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Editorial Office: Department of Logic, University of Łódź
ul. Lindleya 3/5, 90–131 Łódź, Poland
e-mail: bulletin@uni.lodz.pl

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ROUGHNESS OF FILTERS IN EQUALITY ALGEBRAS

Abstract

Rough set theory is an excellent mathematical tool for the analysis of a vague description of actions in decision problems. Now, in this paper by considering the notion of an equality algebra, the notion of the lower and the upper approximations are introduced and some properties of them are given. Moreover, it is proved that the lower and the upper approximations define an interior operator and a closure operator, respectively. Also, using D-lower and D-upper approximation, conditions for a nonempty subset to be definable are provided and investigated that under which condition D-lower and D-upper approximation can be filter.

Keywords: equality algebra, approximation space, D-lower approximation, D-upper approximation, filter, D-lower filter, D-upper filter.

2020 Mathematical Subject Classification: 03G25, 06B10, 06B99.

1. Introduction

The rough sets theory introduced by Pawlak in [11] has often proved to be an excellent mathematical tool for the analysis of a vague description of objects called actions in decision problems. Many different problems

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can be addressed by rough sets theory. During the last few years some mathematicians studied about roughness theory in different fields of mathematics. For example an algebraic approach to rough sets has been given by Iwinski in [2]. Rough set theory is applied to semigroups and groups see [8, 9]. In 1994, Biswas and Nanda in [1] introduced and discussed the concept of rough groups and rough subgroups. Jun in [6] applied rough set theory to BCK-algebras. Recently, Rasouli in [12] introduced and studied the notion of roughness in MV-algebras. A new structure, called equality algebras, is introduced by Jenei in [4] and it is continued in [3, 5]. The study of equality algebras is motivated by EQ-algebras of Novák et al. in [10]. The equality algebra has two connectives, a meet operation and an equivalence, and a constant. Novák et al. in [10] introduced a closure operator in the class of equality algebras, and discussed relations between equality algebras and BCK-algebras.

Zebardast et al. in [13] have shown that there are relations among equality algebras and some of other logical algebras such as residuated lattice, MTL-algebra, BL-algebra, MV-algebra, Hertz-algebra, Heyting-algebra, Boolean-algebra, EQ-algebra and hoop-algebra. They found that under which conditions, equality algebras are equivalent to these logical algebras. Zebardast et al. in [13] also studied commutative equality algebras. They considered characterizations of commutative equality algebras.

In this paper we discuss the roughness of filter of an equality algebra. Using a filter D of an equality algebra E, we first define a congruence relation, so called a D-congruence relation, on E, and construct a D-lower and D-upper approximation and a D-approximation space. We investigate several properties of D-lower and D-upper approximation. We show that a D-lower (resp., D-upper) approximation is an interior (resp., closure) operator. In a D-approximation space, we define the notions of D-lower (resp. a D-upper) rough filter, and show that every filter containing D is both a D-lower and a D-upper rough filter. We provide a characterization of the definable subsets by using D-lower and D-upper approximation.

2. Preliminaries

In this section, we recollect some definitions and results which will be used in this paper.

Definition 2.1. [4] An algebraic structure $(E, \wedge, \sim, 1)$ is called an *equality algebra*, if for any $u, v, w \in E$ it satisfies in the following conditions.

(E1) $(E, \wedge, 1)$ is a commutative idempotent integral monoid,

(E2) the operation "$\sim$" is commutative,

(E3) $u \sim u = 1$,

(E4) $u \sim 1 = u$,

(E5) if $u \leq v \leq w$, then $u \sim w \leq v \sim w$ and $u \sim w \leq u \sim v$,

(E6) $u \sim v \leq (u \wedge w) \sim (v \wedge w)$,

(E7) $u \sim v \leq (u \sim w) \sim (v \sim w)$,

where $u \leq v$ if and only if $u \wedge v = u$.

In an equality algebra $(E, \wedge, \sim, 1)$, for any $u, v \in E$, we define an operation $\rightarrow$ (*implication*) on E by $u \rightarrow v := u \sim (u \wedge v)$.

PROPOSITION 2.2 ([4]). Let $(E, \wedge, \sim, 1)$ be an equality algebra. Then for any $u, v, w \in E$ the following assertions are valid.
(*i*) $u \rightarrow v = 1$ if and only if $u \leq v$,
(*ii*) $u \sim v = 1$ if and only if $u = v$,
(*iii*) $u \rightarrow (v \rightarrow w) = v \rightarrow (u \rightarrow w)$,
(*iv*) $1 \rightarrow u = u$, $u \rightarrow 1 = 1$ and $u \rightarrow u = 1$,
(*v*) $u \leq v \rightarrow w$ if and only if $v \leq u \rightarrow w$,
(*vi*) $u \leq v \rightarrow u$,
(*vii*) $u \leq (u \rightarrow v) \rightarrow v$,
(*viii*) $u \rightarrow v \leq (v \rightarrow w) \rightarrow (u \rightarrow w)$,
(*ix*) if $v \leq u$, then $u \leftrightarrow v = u \rightarrow v = u \sim v$,
(*x*) if $u \leq v$, then $v \rightarrow w \leq u \rightarrow w$ and $w \rightarrow u \leq w \rightarrow v$,
(*xi*) $((u \rightarrow v) \rightarrow v) \rightarrow v = u \rightarrow v$.

An equality algebra E is *bounded* if there exists an element $0 \in E$ such that $0 \leq u$, for all $u \in E$. In a bounded equality algebra E, we define the negation "$\,'\,$" on E by $u' = u \rightarrow 0 = u \sim 0$, for all $u \in E$.

A subset D of E is called a *deductive system* (or *filter*) of E if for any $u, v \in E$, it satisfies in the following statements:

$(F1)$ If $u \leq v$ such that $u \in D$, then $v \in D$,
$(F2)$ If $u \in D$ and $u \sim v \in D$, then $v \in D$.

Denote by $\mathcal{DS}(E)$ the set of all deductive systems of E (see [5]).

Lemma 2.3. [3] *Let* $(E, \sim, \wedge, 1)$ *be an equality algebra. A subset* D *of* E *is a deductive system of* E *if and only if* $1 \in D$ *and for any* $u, v \in E$ *if* $u \in D$ *and* $u \to v \in D$*, then* $v \in D$*.*

Definition 2.4. [13] An equality algebra $(E, \wedge, \sim, 1)$ is called *commutative*, if for any $u, v \in E$,

$$(u \to v) \to v = (v \to u) \to u.$$

Let ϱ be an equivalence relation on a set E and let $\mathcal{P}(E)$ denote the power set of E. For all $x \in E$, let $[x]_\varrho$ denote the equivalence class of x with respect to ϱ. Let ϱ_* and ϱ^* be mappings from $\mathcal{P}(E)$ to $\mathcal{P}(E)$ defined by

$$\varrho_* : \mathcal{P}(E) \to \mathcal{P}(E),\ D \mapsto \{x \in E \mid [x]_\varrho \subseteq D\}$$

and

$$\varrho^* : \mathcal{P}(E) \to \mathcal{P}(E),\ D \mapsto \{x \in E \mid [x]_\varrho \cap D \neq \emptyset\},$$

respectively. The pair (E, ϱ) is called an *approximation space* based on ϱ. A subset D of E is called *definable* if $\varrho_*(D) = \varrho^*(D)$, and *rough* otherwise. The set $\varrho_*(D)$ (resp., $\varrho^*(D)$) is called the *lower* (resp. *upper*) *approximation.*

Notation. In the following, we suppose $(E, \wedge, \sim, 1)$ is an equality algebra with the induced operation "$\to$" (or simply denoted by E) and D is a filter of E, unless otherwise stated.

3. Roughness of filters

In this section, we define the notion of the lower and the upper approximations on equality algebras and investigate some properties of them. Also, we show that the lower and the upper approximations form an interior operator and a closure operator, respectively.

Let $\cong_D$ be a relation on E which is defined by

$$x \cong_D y \text{ if and only if } x \sim y \in D.$$

By routine caculation, it is clear that $\cong_D$ is an equivalence relation on E related to D. Further, we know that $\cong_D$ satisfies the following condition:

if $u \cong_D v$ and $x \cong_D y$, then $(u \sim x) \cong_D (v \sim y)$ and $(u \wedge x) \cong_D (v \wedge y)$.

Thus $\cong_D$ is a congruence relation on E and we say $\cong_D$ is the *D-congruence relation* on E. Denote by E/D the collection of all equivalence classes, that is, $E/D = \{D[x] \mid x \in E\}$. Then $D[1] = D$. For any $D[x], D[y] \in E/D$, define two binary operations "$\sqcap$" and "$\approx$" on E/D as follows:

$$D[x] \sqcap D[y] = D[x \wedge y] \quad \text{and} \quad D[x] \approx D[y] = D[x \sim y].$$

It is routine to verify that $(E/D, \sqcap, \approx, D[1])$ is an equality algebra, and for any $D[x], D[y] \in E/D$, the implication "$\rightsquigarrow$" on E/D is given by,

$$D[x] \rightsquigarrow D[y] = D[x \to y].$$

For the D-congruence relation $\cong_D$ on E, consider the mappings

$$\underline{appr}_D : \mathcal{P}(E) \to \mathcal{P}(E),\ L \mapsto \{x \in E \mid D[x] \subseteq L\},$$
$$\overline{appr}_D : \mathcal{P}(E) \to \mathcal{P}(E),\ L \mapsto \{x \in E \mid D[x] \cap L \neq \emptyset\},$$

which are called the *D-lower approximation* and the *D-upper approximation* of L, respectively. Then $(E, \cong_D)$ is an approximation space based on the filter D of E (briefly, *D-approximation space*), and it is denoted by (E, D). A subset L of E is said to be *definable* with respect to D if $\underline{appr}_D(L) = \overline{appr}_D(L)$, and *rough* otherwise.

The next proposition is similar to the Proposition 3.3 in [7].

PROPOSITION 3.1. [7] Let (E, D) be a D-approximation space. For any $L, M \in \mathcal{P}(E)$, we have

(i) $\underline{appr}_D(L) \subseteq L \subseteq \overline{appr}_D(L)$,

(ii) $\underline{appr}_D(L \cap M) = \underline{appr}_D(L) \cap \underline{appr}_D(M)$,

(iii) $\underline{appr}_D(L) \cup \underline{appr}_D(M) \subseteq \underline{appr}_D(L \cup M)$,

(iv) $\overline{appr}_D(L \cap M) \subseteq \overline{appr}_D(L) \cap \overline{appr}_D(M)$,

(v) $\overline{appr}_D(L) \cup \overline{appr}_D(M) = \overline{appr}_D(L \cup M)$,

(vi) $\underline{appr}_D(\overline{appr}_D(L)) \subseteq \overline{appr}_D(\overline{appr}_D(L))$,

(vii) $\underline{appr}_D(\underline{appr}_D(L)) \subseteq \overline{appr}_D(\underline{appr}_D(L))$,

(viii) $\underline{appr}_D(L^c) = (\overline{appr}_D(L))^c$,

(ix) $\overline{appr}_D(L^c) = \left(\underline{appr}_D(L)\right)^c$,

(x) $\underline{appr}_D(L) = \emptyset$ for $L \neq E$,

(xi) $\overline{appr}_D(L) = L$ for $L \neq \emptyset$,

(xii) $\underline{appr}_D(L) = L$ if and only if $\overline{appr}_D(L^c) = L^c$.

Definition 3.2. Suppose S is a set. A function $C : \mathcal{P}(S) \to \mathcal{P}(S)$ is called a *closure operator* on a set S if for all subsets $X, Y \subseteq S$, the following conditions hold:

(C_1) $X \subseteq C(X)$,

(C_2) if $X \subseteq Y$, then $C(X) \subseteq C(Y)$,

(C_3) $C(C(X)) = C(X)$.

Definition 3.3. Suppose S is a set. A function $int : \mathcal{P}(S) \to \mathcal{P}(S)$ is said to be an *interior operator* on a set S if for all subsets $X, Y \subseteq S$, the following conditions hold:

(i) $int(X) \subseteq X$,

(ii) if $X \subseteq Y$, then $int(X) \subseteq int(Y)$,

(iii) $int(int(X)) = int(X)$.

Theorem 3.4. *Let (E, D) be a D-approximation space. Then $\underline{appr}_D$ and $\overline{appr}_D$ are an interior operator and a closure operator, respectively.*

Proof: The proof is clear. □

Proposition 3.5. Let (E, D) be a D-approximation space. Then $D[x]$ is definable with respective to D, for all $x \in E$.

Proof: By Proposition 3.1(i), it is clear that $\underline{appr}_D(D[x]) \subseteq \overline{appr}_D(D[x])$, for all $x \in E$. Let $y \in \overline{appr}_D(D[x])$. Then $D[y] \cap D[x] \neq \emptyset$, and so

$D[x] = D[y]$. Thus $y \in \underline{appr}_D(D[x])$. Therefore, $D[x]$ is definable with respective to D for all $x \in E$. $\square$

PROPOSITION 3.6. Let (E, D) be a D-approximation space with $D = \{1\}$. Then every subset of E is definable.

PROOF: The proof is straightforward. $\square$

COROLLARY 3.7. Every equality algebra is definable with respect to any filter.

PROOF: The proof is clear. $\square$

PROPOSITION 3.8. Let $\cong_D$ and $\cong_B$ be equivalence relations on E related to filters D and B respectively. If $D \subseteq B$, then $\cong_D \subseteq \cong_B$.

PROOF: Let $x, y \in E$ such that $x \cong_D y$. Then $x \sim y \in D \subseteq B$, which implies that $x \cong_B y$. Hence $\cong_D \subseteq \cong_B$. $\square$

For any subsets D and B of E, we define

$$D \wedge B = \{u \wedge v \mid u \in D, v \in B\}, \quad D \sim B = \{u \sim v \mid u \in D, v \in B\},$$

$$\text{and } D \to B = \{u \to v \mid u \in D, v \in B\}.$$

If either D or B is empty, then we define $D \wedge B = \emptyset$, $D \sim B = \emptyset$ and $D \to B = \emptyset$. It is clear that $D \to B = (D \wedge B) \sim D$.

PROPOSITION 3.9. Let (E, D) be a D-approximation space. Given a D-congruence relation $\cong_D$ on E, if $L, M \in \mathcal{P}(E)$, then

(i) $\overline{appr}_D(L) \to \overline{appr}_D(M) \subseteq \overline{appr}_D(L \to M)$,

(ii) $\overline{appr}_D(L) \wedge \overline{appr}_D(M) \subseteq \overline{appr}_D(L \wedge M)$,

(iii) $\overline{appr}_D(L) \sim \overline{appr}_D(M) \subseteq \overline{appr}_D(L \sim M)$.

PROOF: (i) Let $w \in \overline{appr}_D(L) \to \overline{appr}_D(M)$. Then $w = u \to v$ for some $u \in \overline{appr}_D(L)$ and $v \in \overline{appr}_D(M)$, and so $D[u] \cap L \neq \emptyset$ and $D[v] \cap M \neq \emptyset$. It follows that there are $x, y \in E$ such that $x \in D[u] \cap L$ and $y \in D[v] \cap M$. Since $\cong_D$ is a D-congruence relation on E, we have

$$x \to y \in D[u] \to D[v] = D[u \to v] = D[w].$$

Since $x \to y \in L \to M$, it follows that $x \to y \in D[w] \cap (L \to M)$, and so $w \in \overline{appr}_D(L \to M)$. Hence, (i) is valid.

(ii) Let $w \in \overline{appr}_D(L) \wedge \overline{appr}_D(M)$. Then $w = u \wedge v$ for some $u \in \overline{appr}_D(L)$ and $v \in \overline{appr}_D(M)$. Since $u \in \overline{appr}_D(L)$ and $v \in \overline{appr}_D(M)$, there exist $x \in D[u] \cap L$ and $y \in D[v] \cap M$. It follows that $x \cong_D u$ and $y \cong_D v$. Since $\cong_D$ is a congruence relation on E, we have $x \wedge y \cong_D u \wedge v = w$. Then $x \wedge y \in D[u \wedge v] = D[w]$ and $x \wedge y \in L \wedge M$. Hence $x \wedge y \in D[w] \cap (L \wedge M)$, that is, $D[w] \cap (L \wedge M) \neq \emptyset$, and so $w \in \overline{appr}_D(L \wedge M)$. Therefore

$$\overline{appr}_D(L) \wedge \overline{appr}_D(M) \subseteq \overline{appr}_D(L \wedge M).$$

(iii) The proof is similar to the proof of (ii). □

Proposition 3.10. For a D-approximation space (E, D) and any $L, M \in \mathcal{P}(E)$, we have
(i) $\underline{appr}_D(L) \to \underline{appr}_D(M) \subseteq \underline{appr}_D(L \to M)$.
(ii) $\underline{appr}_D(L) \wedge \underline{appr}_D(M) \subseteq \underline{appr}_D(L \wedge M)$.
(iii) $\underline{appr}_D(L) \sim \underline{appr}_D(M) \subseteq \underline{appr}_D(L \sim M)$.

Proof: (i) Let $w \in \underline{appr}_D(L) \to \underline{appr}_D(M)$. Then $w = u \to v$ for some $u \in \underline{appr}_D(L)$ and $v \in \underline{appr}_D(M)$. Hence $D[u] \subseteq L$ and $D[v] \subseteq M$. It follows that

$$D[u \to v] = D[u] \to D[v] \subseteq L \to M.$$

Then $w = u \to v \in \underline{appr}_D(L \to M)$.

(ii) If $x \in \underline{appr}_D(L) \wedge \underline{appr}_D(M)$, then there exist $u \in \underline{appr}_D(L)$ and $v \in \underline{appr}_D(M)$ such that $x = u \wedge v$, $D[u] \subseteq L$ and $D[v] \subseteq M$. It follows that

$$D[x] = D[u \wedge v] = D[u] \wedge D[v] \subseteq L \wedge M.$$

Hence $x \in \underline{appr}_D(L \wedge M)$, and therefore

$$\underline{appr}_D(L) \wedge \underline{appr}_D(M) \subseteq \underline{appr}_D(L \wedge M).$$

(iii) Let $x \in \underline{appr}_D(L) \sim \underline{appr}_D(M)$. Then $x = u \sim v$ for some $u \in \underline{appr}_D(L)$ and $v \in \underline{appr}_D(M)$. Thus $D[u] \subseteq L$ and $D[v] \subseteq M$, which imply that

$$D[x] = D[u \sim v] = D[u] \sim D[v] \subseteq L \sim M.$$

Hence $x \in \underline{appr}_D(L \sim M)$. □

Proposition 3.11. Let (E, D) be a D-approximation space and $L, M \in \mathcal{P}(E)$. If $\underline{appr}_D(L \sim M) = \emptyset$ (resp., $\underline{appr}_D(L \wedge M) = \emptyset$ and $\underline{appr}_D(L \to M) = \emptyset$), then $\underline{appr}_D(L) = \emptyset$ or $\underline{appr}_D(M) = \emptyset$.

Proof: Let $L, M \in \mathcal{P}(E)$ such that $\underline{appr}_D(L) \neq \emptyset$ and $\underline{appr}_D(M) \neq \emptyset$. Then there exist $u \in \underline{appr}_D(L)$ and $v \in \underline{appr}_D(M)$, such that $D[u] \subseteq L$ and $D[v] \subseteq M$. Since $u \in D[u]$ and $v \in D[v]$, we have $u \in L$ and $v \in M$. Then $u \sim v \in L \sim M$, and so

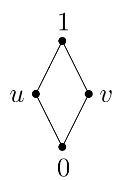
$$u \sim v \in D[u \sim v] = D[u] \sim D[v] \subseteq L \sim M.$$

Hence $\underline{appr}_D(L \sim M) \neq \emptyset$, which is a contradiction. Therefore, $\underline{appr}_D(L) = \emptyset$ or $\underline{appr}_D(M) = \emptyset$.

The proof of other cases is similar. □

Definition 3.12. Let (E, D) be a D-approximation space. A subset L of E is called a D-*lower* (resp. a D-*upper*) *rough filter* of E if $\underline{appr}_D(L)$ (resp., $\overline{appr}_D(L)$) is a filter of E. If L is both a D-lower and a D-upper filters of E, then L is called a D-*rough filter* of E.

Example 3.13. Let $E = \{0, u, v, 1\}$ be a set with the following Hasse diagram.



Then $(E, \wedge, 1)$ is a commutative idempotent integral monoid. We define a binary operation "$\sim$" on E by the following table.

$\sim$	0	u	v	1
0	1	v	u	0
u	v	1	0	u
v	u	0	1	v
1	0	u	v	1

Then $\mathcal{E} = (E, \wedge, \sim, 1)$ is an equality algebra, and the implication "$\rightarrow$" is given by the following Cayley table.

$\rightarrow$	0	u	v	1
0	1	1	1	1
u	v	1	v	1
v	u	u	1	1
1	0	u	v	1

Consider a D-approximation space (E, D) where $D = \{u, 1\}$ is a filter of E. Then $D[u] = D[1] = \{u, 1\}$ and $D[v] = D[0] = \{v, 0\}$. For a subset $L = \{0, u, 1\}$ of E, we have

$$\underline{appr}_D(L) = \{x \in E \mid D[x] \subseteq \{0, u, 1\}\} = \{u, 1\},$$

and

$$\overline{appr}_D(L) = \{x \in E \mid D[x] \cap \{0, u, 1\} \neq \emptyset\} = \{0, u, v, 1\},$$

are filters of E. Hence D is a D-rough filter of E. If we take a subset $M = \{v\}$ of E, then $\underline{appr}_D(M) = \emptyset$ and $\overline{appr}_D(M) = \{0, v\}$ are not filters of E. Hence D is not a D-rough filter of E. Also, if we take a subset $K = \{u, 1\}$ of E, then $\underline{appr}_D(K) = \emptyset$ that is not a filter of E and $\overline{appr}_D(K) = \{u, 1\}$ is a filter of E. Hence D is a D-upper rough filter of E.

The extension theorem of D-upper rough filter of E is obtained from the following theorem.

Theorem 3.14. *Let (E, D) be a D-approximation space. Then every filter L of E which contains D is a D-upper rough filter of E.*

Proof: Let L be a filter of E such that $D \subseteq L$. Then $D[1] \cap L \neq \emptyset$, and so $1 \in \overline{appr}_D(L)$. Suppose $x, y \in E$ such that $x \in \overline{appr}_D(L)$ and $x \sim y \in \overline{appr}_D(L)$. Then $D[x] \cap L \neq \emptyset$ and $D[x \sim y] \cap L \neq \emptyset$, which imply that there exist $u, v \in L$ such that $u \in D[x]$ and $v \in D[x \sim y]$. Hence $u \cong_D x$ and $v \cong_D (x \sim y)$. It follows that $u \sim x \in D \subseteq L$ and $v \sim (x \sim y) \in D \subseteq L$. Since $u, v \in L$ and L is a filter of E, we have $x \in L$ and $x \sim y \in L$, and so $y \in L$. Note that $y \in D[y]$, and so $y \in D[y] \cap L$. Hence $y \in \overline{appr}_D(L)$, and therefore $\overline{appr}_D(L)$ is a filter of E, that is, L is a D-upper rough filter of E. $\square$

Corollary 3.15. Let (E, D) be a D-approximation space with $D = \{1\}$. Then every filter L of E is a D-upper rough filter of E.

In the following example we show that the converse of Theorem 3.14 is not true, in general.

Example 3.16. Let E be the equality algebra as in Example 3.13 and (E, D) be a D-approximation space of E. Suppose $D = \{u, 1\}$ is a filter of E and $\cong_D$ is an equivalence relation on E related to D. Then $D[0] = \{0, v\} = D[v]$ and $D[u] = D = D[1]$. Let $L = \{v, 1\}$ be a subset of E. Then L does not contain D and

$$\overline{appr}_D(L) = \{x \in E \mid D[x] \cap L \neq \emptyset\} = E.$$

Thus L is a D-upper rough filter of E.

THEOREM 3.17. *Let (E, D) be a D-approximation space. Then every filter L of E which contains D is a D-lower rough filter of E.*

PROOF: Let L be a filter of E such that $D \subseteq L$. Since $D = D[1]$, if $x \in D[1]$, then $x \in D \subseteq L$, and so $D[1] \subseteq L$. Hence $1 \in \underline{appr}_D(L)$. Let $x, y \in E$ such that $x \in \underline{appr}_D(L)$ and $x \sim y \in \underline{appr}_D(L)$. Then $D[x] \subseteq L$ and $D[x] \sim D[y] = D[x \sim y] \subseteq L$. Let $u \in D[x]$ and $v \in D[y]$. Then $u \cong_D x$ and $v \cong_D y$, which imply that $(u \sim v) \cong_D (x \sim y)$, that is, $u \sim v \in D[x \sim y] \subseteq L$. Since $u \in L$ and L is a filter of E, we get $v \in L$ and $D[y] \subseteq L$. Thus $y \in \underline{appr}_D(L)$, and therefore $\underline{appr}_D(L)$ is a filter of E. Consequently, L is a D-lower rough filter of E. □

COROLLARY 3.18. Let (E, D) be a D-approximation space such that $D = \{1\}$. Then every filter L of E is a D-lower rough filter of E.

PROPOSITION 3.19. Let (E, D) be a D-approximation space. For any subset L of E, we have

(i) $D \subseteq L$ if and only if $D \subseteq \underline{appr}_D(L)$.

(ii) $L \subseteq D$ if and only if $\overline{appr}_D(L) = D$.

PROOF: (i) Assume that $D \subseteq L$. If $x \in D$, then $D[x] = D \subseteq L$. Hence $x \in \underline{appr}_D(L)$, and so $D \subseteq \underline{appr}_D(L)$. By Proposition 3.1(i), the proof of converse is clear.

(ii) Suppose $L \subseteq D$ and $x \in \overline{appr}_D(L)$. Then $D[x] \cap L \neq \emptyset$, and thus there exists $y \in D[x] \cap L$ which implies that $D[x] = D[y]$ and $y \in L$. Hence $D[y] = D$, and so $x \in D$. This shows that $\overline{appr}_D(L) \subseteq D$. Let $z \in D$.

Then $D[z] = D$ and so $D[z] \cap L = D \cap L \neq \emptyset$. Thus $z \in \overline{appr}_D(L)$, that is, $D \subseteq \overline{appr}_D(L)$. By Proposition 3.1(i), the proof of converse is clear. □

Corollary 3.20. Let (E, D) be a D-approximation space. If L is a filter of E such that $L \subseteq D$, then L is a D-upper rough filter of E.

Theorem 3.21. *If L is a filter in a D-approximation space (E, D), then*

(i) $D \subseteq \overline{appr}_D(L)$.

(ii) $D \subseteq L$ *if and only if* $\underline{appr}_D(L) \subseteq L = \overline{appr}_D(L)$.

Proof: (i) Let $x \in D$. Since $x \in D[x]$, it is clear that $1 \in D[x]$. Moreover, since L is a filter in a D-approximation space (E, D), we have $1 \in L$ and so $1 \in D[x] \cap L$. Hence $x \in \overline{appr}_D(L)$, and therefore $D \subseteq \overline{appr}_D(L)$.

(ii) Assume that $D \subseteq L$. Then by Proposition 3.1(i), $\underline{appr}_D(L) \subseteq L \subseteq \overline{appr}_D(L)$. Let $x \in \overline{appr}_D(L)$. Then $D[x] \cap L \neq \emptyset$ and thus there exists $u \in L$ such that $u \in D[x]$. Since $D \subseteq L$, it follows that $u \sim x \in D \subseteq L$. Hence $x \in L$ and so $\overline{appr}_D(L) \subseteq L$.

Conversely, suppose $\underline{appr}_D(L) \subseteq L = \overline{appr}_D(L)$ and $x \in D$. Since D and L are filters, we get $1 \in D \cap L = D[x] \cap L$. Hence $x \in \overline{appr}_D(L) = L$. Therefore $D \subseteq L$. □

Corollary 3.22. If L is a filter of a D-approximation space (E, D), then

$$\underline{appr}_D(L) = L = \overline{appr}_D(L),$$

and L is a D-rough filter of E.

For any nonempty subset L of E, we let $L' = \{x' \mid x \in L\}$. It is clear that if L and M are nonempty subsets of E, then $L \subseteq M$ staisfies $L' \subseteq M'$.

Proposition 3.23. In a D-approximation space (E, D), for any $L \in \mathcal{P}(E) \setminus \{\emptyset\}$, we have $(\overline{appr}_D(L))' \subseteq \overline{appr}_D(L')$.

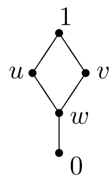
Proof: Let $u \in (\overline{appr}_D(L))'$ for any nonempty subset L of E. Then $u = x'$ for some $x \in \overline{appr}_D(L)$ and so $D[x] \cap L \neq \emptyset$. It follows that there exists $v \in L$ such that $v \in D[x]$, which implies that $v' \in L'$ and $v \sim x \in D$. By (E2) and (E7) we have

$$v \sim x = x \sim v \leq (x \sim 0) \sim (v \sim 0) = x' \sim v'.$$

Since D is a filter of E and $u = x'$, it follows that $u \sim v' = x' \sim v' \in D$. Hence $v' \in D[u] \cap L'$, that is, $D[u] \cap L' \neq \emptyset$. Therefore $u \in \overline{appr}_D(L')$ which shows that $(\overline{appr}_D(L))' \subseteq \overline{appr}_D(L')$. □

The next example shows that the converse of Proposition 3.23 is not true in general.

Example 3.24. Let $E = \{0, u, v, w, 1\}$ be a set with the following Hasse diagram.



Then $(E, \wedge, 1)$ is a commutative idempotent integral monoid. We define a binary operation "$\sim$" on E by Table 1.

Table 1. Table of the implication "$\sim$"

$\sim$	0	u	v	w	1
0	1	0	0	0	0
u	0	1	w	v	u
v	0	w	1	u	v
w	0	v	u	1	w
1	0	u	v	w	1

Then $\mathcal{E} = (E, \wedge, \sim, 1)$ is an equality algebra, and the implication "$\rightarrow$" is given by Table 2.

Table 2. Table of the implication "$\rightarrow$"

$\rightarrow$	0	u	v	w	1
0	1	1	1	1	1
u	0	1	v	v	1
v	0	u	1	u	1
w	0	1	1	1	1
1	0	u	v	w	1

Let $D = \{u, 1\}$. It is clear that D is a filter of E. Let $\cong_D$ be an equivalence relation on E related to D. Then $D[1] = D[u] = \{u, 1\}$, $D[w] = D[v] = \{v, w\}$ and $D[0] = \{0\}$. If $L = \{0, u\}$, then $L' = \{0, 1\}$. Thus

$$\overline{appr}_D(L') = \{0, u, 1\} \quad , \quad \overline{appr}_D(L) = \{0, u, 1\}.$$

But $(\overline{appr}_D(L))' = (\{0, u, 1\})' = \{0, 1\}$. Hence $\overline{appr}_D(L') \nsubseteq (\overline{appr}_D(L))'$.

In the following example, we show that there exists a nonempty subset L of E such that $\underline{appr}_D(L') \nsubseteq (\underline{appr}_D(L))'$.

Example 3.25. Let (E, D) be a D-approximation space where E be the equality algebra as in Example 3.13 and $D = \{u, 1\}$ be a filter of E. If $L = \{u, 0\}$, then $L' = \{0, 1\}$. Thus $\underline{appr}_D(L') = \{0\}$ and $\underline{appr}_D(L) = \{0\}$, and so $(\underline{appr}_D(L))' = \{1\}$. Hence $\underline{appr}_D(L') \nsubseteq (\underline{appr}_D(L))'$.

Proposition 3.26. Let (E, D) be a D-approximation space and L be a nonempty subset of E. Then
(i) $\mathcal{R}(E) \cap \overline{appr}_D(L') \subseteq (\overline{appr}_D(L''))'$,
(ii) $\mathcal{R}(E) \cap \overline{appr}_D((L \cap \mathcal{R}(E))') \subseteq (\overline{appr}_D(L))'$,
where $\mathcal{R}(E) := \{x \in E \mid x'' = x\}$.

Proof: (i) Let $z \in \mathcal{R}(E) \cap \overline{appr}_D(L')$. Then $z'' = z$ and $D[z] \cap L' \neq \emptyset$, which imply that there exists $x \in L$ such that $D[x'] = D[z]$. Hence

$$D[z'] \cap L'' = D[x''] \cap L'' \neq \emptyset,$$

i.e., $z' \in \overline{appr}_D(L'')$. Therefore $z \in (\overline{appr}_D(L''))'$.
(ii) Let $u \in \mathcal{R}(E) \cap \overline{appr}_D((L \cap \mathcal{R}(E))')$. Then $u'' = u$ and $D[u] \cap (L \cap \mathcal{R}(E))' \neq \emptyset$. It follows that there exists $x \in L \cap \mathcal{R}(E)$ such that $D[u] = D[x']$ and $x'' = x$. Hence

$$D[u'] \cap L = D[x''] \cap L = D[x] \cap L \neq \emptyset,$$

and so $u' \in \overline{appr}_D(L)$, i.e., $u \in (\overline{appr}_D(L))'$. Therefore

$$\mathcal{R}(E) \cap \overline{appr}_D((L \cap \mathcal{R}(E))') \subseteq (\overline{appr}_D(L))'. \qquad \square$$

Lemma 3.27. *If E is a bounded equality algebra, then the set*

$$\mathcal{E}(E) := \{x \in E \mid x' = 0\},$$

is a filter of E.

PROOF: Obviously $1 \in \mathcal{E}(E)$. Let $x, y \in E$ such that $x \in \mathcal{E}(E)$ and $x \to y \in \mathcal{E}(E)$. Then $x' = 0$ and $(x \to y)' = 0$. Since $y \leq y''$, by Proposition 2.2(x), we get $x \to y \leq x \to y'' = y' \to x'$. Hence

$$y' = y''' = (y' \to 0)' = (y' \to x')' \leq (x \to y)' = 0$$

and so $y' = 0$, that is, $y \in \mathcal{E}(E)$. Therefore $\mathcal{E}(E)$ is a filter of E. $\square$

PROPOSITION 3.28. Let (E, D) be a D-approximation space and L be a nonempty subset of E. Then

$$D \subseteq \overline{appr}_D(\mathcal{E}(E)) \subseteq \{y \in E \mid y'' \in D\}. \tag{3.1}$$

PROOF: Using Lemma 3.27 and Theorem 3.21(i), we get $D \subseteq \overline{appr}_D(\mathcal{E}(E))$. Let $x \in \overline{appr}_D(\mathcal{E}(E))$. Then $D[x] \cap \mathcal{E}(E) \neq \emptyset$ and so there exists $u \in D[x]$ such that $u' = 0$. Thus $u \sim x \in D$. By (E2) and (E7), $u \sim x \leq (x \sim 0) \sim (u \sim 0) = x' \sim u'$ and D is a filter of E, we have $x' \sim u' \in D$. Thus by (E2), $x'' = 0 \sim x' = u' \sim x' \in D$. Therefore $\overline{appr}_D(\mathcal{E}(E)) \subseteq \{y \in E \mid y'' \in D\}$. $\square$

We provide conditions for a nonempty subset to be definable.

THEOREM 3.29. *Let (E, D) be a D-approximation space. Then a nonempty subset L of E is definable with respect to D if and only if $\underline{appr}_D(L) = L$ or $\overline{appr}_D(L) = L$.*

PROOF: Assume that L is definable with respect to D. Then $L \subseteq \overline{appr}_D(L) = \underline{appr}_D(L) \subseteq L$ and so

$$\overline{appr}_D(L) = \underline{appr}_D(L) = L.$$

Conversely, suppose that $\underline{appr}_D(L) = L$ or $\overline{appr}_D(L) = L$. For the case $\underline{appr}_D(L) = L$, let $x \in \overline{appr}_D(L)$. Then $D[x] \cap L \neq \emptyset$ which implies that $D[x] = D[z]$ for some $z \in L$. It follows from $\underline{appr}_D(L) = L$ that $D[x] = D[z] \subseteq L$. Hence $x \in L$, and therefore $\overline{appr}_D(L) \subseteq L$. Consequently, $\overline{appr}_D(L) = L$. Suppose that $\overline{appr}_D(L) = L$. For any $x \in L$ let $z \in D[x]$. Then $D[z] \cap L = D[x] \cap L \neq \emptyset$ and so $z \in \overline{appr}_D(L) = L$. This shows that $D[x] \subseteq L$, that is, $x \in \underline{appr}_D(L)$. Hence $L \subseteq \underline{appr}_D(L)$, and so $\underline{appr}_D(L) = L$. Therefore L is definable with respect to D. $\square$

4. Conclusions and future works

In this paper the notion of the lower and the upper approximations are introduced on equality algebras and some properties of them are investigated. Moreover, the relation among the lower and the upper approximations with an interior operator and a closure operator are investigated. Also, the conditions for a nonempty subset to be definable are provided. Also, due to the importance of this subject in the field of decision making, we decided to introduce these concepts on equality algebras in order to introduce concepts related to rough soft and soft rough equality algebras and fuzzification of them in the future. Moreover, in the future further study is possible in the direction of roughness with different types of filters and ideals in equality algebras.

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References

[1] R. Biswas, S. Nanda, *Rough groups and rough subgroups*, **Bulletin of the Polish Academy of Sciences Mathematics**, vol. 42(3) (1994), pp. 251–254, DOI: https://doi.org/10.1007/11548669_1.

[2] T. B. Iwiński, *Algebraic approach to rough sets*, **Bulletin of the Polish Academy of Sciences**, vol. 35 (1987), pp. 673–683, DOI: https://doi.org/10.1007/11548669_14.

[3] S. Jenei, *Equality algebras*, **Studia Logica**, vol. 56(2) (2010), pp. 183–186, DOI: https://doi.org/10.1109/CINTI.2010.5672249.

[4] S. Jenei, *Equality algebras*, **Studing Logics**, vol. 100 (2012), pp. 1201–1209.

[5] S. Jenei, Kóródi, *On the variety of equality algebras*, **Fuzzy Logic and Technology**, (2011), pp. 153–155, DOI: https://doi.org/10.2991/eusflat.2011.1.

[6] Y. B. Jun, *Roughness of ideals in BCK-algebras*, **Scientiae Mathematicae Japonicae**, vol. 7 (2002), pp. 115–119.

[7] Y. B. Jun, K. H. Kim, *Rough set theory applied to BCC-algebras*, **International Mathematical Forum**, vol. 2(41–44) (2007), pp. 2023–2029, DOI: https://doi.org/10.12988/imf.2007.07182.

[8] N. Kuroki, *Rough ideals in semigroups*, **Information Sciences**, vol. 100 (1997), pp. 139–163, DOI: https://doi.org/10.1016/S0020-0255(96)00274-5.

[9] N. Kuroki, J. N. Mordeson, *Structure of rough sets and rough groups*, **Journal of Fuzzy Mathematics**, vol. 5 (1997), pp. 183–191.

[10] V. Novák, B. D. Baets, *EQ-algebras*, **Fuzzy Sets and Systems**, vol. 160(20) (2009), pp. 2956–2978, DOI: https://doi.org/10.1016/j.fss.2009.04.010.

[11] Z. Pawlak, *Rough sets*, **International Journal of Computer and Information Sciences**, vol. 11(5) (1982), pp. 341–356, DOI: https://doi.org/10.1007/BF01001956.

[12] S. Rasouli, B. Davvaz, *Roughness in MV-algebras*, **Information Sciences**, vol. 180(5) (2010), pp. 737–747, DOI: https://doi.org/10.1016/j.ins.2009.11.008.

[13] F. Zebardast, R. A. Borzooei, M. A. Kologani, *Results on equality algebras*, **Information Sciences**, vol. 381 (2017), pp. 270–282, DOI: https://doi.org/10.1016/j.ins.2016.11.027.

Gholam Reza Rezaei
University of Sistan and Baluchestan
Department of Mathematics
98167-45845, Daneshjoo Boulevard
Zahedan, Iran
e-mail: grezaei@math.usb.ac.ir

Rajab Ali Borzooei
Shahid Beheshti University
Faculty of Mathematical Sciences, Department of Mathematics
1983969411, Daneshjoo Boulevard
Tehran, Iran
e-mail: borzooei@sbu.ac.ir

Mona Aaly Kologhani
Shahid Beheshti University
Faculty of Mathematical Sciences, Department of Mathematics
1983969411, Daneshjoo Boulevard
Tehran, Iran
e-mail: mona4011@gmail.com

Young Bae Jun
Gyeongsang National University
Department of Mathematics Education
Jinju 52828
Korea
e-mail: skywine@gmail.com

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Beza Lamesgin Derseh
Berhanu Assaye Alaba
Yohannes Gedamu Wondifraw

ON HOMOMORPHISM AND CARTESIAN PRODUCTS OF INTUITIONISTIC FUZZY PMS-SUBALGEBRA OF A PMS-ALGEBRA

Abstract

In this paper, we introduce the notion of intuitionistic fuzzy PMS-subalgebras under homomorphism and Cartesian product and investigate several properties. We study the homomorphic image and inverse image of the intuitionistic fuzzy PMS-subalgebras of a PMS-algebra, which are also intuitionistic fuzzy PMS-subalgebras of a PMS-algebra, and find some other interesting results. Furthermore, we also prove that the Cartesian product of intuitionistic fuzzy PMS-subalgebras is again an intuitionistic fuzzy PMS-subalgebra and characterize it in terms of its level sets. Finally, we consider the strongest intuitionistic fuzzy PMS-relations on an intuitionistic fuzzy set in a PMS-algebra and demonstrate that an intuitionistic fuzzy PMS-relation on an intuitionistic fuzzy set in a PMS-algebra is an intuitionistic fuzzy PMS-subalgebra if and only if the corresponding intuitionistic fuzzy set in a PMS-algebra is an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra.

Keywords: PMS-algebra, intuitionistic fuzzy PMS-subalgebra, homomorphism, cartesian product and strongest intuitionistic fuzzy relation.

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1. Introduction

In 1965, Zadeh [12] introduced the fundamental concept of a fuzzy set as an extension of the classical set theory for representing uncertainties in a physical world. Following the introduction of a fuzzy set, several researchers undertook a large number of studies on the extension of a fuzzy set. Atanassov [2, 3] investigated an intuitionistic fuzzy set as an extension of a fuzzy set to deal with uncertainties more efficiently in the actual situation. In 2007, Panigrahi and Nanda [5] introduced the idea of an intuitionistic fuzzy relation between any two intuitionistic fuzzy subsets defined in the given universal sets. In 2011, Anitha and Arjunan [1] studied the strongest intuitionistic fuzzy relations on intuitionistic fuzzy ideals of Hemirings and obtained some interesting results. In 2016, Sithar Selvam and Nagalakshmi [8] introduced a new class of algebra called PMS-algebra. Sithar Selvam and Nagalakshmi [7] fuzzified PMS-subalgebras and PMS ideals in PMS-algebra. In the same year, Sithar Selvam and Nagalakshmi [9] also introduced the concept of homomorphism and Cartesian product of fuzzy PMS-algebra and set up some properties. In our earlier paper [4], we introduced the notion of fuzzy PMS-subalgebra in PMS-algebra and studied some of its properties.
In this paper, we discuss the notion of intuitionistic fuzzy PMS-subalgebras under homomorphism and Cartesian product and investigate several properties. Furthermore, we investigate the homomorphic image and the inverse image of the intuitionistic fuzzy PMS-subalgebras of a PMS-algebra and find some results. Finally, we consider the strongest intuitionistic fuzzy PMS-relations on an intuitionistic fuzzy set in a PMS-algebra and demonstrate that an intuitionistic fuzzy PMS-relation on an intuitionistic fuzzy set in a PMS-algebra is an intuitionistic fuzzy PMS-subalgebra if and only if the corresponding intuitionistic fuzzy set in a PMS-algebra is an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra.

2. Preliminaries

In this section, we recall some basic definitions and results that are used in the study of this paper.

Definition 2.1 ([8]). A nonempty set X with a constant 0 and a binary operation '$*$' is called a PMS-algebra if it satisfies the following axioms.

1. $0 * x = x$
2. $(y * x) * (z * x) = z * y$, for all $x, y, z \in X$.

For $x, y \in X$, we define a binary relation $\leq$ by $x \leq y$ if and only if $x * y = 0$.

Definition 2.2 ([8]). Let S be a nonempty subset of a PMS-algebra X. Then S is called a PMS-subalgebra of X if $x * y \in S$, for all $x, y \in S$.

Definition 2.3 ([7, 9]). Let X and Y be any two PMS- algebras. Then a mapping $f : X \to Y$ is said to be a homomorphism of PMS-algebras if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$. f is called an epimorphism if it is onto and endomorphism if f is a mapping from a PMS-algebra X to itself.

Note: If f is a homomorphism of PMS-algebra, then $f(0) = 0$.

Definition 2.4 ([12]). Let X be a nonempty set. A fuzzy set A in X is characterized by a membership function $\mu_A : X \to [0, 1]$, where $\mu_A(x)$ represents the degree of membership of x in X.

Definition 2.5 ([7]). A fuzzy set A in a PMS-algebra X is called fuzzy PMS-subalgebra of X if $\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\}$, for all $x, y \in X$.

Definition 2.6 ([2, 3]). An intuitionistic fuzzy subset A in a nonempty set X is an object having the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}$, where the functions $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ define the degree of membership and the degree of nonmembership respectively and satisfying the condition $0 \leq \mu_A(x) + \nu_A(x) \leq 1$, for all $x \in X$.

Remark 2.7. Ordinary fuzzy sets over X may be viewed as special intuitionistic fuzzy sets with the nonmembership function $\nu_A(x) = 1 - \mu_A(x)$. So each Ordinary fuzzy set may be written as $\{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle | x \in X\}$ to define an intuitionistic fuzzy set. For the sake of simplicity we write $A = (\mu_A, \nu_A)$ for an intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}$.

Definition 2.8 ([2, 3]). Let A and B be intuitionistic fuzzy subsets of X, where $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle | x \in X\}$, then

1. $A \cap B = \{\langle x, min\{\mu_A(x), \mu_B(x)\}, \max\{\nu_A(x), \nu_B(x)\}\rangle | x \in X\}$
2. $\Box A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle | x \in X\}$
3. $\Diamond A = \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle | x \in X\}$

Definition 2.9 ([4]). An intuitionistic fuzzy subset $A = (\mu_A, \nu_A)$ of a PMS -algebra X is called an intuitionistic fuzzy PMS-subalgebra of X if $\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\}$ and $\nu_A(x * y) \leq \max\{\nu_A(x), \nu_A(y)\}, \forall x, y \in X$

Definition 2.10 ([10]). Let X and Y be any two nonempty sets and $f : X \to Y$ be a mapping. If $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ are intuitionistic fuzzy subsets of X and Y respectively. Then the image of A under f is defined as $f(A) = \{\langle y, \mu_{f(A)}(y), \nu_{f(A)}(y)\rangle \,|y \in Y\}$, where

$$\mu_{f(A)}(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu_A(x) & \text{if} \quad f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

and

$$\nu_{f(A)}(y) = \begin{cases} \inf\limits_{x \in f^{-1}(y)} \nu_A(x) & \text{if} \quad f^{-1}(y) \neq \emptyset \\ 1 & \text{otherwise} \end{cases}$$

The inverse image of B under f is denoted by $f^{-1}(B)$ and is defined as

$$f^{-1}(B)(x) = \left\{\langle x, \mu_{f^{-1}(B)}(x), \nu_{f^{-1}(B)}(x)\rangle \,|x \in X\right\},$$

where $\mu_{f^{-1}(B)}(x) = \mu_B(f(x))$ and $\nu_{f^{-1}(B)}(x) = \nu_B(f(x))$ for all $x \in X$.

Definition 2.11 ([10]). An intuitionistic fuzzy subset A in a nonempty set X with the degree of membership $\mu_A : X \to [0, 1]$ and the degree of non membership $\nu_A : X \to [0, 1]$ is said to have sup-inf property, if for any subset $T \subseteq X$ there exists $x_0 \in T$ such that $\mu_A(x_0) = \sup\limits_{t \in T} \mu_A(t)$ and $\nu_A(x_0) = \inf\limits_{t \in T} \mu_A(t)$

Definition 2.12. [5, 11] Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be any two intuitionistic fuzzy subsets of X and Y respectively. Then the Cartesian product of A and B is defined as

$$A \times B = \{\langle (x, y), \mu_{A \times B}(x, y), \nu_{A \times B}(x, y)\rangle \,|x \in X, y \in Y\},$$

where $\mu_{A \times B}(x, y) = \min\{\mu_A(x), \mu_B(y)\}$ and $\nu_{A \times B}(x, y) = \max\{(\nu_A(x), \nu_B(x))\}$ such that $\mu_{A \times B} : X \times Y \to [0, 1]$ and $\nu_{A \times B} : X \times Y \to [0, 1]$, for all $x \in X$ and $y \in Y$.

Remark 2.13. Let X and Y be PMS-algebras, for all $(x, y), (u, v) \in X \times Y$, we define '$*$' on $X \times Y$ by $(x, y)*(u, v) = (x*u, y*v)$. Clearly $(X \times Y; *, (0, 0))$ is a PMS-algebra.

Definition 2.14. [5] A fuzzy relation A on a nonempty set X is a fuzzy set A with a membership function $\mu_A : X \times X \to [0, 1]$.

Definition 2.15. [6, 5] An intuitionistic fuzzy relation R on a non empty set X is an expression of the form $R = \{\langle (x, y), \mu_R(x, y), \nu_R(x, y)\rangle \,|x, y \in X\}$ where $\mu_R : X \times X \to [0, 1]$ and $\nu_R : X \times X \to [0, 1]$ satisfy the condition $0 \leq \mu_R(x, y) + \nu_R(x, y) \leq 1$ for every $(x, y) \in X \times X$.

Definition 2.16 ([1, 6, 5]). Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy set on a set X and $R = (\mu_R, \nu_R)$ is an intuitionistic fuzzy relation on a set X. Then the strongest intuitionistic fuzzy relation R_A on X, that is, an intuitionistic fuzzy relation R on A whose membership function $\mu_{R_A} : X \times X \to [0, 1]$ and whose nonmembership function $\nu_{R_A} : X \times X \to [0, 1]$ are given by $\mu_{R_A}(x, y) = \min\{\mu_A(x), \mu_A(y)\}$ and $\nu_{R_A}(x, y) = \max\{\nu_A(x), \nu_A(y)\}$.

3. Homomorphism on intuitionistic Fuzzy PMS-subalgebras

In this section, we discuss on intuitionistic fuzzy PMS-subalgebras in a PMS-algebra under homomorphism. The homomorphic image and inverse image of intuitionistic fuzzy PMS-subalgebras of a PMS-algebra, as well as other results, are examined. Unless otherwise stated, X and Y refer to a PMS-algebra throughout this and the following section.

Theorem 3.1. *Let $f : X \to Y$ be an epimorphism of PMS-algebras. If $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of X with sup-inf property, then $f(A)$ is an intuitionistic fuzzy PMS-subalgebra of Y.*

Proof: Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS-subalgebra of X and let $a, b \in Y$ with $x_0 \in f^{-1}(a)$ and $y_0 \in f^{-1}(b)$ such that

$$\mu_A(x_0) = \sup_{x \in f^{-1}(a)} \mu_A(x),\ \mu_A(y_0) = \sup_{x \in f^{-1}(b)} \mu_A(x),$$

and

$$\nu_A(x_0) = \inf_{x \in f^{-1}(a)} \nu_A(x),\ \nu_A(y_0) = \inf_{x \in f^{-1}(b)} \nu_A(x),$$

then by Definition 2.10 and 2.11 we have

$$\begin{aligned}
\mu_{f(A)}(a*b) = \sup_{x\in f^{-1}(a*b)} \mu_A(x) &= \mu_A(x_0*y_0)\\
&\geq \min\{\mu_A(x_0), \mu_A(y_0)\}\\
&= min\{\sup_{x\in f^{-1}(a)} \mu_A(x), \sup_{x\in f^{-1}(b)} \mu_A(x)\}\\
&= min\{\mu_{f(A)}(a), \mu_{f(A)}(b)\}
\end{aligned}$$

and

$$\begin{aligned}
\nu_{f(A)}(a*b) = \inf_{x\in f^{-1}(a*b)} \nu_A(x) &= \nu_A(x_0*y_0)\\
&\leq \max\{\nu_A(x_0), \nu_A(y_0)\}\\
&= \max\{\inf_{x\in f^{-1}(b)} \nu_A(x), \inf_{x\in f^{-1}(b)} \nu_A(x)\}\\
&= \max\{\nu_{f(A)}(a), \nu_{f(A)}(b)\}
\end{aligned}$$

Hence $f(A)$ is an intuitionistic fuzzy PMS-subalgebra of Y. $\square$

THEOREM 3.2. *Let $f : X \to Y$ be a homomorphism of PMS-algebras. If $B = (\mu_B, \nu_B)$ is an intuitionistic fuzzy PMS-subalgebra of Y, then $f^{-1}(B)$ is an intuitionistic fuzzy PMS-subalgebra of X.*

PROOF: Assume that $B = (\mu_B, \nu_B)$ is an intuitionistic fuzzy PMS-subalgebra of Y and let $x, y \in X$. Then,

$$\begin{aligned}
\mu_{f^{-1}(B)}(x*y) = \mu_B(f(x*y)) &= \mu_B(f(x)*f(y))\\
&\geq \min\{\mu_B(f(x)), \mu_B(f(y))\}\\
&= \min\{\mu_{f^{-1}(B)}(x), \mu_{f^{-1}(B)}(y)\}
\end{aligned}$$

and

$$\begin{aligned}
\nu_{f^{-1}(B)}(x*y) = \nu_B(f(x*y)) &= \nu_B(f(x)*f(y))\\
&\leq \max\{\nu_B(f(x)), \nu_B(f(y))\}\\
&= \max\{\nu_{f^{-1}(B)}(x), \nu_{f^{-1}(B)}(y)\}
\end{aligned}$$

Therefore $f^{-1}(B)$ is an intuitionistic fuzzy PMS-subalgebra of X. $\square$

The Converse of the above theorem is true if f is a PMS-epimorphism.

THEOREM 3.3. *Let $f : X \to Y$ be an epimorphism of PMS-algebras and $B = (\mu_B, \nu_B)$ is a fuzzy set in Y. If $f^{-1}(B)$ is an intuitionistic fuzzy PMS-subalgebra of X, then $B = (\mu_B, \nu_B)$ is an intuitionistic fuzzy PMS-subalgebra of Y.*

PROOF: Assume that f is an epimorphism of PMS-algebras and $f^{-1}(B)$ is an intuitionistic fuzzy PMS-subalgebra of X. Let $y_1, y_2 \in Y$. Since f is an epimorphism of PMS-algebras, there exist $x_1, x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Now,

$$\begin{aligned}
\mu_B(y_1 * y_2) &= \mu_B(f(x_1) * f(x_2)) \\
&= \mu_B(f(x_1 * x_2)) \\
&= \mu_{f^{-1}(B)}(x_1 * x_2) \\
&\geq \min\{\mu_{f^{-1}(B)}(x_1), \mu_{f^{-1}(B)}(x_2)\} \\
&= \min\{\mu_B(f(x_1)), \mu_B(f(x_2))\} \\
&= \min\{\mu_B(y_1), \mu_B(y_2)\}
\end{aligned}$$

and

$$\begin{aligned}
\nu_B(y_1 * y_2) &= \nu_B(f(x_1) * f(x_2)) \\
&= \nu_B(f(x_1 * x_2)) \\
&= \nu_{f^{-1}(B)}(x_1 * x_2) \\
&\leq \max\{\nu_{f^{-1}(B)}(x_1), \nu_{f^{-1}(B)}(x_2)\} \\
&= \max\{\nu_B(f(x_1)), \nu_B(f(x_2))\} \\
&= \max\{\nu_B(y_1), \nu_B(y_2)\}
\end{aligned}$$

Hence $B = (\mu_B, \nu_B)$ is an intuitionistic fuzzy PMS-Subalgebra of Y. $\square$

DEFINITION 3.4. Let $f : X \to Y$ be a homomorphism of PMS-algebras for any intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ in Y. We define an intuitionistic fuzzy set $A^f = (\mu_A^f, \nu_A^f)$ in X by $\mu_A^f(x) = \mu_A(f(x))$ and $\nu_A^f(x) = \nu_A(f(x)), \forall x \in X$.

In the next two theorems we characterize an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra using an intuitionistic fuzzy set defined above in Definition 3.4.

Theorem 3.5. *Let $f : X \to Y$ be a homomorphism of PMS-algebras. If the intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of Y, then the intuitionistic fuzzy set $A^f = (\mu_A^f, \nu_A^f)$ in X is an intuitionistic fuzzy PMS-subalgebra of X.*

Proof: Let f be a homomorphism of PMS-algebras and let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS-subalgebra of Y. Let $x, y \in X$. Then

$$\begin{aligned} \mu_A^f(x * y) = \mu_A(f(x * y)) &= \mu_A(f(x) * f(y)) \\ &\geq \min\{\mu_A(f(x)), \mu_A(f(y))\} \\ &= \min\{\mu_A^f(x), \mu_A^f(y)\} \end{aligned}$$

and

$$\begin{aligned} \nu_A^f(x * y) = \nu_A(f(x * y)) &= \nu_A(f(x) * f(y)) \\ &\leq \max\{\nu_A(f(x)), \nu_A(f(y))\} \\ &= \max\{\nu_A^f(x), \nu_A^f(y)\} \end{aligned}$$

Hence $A^f = (\mu_A^f, \nu_A^f)$ is an intuitionistic fuzzy PMS-subalgebra of X. □

The Converse of Theorem 3.5 is also true if f is an epimorphism of PMS-algebras as shown below in Theorem 3.6

Theorem 3.6. *Let $f : X \to Y$ be an epimorphism of PMS-algebra. If $A^f = (\mu_A^f, \nu_A^f)$ is an intuitionistic fuzzy PMS-subalgebra of X, then $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of Y.*

Proof: Let $A^f = (\mu_A^f, \nu_A^f)$ be an intuitionistic fuzzy PMS-subalgebra in X and let $x, y \in Y$. Then there exist $a, b \in X$ such that $f(a) = x$ and $f(b) = y$. Now we have,

$$\begin{aligned} \mu_A(x * y) &= \mu_A(f(a) * f(b)) \\ &= \mu_A(f(a * b)) \\ &= \mu_A^f(a * b) \\ &\geq \min\{\mu_A^f(a), \mu_A^f(b)\} \end{aligned}$$

$$\begin{aligned} &= \min\{\mu_A(f(a)), \mu_A(f(b))\} \\ &= \min\{\mu_A(x), \mu_A(y)\} \end{aligned}$$

and

$$\begin{aligned} \nu_A(x * y) &= \nu_A(f(a) * f(b)) \\ &= \nu_A(f(a * b)) \\ &= \nu_A^f(a * b) \\ &\leq \max\{\nu_A^f(a), \nu_A^f(b)\} \\ &= \max\{\nu_A(f(a)), \nu_A(f(b))\} \\ &= \max\{\nu_A(x), \nu_A(y)\} \end{aligned}$$

Hence $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of Y. □

As a consequence of Theorems 3.5 and 3.6 we obtain the next theorem.

THEOREM 3.7. *Let $f : X \to Y$ be an epimorphism of PMS-algebra. Then $A^f = (\mu_A^f, \nu_A^f)$ is an intuitionistic fuzzy PMS-subalgebra of X if and only if $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of Y.*

4. Cartesian Product of Intuitionistic Fuzzy PMS-subalgebras

In this section, we discuss the concept of Cartesian product and the strongest fuzzy relation on intuitionistic fuzzy PMS-algebras. We prove that the Cartesian product of two intuitionistic fuzzy PMS-subalgebras is again an intuitionistic fuzzy PMS-subalgebra and some other results are also investigated.

LEMMA 4.1. *Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be any two intuitionistic fuzzy PMS-subalgebras of X and Y respectively. Then*

$$\mu_{A\times B}(0,0) \geq \mu_{A\times B}(x,y)$$

and

$$\nu_{A\times B}(0,0) \leq \nu_{A\times B}(x,y), \forall (x,y) \in X \times Y.$$

Proof: Let $(x, y) \in X \times Y$. Then
$\mu_{A\times B}(0, 0) = \min\{\mu_A(0), \mu_B(0)\} \geq \min\{\mu_A(x), \mu_B(y)\} = \mu_{A\times B}(x, y)$ and
$\nu_{A\times B}(0, 0) = \max\{\nu_A(0), \nu_B(0)\} \leq \max\{\nu_A(x), \nu_B(y)\} = \nu_{A\times B}(x, y)$ $\square$

Theorem 4.2. *Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be any two intuitionistic fuzzy PMS-subalgebras of X and Y respectively. Then $A \times B$ is an intuitionistic fuzzy PMS-subalgebra of $X \times Y$.*

Proof: Let $(x_1, y_1), (x_2, y_2) \in X \times Y$. Then

$$\begin{aligned}
\mu_{A\times B}((x_1,y_1)*(x_2,y_2)) &= \mu_{A\times B}(x_1*x_2,y_1*y_2)\\
&= \min\{\mu_A(x_1*x_2),\mu_B(y_1*y_2)\}\\
&\geq \min\{\min\{\mu_A(x_1),\mu_A(x_2)\},\min\{\mu_B(y_1),\mu_B(y_2)\}\}\\
&= \min\{\min\{\mu_A(x_1),\mu_B(y_1)\},\min\{\mu_A(x_2),\mu_B(y_2)\}\}\\
&= \min\{\mu_{A\times B}(x_1,y_1), \mu_{A\times B}(x_2,y_2)\}
\end{aligned}$$

and

$$\begin{aligned}
\nu_{A\times B}((x_1,y_1)*(x_2, y_2)) &= \nu_{A\times B}(x_1 * x_2,y_1 * y_2)\\
&= \max\{\nu_A(x_1*x_2),\nu_B(y_1*y_2)\}\\
&\leq \max\{\max\{\nu_A(x_1),\nu_A(x_2)\},\max\{\nu_B(y_1),\nu_B(y_2)\}\}\\
&= \max\{\max\{\nu_A(x_1),\nu_B(y_1)\},\max\{\nu_A(x_2),\nu_B(y_2)\}\}\\
&= \max\{\nu_{A\times B}(x_1,y_1),\nu_{A\times B}(x_2,y_2)\}
\end{aligned}$$

Hence $A \times B$ is an intuitionistic fuzzy PMS-subalgebra of $X \times Y$. $\square$

Theorem 4.3. *Let A and B be intuitionistic fuzzy subsets of the PMS-algebras X and Y respectively. Suppose that 0 and $0'$ are the constant elements of X and Y respectively. If $A \times B$ is an intuitionistic fuzzy PMS-subalgebras of $X \times Y$, then at least one of the following two statements holds.*

(i) $\mu_A(x) \leq \mu_B(0')$ *and* $\nu_A(x) \geq \nu_B(0')$,*for all* $x \in X$,

(ii) $\mu_B(y) \leq \mu_A(0)$ *and* $\nu_B(y) \geq \nu_A(0)$, *for all* $y \in Y$.

Proof: Let $A \times B$ be an intuitionistic fuzzy PMS-subalgebra of $X \times Y$. Suppose that none of the statements (i) and (ii) holds. Then we can find $x \in X$ and $y \in Y$ such that $\mu_A(x) > \mu_B(0'), \nu_A(x) < \nu_B(0')$ and $\mu_B(y) > \mu_A(0), \nu_B(y) < \nu_A(0)$. Then we have

$$\mu_{A\times B}(x,y) = \min\{\mu_A(x), \mu_B(y)\} > \min\{\mu_B(0'), \mu_A(0)\} = \mu_{A\times B}(0,0')$$

and

$$\nu_{A\times B}(x,y) = \max\{\nu_A(x), \nu_B(y)\} < \max\{\nu_B(0'), \nu_A(0)\} = \nu_{A\times B}(0,0'),$$

which leads to

$$\mu_{A\times B}(x,y) > \mu_{A\times B}(0,0') \text{ and } \nu_{A\times B}(x,y) < \nu_{A\times B}(0,0').$$

This contradicts Lemma 4.1. Hence, either (i) or (ii) holds □

THEOREM 4.4. *Let A and B be intuitionistic fuzzy subsets of PMS-algebras X and Y respectively such that $\mu_A(x) \leq \mu_B(0')$ and $\nu_A(x) \geq \nu_B(0')$ for all $x \in X$, where $0'$ is a constant in Y. If $A \times B$ is an intuitionistic fuzzy PMS-subalgebra of $X \times Y$, then A is an intuitionistic fuzzy PMS-subalgebra of X.*

PROOF: Let $x, y \in X$. Then $(x,0'), (y,0') \in X \times Y$. Since $\mu_A(x) \leq \mu_B(0')$ and $\nu_A(x) \geq \nu_B(0')$ for all $x \in X$, then for all $x, y \in X$ we get,

$$\begin{aligned}
\mu_A(x*y) &= \min\{\mu_A(x*y), \mu_B(0'*0')\} \\
&= \mu_{A\times B}(x*y, 0'*0') \\
&= \mu_{A\times B}((x,0')*(y,0')) \\
&\geq \min\{\mu_{A\times B}(x,0'), \mu_{A\times B}(y,0')\} \\
&= \min\{\min\{\mu_A(x), \mu_B(0')\}, \min\{\mu_A(y), \mu_B(0')\}\} \\
&= \min\{\mu_A(x), \mu_A(y)\}
\end{aligned}$$

and

$$\begin{aligned}
\nu_A(x*y) &= \max\{\nu_A(x*y), \nu_B(0'*0')\} \\
&= \nu_{A\times B}(x*y, 0'*0') \\
&= \nu_{A\times B}((x,0')*(y,0')) \\
&\leq \max\{\nu_{A\times B}(x,0'), \nu_{A\times B}(y,0')\} \\
&= \max\{\max\{\nu_A(x), \nu_B(0')\}, \max\{\nu_A(y), \nu_B(0')\}\} \\
&= \max\{\nu_A(x), \nu_A(y)\}
\end{aligned}$$

Hence $\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\}$ and $\nu_A(x * y) \leq \max\{\nu_A(x), \nu_A(y)\}$
Therefore A is an intuitionistic fuzzy PMS-subalgebra of X. □

THEOREM 4.5. *Let A and B be intuitionistic fuzzy subsets of PMS-algebras X and Y respectively such that $\mu_B(y) \leq \mu_A(0)$ and $\nu_B(y) \geq \nu_A(0)$ for all $y \in Y$, where 0 is a constant in X. If $A \times B$ is an intuitionistic fuzzy PMS-subalgebra of $X \times Y$, then B is an intuitionistic fuzzy PMS-subalgebra of Y.*

PROOF: Let $x, y \in Y$. Then $(0, x), (0, y) \in X \times Y$. Since $\mu_B(y) \leq \mu_A(0)$ and $\nu_B(y) \geq \nu_A(0)$ for all $y \in Y$, then for all $x, y \in Y$ we get,

$$\begin{aligned}
\mu_B(x * y) &= \min\{\mu_A(0 * 0), \mu_B(x * y)\} \\
&= \mu_{A\times B}(0 * 0, x * y) \\
&= \mu_{A\times B}((0, x) * (0, y)) \\
&\geq \min\{\mu_{A\times B}(0, x), \mu_{A\times B}(0, y)\} \\
&= \min\{\min\{\mu_A(0), \mu_B(x)\}, \min\{\mu_A(0), \mu_B(y)\}\} \\
&= \min\{\mu_B(x), \mu_B(y)\}
\end{aligned}$$

and

$$\begin{aligned}
\nu_B(x * y) &= \max\{\nu_A(0 * 0), \nu_B(x * y)\} \\
&= \nu_{A\times B}(0 * 0, x * y) \\
&= \nu_{A\times B}((0, x) * (0, y)) \\
&\leq \max\{\nu_{A\times B}(0, x), \nu_{A\times B}(0, y)\} \\
&= \max\{\max\{\nu_A(0), \nu_B(x)\}, \max\{\nu_A(0), \nu_B(y)\}\} \\
&= \max\{\nu_B(x), \nu_B(y)\}
\end{aligned}$$

Hence $\mu_B(x * y) \geq \min\{\mu_B(x), \mu_B(y)\}$ and $\nu_B(x * y) \leq \max\{\nu_B(x), \nu_B(y)\}$

Therefore B is an intuitionistic fuzzy PMS-subalgebra of Y. □

From Theorems 4.3 , 4.4 and 4.5, we have the following:

COROLLARY 4.6. Let A and B be intuitionistic fuzzy subsets of PMS-algebras X and Y respectively. If $A \times B$ is an intuitionistic fuzzy PMS-subalgebra of $X \times Y$, then either A is an intuitionistic fuzzy PMS-subalgebra of X or B is an intuitionistic fuzzy PMS-subalgebra of Y.

PROOF: Since $A \times B$ is an intuitionistic fuzzy PMS-subalgebra of $X \times Y$,

$$\mu_{A\times B}((x_1, y_1) * (x_2, y_2)) \geq \min\{\mu_{A\times B}(x_1, y_1), \mu_{A\times B}(x_2, y_2)\} \quad (4.1)$$
$$\nu_{A\times B}((x_1, y_1) * (x_2, y_2)) \leq \max\{\nu_{A\times B}(x_1, y_1), \nu_{A\times B}(x_2, y_2)\} \quad (4.2)$$

If we put $x_1 = 0 = x_2$ in (4.1), we get

$$\begin{aligned}
&\mu_{A\times B}((0, y_1)*(0,y_2)) \geq \min\{\mu_{A\times B}(0, y_1),\mu_{A\times B}(0,y_2)\}\\
&\Rightarrow \mu_{A\times B}(0*0,y_1*y_2) \geq \min\{\mu_{A\times B}(0,y_1),\mu_{A\times B}(0,y_2)\}\\
&\Rightarrow \mu_{A\times B}(0,y_1*y_2) \geq \min\{\mu_{A\times B}(0,y_1),\mu_{A\times B}(0,y_2)\}\\
&\Rightarrow \min\{\mu_A(0),\mu_B(y_1*y_2)\} \geq \min\{\min\{\mu_A(0),\mu_B(y_1)\},\min\{\mu_A(0),\mu_B(y_2)\}\}
\end{aligned}$$

Hence, $\mu_B(y_1 * y_2) \geq \min\{\mu_B(y_1), \mu_B(y_2)\}$. Also,if we put $x_1 = 0 = x_2$ in (4.2), we get

$$\begin{aligned}
&\nu_{A\times B}((0,y_1)*(0, y_2)) \leq \max\{\nu_{A\times B}(0, y_1),\nu_{A\times B}(0,y_2)\}\\
&\Rightarrow \nu_{A\times B}(0*0, y_1*y_2) \leq \max\{\nu_{A\times B}(0,y_1),\nu_{A\times B}(0,y_2)\}\\
&\Rightarrow \nu_{A\times B}(0,y_1*y_2) \leq \max\{\nu_{A\times B}(0,y_1),\nu_{A\times B}(0,y_2)\}\\
&\Rightarrow \max\{\nu_A(0), \nu_B(y_1*y_2)\} \leq \max\{\max\{\nu_A(0),\nu_B(y_1)\},\max\{\nu_A(0),\nu_B(y_2)\}\}
\end{aligned}$$

Hence $\nu_B(y_1 * y_2) \leq \max\{\nu_B(y_1), \nu_B(y_2)\}$ and B is an intuitionistic fuzzy PMS-subalgebra of Y.

Similarly, we prove that A is an intuitionistic fuzzy PMS-subalgebra of X by putting $y_1 = 0 = y_2$ in (4.1) and (4.2). □

THEOREM 4.7. *Let A and B be any intuitionistic fuzzy subsets of X and Y respectively. Then $A \times B$ is an intuitionistic fuzzy PMS-subalgebra of $X \times Y$ if and only if $\mu_{A\times B}$ and $\overline{\nu}_{A\times B}$ are fuzzy PMS-subalgebra of $X \times Y$, where $\overline{\nu}_{A\times B}$ is the complement of $\nu_{A\times B}$.*

PROOF: Let $A \times B$ be an intuitionistic fuzzy PMS-subalgebra of $X \times Y$. Then by Definition 2.9 $\mu_{A\times B}((x_1, y_1) * (x_2, y_2)) \geq \min\{\mu_{A\times B}(x_1, y_1), \mu_{A\times B}(x_2, y_2)\}$ and $\nu_{A\times B}((x_1, y_1) * (x_2, y_2)) \leq \max\{\nu_{A\times B}(x_1, y_1), \nu_{A\times B}(x_2, y_2)\}$,$\forall\, (x_1, y_1), (x_2, y_2) \in X \times Y$. Hence $\mu_{A\times B}$ is a fuzzy PMS-subalgebra of $X \times Y$ by Definition 2.5. Now for all $(x_1, y_1), (x_2, y_2) \in X \times Y$.

$$\begin{aligned}\overline{\nu}_{A\times B}((x_1,y_1)*(x_2,y_2)) &= 1-\nu_{A\times B}((x_1,y_1)*(x_2,y_2))\\ &\geq 1-\max\{\nu_{A\times B}(x_1,y_1),\nu_{A\times B}(x_2,y_2)\}\\ &= \min\{1-\nu_{A\times B}(x_1,y_1),1-\nu_{A\times B}(x_2,y_2)\}\\ &= \min\{\overline{\nu}_{A\times B}(x_1,y_1),\overline{\nu}_{A\times B}(x_2,y_2)\}\end{aligned}$$

Hence $\overline{\nu}_{A\times B}((x_1,y_1)*(x_2,y_2)) \geq \min\{\overline{\nu}_{A\times B}(x_1,y_1),\overline{\nu}_{A\times B}(x_2,y_2)\}$
Thus, $\overline{\nu}_{A\times B}$ is a fuzzy PMS-subalgebra of $X\times Y$.

Conversely, assume $\mu_{A\times B}$ and $\overline{\nu}_{A\times B}$ are fuzzy PMS-subalgebra of $X\times Y$. Then we have that $\mu_{A\times B}((x_1,y_1)*(x_2,y_2)) \geq \min\{\mu_{A\times B}(x_1,y_1), \mu_{A\times B}(x_2,y_2)\}$ and $\overline{\nu}_{A\times B}((x_1,y_1)*(x_2,y_2)) \geq \min\{\overline{\nu}_{A\times B}(x_1,y_1), \overline{\nu}_{A\times B}(x_2,y_2)\}$ for all $(x_1,y_1),(x_2,y_2)\in X\times Y$. So we need to show that $\nu_{A\times B}((x_1,y_1)*(x_2,y_2)) \leq \max\{\nu_{A\times B}(x_1,y_1),\nu_{A\times B}(x_2,y_2)\}$ for all $(x_1,y_1),(x_2,y_2)\in X\times Y$.

Now,

$$\begin{aligned}1-\nu_{A\times B}((x_1,y_1)*(x_2,y_2) &= \overline{\nu}_{A\times B}((x_1,y_1)*(x_2,y_2))\\ &\geq \min\{\overline{\nu}_{A\times B}(x_1,y_1),\overline{\nu}_{A\times B}(x_2,y_2)\}\\ &= \min\{1-\nu_{A\times B}(x_1,y_1),1-\nu_{A\times B}(x_2,y_2)\}\\ &= 1-\max\{\nu_{A\times B}(x_1,y_1),\nu_{A\times B}(x_2,y_2)\},\end{aligned}$$

and so $\nu_{A\times B}((x_1,y_1)*(x_2,y_2) \leq \max\{\nu_{A\times B}(x_1,y_1),\nu_{A\times B}(x_2,y_2)\}$. Hence $A\times B$ is an intuitionistic fuzzy PMS-subalgebra of $X\times Y$. $\square$

Theorem 4.8. *Let A and B be any intuitionistic fuzzy subsets of X and Y respectively, then $A\times B$ is an intuitionistic fuzzy PMS-subalgebra of $X\times Y$ if and only if $\square(A\times B)$ and $\diamondsuit(A\times B)$ are intuitionistic fuzzy PMS-subalgebra of $X\times Y$*

Proof: Suppose $A\times B$ is an intuitionistic fuzzy PMS-subalgebra of $X\times Y$. Then $\mu_{A\times B}((x_1,y_1)*(x_2,y_2) \geq \min\{\mu_{A\times B}(x_1,y_1),\mu_{A\times B}(x_2,y_2)\}$ and $\nu_{A\times B}((x_1,y_1)*(x_2,y_2) \leq \max\{\nu_{A\times B}(x_1,y_1),\nu_{A\times B}(x_2,y_2)\}$, for all $(x_1,y_1),(x_2,y_2)\in X\times Y$

(i) To prove $\square(A\times B)$ is an intuitionistic fuzzy PMS-subalgebra of $X\times Y$, it suffices to show that for $(x_1,y_1),(x_2,y_2)\in X\times Y$, $\overline{\mu}_{A\times B}((x_1,y_1)*(x_2,y_2) \leq \min\{\overline{\mu}_{A\times B}(x_1,y_1),\overline{\mu}_{A\times B}(x_2,y_2)\}$. Now let $(x_1,y_1),(x_2,y_2)$ $\in X\times Y$

$$\begin{aligned}\overline{\mu}_{A\times B}((x_1,y_1)*(x_2,y_2)) &= 1-\mu_{A\times B}((x_1,y_1)*(x_2,y_2))\\ &\leq 1-\min\{\mu_{A\times B}(x_1,y_1),\mu_{A\times B}(x_2,y_2)\}\end{aligned}$$

$$= \max\{1-\mu_{A\times B}((x_1,y_1),1-\mu_{A\times B}(x_2,y_2))\}$$
$$= \max\{\overline{\mu}_{A\times B}((x_1,y_1),\overline{\mu}_{A\times B}(x_2,y_2))\},$$

whence $\overline{\mu}_{A\times B}((x_1,y_1)*(x_2,y_2)) \leq \max\{\overline{\mu}_{A\times B}(x_1,y_1),\overline{\mu}_{A\times B}(x_2,y_2)\}$ follows. Hence $\Box(A\times B)$ is an intuitionistic fuzzy PMS-subalgebra of $X\times Y$.

(ii) To prove $\Diamond(A\times B)$ is an intuitionistic fuzzy PMS-subalgebra of $X\times Y$, it suffices to show that $\overline{\nu}_{A\times B}((x_1,y_1)*(x_2,y_2) \geq \min\{\overline{\nu}_{A\times B}(x_1,y_1), \overline{\nu}_{A\times B}(x_2,y_2)\}$. Now let $(x_1,y_1),(x_2,y_2)\in X\times Y$, then

$$\begin{aligned}\overline{\nu}_{A\times B}((x_1,y_1)*(x_2,y_2)) &= 1-\nu_{A\times B}((x_1,y_1)*(x_2,y_2))\\ &\geq 1-\max\{\nu_{A\times B}(x_1,y_1),\nu_{A\times B}(x_2,y_2)\}\\ &= \min\{1-\nu_{A\times B}((x_1,y_1),1-\nu_{A\times B}(x_2,y_2))\}\\ &= \min\{\overline{\nu}_{A\times B}((x_1,y_1),\overline{\nu}_{A\times B}(x_2,y_2))\},\end{aligned}$$

whence $\overline{\nu}_{A\times B}((x_1,y_1)*(x_2,y_2)) \geq \min\{\overline{\nu}_{A\times B}(x_1,y_1),\overline{\nu}_{A\times B}(x_2,y_2)\}$ follows. Hence $\Diamond(A\times B)$ is an intuitionistic fuzzy PMS-subalgebra of $X\times Y$.

The proof of the converse is trivial. □

Definition 4.9. Let $A=(\mu_A,\nu_A)$ and $B=(\mu_B,\nu_B)$ are intuitionistic fuzzy subset of PMS-algebras X and Y reapectively. For $t,s\in[0,1]$ satisfying the condition $t+s\leq 1$, the set $U(\mu_{A\times B},t)=\{(x,y)\in X\times Y|\mu_{A\times B}(x,y)\geq t\}$ is called upper t-level set of $A\times B$ and the set $L(\nu_{A\times B},s)$ $=\{(x,y)\in X\times Y|\nu_{A\times B}(x,y)\leq s\}$ is called lower s-level set of $A\times B$.

Theorem 4.10. *Let $A=(\mu_A,\nu_A)$ and $B=(\mu_B,\nu_B)$ be intuitionistic fuzzy subsets of X and Y reapectively. Then $A\times B$ is an intuitionistic fuzzy PMS-subalgebras of $X\times Y$ if and only if the nonempty upper t-level set $U(\mu_{A\times B},t)$ and the nonempty lower s-level set $L(\nu_{A\times B},s)$ are PMS-subalgebras of $X\times Y$ for any $t,s\in[0,1]$ with $t+s\leq 1$.*

Proof: Let $A=(\mu_A,\nu_A)$ and $B=(\mu_B,\nu_B)$ be intuitionistic fuzzy subsets of X and Y respectively. Let $(x_1,y_1),(x_2,y_2)\in X\times Y$ such that $(x_1,y_1),(x_2,y_2)\in U(\mu_{A\times B},t)$ for $t\in[0,1]$. Then $\mu_{A\times B}(x_1,y_1)\geq t$ and $\mu_{A\times B}(x_2,y_2)\geq t$. Since $A\times B$ is an intuitionistic fuzzy PMS-subalgebra of $X\times Y$, we have

$$\begin{aligned}\mu_{A\times B}((x_1,y_1)*(x_2,y_2)) &\geq \min\{\mu_{A\times B}(x_1,y_1),\mu_{A\times B}(x_2,y_2)\}\\ &\geq \min\{t,t\}=t\end{aligned}$$

Therefore, $(x_1,y_1)*(x_2,y_2)\in U(\mu_{A\times B},t)$. Hence $U(\mu_{A\times B},t)$ is a PMS-subalgebra of $X\times Y$.

Also, Let $(x_1,y_1),(x_2,y_2)\in X\times Y$ such that $(x_1,y_1),(x_2,y_2)\in L(\nu_{A\times B},s)$ for $s\in[0,1]$. Then $\nu_{A\times B}(x_1,y_1)\leq s$ and $\nu_{A\times B}(x_2,y_2)\leq s$. Since $A\times B$ is an intuitionistic fuzzy PMS-subalgebra of $X\times Y$, we have

$$\begin{aligned}\nu_{A\times B}((x_1,y_1)*(x_2,y_2)) &\leq \max\{\nu_{A\times B}(x_1,y_1),\nu_{A\times B}(x_2,y_2)\}\\ &\leq \max\{s,s\}=s\end{aligned}$$

Therefore, $(x_1,y_1)*(x_2,y_2)\in L(\nu_{A\times B},s)$. Hence $L(\nu_{A\times B},s)$ is a PMS-subalgebra of $X\times Y$.

Conversely, Suppose $U(\mu_{A\times B},t)$ and $L(\nu_{A\times B},s)$ are PMS-subalgebra of $X\times Y$ for any $t,s\in[0,1]$ with $t+s\leq 1$. Assume that $A\times B$ is not an intuitionistic fuzzy PMS-subalgebra of $X\times Y$. Then there exist $(x_1,y_1),(x_2,y_2)\in X\times Y$ such that

$$\mu_{A\times B}((x_1,y_1)*(x_2,y_2))<\min\{\mu_{A\times B}(x_1,y_1),\mu_{A\times B}(x_2,y_2)\}.$$

Then by taking $t_0=\frac{1}{2}\{\mu_{A\times B}((x_1,y_1)*(x_2,y_2))+\min\{\mu_{A\times B}(x_1,y_1),\mu_{A\times B}(x_2,y_2)\}\}$, we get $\mu_{A\times B}((x_1,y_1)*(x_2,y_2))<t_0<\min\{\mu_{A\times B}(x_1,y_1),\mu_{A\times B}(x_2,y_2)\}$. Hence, $(x_1,y_1)*(x_2,y_2)\notin U(\mu_{A\times B},t_0)$ but $(x_1,y_1)\in U(\mu_{A\times B},t_0)$ and $(x_2,y_2)\in U(\mu_{A\times B},t_0)$, This implies $U(\mu_{A\times B},t_0)$ is not a PMS-subalgebra of $X\times Y$, which is a contradiction. Therefore $\mu_{A\times B}((x_1,y_1)*(x_2,y_2))\geq\min\{\mu_{A\times B}(x_1,y_1),\mu_{A\times B}(x_2,y_2)\}$.
Similarly, $\nu_{A\times B}((x_1,y_1)*(x_2,y_2))\leq\max\{\mu_{A\times B}(x_1,y_y),\mu_{A\times B}(x_2,y_2)\}$.

Hence $A\times B$ is an intuitionistic fuzzy PMS-subalgebra of $X\times Y$. □

THEOREM 4.11. *Let $A=(\mu_A,\nu_A)$ be an intuitionistic fuzzy subset of PMS-algebra X and let R_A be the strongest intutionistic fuzzy PMS-relation on X. If R_A is an intuitionistic fuzzy PMS-subalgebra of $X\times X$, then $\mu_A(0)\geq\mu_A(x)$ and $\nu_A(0)\leq\nu_A(x)$, for all $x\in X$.*

PROOF: Since R_A is an intuitionistic fuzzy PMS-subalgebra of $X\times X$, it follows from Lemma 4.1 that $\mu_{R_A}(0,0)\geq\mu_{R_A}(x,x)$ and $\nu_{R_A}(0,0)\leq\nu_{R_A}(x,x)$. Then, we have $\min\{\mu_A(0),\mu_A(0)\}=\mu_{R_A}(0,0)\geq\mu_{R_A}(x,x)=$

$\min\{\mu_A(x), \mu_A(x)\}$, where $(0,0) \in X \times X$, which implies $\min\{\mu_A(0), \mu_A(0)\} \geq \min\{\mu_A(x), \mu_A(x)\}$, and so, $\mu_A(0) = \min\{\mu_A(0), \mu_A(0)\} \geq \min\{\mu_A(x), \mu_A(x)\} = \mu_A(x)$. Moreover, $\max\{\nu_A(0), \nu_A(0)\} = \nu_{R_A}(0,0) \leq \nu_{R_A}(x,x) = \max\{\nu_A(x), \nu_A(x)\}$, where $(0,0) \in X \times X$, whence follows $\max\{\nu_A(0), \nu_A(0)\} \leq \max\{\nu_A(x), \nu_A(x)\}$ and further $\nu_A(0) = \max\{\nu_A(0), \nu_A(0)\} \leq \max\{\nu_A(x), \nu_A(x)\} = \nu_A(x)$.

Hence $\mu_A(0) \geq \mu_A(x)$ and $\nu_A(0) \leq \nu_A(x)$, for all $x \in X$. □

THEOREM 4.12. *Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of a PMS-algebra X and let R_A be the strongest intuitionistic fuzzy PMS-relation on X. Then A is an intuitionistic fuzzy PMS-subalgebra of X if and only if R_A is an intuitionistic fuzzy PMS-subalgebra of $X \times X$.*

PROOF: Assume that A is an intuitionistic fuzzy PMS-subalgebra X. Let $(x_1, x_2), (y_1, y_2) \in X \times X$. Then, we have

$$\begin{aligned}
\mu_{R_A}((x_1, x_2) * (y_1, y_2)) &= \mu_{R_A}(x_1 * y_1, x_2 * y_2) \\
&= \min\{\mu_A(x_1 * y_1), \mu_A(x_2 * y_2)\} \\
&\geq \min\{\min\{\mu_A(x_1), \mu_A(y_1)\}, \min\{\mu_A(x_2), \mu_A(y_2)\}\} \\
&= \min\{\min\{\mu_A(x_1), \mu_A(x_2)\}, \min\{\mu_A(y_1), \mu_A(y_2)\}\} \\
&= \min\{\mu_{R_A}(x_1, x_2), \mu_{R_A}(y_1, y_2)\}.
\end{aligned}$$

and

$$\begin{aligned}
\nu_{R_A}((x_1, x_2) * (y_1, y_2)) &= \nu_{R_A}(x_1 * y_1, x_2 * y_2) \\
&= \max\{\nu_A(x_1 * y_1), \nu_A(x_2 * y_2)\} \\
&\leq \max\{\max\{\nu_A(x_1), \nu_A(y_1)\}, \max\{\nu_A(x_2), \nu_A(y_2)\}\} \\
&= \max\{\max\{\nu_A(x_1), \nu_A(x_2)\}, \max\{\nu_A(y_1), \nu_A(y_2)\}\} \\
&= \max\{\nu_{R_A}(x_1, x_2), \nu_{R_A}(y_1, y_2)\}.
\end{aligned}$$

Hence R_A is an intuitionistic fuzzy PMS-subalgebra of $X \times X$.

Conversely, assume R_A is an intuitionistic fuzzy PMS-subalgebra of $X \times X$. Let $(x_1, x_2), (y_1, y_2) \in X \times X$. Then

$$\begin{aligned}
\min\{\mu_A(x_1 * y_1), \mu_A(x_2 * y_2)\} &= \mu_{R_A}(x_1 * y_1, x_2 * y_2) \\
&= \mu_{R_A}((x_1, x_2) * (y_1, y_2)) \\
&\geq \min\{\mu_{R_A}(x_1, x_2), \mu_{R_A}(y_1, y_2)\} \\
&= \min\{\min\{\mu_A(x_1), \mu_A(x_2)\}, \\
&\qquad \min\{\mu_A(y_1), \mu_A(y_2)\}\}
\end{aligned}$$

In particular, if we take, $x_2 = y_2 = 0$ (or respectively $x_1 = y_1 = 0$), then we get $\mu_A(x_1 * y_1) \geq min\{\mu_A(x_1), \mu_A(y_1)\}$ (or resp. $\mu_A(x_2 * y_2) \geq \min\{\mu_A(x_2), \mu_A(y_2)\}$)and

$$\begin{aligned}\max\{\nu_A(x_1{*}y_1), \nu_A(x_2{*}y_2)\} &= \nu_{R_A}(x_1{*}y_1, x_2{*}y_2)\\ &= \nu_{R_A}(x_1, x_2){*}(y_1, y_2)\\ &\leq \max\{\nu_{R_A}(x_1, x_2), \nu_{R_A}(y_1, y_2)\}\\ &= \max\{\max\{\nu_A(x_1), \nu_A(x_2)\},\\ &\qquad \max\{\nu_A(y_1), \nu_A(y_2)\}\}\end{aligned}$$

In particular, if we take, $x_2 = y_2 = 0$ (or respectively $x_1 = y_1 = 0$), then we get $\nu_A(x_1 * y_1) \leq \max\{\nu_A(x_1), \nu_B(y_1)\}$ (or resp. $\nu_A(x_1 * y_1) \leq \max\{\mu_A(x_1), \mu_A(y_1)\}$)

Therefore A is an intuitionistic fuzzy PMS-subalgebra of X □

5. Conclusion

In this paper, we discussed the concept of intuitionistic fuzzy PMS-subalgebra under homomorphism and Cartesian product in a PMS-algebra. We confirmed that the homomorphic image and the homomorphic inverse image of an intuitionistic fuzzy PMS-subalgebra in a PMS-algebra are intuitionistic fuzzy PMS-subalgebras. We also proved that the Cartesian product of the intuitionistic fuzzy PMS-subalgebras of a PMS-algebra is an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra. Furthermore, we characterized the Cartesian products of intuitionistic fuzzy PMS-subalgebras in terms of their level sets. Finally, we discussed the concept of the strongest intuitionistic fuzzy PMS-relation on an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra and investigated some of its properties. We will further extend these concepts to intuitionistic fuzzy PMS-ideals of a PMS-algebra for new results in our future work.

References

[1] N. Anitha, K. Arjunan, *Notes on intuitionistic fuzzy ideals of Hemiring*, **Applied Mathematical Science**, vol. 5(68) (2011), pp. 3393–3402, URL: http://www.m-hikari.com/ams/ams-2011/ams-65-68-2011/anithaAMS65-68-2011.pdf.

[2] K. T. Atanassov, *Intuitionistic fuzzy sets*, **Fuzzy Sets and Systems**, vol. 20(1) (1986), pp. 87–96, DOI: https://doi.org/10.1016/S0165-0114(86)80034-3.

[3] K. T. Atanassov, *New operations defined over the Intuitionistic fuzzy sets*, **Fuzzy Sets and Systems**, vol. 61(2) (1994), pp. 137–142, DOI: https://doi.org/10.1016/0165-0114(94)90229-1.

[4] B. L. Derseh, B. A. Assaye, Y. G. Wondifraw, *Intuitionistic fuzzy PMS-subalgebra of a PMS-algebra*, **Korean Journal of Mathematics**, vol. 29(3) (2021), pp. 563–576, DOI: https://doi.org/10.11568/kjm.2021.29.3.563.

[5] M. Panigrahi, S. Nanda, *Intuitionistic Fuzzy Relations over Intuitionistic Fuzzy Sets*, **Journal of Fuzzy Mathematics**, vol. 15(3) (2007), pp. 675–688.

[6] J. Peng, *Intuitionistic Fuzzy B-algebras*, **Research Journal of Applied Sciences, Engineering and Technology**, vol. 4(21) (2012), pp. 4200–4205, URL: https://maxwellsci.com/print/rjaset/v4-4200-4205.pdf.

[7] P. M. S. Selvam, K. T. Nagalakshmi, *Fuzzy PMS-ideals in PMS-algebras*, **Annals of Pure and Applied Mathematics**, vol. 12(2) (2016), pp. 153–159, DOI: https://doi.org/10.22457/apam.v12n2a6.

[8] P. M. S. Selvam, K. T. Nagalakshmi, *On PMS-algebras*, **Transylvanian Review**, vol. 24(10) (2016), pp. 31–38.

[9] P. M. S. Selvam, K. T. Nagalakshmi, *Role of homomorphism and Cartesian product over Fuzzy PMS-algebra*, **International Journal of Fuzzy Mathematical Archive**, vol. 11(1) (2016), pp. 1622–1628, DOI: https://doi.org/10.22457/ijfma.v11n1a5.

[10] P. K. Sharma, *Homomorphism of intuitionistic fuzzy groups*, **International Mathematical Forum**, vol. 6(64) (2011), pp. 3169–3178, URL: http://www.m-hikari.com/imf-2011/61-64-2011/sharmapkIMF61-64-2011.pdf.

[11] P. K. Sharma, *On the direct product of Intuitionistic fuzzy groups*, **International Mathematical Forum**, vol. 7(11) (2012), pp. 523–530, DOI: https://doi.org/http://www.m-hikari.com/imf/imf-2012/9-12-2012/sharmapkIMF9-12-2012.pdf.

[12] L. A. Zadeh, *Fuzzy sets*, **Information and Control**, vol. 8 (1965), pp. 338–353, DOI: https://doi.org/10.1016/S0019-9958(65)90241-X.

Beza Lamesgin Derseh
Bahir Dar University
College of Science
Department of Mathematics
Bahir Dar, Ethiopia
e-mail: dbezalem@gmail.com

Berhanu Assaye Alaba
Bahir Dar University
College of Science
Department of Mathematics
Bahir Dar, Ethiopia
e-mail: birhanu.assaye290113@gmail.com

Yohannes Gedamu Wondifraw
Bahir Dar University
College of Science
Department of Mathematics
Bahir Dar, Ethiopia
e-mail: yohannesg27@gmail.com

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Daniel O. Martínez-Rivillas
Ruy J. G. B. de Queiroz

THE THEORY OF AN ARBITRARY HIGHER λ-MODEL

Abstract

One takes advantage of some basic properties of every homotopic λ-model (e.g. extensional Kan complex) to explore the higher $\beta\eta$-conversions, which would correspond to proofs of equality between terms of a theory of equality of any extensional Kan complex. Besides, Identity types based on computational paths are adapted to a type-free theory with higher λ-terms, whose equality rules would be contained in the theory of any λ-homotopic model.

Keywords: higher lambda calculus, homotopic lambda model, Kan complex reflexive, higher conversion, homotopy type-free theory.

2020 Mathematical Subject Classification: 03B70.

1. Introduction

In [4] and [5] the initiative is born to search for higher λ-models with non-trivial structure of ∞-groupoid, by using extensional Kan complexes $K \simeq [K \to K]$. In [3] the existence of higher non-trivial models is proved by solving homotopy domain equations.

If we understand an arbitrary higher λ-model as an extensional Kan complex, the following question arises: What would be the syntactic structure of the equality theory of any higher λ-model, i.e., is its equality theory a generalization of the $\beta\eta$-conversions to $(n)\beta\eta$-conversions in a set

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$\Lambda_{n-1}(a,b)$ by $(n)\beta\eta$-contractions induced by the extensionality from a Kan complex?

We shall see some consequences of the equality theory $Th(\mathcal{K})$ of an extensional Kan complex $\mathcal{K}$ with some examples of equality and nonequality of terms. This paves the way for a definition of the $(n)\beta\eta$-conversions, which will belong to the set of n-conversions Λ_n induced by the least theory of equality on all the extensional Kan complexes, here called *Homotopy Type-Free Theory* (*HoTFT*).

On the other hand, we define, from the identity types based on computational paths [1], the untyped theory of higher $\lambda\beta\eta$-equality TH-$\lambda\beta\eta$. We ask about the relationship between TH-$\lambda\beta\eta$ and HoTFT.

In this work we will try to answer these questions according to the following sections: In section 2, we explore the theory of any extensional Kan complex in order to generalize the $\beta\eta$-conversions to $(n)\beta\eta$-conversions in a set $\Lambda_{n-1}(a,b)$ by $(n)\beta\eta$-contractions induced by the extensionality from a Kan complex. In section 3, the identity types $Id_A(a,b)$ based on computational paths are taken into account, to define a type-free theory of higher $\lambda\beta\eta$-equality TH-$\lambda\beta\eta$ with λ^n-terms and n-redexes in a set $\Lambda^{n-1}(a,b)$ with $n \geq 1$. Finally, we look at the relationship of this TH-$\lambda\beta\eta$ with the least theory of equality on all the extensional Kan complexes HoTFT through the relationship between the sets Λ^n and Λ_n for each $n \geq 0$.

2. Theory of extensional Kan complexes

In this section, we shall see some consequences of the equality theory $Th(\mathcal{K})$ of an extensional Kan complex $\mathcal{K}$ with some examples of equality and nonequality of terms. This shall pave the way for a definition of the $(n)\beta\eta$-conversions, which will belong to the set of n-conversions Λ_n induced by the least theory of equality on all extensional Kan complexes, denoted by HoTFT.

DEFINITION 2.1 (∞-category [2]). An ∞-category is a simplicial set X which has the following property: for any $0 < i < n$, any map $f_0 : \Lambda^n_i \to X$ admits an extension $f : \Delta^n \to X$.

Here the simplicial set K is defined as a presheaf $\Delta^{op} \to Set$, with Δ being the *simplicial indexing category*, whose objects are finite ordinals

$[n] = \{0, 1, \dots, n\}$, and morphisms are the (non strictly) order preserving maps. Δ^n is the *standard n-simplex* defined for each $n \geq 0$ as the simplicial set $\Delta^n := \Delta(-, [n])$. And Λ_i^n is a *horn* defined as largest subobject of Δ^n that does not include the face opposing the i-th vertex.

DEFINITION 2.2. From the definition above, we have the following special cases:

- X is a Kan complex if there is an extension for each $0 \leq i \leq n$.
- X is a category if the extension exists uniquely [6].
- X is a groupoid if the extension exists for all $0 \leq i \leq n$ and is unique [6].

In other words, a Kan complex is an ∞-groupoid; composed of objects, 1-morphisms, 2-morphisms, ..., all those invertible.

Notation. For K a Kan simplex and $n \geq 0$, let $K_n = Fun(\Delta^n, K)$ be the Kan complex of the n-simplexes.

Let Var be the set of all variables of λ-calculus, for all $m, n \geq 0$, each assignment $\rho : Var \to K_n$ ($\rho(t)$ is an n-simplex of K, for each $t \in Var$), $x \in Var$ and $f \in K_m$, denote by $[f/x]\rho$ the assignment $\rho' : Var \to K$ which coincides with ρ, except on x, where ρ' takes the value f.

DEFINITION 2.3 (h.p.o. [5]). Let $\hat{K}$ be an ∞-category. The largest Kan complex $K \subseteq \hat{K}$ is a homotopy partial order (h.p.o.), if for every $x, y \in K$ one has that $\hat{K}(x, y)$ is contractible or empty. Hence, the Kan complex K admits a relation of h.p.o. $\precsim$ defined for each $x, y \in K$ as follows: $x \precsim y$ if $\hat{K}(x, y) \neq \emptyset$, hence the pair $(K, \precsim)$ is a h.p.o. (we denote simply by K).

DEFINITION 2.4 (c.h.p.o. [5]). Let K be an h.p.o.

1. An h.p.o. $X \subseteq K$ is directed if $X \neq \emptyset$ and for each $x, y \in X$, there exists $z \in X$ such that $x \precsim z$ and $y \precsim z$.
2. K is a complete homotopy partial order (c.h.p.o.) if
   (a) There are initial objects, i.e., $\perp \in K$ is a initial object if for each $x \in K$, $\perp \precsim x$.
   (b) For each directed $X \subseteq \mathcal{K}$ the supremum (or colimit) $\curlyvee X \in \mathcal{K}$ exists.

Definition 2.5 (Continuity [3]). Let K and K' be c.h.p.o.'s. A functor $F : K \to K'$ is continuous if $F(\bigvee X) \simeq \bigvee F(X)$, where $F(X)$ is the essential image.

Definition 2.6 ($CHPO$ [3]). Define the subcategory $CHPO \subseteq CAT_\infty$ whose objects are the c.h.p.o.'s and the morphisms are the continuous functors, where CAT_∞ is the ∞-category of the ∞-categories [2].

Definition 2.7 (Reflexive Kan complex[1] [5]). A quadruple $\langle K, F, G, \varepsilon\rangle$ is called a reflexive Kan complex, if K is a c.h.p.o. such that the full subcategory $[K \to K] \subseteq Fun(K, K)$ of the continuous functors is a retract of K, via the functors

$$F : K \to [K \to K], \qquad G : [K \to K] \to K$$

and the natural equivalence $\varepsilon : FG \to 1_{[K\to K]}$. If there is a natural equivalence $\eta : 1_K \to GF$, the quintuple $\langle K, F, G, \varepsilon, \eta\rangle$ represents an extensional Kan complex.

Just as the recursive Domain Equation $X \cong [X \to X]$ (in the category of the c.p.o's) has an implicit recursive definition of data-types, the "Homotopy Domain Equation" [3] $X \simeq [X \to X]$ (in the ∞-category $CHPO$) would also have a recursive definition of data-types. A recursively defined computational object (e.g., a proof by mathematical induction) would be of a higher order relative to the classical case, whose interpretation would be recursively defined by a sequence of partial functors $F_i : K \to K$, over a Kan complex K weakly ordered, which converges to a total functor $F : K \to K$, whose details are not among the objectives of this work, but will be developed in future works, when studying the semantics (case of inductive types) of the version of HoTT based on computational paths.

Example 2.8 ([3]). The c.h.p.o K_∞, which generalizes Dana Scott's c.p.o D_∞, is an extensional Kan complex, since K_∞ is a solution for the Homotopy Domain Equation $X \simeq [X \to X]$ in the ∞-category $CHPO$ of c.h.p.o's and continuous functors.

Thus, intuitively, from the computational point of view, we have that a Kan complex, which satisfies the Homotopy Domain Equation, is not only

[1] In [5] one can also see the relationship between the reflexive Kan complexes and syntactic homotopic λ-models, conceptually introduced in [4], analogously to the semantics of the classic λ-calculus; same for the relationship between complete partial orders (c.p.o.'s) and syntactic λ-models.

capable of verifying the computability of constructions typical of classical programming languages, as D_∞ does it, but it also has the advantage (over D_∞) of verifying the computability of higher constructions, such as a mathematical proof of some proposition, the proof of the equivalence between two proofs of the same proposition, etc.

Besides, in [3], several examples of extensional objects (Kan complexes) are presented in the Kleisli ∞-category $Kl(P)$.

DEFINITION 2.9 ([5]). Let K be a reflexive Kan complex (via the morphisms F, G).

1. For $f, g : \triangle^n \to K$ (or also $f, g \in K_n$) define the n-simplex

$$f \bullet_{\triangle^n} g = F(f)(g).$$

   In particular for vertices $a, b \in K$,

$$a \bullet b = a \bullet_{\triangle^0} b = F(a)(b),$$

   besides, $F(a) \bullet (-) = a \bullet (-)$ and $F(-)(b) = (-) \bullet b$ are functors on K, then for $f \in K_n$ one defines the n-simplexes

$$a \bullet f = F(a)(f), \qquad f \bullet b = F(f)(b).$$

2. For each $n \geq 0$, let ρ be a valuation at K_n. Define the interpretation $[\![\ ]\!]_\rho : \Lambda \to K_n$ by induction as follows
   (a) $[\![x]\!]_\rho = \rho(x)$,
   (b) $[\![MN]\!]_\rho = [\![M]\!]_\rho \bullet [\![N]\!]_\rho$,
   (c) $[\![\lambda x.M]\!]_\rho = G(\boldsymbol{\lambda} f.[\![M]\!]_{[f/x]\rho})$, where $\boldsymbol{\lambda} f.[\![M]\!]_{[f/x]\rho} = [\![M]\!]_{[-/x]\rho} : K \to K_n$.

Remark 2.10. Given $g \in K_n$ and $\rho : Var \to K_n$, the higher β-contraction is interpreted by

$$\begin{aligned} [\![\lambda x.M]\!]_\rho \bullet g &= G(\boldsymbol{\lambda} f.[\![M]\!]_{[f/x]\rho}) \bullet g \\ &= F(G(\boldsymbol{\lambda} f.[\![M]\!]_{[f/x]\rho}))(g) \\ &\xrightarrow{(\varepsilon_{\boldsymbol{\lambda} f.[\![M]\!]_{[f/x]\rho}})_g} (\boldsymbol{\lambda} f.[\![M]\!]_{[f/x]\rho})(g) \\ &= [\![M]\!]_{[g/x]\rho}, \end{aligned}$$

where $\varepsilon_{\boldsymbol{\lambda} f.[\![M]\!]_{[f/x]\rho}}$ is the natural equivalence, induced by ε, between the functors $F(G(\boldsymbol{\lambda} f.[\![M]\!]_{[f/x]\rho}), \boldsymbol{\lambda} f.[\![M]\!]_{[f/x]\rho} : K \to K_n$. Hence $(\varepsilon_{\boldsymbol{\lambda} f.[\![M]\!]_{[f/x]\rho}})_g$ is the equivalence induced by the n-simplex g in K.

Hence, if $\langle K, F, G, \varepsilon, \eta\rangle$ is extensional and $n = 0$, so that the β-contraction is modelled by $\varepsilon : FG \to 1$; the (reverse) η-contraction is modelled by $\eta : 1 \to GF$. Besides, if $n > 0$, we have that the natural equivalences ε and η will induce higher β-contractions and (reverse) η-contractions respectively, as we will see later.

Proposition 2.11. Let x, y, M, N, P be λ-terms. The interpretations of β-reductions

$$\begin{tikzcd}
(\lambda x.M)((\lambda y.N)P) \arrow[r, "1\beta"] \arrow[d, "1\beta"'] & {[(\lambda y.N)P/x]M} \arrow[d, "{[1\beta]}"] \\
(\lambda x.M)([P/y]N) \arrow[r, "1\beta"'] & {[([P/y]N)/x]M}
\end{tikzcd}$$

are equivalent in every reflexive Kan complex $\langle K, F, G, \varepsilon\rangle$.

Proof: Let $a = [\![P]\!]_\rho$, $[\![\lambda y.N]\!]_\rho \bullet a \xrightarrow{f} [\![N]\!]_{[a/y]\rho}$, $R = FG(\boldsymbol{\lambda} f.[\![M]\!]_{[f/x]\rho})$, $L = \boldsymbol{\lambda} f.[\![M]\!]_{[f/x]\rho}$ and $\varepsilon' = \varepsilon_{\boldsymbol{\lambda} f.[\![M]\!]_{[f/x]\rho}}$. One has that the natural equivalence $\varepsilon' : R \to L$ makes the following diagram (weakly) commute:

$$\begin{tikzcd}
R([\![\lambda y.N]\!]_\rho \bullet a) \arrow[r, "\varepsilon'_{[\![\lambda y.N]\!]_\rho \bullet a}"] \arrow[d, "R(f)"'] & L([\![\lambda y.N]\!]_\rho \bullet a) \arrow[d, "L(f)"] \\
R([\![N]\!]_{[a/y]\rho}) \arrow[r, "\varepsilon'_{[\![N]\!]_{[a/y]\rho}}"'] & L([\![N]\!]_{[a/y]\rho})
\end{tikzcd}$$

which, by Remark 2.10, corresponds to the (weakly) commutative diagram

$$\begin{tikzcd}
{[\![\lambda x.M]\!]_\rho \bullet ([\![\lambda y.N]\!]_\rho \bullet a)} \arrow[r, "\varepsilon'_{[\![\lambda y.N]\!]_\rho \bullet a}"] \arrow[d, "R(f)"'] & {[\![M]\!]_{[[\![\lambda y.N]\!]_\rho \bullet a/x]}} \arrow[d, "L(f)"] \\
{[\![\lambda x.M]\!]_\rho \bullet [\![N]\!]_{[a/y]\rho}} \arrow[r, "\varepsilon'_{[\![N]\!]_{[a/y]\rho}}"'] & {[\![M]\!]_{[[\![N]\!]_{[a/y]\rho}/x]}}
\end{tikzcd}$$

$\square$

Example 2.12. The λ-term $(\lambda x.u)((\lambda y.v)z)$ has two β-reductions:

$$\begin{tikzcd}
(\lambda x.u)((\lambda y.v)z) \arrow[r, "1\beta"] \arrow[d, "1\beta"'] & {[(\lambda y.v)z/x]u} \arrow[d, "{[1\beta]}"] \\
(\lambda x.u)([z/y]v) \arrow[r, "1\beta"'] & {[v/x]u}
\end{tikzcd}$$

making $u = M$, $v = N$ and $z = P$, by Proposition 2.11, the interpretations of these β-reductions are equivalent in all reflexive Kan complexes $\langle K, F, G, \varepsilon\rangle$.

Next, we shall give examples where the reductions of λ-terms are not equivalent.

Example 2.13. The λ-term $(\lambda x.(\lambda y.yx)z)v$ has the β-reductions

$$\begin{tikzcd}
(\lambda x.(\lambda y.yx)z)v \arrow[r, "1\beta"] \arrow[d, "1\beta"] & (\lambda y.yv)z \arrow[d, "1\beta"] \\
(\lambda x.zx)v \arrow[r, "1\beta"] & zv
\end{tikzcd}$$

Given a reflexive Kan complex $\langle K, F, G, \varepsilon\rangle$. Let $\rho(v) = c$, $\rho(z) = d$ vertices at K and $R = FG$. The interpretation of the β-reductions of $(\lambda x.(\lambda y.yx)z)v$ depends on solving the diagram equation

$$\begin{tikzcd}
R(\boldsymbol{\lambda} a.R(\boldsymbol{\lambda} b.b \bullet a)(d))(c) \arrow[r, "(\varepsilon_f)_c"] \arrow[d, "(R(?))_c"'] & R(\boldsymbol{\lambda} b.b \bullet c)(d) \arrow[d, "(\varepsilon_{\boldsymbol{\lambda} b.b\bullet c})_d"] \\
R(\boldsymbol{\lambda} a.d \bullet a)(c) \arrow[r, "(\varepsilon_g)_c"'] & d \bullet c
\end{tikzcd}$$

where $f = \boldsymbol{\lambda} a.R(\boldsymbol{\lambda} b.b \bullet a)(d)$ and $g = \boldsymbol{\lambda} a.d \bullet a$ are functors at $[K \to K]$. One has $h_a = (\varepsilon_{\boldsymbol{\lambda} b.b\bullet a})_d : f(a) \to g(a)$ for each vertex $a \in K$, but h_a is not necessarily a functorial equivalence in any reflexive Kan complex $\langle K, F, G, \varepsilon\rangle$ to get the diagram to commute:

$$\begin{tikzcd}
R(f)(c) \arrow[r, "(\varepsilon_f)_c"] \arrow[d, "(R(h?))_c"'] & f(c) \arrow[d, "h_c"] \\
R(g)(c) \arrow[r, "(\varepsilon_g)_c"'] & g(c)
\end{tikzcd}$$

Example 2.14. The λ-term $(\lambda z.xz)y$ has the $\beta\eta$-contractions

$$\begin{tikzcd}
(\lambda z.xz)y \arrow[r, bend left, "1\beta"] \arrow[r, bend right, "1\eta"'] & xy
\end{tikzcd}$$

Take an extensional Kan complex $\langle K, F, G, \varepsilon, \eta\rangle$. Let $\rho(x) = a$ and $\rho(y) = b$ be vertices of K. The interpretation of λ-term is given by: $[\![(\lambda z.xz)y]\!]_\rho = [\![\lambda z.xz]\!]_\rho \bullet b = G(\boldsymbol{\lambda} c.F(a)(c))\bullet b = G(F(a))\bullet b = (FGF)(a)(b)$. The interpretation of the $\beta\eta$-contractions corresponds to the degenerated diagrams

$$\begin{tikzcd}
(FGF)(a)(b) \arrow[d, "(\varepsilon_{F(a)})_b"'] \arrow[dr, "1"] & \\
F(a)(b) \arrow[r, "(F(\eta_a))_b"'] & (FGF)(a)(b)
\end{tikzcd}
\qquad
\begin{tikzcd}
F(a)(b) \arrow[d, "(F(\eta_a))_b"'] \arrow[dr, "1"] & \\
(FGF)(a)(b) \arrow[r, "(\varepsilon_{F(a)})_b"'] & F(a)(b)
\end{tikzcd}$$

But the diagrams do not necessarily commute in every extensional Kan complex $\langle K, F, G, \varepsilon, \eta\rangle$.

For examples of higher extensional λ-models see [3].

It is known that the types of HoTT correspond to ∞-groupoids. Taking advantage of this situation, for a reflexive Kan complex, let us define the theory of equality on that Kan complex (∞-groupoid) as follows.

Definition 2.15 (Theory of an extensional Kan complex). Let $\mathcal{K} = \langle K, F, G, \varepsilon, \eta\rangle$ be an extensional Kan complex. Define the theory of equality of $\mathcal{K}$ as the class

$$Th_1(\mathcal{K}) = \{M = N \mid [\![M]\!]_\rho \simeq [\![N]\!]_\rho \text{ for all } \rho : Var \to K\}$$

where $[\![M]\!]_\rho \simeq [\![N]\!]_\rho$ is the equivalence between vertices of K for some equivalence $[\![s]\!]_\rho : [\![M]\!]_\rho \to [\![N]\!]_\rho$, and "$s$" denotes the conversion between λ-terms M and N induced by $[\![s]\!]_\rho$ for all evaluation ρ.

In the Definition 2.15, notice that the equivalence $[\![M]\!]_\rho \simeq [\![N]\!]_\rho$ for all ρ, induces the intentional equality $M = N$, which can be seen as an identity type based on computational paths [1]; the conversion s may also be seen as a computational proof (a finite sequence of basic rewrites [1] induced by K) of the proposition $M = N$ in the theory $Th_1(\mathcal{K})$.

Remark 2.16. If s is a β-contraction or η-contraction and the functor F is not surjective for objects, the equality $M =_{1\beta} N : K$ or $M =_{1\eta} N : K$ is not necessarily a judgmental equality (as it happens in HoTT); $[\![M]\!]_\rho$ and $[\![N]\!]_\rho$ may be different vertices in K. Thus, the theory $Th_1(\mathcal{K})$ may be seen as the family of all the identity types which are inhabited by paths which are not necessarily equal to the reflexive path $\mathtt{refl}_M$.

Notation. Let M and N be λ-terms ($M, N \in \Lambda_0$) and $\mathcal{K}$ be an extensional Kan complex. Denote by $\Lambda_0(K)(M,N)$ the set of all the 1-conversions from M to N induced by $\mathcal{K}$. We write $\Lambda_1(\mathcal{K}) := \bigcup_{M,N\in\Lambda_0} \Lambda_0(\mathcal{K})(M,N)$ for the family of all 1-conversions induced by $\mathcal{K}$.

Let $s, t \in \Lambda_0(\mathcal{K})(M,N)$. Denote by $\Lambda_0(\mathcal{K})(M,N)(s,t)$ the set of all the 2-conversions from s to t. And let $\Lambda_2(\mathcal{K}) := \bigcup_{s,t\in\Lambda_1} \bigcup_{M,N\in\Lambda_0} \Lambda_0(\mathcal{K})(M,N)(s,t)$ be the family of all 2-conversions induced by $\mathcal{K}$, and so on we keep iterating for the families $\Lambda_3(\mathcal{K})$, $\Lambda_4(\mathcal{K}), \ldots$

Since $\mathcal{K}$ is a reflexive Kan complex, $Th_1(\mathcal{K})$ is an intentional λ-theory of 1-equality which contains the theory $\lambda\beta\eta$. Iterate again, we have the λ-theory of 2-equality

$$Th_2(\mathcal{K}) = \{r = s \mid \forall\rho\,([\![r]\!]_\rho \simeq [\![s]\!]_\rho) \text{ and } r, s \in \Lambda_0(\mathcal{K})(M,N)\}.$$

If we keep iterating, we can see that the reflexive Kan complex $\mathcal{K}$ will certainly induce a λ-theory of higher equality given by the inverse and direct limit

$$Th(\mathcal{K}) = \bigcup_{n\geq 1} Th_n(\mathcal{K}).$$

Just as $Th_1(\mathcal{K})$ contains $\lambda\beta\eta$, $Th(\mathcal{K})$ will contain a (simple version of) 'Homotopy Type-Free Theory', defined as follows.

Definition 2.17 (Homotopy Type-Free Theory). A Homotopy Type-Free Theory (HoTFT) consists of the least theory of equality, that is

$$HoTFT := \bigcap \{Th(\mathcal{K}) \mid \mathcal{K} \text{ is an extensional Kan complex}\}.$$

And for each $n \geq 0$ let

$$\Lambda_n := \bigcap \{\Lambda_n(\mathcal{K}) \,|\, \mathcal{K} \text{ is an extensional Kan complex}\}$$

be the set of $n\beta\eta$-conversions.

For example, let $\mathcal{K} = \langle K, F, G, \varepsilon, \eta \rangle$ be an extensional Kan complex and x, M and N λ-terms. By Definition 2.17, the β-contraction $(\lambda x.M)N \xrightarrow{1\beta} [N/x]M$ inhabits the set $\Lambda_0((\lambda x.M)N, [N/x]M)$;

$$[\![1\beta]\!]_\rho = (\varepsilon_{[\![M]\!]_{[-/x]\rho}})_{[\![N]\!]_\rho} \in K([\![(\lambda x.M)N]\!]_\rho, [\![[N/x]M]\!]_\rho),$$

and the η-contraction $\lambda x.Mx \xrightarrow{1\eta} M$, $x \notin FV(M)$, belongs to $\Lambda_0(K)(\lambda x.Mx, M)$;

$$[\![1\eta]\!]_\rho = \eta_{[\![M]\!]_\rho} \in K([\![\lambda x.Mx]\!]_\rho, [\![M]\!]_\rho).$$

If t is a $\beta\eta$-conversion from λ-term M to N, by Definition 2.17, $t \in \Lambda_0(M, N)$. For x, P λ-terms, we have the vertices $[\![\lambda x.P]\!]_\rho \in K$ and $[\![t]\!]_\rho \in K([\![M]\!]_\rho, [\![N]\!]_\rho)$. Thus, $[\![(\lambda x.P)t]\!]_\rho = [\![\lambda x.P]\!]_\rho \bullet [\![t]\!]_\rho \in K([\![(\lambda x.P)M]\!]_\rho,$ $[\![(\lambda x.P)N]\!]_\rho)$ and $[\![P]\!]_{[[\![t]\!]_\rho/x]\rho} \in K([\![P]\!]_{[[\![M]\!]_\rho/x]\rho}, [\![P]\!]_{[[\![N]\!]_\rho/x]\rho})$, where $[[\![t]\!]_\rho/x]\rho : Var \to K_1$ is an evaluation $\rho'(x) = [\![t]\!]_\rho$ and (n-times degeneration of vertex $\rho(r)$) $\rho'(r) = s^n(\rho(r))$ if $r \neq x$. By Definition 2.17, $(\lambda x.P)t \in \Lambda_0((\lambda x.P)M, (\lambda x.P)N)$ and $[\![P]\!]_{[[\![t]\!]_\rho/x]\rho} \in K([\![P]\!]_{[[\![M]\!]_\rho/x]\rho}, [\![P]\!]_{[[\![N]\!]_\rho/x]\rho})$. But

$$[\![(\lambda x.P)t]\!]_\rho \xrightarrow{(\varepsilon_{[\![P]\!]_{[-/x]\rho}})_{[\![t]\!]_\rho}} [\![P]\!]_{[[\![t]\!]_\rho/x]\rho},$$

So $(\lambda x.P)t = [t/x]P$ and induces the 2β-contraction

$$(\lambda x.P)t \xrightarrow{2\beta_{P,t}} [t/x]P,$$

corresponding to a similar diagram to that of Proposition 2.11, i.e.,

$$\begin{tikzcd}
(\lambda x.P)M \arrow[r, "1\beta_M"] \arrow[d, "(\lambda x.P)t"'] \arrow[dr, phantom, "\Longrightarrow_{2\beta_t}"] & {[M/x]P} \arrow[d, "{[t/x]M}"] \\
(\lambda x.P)N \arrow[r, "1\beta_N"'] & {[N/x]P}
\end{tikzcd}$$

Hence $2\beta_t \in \Lambda_0((\lambda x.P)M, [N/x]P)(\tau(1\beta_M, [t/x]M), \tau((\lambda x.P)t, 1\beta_N))$, where $\tau(r,s)$ is the concatenation of the conversions $r \in \Lambda(a,b)$ and $s \in \Lambda(b,c)$. On the other hand, for $y \notin FV(t)$ one has the equivalence

$$[\![t]\!]_\rho \xrightarrow{\eta_{[\![t]\!]_\rho}} [\![\lambda y.ty]\!]_\rho,$$

that is, $(\lambda y.ty) = t$ and induces the 2η-contraction

$$(\lambda y.ty) \xrightarrow{2\eta_t} t,$$

which corresponds to the diagram

$$\begin{tikzcd}
\lambda y.My \arrow[r, "n\eta_r"] \arrow[d, "\lambda y.ty"'] \arrow[dr, phantom, "\Longrightarrow_{2\eta_t}"] & M \arrow[d, "t"] \\
\lambda y.Ny \arrow[r, "n\eta_s"'] & N
\end{tikzcd}$$

In general, if $t \in \Lambda_{n-1}$, the equivalences

$$[\![(\lambda x.P)t]\!]_\rho \xrightarrow{(\varepsilon_{[\![P]\!]_{[-/x]\rho}})_{[\![t]\!]_\rho}} [\![P]\!]_{[[\![t]\!]_\rho/x]\rho}, \qquad [\![t]\!]_\rho \xrightarrow{\eta_{[\![t]\!]_\rho}} [\![\lambda y.ty]\!]_\rho$$

in every extensional Kan complex K, induce the $(n)\beta\eta$-contractions

$$(\lambda x.P)t \xrightarrow{n\beta_t} [t/x]P, \qquad (\lambda y.ty) \xrightarrow{n\eta_t} t.$$

which explains the following Corollary.

COROLLARY 2.18. If x, y, P be λ-terms, $n \geq 1$ and $t \in \Lambda_n(r,s)$ with $y \notin FV(t)$, then the interpretation from diagrams

$$\begin{tikzcd}
(\lambda x.P)r \arrow[r, "n\beta_r"] \arrow[d, "(\lambda x.P)t"'] & {[r/x]P} \arrow[d, "{[t/x]M}"] \\
(\lambda x.P)s \arrow[r, "n\beta_s"'] & {[s/x]P}
\end{tikzcd} \qquad
\begin{tikzcd}
\lambda y.ry \arrow[r, "n\eta_r"] \arrow[d, "\lambda y.ty"'] & r \arrow[d, "t"] \\
\lambda y.sy \arrow[r, "n\eta_s"'] & s
\end{tikzcd}$$

commutes in every extensional Kan complex K.

Thus, any reflexive Kan complex inductively induces, for each $n \geq 1$, from an $(n)\beta\eta$-conversion t to the $(n+1)\beta\eta$-contractions

$$\begin{tikzcd}[column sep=large]
(\lambda x.P)r \arrow[r, "n\beta_r"] \arrow[d, "(\lambda x.P)t"'] \arrow[dr, phantom, "\Longrightarrow_{(n+1)\beta_t}"] & {[r/x]P} \arrow[d, "{[t/x]M}"] \\
(\lambda x.P)s \arrow[r, "n\beta_s"'] & {[s/x]P}
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=large]
\lambda y.ry \arrow[r, "n\eta_r"] \arrow[d, "\lambda y.ty"'] \arrow[dr, phantom, "\Longrightarrow_{(n+1)\eta_t}"] & r \arrow[d, "t"] \\
\lambda y.sy \arrow[r, "n\eta_s"'] & s
\end{tikzcd}$$

and these, in their turn, define the $(n+1)\beta\eta$-conversions, of $(n)\beta\eta$-conversion, which would inhabit the set Λ_{n+1}.

3. Extensional Kan complexes and Identity types based on higher λ-terms

In this section, we use the extensionality of any extensional Kan complex K to define the set of λ^n-terms $\Lambda^{n-1}(a,b)$ induced by the space $K_{n-1}(\llbracket a \rrbracket_\rho, \llbracket b \rrbracket_\rho)$, which would be a type-free version of the identity type $Id_A(a,b)$ based on computational paths of [1]. And finally we see the relationship between the set Λ^n of all the λ^n-terms and the set Λ_n from the previous section.

By Definition of Cartesian product of simplicial sets one has that for each $n \geq 0$, $(K \times K)_n = K_n \times K_n$. If $\mathcal{K} = \langle K, F, G, \varepsilon, \eta \rangle$ is an extensional Kan complex, then $K_n \times K_n \simeq K_n$, that is $K_n \simeq [K_n \to K_n]$. Hence $\mathcal{K}_n = \langle K_n, F, G, \varepsilon, \eta \rangle$ is an extensional Kan complex for each $n \geq 0$.

For example the case $n = 1$, one has that $\llbracket 1\beta \rrbracket_\rho, \llbracket 1\eta \rrbracket_\rho \in K_1$, that is $1\beta, 1\eta$ would be 'λ^1-terms'. Hence, for any $\beta\eta$-conversion r between λ-terms, $\llbracket r \rrbracket_\rho \in K_1$, i.e., r would be also a 'λ^1-term' (denoted by $r \in \Lambda^1$). If $h(r)$ is a $\beta\eta$-conversion which depends on the $\beta\eta$-conversion r, by extensionality of K_1, one has

$$\llbracket \lambda^1 r.h(r) \rrbracket_\rho := G(\llbracket h(r) \rrbracket_{[-/r]_\rho}) \in K_1,$$

where $\llbracket h(r) \rrbracket_{[-/r]\rho} : K_1 \to K_1$.

Thus, for $m, r \in \Lambda^0(c,d)$ (λ^1-terms from c to d) the 'λ^1-term' $\lambda^1 r.h(r)$ can define the β_2-contraction

$$(\lambda^1 r.h(r))m \xrightarrow{\beta_2} h(m/r)$$

where

$$[\![(\lambda^1 r.h(r))m]\!]_\rho := [\![\lambda^1 r.h(r)]\!]_\rho \bullet_{\Delta^1} [\![m]\!]_\rho = F([\![\lambda^1 r.h(r)]\!]_\rho)([\![m]\!]_\rho) \in K_1,$$

hence, $(\lambda^1 r.h(r))m$ can be seen as a λ^1-term.

The question arises: $[\![\beta_2]\!]_\rho \in K_2$? To answer this question, let us first prove the following proposition.

Proposition 3.1. Let $\mathcal{K} = \langle K, F, G, \varepsilon, \eta \rangle$ be an extensional Kan complex. For each vertex $a, b, c, d \in K$ one has an equivalence of homotopy

$$K(a,b) \simeq [K(c,d) \to K(a \bullet c, b \bullet d)],$$

and in general, for $n \geq 1$ and the vertices $a_{i+1}, b_{i+1} \in K(a_0, b_0) \cdots (a_i, b_i)$ and $c_{i+1}, d_{i+1} \in K(c_0, d_0) \cdots (c_i, d_i)$ with $0 \leq i \leq n-1$, there is an equivalence

$$\begin{aligned} K(a_0, b_0) \cdots (a_n, b_n) \simeq [&K(c_0, d_0) \cdots (c_n, d_n) \to \\ &K(a_0 \bullet c_0, b_0 \bullet d_0) \cdots (a_n \bullet c_n, b_n \bullet d_n)] \end{aligned}$$

Proof: Since $\mathcal{K}$ is extensional, there is the equivalence $F' : K \times K \to K$. Hence

$$K(a,b) \times K(c,d) = (K \times K)((a,c),(b,d)) \simeq K(F'(a,c), F'(b,d)),$$

that is,

$$K(a,b) \simeq [K(c,d) \to K(F(a)(c), F(b)(d))] = [K(c,d) \to K(a \bullet c, b \bullet d)].$$

Let $K_n(p_n, q_n) = K(p_0, q_0) \cdots (p_n, q_n)$ for each $p_i, q_i \in K_i$ with $0 \leq i \leq n$. Given the Induction Hypothesis (IH)

$$K_n(a_n, b_n) \times K_n(c_n, d_n) \simeq K_n(F'(a_n, c_n), F'(b_n, d_n)),$$

for the case $(n+1)$ one has

$$\begin{aligned} &K_{n+1}(a_{n+1}, b_{n+1}) \times K_{n+1}(c_{n+1}, d_{n+1}) = \\ &= K_n(a_n, b_n)(a_{n+1}, b_{n+1}) \times K_n(c_n, d_n)(c_{n+1}, d_{n+1}) \\ &= (K_n(a_n, b_n) \times K_n(c_n, d_n))((a_{n+1}, c_{n+1}), (b_{n+1}, d_{n+1})) \\ &\simeq K_n(F'(a_n, c_n), F'(b_n, d_n))(F'(a_{n+1}, c_{n+1}), F'(b_{n+1}, d_{n+1})) \qquad \text{(by I.H)} \\ &= K_{n+1}(F'(a_{n+1}, c_{n+1}), F'(b_{n+1}, d_{n+1})). \end{aligned}$$

Thus,

$$\begin{aligned}
&K_{n+1}(a_{n+1}, b_{n+1}) \\
&\simeq [K_{n+1}(c_{n+1}, d_{n+1}) \to K_{n+1}(F(a_{n+1})(c_{n+1}), F(b_{n+1})(d_{n+1}))] \\
&= [K_{n+1}(c_{n+1}, d_{n+1}) \to K_{n+1}(a_{n+1} \bullet c_{n+1}, b_{n+1} \bullet d_{n+1})]. \qquad \square
\end{aligned}$$

Therefore, the Proposition 3.1 allows the following definition.

DEFINITION 3.2. Let $\mathcal{K} = \langle K, F, G, \varepsilon, \eta \rangle$ be an extensional Kan complex and ρ be a valuation in K. For the $\beta\eta$-conversions $r, s, h(r)$ such that $[\![r]\!]_\rho \in K(c,d)$, $[\![s]\!]_\rho \in K(a,b)$ and $[\![h(r)]\!]_\rho \in K(a \bullet c, b \bullet d)$, define the interpretation by induction as follows

1. $[\![r]\!]_\rho \in K(c,d)$ is a concatenation of morphisms

$$c \xrightarrow{f_1} c_1 \xrightarrow{f_2} c_2 \xrightarrow{f_3} \cdots \xrightarrow{f_m} d$$

   where each f_i depends on: $(\varepsilon_g)_a : F(G(g))(a) \to g(a)$ (interprets each β-contraction of r) or $\eta_b : b \to G(F(b))$ (interprets each inverted η-contraction of r), with $g \in [K \to K]$ and $a, b \in K$,

2. $[\![sr]\!]_\rho = [\![s]\!]_\rho \bullet_{\Delta^1} [\![r]\!]_\rho = F([\![s]\!]_\rho)([\![r]\!]_\rho) \in K(a \bullet c, b \bullet d)$,

3. $[\![\lambda^1 r.h(r)]\!]_\rho = G([\![h(r)]\!]_{[-/r]\rho}) \in K(a,b)$ where $[\![h(r)]\!]_{[-/r]\rho} : K(c,d) \to K(a \bullet c, b \bullet d)$.

   Take $n \geq 2$. For the $(\beta\eta)_n$-conversions (Definition 3.4) r, s and $h(r)$ such that $[\![r]\!]_\rho \in K_{n-1}(c_{n-1}, d_{n-1})$, $[\![s]\!]_\rho \in K_{n-1}(a_{n-1}, b_{n-1})$ and $[\![h(r)]\!]_\rho \in K_{n-1}(a_{n-1} \bullet c_{n-1}, b_{n-1} \bullet d_{n-1})$, define the interpretation

4. $[\![r]\!]_\rho \in K_{n-1}(c_{n-1}, d_{n-1})$ is a concatenation of n-simplexes

$$c_{n-1} \xrightarrow{f_1} s_1 \xrightarrow{f_2} s_2 \xrightarrow{f_3} \cdots \xrightarrow{f_m} d_{n-1}$$

   where each f_i depends on: $(\varepsilon_g)_e : F(G(g))(e) \to g(e)$ (interprets each β_n-contraction of r) or $\eta_{e'} : e' \to G(F(e'))$ (interprets each inverted η_n-contraction of r), with

   $g : K_{n-1}(c_{n-1}, d_{n-1}) \to K_{n-1}(a_{n-1} \bullet c_{n-1}, b_{n-1} \bullet d_{n-1})$, $e \in K_{n-1}(c_{n-1}, d_{n-1})$ and $e' \in K_{n-1}(a_{n-1}, b_{n-1})$,

5. $[\![sr]\!]_\rho = [\![s]\!]_\rho \bullet_{\Delta^n} [\![r]\!]_\rho = F([\![s]\!]_\rho)([\![r]\!]_\rho) \in K(a \bullet c, b \bullet d)$,

6. $[\![\lambda^n r.h(r)]\!]_\rho = G([\![h(r)]\!]_{[-/r]_\rho}) \in K(a,b)$ where
   $[\![h(r)]\!]_{[-/r]_\rho} : K_{n-1}(c_{n-1}, d_{n-1}) \to K_{n-1}(a_{n-1} \bullet c_{n-1}, b_{n-1} \bullet d_{n-1})$.

Going back to the question: $[\![\beta_2]\!]_\rho \in K_2$? Since $[\![\lambda^1 r.h(r)]\!]_\rho \in K_1$, so there are vertices $a, b \in K$ such that $[\![\lambda^1 r.h(r)]\!]_\rho \in K(a,b)$. If $[\![r]\!]_\rho, [\![m]\!]_\rho \in K(c,d)$, by Definition 3.2 (2), $[\![(\lambda^1 r.h(r))m]\!]_\rho, [\![h(m/r)]\!]_\rho \in K(a \bullet c, b \bullet d)$. Hence,

$$[\![\beta_2]\!]_\rho \in K(a \bullet c, b \bullet d)(a_1, b_1) \subseteq K_2,$$

where $a_1 = [\![(\lambda^1 r.h(r))m]\!]_\rho$ and $b_1 = [\![h(m/r)]\!]_\rho$.

For the question: $[\![\eta_2]\!]_\rho \in K_2$? Let $e \in K(a,b)$ which does not depend on $r \in K(c,d)$. By Definition 3.2 (2), $[\![er]\!]_\rho \in K(a \bullet c, b \bullet d)$. By Definition 3.2 (3), $[\![\lambda^1 r.er]\!]_\rho \in K(a,b)$. Then,

$$[\![\eta_2]\!]_\rho \in K(a,b)(a_1, b_1) \subseteq K_2,$$

where $a_1 = [\![\lambda^1 r.er]\!]_\rho$ and $b_1 = [\![e]\!]_\rho$.

Therefore, the $(\beta\eta)_2$-conversions are λ^2-terms, which in turn define inductively other λ^2-terms by application and abstraction. We can continue iterating and have the following proposition, to prove that the Definition 3.2 (4) is well defined for all $n \geq 2$.

Proposition 3.3. Let K be an extensional Kan complex and $\rho : Var \to K$ be an evaluation. For each $n \geq 1$, $[\![\beta_n]\!]_\rho, [\![\eta_n]\!]_\rho \in K_n$.

Proof: If $n = 1$, one has that $[\![\beta_1]\!]_\rho = [\![1\beta]\!]_\rho \in K_1$ and $[\![\eta_1]\!]_\rho = [\![1\eta]\!]_\rho \in K_1$. Suppose that $[\![\beta_n]\!]_\rho, [\![\eta_n]\!]_\rho \in K_n$. So, induce the λ^n-terms: $r, m \in \Lambda_{n-1}(c_{n-1}, d_{n-1})$ and $\lambda^n r.h(r) \in \Lambda_{n-1}(a_{n-1}, b_{n-1})$. By Proposition 3.1 and Definition 3.2 (5), $[\![(\lambda^n r.h(r))m]\!]_\rho, [\![h(m/r)]\!]_\rho \in K_{n-1}(a_{n-1} \bullet c_{n-1}, b_{n-1} \bullet d_{n-1})$. Thus,

$$[\![\beta_{n+1}]\!]_\rho \in K_{n-1}(a_{n-1} \bullet c_{n-1}, b_{n-1} \bullet d_{n-1})(a_n, b_n) \subseteq K_{n+1},$$

where $a_n = [\![(\lambda^n r.h(r))m]\!]_\rho$ and $b_n = [\![h(m/r)]\!]_\rho$.

By I.H, let the λ^n-term: $[\![e]\!]_\rho \in K_{n-1}(a_{n-1}, b_{n-1})$ which does not depend on $[\![r]\!]_\rho \in K_{n-1}(c_{n-1}, d_{n-1})$. By Definition 3.2 (5), $[\![er]\!]_\rho \in K_{n-1}(a_{n-1} \bullet c_{n-1}, b_{n-1} \bullet d_{n-1})$. By Definition 3.2 (6), $[\![\lambda^n r.er]\!]_\rho \in K_{n-1}(a_{n-1}, b_{n-1})$. So,

$$[\![\eta_{n+1}]\!]_\rho \in K_{n-1}(a_{n-1}, b_{n-1})(a_n, b_n) \subseteq K_{n+1},$$

where $a_n = [\![\lambda^n r.er]\!]_\rho$ and $b_n = [\![e]\!]_\rho$. $\square$

Of course, Definition 3.2 depends on the syntax of higher lambda-terms. Next, we define a 'Theory of higher $\lambda\beta\eta$-equality' as a type-free version of the computational paths of [1].

Definition 3.4 (Theory of higher $\lambda\beta\eta$-equality). A theory of higher $\lambda\beta\eta$-equality (TH-$\lambda\beta\eta$) consists of rules and axioms of the theory of $\beta\eta$-equality ($\beta\eta$-conversions or in our case we write $(\beta\eta)_1$-conversions) between λ-terms, whose set we denote here by Λ^0, and the rules which define the higher $\beta\eta$-conversions in the following sense:

- (1-introduction and 1-formation rules). s is a $(\beta\eta)_1$-conversion from λ-term a to λ-term b (denoted by $a =_s b \in \Lambda^0$) if s is a usual $(\beta\eta)$-conversion from a to b, and we say that all $(\beta\eta)_1$-conversion is a λ^1-term.

  Let $c =_m d \in \Lambda^0$ and $[c =_r d \in \Lambda^0]$ $ac =_{h(r)} bd \in \Lambda_0$. Then $\lambda^1 r.h(r)$ is a λ^1-term from a to b, i.e., $\lambda^1 r.h(r) \in \Lambda^0(a,b)$ and $(\lambda^1 r.h(r))m$ is a λ^1-term from ac to bd, i.e., $(\lambda^1 r.h(r))m \in \Lambda^0(ac,bd)$. Let Λ^1 the set of the λ^1-terms.

- (Reduction rule). Let the λ^{n+1}-terms $m \in \Lambda^n(c,d)$, $[r \in \Lambda^n(c,d)]$ and $h(r) \in \Lambda^n(ac,ad)$. Define the λ^{n+1}-term: $\lambda^{n+1} r.h(r) \in \Lambda^n(a,b)$ and the β_{n+2}-contraction

$$(\lambda^{n+1} r.h(r))m \xrightarrow{\beta_{n+2}} h(m/r) \in \Lambda^n(ac,bd).$$

- (Induction rule). If $t \in \Lambda^n(c,d)$ and $e \in \Lambda^n(a,b)$, then η_{n+2}-contraction is given by

$$\lambda^{n+1} t.et \xrightarrow{\eta_{n+2}} e \in \Lambda^n(a,b),$$

  where e does not depend on t.

- ($(n+2)$-Introduction and $(n+2)$-formation rules). If s is a $(\beta\eta)_{n+2}$-conversion (sequence, it can be empty, of β_{n+2}-contractions or reversed β_{n+2}-contractions or η_{n+2}-contractions or reversed η_{n+2}-contractions) from a to b in Λ^{n+1}, that is $a =_s b \in \Lambda^{n+1}$, then $s \in \Lambda^{n+1}(a,b)$. We say that s is a λ^{n+2}-term if it is a $(\beta\eta)_{n+2}$-conversion.

  Let $m \in \Lambda^{n+1}(c,d)$ and $[c =_r d \in \Lambda^{n+1}]$. Then one has the λ^{n+2}-terms: $\lambda^{n+2} r.h(r) \in \Lambda^{n+1}(a,b)$ and $(\lambda^{n+2} r.h(r))m \in \Lambda^{n+1}(ac,bd)$. Let Λ^{n+2} be the set of the λ^{n+2}-terms.

Proposition 3.5. Let $\mathcal{K} = \langle K, F, G, \varepsilon, \eta\rangle$ be an extensional Kan complex and $\rho : Var \to K$ be an evaluation. The $(n+1)$-simplexes space $K_n([\![p]\!]_\rho, [\![q]\!]_\rho)$ models the set of λ^{n+1}-terms $\Lambda^n(p, q)$.

Proof:

- (1-Formation and 1-introduction rules). Since K is a Kan complex and $p, q \in \Lambda^0$, then $[\![p]\!]_\rho, [\![q]\!]_\rho \in K$ (vertices of K) and $K(p, q)$ is also a Kan complex.

  Let $p =_s q \in \Lambda^0$ be a $(\beta\eta)_1$-conversion. Since K is an extensional Kan complex, by Definition 3.2 the interpretation

  $$[\![s]\!]_\rho : [\![p]\!]_\rho \xrightarrow{f_1} [\![p^1]\!]_\rho \xrightarrow{f_2} [\![p^2]\!]_\rho \xrightarrow{f_3} \cdots \xrightarrow{f_m} [\![q]\!]_\rho$$

  is a concatenation of morphisms in K such that each f_i corresponds to a morphism which depends on a map of the form: $(\varepsilon_g)_a : F(G(g))(a) \to g(a)$ (models the β_1-contraction) or $\eta_b : b \to G(F(b))$ (models the reversed η_1-contraction), where $a, b \in K$ and $g \in [K \to K]$. Thus $[\![s]\!]_\rho \in K([\![p]\!]_\rho, [\![q]\!]_\rho)$.

  Let $m \in \Lambda^0(s, t)$ and $[s =_r t \in \Lambda^0]\ \lambda^1 r.h(r) \in \Lambda^0(p, q)$. Since $\mathcal{K}$ is extensional, by Definition 3.2

  $$[\![(\lambda^1 r.h(r))m]\!]_\rho = F(G([\![h(r)]\!]_{[-/r]\rho}))([\![m]\!]_\rho) \in K([\![ps]\!]_\rho, [\![qt]\!]_\rho).$$

- (Reduction rule). Let $m \in \Lambda^n(s, t)$ and $[s =_r t \in \Lambda^n]\ \lambda^{n+1} r.h(r) \in \Lambda^n(p, q)$. Since K is extensional, the β_{n+2}-contraction

  $$(\lambda^{n+1} r.h(r))m \xrightarrow{\beta_{n+2}} h(m/r) \in \Lambda^n(ps, qt)$$

  corresponds to morphism in $K_n([\![ps]\!]_\rho, [\![qt]\!]_\rho)$ ($(n+2)$-simplex at K):

  $$F(G([\![h(r)]\!]_{[-/r]\rho}))([\![m]\!]_\rho) \xrightarrow{(\varepsilon_{[\![h(r)]\!]_{[-/r]\rho}})_{[\![m]\!]_\rho}} [\![h(m/r)]\!]_\rho.$$

- (Induction rule). Let $r \in \Lambda^n(p, q)$ and $e \in \Lambda^n(p, q)$. Since K is extensional, the η_{n+2}-contraction

  $$\lambda^{n+1} t.et \xrightarrow{\eta_{n+2}} e \in \Lambda^n(p, q)$$

  corresponds to morphism in $K_n([\![p]\!]_\rho, [\![q]\!]_\rho)$:

$$G(F(\llbracket e \rrbracket_\rho)) \xrightarrow{\tilde{\eta}_{\llbracket e \rrbracket_\rho}} \llbracket e \rrbracket_\rho,$$

where $\tilde{\eta}_{\llbracket e \rrbracket_\rho}$ is an inverse (up to homotopy) from $(n+2)$-simplex $\eta_{\llbracket e \rrbracket_\rho}$ in K.

- ($(n+2)$-Introduction and $(n+2)$-Formation rules). Take the $(\beta\eta)_{n+2}$-conversion $s =_r t \in \Lambda^{n+1}$. Since K is an extensional Kan complex, by Definition 3.2 the interpretation

$$\llbracket r \rrbracket_\rho : \llbracket s \rrbracket_\rho \xrightarrow{f_1} \llbracket s^1 \rrbracket_\rho \xrightarrow{f_2} \llbracket s^2 \rrbracket_\rho \xrightarrow{f_3} \cdots \xrightarrow{f_m} \llbracket t \rrbracket_\rho$$

  is a concatenation of morphisms in K_{n+1} such that each f_i corresponds to a morphism which depends on a map of the form: $(\varepsilon_g)_e : F(G(g))(e) \to g(e)$ (models the β_{n+2}-contraction) or $\eta_{e'} : e' \to G(F(e'))$ (models the reversed η_{n+2}-contraction), where $e \in K_{n+1}(c_n, d_n)$, $e' \in K_{n+1}(a_n, b_n)$ and $g : K_{n+1}(c_n, d_n) \to K_{n+1}(a_n \bullet c_n, b_n \bullet d_n)$. Thus $\llbracket r(s,t) \rrbracket_\rho \in K_{n+1}(\llbracket s \rrbracket_\rho, \llbracket t \rrbracket_\rho)$.

  Let $m \in \Lambda^{n+1}(s,t)$ and $[s =_r t : A]\ \lambda^{n+2} r.h(r) \in \Lambda^{n+1}(p,q)$. Since $\mathcal{K}$ is extensional, by Definition 3.2

$$\llbracket (\lambda^{n+2} r.h(r))m \rrbracket_\rho = F(G(\llbracket h(r) \rrbracket_{[-/r]_\rho}))(\llbracket m \rrbracket_\rho) \in K_{n+1}(\llbracket ps \rrbracket_\rho, \llbracket qt \rrbracket_\rho).$$

□

Example 3.6. Let $c =_m d \in \Lambda^0$ and $[c =_r d \in \Lambda^0]\ ac =_{h(r)} bd \in \Lambda^0$, thus $\lambda^1 r.h(r) \in \Lambda^0(a,b)$. The β_2-contraction is 2-dimensional. It can be represented by the diagram

$$\begin{tikzcd} ac \arrow[r, "1"] \arrow[d, "(\lambda^1 r.h(r))m"'] \arrow[dr, phantom, "\Longrightarrow_{\beta_2}"] & ac \arrow[d, "h(m/r)"] \\ bd \arrow[r, "1"'] & bd \end{tikzcd}$$

Since the interpretation of $\lambda^1 r.h(r) \in \Lambda^0(a,b)$ is given by

$$\llbracket \lambda^1 r.h(r) \rrbracket_\rho = G(\llbracket h(r) \rrbracket_{[-/r]\rho}) \in K(\llbracket a \rrbracket_\rho, \llbracket b \rrbracket_\rho)$$

for every extensional Kan complex $\mathcal{K}$ and ρ, by Definition 2.17 one has $\lambda^1 r.h(r) \in \Lambda_0(a,b)$. And the interpretation of the application $\lambda^1 r.h(r))m$ is given by

$$\begin{aligned}[\![(\lambda^1 r.h(r))m]\!]_\rho &= [\![\lambda^1 r.h(r)]\!]_\rho \bullet_{\Delta^1} [\![m]\!]_\rho \\ &= F([\![\lambda^1 r.h(r)]\!]_\rho)([\![m]\!]_\rho) \in K([\![ac]\!]_\rho, [\![bd]\!]_\rho)\end{aligned}$$

for all extensional Kan complex $\mathcal{K}$ and ρ. By Definition 2.17 $(\lambda^1 r.h(r))m \in \Lambda_0(ac, bd)$. Therefore $\Lambda^1 = \Lambda_1$.

Follow the question: $\beta_2 \in \Lambda_2$? By Proposition 3.5 (Reduction rule for $n = 0$) the β_2-contraction is interpreted by the 2-simplex

$$F(G([\![h(r)]\!]_{[-/r]\rho}))([\![m]\!]_\rho) \xrightarrow{(\varepsilon_{[\![h(r)]\!]_{[-/r]\rho}})_{[\![m]\!]_\rho}} [\![h(m/r)]\!]_\rho \in K([\![ac]\!]_\rho, [\![bd]\!]_\rho)$$

for all extensional Kan complex $\mathcal{K}$ and evaluation ρ. By Definition 2.17 one has $\beta_2 \in \Lambda_0(ac, bd)(\lambda^1 r.h(r))m, h(m/r))$. Hence $\beta_2 \in \Lambda_2$.

One the other hand, by Proposition 3.5 (Induction rule for $n = 0$) and the same reasoning from previous example, it can be proved that $\eta_2 \in \Lambda_2$, so $\Lambda^2 \subseteq \Lambda_2$. Thus making use of Definitions 2.9 and 2.17 and Proposition 3.5 we can prove in the same way as the previous example, the following proposition.

PROPOSITION 3.7. For each $n \geq 0$, $\Lambda^n \subseteq \Lambda_n$. Hence TH-$\lambda\beta\eta \subseteq HoTFT$.

4. Conclusion

We define the interpretation of the $\beta\eta$-contractions in an extensional Kan complex, whose ∞-groupoid structure induces higher $\beta\eta$-contractions, which consolidate a type-free version of HoTT, which we call HoTFT (Homotopy Type-Free Theory), which could have the advantage of rescuing the $\beta\eta$-conversions as relations of intentional equality and not as relations of judgmental equality as is the case in HoTT.

Besides, we define, from the identity types based on computational paths, the untyped theory of higher $\lambda\beta\eta$-equality TH-$\lambda\beta\eta$, which is contained in HoTFT.

References

[1] R. de Queiroz, A. de Oliveira, A. Ramos, *Propositional equality, identity types, and direct computational paths*, **South American Journal of Logic**, vol. 2(2) (2016), pp. 245–296.

[2] J. Lurie, **Higher Topos Theory**, Princeton University Press, Princeton and Oxford (2009), DOI: https://doi.org/10.1515/9781400830558.

[3] D. Martínez-Rivillas, R. de Queiroz, *Solving Homotopy Domain Equations*, **arXiv:2104.01195**, (2021).

[4] D. Martínez-Rivillas, R. de Queiroz, *The ∞-groupoid generated by an arbitrary topological λ-model*, **Logic Journal of the IGPL (also arXiv:1906.05729)**, vol. 30 (2022), pp. 465–488, DOI: https://doi.org/10.1093/jigpal/jzab015.

[5] D. Martínez-Rivillas, R. de Queiroz, *Towards a Homotopy Domain Theory*, **Archive for Mathematical Logic (also arXiv 2007.15082)**, (2022), DOI: https://doi.org/10.1007/s00153-022-00856-0.

[6] C. Rezk, **Introduction to Quasicategories**, Lecture Notes for course at University of Illinois at Urbana-Champaign (2022), URL: https://faculty.math.illinois.edu/~{}rezk/quasicats.pdf.

Daniel O. Martínez-Rivillas
Universidade Federal de Pernambuco
Centro de Informática
Av. Jornalista Aníbal Fernandes, s/n
Recife, Pernambuco, Brazil
e-mail: domr@cin.ufpe.br

Ruy J. G. B. de Queiroz
Universidade Federal de Pernambuco
Centro de Informática
Av. Jornalista Aníbal Fernandes, s/n
Recife, Pernambuco, Brazil
e-mail: ruy@cin.ufpe.br

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Zalán Gyenis
Zalán Molnár
Övge Öztürk

THE MODELWISE INTERPOLATION PROPERTY OF SEMANTIC LOGICS

Abstract

In this paper we introduce the modelwise interpolation property of a logic that states that whenever $\models \phi \to \psi$ holds for two formulas ϕ and ψ, then for every model $\mathfrak{M}$ there is an interpolant formula χ formulated in the intersection of the vocabularies of ϕ and ψ, such that $\mathfrak{M} \models \phi \to \chi$ and $\mathfrak{M} \models \chi \to \psi$, that is, the interpolant formula in Craig interpolation may vary from model to model. We compare the modelwise interpolation property with the standard Craig interpolation and with the local interpolation property by discussing examples, most notably the finite variable fragments of first order logic, and difference logic. As an application we connect the modelwise interpolation property with the local Beth definability, and we prove that the modelwise interpolation property of an algebraizable logic can be characterized by a weak form of the superamalgamation property of the class of algebras corresponding to the models of the logic.

Keywords: interpolation, algebraic logic, amalgamation, superamalgamation.

1. The modelwise interpolation property

Interpolation properties have been intensively studied in the literature of (algebraic) logic ever since Craig proved that in classical propositional and first order logic, whenever $\models \phi \to \psi$ holds for two formulas ϕ and

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ψ formulated respectively using the vocabularies (signatures) $\mathrm{Voc}(\phi)$ and $\mathrm{Voc}(\psi)$, then there is an interpolant formula χ formulated in the vocabulary $\mathrm{Voc}(\phi) \cap \mathrm{Voc}(\psi)$ such that $\models \phi \to \chi$ and $\models \chi \to \psi$ hold.

This paper introduces the *modelwise interpolation* property of a logic which states that whenever $\models \phi \to \psi$ holds, then one can find an interpolant formula in every model, that is, the interpolant formula in Craig interpolation may vary from model to model. In order to make sense of this notion we have to work with logics that are semantically defined, e.g. a notion of model should be built in the definition of the logic.[1]

We discuss the relations between the modelwise interpolation, Craig interpolation, and local interpolation properties by providing examples in all logically possible combinations. Most importantly, we prove that while difference logic and the n-variable fragment of first-order logic ($n \geq 2$) lack the standard Craig interpolation property, the former has, while the latter does not have the the modelwise interpolation property. Using the case of difference logic as an example, we show that the modelwise interpolation property implies the local Beth definability property for difference logic.

The modelwise interpolation property might have possible further applications in philosophy of science. Craig original interpolation property (for first-order logic) stemmed from the question of using logic to clarify the relationship between theoretical constructs and observed data: the interpolant formula gives an axiomatization of the observational consequences of the theory in which only symbols of the observational vocabulary occur (cf. [24]). Scientific theories are sometimes axiomatized by logics other than classical first-order logic, for example, in [2] modal logic is used to axiomatize relativity theory (cf. [21]). Such logics may or may not have the Craig interpolation property. If the logic we make use has no Craig interpolation but turns out to have the modelwise interpolation property, and our scientific theories are formulated in this logic and evaluated in a model, then changing our background logic from first-order logic to this new logic still allows us to carry out arguments inside models similar to

[1] While providing the definitions and discussing examples, we employ a rather general notion of a logic. But in the last section of the paper when we provide the algebraic characterization, we adopt the Andréka–Németi–Sain approach [3, 1], cf. [20, 17, 18] which focuses on the semantic aspects of logics. The more mainstream Blok–Pigozzi framework (cf. [8, 10, 29, 9] and Czelakowski [14]) seems not to be (directly) applicable as in that approach the focus is rather on the relation $\vdash$ between sets of formulas and is missing the general notion of models.

Craig's. The previously introduced *local interpolation property* (the definition is provided later below) was motivated by similar considerations, however even very basic logics such as sentential logic, propositional modal logics, finite variable fragments of first order logic, etc. do not have the local interpolation property. Cf. the examples below. Also, the Craig interpolation (resp. modelwise interpolation) has a strong connection with Beth definability (resp. local Beth definability). The local interpolation property does not have such connections. In this respect, the modelwise interpolation property seems to be a "more interesting" property than the local interpolation property. We do not pursue these philosophical issues in this paper.

Interpolation properties of a logic are strongly related to various amalgamation properties of the classes of algebras corresponding to the logic. We refer to [12], [13], [23, 22], [37], [25], [28], [35], [32], [3]. In the last section we show that the modelwise interpolation property of an algebraizable logic can be characterized by a weak form of the superamalgamation property of the class of algebras corresponding to the models of the logic.

* * *

By a logic we understand a tuple $\mathcal{L}(P, Cn) = \langle F, M, \models \rangle$, where

- P is a set, called the set of atomic formulas, and Cn is a set of logical connectives, i.e. function symbols of finite arity.
- F, called the set of formulas, is the universe of the absolutely free algebra generated by P in similarity type Cn.
- M is an abstract, non-empty class, called the class of models.
- $\models$ is a relation between models and formulas: $\models \subset M \times F$. For $\mathfrak{M} \in M$ and $\phi \in F$ we write $\mathfrak{M} \models \phi$ instead of $(\mathfrak{M}, \phi) \in \models$.

As it is standard in logic we extend the consequence relation $\models$ to a relation in between (sets) of formulas: For $\Gamma, \{\phi\} \subseteq F$ we write $\Gamma \models \phi$ if whenever $\mathfrak{M} \models \Gamma$ for a model $\mathfrak{M} \in M$, then $\mathfrak{M} \models \phi$ as well. When it is clear from the context, we simply write $\mathcal{L}$ in place of $\mathcal{L}(P, Cn)$. For a formula $\alpha \in F$, the vocabulary of α, $\mathrm{Voc}(\alpha)$ denotes the set of atomic formulas occurring in α, i.e. the smallest subset of P such that α belongs to the absolutely free algebra generated by $\mathrm{Voc}(\alpha)$ in similarity type Cn.

For our main definition 1.2 below we assume that there is a distinguished binary (derived) connective $\rightsquigarrow$ and we write $(\star)_{\phi,\psi}$ for the property

$$\{\chi \in F : \mathrm{Voc}(\chi) \subseteq \mathrm{Voc}(\phi) \cap \mathrm{Voc}(\psi)\} \neq \emptyset \qquad (\star)_{\phi,\psi}$$

Recall (e.g. from [3, Def.6.13]) that the Craig interpolation property ($\mathrm{IP}^{\rightsquigarrow}$, for short) is the property that whenever $\phi, \psi \in F$ for which $(\star)_{\phi,\psi}$ holds, if $\models \phi \rightsquigarrow \psi$, then there exists $\chi \in F$ with $\mathrm{Voc}(\chi) \subseteq \mathrm{Voc}(\phi) \cap \mathrm{Voc}(\psi)$ such that $\models \phi \rightsquigarrow \chi$ and $\models \chi \rightsquigarrow \psi$.

Remark 1.1. The extra condition $(\star)_{\phi,\psi}$ can be satisfied in two ways: either there is a constant connective in the language, or $\mathrm{Voc}(\phi) \cap \mathrm{Voc}(\psi)$ is not empty. Consider classical propositional logic with connectives $\{\vee, \neg\}$ and with two atomic formulas p and q. As usual, $\phi \to \psi$ abbreviates $\neg\phi \vee \psi$. There is no interpolant for the tautology $\models p \to (q \to q)$, as $\mathrm{Voc}(p) \cap \mathrm{Voc}(q \to q)$ is empty, and there are no formulas over the empty vocabulary (we did not allowed $\perp$ or $\top$ as constants in the language). However, if $(\star)_{\phi,\psi}$ is satisfied, then $\models \phi \to \psi$ will always have an interpolant in this logic.

Let us now define the modelwise interpolation property.

Definition 1.2. We say that the logic $\mathcal{L} = \langle F, M, \models\rangle$ has the *modelwise interpolation property* ($\mathrm{mIP}^{\rightsquigarrow}$, for short) if for every formulas $\phi, \psi \in F$ for which $(\star)_{\phi,\psi}$ holds, if $\models \phi \to \psi$, then for all models $\mathfrak{M} \in M$ there exists $\chi \in F$ with $\mathrm{Voc}(\chi) \subseteq \mathrm{Voc}(\phi) \cap \mathrm{Voc}(\psi)$ such that $\mathfrak{M} \models \phi \to \chi$ and $\mathfrak{M} \models \chi \to \psi$.

The $\mathrm{mIP}^{\rightsquigarrow}$ thus differs from the $\mathrm{IP}^{\rightsquigarrow}$ in that the interpolant formula may vary from model to model. Note that it is crucial for the definition of $\mathrm{mIP}^{\rightsquigarrow}$ to have a notion of model built in the definition of the logic $\mathcal{L}$.

Motivated by model theoretic investigations of homogeneous structures [15, 27] the local interpolation property ($\mathrm{lIP}^{\rightsquigarrow}$, for short) has been introduced in [16] as the property that whenever $\phi, \psi \in F$ for which $(\star)_{\phi,\psi}$ holds, for all $\mathfrak{M} \in M$ if $\mathfrak{M} \models \phi \rightsquigarrow \psi$, then there exists $\chi \in F$ with $\mathrm{Voc}(\chi) \subseteq \mathrm{Voc}(\phi) \cap \mathrm{Voc}(\psi)$ such that $\mathfrak{M} \models \phi \rightsquigarrow \chi$ and $\mathfrak{M} \models \chi \rightsquigarrow \psi$. Notice that the $\mathrm{lIP}^{\rightsquigarrow}$ differs from the $\mathrm{mIP}^{\rightsquigarrow}$ in that in the former the implication $\phi \rightsquigarrow \psi$ is also "localized" to models, making it a rather weak property of a logic.

Claim 1.3. Both the $\mathrm{IP}^{\rightsquigarrow}$ and the $\mathrm{lIP}^{\rightsquigarrow}$ imply the $\mathrm{mIP}^{\rightsquigarrow}$.

PROOF: Straightforward from the definitions. □

Remark 1.4. We note that the modelwise interpolation property could be defined for many other types of logics too. For example, one could allow for infinite formulas, or infinite connectives, or restrictions on the syntactic shape of formulas, etc. Adapting the definition to such cases seems to be straightforward and thus we do not pursue such a generalization. Also, all our examples, and in fact the most traditional propositional and first-order logics, fit to the notion of logic given above.

In the rest of this section we give examples for logics having or not having the discussed interpolation properties in all possible combinations. Even thought our definitions so far were employed for logics in a very broad sense, our examples below are all algebraizable and in fact well-studied in the literature (except for $\mathcal{L}_\infty$ which is algebraizable but not well-studied). The following table summarizes the examples given below.

	$\mathrm{IP}^{\rightarrow}$	$\mathrm{lIP}^{\rightarrow}$	$\mathrm{mIP}^{\rightarrow}$
$\mathcal{L}_{Prop}$	✓	✓	✓
$\mathcal{L}_{Sent}$	✓	×	✓
$\mathcal{L}_\infty$	×	✓	✓
$\mathcal{L}_D$	×	×	✓
$\mathcal{L}_n, \mathfrak{L}_n,\ n > 2$	×	×	×

Note that there are 8 theoretically possible combinations of the three logical properties, but Claim 1.3 rules out three of them. This is why the table above consists of 5 rows only.

Propositional logic $\mathcal{L}_{Prop}$. Let P be an arbitrary set of propositional letters. Let $Cn(\mathcal{L}_{Prop}) = \{\wedge, \neg, \bot\}$ be the set of connectives and let F be the set of formulas generated by P in type $Cn(\mathcal{L}_{Prop})$. Models are evaluations $\mathfrak{M} : P \to \{0, 1\}$ that extend to the set of formulas by the usual $\mathfrak{M}(\bot) = 0$, $\mathfrak{M}(\phi \wedge \psi) = \mathfrak{M}(\phi) \cdot \mathfrak{M}(\psi)$, and $\mathfrak{M}(\neg\phi) = 1 - \mathfrak{M}(\phi)$. The validity relation is defined as

$$\mathfrak{M} \models \varphi \quad \Leftrightarrow \quad \mathfrak{M}(\varphi) = 1. \tag{1.1}$$

We use the derived connectives $\vee$, $\rightarrow$ and $\top$ in the standard way. By Craig's result, $\mathcal{L}_{Prop}$ has the $\mathrm{IP}^{\rightarrow}$ and thus the $\mathrm{mIP}^{\rightarrow}$ as well. That $\mathcal{L}_{Prop}$

has the lIP$^{\to}$ follows from that whenever $\mathfrak{M} \models \phi \to \psi$, then either $\bot$ or $\top$ is a suitable interpolant formula inside the model $\mathfrak{M}$.

Sentential logic $\mathcal{L}_{Sent}$. The set of connectives and the set of formulas are as in the previous case. The class of models is

$$M = \{\langle W, V\rangle : W \neq \emptyset, \ V : P \to \mathcal{P}(W)\}. \tag{1.2}$$

For a model $\mathfrak{M} = \langle W, V\rangle$, $w \in W$ and a formula φ one defines $\mathfrak{M}, w \Vdash \varphi$ by

$$\begin{aligned} \mathfrak{M}, w &\nVdash \bot && (1.3)\\ \mathfrak{M}, w \Vdash p \quad &\Leftrightarrow \quad w \in V(p) && (1.4)\\ \mathfrak{M}, w \Vdash \phi \wedge \psi \quad &\Leftrightarrow \quad \mathfrak{M}, w \Vdash \phi \text{ and } \mathfrak{M}, w \Vdash \psi && (1.5)\\ \mathfrak{M}, w \Vdash \neg\phi \quad &\Leftrightarrow \quad \mathfrak{M}, w \nVdash \phi. && (1.6) \end{aligned}$$

Finally, we set

$$\mathfrak{M} \models \varphi \quad \Leftrightarrow \quad \{w \in W : \mathfrak{M}, w \Vdash \phi\} = W. \tag{1.7}$$

Craig's original result applies to this presentation of classical logic too, i.e. $\mathcal{L}_{Sent}$ has the IP$^{\to}$ and thus the mIP$^{\to}$ too. In contrast, however, $\mathcal{L}_{Sent}$ does not have the lIP$^{\to}$ in general. For, assume that there are (at least) two atomic formulas p and q. Take a model $\mathfrak{M}$ in which $\emptyset \neq V(p) \subsetneq V(q) \neq W$ holds for the atomic propositions p and q. Then $\mathfrak{M} \models p \to q$ holds by the definition of truth in a model. However, $\mathrm{Voc}(p) \cap \mathrm{Voc}(q)$ is empty, therefore the possible interpolant formulas are Boolean combinations of the constant symbol $\bot$. Each such formula is equivalent either to $\bot$ or to $\top$, but neither can be an interpolant in the model $\mathfrak{M}$, as p is not false in $\mathfrak{M}$, and q is not true in $\mathfrak{M}$.

Difference logic $\mathcal{L}_D$. Difference logic is discussed e.g. in Sain [33, 34], Venema [38], Roorda [31], but see also Segerberg [36] who traces this logic back to von Wright. The set of connectives of difference logic is $\{\wedge, \neg, D, \bot\}$. The set of formulas is defined as that of propositional logic together with the following clause: if $\phi \in F$, then $D\phi \in F$. The class of models and the definition of $\mathfrak{M}, w \Vdash \varphi$ are the same as in the sentential case but we also have the case of D:

$$\mathfrak{M}, w \Vdash D\phi \quad \Leftrightarrow \quad (\exists w' \in W \smallsetminus \{w\})\ \mathfrak{M}, w' \Vdash \phi. \tag{1.8}$$

Truth in a model is defined in the same way as in the sentential case:[2]

$$\mathfrak{M} \models \varphi \quad \Leftrightarrow \quad \{w \in W : \ \mathfrak{M}, w \Vdash \phi\} = W. \tag{1.9}$$

That difference logic does not have the lIP$^{\rightarrow}$ can be seen exactly in the same way as in the case of sentential logic: Assuming p and q are atomic formulas, take a model $\mathfrak{M}$ in which p is not false, q is not true, and p implies q, that is, $\emptyset \neq V(p) \subsetneq V(q) \neq W$ holds. The common vocabulary of p and q is empty. Now, every formula of difference logic over the empty vocabulary is either true or false in a model: As for the Boolean combinations this is straightforward. As for the difference operator, it is enough to check that $D\bot$ cannot be satisfied in any world, and $D\top$ is true in all worlds (provided there are at least two worlds).

It is known that $\mathcal{L}_D$ does not have the IP$^{\rightarrow}$ either (see e.g. [11]). Let us briefly recall the argument. Let $E\phi$ abbreviate $\phi \vee D\phi$. The following implication is a logical validity of difference logic:

$$\models_{\mathcal{L}_D} \big(Dp \wedge D\neg p\big) \ \longrightarrow\ \big(E(r \wedge \neg Dr) \ \rightarrow\ E(\neg r \wedge D\neg r)\big). \tag{1.10}$$

The reason is that in a model $\mathfrak{M}$ and a world w, $w \Vdash Dp \wedge D\neg p$ implies that there are at least two other worlds not equal to w, while $E(r \wedge \neg Dr) \rightarrow E(\neg r \wedge D\neg r)$ expresses that if there is only one world satisfying r, then there must be at least two different worlds satisfying $\neg r$. The common vocabulary of the subformulas on the two sides of the implication is empty, and it is not hard to check that neither $\top$ nor $\bot$ nor any formulas built up from $\top$ and $\bot$ can be a global interpolant ([11] contains a detailed proof).

However, $\mathcal{L}_D$ has the modelwise interpolation property as the following theorem shows.

Theorem 1.5. *Difference logic has the mIP$^{\rightarrow}$.*

Proof: Suppose $\models \phi(\vec{p}, \vec{q}) \longrightarrow \psi(\vec{q}, \vec{r})$ is a logical validity where the formulas ϕ and ψ use the atomic formulas $\vec{p}$, $\vec{q}$ and $\vec{r}$ as denoted. We need to find an interpolant formula using the atomic formulas $\vec{q}$ only. Write $\vec{q} = \langle q_0, \ldots, q_{n-1}\rangle$ and $\vec{p} = \langle p_0, \ldots, p_{m-1}\rangle$. Take any model $\mathfrak{M} = \langle W, V\rangle$.

[2] Thus, $\mathfrak{M} \models \phi$ is what is standardly called "global truth" in modal logic (cf. [7, Def.1.21]).

Two worlds $v, w \in W$ are said to be $\vec{q}$-equivalent ($v \sim w$ in symbols) if for all $i < n$ we have

$$\mathfrak{M}, v \Vdash q_i \qquad \Longleftrightarrow \qquad \mathfrak{M}, w \Vdash q_i \tag{1.11}$$

Claim 1.6. If $\mathfrak{M}, v \Vdash \phi$ and $w \sim v$, then $\mathfrak{M}, w \Vdash \psi$.

PROOF: Assume $\mathfrak{M}, v \Vdash \phi$ and define a new model $\mathfrak{M}' = \langle W, V' \rangle$ on the same set of possible worlds as follows. For a world $u \in W$ let us use the notation

$$u' = \begin{cases} v & \text{if } u = w \\ w & \text{if } u = v \\ u & \text{if } u \neq v, u \neq w, \end{cases} \tag{1.12}$$

that is, we exchange v with w but keep everything fixed. Define the new evaluation V' by $V'(q_i) = V(q_i)$, $V'(r_i) = V(r_i)$ and

$$V'(p_i) = \{u' : \ u \in V(p_i)\}. \tag{1.13}$$

LEMMA 1.7. *For any formula $\theta(\vec{p}, \vec{q})$ and world $u \in W$ we have*

$$\mathfrak{M}, u \Vdash \theta \quad \Leftrightarrow \quad \mathfrak{M}', u' \Vdash \theta.$$

PROOF: Induction on the complexity of θ.

- For atomic propositions q_i: As $V'(q_i) = V(q_i)$, if $u \neq v$ and $u \neq w$, then $u = u'$ and thus the statement holds. For $u = v$ or $u = w$ we obtain the result by the assumption $v \sim w$.
- For atomic propositions p_i the statement follows directly from the definition of V': $\mathfrak{M}, u \Vdash p_i$ if and only if $\mathfrak{M}', u' \Vdash p_i$.
- For Boolean combinations the induction is straightforward.
- For formulas of the form $D\theta$: Assume (inductive hypothesis) that the statement holds for θ. Then

$$\begin{aligned} \mathfrak{M}, u \Vdash D\theta \quad &\Leftrightarrow \quad (\exists x \neq u)\ \mathfrak{M}, x \Vdash \theta &(1.14)\\ &\Leftrightarrow \quad (\exists x \neq u)\ \mathfrak{M}', x' \Vdash \theta &(1.15)\\ &\Leftrightarrow \quad (\exists x' \neq u')\ \mathfrak{M}', x' \Vdash \theta &(1.16)\\ &\Leftrightarrow \quad \mathfrak{M}', u' \Vdash D\theta. \quad \square &(1.17)\end{aligned}$$

Applying the lemma to v and ϕ we obtain $\mathfrak{M}', w \Vdash \phi$. As $\models \phi \to \psi$ holds we get $\mathfrak{M}', w \Vdash \psi$. But note that V and V' coincide on the elements of $\vec{q}$ and $\vec{r}$, therefore $\mathfrak{M}, u \Vdash \psi$ if and only if $\mathfrak{M}', u \Vdash \psi$ for any $u \in W$. It follows that $\mathfrak{M}, w \Vdash \psi$, completing the proof of the claim. $\square$

In what follows we use the notation $q^1 = q$ and $q^0 = \neg q$. For $v \in W$ write

$$\chi_v \;=\; \bigwedge_{i<n} q_i^{\varepsilon_i}, \tag{1.18}$$

where

$$\varepsilon_i = \begin{cases} 1 & \text{if } \mathfrak{M}, v \Vdash q_i \\ 0 & \text{if } \mathfrak{M}, v \Vdash \neg q_i \end{cases} \tag{1.19}$$

By the claim above for each v for which $\mathfrak{M}, v \Vdash \phi$ holds, the equivalence class $v/\!\sim$ is a subset of $\{u \in W : \mathfrak{M}, u \Vdash \psi\}$. As $\vec{q}$ is finite, there are only finitely many $\sim$ equivalence classes. Let $v_0, \ldots, v_\ell$ be representative elements of all the different equivalence classes such that $\mathfrak{M}, v_i \Vdash \phi$ and write

$$\chi \;=\; \bigvee_{i<\ell} \chi_{v_i}\,. \tag{1.20}$$

Then $\mathfrak{M} \models \phi \to \chi$ and $\mathfrak{M} \models \chi \to \psi$, that is, χ is a desired interpolant formula in $\mathfrak{M}$. $\square$

First-order logic with n variables $\mathcal{L}_n$. Let $\mathcal{L}_n$ denote standard first-order logic with the restriction that we are allowed to use n variables only (n is finite). It is not hard to see that given any first-order similarity type, $\mathcal{L}_n$ fits into our definition of a logic. The connectives are the standard $\wedge$, $\neg$, $\exists x$ (unary) and $x = y$ (constant) for variables x, y, and the set P is the set of first-order atomic formulas. Models, evaluations, $\models$, etc. are the usual.

For $n \geq 2$, $\mathcal{L}_n$ does not admit Craig's interpolation theorem $\mathrm{IP}^{\to}$, in general.[3] A proof can be found in [5, Theorem 3.5.1], here we briefly

[3] That is, there are similarity types for which the n-variables fragment of first-order logic does not have the Craig interpolation. [5, Theorem 3.5.1] shows the failure of interpolation with monadic predicates; [4] shows that interpolation still fails with one

sketch the argument. Let $n \geq 2$ and let $p_1, \ldots, p_n$ be unary predicates. The formula ϕ that states that there is a one-one correspondence between the elements of the domain of a model and the relations p_i can be expressed by the conjunction of the following formulas:

$$\forall x \bigvee_i p_i(x), \quad \bigwedge_i \exists x p_i(x), \quad \forall x \bigwedge_{i \neq j} (p_i(x) \to \neg p_j(x)), \tag{1.21}$$

$$\forall x \forall y \big(\bigwedge_i (x \neq y) \wedge p_i(x) \to \neg p_i(y) \big). \tag{1.22}$$

Thus, if ϕ is true in a model $\mathfrak{M}$, then $\mathfrak{M}$ has exactly n elements. Let ψ be a similar formula using relation symbols $r_1, \ldots, r_{n+1}$ expressing that the model has $n+1$ elements. Then clearly $\models \phi \to \neg\psi$, but there can be no interpolant formula as no n-variable formula using equality only can distinguish between n and $n+1$ elements. This latter statement follows from e.g. a standard back and forth argument to be recalled in the proof of Theorem 1.8 below.

In the next theorem we adapt this construction[4] to show that $\mathcal{L}_n$ does not always have the modelwise interpolation property, for $n \geq 3$. The $n = 2$ case remains open.

THEOREM 1.8. *For $n \geq 3$, $\mathcal{L}_n$ does not have the mIP$^{\to}$, in general.*

PROOF: Assume there are unary relation symbols p_1, …, p_n, and r_1, …, r_{n+1} and a binary relation symbol e in the similarity type.

Let $\phi(x)$ be the conjunction of the following formulas, having free variable x, using the relation symbols e, p_1, …, p_n only:

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binary and two unary relation symbols. With only two non-logical symbols the question is open. The cases $n = 0$ and $n = 1$ can basically be reduced respectively to propositional logic and modal logic $S5$; both have the Craig interpolation property. Cf. p. 107 in [3].

[4]We would like to thank László Csirmaz for a similar idea.

$$\forall y\, \neg e(y,y), \tag{1.23}$$

$$\forall y\big(e(x,y) \to \bigvee_i p_i(y)\big), \tag{1.24}$$

$$\bigwedge_i \exists y(e(x,y) \wedge p_i(y)), \tag{1.25}$$

$$\forall y\big(e(x,y) \to \bigwedge_{i\neq j}(p_i(y) \to \neg p_j(y))\big), \tag{1.26}$$

$$\forall y \forall z\big(y \neq z \wedge e(x,y) \wedge e(x,z) \to \bigwedge_i (p_i(y) \to \neg p_i(z))\big). \tag{1.27}$$

In a model $\mathfrak{M}$, $e^{\mathfrak{M}}$ is a simple graph, and if $\mathfrak{M} \models \phi[a]$ holds for $a \in \mathfrak{M}$, then a has exactly n neighbours, as there is a bijection between the neighbours of a and the p_i's.

Let $\psi(x)$ be the similar formula but with the relation symbols $r_1, \ldots, r_{n+1}$ in place of the p_i's. Clearly, if $\mathfrak{M} \models \psi[a]$ holds for $a \in \mathfrak{M}$, then a has exactly $n+1$ neighbours.

As no vertex in a graph can have n and $n+1$ neighbours at the same time, we have $\models \phi \to \neg\psi$. The common vocabulary of the formulas ϕ and ψ contains the relation symbol e and the equalities only.

In what follows $\mathfrak{A}$ and $\mathfrak{B}$ denotes the following graphs:

$$A = \{a, c_1, \ldots, c_n\}, \quad e^{\mathfrak{A}} = \{(a, c_i) :\ 1 \leq i \leq n\} \tag{1.28}$$

$$B = \{b, d_1, \ldots, d_{n+1}\}, \quad e^{\mathfrak{B}} = \{(b, d_i) :\ 1 \leq i \leq n+1\}, \tag{1.29}$$

that is, $\mathfrak{A}$ is a "star" with center a, having n neighbours $c_1, \ldots, c_n$; and similarly, $\mathfrak{B}$ is a star with center b, having $n+1$ neighbours $d_1, \ldots, d_{n+1}$. We assume that A and B are disjoint.

Let $\mathfrak{M}$ be the disjoint union of the graphs $\mathfrak{A}$ and $\mathfrak{B}$, and interpret the relation symbols p_i and r_j as the respective neighbours of a and b:

$$M = A \cup B, \quad e^{\mathfrak{M}} = e^{\mathfrak{A}} \cup e^{\mathfrak{B}}, \tag{1.30}$$

$$p_j^{\mathfrak{M}} = \{c_j\} \quad \text{for } 1 \leq j \leq n, \tag{1.31}$$

$$r_k^{\mathfrak{M}} = \{d_k\} \quad \text{for } 1 \leq k \leq n+1. \tag{1.32}$$

The neighbours of a are in one-one correspondence with the p_j's, and the neighbours of b are in one-one correspondence with the r_k's. In this model, we have

$$\phi^{\mathfrak{M}} = \{m \in M :\ \mathfrak{M} \models \phi[m]\} = \{a\}, \tag{1.33}$$

$$\psi^{\mathfrak{M}} = \{m \in M :\ \mathfrak{M} \models \psi[m]\} = \{b\}, \tag{1.34}$$

$$(\neg\psi)^{\mathfrak{M}} = M \setminus \{b\}. \tag{1.35}$$

Suppose χ is an interpolant for $\models \phi \to \neg\psi$ in the model $\mathfrak{M}$, formulated in the language using equality and e only. As $\phi^{\mathfrak{M}}$ is not empty, χ cannot be false in $\mathfrak{M}$. Similarly, as $(\neg\psi)^{\mathfrak{M}}$ is non-empty, χ cannot be true in $\mathfrak{M}$. Observe, that the set

$$I = \{g :\ g \subseteq f \text{ for some partial isomorphism} \tag{1.36}$$

$$f : \mathfrak{M} \to \mathfrak{M} \text{ with } f(a) = b\} \tag{1.37}$$

is an n-back-and-forth system between $\mathfrak{M}$ and $\mathfrak{M}$: it satisfies the properties

(i) $g \subseteq f \in I$ implies $g \in I$, and

(ii) if $f \in I$ and $|f| < n$, then for all $x \in A$ (resp. $y \in B$) there is a $g \in I$ with $f \subseteq g$ and $x \in \mathrm{dom}(g)$ (resp. $y \in \mathrm{ran}(g)$).

Therefore, by a standard back-and-forth argument (see e.g. Theorem 2.4 in [6]) $a \in M$ and $b \in M$ satisfy the same formulas with at most n variables.

It follows that no formula χ in the language of equality and e only can make a distinction between the elements a and b of $\mathfrak{M}$: either both or none of them satisfy χ in $\mathfrak{M}$. Consequently, χ cannot be the desired interpolant formula. $\square$

In the light of Claim 1.3, Theorem 1.8 gives an alternative proof for that $\mathcal{L}_n$ does not admit Craig's interpolation theorem IP$^{\to}$, and that it does not have the lIP$^{\to}$ either.

Łukasiewicz's $Ł_n$ for $n > 2$. Let $n > 2$ be finite and consider the n-element algebra

$$\mathfrak{A}_n = \left\langle \left\{\frac{i}{n-1} :\ i < n\right\}, \wedge, \vee, \neg, \to, 1 \right\rangle, \tag{1.38}$$

where the operations are given by

$$x \wedge y = \min\{x, y\}, \quad x \vee y = \max\{x, y\}, \tag{1.39}$$

$$\neg x = 1 - x, \quad x \to y = \min\{1, 1 - x + y\}. \tag{1.40}$$

Łukasiewicz's logic $Ł_n$ is defined as follows (cf. e.g. [30, 7.3.9]). The connectives $Cn(Ł_n) = \{\wedge, \vee, \neg, \rightarrow, \top\}$ are the usual. If P is a set of propositional variables, then the set of formulas F is generated by P using the connectives. Write $\mathcal{F}$ for the absolutely free formula algebra $\mathcal{F} = \langle F, \wedge, \vee, \neg, \rightarrow, \top\rangle$. The class of models is

$$M = \big\{ h : \mathcal{F} \rightarrow \mathfrak{A}_n \ : \ h \text{ is a homomorphism} \big\}. \tag{1.41}$$

In a model $h \in M$, $h \models \phi$ holds if $h(\phi) = 1$. The definition of logical validity is then

$$\models_{Ł_n} \phi \quad \Longleftrightarrow \quad (\forall h \in M)\ h(\phi) = 1. \tag{1.42}$$

Assume that there are at least two atomic formulas in P. The paper [19] showed that $Ł_n$ does not have the Craig interpolation property $\mathrm{IP}^{\rightarrow}$. A similar argument below reveals that $Ł_n$ does not have the $\mathrm{mIP}^{\rightarrow}$. Then, by Claim 1.3 then it cannot have the $\mathrm{lIP}^{\rightarrow}$ either.

Truth tables show that the implication

$$\models_{Ł_n} p \wedge \neg p \ \longrightarrow \ q \vee \neg q \tag{1.43}$$

holds for any propositional variables $p, q \in P$. Every formula in the empty vocabulary is a Boolean combination of $\perp$ and $\top$, and therefore is equivalent to either $\perp$ or $\top$. However, in the model where both p and q are evaluated to $\frac{\lfloor n/2 \rfloor}{n-1}$ neither $\top$ not $\perp$ can be an interpolant. This is because the truth value $\frac{\lfloor n/2 \rfloor}{n-1}$ is neither 0 nor 1 if $n > 2$.

The same argument carries over to the infinite Łukasiewicz logic $Ł_\infty$. (for this logic, see [26]).

The logic $\mathcal{L}_\infty$. We design the logic $\mathcal{L}_\infty$ for the sake of giving an example for the case where the $\mathrm{IP}^{\rightarrow}$ fails but the $\mathrm{lIP}^{\rightarrow}$ and thus the $\mathrm{mIP}^{\rightarrow}$ hold.

Let ω denote the ordered set of natural numbers and let ω^* be the reverse ordering. Consider the ordering $\omega + \omega^*$. We write $n \in \omega$ and $n \in \omega^*$ to denote that n belong to the ω or the ω^* part of the ordering $\omega + \omega^*$. Particularly, $0 \in \omega$ is the smallest element, and $0 \in \omega^*$ is the largest element of the ordering. Define the algebra

$$\mathfrak{A} = \big\langle \omega + \omega^*, E, L, \rightarrow, c_i \big\rangle_{i \in \omega + \omega^*}, \tag{1.44}$$

where E and L are the unary functions

$$E(n) = \begin{cases} n & \text{if } n \in \omega \\ 0 \in \omega & \text{if } n \in \omega^*, \end{cases} \qquad L(n) = \begin{cases} n & \text{if } n \in \omega^* \\ 0 \in \omega^* & \text{if } n \in \omega, \end{cases} \tag{1.45}$$

the binary $\rightarrow$ is given by

$$x \rightarrow y = \begin{cases} 0 \in \omega^* & \text{if } x \leq^{\omega+\omega^*} y \\ 0 \in \omega & \text{otherwise,} \end{cases} \tag{1.46}$$

and each c_i is a constant with value i for $i \in \omega + \omega^*$.

The connectives of the logic $\mathcal{L}_\infty$ are $\{E, L, \rightarrow, c_i\}_{i\in\omega+\omega^*}$. If P is a set of propositional variables, then the set of formulas F is generated by P using the connectives. Write $\mathcal{F}$ for the absolutely free formula algebra. The class of models is

$$M = \big\{h : \mathcal{F} \rightarrow \mathfrak{A} \;:\; h \text{ is a homomorphism}\big\}. \tag{1.47}$$

For $h \in M$ we let the meaning function mng_h to be equal to h. In a model $h \in M$, $h \models \phi$ holds if $h(\phi) = 0 \in \omega^*$. The definition of logical validity is then

$$\models_{\mathcal{L}_\infty} \phi \quad \Longleftrightarrow \quad (\forall h \in M)\; h(\phi) = 0 \in \omega^*. \tag{1.48}$$

It is easy to check that the implication

$$\models_{\mathcal{L}_\infty} E(p) \rightarrow L(q) \tag{1.49}$$

holds for any propositional variables $p, q \in P$. Every formula in the empty vocabulary is equivalent to one of the constants c_i, therefore in order to see that $\mathcal{L}_\infty$ has no $\mathrm{IP}^{\rightarrow}$, it is enough to check that none of the constants c_i can be a (global) interpolant for the formula $E(p) \rightarrow L(q)$. Indeed, for any c_i take a model h in which $h(c_i) < h(Ep)$ or $h(Lq) < h(c_i)$ holds. Then either $h \not\models E(p) \rightarrow c_i$ or $h \not\models c_i \rightarrow L(q)$.

However, $\mathcal{L}_\infty$ has the $\mathrm{lIP}^{\rightarrow}$ (and thus the $\mathrm{mIP}^{\rightarrow}$) because in any model h the formula $c_{h(E(p))}$ is a suitable interpolant.

2. Applications

The local Beth property of a logic $\mathcal{L}$ states that every implicitly definable relation is locally explicitly definable, that is, the explicit definition may vary from model to model (see [3, Definition 6.9]). To be more precise, let $\mathcal{L} = \langle F, M, \models \rangle$ be a logic, and write F^P to denote the set of formulas of the logic $\mathcal{L}$ that are generated by the propositional letters P, that is, $F^P = \{\phi \in F : \text{Voc}(\phi) \subseteq P\}$, and let $\leftrightarrow$ be a distinguished binary connective. For a set of propositional letters R let R' be a disjoint copy of R and for $\Sigma \subseteq F^R$ we write Σ' to denote the formulas obtained from Σ be replacing each $r \in R$ by the corresponding $r' \in R'$. We say that $\Sigma \subseteq F^{P \cup R}$ *defines* R *implicitly* in terms of P if and only if $\Sigma \cup \Sigma' \models r \leftrightarrow r'$ for every $r \in R$. Further, Σ *defines* R *locally explicitly* in terms of P if for every model $\mathfrak{M} \models \Sigma$, for all $r \in R$ there is $\varphi_r \in F^P$ such that $\mathfrak{M} \models r \leftrightarrow \varphi_r$. That is, the usual explicit definition may vary from model to model.

We show that the modelwise interpolation property implies the local Beth definability property for a wide range of logics. In what follows we work with logics that extend classical propositional logic in the sense that the connectives $\wedge$ and $\rightarrow$ are available and satisfy

$$\models (\phi \wedge \psi) \rightarrow \theta \;\text{ iff }\; \models \phi \rightarrow (\psi \rightarrow \theta) \;\text{ iff }\; \models \psi \rightarrow (\phi \rightarrow \theta) \tag{2.1}$$

The logic $\mathcal{L}$ is said to be *consequence compact* if for every $\Gamma, \{\phi\} \subseteq F$, if $\Gamma \models \phi$, then there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \models \phi$. $\mathcal{L}$ is *conjunctive* if for any $\phi, \psi \in F$ we have

$$\{\theta : \phi, \psi \models \theta\} = \{\theta : \phi \wedge \psi \models \theta\}. \tag{2.2}$$

We say that $\mathcal{L}$ has deduction theorem if for all $\phi, \psi, \theta \in F$ we have

$$\phi, \psi \models \theta \quad \text{if and only if} \quad \phi \models \psi \rightarrow \theta. \tag{2.3}$$

THEOREM 2.1. *Suppose $\mathcal{L}$ is consequence compact, conjunctive, and has deduction theorem. If $\mathcal{L}$ has the mIP$^{\rightarrow}$ then it has the local Beth definability property.*

PROOF: The proof is standard. Suppose that $\Sigma \subseteq F^{P \cup \{r\}}$ defines r implicitly, that is

$$\Sigma \cup \Sigma' \models r \leftrightarrow r'. \tag{2.4}$$

By consequence compactness one can take a finite subset $\Sigma_0 \subset \Sigma$ such that $\Sigma_0 \cup \Sigma_0' \models r \leftrightarrow r'$, and by conjunctiveness if ϕ is the conjunction of the formulas in Σ_0, then

$$\phi, \phi' \models r \leftrightarrow r'. \tag{2.5}$$

By deduction and conjunctiveness

$$\models (\phi \wedge \phi') \to (r \leftrightarrow r'). \tag{2.6}$$

Using (2.1), from (2.6) we get the equivalent

$$\models \phi \to (\phi' \to (r \to r')) \tag{2.7}$$

$$\models \phi \to (r \to (\phi' \to r')) \tag{2.8}$$

$$\models (\phi \wedge r) \to (\phi' \to r') \tag{2.9}$$

For any model $\mathfrak{M}$, by mIP$^{\to}$, there is an interpolant formula $\theta_{\mathfrak{M}} \subseteq F^P$ such that

$$\mathfrak{M} \models (\phi \wedge r) \to \theta_{\mathfrak{M}}, \quad \text{and} \quad \mathfrak{M} \models \theta_{\mathfrak{M}} \to (\phi' \to r'), \tag{2.10}$$

hence, using (2.1) again, we get

$$\mathfrak{M} \models \phi \to (r \leftrightarrow \theta_{\mathfrak{M}}). \tag{2.11}$$

By deduction, for every $\mathfrak{M} \models \Sigma$ one has $\mathfrak{M} \models r \leftrightarrow \theta_{\mathfrak{M}}$, that is, Σ locally explicitly defines r. □

Corollary 2.2. Difference logic $\mathcal{L}_D$ has the local Beth definability property.

Proof: Combine Theorems 1.5 and Theorem 2.1. □

Next, we give an algebraic characterization of the modelwise interpolation property in terms of amalgamation of algebras. Algebraic characterizations of the IP and the lIP have been done respectively in the papers [20] and [16]. The definition of logic employed so far is too general to have an algebraic counterpart. Therefore we restrict our attention to a subclass of logics that are algebraizable. From now on in this section we work with algebraizable logics as defined in the Andréka–Németi–Sain framework [3]. We recall the indispensable definitions below, and for a brief and self-contained summary we refer the reader to [3], or [20, 17].

* * *

By an *algebraizable logic* we understand a tuple $\mathcal{L} = \langle F, M, \mathrm{mng}, \models\rangle$ that satisfies the following requirements.

- $\langle F, M, \models\rangle$ is a logic as described at the beginning of the present paper. That is, the set of formulas F is the universe of the free algebra $\mathcal{F}$ generated by some set P of atomic formulas in similarity type Cn. M is a non-empty class of models, and $\models$ is a relation between models and formulas.
- mng, called the meaning function, is a function with domain $M \times F$. We write $\mathrm{mng}_{\mathfrak{M}}(\phi)$ in place of $\mathrm{mng}(\mathfrak{M}, \phi)$ and require that $(\forall \phi, \psi \in F)$ $(\forall \mathfrak{M} \in M)$
$$\big(\mathrm{mng}_{\mathfrak{M}}(\phi) = \mathrm{mng}_{\mathfrak{M}}(\psi) \text{ and } \mathfrak{M} \models \phi\big) \Longrightarrow \mathfrak{M} \models \psi. \tag{2.12}$$
- **Compositionality**: For every model $\mathfrak{M}$, the meaning function $\mathrm{mng}_{\mathfrak{M}}$ is a homomorphism from the formula algebra $\mathcal{F}$ into some algebra.
- **Filter property**: There are connectives $\leftrightarrow$ (binary) and $\top$ (constant) such that
$$\mathfrak{M} \models \phi \leftrightarrow \psi \quad \text{iff} \quad \mathrm{mng}_{\mathfrak{M}}(\phi) = \mathrm{mng}_{\mathfrak{M}}(\psi) \tag{2.13}$$
and
$$\mathfrak{M} \models \phi \quad \text{iff} \quad \mathfrak{M} \models \phi \leftrightarrow \top. \tag{2.14}$$
- **Substitution property**: For every model $\mathfrak{M}$ and homomorphism $h : \mathcal{F} \to \mathrm{mng}_{\mathfrak{M}}(\mathcal{F})$ there is a model $\mathfrak{N}$ (called the substituted version of $\mathfrak{M}$) such that $\mathrm{mng}_{\mathfrak{N}} = h$.
- **Patchwork property**: Suppose $\mathfrak{M}, \mathfrak{N}$ are models and A and B are sets of atomic formulas. If $\mathrm{mng}_{\mathfrak{M}}$ and $\mathrm{mng}_{\mathfrak{N}}$ agree on formulas using vocabulary $A \cap B$, then there is a model $\mathfrak{P}$ such that $\mathrm{mng}_{\mathfrak{P}}$ agrees with $\mathrm{mng}_{\mathfrak{M}}$ on formulas over the vocabulary A, and $\mathrm{mng}_{\mathfrak{P}}$ agrees with $\mathrm{mng}_{\mathfrak{N}}$ on formulas over the vocabulary B.

We note that all our examples $\mathcal{L}_{Prop}$, $\mathcal{L}_{Sent}$, $\mathcal{L}_D$ and $\mathcal{L}_n$ are algebraizable logics with a proper choice of the meaning function. For a detailed discussion and for more examples we refer to [3]. We write

$$\begin{aligned}
\mathsf{Alg}_m(\mathcal{L}) &= \{\mathrm{mng}_{\mathfrak{M}}(\mathcal{F}') : \ \mathfrak{M} \in M, \mathcal{F}' \text{ is a subalgebra of } \mathcal{F}\} \\
\mathsf{Alg}_{\models}(\mathcal{L}) &= \{\mathfrak{A} : \ \mathfrak{A} \cong \mathcal{F}/{\sim_K}, K \subseteq M\}, \text{ where } \phi \sim_K \psi \text{ iff } K \models \phi \leftrightarrow \psi,
\end{aligned}$$

for the class of meaning algebras and the class of Lindenbaum-Tarski algebras, respectively.

* * *

Let t be an algebraic similarity type. Given a set of equations $e(x, y)$ of type t and a t-type algebra $\mathfrak{A}$ we write

$$\leq_e^{\mathfrak{A}} \;=\; \{\langle a, b\rangle \in A \times A :\ \mathfrak{A} \models e(a, b)\} \tag{2.15}$$

Many cases $e(x, y)$ is a single equation, consider for example the Boolean case, where $x \leq y$ corresponds to the equation $x \wedge y = x$. Note that $\leq_e^{\mathfrak{A}}$ need not be a partial ordering, in general.

Next we define a variant of the superamalgamation property. The original superamalgamation property goes back to Maksimova [23, 22] and a slightly modified version of it has been introduced in [20]. For a class K of algebras and a set X, $\mathfrak{Fr}_{\mathsf{K}}(X)$ denotes the K-free algebra generated by X. For algebras $\mathfrak{A}$ and $\mathfrak{B}$ the relation $\mathfrak{A} \subseteq \mathfrak{B}$ means that $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$.

Definition 2.3. Let $e(x, y)$ be a set of equations. We say that K has the SUP_e (weak superamalgamation property) if for every $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2 \in \mathsf{K}$ with $\mathfrak{A}_0 \subseteq \mathfrak{A}_1$ and $\mathfrak{A}_0 \subseteq \mathfrak{A}_2$ there exists $\mathfrak{A}_3 \in \mathsf{K}$ such that $\mathfrak{A}_1 \subseteq \mathfrak{A}_3$, $\mathfrak{A}_2 \subseteq \mathfrak{A}_3$ and whenever the diagram below commutes (for arbitrary sets X and Y),

$$\begin{tikzcd}
 & \mathfrak{Fr}_{\mathsf{K}}(X) \arrow[dr, tail] \arrow[rrr, dashed, two heads] & & & \mathfrak{A}_1 \arrow[dr, tail] & \\
\mathfrak{Fr}_{\mathsf{K}}(X\cap Y) \arrow[ur, tail] \arrow[dr, tail] \arrow[rrr, dashed, two heads, bend left] & & \mathfrak{Fr}_{\mathsf{K}}(X\cup Y) \arrow[rrr, dashed, two heads, bend right] & \mathfrak{A}_0 \arrow[ur, tail] \arrow[dr, tail, "h"'] & & \mathfrak{A}_3 \\
 & \mathfrak{Fr}_{\mathsf{K}}(Y) \arrow[ur, tail] \arrow[rrr, dashed, two heads] & & & \mathfrak{A}_2 \arrow[ur, tail] &
\end{tikzcd}$$

then $\forall x \in \mathfrak{Fr}_{\mathsf{K}}(X)$ and $\forall y \in \mathfrak{Fr}_{\mathsf{K}}(Y)$ we have

$$\Big(x \;\leq_e^{\mathfrak{Fr}_{\mathsf{K}}(X\cup Y)}\; y \quad\Longrightarrow\quad (\exists z \in \mathfrak{A}_0)(h(x) \;\leq_e^{\mathfrak{A}_3}\; z \text{ and } z \;\leq_e^{\mathfrak{A}_3}\; h(y))\Big)$$

(Here the embeddings between the K-free algebras are the embeddings induced by the inclusion maps between the sets of generators).

Theorem 2.4. *Let $\mathcal{L}$ be an algebraizable logic. Assume $\mathcal{L}$ has a derived binary connective $\rightsquigarrow$ and let $e(x,y)$ denote the equations $x \rightsquigarrow y = \top$. Then*

$$\mathcal{L} \text{ has the mIP}^{\rightsquigarrow} \iff \mathsf{Alg}_m(\mathcal{L}) \text{ has the SUP}_e. \tag{2.16}$$

Proof: ($\Leftarrow$) Assume $\mathsf{Alg}_m(\mathcal{L})$ has the SUP$_e$ and let $\phi, \psi \in F$ be such that $\models \phi \rightsquigarrow \psi$. We need to find, for every $\mathfrak{M} \in M$, a formula $\chi \in F$ with $\mathrm{Voc}(\chi) \subseteq \mathrm{Voc}(\phi) \cap \mathrm{Voc}(\psi)$ such that $\mathfrak{M} \models \phi \rightsquigarrow \chi$ and $\mathfrak{M} \models \chi \rightsquigarrow \psi$. In what follows, $\mathcal{F}^V$ denotes the set of formulas in F whose vocabulary is in V. Let $\mathfrak{M} \in M$ be an arbitrary model, write $V = \mathrm{Voc}(\phi)$, $W = \mathrm{Voc}(\psi)$ and consider the following meaning algebras: $\mathfrak{A}_3 = \mathrm{mng}_{\mathfrak{M}}(\mathcal{F}^{V\cup W})$, $\mathfrak{A}_1 = \mathrm{mng}_{\mathfrak{M}}(\mathcal{F}^{V})$, $\mathfrak{A}_2 = \mathrm{mng}_{\mathfrak{M}}(\mathcal{F}^{W})$, $\mathfrak{A}_0 = \mathrm{mng}_{\mathfrak{M}}(\mathcal{F}^{V\cap W})$. Now, $\models \phi \rightsquigarrow \psi$ implies (see [3, Corollary 5.5])

$$\mathsf{Alg}_{\models}(\mathcal{L}) \models \phi \rightsquigarrow \psi = \top, \tag{2.17}$$

hence, considering ϕ and ψ as elements of the free algebra $\mathfrak{Fr}_{\mathsf{Alg}(\mathcal{L})}(V \cup W)$, we have

$$\phi \leq_e^{\mathfrak{Fr}(V\cup W)} \psi. \tag{2.18}$$

We note that free algebras of $\mathsf{Alg}_{\models}(\mathcal{L})$ and that of $\mathsf{Alg}_m(\mathcal{L})$ are the same as $\mathbf{SP}\mathsf{Alg}_m(\mathcal{L}) \supseteq \mathsf{Alg}_{\models}(\mathcal{L})$ (see [3, Thm 5.3]). Consider the diagram in Definition 2.3. By SUP$_e$ there must exist $z \in \mathfrak{A}_0$ such that $h(\phi) \leq_e^{\mathfrak{A}_3} z$ and $z \leq_e^{\mathfrak{A}_3} h(\psi)$. As $z \in \mathfrak{A}_0$ there is $\chi \in F^{V\cap W}$ with $z = \mathrm{mng}_{\mathfrak{M}}(\chi)$. Then $h(\phi) \leq_e^{\mathfrak{A}_3} z$ implies $\mathfrak{M} \models \phi \rightsquigarrow \chi$ and $z \leq_e^{\mathfrak{A}_3} h(\psi)$ implies $\mathfrak{M} \models \chi \rightsquigarrow \psi$.

($\Rightarrow$) Assume that $\mathcal{L}$ has the mIP$^{\rightsquigarrow}$. To show that $\mathsf{Alg}_m(\mathcal{L})$ has the SUP$_e$, take algebras $\mathfrak{A}_0$, $\mathfrak{A}_1$, $\mathfrak{A}_2 \in \mathsf{Alg}_m(\mathcal{L})$ such that $\mathfrak{A}_0 \subseteq \mathfrak{A}_1$ and $\mathfrak{A}_0 \subseteq \mathfrak{A}_2$.

Lemma 2.5. *For every $\mathfrak{A}_0$, $\mathfrak{A}_1$, $\mathfrak{A}_2 \in \mathsf{Alg}_m(\mathcal{L})$ with $\mathfrak{A}_0 \subseteq \mathfrak{A}_1$ and $\mathfrak{A}_0 \subseteq \mathfrak{A}_2$ there is $\mathfrak{A}_3 \in \mathsf{Alg}_m(\mathcal{L})$ such that $\mathfrak{A}_1 \subseteq \mathfrak{A}_3$ and $\mathfrak{A}_2 \subseteq \mathfrak{A}_3$.*

Proof: Suppose $\mathfrak{A}_0$, $\mathfrak{A}_1$, $\mathfrak{A}_2 \in \mathsf{Alg}_m(\mathcal{L})$ are such that $\mathfrak{A}_0 \subseteq \mathfrak{A}_1$ and $\mathfrak{A}_0 \subseteq \mathfrak{A}_2$. Let $f : A_1 \to A_1$ and $g : A_2 \to A_2$ be the identity mappings. Then f and g extend to homomorphisms $\bar{f} : \mathcal{F}^{A_1} \to \mathfrak{A}_1$ and $\bar{g} : \mathcal{F}^{A_2} \to \mathfrak{A}_2$. By the substitution property of $\mathcal{L}$ there are models $\mathfrak{M} \in M$ and $\mathfrak{N} \in M$ so that $\bar{f} = \mathrm{mng}_{\mathfrak{M}}$ and $\bar{g} = \mathrm{mng}_{\mathfrak{N}}$. By the patchwork property, for some model $\mathfrak{D} \in M$ we have $\mathrm{mng}_{\mathfrak{D}} \restriction F^{A_1} = \mathrm{mng}_{\mathfrak{M}}$ and $\mathrm{mng}_{\mathfrak{D}} \restriction F^{A_2} = \mathrm{mng}_{\mathfrak{N}}$. It follows that $\mathfrak{A}_1 = \mathrm{mng}_{\mathfrak{M}}(\mathcal{F}^{A_1}) \subseteq \mathrm{mng}_{\mathfrak{D}}(\mathcal{F}^{A_1\cup A_2})$ and $\mathfrak{A}_2 = \mathrm{mng}_{\mathfrak{N}}(\mathcal{F}^{A_2}) \subseteq \mathrm{mng}_{\mathfrak{D}}(\mathcal{F}^{A_1\cup A_2})$. $\square$

Let $\mathfrak{A}_3$ be as in Lemma 2.5. As $\mathfrak{A}_3 \in \mathsf{Alg}_m(\mathcal{L})$ it is the image of the meaning function with respect to some model $\mathfrak{M}$, i.e. $\mathfrak{A}_3 = \mathrm{mng}_{\mathfrak{M}}(\mathcal{F}^{A_3})$. Then $\mathfrak{A}_1 = \mathrm{mng}_{\mathfrak{M}}(\mathcal{F}^{A_1})$, $\mathfrak{A}_2 = \mathrm{mng}_{\mathfrak{M}}(\mathcal{F}^{A_2})$ and $\mathfrak{A}_0 = \mathrm{mng}_{\mathfrak{M}}(\mathcal{F}^{A_0})$.

Consider the diagram in Definition 2.3 and suppose that for $x \in \mathfrak{Fr}(X)$ and $y \in \mathfrak{Fr}(Y)$ we have $\mathfrak{Fr}(X \cup Y) \models x \leq_e y$. There are formulas $\phi \in F^{A_1}$ and $\psi \in F^{A_2}$ such that $\mathrm{mng}_{\mathfrak{M}}(\phi) = h(x)$ and $\mathrm{mng}_{\mathfrak{M}}(\psi) = h(y)$. By the filter property, $\mathfrak{A}_3 \models h(x) \leq_e h(y)$ is equivalent to $\mathfrak{M} \models \phi \rightsquigarrow \psi$. Using the mIP$^{\rightsquigarrow}$ one finds a formula $\chi \in F^{A_1 \cap A_2}$ such that $\mathfrak{M} \models \phi \rightsquigarrow \chi$ and $\mathfrak{M} \models \chi \rightsquigarrow \psi$. Clearly, $z = \mathrm{mng}_{\mathfrak{M}}(\chi) \in \mathfrak{A}_0$ and it follows that $h(x) \leq_e^{\mathfrak{A}_3} z$ and $z \leq_e^{\mathfrak{A}_3} h(y)$. $\square$

The weak superamalgamation property is kind of a direct translation of the modelwise interpolation property into an algebraic setting. Even thought this translation is very direct, nevertheless it needed a justification (the proof of Theorem 2.4). As the weak superamalgamation property explicitly mentions free algebras, the correspondence might not be as strong as one would expect. On the other hand, let us note that the algebraic characterization of the regular Craig interpolation property also directly mentions free algebras, as it is equivalent to the superamalgamation property of free algebras (see [20, Def.4.4] for the definition of "Free SUPAP" and [20, Prop.4.6] for the equivalence between the Free SUPAP and the Craig interpolation property). It is "only" certain varieties of Boolean algebras with operators where the free superamalgamation property implies a more general amalgamation property of the variety (for such results we also refer to Madarász [20]). We do not yet know whether our weak superamalgamation property can be strengthened in classes of algebras having additional properties.

There are several variants of the interpolation property, such as Lyndon's interpolation, uniform interpolation, etc. It could be interesting to see to what extent the "modelwise" variants of these properties are meaningful or useful. We did not make effort to investigate this systematically, but it could serve a possible direction for further research.[5]

[5] We would like to thank the anonymous referee for suggesting us to mention such possible further directions.

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References

[1] H. Andréka, Z. Gyenis, I. Németi, I. Sain, **Universal Algebraic Logic**, Birkhauser (2022), DOI: https://doi.org/10.1007/978-3-031-14887-3.

[2] H. Andréka, J. X. Madarász, I. Németi, *Logic of Space-Time and Relativity Theory*, [in:] M. Aiello, I. Pratt-Hartmann, J. Van Benthem (eds.), **Handbook of Spatial Logics**, Springer Netherlands, Dordrecht (2007), pp. 607–711, DOI: https://doi.org/10.1007/978-1-4020-5587-4_11.

[3] H. Andréka, I. Németi, I. Sain, *Algebraic logic*, [in:] **Handbook of Philosophical Logic**, vol. 2, Kluwer Academic Publishers, Dordrecht (2001), pp. 133–247.

[4] H. Andréka, I. Németi, J. van Benthem, *Interpolation and Definability Properties of Finite Variable Fragments*, **Reports of the Mathematical Institute, Hungarian Academy of Sciences**, (1993).

[5] H. Andréka, I. Németi, J. van Benthem, *Modal languages and bounded fragments of predicate logic*, **Journal of Philosophical Logic**, vol. 27(3) (1998), pp. 217–274, DOI: https://doi.org/10.1023/A:1004275029985.

[6] J. Barwise, *On Moschovakis closure ordinals*, **Journal of Symbolic Logic**, vol. 42(2) (1977), pp. 292–296, DOI: https://doi.org/10.2307/2272133.

[7] P. Blackburn, M. d. Rijke, Y. Venema, **Modal Logic**, Cambridge Tracts in Theoretical Computer Science, Cambridge University Press (2001), DOI: https://doi.org/10.1017/CBO9781107050884.

[8] W. J. Blok, D. Pigozzi, *Algebraizable logics*, **Memoirs of the American Mathematical Society**, vol. 77(396) (1989), pp. vi+78, DOI: https://doi.org/10.1090/memo/0396.

[9] W. J. Blok, D. Pigozzi, *Abstract Algebraic Logic*, **Lecture Notes of the Summer School "Algebraic Logic and the Methodology of Applying it", Budapest**, (1994).

[10] W. J. Blok, D. L. Pigozzi, *Local deduction theorems in algebraic logic*, [in:] **Algebraic logic (Budapest, 1988)**, vol. 54 of Colloquia Mathematica Societatis János Bolyai, North-Holland, Amsterdam (1991), pp. 75–109.

[11] W. Conradie, **Definability and changing perspectives**, Master's thesis, University of Amsterdam (2002).

[12] J. Czelakowski, *Logical matrices and the amalgamation property*, **Studia Logica**, vol. 41(4) (1982), pp. 329–341 (1983), DOI: https://doi.org/10.1007/BF00403332.

[13] J. Czelakowski, D. Pigozzi, *Amalgamation and interpolation in abstract algebraic logic*, [in:] **Models, algebras, and proofs (Bogotá, 1995)**, vol. 203 of Lecture Notes in Pure and Applied Mathematics, Dekker, New York (1999), pp. 187–265, DOI: https://doi.org/10.1201/9780429332890.

[14] J. Czelakowski, D. Pigozzi, *Fregean logics*, **Annals of Pure and Applied Logic**, vol. 127(1–3) (2004), pp. 17–76, DOI: https://doi.org/10.1016/j.apal.2003.11.008.

[15] Z. Gyenis, *Interpolation property and homogeneous structures*, **Logic Journal of the IGPL**, vol. 22(4) (2014), pp. 597–607, DOI: https://doi.org/10.1093/jigpal/jzt051.

[16] Z. Gyenis, *Algebraic characterization of the local Craig interpolation property*, **Bulletin of the Section of Logic**, vol. 47(1) (2018), pp. 45–58, DOI: https://doi.org/10.18778/0138-0680.47.1.04.

[17] E. Hoogland, **Algebraic characterizations of two Beth definability properties**, Master's thesis, Universiteit van Amsterdam (1996).

[18] E. Hoogland, **Definability and Interpolation, model-theoretic investigations**, Ph.D. thesis, Institute for Logic, Language and Computation, Universiteit van Amsterdam (2001).

[19] P. Krzystek, S. Zachorowski, *Lukasiewicz logics have not the interpolation property*, **Reports on Mathematical Logic**, vol. 9 (1977), pp. 39–40.

[20] J. X. Madarász, *Interpolation and amalgamation; pushing the limits. I*, **Studia Logica**, vol. 61(3) (1998), pp. 311–345, DOI: https://doi.org/10.1023/A:1005064504044.

[21] J. X. Madarász, I. Németi, G. Székely, *First-Order Logic Foundation of Relativity Theories*, [in:] D. M. Gabbay, M. Zakharyaschev, S. S. Goncharov (eds.), **Mathematical Problems from Applied Logic II: Logics for the XXIst Century**, Springer New York, New York, NY (2007), pp. 217–252, DOI: https://doi.org/10.1007/978-0-387-69245-6_4.

[22] L. Maksimova, *Amalgamation and interpolation in normal modal logic*, **Studia Logica**, vol. 50(3–4) (1991), pp. 457–471, DOI: https://doi.org/10.1007/BF00370682, algebraic logic.

[23] L. L. Maksimova, *Interpolation theorems in modal logics and amalgamable varieties of topological Boolean algebras*, **Algebra i Logika**, vol. 18(5) (1979), pp. 556–586, 632.

[24] P. Mancosu, *Introduction: Interpolations—essays in honor of William Craig*, **Synthese**, vol. 164(3) (2008), pp. 313–319, DOI: https://doi.org/10.1007/s11229-008-9350-6.

[25] G. Metcalfe, F. Montagna, C. Tsinakis, *Amalgamation and interpolation in ordered algebras*, **Journal of Algebra**, vol. 402 (2014), pp. 21–82, DOI: https://doi.org/10.1016/j.jalgebra.2013.11.019.

[26] D. Mundici, *Consequence and Interpolation in Lukasiewicz Logic*, **Studia Logica**, vol. 99(1/3) (2011), pp. 269–278, URL: http://www.jstor.org/stable/41475204.

[27] D. Nyiri, *The Robinson property and amalgamations of higher arities*, **Mathematical Logic Quarterly**, vol. 62(4–5) (2016), pp. 427–433, DOI: https://doi.org/10.1002/malq.201500027.

[28] D. Pigozzi, *Amalgamation, congruence-extension, and interpolation properties in algebras*, **Algebra Universalis**, vol. 1 (1971/72), pp. 269–349, DOI: https://doi.org/10.1007/BF02944991.

[29] D. J. Pigozzi, *Fregean algebraic logic*, [in:] **Algebraic logic (Budapest, 1988)**, vol. 54 of Colloquia Mathematica Societatis János Bolyai, North-Holland, Amsterdam (1991), pp. 473–502.

[30] G. Priest, **An Introduction to Non-Classical Logic: From If to Is**, Cambridge University Press (2008).

[31] D. Roorda, **Resource Logics. Proof-Theoretical Investigations**, Ph.D. thesis, Institute for Logic, Language and Computation, University of Amsterdam (1991).

[32] G. Sági, S. Shelah, *On weak and strong interpolation in algebraic logics*, **The Journal of Symbolic Logic**, vol. 71(1) (2006), pp. 104–118, DOI: https://doi.org/10.2178/jsl/1140641164.

[33] I. Sain, *Successor axioms for time increase the program verifying power of full computational induction*, **Mathematical Institute if the Hungarian Academy of Sciences**, vol. 23 (1983).

[34] I. Sain, *Is "some-other-time" sometimes better than "sometime" for proving partial correctness of programs?*, **Studia Logica**, vol. 47(3) (1988), pp. 279–301, DOI: https://doi.org/10.1007/BF00370557.

[35] I. Sain, *Beth's and Craig's properties via epimorphisms and amalgamation in algebraic logic*, [in:] **Algebraic Logic and Universal Algebra in Computer Science (Ames, IA, 1988)**, vol. 425 of Lecture Notes in Computer Science, Springer, Berlin (1990), pp. 209–225, DOI: https://doi.org/10.1007/BFb0043086.

[36] K. Segerberg, *"Somewhere else" and "Some other time"*, [in:] **Wright and Wrong—Mini essays in honor of Georg Henrik von Wright**, vol. 3 of Publications, Group in Logic and Methodology of Real Finland (1976), pp. 61–64.

[37] S. J. van Gool, G. Metcalfe, C. Tsinakis, *Uniform interpolation and compact congruences*, **Annals of Pure and Applied Logic**, vol. 168(10) (2017), pp. 1927–1948, DOI: https://doi.org/10.1016/j.apal.2017.05.001.

[38] Y. Venema, **Many-dimensional Modal Logic**, Ph.D. thesis, Institute for Logic, Language and Computation, University of Amsterdam (1992).

Zalán Gyenis
Jagiellonian University
Institute of Philosophy
ul. Grodzka 52
31-044 Kraków, Poland
e-mail: zalan.gyenis@uj.edu.pl

Zalán Molnár
Eötvös Loránd University
Department of Logic
Múzeum krt. 4.
1088 Budapest, Hungary
e-mail: mozaag@gmail.com

Övge Öztürk
Eötvös Loránd University
Department of Logic
Múzeum krt. 4.
1088 Budapest, Hungary
e-mail: ovgeovge@gmail.com

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Tore Fjetland Øgaard

THE WEAK VARIABLE SHARING PROPERTY

Abstract

An algebraic type of structure is shown forth which is such that if it is a characteristic matrix for a logic, then that logic satisfies Meyer's weak variable sharing property. As a corollary, it is shown that **RM** and all its odd-valued extensions $\mathbf{RM}_{2n-1}$ satisfy the weak variable sharing property. It is also shown that a proof to the effect that the "fuzzy" version of the relevant logic **R** satisfies the property is incorrect.

Keywords: characteristic matrix, relevant logics, variable sharing properties.

1. Introduction

The variable sharing property—that $A \to B$ is a logical theorem of a logic only if A and B share a propositional variable—is a hallmark of relevant logics. The property was first shown to hold for the logic **E**—Anderson and Belnap's logic of entailment—as well as Ackermann's logic of "rigorous implication" by Belnap in [2]. One of the logics that this property rather surprisingly turned out *not* to hold for is the logic **RM**—Anderson and Belnap's logic **R** augmented by the *mingle* axiom $A \to (A \to A)$; Meyer and Dunn discovered that $\sim(A \to A) \to (B \to B)$ is a theorem of **RM** (cf. [6]).

Even though Meyer did acknowledge that such theorems do undermine the *raison d'être* of the enterprise of relevant logics, Meyer thought that **RM** was "good enough, when some relevance is desirable" [1, p. 393]. Relevant logics allow for no relevance exceptions: If $A \to B$ is a logical theorem, then A must be relevant to B in the sense that A and B must share a propositional variable. Logics like classical logic, on the other

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hand, allow for exceptions: As a consequence of the interpolation theorem we have that if $A \supset B$ is a logical theorem of classical logic, then either A and B will share a propositional variable, or either $\sim A$ or B are logical theorems. The notion of relevance ensured to hold for logics like **RM** is somewhere in between these two, and is brought out by the *weak variable sharing property* (WVSP), that if $A \to B$ is a logical theorem, then either A and B share a propositional variable, or *both* $\sim A$ and B are logical theorems. This property, then, allows for relevance exceptions, but only for antecedents and consequents which are, respectively, logically rejected and logically forced, as it were.

Meyer showed that **RM** does indeed satisfy (WVSP). Unlike Belnap's original variable sharing property, however, (WVSP) does not automatically extend to any sublogic of a logic for which it holds. Neither does Meyer's original proof of the property easily generalize to other logics. Between classical logic and **RM** there are the n-valued logics $\mathbf{RM}_n$, where $n > 2$. In fact, classical logic can be identified as $\mathbf{RM}_2$. Dunn showed in [5] that any such logic $\mathbf{RM}_n$ for even n's, fail to satisfy (WVSP), and stated, albeit without giving a proof, that every odd-valued $\mathbf{RM}_n$ satisfy (WVSP). Robles and Méndez gave a (WVSP)-proof in [9] which covers the four-valued logic $\mathbf{BN}_4$ as well as an "entailment" version of that logic.[1] This paper generalizes that proof so as to make it also apply to **RM** and all the odd-valued $\mathbf{RM}_n$'s (as well as other logics satisfying certain conditions).

There are two interesting sublogics of **RM** which both fail to satisfy the variable sharing property, but for which the status of the weak version is unsettled, namely **RUE** and **RD**—**R** augmented by, respectively, the axiom $A \wedge \sim A \to B \vee \sim B$ and $(A \to B) \vee (B \to A)$.[2] A proof to the effect that **RD**—"fuzzy **R**"—satisfies (WVSP) was put forth by Yang in [14]. That proof, however, is faulty. This paper ends inconclusively by pointing out the error and thus reopens the question as to whether **RD** satisfies (WVSP). In light of the general (WVSP)-proof, however, one way of making progress on whether **RUE** and **RD** do satisfy (WVSP) is pointed out as interesting.

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[1] I am very grateful to Yaroslav Shramko who pointed out that my original proof was quite similar to that given in [9].

[2] The first axiom is sometimes called the axiom of *unrelated extremes*, hence the name ***RUE***, whereas $(A \to B) \vee (B \to A)$ is often called *Dummett's axiom*, hence the name ***RD***.

Table 1. RM and three related logics

R	Ax1–Ax12; R1–R2	**RUE**	**R** +Ax13
RD	**R** +A14	**RM**	**R** +Ax15

2. Logics defined

The consequence relation dealt with in this paper is exclusively the standard Hilbertian one. The following list of axioms and rules are used to define some of the logics in the vicinity of **RM**. Their defining details are found in Tab. 1.

Ax1	$A \to A$
Ax2	$A \to A \vee B$ and $B \to A \vee B$
Ax3	$A \wedge B \to A$ and $A \wedge B \to B$
Ax4	$\neg\neg A \to A$
Ax5	$A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$
Ax6	$(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$
Ax7	$(A \to C) \wedge (B \to C) \to (A \vee B \to C)$
Ax8	$(A \to \neg B) \to (B \to \neg A)$
Ax9	$(A \to B) \to ((B \to C) \to (A \to C))$
Ax10	$(A \to B) \to ((C \to A) \to (C \to B))$
Ax11	$A \to ((A \to B) \to B)$
Ax12	$(A \to (A \to B)) \to (A \to B)$
Ax13	$A \wedge {\sim}A \to B \vee {\sim}B$
Ax14	$(A \to B) \vee (B \to A)$
Ax15	$A \to (A \to A)$
R1	$A, B \vdash A \wedge B$
R2	$A, A \to B \vdash B$

Schechter showed in [10] that **R** $\prec$ **RUE** $\prec$ **RD** $\prec$ **RM**, where $\prec$ is the strict sublogic relation. For instance, **R** does not have **RUE**'s defining axiom (Ax13) as a logical theorem, whereas **RD** does, but does not suffice for the "mingle axiom" (Ax15). Lastly, **RM** suffices for deriving Dummett's axiom (Ax14), but **RUE** does not. Only **R** amongst these logics, then, satisfies Belnap's variable sharing property since (Ax13) is an obvious example of a theorem which violates it.

Definition 2.1. The Dugundji sentences (cf. [4, 5, p. 10]) are the following formulas where any p_i is distinct from p_k for $i \neq k$.

$$\begin{array}{ll}
(P_2) & (p_1 \leftrightarrow p_2) \\
(P_3) & (p_1 \leftrightarrow p_2) \vee (p_1 \leftrightarrow p_3) \vee (p_2 \leftrightarrow p_3) \\
(P_4) & (p_1 \leftrightarrow p_2) \vee (p_1 \leftrightarrow p_3) \vee (p_1 \leftrightarrow p_4) \vee \\
& \quad (p_2 \leftrightarrow p_3) \vee (p_2 \leftrightarrow p_4) \vee \\
& \qquad (p_3 \leftrightarrow p_4) \\
\vdots & \qquad \vdots \\
(P_n) & \bigvee_{1 \leq i < k \leq n} (p_i \leftrightarrow p_k)
\end{array}$$

Definition 2.2. The logic $\mathbf{RM}_n$ for $n \geq 1$ is obtained from $\mathbf{RM}$ by adding every substitutional instance of (P_{n+1}).

Logics in the vicinity of $\mathbf{RM}$ are sometimes outfitted with truth-constants like the *Church constants* $\perp$ and $\top$, or the *Ackermann constants* $\mathbf{t}$ and $\mathbf{f}$. This paper follows the common practice of defining variable sharing properties for the truth-constant-free fragment of the language.[3]

Definition 2.3. A logic $\mathbf{L}$ has the Weak Variable Sharing Property (WVSP) just in case for every truth-constant-free formula A and B, $\vdash_{\mathbf{L}} A \to B$ only if either A and B share a propositional variable, or both $\vdash_{\mathbf{L}} {\sim}A$ and $\vdash_{\mathbf{L}} B$.

To non-trivially satisfy the (WVSP), a logic must have a conditional as a logical constant, and if it is to satisfy (WVSP) while *not* satisfying the full variable sharing property, it must also have a negation. Since the main aim of the paper is to determine some general conditions which are sufficient for a logic to satisfy (WVSP), I have tried to keep the assumptions of the main theorem and lemma to a minimal so that they will also apply to logics with other sets of logical constants.

3. Matrices fit for weak variable sharing

Algebraic structures are in this paper used to provide interpretations for logics, and to do so such structures must provide interpretations for all the logical constants of the logic at hand. A m-ary logical constant $\flat$ will be

[3] See [13] for a different approach, and [7, § 6] for a discussion.

interpreted using a m-ary function $\natural$ on the algebra in question. The arity of such constants and functions will be left to context.

DEFINITION 3.1. A MATRIX for a logic $\mathbf{L}$ with logical constants

$$\langle \sim, \rightarrow, \flat_1, \ldots, \flat_n \rangle,$$

is a structure

$$\mathfrak{A} = \langle \mathcal{K}, \mathcal{D}, \neg, \rightsquigarrow, \natural_1, \ldots, \natural_n \rangle$$

for which

- $\varnothing \neq \mathcal{D} \subseteq \mathcal{K}$
- $\neg$ is a unary function on $\mathcal{K}$
- $\rightsquigarrow$ a binary function on $\mathcal{K}$
- If $\flat_i$ is a m-ary logical constant, then $\natural_i$ is a m-ary function on $\mathcal{K}$.

The elements in $\mathcal{D}$ are the *designated* or "true" elements of $\mathfrak{A}$'s value-space $\mathcal{K}$. $\neg, \rightsquigarrow, \natural_1, \ldots, \natural_n$ are the *defined propositional functions* on $\mathfrak{A}$.

DEFINITION 3.2. An ASSIGNMENT FUNCTION for a matrix $\mathfrak{A}$ is a function I such that for any propositional variable p, $I(p) \in \mathcal{K}$. I is extended to an INTERPRETATION on $\mathfrak{A}$ by letting

$$\begin{aligned} I(\sim A) &=_{df} \neg I(A) \\ I(A \rightarrow B) &=_{df} I(A) \rightsquigarrow I(B) \\ I(\flat_i(A_1, \ldots, A_m)) &=_{df} \natural_i(I(A_1), \ldots, I(A_m)) \end{aligned}$$

- A formula A IS TRUE IN $\mathfrak{A}$ UNDER I just in case $I(A) \in \mathcal{D}$.
- A formula A IS VALID IN $\mathfrak{A}$ just in case it is true in $\mathfrak{A}$ under every assignment function I.

DEFINITION 3.3. A matrix $\mathfrak{A}$ is called a CHARACTERISTIC MATRIX for a logic $\mathbf{L}$ just in case $\vdash_{\mathbf{L}} A$ if and only if A is valid in $\mathfrak{A}$.

DEFINITION 3.4. A WVSP-MATRIX $\mathfrak{W}$ for a logic $\mathbf{L}$ is a matrix for $\mathbf{L}$ for which there exists sets $\mathcal{S}_1$ and $\mathcal{S}_2$ such that

- $\varnothing \neq \mathcal{S}_i \subseteq \mathcal{K}$, for $i \in \{1, 2\}$
- $\mathcal{S}_1$ and $\mathcal{S}_2$ are both closed under all the defined propositional functions of $\mathfrak{W}$
- $a \in \mathcal{R}$ & $b \in \mathcal{S}_1 \Longrightarrow a \rightsquigarrow b \in \mathcal{U}$
- $a \in \mathcal{S}_2$ & $b \in \mathcal{U} \Longrightarrow a \rightsquigarrow b \in \mathcal{U}$,

where $\mathcal{R} =_{df} \{x \in \mathcal{K} \mid \neg x \notin \mathcal{D}\}$ and $\mathcal{U} =_{df} \mathcal{K} \setminus \mathcal{D}$.

Theorem 3.5. *If a logic has a WVSP-matrix as a characteristic matrix, then it satisfies (WVSP).*

Proof: Assume that $\mathbf{L}$ has $\mathfrak{W}$ as a characteristic WVSP-matrix. Furthermore, let $\vdash_{\mathbf{L}} A \to B$, where A and B are truth-constant free formulas which share no propositional variables. For contradiction, then, assume that either $\nvdash_{\mathbf{L}} {\sim}A$ or $\nvdash_{\mathbf{L}} B$. The theorem is proven by showing that both disjuncts lead to a contradiction.

Assume first that $\nvdash_{\mathbf{L}} {\sim}A$. Since $\mathfrak{W}$ is a characteristic matrix for $\mathbf{L}$, there is an assignment function I such that $I({\sim}A) \notin \mathcal{D}$. It follows that $I(A) \in \mathcal{R}$. Let I' be just like I, except that $I'(p) \in \mathcal{S}_1$ for every propositional variable p occurring in B. Since $\mathcal{S}_1$ is closed under every propositional function, it follows by an easy induction that $I'(B) \in \mathcal{S}_1$. I' is well-defined since A and B do not share any propositional variables. Furthermore, $I'(A) = I(A)$. Since, then, $I'(A) \in \mathcal{R}$ and $I'(B) \in \mathcal{S}_1$, it follows by the definition of a WVSP-matrix that $I'(A) \rightsquigarrow I'(B) \in \mathcal{U}$, and so $A \to B$ is not true in $\mathfrak{W}$ under I'. However, $A \to B$ is a logical theorem of $\mathbf{L}$ and so valid in $\mathfrak{W}$. Contradiction.

Secondly, assume that $\nvdash_{\mathbf{L}} B$. Since $\mathfrak{W}$ is a characteristic matrix there is an assignment function I such that $I(B) \notin \mathcal{D}$. By definition, then, $I(B) \in \mathcal{U}$. Let I' be just like I, except that $I'(p) \in \mathcal{S}_2$ for every propositional variable p occurring in A. As above it follows from the fact that $\mathcal{S}_2$ is closed under every propositional function, that $I'(A) \in \mathcal{S}_2$. I' is well-defined since A and B do not share any propositional variables. Furthermore, $I'(B) = I(B)$. Since, then, $I'(A) \in \mathcal{S}_2$ and $I'(B) \in \mathcal{U}$, it follows by the definition of a WVSP-matrix that $I'(A) \rightsquigarrow I'(B) \in \mathcal{U}$, and so $A \to B$ is not true in $\mathfrak{W}$ under I'. However, $A \to B$ is a logical theorem of $\mathbf{L}$ and so valid in $\mathfrak{W}$. Contradiction. □

Definition 3.6. A propositional fixed-point of a matrix

$$\mathfrak{A} = \langle \mathcal{K}, \mathcal{D}, \neg, \rightsquigarrow, \natural_1, \ldots, \natural_n \rangle$$

is any point $\mathfrak{f} \in \mathcal{K}$ such that

$$\begin{array}{ll} (1) & \neg\mathfrak{f} = \mathfrak{f} \\ (2) & \mathfrak{f} \rightsquigarrow \mathfrak{f} = \mathfrak{f} \\ (3) & \natural_i(\mathfrak{f}, \ldots, \mathfrak{f}) = \mathfrak{f} \quad (i \leq n) \end{array}$$

Lemma 3.7. *A matrix $\mathfrak{W}$ is a WVSP-matrix if it satisfies the following three conditions, where a, b are any elements in $\mathcal{K}$:*

- *(Fixed-point) There exists a propositional fixed-point $\mathfrak{f}$ such that $\mathfrak{f} \in \mathcal{D}$*
- *($MT_{\mathfrak{f}}$) $a \rightsquigarrow \mathfrak{f} \in \mathcal{D} \Longrightarrow \neg a \in \mathcal{D}$*
- *($MP_{\mathfrak{f}}$) $\mathfrak{f} \rightsquigarrow b \in \mathcal{D} \Longrightarrow b \in \mathcal{D}$*

Proof: Let $\mathcal{S}_1 = \mathcal{S}_2 = \{\mathfrak{f}\}$. We then only need to show that if $a \in \mathcal{R}$, then $a \rightsquigarrow \mathfrak{f} \in \mathcal{U}$, and that $\mathfrak{f} \rightsquigarrow b \in \mathcal{U}$ for every $b \in \mathcal{U}$.

Assume first, then, that $a \in \mathcal{R} =_{df} \{x \in \mathcal{K} \mid \neg x \notin \mathcal{D}\}$. If $a \rightsquigarrow \mathfrak{f} \notin \mathcal{U}$, then by definition $a \rightsquigarrow \mathfrak{f} \in \mathcal{D}$. It follows then from $(MT_{\mathfrak{f}})$ that $\neg a \in \mathcal{D}$ which contradicts the assumption that $a \in \mathcal{R}$.

Assume now that $b \in \mathcal{U}$. If $\mathfrak{f} \rightsquigarrow b \notin \mathcal{U}$, $\mathfrak{f} \rightsquigarrow b \in \mathcal{D}$. It then follows from $(MP_{\mathfrak{f}})$ that $b \in \mathcal{D}$. This, however, contradicts the assumption that $b \in \mathcal{U} =_{df} \mathcal{K} \setminus \mathcal{D}$. □

The above lemma, then, captures three properties which together are sufficient for making a matrix into a WVSP-matrix, namely the existence of a designated propositional fixed-point, and that the algebraic equivalent of both modus ponens and modus tollens are validated at least with regards to the propositional fixed-point. These properties, as we shall see, are satisfied by one of the characteristic matrices for **RM** as well as the characteristic matrices for its odd-valued extensions.

Definition 3.8. Let $n \geq 1$. The $2n$-element Sugihara matrix $\mathfrak{S}_{2n}$ consists of the elements $\mathcal{K} = \{-n, \ldots, -1, 1, \ldots, n\}$. The $2n-1$-element Sugihara matrix $\mathfrak{S}_{2n-1}$, on the other hand, has value-space

$$\mathcal{K} = \{-(n-1), \ldots, -1, 0, 1, \ldots, n-1\}.$$

The $\mathbb{Z}$-element Sugihara matrix $\mathfrak{S}_{\mathbb{Z}}$ has $\mathcal{K} = \mathbb{Z}$. The set of designated elements is in each case defined as $\mathcal{D} =_{df} \{n \in \mathcal{K} \mid 0 \leq n\}$. The propositional functions $\neg, \rightsquigarrow, \sqcap, \sqcup$ are for every Sugihara matrix defined as follows:

$$\begin{array}{lll} \neg a & =_{df} & -a \\ a \sqcap b & =_{df} & min\{a, b\} \\ a \sqcup b & =_{df} & max\{a, b\} \\ a \rightsquigarrow b & =_{df} & \begin{cases} \neg a \sqcup b & \text{if} \quad a \leq b \\ \neg a \sqcap b & \text{else} \end{cases} \end{array}$$

Dunn showed in [5] that each $\mathbf{RM}_n$, for $n \geq 1$, has the n-valued Sugihara matrix as a characteristic matrix (cf. [5, thm. 9 & cor. 2]).[4] Furthermore, Meyer showed that $\mathfrak{S}_{\mathbb{Z}}$ is a characteristic matrix for **RM** (cf. [1, p. 415, thm. 4]).[5]

As noted in [5, p. 10], each Dugundji sentence P_n, for $n \geq 2$, is invalid in $\mathfrak{S}_i$ for $i \geq n$. Furthermore, it is easy to verify that $\mathfrak{S}_2$ is in fact the two-element Boolean algebra, and so $\mathbf{RM}_2$ simply amounts to classical logic. $\mathbf{RM}_1$, on the other hand, amounts to the trivial logic since every substitutional instance of $p_1 \leftrightarrow p_2$ is a logical axiom of $\mathbf{RM}_1$, and the logic validates modus ponens. It follows, then, that there are infinitely many **RM**-logics which can be ordered according to strength as follows:

$$\mathbf{RM} \prec \ldots \mathbf{RM}_n \prec \mathbf{RM}_{n-1} \prec \ldots \prec \mathbf{RM}_1.$$

Dunn showed that for (WVSP) fails to hold for every $\mathbf{RM}_{2n}$ $(n \geq 1)$ on account of

$$(p \wedge {\sim} p) \rightarrow (q_1 \vee (q_1 \rightarrow q_2) \vee (q_2 \rightarrow q_3) \vee \ldots \vee (q_{n-1} \rightarrow q_n))$$

being valid in $\mathfrak{S}_{2n}$. It is easy to verify that the consequent is not valid in $\mathfrak{S}_{2n}$, however: By assigning $-n$ to q_n, the consequent will be evaluated to -1. Since the antecedent and consequent do not share any propositional

[4] $\mathbf{RM}_3$ is often axiomatized as **RM** augmented by the axiom $A \vee (A \rightarrow B)$. That these axiomatizations, then, are equivalent, follows from Dunn's result, and Brady's result in [3] that $\mathfrak{S}_3$ is characteristic also for $\mathbf{RM}_3$ axiomatized with the other axiom.

[5] Dunn, modifying an example by Meyer, showed that $\mathfrak{S}_{\mathbb{Z}}$ is not *strongly* characteristic for **RM**. Thus **RM** is not strongly complete with regards to interpretations over $\mathfrak{S}_{\mathbb{Z}}$. He showed, however, that the Sugihara matrix over $\mathbb{Q}$ is strongly characteristic for **RM** (cf. [5, p. 12]).

variables, it follows, therefore, that $\mathbf{RM}_{2n}$—all the even-valued extensions of **RM**—cannot satisfy (WVSP).

Dunn also stated, albeit without proof, that the odd-valued extensions $\mathbf{RM}_{2n+1}$ for $n \geq 1$ satisfy (WVSP) (cf. [5, cor. 5]).[6] That this is indeed correct, is an easy consequence of the above lemma and theorem:

Corollary 3.9. **RM** and every $\mathbf{RM}_{2n-1}$, satisfy (WVSP).

Proof: 0 is a propositional fixed-point for $\mathfrak{S}_{\mathbb{Z}}$ as well as of each $\mathfrak{S}_{2n-1}$, where $n \geq 1$. Furthermore, every such Sugihara matrix validates both modus ponens and modus tollens generally, and so also with regards to the propositional fixed-point. By Lem. 3.7, then, these matrices are WVSP-matrices. Since they are also characteristic matrices for **RM** and $\mathbf{RM}_{2n-1}$, it follows from Thm. 3.5 that these logics satisfy (WVSP). □

3.1. Meyer's WVSP-proof in comparison

As we shall soon see, there are **RM**-related logics for which it is currently unknown whether (WVSP) holds. With that in mind it is important to get clear on which features are utilized in the two types of WVSP-proof available—the one displayed in this paper, and that used in Meyer's original proof for **RM**.[7] This subsection briefly outlines Meyer's proof and compares it with the one displayed in this paper.

As already mentioned, the method used in above theorem is a generalization of that found in Robles and Méndez' [9, prop. 8.5].[8] The above

[6]Dunn, however, stated that (WVSP) *fails* to hold for $\mathbf{RM}_1$ (cf. [5, cor. 5]). This is evidently incorrect since $\vdash_{\mathbf{RM}_1} A$ for *every* formula A.

[7]Meyer's proof can be found as *RM84* in [1, p. 417].

[8]I should also mention that Robles' gave in [8] a proof that $\mathbf{RM}_3$ satisfies (WVSP) which also uses the same type of approach as in [9]. That proof, however, contains a regrettable flaw. The following (nitpickingly) explains the error:

Robles' proof is a proof by contradiction wherein it is assumes (1) that $\vdash_{\mathbf{RM}_3} A \to B$, and (2) that A and B are such as to share no propositional variable, yet (3) either $\nvdash_{\mathbf{RM}_3} {\sim}A$ or $\nvdash_{\mathbf{RM}_3} B$. The heart of her error is that she takes the latter assumption to yield that there are interpretations I and I' over the $\mathbf{RM}_3$ matrix such that either $I({\sim}A) \notin \mathcal{D}$ or $I'(B) \notin \mathcal{D}$. The proof is then split into two cases with the latter one left to the reader. In the first, however—where $I({\sim}A) \notin \mathcal{D}$ is the leading assumption—she uses *both* I and I' to construct an interpretation I'' which is such that $I''(A \to B) \notin \mathcal{D}$ where the fact appealed to is that $I(A) = 1$ *and* $I'(B) = -1$. The existence of I, however, is conditioned upon $\nvdash_{\mathbf{RM}_3} {\sim}A$ being the case, and the existence of I' is similarly conditioned upon $\nvdash_{\mathbf{RM}_3} B$ being the case, and so unless both these hold, one cannot assume that both I and I' exist.

corollary shows, then, that the method is quite powerful as it generalizes to cover many logics. This contrasts to Meyer's original proof which so far at least, has not been made to work for other logics.

The method used here relies on the availability of propositionally closed substructure—subsets of the value-space of the algebra which are closed under all the operations used for interpreting the propositional connectives of our language. In the case of the **RM**-logics, this is realized by the presence of a fixed-point: 0 is a fixed-point for every propositional function in both $\mathfrak{S}_{\mathbb{Z}}$ as well as in the odd-numbered Sugihara matrices. Meyer's original proof that **RM** satisfies (WVSP) in contrast, does not rely on such a fixed-point. Rather, it relies on a certain sort of translation being possible. As I will show, however, it can be seen as a variant of the main theorem presented in this paper.

As in the main theorem, Meyer proof relies on the logic having a characteristic matrix. $\mathfrak{S}_{\mathbb{Z}^*}$, Meyer showed, is yet another characteristic matrix for **RM**, where $\mathbb{Z}^*$ is $\mathbb{Z} \setminus \{0\}$. An outline of Meyer's proof, then, goes as follows: Assume that $A \to B$ is a logical theorem and that A and B fail to share any propositional variables. For contradiction it is then assumed that there is some assignment function which makes A true, i.e., that there is some I such that $I(A) \geq 1$. From I a new interpretation I' is defined which assigns to any propositional variable not occurring in A the value 1, and to any p occurring in A the value $I(p) + I(p)$. A little calculation will then show that $I'(A) > 1$ and $I'(B) = \pm 1$, and therefore that $I'(A \to B) = \neg I'(A) \sqcap I'(B) = \neg I'(A) < -1$ contradicting the assumption that $A \to B$ is a logical theorem and hence valid in $\mathfrak{S}_{\mathbb{Z}^*}$. "By parity of reasoning," as Meyer put it, one similarly obtains a contradiction from the assumption that there is some I which fails to make B true.

Notice that $\mathcal{S}_1 =_{df} \{-1, 1\}$ and $\mathcal{S}_2 =_{df} \mathbb{Z}^* \setminus \mathcal{S}_1$ are both closed under the propositional function corresponding to all the logical constants of **RM**. As in the above theorem, let

$$\mathcal{U} =_{df} \mathcal{K} \setminus \mathcal{D} = \mathbb{Z}^* \setminus \{x \in \mathbb{Z}^* \mid x \geq 1\} = \{x \in \mathbb{Z}^* \mid x \leq -1\},$$

and let $\mathcal{U}' =_{df} \mathcal{U} \setminus \mathcal{S}_1$ and $\mathcal{R}' =_{df} \mathcal{R} \setminus \mathcal{S}_1 = \{x \in \mathbb{Z}^* \mid x \geq 2\}$. It is then easy to verify that if $a \in \mathcal{R}'$ and $b \in \mathcal{S}_1$, then $a \rightsquigarrow b \in \mathcal{U}$, and that if $a \in \mathcal{S}_2$ and $b \in \mathcal{U}'$, then $a \rightsquigarrow b \in \mathcal{U}$.

Meyer's proof, then, relies on the fact that if $I(A) \in \mathcal{R}$, then by *translating* the interpretation I by setting $I'(p) = I(p) + I(p)$, $I'(A) \in \mathcal{R}'$.

Similarly, if $I(B) \in \mathcal{U}$, one needs to prove that the translated interpretation I' is such that $I'(B) \in \mathcal{U}'$. Of course, translating thus does work in case of $\mathfrak{S}_{\mathbb{Z}^*}$, but it is not evident that such a translation will work in other cases. A case in point is the finite Sugihara matrices for which $I(p) + I(p)$ will simply not be an element of the matrix in many cases.

Meyer's proof, then, is very much alike the one shown forth in this paper. Whereas the latter, however, works effortlessly when the matrix in question has a propositional fixed-point, a Meyer-type translation may make the presence of such a point redundant. In the search for a suitable characteristic matrix for a logic, however, it might at least be easier to try to find one with a propositional fixed-point, rather than one admitting of Meyer's type of translation.[9]

Although the proof offered here does contribute towards a more general way of proving that a logic satisfies (WVSP), the fact that **RM** and its odd-valued extensions satisfy (WVSP) is not news. What is a more recent claim, however, is that the weaker logic **RD** also satisfied (WVSP). The next section goes through an incorrect WVSP-proof and affirms the unsettled nature of the question as to whether either **RUE** or **RD** do in fact satisfy the weak variable sharing property.

4. An incorrect WVSP-proof

Yang has offered a proof to the effect that **RD** satisfies (WVSP). This section explains why that proof is incorrect.

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[9] A further cause for thinking that making Meyer's translation-approach work for other logics will be difficult is the fact that the propositionally closed substructure $\{-1, 1\}$ of $\mathfrak{S}_{\mathbb{Z}^*}$ contains the values any assignment function must assign to the Ackermann constant $\mathbf{t}$ and its negation $\mathbf{f}$. The Ackermann constant is axiomatized using the axioms $\mathbf{t}$ and $\mathbf{t} \to (A \to A)$. A characteristic matrix for a logic will suffices for showing that $\mathbf{t}$ can be added conservatively, and so one might hope that $\{I(\mathbf{f}), I(\mathbf{t})\}$ would be the needed propositionally closed substructure of a characteristic matrix for, say, **RD** as well. However, it cannot be a propositionally closed substructure of the characteristic matrix for *any* logic weaker than **RM** yet contained in **R** as it would require that $\mathbf{f} \to \mathbf{t}$ be a logical theorem of the logic, and adding $\mathbf{f} \to \mathbf{t}$ as a logical axiom to **R** yields the logic **RM** ($\mathbf{f} \to \mathbf{t}$ yields in **R** $\sim(A \to B) \to (B \to A)$ (cf. [12, p. 33]), which yields the mingle axiom $A \to (A \to A)$ if added to **R** (cf. [10, pp. 122f])). Thus the propositionally closed substructure needed to make Meyer's proof work cannot be identified as $\{I(\mathbf{f}), I(\mathbf{t})\}$ which makes the search for a suitable translation even harder.

Yang's proof can be found as theorem 2.ii in [14]. As it stands it *is* correct had it only been claimed to hold for $\mathbf{RM}_3$ rather than for **RD**.[10] Yang notes that the axioms of **RD** are all true on every interpretation over the $\mathbf{RM}_3$ algebra, which is true, but insufficient for deriving the wanted conclusion. Yang assumes that A and B are formulas which do not share any propositional variables and that either $\nvdash_{\mathbf{RD}} {\sim}A$ or $\nvdash_{\mathbf{RD}} B$. The goal, then, is to show that there is an interpretation in which $A \to B$ fails to be true, and therefore that $A \to B$ fails to be a theorem of the logic. The proof is split into three cases with all of them making the same mistake: from the assumption that $\nvdash_{\mathbf{RD}} C$ to infer that there is a $\mathbf{RM}_3$-interpretation I such that $I(C) = -1$. The proof, then, fails to provide an interpretation in which $A \to B$ fails to hold, and therefore also that $A \to B$ fails to be a theorem of **RD**.

Let's briefly look at an example where Yang's proof goes wrong: Let A be the formula $r \wedge {\sim}r$ and B the formula ${\sim}(p \to p) \to (q \to q)$, where r, then, is distinct from both p and q. Now it is easy to verify that $\nvdash_{\mathbf{RD}} {\sim}(p \to p) \to (q \to q)$ for distinct propositional variables p and q.[11] However, there are no $\mathbf{RM}_3$-interpretation I such that $I({\sim}(p \to p) \to (q \to q)) = -1$, nor any I' such that $I'((r \wedge {\sim}r) \to ({\sim}(p \to p) \to (q \to q))) = -1$ since both these formulas are theorems of **RM** and so are both valid in the $\mathbf{RM}_3$-matrix.

This, then, reopens the question whether logics like **RD**, as well as the other logics [14] calls "relevant fuzzy logics," do in fact satisfy (WVSP). Additionally, whether **RUE** satisfies (WVSP) is also an open question.

The heart of the error in Yang's proof is easily seen to be that the $\mathbf{RM}_3$-matrix is not a *characteristic* matrix of **RD**. Both Meyer's original

[10] I should note that Yang's definition of **RD**—his name for it is ***FR***, "fuzzy **R**"—is different in that Yang defines it as including the Ackermann constants $\mathbf{t}$ and $\mathbf{f}$ and defines ${\sim}A$ as $A \to \mathbf{f}$. If one only allows $\mathbf{f}$ to occur thus, it is easy to show, however, that the logics are theorem-wise identical. Yang also states the linearity axiom as $((A \to B) \wedge \mathbf{t}) \vee ((B \to A) \wedge \mathbf{t})$, but notes (cf. [14, prop. 2.iii.3]) that $(A \to B) \vee (B \to A)$ is a theorem of all the logics that he considers. Yang also defines the logics to have the fusion connective as a primitive one. In **RD**, however, it is definable using negation and the conditional, and so adding it yields a conservative extension. Lastly, I should also note that his proof is stated to hold not only for **RD**, but for eight different logics in total—see [14, def. 5]—amongst them **RM** and its distributionless variant. His proof does not hold for any of these logics for the same reason as it doesn't work for **RD**.

[11] A model is easily found using `MaGIC`—an acronym for *Matrix Generator for Implication Connectives*—which is an open source computer program created by John K. Slaney [11].

proof, as well as that shown forth in this paper rely on the logic in question having a characteristic matrix of a certain sort. As far as I know, neither **RD** nor **RUE** have been shown to have a characteristic matrix. As noted above, then, finding one with a propositional fixed-point would suffice to show that the logic in question satisfies (WVSP). Neither of the available WVSP-proofs, I should stress, indicate that such a characteristic matrix is *required* for the property to hold true, and so it might be possible to find a WVSP-proof which utilizes different properties. Alas, this paper must end inconclusively on this matter, but leaves both the status of a characteristic matrix and that of (WVSP) for both **RUE** and **RD** as interesting open questions for further research.

5. Summary

This paper has shown forth a certain algebraic structure which was used to prove Meyer's weakened version of the variable sharing property—that if $A \to B$ is a logical truth then *either* do A and B share a propositional variable, or both $\sim A$ and B are logical theorems. It was shown that if a logic has such a structure as its characteristic matrix, then it satisfies Meyer's property. As a consequence of results by Meyer and Dunn for the logics **RM** as well as its odd-valued extensions $\mathbf{RM}_{2n-1}$ (for $n \geq 1$), it was then shown that these logics have such algebraic structures as their characteristic matrices and therefore satisfy Meyer's property. The paper also showed that a proof of Meyer's property for the "fuzzy" extension of the relevant logic **R** is incorrect.

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References

[1] A. R. Anderson, N. D. Belnap, **Entailment: The Logic of Relevance and Necessity**, vol. 1, Princeton University Press, Princeton (1975).

[2] N. D. Belnap, *Entailment and Relevance*, **Journal of Symbolic Logic**, vol. 25(2) (1960), pp. 144–146, DOI: https://doi.org/10.2307/2964210.

[3] R. T. Brady, *Completeness Proofs for the Systems RM3 and BN4*, **Logique et Analyse**, vol. 25(97) (1982), pp. 9–32.

[4] J. Dugundji, *Note on a Property of Matrices for Lewis and Langford's Calculi of Propositions*, **The Journal of Symbolic Logic**, vol. 5(4) (1940), pp. 150–151, DOI: https://doi.org/10.2307/2268175.

[5] J. M. Dunn, *Algebraic Completeness Results for R-Mingle and its Extensions*, **Journal of Symbolic Logic**, vol. 35(1) (1970), pp. 1–13, DOI: https://doi.org/10.1017/S0022481200092161.

[6] R. K. Meyer, *R-Mingle and Relevant Disjunction*, **Journal of Symbolic Logic**, vol. 36(2) (1971), p. 366, DOI: https://doi.org/10.2307/2270323.

[7] T. F. Øgaard, *Non-Boolean Classical Relevant Logics II: Classicality Through Truth-Constants*, **Synthese**, vol. 199 (2021), pp. 6169–6201, DOI: https://doi.org/10.1007/s11229-021-03065-z.

[8] G. Robles, *The Quasi-Relevant 3-Valued Logic RM3 and some of its Sublogics Lacking the Variable-Sharing Property*, **Reports on Mathematical Logic**, vol. 51 (2016), pp. 105–131, DOI: https://doi.org/10.4467/20842589RM.16.008.5285.

[9] G. Robles, J. M. Méndez, *A Companion to Brady's 4-Valued Relevant Logic BN4: The 4-Valued Logic of Entailment E4*, **Logic Journal of the IGPL**, vol. 24(5) (2016), pp. 838–858, DOI: https://doi.org/10.1093/jigpal/jzw011.

[10] E. Schechter, *Equivalents of Mingle and Positive Paradox*, **Studia Logica**, vol. 77(1) (2004), pp. 117–128, DOI: https://doi.org/10.2307/20016611.

[11] J. Slaney, *MaGIC, Matrix Generator for Implication Connectives: Release 2.1 Notes and Guide*, *Tech. Rep. TR-ARP-11/95*, Automated Reasoning Project, Australian National University (1995).

[12] J. K. Slaney, **Computers and Relevant Logic: A Project in Computing Matrix Model Structures for Propositional Logics**, Ph.D. thesis, Australian National University (1980), DOI: https://doi.org/10.25911/5d7396e3ab2c0.

[13] E. Yang, *R and Relevance Principle Revisited*, **Journal of Philosophical Logic**, vol. 42(5) (2013), pp. 767–782, DOI: https://doi.org/10.1007/s10992-012-9247-1.

[14] E. Yang, *Substructural Fuzzy-Relevance Logic*, **Notre Dame Journal of Formal Logic**, vol. 56(3) (2015), pp. 471–491, DOI: https://doi.org/10.1215/00294527-3132824.

Tore Fjetland Øgaard
University of Bergen
Department of Philosophy
Postboks 7805, 5020 Bergen
Bergen, Norway
e-mail: Tore.Ogaard@uib.no

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