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INTRODUCTION: BILATERALISM AND PROOF-THEORETIC SEMANTICS (PART I)

Most of the papers contained in this special issue[1] are results from contributions at a conference on this topic, which took place at the Ruhr University Bochum in March 2022. Since the topic of proof-theoretic semantics (PTS) can by now be considered as well-established in the logic community and has been exclusively dealt with at several conferences and in many publications[2], this introduction's focus will be on the part of logical bilateralism. Before summarizing the content of this special issue, a brief overview of the development in the field will be given, though this is not meant and does not aim to be an exhaustive account of the existing literature.[3]

There are rather different approaches branded as *bilateralism* in the literature, whose differences are mostly not made explicit, though. Although the origin of bilateralism is Rumfitt's [19] seminal paper in the sense that the concrete term and idea are introduced therein and spelled out thoroughly, there are some predecessors to the general idea that are frequently cited, like [11], [21], and [7].[4] The most frequent characterization that is

[1] For editorial reasons it was decided to have actually two issues on this topic, which is why this introduction will appear in both parts and only differ in the presentation of the papers contained in the respective issue.

[2] See, e.g., [20, 3, 8, 10].

[3] Parts of the following paragraphs can also be found in a joint paper by Heinrich Wansing and myself on the topic of multilateralism [26]. In its introductory part we give an overview of the literature on bilateralism as well as of the existing but scarce literature extending this concept to multilateralism.

[4] A paper which is *not* often mentioned in this context, probably due to the fact that it was written in German, but which deserves recognition in this context is [23]. Von Kutschera is concerned with the relation between the notions of proof and refutation and claims, e.g., that it is not necessary to define the latter in terms of the former but

used for bilateralism is that it is a theory of meaning displaying a symmetry between certain notions (or often rather: conditions governing these notions), which have not been considered being on a par by 'conventional' theories of meaning. The relevant notions are most often *assertion* and *denial*, or *assertibility* and *deniability*, sometimes also *acceptance* and *rejection*.[5] While the former are usually taken to describe speech acts, the latter are usually – though not always (see [18] for a thorough distinction) – considered to describe the corresponding internal cognitive states or attitudes. 'Assertibility' and 'deniability', on the other hand, are of a third kind, since they can be seen to describe something like properties of propositions. The symmetry between these respective concepts is often described with expressions like "both being primitive", "not reducible to each other", "being on a par", and "of equal importance". Another point to characterize bilateralism, which is often mentioned, though not as frequent or central as the former point,[6] is that in a bilateral approach the denial of A is not interpreted in terms of, or *as* the assertion of the negation of A but that it is the other way around: In bilateralism rejection and/or denial are usually considered as conceptually prior to negation.

Ripley [17, 18] distinguishes two camps of bilateral theories of meaning in terms of "what kinds of condition on assertion and denial they appeal to" [18, p. 50]: a warrant-based approach and a coherence-based approach, for the latter of which he himself argues [16] and which was firstly devised by Restall [12, 13].[7] As references for the first camp, which Ripley calls the 'orthodox' bilateralism, [11], [21], and [19] are given. Warrant-based bilateralism takes the relevant conditions to be the ones under which propositions can be *warrantedly* asserted or denied. Coherence-based bilateralism, on the other hand, takes the relevant conditions to be the conditions under

that it could just as well be done the other way around, or, although in the paper he does differently, that both could be seen as primitive. Thus, it seems that he voices quite bilateralist ideas.

[5] To give some examples of references using a characterization of essentially this flavor: [4, 6, 9, 15, 19, 25].

[6] The following use this as an additional characterization (while also using the essential characterization that the references in fn. 4 use): [1, 2, 16, 22]. This is not to say that this point does not occur in other works on bilateralism but that it is not used as a *characterizing feature* of bilateralism there.

[7] In [18] this one is called the "bounds-based bilateralism". Interestingly, Restall does not use the expression "bilateralism" at all in the cited works, only later does this term become part of his terminology, e.g., in [14].

which *collections* of propositions can be *coherently* asserted and/or denied together.

What the two approaches have in common is that they were both meant, as they were originally devised, to motivate a PTS approach using *classical* instead of intuitionistic logic. What they tend to differ in, though, is their design and interpretations of proof systems. Rumfitt [19] uses a natural deduction system with signed formulas for assertion and denial, i.e., rules do not apply to propositions but to speech acts. He argues that the shortcomings that a classical natural deduction calculus has from a PTS point of view are overcome once we consider a calculus containing introduction and elimination rules determining not only the assertion conditions for formulas containing the connective in question but also the denial conditions. Thus, he means to give a motivation how the rules of classical logic lay down the meaning of the connectives.[8]

Restall [12], opting for the coherence-based approach, does the same but comes from another direction in suggesting a bilateral reading of classical sequent calculus (i.e., with multiple conclusions) incorporating the speech acts of assertion and denial. In a nutshell, he proposes that having the derivation of a sequent $\Gamma \vdash \Delta$, means that the position of asserting each of the members of Γ while simultaneously denying each of the members of Δ would be 'out of bounds'. In a recent paper, though, Restall [14] seems convinced by Steinberger's [22] criticism of multiple-conclusion systems as not adhering to our natural inferential practice and he considers an approach using a natural deduction system instead, which does not employ signed formulas but rather uses different positions for certain commitments from which the inference is drawn to the conclusion.[9]

What Ripley [18] mentions in a footnote is that there are also other kinds of bilateralism, which do not fit into either camp because they do not consider speech acts (i.e., assertion and denial) as the primary notions to act upon in the context of PTS but rather notions being on a par with proof, provability, or verification, i.e., refutation, refutability, or falsification, respectively. The point of interest is, thus, to implement different derivability relations in a proof-theoretic framework expressing a duality

[8] For critical assessments of that paper, see, e.g., [5, 1, 9, 4].

[9] The motivation is still to make a case for classical logic being usable in a PTS framework, although Restall does not seem too dogmatic about anything being 'the best' logic. He also wants to show how such a system can be used for substructural logics.

between different inferential relationships, which has been devised, e.g., in [24, 25].

These different varieties of bilateralism depicted above are actually very well represented in this special issue. It is even the majority of the contributions dealing with what can be called – in one way or another – 'unorthodox' bilateralism.

Greg Restall's paper "Structural rules in natural deduction with alternatives" explores features of a special kind of bilateralist natural deduction system, namely with *alternatives*. These are 'negative assumptions' with which a natural deduction system of Gentzen-Prawitz-style is extended; otherwise, the rules for the connectives are not changed from the usual ones of such a system, i.e., as Restall notes, the extension is of purely structural nature. What is shown for this system is that the rule of explosion and the rule of allowing vacuous discharge, both being principles introducing irrelevance to the system, can actually be seen as corresponding principles. Restall shows how with the shift to what he calls a *mildly* bilateralist system this extension of Gentzen-Prawitz-style natural deduction can not only be used to give an account for classical logic but also for substructural systems, such as linear, relevant and affine logic. It is only 'mildly' bilateralist because neither is every formula in the system signed to be of either some positive or negative force nor are any operational rules added to the system, as it is done in one way or another in what he calls 'fully' bilateralist systems.

The paper "Core type theory" by Emma van Dijk, David Ripley and Julian Gutierrez also deals with a system which may not strike one as 'obviously' bilateralist but which nevertheless can be seen as one in an interesting way. In the paper a slightly modified version of Tennant's natural deduction proof system for his core logic is presented and used as a type theory. It is shown that strong normalization can be proven for this system, while it cannot for Tennant's original system. Although there are no signed formulas or derivability relations in this system, it is bilateralist in the sense that it is a system in which both proofs and refutations can be constructed and neither concept is taken to be reducible to the other. For this reason, the authors connect the spirit of bilateralism inherent in core logic to the type of bilateralism that is put forth in [24, 25].

Implementing bilateralism on the level of derivational constructions is also advocated in the paper "On synonymy in proof-theoretic semantics.

The case of `2Int`” by myself and Heinrich Wansing. We present a `G3`-style sequent calculus, `SC2Int`, for the bi-intuitionistic logic `2Int`, which is bilateral in that two kinds of signed sequents are used, one representing proofs, the other representing refutations and for which the structural rules are shown to be admissible. Then, by defining and using so-called *interaction rules*, which allow switching from proofs to refutations, and vice versa, an approach to propositional synonymy in a bilateralist PTS setting is devised. This concept relies on the notion of *inherited identity* between derivations and, applied to `SC2Int`, leads to notions of positive and negative synonymy of formulas.

Another special form of PTS and bilateralism is explored by Alexander V. Gheorghiu and David J. Pym in “Definite formulae, negation-as-failure, and the base-extension semantics of intuitionistic propositional logic”. They analyze a base-extension semantics for intuitionistic propositional logic – that is, a semantics building upon sets of inference rules for atomic sentences – in the context of logic programming. The bases are interpreted as programs, i.e., collections of definite formulas, and investigated using an operational reading. The paper recovers the completeness of intuitionistic propositional logic through this perspective. Significantly, in logic programming, assertion and denial are understood in terms of the success and failure to find a proof. Using the *negation-as-failure protocol*, the paper provides an interpretation of negation in a PTS for intuitionistic propositional logic as denial, meaning that the latter is – in accordance with a bilateralist conception – conceptionally prior to the former.

Acknowledgements. I would like to thank the speakers and participants of the conference “Bilateralism and Proof-Theoretic Semantics” for the lively discussion, the vital input on the presented works and for contributing to this issue. I am also very grateful to the editors of this journal for their kind help with administrative and technical questions, to all the reviewers for their constructive feedback on the contributions and last but not least, to Heinrich Wansing for helping out with all of the above and much more.

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Greg Restall

STRUCTURAL RULES IN NATURAL DEDUCTION WITH ALTERNATIVES

Abstract

Natural deduction with alternatives extends Gentzen–Prawitz-style natural deduction with a single structural addition: negatively signed assumptions, called *alternatives*. It is a mildly bilateralist, single-conclusion natural deduction proof system in which the connective rules are unmodified from the usual Prawitz introduction and elimination rules — the extension is purely structural. This framework is general: it can be used for (1) classical logic, (2) relevant logic without distribution, (3) affine logic, and (4) linear logic, keeping the connective rules fixed, and varying purely structural rules.

The key result of this paper is that the two principles that introduce kinds of *irrelevance* to natural deduction proofs: (a) the rule of explosion (from a contradiction, anything follows); and (b) the structural rule of vacuous discharge; are shown to be two sides of a single coin, in the same way that they correspond to the structural rule of weakening in the sequent calculus. The paper also includes a discussion of assumption classes, and how they can play a role in treating additive connectives in substructural natural deduction.

Keywords: proof, natural deduction, classical logic, bilateralism, substructural logics.

2020 Mathematical Subject Classification: 03A05, 03F03, 03F52.

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1. Proofs and sequents

Gentzen–Prawitz-style natural deduction is an elegant way to present proofs. In this proof calculus, each connective is governed by introduction and elimination rules, and the structural features of proofs — the conditions governing propositions as such, unlike connective rules, which govern propositions of particular forms — are given by the proof's tree structure, together with rules governing discharge of assumptions [11]. To illustrate, consider the natural deduction system for intuitionistic linear logic. The simplest proof in this system is a formula standing on its own:

$$A$$

This is the *identity proof*, in which the conclusion is identical to the undischarged assumption, A. In this limiting case, the first thing that follows from the assumption of A is A itself, in zero inference steps. To keep matters simple, let's consider two connectives, the conditional ($\to$) and negation ($\neg$).

$$\dfrac{\begin{array}{c}[A]^i\\ \Pi\\ B\end{array}}{A \to B}{\to}I^i \qquad \dfrac{\begin{array}{c}\Pi\\ A \to B\end{array} \quad \begin{array}{c}\Pi'\\ A\end{array}}{B}{\to}E \qquad \dfrac{\begin{array}{c}[A]^i\\ \Pi\\ \sharp\end{array}}{\neg A}\neg I^i \qquad \dfrac{\begin{array}{c}\Pi\\ \neg A\end{array} \quad \begin{array}{c}\Pi'\\ A\end{array}}{\sharp}\neg E$$

Each inference rule builds a larger proof from smaller proofs (here marked with a Π or a Π'). (To be precise, in these statements of rules, a Π with a formula below it represents a proof *with that formula as its conclusion*. If, in addition, it has a formula above it (perhaps surrounded with brackets), it represents the proof with that formula among its assumptions.) In the elimination rules $\to E$ and $\neg E$, we form a proof by combining two proofs. For $\to E$ we combine one proof (Π) of $A \to B$ with another (Π') of A to form a proof of B. The resulting proof has, as its assumptions, all those assumptions used in Π, together with those used in Π'. For $\neg E$, we combine a proof of $\neg A$ with a proof of A. This introduces a new kind of conclusion, the symbol '$\sharp$'. This is not a formula, but is a punctuation mark,[1] indicating that the proof has reached a contradiction, because we have proved (from the assumptions granted in Π and Π') a contradictory pair of conclusions.

[1] This treatment of falsity follows Neil Tennant [20].

The mark '$\sharp$' is exploited in the rule $\neg I$ which allows us to backtrack when we have reached such a contradiction, by 'blaming' it on one of the undischarged assumptions — by discharging it and concluding its negation. A similar sort of move is made in the conditional introduction rule $\to I$. Here we prove a condition $A \to B$ by first proving B on the basis of the assumption A. We *discharge* the assumption A to conclude $A \to B$ on the basis of the remaining assumptions.

Here is an example proof, illustrating the use of all four rules:

$$\dfrac{\dfrac{\dfrac{\dfrac{p \to \neg q \qquad [p]^1}{\neg q}{\to}E \qquad\qquad [q]^2}{\sharp}\neg E}{\neg p}\neg I^1}{q \to \neg p}{\to}I^2$$

This proof represents the process of reasoning from the premise $p \to \neg q$ as follows: we assume p to derive $\neg q$. We assume q and get a contradiction. We 'blame' that contradiction on the assumption of p, discharging it, to conclude $\neg p$, and so, we have proved $\neg p$ having assumed q, so we discharge that assumption, to conclude $q \to \neg p$.

The undischarged assumption of the proof, $p \to \neg q$, stands unbracketed as a leaf of the tree, and the conclusion, $q \to \neg p$ is at the root. Each transition in the proof is governed by an introduction or elimination rule. The two introduction steps discharge one assumption: the negation introduction discharges the assumption p (tagged with a '1') while the conditional introduction discharges the q (tagged with a '2').

The system with these rules models the implication/negation fragment of intuitionistic linear logic. (See Girard's fundamental paper [3] for an introduction to linear logic, and Troelstra's *Lectures on Linear Logic* [21] for a presentation of natural deduction for intuitionistic linear logic in a *sequent* format.) It is *intuitionistic* linear logic because (as is familiar) the proof system provides no way to prove p from $\neg\neg p$. It is *linear* because each introduction step is restricted to discharge one and only one occurrence of an assumption. As a result, we cannot (for example) prove $p \to q$ from the assumption of $p \to (p \to q)$ (that would require discharging *two* copies of p) and neither can we prove $p \to q$ from the assumption p (that would require discharging *zero* copies of p).

* * *

To extend this system to stronger logics, including intuitionistic logic, we can keep these connective rules largely unchanged, by adding purely *structural* rules to the calculus, managing the assumption classes used in the discharging rules $\neg I$ and $\to I$. To extend the system first to the system of *relevant* implication, we allow for more than one assumption instance to be discharged at once, we allow for proofs like this:

$$\cfrac{\cfrac{\cfrac{p \to (p \to q) \qquad [p]^1}{p \to q}{\to}E \qquad [p]^1}{q}{\to}E}{p \to q}{\to}I$$

Proofs such as this allow for *duplicate* discharge, in which the set of discharged formula instances has size at least two.

To extend the system to *minimal* logic, we modify the discharge policy further, by allowing for for *any* number of instances of the indicated assumption to be discharged, including *zero*. With this in place, we have a very short proof of $p \to (q \to p)$.

$$\cfrac{\cfrac{[p]^1}{q \to p}{\to}I}{p \to (q \to p)}{\to}I^1$$

Here, *zero* instances of the assumption q are discharged at the first $\to E$ step. Let's say that a policy for discharging assumption classes allows for *vacuous* discharging if and only if it allows for proofs like this, where the set of discharged assumptions is empty.

Given that we can either allow or ban vacuous discharge, and allow or ban duplicate discharge, we have four different proof systems in one, given this simple set of rules. This is a natural deduction proof system for intuitionistic logic if we allow both vacuous and duplicate discharge. If we ban duplicate discharge while allowing vacuous discharge, we get *affine logic*. If we allow duplicate discharge while banning vacuous discharge, we get relevant logic, and if we ban both vacuous and duplicate discharge, we get *linear* logic.

Well, *almost*. There is one small wrinkle in this simple story. There is no vacuous discharge in the following proof, from $\neg p$ and p to q. This proof is not allowed in linear logic or in relevant logic, but it is intuitionistically acceptable:

$$\dfrac{\dfrac{\neg p \qquad p}{\sharp}\,\neg E}{q}\,\sharp E$$

This proof does not use vacuous discharge (there is no discharge at all in the $\neg E$ inference). Instead, it uses the new primitve inference rule, an elimination principle for $\sharp$:

$$\dfrac{\begin{array}{c}\Pi\\ \sharp\end{array}}{A}\,\sharp E$$

The $\sharp E$ rule is another properly structural proof principle, governing the logical power of reaching an inconisistent state, and not governing any connective in particular. To extend our proof system all the way up to intuitionistic logic, we need to add $\sharp E$ as well as allowing duplicate and vacuous discharge. Conversely, to convert intuitionistic logic into a properly *relevant* logic, we must not only ban vacuous discharge—you must also ban $\sharp E$.

* * *

It is straightforward to verify that the natural deduction system with no vacuous discharge and no duplicate discharge gives us proofs for the implication/negation fragment of intuitionistic linear logic. This logic is given by the following single-conclusion sequent system, in which sequents consist of a *multiset* of formulas on the left and either a single formula on the right, or an *empty* right hand side. (The empty right hand side plays the same sort of role in a sequent as the contradiction marker $\sharp$ does in the conclusion of a proof.) We use 'C' to range over possible inhabitants of the conclusion position, so here, 'C' is either a formula or the empty RHS, while 'A' and 'B' always stand for formulas, and 'X' and 'X'' range over arbitrary multisets of formulas. The structural rules are *Id* and *Cut*:

$$A \succ A \quad Id \qquad \dfrac{X \succ A \qquad X', A \succ C}{X, X' \succ C}\, Cut$$

The connectives are governed by the expected left and right rules.

$$\frac{X \succ A \qquad X', B \succ C}{X, X', A \to B \succ C}{\to}L \qquad \frac{X, A \succ B}{X \succ A \to B}{\to}R \qquad \frac{X \succ A}{X, \neg A \succ}\neg L \qquad \frac{X, A \succ}{X \succ \neg A}\neg R$$

Fact 1.1. There is a linear natural deduction proof from the premises X to the conclusion C if and only if there is a derivation of the sequent $X \succ C$ in the linear sequent calculus. (Following our convention concerning 'C', this means that there is a sequent derivation of $X \succ$ if and only if there is a natural deduction proof from X to $\sharp$. We allow 'C' to take the appropriate form whether occuring as a conclusion of a proof or the RHS of a sequent.)

Proof: From left to right, this can be verified by a simple induction on the construction of the proof. The base case proof is the identity proof A, which corresponds exactly to the identity sequent $A \succ A$. Now, for the induction steps, consider the ways to generate new proofs from old. For ${\to}I$, suppose we have a proof from assumptions X together with one given occurrence of the assumption A to conclusion B and we discharge that occurrence of A in a ${\to}I$ step to deduce $A \to B$. The induction hypothesis delivers us a derivation of $X, A \succ B$, which can be extended to a derivation of $X \succ A \to B$, by ${\to}R$, as desired.

For ${\to}E$, suppose we have a proof from X to $A \to B$ and another from X' to A, and we combine these into a proof from X, X' to B. The induction hypothesis delivers us derivations of $X \succ A \to B$ and $X' \succ A$. Using *Cut* and ${\to}R$ we can construct the desired derivation of $X, X' \succ B$ like this:

$$\frac{X \succ A \to B \qquad \dfrac{X' \succ A \qquad \dfrac{}{B \succ B}\mathit{Id}}{X', A \to B \succ B}{\to}L}{X, X' \succ B}\mathit{Cut}$$

The cases for the negation rules parallel the conditional rules precisely, so leaving these as an exercise, I will declare this part of the proof done.

For the right-to-left direction of the equivalence, we show how we can construct a proof from X to C, given a derivation of $X \succ C$ (whether C is a formula or $\sharp$). If our derivation is a simple appeal to *Id* ($A \succ A$) we have the atomic proof featuring the assumption A standing alone as both assumption and conclusion. For *Cut*, we paste together a proof from X to A to a proof from X', A to C to construct the combined proof from X and X' to C, going through A as an intermediate step.[2]

$$\begin{array}{c}
\quad X \\
\quad \Pi_1 \\
\underline{X' \;\; \quad A} \\
\Pi_2 \\
C
\end{array}$$

The connective rules on the left and right correspond neatly to the corresponding applications of the elimination and introduction rules. For $\to L$, suppose we already have a proof Π_1 from X to A and a proof Π_2 from X', B to C we construct a proof from $X, X', A \to B$ to C like this:

$$\begin{array}{c}
\begin{array}{cc} X' & \dfrac{A \to B \qquad \begin{array}{c} X \\ \Pi_1 \\ A \end{array}}{B}{\scriptstyle \to E} \end{array} \\
\hdashline
\Pi_2 \\
C
\end{array}$$

Similarly, given a proof from X, A to B, we can discharge that instance of A in the assumptions in one $\to I$ step to construct a proof from X to $A \to B$. The reasoning for the negation rules has the same shape, so again, we can declare the proof complete. □

So, we can see that the sequent calculus and the natural deduction system for linear implication and negation mirror each other.

To extend the sequent calculus to model relevant logic, affine logic and intuitionistic logic, we can add the structural rules of contraction (on the left) and weakening (both on the left and on the right), like so:[3]

$$\frac{X, A, A \succ C}{X, A \succ C}\, W \qquad \frac{X \succ C}{X, A \succ C}\, KL \qquad \frac{X \succ}{X \succ B}\, KR$$

---

[2]Here, the dashed line above Π_2 indicates that the subproof Π_2 has the formulas listed in X' and A together as its undischarged leaves.

[3]We use 'W' for contraction and 'K' for weakening, following the names from Combinatory Logic. Haskell Curry named the contraction combinator 'W' (for the combinator satisfying $(\mathsf{W}xy) = (xyy)$), since 'W' is reminiscent of repetition [1]; while Schönfinkel's 'K' (for the combinator satisfying $(\mathsf{K}xy) = x$) stands for '*Konstanzfunktion*' [17].

Using *contraction* (W), we can implement in the sequent calculus the behaviour of *duplicate discharge* in natural deduction. If we wish to discharge more than one instance of the assumption formula A in a $\rightarrow I$ step, then in the derivation, you may contract those copies of A in the left of the sequent down to one, with W, and then you are in a position to apply $\rightarrow R$. Using *weakening* on the left (KL), we can do the work of *vacuous* discharge in natural deduction. Wherever we would vacuously discharge an assumption formula in some inference, in the sequent calculus we insert that formula using KL to be in a position to apply the right rule, introducing a conditional or a negation.

However, once we add these structural rules, the parallel between the sequent calculus and natural deduction is less direct and straightforward than it is in the linear case. Consider the following derivation of the sequent $p \rightarrow (p \rightarrow q), p \succ q$, using contraction:

$$\dfrac{\dfrac{p \succ p \qquad \dfrac{p \succ p \qquad q \succ q}{p \rightarrow q, p \succ q}\rightarrow L}{p \rightarrow (p \rightarrow q), p, p \succ q}\rightarrow L}{p \rightarrow (p \rightarrow q), p \succ q}\,W$$

This sequent derivation in some sense 'says' that there is a proof of q from $p \rightarrow (p \rightarrow q)$ and p—from one copy of each. There indeed *is* a natural deduction proof from $p \rightarrow (p \rightarrow q)$ and p to q, but there is no such proof that simply uses two steps of $\rightarrow E$, in the way that this derivation uses two steps of $\rightarrow L$. In our natural deduction system, the job of contraction is accomplished at the points where we *discharge* assumptions, in $\rightarrow I$ and $\neg I$ inferences. Our proof which uses only one copy of p among the assumptions goes like this:

$$\dfrac{\dfrac{\dfrac{\dfrac{p \rightarrow (p \rightarrow q) \qquad [p]^1}{p \rightarrow q}\rightarrow E \qquad [p]^1}{q}\rightarrow E}{p \rightarrow q}\rightarrow I^1 \qquad p}{q}\rightarrow E$$

This proof manages to get to the conclusion q from one copy each of the premises $p \rightarrow (p \rightarrow q)$ and p, but it does so at the cost of making an initial detour, constructing $p \rightarrow q$ and immediately breaking it down again. It does more work than seems appropriate in deriving q from those premises. This is our first hint that we may not yet have the clearest understanding of the behaviour of structural rules, like weakening and contraction, in Prawitz-style natural deduction.

* * *

However, there is a more pressing issue concerning the behaviour of structural rules in natural deduction, and that is the extension of our simple natural deduction system to extend to *classical* logic, and to the classical variants of the implication/negation fragments of linear logic, relevant logic and affine logic. If we extend the sequent calculus to allow for more than one formula on the right, like this —

$$A \succ A \; \textit{Id} \qquad \frac{X \succ A, Y \qquad X', A \succ Y'}{X, X' \succ Y, Y'}\, \textit{Cut}$$

$$\frac{X \succ A, Y \qquad X', B \succ Y'}{X, Y, A \to B \succ Y, Y'} \to\! L \qquad \frac{X, A \succ B, Y}{X \succ A \to B, Y} \to\! R$$

$$\frac{X \succ A, Y}{X, \neg A \succ Y} \neg L \qquad \frac{X, A \succ Y}{X \succ \neg A, Y} \neg R$$

— it is well known that we get fully dualising behaviour from these rules. For example, we can derive *double negation elimination* as well as *introduction*. The fully left–right symmetric sequent calculus allows for this symmetric pair of derivations:

$$\frac{\dfrac{p \succ p}{p, \neg p \succ} \neg L}{p \succ \neg\neg p} \neg R \qquad \frac{\dfrac{p \succ p}{\succ \neg p, p} \neg R}{\neg\neg p \succ p} \neg L$$

Can we extend Prawitz-style natural deduction with purely structural rules, so as to do justice to derivations like these, which make use of more than one formula on the right hand side? This is one motivation for a *bilateralist* proof system, in which there is a full symmetry between premise and conclusion, between assertion and denial, and between left and right. The most direct attempts to expand natural deduction in this fully symmetric direction is to propose proof systems with *multiple conclusions* [18, 15, 13, 14], in addition to the multiple premises available in a Gentzen–Prawitz-style proof. This extension of natural deduction in a fully bilateral format is well-motivated, but to get the details correct, one must move beyond tree-like structures to graphs [18, see Parts I and II], and the correspondence with natural deduction becomes less direct.[4]

Another way to extend natural deduction in a bilateral direction is to allow for *negatively* decorated formulas (for *rejection* or *denial*), as well as *positive* formulas [19]. In modern renderings of this kind of bilateralist natural deduction, we assign every formula in a proof a *sign*, either '+' or '−', for assertion and denial respectively [4, 16]. This provides a neat way to pair full symmetry between positive and negative position, in a structure with many *premises* and a single *conclusion*. A proof from $+A$, $-B$, $+C$ to $+D$ can do duty for the sequent $A, C \succ B, D$, since it reassures us that there is no way for A and C to be *true* while B and D are *false*, or equally, it is inconsistent to *accept* A and C and *reject* B and D, or to put things in terms of speech acts, to *assert* A and C and *deny* B and D. This sequent corresponds to other proofs, too, such as a proof from $-B$, $-D$, $+C$ to $-A$. By decorating formulas with '+' or '−', we can move them between premise to conclusion position in a proof as desired. Since formulas can appear both positively and negatively signed, instead of each connective being defined by *two* rules, they have *four*: introduction and elimination rules for both positively and negatively signed occurrences. Such *fully bilateralist* natural deduction systems are interesting and powerful, but as we will see, they add to the natural deduction framework more than is strictly necessary to ford the chasm between intuitionist and classical natural deduction, and the substructural variants thereof. It is possible to be bilateralist in a much less drastic manner, and to still get all the power of classical reasoning. In the rest of this paper, we will see how.

* * *

Before introducing the structural addition to proofs that suffices for mild bilateralism, there is one more modification to natural deduction that is worth mentioning, the *Restart* rule of Michael and Murdoch Gabbay [2]. The restart rule:

[4] A simple case that shows the problem is this. If we would like a downward branching disjunction rule (from $A \vee B$ you branch to two conclusions, one A and the other B) in parallel to the upward branching conjunction rule (infer $A \wedge B$ from two premises, one A and one B), then there seems to be no way to construct a proof from $p \vee p$ to $p \wedge p$ without in some way breaking the tree shape of proofs.

$$\frac{A}{B}\,\textit{Restart}$$

is an addition to natural deduction for intuitionistic logic that is indeed sufficient to capture classical logic. Of course the rule does not apply without restrictions. A proof using the *Restart* rule is complete only when *below* every application of a *Restart* inference from A to B there is at least one further occurrence of A. Surprising as it is, natural deduction extended with this rule is indeed sound and complete for classical logic. Before explaining why, let's see a complete proof of the classical tautology $((p \to q) \to p) \to p$ using *Restart*:

$$\dfrac{\dfrac{[(p \to q) \to p]^2 \qquad \dfrac{\dfrac{[p]^1}{q}\,\textit{Restart}}{p \to q}\,{\to}I^1}{p}\,{\to}E}{((p \to q) \to p) \to p}\,{\to}I^2$$

In this proof, the *Restart* in the first inference is paid off when we return to p in the second last inference. Here is why the restart rule is sound and complete for classical logic. Suppose have a proof from premises X to a conclusion A. So, we have $X \succ A$. Then, if we *restart* to introduce B, the 'score' is now $X \succ B, A$. The A does not *go away*, as it were. We just set it aside (as an alternative conclusion) to insert another conclusion in its place. The *Restart* rule at the point of application is a kind of weakening on the right (KR). To make explicit the idea that the proof still has a single formula in the conclusion, let's represent the sequent in the form $X \succ C; Y$ where C is the formula (or $\sharp$, perhaps) in conclusion position, and Y collects together the other conclusions we have discarded along the way whenever we have applied *Restart*.

What, then, is the point of the side condition to the effect that we must return to the discarded formula A? When we return to a previously discarded conclusion, A, the score is $X \succ A; A, Y$. We declare the restart step *complete* and the formula is removed from the discard pile: so the score is then $X \succ A; Y$. This side condition, therefore, is an application of *contraction* on the right hand side of the sequent (WR). If we complete *every* restart step in a proof, the discard pile is empty, the score has the shape $X \succ A;$ —and the proof is indeed a justificiation of the conclusion on the basis of the undischarged assumptions.

The *Restart* rule is an ingenious addition to natural deduction that happens to be tailor-made for classical logic. However, the rule encodes both contraction and weakening, so it is ill-suited to substructural variants of classical logic. Furthermore, it is difficult to see how it can be motivated on explicitly *bilateralist* lines. Nonetheless, it contains the kernel of the idea of how we can make a small structural modification of natural deduction that suffices for this range of logical systems, and as we will see, this modification can be motivated by bilateralist considerations.

2. Natural deduction with alternatives

In any natural deduction proof, we have some collection X (possibly empty) of undischarged assumptions, and a concluding formula B, or a contradiction marker $\sharp$. If we wish the 'score' of our proof to encompass the whole range of sequents of the form $X \succ Y$ (as seems to be desirable, in order to match our classical systems), then if the conclusion formula is selected from the collection Y of formulas on the right hand side, we need some way to take care of the *remaining* formulas on the right, if there are any.

Let's use the notation that seemed natural when considering the restart rule, and think of the score in our proof as taking the shape $X \succ C; Y$ where C is the conclusion of the proof (whether a formula or $\sharp$), X collects together the undischarged assumptions, and Y is yet to be accounted for. The distinguished position in the right hand side of the sequent is the *focus.* At any stage of a proof, there is either a formula in the focus position (the conculding formula of the proof), or the focus is empty, in which case the proof concludes in $\sharp$. The restart rule manipulated the score by allowing us to remove a formula from the focus, and to place something else in its place (in this case, any other formula we please). If we wish to model any of our substructural logics, this is altogether too generous, since this corresponds to weakening our sequent by adding a new formula to the RHS. If we wish to move a formula out of focus, there is only one thing, in general, we can put in its place, if we wish to refrain from weakening. That is $\sharp$, or in sequent vocabulary, *nothing.*

The appropriate sequent rule to remove a formula from conclusion position has the following shape:

$$\frac{X \succ A; Y}{X \succ \ ; A, Y}\uparrow$$

Here, there is no contraction or weakening. We simply remove a formula from the focus position, and leaving nothing in its place. Formula occurrences are neither deleted (as happens in *contraction*) nor added (in *weakening*). A natural mate for the $\uparrow$ rule is its converse:

$$\frac{X \succ\ ; A, Y}{X \succ A; Y}\downarrow$$

This rule takes a formula out of the discard pile to return it to focus. Again, there is no implicit contraction or weakening involved.

Let's now consider how we can achieve the effect of these moves in a natural deduction framework. First, for the $\uparrow$ step, we move from a proof in which a given formula A is the conclusion, to a proof in which the conclusion is now $\sharp$, a contradiction. In this new proof, the formula A is now added to the discard pile, or the collection of alternative conclusions. In natural deduction proofs, one option to represent this formula A is among the leaves of the proof (the context against which the conclusion is derived), but we must find some way to distinguish this former conclusion — now set aside — from the other undischarged assumptions, also in the leaves of the proof tree. We do this with a *sign*, as with other bilateralist natural deduction systems. To emphasise the *negative* role played by these formulas, we will use a slash for the sign. (The slash through the entire formula should also bring to mind that it is not another connective, able to be composed with other connectives.) The corresponding proof step then takes this form:

$$\frac{\begin{array}{c}\Pi \\ A\end{array} \quad \cancel{A}}{\sharp}\uparrow$$

This looks rather like the $\neg E$ rule in that a contradiction is derived from A and a negative version of A. However, there are two differences. The first is obvious: negation is an embeddable, composable content of a judgement — the negation of a formula can occur inside other formulas — while the slash here is a structural feature of proofs, and cannot be so embedded. The second is more subtle, but no less important: the negation elimination rule composes *two* proofs, one for A and the other for $\neg A$, into a single refutation, a proof ending in $\sharp$. The $\uparrow$ rule, on the other hand, does not compose two proofs. There is no proof ending in $\cancel{A}$. In this proof calculus, slashed formulas will appear only in leaves, and never as the conclusion of a proof. These formulas represent the conclusions we have temporarily set aside, and are stored among the leaves. Furthermore, unlike $\neg E$ which dictates the behaviour of a specific kind of formula, the $\uparrow$ rule is purely structural, allowing for the rearrangement of information around the proof structure, independently of the particular content or shape of the formula involved.

Why is this rule labelled with ‘$\uparrow$’? When we apply it, the formula A—which was the conclusion of the proof—is lifted up from the conclusion and stored among the leaves of the proof, where it takes its place as part of the context against which the conclusion is proved. For this reason, we also call it the *store* rule, and the conclusion formulas, temporarily stored up in the leaves are also called *alternatives*, since they are alternative candidates for conclusion, temporarily set aside for the sake of the argument. The converse of the *store* rule must do the reverse. It must *retrieve* an item kept in storage, to return it to the focus of the proof, its conclusion. Here is the appropriate shape in natural deduction:

$$\begin{array}{c} [\cancel{A}]^i \\ \Pi \\ \dfrac{\sharp}{A}\,{\downarrow^i} \end{array}$$

Once we have proved a contradiction, we are in a position to select a stored formula (one instance only, in linear natural deduction) and discharging it, we return it to the conclusion. Before the retrieve step, the score was $X \succ\, ; A, Y$, and after, it is $X \succ A; Y$, when the A is retrieved from the storehouse of alternatives, to return to its place as a conclusion.

With these rules, we can mimic multiple-conclusion sequent derivations, despite the asymmetric shape of tree proofs. Here are proofs of double negation elimination, and Peirce's Law, the latter now making explicit how weakening ($\sharp E$) and contraction (duplicate discharge) play a role:

$$\begin{prooftree}
\AxiomC{$\neg\neg p$}
\AxiomC{$[p]^1$}
\AxiomC{$[\cancel{p}]^2$}
\RightLabel{$\uparrow$}
\BinaryInfC{$\sharp$}
\RightLabel{$\neg I^1$}
\UnaryInfC{$\neg p$}
\RightLabel{$\neg E$}
\BinaryInfC{$\sharp$}
\RightLabel{$\downarrow^2$}
\UnaryInfC{p}
\end{prooftree}
\qquad
\begin{prooftree}
\AxiomC{$[(p \to q) \to p]^3$}
\AxiomC{$[p]^1$}
\AxiomC{$[\cancel{p}]^2$}
\RightLabel{$\uparrow$}
\BinaryInfC{$\sharp$}
\RightLabel{$\sharp E$}
\UnaryInfC{q}
\RightLabel{$\to I^1$}
\UnaryInfC{$p \to q$}
\RightLabel{$\to E$}
\BinaryInfC{p}
\AxiomC{$[\cancel{p}]^2$}
\RightLabel{$\uparrow$}
\BinaryInfC{$\sharp$}
\RightLabel{$\downarrow^2$}
\UnaryInfC{p}
\RightLabel{$\to I^3$}
\UnaryInfC{$((p \to q) \to p) \to p$}
\end{prooftree}$$

This proof system is a purely structural extension of Prawitz-style natural deduction, changing it only with the addition of two structural rules, *store* and *retrieve*. This calculus is *bilateralist* because modifying the rules in this way allows for the *context* in which a formula is proved to have a twofold structure. A proof of A from the assumption formulas X and the alternatives Y is a proof corresponding to the sequent $X \succ A; Y$, and the intuitive interpretation is that A follows, provided that we have the means to rule X *in* and rule Y *out*.

Although this natural deduction calculus is bilateralist, it is bilateralist in a much milder manner than other bilateralist generalisations of natural deduction. We do not tag *every* formula in the proof, or add to the connective rules, and neither have we had to change the topology of proofs from the familiar tree structure. The *context* against which formulas are proved has been enlarged, but the remaining rules of the familiar natural deduction calculus are unchanged.

Although I have presented this natural deduction system as a more flexible sibling of Gabbay and Gabbay's natural deduction with restart, its origins go back further than their work. The proof system here is derived from Michel Parigot's $\lambda\mu$-calculus for classical logic [8, 9, 10]. The original contribution of this paper is twofold: first, rewriting the rules to make the connection with natural deduction and the sequent calculus more explicit, and second, formulating the store and retrieve rules so that the formulation applies equally to substructural systems of natural deduction. It is to the consideration of structural rules that we will now return, before finishing this paper with an indication of how rules for other connectives can be formulated, and a proof that the rules are indeed sound and complete for the substuctural multiple-conclusion sequent logics in question.

3. Weakening and explosion

We have already seen that adding *irrelevance* to linear natural deduction comes in two distinct ways. Vacuous discharge, and $\sharp E$.

$$\dfrac{\dfrac{[p]^1}{q \to p}{\to}I}{p \to (q \to p)}{\to}I^1 \qquad\qquad \dfrac{\dfrac{\neg p \quad p}{\sharp}\neg E}{q}\sharp E$$

These are distinct features of the natural deduction calculus. They are so distinct that we can have a proof system (for *minimal logic*) in which we have vacuous discharge without $\sharp E$. This is no longer so in the classical setting, in the presence of the store and retrieve rules. Given the retrieve rule, $\sharp E$ is no longer a separate distinct rule — it is simply the *vacuous* case of the retrieve inference. We can step from $\sharp$ to any given formula A by retrieving *zero* copies of the stored formula A. The proof from $\neg p$ and p to q now takes this form:

$$\cfrac{\cfrac{\neg p \qquad p}{\sharp}\,\neg E}{q}\downarrow$$

The natural deduction system with alternative rules *unifies* these two distinct kinds of irrelevance, by showing that they both count as forms of vacuous discharge.

The connection between $\sharp E$ and vacuous retrieval is a tight one, since if we have the store and retrieve rules with vacuous discharge of assumptions then we get the effect of $\sharp E$ whether we add vacuous discharge of *alternatives* as a primitive rule or not. Vacuous discharge comes as a package deal, in the presence of the store and retrieve rules. It is well known from minimal logic that from a contradiction we can infer an arbitary *negation*, including $\neg\neg q$ by vacuous discharge of the assumption $\neg q$, and so, using a store and retrieve two-step, we can infer the arbitrary q anyway:

$$\cfrac{\cfrac{\cfrac{\cfrac{\neg p \qquad p}{\sharp}\,\neg E}{\neg\neg q}\,\neg I \qquad \cfrac{\cfrac{[q]^1 \qquad [\cancel{q}]^2}{\sharp}\,\uparrow}{\neg q}\,\neg I^1}{\sharp}\,\neg E}{q}\downarrow^2$$

So, the store and retrieve rules of natural deduction with alternatives gives us a vantage point from which we can see the phenomena of *irrelevance* arising from one single source, the vacuous appeal to context, whether positive or negative.

4. Varieties of conjunction

Let's add conjunction to our natural deduction system. It is well known that if we use the familiar Prawitz rules $\&I$ and $\&E$, we see that we can get the effect of vacuous discharge, by laundering our unused assumption (here q through an $\&I/\&E$ two-step).

$$\dfrac{\dfrac{\dfrac{p \qquad [q]^1}{p \mathbin{\&} q}\,\&I}{p}\,\&E}{q \to p}\,{\to}I^1$$

So, if we wish to do without weakening, we should not use $\&I$ together with $\&E$. One option is to start with the rule $\&I$ and to scout around for a rule that fits neatly with it, whether contraction or weakening are present or absent. The resulting connective is a *multiplicative* conjunction, and we will write '$\otimes$' to set multiplicative conjunction apart from other conjunctions. Given the familiar introduction rule $\otimes I$, the matching elimination rule is natural:

$$\dfrac{\begin{matrix}\Pi_1 \\ A\end{matrix} \qquad \begin{matrix}\Pi_2 \\ B\end{matrix}}{A \otimes B}\,\otimes I \qquad\qquad \dfrac{\begin{matrix}\Pi_1 \\ A \otimes B\end{matrix} \qquad \begin{matrix}[A]^i, [B]^j \\ \Pi_2 \\ C\end{matrix}}{C}\,\otimes E^{i,j}$$

To eliminate a conjuction $A \otimes B$ we can derive anything we can derive from the conjuncts individually. In a linear context, we discharge one copy each of each conjunct. In the presence of contraction, we may discharge more copies. In the presence of weakening, we may discharge *zero* copies. The result is the expected behaviour of multiplicative conjunction in our systems, and we need not spend any time considering its distinctive behaviour, because in a sense, it brings nothing new to the table. Multiplicative conjunction is *definable* in terms of negation and the conditional in the way you expect: $A \otimes B$ is equivalent to $\neg(A \to \neg B)$, and the inference rules are derivable from the rules at hand. First, we can reconstruct the $\otimes$ introduction rule by combining two elimination steps with one introduction:

$$\dfrac{\begin{matrix}\Pi_1 \\ A\end{matrix} \qquad \begin{matrix}\Pi_2 \\ B\end{matrix}}{A \otimes B}\,\otimes I \qquad\qquad \dfrac{\dfrac{\dfrac{[A \to \neg B]^1 \qquad \begin{matrix}\Pi_1 \\ A\end{matrix}}{\neg B}\,{\to}E \qquad \begin{matrix}\Pi_2 \\ B\end{matrix}}{\sharp}\,\neg E}{\neg(A \to \neg B)}\,\neg I^1$$

Dually, the job of the $\otimes$ elimination rule can be performed by two introduction steps with one elimination, combined with one storage and one retrieval:

$$\dfrac{\begin{matrix}\Pi_1\\ A\otimes B\end{matrix} \qquad \begin{matrix}[A]^i,[B]^j\\ \Pi_2\\ C\end{matrix}}{C}\,{\otimes E^{i,j}} \qquad\qquad \dfrac{\dfrac{\begin{matrix}\Pi_1\\ \neg(A\to\neg B)\end{matrix} \qquad \dfrac{\dfrac{\dfrac{\begin{matrix}[A]^i,[B]^j\\ \Pi_2\\ C\end{matrix} \qquad [\cancel{C}]^k}{\sharp}\,{\uparrow}}{\neg B}\,{\neg I^j}}{A\to\neg B}\,{\to I^i}}{\sharp}\,{\neg E}}{C}\,{\downarrow^k}$$

So, adding muliplicative conjunction gives us no increase in expressive power, over and above the rules already at hand.[5]

* * *

So, what of the other kind of conjunction, the *additive* conjunction, which is found when we start with Prawitz's elimination rule? Here the elimination rules are trivial, but the corresponding introduction rule is harder to find. At the level of *sequents* the target rules are straightforward:

$$\dfrac{X, A \succ C, Y}{X, A\wedge B \succ C, Y}\,{\wedge L} \qquad \dfrac{X, B \succ C, Y}{X, A\wedge B \succ C, Y}\,{\wedge L} \qquad \dfrac{X \succ A, Y \qquad X \succ B, Y}{X \succ A\wedge B, Y}\,{\wedge R}$$

The left rules correspond to the expected elimination rules for conjunction: if we can prove something from A we could have proved it from $A\wedge B$ instead — and the same goes if we could have proved it from B. The *right*

[5] We do not have space to consider normalisation of proofs here, but indeed, the expected normalisation behaviour for $\otimes I/E$ detours follows from the normalisation rules for the other connectives, together with $\uparrow$ and $\downarrow$. Cut elimination for the linear sequent calculus is very easy to show (in the absence of contraction, each cut reduction shrinks a derivation), and cut elimination for the extensions with contraction or weakening follows from standard techniques [7, 12]. Parigot shows strong normalisation for his classical natural deduction calculus (which differs slightly in structural rules from the calculus presented here, in ways that make no difference in the presence of contraction and weakening), and a close analysis of the reduction steps in Parigot's argument can apply in the four natural deduction systems presented here [8, 10]. However, a detailed consideration of normalisation must wait for another occasion.

rule, on the other hand, is hard to model in natural deduction. The intended behaviour is that if we can prove A and prove B *from the same context of assumptions* (whether positive or negative), then we can prove the conjunction $A \wedge B$ from that same context of assumptions. This is hard to model in the usual tree structure of natural deduction proofs. Consider the usual introduction rule:

$$\dfrac{\begin{matrix}\Pi_1 \\ A\end{matrix} \qquad \begin{matrix}\Pi_2 \\ B\end{matrix}}{A \wedge B}\wedge I$$

Here, the rule *combines* the assumptions from from Π_1 and Π_2 into the larger collection of assumptions for the new proof. This does not have the desired effect, in the linear context.

One option, explored by Ernst Zimmermann [23], is to constrain $\wedge I$ in such a way as to require that the contexts in Π_1 and Π_2 are identical, but to then choose one side to *discharge* all assumptions in the context at the application of $\wedge I$:

$$\dfrac{\begin{matrix}X \\ \Pi_1 \\ A\end{matrix} \qquad \begin{matrix}[X]^i \\ \Pi_2 \\ B\end{matrix}}{A \wedge B}\wedge I^i$$

A rule of this form certainly has the desired shape: if we can prove A and prove B from the same context, then the result will be a proof of $A \wedge B$ from the very same context. However, the rule has one structural shortcoming, and this is that proofs no longer *compose*. That is, the following two proofs are acceptable:

$$\dfrac{p \wedge q}{p}\wedge E \qquad \dfrac{p \qquad [p]^1}{p \wedge p}\wedge I^1$$

However, we cannot compose these two proofs to form a proof from $p \wedge q$ to $p \wedge p$.

$$\dfrac{\dfrac{p \wedge q}{p}\wedge E \qquad [p]^1}{p \wedge p}\cancel{\wedge I^1}$$

This is not a proof, since the conjunction introduction rule is no longer a correct application in context, since the proofs of p no longer come from the same context.[6] So, while Zimmermann's discharging rule for additive conjunction is ingenious, I will set it aside for another option.

* * *

It will help to return to the discussion of structural rules from the first section, and to pay closer attention to the behaviour of assumptions in natural deduction proofs. An assumption class is a collection of occurrences of assumptions (of the same formula) in a proof, which are discharged together in one inference steps [5]. In our linear natural deduction system for $\to$ and $\neg$, assumption classes are always single formula occurrences. In the presence of multiple discharge, we allow for larger assumption classes, and in the presence of vacuous discharge, we allow for assumption classes to be empty. In proofs, we indicate assumption classes, where necessary, by superscript numerals. To treat additive conjunction — and to give a more detailed analysis of the behaviour of the structural rules — we will more closely examine this behaviour, by splitting the treatment of multiple discharge into two distinct phases. The first is the merging of two assumption classes into one, and the second is the discharge of that single assumption class. In this way, we will have the intermediate phase of the single assumption class occurring undischarged at two places in the proof. Since we indicate discharge with a notation with two components (the brackets and the superscript), we will use one component (the superscript) to indicate the assumption class, and the other (the brackets) to indicate discharge. With this notation in mind, consider the following two proofs, which differ only in one respect:

$$\dfrac{\dfrac{p \to (p \to q) \qquad p}{p \to q}{\to}E \qquad p}{q}{\to}E \qquad\qquad \dfrac{\dfrac{p \to (p \to q) \qquad p^1}{p \to q}{\to}E \qquad p^1}{q}{\to}E$$

In the first, the two occurrences of p occur in different assumption classes. In the second, the two occurrences are members of the same assumption class. In this second proof, the one act of assumption (an assumption that p) has been used *twice* in two separate $\to E$ inferences. In

[6] Notice that the corresponding sequent derivation with a *Cut*, composing the derivation of $p \wedge q \succ p$ with that of $p \succ p \wedge p$ is unproblematic.

$$\dfrac{\dfrac{p \succ p}{p \wedge q \succ p}\wedge L \qquad \dfrac{p \succ p \qquad p \succ p}{p \succ p \wedge p}\wedge R}{p \wedge q \succ p \wedge p}\,Cut$$

the first proof, the two assumptions p may have been assumed in two separate acts, or they may be justified by two separate processes.[7] The first proof is for the sequent $p \to (p \to q), p, p \succ q$, while the second is for $p \to (p \to q), p \succ q$, since there is only a single assumption class for p in use. The lacuna mentioned in Section 1 concerning contraction and natural deduction is now dealt with in a new way, using assumption classes.

One key feature of this treatment of assumption classes is their interaction with proof composition. If I compose my proof from $p \to (p \to q), p$ to q with another proof, say, from $p \wedge r$ to p, the composition should be a proof from $p \to (p \to q), p \wedge r$ to q, since we replace the assumption of p by the proof of p from the new assumption $p \wedge r$. Writing out the whole proof, we get this:

$$\dfrac{\dfrac{p \to (p \to q) \qquad \dfrac{p \wedge r^1}{p}\,{\wedge}E}{p \to q}\,{\to}E \qquad \dfrac{p \wedge r^1}{p}\,{\wedge}E}{q}\,{\to}E$$

Here, the tree format requires that we insert the new proof at two places, and now the two occurrences of the new assumption $p \wedge r$ come from a single assumption class. In this way, we can compose proofs naturally, and without restriction.

With the addition of our explicit treatment of assumption classes, we need to revisit the formulation of each of our rules. The introduction rules $\to I$, $\neg I$ and the retrieve rule $\downarrow$ make use of assumption classes directly. In each case, each formula occurrence (unslashed assumptions in the case of the introduction rules, and slashed alternatives in the case of the retrieve rule) in a single assumption class is discharged, while the remaining assumption classes in the proof are undisturbed.

$$\dfrac{\begin{array}{c}[A]^i \\ \Pi \\ B\end{array}}{A \to B}\,{\to}I^i \qquad \dfrac{\begin{array}{c}[A]^j \\ \Pi \\ \sharp\end{array}}{\neg A}\,\neg I^j \qquad \dfrac{\begin{array}{c}[\cancel{A}]^k \\ \Pi \\ \sharp\end{array}}{A}\,{\downarrow}^k$$

[7] In a *type* theory, in which all formulas are types of terms, the difference is recorded by the identity or difference of variables used in assumptions. In the first proof, the formula p types two distinct variables, while in the latter, it types one variable, occurring twice in the proof.

In addition, in the $\to E$ rule or a $\neg E$, in which two proofs are combined into one, if *contraction* is not in use, the assumption classes in the context of both proofs are kept separate. For example, a proof from X and Y to $A \to B$, combined with a proof from X' and Y' to A, using a $\to E$ step, gives us a proof from X, X' and Y, Y' to B. The assumption classes are not combined. If we are allowing contraction, in our proof we allow for some merging of assumption classes, as we have seen in the proof from $p \to (p \to q)$, p to q, in which two assumption occurrences of p are merged into the one assumption class. To represent this operation on assumption classes, let us use $\mathcal{C}$ and $\mathcal{C}'$ as natural deduction contexts (of assumptions and alternatives, grouped into classes). These inference steps take the form:

$$\frac{\begin{matrix}\mathcal{C}\\ \Pi\\ A \to B\end{matrix} \qquad \begin{matrix}\mathcal{C}'\\ \Pi'\\ A\end{matrix}}{B}\,{\to}E \qquad\qquad \frac{\begin{matrix}\mathcal{C}\\ \Pi\\ \neg A\end{matrix} \qquad \begin{matrix}\mathcal{C}'\\ \Pi'\\ A\end{matrix}}{\sharp}\,\neg E$$

Here, the context of the whole proof has the form $\mathcal{C} + \mathcal{C}'$ where this is the *disjoint* union of context classes in the case of linear natural deduction. If *contraction* is allowed, the requirement that this be a *disjoint* union is dropped: an *arbitrary* union is allowed.

With this treatment of assumption classes in hand, we can return to the additive conjunction rules. The rules take this format:

$$\frac{\begin{matrix}\mathcal{C}\\ \Pi_1\\ A\end{matrix} \qquad \begin{matrix}\mathcal{C}\\ \Pi_2\\ B\end{matrix}}{A \wedge B}\,{\wedge}I \qquad \frac{\begin{matrix}\Pi\\ A \wedge B\end{matrix}}{A}\,{\wedge}E \quad \frac{\begin{matrix}\Pi\\ A \wedge B\end{matrix}}{B}\,{\wedge}E$$

where the condition in the introduction rule is that the assumption classes in Π_1 and Π_2 are *identical*, and after the $\wedge I$ step, the assumption classes are *combined*, so that the assumption class for the whole proof remains $\mathcal{C}$. (Rules for additive connectives in substructural natural deduction of this form are given by Sara Negri [6], but the discussion of the behaviour in terms of assumption classes and distinguishing the phases of *identification* and *discharge* is original to this paper.)

Here is an example proof, from the premise $(p \to q) \land (p \to r)$ to the conclusion $p \to (q \land r)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{(p \to q) \land (p \to r)^1}{p \to q}{\land E} \quad [p]^2}{q}{\to E} \quad \dfrac{\dfrac{(p \to q) \land (p \to r)^1}{p \to r}{\land E} \quad [p]^2}{r}{\to E}}{q \land r}{\land I}}{p \to (q \land r)}{\to I^2}$$

In this proof, at the $\land I$ step, we have two subproofs, each from $(p \to q) \land (p \to r)$ and p, and the assumption classes of both of these subproofs are combined, using the labels 1 and 2. So the rule is appropriately applied, and in addition, we discharge the single assumption class for p to derive $p \to (q \land r)$ in the last inference step.

We have considered how assumption classes can be combined in the presence of contraction. It remains to consider the role of weakening. In the simple natural deduction proof from p to $q \to p$, with one $\to I$ inference, *zero* instances of q are discharged. This means that in proofs with weakening, we must allow assumption classes to be *empty*. Once assumption classes can be empty, there will be many more ways for different proofs to come from the same context. Consider the following sequent derivation, using *weakening*, to derive $p, q \succ p \land q$.

$$\dfrac{\dfrac{p \succ p}{p, q \succ p}{KL} \quad \dfrac{q \succ q}{p, q \succ q}{KL}}{p, q \succ p \land q}{\land R}$$

What proof might correspond to this sequent derivation? The proof we might expect should have the shape

$$\dfrac{p \quad q}{p \land q}{\land I}$$

but for this to be a correct proof, we must understand the sense in which the two subproofs (the atomic proofs of p and of q) have the same context. In the presence of weakening, the atomic proof of p is indeed a proof of p from that occurrence of p, but it is also a proof of p from p, q, where the assumption class for q is *empty*, while the assumption class for p has one inhabitant. In the presence of weakening, a proof is not only a proof from a single context $\mathcal{C}$, but also any *extension* of $\mathcal{C}$ by any finite number of empty

$$\frac{\begin{array}{c}[A]^i\\ \Pi\\ B\end{array}}{A \to B}\,{\to}I^i \qquad \frac{\begin{array}{c}\mathcal{C}\\ \Pi\\ A \to B\end{array} \quad \begin{array}{c}\mathcal{C}'\\ \Pi'\\ A\end{array}}{B}\,{\to}E \qquad \frac{\begin{array}{c}[A]^j\\ \Pi\\ \sharp\end{array}}{\neg A}\,\neg I^j \qquad \frac{\begin{array}{c}\mathcal{C}\\ \Pi\\ \neg A\end{array} \quad \begin{array}{c}\mathcal{C}'\\ \Pi'\\ A\end{array}}{\sharp}\,\neg E$$

$$\frac{\begin{array}{c}\mathcal{C}\\ \Pi\\ A\end{array} \quad \cancel{A}}{\sharp}\,\uparrow \qquad \frac{\begin{array}{c}[\cancel{A}]^k\\ \Pi\\ \sharp\end{array}}{A}\,\downarrow^k \qquad \frac{\begin{array}{c}\mathcal{C}\\ \Pi_1\\ A\end{array} \quad \begin{array}{c}\mathcal{C}\\ \Pi_2\\ B\end{array}}{A \wedge B}\,\wedge I \qquad \frac{\begin{array}{c}\mathcal{C}\\ \Pi\\ A \wedge B\end{array}}{A}\,\wedge E \qquad \frac{\begin{array}{c}\mathcal{C}\\ \Pi\\ A \wedge B\end{array}}{B}\,\wedge E$$

Figure 1. The Natural Deduction Rules

assumption classes. In this way, an atomic proof p corresponds not only to the sequent $p \succ p$, but $p, q \succ p$ (adding the empty positive assumption class q), $p \succ p, r$ (adding the empty class of occurrences of $\cancel{r}$), and any other sequent of the form $X, p \succ p, Y$ for finite X and Y. The effect of this condition in natural deduction proofs is twofold: first, in the discharging inferences ${\to}I$, $\neg I$ and $\downarrow$, in which an empty class of occurrencs may be discharged, as expected. Second, as we have seen in the above example, it may also play a role in the $\wedge I$ rule, which can apply even when the *non-empty* assumption classes occurring in the proofs of A and of B are not identical, since we can add extra *empty* assumption classes to either proof, until the contexts match.

Figure 1 compiles the rules for our natural deduction system. These rules can be read in four different ways, depending on the presence or absence of contraction and weakening.

- If contraction is *absent*, the contexts $\mathcal{C}$ and $\mathcal{C}'$ in the ${\to}E$ and $\neg E$ rules are required to be *disjoint*. If contraction is *present*, at each ${\to}E$ and $\neg E$ step, some assumption classes are permitted to be *merged*.

- If weakening is *absent*, the assumption classes in discharge rules are non-empty, and each proof has a *unique* context, of non-empty classes appearing in the leaves of the proof. If weakening is *present*, each proof has not only a minimal context $\mathcal{C}$ of formula occurrences present in the leaves, but is also a proof from any *wider* context $\mathcal{C}'$ with empty assumption classes added. As a result, any two proofs can be combined in a $\wedge I$ inference, by the addition of empty assumption classes to each side, to ensure that the assumption classes match.

5. Soundness and completeness

It remains to show that this system of natural deduction with alternatives corresponds tightly with the traditional sequent calculi, and it is to this result that we turn. For clarity, we will split this result into two cases. First, for the *linear* calculus, and then we will end with the result for calculi with structural rules.

FACT 5.1. There is a linear natural deduction proof with alternatives, from the premises X and alternatives Y to the conclusion C if and only if there is a derivation of the sequent $X \succ C, Y$ in the classical linear sequent calculus.

FACT 5.2. There is a natural deduction proof with alternatives (a) *using duplicate discharge*, or (b) *using vacuous discharge* from the premises X and alternatives Y to the conclusion C if and only if there is a derivation of the sequent $X \succ C, Y$ in the classical linear sequent calculus with the addition of (a) *contraction*, or (b) *weakening*.

To verify both of these facts, it is useful to draw out a simple lemma, which has the effect that we treat natural deduction proofs as representing sequents of the form $X \succ Y$, in which we disregard which formula is in focus, since focus can be moved freely.

LEMMA 5.3 (*Focus Shift*). *There is a proof from assumptions X and alternatives Y to conclusion A iff there is a proof from assumptions X and alternatives A, Y to conclusion $\sharp$. (The second proof uses vacuous discharge or duplicate discharge if and only if the first proof does.) Similarly, there is a proof from X and A, Y to B iff there is a proof from X and B, Y to A.*

PROOF: This is an immediate application of the store and retrieve rules. Any proof from X and Y to A, extended with one $\uparrow$ step is a proof from X and A, Y to $\sharp$. Conversely, any proof from X and A, Y to $\sharp$, extended with one $\downarrow$ step, is a proof from X and Y to A. If we have a proof from X and

A, Y to B, on one $\uparrow$ step this is a proof from X and A, B, Y to $\sharp$, which in one $\downarrow$ step is a proof from X and B, Y to A. $\square$

With the Focus Shift Lemma at hand, we can complete the proof of Fact 5.1. This proof follows the structure of the proof of Fact 1.1 (see page 114) directly, except we allow for the presence of alternatives (on the proof side) and sequents with more than one formula on the right (on the derivation side) and we add cases for the new rules in each system.

Proof: The left-to-right direction is an induction on the construction of the proof from X and Y to C. The base case is unchanged from our earlier reasoning: a proof of A corresponds to the identity derivation $A \succ A$. For the induction steps, we suppose we are generating a new proof, by some inference step, from proofs for which the induction hypothesis holds. For the connective rules for the conditional and negation, the argument is exactly the same as in our earlier reasoning, except we have to verify that the derivation steps corresponding to natural deduction inferences are correct in the presence of proofs with alternatives. Consider the case for $\rightarrow E$. This step is applied in a natural deduction proof when we have a proof from X and Y to $A \rightarrow B$ and a proof from X' and Y' to A, which we combine, to produce a proof from X, X' and Y, Y' to B. The induction hypothesis ensures we have derivations of $X \succ A \rightarrow B, Y$ and $X' \succ A, Y'$. Using *Cut* and $\rightarrow L$ we can construct the desired derivation of $X, X' \succ B, Y, Y'$ like this:

$$\dfrac{X \succ A \rightarrow B, Y \qquad \dfrac{X' \succ A, Y' \qquad \dfrac{}{B \succ B}\,\mathit{Id}}{X', A \rightarrow B \succ B, Y'}\,{\rightarrow}L}{X, X' \succ B, Y, Y'}\,\mathit{Cut}$$

The cases for the other rules for the conditional and negation follow in just the same manner as this, making the obvious changes to allow for sequents with a more general RHS.

Next, consider the rules for additive conjunction. If we extend our proof with a $\wedge E$ step, we extend a proof from X, Y to $A \wedge B$ to a proof from the same context to the conclusion A (or B). The induction hypothesis ensures that we have a derivation of $X \succ A \wedge B, Y$. This can be extended to derivations of $X \succ A, Y$ and $X \succ B, Y$ straightforwardly:

$$\dfrac{X \succ A \wedge B, Y \qquad \dfrac{\dfrac{}{A \succ A}\,Id}{A \wedge B \succ A}\,{\wedge}L}{X \succ A, Y}\,Cut \qquad \dfrac{X \succ A \wedge B, Y \qquad \dfrac{\dfrac{}{B \succ B}\,Id}{A \wedge B \succ B}\,{\wedge}L}{X \succ B, Y}\,Cut$$

If our proof ends in a $\wedge I$ inference, with conclusion $A \wedge B$, from context X, Y, then we have two proofs, one to A and the other, to B, from the same context X, Y. This means we have two derivations, one of $X \succ A, Y$, and the other, of $X \succ B, Y$. They can be extended like this

$$\dfrac{X \succ A, Y \qquad X \succ B, Y}{X \succ A \wedge B, Y}\,{\wedge}R$$

to give us the derivation we need. So, we have completed the cases for the connective rules for the left-to-right part of our fact. It remains to consider the structural *store* and *retrieve* rules. If our proof ends in a *store* ($\uparrow$) step, we convert a proof from X and Y to A to a proof from X and A, Y to $\sharp$. The induction hypothesis delivers us a derivation of $X \succ A, Y$, and we want a derivation corresponding to our new proof from X and A, Y to $\sharp$, which is *also* a derivation from $X \succ A, Y$, so the *store* rule is inert at the level of sequent derivations without focus. (This is one lesson of the *focus shift* lemma.) So, too, is the *retrieve* ($\downarrow$) rule, which simply reverses the effect of a *store* step. So, with this noted, we complete the proof of the left-to-right direction of our fact.

For the right-to-left direction of the equivalence, we show how we can construct a proof from context X, Y to C, given a derivation of $X \succ C, Y$ (whether C is a formula or $\sharp$). As before, if our derivation is a simple appeal to *Id* ($A \succ A$) we have the atomic proof featuring the assumption A standing alone as both assumption and conclusion. Or, given that $A \succ A$ is a derivation corresponding to a proof of $\sharp$ from the context A and A, this derivation *also* corresponds to the proof

$$\dfrac{A \qquad \cancel{A}}{\sharp}\,{\uparrow}$$

consisting of a single *store* inference. Notice that this proof is found by a simple modification of the original identity proof of A. We could, here, appeal to the *focus shift* lemma instead, rather than explicitly constructing every focus variant of our first proof.

For the other structural rule, *Cut*, we have derivations of $X \succ A, Y$ and $X', A \succ Y'$. By the induction hypothesis, we have a proof from context X and Y to A, and a proof from context X', A and Y' to $\sharp$.[8] We paste these proofs together to construct the combined proof from X, X' and Y, Y' to $\sharp$, going through A as an intermediate step, just as we did in the proof of Fact 1.1.

$$\begin{array}{ccc} & & X \ \cancel{Y} \\ & & \Pi_1 \\ X' \ \cancel{Y'} & & A \\ & \Pi_2 & \\ & \sharp & \end{array}$$

For the proofs corresponding to the remaining focusings of the sequent $X, X' \succ Y, Y'$, we appeal to the focus shift lemma.

As before, the connective rules on the left and right correspond neatly to the corresponding applications of the elimination and introduction rules. For $\to L$, suppose we already have a proof Π_1 from X and Y to A and a proof Π_2 from X', B and Y' to C. We construct a proof from $X, X', A \to B$ and Y, Y' to C like this:

$$\begin{array}{ccc} & & X \ \cancel{Y} \\ & & \Pi_1 \\ & \dfrac{A \to B \qquad A}{B}{\scriptstyle \to E} & \\ X' \ \cancel{Y'} & & \\ & \Pi_2 & \\ & C & \end{array}$$

(Again, if we wished to construct a proof of a different conclusion, shifting the focus, we appeal to the focus shift lemma.) Similarly, given a proof from X, A and Y to B, we can discharge that assumption class of instances A in one $\to I$ step to construct a proof from X to $A \to B$. The reasoning for the negation rules has the same shape, so it remains only to consider the additive conjunction rules. For $\wedge L$, we have a derivation of $X, A \succ Y$ (and so, a proof of $\sharp$ from X, A and Y), and we extend this to a derivation of $X, A \wedge B \succ Y$. So, we want a proof of $\sharp$ from $X, A \wedge B$ to Y. This is trivial, since we can extend our proof by replacing every instance A in the

[8] We could pick out a given formula from the family Y' of alternatives, if Y' is non-empty, but allowing the focus to remain on $\sharp$ is the general case, so we use this case here.

indicated assumption class by $\wedge E$ inference from $A \wedge B$ to A, being careful to merge each assumption $A \wedge B$ into one assumption class. The result is a proof from $X, A \wedge B$ and Y to $\sharp$, as desired:

$$\begin{array}{c} X \quad \cancel{Y} \qquad \dfrac{A \wedge B}{A}\wedge E \\ \text{- - - - - - - - - - -} \\ \Pi \\ \sharp \end{array}$$

The same goes for a derivation from $X, B \succ Y$ to $X, A \wedge B \succ Y$, using the $\wedge E$ step from $A \wedge B$ to B. (The focus shift lemma deals with the proofs corresponding to different selections of the conclusion from the context.)

Our final case is the conjunction *right* rule, for which we have derivations of $X \succ A, Y$ and of $X \succ B, Y$, which we extend into a derivation of $X \succ A \wedge B, Y$. By hypothesis, we have proofs of A from X and Y and of B from the same context, X and Y. So, we can extend these in one $\wedge I$ step, in which we *identify* the assumption classes, pairing each assumption class from the context of the proof of A with exactly one assumption class from the context of the proof of B. The result is a proof of $A \wedge B$ from exactly the same context X and Y as desired, and we can declare our proof complete, modulo another appeal to the focus shift lemma. □

The only remaining item is to prove Fact 5.2, which requires attention to the conditions for *contraction* and *weakening* in proofs and in sequent derivations.

PROOF: We extend the reasoning of the previous proof, first by considering what additions we need to make to account for *contraction*, and then, for *weakening*. First, let's consider *contraction*. For the left-to-right direction, we wish to constuct a sequent derivation (perhaps using the contraction rule) of $X \succ A, Y$ from a natural deduction proof of A from the context X and Y in which we allow for the merging of assumption classes in the inferences $\to E$ and $\neg E$. The reasoning for atomic proofs is the same as before, since no contraction can take place with only one formula in the context. Take a proof ending in a $\to E$ step in which some classes are merged. We have a proof from X and Y to $A \to B$ and another, from X' and Y' to A, and by induction hypothesis, we have a derivation of $X \succ A \to B, Y$ and of $X' \succ A, Y'$. As in the proof of the previous fact, we have a derivation of $X, X' \succ B, Y, Y'$, by way of a $\to L$ inference and a

Cut. The context X, X' and Y, Y' is too large, because this is the disjoint combination of the two contexts. The application of some contraction steps is enough to pare down the context so there is a member of the multiset on the LHS and that on the RHS for each assumption class in the proof. This is the only change required to produce a sequent derivation using contraction, and we can declare the left-to-right direction of this part of our proof complete.

For the right-to-left case, we show that from any derivation of $X \succ Y$ we can construct a proof of $\sharp$ from the context X and Y, as well as any focus shift of that proof. Notice that contraction steps can occur at *any* point of a derivation, not only at the steps immediately before $\to E$ and $\neg E$ inferences. To take account of that, we prove a more general fact, that from any derivation of $X \succ Y$ we can construct a proof of $\sharp$ from the context X and Y as well as any *contraction* of that context (in which assumption classes are merged), as well as any focus shift of such a proof. The base case, corresponding to the sequent $A \succ A$ corresponds to the atomic proof of A and the proof of $\sharp$ from $A, \cancel{A}$, neither of which may be contracted.

For the other structural rule, *Cut*, we have derivations of $X \succ A, Y$ and $X', A \succ Y'$. By the induction hypothesis, we have a proof from context X and Y to A (and of any contraction X^* and Y^* of that context), and a proof from context X', A and Y' to $\sharp$ (and from any contraction $X^{*\prime}, A$ and $Y^{*\prime}$ of that context). We paste these proofs together to construct the combined proof from $X^*, X^{*\prime}$ and $Y^*, Y^{*\prime}$ to $\sharp$, going through A as an intermediate step, just as we did in the proof of Fact 5.1.

$$
\dfrac{X^{*\prime}\ \cancel{Y^{*\prime}} \qquad \dfrac{X^*\ \cancel{Y^*}}{\begin{array}{c}\Pi_1\\ A\end{array}}}{\begin{array}{c}\Pi_2\\ \sharp\end{array}}
$$

Perhaps the new context $X^*, X^{*\prime}$ and $Y^*, Y^{*\prime}$ may be contracted further. If so, there is a point in the proof (either in an inference in Π_1 or an inference in Π_2) where the two distinct assumption classes to be contracted first enter the proof. This must be in either a $\to E$ step or a $\neg E$ step, because in the other inference steps, we do not join proofs with different assumption classes. At this inference, then, we can contract the desired assumption classes, to ensure that in the whole proof we have contracted the context to the desired extent. The reasoning in the *Cut* rule can apply to the other steps in a derivation where different contexts are combined. These rules are $\rightarrow L$ and $\neg L$, and the corresponding proofs have $\rightarrow E$ and $\neg E$ steps, at which we can contract the corresponding assumption classes, as desired. With this modification, contractions in our derivations can be dealt with directly. If our derivation moves to $X, A \succ Y$ from $X, A, A \succ Y$, the induction hypothesis ensures that we have a proof of $\sharp$ from X, A, A and Y and any contraction of this context. This means it is immediate that we have a proof of $\sharp$ from X, A and Y and any contraction of *this* context, too. The same reasoning applies to contraction on the left, and we can declare the right-to-left case for contraction complete.

Now consider proofs with the *weakening* conditions in force. To confirm the left-to-right direction of our fact, we wish to construct, for any C from context X and Y, a derivation of $X \succ C, Y$. The atomic case of a proof consisting of the lone assumption A now counts as a proof of A from the context X, A and Y for any finite X and Y. We have a derivation of $X, A \succ A, Y$ in our sequent calculus by applying weakening on the left and the right the appropriate number of times from the identity sequent $A \succ A$. With the atomic case dealt with, the remaining proof steps are straightforward. The only modifications needed for our earlier argument (whether the linear calclulus, or the calculus with contraction) is to note that we allow for discharging of empty assumption classes in the $\rightarrow I$, $\neg I$ and $\downarrow$ inferences. So for the connective rules, at the corresponding $\rightarrow R$, and $\neg R$ steps in the sequent calculus we must weaken in the vacuously discharged formula before applying the rule. For the structural rule $\downarrow$, a vacuous application corresponds in the sequent calculus to an explicit step of weakening on the right. Finally, consider the $\wedge I$ inference. Suppose we have a proof of A from the context $\mathcal{C}$ and a proof of B from the same context, with the weakening conditions in play, and we extend this proof to conclude $A \wedge B$ from the same context. This means we have some derivation of a sequent $X \succ A, Y$ and another derivation of a sequent $X' \succ B, Y'$ where the contexts X, Y and X', Y' are the assumption classes explicitly appearing in the proof. However, we add new empty assumption classes to both contexts, sufficient to allow the contexts to match. That is, we have the wider context $\mathcal{C} = X'', Y''$ where X'' subsumes both X and X', and similarly, Y'' subsumes Y and Y'. By hypothesis, we have derivations for $X'' \succ A, Y''$ and $X'' \succ B, Y''$, and so, by $\wedge R$ this may be extended to a derivation for $X'' \succ A \wedge B, Y''$ as desired. The reasoning for the other rules works in the same way.

For the right-to-left reasoning, we wish to show that for any derivation of a sequent $X \succ Y$ (using the structural rule of weakening) we have a natural deduction proof (using the weakening conditions) of $\sharp$ from the context X and Y, as well as any focus shift of that proof. Here, the proof is quick because we have defined natural deduction proofs with weakening in such a way that if we have a derivation of some conclusion from the context X, Y it counts as a proof from any weakened context, too. So, any appeal to the structural rule of weakening in the derivation is *inert* at the level of the natural deduction proof. (The atomic proof A counts as a proof from A to A as well as a proof from A, B to A in which the B is unused.) It is straightforward to check that the process for defining a natural deduction proof from a sequent derivation will — if we simply do not attempt to translate the appeals to weakening into the application of any particular rule — generate a natural deduction proof in which the weakening conditions are applied, and with that, we can declare this result proved. $\square$

So, with this result established, we can see that with the shift from a *unilateral* context X (of things *positively* granted) to a *bilateral* context X and Y (where some things have been ruled *in* and others ruled *out*) we have a simple extension of Gentzen–Prawitz-style natural deduction, sufficient to give an account not only of classical proof, but of proof in classical flavours of linear, relevant and affine logic, too.

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CORE TYPE THEORY

Abstract

Neil Tennant's core logic is a type of bilateralist natural deduction system based on proofs and refutations. We present a proof system for propositional core logic, explain its connections to bilateralism, and explore the possibility of using it as a type theory, in the same kind of way intuitionistic logic is often used as a type theory. Our proof system is not Tennant's own, but it is very closely related. The difference matters for our purposes, and we discuss this. We then turn to the question of strong normalization, showing that although Tennant's proof system for core logic is not strongly normalizing, our modified system is.

Keywords: core logic, type theory, strong normalization.

2020 Mathematical Subject Classification: 03A05, 03B38, 03B47, 03F05.

1. Introduction

Neil Tennant's core logic is a type of bilateralist natural deduction system based on proofs and refutations. We present a proof system for propositional core logic, explain its connections to bilateralism, and explore the possibility of using it as a type theory, in the same kind of way intuitionistic logic is often used as a type theory. Our proof system is not Tennant's own, but it is very closely related. The difference matters for our purposes, and we discuss this. We then turn to the question of strong normalization, showing that although Tennant's proof system for core logic is not strongly normalizing, our modified system is.

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2. Core logic

We open by presenting a natural deduction system for core logic. This is not Tennant's own system, although it is closely related. (As the paper progresses, we'll get more and more perspective on the differences; we discuss them in Sections 2.4, 3.5 and 5.1.) The language is an ordinary propositional language with connectives $\wedge, \vee, \rightarrow, \neg$ of arities 2, 2, 2, 1, respectively. We use $p, q, r, \ldots$ for atomic formulas and $\varphi, \psi, \theta, \ldots$ for arbitrary formulas. We suppress parentheses according to the following conventions: the connectives $\wedge$ and $\vee$ bind more tightly than $\rightarrow$, and $\neg$ more tightly still; and $\rightarrow$ associates to the right. Thus $\neg p \wedge q \rightarrow r \vee s \rightarrow t$ is $((\neg p) \wedge q) \rightarrow ((r \vee s) \rightarrow t)$.

2.1. Natural deduction

We first present core logic via a natural deduction system, following presentations such as [15, 21, 22]. This proceeds in the style of [5, 12], with an important modification: not every node in a derivation needs to be a formula. There is one additional symbol ☹ that can also occupy nodes in a derivation. It is important to keep in mind, though, that ☹ is *not* a formula, and does not enter into formula construction. As a result, things like '¬☹' and '☹ $\wedge\, p$' make no sense.[1]

We will call the things that can stand at nodes of a derivation *hats* (for reasons that will emerge). That is, a hat is either a formula or else ☹. Recall that we use $\varphi, \psi, \theta, \ldots$ for arbitrary *formulas*; for arbitrary *hats*, we use $\mathfrak{C}, \mathfrak{D}$. There is an important partial order on hats: $\mathfrak{C} \leq \mathfrak{D}$ iff either $\mathfrak{C}$ is ☹ or $\mathfrak{C} = \mathfrak{D}$. That is, any two distinct formulas are $\leq$-incomparable, and ☹ is $\leq$-below all formulas. We will also use the maximum $\max(\mathfrak{C}, \mathfrak{D})$ of two hats $\mathfrak{C}, \mathfrak{D}$ according to this order; note that this is only defined when either $\mathfrak{C} = \mathfrak{D}$ or one of $\mathfrak{C}, \mathfrak{D}$ is ☹. A *sequent*, as we use the term, is a set of premise *formulas* and a conclusion *hat*; we write $\Gamma \succ \mathfrak{C}$ for the sequent with premises Γ and conclusion $\mathfrak{C}$. We draw a distinction between sequents and arguments: an *argument* is a sequent with a formula as its conclusion.

The role of ☹ in these systems is not to carry content, the way a formula might. Rather, when it occurs in a derivation, it should be seen as part of the structure of that derivation, the surrounds that the content-bearing

[1] Tennant uses the symbol $\perp$ for this purpose; we use ☹ instead because $\perp$ is in common use in other work as a formula. To reduce potential confusion, we've chosen a symbol that is not usually used as a formula.

formulas fit into. It plays, then, the same kind of role in a derivation as the horizontal bar separating nodes from each other, or the rule labels decorating such bars, or markers of which assumptions are discharged; it indicates (in concert with other such apparatus) relations between the formulas in play.

Assumptions work as usual in these natural deduction systems, and in particular only *formulas* may be assumed. Any derivation, then, has a set Γ of open assumptions, all of which are formulas, and it has a conclusion node, which is a hat $\mathfrak{C}$. We refer to $\Gamma \succ \mathfrak{C}$ as *the sequent of* the derivation, and the derivation as *a derivation of* its sequent. What we understand a derivation as telling us depends on whether the derivation's sequent is an argument or not. A derivation with sequent $\Gamma \succ \varphi$ should be understood as a proof of φ from the assumptions Γ, or, as we will also say, a proof of the argument $\Gamma \succ \varphi$. On the other hand, a derivation with sequent $\Gamma \succ ☹$ should be understood as a *refutation* of the set Γ. It is very much not a proof of ☹—that wouldn't make sense, as ☹ does not carry content. We have here two fundamentally different roles for a derivation to play: a proof of an argument, or a refutation of a set of formulas.

This is the bilateralism in core logic: a bilateralism of proofs and refutations. In this setting, it would not be right to understand either proofs or refutations as a special kind of the other. The rules of derivation allow us to build proofs and refutations both, from components that themselves may be proofs and refutations both. In this sense, then, core logic derivations are bilateralist: based on two core notions, one positive and one negative, neither of which should be understood as a special case of the other. In this regard, the bilateralism in core logic is like the bilateralisms explored in [1, 23, 24, 25]. Tennant's discussion of these issues in [19] is useful here.

To forestall any misunderstandings, however, we note that core logic is not at all *symmetrical* in the way that many bilateralist theories are. Proofs and refutations in these systems are not at all each other's mirror image. Even before we present the rules, we can see this already, as they apply to different things. A proof is a proof of an *argument*: a pair of a set of premises and a single conclusion; while a refutation is a refutation of just a set of formulas. Both are species of derivation, to be sure, but neither is reducible to the other.

2.2. Rules for core logic

With that understood, derivations are otherwise relatively standard. What makes core logic distinctive, other than some care about the difference between formulas and hats, is its use of mostly *general* eliminations (see for example [17] or [10, Ch. 8]), and a bit of fuss around discharge policies.

Derivations begin, as usual, from *assumptions*. Any formula may be assumed; recall that ☹, which is not a formula, may not be assumed. An assumption of φ counts as a proof of $\varphi \succ \varphi$: a proof of φ from the open assumption φ.

2.2.1. Conjunction

From here, rules proceed connective by connective, with introduction and elimination rules for each connective. Each elimination rule has a major premise, which will be indicated as we proceed. Many of these rules have particular restrictions against certain kinds of vacuous discharge, which we will describe as we go.

$$\wedge\mathrm{I}\ \frac{\varphi \qquad \psi}{\varphi \wedge \psi} \qquad\qquad \wedge\mathrm{E}^n\ \frac{\varphi \wedge \psi \qquad \begin{array}{c}[\varphi, \psi]^n \\ \vdots \\ \mathfrak{C}\end{array}}{\mathfrak{C}}$$

Discharged assumptions are marked with [square brackets]; other assumptions, including other occurrences of these discharged formulas, may also occur as assumptions.[2] We use numeral annotations (here schematized as ^{n}) to indicate which rule discharges which discharged assumption: in any derivation, we assume that each occurrence of each discharging rule wears a distinct discharge numeral, and that each discharged assumption wears the numeral corresponding to the rule occurrence that discharged it.

Discharge restriction: in $\wedge$E, the discharge $[\varphi, \psi]$ may not be completely vacuous. That is, it must discharge at least one occurence of φ *or* at least one occurrence of ψ. The major premise of $\wedge$E is $\varphi \wedge \psi$.

[2]See Section 2.4 for discussion.

2.2.2. Disjunction

$$\vee\mathrm{I}_l\ \frac{\varphi}{\varphi \vee \psi} \qquad \vee\mathrm{I}_r\ \frac{\psi}{\varphi \vee \psi} \qquad \vee\mathrm{E}^n\ \frac{\varphi \vee \psi \qquad \begin{matrix}[\varphi]^n \\ \vdots \\ \mathfrak{C}\end{matrix} \qquad \begin{matrix}[\psi]^n \\ \vdots \\ \mathfrak{D}\end{matrix}}{\max(\mathfrak{C}, \mathfrak{D})}$$

Discharge restriction: in $\vee$E, *neither* discharge $[\varphi]$ nor $[\psi]$ may be vacuous. Recall as well that $\max(\mathfrak{C}, \mathfrak{D})$ is only defined when either $\mathfrak{C} = \mathfrak{D}$ or at least one of $\mathfrak{C}, \mathfrak{D}$ is ☹; in other cases the rule $\vee$E is not applicable. The major premise of $\vee$E is $\varphi \vee \psi$.

2.2.3. Implication

$$\to\mathrm{I}^n\ \frac{\begin{matrix}[\varphi]^n \\ \vdots \\ \mathfrak{C}\end{matrix}}{\varphi \to \psi} \qquad \to\mathrm{E}^n\ \frac{\varphi \to \psi \qquad \varphi \qquad \begin{matrix}[\psi]^n \\ \vdots \\ \mathfrak{C}\end{matrix}}{\mathfrak{C}}$$

In the rule $\to$I, we must have $\mathfrak{C} \leq \psi$. In addition, *if* $\mathfrak{C}$ is ☹, then the discharge of $[\varphi]$ must not be vacuous. However, in cases where $\mathfrak{C}$ is ψ itself, the discharge $[\varphi]$ may be vacuous. In $\to$E, the discharge $[\psi]$ may not be vacuous. The major premise of $\to$E is $\varphi \to \psi$.

2.2.4. Negation

$$\neg\mathrm{I}^n\ \frac{\begin{matrix}[\varphi]^n \\ \vdots \\ ☹\end{matrix}}{\neg\varphi} \qquad \neg\mathrm{E}\ \frac{\neg\varphi \qquad \varphi}{☹}$$

Discharge restriction: in $\neg$I, the discharge $[\varphi]$ may not be vacuous. The major premise of $\neg$E is $\neg\varphi$.

2.3. Core derivations and core logic

What we have in view so far is in fact a proof system for *intuitionistic* logic, not core logic. That is, an argument $\Gamma \succ \varphi$ is provable in this system iff it is intuitionistically valid, and a set Γ of formulas is refutable in this system iff it is intuitionistically inconsistent.[3]

To get to core logic, we use the notion of a *core derivation*, which we now present. A derivation is *core* iff every major premise of every elimination rule in it is an assumption, and a sequent is *core derivable* iff it is the sequent of some core derivation. We say that an argument is *core provable* iff it has a proof that is core, and that a set of formulas is *core refutable* iff it has a refutation that is core.

Not every provable argument is core provable. For example, $\neg p, p \succ q$ is provable as follows:

$$
\begin{prooftree}
\AxiomC{$\neg p$}
\AxiomC{$[p]^1$}
\LeftLabel{$\neg\mathrm{E}$}
\BinaryInfC{$\frownie$}
\LeftLabel{$\to\mathrm{I}^1$}
\UnaryInfC{$p \to q$}
\AxiomC{p}
\AxiomC{$[q]^2$}
\LeftLabel{$\to\mathrm{E}^2$}
\TrinaryInfC{q}
\end{prooftree}
$$

This derivation is not core, as the major premise of $\to$E in it is the conclusion of a step of $\to$I rather than an assumption. And indeed there is no core proof of $\neg p, p \succ q$. To see this, note (by checking the rules) that in a core derivation, every formula that occurs must be a subformula either of some open assumption or of the conclusion. That gives very little room to work with when attempting to prove $\neg p, p \succ q$, and it's not hard to see that the task can't be done. The closest we can get is instead a core refutation of the set $\{\neg p, p\}$:

$$
\begin{prooftree}
\AxiomC{$\neg p$}
\AxiomC{p}
\LeftLabel{$\neg\mathrm{E}$}
\BinaryInfC{$\frownie$}
\end{prooftree}
$$

Similarly, not every refutable set of formulas is core refutable. For example, the set $\{\neg p, p, q\}$ is refutable as follows:

$$
\begin{prooftree}
\AxiomC{$\neg p$}
\AxiomC{p}
\AxiomC{q}
\LeftLabel{$\wedge\mathrm{I}$}
\BinaryInfC{$p \wedge q$}
\AxiomC{$[p]^1$}
\LeftLabel{$\wedge\mathrm{E}^1$}
\BinaryInfC{p}
\LeftLabel{$\neg\mathrm{E}$}
\BinaryInfC{$\frownie$}
\end{prooftree}
$$

[3] For discussion of this point, see [13, 20].

However, this set has no core refutation, by similar reasoning to the above. Again, the closest we can get is a core refutation of the distinct set $\{\neg p, p\}$.

One way to see core logic as a consequence relation is this: say that a sequent $\Gamma \succ \mathfrak{C}$ is in core logic iff it is core derivable. As we've just seen, then, neither $\neg p, p \succ q$ nor $\neg p, p, q \succ \frownie$ is in core logic, but $\neg p, p \succ \frownie$ is in core logic. In this sense, then, core logic is nonmonotonic on both sides: neither $\subseteq$ on the left nor $\leq$ on the right preserves core derivability.

Core logic is probably best known for not admitting *cut*: there are cases where both $\Gamma \succ \varphi$ and $\varphi, \Delta \succ \mathfrak{C}$ are in core logic, but where $\Gamma, \Delta \succ \mathfrak{C}$ is not. For example, $p \succ p \vee q$ and $\neg p, p \vee q \succ q$ are both core derivable, but we've just seen that $\neg p, p \succ q$ is not. What holds instead is a property Tennant calls *epistemic gain*: whenever both $\Gamma \succ \varphi$ and $\varphi, \Delta \succ \mathfrak{C}$ are in core logic, then there is some $\Sigma \succ \mathfrak{D}$ in core logic such that $\Sigma \subseteq \Gamma \cup \Delta$ and $\mathfrak{D} \leq \mathfrak{C}$. Tennant appeals to epistemic gain to defuse criticisms of core logic based on its not admitting cut, and we will depend on epistemic gain in much of our reasoning that follows. It's not our purpose here, however, to evaluate core logic, so we don't discuss such defenses further; our purposes just involve noting that this epistemic gain property holds.

2.4. The Prawitz restriction

That, then, is the natural deduction system we will work with in what follows. It differs from Tennant's own systems for core logic and its relatives in one important respect, which is the topic of this subsection and Sections 3.5 and 5.1. Tennant's systems, as we interpret them, impose a further restriction on discharges, one that we do not impose: that whenever a rule application *can* discharge an occurrence of an open assumption, it *must* discharge that occurrence.

The first thing to note about this restriction is that it has nothing special to do with core logic. Restrictions like this can be imposed, or not, in ordinary natural deduction systems for logics of all sorts. For example, Gentzen's original system NJ (in [5]) for intuitionistic logic does not impose any such restriction; but Prawitz's closely-related system I (in [12]) for intuitionistic logic adds this restriction. Accordingly, we call this restriction 'the Prawitz restriction', and call a derivation 'Prawitz' when it obeys this restriction.[4]

[4]For Tennant's imposing this restriction, see, e.g., [16, p. 674], [22, §§2.3.2, 4.6].
In some other places, however, Tennant is less explicit. For example, [21, p. 454] imposes the restriction explicitly only for those cases of $\to$I where vacuous discharge

2.4.1. Keeping track of discharge

The main reason to impose the Prawitz restriction, as we see it, is that it saves on some bookkeeping. (This is discussed in [12, § I.4].) With the restriction imposed, there is no need to mark separately in a derivation which assumptions are discharged, and no need to mark what rules do the discharging work. In a Prawitz derivation, each assumption is discharged if and only if it can be, and discharged by the earliest rule that could have done the discharging.[5]

For example, take our above-presented natural deduction system. Now consider this:

$$\dfrac{\dfrac{\dfrac{p \qquad p}{p \wedge p}\wedge\mathrm{I}}{p \to p \wedge p}\to\mathrm{I}}{p \to p \to p \wedge p}\to\mathrm{I}$$

If this is to be understood as a Prawitz derivation, both assumptions of p must in fact be discharged—despite the fact that these occurrences of $\to$I allow for vacuous discharges. This is because the Prawitz restriction requires every rule to discharge every assumption it can. Since these occurrences of $\to$I introduce formulas with antecedent p, they can discharge assumptions of p; and so they must discharge any such assumptions not already discharged. This means, in addition, that both assumptions of p must be discharged by the *upper* instance of $\to$I. The lower instance, then, does feature vacuous discharge, since by the time it is reached there are no further open assumptions.

It is the Prawitz restriction that allows us to conclude all this from the structure above. Without the Prawitz restriction in place, there are

would be permissible; and [20] does not state any explicit policy, but on p. 315 includes discussion that seems to require the Prawitz restriction. We (tentatively) think it's probably best to interpret these sources too as imposing the restriction.

[5] An anonymous referee suggests that another motivation for the Prawitz restriction might come from searching for derivations of a given sequent, because the restriction 'allows for faster breakdown in the complexity of sequents for which proofs are being sought'.

However, we think that imposing the Prawitz restriction simply cannot be an aid to finding derivations of a given sequent. Any derivation-search strategy that succeeds in finding a Prawitz derivation thereby succeeds in finding a derivation. So any strategy that works in the presence of the Prawitz restriction will work exactly as well in its absence.

options. Since these uses of $\to$I both allow vacuous discharge, each assumption of p might be discharged by the upper $\to$I, by the lower $\to$I, or not at all; and these choices can be made independently. This means that the above display, read as containing no information about discharges, corresponds to nine distinct derivations.[6]

Working in systems without the Prawitz restriction, then, more bookkeeping is needed to indicate which assumptions are discharged and which are not, and to indicate which rules do the discharging. Our convention is a usual one: every occurrence of a discharging rule in a derivation must be annotated with a distinct numeral, and every discharged assumption in a derivation must appear surrounded by [square brackets] and annotated with the numeral of the rule that discharged it.

Using this convention, we could indicate the Prawitz derivation described above like so:

$$\dfrac{\dfrac{\dfrac{[p]^1 \qquad [p]^1}{p \wedge p}\,\wedge\mathrm{I}}{p \to (p \wedge p)}\,{\to}\mathrm{I}^1}{p \to p \to (p \wedge p)}\,{\to}\mathrm{I}^2$$

However, we can also use this convention to indicate non-Prawitz derivations, for example this one:

$$\dfrac{\dfrac{\dfrac{[p]^2 \qquad [p]^1}{p \wedge p}\,\wedge\mathrm{I}}{p \to p \wedge p}\,{\to}\mathrm{I}^1}{p \to p \to p \wedge p}\,{\to}\mathrm{I}^2$$

Indeed, one of the key reasons we do not impose the Prawitz restriction is because we want to study derivations like this latter example. Already, though, we can see one important effect of the restriction on Tennant's own natural deduction systems: the property of *being a Prawitz derivation* is not closed under substitution of arbitrary formulas for atomic formulas. To see this, return to the most recent displayed derivation, the non-Prawitz

[6] According to some conventions, this display would be read as *containing* the information that no discharges have occurred, thus picking out a particular one of these nine.

one, and note that it is a substitution instance (substituting p for q) of the following derivation, which is Prawitz:

$$\dfrac{\dfrac{\dfrac{[p]^2 \qquad [q]^1}{p \wedge q}\wedge\mathrm{I}}{q \to p \wedge q}{\to}\mathrm{I}^1}{p \to q \to p \wedge q}{\to}\mathrm{I}^2$$

By dropping the Prawitz restriction, we ensure that our derivations are closed under substitutions. We will look at some other reasons for dropping this restriction in Sections 3.5 and 5.1.

2.4.2. Prawitz derivations and Prawitz derivability

Before moving on, we pause to explore the effects of the Prawitz restriction on derivability and on core derivability.[7] It turns out that for simple derivability, imposing the Prawitz restriction or not makes no difference:

PROPOSITION 2.1. If a sequent has a derivation, it has a Prawitz derivation.

PROOF: Take a sequent with a derivation D. If D itself is Prawitz, we're done. If D is not Prawitz, suppose that all of D's proper subderivations are Prawitz. (By induction on D, it is enough to consider this situation only.)

For example, suppose D ends in an application of →I:

$$\dfrac{\begin{matrix}[\varphi]^n \\ \vdots \\ \mathfrak{C}\end{matrix}}{\varphi \to \psi}{\to}\mathrm{I}^n$$

If D is not Prawitz, but all its proper subderivations are, then this final →I leaves some assumptions of φ undischarged. D is then a derivation of $\varphi, \Gamma \succ \varphi \to \psi$, for some set Γ that does not contain φ. By modifying D to discharge all open assumptions of φ at this final step, we reach a Prawitz derivation D' of $\Gamma \succ \varphi \to \psi$. We can then extend D' as follows (with fresh discharge numerals m, o):

[7] Thanks to an anonymous referee for encouraging us to develop this material.

$$
\dfrac{\dfrac{\begin{matrix} D' \\ \varphi \to \psi \end{matrix}}{\varphi \to \varphi \to \psi}\;\to\mathrm{I}^m \qquad \varphi \qquad [\psi]^o}{\varphi \to \psi}\;\to\mathrm{E}^o
$$

Note that the discharge labeled m is vacuous, as we know that there are no open assumptions of φ in D'. This resulting derivation is Prawitz, and is a derivation of $\varphi, \Gamma \succ \varphi \to \psi$, just as D itself was.

This strategy works in general: if D is not Prawitz at its final rule occurrence, it must be because this occurrence leaves some assumption open that it could have discharged. So we first modify D to a Prawitz D' that does discharge everything it can at this final step, and then use $\to$I and $\to$E in tandem to restore the needed open assumptions. $\square$

So removing the Prawitz restriction has no effect on which sequents are derivable, and thus no effect on provability or refutability. Since derivability itself is closed under substitutions, then, it follows that Prawitz derivability is also closed under substitutions, even though the property of being a Prawitz derivation is not.

The strategy adopted in the above proof, however, produces non-core derivations, even starting from a core derivation. And indeed, the situation is different when it comes to core derivability: there are sequents that have core derivations but no Prawitz core derivations. For example, consider $p \succ p \to p \wedge p$; this has the following core derivation:

$$
\dfrac{\dfrac{p \qquad [p]^1}{p \wedge p}\;\wedge\mathrm{I}}{p \to p \wedge p}\;\to\mathrm{I}^1
$$

It does not, however, have any Prawitz core derivation. To see this, note that any core derivation of $p \succ p \to p \wedge p$ must end in a step of $\to$I; no elimination rule is possible as a last step, since the major premise of that elimination rule would have to be an open assumption, and p cannot stand as a major premise of any elimination rule. This final step of $\to$I, however, is able to discharge any open assumptions of p in the derivation, so in a Prawitz derivation it must do so; p cannot stand as an open assumption at the end of such a derivation. Accordingly, there is no Prawitz core derivation of $p \succ p \to p \wedge p$.

So imposing the Prawitz restriction or not *does* make a difference as to which sequents are core derivable. Moreover, Prawitz core derivability is not closed under substitution: witness the following Prawitz core derivation of $p \succ q \to p \wedge q$.

$$\dfrac{\dfrac{p \qquad [q]^1}{p \wedge q}\,\wedge\mathrm{I}}{q \to p \wedge q}\,{\to}\mathrm{I}^1$$

Since Tennant's own version of core logic imposes the Prawitz restriction, then, it is not closed under substitutions. However, our liberalized version, which does not impose the Prawitz restriction, is.

3. Terms and reductions

Here, we define a language of terms, and consider reduction relations on these terms. The motivating idea is to develop, for the above natural deduction system, a term calculus that corresponds to it in the usual Curry-Howard way, the way that the calculus of [8] corresponds to a more usual intuitionistic natural deduction system. (This work is begun in [13], which explores the $\neg, \to$ fragment of core logic in this way; this section extends that work to take account of $\wedge, \vee$ as well.) The usual Curry-Howard correspondence allows us to see intuitionistic proofs as programs in a simply-typed lambda calculus, and reduction on proofs as execution of those programs. Similarly, the system presented here allows us to see derivations in the above-presented proof system as programs, and reduction of those derivations as execution.[8]

Our *types* for this system are the formulas of our language. *Hats* are as before: a hat is either a type or ☹.

3.1. Terms and eliminators

We use a mutual induction to define terms, eliminators, and the free variables in a term or eliminator. We use M, N, O, etc for terms; each term M wears a hat $\mathfrak{C}$, indicated as $M^{\mathfrak{C}}$. Every term is either *typed* or *exceptional*, according to its hat: if its hat is a type, the term is typed; and if its hat is ☹, the term is exceptional. We use $\mathcal{E}, \mathcal{F}$, etc for eliminators;

[8]For background and details, see for example [6, 14].

each eliminator $\mathcal{E}$ wears both a type φ and a separate hat $\mathfrak{C}$, indicated as ${}_{\varphi}\mathcal{E}^{\mathfrak{C}}$. We sometimes have use for metavariables that can be either terms or eliminators; for this purpose we use $\mathbb{X}, \mathbb{Y}$, etc. For every *type* φ we assume denumerably many variables x^{φ}, y^{φ}, etc; there are no variables with hat ☹. For any term or eliminator $\mathbb{X}$ there is a set $\mathtt{FV}(\mathbb{X})$ of variables that are $\mathbb{X}$'s *free variables.*

Definition 3.1. (Terms and eliminators)
Terms:

- All variables are terms; for any variable x, we have $\mathtt{FV}(x) = \{x\}$.
- For any terms M^{φ} and N^{ψ}, there is a term $\langle M, N\rangle^{\varphi\wedge\psi}$. We have $\mathtt{FV}(\langle M, N\rangle) = \mathtt{FV}(M) \cup \mathtt{FV}(N)$.
- For any term M^{φ} and type ψ, there are terms $(\mathsf{inl}(M))^{\varphi\vee\psi}$ and $(\mathsf{inr}(M))^{\psi\vee\varphi}$. We have $\mathtt{FV}(\mathsf{inl}(M)) = \mathtt{FV}(\mathsf{inr}(M)) = \mathtt{FV}(M)$.
- For any term $M^{☹}$ with $x^{\varphi} \in \mathtt{FV}(M)$, there is a term $(\lambda^{\neg}x.M)^{\neg\varphi}$, and in addition for each type ψ a term $(\lambda^{\rightarrow}x.M)^{\varphi\rightarrow\psi}$. We have $\mathtt{FV}(\lambda^{\neg}x.M) = \mathtt{FV}(\lambda^{\rightarrow}x.M) = \mathtt{FV}(M) \setminus \{x\}$.
- For any term M^{ψ} and variable x^{φ}, there is a term $(\lambda^{\rightarrow}x.M)^{\varphi\rightarrow\psi}$. Again, $\mathtt{FV}(\lambda^{\rightarrow}x.M) = \mathtt{FV}(M) \setminus \{x\}$.
- For any term M^{φ} and eliminator ${}_{\varphi}\mathcal{E}^{\mathfrak{C}}$, there is a term $(M\mathcal{E})^{\mathfrak{C}}$. We have $\mathtt{FV}(M\mathcal{E}) = \mathtt{FV}(M) \cup \mathtt{FV}(\mathcal{E})$.

Eliminators:

- For any term $N^{\mathfrak{C}}$ with $\{x^{\varphi}, y^{\psi}\} \cap \mathtt{FV}(M) \neq \emptyset$, there is an eliminator ${}_{\varphi\wedge\psi}(\!|\langle x, y\rangle.N|\!)^{\mathfrak{C}}$. We have $\mathtt{FV}((\!|\langle x, y\rangle.N|\!)) = \mathtt{FV}(N) \setminus \{x, y\}$.
- For any terms $N^{\mathfrak{C}}$ and $O^{\mathfrak{D}}$ with $x^{\varphi} \in \mathtt{FV}(N)$ and $y^{\psi} \in \mathtt{FV}(O)$, such that either $\mathfrak{C} \leq \mathfrak{D}$ or $\mathfrak{D} \leq \mathfrak{C}$, there is an eliminator ${}_{\varphi\vee\psi}(\!|x.N, y.O|\!)^{\max(\mathfrak{C},\mathfrak{D})}$. We have $\mathtt{FV}((\!|x.N, y.O|\!)) = (\mathtt{FV}(N) \setminus \{x\}) \cup (\mathtt{FV}(O) \setminus \{y\})$.
- For any terms N^{φ} and $O^{\mathfrak{C}}$ with $x^{\psi} \in \mathtt{FV}(O)$, there is an eliminator ${}_{\varphi\rightarrow\psi}(\!|N, x.O|\!)^{\mathfrak{C}}$. We have $\mathtt{FV}((\!|N, x.O|\!)) = \mathtt{FV}(N) \cup (\mathtt{FV}(O) \setminus \{x\})$.
- For any term N^{φ}, there is an eliminator ${}_{\neg\varphi}(\!|N|\!)^{☹}$. We have $\mathtt{FV}((\!|N|\!)) = \mathtt{FV}(N)$.

All terms and eliminators are identified up to change in bound variables, and we make free use of this identification without further comment. As you may have noticed in the above definition, we often omit hats, either where they can be inferred or where we are generalizing.

By comparing the above definitions to the natural deduction system, you can see the following correspondences:

Open assumption of φ	Free variable of type φ
Discharging an assumption of φ	Binding a variable of type φ
Derivation of the sequent $\Gamma \succ \mathfrak{C}$	Term $M^{\mathfrak{C}}$ with $\mathtt{FV}(M)$ having types in Γ

Let's look at two examples, to get the flavour. First, our earlier proof of $\neg p, p \succ q$:

$$\dfrac{\dfrac{\dfrac{\neg p \qquad [p]^1}{\frown}\ \neg\mathrm{E}}{p \to q}\ {\to}\mathrm{I}^1 \qquad p \qquad [q]^2}{q}\ {\to}\mathrm{E}^2$$

We can annotate this derivation as follows:

$$\dfrac{\dfrac{\dfrac{w : \neg p \qquad [x : p]^1}{w(\!|x|\!) : \frown}\ \neg\mathrm{E}}{\lambda^{\to} x.w(\!|x|\!) : p \to q}\ {\to}\mathrm{I}^1 \qquad y : p \qquad [z : q]^2}{(\lambda^{\to} x.w(\!|x|\!))(\!|y, z.z|\!) : q}\ {\to}\mathrm{E}^2$$

This derivation thus corresponds to the term $(\lambda^{\to} x.w(\!|x|\!))(\!|y, z.z|\!)$, which, fully spelled out with all hats visible, is

$$(\lambda^{\to} x^p.(w^{\neg p}(\!|x^p|\!))^{\frown})^{p \to q}({}_{p \to q}(\!|y^p, z^q.z^q|\!)^q)^q.$$

Second, our earlier example of a derivation that violates the Prawitz restriction:

$$\dfrac{\dfrac{\dfrac{[p]^2 \qquad [p]^1}{p \wedge p}\ \wedge\mathrm{I}}{p \to (p \wedge p)}\ {\to}\mathrm{I}^1}{p \to p \to (p \wedge p)}\ {\to}\mathrm{I}^2$$

We can annotate this derivation as follows:

$$\dfrac{\dfrac{\dfrac{[x:p]^2 \qquad [y:p]^1}{\langle x,y\rangle : p\wedge p}\wedge\mathrm{I}}{\lambda^{\to}y.\langle x,y\rangle : p\to(p\wedge p)}\to\mathrm{I}^1}{\lambda^{\to}x.\lambda^{\to}y.\langle x,y\rangle : p\to p\to(p\wedge p)}\to\mathrm{I}^2$$

This derivation thus corresponds to the term $(\lambda^{\to}x.\lambda^{\to}y.\langle x,y\rangle)$, which, fully spelled out, is $(\lambda^{\to}x^p.(\lambda^{\to}y^p.(\langle x^p,y^p\rangle)^{p\wedge p})^{p\to p\wedge p})^{p\to p\to p\wedge p}$. Hopefully it is by now apparent why we often suppress hats where they are not needed!

3.2. Terminology

Terms of the form $\langle M,N\rangle$, $\mathsf{inl}(M)$, $\mathsf{inr}(M)$, $\lambda^{\to}x.M$, or $\lambda^{\neg}x.M$ are *introductions*. Terms of the form $M\mathcal{E}$ are *eliminations*. So every term is a variable, an introduction, or an elimination.

Variables have no *immediate subterms*. The immediate subterms of an introduction or an eliminator are what you'd expect. (For example, the immediate subterms of $(\!|N,x.O|\!)$ are N and O.) The immediate subterms of an elimination $M\mathcal{E}$ are M and the immediate subterms of $\mathcal{E}$. The subterm relation is the reflexive transitive closure of the immediate subterm relation.

All immediate subterms of an eliminator are *minor* subterms of that eliminator. In eliminators of the form $(\!|\langle x,y\rangle.N|\!)$ or $(\!|x.N,y.O|\!)$, these minor subterms are also *commuting* subterms. In eliminators of the form $(\!|N,x.O|\!)$, only O is a commuting subterm. And in eliminators of the form $(\!|N|\!)$, there are no commuting subterms. The minor and commuting subterms of an elimination $M\mathcal{E}$ are those of the eliminator $\mathcal{E}$. The *major* subterm of an elimination $M\mathcal{E}$ is M. Note that every immediate subterm of an elimination is either major or minor.

3.3. Composition of eliminators

Given two eliminators ${}_{\varphi}\mathcal{E}^{\psi}$ and ${}_{\psi}\mathcal{F}^{\mathfrak{C}}$, the eliminator ${}_{\varphi}(\!|\mathcal{E}\mathcal{F}|\!)^{\mathfrak{C}}$ is the eliminator like $\mathcal{E}$, but with each commuting subterm P of $\mathcal{E}$ replaced with $P\mathcal{F}$.[9] For example, if $\mathcal{E}$ is ${}_{\varphi\to\psi}(\!|N^{\varphi},x.O^{\theta\wedge\rho}|\!)^{\theta\wedge\rho}$ and $\mathcal{F}$ is ${}_{\theta\wedge\rho}(\!|\langle y,z\rangle.P^{\mathfrak{C}}|\!)^{\mathfrak{C}}$, then $(\!|\mathcal{E}\mathcal{F}|\!)$ is $(\!|N,x.O\mathcal{F}|\!)$. As the commuting subterms of an eliminator always wear the same hat as the eliminator's right (output) hat, this is well-defined.

[9] Change to bound variables in $\mathcal{E}$ might be needed here to avoid capturing any variables free in $\mathcal{F}$.

3.4. Substitution

Capture-avoiding substitution of terms for variables in this calculus works as it does in similar calculi; there's nothing particularly remarkable about it. We pause to go through the details nonetheless; many aspects of core type theory do *not* work as usual, so it's worth checking the details even of those aspects that do.

Where $x_1^{\varphi_1}, \ldots, x_n^{\varphi_n}$ are distinct variables and $N_1^{\varphi_1}, \ldots, N_n^{\varphi_n}$ terms of corresponding types, then $[x_1 \mapsto N_1, \ldots, x_n \mapsto N_n]$ is a *substitution*. (Note that all substitutions are finite.) Given a substitution σ, the substitution $\sigma^{\downarrow y}$ is just like σ except that it does not substitute anything for the variable y. That is, $[x_1 \mapsto N_1, \ldots, x_n \mapsto N_n]^{\downarrow x_i}$ is $[x_1 \mapsto N_1, \ldots, x_{i-1} \mapsto N_{i-1}, x_{i+1} \mapsto N_{i+1}, \ldots, x_n \mapsto N_n]$; and $[x_1 \mapsto N_1, \ldots, x_n \mapsto N_n]^{\downarrow y}$ is just $[x_1 \mapsto N_1, \ldots, x_n \mapsto N_n]$ if y is not one of the x_is. Say that a variable y is *free* in $[x_1 \mapsto N_1, \ldots, x_n \mapsto N_n]$ iff it is free in some N_i; and say that y is *acted on* by $[x_1 \mapsto N_1, \ldots, x_n \mapsto N_n]$ iff it is one of the x_i.

Given a term or eliminator, capture-avoiding substitution works as usual:

- $x_i[x_1 \mapsto N_1, \ldots, x_n \mapsto N_n] = N_i$;
- $y[x_1 \mapsto N_1, \ldots, x_n \mapsto N_n] = y$, where y is not one of the x_is;
- $\langle M, N\rangle\sigma = \langle M\sigma, N\sigma\rangle$;
- $\mathsf{inl}(M)\sigma = \mathsf{inl}(M\sigma)$; $\mathsf{inr}(M)\sigma = \mathsf{inr}(M\sigma)$;
- $(\lambda^{\rightarrow} y.M)\sigma = \lambda^{\rightarrow} y.(M\sigma^{\downarrow y})$, assuming y is not free in σ;[10]
- $(\lambda^{\neg} y.M)\sigma = \lambda^{\neg} y.(M\sigma^{\downarrow y})$, assuming y is not free in σ;
- $(M\mathcal{E})\sigma = (M\sigma)(\mathcal{E}\sigma)$;
- ${}_{\neg\varphi}(\!|M|\!)\sigma = {}_{\neg\varphi}(\!|M\sigma|\!)$;
- ${}_{\varphi\wedge\psi}(\!|\langle x, y\rangle.M|\!)\sigma = {}_{\varphi\wedge\psi}(\!|\langle x, y\rangle.M\sigma^{\downarrow x\downarrow y}|\!)$, assuming neither x nor y is free in σ;

[10] Recall that we identify terms up to change of bound variable. So if y *is* free in σ, we first change the bound variable y in $\lambda^{\rightarrow} y.M$ to some variable that is *not* free in σ. (Since all substitutions are finite, there is always some such.) All similar assumptions in this definition should be read the same way.

- ${}_{\varphi\vee\psi}(\!|x.M, y.N|\!)\sigma = {}_{\varphi\vee\psi}(\!|x.N\sigma^{\downarrow x}, y.O\sigma^{\downarrow y}|\!)$, assuming neither x nor y is free in σ; and
- ${}_{\varphi\to\psi}(\!|M, x.N|\!)\sigma = {}_{\varphi\to\psi}(\!|M\sigma, x.N\sigma^{\downarrow x}|\!)$, assuming x is not free in σ.

Note two things: first that, since there are no variables with hat $\frownie$, that $M[x \mapsto N^{\frownie}]$ is never defined; and second that substitution never affects hats: that is, the hat on $M^{\mathfrak{C}}[x \mapsto N]$ is always exactly $\mathfrak{C}$.

Substitution interacts pleasantly with composition of eliminators:

Lemma 3.2. *Given eliminators $\mathcal{E}$ and $\mathcal{F}$ such that $(\!|\mathcal{E}\mathcal{F}|\!)$ is defined, and a substitution σ, the eliminator $(\!|(\mathcal{E}\sigma)(\mathcal{F}\sigma)|\!)$ is $(\!|\mathcal{E}\mathcal{F}|\!)\sigma$.*

Proof: Unpacking definitions. □

3.5. The Prawitz restriction on terms

Recall that the Prawitz restriction on derivations requires that when any rule application in a derivation *can* discharge any open assumption, it *must* discharge that open assumption. The corresponding restriction on terms is this: that whenever a component of a term binds a variable of type φ, it binds *all* free variables of type φ in its scope. Equivalently, the Prawitz restriction corresponds to a term system with a *single* variable of each type, rather than the denumerably many variables of each type that we have assumed.[11]

We noted in Section 2.4 that there are many derivations in our system that do not obey the Prawitz restriction, such as the derivation repeated here:

$$\to\mathrm{I}^2\ \dfrac{\to\mathrm{I}^1\ \dfrac{\wedge\mathrm{I}\ \dfrac{[p]^2 \qquad [p]^1}{p \wedge p}}{p \to p \wedge p}}{p \to p \to p \wedge p}$$

This derivation corresponds to the term $(\lambda^{\to}x^p.\lambda^{\to}y^p.(\langle x, y\rangle)^{p\wedge p})^{p\to p\to p\wedge p}$. This term requires two distinct variables of type p. This is because $\lambda^{\to}y$

[11]Term systems like this are not often explored, because they do not allow for a definition of capture-avoiding substitution; our definition in Section 3.4, like other definitions, relies crucially on being able to draw on fresh variables of a given type to avoid clashes between free and bound variables. (As we will see in Section 5.1, this interference with substitution also blocks strong normalization.)

must bind the y in $\langle x^p, y^p \rangle$ *without* binding the x, so that the outer $\lambda^{\rightarrow} x$ can bind the x instead.

This brings us to the main reason we've chosen to go without the Prawitz restriction: the terms it excludes include terms with natural and important computational behaviour. The term $\lambda^{\rightarrow} x.\lambda^{\rightarrow} y.\langle x, y \rangle$ is a very simple pairing function, a function that takes inputs x and y and returns their ordered pair.[12] Imposing the Prawitz restriction would allow us to define this function only in the case where the two inputs have distinct types, but it is also perfectly natural to want to pair up two pieces of data that have the same type.

Indeed, the Prawitz restriction prevents us from defining *any* functions that take multiple inputs of the same type: the binding required for the final input is required by the Prawitz restriction to bind all free variables of that type; any outer bindings of that same type turn out vacuous. It would be impossible, for example, to build basic arithmetic on the Church numerals (see [7, Ch. 4]) in a system obeying the Prawitz restriction, since this requires defining addition and multiplication functions, each of which takes two inputs of the same (numeric) type.

We take it, then, that most standard term systems work without the Prawitz restriction for good reason, and so we develop core type theory without any such restriction.

4. Reduction

In this section, we define two relations of *reduction* on terms of our calculus: what we call *principal reduction* and *full reduction*. The difference is that full reduction includes commuting conversions; principal reduction does not. We then prove a number of lemmas about these reduction relations, in the leadup to Section 5, where we prove that principal reduction is strongly normalizing. We conjecture that full reduction is also strongly normalizing, but leave that question for future work.

4.1. Redexes and reducts

Both reduction relations are defined by identifying a class of special terms called *redexes*, and assigning to each redex a term called its *reduct*. The

[12] This is the function written `(,)` in Haskell, for example.

difference between principal reduction and full reduction is entirely in which terms are redexes. Then, given a chosen notion of redex, for any term M that contains a redex R as a subterm, we define a specific term as the *one-step reduction of M at R*. The move from redexes to one-step reduction is very much *not* as usual; this is one of the more distinctive features of core type theory, and it is a key motivation of this work to explore this nonstandard notion. Let's dive in.

4.1.1. Principal redexes

The following table displays the forms of all *principal redexes* and their corresponding reducts.

Redex	Reduct
$\langle M, N\rangle \llparenthesis \langle x, y\rangle . O \rrparenthesis$	$O[x \mapsto M, y \mapsto N]$
$\mathsf{inl}(M)\llparenthesis x.N, y.O \rrparenthesis$	$N[x \mapsto M]$
$\mathsf{inr}(M)\llparenthesis x.N, y.O \rrparenthesis$	$O[y \mapsto M]$
$(\lambda^{\rightarrow} x.(M^{\psi}))\llparenthesis N, y.O \rrparenthesis$	$O[y \mapsto M[x \mapsto N]]$
$(\lambda^{\rightarrow} x.(M^{\frownie}))\llparenthesis N, y.O \rrparenthesis$	$M[x \mapsto N]$
$(\lambda^{\neg} x.M)\llparenthesis N \rrparenthesis$	$M[x \mapsto N]$

In defining *principal reduction*, all and only the principal redexes count as redexes.

4.1.2. Commuting redexes

Any term of the form $(M\mathcal{E})\mathcal{F}$ is a *commuting redex*; its reduct is $M\llparenthesis \mathcal{E}\mathcal{F} \rrparenthesis$. Note that $\llparenthesis \mathcal{E}\mathcal{F} \rrparenthesis$ is defined, and $M\llparenthesis \mathcal{E}\mathcal{F} \rrparenthesis$ well-formed, whenever $(M\mathcal{E})\mathcal{F}$ is well-formed. Note as well that no commuting redex is a principal redex, so given a redex (of either kind), the reduct of that redex is unambiguously determined. In defining *full reduction*, both principal redexes and commuting redexes count as redexes.

Since we focus on principal reduction rather than full reduction in Section 5, we don't linger specifically on commuting redexes. However, the definitions and lemmas in this section don't care about the difference; when we speak of 'reduction' unqualified, we are making a definition or claim that applies to both principal and full reduction.[13]

4.2. One-step reduction

Using these redexes and their reducts, we define a relation of *one-step reduction* between terms. (Since we have two different choices for what counts as a redex—principal only or principal plus commuting—we end up with two different choices for a one-step reduction relation: principal or full.) Given any term that contains an occurrence of a redex at a subterm, we define the unique result of reducing that term at that redex occurrence. That much is as usual for term systems like this.

However—and this is not usual—reduction in this system is not a *compatible* relation. That is, we do not always simply replace a redex with its reduct in place, leaving its context alone. Such a procedure could not work in core type theory. The reason is that the result of such a procedure is not always well-formed in this system.

For example, consider the redex $((\lambda^{\to} y^{\varphi}.x^{\psi})w^{\varphi})^{\psi}$ with reduct x^{ψ} as it occurs in the term $(\lambda^{\to} w.(z^{\neg\psi}(\!|(\lambda^{\to} y.x)w|\!))^{☹})^{\varphi\to\theta}$. Replacing this redex with its reduct would yield $(\lambda^{\to} w.(z^{\neg\psi}(\!|x^{\psi}|\!))^{☹})^{\varphi\to\theta}$. This latter, however, is not a term, as it violates a restriction on $\lambda^{\to}$, which may not bind w vacuously in this situation. (This restriction corresponds to the restrictions against certain cases of vacuous discharge in the rule $\to$I.)

This is an example of the following. Many of our formation rules (in the above example, using $\lambda^{\to}$ to bind into an exceptional term) require certain variables to appear free; but some redexes, because they themselves involve vacuous binding, contain free variables that are not contained in their reducts. That is, core type theory allows vacuous binding in some

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[13] There are two more potential sources of redexes that might come to mind, although we use neither in this paper.

First, uses of an explosion rule like typical $\bot$E in natural deduction systems create possible violations of the subformula property, and so reduction steps are sometimes introduced to prevent these violations, as in [12, p. 40]. However, core logic contains no such explosion rules, so no such reduction steps are needed or even possible.

Second, [18] considers a type of reduction there called 'shrinking', which in effect allows a one-step reduction directly from $M^{\mathfrak{C}}$ to $N^{\mathfrak{C}}$ whenever N is a subterm of M. This makes havoc for computational interpretations of the term language, for reasons discussed in [11]; we leave it aside here.

circumstances but not all, and it is the interaction between these circumstances that creates the phenomenon of interest.[14]

For a different kind of example, consider the redex

$$((\lambda^{\to} y^{\varphi}.(z^{\neg\varphi} y)^{\circledS})^{\varphi\to\psi} (\!|x^{\varphi}, w^{\varphi}.w|\!))^{\psi}$$

with redex $(zx)^{\circledS}$ as it occurs in the term $(\langle(\lambda^{\to} y.zy)(\!|x, w.w|\!), v^{\theta}\rangle)^{\psi\wedge\theta}$. Replacing this redex with its reduct would yield $\langle(zx)^{\circledS}, w\rangle$. This latter, however, is not a term, as the constructor $\langle\ ,\ \rangle$ requires two *typed* subterms, and $(zx)^{\circledS}$ is exceptional. This corresponds to the rule $\wedge$I's requiring formulas as premises.

This is an example of a different kind of phenomenon. Many of our formation rules for terms (in the above example, using $\langle\ ,\ \rangle$) require terms to be typed; but some redexes are typed and yet have exceptional reducts. Reducing such a redex in place, then, yields a nonsensical result.

The troubles with reducing in place, then, are twofold: moving from a redex to its reduct can drop free variables, and it can move from a typed term to an exceptional one. But these reductions can happen in places where free variables or types are required. Leaving everything else in place, then, won't do in general. In what follows, we show how to handle these problems. We start by noting two important facts about redexes and their reducts: for any redex $R^{\mathfrak{C}}$ with reduct $R'^{\mathfrak{D}}$, we always have $\mathtt{FV}(R') \subseteq \mathtt{FV}(R)$ and $\mathfrak{D} \leq \mathfrak{C}$. That is, free variables and hats do not always remain constant between a redex and its reduct, but they cannot change freely; when there is a change, it is always in the same direction. We repeatedly use this constraint—which is the term-level reflection of epistemic gain—in what follows.

Basically, our strategy works like this: where we can get away with reducing in place, leaving the immediate context alone, that's what we do. Where the result would not be well-formed, we simply drop the immediate context altogether. That's the intuition, anyhow; here's the precise definition of one-step reduction.

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[14] Contrast a usual simply-typed lambda calculus, where vacuous binding is always allowed; but also contrast the lambda calculus of [3], standardly now called the λI calculus, where vacuous binding is never allowed; also see [2, Ch. 9]. In this calculus, redexes and their corresponding reducts always have exactly the same free variables (see [2, Lemma 9.1.2]), so any nonvacuous binding into a redex remains nonvacuous into its reduct.

Definition 4.1 (One-step reduction). First, if R is a redex and S its reduct, then R reduces to S in one step; as we write, $R \rightsquigarrow_1 S$. The rest of the definition contains a number of conditions. These are expressed in the form:

$$\frac{\mathbb{X} \rightsquigarrow_1 \mathbb{Y}}{\mathbb{Z} \rightsquigarrow_1 \mathbb{W}}$$

Here is how such a condition should be read. We only apply it if $\mathbb{X}, \mathbb{Y}, \mathbb{Z}$ are each well-formed, without any assumption that $\mathbb{W}$ is well-formed. Under these conditions, if $\mathbb{X} \rightsquigarrow_1 \mathbb{Y}$ *and* $\mathbb{W}$ *is well-formed*, then $\mathbb{Z} \rightsquigarrow_1 \mathbb{W}$; on the other hand, if $\mathbb{X} \rightsquigarrow_1 \mathbb{Y}$ and $\mathbb{W}$ is *not* well-formed, then $\mathbb{Z} \rightsquigarrow_1 \mathbb{Y}$ instead.

This fallback condition—that when $\mathbb{W}$ is not well-formed we have $\mathbb{Z} \rightsquigarrow_1 \mathbb{Y}$—is what gives one-step core reduction its distinctive flavour. Note that there is no indeterminism or choice introduced here: if $\mathbb{W}$ *is* well-formed we do not have $\mathbb{Z} \rightsquigarrow_1 \mathbb{Y}$ from such a condition. Only in the case that $\mathbb{W}$ is not well-formed do we fall back to $\mathbb{Z} \rightsquigarrow_1 \mathbb{Y}$. Here, then, are the conditions:

$$\frac{M \rightsquigarrow_1 M'}{M\mathcal{E} \rightsquigarrow_1 M'\mathcal{E}} \quad \frac{\mathcal{E} \rightsquigarrow_1 \mathcal{E}'}{M\mathcal{E} \rightsquigarrow_1 M\mathcal{E}'} \quad \frac{\mathcal{E} \rightsquigarrow_1 N}{M\mathcal{E} \rightsquigarrow_1 N}$$

$$\frac{M \rightsquigarrow_1 M'}{\langle M, N\rangle \rightsquigarrow_1 \langle M', N\rangle} \quad \frac{N \rightsquigarrow_1 N'}{\langle M, N\rangle \rightsquigarrow_1 \langle M, N'\rangle}$$

$$\frac{M \rightsquigarrow_1 M'}{\mathsf{inl}(M) \rightsquigarrow_1 \mathsf{inl}(M')} \quad \frac{M \rightsquigarrow_1 M'}{\mathsf{inr}(M) \rightsquigarrow_1 \mathsf{inr}(M')}$$

$$\frac{M \rightsquigarrow_1 M'}{\lambda^{\rightarrow}x.M \rightsquigarrow_1 \lambda^{\rightarrow}x.M'} \quad \frac{M \rightsquigarrow_1 M'}{\lambda^{\neg}x.M \rightsquigarrow_1 \lambda^{\neg}x.M'}$$

$$\frac{M \rightsquigarrow_1 M'}{(\!| M |\!) \rightsquigarrow_1 (\!| M' |\!)} \quad \frac{M \rightsquigarrow_1 M'}{(\!| \langle x, y\rangle.M |\!) \rightsquigarrow_1 (\!| \langle x, y\rangle.M' |\!)}$$

$$\frac{M \rightsquigarrow_1 M'}{(\!|M, x.N|\!) \rightsquigarrow_1 (\!|M', x.N|\!)} \qquad \frac{N \rightsquigarrow_1 N'}{(\!|M, x.N|\!) \rightsquigarrow_1 (\!|M, x.N'|\!)}$$

$$\frac{M \rightsquigarrow_1 M'}{(\!|x.M, y.N|\!) \rightsquigarrow_1 (\!|x.M', y.N|\!)} \qquad \frac{N \rightsquigarrow_1 N'}{(\!|x.M, y.N|\!) \rightsquigarrow_1 (\!|x.M, y.N'|\!)}$$

Expressed in this way, these conditions might *look* like usual reduce-in-place conditions. But recall our distinctive way of reading these, involving fallback in case the lower-right component is not well-formed; this is the key to the definition.

Since this is an unusual way to handle one-step reduction, let's look at an example. Consider the condition for $\mathsf{inl}()$, reproduced here:

$$\frac{M \rightsquigarrow_1 M'}{\mathsf{inl}(M) \rightsquigarrow_1 \mathsf{inl}(M')}$$

Suppose first that M^ψ is $(\lambda^\to x^\varphi.y^\psi)(\!|z, v.v|\!)$. Then M is a redex, with reduct y. So, according to the condition for $\mathsf{inl}()$, we can conclude that $\mathsf{inl}(M)^{\psi\vee\theta}$ can be reduced in one step to $\mathsf{inl}(y)$. So far, so normal.

Suppose instead, though, that M^ψ is $(\lambda^\to x^\varphi.y^{\neg\varphi}(\!|x|\!))(\!|z, v.v|\!)$. Then M is again a redex, now with reduct $(y(\!|z|\!))^{☹}$. By the same condition, then, $\mathsf{inl}(M)^{\psi\vee\theta}$ can be reduced. However, note that $\mathsf{inl}(y(\!|z|\!))$ is not well-formed; $\mathsf{inl}()$ can only be applied to *typed* terms, and $y(\!|z|\!)$ is exceptional. Thus, $\mathsf{inl}(M)$ cannot reduce to $\mathsf{inl}(y(\!|z|\!))$, since the latter isn't a term at all. So, according to the condition for $\mathsf{inl}()$, we conclude that $\mathsf{inl}(M)$ reduces in one step directly to $y(\!|z|\!)$.

Three important facts about one-step reduction. First, terms always reduce to terms, while eliminators sometimes reduce to eliminators and sometimes to terms. Second, if $M^{\mathfrak{C}} \rightsquigarrow_1 N^{\mathfrak{D}}$, then $\mathfrak{D} \leq \mathfrak{C}$. Finally, if $M \rightsquigarrow_1 N$, then $\mathtt{FV}(N) \subseteq \mathtt{FV}(M)$. (All these can be shown by induction on the above definition.)

Let's look at an example that demonstrates some of these complexities. Consider the term $M^{\neg(\varphi\wedge\psi)} = (\lambda^\neg x^{\varphi\wedge\psi}.(w^{\neg\theta}(\!|x(\!|\langle y^\varphi, z^\psi\rangle.(\lambda^\to v^\varphi.u^\theta)y^\varphi|\!)|\!))^{☹})$. The free variables of this term are $w^{\neg\theta}$ and u^θ, and so this term corresponds to a derivation of the sequent $\neg\theta, \theta \succ \neg(\varphi\wedge\psi)$. It contains a redex $(\lambda^\to v.u)y$ with reduct u, inside the eliminator $(\!|\langle y, z\rangle.(\lambda^\to v.u)y|\!)$. Let's go through the one-step reduction of M at this redex.

First, we note that $(\!|\langle y, z\rangle.u|\!)$ is not well-formed, since a conjunction eliminator cannot bind fully vacuously; so we reduce $(\!|\langle y, z\rangle.(\lambda^{\rightarrow} v.u)y|\!)$ directly to u itself. Having done this, we note that $x^{\varphi\wedge\psi}u^{\theta}$ is also not well-formed; no rule allows us to juxtapose two terms at all. So we reduce $x(\!|\langle y, z\rangle.(\lambda^{\rightarrow} v.u)y|\!)$ also directly to u. The next two layers do work in place, so we reduce $w(\!|x(\!|\langle y, z\rangle.(\lambda^{\rightarrow} v.u)y|\!)|\!)$ to $w(\!|u|\!)$. The final layer, however, runs into trouble again; as x is not free in $w(\!|u|\!)$, the binder $\lambda^{\neg}x$ may not bind into $w(\!|u|\!)$. So M itself reduces to $(w(\!|u|\!))^{\circledcirc}$. Although we have here worked through this reduction layer by layer, we emphasize that this is *one-step* reduction; this is the result of reducing a single term at a single redex.

4.3. Reduction concepts

Definition 4.2 (Reduction paths). Given a relation $\rightsquigarrow_1$ of one-step reduction, a *reduction path from* $\mathbb{X}$ is a sequence (finite or infinite) $\mathbb{X}_0, \ldots, \mathbb{X}_n, \ldots$ such that $\mathbb{X}_0 = \mathbb{X}$, and for each n, $\mathbb{X}_n \rightsquigarrow_1 \mathbb{X}_{n+1}$. For a finite reduction path $\mathbb{X}_0, \ldots, \mathbb{X}_n$, we say it is a reduction path *from* $\mathbb{X}_0$ *to* $\mathbb{X}_n$, and its length is the number n of reduction steps in it.

Definition 4.3 (Normal, strongly normalizing). A term or eliminator is *normal* iff all reduction paths from it have length 0. A term or eliminator is *strongly normalizing* iff all reduction paths from it are finite.

If a term M is strongly normalizing, then $|M|$ is the length of its longest reduction path. (If M is not strongly normalizing, $|M|$ is not defined.) We also define $|\mathcal{E}|$ for eliminators $\mathcal{E}$, but slightly differently: $|\mathcal{E}|$ is the total of all $|N|$ for $\mathcal{E}$'s immediate subterms N, and is undefined if any such $|N|$ is undefined.

Definition 4.4 (Multistep reductions). We say $\mathbb{X}$ *reduces to* $\mathbb{Y}$, written $\mathbb{X} \rightsquigarrow \mathbb{Y}$, iff there is a (necessarily finite) reduction path from $\mathbb{X}$ to $\mathbb{Y}$. We say $\mathbb{X}$ *properly reduces* to $\mathbb{Y}$, written $\mathbb{X} \rightsquigarrow^{+} \mathbb{Y}$, iff there is a reduction path from $\mathbb{X}$ to $\mathbb{Y}$ with length at least 1.

Note, now by induction on reduction paths, that if $M^{\mathfrak{C}} \rightsquigarrow N^{\mathfrak{D}}$ (and so also if $M \rightsquigarrow^{+} N$), then $\mathfrak{D} \leq \mathfrak{C}$ and $\mathtt{FV}(N) \subseteq \mathtt{FV}(M)$.

Since we have two different notions of reduction in view (principal and full), we also have two different notions of normal form, strongly normalizing, etc. It's worth pausing here to think a bit about relations between these. Since full reduction is defined in terms of all the principal redexes (and then some), we have that any principal reduction path is also a full reduction path. This gives us that any term in full normal form is also in principal normal form, and that any term that is fully strongly normalizing is also principally strongly normalizing.[15]

We also note that the full normal forms are exactly the *core* terms. Corresponding to our definition of core derivations, we say that a term is *core* iff in all its subterms of the form $M\mathcal{E}$, the term M is a variable. This is also what it takes to be a full normal form: M is an introduction iff $M\mathcal{E}$ is a principal redex, and M is an elimination iff $M\mathcal{E}$ is a commuting redex.

4.4. Reduction lemmas

Here we prove a number of facts about reduction, and about interactions between reduction and substitution, that will be used in Section 5. These facts hold for both principal and full reduction.

Lemma 4.5. *All the clauses of Definition 4.1 hold as well for $\rightsquigarrow$. That is, where*

$$\frac{\mathbb{X} \rightsquigarrow_1 \mathbb{Y}}{\mathbb{Z}(\mathbb{X}) \rightsquigarrow_1 \mathbb{Z}(\mathbb{Y})}$$

is a condition appearing in Definition 4.1, for any terms or eliminators $\mathbb{X}, \mathbb{Y}, \mathbb{Z}(\mathbb{X})$ such that $\mathbb{X} \rightsquigarrow \mathbb{Y}$: if $\mathbb{Z}(\mathbb{Y})$ is well-formed we have $\mathbb{Z}(\mathbb{X}) \rightsquigarrow \mathbb{Z}(\mathbb{Y})$, and if $\mathbb{Z}(\mathbb{Y})$ is not well-formed we have $\mathbb{Z}(\mathbb{X}) \rightsquigarrow \mathbb{Y}$.[16]

Proof: Induction on the reduction path from $\mathbb{X}$ to $\mathbb{Y}$. At each step, we need to know that if $\mathbb{Z}(\mathbb{Y})$ is well-formed and $\mathbb{W} \rightsquigarrow_1 \mathbb{Y}$, then $\mathbb{Z}(\mathbb{W})$ is also well-formed—this way, if $\mathbb{Z}(\mathbb{Y})$ is well-formed, we can ensure that all the needed intermediate links from $\mathbb{Z}(\mathbb{X})$ to $\mathbb{Z}(\mathbb{Y})$ are also well-formed. This holds, though, because of what we know about how reduction affects hats and free variables. □

[15] We do not consider in this paper, outside this footnote, the notion of *weak* normalization, where a term M counts as weakly normalizing iff there is some normal form N with $M \rightsquigarrow N$. In general, when we have two notions of reduction $\rightsquigarrow_a \subseteq \rightsquigarrow_b$, like our principal and full reductions, nothing useful follows about a relationship between weak normalization for a and b. In this regard, weak normalization is unlike both strong normalization and normal forms.

[16] Here, $\mathbb{Z}(\mathbb{X})$ should be understood as a term or eliminator with $\mathbb{X}$ as an immediate constituent, and similarly for $\mathbb{Z}(\mathbb{Y})$.

Lemma 4.6. *If $N \rightsquigarrow_1 N'$ and N is a subterm of M, then there is some M' with $M \rightsquigarrow_1 M'$ and N' a subterm of M'.*

Proof: Induction on N's being a subterm of M.

- If $N = M$ then reducing the same way yields $M' = N'$ and we're done.
- Otherwise, let O be the immediate subterm of M that contains N. By the induction hypothesis, there is some O' with $O \rightsquigarrow_1 O'$ and N' a subterm of O'. By inspecting the one-step reduction rules, we can see that there is some M' with $M \rightsquigarrow_1 M'$ and O' as a subterm. □

Lemma 4.7. *If there is a reduction path of length n from N to N' and N is a subterm of M, then there is a reduction path of length n from M to some M' such that N' is a subterm of M'.*

Proof: Induction on the reduction path from N to N', using Lemma 4.6 at each step. □

Lemma 4.8. *If M is strongly normalizing and N is a subterm of M, then N is also strongly normalizing, and $|N| \leq |M|$.*

Proof: Immediate from Lemma 4.7. □

Lemma 4.9. *If M is strongly normalizing and $M \rightsquigarrow^+ M'$, then M' is strongly normalizing and $|M'| < |M|$.*

Proof: Immediate from definitions. □

Lemma 4.10 (Substitution lemma (see [2, 2.1.16])).
Let $\sigma = [x_1 \mapsto P_1, \ldots, x_m \mapsto P_m]$ and $\tau = [y_1 \mapsto Q_1, \ldots, y_n \mapsto Q_n]$ be substitutions such that all x_i are distinct from all y_j and no x_i occurs free in any Q_j. Let (σ^τ) be the substitution $[x_1 \mapsto P_1\tau, \ldots, x_m \mapsto P_m\tau]$. Then $\mathbb{X}\sigma\tau = \mathbb{X}\tau(\sigma^\tau)$.

Proof: Induction on $\mathbb{X}$.

- $\mathbb{X}$ is a variable. If $\mathbb{X}$ is no x_i or y_j, then both sides are M. If $\mathbb{X}$ is x_i, then both sides are $P_i\tau$. if $\mathbb{X}$ is y_j, then both sides are Q_j.
- $\mathbb{X}$ is $(\!|O|\!)$ or $\langle N, O\rangle$ or $\mathsf{inl}(N)$ or $\mathsf{inr}(N)$ or $N\mathcal{E}$. These cases follow immediately from the induction hypothesis.

- $\mathbb{X}$ is $\lambda^{\to}z.N$. Set up $\lambda^{\to}z.N$'s bound variables so that z is no x_i or y_j, and so that z is not free in any P_i or Q_j. The the induction hypothesis suffices, since $\mathbb{X}\sigma\tau = \lambda^{\to}z.(N\sigma\tau)$ and $\mathbb{X}\tau(\sigma^{\tau}) = \lambda^{\to}z.(N\tau(\sigma^{\tau}))$.

- $\mathbb{X}$ is a $\lambda^{\neg}$ term or an eliminator other than $(\!|N|\!)$. The reasoning in these cases is parallel to the $\lambda^{\to}$ case. $\square$

LEMMA 4.11 (Substitution in redexes). *If R is a redex and R' is its reduct, then $R[x_1 \mapsto P_1, \ldots, x_n \mapsto P_n]$ is a redex and $R'[x_1 \mapsto P_1, \ldots, x_n \mapsto P_n]$ is its reduct.*

PROOF: Verifying is a matter of checking each kind of redex in turn. That substitution preserves redexhood is relatively straightforward, so we turn to the second part of the claim. Let σ be the substitution $[x_1 \mapsto P_1, \ldots, x_n \mapsto P_n]$, and change bound variables in R so that no x_i is bound in R and no variable free in any P_i is bound in R.

Principal redexes:

- If R is $(\lambda^{\to}x.(M^{\psi}))(\!|N, y.O|\!)$, then R' is $O[y \mapsto M[x \mapsto N]]$. By setting up R's bound variables (which certainly include x and y) as we have, $R\sigma = (\lambda^{\to}x.M\sigma)(\!|N\sigma, y.O\sigma|\!)$, and so its reduct is $O\sigma[y \mapsto M\sigma[x \mapsto N\sigma]]$. By Lemma 4.10 (twice) this is $O[y \mapsto M[x \mapsto N]]\sigma$, which is $R'\sigma$.

- If R is $(\lambda^{\to}x.(M^{☹}))(\!|N, y.O|\!)$, then R' is $M[x \mapsto N]$. By setting up bound variables as we have, $R\sigma = (\lambda^{\to}x.M\sigma)(\!|N\sigma, y.O\sigma|\!)$, and so its reduct is $M\sigma[x \mapsto N\sigma]$. By Lemma 4.10, this is $M[x \mapsto N]\sigma$, which is $R'\sigma$.

- If R is $\langle M, N\rangle(\!|\langle x, y\rangle.O|\!)$, then R' is $O[x \mapsto M, y \mapsto N]$. By setting up bound variables as we have, $R\sigma = \langle M\sigma, N\sigma\rangle(\!|\langle x, y\rangle.O\sigma|\!)$, and so its reduct is $O\sigma[x \mapsto M\sigma, y \mapsto N\sigma]$. By Lemma 4.10 this is $O[x \mapsto M, y \mapsto N]\sigma$, which is $R'\sigma$.

- If R is $\mathsf{inl}(M)(\!|x.N, y.O|\!)$ or $\mathsf{inr}(M)(\!|x.N, y.O|\!)$ or $(\lambda^{\neg}x.M)(\!|N|\!)$, the reasoning is parallel to the above cases.

As for commuting redexes: If R is $(M\mathcal{E})\mathcal{F}$, then R' is $M(\!|\mathcal{E}\mathcal{F}|\!)$, and $R\sigma = ((M\sigma)(\mathcal{E}\sigma))(\mathcal{F}\sigma)$. The reduct of $R\sigma$ is thus $(M\sigma)(\!|(\mathcal{E}\sigma)(\mathcal{F}\sigma)|\!)$. By Lemma 3.2 this is $M\sigma((\!|\mathcal{E}\mathcal{F}|\!)\sigma)$; and by Lemma 4.10 this is in turn $(M(\!|\mathcal{E}\mathcal{F}|\!))\sigma$, which is $R'\sigma$. $\square$

Lemma 4.12 (Substitution and reduction).
If $\mathbb{X} \rightsquigarrow \mathbb{Y}$, *then* $\mathbb{X}[x_1 \mapsto P_1, \ldots, x_n \mapsto P_n] \rightsquigarrow \mathbb{Y}[x_1 \mapsto P_1, \ldots, x_n \mapsto P_n]$.

Proof: Because of the complications in our notion of one-step reduction, Lemma 4.11 does not immediately suffice for this claim; it needs to be worked through.

It suffices to show that if $\mathbb{X} \rightsquigarrow_1 \mathbb{Y}$, then for all substitutions σ we have $\mathbb{X}\sigma \rightsquigarrow_1 \mathbb{Y}\sigma$. This we show by induction on the formation of $\mathbb{X}$, explicitly stating only some representative cases. (Recall that all substitutions preserve hat exactly.)

- If $\mathbb{X}$ is a variable x, then there's nothing to show, since it's false that $x \rightsquigarrow_1 \mathbb{Y}$.

- If $\mathbb{X}$ is $N\mathcal{E}$, there are three possibilities for $\mathbb{X} \rightsquigarrow_1 \mathbb{Y}$: the redex is in N, in $\mathcal{E}$, or is $N\mathcal{E}$ itself.

  - If the redex is inside N, let N' be the result of reducing N at that redex. Applying the induction hypothesis, $N\sigma \rightsquigarrow_1 N'\sigma$; moreover, N' and $N'\sigma$ have the same hat.

    * If this hat is ☹, then $\mathbb{Y} = N'$, and so $\mathbb{X}\sigma = (N\sigma)(\mathcal{E}\sigma) \rightsquigarrow_1 N'\sigma = \mathbb{Y}\sigma$.
    * If it is some φ, then $\mathbb{Y} = N'\mathcal{E}$, and so $\mathbb{X}\sigma = (N\sigma)(\mathcal{E}\sigma) \rightsquigarrow_1 (N'\sigma)(\mathcal{E}\sigma) = \mathbb{Y}\sigma$.

  - If the redex is inside $\mathcal{E}$, the reasoning is parallel, except instead of concern for hats, we are concerned whether $\mathcal{E}$ reduces at this redex to an eliminator or a term.

  - If the redex is $N\mathcal{E}$ itself, we're covered by Lemma 4.11.

- If $\mathbb{X}$ is $\lambda^{\rightarrow}x.N$, change its bound variables so that x is not among the x_i and not free in any P_i. The redex securing $\mathbb{X} \rightsquigarrow_1 \mathbb{Y}$ must be inside N. Let N' be the result of reducing N at that redex. Applying the induction hypothesis, $N\sigma \rightsquigarrow_1 N'\sigma$. Moreover, N' and $N'\sigma$ have the same hat, and x is free in N' iff it is free in $N'\sigma$. Thus, $\lambda^{\rightarrow}x.N'$ is well-formed iff $\lambda^{\rightarrow}x.(N'\sigma)$ is.

  - If they are well-formed, then $\mathbb{Y} = \lambda^{\rightarrow}x.N'$, and so $\mathbb{X}\sigma = \lambda^{\rightarrow}x.(N\sigma) \rightsquigarrow_1 \lambda^{\rightarrow}x.(N'\sigma) = \mathbb{Y}\sigma$.

– If they are not, then $\mathbb{Y} = N'$, and so $\mathbb{X}\sigma = \lambda^{\to}x.(N\sigma) \rightsquigarrow_1 N'\sigma = \mathbb{Y}\sigma$.

- Other cases without bound variables are like the case of $N\mathcal{E}$; other cases with bound variables are like the case of $\lambda^{\to}x.N$. □

5. Strong normalization

The foregoing discussion covers both principal and full reduction. In this section, we narrow our attention to principal reduction only, and show that every term in our system is (principally) strongly normalizing. In this, we closely follow the approach of [4]. (Again, we conjecture that full reduction is also strongly normalizing, but leave that question, which requires different techniques, for future work.)

5.1. The Prawitz restriction revisited

First, however, we return briefly to the topic of Sections 2.4 and 3.5: the Prawitz restriction, which Tennant imposes and we do not. In Section 2.4 we saw that the Prawitz restriction rules out a range of derivations that we allow, and in Section 3.5 we saw that these derivations include some with important computational interpretations. That much alone, we think, motivates our dropping the Prawitz restriction. However, there is another interesting effect of the restriction, which we point out here: it blocks strong normalization, even for principal reduction (and therefore for full reduction as well). To show this, we use a (slightly modified) example of [9]. (Spelling this out in our term language would save space, but at the cost of even lower readability, so we return to derivations for the example.)

Look to the three derivations in Figure 1. Note that the first principally reduces (at the redex indicated with $\star$) to the second, and the second principally reduces (at the redex indicated with $\star$) to the third. Note also that the first and second obey the Prawitz restriction, but the third does not; the step of $\to$I indicated with † in the third derivation can discharge open assumptions of p, and indeed there are two open assumptions of p in scope at that step in the derivation, also indicated with †.

Reduction in a system obeying the Prawitz restriction, then, could not reduce the second derivation here to the third, since the third does not belong in such a system. Rather, it would reduce the second derivation

$$
{\to}\mathrm{E}\;\dfrac{{\to}\mathrm{I}^{1}\;\dfrac{{\to}\mathrm{E}\;\dfrac{{\to}\mathrm{I}\;\dfrac{p}{q \to p} \qquad {\to}\mathrm{E}\;\dfrac{{\to}\mathrm{I}\;\dfrac{[q]^{1}}{p \to q} \qquad {\to}\mathrm{E}\;\dfrac{{\to}\mathrm{I}\;\dfrac{p}{q \to p} \qquad [q]^{1}}{p}}{q}}{p}}{q \to p} \qquad {\to}\mathrm{E}\;\dfrac{{\to}\mathrm{I}\;\dfrac{q}{p \to q} \qquad {\to}\mathrm{E}\;\dfrac{{\to}\mathrm{I}\;\dfrac{p}{q \to p} \qquad {\to}\mathrm{E}\;\dfrac{{\to}\mathrm{I}\;\dfrac{q}{p \to q} \qquad {\to}\mathrm{E}{\star}\;\dfrac{{\to}\mathrm{I}{\star}\;\dfrac{p}{q \to p} \qquad q}{p}}{q}}{p}}{q}}{p}
$$

$$
{\to}\mathrm{E}{\star}\;\dfrac{{\to}\mathrm{I}^{1}{\star}\;\dfrac{{\to}\mathrm{E}\;\dfrac{{\to}\mathrm{I}\;\dfrac{p}{q \to p} \qquad {\to}\mathrm{E}\;\dfrac{{\to}\mathrm{I}\;\dfrac{[q]^{1}}{p \to q} \qquad {\to}\mathrm{E}\;\dfrac{{\to}\mathrm{I}\;\dfrac{p}{q \to p} \qquad [q]^{1}}{p}}{q}}{p}}{q \to p} \qquad {\to}\mathrm{E}\;\dfrac{{\to}\mathrm{I}\;\dfrac{q}{p \to q} \qquad {\to}\mathrm{E}\;\dfrac{{\to}\mathrm{I}\;\dfrac{p}{q \to p} \qquad {\to}\mathrm{E}\;\dfrac{{\to}\mathrm{I}\;\dfrac{q}{p \to q} \qquad p}{q}}{p}}{q}}{p}
$$

$$
{\to}\mathrm{E}\;\dfrac{{\to}\mathrm{I}\;\dfrac{p}{q \to p} \qquad {\to}\mathrm{E}\;\dfrac{{\to}\mathrm{I}^{\dagger}\;\dfrac{{\to}\mathrm{E}\;\dfrac{{\to}\mathrm{I}\;\dfrac{q}{p \to q} \qquad {\to}\mathrm{E}\;\dfrac{{\to}\mathrm{I}\;\dfrac{p^{\dagger}}{q \to p} \qquad {\to}\mathrm{E}\;\dfrac{{\to}\mathrm{I}\;\dfrac{q}{p \to q} \qquad p^{\dagger}}{q}}{p}}{q}}{p \to q} \qquad {\to}\mathrm{E}\;\dfrac{{\to}\mathrm{I}\;\dfrac{p}{q \to p} \qquad {\to}\mathrm{E}\;\dfrac{{\to}\mathrm{I}\;\dfrac{q}{p \to q} \qquad {\to}\mathrm{E}\;\dfrac{{\to}\mathrm{I}\;\dfrac{p}{q \to p} \qquad {\to}\mathrm{E}\;\dfrac{{\to}\mathrm{I}\;\dfrac{q}{p \to q} \qquad p}{q}}{p}}{q}}{p}}{q^{\ddagger}}}{p}
$$

Figure 1. Strong normalization fails in Tennant's original system

here to a derivation much like the third, but which discharges the indicated open assumptions of p at the indicated step of $\to$I.

That, in turn, would defeat strong normalization: look to the q node indicated with ‡ in the third derivation, and consider the subderivation from that node upwards. With the binding in place needed to meet the Prawitz restriction, this subderivation is isomorphic to the original derivation, just with the roles of p and q switched. So we can repeat the cycle endlessly, producing an infinite reduction path.

Without the Prawitz restriction, on the other hand, the second derivation reduces to the third, with no additional binding needed. No cycle is created. And as we now show, indeed strong normalization does hold for our system.

5.2. Proving strong normalization

Definition 5.1. We define a notion of *strongly computable term* (SC term) by induction on hats:

- For an atomic type p, a term M^p is SC iff it is strongly normalizing;
- A term $M^{\circledcirc}$ is SC iff it is strongly normalizing;
- A term $M^{\varphi\wedge\psi}$ is SC iff it is strongly normalizing and whenever it reduces to a term $\langle N, O\rangle$, both N and O are SC;
- A term $M^{\varphi\vee\psi}$ is SC iff it is strongly normalizing and whenever it reduces to either $\mathsf{inl}(N)$ or $\mathsf{inr}(N)$, then N is SC; and
- A term $M^{\varphi\to\psi}$ is SC iff it is strongly normalizing and whenever it reduces to a term $\lambda^{\to}x.N$, then for all SC terms O^{φ}, the term $N[x \mapsto O]$ is SC.[17]
- A term $M^{\neg\varphi}$ is SC iff it is strongly normalizing and whenever it reduces to a term $\lambda^{\neg}x.N$, then for all SC terms O^{φ}, the term $N[x \mapsto O]$ is SC.

It is clear from this definition that every SC term is strongly normalizing. Then we show by induction on *terms* that every term is SC. This

[17] [13], which features a similar proof, has a slightly different definition here, following [7, Appendix A3], but that doesn't consider conjunction or disjunction. Here, we follow [4].

works because the inductive structures of terms and of types do not align, so we can play them off against each other.

Lemma 5.2 (Variables). *For any type φ, every variable of type φ is SC.*

Proof: All variables x^φ do not contain any redexes as subterms, thus do not have any one-step reductions, and hence all reduction paths from x^φ are of length 0, so finite. When φ is complex, the additional conditions following "whenever it reduces" are vacuously fulfilled, as variables never reduce to such forms. So all variables are SC. $\square$

Lemma 5.3 (Closure by reduction). *If M is SC and $M \rightsquigarrow N$, then N is SC.*[18]

Proof: Note first that if M is strongly normalizing and $M \rightsquigarrow N$, then N too must be strongly normalizing; any infinite reduction path starting from N would give rise to an infinite reduction path starting from M. Since M is SC, it must be strongly normalizing, so N too must be strongly normalizing.

It remains only to check the additional requirements for N to be SC, according to N's hat. Recall that if N is N^φ, then M must be M^φ.

- If N is $N^{☹}$, then there are no additional requirements, and we're done.
- If N is N^p for an atomic type p, then there are no additional requirements, and we're done.
- If $M^{\varphi\wedge\psi} \rightsquigarrow N^{\varphi\wedge\psi}$, then if $N^{\varphi\wedge\psi}$ reduces to $\langle O, P\rangle$ so does M. Since M is SC, in this case O and P must be SC, so the additional requirement on N is met.
- If $M^{\varphi\vee\psi} \rightsquigarrow N^{\varphi\vee\psi}$, then if $N^{\varphi\vee\psi}$ reduces to $\mathsf{inl}(O)$ or $\mathsf{inr}(O)$ so does M. Since M is SC, in these cases O must be SC, so the additional requirement on N is met.
- If $M^{\varphi\to\psi} \rightsquigarrow N^{\varphi\to\psi}$, then if N reduces to $\lambda^{\to}x.O$ so does M. Since M is SC, in these cases it must be that for all SC terms P^φ, the term $O[x \mapsto P]$ is SC. So the additional requirement on N is met.

[18] Note that M and N needn't have the same hat, so this claim precisely as stated in [4] would be false.

- If $M^{\neg\varphi} \rightsquigarrow N^{\neg\varphi}$, then if N reduces to $\lambda^{\neg}x.O$ so does M. Since M is SC, in these cases it must be that for all SC terms P^{φ}, the term $O[x \mapsto P]$ is SC. So the additional requirement on N is met. □

LEMMA 5.4 (Girard's lemma). *Let M be a term that is not an introduction, such that for all N with $M \rightsquigarrow_1 N$, N is SC. Then M is SC.*

PROOF: If there does not exist such an N then M is SC because M does not have any one-step reductions, hence all reduction paths from M are of finite 0 length and additional requirements depending on hat do not apply.

Since N is SC, every reduction path is finite from N, hence M is strongly normalizing because M reduces finitely in one step to N.

- If all N have hat ☹, then M is SC because M is SN and additional requirements depending on hat don't apply because M does not reduce to any introductions.

- If there exists N with an atomic hat, then M has an atomic hat and is SC because M is SN.

Since M is not an introduction, it is not, in reduction to itself, required to satisfy the additional conditions for M to be SC for the following hats:

- If there exists N with a hat of the form $\varphi \wedge \psi$, then M has hat $\varphi \wedge \psi$. If $M \rightsquigarrow_1 N \rightsquigarrow \langle O, P \rangle$, O and P are SC because N is SC. Since M is strongly normalizing and whenever M reduces to a term $\langle O, P \rangle$, O and P are SC, M is SC.

- If there exists N with a hat of the form $\varphi \vee \psi$, then M has hat $\varphi \vee \psi$. If $M \rightsquigarrow_1 N \rightsquigarrow \mathsf{inl}(O)$ or $M \rightsquigarrow_1 N \rightsquigarrow \mathsf{inr}(O)$, O is SC because N is strongly normalizing. Since M is SN and whenever M reduces to a term $\mathsf{inl}(O)$ or $\mathsf{inr}(O)$, O is SC, M is SC.

- If there exists N with hat $\varphi \to \psi$, then M has hat $\varphi \to \psi$. If $M \rightsquigarrow_1 N \rightsquigarrow \lambda^{\to}x.O$, for all SC terms P^{φ}, the term $O[x \mapsto P]$ is SC. Since M is strongly normalizing and whenever M reduces to a term $\lambda^{\to}x.O$, for all SC terms P^{φ}, the term $O[x \mapsto P]$ is SC, M is SC.

- If there exists N with hat $\neg\varphi$, then M has hat $\neg\varphi$. If $M \rightsquigarrow_1 N \rightsquigarrow \lambda^{\neg}x.O$, for all SC terms P^{φ}, the term $O[x \mapsto P]$ is SC. Since M is strongly normalizing and whenever M reduces to a term $\lambda^{\neg}x.O$, for all SC terms P^{φ}, the term $O[x \mapsto P]$ is SC, M is SC. □

LEMMA 5.5 (Adequacy of λ (I)). *If for all SC M^{φ} we have $N^{\psi}[x \mapsto M]$ is SC, then $(\lambda^{\rightarrow}x.N)^{\varphi\rightarrow\psi}$ is SC.*

PROOF: By Lemma 5.2 , all variables are SC. Let $M := x$, $N[x \mapsto x] = N$ is SC and hence N is strongly normalizing. Thus, $\lambda^{\rightarrow}x.N$ is strongly normalizing because the only possible reductions involve reducing N within the term or reduction to an exceptional term. Thus, the reduction paths of N bind the reduction paths of $\lambda^{\rightarrow}x.N$.

If $\lambda^{\rightarrow}x.N \rightsquigarrow \lambda^{\rightarrow}x.N'$, then $N \rightsquigarrow N'$ by the reduction rules. By Lemma 4.12, $N[x \mapsto M] \rightsquigarrow N'[x \mapsto M]$ and $N'[x \mapsto M]$ is SC by Lemma 5.3.

Thus, $\lambda^{\rightarrow}x.N$ is SC because it is strongly normalizing and whenever it reduces to $\lambda^{\rightarrow}x.N'$, for any SC M^{φ}, $N'[x \mapsto M]$ is SC. □

LEMMA 5.6 (Adequacy of λ (II)). *If for all SC M^{φ} we have $N^{\frownie}[x \mapsto M]$ is SC (and so long as $x \in \mathtt{FV}(N)$), then $(\lambda^{\rightarrow}x.N)^{\varphi\rightarrow\psi}$ and $(\lambda^{\neg}x.N)^{\neg\varphi}$ are both SC.*

PROOF: By Lemma 5.2 , all variables are SC. Let $M := x$, $N[x \mapsto x] = N$ is SC and hence N is strongly normalizing. Thus, both $\lambda^{\rightarrow}x.N$ and $\lambda^{\neg}x.N$ are strongly normalizing because the only possible reductions involve reducing N within the term or reduction to an exceptional term. Thus, the reduction paths of N bind the reduction paths of $\lambda^{\rightarrow}x.N$ and $\lambda^{\neg}x.N$.

If $\lambda^{\rightarrow}x.N \rightsquigarrow \lambda^{\rightarrow}x.N'$ or $\lambda^{\neg}x.N \rightsquigarrow \lambda^{\neg}x.N'$, then $N \rightsquigarrow N'$ by the reduction rules. By Lemma 4.12, $N[x \mapsto M] \rightsquigarrow N'[x \mapsto M]$ and $N'[x \mapsto M]$ is SC by Lemma 5.3.

Thus, $\lambda^{\rightarrow}x.N$ and $\lambda^{\neg}x.N$ are SC because they are strongly normalizing and whenever they respectively reduce to $\lambda^{\rightarrow}x.N'$ and $\lambda^{\neg}x.N'$, for any SC M^{φ}, $N'[x \mapsto M]$ is SC. □

LEMMA 5.7 (Adequacy of $\langle,\rangle$). *If M^{φ} and N^{ψ} are both SC, then $\langle M, N\rangle^{\varphi\wedge\psi}$ is SC.*

PROOF: $\langle M, N\rangle$ is strongly normalizing because the only possible reductions involve reducing M and N within the term or reduction to an exceptional term. Thus, since M and N are strongly normalizing, their reduction paths bind the reduction paths of $\langle M, N\rangle$.

By Lemma 5.3, if $M \rightsquigarrow M'$ and $N \rightsquigarrow N'$ then M' and N' are SC.

Whenever $\langle M, N\rangle$ reduces to an introduction $\langle M', N'\rangle$, M' and N' are SC, thus, since $\langle M, N\rangle$ is also strongly normalizing, by Definition 5.1 it is SC. $\square$

LEMMA 5.8 (Adequacy of inl, inr). *If M^{φ} is SC, then $\mathsf{inl}(M)$ and $\mathsf{inr}(M)$ are both SC.*

PROOF: Wlog, we consider just $\mathsf{inl}(M)$.

$\mathsf{inl}(M)$ is strongly normalizing because the only possible reductions involve reducing M within the term or reduction to an exceptional term. Thus, since M is strongly normalizing, reduction paths from $\mathsf{inl}(M)$ are bound by reduction paths of M.

By Lemma 5.3 if $M \rightsquigarrow M'$, then M' is SC.

Whenever $\mathsf{inl}(M)$ reduces to an introduction $\mathsf{inl}(M')$, M' is SC, thus, since $\mathsf{inl}(M)$ is also strongly normalizing, by Definition 5.1 it is SC. $\square$

LEMMA 5.9 (Adequacy of application (I)). *If $M^{\varphi\to\psi}$ is SC, N^{φ} is SC, and for all SC Q^{ψ}, $O[x \mapsto Q]$ is SC, then $M(\!|N, x.O|\!)$ is SC.*

PROOF: Let $Q = x$ where x is SC by Lemma 5.2, thus $O[x \mapsto x] = O$ is SC. Since M, N and O are SC, they are strongly normalising and hence $|M|$, $|N|$ and $|O|$ are defined. We proceed by induction on $|M|+|N|+|O|$. By Lemma 5.4, to prove that $M(\!|N, x.O|\!)$ is SC, we need to prove that all one-step reducts are SC. Given $M \rightsquigarrow_1 M'$ or $N \rightsquigarrow_1 N'$ or $O \rightsquigarrow_1 O'$ where M', N', and O' are SC by Lemma 5.3:

- If $M(\!|N, x.O|\!) \rightsquigarrow_1 M'(\!|N, x.O|\!)$ or $M(\!|N, x.O|\!) \rightsquigarrow_1 M(\!|N', x.O|\!)$ or $M(\!|N, x.O|\!) \rightsquigarrow_1 M(\!|N, x.O'|\!)$, then we apply the induction hypothesis and Lemma 4.9 to obtain $|M| + |N| + |O| > |M'| + |N| + |O|$, $|M|+|N|+|O| > |M|+|N'|+|O|$ or $|M|+|N|+|O| > |M'|+|N|+|O'|$.

- If $M(\!|N, x.O|\!) \rightsquigarrow_1 M'^{☹}$ or $M(\!|N, x.O|\!) \rightsquigarrow_1 N'^{☹}$ or $M(\!|N, x.O|\!) \rightsquigarrow_1 O'^{☹}$, then we already have M', N', or O' SC.

- If $M(\!|N, x.O|\!)$ is a principal redex, then M is of the form $\lambda^{\to}y.P^{\mathfrak{D}}$. If $\mathfrak{D} = ☹$, then $M(\!|N, x.O|\!) \rightsquigarrow_1 P[y \mapsto N]$ which is SC by Definition 5.1. Otherwise $M(\!|N, x.O|\!) \rightsquigarrow_1 O[x \mapsto P[y \mapsto N]]$ which is SC by the lemma statement. $\square$

Lemma 5.10 (Adequacy of application (II)). *If $M^{\neg\varphi}$ is SC and N^{φ} is SC, then $M(\!|N|\!)$ is SC.*

Proof: Since M and N are SC, they are strongly normalising and hence $|M|$ and $|N|$ are defined. We proceed by induction on $|M| + |N|$. By Lemma 5.4, to prove that $M(\!|N|\!)$ is SC, we need to prove that all one-step reducts are SC. Given $M \rightsquigarrow_1 M'$ or $N \rightsquigarrow_1 N'$ where M' and N' are SC by Lemma 5.3:

- If $M(\!|N|\!) \rightsquigarrow_1 M'(\!|N|\!)$ or $M(\!|N|\!) \rightsquigarrow_1 M(\!|N'|\!)$ then we apply the induction hypothesis and Lemma 4.9 to obtain $|M| + |N| > |M'| + |N|$ or $|M| + |N| > |M| + |N'|$.
- If $M(\!|N|\!) \rightsquigarrow_1 M'^{☹}$ or $M(\!|N|\!) \rightsquigarrow_1 N'^{☹}$, then we already have M' or N' SC.
- If $M(\!|N|\!)$ is a principal redex, then M is of the form $\lambda^{\neg}x.O$, and $M(\!|N|\!) \rightsquigarrow_1 O[x \mapsto N]$ which is SC by Definition 5.1. □

Lemma 5.11 (Adequacy of Conjunction elimination). *If $M^{\varphi\wedge\psi}$ is SC, and for all SC P^{φ}, Q^{ψ} the term $N[x \mapsto P, y \mapsto Q]$ is SC, then $M(\!|\langle x, y\rangle.N|\!)$ is SC (if well-formed).*

Proof: Let $P = x$ and $Q = y$ where x and y are SC by Lemma 5.2, thus $N[x \mapsto x, y \mapsto y] = N$ is SC. We proceed by induction on $|M| + |N|$. By Lemma 5.4, to prove that $M(\!|\langle x, y\rangle.N|\!)$ is SC, we need to prove that all one-step reducts are SC. Given $M \rightsquigarrow_1 M'$ and $N \rightsquigarrow_1 N'$ where M' and N' are SC by Lemma 5.3:

- If $M(\!|\langle x, y\rangle.N|\!) \rightsquigarrow_1 M'(\!|\langle x, y\rangle.N|\!)$ or $M(\!|\langle x, y\rangle.N|\!) \rightsquigarrow_1 M(\!|\langle x, y\rangle.N'|\!)$ then we apply the induction hypothesis and Lemma 4.9 to obtain $|M| + |N| > |M'| + |N|$ or $|M| + |N| > |M| + |N'|$.
- If $M(\!|\langle x, y\rangle.N|\!) \rightsquigarrow_1 M'^{☹}$ or $M(\!|\langle x, y\rangle.N|\!) \rightsquigarrow_1 N'^{☹}$, then we already have M' and N' SC.
- If $M(\!|\langle x, y\rangle.N|\!)$ is a principal redex, then M is of the form $\langle R, S\rangle$ and $M(\!|\langle x, y\rangle.N|\!) \rightsquigarrow_1 N[x \mapsto R, y \mapsto S]$ which is SC by the lemma statement and Definition 5.1. □

LEMMA 5.12 (Adequacy of Disjunction elimination). *If $M^{\varphi\vee\psi}$ is SC, and for all SC P^{φ} the term $N[x \mapsto P]$ is SC, and for all SC Q^{ψ} the term $O[y \mapsto Q]$ is SC, then $M(\!|x.N, y.O|\!)$ is SC (if well-formed).*

PROOF: Let $P = x$ and $Q = y$ where x and y are SC by Lemma 5.2, thus $N[x \mapsto x] = N$ and $O[y \mapsto y] = O$ are SC. Since M, N and O are SC, they are strongly normalising and hence $|M|$, $|N|$ and $|O|$ are defined. We proceed by induction on $|M| + |N| + |O|$. By Lemma 5.4, to prove that $M(\!|x.N, y.O|\!)$ is SC, we need to prove that all one-step reducts are SC. Given $M \rightsquigarrow_1 M'$ or $N \rightsquigarrow_1 N'$ or $O \rightsquigarrow_1 O'$ where M', N', and O' are SC by Lemma 5.3:

- If $M(\!|x.N, y.O|\!) \rightsquigarrow_1 M'(\!|x.N, y.O|\!)$ or $M(\!|x.N, y.O|\!) \rightsquigarrow_1 M(\!|x.N', y.O|\!)$ or $M(\!|x.N, y.O|\!) \rightsquigarrow_1 M(\!|x.N, y.O'|\!)$, then we apply the induction hypothesis and Lemma 4.9 to obtain $|M|+|N|+|O| > |M'|+|N|+|O|$, $|M|+|N|+|O| > |M|+|N'|+|O|$ or $|M|+|N|+|O| > |M'|+|N|+|O'|$.

- If $M(\!|x.N, y.O|\!) \rightsquigarrow_1 M'$ or $M(\!|x.N, y.O|\!) \rightsquigarrow_1 N'$ or $M(\!|x.N, y.O|\!) \rightsquigarrow_1 O'$, then we already have M', N', or O' SC.

- If $M(\!|x.N, y.O|\!)$ is a principal redex, then M is of the form $\mathsf{inl}(R)$ or $\mathsf{inr}(R)$ and $M(\!|x.N, y.O|\!) \rightsquigarrow_1 N[x \mapsto R]$ or $M(\!|x.N, y.O|\!) \rightsquigarrow_1 O[y \mapsto R]$ which are both SC by the lemma statement and Definition 5.1. □

DEFINITION 5.13. A substitution $[x_1 \mapsto P_1, \ldots, x_n \mapsto P_n]$ is *SC* iff $P_1, \ldots, P_n$ are all SC. A term M is *SC under substitution* iff for all SC substitutions σ, the term $M\sigma$ is SC.

THEOREM 5.14. *All terms are SC under substitution.*

PROOF: Take any term M. To see that M is SC under substitution, proceed by induction on M's formation.

- If M is x^{φ} then any substitution for x will be a variable and Lemma 5.2 applies.

- If M is $\langle N, O\rangle$: take any SC substitution σ. By the induction hypothesis, N and O are SC under substitution, so $N\sigma$ and $O\sigma$ are SC. Thus, by Lemma 5.7, $\langle N\sigma, O\sigma\rangle$ is SC; but this is just $M\sigma$.

- If M is $\mathsf{inl}(N)$ or $\mathsf{inr}(N)$, the reasoning is similar to the $\langle,\rangle$ case.

- If M is $\lambda^{\to} x^{\varphi}.N$: take any SC substitution σ, and change M's bound variables so that x is neither acted on by σ nor free in σ. By the induction hypothesis, N is SC under substitution, so for any SC term P^{φ}, we have that $N\sigma[x \mapsto P]$ is SC. Thus, by Lemma 5.5 and Lemma 5.6, $\lambda^{\to} x.(N\sigma)$ is SC; but this is just $M\sigma$.

- If M is $\lambda^{\neg} x.M$, the reasoning is similar to the $\lambda^{\to}$ case.

- If M is $N(\!|O, x.P|\!)$: take any SC substitution σ, and change M's bound variables so that x is neither acted on by σ nor free in σ. By the induction hypothesis, N, O and P are SC under substitution, so $N\sigma$, $O\sigma$ and $P\sigma$ are SC. Given SC Q^{φ}, we have $P\sigma[x \mapsto Q]$ is SC. Thus, by Lemma 5.9, $N\sigma(\!|O\sigma, x.P\sigma|\!)$ is SC; but this is just $M\sigma$.

- If M is $N(\!|O|\!)$: take any SC substitution σ. By the induction hypothesis, N and O are SC under substitution, so $N\sigma$ and $O\sigma$ are SC. Thus, by Lemma 5.10, $N\sigma(\!|O\sigma|\!)$ is SC; but this is just $M\sigma$.

- If M is $N(\!|\langle x, y\rangle.O|\!)$: take any SC substitution σ, and change M's bound variables so that x and y are neither acted on by σ nor free in σ. By the induction hypothesis, N and O are SC under substitution, so $N\sigma$ and $O\sigma$ are SC. Given SC P^{φ} and Q^{ψ}, $O[x \mapsto P, y \mapsto Q]$ is SC. Thus, by Lemma 5.11, $N\sigma(\!|\langle x, y\rangle.O\sigma|\!)$ is SC; but this is just $M\sigma$.

- If M is $N(\!|x.O, y.P|\!)$: take any SC substitution σ, and change M's bound variables so that x and y are neither acted on by σ nor free in σ. By the induction hypothesis, N, O and P are SC under substitution, so $N\sigma$, $O\sigma$ and $P\sigma$ are SC. Given SC Q^{φ} and R^{ψ}, $O\sigma[x \mapsto Q]$ and $P\sigma[y \mapsto R]$ are SC. Thus, by Lemma 5.12, $N\sigma(\!|x.O\sigma, y.P\sigma|\!)$ is SC; but this is just $M\sigma$. □

Corollary 5.15. All terms are strongly normalizing.

Proof: Take any term M. By Theorem 5.14, M is SC under substitution; clearly, then, M is SC. (Consider the substitution $[x^{\varphi} \mapsto x^{\varphi}]$.) By Definition 5.1, then, M is strongly normalizing. □

6. Conclusion

In this paper, we've presented a natural deduction system for core logic, and developed a term calculus that corresponds to this natural deduction system. We've defined two reduction relations on this term calculus—principal and full reduction—and explored the ways that core logic's restrictions make reduction somewhat different from reduction in more familiar term calculi. We've discussed the Prawitz restriction and our reasons for dropping it. And finally, we've shown that principal reduction in this system is strongly normalizing (although it would not be with the Prawitz restriction in place). In future work, we hope to extend this strong normalization to full reduction as well, but as that will require different techniques, only time will tell.

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ON SYNONYMY IN PROOF-THEORETIC SEMANTICS. THE CASE OF 2Int

Abstract

We consider an approach to propositional synonymy in proof-theoretic semantics that is defined with respect to a bilateral G3-style sequent calculus SC2Int for the bi-intuitionistic logic 2Int. A distinctive feature of SC2Int is that it makes use of two kinds of sequents, one representing proofs, the other representing refutations. The structural rules of SC2Int, in particular its cut rules, are shown to be admissible. Next, interaction rules are defined that allow transitions from proofs to refutations, and vice versa, mediated through two different negation connectives, the well-known *implies-falsity negation* and the less well-known *co-implies-truth negation* of 2Int. By assuming that the interaction rules have no impact on the identity of derivations, the concept of inherited identity between derivations in SC2Int is introduced and the notions of positive and negative synonymy of formulas are defined. Several examples are given of distinct formulas that are either positively or negatively synonymous. It is conjectured that the two conditions cannot be satisfied simultaneously.

Keywords: bilateralism, bi-intuitionistic logic 2Int, cut-elimination, identity of derivations, synonymy.

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1. Introduction

This paper is a sequel to [18], where an approach to synonymy of formulas in proof-theoretic semantics is explored that is not based on a structural isomorphism between derivation trees and that departs from the idea of identifying only derivations of one and the same formula. The approach is *bilateral* in the sense that a distinction is drawn between two kinds of derivations, namely proofs and refutations.[1] The identification of derivations of distinct formulas is arrived at in particular by considering any proof of a formula A as a refutation of A's negation, $\sim A$, and identifying a refutation of A with a proof of $\sim A$. Such a direct relationship between proofs and refutations understood as disproofs is given in the constructive paraconsistent logic `N4` with strong negation due to Almukdad and Nelson [1], a system that was independently studied already by Prawitz (very briefly in Appendix B of [12]) and von Kutschera [14]. In [18] the notion of *inherited identity* of derivations is introduced for derivations in a cut-free sequent system for `N4` with two kinds of sequents by considering sequent rules the application of which leaves the identity of derivations untouched. The relation of inherited identity is used to define a bilateralist notion of synonymy between formulas, which is a relation drawing more fine-grained distinctions between formulas than the relation of strong equivalence that obtains between two formulas A and B if both A and B and their strong negations $\sim A$ and $\sim B$ are interderivable.

In [18] the problem was left open, whether and how the explored bilateralist conception of propositional synonymy in proof-theoretic semantics can be applied to a system closely related to `N4`, namely the bi-intuitionistic system `2Int` from [15], see also [4]. Like in the proof theory of `N4`, in proof systems for `2Int` a distinction can be drawn between proofs and refutations, there called "dual proofs". Now, however, the relationship between proofs and refutations is more intricate since the transition between them is reflected in the logical vocabulary not by the presence of a single strong negation connective, but by making use of two negation operations, the familiar *implies-falsity negation* known from intuitionistic logic and the *co-implies-truth negation* from `2Int`. These are defined on the basis of two

[1] In [19] we discuss the existing notions of bilateralism in the context of proof-theoretic semantics and propose, based on our understanding of bilateralism, an extension to logical *multilateralism* as a theory of multiple derivability relations, more specifically, as a theory of sequent calculi that make use of multiple sequent arrows.

dual implications, namely the intuitionistic conditional and a so-called "co-implication", which can be seen as the object language realizations of the two derivability relations.

This paper is devoted to applying the bilateralist approach of [18] to 2Int. For that purpose, first of all a suitable proof-theoretic presentation of 2Int is needed, and this is a central contribution of the present paper. For motivation of the bilateralist approach in the case of N4 based on a proof/disproof interpretation that amends the Brouwer-Heyting-Kolmogorov interpretation of the intuitionistic connectives and a comparison to other approaches to propositional synonymy in proof-theoretic terms we refer to [18]. In the present paper we introduce the basic ideas only to the extent of keeping the paper self-contained. In Section 2 we first present the bilateralist sequent calculus SC2Int for 2Int. Next, in Section 3, the admissibility of the structural rules of SC2Int is dealt with. A detailed proof of cut-elimination for SC2Int is given in the appendix, Section 6. Section 4 is devoted to inherited identity of derivations in SC2Int and the definition of propositional synonymy. We conclude the paper with a brief summary and outlook in Section 5.

2. The calculus SC2Int

The purpose of this section is to introduce a bi-intuitionistic sequent calculus and to give proofs of admissibility for its structural rules. The calculus we will present, called SC2Int, is a sequent calculus for the bi-intuitionistic logic 2Int from [15]. There a natural deduction system for this logic, N2Int, is given to which SC2Int is equivalent in terms of what is derivable. We spell out below what this amounts to exactly. What is important is that these calculi represent a kind of bilateralist reasoning, since they do not only internalize processes of verification or provability but also the dual processes in terms of falsification or what is called *dual provability*. In [17] a normal form theorem for N2Int is stated, here, we want to prove a cut-elimination theorem for SC2Int, which goes beyond the results existing so far.

The language $\mathscr{L}_{2Int}$ of 2Int, as given in [15], is defined in Backus-Naur form as follows:

$$A ::= p \mid \bot \mid \top \mid (A \wedge A) \mid (A \vee A) \mid (A \to A) \mid (A \prec A).$$

As can be seen, we have a non-standard connective in this language, namely the operator of co-implication, $\prec$, which acts as a dual to implication, just like conjunction and disjunction can be seen as dual connectives.[2] With that, we are in the realms of so-called bi-intuitionistic logic, which is a conservative extension of intuitionistic logic with co-implication.[3] We read $A \prec B$ as 'B co-implies A'.

The general design of `SC2Int` resembles the intuitionistic sequent calculus `G3ip`. The distinguishing features of this calculus consist in the shared contexts for all the logical rules, the axiom (in our calculus the reflexivity rules) being restricted to atomic formulas and the admissibility of *all* structural rules (cf. [10, pp. 28–30] for more information about the origins of this calculus). Another distinguishing feature is the repetition of $A \to B$ in the left premise of the left introduction rule for implication, which is necessary for the proof of admissibility of contraction. Here, this happens in $\to L^a$ as well as with $A \prec B$ in $\prec L^c$.

We will use $p, q, r, ...$ for atomic formulas, $A, B, C, ...$ for arbitrary formulas, and Γ, Δ, Γ', ... for multisets of formulas. For a singleton multiset $\{A\}$ we usually write just A, and A, Γ as well as Γ, A (Δ, Γ as well as Γ, Δ) designates the union of the multisets Γ and $\{A\}$ (Δ and Γ). Sequents are of the form $(\Gamma; \Delta) \vdash^* C$ (with Γ and Δ being finite, possibly empty multisets), which are read as "From the verification of all formulas in Γ and the falsification of all formulas in Δ one can derive the verification (resp.

[2]An anonymous reviewer raised the question whether co-implication as the dual of implication is again an implication (and the co-negation defined in section 4 is indeed a negation), whereas conjunction as the dual of disjunction is not a disjunction, and disjunction as the dual of conjunction is not a conjunction. Thus, is the dual of a logical operation of a kind different from the kind of operation from which it is a dual? We cannot address this general question here, or the questions "What is an implication?" and "What is a negation?". As far as `2Int` is concerned, there is a clear sense in which implication, $\to$, and co-implication, $\prec$, are of the same kind. In a two-sorted term calculus for `2Int`, see [2], the rule for introducing $\to$ on the right of a sequent arrow in proofs (for introducing $\prec$ on the right of a sequent arrow in dual proofs) comes with λ-abstraction, and the rule for introducing $\to$ on the left of a sequent arrow in proofs (for introducing $\prec$ on the left of a sequent arrow in dual proofs) comes with functional application. The same holds for negation and co-negation in `2Int`.

[3]Note that there is also a use of *bi-intuitionistic logic* in the literature to refer to a specific system, namely `BiInt`, also called *Heyting-Brouwer logic* (e.g. [13, 6, 11, 9, 5]). Co-implication is there to be understood to internalize the preservation of non-truth from the conclusion to the premises in a valid inference. The system 2Int, which is treated here, uses the same language as `BiInt`, but the meaning of co-implication differs (cf. [17, p. 30f.] and [15, 16, 4]).

falsification) of C for $* = +$ (resp. $* = -$)".[4] Thus, we have a calculus in which more than one derivability relation is considered, not only the one of verification but also the one of falsification (or refutation).[5] The formulas in Γ can then be understood as *assumptions*, while the formulas in Δ can be understood as *counterassumptions*. SC2Int is equivalent to N2Int in that we have a proof in N2Int of A from the pair $(\Gamma; \Delta)$ of assumptions Γ and counterassumptions Δ, iff the sequent $(\Gamma; \Delta) \vdash^+ A$ is derivable in SC2Int and we have a dual proof of A from the pair $(\Gamma; \Delta)$ of assumptions Γ and counterassumptions Δ, iff the sequent $(\Gamma; \Delta) \vdash^- A$ is derivable in SC2Int.

In contrast to G3ip, there will be no distinction between axioms and logical rules but within the logical rules the zero-premise rules, which comprise Rf^+, Rf^-, $\bot L^a$, $\top L^c$, $\bot R^-$, and $\top R^+$, are distinguished from the non-zero-premise rules due to the special role of the former for the admissibility proofs below. Each of the logical rules has a *context* designated by Γ and Δ, *active formulas* designated by A and B and a *principal formula*, which is the one introduced on the left or right side of $\vdash^*$. Within the right introduction rules we need to distinguish whether the derivability relation expresses verification or falsification by using the superscripts $+$ and $-$. Within the left rules this is not necessary, but what is needed here is distinguishing an introduction of the principal formula into the *assumptions* from an introduction into the *counterassumptions*. The former are indexed by superscript a, while the latter are indexed by superscript c. The set of R^+ and L^a rules are the *proof rules*; the set of R^- and L^c rules are the *dual proof rules*.

SC2Int

For $* \in \{+, -\}$:

$$\frac{}{(\Gamma, p; \Delta) \vdash^+ p}\ Rf^+ \qquad \frac{}{(\Gamma; \Delta, p) \vdash^- p}\ Rf^-$$

[4] Note that the notation for sequents in [18] is different and follows the presentation of the subformula calculus for N4 in [7, 8]. In particular, expressions $\Gamma : \Delta \Rightarrow^* C$ (with Γ and Δ being finite, possibly empty multisets) are read as "From the falsification of all formulas in Γ and the verification of all formulas in Δ one can derive the verification (resp. falsification) of C for $* = +$ (resp. $* = -$)". The notation in the present paper is taken from [3]

[5] In N2Int this is indicated by using single lines for verification and double lines for falsification.

$$\frac{}{(\Gamma, \bot; \Delta) \vdash^* C}\ \bot L^a \qquad \frac{}{(\Gamma; \Delta, \top) \vdash^* C}\ \top L^c$$

$$\frac{}{(\Gamma; \Delta) \vdash^- \bot}\ \bot R^- \qquad \frac{}{(\Gamma; \Delta) \vdash^+ \top}\ \top R^+$$

$$\frac{(\Gamma; \Delta) \vdash^+ A \qquad (\Gamma; \Delta) \vdash^+ B}{(\Gamma; \Delta) \vdash^+ A \wedge B}\ \wedge R^+ \qquad \frac{(\Gamma, A, B; \Delta) \vdash^* C}{(\Gamma, A \wedge B; \Delta) \vdash^* C}\ \wedge L^a$$

$$\frac{(\Gamma; \Delta) \vdash^- A}{(\Gamma; \Delta) \vdash^- A \wedge B}\ \wedge R^-_1 \qquad \frac{(\Gamma; \Delta) \vdash^- B}{(\Gamma; \Delta) \vdash^- A \wedge B}\ \wedge R^-_2$$

$$\frac{(\Gamma; \Delta, A) \vdash^* C \qquad (\Gamma; \Delta, B) \vdash^* C}{(\Gamma; \Delta, A \wedge B) \vdash^* C}\ \wedge L^c$$

$$\frac{(\Gamma; \Delta) \vdash^+ A}{(\Gamma; \Delta) \vdash^+ A \vee B}\ \vee R^+_1 \qquad \frac{(\Gamma; \Delta) \vdash^+ B}{(\Gamma; \Delta) \vdash^+ A \vee B}\ \vee R^+_2$$

$$\frac{(\Gamma, A; \Delta) \vdash^* C \qquad (\Gamma, B; \Delta) \vdash^* C}{(\Gamma, A \vee B; \Delta) \vdash^* C}\ \vee L^a$$

$$\frac{(\Gamma; \Delta) \vdash^- A \qquad (\Gamma; \Delta) \vdash^- B}{(\Gamma; \Delta) \vdash^- A \vee B}\ \vee R^- \qquad \frac{(\Gamma; \Delta, A, B) \vdash^* C}{(\Gamma; \Delta, A \vee B) \vdash^* C}\ \vee L^c$$

$$\frac{(\Gamma, A; \Delta) \vdash^+ B}{(\Gamma; \Delta) \vdash^+ A \to B}\ {\to}R^+ \qquad \frac{(\Gamma, A \to B; \Delta) \vdash^+ A \quad (\Gamma, B; \Delta) \vdash^* C}{(\Gamma, A \to B; \Delta) \vdash^* C}\ {\to}L^a$$

$$\frac{(\Gamma; \Delta) \vdash^+ A \qquad (\Gamma; \Delta) \vdash^- B}{(\Gamma; \Delta) \vdash^- A \to B}\ {\to}R^- \qquad \frac{(\Gamma, A; \Delta, B) \vdash^* C}{(\Gamma; \Delta, A \to B) \vdash^* C}\ {\to}L^c$$

$$\frac{(\Gamma; \Delta) \vdash^+ A \qquad (\Gamma; \Delta) \vdash^- B}{(\Gamma; \Delta) \vdash^+ A \prec B}\ {\prec}R^+ \qquad \frac{(\Gamma, A; \Delta, B) \vdash^* C}{(\Gamma, A \prec B; \Delta) \vdash^* C}\ {\prec}L^a$$

$$\frac{(\Gamma; \Delta, B) \vdash^- A}{(\Gamma; \Delta) \vdash^- A \prec B}\ {\prec}R^- \qquad \frac{(\Gamma; \Delta, A \prec B) \vdash^- B \qquad (\Gamma; \Delta, A) \vdash^* C}{(\Gamma; \Delta, A \prec B) \vdash^* C}\ {\prec}L^c$$

Note that the rules for $\wedge L^a$, $\vee L^c$, $\to L^c$ and $\prec L^a$ could also be given in the form of two rules, each with only one active formula A or B, as it is for example done in Gentzen's original calculus for the left conjunction rule. We need this single rule formulation, however, in order to get the invertibility of these rules (cf. Lemma 3.3 below), which is important for the proof of admissibility of contraction. As said above, the structural rules do not have to be taken as primitive in the calculus but can be shown to be admissible.

We want to consider rules for weakening, contraction and cut. Due to the dual nature of the calculus, we need two rules for each of these rules:

$$\frac{(\Gamma; \Delta) \vdash^* C}{(\Gamma, A; \Delta) \vdash^* C}\ W^a \qquad \frac{(\Gamma; \Delta) \vdash^* C}{(\Gamma; \Delta, A) \vdash^* C}\ W^c$$

$$\frac{(\Gamma, A, A; \Delta) \vdash^* C}{(\Gamma, A; \Delta) \vdash^* C}\ C^a \qquad \frac{(\Gamma; \Delta, A, A) \vdash^* C}{(\Gamma; \Delta, A) \vdash^* C}\ C^c$$

$$\frac{(\Gamma; \Delta) \vdash^+ D \qquad (\Gamma', D; \Delta') \vdash^* C}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^* C}\ Cut^a$$

$$\frac{(\Gamma; \Delta) \vdash^- D \qquad (\Gamma'; \Delta', D) \vdash^* C}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^* C}\ Cut^c$$

3. Proving admissibility of the structural rules

3.1. Preliminaries

The proofs of admissibility of the structural rules and especially of cut-elimination are conducted analogously to the respective proofs of [10, pp. 30–40] for `G3ip`. The proofs will use induction on **weight of formulas** and **height of derivations**.

DEFINITION 3.1. The weight w(A) of a formula A is defined inductively by
$w(\bot) = w(\top) = 0$,
$w(p) = 1$ for atoms p,
$w(A \# B) = w(A) + w(B) + 1$ for $\# \in \{\wedge, \vee, \to, \prec\}$.

DEFINITION 3.2. A derivation in `SC2Int` is either an instance of a zero-premise rule, or an application of a logical rule to derivations concluding its premises. The *height of a derivation* is the greatest number of successive applications of rules in it, where zero-premise rules have height 0.

First, we will show that the reflexivity rules can be generalized to instances with arbitrary formulas, not only atomic formulas.

LEMMA 3.3. *The sequents* $(\Gamma, C; \Delta) \vdash^+ C$ *and* $(\Gamma; \Delta, C) \vdash^- C$ *are derivable for an arbitrary formula* C *and arbitrary context* $(\Gamma; \Delta)$.

PROOF: The proof is by induction on weight of C. If $w(C) \leq 1$, we have the 19 cases listed below. Note that for some of the derivations there is more than one possibility to derive the desired sequent and also some of the conclusions of zero-premise rules are conclusions of more than one of those rules. We will just show one exemplary derivation for each case, since this is enough for the proof.

$C = \bot$. Then $(\Gamma, C; \Delta) \vdash^+ C$ is an instance of $\bot L^a$ and $(\Gamma; \Delta, C) \vdash^- C$ is an instance of $\bot R^-$.

$C = \top$. Then $(\Gamma, C; \Delta) \vdash^+ C$ is an instance of $\top R^+$ and $(\Gamma; \Delta, C) \vdash^- C$ is an instance of $\top L^c$.

$C = p$ for some atom p. Then $(\Gamma, C; \Delta) \vdash^+ C$ is an instance of Rf^+ and $(\Gamma; \Delta, C) \vdash^- C$ is an instance of Rf^-.

$C = \bot \wedge \bot$. Then $(\Gamma, C; \Delta) \vdash^+ C$ and $(\Gamma; \Delta, C) \vdash^- C$ are derived by

$$\dfrac{\dfrac{}{(\Gamma, \bot, \bot; \Delta) \vdash^+ \bot \wedge \bot}\ \bot L^a}{(\Gamma, \bot \wedge \bot; \Delta) \vdash^+ \bot \wedge \bot}\ \wedge L^a \quad \text{and} \quad \dfrac{\dfrac{}{(\Gamma; \Delta, \bot \wedge \bot) \vdash^- \bot}\ \bot R^-}{(\Gamma; \Delta, \bot \wedge \bot) \vdash^- \bot \wedge \bot}\ \wedge R^-$$

$C = \bot \vee \bot$. Then $(\Gamma, C; \Delta) \vdash^+ C$ and $(\Gamma; \Delta, C) \vdash^- C$ are derived by

$$\dfrac{\dfrac{}{(\Gamma, \bot; \Delta) \vdash^+ \bot \vee \bot}\ \bot L^a \qquad \dfrac{}{(\Gamma, \bot; \Delta) \vdash^+ \bot \vee \bot}\ \bot L^a}{(\Gamma, \bot \vee \bot; \Delta) \vdash^+ \bot \vee \bot}\ \vee L^a \qquad \text{and}$$

$$\dfrac{\dfrac{}{(\Gamma; \Delta, \bot \vee \bot) \vdash^- \bot}\ \bot R^- \qquad \dfrac{}{(\Gamma; \Delta, \bot \vee \bot) \vdash^- \bot}\ \bot R^-}{(\Gamma; \Delta, \bot \vee \bot) \vdash^- \bot \vee \bot}\ \vee R^-$$

$C = \bot \to \bot$. Then $(\Gamma, C; \Delta) \vdash^+ C$ and $(\Gamma; \Delta, C) \vdash^- C$ are derived by

$$\dfrac{\dfrac{}{(\Gamma, \bot \to \bot, \bot; \Delta) \vdash^+ \bot}\ \bot L^a}{(\Gamma, \bot \to \bot; \Delta) \vdash^+ \bot \to \bot}\ \to R^+ \quad \text{and} \quad \dfrac{\dfrac{}{(\Gamma, \bot; \Delta, \bot) \vdash^- \bot \to \bot}\ \bot L^a}{(\Gamma; \Delta, \bot \to \bot) \vdash^- \bot \to \bot}\ \to L^c$$

$C = \bot \prec \bot$. Then $(\Gamma, C; \Delta) \vdash^+ C$ and $(\Gamma; \Delta, C) \vdash^- C$ are derived by

$$\dfrac{\dfrac{}{(\Gamma,\bot;\Delta,\bot)\vdash^{+}\bot\prec\bot}\,{\scriptstyle \bot L^{a}}}{(\Gamma,\bot\prec\bot;\Delta)\vdash^{+}\bot\prec\bot}\,{\scriptstyle \prec L^{a}}\quad\text{and}\quad\dfrac{\dfrac{}{(\Gamma;\Delta,\bot\prec\bot,\bot)\vdash^{-}\bot}\,{\scriptstyle \bot R^{-}}}{(\Gamma;\Delta,\bot\prec\bot)\vdash^{-}\bot\prec\bot}\,{\scriptstyle \prec R^{-}}$$

$C=\bot\wedge\top$. Then $(\Gamma,C;\Delta)\vdash^{+}C$ and $(\Gamma;\Delta,C)\vdash^{-}C$ are derived by

$$\dfrac{\dfrac{}{(\Gamma,\bot,\top;\Delta)\vdash^{+}\bot\wedge\top}\,{\scriptstyle \bot L^{a}}}{(\Gamma,\bot\wedge\top;\Delta)\vdash^{+}\bot\wedge\top}\,{\scriptstyle \wedge L^{a}}\quad\text{and}\quad\dfrac{\dfrac{}{(\Gamma;\Delta,\bot\wedge\top)\vdash^{-}\bot}\,{\scriptstyle \bot R^{-}}}{(\Gamma;\Delta,\bot\wedge\top)\vdash^{-}\bot\wedge\top}\,{\scriptstyle \wedge R_{1}^{-}}$$

$C=\bot\vee\top$. Then $(\Gamma,C;\Delta)\vdash^{+}C$ and $(\Gamma;\Delta,C)\vdash^{-}C$ are derived by

$$\dfrac{\dfrac{}{(\Gamma,\bot\vee\top;\Delta)\vdash^{+}\top}\,{\scriptstyle \top R^{+}}}{(\Gamma,\bot\vee\top;\Delta)\vdash^{+}\bot\vee\top}\,{\scriptstyle \vee R_{2}^{+}}\quad\text{and}\quad\dfrac{\dfrac{}{(\Gamma;\Delta,\bot,\top)\vdash^{-}\bot\vee\top}\,{\scriptstyle \top L^{c}}}{(\Gamma;\Delta,\bot\vee\top)\vdash^{-}\bot\vee\top}\,{\scriptstyle \vee L^{c}}$$

$C=\bot\to\top$. Then $(\Gamma,C;\Delta)\vdash^{+}C$ and $(\Gamma;\Delta,C)\vdash^{-}C$ are derived by

$$\dfrac{\dfrac{}{(\Gamma,\bot\to\top,\bot;\Delta)\vdash^{+}\top}\,{\scriptstyle \top R^{+}}}{(\Gamma,\bot\to\top;\Delta)\vdash^{+}\bot\to\top}\,{\scriptstyle \to R^{+}}\quad\text{and}\quad\dfrac{\dfrac{}{(\Gamma,\bot;\Delta,\top)\vdash^{-}\bot\to\top}\,{\scriptstyle \top L^{c}}}{(\Gamma;\Delta,\bot\to\top)\vdash^{-}\bot\to\top}\,{\scriptstyle \to L^{c}}$$

$C=\bot\prec\top$. Then $(\Gamma,C;\Delta)\vdash^{+}C$ and $(\Gamma;\Delta,C)\vdash^{-}C$ are derived by

$$\dfrac{\dfrac{}{(\Gamma,\bot;\Delta,\top)\vdash^{+}\bot\prec\top}\,{\scriptstyle \bot L^{a}}}{(\Gamma,\bot\prec\top;\Delta)\vdash^{+}\bot\prec\top}\,{\scriptstyle \prec L^{a}}\quad\text{and}\quad\dfrac{\dfrac{}{(\Gamma;\Delta,\bot\prec\top,\top)\vdash^{-}\bot}\,{\scriptstyle \top L^{c}}}{(\Gamma;\Delta,\bot\prec\top)\vdash^{-}\bot\prec\top}\,{\scriptstyle \prec R^{-}}$$

$C=\top\wedge\bot$. Then $(\Gamma,C;\Delta)\vdash^{+}C$ and $(\Gamma;\Delta,C)\vdash^{-}C$ are derived by

$$\dfrac{\dfrac{}{(\Gamma,\top,\bot;\Delta)\vdash^{+}\top\wedge\bot}\,{\scriptstyle \bot L^{a}}}{(\Gamma,\top\wedge\bot;\Delta)\vdash^{+}\top\wedge\bot}\,{\scriptstyle \wedge L^{a}}\quad\text{and}\quad\dfrac{\dfrac{}{(\Gamma;\Delta,\top\wedge\bot)\vdash^{-}\bot}\,{\scriptstyle \bot R^{-}}}{(\Gamma;\Delta,\top\wedge\bot)\vdash^{-}\top\wedge\bot}\,{\scriptstyle \wedge R_{2}^{-}}$$

$C=\top\vee\bot$. Then $(\Gamma,C;\Delta)\vdash^{+}C$ and $(\Gamma;\Delta,C)\vdash^{-}C$ are derived by

$$\dfrac{\dfrac{}{(\Gamma,\top\vee\bot;\Delta)\vdash^{+}\top}\,{\scriptstyle \top R^{+}}}{(\Gamma,\top\vee\bot;\Delta)\vdash^{+}\top\vee\bot}\,{\scriptstyle \vee R_{1}^{+}}\quad\text{and}\quad\dfrac{\dfrac{}{(\Gamma;\Delta,\top,\bot)\vdash^{-}\top\vee\bot}\,{\scriptstyle \top L^{c}}}{(\Gamma;\Delta,\top\vee\bot)\vdash^{-}\top\vee\bot}\,{\scriptstyle \vee L^{c}}$$

$C=\top\to\bot$. Then $(\Gamma,C;\Delta)\vdash^{+}C$ and $(\Gamma;\Delta,C)\vdash^{-}C$ are derived by

$$\dfrac{\dfrac{}{(\Gamma,\top\to\bot;\Delta)\vdash^{+}\top}\,{\scriptstyle \top R^{+}}\qquad\dfrac{}{(\Gamma,\bot;\Delta)\vdash^{+}\top\to\bot}\,{\scriptstyle \bot L^{a}}}{(\Gamma,\top\to\bot;\Delta)\vdash^{+}\top\to\bot}\,{\scriptstyle \to L^{a}}\quad\text{and}$$

$$\dfrac{\dfrac{}{(\Gamma;\Delta,\top\to\bot)\vdash^{+}\top}\,{\top R^{+}} \qquad \dfrac{}{(\Gamma;\Delta,\top\to\bot)\vdash^{-}\bot}\,{\bot R^{-}}}{(\Gamma;\Delta,\top\to\bot)\vdash^{-}\top\to\bot}\,{\to R^{-}}$$

$C = \top \prec \bot$. Then $(\Gamma, C; \Delta) \vdash^{+} C$ and $(\Gamma; \Delta, C) \vdash^{-} C$ are derived by

$$\dfrac{\dfrac{}{(\Gamma,\top\prec\bot;\Delta)\vdash^{+}\top}\,{\top R^{+}} \qquad \dfrac{}{(\Gamma,\top\prec\bot;\Delta)\vdash^{-}\bot}\,{\bot R^{-}}}{(\Gamma,\top\prec\bot;\Delta)\vdash^{+}\top\prec\bot}\,{\prec R^{+}} \quad \text{and}$$

$$\dfrac{\dfrac{}{(\Gamma;\Delta,\top\prec\bot)\vdash^{-}\bot}\,{\bot R^{-}} \qquad \dfrac{}{(\Gamma;\Delta,\top)\vdash^{-}\top\prec\bot}\,{\top L^{c}}}{(\Gamma;\Delta,\top\prec\bot)\vdash^{-}\top\prec\bot}\,{\prec L^{c}}$$

$C = \top \wedge \top$. Then $(\Gamma, C; \Delta) \vdash^{+} C$ and $(\Gamma; \Delta, C) \vdash^{-} C$ are derived by

$$\dfrac{\dfrac{}{(\Gamma,\top\wedge\top;\Delta)\vdash^{+}\top}\,{\top R^{+}} \qquad \dfrac{}{(\Gamma,\top\wedge\top;\Delta)\vdash^{+}\top}\,{\top R^{+}}}{(\Gamma,\top\wedge\top;\Delta)\vdash^{+}\top\wedge\top}\,{\wedge R^{+}} \quad \text{and}$$

$$\dfrac{\dfrac{}{(\Gamma;\Delta,\top)\vdash^{-}\top\wedge\top}\,{\top L^{c}} \qquad \dfrac{}{(\Gamma;\Delta,\top)\vdash^{-}\top\wedge\top}\,{\top L^{c}}}{(\Gamma;\Delta,\top\wedge\top)\vdash^{-}\top\wedge\top}\,{\wedge L^{c}}$$

$C = \top \vee \top$. Then $(\Gamma, C; \Delta) \vdash^{+} C$ and $(\Gamma; \Delta, C) \vdash^{-} C$ are derived by

$$\dfrac{\dfrac{}{(\Gamma,\top\vee\top;\Delta)\vdash^{+}\top}\,{\top R^{+}}}{(\Gamma,\top\vee\top;\Delta)\vdash^{+}\top\vee\top}\,{\vee R^{+}} \quad \text{and} \quad \dfrac{\dfrac{}{(\Gamma;\Delta,\top,\top)\vdash^{-}\top\vee\top}\,{\top L^{c}}}{(\Gamma;\Delta,\top\vee\top)\vdash^{-}\top\vee\top}\,{\vee L^{c}}$$

$C = \top \to \top$. Then $(\Gamma, C; \Delta) \vdash^{+} C$ and $(\Gamma; \Delta, C) \vdash^{-} C$ are derived by

$$\dfrac{\dfrac{}{(\Gamma,\top\to\top,\top;\Delta)\vdash^{+}\top}\,{\top R^{+}}}{(\Gamma,\top\to\top;\Delta)\vdash^{+}\top\to\top}\,{\to R^{+}} \quad \text{and} \quad \dfrac{\dfrac{}{(\Gamma,\top;\Delta,\top)\vdash^{-}\top\to\top}\,{\top L^{c}}}{(\Gamma;\Delta,\top\to\top)\vdash^{-}\top\to\top}\,{\to L^{c}}$$

$C = \top \prec \top$. Then $(\Gamma, C; \Delta) \vdash^{+} C$ and $(\Gamma; \Delta, C) \vdash^{-} C$ are derived by

$$\dfrac{\dfrac{}{(\Gamma,\top;\Delta,\top)\vdash^{+}\top\prec\top}\,{\top L^{c}}}{(\Gamma,\top\prec\top;\Delta)\vdash^{+}\top\prec\top}\,{\prec L^{a}} \quad \text{and} \quad \dfrac{\dfrac{}{(\Gamma;\Delta,\top\prec\top,\top)\vdash^{-}\top}\,{\top L^{c}}}{(\Gamma;\Delta,\top\prec\top)\vdash^{-}\top\prec\top}\,{\prec R^{-}}$$

The inductive hypothesis is that $(\Gamma, C; \Delta) \vdash^+ C$ and $(\Gamma; \Delta, C) \vdash^- C$ are derivable for all formulas C with $w(C) \leq n$, and we have to show that $(\Gamma, D; \Delta) \vdash^+ D$ and $(\Gamma; \Delta, D) \vdash^- D$ are derivable for formulas D of weight $\leq n + 1$. There are four cases:

$D = A \wedge B$. By the definition of weight and our inductive hypothesis, $w(A) \leq n$ and $w(B) \leq n$.
We can derive $(\Gamma, A \wedge B; \Delta) \vdash^+ A \wedge B$ by

$$\dfrac{\dfrac{(\Gamma, A, B; \Delta) \vdash^+ A}{(\Gamma, A \wedge B; \Delta) \vdash^+ A}\ {\scriptstyle \wedge L^a} \qquad \dfrac{(\Gamma, A, B; \Delta) \vdash^+ B}{(\Gamma, A \wedge B; \Delta) \vdash^+ B}\ {\scriptstyle \wedge L^a}}{(\Gamma, A \wedge B; \Delta) \vdash^+ A \wedge B}\ {\scriptstyle \wedge R^+}$$

and $(\Gamma; \Delta, A \wedge B) \vdash^- A \wedge B$ by

$$\dfrac{\dfrac{(\Gamma; \Delta, A) \vdash^- A}{(\Gamma; \Delta, A) \vdash^- A \wedge B}\ {\scriptstyle \wedge R_1^-} \qquad \dfrac{(\Gamma; \Delta, B) \vdash^- B}{(\Gamma; \Delta, B) \vdash^- A \wedge B}\ {\scriptstyle \wedge R_2^-}}{(\Gamma; \Delta, A \wedge B) \vdash^- A \wedge B}\ {\scriptstyle \wedge L^c}$$

$(\Gamma; \Delta, A) \vdash^- A$ and $(\Gamma; \Delta, B) \vdash^- B$ are derivable by the inductive hypothesis and since the context is arbitrary, so are $(\Gamma', A; \Delta) \vdash^+ A$ and $(\Gamma'', B; \Delta) \vdash^+ B$, for $\Gamma' = \Gamma, B$ and $\Gamma'' = \Gamma, A$.

$D = A \vee B$. As before, $w(A) \leq n$ and $w(B) \leq n$.
We can derive $(\Gamma, A \vee B; \Delta) \vdash^+ A \vee B$ by

$$\dfrac{\dfrac{(\Gamma, A; \Delta) \vdash^+ A}{(\Gamma, A; \Delta) \vdash^+ A \vee B}\ {\scriptstyle \vee R_1^+} \qquad \dfrac{(\Gamma, B; \Delta) \vdash^+ B}{(\Gamma, B; \Delta) \vdash^+ A \vee B}\ {\scriptstyle \vee R_2^+}}{(\Gamma, A \vee B; \Delta) \vdash^+ A \vee B}\ {\scriptstyle \vee L^a}$$

and $(\Gamma; \Delta, A \vee B) \vdash^- A \vee B$ by

$$\dfrac{\dfrac{(\Gamma; \Delta, A, B) \vdash^- A}{(\Gamma; \Delta, A \vee B) \vdash^- A}\ {\scriptstyle \vee L^c} \qquad \dfrac{(\Gamma; \Delta, A, B) \vdash^- B}{(\Gamma; \Delta, A \vee B) \vdash^- B}\ {\scriptstyle \vee L^c}}{(\Gamma; \Delta, A \vee B) \vdash^- A \vee B}\ {\scriptstyle \vee R^-}$$

Again, by inductive hypothesis we get the derivability of $(\Gamma, A; \Delta) \vdash^+ A$ and $(\Gamma, B; \Delta) \vdash^+ B$ and since the context is arbitrary, $(\Gamma; \Delta', A) \vdash^- A$ and $(\Gamma; \Delta'', B) \vdash^- B$ are derivable, for $\Delta' = \Delta, B$ and $\Delta'' = \Delta, A$.

$D = A \to B$. As before, $w(A) \leq n$ and $w(B) \leq n$.
We can derive $(\Gamma, A \to B; \Delta) \vdash^+ A \to B$ by

$$\dfrac{\dfrac{(\Gamma, A, A \to B; \Delta) \vdash^+ A \qquad (\Gamma, A, B; \Delta) \vdash^+ B}{(\Gamma, A, A \to B; \Delta) \vdash^+ B}\,{\to}L^a}{(\Gamma, A \to B; \Delta) \vdash^+ A \to B}\,{\to}R^+$$

and $(\Gamma; \Delta, A \to B) \vdash^- A \to B$ by

$$\dfrac{\dfrac{(\Gamma, A; \Delta, B) \vdash^+ A \qquad (\Gamma, A; \Delta, B) \vdash^- B}{(\Gamma, A; \Delta, B) \vdash^- A \to B}\,{\to}R^-}{(\Gamma; \Delta, A \to B) \vdash^- A \to B}\,{\to}L^c$$

The case of $(\Gamma, A, B; \Delta) \vdash^+ B$ was already mentioned in the case of conjunction and with the same reasoning $(\Gamma', A; \Delta) \vdash^+ A$ for $\Gamma' = \Gamma, A \to B$, $(\Gamma, A; \Delta') \vdash^+ A$ for $\Delta' = \Delta, B$ as well as $(\Gamma'; \Delta, B) \vdash^- B$ for $\Gamma' = \Gamma, A$ are derivable.

$D = A \prec B$. As before, $w(A) \leq n$ and $w(B) \leq n$.
We can derive $(\Gamma, A \prec B; \Delta) \vdash^+ A \prec B$ by

$$\dfrac{\dfrac{(\Gamma, A; \Delta, B) \vdash^+ A \qquad (\Gamma, A; \Delta, B) \vdash^- B}{(\Gamma, A; \Delta, B) \vdash^+ A \prec B}\,{\prec}R^+}{(\Gamma, A \prec B; \Delta) \vdash^+ A \prec B}\,{\prec}L^a$$

and $(\Gamma; \Delta, A \prec B) \vdash^- A \prec B$ by

$$\dfrac{\dfrac{(\Gamma; \Delta, B, A \prec B) \vdash^- B \qquad (\Gamma; \Delta, A, B) \vdash^- A}{(\Gamma; \Delta, B, A \prec B) \vdash^- A}\,{\prec}L^c}{(\Gamma; \Delta, A \prec B) \vdash^- A \prec B}\,{\prec}R^-$$

With the same reasoning as above $(\Gamma; \Delta', B) \vdash^- B$ is derivable for $\Delta' = \Delta, A \prec B$ and all other cases are already dealt with above. □

3.2. Admissibility of weakening

We will now start with the proof of admissibility of weakening by induction on height of derivations. The general procedure when proving admissibility of a rule with this is to prove it for applications of this rule to conclusions of zero-premise rules and then generalize by induction on the number of applications of the rule to arbitrary derivations. Thus, we can assume that there is only one instance – as the last step in the derivation – of the rule in question.

THEOREM 3.4 (Height-preserving weakening). *If* $(\Gamma; \Delta) \vdash^* C$ *is derivable with a height of derivation at most* n*, then* $(\Gamma, D; \Delta) \vdash^* C$ *and* $(\Gamma; \Delta, D) \vdash^* C$ *are derivable with a height of derivation at most* n *for arbitrary* D*.*

PROOF: If $n = 0$, then $(\Gamma; \Delta) \vdash^* C$ is a zero-premise rule, which means that one of the following six cases holds. C is an atom and 1) a formula in Γ with $* = +$ or 2) a formula in Δ with $* = -$. Otherwise it can be the case that 3) C is $\top$ with $* = +$ or 4) C is $\bot$ with $* = -$. Lastly, it could be that 5) $\bot$ is a formula in Γ or 6) $\top$ a formula in Δ. In either case, $(\Gamma, D; \Delta) \vdash^* C$ *and* $(\Gamma; \Delta, D) \vdash^* C$ are conclusions of the respective zero-premise rules. Our inductive hypothesis is now that height-preserving weakening is admissible up to derivations of height $\leq n$. Let $(\Gamma; \Delta) \vdash^* C$ be derivable with a height of derivation at most $n + 1$.
If the last rule applied is $\wedge L^a$, then $\Gamma = \Gamma', A \wedge B$ and the last step is

$$\frac{(\Gamma', A, B; \Delta) \vdash^* C}{(\Gamma', A \wedge B; \Delta) \vdash^* C} \wedge L^a$$

So $(\Gamma', A, B; \Delta) \vdash^* C$ is derivable in $\leq n$ steps. By inductive hypothesis, also $(\Gamma', A, B, D; \Delta) \vdash^* C$ and $(\Gamma', A, B; \Delta, D) \vdash^* C$ are derivable in $\leq n$ steps. Thus, the application of $\wedge L^a$ gives a derivation of $(\Gamma', A \wedge B, D; \Delta) \vdash^* C$ and $(\Gamma', A \wedge B; \Delta, D) \vdash^* C$ in $\leq n + 1$ steps.
If the last rule applied is $\wedge L^c$, then $\Delta = \Delta', A \wedge B$ and the last step is

$$\frac{(\Gamma; \Delta', A) \vdash^* C \qquad (\Gamma; \Delta', B) \vdash^* C}{(\Gamma; \Delta', A \wedge B) \vdash^* C} \wedge L^c$$

So $(\Gamma; \Delta', A) \vdash^* C$ and $(\Gamma; \Delta', B) \vdash^* C$ are derivable in $\leq n$ steps. By inductive hypothesis, also $(\Gamma, D; \Delta', A) \vdash^* C$, $(\Gamma; \Delta', A, D) \vdash^* C$, $(\Gamma, D; \Delta', B) \vdash^* C$ and $(\Gamma; \Delta', B, D) \vdash^* C$ are derivable in $\leq n$ steps. Thus, the application of $\wedge L^c$ to the first and the third premise and to the second and the fourth premise gives a derivation of $(\Gamma, D; \Delta', A \wedge B) \vdash^* C$ and $(\Gamma; \Delta', A \wedge B, D) \vdash^* C$, respectively, in $\leq n + 1$ steps.
If the last rule applied is $\wedge R^+$, then $C = A \wedge B$ and the last step is

$$\frac{(\Gamma; \Delta) \vdash^+ A \qquad (\Gamma; \Delta) \vdash^+ B}{(\Gamma; \Delta) \vdash^+ A \wedge B} \wedge R^+$$

So $(\Gamma;\Delta) \vdash^+ A$ and $(\Gamma;\Delta) \vdash^+ B$ are derivable in $\leq n$ steps. By inductive hypothesis, also $(\Gamma, D;\Delta) \vdash^+ A$, $(\Gamma;\Delta, D) \vdash^+ A$, $(\Gamma, D;\Delta) \vdash^+ B$ and $(\Gamma;\Delta, D) \vdash^+ B$ are derivable in $\leq n$ steps. Thus, the application of $\wedge R^+$ to the first and the third premise and to the second and the fourth premise gives a derivation of $(\Gamma, D;\Delta) \vdash^+ A \wedge B$ and $(\Gamma;\Delta, D) \vdash^+ A \wedge B$, respectively, in $\leq n+1$ steps.
If the last rule applied is $\wedge R_1^-$, then $C = A \wedge B$ and the last step is

$$\frac{(\Gamma;\Delta) \vdash^- A}{(\Gamma;\Delta) \vdash^- A \wedge B}\ \wedge R_1^-$$

So $(\Gamma;\Delta) \vdash^- A$ is derivable in $\leq n$ steps. By inductive hypothesis, also $(\Gamma, D;\Delta) \vdash^- A$ and $(\Gamma;\Delta, D) \vdash^- A$ are derivable in $\leq n$ steps. Thus, the application of $\wedge R_1^-$ gives a derivation of $(\Gamma, D;\Delta) \vdash^- A \wedge B$ and $(\Gamma;\Delta, D) \vdash^- A \wedge B$ in $\leq n+1$ steps.

For the other logical rules the same can be shown with similar steps. □

Now we want to show one other thing related to weakening because we will need this result later in our proof for the admissibility of the cut rules, namely that for the special case that the weakening formula is $\top$ for W^a and respectively $\bot$ for W^c, the weakening rules are invertible, i.e.:

$$\frac{(\Gamma, \top;\Delta) \vdash^* C}{(\Gamma;\Delta) \vdash^* C}\ W^{\top}_{inv} \qquad \frac{(\Gamma;\Delta, \bot) \vdash^* C}{(\Gamma;\Delta) \vdash^* C}\ W^{\bot}_{inv}$$

Lemma 3.5 (Special case of inverted weakening). *If $(\Gamma, \top;\Delta) \vdash^* C$ or $(\Gamma;\Delta, \bot) \vdash^* C$ are derivable with a height of derivation at most n, then so is $(\Gamma;\Delta) \vdash^* C$.*

Proof: If $n = 0$, then exactly the same reasoning as for Theorem 3.4 can be applied here.
Now we assume height-preserving invertibility for these two special cases of weakening up to height n, and let $(\Gamma, \top;\Delta) \vdash^* C$ and $(\Gamma;\Delta, \bot) \vdash^* C$ be derivable with a height of derivation $\leq n+1$. The proof works correspondingly to the proof of height-preserving weakening above. We will show it for the case of the $\to L^c$-rule this time, just to choose one that is not familiar in 'usual' calculi, but it works similar for all logical connectives and their rules.
If the last rule applied is $\to L^c$, then we have $\Delta = \Delta', A \to B$ and the last step is

$$\frac{(\Gamma, A, \top; \Delta', B) \vdash^* C}{(\Gamma, \top; \Delta', A \to B) \vdash^* C} \to L^c \text{ or respectively } \frac{(\Gamma, A; \Delta', B, \bot) \vdash^* C}{(\Gamma; \Delta', A \to B, \bot) \vdash^* C} \to L^c$$

So, $(\Gamma, A, \top; \Delta', B) \vdash^* C$ and $(\Gamma, A; \Delta', B, \bot) \vdash^* C$ are derivable in $\leq n$ steps. Then by inductive hypothesis, $(\Gamma, A; \Delta', B) \vdash^* C$ is derivable in $\leq n$ steps. If we apply $\to L^c$ to this, this gives us $(\Gamma; \Delta', A \to B) \vdash^* C$ in $\leq n + 1$ steps. □

3.3. Admissibility of contraction

Before we can prove the admissibility of the contraction rules, we need to prove the following lemma about the invertibility of premises and conclusions of the logical rules for the left introduction of formulas. Note that for $\to L^a$ and $\prec L^c$ the invertibility only holds for the right premises.[6]

Lemma 3.6 (Inversion).

- (i_1) *If* $(\Gamma, A \wedge B; \Delta) \vdash^* C$ *is derivable with a height of derivation at most* n*, then* $(\Gamma, A, B; \Delta) \vdash^* C$ *is derivable with a height of derivation at most* n*.*
- (i_2) *If* $(\Gamma; \Delta, A \wedge B) \vdash^* C$ *is derivable with a height of derivation at most* n*, then* $(\Gamma; \Delta, A) \vdash^* C$ *and* $(\Gamma; \Delta, B) \vdash^* C$ *are derivable with a height of derivation at most* n*.*
- (ii_1) *If* $(\Gamma, A \vee B; \Delta) \vdash^* C$ *is derivable with a height of derivation at most* n*, then* $(\Gamma, A; \Delta) \vdash^* C$ *and* $(\Gamma, B; \Delta) \vdash^* C$ *are derivable with a height of derivation at most* n*.*
- (ii_2) *If* $(\Gamma; \Delta, A \vee B) \vdash^* C$ *is derivable with a height of derivation at most* n*, then* $(\Gamma; \Delta, A, B) \vdash^* C$ *is derivable with a height of derivation at most* n*.*
- (iii_1) *If* $(\Gamma, A \to B; \Delta) \vdash^* C$ *is derivable with a height of derivation at most* n*, then* $(\Gamma, B; \Delta) \vdash^* C$ *is derivable with a height of derivation at most* n*.*
- (iii_2) *If* $(\Gamma; \Delta, A \to B) \vdash^* C$ *is derivable with a height of derivation at most* n*, then* $(\Gamma, A; \Delta, B) \vdash^* C$ *is derivable with a height of derivation at most* n*.*

[6] [10, p. 33] give a counterexample for the implication rule. The analogous counterexamples for `SC2Int` would be the derivability of the sequents $(\bot \to \bot; \varnothing) \vdash^+ \bot \to \bot$ and $(\varnothing; \top \prec \top) \vdash^- \top \prec \top$.

(iv_1) *If* $(\Gamma, A \prec B; \Delta) \vdash^* C$ *is derivable with a height of derivation at most* n*, then* $(\Gamma, A; \Delta, B) \vdash^* C$ *is derivable with a height of derivation at most* n*.*

(iv_2) *If* $(\Gamma; \Delta, A \prec B) \vdash^* C$ *is derivable with a height of derivation at most* n*, then* $(\Gamma; \Delta, A) \vdash^* C$ *is derivable with a height of derivation at most* n*.*

Proof: The proof is by induction on n.
1) If $(\Gamma, A \# B; \Delta) \vdash^* C$ with $\# \in \{\wedge, \vee, \rightarrow, \prec\}$ is the conclusion of a zero-premise rule, then so are $(\Gamma, A, B; \Delta) \vdash^* C$, $(\Gamma, A; \Delta) \vdash^* C$, $(\Gamma, B; \Delta) \vdash^* C$, $(\Gamma; \Delta, B) \vdash^* C$ since $A \# B$ is neither atomic nor $\bot$ nor $\top$.
Now we assume height-preserving inversion up to height n, and let $(\Gamma, A \# B; \Delta) \vdash^* C$ be derivable with a height of derivation $\leq n + 1$.

(i_1) Either $A \wedge B$ is principal in the last rule or not. If $A \wedge B$ is the principal formula, the premise $(\Gamma, A, B; \Delta) \vdash^* C$ has a derivation of height n. If $A \wedge B$ is not principal in the last rule, then there must be one or two premises $(\Gamma', A \wedge B; \Delta') \vdash^* C'$, $(\Gamma'', A \wedge B; \Delta'') \vdash^* C''$ with a height of derivation $\leq n$. Then, by inductive hypothesis, also $(\Gamma', A, B; \Delta') \vdash^* C'$, $(\Gamma'', A, B; \Delta'') \vdash^* C''$ are derivable with a height of derivation $\leq n$. Now the last rule can be applied to these premises to conclude $(\Gamma, A, B; \Delta) \vdash^* C$ in at most $n + 1$ steps.

(ii_1) Either $A \vee B$ is principal in the last rule or not. If $A \vee B$ is the principal formula, the premises $(\Gamma, A; \Delta) \vdash^* C$ and $(\Gamma, B; \Delta) \vdash^* C$ have a derivation of height $\leq n$. If $A \vee B$ is not principal in the last rule, then there must be one or two premises $(\Gamma', A \vee B; \Delta') \vdash^* C'$, $(\Gamma'', A \vee B; \Delta'') \vdash^* C''$ with a height of derivation $\leq n$. Then, by inductive hypothesis, also $(\Gamma', A; \Delta') \vdash^* C'$, $(\Gamma', B; \Delta') \vdash^* C'$ and $(\Gamma'', A; \Delta') \vdash^* C''$, $(\Gamma'', B; \Delta'') \vdash^* C''$ are derivable with a height of derivation $\leq n$. Now the last rule can be applied to the first and third premise to conclude $(\Gamma, A; \Delta) \vdash^* C$ and to the second and fourth premise to conclude $(\Gamma, B; \Delta) \vdash^* C$ in at most $n + 1$ steps.

(iii_1) Either $A \rightarrow B$ is principal in the last rule or not. If $A \rightarrow B$ is the principal formula, the premise $(\Gamma, B; \Delta) \vdash^* C$ has a derivation of height $\leq n$. If $A \rightarrow B$ is not principal in the last rule, then there must be one or two premises $(\Gamma', A \rightarrow B; \Delta') \vdash^* C'$, $(\Gamma'', A \rightarrow B; \Delta'') \vdash^* C''$ with a height of derivation $\leq n$. Then, by inductive hypothesis, also $(\Gamma', B; \Delta') \vdash^* C'$, $(\Gamma'', B; \Delta'') \vdash^* C''$ are derivable with a height of derivation $\leq n$. Now the last rule can be applied to these premises to conclude $(\Gamma, B; \Delta) \vdash^* C$ in at most $n+1$ steps.

(iv_1) Either $A \prec B$ is principal in the last rule or not. If $A \prec B$ is the principal formula, then the premise $(\Gamma, A; \Delta, B) \vdash^* C$ has a derivation of height n. If $A \prec B$ is not principal in the last rule, then there must be one or two premises $(\Gamma', A \prec B; \Delta') \vdash^* C'$, $(\Gamma'', A \prec B; \Delta'') \vdash^* C''$ with a height of derivation $\leq n$. Then, by inductive hypothesis, also $(\Gamma', A; \Delta', B) \vdash^* C'$, $(\Gamma'', A; \Delta'', B) \vdash^* C''$ are derivable with a height of derivation $\leq n$. Now the last rule can be applied to these premises to conclude $(\Gamma, A; \Delta, B) \vdash^* C$ in at most $n+1$ steps.

2) If $(\Gamma; \Delta, A \# B) \vdash^* C$ with $\# \in \{\wedge, \vee, \rightarrow, \prec\}$ is the conclusion of a zero-premise rule, then so are $(\Gamma; \Delta, A) \vdash^* C$, $(\Gamma; \Delta, B) \vdash^* C$, $(\Gamma; \Delta, A, B) \vdash^* C$, $(\Gamma, A; \Delta) \vdash^* C$ since $A \# B$ is neither atomic nor $\bot$ nor $\top$.
Now we assume height-preserving inversion up to height n, and let $(\Gamma; \Delta, A \# B) \vdash^* C$ be derivable with a height of derivation $\leq n+1$.

(i_2) Either $A \wedge B$ is principal in the last rule or not. If $A \wedge B$ is the principal formula, the premises $(\Gamma; \Delta, A) \vdash^* C$ and $(\Gamma; \Delta, B) \vdash^* C$ have a derivation of height $\leq n$. If $A \wedge B$ is not principal in the last rule, then there must be one or two premises $(\Gamma'; \Delta', A \wedge B) \vdash^* C'$, $(\Gamma''; \Delta'', A \wedge B) \vdash^* C''$ with a height of derivation $\leq n$. Then, by inductive hypothesis, also $(\Gamma'; \Delta', A) \vdash^* C'$, $(\Gamma'; \Delta', B) \vdash^* C'$, $(\Gamma''; \Delta'', A) \vdash^* C''$, $(\Gamma''; \Delta'', B) \vdash^* C''$ are derivable with a height of derivation $\leq n$. Now the last rule can be applied to the first and third premise to conclude $(\Gamma; \Delta, A) \vdash^* C$ and to the second and fourth premise to conclude $(\Gamma; \Delta, B) \vdash^* C$ in at most $n+1$ steps.

(ii_2) Either $A \vee B$ is principal in the last rule or not. If $A \vee B$ is the principal formula, the premise $(\Gamma; \Delta, A, B) \vdash^* C$ has a derivation of height n. If $A \vee B$ is not principal in the last rule, then there must be one or two premises $(\Gamma'; \Delta', A \vee B) \vdash^* C'$, $(\Gamma''; \Delta'', A \vee B) \vdash^* C''$ with a height of derivation $\leq n$. Then, by inductive hypothesis, also $(\Gamma'; \Delta', A, B) \vdash^* C'$, $(\Gamma''; \Delta'', A, B) \vdash^* C''$ are derivable with a height of derivation $\leq n$. Now the last rule can be applied to these premises to conclude $(\Gamma; \Delta, A, B) \vdash^* C$ in at most $n+1$ steps.

(iii_2) Either $A \to B$ is principal in the last rule or not. If $A \to B$ is the principal formula, the premise $(\Gamma, A; \Delta, B) \vdash^* C$ has a derivation of height n. If $A \to B$ is not principal in the last rule, then there must be one or two premises $(\Gamma'; \Delta', A \to B) \vdash^* C'$, $(\Gamma''; \Delta'', A \to B) \vdash^* C''$ with a height of derivation $\leq n$. Then, by inductive hypothesis, also $(\Gamma', A; \Delta', B) \vdash^* C'$, $(\Gamma'', A; \Delta'', B) \vdash^* C''$ are derivable with a height of derivation $\leq n$. Now the last rule can be applied to these premises to conclude $(\Gamma, A; \Delta, B) \vdash^* C$ in at most $n + 1$ steps.

(iv_2) Either $A \prec B$ is principal in the last rule or not. If $A \prec B$ is the principal formula, the premise $(\Gamma; \Delta, A) \vdash^* C$ has a derivation of height $\leq n$. If $A \prec B$ is not principal in the last rule, then there must be one or two premises $(\Gamma'; \Delta', A \prec B) \vdash^* C'$, $(\Gamma''; \Delta'', A \prec B) \vdash^* C''$ with a height of derivation $\leq n$. Then, by inductive hypothesis, also $(\Gamma'; \Delta', A) \vdash^* C'$, $(\Gamma''; \Delta'', A) \vdash^* C''$ are derivable with a height of derivation $\leq n$. Now the last rule can be applied to these premises to conclude $(\Gamma; \Delta, A) \vdash^* C$ in at most $n + 1$ steps. $\square$

Next, we will prove the admissibility of the contraction rules in `SC2Int`.

THEOREM 3.7 (Height-preserving contraction). *If* $(\Gamma, D, D; \Delta) \vdash^* C$ *is derivable with a height of derivation at most* n*, then* $(\Gamma, D; \Delta) \vdash^* C$ *is derivable with a height of derivation at most* n *and if* $(\Gamma; \Delta, D, D) \vdash^* C$ *is derivable with a height of derivation at most* n*, then* $(\Gamma; \Delta, D) \vdash^* C$ *is derivable with a height of derivation at most* n*.*

PROOF: The proof is again by induction on the height of derivation n.
If $(\Gamma, D, D; \Delta) \vdash^* C$ (resp. $(\Gamma; \Delta, D, D) \vdash^* C$) is the conclusion of a zero-premise rule, then either C is an atom and contained in the antecedent, in the assumptions for $\vdash^+$ or in the counterassumptions for $\vdash^-$, or $\bot$ is part of the assumptions, or $\top$ is part of the counterassumptions, or $C = \top$ for $\vdash^+$, or $C = \bot$ for $\vdash^-$. In either case, also $(\Gamma, D; \Delta) \vdash^* C$ (resp. $(\Gamma; \Delta, D) \vdash^* C$) is a conclusion of the respective zero-premise rule.
Let contraction be admissible up to derivation height n and let $(\Gamma, D, D; \Delta) \vdash^* C$ (resp. $(\Gamma; \Delta, D, D) \vdash^* C$) be derivable in at most $n + 1$ steps. Either the contraction formula is not principal in the last inference step or it is principal.
If D is not principal in the last rule concluding the premise of contraction $(\Gamma, D, D; \Delta) \vdash^* C$, there must be one or two premises $(\Gamma', D, D; \Delta') \vdash^* C'$,

$(\Gamma'', D, D; \Delta'') \vdash^* C'$ with a height of derivation $\leq n$. So by inductive hypothesis, we can derive $(\Gamma', D; \Delta') \vdash^* C'$, $(\Gamma'', D; \Delta'') \vdash^* C'$ with a height of derivation $\leq n$. Now the last rule can be applied to these premises to conclude $(\Gamma, D; \Delta) \vdash^* C$ in at most $n + 1$ steps. For the case of $(\Gamma; \Delta, D, D) \vdash^* C$ being the premise of contraction, the same argument applies respectively.
If D is principal in the last rule, we have to consider four cases for each contraction rule according to the form of D. We will show the cases for C^c this time; for C^a the same arguments apply respectively.

$D = A \wedge B$. Then the last rule applied must be $\wedge L^c$ and we have as premises $(\Gamma; \Delta, A \wedge B, A) \vdash^* C$ and $(\Gamma; \Delta, A \wedge B, B) \vdash^* C$ with a derivation height $\leq n$. By the inversion lemma this means that $(\Gamma; \Delta, A, A) \vdash^* C$ and $(\Gamma; \Delta, B, B) \vdash^* C$ are also derivable with a derivation height $\leq n$. Then by inductive hypothesis, we get $(\Gamma; \Delta, A) \vdash^* C$ and $(\Gamma; \Delta, B) \vdash^* C$ with a height of derivation $\leq n$ and by applying $\wedge L^c$ we can derive $(\Gamma; \Delta, A \wedge B) \vdash^* C$ in at most $n + 1$ steps.

$D = A \vee B$. Then the last rule applied must be $\vee L^c$ and $(\Gamma; \Delta, A \vee B, A, B) \vdash^* C$ is derivable with a height of derivation $\leq n$. By the inversion lemma, also $(\Gamma; \Delta, A, B, A, B) \vdash^* C$ is derivable with a derivation height $\leq n$. Then by inductive hypothesis (applied twice), we get $(\Gamma; \Delta, A, B) \vdash^* C$ with a height of derivation $\leq n$ and by applying $\vee L^c$ we can derive $(\Gamma; \Delta, A \vee B) \vdash^* C$ in at most $n + 1$ steps.

$D = A \to B$. Then the last rule applied must be $\to L^c$ and accordingly $(\Gamma, A; \Delta, B, A \to B) \vdash^* C$ is derivable with a height of derivation $\leq n$. By the inversion lemma, then also $(\Gamma, A, A; \Delta, B, B) \vdash^* C$ is derivable with a derivation height $\leq n$. By inductive hypothesis (applied twice), we get $(\Gamma, A; \Delta, B) \vdash^* C$ with a height of derivation $\leq n$ and by applying $\to L^c$ we can derive $(\Gamma; \Delta, A \to B) \vdash^* C$ in at most $n + 1$ steps.

$D = A \prec B$. Then the last rule applied must be $\prec L^c$ and we have as premises $(\Gamma; \Delta, A \prec B, A \prec B) \vdash^- B$ and $(\Gamma; \Delta, A \prec B, A) \vdash^* C$ with a derivation height $\leq n$. The inductive hypothesis applied to the first, gives us $(\Gamma; \Delta, A \prec B) \vdash^- B$ with a derivation height $\leq n$ and the inversion lemma applied to the second, also $(\Gamma; \Delta, A, A) \vdash^* C$ and again by inductive hypothesis $(\Gamma; \Delta, A) \vdash^* C$ with a derivation height $\leq n$. By applying $\prec L^c$ we can now derive $(\Gamma; \Delta, A \prec B) \vdash^* C$ in at most $n + 1$ steps. $\square$

3.4. Admissibility of cut

Now, we will come to the main result of this section, the proof of cut-elimination. The proof shows that cuts can be permuted upward in a derivation until they reach one of the zero-premise rules the derivation started with. When cut has reached zero-premise rules, the derivation can be transformed into one beginning with the conclusion of the cut, which can be shown by the following reasoning.

When both premises of cut are conclusions of a zero-premise rule, then the conclusion of cut is also a conclusion of one of these rules: If the left premise is $(\Gamma, \bot; \Delta) \vdash^* D$, then the conclusion also has $\bot$ in the assumptions of the antecedent. If the left premise is $(\Gamma; \Delta, \top) \vdash^* D$, then the conclusion also has $\top$ in the counterassumptions of the antecedent. If the left premise of Cut^a is $(\Gamma; \Delta) \vdash^+ \top$ or the left premise of Cut^c is $(\Gamma; \Delta) \vdash^- \bot$, then the right premise is $(\Gamma', \top; \Delta') \vdash^* C$ or $(\Gamma'; \Delta', \bot) \vdash^* C$ respectively. These are conclusions of zero-premise rules only in one of the following cases:

- C is an atom in Γ' for $* = +$ or C is an atom in Δ' for $* = -$
- $C = \top$ for $* = +$ or $C = \bot$ for $* = -$
- $\bot$ is in Γ' or $\top$ is in Δ'

In each case the conclusion of cut $(\Gamma, \Gamma'; \Delta, \Delta') \vdash^* C$ is also a conclusion of the same zero-premise rule as the right premise. The last two possibilities are that the left premise is $(\Gamma, p; \Delta) \vdash^+ p$ for Cut^a or $(\Gamma; \Delta, p) \vdash^- p$ for Cut^c respectively. For the former case this means that the right premise is $(\Gamma', p; \Delta') \vdash^* C$. This is the conclusion of a zero-premise rule only in one of the following cases:

- For $* = +$: $C = p$, or C is an atom in Γ', or $C = \top$
- For $* = -$: C is an atom in Δ', or $C = \bot$
- $\bot$ is in Γ', or $\top$ is in Δ'

In each case the conclusion of cut $(\Gamma, p, \Gamma'; \Delta, \Delta') \vdash^* C$ is also a conclusion of the same zero-premise rule as the right premise. For the latter case this means that the right premise is $(\Gamma'; \Delta', p) \vdash^* C$. This is the conclusion of a zero-premise rule only in one of the following cases:

- For $* = +$: C is an atom in Γ', or $C = \top$
- For $* = -$: $C = p$, or C is an atom in Δ', or $C = \bot$
- $\bot$ is in Γ', or $\top$ is in Δ'

In each case the conclusion of cut $(\Gamma, \Gamma'; \Delta, p, \Delta') \vdash^* C$ is also a conclusion of the same zero-premise rule as the right premise. So, when cut has reached zero-premise rules as premises, the derivation can be transformed into one beginning with the conclusion of the cut by deleting the premises.

The proof is – as before – conducted in a manner corresponding to the proof of cut-elimination for `G3ip` by [10], which means that it is by induction on the weight of the cut formula and a subinduction on the cut-height, the sum of heights of derivations of the two premises of cut.

Definition 3.8. The *cut-height* of an application of one of the rules of cut in a derivation is the sum of heights of derivation of the two premises of the rule.

In the proof permutations are given that always reduce the weight of the cut formula or the cut-height of instances of the rules. When the cut formula is not principal in at least one (or both) of the premises of cut, cut-height is reduced. In the other cases, i.e. in which the cut formula is principal in both premises, it is shown that cut-height and/or the weight of the cut formula can be reduced. This process terminates since atoms cannot be principal formulas.

The difference between the height of a derivation and cut-height needs to be emphasized here, because it is essential to understand that if there are two instances of cut, one occurring below the other in the derivation, this does not necessarily mean that the lower instance has a greater cut-height than the upper. Let us suppose the upper instance of cut occurs in the derivation of the left premise of the lower cut. The upper instance can have a cut-height which is greater than the height of either its premises because the *sum* of the premises is what matters. However, the lower instance can have as a right premise one with a much shorter derivation height than either of the premises of the upper cut, making the sum of the derivation heights of those two premises lesser than the one from the upper cut. So, what follows is that it is not enough to show that occurrences of cut can be permuted upward in a derivation in order to show that cut-height decreases, but we need to calculate exactly the cut-height of each derivation in our proof. As before, it can be assumed that in a given derivation the last instance is the one and only occurrence of cut.

THEOREM 3.9. *The cut rules*

$$\frac{(\Gamma;\Delta)\vdash^{+} D \qquad (\Gamma', D;\Delta')\vdash^{*} C}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^{*} C}\ Cut^{a} \quad \textit{and} \quad \frac{(\Gamma;\Delta)\vdash^{-} D \qquad (\Gamma';\Delta', D)\vdash^{*} C}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^{*} C}\ Cut^{c}$$

are admissible in **SC2Int**.

The proof is organized as follows. First, we consider the case that at least one premise in a cut is a conclusion of one of the zero-premise rules and show how cut can be eliminated in these cases. Otherwise three cases can be distinguished: 1) The cut formula is not principal in either premise of cut, 2) the cut formula is principal in just one premise of cut, and 3) the cut formula is principal in both premises of cut. The proof is presented in detail in the appendix, Section 6.

COROLLARY 3.10. (Subformula property) If $(\Gamma;\Delta)\vdash^{*} A$ $(* \in \{+,-\})$ is derivable in SC2Int, then all subformulas occurring in the derivation are subformulas of Γ or Δ.
(Decidability) Derivability of sequents $(\Gamma;\Delta)\vdash^{*} A$ $(* \in \{+,-\})$ in SC2Int is decidable.

4. Synonymy of formulas through inherited identity between derivations

In order to define a certain notion of identity between derivations that is inspired by the bilateralist distinction between proofs and their duals, we consider (i) the following two negation operations defined in terms of implication and co-implication:

$$\neg A := A \to \bot \text{ (negation)}, \qquad -A := \top \prec A \text{ (co-negation)}.$$

and (ii) the following rules that state an interaction between proofs and dual proofs mediated through the two negation connectives:

$$\dfrac{(\Gamma;\Delta,A)\vdash^{*} B}{(-A,\Gamma;\Delta)\vdash^{*} B}\,-ai \qquad \dfrac{(-A,\Gamma;\Delta)\vdash^{*} B}{(\Gamma;\Delta,A)\vdash^{*} B}\,-ae$$

$$\dfrac{(\Gamma;\Delta)\vdash^{-} A}{(\Gamma;\Delta)\vdash^{+} -A}\,-ci \qquad \dfrac{(\Gamma;\Delta)\vdash^{+} -A}{(\Gamma;\Delta)\vdash^{-} A}\,-ce$$

$$\dfrac{(\Gamma,A;\Delta)\vdash^{*} B}{(\Gamma;\neg A,\Delta)\vdash^{*} B}\,\neg ai \qquad \dfrac{(\Gamma;\neg A,\Delta)\vdash^{*} B}{(\Gamma,A;\Delta)\vdash^{*} B}\,\neg ae$$

$$\dfrac{(\Gamma;\Delta)\vdash^{+} A}{(\Gamma;\Delta)\vdash^{-} \neg A}\,\neg ci \qquad \dfrac{(\Gamma;\Delta)\vdash^{-} \neg A}{(\Gamma;\Delta)\vdash^{+} A}\,\neg ce$$

One idea behind these interaction rules is that they are rules the application of which has no effect on the identity of derivations, so that a proof of A *is* a refutation of $\neg A$, and vice versa, and a refutation of A *is* a proof of $-A$, and vice versa. Whereas in the case of the sequent calculus for `N4` in [18], it is possible to identify derivations of different formulas because the strong negation marks a back and forth between proofs and refutations, in the case of the interaction rules of the sequent calculus `SCInt`, derivations of different formulas are identified because proving (refuting) A is seen as amounting to refuting (proving) $\neg A$ ($-A$). As mentioned in the introduction, we shall not delve into elaborating a motivation for this approach but are content to apply the idea of interaction rules having no effect on the identity of derivations to `SC2Int`.

The interaction rules are admissible in `SC2Int`:

$$\dfrac{\dfrac{(\Gamma;\Delta,A)\vdash^{*} B}{(\Gamma,\top;\Delta,A)\vdash^{*} B}\,W^{a}}{(\top\prec A,\Gamma;\Delta)\vdash^{*} B}\,{\prec}L^{a}$$

$$\dfrac{\dfrac{\dfrac{}{(\varnothing;A)\vdash^{+}\top}\,\top R^{+} \quad \dfrac{}{(\varnothing;A)\vdash^{-} A}\,\textit{Lemma 3.3}}{(\varnothing;A)\vdash^{+}\top\prec A}\,{\prec}R^{+} \qquad (\top\prec A,\Gamma;\Delta)\vdash^{*} B}{(\Gamma;\Delta,A)\vdash^{*} B}\,Cut^{a}$$

$$\dfrac{\dfrac{}{(\Gamma;\Delta)\vdash^{+}\top}\,\top R^{+} \quad (\Gamma;\Delta)\vdash^{-} A}{(\Gamma;\Delta)\vdash^{+}\top\prec A}\,{\prec}R^{+} \qquad \dfrac{(\Gamma;\Delta)\vdash^{+}\top\prec A \quad \dfrac{\dfrac{}{(\top;A)\vdash^{-} A}\,\textit{Lemma 3.3}}{(\top\prec A;\varnothing)\vdash^{-} A}\,{\prec}L^{a}}{(\Gamma;\Delta)\vdash^{-} A}\,Cut^{a}$$

$$\dfrac{\dfrac{(\Gamma, A; \Delta) \vdash^* B}{(\Gamma, A; \Delta, \bot) \vdash^* B}\, W^c}{(\Gamma; \Delta, A \to \bot) \vdash^* B}\, {\to} L^c$$

$$\dfrac{\dfrac{\dfrac{}{(A;\varnothing)\vdash^+ A}\,\textit{Lemma 3.3} \qquad \dfrac{}{(A;\varnothing)\vdash^- \bot}\,\bot R^-}{(A;\varnothing)\vdash^- A\to\bot}\,{\to}R^- \qquad (\Gamma; A\to\bot,\Delta)\vdash^* B}{(\Gamma, A;\Delta)\vdash^* B}\, Cut^c$$

$$\dfrac{(\Gamma;\Delta)\vdash^+ A \qquad \dfrac{}{(\Gamma;\Delta)\vdash^- \bot}\,\bot R^-}{(\Gamma;\Delta)\vdash^- A\to\bot}\,{\to}R^- \qquad \dfrac{(\Gamma;\Delta)\vdash^- A\to\top \qquad \dfrac{\dfrac{}{(A;\bot)\vdash^+ A}\,\textit{Lemma 3.3}}{(\varnothing; A\to\bot)\vdash^+ A}\,{\to}L^c}{(\Gamma;\Delta)\vdash^+ A}\, Cut^c$$

In what follows, we will consider `SC2Int` without the admissible structural rules of contraction, weakening, and cut. We use $s, s_1, s_2, \ldots$ to stand for sequents. If $\mathscr{D}$ and $\mathscr{D}'$ are derivations in `SC2Int`, we shall write $\mathscr{D} \equiv \mathscr{D}'$ to express that $\mathscr{D}$ and $\mathscr{D}'$ are syntactically identical (as types of expressions, not as tokens).

Definition 4.1. The relation $\approx$ of inherited identity (in-identity) between derivations $\mathscr{D}_1$ and $\mathscr{D}_2$ in `SC2Int` is defined inductively. It is the smallest binary relation on the set of derivations in `SC2Int` such that:

1. $\mathscr{D}_1 \approx \mathscr{D}_2$ if $\mathscr{D}_1 \equiv \mathscr{D}_2$.
2. $\mathscr{D}_1 \approx \mathscr{D}_2$ if either $\mathscr{D}_1 \approx \mathscr{D}$ and $\mathscr{D}_2 \equiv \dfrac{\mathscr{D}}{s}$ or $\mathscr{D}_2 \approx \mathscr{D}$ and $\mathscr{D}_1 \equiv \dfrac{\mathscr{D}}{s}$, where s is obtained from $\mathscr{D}$ by an application of an (instance of an) interaction rule.
3. $\mathscr{D}_1 \approx \mathscr{D}_2$ if $\mathscr{D}_1 \equiv \dfrac{\mathscr{D}^1_1 \ldots \mathscr{D}^1_n}{s_1}$, $\mathscr{D}_2 \equiv \dfrac{\mathscr{D}^2_1 \ldots \mathscr{D}^2_n}{s_2}$, and $\mathscr{D}^1_i \approx \mathscr{D}^2_i$ $(1 \leq i \leq n \leq 2)$.

As in [18] it can be shown that the relation $\approx$ is an equivalence relation. Note that the third clause of Definition 4.1 allows one to identify, for example, proofs of $(A \vee B)$ and $(A \vee C)$, which is in accordance with the Brouwer-Heyting-Kolmogorov interpretation allowing for one and the same construction being a proof of both $(A \vee B)$ and $(A \vee C)$. Moreover, it is obvious that not any two cut-free derivations $\mathscr{D}_1$ and $\mathscr{D}_2$ in SC2Int of a formula A are in-identical. There are, e.g., syntactically distinct cut-free derivations of the sequent $(\varnothing;\varnothing) \vdash^+ (p \wedge q) \to (p \vee q)$ that are not in-identical. We shall give examples of in-identical derivations in the proof of Proposition 4.3.

DEFINITION 4.2. Two formulas A and B are said to be synonymous with respect to SC2Int iff

1. (positive condition) there exists a derivation $\mathscr{D}$ of $(A;\varnothing) \vdash^+ B$ and a derivation $\mathscr{D}'$ of $(B;\varnothing) \vdash^+ A$ with $\mathscr{D} \approx \mathscr{D}'$,
2. (negative condition) there exists a derivation $\mathscr{D}$ of $(\varnothing;A) \vdash^- B$ and a derivation $\mathscr{D}'$ of $(\varnothing;B) \vdash^- A$ with $\mathscr{D} \approx \mathscr{D}'$.

If the positive (negative) condition is satisfied, A and B are said to be positively (negatively) synonymous.

Accomplishing the interaction between proofs and refutations by means of two different negation connectives instead of a single strong negation, $\sim$, as in the sequent calculus SN4 from [18], has a considerable effect on the notion of synonymy stated in Definition 4.2. While in N4 all double negation and De Morgan laws hold and, for example, the following pairs of formulas turn out to be synonymous with respect to cut-free SN4

1. p and $\sim\sim p$,
2. $(p \wedge q)$ and $\sim(\sim p \vee \sim q)$,
3. $(p \vee q)$ and $\sim(\sim p \wedge \sim q)$,

not all double negation and De Morgan laws hold for $\neg$ and $-$ in SC2Int. We can observe a number of cases of positive or negative synonymy with respect to SC2Int.

PROPOSITION 4.3. The following pairs of formulas are positively synonymous with respect to SC2Int:

1. p and $-\neg p$,
2. $-(p \to q)$ and $(p \wedge -q)$,
3. $-(\neg p \vee q)$ and $(p \wedge -q)$,
4. $-(p \to q)$ and $-(\neg p \vee q)$,

whereas the following pairs are negatively synonymous:

5. p and $\neg - p$,
6. $\neg(p \prec q)$ and $(\neg p \vee q)$,
7. $\neg(p \wedge -q)$ and $(\neg p \vee q)$,
8. $\neg(p \prec q)$ and $\neg(p \wedge -q)$.

Proof: 1. and 5.: The following pairs of derivations are in-identical by the first clause of Definition 4.1:

$$\dfrac{\dfrac{(p;\varnothing) \vdash^{+} p}{(p;\varnothing) \vdash^{-} \neg p}\,\neg ci}{(p;\varnothing) \vdash^{+} -\neg p}\,-ci \qquad \dfrac{\dfrac{(p;\varnothing) \vdash^{+} p}{(\varnothing;\neg p) \vdash^{+} p}\,\neg ai}{(-\neg p;\varnothing) \vdash^{+} p}\,-ai$$

$$\dfrac{\dfrac{(\varnothing;p) \vdash^{-} p}{(\varnothing,p) \vdash^{+} -p}\,-ci}{(\varnothing;p) \vdash^{-} \neg - p}\,\neg ci \qquad \dfrac{\dfrac{(\varnothing;p) \vdash^{-} p}{(-p;\varnothing) \vdash^{-} p}\,-ai}{(\varnothing;\neg - p) \vdash^{-} p}\,\neg ai$$

2. We shall demonstrate the in-identity of the following two derivations in detail. The demonstration for the cases 3., 6., and 7. is similar and left to the reader.

$$\dfrac{\dfrac{\dfrac{(p;q) \vdash^{+} p}{(\varnothing;(p \to q)) \vdash^{+} p}}{(-(p \to q);\varnothing) \vdash^{+} p} \quad \dfrac{\dfrac{\dfrac{(p;q) \vdash^{-} q}{(\varnothing;(p \to q)) \vdash^{-} q}}{(-(p \to q);\varnothing) \vdash^{-} q}}{(-(p \to q);\varnothing) \vdash^{+} -q}}{(-(p \to q);\varnothing) \vdash^{+} (p \wedge -q)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{(p;q) \vdash^{+} p}{(p,-q;\varnothing) \vdash^{+} p}}{((p \wedge -q);\varnothing) \vdash^{+} p} \quad \dfrac{\dfrac{(p;q) \vdash^{-} q}{(p,-q;\varnothing) \vdash^{-} q}}{((p \wedge -q);\varnothing) \vdash^{-} q}}{((p \wedge -q);\varnothing) \vdash^{-} (p \to q)}}{((p \wedge -q);\varnothing) \vdash^{+} -(p \to q)}$$

Let $\mathscr{D}_1$ and $\mathscr{D}_2$ be the derivations

$$\dfrac{(p;q) \vdash^{+} p}{(\varnothing;(p \to q)) \vdash^{+} p} \quad \text{and} \quad \dfrac{(p;q) \vdash^{+} p}{(p,-q;\varnothing) \vdash^{+} p}$$

and let $\mathscr{D}_3$ and $\mathscr{D}_4$ be the derivations

$$\dfrac{\mathscr{D}_1}{(-(p \to q);\varnothing) \vdash^{+} p} \quad \text{and} \quad \dfrac{\mathscr{D}_2}{(p \wedge -q);\varnothing) \vdash^{+} p.}$$

By clauses 1. and 3. of Definition 4.1, $\mathscr{D}_1 \approx \mathscr{D}_2$, and by clause 3. of Definition 4.1, $\mathscr{D}_3 \approx \mathscr{D}_4$. Let $\mathscr{D}_5$ and $\mathscr{D}_6$ be the derivations

$$\dfrac{(p; q) \vdash^- q}{(\varnothing; (p \to q)) \vdash^- q} \quad \text{and} \quad \dfrac{(p; q) \vdash^- q}{(p, -q; \varnothing) \vdash^- q}$$

and let $\mathscr{D}_7$ and $\mathscr{D}_8$ be the derivations

$$\dfrac{\dfrac{\mathscr{D}_5}{(-(p \to q); \varnothing) \vdash^- q}}{(-(p \to q); \varnothing) \vdash^+ -q} \quad \text{and} \quad \dfrac{\mathscr{D}_6}{(p \wedge -q); \varnothing) \vdash^- q.}$$

By clauses 1. and 3. of Definition 4.1, $\mathscr{D}_5 \approx \mathscr{D}_6$, and by clauses 3. and 2. of Definition 4.1, $\mathscr{D}_7 \approx \mathscr{D}_8$. Then, by clause 3. of Definition 4.1, we obtain that for the derivations $\mathscr{D}_9$ and $\mathscr{D}_{10}$, namely,

$$\dfrac{\mathscr{D}_3 \qquad \mathscr{D}_7}{(-(p \to q); \varnothing) \vdash^+ (p \wedge -q)} \quad \text{and} \quad \dfrac{\mathscr{D}_4 \qquad \mathscr{D}_8}{(p \wedge -q); \varnothing) \vdash^- (p \to q)}$$

it holds that $\mathscr{D}_9 \approx \mathscr{D}_{10}$. Let $\mathscr{D}_{11}$ be

$$\dfrac{\mathscr{D}_{10}}{(p \wedge -q); \varnothing) \vdash^+ -(p \to q).}$$

By clause 2. of Definition 4.1, $\mathscr{D}_9 \approx \mathscr{D}_{11}$.
3.:

$$\dfrac{\dfrac{\dfrac{\dfrac{(p; q) \vdash^+ p}{(\varnothing; \neg p, q) \vdash^+ p}}{(\varnothing; (\neg p \vee q)) \vdash^+ p}}{(-(\neg p \vee q); \varnothing) \vdash^+ p} \quad \dfrac{\dfrac{\dfrac{\dfrac{(p; q) \vdash^- q}{(\varnothing; \neg p, q) \vdash^- q}}{(\varnothing; (\neg p \vee q)) \vdash^- q}}{(-(\neg p \vee q); \varnothing) \vdash^- q}}{(-(\neg p \vee q); \varnothing) \vdash^+ -q}}{(-(\neg p \vee q); \varnothing) \vdash^+ (p \wedge -q)} \approx \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(p; q) \vdash^+ p}{((p, -q); \varnothing) \vdash^+ p}}{((p \wedge -q); \varnothing) \vdash^+ p}}{((p \wedge -q); \varnothing) \vdash^- \neg p} \quad \dfrac{\dfrac{(p; q) \vdash^- q}{(p, -q; \varnothing) \vdash^- q}}{((p \wedge -q); \varnothing) \vdash^- q}}{((p \wedge -q); \varnothing) \vdash^- (\neg p \vee q)}}{((p \wedge -q); \varnothing) \vdash^+ -(\neg p \vee q)}$$

4.: By 2., 3., and the transitivity of $\approx$.
6.:

$$\dfrac{\dfrac{\dfrac{\dfrac{(p; q)) \vdash^+ p}{((p \prec q); \varnothing) \vdash^+ p}}{(\varnothing; \neg(p \prec q)) \vdash^+ p}}{(\varnothing; \neg(p \prec q)) \vdash^- \neg p} \quad \dfrac{\dfrac{(p; q) \vdash^- q}{((p \prec q); \varnothing) \vdash^- q}}{(\varnothing; \neg(p \prec q)) \vdash^- q}}{(\varnothing; \neg(p \prec q)) \vdash^- (\neg p \vee q)} \approx \dfrac{\dfrac{\dfrac{\dfrac{(p; q) \vdash^+ p}{(\varnothing; \neg p, q) \vdash^+ p}}{(\varnothing; (\neg p \vee q)) \vdash^+ p} \quad \dfrac{\dfrac{(p; q) \vdash^- q}{(\varnothing; \neg p, q) \vdash^- q}}{(\varnothing; (\neg p \vee q)) \vdash^- q}}{(\varnothing; (\neg p \vee q)) \vdash^+ (p \prec q)}}{(\varnothing; (\neg p \vee q)) \vdash^- \neg(p \prec q)}$$

7.:

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{(p;q) \vdash^{+} p}{(p,-q;\varnothing) \vdash^{+} p}}{((p \wedge -q);\varnothing) \vdash^{+} p}}{(\varnothing;\neg(p \wedge -q)) \vdash^{+} p}}{(\varnothing;\neg(p \wedge -q)) \vdash^{-} \neg p}
\quad
\dfrac{\dfrac{\dfrac{(p;q) \vdash^{-} q}{(p,-q;\varnothing) \vdash^{-} q}}{((p \wedge -q);\varnothing) \vdash^{-} q}}{(\varnothing;\neg(p \wedge -q)) \vdash^{-} q}
}{(\varnothing;\neg(p \wedge -q)) \vdash^{-} (\neg p \vee q)}
\approx
\dfrac{\dfrac{
\dfrac{\dfrac{(p;q) \vdash^{+} p}{(\varnothing;\neg p, q) \vdash^{+} p}}{(\varnothing;(\neg p \vee q)) \vdash^{+} p}
\quad
\dfrac{\dfrac{\dfrac{(p;q) \vdash^{-} q}{(\varnothing;\neg p, q) \vdash^{-} q}}{(\varnothing;(\neg p \vee q)) \vdash^{-} q}}{(\varnothing;(\neg p \vee q)) \vdash^{+} -q}
}{(\varnothing;(\neg p \vee q)) \vdash^{+} (p \wedge -q)}}{(\varnothing;(\neg p \vee q)) \vdash^{-} \neg(p \wedge -q)}
$$

8.: By 6., 7., and the transitivity of $\approx$. □

Since we have not been able to find pairs of distinct formulas in the language of 2Int that are synonymous with respect to SC2Int in the sense of Definition 4.2, we are led to conjecture that there are no such pairs of formulas.

Conjecture 4.4. There exist no two distinct formulas A, B in the language of 2Int that are synonymous with respect to SC2Int in the sense of Definition 4.2.

If that conjecture is true, then synonymy based on in-identity with respect to SC2Int trivializes in the sense that it seems to be an empty concept. However, this is neither really surprising if we reconsider the differences between 2Int and N4 nor does it have to be seen as a defect of in-identity or SC2Int. While in N4 there is one negation, which is firstly primitive and secondly serves as a toggle between proofs and refutations, in 2Int we have two negations, which are mere results from having two *implications*, which in turn are the object language manifestation of having two *derivability relations*. With that in mind it does not seem odd that there are no two (distinct) synonymous formulas interderivable w.r.t. to *both* derivability relations. After all, the interaction rules solely work with the two negations but do not allow a 'toggling' back and forth between proofs and refutations. So, in order to get an interderivability w.r.t. the positively signed derivability relation using the interaction rules, it seems that we will always have to use the $-$ci and the $-$ai rule as the last interaction rules in the derivation. This is not to say that one of them has to be the very last rule applied in the derivation and also not to say that other interaction rules cannot appear within the derivation. As we see in the exemplary derivations above, of course, the very last rule can be a normal operational rule and of course, there can be other interaction rules like the ones for $\neg$. But the last of the interaction rules to occur, must always be $-$ci and $-$ai (the order between those two does not matter). This is just because *if* interaction rules are to be used, then these are the ones getting a formula into the assumptions and switching the derivability relation from $-$ to $+$, which is the result we need for derivations of the form $(A; \varnothing) \vdash^+ B$ and $(B; \varnothing) \vdash^+ A$. The same holds for interderivability w.r.t. the negatively signed derivability relation and the use of the interaction rules $\neg$ai and $\neg$ci. Since applying these rules results in different formulas, though, namely in formulas having $-$, resp. $\neg$ as main operator, it simply does not seem possible to have both interderivabilities for the same pair of formulas.

So, this result can be regarded as an interesting consequence of the basics of `SC2Int` because what we obtain by having bilateralist concepts also overtly realized in the connectives is an exclusive division between positive and negative synonymy. It highlights the bilateralist principle of verifications (proofs) and falsifications (refutations) being two primitive kinds of derivations in their own right.

5. Conclusion and outlook

By applying the proof methods that [10] use for their calculus `G3ip`, we were able to show the admissibility of the structural rules of weakening, contraction, and cut in the sequent calculus `SC2Int` for the bi-intuitionistic logic `2Int`. With `SC2Int` at hand, we could apply the definition of inherited identity of derivations from [18] to define the notion of propositional synonymy of formulas with respect to `SC2Int` as the combination of two concepts of positive and negative synonymy. We were able to present various pairs of distinct formulas that are either positively or negatively synonymous with respect to `SC2Int`, and we conjectured that there exist no pairs of distinct formulas that are both positively and negatively synonymous with respect to `SC2Int`.

An obvious task is to decide Conjecture 4.4. Moreover, as already indicated in [18], it would be interesting to encode derivations in a bilateral sequent calculus that accommodates proofs as well as refutations, such as `SC2Int`, in a suitable two-sorted typed λ-calculus with terms of one sort denoting proofs and terms of a second sort denoting dual proofs, refutations. This is currently work in progress by one of the authors (cf. [2]). There, it will be pondered what other ways of understanding the concept of identity between proofs and refutations are available and sensible in the light of identifying lambda-term constructions.

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6. Appendix

We present a proof of Theorem 3.9 by considering the mentioned case distinction.

Cut with a conclusion of a zero-premise rule as premise

Cut with a conclusion of Rf^+, Rf^-, $\bot L^a$, $\top L^c, \bot R^-$, or $\top R^+$ as premise

If at least one of the premises of cut is a conclusion of one of the zero-premise rules, we distinguish three cases for both cut rules:

-1- Cut^a

-1.1- The left premise $(\Gamma; \Delta) \vdash^+ D$ is a conclusion of a zero-premise-rule. There are four subcases:

(a) The cut formula D is an atom in Γ. Then the conclusion $(\Gamma, \Gamma'; \Delta, \Delta') \vdash^* C$ is derived from $(\Gamma', D; \Delta') \vdash^* C$ by W^a and W^c.

(b) $\bot$ is a formula in Γ. Then $(\Gamma, \Gamma'; \Delta, \Delta') \vdash^* C$ is also a conclusion of $\bot L^a$.

(c) $\top$ is a formula in Δ. Then $(\Gamma, \Gamma'; \Delta, \Delta') \vdash^* C$ is also a conclusion of $\top L^c$.

(d) $\top =$ D. Then the right premise is $(\Gamma', \top; \Delta') \vdash^* C$ and $(\Gamma, \Gamma'; \Delta, \Delta') \vdash^* C$ follows by $W^{\top}_{inv}$ as well as W^a and W^c.

-1.2- The right premise $(\Gamma', D; \Delta') \vdash^{+} C$ is a conclusion of a zero-premise rule. There are six subcases:

(a) C is an atom in Γ'. Then $(\Gamma, \Gamma'; \Delta, \Delta') \vdash^{+} C$ is also a conclusion of Rf^{+}.

(b) $C = D$. Then the left premise is $(\Gamma; \Delta) \vdash^{+} C$ and $(\Gamma, \Gamma'; \Delta, \Delta') \vdash^{+} C$ follows by W^{a} and W^{c}.

(c) $\bot$ is in Γ'. Then $(\Gamma, \Gamma'; \Delta, \Delta') \vdash^{+} C$ is also a conclusion of $\bot L^{a}$.

(d) $\bot$ = D. Then the left premise is $(\Gamma; \Delta) \vdash^{+} \bot$ and is either a conclusion of $\bot L^{a}$ or $\top L^{c}$ (in which case cf. 1.1 (b) or 1.1 (c)) or it has been derived by a left rule. There are eight cases according to the rule used which can be transformed into derivations with lesser cut-height. We will not show this here, since this is only a special case of the cases 3.1-3.8 below.

(e) $\top$ is in Δ'. Then $(\Gamma, \Gamma'; \Delta, \Delta') \vdash^{+} C$ is also a conclusion of $\top L^{c}$.

(f) $\top$ = C. Then $(\Gamma, \Gamma'; \Delta, \Delta') \vdash^{+} C$ is also a conclusion of $\top R^{+}$.

-1.3- The right premise $(\Gamma', D; \Delta') \vdash^{-} C$ is a conclusion of a zero-premise rule. There are five subcases:

(a) C is an atom in Δ'. Then $(\Gamma, \Gamma'; \Delta, \Delta') \vdash^{-} C$ is also a conclusion of Rf^{-}.

(b) $\bot$ is in Γ'. Then $(\Gamma, \Gamma'; \Delta, \Delta') \vdash^{-} C$ is also a conclusion of $\bot L^{a}$.

(c) $\bot$ = D. Then the left premise is $(\Gamma; \Delta) \vdash^{+} \bot$ and the same as mentioned in 1.2 (d) holds.

(d) $\top$ is in Δ'. Then $(\Gamma, \Gamma'; \Delta, \Delta') \vdash^{-} C$ is also a conclusion of $\top L^{c}$.

(e) $\bot$ = C. Then $(\Gamma, \Gamma'; \Delta, \Delta') \vdash^{-} C$ is also a conclusion of $\bot R^{-}$.

-2- Cut^{c}

-2.1- The left premise $(\Gamma; \Delta) \vdash^{-} D$ is a conclusion of a zero-premise rule. There are four subcases:

(a) The cut formula D is an atom in Δ. Then the conclusion $(\Gamma, \Gamma'; \Delta, \Delta') \vdash^{*} C$ is derived from $(\Gamma'; \Delta', D) \vdash^{*} C$ by W^{a} and W^{c}.

(b) $\bot$ is in Γ. Then $(\Gamma, \Gamma'; \Delta, \Delta') \vdash^* C$ is also a conclusion of $\bot L^a$.

(c) $\top$ is in Δ. Then $(\Gamma, \Gamma'; \Delta, \Delta') \vdash^* C$ is also a conclusion of $\top L^c$.

(d) $\bot =$ D. Then the right premise is $(\Gamma'; \Delta', \bot) \vdash^* C$ and $(\Gamma, \Gamma'; \Delta, \Delta') \vdash^* C$ follows by $W^{\bot}_{inv}$ as well as W^a and W^c.

-2.2- The right premise $(\Gamma'; \Delta', D) \vdash^+ C$ is a conclusion of a zero-premise rule. There are five subcases:

(a) C is an atom in Γ'. Then $(\Gamma, \Gamma'; \Delta, \Delta') \vdash^+ C$ is also a conclusion of Rf^+.

(b) $\bot$ is in Γ'. Then $(\Gamma, \Gamma'; \Delta, \Delta') \vdash^+ C$ is also a conclusion of $\bot L^a$.

(c) $\top$ is in Δ'. Then $(\Gamma, \Gamma'; \Delta, \Delta') \vdash^+ C$ is also a conclusion of $\top L^c$.

(d) $\top =$ D. Then the left premise is $(\Gamma; \Delta) \vdash^- \top$ and the same as mentioned in 1.2 (d) holds.

(e) $\top =$ C. Then $(\Gamma, \Gamma'; \Delta, \Delta') \vdash^+ C$ is also a conclusion of $\top R^+$.

-2.3- The right premise $(\Gamma'; \Delta', D) \vdash^- C$ is a conclusion of a zero-premise rule. There are six subcases:

(a) C is an atom in Δ'. Then $(\Gamma, \Gamma'; \Delta, \Delta') \vdash^- C$ is also a conclusion of Rf^-.

(b) $C = D$. Then the left premise is $(\Gamma; \Delta) \vdash^- C$ and $(\Gamma, \Gamma'; \Delta, \Delta') \vdash^- C$ follows by W^a and W^c.

(c) $\bot$ is in Γ'. Then $(\Gamma, \Gamma'; \Delta, \Delta') \vdash^- C$ is also a conclusion of $\bot L^a$.

(d) $\top$ is in Δ'. Then $(\Gamma, \Gamma'; \Delta, \Delta') \vdash^- C$ is also a conclusion of $\top L^c$.

(e) $\top =$ D. Then the left premise is $(\Gamma; \Delta) \vdash^- \top$ and the same as mentioned in 1.2 (d) holds.

(f) $\bot =$ C. Then $(\Gamma, \Gamma'; \Delta, \Delta') \vdash^- C$ is also a conclusion of $\bot R^-$.

Cut with neither premise a conclusion of a zero-premise rule

We distinguish the cases that a left rule is used to derive the left premise (cf. 3), a right rule is used to derive the left premise (cf. 5), a right or a left rule is used to derive the right premise with the cut formula not being principal there (cf. 4), and that a left rule is used to derive the right premise with the cut formula being principal (cf. 5). These cases can be subsumed in a more compact form as categorized below. We assume, like [10], that in the derivations the topsequents, from left to right, have derivation heights n, m, k,...

-3- Cut not principal in the left premise

If the cut formula D is not principal in the left premise, this means that this premise is derived by a left introduction rule. By permuting the order of the rules for the logical connectives with the cut rules, cut-height can be reduced in each of the following eight cases:

-3.1- $\wedge L^a$ is the last rule used to derive the left premise with $\Gamma = \Gamma'', A \wedge B$. The derivations for Cut^a and Cut^c with cuts of cut-height $n+1+m$ are

$$\dfrac{\dfrac{(\Gamma'', A, B; \Delta) \vdash^+ D}{(\Gamma'', A \wedge B; \Delta) \vdash^+ D}\,\wedge L^a \quad (\Gamma', D; \Delta') \vdash^* C}{(\Gamma'', A \wedge B, \Gamma'; \Delta, \Delta') \vdash^* C}\,Cut^a$$

$$\dfrac{\dfrac{(\Gamma'', A, B; \Delta) \vdash^- D}{(\Gamma'', A \wedge B; \Delta) \vdash^- D}\,\wedge L^a \quad (\Gamma'; \Delta', D) \vdash^* C}{(\Gamma'', A \wedge B, \Gamma'; \Delta, \Delta') \vdash^* C}\,Cut^c$$

These can be transformed into derivations with cuts of cut-height $n+m$:

$$\dfrac{\dfrac{(\Gamma'', A, B; \Delta) \vdash^+ D \quad (\Gamma', D; \Delta') \vdash^* C}{(\Gamma'', A, B, \Gamma'; \Delta, \Delta') \vdash^* C}\,Cut^a}{(\Gamma'', A \wedge B, \Gamma'; \Delta, \Delta') \vdash^* C}\,\wedge L^a \qquad \dfrac{\dfrac{(\Gamma'', A, B; \Delta) \vdash^- D \quad (\Gamma'; \Delta', D) \vdash^* C}{(\Gamma'', A, B, \Gamma'; \Delta, \Delta') \vdash^* C}\,Cut^c}{(\Gamma'', A \wedge B, \Gamma'; \Delta, \Delta') \vdash^* C}\,\wedge L^a$$

-3.2- $\wedge L^c$ is the last rule used to derive the left premise with $\Delta = \Delta'', A \wedge B$. The derivations with cuts of cut-height $max(n, m) + 1 + k$ are

$$\dfrac{\dfrac{(\Gamma; \Delta'', A) \vdash^+ D \quad (\Gamma; \Delta'', B) \vdash^+ D}{(\Gamma; \Delta'', A \wedge B) \vdash^+ D}\,\wedge L^c \quad (\Gamma', D; \Delta') \vdash^* C}{(\Gamma, \Gamma'; \Delta'', A \wedge B, \Delta') \vdash^* C}\,Cut^a$$

$$\dfrac{\dfrac{(\Gamma; \Delta'', A) \vdash^- D \quad (\Gamma; \Delta'', B) \vdash^- D}{(\Gamma; \Delta'', A \wedge B) \vdash^- D}\,\wedge L^c \quad (\Gamma'; \Delta', D) \vdash^* C}{(\Gamma, \Gamma'; \Delta'', A \wedge B, \Delta') \vdash^* C}\,Cut^c$$

These can be transformed into derivations each with two cuts of cut-height $n+k$ and $m+k$, respectively:

$$\dfrac{\dfrac{(\Gamma;\Delta'',A)\vdash^{+} D \quad (\Gamma',D;\Delta')\vdash^{*} C}{(\Gamma,\Gamma';\Delta'',A,\Delta')\vdash^{*} C}\,{\scriptstyle Cut^a} \qquad \dfrac{(\Gamma;\Delta'',B)\vdash^{+} D \quad (\Gamma',D;\Delta')\vdash^{*} C}{(\Gamma,\Gamma';\Delta'',B,\Delta')\vdash^{*} C}\,{\scriptstyle Cut^a}}{(\Gamma,\Gamma';\Delta'',A\wedge B,\Delta')\vdash^{*} C}\,{\scriptstyle \wedge L^c}$$

$$\dfrac{\dfrac{(\Gamma;\Delta'',A)\vdash^{-} D \quad (\Gamma';\Delta',D)\vdash^{*} C}{(\Gamma,\Gamma';\Delta'',A,\Delta')\vdash^{*} C}\,{\scriptstyle Cut^c} \qquad \dfrac{(\Gamma;\Delta'',B)\vdash^{-} D \quad (\Gamma';\Delta',D)\vdash^{*} C}{(\Gamma,\Gamma';\Delta'',B,\Delta')\vdash^{*} C}\,{\scriptstyle Cut^c}}{(\Gamma,\Gamma';\Delta'',A\wedge B,\Delta')\vdash^{*} C}\,{\scriptstyle \wedge L^c}$$

-3.3- $\vee L^a$ is the last rule used to derive the left premise with $\Gamma=\Gamma'',A\vee B$. The derivations with cuts of cut-height $max(n,m)+1+k$ are

$$\dfrac{\dfrac{(\Gamma'',A;\Delta)\vdash^{+} D \quad (\Gamma'',B;\Delta)\vdash^{+} D}{(\Gamma'',A\vee B;\Delta)\vdash^{+} D}\,{\scriptstyle \vee L^a} \quad (\Gamma',D;\Delta')\vdash^{*} C}{(\Gamma'',A\vee B,\Gamma';\Delta,\Delta')\vdash^{*} C}\,{\scriptstyle Cut^a}$$

$$\dfrac{\dfrac{(\Gamma'',A;\Delta)\vdash^{-} D \quad (\Gamma'',B;\Delta)\vdash^{-} D}{(\Gamma'',A\vee B;\Delta)\vdash^{-} D}\,{\scriptstyle \vee L^a} \quad (\Gamma';\Delta',D)\vdash^{*} C}{(\Gamma'',A\vee B,\Gamma';\Delta,\Delta')\vdash^{*} C}\,{\scriptstyle Cut^c}$$

These can be transformed into derivations each with two cuts of cut-height $n+k$ and $m+k$, respectively:

$$\dfrac{\dfrac{(\Gamma'',A;\Delta)\vdash^{+} D \quad (\Gamma',D;\Delta')\vdash^{*} C}{(\Gamma'',A,\Gamma';\Delta,\Delta')\vdash^{*} C}\,{\scriptstyle Cut^a} \qquad \dfrac{(\Gamma'',B;\Delta)\vdash^{+} D \quad (\Gamma',D;\Delta')\vdash^{*} C}{(\Gamma'',B,\Gamma';\Delta,\Delta')\vdash^{*} C}\,{\scriptstyle Cut^a}}{(\Gamma'',A\vee B,\Gamma';\Delta,\Delta')\vdash^{*} C}\,{\scriptstyle \vee L^a}$$

$$\dfrac{\dfrac{(\Gamma'',A;\Delta)\vdash^{-} D \quad (\Gamma';\Delta',D)\vdash^{*} C}{(\Gamma'',A,\Gamma';\Delta,\Delta')\vdash^{*} C}\,{\scriptstyle Cut^c} \qquad \dfrac{(\Gamma'',B;\Delta)\vdash^{-} D \quad (\Gamma';\Delta',D)\vdash^{*} C}{(\Gamma'',B,\Gamma';\Delta,\Delta')\vdash^{*} C}\,{\scriptstyle Cut^c}}{(\Gamma'',A\vee B,\Gamma';\Delta,\Delta')\vdash^{*} C}\,{\scriptstyle \vee L^a}$$

-3.4- $\vee L^c$ is the last rule used to derive the left premise with $\Delta=\Delta'',A\vee B$. The derivations with cuts of cut-height $n+1+m$ are

$$\dfrac{\dfrac{(\Gamma;\Delta'',A,B)\vdash^{+} D}{(\Gamma;\Delta'',A\vee B)\vdash^{+} D}\,{\scriptstyle \vee L^c} \quad (\Gamma',D;\Delta')\vdash^{*} C}{(\Gamma,\Gamma';\Delta'',A\vee B,\Delta')\vdash^{*} C}\,{\scriptstyle Cut^a}$$

$$\dfrac{\dfrac{(\Gamma;\Delta'',A,B)\vdash^{-} D}{(\Gamma;\Delta'',A\vee B)\vdash^{-} D}\,{\vee L^c} \qquad (\Gamma';\Delta',D)\vdash^{*} C}{(\Gamma,\Gamma';\Delta'',A\vee B,\Delta')\vdash^{*} C}\,{Cut^c}$$

These can be transformed into derivations with cuts of cut-height $n+m$:

$$\dfrac{\dfrac{(\Gamma;\Delta'',A,B)\vdash^{+} D \qquad (\Gamma',D;\Delta')\vdash^{*} C}{(\Gamma,\Gamma';\Delta'',A,B,\Delta')\vdash^{*} C}\,{Cut^a}}{(\Gamma,\Gamma';\Delta'',A\vee B,\Delta')\vdash^{*} C}\,{\vee L^c} \qquad \dfrac{\dfrac{(\Gamma;\Delta'',A,B)\vdash^{-} D \qquad (\Gamma';\Delta',D)\vdash^{*} C}{(\Gamma,\Gamma';\Delta'',A,B,\Delta')\vdash^{*} C}\,{Cut^c}}{(\Gamma,\Gamma';\Delta'',A\vee B,\Delta')\vdash^{*} C}\,{\vee L^c}$$

-3.5- $\rightarrow L^a$ is the last rule used to derive the left premise with $\Gamma = \Gamma'', A \rightarrow B$. The derivations with cuts of cut-height $max(n,m)+1+k$ are

$$\dfrac{\dfrac{(\Gamma'',A\rightarrow B;\Delta)\vdash^{+} A \qquad (\Gamma'',B;\Delta)\vdash^{+} D}{(\Gamma'',A\rightarrow B;\Delta)\vdash^{+} D}\,{\rightarrow L^a} \qquad (\Gamma',D;\Delta')\vdash^{*} C}{(\Gamma'',A\rightarrow B,\Gamma';\Delta,\Delta')\vdash^{*} C}\,{Cut^a}$$

$$\dfrac{\dfrac{(\Gamma'',A\rightarrow B;\Delta)\vdash^{+} A \qquad (\Gamma'',B;\Delta)\vdash^{-} D}{(\Gamma'',A\rightarrow B;\Delta)\vdash^{-} D}\,{\rightarrow L^a} \qquad (\Gamma';\Delta',D)\vdash^{*} C}{(\Gamma'',A\rightarrow B,\Gamma';\Delta,\Delta')\vdash^{*} C}\,{Cut^c}$$

These can be transformed into derivations with cuts of cut-height $m+k$:

$$\dfrac{\dfrac{(\Gamma'',A\rightarrow B;\Delta)\vdash^{+} A}{(\Gamma'',A\rightarrow B,\Gamma';\Delta,\Delta')\vdash^{+} A}\,{W^{a/c}} \qquad \dfrac{(\Gamma'',B;\Delta)\vdash^{+} D \qquad (\Gamma',D;\Delta')\vdash^{*} C}{(\Gamma'',B,\Gamma';\Delta,\Delta')\vdash^{*} C}\,{Cut^a}}{(\Gamma'',A\rightarrow B,\Gamma';\Delta,\Delta')\vdash^{*} C}\,{\rightarrow L^a}$$

$$\dfrac{\dfrac{(\Gamma'',A\rightarrow B;\Delta)\vdash^{+} A}{(\Gamma'',A\rightarrow B,\Gamma';\Delta,\Delta')\vdash^{+} A}\,{W^{a/c}} \qquad \dfrac{(\Gamma'',B;\Delta)\vdash^{-} D \qquad (\Gamma';\Delta',D)\vdash^{*} C}{(\Gamma'',B,\Gamma';\Delta,\Delta')\vdash^{*} C}\,{Cut^c}}{(\Gamma'',A\rightarrow B,\Gamma';\Delta,\Delta')\vdash^{*} C}\,{\rightarrow L^a}$$

-3.6- $\rightarrow L^c$ is the last rule used to derive the left premise with $\Delta = \Delta'', A \rightarrow B$. The derivations with cuts of cut-height $n+1+m$ are

$$\dfrac{\dfrac{(\Gamma,A;\Delta'',B)\vdash^{+} D}{(\Gamma;\Delta'',A\rightarrow B)\vdash^{+} D}\,{\rightarrow L^c} \qquad (\Gamma',D;\Delta')\vdash^{*} C}{(\Gamma,\Gamma';\Delta'',A\rightarrow B,\Delta')\vdash^{*} C}\,{Cut^a}$$

$$\cfrac{\cfrac{(\Gamma, A; \Delta'', B) \vdash^- D}{(\Gamma; \Delta'', A \to B) \vdash^- D}\ {\to}L^c \qquad (\Gamma'; \Delta', D) \vdash^* C}{(\Gamma, \Gamma'; \Delta'', A \to B, \Delta') \vdash^* C}\ Cut^c$$

These can be transformed into derivations with cuts of cut-height $n + m$:

$$\cfrac{\cfrac{(\Gamma, A; \Delta'', B) \vdash^+ D \qquad (\Gamma', D; \Delta') \vdash^* C}{(\Gamma, A, \Gamma'; \Delta'', B, \Delta') \vdash^* C}\ Cut^a}{(\Gamma, \Gamma'; \Delta'', A \to B, \Delta') \vdash^* C}\ {\to}L^c \qquad \cfrac{\cfrac{(\Gamma, A; \Delta'', B) \vdash^- D \qquad (\Gamma'; \Delta', D) \vdash^* C}{(\Gamma, A, \Gamma'; \Delta'', B, \Delta') \vdash^* C}\ Cut^c}{(\Gamma, \Gamma'; \Delta'', A \to B, \Delta') \vdash^* C}\ {\to}L^c$$

-3.7- ${\prec}L^a$ is the last rule used to derive the left premise with $\Gamma = \Gamma'', A{\prec}B$. The derivations with cuts of cut-height $n + 1 + m$ are

$$\cfrac{\cfrac{(\Gamma'', A; \Delta, B) \vdash^+ D}{(\Gamma'', A \prec B; \Delta) \vdash^+ D}\ {\prec}L^a \qquad (\Gamma', D; \Delta') \vdash^* C}{(\Gamma'', A \prec B, \Gamma'; \Delta, \Delta') \vdash^* C}\ Cut^a$$

$$\cfrac{\cfrac{(\Gamma'', A; \Delta, B) \vdash^- D}{(\Gamma'', A \prec B; \Delta) \vdash^- D}\ {\prec}L^a \qquad (\Gamma'; \Delta', D) \vdash^* C}{(\Gamma'', A \prec B, \Gamma'; \Delta, \Delta') \vdash^* C}\ Cut^c$$

These can be transformed into derivations with cuts of cut-height $n + m$:

$$\cfrac{\cfrac{(\Gamma'', A; \Delta, B) \vdash^+ D \qquad (\Gamma', D; \Delta') \vdash^* C}{(\Gamma'', A, \Gamma'; \Delta, B, \Delta') \vdash^* C}\ Cut^a}{(\Gamma'', A \prec B, \Gamma'; \Delta, \Delta') \vdash^* C}\ {\prec}L^a \qquad \cfrac{\cfrac{(\Gamma'', A; \Delta, B) \vdash^- D \qquad (\Gamma'; \Delta', D) \vdash^* C}{(\Gamma'', A, \Gamma'; \Delta, B, \Delta') \vdash^* C}\ Cut^c}{(\Gamma'', A \prec B, \Gamma'; \Delta, \Delta') \vdash^* C}\ {\prec}L^a$$

-3.8- ${\prec}L^c$ is the last rule used to derive the left premise with $\Delta = \Delta'', A{\prec}B$. The derivations with cuts of cut-height $max(n, m) + 1 + k$ are

$$\cfrac{\cfrac{(\Gamma; \Delta'', A \prec B) \vdash^- B \quad (\Gamma; \Delta'', A) \vdash^+ D}{(\Gamma; \Delta'', A \prec B) \vdash^+ D}\ {\prec}L^c \qquad (\Gamma', D; \Delta') \vdash^* C}{(\Gamma, \Gamma'; \Delta'', A \prec B, \Delta') \vdash^* C}\ Cut^a$$

$$\cfrac{\cfrac{(\Gamma; \Delta'', A \prec B) \vdash^- B \quad (\Gamma; \Delta'', A) \vdash^- D}{(\Gamma; \Delta'', A \prec B) \vdash^- D}\ {\prec}L^c \qquad (\Gamma'; \Delta', D) \vdash^* C}{(\Gamma, \Gamma'; \Delta'', A \prec B, \Delta') \vdash^* C}\ Cut^c$$

These can be transformed into derivations with cuts of cut-height $m + k$:

$$\dfrac{\dfrac{(\Gamma; \Delta'', A \prec B) \vdash^- B}{(\Gamma, \Gamma'; \Delta'', A \prec B, \Delta') \vdash^- B}\, W^{a/c} \quad \dfrac{(\Gamma; \Delta'', A) \vdash^+ D \quad (\Gamma', D; \Delta') \vdash^* C}{(\Gamma, \Gamma'; \Delta'', A, \Delta') \vdash^* C}\, Cut^a}{(\Gamma, \Gamma'; \Delta'', A \prec B, \Delta') \vdash^* C}\, {\prec} L^c$$

$$\dfrac{\dfrac{(\Gamma; \Delta'', A \prec B) \vdash^- B}{(\Gamma, \Gamma'; \Delta'', A \prec B, \Delta') \vdash^- B}\, W^{a/c} \quad \dfrac{(\Gamma; \Delta'' A) \vdash^- D \quad (\Gamma'; \Delta', D) \vdash^* C}{(\Gamma, \Gamma'; \Delta'', A, \Delta') \vdash^* C}\, Cut^c}{(\Gamma, \Gamma'; \Delta'', A \prec B, \Delta') \vdash^* C}\, {\prec} L^c$$

As said above, cut-height is reduced in all cases.

-4- Cut formula D principal in the left premise only

The cases distinguished here concern the way the right premise is derived. We can distinguish 16 cases and show for each case that the derivation of the right premise can be transformed into one containing only occurrences of cut with a reduced cut-height.

-4.1- $\wedge L^a$ is the last rule used to derive the right premise with $\Gamma' = \Gamma'', A \wedge B$. The derivations with cuts of cut-height $n + m + 1$ are

$$\dfrac{(\Gamma; \Delta) \vdash^+ D \quad \dfrac{(\Gamma'', A, B, D; \Delta') \vdash^* C}{(\Gamma'', A \wedge B, D; \Delta') \vdash^* C}\, \wedge L^a}{(\Gamma, \Gamma'', A \wedge B; \Delta, \Delta') \vdash^* C}\, Cut^a$$

$$\dfrac{(\Gamma; \Delta) \vdash^- D \quad \dfrac{(\Gamma'', A, B; \Delta', D) \vdash^* C}{(\Gamma'', A \wedge B; \Delta', D) \vdash^* C}\, \wedge L^a}{(\Gamma, \Gamma'', A \wedge B; \Delta, \Delta') \vdash^* C}\, Cut^c$$

These can be transformed into derivations with cuts of cut-height $n + m$:

$$\dfrac{\dfrac{(\Gamma; \Delta) \vdash^+ D \quad (\Gamma'', A, B, D; \Delta') \vdash^* C}{(\Gamma, \Gamma'', A, B; \Delta, \Delta') \vdash^* C}\, Cut^a}{(\Gamma, \Gamma'', A \wedge B; \Delta, \Delta') \vdash^* C}\, \wedge L^a \qquad \dfrac{\dfrac{(\Gamma; \Delta) \vdash^- D \quad (\Gamma'', A, B; \Delta', D) \vdash^* C}{(\Gamma, \Gamma'', A, B; \Delta, \Delta') \vdash^* C}\, Cut^c}{(\Gamma, \Gamma'', A \wedge B; \Delta, \Delta') \vdash^* C}\, \wedge L^a$$

-4.2- $\wedge L^c$ is the last rule used to derive the right premise with $\Delta' = \Delta'', A \wedge B$. The derivations with cuts of cut-height $n + max(m, k) + 1$ are

$$\dfrac{(\Gamma; \Delta) \vdash^+ D \quad \dfrac{(\Gamma', D; \Delta'', A) \vdash^* C \quad (\Gamma', D; \Delta'', B) \vdash^* C}{(\Gamma', D; \Delta'', A \wedge B) \vdash^* C}\, \wedge L^c}{(\Gamma, \Gamma'; \Delta, \Delta'', A \wedge B) \vdash^* C}\, Cut^a$$

$$\dfrac{(\Gamma;\Delta)\vdash^- D \qquad \dfrac{(\Gamma';\Delta'',A,D)\vdash^* C \quad (\Gamma';\Delta'',B,D)\vdash^* C}{(\Gamma';\Delta'',A\wedge B,D)\vdash^* C}\,\wedge L^c}{(\Gamma,\Gamma';\Delta,\Delta'',A\wedge B)\vdash^* C}\,Cut^c$$

These can be transformed into derivations each with two cuts of cut-height $n+m$ and $n+k$, respectively:

$$\dfrac{\dfrac{(\Gamma;\Delta)\vdash^+ D \quad (\Gamma',D;\Delta'',A)\vdash^* C}{(\Gamma,\Gamma';\Delta,\Delta'',A)\vdash^* C}\,Cut^a \qquad \dfrac{(\Gamma;\Delta)\vdash^+ D \quad (\Gamma',D;\Delta'',B)\vdash^* C}{(\Gamma,\Gamma';\Delta,\Delta'',B)\vdash^* C}\,Cut^a}{(\Gamma,\Gamma';\Delta,\Delta'',A\wedge B)\vdash^* C}\,\wedge L^c$$

$$\dfrac{\dfrac{(\Gamma;\Delta)\vdash^- D \quad (\Gamma';\Delta'',A,D)\vdash^* C}{(\Gamma,\Gamma';\Delta,\Delta'',A)\vdash^* C}\,Cut^c \qquad \dfrac{(\Gamma;\Delta)\vdash^- D \quad (\Gamma';\Delta'',B,D)\vdash^* C}{(\Gamma,\Gamma';\Delta,\Delta'',B)\vdash^* C}\,Cut^c}{(\Gamma,\Gamma';\Delta,\Delta'',A\wedge B)\vdash^* C}\,\wedge L^c$$

-4.3- $\vee L^a$ is the last rule used to derive the right premise with $\Gamma' = \Gamma'', A\vee B$. The derivations with cuts of cut-height $n + max(m,k) + 1$ are

$$\dfrac{(\Gamma;\Delta)\vdash^+ D \qquad \dfrac{(\Gamma'',A,D;\Delta')\vdash^* C \quad (\Gamma'',B,D;\Delta')\vdash^* C}{(\Gamma'',A\vee B,D;\Delta')\vdash^* C}\,\vee L^a}{(\Gamma,\Gamma'',A\vee B;\Delta,\Delta')\vdash^* C}\,Cut^a$$

$$\dfrac{(\Gamma;\Delta)\vdash^- D \qquad \dfrac{(\Gamma'',A;\Delta',D)\vdash^* C \quad (\Gamma'',B;\Delta',D)\vdash^* C}{(\Gamma'',A\vee B;\Delta',D)\vdash^* C}\,\vee L^a}{(\Gamma,\Gamma'',A\vee B;\Delta,\Delta')\vdash^* C}\,Cut^c$$

These can be transformed into derivations each with two cuts of cut-height $n+m$ and $n+k$, respectively:

$$\dfrac{\dfrac{(\Gamma;\Delta)\vdash^+ D \quad (\Gamma'',A,D;\Delta')\vdash^* C}{(\Gamma,\Gamma'',A;\Delta,\Delta')\vdash^* C}\,Cut^a \qquad \dfrac{(\Gamma;\Delta)\vdash^+ D \quad (\Gamma'',B,D;\Delta')\vdash^* C}{(\Gamma,\Gamma'',B;\Delta,\Delta')\vdash^* C}\,Cut^a}{(\Gamma,\Gamma'',A\vee B;\Delta,\Delta')\vdash^* C}\,\vee L^a$$

$$\dfrac{\dfrac{(\Gamma;\Delta)\vdash^- D \quad (\Gamma'',A;\Delta',D)\vdash^* C}{(\Gamma,\Gamma'',A;\Delta,\Delta')\vdash^* C}\,Cut^c \qquad \dfrac{(\Gamma;\Delta)\vdash^- D \quad (\Gamma'',B;\Delta',D)\vdash^* C}{(\Gamma,\Gamma'',B;\Delta,\Delta')\vdash^* C}\,Cut^c}{(\Gamma,\Gamma'',A\vee B;\Delta,\Delta')\vdash^* C}\,\vee L^a$$

-4.4- $\vee L^c$ is the last rule used to derive the right premise with $\Delta' = \Delta'', A\vee B$. The derivations with cuts of cut-height $n+m+1$ are

$$\dfrac{(\Gamma;\Delta)\vdash^+ D \qquad \dfrac{(\Gamma',D;\Delta'',A,B)\vdash^* C}{(\Gamma',D;\Delta'',A\vee B)\vdash^* C}\,\vee L^c}{(\Gamma,\Gamma';\Delta,\Delta'',A\vee B)\vdash^* C}\,Cut^a$$

$$\dfrac{(\Gamma;\Delta)\vdash^- D \qquad \dfrac{(\Gamma';\Delta'',A,B,D)\vdash^* C}{(\Gamma';\Delta'',A\vee B,D)\vdash^* C}\,\vee L^c}{(\Gamma,\Gamma';\Delta,\Delta'',A\vee B)\vdash^* C}\,Cut^c$$

These can be transformed into derivations with cuts of cut-height $n+m$:

$$\dfrac{\dfrac{(\Gamma;\Delta)\vdash^+ D \qquad (\Gamma',D;\Delta'',A,B)\vdash^* C}{(\Gamma,\Gamma';\Delta,\Delta'',A,B)\vdash^* C}\,Cut^a}{(\Gamma,\Gamma';\Delta,\Delta''A\vee B)\vdash^* C}\,\vee L^c \qquad \dfrac{\dfrac{(\Gamma;\Delta)\vdash^- D \qquad (\Gamma';\Delta'',A,B,D)\vdash^* C}{(\Gamma,\Gamma';\Delta,\Delta'',A,B)\vdash^* C}\,Cut^c}{(\Gamma,\Gamma';\Delta,\Delta'',A\vee B)\vdash^* C}\,\vee L^c$$

-4.5- $\to L^a$ is the last rule used to derive the right premise with $\Gamma' = \Gamma'', A\to B$. The derivations with cuts of cut-height $n+max(m,k)+1$ are

$$\dfrac{(\Gamma;\Delta)\vdash^+ D \qquad \dfrac{(\Gamma'',A\to B,D;\Delta')\vdash^+ A \quad (\Gamma'',B,D;\Delta')\vdash^* C}{(\Gamma'',A\to B,D;\Delta')\vdash^* C}\,\to L^a}{(\Gamma,\Gamma'',A\to B;\Delta,\Delta')\vdash^* C}\,Cut^a$$

$$\dfrac{(\Gamma;\Delta)\vdash^- D \qquad \dfrac{(\Gamma'',A\to B;\Delta',D)\vdash^+ A \quad (\Gamma'',B;\Delta',D)\vdash^* C}{(\Gamma'',A\to B;\Delta',D)\vdash^* C}\,\to L^a}{(\Gamma,\Gamma'',A\to B;\Delta,\Delta')\vdash^* C}\,Cut^c$$

These can be transformed into derivations each with two cuts of cut-height $n+m$ and $n+k$, respectively:

$$\dfrac{\dfrac{(\Gamma;\Delta)\vdash^+ D \quad (\Gamma'',A\to B,D;\Delta')\vdash^+ A}{(\Gamma,\Gamma'',A\to B;\Delta,\Delta')\vdash^+ A}\,Cut^a \qquad \dfrac{(\Gamma;\Delta)\vdash^+ D \quad (\Gamma'',B,D;\Delta')\vdash^* C}{(\Gamma,\Gamma'',B;\Delta,\Delta')\vdash^* C}\,Cut^a}{(\Gamma,\Gamma'',A\to B;\Delta,\Delta')\vdash^* C}\,\to L^a$$

$$\dfrac{\dfrac{(\Gamma;\Delta)\vdash^- D \quad (\Gamma'',A\to B;\Delta',D)\vdash^+ A}{(\Gamma,\Gamma'',A\to B;\Delta,\Delta')\vdash^+ A}\,Cut^c \qquad \dfrac{(\Gamma;\Delta)\vdash^- D \quad (\Gamma'',B;\Delta',D)\vdash^* C}{(\Gamma,\Gamma'',B;\Delta,\Delta')\vdash^* C}\,Cut^c}{(\Gamma,\Gamma'',A\to B;\Delta,\Delta')\vdash^* C}\,\to L^a$$

-4.6- $\to L^c$ is the last rule used to derive the right premise with $\Delta' = \Delta'', A\to B$. The derivations with cuts of cut-height $n+m+1$ are

$$\dfrac{(\Gamma;\Delta)\vdash^+ D \qquad \dfrac{(\Gamma',A,D;\Delta'',B)\vdash^* C}{(\Gamma',D;\Delta'',A\to B)\vdash^* C}\,\to L^c}{(\Gamma,\Gamma';\Delta,\Delta'',A\to B)\vdash^* C}\,Cut^a$$

$$\dfrac{(\Gamma;\Delta)\vdash^- D \qquad \dfrac{(\Gamma',A;\Delta'',B,D)\vdash^* C}{(\Gamma';\Delta'',A\to B,D)\vdash^* C}\,\to L^c}{(\Gamma,\Gamma';\Delta,\Delta'',A\to B)\vdash^* C}\,Cut^c$$

These can be transformed into derivations with cuts of cut-height $n+m$:

$$\dfrac{\dfrac{(\Gamma;\Delta)\vdash^{+} D \qquad (\Gamma',A,D;\Delta'',B)\vdash^{*} C}{(\Gamma,\Gamma',A;\Delta,\Delta'',B)\vdash^{*} C}\,Cut^{a}}{(\Gamma,\Gamma';\Delta,\Delta'',A\rightarrow B)\vdash^{*} C}\,{\rightarrow}L^{c} \qquad \dfrac{\dfrac{(\Gamma;\Delta)\vdash^{-} D \qquad (\Gamma',A;\Delta'',B,D)\vdash^{*} C}{(\Gamma,\Gamma',A;\Delta,\Delta'',B)\vdash^{*} C}\,Cut^{c}}{(\Gamma,\Gamma';\Delta,\Delta'',A\rightarrow B)\vdash^{*} C}\,{\rightarrow}L^{c}$$

-4.7- ${\prec}L^a$ is the last rule used to derive the right premise with $\Gamma'=\Gamma'',A\prec B$. The derivations with cuts of cut-height $n+m+1$ are

$$\dfrac{(\Gamma;\Delta)\vdash^{+} D \qquad \dfrac{(\Gamma'',A,D;\Delta',B)\vdash^{*} C}{(\Gamma'',A\prec B,D;\Delta')\vdash^{*} C}\,{\prec}L^{a}}{(\Gamma,\Gamma'',A\prec B;\Delta,\Delta')\vdash^{*} C}\,Cut^{a}$$

$$\dfrac{(\Gamma;\Delta)\vdash^{-} D \qquad \dfrac{(\Gamma'',A;\Delta',B,D)\vdash^{*} C}{(\Gamma'',A\prec B;\Delta',D)\vdash^{*} C}\,{\prec}L^{a}}{(\Gamma,\Gamma'',A\prec B;\Delta,\Delta')\vdash^{*} C}\,Cut^{c}$$

These can be transformed into derivations with cuts of cut-height $n+m$:

$$\dfrac{\dfrac{(\Gamma;\Delta)\vdash^{+} D \qquad (\Gamma'',A,D;\Delta',B)\vdash^{*} C}{(\Gamma,\Gamma'',A;\Delta,\Delta',B)\vdash^{*} C}\,Cut^{a}}{(\Gamma,\Gamma'',A\prec B;\Delta,\Delta')\vdash^{*} C}\,{\prec}L^{a} \qquad \dfrac{\dfrac{(\Gamma;\Delta)\vdash^{-} D \qquad (\Gamma'',A;\Delta',B,D)\vdash^{*} C}{(\Gamma,\Gamma'',A;\Delta,\Delta',B)\vdash^{*} C}\,Cut^{c}}{(\Gamma,\Gamma'',A\prec B;\Delta,\Delta')\vdash^{*} C}\,{\prec}L^{a}$$

-4.8- ${\prec}L^c$ is the last rule used to derive the right premise with $\Delta'=\Delta'',A\prec B$. The derivations with cuts of cut-height $n+max(m,k)+1$ are

$$\dfrac{(\Gamma;\Delta)\vdash^{+} D \qquad \dfrac{(\Gamma',D;\Delta'',A\prec B)\vdash^{-} B \quad (\Gamma',D;\Delta'',A)\vdash^{*} C}{(\Gamma',D;\Delta'',A\prec B)\vdash^{*} C}\,{\prec}L^{c}}{(\Gamma,\Gamma';\Delta,\Delta'',A\prec B)\vdash^{*} C}\,Cut^{a}$$

$$\dfrac{(\Gamma;\Delta)\vdash^{-} D \qquad \dfrac{(\Gamma';\Delta'',A\prec B,D)\vdash^{-} B \quad (\Gamma';\Delta'',A,D)\vdash^{*} C}{(\Gamma';\Delta'',A\prec B,D)\vdash^{*} C}\,{\prec}L^{c}}{(\Gamma,\Gamma';\Delta,\Delta'',A\prec B)\vdash^{*} C}\,Cut^{c}$$

These can be transformed into derivations each with two cuts of cut-height $n+m$ and $n+k$, respectively:

$$\dfrac{\dfrac{(\Gamma;\Delta)\vdash^{+} D \quad (\Gamma',D;\Delta'',A\prec B)\vdash^{-} B}{(\Gamma,\Gamma';\Delta,\Delta'',A\prec B)\vdash^{-} B}\,Cut^{a} \qquad \dfrac{(\Gamma;\Delta)\vdash^{+} D \quad (\Gamma',D;\Delta'',A)\vdash^{*} C}{(\Gamma,\Gamma';\Delta,\Delta'',A)\vdash^{*} C}\,Cut^{a}}{(\Gamma,\Gamma';\Delta,\Delta'',A\prec B)\vdash^{*} C}\,{\prec}L^{c}$$

$$\dfrac{\dfrac{(\Gamma;\Delta)\vdash^{-} D \quad (\Gamma';\Delta'',A\prec B,D)\vdash^{-} B}{(\Gamma,\Gamma';\Delta,\Delta'',A\prec B)\vdash^{-} B}\,{\scriptstyle Cut^c} \quad \dfrac{(\Gamma;\Delta)\vdash^{-} D \quad (\Gamma';\Delta'',A,D)\vdash^{*} C}{(\Gamma,\Gamma';\Delta,\Delta'',A)\vdash^{*} C}\,{\scriptstyle Cut^c}}{(\Gamma,\Gamma';\Delta,\Delta'',A\prec B)\vdash^{*} C}\,{\scriptstyle \prec L^c}$$

-4.9- $\wedge R^{+}$ is the last rule used to derive the right premise with $C = A\wedge B$. The derivations with cuts of cut-height $n + max(m, k) + 1$ are

$$\dfrac{(\Gamma;\Delta)\vdash^{+} D \quad \dfrac{(\Gamma',D;\Delta')\vdash^{+} A \quad (\Gamma',D;\Delta')\vdash^{+} B}{(\Gamma',D;\Delta')\vdash^{+} A\wedge B}\,{\scriptstyle \wedge R^{+}}}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^{+} A\wedge B}\,{\scriptstyle Cut^a}$$

$$\dfrac{(\Gamma;\Delta)\vdash^{-} D \quad \dfrac{(\Gamma';\Delta',D)\vdash^{+} A \quad (\Gamma';\Delta',D)\vdash^{+} B}{(\Gamma';\Delta',D)\vdash^{+} A\wedge B}\,{\scriptstyle \wedge R^{+}}}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^{+} A\wedge B}\,{\scriptstyle Cut^c}$$

These can be transformed into derivations each with two cuts of cut-height $n + m$ and $n + k$, respectively:

$$\dfrac{\dfrac{(\Gamma;\Delta)\vdash^{+} D \quad (\Gamma',D;\Delta')\vdash^{+} A}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^{+} A}\,{\scriptstyle Cut^a} \quad \dfrac{(\Gamma;\Delta)\vdash^{+} D \quad (\Gamma',D;\Delta')\vdash^{+} B}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^{+} B}\,{\scriptstyle Cut^a}}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^{+} A\wedge B}\,{\scriptstyle \wedge R^{+}}$$

$$\dfrac{\dfrac{(\Gamma;\Delta)\vdash^{-} D \quad (\Gamma';\Delta',D)\vdash^{+} A}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^{+} A}\,{\scriptstyle Cut^c} \quad \dfrac{(\Gamma;\Delta)\vdash^{-} D \quad (\Gamma';\Delta',D)\vdash^{+} B}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^{+} B}\,{\scriptstyle Cut^c}}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^{+} A\wedge B}\,{\scriptstyle \wedge R^{+}}$$

-4.10.1- $\wedge R_1^{-}$ is the last rule used to derive the right premise with $C = A\wedge B$. The derivations with cuts of cut-height $n + m + 1$ are

$$\dfrac{(\Gamma;\Delta)\vdash^{+} D \quad \dfrac{(\Gamma',D;\Delta')\vdash^{-} A}{(\Gamma',D;\Delta')\vdash^{-} A\wedge B}\,{\scriptstyle \wedge R_1^{-}}}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^{-} A\wedge B}\,{\scriptstyle Cut^a} \qquad \dfrac{(\Gamma;\Delta)\vdash^{-} D \quad \dfrac{(\Gamma';\Delta',D)\vdash^{-} A}{(\Gamma';\Delta',D)\vdash^{-} A\wedge B}\,{\scriptstyle \wedge R_1^{-}}}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^{-} A\wedge B}\,{\scriptstyle Cut^c}$$

These can be transformed into derivations with cuts of cut-height $n + m$:

$$\dfrac{\dfrac{(\Gamma;\Delta)\vdash^{+} D \quad (\Gamma',D;\Delta')\vdash^{-} A}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^{-} A}\,{\scriptstyle Cut^a}}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^{-} A\wedge B}\,{\scriptstyle \wedge R_1^{-}} \qquad \dfrac{\dfrac{(\Gamma;\Delta)\vdash^{-} D \quad (\Gamma';\Delta',D)\vdash^{-} A}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^{-} A}\,{\scriptstyle Cut^c}}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^{-} A\wedge B}\,{\scriptstyle \wedge R_1^{-}}$$

-4.10.2- $\wedge R_2^-$ is the last rule used to derive the right premise with $C = A \wedge B$. The derivations with cuts of cut-height $n + m + 1$ are

$$\dfrac{(\Gamma;\Delta) \vdash^+ D \qquad \dfrac{(\Gamma', D; \Delta') \vdash^- B}{(\Gamma', D; \Delta') \vdash^- A \wedge B}\ {\scriptstyle \wedge R_2^-}}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^- A \wedge B}\ {\scriptstyle Cut^a} \quad \dfrac{(\Gamma;\Delta) \vdash^- D \qquad \dfrac{(\Gamma'; \Delta', D) \vdash^- B}{(\Gamma'; \Delta', D) \vdash^- A \wedge B}\ {\scriptstyle \wedge R_2^-}}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^- A \wedge B}\ {\scriptstyle Cut^c}$$

These can be transformed into derivations with cuts of cut-height $n + m$:

$$\dfrac{\dfrac{(\Gamma;\Delta) \vdash^+ D \qquad (\Gamma', D; \Delta') \vdash^- B}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^- B}\ {\scriptstyle Cut^a}}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^- A \wedge B}\ {\scriptstyle \wedge R_2^-} \qquad \dfrac{\dfrac{(\Gamma;\Delta) \vdash^- D \qquad (\Gamma'; \Delta', D) \vdash^- B}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^- B}\ {\scriptstyle Cut^c}}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^- A \wedge B}\ {\scriptstyle \wedge R_2^-}$$

-4.11.1- $\vee R_1^+$ is the last rule used to derive the right premise with $C = A \vee B$. The derivations with cuts of cut-height $n + m + 1$ are

$$\dfrac{(\Gamma;\Delta) \vdash^+ D \qquad \dfrac{(\Gamma', D; \Delta') \vdash^+ A}{(\Gamma', D; \Delta') \vdash^+ A \vee B}\ {\scriptstyle \vee R_1^+}}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^+ A \vee B}\ {\scriptstyle Cut^a} \quad \dfrac{(\Gamma;\Delta) \vdash^- D \qquad \dfrac{(\Gamma'; \Delta', D) \vdash^+ A}{(\Gamma'; \Delta', D) \vdash^+ A \vee B}\ {\scriptstyle \vee R_1^+}}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^+ A \vee B}\ {\scriptstyle Cut^c}$$

These can be transformed into derivations with cuts of cut-height $n + m$:

$$\dfrac{\dfrac{(\Gamma;\Delta) \vdash^+ D \qquad (\Gamma', D; \Delta') \vdash^+ A}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^+ A}\ {\scriptstyle Cut^a}}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^+ A \vee B}\ {\scriptstyle \vee R_1^+} \qquad \dfrac{\dfrac{(\Gamma;\Delta) \vdash^- D \qquad (\Gamma'; \Delta', D) \vdash^+ A}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^+ A}\ {\scriptstyle Cut^c}}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^+ A \vee B}\ {\scriptstyle \vee R_1^+}$$

-4.11.2- $\vee R_2^+$ is the last rule used to derive the right premise with $C = A \vee B$. The derivations with cuts of cut-height $n + m + 1$ are

$$\dfrac{(\Gamma;\Delta) \vdash^+ D \qquad \dfrac{(\Gamma', D; \Delta') \vdash^+ B}{(\Gamma', D; \Delta') \vdash^+ A \vee B}\ {\scriptstyle \vee R_2^+}}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^+ A \vee B}\ {\scriptstyle Cut^a} \quad \dfrac{(\Gamma;\Delta) \vdash^- D \qquad \dfrac{(\Gamma'; \Delta', D) \vdash^+ B}{(\Gamma'; \Delta', D) \vdash^+ A \vee B}\ {\scriptstyle \vee R_2^+}}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^+ A \vee B}\ {\scriptstyle Cut^c}$$

These can be transformed into derivations with cuts of cut-height $n + m$:

$$\dfrac{\dfrac{(\Gamma;\Delta) \vdash^+ D \qquad (\Gamma', D; \Delta') \vdash^+ B}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^+ B}\ {\scriptstyle Cut^a}}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^+ A \vee B}\ {\scriptstyle \vee R_2^+} \qquad \dfrac{\dfrac{(\Gamma;\Delta) \vdash^- D \qquad (\Gamma'; \Delta', D) \vdash^+ B}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^+ B}\ {\scriptstyle Cut^c}}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^+ A \vee B}\ {\scriptstyle \vee R_2^+}$$

-4.12- $\vee R^-$ is the last rule used to derive the right premise with $C = A \vee B$. The derivations with cuts of cut-height $n + max(m, k) + 1$ are

$$\dfrac{(\Gamma; \Delta) \vdash^+ D \qquad \dfrac{(\Gamma', D; \Delta') \vdash^- A \quad (\Gamma', D; \Delta') \vdash^- B}{(\Gamma', D; \Delta') \vdash^- A \vee B}\, \vee R^-}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^- A \vee B}\, Cut^a$$

$$\dfrac{(\Gamma; \Delta) \vdash^- D \qquad \dfrac{(\Gamma'; \Delta', D) \vdash^- A \quad (\Gamma'; \Delta', D) \vdash^- B}{(\Gamma'; \Delta', D) \vdash^- A \vee B}\, \vee R^-}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^- A \vee B}\, Cut^c$$

These can be transformed into derivations each with two cuts of cut-height $n + m$ and $n + k$, respectively:

$$\dfrac{\dfrac{(\Gamma; \Delta) \vdash^+ D \quad (\Gamma', D; \Delta') \vdash^- A}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^- A}\, Cut^a \qquad \dfrac{(\Gamma; \Delta) \vdash^+ D \quad (\Gamma', D; \Delta') \vdash^- B}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^- B}\, Cut^a}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^- A \vee B}\, \vee R^-$$

$$\dfrac{\dfrac{(\Gamma; \Delta) \vdash^- D \quad (\Gamma'; \Delta', D) \vdash^- A}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^- A}\, Cut^c \qquad \dfrac{(\Gamma; \Delta) \vdash^- D \quad (\Gamma'; \Delta', D) \vdash^- B}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^- B}\, Cut^c}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^- A \vee B}\, \vee R^-$$

-4.13- $\to R^+$ is the last rule used to derive the right premise with $C = A \to B$. The derivations with cuts of cut-height $n + m + 1$ are

$$\dfrac{(\Gamma; \Delta) \vdash^+ D \qquad \dfrac{(\Gamma', A, D; \Delta') \vdash^+ B}{(\Gamma', D; \Delta') \vdash^+ A \to B}\, {\to}R^+}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^+ A \to B}\, Cut^a \qquad \dfrac{(\Gamma; \Delta) \vdash^- D \qquad \dfrac{(\Gamma', A; \Delta', D) \vdash^+ B}{(\Gamma'; \Delta', D) \vdash^+ A \to B}\, {\to}R^+}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^+ A \to B}\, Cut^c$$

These can be transformed into derivations with cuts of cut-height $n + m$:

$$\dfrac{\dfrac{(\Gamma; \Delta) \vdash^+ D \qquad (\Gamma', A, D; \Delta') \vdash^+ B}{(\Gamma, \Gamma', A; \Delta, \Delta') \vdash^+ B}\, Cut^a}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^+ A \to B}\, {\to}R^+ \qquad \dfrac{\dfrac{(\Gamma; \Delta) \vdash^- D \qquad (\Gamma', A; \Delta', D) \vdash^+ B}{(\Gamma, \Gamma', A; \Delta, \Delta') \vdash^+ B}\, Cut^c}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^+ A \to B}\, {\to}R^+$$

-4.14- $\to R^-$ is the last rule used to derive the right premise with $C = A \to B$. The derivations with cuts of cut-height $n + max(m, k) + 1$ are

$$\dfrac{(\Gamma; \Delta) \vdash^+ D \qquad \dfrac{(\Gamma', D; \Delta') \vdash^+ A \quad (\Gamma', D; \Delta') \vdash^- B}{(\Gamma', D; \Delta') \vdash^- A \to B}\, {\to}R^-}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^- A \to B}\, Cut^a$$

$$\dfrac{(\Gamma;\Delta)\vdash^- D \qquad \dfrac{(\Gamma';\Delta',D)\vdash^+ A \quad (\Gamma';\Delta',D)\vdash^- B}{(\Gamma';\Delta',D)\vdash^- A\to B}\,{\to}R^-}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^- A\to B}\,Cut^c$$

These can be transformed into derivations each with two cuts of cut-height $n+m$ and $n+k$, respectively:

$$\dfrac{\dfrac{(\Gamma;\Delta)\vdash^+ D \quad (\Gamma',D;\Delta')\vdash^+ A}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^+ A}\,Cut^a \qquad \dfrac{(\Gamma;\Delta)\vdash^+ D \quad (\Gamma',D;\Delta')\vdash^- B}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^- B}\,Cut^a}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^- A\to B}\,{\to}R^-$$

$$\dfrac{\dfrac{(\Gamma;\Delta)\vdash^- D \quad (\Gamma';\Delta',D)\vdash^+ A}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^+ A}\,Cut^c \qquad \dfrac{(\Gamma;\Delta)\vdash^- D \quad (\Gamma';\Delta',D)\vdash^- B}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^- B}\,Cut^c}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^- A\to B}\,{\to}R^-$$

-4.15- ${\prec}R^+$ is the last rule used to derive the right premise with $C = A\prec B$. The derivations with cuts of cut-height $n+max(m,k)+1$ are

$$\dfrac{(\Gamma;\Delta)\vdash^+ D \qquad \dfrac{(\Gamma',D;\Delta')\vdash^+ A \quad (\Gamma',D;\Delta')\vdash^- B}{(\Gamma',D;\Delta')\vdash^+ A\prec B}\,{\prec}R^+}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^+ A\prec B}\,Cut^a$$

$$\dfrac{(\Gamma;\Delta)\vdash^- D \qquad \dfrac{(\Gamma';\Delta',D)\vdash^+ A \quad (\Gamma';\Delta',D)\vdash^- B}{(\Gamma';\Delta',D)\vdash^+ A\prec B}\,{\prec}R^+}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^+ A\prec B}\,Cut^c$$

These can be transformed into derivations each with two cuts of cut-height $n+m$ and $n+k$, respectively:

$$\dfrac{\dfrac{(\Gamma;\Delta)\vdash^+ D \quad (\Gamma',D;\Delta')\vdash^+ A}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^+ A}\,Cut^a \qquad \dfrac{(\Gamma;\Delta)\vdash^+ D \quad (\Gamma',D;\Delta')\vdash^- B}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^- B}\,Cut^a}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^+ A\prec B}\,{\prec}R^+$$

$$\dfrac{\dfrac{(\Gamma;\Delta)\vdash^- D \quad (\Gamma';\Delta',D)\vdash^+ A}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^+ A}\,Cut^c \qquad \dfrac{(\Gamma;\Delta)\vdash^- D \quad (\Gamma';\Delta',D)\vdash^- B}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^- B}\,Cut^c}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^+ A\prec B}\,{\prec}R^+$$

-4.16- $\prec R^-$ is the last rule used to derive the right premise with $C = A \prec B$. The derivations with cuts of cut-height $n+m+1$ are

$$\dfrac{(\Gamma;\Delta) \vdash^+ D \qquad \dfrac{(\Gamma', D; \Delta', B) \vdash^- A}{(\Gamma', D; \Delta') \vdash^- A \prec B}\prec R^-}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^- A \prec B}Cut^a \quad \dfrac{(\Gamma;\Delta) \vdash^- D \qquad \dfrac{(\Gamma'; \Delta', B, D) \vdash^- A}{(\Gamma'; \Delta', D) \vdash^- A \prec B}\prec R^-}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^- A \prec B}Cut^c$$

These can be transformed into derivations with cuts of cut-height $n+m$:

$$\dfrac{\dfrac{(\Gamma;\Delta) \vdash^+ D \qquad (\Gamma', D; \Delta', B) \vdash^- A}{(\Gamma, \Gamma'; \Delta, \Delta', B) \vdash^- A}Cut^a}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^- A \prec B}\prec R^- \quad \dfrac{\dfrac{(\Gamma;\Delta) \vdash^- D \qquad (\Gamma'; \Delta', B, D) \vdash^- A}{(\Gamma, \Gamma'; \Delta, \Delta', B) \vdash^- A}Cut^c}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^- A \prec B}\prec R^-$$

It is shown that cut-height is reduced in all cases.

-5- Cut formula D principal in both premises

For each cut rule four cases can be distinguished. Here, it can be shown for each case that the derivations can be transformed into ones in which the occurrences of cut have a reduced cut-height or the cut formula has a lower weight (or both).

-5.1- $D = A \wedge B$. The derivation for Cut^a with a cut of cut-height $max(n, m) + 1 + k + 1$ is

$$\dfrac{\dfrac{(\Gamma;\Delta) \vdash^+ A \quad (\Gamma;\Delta) \vdash^+ B}{(\Gamma;\Delta) \vdash^+ A \wedge B}\wedge R^+ \qquad \dfrac{(\Gamma', A, B; \Delta') \vdash^* C}{(\Gamma', A \wedge B; \Delta') \vdash^* C}\wedge L^a}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^* C}Cut^a$$

and can be transformed into a derivation with two cuts of cut-height (from top to bottom) $n + k$ and $m + max(n, k) + 1$:

$$\dfrac{\dfrac{(\Gamma;\Delta) \vdash^+ B \qquad \dfrac{(\Gamma;\Delta) \vdash^+ A \qquad (\Gamma', A, B; \Delta') \vdash^* C}{(\Gamma, \Gamma', B; \Delta, \Delta') \vdash^* C}Cut^a}{(\Gamma, \Gamma, \Gamma'; \Delta, \Delta, \Delta') \vdash^* C}Cut^a}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^* C}C_{a/c}$$

Note that in both cases the weight of the cut formula is reduced. The upper cut is also reduced in height, while with the lower cut we have a case where cut-height is not necessarily reduced.

The possible derivations for Cut^c with a cut of cut-height $n+1+max(m,k)+1$ are

$$\dfrac{\dfrac{(\Gamma;\Delta)\vdash^- A}{(\Gamma;\Delta)\vdash^- A\wedge B}\wedge R_1^- \qquad \dfrac{(\Gamma';\Delta',A)\vdash^* C \quad (\Gamma';\Delta',B)\vdash^* C}{(\Gamma';\Delta',A\wedge B)\vdash^* C}\wedge L^c}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^* C}\, Cut^c$$

or

$$\dfrac{\dfrac{(\Gamma;\Delta)\vdash^- B}{(\Gamma;\Delta)\vdash^- A\wedge B}\wedge R_2^- \qquad \dfrac{(\Gamma';\Delta',A)\vdash^* C \quad (\Gamma';\Delta',B)\vdash^* C}{(\Gamma';\Delta',A\wedge B)\vdash^* C}\wedge L^c}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^* C}\, Cut^c$$

and those can be transformed into derivations with cuts of cut-height $n+m$ or $n+k$, respectively:

$$\dfrac{(\Gamma;\Delta)\vdash^- A \qquad (\Gamma';\Delta',A)\vdash^* C}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^* C}\, Cut^c \quad \dfrac{(\Gamma;\Delta)\vdash^- B \qquad (\Gamma';\Delta',B)\vdash^* C}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^* C}\, Cut^c$$

Here, both cut-height and weight of the cut formulas are reduced.

-5.2- $D = A\vee B$. The possible derivations for Cut^a with a cut of cut-height $n+1+max(m,k)+1$ are

$$\dfrac{\dfrac{(\Gamma;\Delta)\vdash^+ A}{(\Gamma;\Delta)\vdash^+ A\vee B}\vee R_1^+ \qquad \dfrac{(\Gamma',A;\Delta')\vdash^* C \quad (\Gamma',B;\Delta')\vdash^* C}{(\Gamma',A\vee B;\Delta')\vdash^* C}\vee L^a}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^* C}\, Cut^a$$

or

$$\dfrac{\dfrac{(\Gamma;\Delta)\vdash^+ B}{(\Gamma;\Delta)\vdash^+ A\vee B}\vee R_2^+ \qquad \dfrac{(\Gamma',A;\Delta')\vdash^* C \quad (\Gamma',B;\Delta')\vdash^* C}{(\Gamma',A\vee B;\Delta')\vdash^* C}\vee L^a}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^* C}\, Cut^a$$

and those can be transformed into derivations with cuts of cut-height $n+m$ and $n+k$, respectively:

$$\dfrac{(\Gamma;\Delta)\vdash^+ A \qquad (\Gamma',A;\Delta')\vdash^* C}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^* C}\, Cut^a \quad \dfrac{(\Gamma;\Delta)\vdash^+ B \qquad (\Gamma',B;\Delta')\vdash^* C}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^* C}\, Cut^a$$

Again, both cut-height and weight of the cut formulas are reduced.

The derivation for Cut^c with a cut of cut-height $max(n,m)+1+k+1$ is

$$\dfrac{\dfrac{(\Gamma;\Delta)\vdash^- A \quad (\Gamma;\Delta)\vdash^- B}{(\Gamma;\Delta)\vdash^- A\vee B}\vee R^- \qquad \dfrac{(\Gamma';\Delta',A,B)\vdash^* C}{(\Gamma';\Delta',A\vee B)\vdash^* C}\vee L^c}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^* C}\, Cut^c$$

and can be transformed into a derivation with two cuts of cut-height $n+k$ and $m+max(n,k)+1$:

$$\dfrac{\dfrac{(\Gamma;\Delta)\vdash^{-} B \qquad \dfrac{(\Gamma;\Delta)\vdash^{-} A \qquad (\Gamma';\Delta',A,B)\vdash^{*} C}{(\Gamma,\Gamma';\Delta,\Delta',B)\vdash^{*} C}\,Cut^{c}}{(\Gamma,\Gamma,\Gamma';\Delta,\Delta,\Delta')\vdash^{*} C}\,Cut^{c}}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^{*} C}\,C^{a/c}$$

Note that again, in the case of the lower cut, although the cut-height might increase, the weight of the cut formula is reduced. For the upper cut both cut-height and weight of the cut formula is reduced.

-5.3- $D = A \to B$. The derivation for Cut^a with a cut of cut-height $n+1+max(m,k)+1$ is

$$\dfrac{\dfrac{(\Gamma,A;\Delta)\vdash^{+} B}{(\Gamma;\Delta)\vdash^{+} A\to B}\,{\to}R^{+} \qquad \dfrac{(\Gamma',A\to B;\Delta')\vdash^{+} A \quad (\Gamma',B;\Delta')\vdash^{*} C}{(\Gamma',A\to B;\Delta')\vdash^{*} C}\,{\to}L^{a}}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^{*} C}\,Cut^{a}$$

and this can be transformed into a derivation with three cuts of cut-height (from left to right and from top to bottom) $n+1+m$, $n+k$, and $max(n+1,m)+1+max(n,k)+1$ respectively:

$$\dfrac{\dfrac{\dfrac{\dfrac{(\Gamma,A;\Delta)\vdash^{+} B}{(\Gamma;\Delta)\vdash^{+} A\to B}\,{\to}R^{+} \quad (\Gamma',A\to B;\Delta')\vdash^{+} A}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^{+} A}\,Cut^{a} \qquad \dfrac{(\Gamma,A;\Delta)\vdash^{+} B \quad (\Gamma',B;\Delta')\vdash^{*} C}{(\Gamma,A,\Gamma';\Delta,\Delta')\vdash^{*} C}\,Cut^{a}}{(\Gamma,\Gamma,\Gamma',\Gamma';\Delta,\Delta,\Delta',\Delta')\vdash^{*} C}\,Cut^{a}}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^{*} C}\,C^{a/c}$$

In the first case cut-height is reduced, in the second case cut-height and weight of the cut formula is reduced and in the third case weight of the cut formula is reduced.

The derivation for Cut^c with a cut of cut-height $max(n,m)+1+k+1$ is

$$\dfrac{\dfrac{(\Gamma;\Delta)\vdash^{+} A \quad (\Gamma;\Delta)\vdash^{-} B}{(\Gamma;\Delta)\vdash^{-} A\to B}\,{\to}R^{-} \qquad \dfrac{(\Gamma',A;\Delta',B)\vdash^{*} C}{(\Gamma';\Delta',A\to B)\vdash^{*} C}\,{\to}L^{c}}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^{*} C}\,Cut^{c}$$

This can be transformed into a derivation with two cuts of cut-height $n+k$ and $m+max(n,k)+1$:

$$\dfrac{\dfrac{(\Gamma;\Delta)\vdash^{-} B \qquad \dfrac{(\Gamma;\Delta)\vdash^{+} A \qquad (\Gamma', A;\Delta', B)\vdash^{*} C}{(\Gamma,\Gamma';\Delta,\Delta', B)\vdash^{*} C}\, Cut^{a}}{(\Gamma,\Gamma,\Gamma';\Delta,\Delta,\Delta')\vdash^{*} C}\, Cut^{c}}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^{*} C}\, C^{a/c}$$

In the first case cut-height and weight of the cut formula is reduced, while in the second case the weight of the cut formula is reduced. Here we can observe a result specific for this calculus due to the mixture of derivability relations $\vdash^{+}$ and $\vdash^{-}$ in $\rightarrow R^{-}$ and the position of the active formulas in the assumptions *and* in the counterassumptions in $\rightarrow L^{c}$: Derivations containing instances of Cut^{c} are not necessarily transformed into derivations with a lesser cut-height or a reduced weight of the cut formula of another instance of Cut^{c} but it can also happen that Cut^{c} is replaced by Cut^{a}.

-5.4- $D = A \prec B$. The derivation for Cut^{a} with a cut of cut-height $max(n,m)+1+k+1$ is

$$\dfrac{\dfrac{(\Gamma;\Delta)\vdash^{+} A \quad (\Gamma;\Delta)\vdash^{-} B}{(\Gamma;\Delta)\vdash^{+} A \prec B}\,{\prec}R^{+} \qquad \dfrac{(\Gamma', A;\Delta', B)\vdash^{*} C}{(\Gamma', A\prec B;\Delta')\vdash^{*} C}\,{\prec}L^{a}}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^{*} C}\, Cut^{a}$$

This can be transformed into a derivation with two cuts of cut-height $n+k$ and $m+max(n,k)+1$:

$$\dfrac{\dfrac{(\Gamma;\Delta)\vdash^{-} B \qquad \dfrac{(\Gamma;\Delta)\vdash^{+} A \qquad (\Gamma', A;\Delta', B)\vdash^{*} C}{(\Gamma,\Gamma';\Delta,\Delta', B)\vdash^{*} C}\, Cut^{a}}{(\Gamma,\Gamma,\Gamma';\Delta,\Delta,\Delta')\vdash^{*} C}\, Cut^{c}}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^{*} C}\, C^{a/c}$$

Again, due to the mixture of derivability relations $\vdash^{+}$ and $\vdash^{-}$ in ${\prec}R^{+}$ and the presence of the active formulas both in assumptions and counterassumptions in ${\prec}L^{a}$, in this case Cut^{a} can be replaced by instances of Cut^{c} with a reduced weight of the cut formula. In the upper cut we have a reduction of both cut-height and weight of the cut formula.

The derivation for Cut^{c} with a cut of cut-height $n+1+max(m,k)+1$ is

$$\dfrac{\dfrac{(\Gamma;\Delta, B)\vdash^{-} A}{(\Gamma;\Delta)\vdash^{-} A\prec B}\,{\prec}R^{-} \qquad \dfrac{(\Gamma';\Delta', A\prec B)\vdash^{-} B \quad (\Gamma';\Delta', A)\vdash^{*} C}{(\Gamma';\Delta', A\prec B)\vdash^{*} C}\,{\prec}L^{c}}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^{*} C}\, Cut^{c}$$

and this can be transformed into a derivation with three cuts of cut-height (from left to right and from top to bottom) $n+1+m$, $n+k$, and $max(n+1,m)+1+max(n,k)+1$ respectively:

$$
\dfrac{\dfrac{\dfrac{\dfrac{(\Gamma;\Delta,B)\vdash^{-} A}{(\Gamma;\Delta)\vdash^{-} A\prec B}{\scriptstyle \prec R^{-}} \quad (\Gamma';\Delta',A\prec B)\vdash^{-} B}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^{-} B}{\scriptstyle Cut^{c}} \quad \dfrac{(\Gamma;\Delta,B)\vdash^{-} A \quad (\Gamma';\Delta',A)\vdash^{*} C}{(\Gamma,\Gamma';\Delta,B,\Delta')\vdash^{*} C}{\scriptstyle Cut^{c}}}{(\Gamma,\Gamma,\Gamma',\Gamma';\Delta,\Delta,\Delta',\Delta')\vdash^{*} C}{\scriptstyle Cut^{c}}}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^{*} C}{\scriptstyle C^{a/c}}
$$

In the first case cut-height is reduced, in the second case cut-height and weight of the cut formula and in the third case weight of the cut formula. □

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DEFINITE FORMULAE, NEGATION-AS-FAILURE, AND THE BASE-EXTENSION SEMANTICS OF INTUITIONISTIC PROPOSITIONAL LOGIC

Abstract

Proof-theoretic semantics (P-tS) is the paradigm of semantics in which meaning in logic is based on proof (as opposed to truth). A particular instance of P-tS for intuitionistic propositional logic (IPL) is its *base-extension semantics* (B-eS). This semantics is given by a relation called support, explaining the meaning of the logical constants, which is parameterized by systems of rules called *bases* that provide the semantics of atomic propositions. In this paper, we interpret bases as collections of definite formulae and use the operational view of them as provided by uniform proof-search—the proof-theoretic foundation of logic programming (LP)—to establish the completeness of IPL for the B-eS. This perspective allows negation, a subtle issue in P-tS, to be understood in terms of the *negation-as-failure* protocol in LP. Specifically, while the denial of a proposition is traditionally understood as the assertion of its negation, in B-eS we may understand the denial of a proposition as the failure to find a proof of it. In this way, assertion and denial are both prime concepts in P-tS.

Keywords: logic programming, proof-theoretic semantics, bilateralism, negation-as-failure.

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1. Introduction

The definition of a system of logic may be given *proof-theoretically* as a collection of rules of inference that, when composed, determine proofs; that is, formal constructions of arguments that establish that a conclusion is a consequence of some assumptions:

$$\frac{\text{Established Premiss}_1 \quad \dots \quad \text{Established Premiss}_k}{\text{Conclusion}} \Downarrow$$

The systematic use of symbolic and mathematical techniques to determine the forms of valid deductive argument defines *deductive logic*: conclusions are inferred from assumptions.

This is all very well as a way of defining what proofs are, but it relatively rarely reflects either how logic is used in practical reasoning problems or the method by which proofs are found. Rather, proofs are more often constructed by starting with a desired, or putative, conclusion and applying the rules of inference 'backwards'. In this usage, the rules are sometimes called *reduction operators*, read from conclusion to premisses, and denoted

$$\frac{\text{Sufficient Premiss}_1 \quad \dots \quad \text{Sufficient Premiss}_k}{\text{Putative Conclusion}} \Uparrow$$

Constructions in a system of reduction operators are called *reductions*. This paradigm is known as *reductive logic*. The space of reductions of a putative conclusion is larger than its space of proofs, including also failed searches—Pym and Ritter [22] have studied the reductive logic for intuitionistic and classical logic in which such objects are meaningful entities.

As one fixes more and more control structure relative to a set of reduction operators, which determining what reductions are made at what time, one increasingly delegates work to a machine. The extreme case is *logic programming* (LP) in which such controls are fully specified. This view is, perhaps, somewhat obscured by the usual presentation of Horn-clause LP with SLD-resolution—see, for example, Kowalski [14] and Lloyd [17]—but it is explicit in work by Miller et al. [19, 20]. What makes this work is that one restricts to the *hereditary Harrop fragment* of a logic in which contexts contain only *definite formulae*—essentially, formulae in which disjunction only appears negatively. In LP, one typically thinks of the formulae in the context of a sequent as *definional*, which underpins its use in symbolic artificial intelligence.

While deductive logic is suitable for considering the validity of propositions relative to sets of axioms, reductive logic is suitable for considering the meaning of propositions relative to *systems of inference*. That the semantics of a statement is determined by its inferential behaviour is known as *inferentialism* (see Brandom [2]), which has a mathematical realization as *proof-theoretic semantics* (P-tS).

In P-tS, the meaning of the logical connectives is usually derived from the rules of a natural deduction system for the logic—for example, typically, one uses Gentzen's [32] NJ for intuitionistic logic. Meanwhile, the meanings of atomic propositions is supplied by an *atomic system*—a set of rules over atomic propositions. For example, taken from Sandqvist [26], the meaning of the proposition 'Tammy is a vixen' can be understood as arising from the following rule:

$$\frac{\text{Tammy is a fox} \quad \text{Tammy is female}}{\text{Tammy is a vixen}}$$

Sandqvist [29] gave a P-tS for intuitionistic propositional logic (IPL) called *base-extension semantics* (B-eS). It proceeds by a judgement called *support*, parameterized by atomic systems, that defines the logical constants whose base case, the meaning of atoms, is given by derivability in an atomic system.

There is an intuitive relationship between P-tS and LP: the way in bases are *definitional* in P-tS is precisely how sets of definite formulae are *definitional* in LP. Schroeder-Heister and Hallnäs [9, 10] have used this relationship to address questions of *harmony* and *inversion* in P-tS.

In this paper, we show that the completeness of IPL for the B-eS can be understood in terms of the operational view of definite formulae. Miller [19] gave this operational view of the hereditary Harrop fragment of IPL a proof-theoretic denotational semantics which proceeds by a least fixed point construction over the Herbrand base. A set of definite formulae parameterizes the construction. By thinking of this set as a base, we prove the completeness of IPL for the aforementioned B-eS by passing through the denotational semantics.

This work exposes an interpretation of negation in P-tS as a manifestation of the *negation-as-failure* (NAF) protocol. The P-tS of negation is a subtle issue—see, for example, Kürbis [16]. Meanwhile, in LP, the relationship between provability and refutation is made through NAF: a statement

$\neg\varphi$ is established precisely when the system fails to find a proof for φ. The completeness argument for IPL in this paper shows that negation in B-eS can be understood in terms of the failure to find a proof. Hence, from the perspective of B-eS, it is not the case, as advanced by Frege [6] and endorsed by Dummett [4], that denying a statement φ is equal to asserting the negation of φ. Instead, denial in P-tS is conceptually prior to negation. In this way, through the lens of reductive logic, P-tS may be regarded as practising a form of *bilateralism*—the philosophical practice of giving equal consideration to dual concepts such as assertion and denial, truth and falsity, and so on. Of course, bilateralism with respect to negation in logic is a subject that received serious attention in the literature—see, for example, Smiley [31], Rumfitt [25], Francez [5], Wansing [35], and Kürbis [16].

The paper brings together the following fields: proof-theoretic semantics, reductive logic, and logic programming. Some such connexions have already been witnessed in the literature—see, for example, Hallnäs and Schroeder-Heister [9, 10]. The value is that we can mutually use one to explicate phenomena in the other, such as understanding the meaning of negation in terms of NAF. That is not to argue in favour of NAF as an explanation of negation, but only that it manifests in the operational account of B-eS provided by the LP perspective.

The paper has three parts. In the first part, Section 2, we give the relevant background on IPL: Section 2.1 contains the syntax and terminology that we adopt for IPL; Section 2.2 defines the hereditary Harrop fragment (i.e., definite formulae) and gives their operational reading. In the second part, Section 3, we summarize the B-eS for IPL as given by Sandqvist [29]: in Section 3.1 we define the support relation giving the semantics, and in Section 3.2 we summarize the existing proof of completeness. In the third part, Section 4, we study B-eS from the perspective of the operational reading of definite formulae: Section 4.1 relates atomic systems and sets of definite formulae; Section 4.2 proves completeness argument for IPL for the B-eS through the operational reading of definite formulae; and, Section 4.3 discusses how this perspective manifests negation-as-failure as an explanation of the proof-theoretic meaning of negation. The paper concludes in Section 5 with a summary of our results and a discussion of future work.

2. Intuitionistic propositional logic

2.1. Syntax and consequence

There are various presentation of intuitionistic propositional logic (IPL) in the literature. We begin by fixing the relevant concepts and terminology used in this paper.

Definition 2.1 (Formulae). Fix a (denumerable) set of atomic propositions $\mathbb{A}$. The set of formulae $\mathbb{F}$ (over $\mathbb{A}$) is constructed by the following grammar:

$$\varphi ::= \mathrm{p} \in \mathbb{A} \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \varphi \to \varphi \mid \bot$$

Definition 2.2 (Sequent). A sequent is a pair $\Gamma \triangleright \varphi$ in which Γ is a (countable) set of formulae and φ is a formula.

We use $\vdash$ as the consequence judgement relation defining IPL—that is, $\Gamma \vdash \varphi$ denotes that the sequent $\Gamma \triangleright \varphi$ is a consequence of IPL. We may write $\vdash \varphi$ to abbreviate $\varnothing \vdash \varphi$.

Throughout, we assume familiarity with the standard natural deduction system NJ for IPL as introduced by Gentzen [32]—see, for example, van Dalen [34] and Troelstra and Schwichtenberg [33]. Nonetheless we provide the relevant definitions in quick succession to keep the paper self-contained

Definition 2.3 (Natural Deduction Argument). A natural deduction argument is a rooted tree of formulas in which some (possibly no) leaves are marked as discharged. An argument is open if it has undischarged assumptions; otherwise, it is closed.

The leaves of an argument are its *assumptions*, the root is its *conclusion*. That $\mathcal{A}$ has open assumptions Γ, closed assumptions Δ, and conclusion φ may be denoted as follows:

$$\begin{array}{c} \mathcal{A} \\ \varphi \end{array} \qquad \begin{array}{c} \Gamma, [\Delta] \\ \mathcal{A} \end{array} \qquad \begin{array}{c} \Gamma, [\Delta] \\ \mathcal{A} \\ \varphi \end{array}$$

Definition 2.4 (Natural Deduction System NJ). The natural deduction system NJ is composed of the rules in Figure 1.

Definition 2.5 (NJ-Derivation). The set of NJ-derivations is defined inductively as follows:

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi}\ \wedge_{\mathsf{I}} \qquad \frac{\varphi \wedge \psi}{\varphi}\ \wedge^1_{\mathsf{E}} \quad \frac{\varphi \wedge \psi}{\psi}\ \wedge^2_{\mathsf{E}}$$

$$\frac{\varphi}{\varphi \vee \psi}\ \vee^1_{\mathsf{I}} \quad \frac{\psi}{\varphi \vee \psi}\ \vee^2_{\mathsf{I}} \qquad \frac{\varphi \vee \psi \quad \begin{matrix}[\varphi]\\ \chi\end{matrix} \quad \begin{matrix}[\psi]\\ \chi\end{matrix}}{\chi}\ \vee_{\mathsf{E}}$$

$$\frac{\begin{matrix}[\varphi]\\ \psi\end{matrix}}{\varphi \to \psi}\ \to_{\mathsf{I}} \qquad \frac{\varphi \quad \varphi \to \psi}{\psi}\ \to_{\mathsf{E}} \qquad \frac{\bot}{\varphi}\ \bot_{\mathsf{E}}$$

Figure 1. Calculus NJ

– Base Case. If φ is a formula, then the one element tree φ is an NJ-derivation.

– Inductive Step. Let r be a rule in NJ and $\mathcal{D}_1, ..., \mathcal{D}_n$ be a (possibly empty) list of NJ-derivations. If $\mathcal{D}$ is an argument arising from applying r to $\mathcal{D}_1, ..., \mathcal{D}_n$, then $\mathcal{D}$ is an NJ-derivation.

If $\mathcal{D}$ is an NJ-derivation with undischarged leaves composing the set Γ and root φ, then it is an argument for the sequent $\Gamma \triangleright \varphi$. In this paper, we characterize IPL by NJ:

$$\Gamma \vdash \varphi \qquad \text{iff} \qquad \text{there is an NJ-derivation for } \Gamma \triangleright \varphi$$

2.2. The hereditary Harrop fragment

The hereditary Harrop fragment of IPL admits an operational reading that we use to deliver the completeness of a proof-theoretic semantics for IPL. This section closely follows work by Miller [19] (see also Harland [11]).

The propositional hereditary Harrop formulae are generated by the following grammar in which $A \in \mathbb{A}$ is an atomic proposition, D is a *definite formula*, and G is a *goal* formula:

$$\begin{array}{rcl} D & ::= & A \mid G \to A \mid D \wedge D \\ G & ::= & A \mid D \to G \mid G \wedge G \mid G \vee G \end{array}$$

A set of definite formulae $\mathscr{P}$ is a *program*—typically, it is a finite set, but we shall have cause to consider infinite sets. The set of all programs is $\mathbb{P}$.

$\mathscr{P} \vdash A$	if	$A \in \mathsf{cl}(\mathscr{P})$	(IN)
$\mathscr{P} \vdash A$	if	$G \to A \in \mathsf{cl}(\mathscr{P})$ and $\mathscr{P} \vdash G$	(CLAUSE)
$\mathscr{P} \vdash G_1 \vee G_2$	if	$\mathscr{P} \vdash G_1$ or $\mathscr{P} \vdash G_2$	(OR)
$\mathscr{P} \vdash G_1 \wedge G_2$	if	$\mathscr{P} \vdash G_1$ and $\mathscr{P} \vdash G_2$	(AND)
$\mathscr{P} \vdash D \to G$	if	$\mathscr{P} \cup \{D\} \vdash G$	(LOAD)

Figure 2. Operational Semantics for hHLP

We call a sequent $\mathscr{P} \triangleright G$, in which $\mathscr{P}$ is a program and G is a goal, a *query*.

The hereditary Harrop fragment of IPL admits an operational reading which renders it a logic programming language, here called hHLP. The operational semantics of hHLP is given by *uniform* proof-search for $\mathscr{P} \triangleright G$ in a sequent calculus for IPL—see Miller et al. [20].

For purely technical reasons, we require a decomposition function $\mathsf{cl}(-) : \mathbb{P} \to \mathbb{P}$ that will unpack conjunctions. Let $\mathsf{cl}(\mathscr{P})$ be the least set satisfying the following:

- $\mathscr{P} \subseteq \mathsf{cl}(\mathscr{P})$
- If $D_1 \wedge D_2 \in \mathsf{cl}(\mathscr{P})$, then $D_1 \in \mathsf{cl}(\mathscr{P})$ and $D_2 \in \mathsf{cl}(\mathscr{P})$.

Definition 2.6 (Operational Semantics for hHLP). The operational semantics for hHLP is given by the clauses in Figure 2.

Importantly, hHLP language is complete for the hereditary Harrop fragment of IPL; that is, $\mathscr{P} \triangleright G$ has a successful execution iff it is a consequence of IPL—see Miller [20].

The standard frame semantics for IPL by Kripke [15] forms a model-theoretic semantics for hHLP. However, the hereditary Harrop fragment is sufficiently restrictive that we may simplify the semantics in a useful way.

Definition 2.7 (Interpretation). An interpretation is a mapping $I : \mathbb{P} \to \mathcal{P}(\mathbb{A})$ such that $\mathscr{P} \subseteq \mathscr{Q}$ implies $I(\mathscr{P}) \subseteq I(\mathscr{Q})$.

Definition 2.8 (Satisfaction). The satisfaction judgement is given by the clauses of Figure 3.

$$
\begin{array}{lll}
I, \mathscr{P} \vDash A & \text{iff} & A \in I(\mathscr{P}) \\
I, \mathscr{P} \vDash G_1 \vee G_2 & \text{iff} & I, \mathscr{P} \vDash G_1 \text{ or } I, \mathscr{P} \vDash G_2 \\
I, \mathscr{P} \vDash G_1 \wedge G_2 & \text{iff} & I, \mathscr{P} \vDash G_1 \text{ and } I, \mathscr{P} \vDash G_2 \\
I, \mathscr{P} \vDash D \to G & \text{iff} & I, \mathscr{P} \cup \{D\} \vDash G
\end{array}
$$

Figure 3. Denotational Semantics for hHLP

We desire a particular interpretation J such that the following holds:

$$J, \mathscr{P} \vDash G \qquad \text{iff} \qquad \mathscr{P} \vdash G$$

To this end, we consider a function T from interpretations to interpretations that corresponds to unfolding derivability in a base:

$$
\begin{array}{lll}
T(I)(\mathscr{P}) & := & \{A \mid A \in \mathsf{cl}(\mathscr{P})\} \cup \\
& & \{A \mid (G \to A) \in \mathsf{cl}(\mathscr{P}) \text{ and } I, \mathscr{P} \vDash G\}
\end{array}
$$

Interpretations form a lattice under point-wise union ($\sqcup$), point-wise intersection ($\sqcap$), and point-wise subset ($\sqsubseteq$); the bottom of the lattice is given by $I_\perp : \mathscr{P} \mapsto \varnothing$. It is easy to see that T is monotonic and continuous on this lattice, and, by the Knaster-Tarski Theorem [1], its least fixed-point is given as follows:

$$T^\omega I_\perp := I_\perp \sqcup T(I_\perp) \sqcup T^2(I_\perp) \sqcup \ldots$$

Intuitively, each application of T concerns the application of a clause so that $T^\omega I_\perp$ corresponds to arbitrarily many applications.

LEMMA 2.9. *For any program $\mathscr{P}$ and goal G,*

$$T^\omega I_\perp, \mathscr{P} \vDash G \qquad \textit{iff} \qquad \mathscr{P} \vdash G$$

PROOF: The result was proved by Miller [19]—see also Harland [11]. $\square$

3. Base-extension semantics

In this section, we give a brief, but complete, synopsis of the base-extension semantics (B-eS) for IPL as introduced by Sandqvist [29]. The semantics proceeds through a *support* relation parametrized by certain atomic systems, called *bases*. There are related base-extension semantics for classical logic—see Sandqvist [27, 28] and Makinson [18].

We differ slightly in presentation from Sandqvist [29]. First, we refer to more the possibility of more general definitions (e.g., considering nth level atomic systems for $n > 2$). Second, we make use of derivations as mathematical objects. Third, we parameterize support over a notion of base called a *basis*, a class of atomic systems. These differences help bridge the gap between the earlier work and the connexions to logic programming in this paper. It also sets the B-eS for IPL within the wider literature of P-tS from which we draw the generalizations.

3.1. Support in a base

A common idea in proof-theoretic semantics—the paradigm of meaning in which B-eS operates—is that the meaning of atomic propositions is given by sets of atomic rules governing their inferential behaviour. Piecha and Schroeder-Heister [30, 21] have given a useful inductive hierarchy of them.

Definition 3.1 (Atomic Rule). An nth-level atomic rule is defined as follows:

- A zeroth-level atomic rule is a rule of the following form in which $\mathrm{c} \in \mathbb{A}$:

$$\overline{\mathrm{c}}$$

- A first-level atomic rule is a rule of the following form in which $\mathrm{p}_1, ..., \mathrm{p}_n, \mathrm{c} \in \mathbb{A}$,

$$\frac{\mathrm{p}_1 \quad \dots \quad \mathrm{p}_n}{\mathrm{c}}$$

- An $(n+1)$th-level atomic rule is a rule of the following form in which $\mathrm{p}_1, ..., \mathrm{p}_n, \mathrm{c} \in \mathbb{A}$ and $\Sigma_1, ..., \Sigma_n$ are (possibly empty) sets of nth-level atomic rules:

$$\frac{\begin{matrix}[\Sigma_1] \\ \mathrm{p}_1\end{matrix} \quad \dots \quad \begin{matrix}[\Sigma_n] \\ \mathrm{p}_n\end{matrix}}{\mathrm{c}}$$

We take that premisses may be empty such that an mth-level atomic rule is an nth-level atomic rule for any $n > m$. Having sets of atomic rule as hypotheses is more general than have sets of atomic propositions as hypotheses; the latter is captured by the former by taking zeroth-order atomic rules. Nonetheless, the generalization is, perhaps, unexpected. We discuss it further in Section 4.2.

Definition 3.2 (Atomic System). An atomic system is a set of atomic rules.

Atomic systems may have infinitely many rules but they are at most countably infinite. They are used to base validity in P-tS on proof. The definition of a derivation is a generalization of natural deduction *à la* Gentzen [32], which was given by Piecha and Schroeder-Heister [30, 21].

Definition 3.3 (Derivation in an Atomic System). Let $\mathscr{A}$ be an atomic system. The set of $\mathscr{A}$-derivations is defined inductive as follows:

- Base Case. If $\mathscr{A}$ contains a zeroth-level rule concluding c, then the natural deduction argument consisting of just the node c is a $\mathscr{A}$-derivation.
- Induction Step. Suppose $\mathscr{A}$ contains an $(n+1)$th-level rule r of the following form:

$$\frac{\begin{matrix}[\Sigma_1] \\ \mathrm{p}_1\end{matrix} \quad \dots \quad \begin{matrix}[\Sigma_n] \\ \mathrm{p}_n\end{matrix}}{\mathrm{c}}$$

  And suppose that for each $1 \leq i \leq n$ there is a $\mathscr{A}$-derivation $\mathcal{D}_i$ of the following form:

$$\begin{matrix}\Gamma_i, \Sigma_i \\ \mathcal{D}_i \\ \mathrm{p}_i\end{matrix}$$

  Then the natural deduction argument with root c and immediate sub-trees $\mathcal{D}_1$,...,$\mathcal{D}_n$ is a $\mathscr{A}$-argument from $\Gamma_1 \cup \ldots \cup \Gamma_n$ to c.

An atom c is derivable from Γ in $\mathscr{A}$—denoted $\Gamma \vdash_{\mathscr{A}} \mathrm{c}$—iff there is a $\mathscr{A}$-derivation from Γ to c.

Typically, we do not consider all atomic systems, but restrict attention to some particular class.

$\Gamma \Vdash_{\mathscr{B}} \varphi$	iff	for any $\mathscr{C} \in \mathfrak{B}$ such that $\mathscr{B} \subseteq \mathscr{C}$, if $\Vdash_{\mathscr{C}} \psi$ for all $\psi \in \Gamma$, then $\Vdash_{\mathscr{C}} \varphi$	$(\Rightarrow)$
$\Vdash_{\mathscr{B}} \mathrm{p}$	iff	$\vdash_{\mathscr{B}} \mathrm{p}$	$(\mathbb{A})$
$\Vdash_{\mathscr{B}} \varphi \to \psi$	iff	$\varphi \Vdash_{\mathscr{B}} \psi$	$(\to)$
$\Vdash_{\mathscr{B}} \varphi \wedge \psi$	iff	$\Vdash_{\mathscr{B}} \varphi$ and $\Vdash_{\mathscr{B}} \psi$	$(\wedge)$
$\Vdash_{\mathscr{B}} \varphi \vee \psi$	iff	for any $\mathscr{C} \in \mathfrak{B}$ such that $\mathscr{B} \subseteq \mathscr{C}$ and any $\mathrm{p} \in \mathbb{A}$, if $\varphi \Vdash_{\mathscr{C}} \mathrm{p}$ and $\psi \Vdash_{\mathscr{C}} \mathrm{p}$, then $\Vdash_{\mathscr{C}} \mathrm{p}$	$(\vee)$
$\Vdash_{\mathscr{B}} \bot$	iff	$\Vdash_{\mathscr{B}} \mathrm{p}$ for any $\mathrm{p} \in \mathbb{A}$	$(\bot)$

Figure 4. Support in a Base

Definition 3.4 (Basis). A basis is a set of atomic systems.

Having fixed a basis $\mathfrak{B}$, an atomic system $\mathscr{B} \in \mathfrak{B}$ is called a *base*. A base-extension semantics is formulated relative to a basis via a support relation.

Definition 3.5 (Support in a Base). Fix a basis $\mathfrak{B}$. Support over $\mathscr{B}$ is the least relation $\Vdash_{\_}$ on sequents and bases in $\mathfrak{B}$ defined by the clause of Figure 4. The validity judgement over $\mathfrak{B}$ is the following relation $\Vdash$ one sequent:

$$\Gamma \Vdash \varphi \qquad \text{iff} \qquad \Gamma \Vdash_{\mathscr{B}} \varphi \text{ for any } \mathscr{B} \in \mathfrak{B}$$

Sandqvist [27] gave this semantics with a basis $\mathfrak{S}$ consisting of atomic rules that are *properly* second-level; that is, rules of the form

$$\frac{\begin{matrix}[\Sigma_1] \\ \mathrm{p}_1\end{matrix} \quad \dots \quad \begin{matrix}[\Sigma_n] \\ \mathrm{p}_n\end{matrix}}{\mathrm{c}}$$

in which Σ_1,...,Σ_n are sets of atoms.

Theorem 3.6 (Soundness & Completeness). $\Gamma \vdash \varphi$ *iff* $\Gamma \Vdash \varphi$ *over* $\mathfrak{S}$.

Proof: Proved by Sandqvist [29]—see Section 3.2. □

The support relation satisfies some important expected properties, such as the following:

Lemma 3.7. *If* $\Gamma \Vdash_{\mathscr{B}} \varphi$ *and* $\mathscr{C} \supseteq \mathscr{B}$, *then* $\Gamma \Vdash_{\mathscr{C}} \varphi$.

Proof: Proved by Sandqvist [29] by induction on support in a base. □

This summarizes the B-eS for IPL Sandqvist [29] proved the soundness of IPL for the B-eS by showing that validity admits all the rules of NJ. His proof of completeness is more complex. In essence, Sandqvist [29] proved completeness of IPL for the B-eS by constructing a bespoke atomic system $\mathscr{N}$ to a given validity judgement that allows us to *simulate* an NJ-derivation for the sequent in question. We present the main ideas here as we refer to them in Section 4.2.

3.2. Completeness of IPL via a natural base

We want to show that $\Gamma \Vdash \gamma$ implies $\Gamma \vdash \gamma$. We understand the latter in terms of provability in NJ. Therefore, we associate to each formula ρ in the sequent $\Gamma \triangleright \gamma$ a unique atom r and construct a base $\mathscr{N}$ emulating NJ such that r behaves in $\mathscr{N}$ as ρ behaves in NJ.

For example, let $\Gamma \triangleright \gamma$ contain $\rho := \mathrm{p} \wedge \mathrm{q}$. The rules governing ρ are the conjunction introduction and elimination rules of NJ, so we require $\mathscr{N}$ to contain the following rules in which r is alien to $\Gamma \triangleright \gamma$:

$$\frac{\mathrm{p} \quad \mathrm{q}}{\mathrm{r}} \qquad \frac{\mathrm{r}}{\mathrm{p}} \quad \frac{\mathrm{r}}{\mathrm{q}}$$

These rules are designed such that r behaves in $\mathscr{N}$ precisely as ρ does in NJ. That is, they emulate the conjunction rules. The shorthand for r is $(\mathrm{p} \wedge \mathrm{q})^\flat$—that is $r = \rho^\flat$—so that the above rules may be expressed more clearly as follows:

$$\frac{\mathrm{p} \quad \mathrm{q}}{(\mathrm{p} \wedge \mathrm{q})^\flat} \qquad \frac{(\mathrm{p} \wedge \mathrm{q})^\flat}{\mathrm{p}} \quad \frac{(\mathrm{p} \wedge \mathrm{q})^\flat}{\mathrm{q}}$$

For clarity, we give another example. Suppose $\Gamma \triangleright \gamma$ also contains $\sigma := \mathrm{p} \to \mathrm{q}$, then $\mathscr{N}$ contains rules that emulate the implication introduction and elimination rules of NJ for σ using an atom $s := \sigma^\flat := (\mathrm{p} \to \mathrm{q})^\flat$ alien to Γ and γ. That is, $\mathscr{N}$ contains the following rules:

$$\frac{\varphi^\flat \quad \psi^\flat}{(\varphi \wedge \psi)^\flat}\wedge_\mathsf{I}{}^\flat \qquad \frac{(\varphi \wedge \psi)^\flat}{\varphi^\flat}\wedge_\mathsf{E}{}^\flat \quad \frac{(\varphi \wedge \psi)^\flat}{\psi^\flat}\wedge_\mathsf{E}{}^\flat$$

$$\frac{\varphi^\flat}{(\varphi \vee \psi)^\flat}\vee_\mathsf{I}{}^\flat \quad \frac{\psi^\flat}{(\varphi \vee \psi)^\flat}\vee_\mathsf{I}{}^\flat \qquad \frac{(\varphi \vee \psi)^\flat \quad \begin{array}{c}[\varphi^\flat]\\ \mathrm{p}\end{array} \quad \begin{array}{c}[\psi^\flat]\\ \mathrm{p}\end{array}}{\mathrm{p}}\vee_\mathsf{E}{}^\flat$$

$$\frac{\begin{array}{c}[\varphi^\flat]\\ \psi^\flat\end{array}}{(\varphi \to \psi)^\flat}\to_\mathsf{I}{}^\flat \qquad \frac{\varphi^\flat \quad (\varphi \to \psi)^\flat}{\psi^\flat}\to_\mathsf{E}{}^\flat \qquad \frac{\bot^\flat}{\mathrm{p}}\bot_\mathsf{E}{}^\flat$$

Figure 5. Atomic System $\mathscr{N}$

$$\frac{\begin{array}{c}[\mathrm{p}]\\ \mathrm{q}\end{array}}{(\mathrm{p} \to \mathrm{q})^\flat} \qquad \frac{\mathrm{p} \quad (\mathrm{p} \to \mathrm{q})^\flat}{\mathrm{q}}$$

The details of how $\mathscr{N}$ is constructed and how it delivers completeness are below.

Fix a sequent $\Gamma \triangleright \gamma$. To every sub-formula φ of $\Gamma \triangleright \gamma$ associate a unique atomic proposition $\varphi^\flat$ as follows:

- if $\varphi \notin \mathbb{A}$, then $\varphi^\flat$ is an atom that does not occur in $\Gamma \triangleright \gamma$;
- if $\varphi \in \mathbb{A}$, then $\varphi^\flat = \varphi$.

The right-inverse of $-^\flat$ is $-^\natural$ and both functions act on sets point-wise,

$$\Sigma^\flat := \{\varphi^\flat \mid \varphi \in \Sigma\} \qquad P^\natural := \{\mathrm{p}^\natural \mid \mathrm{p} \in P\}$$

Let $\mathscr{N}$ be the atomic system containing precisely the rules of Figure 5 for any φ, ψ occurring in $\Gamma \triangleright \gamma$ and any $\mathrm{p} \in \mathbb{A}$. These rules are precisely such that $\varphi^\flat$ behaves in $\mathscr{N}$ as φ does in NJ. Note that, for any validity judgement, the atomic system $\mathscr{N}$ thus generated is indeed a Sandqvist base.

In this set-up, Sandqvist [29] establishes three properties that collectively deliver completeness.

Lemma 3.8. *Let* $\mathrm{P} \subseteq \mathbb{A}$ *and* $\mathrm{p} \in \mathbb{A}$ *and let* $\mathscr{B} \in \mathfrak{S}$*,*

$$\mathrm{P} \Vdash_{\mathscr{B}} \mathrm{p} \qquad \textit{iff} \qquad \mathrm{P} \vdash_{\mathscr{B}} \mathrm{p}$$

This claim is a basic completeness result in which the context Σ is restricted to a set of atomic propositions and the extract p is an atomic proposition.

Lemma 3.9. *For every* φ *occurring in* $\Gamma \triangleright \gamma$ *and any* $\mathscr{N}' \supseteq \mathscr{N}$*,*

$$\Vdash_{\mathscr{N}'} \varphi^{\flat} \qquad \textit{iff} \qquad \Vdash_{\mathscr{N}'} \varphi$$

In other words, $\varphi^{\flat}$ and φ are equivalent in $\mathscr{N}$—that is, $\varphi^{\flat} \Vdash_{\mathscr{N}} \varphi$ and $\varphi \Vdash_{\mathscr{N}} \varphi^{\flat}$. The property allows us to move between the basic case (i.e., the set-up of Lemma 3.8) and the general case (i.e., completeness—Theorem 3.6). This is the crucial step in the proof of completeness. In Section 4.2, we study it in terms of the operational account of definite formulae given in Section 2.2.

Lemma 3.10. *Let* $\mathrm{P} \subseteq \mathbb{A}$ *and* $\mathrm{p} \in \mathbb{A}$*,*

$$\mathrm{P} \Vdash_{\mathscr{N}} p \qquad \textit{implies} \qquad \mathrm{P}^{\natural} \vdash p^{\natural}$$

This property is the simulation statement. It allows us to make the final move from derivability in $\mathscr{N}$ to derivability in NJ.

These lemmas collectively suffice for completeness:

Proof: Theorem 3.6—Completeness. Let $\mathscr{N}$ be the bespoke base for $\Gamma \triangleright \varphi$. By 3.9, for any $\mathscr{N}' \supseteq \mathscr{N}$ we have $\Gamma^{\flat} \Vdash_{\mathscr{N}'} \varphi^{\flat}$. Since $\mathscr{N} \supseteq \mathscr{N}$, we infer $\Gamma^{\flat} \Vdash_{\mathscr{N}} \varphi^{\flat}$. Therefore, by 3.8, we have $\Gamma^{\flat} \vdash \mathscr{N} \varphi^{\flat}$. Finally, by 3.10, $\Gamma \vdash \varphi$, as required. □

In the next section, we show that the completeness follows intuitively from regarding $\mathscr{N}$ as a program capturing the inferential content of NJ. In general, a base may be regarded as a program, so that the application of a rule in the base corresponds to the use of a clause in the program. We demonstrate that the validity of a formula φ in the base $\mathscr{N}$ emulates the execution of a goal $\varphi^{\flat}$ relative to the program $\mathscr{N}$. By construction of $\mathscr{N}$, such executions simulate the construction of an NJ proof of φ. Hence, IPL is complete with respect to the B-eS.

4. Definite formulae, proof-search, and completeness

There is an intuitive encoding of atomic rules as formulae. More precisely, as *definite* formulae. Under this encoding, the bases which deliver B-eS live within the hereditary Harrop fragment of IPL. The latter has a simple operational reading via proof-search for uniform proofs (see Section 2.2) that enables a proof-theoretic denotational semantics—the least fixed point construction. We use this well-understood phenomenon to deliver the completeness of IPL with respect to Sandqvist's B-eS [29]—see Section 3.

Doing this reveals a subtle interpretation of the meaning of negation in terms of the negation-as-failure protocol. A reductive logic view of the denial of a formula is the failure to find a proof of it. Thus, according to the view of B-eS arising from the account passing through the operational reading of definite formulae, in B-eS denial is conceptionally prior to negation and both require equal consideration.

4.1. Atomic systems vs. programs

Intuitively, atomic systems in B-eS are definitional in precisely the same way as programs in hHLP are definitional. To illustrate this, we must systematically move between them, which we do by encoding atomic systems as programs.

Let $\lfloor - \rfloor$ be as follows:

– The encoding of zeroth-level rule is as follows:

$$\left\lfloor \frac{}{\mathrm{c}} \right\rfloor := \mathrm{c}$$

– The encoding of a first-level rule is as follows:

$$\left\lfloor \frac{\mathrm{p}_1 \quad \dots \quad \mathrm{p}_n}{\mathrm{c}} \right\rfloor := (\mathrm{p}_1 \wedge \dots \wedge \mathrm{p}_n) \to \mathrm{c}$$

– The encoding of an nth-level rule is as follows:

$$\left\lfloor \frac{\begin{matrix}[\Sigma_1] & & [\Sigma_n] \\ \mathrm{p}_1 & \dots & \mathrm{p}_n\end{matrix}}{\mathrm{c}} \right\rfloor := \big((\lfloor \Sigma_1 \rfloor \to \mathrm{p}_1) \wedge \dots \wedge (\lfloor \Sigma_n \rfloor \to \mathrm{p}_n)\big) \to \mathrm{c}$$

For example, $\rightarrow_{\mathsf{I}}{}^{\flat}$ in Figure 5 yields the following schematically:

$$(\varphi^\flat \to \psi^\flat) \to (\varphi \to \psi)^\flat$$

The hierarchy of atomic system provided by Piecha and Schroeder-Heister [30, 21] (Definition 3.1) precisely corresponds to the inductive depth of the grammar for hereditary Harrop formulae—that is, if $\mathscr{A}$ is an n-th level atomic system, then

$$\vdash_{\mathscr{A}} \mathrm{p} \qquad \text{iff} \qquad \lfloor \mathscr{A} \rfloor \vdash \mathrm{p}$$

Therefore, we may suppress the encoding function, and henceforth use atomic systems and programs interchangeably—that is, we may write $\mathscr{A} \vdash \mathrm{p}$ to denote $\lfloor \mathscr{A} \rfloor \vdash \mathrm{p}$.

Of course, in the Sanqvist basis, we are limited to properly second-level atomic systems, but the grammar of definite clauses can handle considerably more. Indeed, the work below suggests that completeness holds for nth-level atomic systems for $n \geq 2$.

Formally, to say that bases are definitional in the sense of programs, we mean the following:

$$\Vdash_{\mathscr{B}} \varphi \qquad \text{iff} \qquad \mathscr{N} \cup \mathscr{B} \vdash \varphi^\flat \tag{$*$}$$

Here $\mathscr{N}$ contains rules governing φ when the formula is complex—that is, φ is a sub-formula of a sequent $\Gamma \triangleright \psi$ which generates $\mathscr{N}$—and arbitrary otherwise.

It is important that we use $\varphi^\flat$ rather than φ in $(*)$. It is certainly *not* the case that bases behave exactly as contexts; that is, we do *not* have the following equivalence:

$$\Vdash_{\mathscr{B}} \varphi \qquad \text{iff} \qquad \mathscr{B} \vdash \varphi \tag{$**$}$$

That this generalization fails is shown by the following counter-example:

Example 4.1. Consider the following formula:

$$\varphi := (\mathrm{a} \to \mathrm{b} \vee \mathrm{c}) \to \big((\mathrm{a} \to \mathrm{b}) \vee (\mathrm{a} \to \mathrm{c})\big)$$

The formula φ is not a consequence of IPL; hence, by completeness of IPL with respect to the B-eS we have $\Vdash_{\mathscr{B}} (\mathrm{a} \to \mathrm{b} \vee \mathrm{c})$ and $\nVdash_{\mathscr{B}} (\mathrm{a} \to \mathrm{b}) \vee (\mathrm{a} \to \mathrm{c})$, for some $\mathscr{B}$. However, assuming $(**)$, the second judgment obtains whenever the the first obtains – that is, $\Vdash_{\mathscr{B}} (\mathrm{a} \to \mathrm{b} \vee \mathrm{c})$ implies

$\Vdash_{\mathscr{B}} (\mathrm{a} \to \mathrm{b}) \vee (\mathrm{a} \to \mathrm{c})$, for any $\mathscr{B}$! This is witnessed by the following computation in hHLP:

$$
\begin{array}{llr}
\Vdash_{\mathscr{B}} \mathrm{a} \to \mathrm{b} \vee \mathrm{c} \quad \text{implies} & \mathscr{B} \vdash \mathrm{a} \to \mathrm{b} \vee \mathrm{c} & (**) \\
\text{implies} & \mathscr{B} \cup \{\mathrm{a}\} \vdash \mathrm{b} \vee \mathrm{c} & (\mathsf{LOAD}) \\
\text{implies} & \mathscr{B} \cup \{\mathrm{a}\} \vdash \mathrm{b} \text{ or } \mathscr{B} \cup \{\mathrm{a}\} \vdash \mathrm{c} & (\mathsf{OR}) \\
\text{implies} & \mathscr{B} \vdash \mathrm{a} \to \mathrm{b} \text{ or } \mathscr{B} \vdash \mathrm{a} \to \mathrm{c} & (\mathsf{LOAD}) \\
\text{implies} & \mathscr{B} \vdash (\mathrm{a} \to \mathrm{b}) \vee (\mathrm{a} \to \mathrm{c}) & (\mathsf{OR}) \\
\text{implies} & \Vdash_{\mathscr{B}} (\mathrm{a} \to \mathrm{b}) \vee (\mathrm{a} \to \mathrm{c}) & (**)
\end{array}
$$

That LOAD and OR may be used invertibly is justified by case-analysis on the structure of the goal formula with respect to the operational semantics (Figure 2) – it can also be seen by Lemma 2.9.

Example 4.2. By Theorem 3.6, we have $\Vdash_{\varnothing} \mathrm{a} \vee \mathrm{b} \to \mathrm{b} \vee \mathrm{a}$. That $\mathscr{N} \vdash (\mathrm{a} \vee \mathrm{b} \to \mathrm{b} \vee \mathrm{a})^\flat$ indeed obtains is witnessed by the computation,

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\mathscr{N}, (\mathrm{a} \vee \mathrm{b})^\flat \vdash (\mathrm{a} \vee \mathrm{b})^\flat} \Uparrow \mathsf{IN} \quad \mathcal{R}_a \quad \mathcal{R}_b}{\mathscr{N}, (\mathrm{a} \vee \mathrm{b})^\flat \vdash (\mathrm{a} \vee \mathrm{b})^\flat \wedge (\mathrm{a} \to (\mathrm{b} \vee \mathrm{a})^\flat) \wedge (\mathrm{b} \to (\mathrm{b} \vee \mathrm{a})^\flat)} \Uparrow \mathsf{AND}}{\mathscr{N}, (\mathrm{a} \vee \mathrm{b})^\flat \vdash (\mathrm{b} \vee \mathrm{a})^\flat} \Uparrow \mathsf{CLAUSE}\,(\vee_{\mathsf{E}})^\flat}{\mathscr{N}, (\mathrm{a} \vee \mathrm{b})^\flat \vdash (\mathrm{b} \vee \mathrm{a})^\flat} \Uparrow \mathsf{LOAD}}{\mathscr{N} \vdash (\mathrm{a} \vee \mathrm{b} \to \mathrm{b} \vee \mathrm{a})^\flat} \Uparrow \mathsf{CLAUSE}\,(\to_{\mathsf{I}})^\flat
$$

where $\mathcal{R}_x$ for $x \in \{a, b\}$ is

$$
\dfrac{\dfrac{\dfrac{}{\mathscr{N}, (\mathrm{b} \vee \mathrm{a})^\flat, \mathrm{x} \vdash \mathrm{x}} \Uparrow \mathsf{IN}}{\mathscr{N}, (\mathrm{b} \vee \mathrm{a})^\flat, \mathrm{x} \vdash (\mathrm{b} \vee \mathrm{a})^\flat} \Uparrow \mathsf{CLAUSE}\,(\vee_{\mathsf{I}})^\flat}{\mathscr{N}, (\mathrm{b} \vee \mathrm{a})^\flat \vdash \mathrm{x} \to (\mathrm{b} \vee \mathrm{a})^\flat} \Uparrow \mathsf{LOAD}
$$

In the next section, we use the relationship between atomic systems and programs to prove completeness of IPL with respect to the B-eS.

4.2. Completeness of IPL via logic programming

We may prove completeness of IPL with respect to the B-eS by passing through hHLP as follows:

$$
\begin{tikzcd}
T^\omega I_\bot, \mathscr{N} \vDash \varphi^\flat \arrow[r, leftrightarrow] & \mathscr{N} \vdash \varphi^\flat \arrow[d] \\
\Vdash_{\mathscr{N}} \varphi \arrow[u] & \vdash \varphi
\end{tikzcd}
$$

The diagram requires three claims, the middle one of which is Lemma 2.9. The other two are Lemma 4.3 and Lemma 4.4, respectively, reading in the direction of the arrows.

The intuition of the completeness argument is two-fold: firstly, that $\mathscr{N}$ is to $\varphi^\flat$ as NJ is to φ; secondly, the use of a rule in a base corresponds to the use of a clause in the corresponding program; thirdly, execution in $\mathscr{N}$ corresponds to proof(-search) in NJ. In this set-up, the T^ω construction captures the construction of a proof: the application of a rule corresponds to a use of T, the iterative application of rules corresponds to the iterative application of T—that is, to T^ω.

It remains to prove the claims and completeness. Fix a sequent $\Gamma \triangleright \varphi$ and let $-^\flat$ and $\mathscr{N}$ be constructed as in Section 3.2 for this sequent. Let Δ be an arbitrary set of sub-formulae of the sequent and ψ an arbitrary subformula of the sequent.

Lemma 4.3 (Emulation). *If $\Vdash_{\mathscr{N}} \psi$, then $T^\omega I_\bot, \mathscr{N} \vDash \psi^\flat$.*

Proof: We prove a stronger proposition: for any $\mathscr{N}' \supseteq \mathscr{N}$, if $\Vdash_{\mathscr{N}'} \psi$, then $T^\omega I_\bot, \mathscr{N}' \vDash \psi^\flat$. We proceed by induction on support in a base according to the various cases of Figure 4. As above, for the sake of economy, we combine the clauses $\Rightarrow$ and $\to$.

- $\psi \in \mathbb{A}$. Note $\psi^\flat = \psi$, by definition. Therefore, if $\Vdash_{\mathscr{N}'} \psi$, then $\vdash_{\mathscr{N}'} \psi$, but this is precisely emulated by application of T. Hence, $T^\omega I_\bot, \mathscr{N}' \vDash \psi$.

- $\psi = \bot$. If $\Vdash_{\mathscr{N}'} \bot$, then $\Vdash_{\mathscr{N}'} \mathrm{p}$, for every $\mathrm{p} \in \mathbb{A}$. By the induction hypothesis (IH), $T^\omega I_\bot, \mathscr{N}' \vDash \mathrm{p}$ for every $\mathrm{p} \in \mathbb{A}$. It follows that $T^\omega I_\bot, \mathscr{N}' \vDash \bot^\flat$.

– $\psi := \psi_1 \wedge \psi_2$. By the $\wedge$-clause for support, $\Vdash_{\mathscr{N}'} \psi_1$ and $\Vdash_{\mathscr{N}'} \psi_2$. Hence, by the IH, $T^\omega I_\perp, \mathscr{N}' \vDash \psi_1^\flat$ and $T^\omega I_\perp, \mathscr{N}' \vDash \psi_2^\flat$. By $\wedge$-clause for satisfaction, $T^\omega I_\perp, \mathscr{N}' \vDash \psi_1^\flat \wedge \psi_2^\flat$. The result follows by $\wedge_{\mathsf{I}}{}^\flat$-schema.

– $\psi := \psi_1 \vee \psi_2$. By Lemma 3.9, $\psi_1 \Vdash_{\mathscr{N}'} \psi_1^\flat$ and $\psi_2 \Vdash_{\mathscr{N}'} \psi_2^\flat$. By the $\vee_{\mathsf{I}}$-scheme in $\mathscr{N}'$, both $\psi_1^\flat \Vdash (\psi_1 \vee \psi_2)^\flat$ and $\psi_2^\flat \Vdash (\psi_1 \vee \psi_2)^\flat$. Therefore, by $\Rightarrow$-clause for support, we have $\psi_1 \Vdash_{\mathscr{N}'} (\psi_1 \vee \psi_2)^\flat$ and $\psi_2 \Vdash_{\mathscr{N}'} (\psi_1 \vee \psi_2)^\flat$. Using the $\vee$-clause for support on the assumption $\Vdash_{\mathscr{N}'} \psi_1 \vee \psi_2$ with these results means that $\Vdash_{\mathscr{N}'} (\psi_1 \vee \psi_2)^\flat$. That is, $T^\omega, \mathscr{N}' \vDash (\psi_1 \vee \psi_2)^\flat$, as required.

– $\psi := \psi_1 \to \psi_2$. By the $\to$-clause for satisfaction, $\psi_1 \Vdash_{\mathscr{N}'} \psi_2$. So, by the $\Rightarrow$-clause for satisfaction, $\Vdash_{\mathscr{N}''} \psi_1$ implies $\Vdash_{\mathscr{N}''} \psi_2$ for any $\mathscr{N}'' \supseteq \mathscr{N}'$. Let $\mathscr{N}'' := \mathscr{N}' \cup \{\psi_1^\flat\}$. Since $\Vdash_{\mathscr{N}', \psi^\flat} \psi^\flat$, by Lemma 3.9, we have $\Vdash_{\mathscr{N}', \psi^\flat} \psi$, hence we infer $\Vdash_{\mathscr{N}', \psi^\flat} \psi_2$. By the IH, $T^\omega I_\perp, \mathscr{N}' \cup \{\psi_1^\flat\} \vDash \psi_2^\flat$. Hence, $T^\omega I_\perp, \mathscr{N}' \vDash \psi_1^\flat \to \psi_2^\flat$. By the $\to_{\mathsf{I}}{}^\flat$-scheme, $T^\omega I_\perp \mathscr{N}' \vDash (\psi_1 \to \psi_2)^\flat$, as required.

This completes the induction. □

Lemma 4.4 (Simulation). *If $\mathscr{N} \cup \Delta^\flat \vdash \psi^\flat$, then $\Delta \vdash \psi$.*

Proof: We proceed by induction on the length of execution. Intuitively, the execution of $\mathscr{N} \cup \Delta^\flat \vdash \psi^\flat$ simulates the reductive construction of a proof of ψ from Δ in NJ—that is, a proof-search. We proceed by induction on the length of the execution.

Base Case: It must be that $\psi \in \Delta$, so $\Delta \vdash \psi$ is immediate.

Inductive Step: By construction of $\mathscr{N}$, the execution concludes by CLAUSE applied to a definite clause ρ simulating a rule $\mathrm{r} \in \mathsf{NJ}$; that is, $\mathscr{N} \cup \Delta^\flat \vdash \psi_i^\flat$ for ψ_i such that $\psi_1^\flat \wedge .... \wedge \psi_n^\flat \to \psi^\flat$. By the induction hypothesis (IH), $\Delta \vdash \psi_i$ for $1 \leq i \leq n$. It follows that $\Delta \vdash \psi$ by applying $r \in \mathsf{NJ}$.

For example, if the execution concludes by CLAUSE applied to the clause for $\wedge$-introduction (i.e., $\psi^\flat \wedge \psi^\flat \to (\psi \wedge \psi)^\flat$), then the trace is as follows:

$$\dfrac{\dfrac{\dfrac{\vdots}{\mathscr{N} \cup \Delta^\flat \vdash \psi^\flat} \quad \dfrac{\vdots}{\mathscr{N} \cup \Delta^\flat \vdash \psi^\flat}}{\mathscr{N} \cup \Delta^\flat \vdash \psi^\flat \wedge \psi^\flat}}{\mathscr{N} \cup \Delta^\flat \vdash (\psi \wedge \psi)^\flat}$$

By the induction hypothesis, we have proofs witnessing $\Delta \vdash \psi$ and $\Delta \vdash \psi$, and by $\wedge$-introduction:

$$\dfrac{\begin{matrix}\vdots \\ \psi\end{matrix} \quad \begin{matrix}\vdots \\ \psi\end{matrix}}{\psi \wedge \psi}$$

This completes the induction. □

Following the diagram, we have the completeness of IPL with respect to the B-eS:

PROOF: Theorem 3.6—Completeness. We require to show that $\Vdash \varphi$ implies $\Vdash_{\mathscr{N}} \varphi$ for arbitrary φ. To this end, assume $\Vdash \varphi$. Let $\mathscr{N}$ be the natural base generated by φ. By definition, from the assumption, we have $\Vdash_{\mathscr{N}} \varphi$. Hence, by Lemma 4.3, it follows that $T^\omega I_\perp, \mathscr{N} \vDash \varphi^\flat$. Whence, by Lemma 2.9, we obtain $\mathscr{N} \vdash \varphi^\flat$. Thus, by Lemma 4.4, $\vdash \varphi$, as required. □

In the following section, we discuss how reductive logic delivers the completeness proof above and the essential role played by both proofs and refutations.

4.3. Negation-as-failure

A reduction in a proof system is constructed co-recursively by applying the rules of inference backwards. Even though each step corresponds to the application of a rule, the reduction can fail to be a proof as the computation arrives at an irreducible sequent that is not an instance of an axiom in the logic. For example, in hHLP, one may compute the following:

$$\dfrac{\dfrac{\mathrm{p} \triangleright \mathrm{q}}{\mathrm{p} \triangleright \mathrm{p} \vee \mathrm{q}}\ \Uparrow \mathsf{LOAD}}{\varnothing \triangleright \mathrm{p} \to (\mathrm{p} \vee \mathrm{q})}\ \Uparrow \mathsf{OR}$$

This reduction fails to be a proof, despite every step being a valid inference, since the initial sequent is not an instance of IN. In reductive logic, such failed attempts at constructing proofs are not meaningless: Pym and Ritter [22] have provided a semantics of the reductive logic of IPL in which such reductions are given meaning by using hypothetical rules—that is, the construction would succeed in the presence of the following rule:

$$\frac{\mathrm{p}}{\mathrm{q}}$$

The categorical treatment of this semantics has them as *indeterminates* in a polynomial category—this adumbrates current work by Pym et al. [23], who have shown that the B-eS is entirely natural from the perspective of categorical logic. The use of such additional rules to give semantics to constructions that are not proofs directly corresponds to the use of atomic systems in the B-eS for IPL; for example, let $\mathscr{A}$ be the atomic system containing the rule above, then the judgement $p \Vdash_{\mathscr{A}} q$ obtains. Altogether, this suggests a close relationship between B-eS and reductive logic, which manifests with the operational reading of definite clauses and their relationship to atomic rules in Section 4.

Within P-tS, negation is a subtle issue—see Kürbis [16]. We may use the perspective of LP developed herein to review the meaning of absurdity ($\bot$).

There is no introduction rule for $\bot$ in NJ. One may not construct a proof of absurdity without it already being, *in some sense*, assumed; for example, $\varphi, \varphi \to \bot \vdash \bot$ obtains because the context $\{\varphi, \varphi \to \bot\}$ is already, in some sense, absurd. We may use LP to understand what that sense is. To simplify matters, observe that the judgement $\Gamma \vdash \bot$ is equivalent to $\vdash \varphi \to \bot$ for some formula φ. Therefore, we may restrict attention to negations of this kind to understand the meaning of absurdity.

By Theorem 3.6 (Soundness) and Lemma 4.4 (Simulation), we see that the converse of Theorem 4.3 holds. Therefore,

$$\Vdash \neg\varphi \qquad \text{iff} \qquad T^{\omega} I_{\bot}, \mathscr{N} \vdash (\neg\varphi)^{\flat}$$

Unfolding the semantics, this is equivalent to $T^{\omega} I_{\bot}, \mathscr{N} \cup \{\varphi^{\flat}\} \vdash \bot^{\flat}$. Thus, the sense in which φ is absurd is that its interpretation under $T^{\omega} I_{\bot}$ contains absurdity; that is, φ is absurd iff $\bot^{\flat} \in T^{\omega} I_{\bot}(\varphi)$. What does this tell us about the meaning of $\neg\varphi$? Since there is no proof of $\bot^{\flat}$, we have that the meaning of $\neg\varphi$ is that there is no proof of $(\varphi)^{\flat}$ in $\mathscr{N}$. This is the *negation-as-failure* principle. How does it yield the clause for $\bot$ in Figure 4?

Passing through (∗) in Section 4.1,

$$\Vdash_{\mathscr{B}} \bot \qquad \text{iff} \qquad \mathscr{N} \cup \mathscr{B} \vdash \bot^{\flat}$$

Since there is no introduction rule for $\bot^{\flat}$ in $\mathscr{N}$, it must be that $\mathscr{B}$ derives it. Thus, there is rule in $\mathscr{B}$ of the following form:

$$\frac{\begin{matrix}[\Sigma_1] & & [\Sigma_n] \\ \mathrm{p}_1 & \dots & \mathrm{p}_n\end{matrix}}{\bot^{\flat}}$$

To simplify matters, we introduce alien q and $\bar{\mathrm{q}}$ as 'conjunctions' of some subset $q_1, ..., q_k$ and $q_{k+1}, ..., q_n$ of $\mathrm{p}_1, ..., \mathrm{p}_n$ in the inferentialist sense. That is, we introduce the following, where $\Pi_i = \Sigma_j$ iff $\mathrm{q}_i = \mathrm{p}_i$ for $i, j \in \{1, ..., n\}$:

$$\frac{\begin{matrix}[\Pi_1] & & [\Pi_n] \\ \mathrm{q}_1 & \dots & \mathrm{q}_n\end{matrix}}{\mathrm{q}} \qquad \frac{\begin{matrix}[\Pi_{k+1}] & & [\Pi_n] \\ \mathrm{q}_{k+1} & \dots & \mathrm{q}_n\end{matrix}}{\bar{\mathrm{q}}}$$

Doing this allows us to replace the above rule with the following:

$$\frac{\mathrm{q} \quad \bar{\mathrm{q}}}{\bot^{\flat}}$$

In this case, the inferential behaviour of q and $\bar{\mathrm{q}}$ is that they are contradictory propositions: together, they infer absurdity.

In this way, negation is implicit in atoms. What is significant from this analysis is that the semantics of $\bot$ requires us to observe that there is no proof of it and thus extend the space with proofs of contradictory q and $\bar{\mathrm{q}}$. If they are proved in $\mathscr{B}$, then one has proved absurdity; if $\mathscr{B}$ has proved absurdity, then one has proofs for each of these. The subtlety is that since we do not have negation *explicit* in our atoms, we only admit the principle that some atoms are contradictory. If we prove all atoms, then we prove these contradictory atoms; and, if we prove these contradictory atoms, then we have proved absurdity. This justifies the clause for $\bot$,

$$\Vdash_{\mathscr{B}} \bot \qquad \text{iff} \qquad \Vdash_{\mathscr{B}} \mathrm{p} \text{ for any } \mathrm{p} \in \mathbb{A}$$

Piecha and Schroeder-Heister [30, 21] have argued that there are two perspectives on atomic systems: the knowledge view and the definitional view. This becomes clear according to various ways in which a program may be regarded in LP. The negation-as-failure protocol makes use of the definitional perspective; its analogue in terms of knowledge is the *closed-world assumption*. In this case, a knowledge base treats everything that is not known to be valid as invalid. There is significant literature about the closed-world assumption that may be useful for understanding P-tS and what it tells us about reasoning—see, for example, Clark [3], Reiter [24], and Kowalski [14, 13], and Harland [11, 12].

5. Conclusion

Proof-theoretic semantics is the paradigm of meaning based on proof (as opposed to truth). Essential to this approach is the use of atomic systems, which give meaning to atomic propositions. Base-extension semantics is a particular instance of proof-theoretic semantics that proceeds by an inductively defined judgement whose base case is given by provability in an atomic system. It may be regarded as capturing the declarative content of proof-theoretic semantics in the Dummett-Prawitz tradition—see Gheorghiu and Pym [8]. Sandqvist [27] has given a base-extension semantics for intuitionistic propositional logic. Completeness follows by constructing a special bespoke base in which the validity of a complex proposition simulates a natural deduction proof of that formula.

In the base-extension semantics, the meaning of the logical constants is derived from the rules of NJ, while the atomic systems give the meaning of atomic propositions. These atomic systems, which include Sandqvist's special bases that delivers completeness, all sit within the hereditary Harrop fragment of IPL. The significance of this is that an effective operational reading of definite formulae renders them meaning-conferring in a sense analogous to the use of atomic systems. Moreover, this operational account coheres with the independently conceived notion of derivability in an atomic system. Of course, that atomic systems and programs are intimately related has been studied before—see Schroeder-Heister and Hallnäs [9, 10].

Significantly, the operational reading of the definite formulae allows from a simple proof-theoretic model-theoretic semantics that captures the idea of *unfolding* the inferential content of a set of definite clauses or an atomic system. In this paper, we have used the operational account of definite formulae to prove the completeness of intuitionistic propositional logic with respect to its base-extension semantics. The aforementioned special base is interpreted as a program so that completeness follows immediately from the existing completeness result of the model-theoretic semantics of the logic programming language. Doing this reveals the subtle meaning of negation in proof-theoretic semantics.

Historically, the negation of a formula is understood as the denial of the formula itself. This is indeed the case in the model-theoretic semantics of IPL—see Kripke [15]. Using the connection to logic programming in this paper, we see that in base-extension semantics, negation is defined by the failure for there to be a proof. Thus, denial is conceptionally prior to negation. In short, base-extension semantics consider the space of reductions, which is larger than the space of proofs, including failed searches. As illustrated above, the connection between logic programming and base-extension semantics is quite intuitive and useful. More specifically, the T operator delivering the semantics of logic programming corresponds to the application of a rule in a proof system; hence, the T^ω construction is fundamental to proof-theoretic semantics. Since logic programming has been studied for various logics (see, for example, the treatment of BI in Gheorghiu et al. [7]), this suggests the possibility for uniform approaches to setting up base-extension semantics for logics by studying their proof-search behaviours. In particular, work by Harland [11, 12] on handling negation in logic programming may be used to address the difficulties posed by the connective—see Kürbis [16].

It remains to investigate further the connection between proof-theoretic semantics and reductive logic, in general, and base-extension semantics and logic programming, in particular.

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