}_2\\ -B\to C\end{matrix}}{+B}\,(-\to\mathrm{E})}{+B}\,(\mathrm{bem})^1 \qquad \dfrac{\dfrac{\begin{matrix}[+A]^2\\ \mathcal{D}_1\\ -B\to C\end{matrix}}{-C}\,(-\to\mathrm{E}) \qquad \dfrac{\begin{matrix}[-A]^2\\ \mathcal{D}_2\\ -B\to C\end{matrix}}{-C}}{-C}\,(\mathrm{bem})^2}{-B\to C}\,(-\to\mathrm{I})$$

Disjunction: Analogous to implication.

4.4. Ancillary reductions

As before, the α_i range over arbitrary *atoms*, and $\overline{\alpha_i}$ denotes the conjugate of α_i.
Clause **(iii)**(a):

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ +p\end{matrix}\qquad\begin{matrix}\mathcal{D}_2\\ -p\end{matrix}}{\alpha_1}\,(\mathrm{ex})\qquad\begin{matrix}\mathcal{D}_3\\ \overline{\alpha}_1\end{matrix}}{\alpha_2}\,(\mathrm{ex})\qquad\rightsquigarrow\qquad\dfrac{\begin{matrix}\mathcal{D}_1\\ +p\end{matrix}\qquad\begin{matrix}\mathcal{D}_2\\ -p\end{matrix}}{\alpha_2}\,(\mathrm{ex})$$

Clause **(iii)**(b):

$$(\mathrm{bem})^n\,\dfrac{\begin{matrix}\dfrac{\begin{matrix}\dfrac{[\alpha_1]^1\qquad[\overline{\alpha}_1]^n}{\alpha_2}\,(\mathrm{ex})\\ \mathcal{D}_1\\ \alpha_3\end{matrix}\qquad\begin{matrix}[\overline{\alpha}_1]^1\\ \mathcal{D}_2\\ \alpha_3\end{matrix}}{\alpha_3}\,(\mathrm{bem})^1\\ \mathcal{D}_3\\ \alpha_4\end{matrix}\qquad\begin{matrix}[\alpha_1]^n\\ \mathcal{D}_4\\ \alpha_4\end{matrix}}{\alpha_4}\qquad\rightsquigarrow\qquad\dfrac{\begin{matrix}[\overline{\alpha}_1]^1\\ \mathcal{D}_2\\ \alpha_3\\ \mathcal{D}_3\\ \alpha_4\end{matrix}\qquad\begin{matrix}[\alpha_1]^1\\ \mathcal{D}_4\\ \alpha_4\end{matrix}}{\alpha_4}\,(\mathrm{bem})^1$$

Note that in this last reduction we have assumed that the left premise of the application of (ex) is discharged before the right one. This is unimportant: if it is the other way around, the appropriate reduction is analogous.

Clause **(iii)**(c):

$$(\mathrm{ex})\,\dfrac{\dfrac{\begin{matrix}[+p]^1\\ \mathcal{D}_1\\ \alpha_1\end{matrix}\qquad\begin{matrix}[-p]^1\\ \mathcal{D}_2\\ \alpha_1\end{matrix}}{\alpha_1}\,(\mathrm{bem})^1\qquad\begin{matrix}\mathcal{D}_3\\ \overline{\alpha}_1\end{matrix}}{\alpha_2}\qquad\rightsquigarrow\qquad\dfrac{\dfrac{\begin{matrix}[+p]^1\\ \mathcal{D}_1\\ \alpha_1\end{matrix}\qquad\begin{matrix}\mathcal{D}_3\\ \overline{\alpha}_1\end{matrix}}{\alpha_2}\,(\mathrm{ex})\qquad\dfrac{\begin{matrix}[-p]^1\\ \mathcal{D}_2\\ \alpha_1\end{matrix}\qquad\begin{matrix}\mathcal{D}_3\\ \overline{\alpha}_1\end{matrix}}{\alpha_2}\,(\mathrm{ex})}{\alpha_2}\,(\mathrm{bem})^1$$

Applications of (ex) like the one above on the left, where at least one of the premises is a conclusion of (bem), are called **peaks**. The **size** of a peak is the sum of the length of the maximal segments that the premises of (ex) belong to (if a premise is not part of a maximal segment, we assign it length 0). In the normalisation process we will assign to each derivation a **peak**

rank (j, k), where j is the greatest size of a peak in the derivation, k the number of peaks of greatest size. The reader can check that the reduction above, when applied to a maximal segment such that there are no longer maximal segments above it, side connected with it, or above a formula side connected with it, strictly reduces the peak rank of a derivation.

4.5. Normalisation and corollaries

Definition 4.2 (Segment). A segment σ in a branch π is a sequence of formula occurrences $\sigma_1, ..., \sigma_n$ in π such that: **(i)** σ_1 is *not* the conclusion of an application of (bem). **(ii)** Each σ_i for $i < n$ is a premise of (bem), and σ_{i+1} stands immediately below σ_i. **(iii)** σ_n is *not* a premise of an application of (bem).

Definition 4.2 entails that all the elements of a segment are occurrences of the same formula. The **length of a segment** is the number of formula occurrences in it. A segment is called **maximal** if it ends in an application of (ex). This means that maximal coordination formulae that infringe clause **(iii)**(c) of Definition 4.1 are always final formula occurrences in maximal segments of length ≥ 1, and maximal coordination formulae that infringe clauses **(iii)**(a) and **(iii)**(b) are always maximal segments of length 1. There are no maximal segments of other types.

Lemma 4.3. *Every branch can be uniquely divided into consecutive segments.*

Proof: By induction on the length of branches. □

Theorem 4.4 (Normalisation). *If there is a derivation $\mathcal{D}$ of φ from Γ then there is a normal derivation $\mathcal{D}'$ of φ from $\Gamma' \subseteq \Gamma$.*

Proof: Analogous to the previous proof of normalisation. Derivations are assigned a coordination and an operational rank, defined as before. We apply first the coordination reductions (Sections 4.1–4.3) and then the operational reductions (Section 3.1), starting always from maximal formulae of maximal complexity such that there are no maximal formulae of maximal complexity above them or above a formula side connected with them. Once a derivation has no coordination or operational maximal formulae we assign it a peak rank, as defined at the end of Section 4.4, and apply the reduction for ancillary formulas of type **(iii)**(c) as indicated there. Once there are no peaks left, the only remaining maximal formulae are those that infringe clauses **(iii)**(a) and **(iii)**(b). They can be eliminated in any order using the appropriate reduction from Section 4.4. □

THEOREM 4.5 (Shape of normal derivations). *Let $\mathcal{D}$ be a derivation in normal form, π a branch in $\mathcal{D}$, and let $\sigma_1, ...\sigma_n$ be the segments in π. Then there is a segment σ_i in π, called the minimum segment, which separates /pi into two (possibly empty) parts, the E-part and the I-part, with the properties:*

1. *For each σ_j in the E-part (i.e. $j < i$), σ_j is a major premise of an E-rule, except possibly σ_{i-1}, which may be a premise of (ex).*

2. *If $i \neq n$, then each formula in the segment σ_i is a premise of (bem) except the last one, which may be a premise of an I-rule.*

3. *For each σ_j in the I-part (i.e. $i < j < n$), σ_j is a premise of an I-rule.*

PROOF: It is easy to see that, in a branch $\pi = \varphi_1, ..., \varphi_n$ of a normal derivation, no formula occurrences that are premises of an Introduction rule precede formula occurrences that are major premises of an Elimination rule, (bem) or (ex), no formula occurrences that are premises of (bem) precede formula occurrences that are premises of (ex) or major premises of an E-rule, and no formula occurrences that are premises of (ex) precede formula occurrences that are major premises of an E-rule or (ex). Now:

If there is no formula occurrence that is a premise of an I-rule or (bem), let $\sigma_i = \varphi_n$. If there is a formula occurrence that is a premise (bem), let φ_i be the first such formula, and let σ_i be the segment starting at φ_i. Finally, if there is no formula occurrence that is a premise (bem), but there is a formula occurrence that is a premise of an I-rule, let φ_i be the first such formula, and let $\sigma_i = \varphi_i$. □

Remark 4.6. An alternative way of phrasing Theorem 4.5 is to say that a branch in a normal derivation consists of three (possibly empty) parts: an E-part, where every formula occurrence is the major premise of an E-rule, a C-part, where every formula occurrence is a premise of a coordination principle – and within which (ex) is applied before (bem) – and an I-part, where every formula occurrence is a premise of an I-rule.

Corollary 4.7 (Subformula property). All the formulae that occur in a normal derivation of φ from Γ are subformulae of some $\gamma \in \Gamma$ or of φ.

Proof: By induction on the order of branches. Let $\pi = \sigma_1, ...\sigma_n$ be a branch of order p, let σ_i be its minimum segment, and assume the result for branches of lower order. Consider first all σ_j with $i \leq j \leq n$. All such formulae are subformulae of φ_n, the formula in the last segment σ_n of the branch. If the branch in question is of order 0 the result immediately follows. If the branch is of order > 0 then φ_n is the minor premise of an application of an E-rule, the major premise ψ of which belongs to a branch of order $p - 1$. But by induction hypothesis the result holds for ψ, and φ_n is a subformula of ψ, so the result follows.

It remains to account for all the σ_j with $j < i$. Note that all such formulae are subformulae of φ_1, the first formula of the branch. If φ_1 is an undischarged assumption the result immediately follows. Similarly, if φ_1 is discharged by an application of an I-rule, then it is a subformula of some formula in an I-part, and the result follows by the above. Finally, suppose that φ_1, is discharged by an application of (bem). Now, φ_1 cannot be the major premise of an elimination rule, since it is an atom. If it is the minor premise of an E-rule, or a premise of an I-rule or (bem), then there are no σ_j with $j < i$ and we are done. The only remaining possibility is that φ_1 is a premise of (ex). Then φ_1 is the only formula before the minimum segment σ_i (in other words, φ_1 is the only formula we still need to account for). Now, φ_1 is a subformula of the other premise $\overline{\varphi}_1$ of the application of (ex) in question, and $\overline{\varphi}_1$ cannot be discharged by (bem). Moreover, $\overline{\varphi}_1$ belongs to a branch of the same order as π. If $\overline{\varphi}_1$ is undischarged, or discharged by a I-rule, the result immediately follows. If it is a consequence of an E-rule, then it is a subformula of the initial formula ψ of its branch. But then ψ is not atomic, and so can only be undischarged or discharged by an I-rule. In either case, the result follows. □

Corollary 4.8 (Separation property). In a normal derivation of φ from Γ only operational rules for connectives in φ and Γ (and perhaps coordination principles) are applied.

Proof: Follows immediately from the previous corollary. □

5. Comparison with other normalisation results

In this section I will briefly compare the present normalisation results with those obtained by Nils Kürbis [6] and Marcello D'Agostino, Dov Gabbay, and Sanjay Modgyl [1], so as to outline the similarities and differences between them.

5.1. $\mathbf{HB}_1$ and Kürbis-normal form

Kürbis proves his normalisation result for Rumfitt's original calculus,[7] which makes the comparison with normal form for $\mathbf{HB}_1$ straightforward. The respective definitions of normal form are:

Kürbis-normal form:	**$\mathbf{HB}_1$-normal form:**
(a) No conclusion of an I-rule is a major premise of an E-rule.	(i) No conclusion of an I-rule is a major premise of an E-rule.
(b) No conclusion of *Smilean reductio* is a major premise of an E-rule.	(ii) Coordination principles are applied only to atoms.
(c) No conclusion of an I-rule is a premise of an application of (Rejection) the other premise of which is also the conclusion of an I-rule.	
(d) No conclusion of *Smilean reductio* is a premise of (Rejection).	(iii) No conclusion of *Smilean reductio* is a premise of (Rejection).
(e) There are no maximal segments.	

Clause (a) of Kürbis-normal form is identical to clause (i) of $\mathbf{HB}_1$-normal form, and the same goes for clauses (d) and (iii). The correlate of clauses (b) and (c) of Kürbis-normal form is clause (ii). Crucially, though, (ii) is strictly stronger that (b) and (c) combined: all derivations that satisfy (ii) satisfy (b) and (c), but the converse does not hold. The segments referred to in clause (e) are defined as usual: sequences of occurrences of the same

[7] That is, the calculus comprising the operational rules without del Valle-Inclan and Schlöder's modifications, plus (Rejection) and *Smilean reductio* as coordination principles.

formula that end in a maximal formula. Because of the modified operational rules of $\mathbf{HB}_1$, maximal segments simply cannot arise in normal derivations. Thus, clause (e) has no correlate in $\mathbf{HB}_1$-normal form. It follows that $\mathbf{HB}_1$-normal form is stronger than Kürbis-normal form, in the sense that all derivations in $\mathbf{HB}_1$-normal form are Kürbis-normal, but the converse does not hold. Rumfitt's calculus, for instance, can be Kürbis-normalised but not $\mathbf{HB}_1$-normalised.

5.2. $\mathbf{HB}_2$ and C-intelim normal form

Marcello D'Agostino, Dov Gabbay, and Sanjay Modgyl prove their normalisation result for a calculus they call C-intelim.[8] The crucial difference between C-intelim and $\mathbf{HB}_2$ is that no operational rule of C-intelim discharges any premises. More precisely, their rules for disjunction are Rumfitt's $(-\vee\ \mathrm{I})$, $(-\vee\ \mathrm{E})$, $(+\vee\ \mathrm{I})$ plus the following two:

$$\frac{+A \vee B \qquad -A}{+B} \qquad \frac{+A \vee B \qquad -B}{+A}$$

Their rules for conjunction are Rumfitt's $(+\wedge\ \mathrm{I})$, $(+\wedge\ \mathrm{E})$, $(-\wedge\ \mathrm{I})$ and:

$$\frac{-A \wedge B \qquad +A}{-B} \qquad \frac{-A \wedge B \qquad +B}{-A}$$

And their rules for conditionals are Rumfitt's $(- \to \mathrm{I})$, $(- \to \mathrm{E})$, $(+ \to \mathrm{E})$ and:

$$\frac{-A}{+A \to B} \qquad \frac{+B}{+A \to B} \qquad \frac{+A \to B \qquad -B}{-A}$$

The coordination principles in C-intelim are essentially *Explosion* and *Bilateral Excluded Middle*, but there is an additional consideration to keep in mind. In C-intelim *Explosion* is reformulated as two distinct rules, namely:

$$\frac{-A \qquad +A}{\perp} \qquad \frac{\perp}{\varphi}$$

with the proviso that '$\perp$' can only occur in the context of these rules, as a punctuation sign [1, p. 302]. Clearly, this makes the difference between (ex)

[8] They present two versions of C-intelim: a bilateral version and a unilateral one, which they regard as a 'practically convenient translation of the rules for signed formulae into an ordinary logical language' ([1], p. 303-4). Here I will only consider the bilateral formulation of the calculus.

and their two-rule combination strictly notational. In order to simplify the comparison with $\mathbf{HB}_2$, then, I will take C-intelim to contain *Explosion* in its single-rule presentation (ex). Nothing substantial hinges on this.

We can finally compare both notions of normal form. Normal derivations in C-intelim have the following shape:

$$\dfrac{\begin{matrix}\dfrac{\begin{matrix}\mathcal{D}_1\\ \varphi\end{matrix}\qquad\begin{matrix}\mathcal{D}_2\\ \varphi\end{matrix}}{\varphi}\,(\text{bem})\\ \mathcal{T}_1\\ \varphi\end{matrix}\qquad \begin{matrix}\cdots\\ \\ \cdots\\ \\ \end{matrix}\qquad \begin{matrix}\dfrac{\begin{matrix}\mathcal{D}_{n-1}\\ \varphi\end{matrix}\qquad\begin{matrix}\mathcal{D}_n\\ \varphi\end{matrix}}{\varphi}\,(\text{bem})\\ \mathcal{T}_k\\ \varphi\end{matrix}}{\varphi}\,(\text{bem})$$

where in the $\mathcal{D}_i$ only operational rules are used, except possibly at the last step, which may be an application of *Explosion*, and the (possibly empty) $\mathcal{T}_i$ consist exclusively of applications of *Bilateral Excludded Middle.* Moreover, in normal derivations the assumptions discharged by (bem) are always subformulae of undischarged premises of the derivation or of its conclusion.

It is obvious that normal $\mathbf{HB}_2$ derivations need not be of this form. More importantly, derivations in $\mathbf{HB}_2$ cannot, in general, be put in C-intelim normal form. The reason is that certain operational rules of $\mathbf{HB}_2$ discharge premisses, which means that it is sometimes unavoidable to use them after an application of *Explosion*, as in the ($\mathbf{HB}_2$-normal) derivation below:

$$\dfrac{\dfrac{\dfrac{+\neg p}{-p}\qquad [+p]^1}{+q}\,(\text{ex})}{+p \to q}\,(+\to \mathrm{I})^1$$

Conversely, derivations in C-intelim cannot in general be put in $\mathbf{HB}_2$-normal form. This is due to the fact that several operational rules of C-intelim do not preserve the coordination principle of *Bilateral Excluded Middle*, and hence are not harmonious in del Valle-Inclan and Schlöder's sense. The following, for instance, is a C-intelim normal derivation where (bem) is applied to complex formulae in a way that cannot be eliminated:

$$\dfrac{\dfrac{[+p]^1}{+p \vee \neg p} \qquad \dfrac{\dfrac{[-p]^1}{+\neg p}}{+p \vee \neg p}}{+p \vee \neg p}\,(\text{bem})^1$$

In summary: C-intelim normal form and $\mathbf{HB}_2$-normal form are neither stronger nor weaker than each other.

Of course, normal forms for different but related calculi need not coincide on every point, so the fact that Rumfitt's calculus and C-intelim do not $\mathbf{HB}_1$ and $\mathbf{HB}_2$-normalise (respectively) is not particularly surprising. Still, given the close connection between the present notions of normal form and bilateral harmony, this can be seen as a symptom of the underlying disharmony of these calculi. Conversely, the results shown by Kürbis and D'Agostino, Gabbay and Modgyl show that, despite their disharmony, C-intelim and Rumfitt's calculus are relatively well-behaved. This emphasises the fact that del Valle-Inclan and Schlöder's notion of bilateral harmony rules out more than the glaring problems raised by connectives like tonk and bink.[9]

6. Concluding remarks

The idea that the operational rules for each connective should be 'balanced' underlies most approaches to proof-theoretic harmony. This idea has a correlate in normalisation proofs, in the requirement that formula occurrences that are the consequence of an introduction rule and the major premise of an elimination should be removed. Del Valle-Inclan and Schlöder's bilateral criterion of harmony suggests a similar requirement for bilateral systems, namely that normal proofs should apply coordination principles to atomic formulae only. These two requirements are enough for a weak notion of normality that remains stable across calculi $\mathbf{HB}_1$ and $\mathbf{HB}_2$, and which guarantees the separation but not the subformula property. In order to guarantee the subformula property a third kind of constraint, regulating how coordination principles are allowed to interact with each other, is needed. These constraints vary across $\mathbf{HB}_1$ and $\mathbf{HB}_2$, as the two calculi

[9]Thanks to an anonymous referee for prompting me to say more about this.

have different coordination principles. The resulting notion of normal form is strictly stronger than Kürbis-normal form in the case of $\mathbf{HB}_1$, and neither stronger nor weaker than C-intelim normal form in the case of $\mathbf{HB}_2$.

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Pedro del Valle-Inclan
Scuola Normale Superiore
Department of Philosophy
56126, Piazza dei Cavalieri 7
Pisa, Italy
e-mail: pedro.delvalleinclan@sns.it

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