UNIVERSITY OF ŁÓDŹ
DEPARTMENT OF LOGIC

BULLETIN
OF THE SECTION OF LOGIC

VOLUME 51, NUMBER 4

ŁÓDŹ, DECEMBER 2022

Layout
Michał Zawidzki

Initiating Editor
Katarzyna Smyczek

Printed directly from camera-ready materials provided
to the Lodz University Press

Published by Lodz University Press
First edition. W.10604.22.0.C

Printing sheets 12.5

Lodz University Press
90-237 Łódź, 34A Jana Matejki St.
www.wydawnictwo.uni.lodz.pl
e-mail: ksiegarnia@uni.lodz.pl
+48 42 635 55 77

The **Bulletin of the Section of Logic** (*BSL*) is a quarterly peer-reviewed journal published with the support from the University of Łódź. Its aim is to act as a forum for a wide and timely dissemination of new and significant results in logic through rapid publication of relevant research papers. *BSL* publishes contributions on topics dealing directly with logical calculi, their methodology, and algebraic interpretation.

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Homepage: https://czasopisma.uni.lodz.pl/bulletin

TABLE OF CONTENTS

Bulletin of the **S**ection of **L**ogic
Volume 51/4 (2022), pp. 455–486
https://doi.org/10.18778/0138-0680.2022.09

Batoul Ganji Saffar
Mona Aaly Kologani
Rajab Ali Borzooei

n-FOLD FILTERS OF EQ-ALGEBRAS

Abstract

In this paper, we apply the notion of n-fold filters to the EQ-algebras and introduce the concepts of n-fold pseudo implicative, n-fold implicative, n-fold obstinate, n-fold fantastic prefilters and filters on an EQ-algebra $\mathcal{E}$. Then we investigate some properties and relations among them. We prove that the quotient algebra $\mathcal{E}/F$ modulo an 1-fold pseudo implicative filter of an EQ-algebra $\mathcal{E}$ is a good EQ-algebra and the quotient algebra $\mathcal{E}/F$ modulo an 1-fold fantastic filter of a good EQ-algebra $\mathcal{E}$ is an IEQ-algebra.

Keywords: EQ-algebra, n-fold pseudo implicative (implicative, obstinate, fantastic) prefilter, n-fold pseudo implicative (implicative, fantastic) EQ-algebra.

2020 Mathematical Subject Classification: 03G25, 06B10, 06B99.

1. Introduction

Recently, a new class of algebras called EQ-algebras has been introduced by Novák in [9]. These algebras are intended to become algebras of truth values for a higher-order fuzzy logic (a fuzzy type theory, FTT). An EQ-algebra has three basic binary operations (meet, multiplication and a fuzzy equality) and a top element. The implication is defined from the fuzzy equality "$\sim$" by the formula $a \to b = (a \wedge b) \sim a$. Its implication and multiplication are no more closely tied by the adjunction and so, this algebra generalizes residuated lattice. From the point of view of potential application,

Presented by: Hanamantagouda P. Sankappanavar
Received: April 8, 2021
Published online: October 14, 2022

it seems interesting that unlike Hájek [5], we can have non-commutativity without the necessity to introduce, two kinds of implication. Novák and De Baets in [10] introduced several kinds of EQ-algebras. El-Zekey in [4], proved that the class of EQ-algebras is a variety. El-Zekey in [4] introduced prelinear good EQ-algebras and proved that a prelinear good EQ-algebra is a distributive lattice. Novák and De Baets in [10] defined the concept of prefilter on EQ-algebras which is the same as filter of other algebraic structures such as residuated lattices, MTL-algebras, and etc. But the binary relation introduced by prefilter is not a congruence relation. To learn more about EQ-algebras, the reader can consult [1, 2, 7, 11, 13, 14]. Filter theory plays an important role in studying logical algebras. From a logical point of view, various filters have a natural interpretation as various sets of provable formulas. In this paper, we introduce n-fold implicative prefilter, n-fold pseudo implicative prefilter, n-fold fantastic prefilter, n-fold obstinate prefilter in EQ-algebra. We prove that the quotient algebra $\mathcal{E}/F$ modulo an 1-fold pseudo implicative filter of an EQ-algebra $\mathcal{E}$ is a good EQ-algebra and the quotient algebra $\mathcal{E}/F$ modulo an 1-fold fantastic filter of good EQ-algebra $\mathcal{E}$ is an involutive EQ-algebra. This paper is organized as follows: In Section 2, the basic definitions, special types of EQ-algebras and their properties are reviewed. In Section 3, n-fold prefilters and n-fold pseudo implicative prefilters of EQ-algebras and EQ_n-algebras are defined and investigated some results about them. We prove that the quotient algebra modulo 1-fold pseudo implicative filter is a good EQ-algebra. In Section 4, n-fold implicative prefilter of EQ-algebra, n-fold implicative EQ-algebra are studied. We show that in good EQ-algebra $\mathcal{E}$ with least element 0, a prefilter F is an n-fold implicative prefilter of $\mathcal{E}$ if and only if $\mathcal{E}/F$ is an n-fold implicative EQ-algebra. In Section 5, n-fold obstinate prefilters, and maximal prefilters of EQ-algebras are investigated. We show that filter $\{1\}$ is an n-fold obstinate filter of residuated EQ-algebra $\mathcal{E}$ if and only if every filter of $\mathcal{E}$ is an n-fold obstinate filter of $\mathcal{E}$ and in a residuated EQ-algebra $\mathcal{E}$, a filter F is an n-fold obstinate filter of $\mathcal{E}$ if and only if every filter of quotient algebra $\mathcal{E}/F$ is an n-fold obstinate filter of $\mathcal{E}/F$. Finally in Section 6, n-fold fantastic prefilters of EQ-algebras and n-fold fantastic EQ-algebras are introduced and studied the relation among the n-fold fantastic prefilters and n-fold fantastic algebras. Then we prove that in any good EQ-algebra, if F is an 1-fold fantastic filter of $\mathcal{E}$, then $\mathcal{E}/F$ is an involutive EQ-algebra, and we show that in any residuated EQ-algebra with least element, F is an n-fold implicative filter of $\mathcal{E}$ if and only if F

is an n-fold pseudo implicative filter and n-fold fantastic filter of $\mathcal{E}$. So we conclude that in any residuated EQ-algebra, $\mathcal{E}$ is an n-fold implicative EQ-algebra if and only if $\mathcal{E}$ is both n-fold pseudo implicative EQ-algebra and n-fold fantastic EQ-algebra.

2. Preliminaries

In this section, we recollect some definitions and results which will be used in the next sections.

Definition 2.1. [4] An *EQ-algebra* is an algebraic structure $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ of type (2, 2, 2, 0) such that, for all $x, y, z, t \in E$ the following conditions hold:

($E1$) $\langle E, \wedge, 1\rangle$ is a commutative idempotent monoid (i.e. $\wedge$-semilattice with top element 1);

($E2$) $\langle E, \otimes, 1\rangle$ is a commutative monoid and $\otimes$ is isotone w.r.t. $\leq$, where $x \leq y$ is defined as $x \wedge y = x$;

($E3$) $x \sim x = 1$; (reflexivity axiom)

($E4$) $((x \wedge y) \sim z) \otimes (t \sim x) \leq z \sim (t \wedge y)$; (substitution axiom)

($E5$) $(x \sim y) \otimes (z \sim t) \leq (x \sim z) \sim (y \sim t)$; (congruence axiom)

($E6$) $(x \wedge y \wedge z) \sim x \leq (x \wedge y) \sim x$; (monotonicity axiom)

($E7$) $x \otimes y \leq x \sim y$. (boundedness axiom)

Proposition 2.2. [10] Let $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ be an EQ-algebra. Define $x \to y := (x \wedge y) \sim x$ and $\bar{x} := x \sim 1$. Then, for all $x, y, z, t \in E$ the following properties hold:

(i) $x \otimes y \leq x, y$ and $x \otimes y \leq x \wedge y$;

(ii) $x \leq y \to x$;

(iii) $x \to y \leq (z \to x) \to (z \to y)$ and $x \to y \leq (y \to z) \to (x \to z)$;

(iv) if $x \leq y$, then $x \sim y = y \to x$, $z \to x \leq z \to y$ and $y \to z \leq x \to z$;

(v) $x \to y \leq (x \wedge z) \to (y \wedge z)$.

Definition 2.3. [10] Let $\mathcal{E}$ be an EQ-algebra. Then $\mathcal{E}$ is called:

(i) *separated* if $x \sim y = 1$, then $x = y$, for all $x, y \in E$, (in other words $x \sim y = 1$ if and only if $x = y$);

(ii) *good* if $x \sim 1 = x = 1 \sim x$, for all $x \in E$;

(iii) *residuated* if $x \leq y \to z$ if and only if $x \otimes y \leq z$, for all $x, y, z \in E$;

(iv) *involutive (IEQ-algebra)* if E contains 0 and $\neg\neg x = x$, for all $x \in E$, where $\neg x = x \sim 0$;

(v) *lattice ordered* if the poset induced by the underlying semilattice of $\mathcal{E}$ is a lattice;

(vi) a lattice *EQ-algebra (ℓEQ-algebra)* if $\mathcal{E}$ is a lattice ordered and for all $x, y, x, t \in E$ the following substitution axiom holds, $((x \vee y) \sim z) \otimes (t \sim x) \leq (z \sim (t \vee y))$.

Proposition 2.4. [10] Each IEQ-algebra is a good, separated and ℓEQ-algebra.

Proposition 2.5. [4] Let $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ be an EQ-algebra. Then, for all $x, y \in E$ the following statements are equivalent:

(i) $\mathcal{E}$ is good;

(ii) $x \otimes (x \sim y) \leq y$;

(iii) $x \otimes (x \to y) \leq y$;

(iv) $1 \to x = x$.

Proposition 2.6. [4] Let $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ be an EQ-algebra. Then, for all $x, y, z \in E$ the following statements are equivalent:

(i) $\mathcal{E}$ is residuated;

(ii) $\mathcal{E}$ is good and $x \to y \leq (x \otimes z) \to (y \otimes z)$;

(iii) $\mathcal{E}$ is good and $x \leq y \to (x \otimes y)$;

(iv) $\mathcal{E}$ is separated and $(x \otimes y) \to z = x \to (y \to z)$.

PROPOSITION 2.7. [4] Let $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ be a good EQ-algebra. Then, for all $x, y, z \in E$ the following properties hold:

(i) $\mathcal{E}$ is residuated if and only if $x \otimes y \leq z$ implies $x \leq y \to z$;

(ii) $x \leq (x \sim y) \sim y$ and $x \leq (x \to y) \to y$;

(iii) $\mathcal{E}$ is separated;

(iv) $x \to (y \to z) = y \to (x \to z)$;

(v) $x \to (y \to z) \leq (x \otimes y) \to z$.

DEFINITION 2.8. [10] Let $\mathcal{E}$ be an EQ-algebra. A nonempty subset $F \subseteq E$ is called a *prefilter* of $\mathcal{E}$, if for all $x, y \in E$,

$(F1)$ $1 \in F$;

$(F2)$ If $x, x \to y \in F$, then $y \in F$.

A prefilter F is said to be a *filter*, if

$(F3)$ $x \to y \in F$ implies $(x \otimes z) \to (y \otimes z) \in F$, for all $x, y, z \in E$.
A proper prefilter F is called a *prime prefilter* of $\mathcal{E}$ if $x \to y \in F$ or $y \to x \in F$, for all $x, y \in E$.

DEFINITION 2.9. [12] A prefilter F of an EQ-algebra $\mathcal{E}$ is called *maximal* if and only if it is proper and no prefilter of $\mathcal{E}$ strictly contains F that is, for each prefilter G of $\mathcal{E}$, if $F \subsetneq G$, then $G = E$.

LEMMA 2.10. [3] *Let F be a prefilter of an EQ-algebra $\mathcal{E}$. Then, for all $x, y, z \in E$ the following statements hold:*

(i) *If $x \in F$ and $x \leq y$, then $y \in F$;*

(ii) *If $x, x \sim y \in F$, then $y \in F$;*

(iii) *If $x, y \in F$, then $x \wedge y \in F$;*

Moreover, if F is a filter of $\mathcal{E}$, we have:

(iv) *If $x, y \in F$, then $x \otimes y \in F$;*

(v) *If $x \to y \in F$ and $y \to z \in F$, then $x \to z \in F$.*

Remark 2.11. By Proposition 2.6 and Lemma 2.10, if $\mathcal{E}$ is a residuated EQ-algebra, then every prefilter of $\mathcal{E}$ is a filter of $\mathcal{E}$.

Definition 2.12. [8] Let $\mathcal{E}$ be an EQ-algebra and X be a nonempty subset of E. Then the smallest prefilter of $\mathcal{E}$ which contains X, i.e.
$\bigcap\{F \mid \text{F is a prefilter of E such that, } X \subseteq F\}$ is said to be a *prefilter of* $\mathcal{E}$ *generated by* X and is denoted by $\langle X\rangle$. If $a \in E$ and $X = \{a\}$, then we denote by $\langle a\rangle$ the prefilter generated by $\{a\}$ ($\langle a\rangle$ is called principal). For prefilter F and $a \in E$, we denote by $F(a) = \langle F \cup \{a\}\rangle$.
It is clear that $a \in F$ implies $F(a) = F$. We can prove

$$F(a) = \{z \in E \mid a \to z \in F\}$$

and
$\langle X\rangle = \{a \in E \mid x_1 \to (x_2 \to (x_3 \to ...(x_n \to a)...)) = 1, \text{ for some } x_i \in X \text{ and } n \in \mathbb{N}\}$.

Definition 2.13. [6] Let F be a prefilter of an EQ-algebra $\mathcal{E}$. Then F is called

(i) an *implicative prefilter* of $\mathcal{E}$, if for all $x, y, z \in E$,

($F4$) $z \to ((x \to y) \to x) \in F$ and $z \in F$ imply $x \in F$.

(ii) a *positive implicative prefilter* of $\mathcal{E}$, if for all $x, y, z \in E$,

($F5$) $x \to (y \to z) \in F$ and $x \to y \in F$ imply $x \to z \in F$.

(iii) a *fantastic prefilter* of $\mathcal{E}$, if for all $x, y \in E$,

($F6$) $y \to x \in F$ implies $((x \to y) \to y) \to x \in F$.

(iv) an *obstinate prefilter* of $\mathcal{E}$,

($F7$) $x, y \notin F$ imply $x \to y \in F$ and $y \to x \in F$.

Proposition 2.14 ([12]). Let $\mathcal{E}$ be a residuated EQ-algebra and F be a fantastic prefilter of $\mathcal{E}$. Then F is an implicative prefilter of $\mathcal{E}$ if and only if F is a positive implicative prefilter of $\mathcal{E}$.

Proposition 2.15 ([12]). Let $\mathcal{E}$ be a residuated EQ-algebra and F be a positive implicative prefilter of $\mathcal{E}$. Then F is an implicative prefilter of E if and only if F is a fantastic prefilter of $\mathcal{E}$.

PROPOSITION 2.16 ([12]). Let $\mathcal{E}$ be a good EQ-algebra and F be a nonempty subset of $\mathcal{E}$. Then F is an implicative prefilter if and only if F is both a positive implicative prefilter and a fantastic prefilter of $\mathcal{E}$.

Let F be a filter of an EQ-algebra $\mathcal{E}$. Then we define a binary relation $\equiv_F$ on E as follows:

$$x \equiv_F y \;\text{ if and only if }\; x \sim y \in F.$$

Then $\equiv_F$ is a congruence relation on E. Denote $E/F := \{[x]_F \mid x \in E\}$ and $[x]_F = \{y \in E \mid x \equiv_F y\}$ and define operations $\wedge_F$, $\otimes_F$, $\sim_F$ and relation $\leq_F$ on E/F as follows:

$$[x]_F \wedge_F [y]_F = [x \wedge y]_F, \;\; [x]_F \otimes_F [y]_F = [x \otimes y]_F, \;\; [x]_F \sim_F [y]_F = [x \sim y]_F,$$

$$[x]_F \leq_F [y]_F \text{ if and only if } x \to y \in F \text{ if and only if } [x]_F \to_F [y]_F = [1]_F.$$

We write $[x]$ instead of $[x]_F$, for short.

THEOREM 2.17 ([4]). *Let F be a filter of an ℓEQ-algebra $\mathcal{E}$. Then the quotient algebra $\mathcal{E}/F = (E/F, \wedge_F, \otimes_F, \sim_F, F)$ is a separated ℓEQ-algebra and the mapping $f : x \to [x]_F$ is an epimorphism.*

3. n-fold pseudo implicative prefilters of EQ-algebras

In this section, we introduce the notions of n-fold prefilters and n-fold pseudo implicative prefilters on EQ-algebras and prove some related results. Also, we prove that the quotient algebra modulo by 1-fold pseudo implicative filter is a good EQ-algebra.

In what follows, let n denotes a positive integer and for any $x \in E$, x^n denotes $x \otimes x \otimes ... \otimes x$, in which x occurs n times and $x^0 = 1$.

DEFINITION 3.1. Let $\mathcal{E}$ be an EQ-algebra. A nonempty subset $F \subseteq E$ is called an *n-fold prefilter* of $\mathcal{E}$, if for all $x, y \in E$,

(i) $1 \in F$;

(ii) If $x^n, x^n \to y \in F$, then $y \in F$.

An n-fold prefilter F is said to be an *n-fold filter* of $\mathcal{E}$, if F satisfies $(F3)$.

Obviously, each prefilter is an n-fold prefilter. But the converse is not true.

Example 3.2. Let $E = \{0, a, b, c, 1\}$ be a chain such that $0 \leq a \leq b \leq c \leq 1$. Define the operations $\wedge, \otimes$ and $\sim$ on E as follows:

$\otimes$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	0	a
b	0	0	0	0	b
c	0	0	0	0	c
1	0	a	b	c	1

$\sim$	0	a	b	c	1
0	1	a	0	0	0
a	a	1	a	a	a
b	0	a	1	b	b
c	0	a	b	1	c
1	0	a	b	c	1

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	a	1	1	1	1
b	0	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

$$x \wedge y = \min\{x, y\}.$$

Then $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ is an EQ-algebra. Let $F = \{1, c\}$. Then F is an n-fold filter of $\mathcal{E}$, for all $n \in \mathbb{N}$.
Let $F = \{1, a\}$. Then F is a 2-fold prefilter of $\mathcal{E}$. Since $a \in F$ and $a \rightarrow b = 1 \in F$ but $b \notin F$, F is not a prefilter of $\mathcal{E}$. Similarly $F = \{1, b\}$ is a 2-fold prefilter but not a prefilter of $\mathcal{E}$.

Definition 3.3. Let $\mathcal{E}$ be an EQ-algebra. A nonempty subset $F \subseteq E$ is called an *n-fold pseudo implicative prefilter* of $\mathcal{E}$, if for all $x, y, z \in E$,

(*i*) $1 \in F$;

(*ii*) $x^n \rightarrow (y \rightarrow z) \in F$ and $x^n \rightarrow y \in F$ imply $x^n \rightarrow z \in F$.

Example 3.4. Let $E = \{0, a, b, 1\}$ be a chain such that $0 \leq a \leq b \leq 1$. Define the operations $\wedge, \otimes$ and $\sim$ on E as follows:

$\otimes$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	b	1
1	0	a	b	1

$\sim$	0	a	b	1
0	1	0	0	0
a	0	1	a	a
b	0	a	1	1
1	0	a	1	1

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	a	1	1
1	0	a	1	1

$$x \wedge y = \min\{x, y\}.$$

Then $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ is an EQ-algebra. Let $F = \{1, b\}$. Then F is an n-fold filter and n-fold pseudo implicative filter of $\mathcal{E}$, for all $n \geq 2$.
If $F = \{1, a\}$, then F is an n-fold pseudo implicative prefilter of $\mathcal{E}$, for all $n \geq 2$. Clearly, F is not a filter of $\mathcal{E}$, since $a^2 = a \in F$ and $a^2 \to b = a \to b = 1 \in F$ but $b \notin F$. In addition, we can see that F is not an n-fold filter of $\mathcal{E}$.

Obviously each pseudo implicative prefilter of $\mathcal{E}$ is an n-fold pseudo implicative prefilter of $\mathcal{E}$, but the converse is not true.

Example 3.5. Let $\mathcal{E}$ be an EQ-algebra as in Example 3.2. Suppose $F = \{1, c\}$. Then F is a 2-fold pseudo implicative filter of $\mathcal{E}$. Since $a \to (a \to 0) = 1 \in F$ and $a \to a = 1 \in F$ but $a \to 0 = a \notin F$, we get F is not a pseudo implicative filter of $\mathcal{E}$.

PROPOSITION 3.6. Let $\mathcal{E}$ be a good EQ-algebra. Then every n-fold pseudo implicative prefilter of $\mathcal{E}$ is an n-fold prefilter of $\mathcal{E}$.

PROOF: Let $x, y \in E$ such that $x^n, x^n \to y \in F$. Then by goodness, $1^n \to x^n, 1^n \to (x^n \to y) \in F$. Hence $1^n \to y = y \in F$. □

Example 3.7. Let $\mathcal{E}$ be the EQ-algebra as in Example 3.4. Since $b \sim 1 = 1 \neq b$, we have $\mathcal{E}$ is not good. Suppose $F = \{1, a\}$. Then F is an n-fold pseudo implicative filter of $\mathcal{E}$, for all $n \geq 2$. Since $a^n = a \in F$ and $a^n \to b = 1 \in F$ but $b \notin F$, we have F is not an n-fold filter of $\mathcal{E}$, for all $n \geq 2$.

COROLLARY 3.8. Let $\mathcal{E}$ be a good EQ-algebra. Then every n-fold pseudo implicative prefilter of $\mathcal{E}$ is a prefilter of $\mathcal{E}$.

PROOF: Let F be an n-fold pseudo implicative prefilter of $\mathcal{E}$, $x \in F$ and $x \to y \in F$. Then $1^n \to (x \to y) \in F$ and $1^n \to x \in F$ and so $1^n \to y = y \in F$. Therefore, F is a prefilter of $\mathcal{E}$. □

PROPOSITION 3.9. Let $\mathcal{E}$ be a good EQ-algebra. Then $\{1\}$ is an n-fold prefilter of $\mathcal{E}$, for all $n \in \mathbb{N}$.

PROOF: Let $x^n \in \{1\}$ and $x^n \to y \in \{1\}$, then $1 \to y = y \in \{1\}$ and so $\{1\}$ is an n-fold prefilter of $\mathcal{E}$, for all $n \in \mathbb{N}$. Now, let $x \to y \in \{1\}$. Then $x \leq y$ and so $x \otimes z \leq y \otimes z$, for all $z \in E$. Hence $(x \otimes z) \to (y \otimes z) = 1 \in \{1\}$. Therefore, $\{1\}$ is an n-fold filter of $\mathcal{E}$. □

Example 3.10. Let $\mathcal{E}$ be the EQ-algebra as in Example 3.4. Since $b \sim 1 = 1 \neq b$, we get $\mathcal{E}$ is not a good EQ-algebra. Since $1^n = 1 \in \{1\}$ and $1^n \to b = 1 \in \{1\}$ but $b \notin \{1\}$, we get $\{1\}$ is not an n-fold filter of $\mathcal{E}$, for all $n \in \mathbb{N}$

In the following theorem, we provide some conditions equivalent to the concept of n-fold pseudo implicative filter.

THEOREM 3.11. *Let $\mathcal{E}$ be a residuated EQ-algebra, F be a filter of $\mathcal{E}$ and $n \in \mathbb{N}$. Then, for all $x, y, z \in E$ the following conditions are equivalent:*

(*i*) *F is an n-fold pseudo implicative filter of $\mathcal{E}$;*

(*ii*) *$x^n \to x^{2n} \in F$;*

(*iii*) *$x^{n+1} \to y \in F$ implies $x^n \to y \in F$;*

(*iv*) *$x^n \to (y \to z) \in F$ implies $(x^n \to y) \to (x^n \to z) \in F$.*

PROOF: $(i) \Longrightarrow (ii)$: By Proposition 2.6(iv), we have $x^n \to (x^n \to x^{2n}) = x^{2n} \to x^{2n} = 1 \in F$. Since $x^n \to x^n = 1 \in F$ by (i), we have $x^n \to x^{2n} \in F$.
$(ii) \Longrightarrow (i)$: Let $x^n \to (y \to z) \in F$ and $x^n \to y \in F$. Then by Propositions 2.6 and 2.5(iii),

$$\begin{aligned}(x^n \to (y \to z)) \otimes (x^n \to y) \otimes x^{2n} &= (x^n \to (y \to z)) \otimes x^n \otimes (x^n \to y) \otimes x^n \\ &\leq (y \to z) \otimes y \leq z.\end{aligned}$$

Thus by Proposition 2.7(i), $(x^n \to (y \to z)) \otimes (x^n \to y) \leq x^{2n} \to z$ and so $x^{2n} \to z \in F$. Since by assumption, F is a filter of $\mathcal{E}$, we get $x^{2n} \to z \in F$. Also, by Proposition 2.2(iii), $x^n \to x^{2n} \leq (x^{2n} \to z) \to (x^n \to z)$. Hence by (ii), $(x^{2n} \to z) \to (x^n \to z) \in F$, and so $x^n \to z \in F$. Therefore, F is an n-fold pseudo implicative filter of $\mathcal{E}$.
$(ii) \Longrightarrow (iii)$: Let $x^{n+1} \to y \in F$. Then by Proposition 2.6(iv), we have $x^{n+1} \to y = x^n \to (x \to y) \in F$. Since $x^n \leq x$, we have $x^n \to x = 1 \in F$. Hence by (i) or equivalently (ii), $x^n \to y \in F$.
$(iii) \Longrightarrow (ii)$: By Proposition 2.6$(iv)$,

$$x^{n+1} \to (x^{n-1} \to x^{2n}) = x^{2n} \to x^{2n} = 1 \in F.$$

Thus by (iii), $x^n \to (x^{n-1} \to x^{2n}) \in F$. Also, we have

$$x^{n+1} \to (x^{n-2} \to x^{2n}) = x^{2n-1} \to x^{2n} = x^n \to (x^{n-1} \to x^{2n}) \in F.$$

Hence by (iii), $x^n \to (x^{n-2} \to x^{2n}) \in F$. By repeating this method n times we get

$$x^n \to (x^0 \to x^{2n}) = x^n \to (1 \to x^{2n}) = x^n \to x^{2n} \in F.$$

$(ii) \Longrightarrow (iv)$: Let $x^n \to (y \to z) \in F$. Then by Propositions 2.2(iii), (iv) and 2.7(iv),

$$\begin{aligned} x^n \to (y \to z) &\leq x^n \to ((x^n \to y) \to (x^n \to z)) \\ &= x^n \to (x^n \to ((x^n \to y) \to z)) \\ &= x^{2n} \to ((x^n \to y) \to z) \in F. \end{aligned}$$

Also, we have $x^{2n} \to ((x^n \to y) \to z)) \leq (x^n \to x^{2n}) \to (x^n \to ((x^n \to y) \to z))$. Thus

$$(x^n \to x^{2n}) \to (x^n \to ((x^n \to y) \to z)) \in F.$$

By (ii), since $x^n \to x^{2n} \in F$, we have

$$x^n \to ((x^n \to y) \to z)) = (x^n \to y) \to (x^n \to z) \in F.$$

$(iv) \Longrightarrow (ii)$: Since $x^n \to (x^n \to x^{2n}) = x^{2n} \to x^{2n} = 1 \in F$ by (iv), we get $(x^n \to x^n) \to (x^n \to x^{2n}) \in F$ and so by goodness, $x^n \to x^{2n} \in F$. $\square$

Proposition 3.12. Let $\mathcal{E}$ be an EQ-algebra and F be a prefilter of $\mathcal{E}$. If F is an 1-fold pseudo implicative prefilter of $\mathcal{E}$, then for all $x, y \in E$ and $n \in \mathbb{N}$ the following properties hold:

(i) $((x^n \wedge (x^n \to y)) \to y) \in F$;

(ii) $((x^n \otimes (x^n \to y)) \to y) \in F$.

Proof: (i): Let F be an 1-fold pseudo implicative prefilter of $\mathcal{E}$. Since $(x^n \wedge (x^n \to y)) \leq x^n \to y, x^n$, we get $((x^n \wedge (x^n \to y)) \to (x^n \to y) = 1 \in F$ and $(x^n \wedge (x^n \to y)) \to x^n = 1 \in F$. Hence, by assumption $(x^n \wedge (x^n \to y)) \to y \in F$.
(ii): By (i), $(x^n \wedge (x^n \to y) \to y) \in F$. Then by Proposition 2.2(i), $x^n \otimes (x^n \to y) \leq x^n \wedge (x^n \to y)$ and so $(x^n \wedge (x^n \to y)) \to y \leq (x^n \otimes (x^n \to y)) \to y$. Hence, $(x^n \otimes (x^n \to y)) \to y \in F$. $\square$

Corollary 3.13. Let $\mathcal{E}$ be an EQ-algebra and F be a prefilter of $\mathcal{E}$. If F is an 1-fold pseudo implicative prefilter of $\mathcal{E}$, then $(1 \to x) \to x \in F$, for all $x \in E$.

Proof: By Proposition 3.12(i), since $1, x \in E$, we have $(1^n \wedge (1^n \to x)) \to x = (1 \to x) \to x \in F$. $\square$

Theorem 3.14. *Let $\mathcal{E}$ be an EQ-algebra and F be a prefilter of $\mathcal{E}$. If F is an 1-fold pseudo implicative filter of $\mathcal{E}$, then $\mathcal{E}/F$ is a good EQ-algebra.*

Proof: By Theorem 2.17, $\mathcal{E}/F$ is a separated EQ-algebra. Then by Corollary 3.13, for any $x \in E$, $(1 \to x) \to x \in F$ and so $[1 \to x] \leq [x]$. Thus $[x] \sim [1] \leq [x]$ and by Proposition 2.2(ii), $[x] \leq [1] \sim [x]$, that is $[1] \sim [x] = [x]$, for all $[x] \in \mathcal{E}/F$. Therefore, $\mathcal{E}/F$ is a good EQ-algebra. $\square$

Theorem 3.15. *Let $\mathcal{E}$ be a residuated EQ-algebra and F be a filter of $\mathcal{E}$. Then the following statements are equivalent:*

(i) *F is an n-fold pseudo implicative filter of $\mathcal{E}$;*

(ii) *$x^m \to (x \to y) \in F$ implies $x^m \to y \in F$, for all $x, y \in F$ and $m \geq n$.*

Proof: $(i) \Longrightarrow (ii)$: Let F be an n-fold pseudo implicative filter of $\mathcal{E}$ and $x^m \to (x \to y) \in F$, for $x, y \in E$. Since $x^m \leq x$, we have $x^m \to x = 1 \in F$ and so by (i), $x^m \to y \in F$.
$(ii) \Longrightarrow (i)$: Let $x^n \to (y \to z) \in F$ and $x^n \to y \in F$. Then by Proposition 2.2(iii), we have

$$x^n \to (y \to z) \leq ((y \to z) \to (x^n \to z)) \to (x^n \to (x^n \to z)),$$

and $x^n \to y \leq (y \to z) \to (x^n \to z)$. Thus $((y \to z) \to (x^n \to z)) \to (x^n \to (x^n \to z)) \in F$ and $(y \to z) \to (x^n \to z) \in F$ and so $x^n \to (x^n \to z) = x^{2n-1} \to (x \to z) \in F$. By (ii), we have $x^{2n-1} \to z \in F$. Since $x^{2n-1} \to z = x^{2n-2} \to (x \to z) \in F$, by (ii), we obtain $x^{2n-2} \to z \in F$. By repeating this method, we have $x^n \to z \in F$. Therefore, F is an n-fold pseudo implicative filter of $\mathcal{E}$. $\square$

Proposition 3.16. Let $\mathcal{E}$ be a residuated EQ-algebra and F be a filter of $\mathcal{E}$. If F is an n-fold pseudo implicative filter of $\mathcal{E}$, then F is an $n+1$-fold pseudo implicative filter of $\mathcal{E}$.

Proof: Let F be an n-fold pseudo implicative filter of $\mathcal{E}$ and $x, y \in E$ such that $x^{n+2} \to y \in F$. Then by Proposition 2.6(iv), $x^{n+2} \to y = (x^{n+1} \otimes x) \to y = x^{n+1} \to (x \to y) \in F$. Thus by Theorem 3.15$(ii)$, $x^{n+1} \to y \in F$ and so F is an $n+1$-fold pseudo implicative filter of $\mathcal{E}$. $\square$

By the following example we show that the converse of Proposition 3.16, is not true.

Example 3.17. Let $\mathcal{E}$ be the EQ-algebra as in Example 3.2. Suppose $F = \{1, c\}$. Then F is a 2-fold pseudo implicative prefilter of $\mathcal{E}$. Since $a \to (a \to 0) = 1 \in F$ and $a \to a = 1 \in F$ but $a \to 0 = a \notin F$, we get F is not a 1-fold pseudo implicative prefilter of $\mathcal{E}$.

Proposition 3.18. Let F and G be two filters of residuated EQ-algebra $\mathcal{E}$ such that $F \subseteq G$. If F is an n-fold pseudo implicative filter of $\mathcal{E}$, then G is an n-fold pseudo implicative filter of $\mathcal{E}$.

Proof: Let F be an n-fold pseudo implicative filter of $\mathcal{E}$. Then by Theorem 3.11(ii), $x^n \to x^{2n} \in F$, for all $x \in E$ and so $x^n \to x^{2n} \in G$. Therefore, G is an n-fold pseudo implicative filter of $\mathcal{E}$. □

We now define a sequence of subvarieties of the variety of EQ-algebras.

Definition 3.19. Let $\mathcal{E}$ be an EQ-algebra. Then $\mathcal{E}$ is called an *EQ_n-algebra*, if for all $x, y \in E$, $x^n \to y = x^{n+1} \to y$.

Example 3.20. Let $\mathcal{E}$ be the EQ-algebra as in Example 3.2. Then $\mathcal{E}$ is an EQ_n-algebra, for all $n \geq 2$.

Proposition 3.21. In any residuated EQ_n-algebra, n-fold filters and n-fold pseudo implicative filters coincide.

Proof: By Proposition 3.6, each n-fold pseudo implicative filter of $\mathcal{E}$ is an n-fold filter of $\mathcal{E}$. Let F be an n-fold filter of $\mathcal{E}$ and $x^{n+1} \to y \in F$. Then by assumption, $x^n \to y \in F$ and so by Theorem 3.11, F is an n-fold pseudo implicative filter of $\mathcal{E}$. □

Proposition 3.22. Let $\mathcal{E}$ be a residuated EQ-algebra. Then $\mathcal{E}$ is an EQ_n-algebra if and only if $\{1\}$ is an n-fold pseudo implicative filter of $\mathcal{E}$.

Proof: Let $\mathcal{E}$ be an EQ_n-algebra and $x^{n+1} \to y \in \{1\}$. Then by Definition 3.19, $x^n \to y \in \{1\}$ and so $\{1\}$ is an n-fold pseudo implicative filter of $\mathcal{E}$.

Conversely, let $\{1\}$ be an n-fold pseudo implicative filter of $\mathcal{E}$. Since

$$1 = x^{2n} \to x^{n+1} = (x^n \otimes x^n) \to x^{n+1} = x^n \to (x^n \to x^{n+1}) \in \{1\}$$

and $x^n \to x^n = 1 \in \{1\}$, then $x^n \to x^{n+1} \in \{1\}$ and so $x^n \leq x^{n+1}$. On the other hands $x^{n+1} = x^n \otimes x \leq x^n$. Hence $x^n = x^{n+1}$. Therefore, $\mathcal{E}$ is an EQ_n-algebra. □

THEOREM 3.23. *Let $\mathcal{E}$ be a residuated EQ-algebra. Then the following conditions are equivalent:*

(i) *$\mathcal{E}$ is an EQ_n-algebra;*

(ii) *$\{1\}$ is an n-fold pseudo implicative filter of $\mathcal{E}$;*

(iii) *Each filter of $\mathcal{E}$ is an n-fold pseudo implicative filter of $\mathcal{E}$;*

(iv) *$x^{2n} = x^n$, for all $x \in E$*

PROOF: $(i) \Longrightarrow (ii)$: By Proposition 3.22, the proof is clear.
$(ii) \Longrightarrow (iii)$: By Proposition 3.18, the proof is clear.
$(iii) \Longrightarrow (i)$: Since $F = \{1\}$ is a filter of $\mathcal{E}$, by (iii) and Proposition 3.22, we have (i).
$(i) \Longrightarrow (iv)$: By (i) or equivalently (ii) and the proof of Proposition 3.22, $x^n = x^{n+1}$. Thus

$$x^{n+2} = x^{n+1} \otimes x = x^n \otimes x = x^{n+1} = x^n.$$

By repeating this method, we have $x^{2n} = x^n$.
$(iv) \Longrightarrow (i)$: Let $x^{2n} = x^n$. Then $x^n \to x^{2n} = 1$ and so $x^n \to x^{2n} \in \{1\}$. Hence by Theorem 3.11(ii), $\{1\}$ is an n-fold pseudo implicative filter of $\mathcal{E}$ and by Proposition 3.22, we have (i). □

THEOREM 3.24. *Let $\mathcal{E}$ be a residuated EQ-algebra and F be a filter of $\mathcal{E}$. Then F is an n-fold pseudo implicative filter of $\mathcal{E}$ if and only if $\mathcal{E}/F$ is an EQ_n-algebra.*

PROOF: By Theorem 3.11(ii), F is an n-fold pseudo implicative filter of $\mathcal{E}$ if and only if $x^n \to x^{2n} \in F$, for all $x \in E$ if and only if $[x]^n \to [x]^{2n} = [x^n \to x^{2n}] = [1]$ if and only if by Theorem 3.23, $\{[1]\}$ is an n-fold pseudo implicative filter of $\mathcal{E}/F$ if and only if $\mathcal{E}/F$ is an EQ_n-algebra. □

4. n-fold implicative prefilters in EQ-algebras

In this section, we introduce the concept of an n-fold implicative prefilters in EQ-algebras and investigate some properties of them. We define an n-fold implicative EQ-algebra and show that in good EQ-algebra $\mathcal{E}$ with least element 0 a prefilter F is an n-fold implicative prefilter of $\mathcal{E}$ if and only if $\mathcal{E}/F$ is an n-fold implicative EQ-algebra.

DEFINITION 4.1. Let $\mathcal{E}$ be an EQ-algebra. A nonempty subset $F \subseteq E$ is called an *n-fold implicative prefilter* of $\mathcal{E}$, if for all $x, y, z \in E$,

(i) $1 \in F$;

(ii) $z \to ((x^n \to y) \to x) \in F$ and $z \in F$ imply $x \in F$.

Obviously each implicative prefilter is an n-fold implicative prefilter (for $n = 1$). But the converse is not true.

Example 4.2.

(i) Let $\mathcal{E}$ be the EQ-algebra as in Example 3.2. Suppose $F = \{1, c\}$. Then F is a 2-fold implicative prefilter of $\mathcal{E}$. Since $1 \to ((a \to 0) \to a) = 1 \in F$ and $1 \in F$ but $a \notin F$, we get F is not an implicative prefilter of $\mathcal{E}$.

(ii) According to Example 3.4, if $F = \{1, a, b\}$, then F is an n-fold implicative filter of $\mathcal{E}$, for all $n \in \mathbb{N}$ and $F = \{1, a\}$ is not an n-fold implicative filter of $\mathcal{E}$, because $1 \to ((b \to 0) \to b) = 1 \in F$ and $1 \in F$ but $b \notin F$.

PROPOSITION 4.3. Let $\mathcal{E}$ be an EQ-algebra and F be an n-fold implicative prefilter of $\mathcal{E}$. Then F is an n-fold prefilter of $\mathcal{E}$, for all $n \in \mathbb{N}$.

PROOF: Let $x^n \in F$ and $x^n \to y \in F$. Since $y \leq 1 \to y$, we have $x^n \to y \leq x^n \to (1 \to y)$ and so $x^n \to (1 \to y) \in F$. Thus $x^n \to ((y^n \to 1) \to y) \in F$. Since F is an n-fold implicative prefilter of $\mathcal{E}$ and $x^n \in F$, we have $y \in F$. Hence F is an n-fold prefilter of $\mathcal{E}$, for all $n \in \mathbb{N}$. $\square$

In the next example, we show that the converse of Proposition 4.3 is not true.

Example 4.4. Let $\mathcal{E}$ be the EQ-algebra as in Example 3.4. Suppose $F = \{1, b\}$. Then F is a 2-fold filter of $\mathcal{E}$. Since $1 \to ((a^2 \to 0) \to a) = 1 \in F$ and $1 \in F$ but $a \notin F$, we get F is not a 2-fold implicative filter of $\mathcal{E}$.

THEOREM 4.5. *Let $\mathcal{E}$ be a good EQ-algebra with least element* 0 *and F be a prefilter of $\mathcal{E}$. Then, for all $x, y \in E$ and $n \in \mathbb{N}$ the following statements are equivalent:*

(i) *F is an n-fold implicative prefilter;*

(ii) *$(x^n \to 0) \to x \in F$ implies $x \in F$;*

(iii) *$(x^n \to y) \to x \in F$ implies $x \in F$.*

Proof: $(i) \Longrightarrow (iii)$: Let F be an n-fold implicative prefilter of $\mathcal{E}$ and $(x^n \to y) \to x \in F$. Then by goodness we have $1 \to ((x^n \to y) \to x) = (x^n \to y) \to x \in F$. Since $1 \in F$, by (i), we get $x \in F$.
$(iii) \Longrightarrow (ii)$: The proof is clear.
$(ii) \Longrightarrow (i)$: Let $x \to ((y^n \to z) \to y) \in F$ and $x \in F$. Since F is a prefilter of $\mathcal{E}$, we get $(y^n \to z) \to y \in F$. Moreover, from $0 \leq z$, we obtain $y^n \to 0 \leq y^n \to z$ and $(y^n \to z) \to y \leq (y^n \to 0) \to y$. Hence $(y^n \to 0) \to y \in F$. Thus by (ii), $y \in F$. Therefore, F is an n-fold implicative prefilter of $\mathcal{E}$. □

Proposition 4.6. Let $\mathcal{E}$ be a good EQ-algebra with least element 0. If F is an n-fold implicative prefilter of $\mathcal{E}$, then F is an $n+1$-fold implicative prefilter of $\mathcal{E}$.

Proof: Let F be an n-fold implicative prefilter of $\mathcal{E}$ such that $(x^{n+1} \to 0) \to x \in F$. Then by Proposition 2.2(iv), from $x^{n+1} \leq x^n$ we have $x^n \to 0 \leq x^{n+1} \to 0$ and so $(x^{n+1} \to 0) \to x \leq (x^n \to 0) \to x$. Since F is prefilter, we have $(x^n \to 0) \to x \in F$ and by assumption $x \in F$. Therefore, F is an $n+1$-fold implicative prefilter of $\mathcal{E}$. □

The next example shows that the converse of Proposition 4.6, is not true.

Example 4.7. Let $\mathcal{E}$ be the EQ-algebra as in Example 3.2. Suppose $F = \{1, c\}$. Then F is a 2-fold implicative prefilter of $\mathcal{E}$. Since $(a \to 0) \to a = 1 \in F$ but $a \notin F$, we get F is not a 1-implicative prefilter of $\mathcal{E}$.

Theorem 4.8. *Let $\mathcal{E}$ be a residuated EQ-algebra. Then each n-fold implicative filter of $\mathcal{E}$ is an n-fold pseudo implicative filter of $\mathcal{E}$.*

Proof: Let F be an n-fold implicative filter of $\mathcal{E}$ and $x^{n+1} \to y \in F$. Then by Propositions 2.2(iii) and 2.7(iv), we have

$$\begin{aligned}
&(x^{n+1} \to y)^n \to (x^n \to y) \\
&= (x^{n+1} \to y)^{n-1} \to ((x^{n+1} \to y) \to (x^n \to y)) \\
&= (x^{n+1} \to y)^{n-1} \to ((x^{n+1} \to y) \to (x^{n-1} \to (x \to y)) \\
&= (x^{n+1} \to y)^{n-1} \to ((x^{n-1} \to ((x^{n+1} \to y) \to (x \to y))) \\
&= (x^{n+1} \to y)^{n-1} \to (x^{n-1} \to ((x \to (x^n \to y)) \to (x \to y))) \\
&\geq (x^{n+1} \to y)^{n-1} \to (x^{n-1} \to ((x^n \to y) \to y))
\end{aligned}$$

$$= (x^{n+1} \to y)^{n-1} \to ((x^n \to y) \to (x^{n-1} \to y))$$
$$= (x^n \to y) \to ((x^{n+1} \to y)^{n-1} \to (x^{n-1} \to y)).$$

Since $x^{n+1} \to y \leq x^{n+1} \to y = x \to (x^n \to y)$, we have $x \otimes (x^{n+1} \to y) \leq x^n \to y$. Then

$$(x^n \to y) \otimes (x^{n+1} \to y)^{n-1} \otimes x^{n-1} \leq (x^n \to y)^2 \otimes (x^{n+1} \to y)^{n-2} \otimes x^{n-2},$$

Hence
$((x^n \to y)^2 \otimes (x^{n+1} \to y)^{n-2} \otimes x^{n-2}) \to y \leq ((x^n \to y) \otimes (x^{n+1} \to y)^{n-1} \otimes x^{n-1}) \to y$ and so
$(x^n \to y)^2 \to ((x^{n+1} \to y)^{n-2} \to (x^{n-2} \to y)) \leq (x^n \to y) \to ((x^{n+1} \to y)^{n-1} \to (x^{n-1} \to y))$, By (4.1), we have

$$(x^{n+1} \to y)^n \to (x^n \to y) \geq (x^n \to y) \to ((x^{n+1} \to y)^{n-1} \to (x^{n-1} \to y)).$$

Then

$$(x^{n+1} \to y)^n \to (x^n \to y)$$
$$\geq (x^n \to y)^2 \to ((x^{n+1} \to y)^{n-2} \to (x^{n-2} \to y)).$$

Hence, by repeating this method n-times we get:

$$\begin{aligned}(x^{n+1} \to y)^n \to (x^n \to y) &\geq (x^n \to y)^2 \to ((x^{n+1} \to y)^{n-2} \to (x^{n-2} \to y))\\ &\vdots\\ &\geq (x^n \to y)^n \to ((x^{n+1} \to y)^0 \to (x^0 \to y))\\ &= (x^n \to y)^n \to (1 \to (1 \to y))\\ &= (x^n \to y)^n \to y.\end{aligned}$$

Thus
$$(x^{n+1} \to y)^n \to (((x^n \to y)^n \to y) \to (x^n \to y)) = 1.$$

Since F is an n-fold filter of $\mathcal{E}$ and $x^{n+1} \to y \in F$, we get $((x^n \to y)^n \to y) \to (x^n \to y) \in F$. Hence, by Theorem 4.5(*iii*), $x^n \to y \in F$. Therefore, by Theorem 3.11, F is an n-fold pseudo implicative filter of $\mathcal{E}$. □

The following example shows that the converse of Theorem 4.8 is not true.

Example 4.9. Let $\mathcal{E}$ be the EQ-algebra as in Example 3.4. Suppose $F = \{1, a\}$. Then F is a 2-fold pseudo implicative prefilter of $\mathcal{E}$. Since $(b^2 \to 0) \to b = 1 \in F$ but $b \notin F$, we have F is not a 2-fold implicative prefilter of $\mathcal{E}$.

Definition 4.10. Let $\mathcal{E}$ be an EQ-algebra. Then $\mathcal{E}$ is called an *n-fold implicative EQ-algebra*, if for all $x, y \in E$, $(x^n \to y) \to x = x$.

Example 4.11.

(i) Let $\mathcal{E}$ be the EQ-algebra as in Example 3.2. Then $\mathcal{E}$ is an n-fold implicative EQ-algebra, for all $n \geq 2$.

(ii) Let $\mathcal{E}$ be the EQ-algebra as in Example 3.4. Since $(a^n \to 0) \to a = 1 \neq a$, we have $\mathcal{E}$ is not an n-fold implicative algebra of $\mathcal{E}$, for all $n \in \mathbb{N}$.

Proposition 4.12. Every n-fold implicative EQ-algebra is an $n+1$-fold implicative EQ-algebra.

Proof: Let $\mathcal{E}$ be an n-fold implicative EQ-algebra. Then $(x^n \to y) \to x = x$, for all $x, y \in E$. Since $x^{n+1} = x^n \otimes x \leq x^n$, by Proposition 2.2($iv$), we have $x^n \to y \leq x^{n+1} \to y$ and so $(x^{n+1} \to y) \to x \leq (x^n \to y) \to x = x$. By Proposition 2.2($ii$), $x \leq (x^{n+1} \to y) \to x$. Hence $(x^{n+1} \to y) \to x = x$ and so $\mathcal{E}$ is an $n+1$-fold implicative EQ-algebra. □

The next example shows that the converse of Proposition 4.12, is not true.

Example 4.13. Let $\mathcal{E}$ be the EQ-algebra as in Example 3.2. Then $\mathcal{E}$ is a 2-fold implicative EQ-algebra. Since $(a \to 0) \to a = 1 \neq a$, we get $\mathcal{E}$ is not a 1-fold implicative EQ-algebra.

Lemma 4.14. *In a good n-fold implicative EQ-algebra concepts of n-fold implicative prefilter and n-fold prefilter are coincide.*

Proof: Let F be an n-fold implicative prefilter of $\mathcal{E}$. Then by Proposition 4.3, F is an n-fold prefilter of $\mathcal{E}$.

Conversely, let F be an n-fold prefilter of $\mathcal{E}$ and $(x^n \to y) \to x \in F$. Then by Definition 4.10, $x \in F$. Hence, F is an n-fold implicative prefilter of $\mathcal{E}$. □

PROPOSITION 4.15. Let $\mathcal{E}$ be a good EQ-algebra with least element 0. Then the following statements are equivalent:

(i) $\mathcal{E}$ is an n-fold implicative EQ-algebra.

(ii) Every n-fold prefilter of $\mathcal{E}$ is an n-fold implicative prefilter of $\mathcal{E}$.

(iii) $\{1\}$ is an n-fold implicative prefilter of $\mathcal{E}$.

PROOF: $(i) \Longrightarrow (ii)$: By Lemma 4.14, the proof is clear.
$(ii) \Longrightarrow (iii)$: By Proposition 3.9, the proof is clear.
$(iii) \Longrightarrow (i)$: Let $\{1\}$ be an n-fold implicative prefilter of $\mathcal{E}$, $x \in E$ and $t = ((x^n \to 0) \to x) \to x$. Then by Propositions 2.2(iii) and 2.7(iv), we have:

$$\begin{aligned}(t^n \to 0) \to t &= (t^n \to 0) \to (((x^n \to 0) \to x) \to x) \\ &= ((x^n \to 0) \to x) \to ((t^n \to 0) \to x) \\ &\geq (t^n \to 0) \to (x^n \to 0) \\ &\geq x^n \to t^n.\end{aligned}$$

By Proposition 2.2(ii), $x \leq (x^n \to 0) \to x = t$ and so $x^n \leq t^n$. Hence $(t^n \to 0) \to t = 1 \in \{1\}$. Then by (iii), $t = ((x^n \to 0) \to x) \to x \in \{1\}$ and so $(x^n \to 0) \to x \leq x$. By Proposition 2.2(ii), $x \leq (x^n \to 0) \to x$. Thus $(x^n \to 0) \to x = x$, for all $x \in E$. Therefore, $\mathcal{E}$ is an n-fold implicative EQ-algebra. $\square$

THEOREM 4.16. *Let $\mathcal{E}$ be a good EQ-algebra with least element 0 and F be a prefilter of $\mathcal{E}$. Then F is an n-fold implicative prefilter of $\mathcal{E}$ if and only if $\mathcal{E}/F$ is an n-fold implicative EQ-algebra.*

PROOF: Let F be an n-fold implicative prefilter of $\mathcal{E}$ and $x \in E$ such that $([x]^n \to [0]) \to [x] = [1]$. Then $(x^n \to 0) \to x \in F$. Thus by Theorem 4.5, $x \in F$, and so $[x] = [1]$. Hence, $\{[1]\}$ is an n-fold implicative prefilter of $\mathcal{E}/F$. Therefore, by Proposition 4.15, $\mathcal{E}/F$ is an n-fold implicative EQ-algebra.

Conversely, let $\mathcal{E}/F$ be an n-fold implicative EQ-algebra and $x \in E$ such that $(x^n \to 0) \to x \in F$. Then $[x] = ([x]^n \to [0]) \to [x] = [(x^n \to 0) \to x] = [1]$ and so $[x] = [1]$, that is $x \in F$. Therefore, F is an n-fold implicative prefilter of $\mathcal{E}$. $\square$

Corollary 4.17. Let F and G be prefilters of good EQ-algebra $\mathcal{E}$ with least element 0 such that $F \subseteq G$ and F be an n-fold implicative prefilter of $\mathcal{E}$. Then G is an n-fold implicative prefilter of $\mathcal{E}$.

Proof: Let $x \in E$ such that $(x^n \to 0) \to x \in G$. Since F is an n-fold implicative prefilter of $\mathcal{E}$, by Theorem 4.16 we have $\mathcal{E}/F$ is an n-fold implicative EQ-algebra. Then $[(x^n \to 0) \to x] = ([x]^n \to [0]) \to [x] = [x]$ and so $((x^n \to 0) \to x) \to x \in F \subseteq G$. Hence, by assumption, $x \in G$. Therefore, G is an n-fold implicative prefilter of $\mathcal{E}$. $\square$

5. n-fold obstinate prefilters in EQ-algebras

In this section, we introduce the concept of n-fold obstinate prefilters in EQ-algebras and investigate some properties. We also show that, a filter $\{1\}$ is an n-fold obstinate filter of residuated EQ-algebra $\mathcal{E}$ if and only if every filter of $\mathcal{E}$ is an n-fold obstinate filter of $\mathcal{E}$ and in each residuated EQ-algebra $\mathcal{E}$, a filter F is an n-fold obstinate filter of $\mathcal{E}$ if and only if every filter of quotient algebra $\mathcal{E}/F$ is an n-fold obstinate filter of $\mathcal{E}$.

Definition 5.1. Let F be a prefilter of EQ-algebra $\mathcal{E}$. Then F is called an *n-fold obstinate prefilter* of $\mathcal{E}$, if $x, y \notin F$ implies $x^n \to y \in F$ and $y^n \to x \in F$.

Example 5.2.

- (i) Let $\mathcal{E}$ be the EQ-algebra as in Example 3.2. Suppose $F = \{1, c\}$. Then F is an n-fold obstinate filter of $\mathcal{E}$, for all $n \geq 2$.
- (ii) Let $\mathcal{E}$ be the EQ-algebra as in Example 3.4. Suppose $F = \{1, b\}$. Then F is a filter and n-fold filter of $\mathcal{E}$, for all $n \in \mathbb{N}$. Since $a, 0 \notin F$ and $a^n \to 0 = a \to 0 = 0 \notin F$, we get F is not an n-fold obstinate filter of $\mathcal{E}$, for all $n \geq 2$.

Proposition 5.3. Let $\mathcal{E}$ be an EQ-algebra. Then every n-fold obstinate prefilter is an $n+1$-fold obstinate prefilter of $\mathcal{E}$.

Proof: Let F be an n-fold obstinate prefilter of $\mathcal{E}$ and $x, y \notin F$. Then $x^n \to y, y^n \to x \in F$. Since $x^{n+1} \leq x^n$ by Proposition 2.2(ii), $x^n \to y \leq x^{n+1} \to y$. Thus $x^{n+1} \to y \in F$ and similarly $y^{n+1} \to x \in F$. Therefore, F is an $n+1$-fold prefilter of $\mathcal{E}$. $\square$

The next example shows that the converse of Proposition 5.3, is not true.

Example 5.4. Let $\mathcal{E}$ be the EQ-algebra as in Example 3.2. Suppose $F = \{1, c\}$. Then F is a 2-fold obstinate filter of $\mathcal{E}$. Since $0, a \notin F$, we have $a \to 0 = a \notin F$. Thus F is not a 1-fold obstinate filter of $\mathcal{E}$.

THEOREM 5.5. *Let $\mathcal{E}$ be an EQ-algebra with least element 0 and F be a prefilter of $\mathcal{E}$. Then F is an n-fold obstinate prefilter if and only if $x \in F$ or $(\neg(x^n))^m \in F$, for all $x \in E$ and some $m \in \mathbb{N}$.*

PROOF: Let F be an n-fold obstinate prefilter of $\mathcal{E}$ such that $x \notin F$. Since F is a filter of $\mathcal{E}$, we have $0 \notin F$. Then $\neg(x^n) = x^n \to 0 \in F$ and $0^n \to x \in F$. Hence, for $m = 1$ we have, $(\neg(x^n))^m \in F$.

Conversely, let $x, y \notin F$. Then $(\neg(x^n))^m \in F$ and $(\neg(y^n))^k \in F$, for some $m, k \in \mathbb{N}$. Thus by Proposition 2.2(i), $(\neg(x^n))^m \leq \neg(x^n)$ and $(\neg(y^n))^k \leq \neg(y^n)$ and so $\neg(x^n), \neg(y^n) \in F$. By Proposition 2.2($iv$), $x^n \to 0 \leq x^n \to y$ and $y^n \to 0 \leq y^n \to x$. Hence, $x^n \to y, y^n \to x \in F$. Therefore, F is an n-fold obstinate prefilter of $\mathcal{E}$. □

THEOREM 5.6. *Let $\mathcal{E}$ be a residuated EQ-algebra with least element 0 and F be a filter of $\mathcal{E}$. Then the following statements are equivalent:*

(i) *F is a maximal filter of $\mathcal{E}$;*

(ii) *For any $x \notin F$, there exists $n \in \mathbb{N}$ such that $\neg(x^n) \in F$.*

PROOF: (i) $\Rightarrow$ (ii): Let F be a maximal filter of $\mathcal{E}$ and $x \notin F$. Then $< F \cup \{x\} >= E$ and so $0 \in < F \cup \{x\} >$. Thus $x \to 0 \in F$. Hence, $\neg x \in F$.
(ii) $\Rightarrow$ (i): Let G be a proper filter of $\mathcal{E}$ such that $F \subsetneq G$. Then there exists $x \in G$ such that $x \notin F$. Thus, there exists $n \in \mathbb{N}$ such that $\neg(x^n) \in F$ or $x \to (x \to (\ldots(x \to 0)\ldots)) \in F \subsetneq G$. Since G is a filter of $\mathcal{E}$, we get $0 \in G$. Hence $G = E$, which is a contradiction. Therefore, F is a maximal filter of $\mathcal{E}$. □

COROLLARY 5.7. Let $\mathcal{E}$ be a residuated EQ-algebra with least element 0. Then every proper n-fold obstinate filter of $\mathcal{E}$ is a maximal filter of $\mathcal{E}$, for all $n \in \mathbb{N}$.

PROOF: Let F be an n-fold obstinate filter of $\mathcal{E}$ and G be a filter of $\mathcal{E}$ such that $F \subseteq G \subseteq E$. If $F \neq G$, then there exists $x \in G$ such that $x \notin F$. Since

$0, x \notin F$, by assumption $x^n \to 0 \in F$ and so $\neg(x^n) \in G$. Hence $0 \in G$ and so $G = E$. Therefore, F is a maximal filter of $\mathcal{E}$. $\square$

Proposition 5.8. Let $\mathcal{E}$ be an EQ-algebra and F be an n-fold obstinate prefilter of $\mathcal{E}$. Then F is an n-fold implicative prefilter of $\mathcal{E}$.

Proof: Let F be an n-fold obstinate prefilter of $\mathcal{E}$ but not an n-fold implicative prefilter of $\mathcal{E}$. Then there exist $x, y \in E$ such that $(x^n \to y) \to x \in F$ but $x \notin F$. Let $y \in F$. Since $y \leq x^n \to y$, we have $x^n \to y \in F$ and so $x \in F$, which is a contradiction. If $y \notin F$, then by assumption $x^n \to y \in F$ and so $x \in F$, which is a contradiction. Therefore, F is an n-fold implicative prefilter of $\mathcal{E}$. $\square$

The following example shows that the converse of Proposition 5.8, is not true.

Example 5.9. Let $\mathcal{E}$ be the EQ-algebra as in Example 3.4. Suppose $F = \{1, a, b\}$. Then F is an n-fold implicative filter of $\mathcal{E}$, for all $n \geq 2$. Since $0, a \notin F$ and $a^n \to 0 = 0 \notin F$, we get F is not an n-fold obstinate filter of $\mathcal{E}$.

Theorem 5.10. *Let $\mathcal{E}$ be a residuated EQ-algebra and F be an n-fold obstinate filter of $\mathcal{E}$. Then F is an n-fold pseudo implicative filter of $\mathcal{E}$.*

Proof: By Theorem 4.8 and Proposition 5.8, the proof is clear. $\square$

The following example shows that the converse of Theorem 5.10, is not true.

Example 5.11. Let $\mathcal{E}$ be the EQ-algebra as in Example 3.4. Suppose $F = \{1, a\}$. Then F is an n-fold pseudo implicative filter of $\mathcal{E}$. Since $0, b \notin F$ and $b^n \to 0 = 0 \notin F$, we have F is not an n-fold obstinate filter of $\mathcal{E}$, for all $n \geq 2$.

Proposition 5.12. Filter $\{1\}$ is an n-fold obstinate filter of residuated EQ-algebra $\mathcal{E}$ if and only if every filter of $\mathcal{E}$ is an n-fold obstinate filter of $\mathcal{E}$.

Proof: Let F be a filter of $\mathcal{E}$ and $x, y \notin F$. Then $x, y \notin \{1\}$ and so $x^n \to y \in \{1\} \subseteq F$ and $y^n \to x \in \{1\} \subseteq F$. Hence, F is an n-fold obstinate filter of $\mathcal{E}$. The proof of the converse is clear. $\square$

THEOREM 5.13. *Let $\mathcal{E}$ be a residuated EQ-algebra and F be a filter of $\mathcal{E}$. Then F is an n-fold obstinate filter of $\mathcal{E}$ if and only if every filter of quotient algebra $\mathcal{E}/F$ is an n-fold obstinate filter of $\mathcal{E}/F$.*

PROOF: Let F be an n-fold obstinate filter of $\mathcal{E}$ and $x \in E$ such that $[x] \neq [1]$. Then $x \notin F$ and so there exists $m \in \mathbb{N}$ such that $(\neg(x^n))^m \in F$ and so $[(\neg(x^n))^m] \in \{[1]\}$. Hence by Theorem 5.5, $\{[1]\}$ is an n-fold obstinate filter of $\mathcal{E}/F$. Therefore, by Proposition 5.12, each filter of the quotient algebra $\mathcal{E}/F$ is an n-fold obstinate filter.

Conversely, let every filter of the quotient algebra $\mathcal{E}/F$ be an n-fold obstinate filter of $\mathcal{E}/F$ and $x \in E$ such that $x \notin F$. Then $[x] \neq [1]$. Since $\{[1]\}$ is a filter of $\mathcal{E}/F$, by assumption, $\{[1]\}$ is an n-fold obstinate filter of $\mathcal{E}$, and so there exists $m \in \mathbb{N}$ such that $[(\neg(x^n))^m] \in \{[1]\}$. Thus $(\neg(x^n))^m \in F$. Hence, by Theorem 5.5, F is an n-fold obstinate filter of $\mathcal{E}$. □

6. n-fold fantastic prefilters in EQ-algebras

In this section, we introduce the concept of n-fold fantastic prefilters in EQ-algebras and investigate some properties about them. Then we prove that in any good EQ-algebra, if F is an 1-fold fantastic filter of $\mathcal{E}$, then $\mathcal{E}/F$ is an IEQ-algebra, and we show that in any residuated EQ-algebra with least element 0, F is an n-fold implicative filter of $\mathcal{E}$ if and only if F is an n-fold pseudo implicative filter and n-fold fantastic filter of $\mathcal{E}$. So we conclude that in any residuated EQ-algebra, $\mathcal{E}$ is an n-fold implicative EQ-algebra if and only if $\mathcal{E}$ is both EQ_n-algebra and n-fold fantastic EQ-algebra.

DEFINITION 6.1. Let $\mathcal{E}$ be an EQ-algebra. A nonempty subset $F \subseteq E$ is called an *n-fold fantastic prefilter* of $\mathcal{E}$, if for all $x, y \in E$,

(i) $1 \in F$;

(ii) $z \to (y \to x) \in F$ and $z \in F$, imply $((x^n \to y) \to y) \to x \in F$.

An n-fold fantastic prefilter F is said to be an *n-fold fantastic filter* if F satisfies in $(F3)$.

Example 6.2. (i) Let $\mathcal{E}$ be the EQ-algebra as in Example 3.2. Suppose $F = \{1, c\}$. Then F is an n-fold fantastic filter of $\mathcal{E}$, for all $n \geq 2$.
(ii) Let $\mathcal{E}$ be the EQ-algebra as in Example 3.4. Suppose $F = \{1, b\}$. Since

$1 \to (0 \to a) = 1 \in F$ and $1 \in F$ but $((a^n \to 0) \to 0) \to a = a \notin F$, we get F is not an n-fold fantastic prefilter of $\mathcal{E}$, for all $n \in \mathbb{N}$.

THEOREM 6.3. *Let F be a prefilter of good EQ-algebra $\mathcal{E}$. Then F is an n-fold fantastic prefilter of $\mathcal{E}$ if and only if $y \to x \in F$ implies $((x^n \to y) \to y) \to x \in F$, for all $x, y \in E$.*

PROOF: Let F be an n-fold fantastic prefilter of $\mathcal{E}$ and $y \to x \in F$. Then $1 \to (y \to x) = y \to x \in F$ and $1 \in F$. Hence $((x^n \to y) \to y) \to x \in F$, for all $x, y \in E$.

Conversely, let $z \to (y \to x) \in F$ and $z \in F$. Since F is a prefilter of $\mathcal{E}$, we get $y \to x \in F$ and so $((x^n \to y) \to y) \to x \in F$. Then F is an n-fold fantastic prefilter of $\mathcal{E}$. $\square$

PROPOSITION 6.4. Each n-fold fantastic prefilter of good EQ-algebra $\mathcal{E}$ is an n-fold prefilter of $\mathcal{E}$.

PROOF: Let $x^n, x^n \to y \in F$. Then $x^n \to y = x^n \to (1 \to y) \in F$. Since F is an n-fold fantastic prefilter of $\mathcal{E}$, we get $((y^n \to 1) \to 1) \to y = y \in F$. $\square$

The next example shows that the converse of Proposition 6.4 is not true and condition of goodness is necessary.

Example 6.5. Let $\mathcal{E}$ be the EQ-algebra as in Example 3.4.

(i) Suppose $F = \{1, b\}$. Then F is a 2-fold filter of $\mathcal{E}$. Since $1 \to (0 \to a) = 1 \in F$ and $1 \in F$ but $((a^n \to 0) \to 0) \to a = a \notin F$, we get F is not a 2-fold fantastic filter of $\mathcal{E}$.

(ii) Since $b \sim 1 \neq b$, we get $\mathcal{E}$ is not a good EQ-algebra. Let $F = \{1, a\}$. Then F is an n-fold fantastic filter of $\mathcal{E}$, for all $n \in \mathbb{N}$. Since $a^n = a \in F$ and $a^n \to b = 1 \in F$ but $b \notin F$, we have F is not an n-fold filter of $\mathcal{E}$, for all $n \in \mathbb{N}$.

PROPOSITION 6.6. Let F and G be two prefilters of good EQ-algebra $\mathcal{E}$ such that $F \subseteq G$. If F is an n-fold fantastic prefilter of $\mathcal{E}$, then so is G.

PROOF: Let $y \to x \in G$ and $k := (y \to x) \to x$. Then

$$y \to k = y \to ((y \to x) \to x) = (y \to x) \to (y \to x) = 1 \in F.$$

Since F is an n-fold fantastic prefilter of $\mathcal{E}$, we have

$$\begin{aligned}(y \to x) \to (((k^n \to y) \to y) \to x) &= ((k^n \to y) \to y) \to ((y \to x) \to x) \\ &= ((k^n \to y) \to y) \to k \in F \subseteq G.\end{aligned}$$

Since G is a filter of $\mathcal{E}$ and $y \to x \in G$, we get $((k^n \to y) \to y) \to x \in G$. Moreover, from $x \leq k = (y \to x) \to x$, we get $k^n \to y \leq x^n \to y$ and so

$$((k^n \to y) \to y) \to x \leq ((x^n \to y) \to y) \to x.$$

Hence $((x^n \to y) \to y) \to x \in G$. Therefore, G is an n-fold fantastic prefilter of $\mathcal{E}$. □

Definition 6.7. Let $\mathcal{E}$ be an EQ-algebra. Then $\mathcal{E}$ is called an *n-fold fantastic EQ-algebra*, if for all $x, y \in E$, $((x^n \to y) \to y) \to x = y \to x$.

Example 6.8. (i) Let $\mathcal{E}$ be the EQ-algebra as in Example 3.2. Then $\mathcal{E}$ is an n-fold fantastic EQ-algebra, for all $n \geq 2$.
(ii) Let $\mathcal{E}$ be the EQ-algebra as in Example 3.4. Since $((a^n \to 0) \to 0) \to a = a \neq 0 \to a = 1$, we have $\mathcal{E}$ is not an n-fold fantastic EQ-algebra.

Proposition 6.9. Let $\mathcal{E}$ be an n-fold fantastic EQ-algebra and F be a prefilter of $\mathcal{E}$. Then F is an n-fold fantastic prefilter of $\mathcal{E}$

Proof: The proof is clear. □

Theorem 6.10. *Let $\mathcal{E}$ be a residuated EQ-algebra $\mathcal{E}$. Then, for all $x, y, z \in E$ the following conditions are equivalent:*

(i) *$\mathcal{E}$ is an n-fold fantastic EQ-algebra;*

(ii) $(x^n \to y) \to y \leq (y \to x) \to x$;

(iii) *If $x^n \to z \leq y \to z$ and $z \leq x$, then $y \leq x$;*

(iv) *If $x^n \to z \leq y \to z$ and $z \leq x, y$, then $y \leq x$;*

(v) *If $y \leq x$, then $(x^n \to y) \to y \leq x$.*

Proof: $(i) \Longrightarrow (ii)$: Let $\mathcal{E}$ be an n-fold fantastic EQ-algebra. Then

$$\begin{aligned}((x^n \to y) \to y) \to ((y \to x) \to x) &= (y \to x) \to (((x^n \to y) \to y) \to x)\\ &= (y \to x) \to (y \to x)\\ &= 1.\end{aligned}$$

Hence by Proposition 2.6, $(x^n \to y) \to y \leq (y \to x) \to x$.
$(ii) \Longrightarrow (i)$: Let $(x^n \to y) \to y \leq (y \to x) \to x$, for all $x, y \in E$. Then

$$\begin{aligned}(y \to x) \to (((x^n \to y) \to y) \to x) &= ((x^n \to y) \to y) \to ((y \to x) \to x)\\ &= 1.\end{aligned}$$

Thus $y \to x \leq ((x^n \to y) \to y) \to x$. Also,

$$\begin{aligned}(((x^n \to y) \to y) \to x) \to (y \to x) &\geq y \to ((x^n \to y) \to y)\\ &= (x^n \to y) \to (y \to y)\\ &= (x^n \to y) \to 1\\ &= 1.\end{aligned}$$

Then $(((x^n \to y) \to y) \to x) \to (y \to x) = 1$ and so $(((x^n \to y) \to y) \to x) \leq y \to x$. Hence $((x^n \to y) \to y) \to x = y \to x$. Therefore, $\mathcal{E}$ is an n-fold fantastic EQ-algebra.
$(ii) \Longrightarrow (iii)$: Let $x^n \to z \leq y \to z$ and $z \leq x$. Then by (ii), we have

$$\begin{aligned}1 = (x^n \to z) \to (y \to z) = y \to ((x^n \to z) \to z) &\leq y \to ((z \to x) \to x)\\ &= y \to (1 \to x)\\ &= y \to x.\end{aligned}$$

Thus $y \to x = 1$ and so $y \leq x$.
$(iii) \Longrightarrow (iv)$: The proof is clear.
$(iv) \Longrightarrow (v)$: Let $y \leq x$. Since $y \leq (x^n \to y) \to y$ and

$$\begin{aligned}(x^n \to y) \to (((x^n \to y) \to y) \to y) &= ((x^n \to y) \to y)) \to ((x^n \to y) \to y)\\ &= 1,\end{aligned}$$

we have $x^n \to y \leq (((x^n \to y)) \to y) \to y$ and so by (iv), $(x^n \to y) \to y \leq x$.
$(v) \Longrightarrow (ii)$: Since $x \leq (y \to x) \to x$, by induction we have $((y \to x) \to x)^n \to y \leq x^n \to y$ and $(x^n \to y) \to y \leq (((y \to x) \to x)^n \to y) \to y$. By Proposition 2.7$(ii)$, we have $y \leq (y \to x) \to x$ and by (v) we get

$$(x^n \to y) \to y \leq (((y \to x) \to x)^n \to y) \to y \leq (y \to x) \to x. \qquad \square$$

Proposition 6.11. Let $\mathcal{E}$ be a residuated EQ-algebra. Then $\mathcal{E}$ is an n-fold fantastic EQ-algebra if and only if $\{1\}$ is an n-fold fantastic filter of $\mathcal{E}$.

Proof: Let $\mathcal{E}$ be an n-fold fantastic EQ-algebra and $y \to x = 1$. Then $((x^n \to y) \to y) \to x = 1$. Hence $\{1\}$ is an n-fold fantastic filter of $\mathcal{E}$.

Conversely, let $\{1\}$ be an n-fold fantastic filter of $\mathcal{E}$ and $k = (y \to x) \to x$. Then

$$y \to k = y \to ((y \to x) \to x) = (y \to x) \to (y \to x) = 1 \in \{1\},$$

and so $((k^n \to y) \to y) \to k = 1$ that is $(k^n \to y) \to y \leq k$. Since $x \leq k$, we get $k^n \to y \leq x^n \to y$ and $(x^n \to y) \to y \leq (k^n \to y) \to y$. Thus $1 = ((k^n \to y) \to y) \to k \leq ((x^n \to y)) \to y) \to k$. Hence $((x^n \to y) \to y) \to k = 1$. So $((x^n \to y) \to y) \to ((y \to x) \to x) = 1$. Thus $(x^n \to y) \to y \leq (y \to x) \to x$. Therefore, by Theorem 6.10, $\mathcal{E}$ is an n-fold fantastic EQ-algebra. □

Lemma 6.12. *Each filter of residuated EQ-algebra $\mathcal{E}$ is an n-fold fantastic filter of $\mathcal{E}$ if and only if $\{1\}$ is an n-fold fantastic filter of $\mathcal{E}$.*

Proof: Let F be a filter of $\mathcal{E}$ and $\{1\}$ be an n-fold fantastic filter of $\mathcal{E}$. Then by Proposition 6.11, $\mathcal{E}$ is an n-fold fantastic EQ-algebra and so by Proposition 6.9, F is an n-fold fantastic filter of $\mathcal{E}$. The proof of the converse is clear. □

Theorem 6.13. *Let $\mathcal{E}$ be a residuated EQ-algebra and F be a filter of $\mathcal{E}$. Then F is an n-fold fantastic filter of $\mathcal{E}$ if and only if every filter of $\mathcal{E}/F$ is an n-fold fantastic filter of $\mathcal{E}/F$.*

Proof: Let F be an n-fold fantastic filter of $\mathcal{E}$ and $[x] \to [y] = [1]$. Then $x \to y \in F$ and so $((y^n \to x) \to x) \to y \in F$. Hence

$$(([y]^n \to [x]) \to [x]) \to [y] = [((y^n \to x) \to x) \to y] = [1].$$

Thus $\{[1]\}$ is an n-fold fantastic filter of $\mathcal{E}/F$. By Lemma 6.12, every filter of $\mathcal{E}/F$ is an n-fold fantastic filter of $\mathcal{E}/F$.

Conversely, let every filter of $\mathcal{E}/F$ be an n-fold fantastic filter of $\mathcal{E}/F$ and let $y \to x \in F$. Then $[y] \to [x] = [y \to x] = [1]$. Since $\{[1]\}$ is an n-fold fantastic filter of $\mathcal{E}/F$, we have

$$[((x^n \to y) \to y) \to x] = (([x]^n \to [y]) \to [y]) \to [x] = [1].$$

Hence $((x^n \to y) \to y) \to x \in F$ and so F is an n-fold fantastic filter of $\mathcal{E}$. □

Theorem 6.14. *Let $\mathcal{E}$ be a good EQ-algebra with least element 0 and F be a filter of $\mathcal{E}$. If F is an 1-fold fantastic filter of $\mathcal{E}$, then $\mathcal{E}/F$ is an IEQ-algebra.*

Proof: By Theorem 2.17, $\mathcal{E}$ is a good EQ-algebra. Since $0 \to x = 1 \in F$, we have

$$((x \to 0) \to 0) \to x = \neg(\neg x) \to x \in F$$

and so $[\neg(\neg x)] \leq [x]$. By Proposition 2.7(*ii*), $[x] \leq [\neg(\neg x)]$. Hence $[\neg(\neg x)] = [x]$ and so $\mathcal{E}/F$ is an IEQ-algebra. $\square$

By Theorem 4.8, we see that in residuated EQ-algebra such as $\mathcal{E}$, every n-fold implicative filter is an n-fold pseudo implicative filter, but the converse is not true. Now, we show that under certain conditions an n-fold pseudo implicative filter of $\mathcal{E}$ is an n-fold implicative filter of $\mathcal{E}$.

Theorem 6.15. *Let $\mathcal{E}$ be a residuated EQ-algebra and F be a filter of $\mathcal{E}$. If F is an n-fold implicative filter of $\mathcal{E}$, then F is an n-fold fantastic filter of $\mathcal{E}$, for all $n \in \mathbb{N}$.*

Proof: Let F be an n-fold implicative filter of $\mathcal{E}$ and $y \to x \in F$. Since $x \leq ((x^n \to y) \to y) \to x$, we have $x^n \leq (((x^n \to y) \to y) \to x)^n$ and $(((x^n \to y) \to y) \to x)^n \to y \leq x^n \to y$. Also, we have

$$\begin{aligned}
y \to x &\leq ((x^n \to y) \to y) \to ((x^n \to y) \to x) \\
&= (x^n \to y) \to (((x^n \to y) \to y) \to x) \\
&\leq ((((x^n \to y) \to y) \to x)^n \to y) \to (((x^n \to y) \to y) \to x).
\end{aligned}$$

Thus

$$((((x^n \to y) \to y) \to x)^n \to y) \to (((x^n \to y) \to y) \to x) \in F$$

and so by Theorem 4.5(*iii*), $((x^n \to y) \to y) \to x \in F$. Therefore, F is an n-fold fantastic filter of $\mathcal{E}$. $\square$

Theorem 6.16. *Let F be a filter of residuated EQ-algebra $\mathcal{E}$ with least element* 0*. Then F is an n-fold implicative filter of $\mathcal{E}$ if and only if F is an n-fold pseudo implicative filter and n-fold fantastic filter of $\mathcal{E}$.*

Proof: Let F be an n-fold pseudo implicative filter and n-fold fantastic filter of $\mathcal{E}$ and $(x^n \to 0) \to x \in F$. Since $x^n \to x^{2n} \leq (x^{2n} \to 0) \to (x^n \to 0)$, by Theorem 3.11, we have $x^n \to x^{2n} \in F$ and so $(x^{2n} \to 0) \to (x^n \to 0) \in F$. Also, F is an n-fold fantastic filter of $\mathcal{E}$ and $(x^n \to 0) \to x \in F$. Thus

$$((x^n \to (x^n \to 0)) \to (x^n \to 0)) \to x = ((x^{2n} \to 0) \to (x^n \to 0)) \to x \in F.$$

On the other hand, since $(x^{2n} \to 0) \to (x^n \to 0) \in F$ and F is a filter of $\mathcal{E}$, we get $x \in F$. Hence, F is an n-fold implicative filter of $\mathcal{E}$.
By Theorems 6.15 and 4.8, the proof of the converse is clear. □

THEOREM 6.17. *Let $\mathcal{E}$ be a residuated EQ-algebra. Then $\mathcal{E}$ is an n-fold implicative EQ-algebra if and only if $\mathcal{E}$ is both n-fold pseudo implicative EQ-algebra and n-fold fantastic EQ-algebra.*

PROOF: Let $\mathcal{E}$ be an n-fold implicative EQ-algebra. Then by Proposition 4.15, $\{1\}$ is an n-fold implicative filter of $\mathcal{E}$. Thus by Proposition 3.9, Theorems 6.15 and 4.8, $\{1\}$ is an n-fold fantastic filter and pseudo implicative filter of $\mathcal{E}$ and so by Propositions 6.11 and 3.22, $\mathcal{E}$ is both n-fold positive implicative EQ-algebra and n-fold fantastic EQ-algebra.

Conversely, let $\mathcal{E}$ be both n-fold pseudo implicative EQ-algebra and n-fold fantastic EQ-algebra and $u = x^n \to y$. Then

$$u = x^n \to y = x^{2n} \to y = x^n \to (x^n \to y) = x^n \to u.$$

By Theorem 6.10(ii), we have

$$((x^n \to y) \to x) \to x = (u \to x) \to x \geq (x^n \to u) \to u = u \to u = 1.$$

Hence $(x^n \to y) \to x \leq x$. By Proposition 2.2(ii), $x \leq (x^n \to y) \to x$. Thus $(x^n \to y) \to x = x$. Therefore, $\mathcal{E}$ is an n-fold implicative EQ-algebra. □

7. Conclusion

In this paper, the notions of n-fold implicative prefilter, n-fold pseudo implicative prefilter, n-fold fantastic prefilter, n-fold obstinate prefilter are introduced and some related results are investigated. At first, equivalent definition of them are studied and the relation between them are investigated. Then by introducing the notions of n-fold (pseudo) implicative EQ-algebra and n-fold fantastic EQ-algebra, some related results are studied. In addition, by using the concept of 1-fold pseudo implicative filter of an EQ-algebra $\mathcal{E}$, it is shown that $\mathcal{E}/F$ is a good EQ-algebra and by using the concept of 1-fold fantastic filter of a good EQ-algebra $\mathcal{E}$, it is shown that $\mathcal{E}/F$ is an IEQ-algebra.

Acknowledgements. The authors are very indebted to the editor and anonymous referees for their careful reading and valuable suggestions which helped to improve the readability of the paper.

8. Compliance with ethical standards

Conflict of interest: The authors declare that there is no conflict of interest.

Human and animal rights: This article does not contain any studies with human participants or animals performed by any of the authors.

References

[1] N. Akhlaghinia, M. Aaly Kologani, R. A. Borzooei, X. L. Xin, *On the Category of EQ-algebras*, **Bulletin of the Section of Logic**, vol. 50(4) (2021), pp. 397–419, DOI: https://doi.org/10.18778/0138-0680.2021.01.

[2] R. A. Borzooei, B. Ganji Saffar, *States on EQ-Algebras*, **Journal of Intelligent and Fuzzy Systems**, vol. 29 (2015), pp. 209–221, DOI: https://doi.org/10.3233/IFS-151588.

[3] M. El-Zekey, *Representable good EQ-algebras*, **Soft Computing**, vol. 14 (2010), pp. 1011–1023, DOI: https://doi.org/10.1007/s00500-009-0491-4.

[4] M. El-Zekey, V. Novák, R. Mesiar, *On good EQ-algebras*, **Fuzzy Sets and Systems**, vol. 178(1) (2011), pp. 1–23, DOI: https://doi.org/10.1016/j.fss.2011.05.011.

[5] P. Hájek, *Fuzzy logics with noncommutative conjuctions*, **Journal of Logic and Computation**, vol. 13(4) (2003), pp. 469–479, DOI: https://doi.org/10.1093/logcom/13.4.469.

[6] L. Z. Liu, X. Y. Zhang, *Implicative and positive implicative prefilters of EQ-algebras*, **Journal of Intelligent & Fuzzy Systems**, vol. 26(5) (2014), pp. 2087–2097, DOI: https://doi.org/10.3233/IFS-130884.

[7] Z. M. Ma, B. Q. Hu, *Fuzzy EQ-filters of EQ-algebras*, Computer Engineering and Information Science, [in:] **Quantitative Logic and Soft Computing**, World Scientific (2012), pp. 528–535, DOI: https://doi.org/10.1142/9789814401531_0071.

[8] N. Mohtashamniya, L. Torkzadeh, *A prefilter generated by a set in EQ-algebras*, [in:] **46th Annual Iranian Mathematics Conference** (2015), pp. 1–4, URL: https://www.sid.ir/En/Seminar/ViewPaper.aspx?ID=18390.

[9] V. Novák, *EQ-algebra: Primary concepts and properties*, [in:] **International Joint, Czech Republic-Japan & Taiwan-Japan Symposium, Kitakyushu, Japan, August 2006** (2006), pp. 219–223, URL: https://irafm.osu.cz/research_report/101_rep101.pdf.

[10] V. Novák, B. D. Baets, *EQ-algebras*, **Fuzzy Sets and Systems**, vol. 160(20) (2009), pp. 2956–2978, DOI: https://doi.org/10.1016/j.fss.2009.04.010.

[11] W. Wang, X. L. Xin, J. T. Wang, *EQ-algebras with internal states*, **Soft Computing**, vol. 22(20) (2018), pp. 2825–2841, DOI: https://doi.org/10.1007/s00500-017-2754-9.

[12] X. L. Xin, P. F. He, Y. W. Yang, *Characterizations of some fuzzy prefilters (filters) in EQ-algebras*, **The Scientific World Journal**, vol. 2014 (2014), p. 829527, DOI: https://doi.org/10.1155/2014/829527.

[13] X. L. Xin, M. Khan, Y. B. Jun, *Generalized states on EQ-algebras*, **Iranian Journal of Fuzzy Systems**, vol. 16(1) (2019), pp. 159–172, URL: https://www.sid.ir/En/Journal/ViewPaper.aspx?ID=697512.

[14] F. Zebardast, R. A. Borzooei, M. Aaly Kologhani, *Results on equality algebras*, **Information Sciences**, vol. 381 (2017), pp. 270–282, DOI: https://doi.org/10.1016/j.ins.2016.11.027.

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Bulletin of the **S**ection of **L**ogic
Volume 51/4 (2022), pp. 487–505
https://doi.org/10.18778/0138-0680.2022.20

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A 2 SET-UP BINARY ROUTLEY SEMANTICS FOR GÖDELIAN 3-VALUED LOGIC G3 AND ITS PARACONSISTENT COUNTERPART $\mathrm{G3}^{\leq}_{\text{Ł}}$

Abstract

G3 is Gödelian 3-valued logic, $\mathrm{G3}^{\leq}_{\text{Ł}}$ is its paraconsistent counterpart and $\mathrm{G3}^{1}_{\text{Ł}}$ is a strong extension of $\mathrm{G3}^{\leq}_{\text{Ł}}$. The aim of this paper is to endow each one of the logics just mentioned with a 2 set-up binary Routley semantics.

Keywords: Binary Routley semantics, 2 set-up binary Routley semantics, 3-valued logics, paraconsistent logics, Gödelian 3-valued logic G3.

1. Introduction

The aim of this paper is to define a 2 set-up binary Routley semantics (2bR-semantics) for each one of the logics G3, $\mathrm{G3}^{\leq}_{\text{Ł}}$ and $\mathrm{G3}^{1}_{\text{Ł}}$. G3 is Gödelian 3-valued logic (cf. [3]), $\mathrm{G3}^{\leq}_{\text{Ł}}$ is the paraconsistent counterpart to G3 and $\mathrm{G3}^{1}_{\text{Ł}}$ is a strong extension of $\mathrm{G3}^{\leq}_{\text{Ł}}$. The logics $\mathrm{G3}^{\leq}_{\text{Ł}}$ and $\mathrm{G3}^{1}_{\text{Ł}}$ were introduced in [6]. Proof-theoretically, they were defined as Hilbert-type systems. Semantically, "two-valued" Belnap-Dunn semantics was the tool to interpret them. Nevertheless, they were endowed with a general Routley-Meyer semantics in [4] and with a binary Routley one in [7]. Recently, Avron (cf. [1]) has provided Gentzen-type calculi equivalent to the Hilbert-type formulations for $\mathrm{G3}^{\leq}_{\text{Ł}}$ and $\mathrm{G3}^{1}_{\text{Ł}}$ defined in [6].

2 set-up Routley-Meyer semantics (2RM-semantics) is introduced in [2], where the logics BN4, RM3 and Łukasiewicz's 3-valued logic Ł3 are interpreted with said semantics. Additionally, the logic E4 is also given a

Presented by: Norihiro Kamide
Received: August 30, 2021
Published online: October 14, 2022

2RM-semantics in [5]. 2RM-semantics is a particular class of the general Routley-Meyer semantics (cf. [10, Chapter 4]) adequate for interpreting some finite many-valued logics. 2RM-models are based upon structures of the type $(K, R, *)$, where K is a 2 set-up set, $*$ is the Routley operator and R is the ternary relation on K characteristic of the general Routley-Meyer semantics.

On the other hand, 2 set-up binary Routley semantics (2bR-semantics) is going to be introduced for the first time in the present paper, to the best of our knowledge. As it is the case with general Routley-Meyer semantics and 2RM-semantics, 2bR-semantics is a particular class of general binary Routley semantics, introduced in [7]. 2bR-semantics is adequate for interpreting some finite many-valued logics. 2bR-models are based upon structures of the type $(K, R, *)$, where K and $*$ are defined similarly as in 2RM-semantics, but R is a binary relation on K, instead of a ternary one.

It is our opinion that a semantic interpretation S of a given logic L alternative to the standard one, especially if it is a simple one, as it is the case with 2bR-semantics, sheds new light not only on the alternatively interpreted logic L, but also on the connection between L and the class of logics SL interpreted with S, as well as on the elements of the class SL itself. In this regard, we hope that the 2bR-semantics for G3, $\mathrm{G3}^{\leq}_{\mathrm{Ł}}$ and $\mathrm{G3}^{1}_{\mathrm{Ł}}$ introduced in the present paper will be useful in the sense just explained, but also in illustrating how the much discussed Routley-Meyer semantics (cf., e.g., [8] and the references therein) behave in the simple setting of a two-element model.

The structure of the paper is as follows. In Section 2, the definition of the logics $\mathrm{G3}^{\leq}_{\mathrm{Ł}}$, $\mathrm{G3}^{1}_{\mathrm{Ł}}$ and G3 is recalled. In Section 3, $\mathrm{G3}^{\leq}_{\mathrm{Ł}}$ is given a 2bR-semantics (a $\mathrm{2bRG3}^{\leq}_{\mathrm{Ł}}$-semantics) and the (strong) soundness theorem w.r.t. $\mathrm{2bRG3}^{\leq}_{\mathrm{Ł}}$-semantics is proved. In Section 4, it is shown that $\mathrm{G3}^{\leq}_{\mathrm{Ł}}$ is (strongly) complete w.r.t. $\mathrm{2bRG3}^{\leq}_{\mathrm{Ł}}$-semantics by using a proof based upon a canonical model construction. In Section 5, (resp., Section 6), we give a $\mathrm{2bRG3}^{1}_{\mathrm{Ł}}$-semantics (resp., a 2bRG3-semantics) for $\mathrm{G3}^{1}_{\mathrm{Ł}}$ (resp., G3). Then, the results in Section 3 and Section 4 are essentially used to prove (strong) soundness and completeness theorems for $\mathrm{G3}^{1}_{\mathrm{Ł}}$ and G3 w.r.t. their respective 2bR-semantics.

2. The logics $\mathbf{G3}_{\mathbf{Ł}}^{\leq}$, $\mathbf{G3}_{\mathbf{Ł}}^{1}$ and G3

In this section, the logics $\mathrm{G3}_{\mathrm{Ł}}^{\leq}$, $\mathrm{G3}_{\mathrm{Ł}}^{1}$ and G3 are defined. Firstly, some preliminary notions are noted. Then, we define the matrices $\mathrm{MG3}_{\mathrm{Ł}}$ and MG3.

Definition 2.1 (Some preliminary notions). The propositional language consists of a denumerable set of propositional variables $p_0, p_1, ..., p_n, ...$, and some or all of the following connectives: $\to$ (conditional), $\wedge$ (conjunction), $\vee$ (disjunction) and $\neg$ (negation). The biconditional ($\leftrightarrow$) and the set of formulas (wffs) are defined in the customary way. A, B, etc, are metalinguisitic variables. Logics are formulated as Hilbert-type axiomatic systems, the notions of "theorem" and "proof from a set of premises" being the usual ones, while the following notions are understood in a fairly standard sense (cf., e.g., [9]): extension and expansion of a given logic; logical matrix M and M-interpretation, M-consequence and M-validity and finally, M-determined logic.

Definition 2.2 (The matrices $\mathrm{MG3}_{\mathrm{Ł}}$ and MG3). The matrix $\mathrm{MG3}_{\mathrm{Ł}}$ is the structure $(\mathcal{V}, D, \mathtt{F})$ where (1) $\mathcal{V}$ is $\{0, \frac{1}{2}, 1\}$ with $0 < \frac{1}{2} < 1$; (2) $D = \{1\}$; (3) $\mathtt{F} = \{f_\to, f_\wedge, f_\vee, f_\neg\}$ where $f_\wedge$ and $f_\vee$ are defined as the glb (or lattice meet) and the lub (or lattice joint), respectively, and $f_\neg$ is an involution with $f_\neg(1) = 0, f_\neg(0) = 1, f_\neg(\frac{1}{2}) = (\frac{1}{2})$, while $f_\to$ is defined according to the following truth-table (tables for $\wedge, \vee$ and $\neg$ are also displayed):

$\to$	0	$\frac{1}{2}$	1
0	1	1	1
$\frac{1}{2}$	0	1	1
1	0	$\frac{1}{2}$	1

$\wedge$	0	$\frac{1}{2}$	1
0	0	0	0
$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$
1	0	$\frac{1}{2}$	1

$\vee$	0	$\frac{1}{2}$	1
0	0	$\frac{1}{2}$	1
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1
1	1	1	1

	$\neg$
0	1
$\frac{1}{2}$	$\frac{1}{2}$
1	0

Then, MG3 is defined exactly as $\mathrm{MG3}_{\mathrm{Ł}}$, except that $f_\neg$ is now interpreted according to the following truth-table:

	$\neg$
0	1
$\frac{1}{2}$	0
1	0

Well then, the logic $G3^{\leq}_{Ł}$ (resp., $G3^{1}_{Ł}$) is determined by the degree of truth-preserving (resp., truth-preserving) consequence relation defined on the matrix $MG3_{Ł}$. On the other hand, Gödelian 3-valued logic G3 is determined by the truth-preserving consequence relation defined on the matrix MG3 (cf. [6] and references therein).

The logics $G3^{\leq}_{Ł}$ and $G3^{1}_{Ł}$ are expansions of positive intuitionistic logic H_{+}, while G3 is an extension of intuitionistic logic H. They are defined as follows (cf. [6], [7] and references therein).

Definition 2.3 (The logic $G3^{\leq}_{Ł}$). The logic $G3^{\leq}_{Ł}$ can be axiomatized as follows:

A1. $A \to (B \to A)$

A2. $[A \to (B \to C)] \to [(A \to B) \to (A \to C)]$

A3. $(A \wedge B) \to A$; $(A \wedge B) \to B$

A4. $A \to [B \to (A \wedge B)]$

A5. $A \to (A \vee B)$; $B \to (A \vee B)$

A6. $(A \to C) \to [(B \to C) \to [(A \vee B) \to C)]]$

A7. $A \to \neg\neg A$

A8. $\neg\neg A \to A$

A9. $(A \vee \neg B) \vee (A \to B)$

A10. $\neg A \to [A \vee (A \to B)]$

A11. $(A \wedge \neg A) \to (B \vee \neg B)$

Rules

Modus Ponens (MP): If $A \to B$ and A, then B.
Contraposition (Con): If $A \to B$ is a theorem, then $\neg B \to \neg A$ is also a theorem.

Remark 2.4 (Rules of inference and rules of proof). A rule r of a logic L is a 'rule of inference' if it can be applied to any premises formulated in the language of L; and r is a 'rule of proof' if it is applied only to theorems of L. Notice that Con is formulated as a rule of proof in $G3^{\leq}_{Ł}$ (cf. [6, Remark 6.23], [8, §1.5] on this important question in logics with weak rules of inference).

Definition 2.5 (The logic $G3^1_{Ł}$). The logic $G3^1_{Ł}$ is defined exactly as $G3^{\leq}_{Ł}$ except that now Con is understood as a rule of inference: If $A \to B$, then $\neg B \to \neg A$.

Definition 2.6 (The logic G3). The logic G3 is axiomatized by adding

$$\text{A12. } (A \to B) \to (\neg B \to \neg A)$$
$$\text{A13. } \neg A \to (A \to B)$$

to A1–A7 and A9 of $G3^{\leq}_{Ł}$. The sole rule of inference is MP (cf. [7, §A2]).

The section is ended by noting some theorems and rules of the logics just defined.

Remark 2.7 (Some theorems and rules of $G3^{\leq}_{Ł}$, $G3^1_{Ł}$ and G3). The following are provable in the three logics defined above (cf. [6, 7] and references therein):

T1. $A \to A$

T2. $[(A \to B) \wedge A] \to B$

T3. $(\neg A \wedge \neg B) \to \neg(A \vee B)$

T4. $\neg B \to [\neg A \vee \neg(A \to B)]$

T5. $\neg(A \to B) \to \neg B$

T6. $[\neg(A \to B) \wedge \neg A] \to A$

T7. $[\neg(A \to B) \wedge (\neg A \wedge B)] \to C$

Efq. If $\neg A$ is a theorem, then $A \to B$ is also a theorem.

In addition, the rule Ecq ("E contradictione quodlibet" —"Any proposition is derivable from a contradiction"), if $A \wedge \neg A$, then B, is provable in $G3^1_{Ł}$, whereas A10 and A11 of $G3^{\leq}_{Ł}$ and Ecq are, of course, provable in G3. (Efq abbreviates "E falso quodlibet": "Any proposition follows from a false proposition").

3. A 2 set-up binary Routley semantics for $\mathbf{G3}_{\mathbf{Ł}}^{\leq}$

In this section, $G3_{Ł}^{\leq}$ is given a 2 set-up binary Routley semantics ($2bRG3_{Ł}^{\leq}$-semantics, for short). Firstly, we define the concept of a model and related notions.

DEFINITION 3.1 ($2bRG3_{Ł}^{\leq}$-models). Let $*$ be an involutive unary operation defined on the set K. That is, for any $x \in K$, $x = x^{**}$, and let K be the two-element set $\{0, 0^*\}$. A 2 set-up binary Routley $G3_{Ł}^{\leq}$-model ($2bRG3_{Ł}^{\leq}$-model, for short) is a structure $(K, R, *, \vDash)$ where (I) R is a reflexive binary relation on K such that $R00^*$ or $R0^*0$, and (II) $\vDash$ is a valuation relation from K to the set of all wffs such that the following conditions (clauses) are satisfied for every propositional variable p, wffs A, B and $a \in K$:

(i) $(Rab \ \ \& \ \ a \vDash p) \Rightarrow b \vDash p$

(ii) $a \vDash A \wedge B$ iff $a \vDash A$ and $a \vDash B$

(iii) $a \vDash A \vee B$ iff $a \vDash A$ or $a \vDash B$

(iv) $a \vDash A \to B$ iff for all $b \in K$, $(Rab$ and $b \vDash A) \Rightarrow b \vDash B$

(v) $a \vDash \neg A$ iff $a^* \nvDash A$

DEFINITION 3.2 ($2bRG3_{Ł}^{\leq}$-consequence, $2bRG3_{Ł}^{\leq}$-validity). For any non-empty set of wffs Γ and wff A, $\Gamma \vDash_{\mathrm{M}} A$ (A is a consequence of Γ in the $2bRG3_{Ł}^{\leq}$-model M) iff for all $a \in K$ in M, $a \vDash_{\mathrm{M}} A$ whenever $a \vDash_{\mathrm{M}} \Gamma$ ($a \vDash_{\mathrm{M}} \Gamma$ iff $a \vDash_{\mathrm{M}} B$ for all $B \in \Gamma$). Then, $\Gamma \vDash_{2bRG3_{Ł}^{\leq}} A$ (A is a $2bRG3_{Ł}^{\leq}$-consequence of Γ) iff $\Gamma \vDash_{\mathrm{M}} A$ for each $2bRG3_{Ł}^{\leq}$-model M. In particular, if $\Gamma = \emptyset$, $\vDash_{\mathrm{M}} A$ (A is true in M) iff $a \vDash_{\mathrm{M}} A$ for all $a \in K$ in M. And $\vDash_{2bRG3_{Ł}^{\leq}} A$ (A is $2bRG3_{Ł}^{\leq}$-valid) iff $\vDash_{\mathrm{M}} A$ in every $2bRG3_{Ł}^{\leq}$-model.

We prove some facts about $2bRG3_{Ł}^{\leq}$-models.

PROPOSITION 3.3 ($0^* \vDash \neg A$ iff $0 \nvDash A$). For any $2bRG3_{Ł}^{\leq}$-model M and wff A, $0^* \vDash_{\mathrm{M}} \neg A$ iff $0 \nvDash_{\mathrm{M}} A$.

PROOF: Immediate by clause (v) in Definition 3.1 and the involutiveness of $*$: $0^* \vDash_{\mathrm{M}} \neg A$ iff (clause (v)) $0^{**} \nvDash_{\mathrm{M}} A$ iff (involutiveness of $*$) $0 \nvDash_{\mathrm{M}} A$. $\Box$

Lemma 3.4 (Hereditary Condition). *For any $2bRG3^{\leq}_{L}$-model M, $a, b \in K$ in M and wff A, $(Rab\ \&\ a \vDash_M A) \Rightarrow b \vDash_M A$.*

Proof: Induction on the structure of A. If A is $B \wedge C$ or $B \vee C$, the proof is immediate. Then, let us prove the cases where A is $B \to C$ and $\neg B$. If $a = b$, the proof is trivial. So, we assume $a \neq b$ (clauses (iv) and (v) in Definition 3.1 are applied without mentioning them).

(I) A is $B \to C$. (Ia) $a = 0$ and $b = 0^*$. Suppose then (1) $R00^*$ and (2) $0 \vDash_{\mathrm{M}} B \to C$. We have to prove $0^* \vDash_{\mathrm{M}} B \to C$. There are two possibilities to consider: $R0^*0^*$ and $R0^*0$. Suppose the first one, that is (3) $R0^*0^*$. Assume also (4) $0^* \vDash_{\mathrm{M}} B$. By 1, 2 and 4, we get (5) $0^* \vDash_{\mathrm{M}} C$, as required. Suppose now the second alternative, that is, (6) $R0^*0$. Assume also (7) $0 \vDash_{\mathrm{M}} B$. By reflexivity of R, we have (8) $R00$, whence by 2 and 7, we get (9) $0 \vDash_{\mathrm{M}} C$, as it was to be proved. (Ib) $a = 0^*$ and $b = 0$. Suppose (1) $R0^*0$ and (2) $0^* \vDash_{\mathrm{M}} B \to C$. We have to prove $0 \vDash_{\mathrm{M}} B \to C$. There are two possibilities to consider: $R00$ and $R00^*$. Then, the proof proceeds similarly as in case Ia.

(II) A is $\neg B$. (IIa) $a = 0$ and $b = 0^*$. Suppose then (1) $R00^*$ and (2) $0 \vDash_{\mathrm{M}} \neg B$ (i.e., $0^* \nvDash_{\mathrm{M}} B$). By the induction hypothesis, 1 and 2, we have (3) $0 \nvDash B$, i.e., $0^* \vDash \neg B$, by Proposition 3.3, as required. (IIb) $a = 0^*$ and $b = 0$. The proof is similar to that of IIa. □

Lemma 3.5 (Entailment Lemma). *For any wffs A, B, $\vDash_{2bRG3^{\leq}_{L}} A \to B$ iff ($a \vDash_M A \Rightarrow a \vDash_M B$, for all $a \in K$ in all $2bRG3^{\leq}_{L}$-models M).*

Proof: ($\Rightarrow$) Let M be a $2\mathrm{bRG3}^{\leq}_{\mathrm{L}}$-model. Suppose (1) $\vDash_{2\mathrm{bRG3}^{\leq}_{\mathrm{L}}} A \to B$ and (2) $0 \vDash_{\mathrm{M}} A$ (resp., $0^* \vDash_{\mathrm{M}} A$). By reflexivity of R, we have (3) $R00$ and $R0^*0^*$. By 1, 2 and 3, we get (4) $0 \vDash_{\mathrm{M}} B$ (resp. $0^* \vDash_{\mathrm{M}} B$) as desired. ($\Leftarrow$) Suppose (1) $a \vDash_{\mathrm{M}} A \Rightarrow a \vDash_{\mathrm{M}} B$, for all $a \in K$ in M. Furthermore, suppose (2) $R0b$ (resp., $R0^*b$) and $b \vDash_{\mathrm{M}} A$ for a given $b \in K$. Then (3) $b \vDash_{\mathrm{M}} B$ trivially follows from 1, as it was required. □

Now, we can prove soundness of $\mathrm{G3}^{\leq}_{\mathrm{L}}$ w.r.t. $2\mathrm{bRG3}^{\leq}_{\mathrm{L}}$-semantics.

Theorem 3.6 (Soundness of $\mathrm{G3}^{\leq}_{\mathrm{L}}$). *For any set of wffs Γ and wff A, if $\Gamma \vdash_{G3^{\leq}_{L}} A$, then $\Gamma \vDash_{2bRG3^{\leq}_{L}} A$.*

Proof: If $A \in \Gamma$, the proof is trivial; and if A has been obtained by applying MP, the proof is immediate by leaning upon the reflexivity of R.

Then, suppose that A has been obtained by an application of Con. In this case, A is of the form (1) $\neg B \to \neg C$ and, by hypothesis, we have (2) $\vDash_{2\mathrm{bRG3}^{\leq}_{\text{Ł}}} C \to B$. We need to prove $\vDash_{2\mathrm{bRG3}^{\leq}_{\text{Ł}}} \neg B \to \neg C$. We use the Entailment Lemma. So, suppose for any arbitrary 2bRG3$^{\leq}_{\text{Ł}}$-model M, (3) $0 \vDash_{\mathrm{M}} \neg B$ (resp., $0^* \vDash_{\mathrm{M}} \neg B$). By clause (v) (resp., Proposition 3.3), we have (4) $0^* \nvDash_{\mathrm{M}} B$ (resp., $0 \nvDash_{\mathrm{M}} B$), whence by the Entailment Lemma and 2, we get (5) $0^* \nvDash_{\mathrm{M}} C$ (resp., $0 \nvDash_{\mathrm{M}} C$) and (6) $0 \vDash_{\mathrm{M}} \neg C$ (resp., $0^* \vDash_{\mathrm{M}} \neg C$) by applying again clause (v) (resp. Proposition 3.3).

Concerning the axioms, we focus on the characteristic MG3$_{\text{Ł}}$ -axioms, that is, A9, A10 and A11. The proof of the validity of A1-A6 as well as that of the double negation axioms A7 and A8 is left to the reader (notice that A7 and A8 are immediate by involutiveness of $*$).

A9, $(A \vee \neg B) \vee (A \to B)$, is 2bRG3$^{\leq}_{\text{Ł}}$-valid. Suppose that M is a 2bRG3$^{\leq}_{\text{Ł}}$-model falsifying A9. Then, for some wffs A, B, either (I) $0 \nvDash_{\mathrm{M}} (A \vee \neg B) \vee (A \to B)$ or (II) $0^* \nvDash_{\mathrm{M}} (A \vee \neg B) \vee (A \to B)$. Case I: We have (1) $0 \nvDash A$, (2) $0 \nvDash \neg B$ (i.e., $0^* \vDash B$) and (3) $0 \nvDash A \to B$. There are two possibilities to consider: (4) $R00$, $0 \vDash A$ and $0 \nvDash B$; and (5) $R00^*$, $0^* \vDash A$ and $0^* \nvDash B$. But 4 contradicts 1, while 5 contradicts 2. Case (II) We have (1) $0^* \nvDash A$, (2) $0^* \nvDash \neg B$ (i.e., $0 \vDash B$) and (3) $0^* \nvDash A \to B$. There are two possibilities to consider: (4) $R0^*0^*$, $0^* \vDash A$ and $0^* \nvDash B$; and (5) $R0^*0$, $0 \vDash A$ and $0 \nvDash B$. But 4 contradicts 1 whereas 5 contradicts 2.

A10, $\neg A \to [A \vee (A \to B)]$, is 2bRG3$^{\leq}_{\text{Ł}}$-valid. Suppose that M is a 2bRG3$^{\leq}_{\text{Ł}}$-model falsifying A10. By the Entailment Lemma, for some wffs A, B, either (I) $0 \vDash_{\mathrm{M}} \neg A$ and $0 \nvDash_{\mathrm{M}} A \vee (A \to B)$ or (II) $0^* \vDash_{\mathrm{M}} \neg A$ and $0^* \nvDash_{\mathrm{M}} A \vee (A \to B)$. Case I: We have (1) $0^* \nvDash A$, (2) $0 \nvDash A$ and (3) $0 \nvDash A \to B$. Now, either (4) $R00$, $0 \vDash A$ and $0 \nvDash B$ or (5) $R00^*$, $0^* \vDash A$ and $0^* \nvDash B$. But 4 contradicts 2, and 5 contradicts 1. Case II is treated similarly.

A11, $(A \wedge \neg A) \to (B \vee \neg B)$, is 2bRG3$^{\leq}_{\text{Ł}}$-valid. The proof is similar to that of A10. □

4. Completeness of G3$^{\leq}_{\text{Ł}}$

Completeness of G3$^{\leq}_{\text{Ł}}$ is proved by using a canonical model construction. We begin by defining the notion of a G3$^{\leq}_{\text{Ł}}$-theory and the classes of G3$^{\leq}_{\text{Ł}}$-theories of interest in the present paper.

Definition 4.1 ($G3^{\leq}_{Ł}$-theories. Classes of $G3^{\leq}_{Ł}$-theories). A $G3^{\leq}_{Ł}$-theory (theory, for short) is a set of formulas containing all $G3^{\leq}_{Ł}$-theorems and closed under Modus Ponens (MP). Let t be a theory. We set: (1) t is prime iff whenever $A \vee B \in t$, then $A \in t$ or $B \in t$; (2) t is trivial iff it contains all wffs; (3) t is a-consistent ('consistent in an absolute sense') iff t is not trivial; (4) t is w-inconsistent ('inconsistent in a weak sense') iff $\neg A \in t$, A being a $G3^{\leq}_{Ł}$-theorem; then t is w-consistent ('consistent in a weak sense') iff t is not w-inconsistent; (5) t is inconsistent iff $A \wedge \neg A \in t$ for some wff A; then t is consistent if it is not inconsistent (cf. [8] and references therein on the notion of w-consistency).

Lemma 4.2 (Extension to prime theories). *Let t be a theory and A a wff such that $A \notin t$. Then, there is a prime theory u such that $t \subseteq u$ and $A \notin u$.*

Proof: We extend t to a maximal theory u such that $A \notin u$. If u is not prime, then there are wffs B, C such that $B \vee C \in u$ but $B \notin u$ and $C \notin u$. Then, we define the sets $[u, B] = \{D \mid B \to D \in u\}$, $[u, C] = \{D \mid C \to D \in u\}$. By using A2, it is shown that (1) $[u, B]$ and $[u, C]$ are closed under MP; by using A1, (2) that they include u. Finally, by T1, (3) that $B \in [u, B]$ and $C \in [u, C]$. Next, by the hypothesis and (1), it follows that neither $[u, B]$ nor $[u, C]$ is included in u, whence we have $A \in [u, B]$ and $A \in [u, C]$ due to the maximality of u. But then, we have (4) $A \in u$ by A6 and the fact that $B \vee C \in u$, contradicting our hypothesis. Consequently, u is prime. $\square$

In what follows, it is shown how canonical $2bRG3^{\leq}_{Ł}$-models are built, Also, we prove some general facts about them.

Let Γ be a set of wffs and A a wff such that $\Gamma \nvdash_{G3^{\leq}_{Ł}} A$. Then, A is not included in the set of consequences derivable from Γ (in symbols, $A \notin Cn\Gamma[G3^{\leq}_{Ł}]$). By the Extension Lemma, there is a prime theory $\mathcal{T}$ such that $Cn\Gamma[G3^{\leq}_{Ł}] \subseteq \mathcal{T}$ and $A \notin \mathcal{T}$. (Notice that $\mathcal{T}$ is a-consistent.) Then, the canonical $2bRG3^{\leq}_{Ł}$-model built upon $\mathcal{T}$ is defined as follows.

Definition 4.3 (Canonical $2bRG3^{\leq}_{Ł}$-models). The canonical $2bRG3^{\leq}_{Ł}$-model built upon $\mathcal{T}$, as this theory has been defined above, is the structure $(K^C, R^C, *^C, \vDash^C)$, where (1) $K^C = \{\mathcal{T}, \mathcal{T}^{*^C}\}$ and for any wffs A, B and

$a, b \in K^C$, we have: (2) $R^C ab$ iff $(A \to B \in a \;\;\&\;\; A \in b) \Rightarrow B \in b$; (3) $a^{*^C} = \{A \mid \neg A \notin a\}$ and (4) $a \vDash^C A$ iff $A \in a$.

We prove some significant and useful facts about canonical 2bRG3$^{\leq}_{\text{Ł}}$-models. By $\mathcal{T}$, we refer to the G3$^{\leq}_{\text{Ł}}$-theory upon which each canonical 2bRG3$^{\leq}_{\text{Ł}}$-model is built (the superscript C above R and $*$ is dropped when there is no risk of confusion).

Proposition 4.4 ($\mathcal{T}$ is a w-consistent G3$^{\leq}_{\text{Ł}}$-theory). The G3$^{\leq}_{\text{Ł}}$-theory $\mathcal{T}$ is a w-consistent G3$^{\leq}_{\text{Ł}}$-theory.

Proof: Suppose $\neg A \in \mathcal{T}$, A being a G3$^{\leq}_{\text{Ł}}$-theorem. By the rule Efq, $\neg A \to B$ is a G3$^{\leq}_{\text{Ł}}$-theorem where B is an arbitrary wff. Then, $B \in \mathcal{T}$, contradicting the a-consistency of $\mathcal{T}$. □

Proposition 4.5 ($\mathcal{T}^{*^C}$ is a prime G3$^{\leq}_{\text{Ł}}$-theory). The $*^C$-image of $\mathcal{T}$, $\mathcal{T}^{*^C}$, is a prime G3$^{\leq}_{\text{Ł}}$-theory.

Proof: (I) $\mathcal{T}^*$ is closed under MP: Suppose (1) $A \to B \in \mathcal{T}^*$ (i.e., $\neg(A \to B) \notin \mathcal{T}$) and (2) $A \in \mathcal{T}^*$ (i.e., $\neg A \notin \mathcal{T}$) but (3) $B \notin \mathcal{T}^*$ (i.e., $\neg B \in \mathcal{T}$). By using the G3$^{\leq}_{\text{Ł}}$-theorem $\neg B \to [\neg A \vee \neg(A \to B)]$ (T4), we have (4) $\neg A \in \mathcal{T}$ or $\neg(A \to B) \in \mathcal{T}$. But 1 and 2 contradict 4. (II) $\mathcal{T}^*$ contains all G3$^{\leq}_{\text{Ł}}$-theorems: Let A be a G3$^{\leq}_{\text{Ł}}$-theorem such that $A \notin \mathcal{T}^*$. Then, $\neg A \in \mathcal{T}$, contradicting the w-consistency of $\mathcal{T}$. (III) $\mathcal{T}^*$ is prime: Suppose (5) $A \vee B \in \mathcal{T}^*$ (i.e., $\neg(A \vee B) \notin \mathcal{T}$) but (6) $A \notin \mathcal{T}^*$ (i.e., $\neg A \in \mathcal{T}$) and (7) $B \notin \mathcal{T}^*$ (i.e., $\neg B \in \mathcal{T}$). By the G3$^{\leq}_{\text{Ł}}$-theorem $(\neg A \wedge \neg B) \to \neg(A \vee B)$ (T3), we have (8) $\neg(A \vee B) \in \mathcal{T}$, contradicting 5. □

Next, an alternative reading of the canonical accessibility relation is provided together with the proof that R^C is a reflexive relation such that $R^C\mathcal{T}\mathcal{T}^{*^C}$ or $R^C\mathcal{T}^{*^C}\mathcal{T}$. Then, it is shown that $*^C$ is an involutive operation in canonical 2bRG3$^{\leq}_{\text{Ł}}$-models. Also, that clauses (i), (ii), (iii) and (v) hold in canonical 2bRG3$^{\leq}_{\text{Ł}}$-models.

Proposition 4.6 ($R^C ab$ iff $a \subseteq b$). For any $a, b \in K^C$, $R^C ab$ iff $a \subseteq b$.

Proof: ($\Rightarrow$) Suppose (1) $R^C ab$ and (2) $A \in a$, and let (3) $B \in b$. By A1 and 2, we have (4) $B \to A \in a$, whence (5) $A \in b$ follows by 1, 3 and 4. ($\Leftarrow$) Suppose (1) $a \subseteq b$. (2) $A \to B \in a$ and (3) $A \in b$. By 1 and 2, we have

(4) $A \to B \in b$. By T2, $[(A \to B) \wedge A] \to B$, 3 and 4, (5) $B \in b$ follows, as it was to be proved. □

PROPOSITION 4.7 ($R^C\mathcal{T}\mathcal{T}^{*^C}$ or $R^C\mathcal{T}^{*^C}\mathcal{T}$). The canonical relation R^C is a reflexive relation such that $R^C\mathcal{T}\mathcal{T}^{*^C}$ or $R^C\mathcal{T}^{*^C}\mathcal{T}$.

PROOF: By Proposition 4.6, it is immediate that R^C is reflexive. On the other hand, suppose that there are A, B such that (1) $A \in \mathcal{T}$, (2) $B \in \mathcal{T}^*$ (i.e., $\neg B \notin \mathcal{T}$), (3) $A \notin \mathcal{T}^*$ (i.e., $\neg A \in \mathcal{T}$) and (4) $B \notin \mathcal{T}$. By $(A \wedge \neg A) \to (B \vee \neg B)$ (A11), we have (5) $B \vee \neg B \in \mathcal{T}$. But 2 and 4 contradict 5. □

PROPOSITION 4.8 ($*^C$ is an involutive operation on K^C). The canonical operation $*^C$ is an involutive operation on K^C.

PROOF: Let $a \in K^C$. Given that a is a $\mathrm{G3}^{\leq}_{\mathrm{Ł}}$-theory, $A \in a$ iff $\neg\neg A \in a$ follows by A7 and A8 Then, we have $A \in a$ iff $A \in a^{**}$ by Definition 4.3(3). □

PROPOSITION 4.9 (Clauses (i), (ii), (iii) and (v) hold canonically). Conditions (i), (ii), (iii) and (v) in Definition 3.1 hold when canonically interpreted according to Definition 4.3.

PROOF: Condition (i) is trivial by Proposition 4.6 and condition (v) by Definition 4.3(4). Then, condition (iii) (resp., condition (ii)) is immediate by A5, A6 and primeness of both $\mathcal{T}$ and $\mathcal{T}^*$ (resp., A3 and A4). □

Concerning clause (iv), we have:

PROPOSITION 4.10 (Clause (iv) holds in the canonical $\mathrm{2bRG3}^{\leq}_{\mathrm{Ł}}$-model). Condition (iv) in Definition 3.1 holds in the canonical $\mathrm{2bRG3}^{\leq}_{\mathrm{Ł}}$-model.

PROOF: ($\Rightarrow$). Let $a \in K^C$ and suppose $a \vDash^C A \to B$ (i.e., $A \to B \in a$), R^Cab (i.e., $a \subseteq b$) and $b \vDash^C A$ (i.e., $A \in b$). Then, $b \vDash^C B$ (i.e., $B \in b$) is immediate by MP.

($\Leftarrow$) We use Proposition 4.7. (I) $\mathcal{T} \subseteq \mathcal{T}^*$. (Ia) Assume $A \to B \notin \mathcal{T}$ (i.e., $\neg(A \to B) \in \mathcal{T}^*$). Given $R\mathcal{T}\mathcal{T}$ and $R\mathcal{T}\mathcal{T}^*$, it suffices to show $[A \in \mathcal{T}\ \ \&\ \ B \notin \mathcal{T}]$ or $[A \in \mathcal{T}^*\ \ \&\ \ B \notin \mathcal{T}^*]$. For reductio, suppose (1) $[A \notin \mathcal{T}\ \ \&\ \ A \notin \mathcal{T}^*]$ or (2) $[A \notin \mathcal{T}\ \ \&\ \ B \in \mathcal{T}^*]$ or (3) $[B \in \mathcal{T}\ \ \&\ \ A \notin \mathcal{T}^*]$ or (4) $[B \in \mathcal{T}\ \ \&\ \ B \in \mathcal{T}^*]$. But 1, 2, 3 and 4 are impossible by $\neg A \to [A \vee (A \to B)]$ (A10), $(A \vee \neg B) \vee (A \to B)$ (A9), $A \to (B \to A)$ (A1) and A1, respectively. (Ib) Assume $A \to B \notin \mathcal{T}^*$ (i.e., $\neg(A \to B) \in \mathcal{T}$). Given

$R\mathcal{T}^*\mathcal{T}^*$, it suffices to show $A \in \mathcal{T}^*$ and $B \notin \mathcal{T}^*$. Suppose, for reductio, (1) $A \notin \mathcal{T}^*$ (i.e., $\neg A \in \mathcal{T}$) or (2) $B \in \mathcal{T}^*$ (i.e, $\neg B \notin \mathcal{T}$). By I and 1, (3) $A \notin \mathcal{T}$ follows. But 1 and 2 are impossible by $[\neg(A \to B) \wedge \neg A] \to A$ (T6) and $\neg(A \to B) \to \neg B$ (T5), respectively.

(II) $\mathcal{T}^* \subseteq \mathcal{T}$. (IIa) Assume $A \to B \notin \mathcal{T}$. Given $R\mathcal{T}\mathcal{T}$, it suffices to show $A \in \mathcal{T}$ and $B \notin \mathcal{T}$. By II and IIa, we have (1) $A \to B \notin \mathcal{T}^*$ (i.e., $\neg(A \to B) \in \mathcal{T}$). Suppose now, for reductio, (2) $A \notin \mathcal{T}$ or (3) $B \in \mathcal{T}$. If 3 obtains, then $A \to B \in \mathcal{T}$ is immediate by A1, contradicting IIa. Let then 2 be the case. By II, we have (4) $A \notin \mathcal{T}^*$ (i.e., $\neg A \in \mathcal{T}$). Next, $[\neg(A \to B) \wedge \neg A] \to A$ (T6) is used. By T6, 1 and 4, (5) $A \in \mathcal{T}$ follows, contradicting 2. (IIb) $A \to B \notin \mathcal{T}^*$ (i.e., $\neg(A \to B) \in \mathcal{T}$). Given $R\mathcal{T}^*\mathcal{T}^*$ and $R\mathcal{T}^*\mathcal{T}$, it suffices to show $[A \in \mathcal{T}^* \ \& \ B \notin \mathcal{T}^*]$ or $[A \in \mathcal{T} \ \& \ B \notin \mathcal{T}]$. Then, the proof is similar to that of Ia by using now $[\neg(A \to B) \wedge \neg A] \to A$ (T6), $[\neg(A \to B) \wedge (\neg A \wedge B)] \to C$ (T7) and $\neg(A \to B) \to \neg B$ (T5).

We remark that the use of A9 (resp., T7) requires the primeness (resp., the a-consistency) of $\mathcal{T}$. □

Remark 4.11 (On the canonical clause (iv)). Suppose that R is required to be only reflexive: it is not demanded of 2bRG$3^{\leq}_{\text{Ł}}$-models that one of $R00^*$ and $R0^*0$ be present. Then, the proof of the canonical validity of clause (iv) would require the theoremhood of disjunctive Peirce's law, $A \vee (A \to B)$.

Once Proposition 4.10 proved, it immediately follows that the canonical 2bRG$3^{\leq}_{\text{Ł}}$-model is indeed a 2bRG$3^{\leq}_{\text{Ł}}$-model.

Lemma 4.12 (The canonical model is indeed a model). *The canonical 2bRG$3^{\leq}_{\text{Ł}}$-model is indeed a 2bRG$3^{\leq}_{\text{Ł}}$-model.*

Proof: (1) By Proposition 4.7, R^C is a reflexive relation such that $R\mathcal{T}\mathcal{T}^*$ or $R\mathcal{T}^*\mathcal{T}$. (2) By Proposition 4.8, $*^C$ is an involutive operation on K^C. (3) Finally, by Propositions 4.9 and 4.10, $\vDash^C$ fulfils conditions (i)–(v) in Definition 3.1. □

Now, we prove completeness.

Theorem 4.13 (Completeness of G$3^{\leq}_{\text{Ł}}$). *For any set of wffs Γ and wff A, if $\Gamma \vDash_{2bRG3^{\leq}_{Ł}} A$, then $\Gamma \vdash_{G3^{\leq}_{Ł}} A$.*

Proof: Suppose $\Gamma \nvdash_{G3^{\leq}_{Ł}} A$. By the Extension Lemma (Lemma 4.2), there is a prime theory $\mathcal{T}$ such that $\Gamma \subseteq \mathcal{T}$ and $A \notin \mathcal{T}$. Then, the canonical 2bRG$3^{\leq}_{\text{Ł}}$-model is defined upon $\mathcal{T}$ as shown in Definition 4.3. By

Lemma 4.12, the canonical $2\mathrm{bRG3}_{\text{Ł}}^{\leq}$-model is a $2\mathrm{bRG3}_{\text{Ł}}^{\leq}$-model. Then, $\Gamma \nvDash^{C} A$, since $\mathcal{T} \vDash^{C} \Gamma$ but $\mathcal{T} \nvDash^{C} A$. Consequently, $\Gamma \nvDash_{2\mathrm{bRG3}_{\text{Ł}}^{\leq}} A$ by Definition 3.2. □

5. A 2 set-up binary Routley semantics for $\mathrm{G3}_{\text{Ł}}^{1}$

In this section, $\mathrm{G3}_{\text{Ł}}^{1}$ is given a 2 set-up binary Routley semantics ($2\mathrm{bRG3}_{\text{Ł}}^{1}$-semantics, for short) and $\mathrm{G3}_{\text{Ł}}^{1}$ is proved strongly sound and complete w.r.t. said semantics (we lean upon the results in Sections 3 and 4).

Definition 5.1 ($2\mathrm{bRG3}_{\text{Ł}}^{1}$-models). A 2-set-up binary $\mathrm{G3}_{\text{Ł}}^{1}$-model ($2\mathrm{bRG3}_{\text{Ł}}^{1}$-model, for short) is a structure $(K, R, *, \vDash)$ where $K, *$ and $\vDash$ are defined exactly as in $2\mathrm{bRG3}_{\text{Ł}}^{\leq}$-models and R is a reflexive relation such that $R00^{*}$, instead of being a reflexive relation such that $R00^{*}$ or $R0^{*}0$, as in $2\mathrm{bRG3}_{\text{Ł}}^{\leq}$-models.

Definition 5.2 ($2\mathrm{bRG3}_{\text{Ł}}^{1}$-consequence, $2\mathrm{bRG3}_{\text{Ł}}^{1}$-validity). The notions of $2\mathrm{bRG3}_{\text{Ł}}^{1}$-consequence and $2\mathrm{bRG3}_{\text{Ł}}^{1}$-validity are defined similarly as the corresponding notions for $\mathrm{G3}_{\text{Ł}}^{\leq}$, except that in each model M they are restricted now to the element 0 in K. Thus, for example, $\Gamma \vDash_{\mathrm{M}} A$ iff $0 \vDash_{\mathrm{M}} A$, whenever $0 \vDash \Gamma$ ($0 \vDash \Gamma$ iff $0 \vDash B$ for all $B \in \Gamma$).

Then, we note that Proposition 3.3 and Lemmas 3.4 and 3.5 still hold for $\mathrm{G3}_{\text{Ł}}^{1}$ and are proved in a similar way as in $\mathrm{G3}_{\text{Ł}}^{\leq}$.

Concerning soundness, the essential point is to prove that Contraposition (Con) holds as a rule of inference.

Proposition 5.3 (Con preserves $2\mathrm{bRG3}_{\text{Ł}}^{1}$-validity). Con (if $A \to B$, then $\neg B \to \neg A$) preserves $2\mathrm{bRG3}_{\text{Ł}}^{1}$-validity.

Proof: Let M be a $2\mathrm{bRG3}_{\text{Ł}}^{1}$-model and A, B wffs such that (1) $0 \vDash A \to B$ but (2) $0 \nvDash \neg B \to \neg A$. There are two possibilities to consider: (3) $R00$, $0 \vDash \neg B$ (i.e., $0^{*} \nvDash B$), $0 \nvDash \neg A$ (i.e., $0^{*} \vDash A$) and (4) $R00^{*}$, $0^{*} \vDash \neg B$ (i.e., $0 \nvDash B$) and $0^{*} \nvDash \neg A$ (i.e., $0 \vDash A$). If 3 obtains, we get (5) $0^{*} \vDash B$ by 1, since $R00^{*}$ holds in M. But 3 and 5 contradict each other. If, on the other hand, 4 is the case. we have (6) $0 \vDash B$ by using again 1, since $R00$ holds in M. But, as in the previous case, a contradiction arises (6 contradicts 4). □

Remark 5.4 (Con cannot be validated w.r.t. K). We note that if $2\mathrm{bRG3}_{\text{Ł}}^{1}$-consequence is defined w.r.t. K instead of w.r.t. only 0 in K, Con as a rule of inference does not preserve $2bRG3^1_Ł$-validity. Consider a $2bRG3^1_Ł$-model M where $R0^*0$ does not follow and, for distinct propositional variables p, q, we have $0 \vDash_M p$ (i.e., $0^* \nvDash_M \neg p$), $0 \nvDash_M q$ (i.e., $0^* \vDash_M \neg q$), $0^* \vDash_M p$ and $0^* \vDash_M q$. Clearly, $0^* \vDash_M p \to q$ but $0^* \nvDash_M \neg q \to \neg p$ as $R0^*0^*$ holds by reflexivity of R. Also, notice that by the Entailment Lemma, this $2bRG3^1_Ł$-model shows that the contraposition axiom, $(A \to B) \to (\neg B \to \neg A)$, is not $2bRG3^1_Ł$-valid.

Now, the proof that MP preserves $2bRG3^1_Ł$-validity and that A1-A11 are $2bRG3^1_Ł$-valid is similar as in $2bRG3^{\leq}_Ł$-models. In fact, it is simpler. If A is an implicative axiom, only the case $R00^*$, not both $R00^*$ and $R0^*0$, as in $2bRG3^{\leq}_Ł$-models, has to be considered. And if A is A9, only truth w.r.t. 0, not w.r.t. both 0 and 0^*, has to be examined. Finally, that MP preserves $2bRG3^1_Ł$-validity is immediate by reflexivity of R, as in $2bRG3^{\leq}_Ł$-models.

As regards completeness, the main difference w.r.t. $G3^{\leq}_Ł$ is that $G3^1_Ł$-theories need now to be closed under Con. Consequently, the Extension Lemma (Lemma 4.2) does not hold, as it stands, in the case of $G3^1_Ł$. Nevertheless, the disjunctive derivability strategy (as it is carried on in e.g., [9] following [2] or [10]) is applicable since disjunctive Con (i.e., if $C \vee (A \to B)$, then $C \vee (\neg B \to \neg A)$) is an admissible rule in $G3^1_Ł$ since it is admissible in $G3_Ł$, that is, the logic containing all and only all $MG3_Ł$-valid wffs (cf. [4, §4.3 and also Remark 6.20]). Consequently, we have an adequate Extension Lemma at our disposal (cf., e.g., [9]), and then the completeness proof can proceed similarly as in $G3^{\leq}_Ł$. However, three points have to be stressed. (1) The $G3^1_Ł$-theory $\mathcal{T}$ upon which the canonical $G3^1_Ł$-model is defined is a consistent $G3^1_Ł$-theory. This is immediate since the a-consistency of $\mathcal{T}$ entails its consistency due to its closure under the rule Ecq (cf. Remark 2.7). (2) The property $R00^*$ holds when interpreted canonically. For suppose for reductio that there is a wff A such that $A \in \mathcal{T}$ but $A \notin \mathcal{T}^{*^C}$. Then, $\neg A \in \mathcal{T}$ contradicting the consistency of $\mathcal{T}$. (3) (I) in Proposition 4.10 suffices for the proof of the canonical validity of the conditional clause, condition (iv) in Definition 3.1.

Based upon the argumentation developed so far in the present section, we think that we are entitled to state the following theorem.

THEOREM 5.5 (Soundness and completeness of $G3^1_Ł$). *For any set of wffs* Γ *and wff* A, $\Gamma \vDash_{2bRG3^1_Ł} A$ *iff* $\Gamma \vdash_{G3^1_Ł} A$.

6. A 2 set-up binary Routley semantics for G3

This section on Gödelian 3-valued logic G3 mirrors the preceding section about the logic $\mathrm{G3}^1_{Ł}$. That is, G3 is endowed with a 2 set-up binary Routley semantics (2bRG3-semantics, for short) w.r.t. which G3 is shown strongly sound and complete.

Definition 6.1 (2bRG3-models). A 2-set-up binary G3-model (2bRG3-model, for short) is a structure $(K, R, *, \vDash)$ where K, R and $\vDash$ are defined exactly as in $2\mathrm{bRG3}^1_{Ł}$-models but $*$ is a quasi-involutive unary operation on the set K, instead of a involutive one as in Definitions 3.1 and 5.1. That is, we now have: for any $x \in K$, $x^* = x^{**}$.

Definition 6.2 (2bRG3-consequence, 2bRG3-validity). The notions of 2bRG3-consequence and 2bRG3-validity are defined w.r.t. the set K (not only w.r.t. 0 in K) similarly as in $2\mathrm{bRG3}^{\leq}_{Ł}$-models (and unlike in $2\mathrm{bRG3}^1_{Ł}$-models).

Regarding Proposition 3.3 and Lemmas 3.4 and 3.5, we note the following facts.

Lemma 3.5 (Entailment Lemma) and conjunction, disjunction and conditional cases in Lemma 3.4 (Hereditary Condition) are proved similarly as in the case of $\mathrm{G3}^{\leq}_{Ł}$, while the negation case in the latter lemma is proved as follows.

Proposition 6.3 (The negation case in Lemma 3.4). The negation case in Lemma 3.4 holds for G3.

Proof: (II) A is $\neg B$. (IIa) $a = 0$ and $b = 0^*$. Suppose (1) $R00^*$ and (2) $0 \vDash_{\mathrm{M}} \neg B$ (i.e., $0^* \nvDash_{\mathrm{M}} B$). By quasi-involutiveness of $*$, we get (3) $0^{**} \nvDash_{\mathrm{M}} B$, whence (4) $0^* \vDash_{\mathrm{M}} \neg B$ follows by clause (v) in Definition 3.1. (IIb) $a = 0^*$ and $b = 0$. Suppose (1) $R0^*0$ and (2) $0^* \vDash_{\mathrm{M}} \neg B$. By clause (v) in Definition 3.1, we have (3) $0^{**} \nvDash B$, whence by involutiveness of $*$ (4) $0^* \nvDash_{\mathrm{M}} B$ follows. Finally, (5) $0 \vDash_{\mathrm{M}} \neg B$ is obtained by applying again clause (v) in Definition 3.1. □

Contrary to what the strategy was in the case of $\mathrm{G3}^{\leq}_{Ł}$, the negation case in Lemma 3.4 has not been proved leaning upon Proposition 3.3, since this proposition only holds from left to right.

Proposition 6.4 ($0^* \vDash \neg A \Rightarrow 0 \nvDash A$). For any 2bRG3-model M and wff A, if $0^* \vDash_{\mathrm{M}} \neg A$, then $0 \nvDash_{\mathrm{M}} A$.

Proof: Suppose (1) $0^* \vDash_{\mathrm{M}} \neg A$. By clause (v) (Definition 3.1), (2) $0^{**} \nvDash_{\mathrm{M}} A$, whence by quasi-involutiveness of $*$, we get (3) $0^* \nvDash_{\mathrm{M}} A$, and finally, (4) $0 \nvDash_{\mathrm{M}} A$ by Lemma 3.4, $R00^*$ and 3. $\square$

As regards soundness, the 2bRG3-validity of the contraposition and Efq axioms ($(A \to B) \to (\neg B \to \neg A)$ (A12) and $\neg A \to (A \to B)$ (A13), respectively) is the point of interest, by comparison to $\mathrm{G3}^{\leq}_{\mathrm{L}}$ and $\mathrm{G3}^{1}_{\mathrm{L}}$, since the rest of the proof proceeds much as the corresponding proofs for the two logics just mentioned. So, let us prove the 2bRG3-validity of A13 as a way of an example.

Proposition 6.5 (Efq is 2bRG3-valid). The Efq axiom $\neg A \to (A \to B)$ (A13) is 2bRG3-valid.

Proof: A13, $\neg A \to (A \to B)$, is 2bRG3-valid. Suppose that M is a 2bRG3-model falsifying A13. By the Entailment Lemma, for some wffs A, B, either (I) $0 \vDash_{\mathrm{M}} \neg A$ and $0 \nvDash_{\mathrm{M}} A \to B$ or (II) $0^* \vDash_{\mathrm{M}} \neg A$ and $0^* \nvDash_{\mathrm{M}} A \to B$. Case I: We have (1) $0^* \nvDash_{\mathrm{M}} A$ and either (2) $R00$, $0 \vDash_{\mathrm{M}} A$ and $0 \nvDash_{\mathrm{M}} B$ or (3) $R00^*$, $0^* \vDash_{\mathrm{M}} A$ and $0^* \nvDash_{\mathrm{M}} B$. But 3 contradicts 1, whereas (4) $0^* \vDash_{\mathrm{M}} A$ follows from $R00^*$ and 2, contradicting again 1. Case II: we have (1) $0^* \vDash_{\mathrm{M}} \neg A$ (i.e., $0^{**} \nvDash A$) and either (2) $R0^*0^*$, $0^* \vDash_{\mathrm{M}} A$ and $0^* \nvDash_{\mathrm{M}} B$ or (3) $R0^*0$, $0 \vDash_{\mathrm{M}} A$ and $0 \nvDash_{\mathrm{M}} B$. If 2 obtains, by quasi-involutiveness of $*$ and 1, we get (4) $0^* \nvDash_{\mathrm{M}} A$, a contradiction. If (3) is the case, by Proposition 6.4, we have (5) $0^* \nvDash_{\mathrm{M}} \neg A$ contradicting 1. $\square$

Turning to completeness, the proof can be carried on similarly as that for $\mathrm{G3}^{\leq}_{\mathrm{L}}$, given that the sole rule of inference is MP and consequently the disjunctive derivability strategy used in the completeness proof for $\mathrm{G3}^{1}_{\mathrm{L}}$ is not needed here. The only worth-remarking differences w.r.t. the completeness proof for $\mathrm{G3}^{\leq}_{\mathrm{L}}$ are the following ones: (1) as it was the case with $\mathrm{G3}^{1}_{\mathrm{L}}$, (a) the theory $\mathcal{T}$ basing the canonical 2bRG3-model is a consistent 2bRG3-theory. (b) The property $R00^*$ is proved to hold when canonically interpreted by using the consistency of $\mathcal{T}$. (2) $*^C$ is now a quasi-involutive operation on K^C (not an involutive one as in the canonical $2\mathrm{bRG3}^{\leq}_{\mathrm{L}}$- and $2\mathrm{bRG3}^{1}_{\mathrm{L}}$-models). The fact is proved by using the consistency of $\mathcal{T}$ and the G3-theorem $\neg A \vee \neg\neg A$. (3) As it happened with $\mathrm{G3}^{1}_{\mathrm{L}}$, (I) in Proposition 4.10 suffices in order to prove the canonical validity of clause (iv).

The end of section mirrors that of the precedent one.

THEOREM 6.6 (Soundness and completeness of G3). *For any set of wffs* Γ *and wff* A, $\Gamma \vDash_{2bRG3} A$ *iff* $\Gamma \vdash_{G3} A$.

7. Concluding remarks

In the present paper, a 2 set-up binary Routley semantics (2bR-semantics) is provided for each one of the logics G3, its paraconsistent counterpart, $G3^{\leq}_{Ł}$, and an extension of the latter, $G3^{1}_{Ł}$. The logics $G3^{\leq}_{Ł}$ and $G3^{1}_{Ł}$ were introduced in [6], where they were given Hilbert-type axiomatic formulations, once having been interpreted with a 'two-valued' Belnap-Dunn semantics. Recently, Gentzen-type calculi equivalent to the Hilbert-type formulations have been defined in [1].

The different 2bR-semantics defined above have been characterized by having one of the two ensuing features listed in 1, 2 and 3 below.

1. Binary relation R. Property (a) $R00^*$ and property (b) $R00^*$ or $R0^*0$, in addition to reflexivity (i.e,, $R00$ and $R0^*0^*$).

2. Unary relation $*$. (a) Involutiveness. (b) Quasi-involutiveness.

3. Definition of validity. (a) W.r.t. the set K of the two points. (b) Only w.r.t. 0 in K.

But there are other possibilities that may be interesting to examine. For example, inclusion of the property $R0^*0$. Of course, if R is such that both $R00^*$ and $R0^*0$ hold, the resulting 2bR-semantics verifies all classical tautologies. But what about $R0^*0$ and involutiveness? Or what about $R0^*0$ and quasi-involutiveness? And which is the notion of validity the 2bR-semantics is going to be defined with? Are there interesting systems characterized by the sketched 2bR-semantics?

Acknowledgements. —We sincerely thank two anonymous reviewers of the Bulletin of the Section of Logic for their comments and suggestion on a previous draft of this paper. —This work is supported by the Spanish Ministry of Science and Innovation (MCIN/AEI/ 10.13039/501100011033) under Grant PID2020-116502GB-I00.

References

[1] A. Avron, *Proof systems for 3-valued Logics based on Gödel's implication*, **Logic Journal of the IGPL**, vol. 30(3) (2021), pp. 437–453, DOI: https://doi.org/10.1093/jigpal/jzab013.

[2] R. T. Brady, *Completeness proofs for the systems RM3 and BN4*, **Logique et Analyse**, vol. 25(97) (1982), pp. 9–32, URL: https://www.jstor.org/stable/44084001.

[3] K. Gödel, *Zum Intuitionistischen Aussagenkalkül*, **Anzeiger Der Akademie Der Wissenschaften in Wien**, vol. 69 (1932), pp. 65–66.

[4] G. Robles, *A Routley-Meyer semantics for Gödel 3-valued logic and its paraconsistent counterpart*, **Logica Universalis**, vol. 7(4) (2013), pp. 507–532, DOI: https://doi.org/10.1007/s11787-013-0088-7.

[5] G. Robles, J. M. Blanco, S. M. López, J. R. Paradela, M. M. Recio, *Relational semantics for the 4-valued relevant logics BN4 and E4*, **Logic and Logical Philosophy**, vol. 25(2) (2016), pp. 173–201, DOI: https://doi.org/10.12775/LLP.2016.006.

[6] G. Robles, J. M. Méndez, *A paraconsistent 3-valued logic related to Gödel logic G3*, **Logic Journal of the IGPL**, vol. 22(4) (2014), pp. 515–538, DOI: https://doi.org/10.1093/jigpal/jzt046.

[7] G. Robles, J. M. Méndez, *A binary Routley semantics for intuitionistic De Morgan minimal logic H_{M} and its extensions*, **Logic Journal of the IGPL**, vol. 23(2) (2015), pp. 174–193, DOI: https://doi.org/10.1093/jigpal/jzu029.

[8] G. Robles, J. M. Méndez, **Routley-Meyer ternary relational semantics for intuitionistic-type negations**, Academic Press, Elsevier, London (2018), DOI: https://doi.org/10.1016/C2015-0-01638-0.

[9] G. Robles, F. Salto, J. M. Méndez, *Belnap-Dunn semantics for natural implicative expansions of Kleene's strong three-valued matrix II*, **Journal of Applied Non-Classical Logics**, vol. 23(3) (2019), pp. 307–325, DOI: https://doi.org/10.1080/11663081.2019.1644079.

[10] R. Routley, R. K. Meyer, V. Plumwood, R. T. Brady, **Relevant logics and their rivals**, vol. 1, Ridgeview Publishing Co., Atascadero, CA (1982).

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Bulletin of the **S**ection of **L**ogic
Volume 51/4 (2022), pp. 507–533
https://doi.org/10.18778/0138-0680.2022.21

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BASIC FOUR-VALUED SYSTEMS OF CYCLIC NEGATIONS[1]

Abstract

We consider an example of four valued semantics partially inspired by quantum computations and negation-like operations occurred therein. In particular we consider a representation of so called square root of negation within this four valued semantics as an operation which acts like a cycling negation. We define two variants of logical matrices performing different orders over the set of truth values. Purely formal logical result of our study consists in axiomatizing the logics of defined matrices as the systems of binary consequence relation and proving correctness and completeness theorems for these deductive systems.

Keywords: Generalized truth values, consequence relation, first degree entailment.

1. Introduction

The study of properties of negation-like connectives constitutes nowadays is a well established area of interdisciplinary research activity, including purely logical investigations (consult collective monographs [13, 26]). Negation often expresses the characteristic features of logical systems acting

[1]This research is conducted as a part of "Brain, cognitive systems and artificial intelligence" Lomonosov Moscow University scientific school project.

Presented by: Andrzej Indrzejczak
Received: June 9, 2022
Published online: October 25, 2022

thereby as a mean for distinguishing and systematizing them (see, for instance, [16] for the treatment of different types of paraconsistent logics in accordance with the properties of negations introduced there). In the literature one can find examples of hierarchic structures over the sets of negations, aimed to reflect their logical properties. Probably the best known is the "kite of negations" proposed by M. Dunn in [8] and refined in the subsequent articles.

In this paper, we are not intent on providing a complete picture of some big family of negation-like operations, instead we concentrate on a particular type of negation which may be characterized as a *cyclic* operation over certain set of truth values. Specifically we are interested in its behaviour in the context of four-valued semantics, the breeding ground of many well known non-classical logics.

Occasionally our research was brought to life with an interest to the problematics of quantum computation and its possible representations within the semantic framework of non-classical logic. In particular the reflections on one of the most unusual quantum gates, the square root of negation, induced a unary operation on the four-element set of truth values. On the syntactic level, we defined two logical systems considerably differing from each other with respect to the set of deductive postulates but sharing "classicality" of double negation. This particular feature is inherent in some other non-classical logics [14, 15, 18, 28, 29].

2. Cyclic negation in the generalized truth values setting

Our interest to studies of cyclic negation stems from the different sources. This kind of negation is primarily known in the field of Post algebras and their logics (see [20, 21]). Another origin can be found in the context of four valued semantics and corresponding logics. According to [17], the first appearance of a cyclic negation in four valued framework can be found in [22], while [17] itself deals with the property of functional completeness for the expansions of Belnap-Dunn logic. In particular Belnap-Dunn logic equipped with cyclic negation in [22] is proved to be functionally complete. In [28], two versions of cycling negation appeared under the names *left* and *right turns* as the specific operations over the set of two-component generalized truth values[2], but they had not been studied there at any extent. Four-valued systems with some relatives of cyclic negation (different from ours) are investigated in [15, 18, 19]. One of the features of the negation-like operations studied there consists in their ability to simulate the properties of classical (and is some cases intuitionistic) negation via composition. It is worth noting that [14] addresses the problem of simulating conventional negations via other unary operations touching upon a cyclic negation.

2.1. The Basics of generalized truth values

The truth values that we concern with throughout this paper can be understood as a kind of generalized truth values. Although we start with the idea of how these truth values arise from the representation of quantum computational logic gates in the framework of four-valued semantics, later we show that the values can be generated in a regular way via elementary set-theoretical operations. Let us discuss this the process of introducing generalized truth values in more details.

Generalized truth values are the result of power-setting (or sometimes taking Cartesian product) of an initial set of truth-values. For example, if we start with the set **2** of classical values $\{t, f\}$, then the first stage of its generalization is the set $\mathscr{P}(\mathbf{2}) = \mathbf{4} = \{\{t, f\}, \{t\}, \{f\}, \varnothing\}$. Ordered by "definiteness-of-truth relation", the set **4** forms a well known lattice $\mathcal{FOUR}_2$ of Belnap's truth values (assuming that $\mathbf{T} = \{t\}$, $\mathbf{B} = \{t, f\}$, $\mathbf{F} = \{f\}$ and $\mathbf{N} = \varnothing$). This structure can also be considered as bilattice when the second, informational order, is taken into account (see Figure 1).

To proceed further, one needs to generalize a valuation function as well, to be a map from the set of propositional variables to the set **4**. If we in a natural way extend valuation to arbitrary formula and define an appropriate consequence relation, we arrive at certain semantic logic. Interestingly, a logic whose consequence relation is defined via the logical ordering is exactly the useful 4-valued relevant logic constructed by [9] and [1, 2].

Generalization procedure has no limits. From **2**, it leads through **4** to P(**4**)=**16** and the trilattice $\mathcal{SIXTEEN}_3$ with three independent orderings. This algebraic structure is a special case of multilattice proposed and discussed in [23]. Moreover, two of these three ordering relations generate

[2] For the detailed account of this kind of compound truth values see [29].

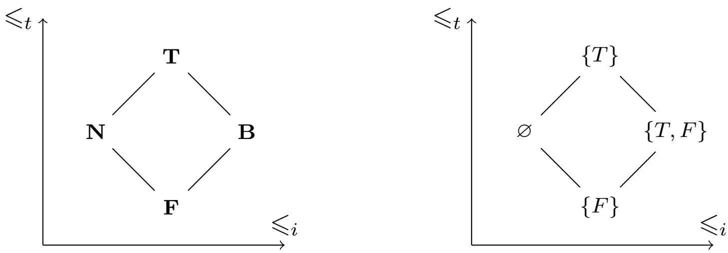

Figure 1. Bilattice $\mathcal{FOUR}_2$ in Belnap's and generalized truth-values setting.

useful 16-valued logics of the first-degree entailment [24]. If one takes the set **3** of strong Kleene's three-valued logic, it gives rise to a valuational system corresponding to the lattice $\mathcal{EIGHT}_3$ with three orderings [27]. And again this valuational structure generates the first-degree relevant logic.

Some constructions of the generalized truth valued might deviate from the paradigm pictured above. For example the values used in [29] are generated from the set $\{t, 1\}$ of two different types of truth, while false (of a certain type) is rendered as just the absence of truth (of the same type).

2.2. Four-valuedness and cyclic negation from quantum computations

Although this paper *does not* concern with quantum computations or their logic at all, some concepts from the field of quantum computational logic have inspired the four-valued semantics underlying the logics discussed below and, specifically, the choice of the unary operation acting over there. This section clarifies the origins of the family of truth values used below.

One of the ideas that motivated this paper, namely, to merge generalized truth values approach and quantum computation in a joint logical framework, was prompted by seminal writings of prominent logicians of past and present, and after all is connected with the search of answers to the question, what (modern) logic is.

The first one was proposed by G. Frege and J. Łukasiewicz many years ago and now enjoys a new lease on life within the project of generalized truth values. The core idea may be expressed in Łukasiewicz's words

– logic is the science of objects of a special kind, namely a science of logical values. Though seems strange, this understanding of logic is coherent with standard conception of logic, because the search for criteria of correct reasoning and argument immediately leads one to truth-(or, designated value-) preserving interpretation of logical inference.

Another conception of logic is due to J. van Benthem, who in [25] develops a program of Logical Dynamics, which presupposes the interpretation of logic as a theory of information-driven agency, being thus the study of explicit informational processes (inference, observation, communication). The latter interpretation may be seen as the other side of the same coin – in words of J. van Benthem, "inference is just one way of producing information, at best on a par, even for logic itself, with others" [3, p. 183], so it is little wonder that "inference and information update are intertwined" [3, p. 189].

One step away from here and just a moment to go, there is an idea to consider quantum logic as logic of quantum computation, where the latter offers a new possibility opened up by quantum gates to deal with information processing procedures being generalizations of reasoning and argument. An additional interest is connected with logical formalization of so called genuine quantum gates "that transform classical registers into quregisters that are superpositions: the square root of the negation and the square root of the identity" [5, p. 298]. According to [6] "logicians are now entitled to propose a new logical operation $\sqrt{\mathtt{NOT}}$. Why? Because a faithful physical model for it exists in nature".

Let us remind some key concepts of quantum computational logic (for more details see, for example, [4]). The unit of representation of quantum information is a qubit (from English "quantum bit"), $a|0\rangle + b|1\rangle$, where $|0\rangle$ and $|1\rangle$ are vectors $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$, respectively, written in so called Dirac notation, while a and b are complex numbers, the *amplitudes*, expressing the probabilities.

Quantum computational logic offers a broad family of operators, quantum logic gates[3], which in some cases can be rendered as the counterparts of classical logic gates and thus give rise to a family of propositional con-

[3]Well known examples are CNOT, TOFFOLI, FREDKIN, SWAP gates which perform reversible computation using some qubits as control registers for governing the actions on target bit. For example, CNOT negates its target bit if and only if the control bit is recognized as 1.

nectives in formal languages of quantum logical systems. But quantum computations provide also examples of non-classical gates. The square root of negation is of the special interest for us. For a qubit $|\varphi\rangle = a|0\rangle + b|1\rangle$, $\sqrt{\mathtt{NOT}}(|\varphi\rangle) = \frac{1}{2}[(1+i)a + (1-i)b]|0\rangle + \frac{1}{2}[(1-i)a + (1+i)b]|1\rangle$, where i is an imaginary unit. While NOT gate transforms $|1\rangle$ into $|0\rangle$ and vice versa, $\sqrt{\mathtt{NOT}}$ does only half of the work.

The key observation here is that the square root of the negation is a kind of "connective with memory". In particular, when applied twice to Truth, it returns Falsity and vice versa. At the same time, the first application to True or False gives intermediate value. Thus, to understand where to go after the first application of the square root of the negation, one should somehow remember the point of departure. The complex nature of generalized truth values allows to yield this peculiarity by preserving the component of the initial value. For example, starting with **T**, the first application of the square root of the negation "adds" uncertainty thus producing **TU**; the second application transforms it to **F**; the third again adds **U** to **F** resulting in **FU**; and finally after the fourth application we arrive at **T**. So we can see that our representation of the square root of negation within four-valued framework is nothing more then *a cyclic negation*.

Thus we have new set of truth values, $\{\mathbf{T}, \mathbf{TU}, \mathbf{FU}, \mathbf{F}\}$, and an open choice of order relation and subset of the designated values. Below we consider two natural variants of partial order over this set with the same two-element subset of designated values, $\{\mathbf{T}, \mathbf{TU}\}$. The choice of this subset seems reasonable for several reasons. It contains Truth itself (**T**) and the the other value (**TU**), having something that we would call *a trace of truth*. Moreover, this subset is one of the two prime filters in lattice $4\mathcal{Q}$ described below.

In this paper, we consider two propositional logics, $\mathbf{CNL_4^2}$ and $\mathbf{CNLL_4^2}$, determined by four-valued matrices (with two-valued matrix filters) constructed over the set of generalized truth values inspired by quantum computations as explained above. Though these logics have much in common, they differ essentially with respect to the properties of negations and their interrelation with conjunction and disjunction.

2.3. Four-valued matrices

For both logics, $\mathbf{CNL_4^2}$ and $\mathbf{CNLL_4^2}$, we subsume the same propositional language $\mathcal{L}_{cnl}$ of the signature $\{\wedge, \vee, \neg\}$ over denumerable set of variables $\mathtt{Var}$ with the set of complex formulas $\mathtt{For}$ constructed according to the standard inductive definition.

On the basis of the set $\mathscr{U} = \{\mathbf{T}, \mathbf{TU}, \mathbf{FU}, \mathbf{F}\}$ we define two distinct matrices, $\mathcal{M}^{\mathbf{CNL_4^2}}$ and $\mathcal{M}^{\mathbf{CNLL_4^2}}$, over this set with the same subset of designated values $\mathcal{D} = \{\mathbf{T}, \mathbf{TU}\}$ and the same definition of unary operation $\mathcal{O} = \{\sim, \wedge, \vee\}$ differing with respect to meet and join in the lattice reducts of these matrices.

Tableau definitions for the binary operations $\wedge$ and $\vee$ can be easily imported from the order relations over the set of truth values represented via Hasse diagrams, depicted in Figure 2. Evidently these ordered sets of truth values constitute two simple lattices, $4\mathcal{Q}$ (left diagram) and $4\mathcal{LQ}$.

Definition 2.1. $\mathcal{M}^{\mathbf{CNL_4^2}}$ matrix is a structure $\langle \mathscr{U}, \{f_c\}_{c \in \mathcal{O}}, \mathcal{D}\rangle$, where the operations $f_\wedge$ and $f_\vee$ are defined as meet and join in $4\mathcal{Q}$, $f_\sim$ is defined via the following table:

x	$f_\sim(x)$
$\mathbf{T}$	$\mathbf{TU}$
$\mathbf{TU}$	$\mathbf{F}$
$\mathbf{F}$	$\mathbf{FU}$
$\mathbf{FU}$	$\mathbf{T}$

Definition 2.2. $\mathcal{M}^{\mathbf{CNLL_4^2}}$ matrix is a structure $\langle \mathscr{U}, \{g_c\}_{c \in \mathcal{O}}, \mathcal{D}\rangle$, where the operations $g_\wedge$ and $g_\vee$ are defined as meet and join in $4\mathcal{LQ}$, $g_\sim$ is defined via the same table as $f_\sim$.

A valuation v is a mapping $\mathtt{Var} \mapsto \mathscr{U}$. An extension of v to the set $\mathtt{For}$ depends on a matrix assumed. For example, in case of $\mathcal{M}^{\mathbf{CNL_4^2}}$ we define extension v_2 of v via following expressions for all $A, B \in \mathtt{For}$: $v_2(A \wedge B) = f_\wedge(v_2(A), v_2(B))$, $v_2(A \vee B) = f_\vee(v_2(A), v_2(B))$, $v_2(\neg A) = f_\sim(v_2(A))$. In the same manner we define an extension v_3 of a valuation over $\mathbf{CNLL_4^2}$ matrix, using operations $g_\sim$, $g_\wedge$ and $g_\vee$.

The semantic consequence relation is defined via preservation of a designated truth value and again relies on a matrix assumed:

Definition 2.3. For all $A, B \in \mathtt{For}$,

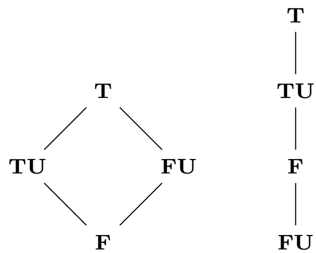

Figure 2. Lattices $4\mathcal{Q}$ and $4\mathcal{LQ}$.

(1) $A \vDash_{\mathbf{CNL^2_4}} B \iff v(A) \in \mathcal{D} \Rightarrow v(B) \in \mathcal{D}$, for each $\mathbf{CNL^2_4}$-valuation v,

(2) $A \vDash_{\mathbf{CNLL^2_4}} B \iff v(A) \in \mathcal{D} \Rightarrow v(B) \in \mathcal{D}$, for each $\mathbf{CNLL^2_4}$-valuation v.

It is instructive to examine set $\mathscr{U}$ from the generalized truth values perspective. A common way to construct a set of generalized truth values is to get powerset over some semantic basis. So, let us choose the basic set $\{\mathbf{T}, \mathbf{U}\}$, consisting of Truth and Uncertainty values, obtaining thereby the set of generalized truth values $\{\{\mathbf{T}, \mathbf{U}\}, \{\mathbf{T}\}, \{\mathbf{U}\}, \varnothing\}$. It is natural to think of $\{\mathbf{T}\}$ as just $\mathbf{T}$, while $\{\mathbf{T}, \mathbf{U}\}$ as our $\mathbf{TU}$. Then $\mathbf{U}$ is just "uncertainty without being true". Recall that the absence of truth can be understood as just being false. This suggests that $\mathbf{U}$ can be thought as $\mathbf{FU}$; likewise $\varnothing$ is just $\mathbf{F}$.

3. Binary consequence systems for $\mathbf{CNL^2_4}$ and $\mathbf{CNLL^2_4}$

To formalize semantically defined consequence relation we will use a specific variant of a logical calculus, "a binary consequence system"[4], which is typical of all **FDE**-related logics. The term "binary" means that a sequent[5]

[4]See [10, Chapter 6] for a discussion of terminology concerning to different presentations of logical systems. In particular our approach is called "binary implicational system" there.

[5]We use the term 'sequent' in a broad sense, not reffering here to the apparatus of Gentzen calculi.

is an expression of a form $A \vdash B$ which contains exactly one formula in the antecedent or consequent position. We take some *schemata* of sequents regarded as the axiomatic schemata. A sequent is an axiom if it is a particular instance of a schema. To make the presentation succinct we abbreviate $\sim\sim$ as $\sim^2$, $\sim\sim\sim$ as $\sim^3$ and so on.

DEFINITION 3.1. A sequent $A \vdash B$ is called $\mathbf{CNL_4^2}$-valid ($\mathbf{CNLL_4^2}$-valid) $\Longleftrightarrow$

$$A \vDash_{\mathbf{CNL_4^2}} B \quad (A \vDash_{\mathbf{CNLL_4^2}} B).$$

DEFINITION 3.2. A $\mathbf{CNL_4^2}$-proof (a $\mathbf{CNLL_4^2}$-proof) as a list of sequents each of them is whether an axiom of $\mathbf{CNL_4^2}$(an axiom of $\mathbf{CNLL_4^2}$) or derived from the previous items of the list using some rule of inference. A $\mathbf{CNL_4^2}$-proof ($\mathbf{CNLL_4^2}$-proof) for a sequent $A \vdash B$ is a $\mathbf{CNL_4^2}$-proof ($\mathbf{CNLL_4^2}$-proof) the last item of which coincides with $A \vdash B$. A sequent $A \vdash B$ is called $\mathbf{CNL_4^2}$-provable ($\mathbf{CNLL_4^2}$-provable) if there is a $\mathbf{CNL_4^2}$-proof ($\mathbf{CNLL_4^2}$-proof) for $A \vdash B$.

To indicate that a sequent $A \vdash B$ is $\mathbf{CNL_4^2}$-provable ($\mathbf{CNLL_4^2}$-provable) we also adopt the expression $A \vdash_{\mathbf{CNL_4^2}} B$ ($A \vdash_{\mathbf{CNLL_4^2}} B$).

$\mathbf{CNL_4^2}$ & $\mathbf{CNLL_4^2}$ COMMON AXIOMATIC SCHEMATA AND RULES OF INFERENCE:

(a1) $A \wedge B \vdash A$,

(a2) $A \wedge B \vdash B$,

(a3) $B \vdash A \vee B$,

(a4) $A \vdash A \vee B$,

(a5) $\sim A \wedge \sim B \vdash \sim(A \wedge B)$,

(a6) $\sim(A \vee B) \vdash \sim A \vee \sim B$,

(a7) $A \wedge \sim^2 A \vdash B$,

(a8) $A \wedge (B \vee C) \vdash (A \wedge B) \vee (A \wedge C)$

(a9) $A \vdash \sim^4 A$,

(a10) $\sim^4 A \vdash A$.

(r1) $A \vdash B,\ B \vdash C\ /\ A \vdash C$,

(r2) $A \vdash B,\ A \vdash C\ /\ A \vdash B \wedge C$,

(r3) $A \vdash C,\ B \vdash C\ /\ A \vee B \vdash C$,

(r4) $A \vdash B\ /\ \sim^2 B \vdash \sim^2 A$.

$\mathbf{CNL_4^2}$ ADDITIONAL AXIOMATIC SCHEMATA:

(b1) $\sim(A \wedge B) \vdash \sim A \wedge \sim B$, (b2) $\sim A \vee \sim B \vdash \sim(A \vee B)$.

$\mathbf{CNLL_4^2}$ ADDITIONAL AXIOMATIC SCHEMATA:

(c1) $\sim A \wedge \sim B \vdash \sim(A \vee B)$,
(c2) $\sim(A \wedge B) \vdash \sim A \vee \sim B$,
(c3) $\sim A \wedge \sim^2 A \vdash \sim(A \wedge B)$
(c4) $A \wedge \sim A \vdash \sim(A \vee B)$,
(c5) $\sim(A \vee B) \vdash \sim A \vee B$,
(c6) $\sim(A \vee B) \vdash \sim(B \vee A)$,
(c7) $\sim(A \wedge B) \vdash \sim(B \wedge A)$,
(c8) $(\sim(A \vee B) \wedge \sim(A \wedge B)) \vdash \sim A \wedge \sim B$.

PROPOSITION 3.3. The following sequents are provable in $\mathbf{CNL_4^2}$:

(1) $\sim A \wedge \sim B \vdash \sim(A \vee B)$,
(2) $\sim(A \wedge B) \vdash \sim A \vee \sim B$.

PROPOSITION 3.4. The following sequents are provable in both $\mathbf{CNL_4^2}$ and $\mathbf{CNLL_4^2}$.

(Id) $A \vdash A$
(De1) $\sim^2 A \wedge \sim^2 B \dashv\vdash \sim^2(A \vee B)$,
(De2) $\sim^2 A \vee \sim^2 B \dashv\vdash \sim^2(A \wedge B)$,
(T) $B \vdash A \vee \sim^2 A$.

PROOF: Let us show the proof for (T) only:

1. $A \wedge \sim^2 A \vdash \sim^2 B$ (a7)
2. $\sim^4 B \vdash \sim^2(A \wedge \sim^2 A)$ 1, (r4)
3. $\sim^2(A \wedge \sim^2 A) \vdash \sim^2 A \vee \sim^4 A$ (De2)
4. $\sim^2 A \vee \sim^4 A \vdash A \vee \sim^2 A$ (Id), (a3), (a4), (a10), (r1), (r3)
5. $B \vdash \sim^4 B$ (a9)
6. $B \vdash A \vee \sim^2 A$ 2, 3, 4, 5, (r1)

□

4. Systems of cyclic negation and classical logic

Systems $\mathbf{CNL_4^2}$ and $\mathbf{CNLL_4^2}$ have much in common with classical logic. Indeed, if we were intended to represent classical logic as a binary consequence system, we would take (a1)–(a4), (a8)–(a10) and (r1)–(r4), adding paradoxical postulates like (a7) (then, of course, a pair $\sim\sim$ should be treated as classical $\neg$). Is is well known that an alternative formulation of classical system is obtained by replacing contraposition rule with a full collection of De Morgan laws (but then both $A \wedge \neg A \vdash_{\mathbf{Cl}} B$ and $A \vdash_{\mathbf{Cl}} B \vee \neg B$ are needed, where $\vdash_{\mathbf{Cl}}$ stands for classical binary consequence relation) as axiomatic schemas. For further references we will denote this system as **Cl**.

As mentioned above, double $\sim$ have all these features of classical negation. Thus a kind of *intrinsic* classicality present in both our systems. More precisely we can represent this fact via translation function Φ from the language of classical logic $\mathcal{L}_{cl}$ (over the signature $\{\wedge, \vee, \neg\}$, with the set of formulas denoted as $\mathtt{For}_{cl}$) to the language of the present systems (with the proviso that both languages share the same denumerable set of propositional variables $\mathtt{Var} = \{p_1, p_2, \ldots\}$):

$$\begin{aligned} \Phi(p) &= p, \quad p \in \mathtt{Var}, \\ \Phi(A \circ B) &= \Phi(A) \circ \Phi(B), \quad \circ \in \{\wedge, \vee\}, \\ \Phi(\neg A) &= {\sim\sim}\Phi(A), \quad A, B \in \mathtt{For}_{cl}. \end{aligned}$$

We would like to show, that Φ is not only a translation, but an embedding function as well[6]. We prove this statement via semantic argument. Let us consider an expression $A \vDash_{\mathbf{Cl}} B$ as an assertion about classical consequence relation according to a standard definition of a classical consequence relation.

Given a valuation $v\colon \mathtt{Var} \mapsto \mathscr{U}$ we define a corresponding classical valuation v^*:

$$v^*(p) = \begin{cases} t, & \text{if } v(p) \in \mathcal{D}, \\ f & \text{otherwise}, \end{cases}$$

[6]In the context of the current research a translation function Φ from the language of a binary consequence system S_1 to the language of a binary consequence system S_2 is an *embedding* when it holds that $A \vDash_{S_1} B \iff \Phi(A) \vDash_{S_2} \Phi(B)$. The are some other terms for similar kind of translations in the literature, see eg. [11, 7, 12].

where $p \in \mathtt{Var}$. Now let v_1, v_2 and v_3 be extensions of classical-, $\mathbf{CNL_4^2}$- and $\mathbf{CNLL_4^2}$-valuations correspondingly (in the sequel we tacitly assume that a valuation v_1, v_2 or v_3 is an extended one when applied to formulas). It is not difficult to verify that the following lemma holds (in what follows 't.c.' stands for 'truth conditions', 'IH' for 'induction hypothesis').

Lemma 4.1. *For any formula $A \in \mathtt{For}_{cl}$, any valuation v_2 (valuation v_3) there is a valuation v_1 such that $v_1(A) = t \iff v_2(\Phi(A)) \in \mathcal{D}$, ($v_1(A) = t \iff v_3(\Phi(A)) \in \mathcal{D}$).*

Proof: Simple reasoning by complexity of a formula A. Let us consider some cases, focusing on a valuation v_2 only.

Case $A = \neg B$.

$v_1(\neg B) = t \overset{\text{t.c.}\,\neg}{\iff} v_1(B) \neq t \overset{\text{IH}}{\iff} v_2(\Phi(B)) \notin \mathcal{D} \overset{\text{lem. 5.7}}{\iff} v_2({\sim}{\sim}\Phi(B)) \in \mathcal{D}$.

Case $A = B \wedge C$.

$v_1(B \wedge C) = t \overset{\text{t.c.}\,\wedge}{\iff} v_1(B) = t$ and $v_1(C) = t \overset{\text{IH}}{\iff} v_2(\Phi(B)) \in \mathcal{D}$ and $v_2(\Phi(C)) \in \mathcal{D} \overset{\text{lem. 5.7}}{\iff} v_2(\Phi(B \wedge C)) \in \mathcal{D}$. □

We also need the converse of the previous lemma. Given that $v_1(p) = t$ for some $p \in \mathtt{Var}$ we can choose a valuation v_2 (a valuation v_3) such that $v_2(p) \in \mathcal{D}$ ($v_3(p) \in \mathcal{D}$). Then it is easy to get the following lemma.

Lemma 4.2. *For any formula $A \in \mathtt{For}_{cl}$, a classical valuation v_1, there exists a valuation v_2 (resp. a valuation v_3) such that*

$$v_1(A) = t \iff v_2(\Phi(A)) \in \mathcal{D} \quad (\textit{resp. } v_3(\Phi(A)) \in \mathcal{D}).$$

Lemma 4.3. *For all formulas $A, B \in \mathtt{For}_{cl}$*

(1) $A \vDash_{\mathbf{Cl}} B \iff \Phi(A) \vDash_{\mathbf{CNL_4^2}} \Phi(B)$

(2) $A \vDash_{\mathbf{Cl}} B \iff \Phi(A) \vDash_{\mathbf{CNLL_4^2}} \Phi(B)$.

Proof: We consider $\mathbf{CNL_4^2}$ part. Let $A \models_{\mathbf{Cl}} B$, but $\Phi(A) \not\models_{\mathbf{CNL_4^2}} \Phi(B)$. Then there is a valuation v_2 such that $v_2(\Phi(A)) \in \mathcal{D}$, $v_2(\Phi(B)) \notin \mathcal{D}$. Applying lemma 4.1 we find a classical valuation v_1 such that $v_1(A) = t$, $v_1(B) \neq t$. The other direction is also clear. □

Corollary 4.4. Φ is an embedding of $\mathbf{Cl}$ into $\mathbf{CNL_4^2}$ ($\mathbf{CNLL_4^2}$).

What about the converse? Can we non-trivially translate our systems of cyclic negation to classical logic? To address this question let us define the following function Ψ, where i is a positive integer, A and B are formulas of the language $\mathcal{L}_{cnl}$:

$$\begin{aligned}
\Psi(p_i) &= p_{2i-1},\\
\Psi({\sim}p_i) &= p_{2i},\\
\Psi({\sim}{\sim}A) &= \neg\Psi(A),\\
\Psi(A \circ B) &= \Psi(A) \circ \Psi(B), \quad \circ \in \{\wedge, \vee\},\\
\Psi({\sim}(A \circ B)) &= \Psi({\sim}A) \circ \Psi({\sim}B), \quad \circ \in \{\wedge, \vee\}.
\end{aligned}$$

Similarly to the construction of a classical valuation v^* that has been used before, here we define (where i is a positive integer, $v\colon \mathtt{Var} \mapsto \mathscr{U}$)

$$v^*(p_i) = \begin{cases} t, & \text{if } i \text{ is odd and } v(p_{\frac{i+1}{2}}) \in \mathcal{D},\\ t, & \text{if } i \text{ is even and } v(p_{\frac{i}{2}}) \in \{\mathbf{T}, \mathbf{FU}\},\\ f & \text{otherwise.}\end{cases}$$

We proceed with the following

Lemma 4.5. *For every* $\mathbf{CNL^2_4}$*-valuation* v_2 *and a formula* $A \in \mathtt{For}$ *there exists a classical valuation* v_1 *such that*

$$v_1(\Psi(A)) = t \iff v_2(A) \in \mathcal{D}.$$

Proof: Let us consider firstly the case when $\Psi(A)$ is a propositional variable, say p_k. If k is odd index, then the statement follows from definition of v^*. If k is even, then suppose that $v_1(p_k) = v^*(p_k) = t$. Since preimage of p_k is ${\sim}p_{\frac{k}{2}}$ and $v_2(p_{\frac{k}{2}}) = v(p_{\frac{k}{2}}) \in \{\mathbf{T}, \mathbf{FU}\}$, $v_2({\sim}p_k) \in \mathcal{D}$. Other direction is evident.

Next let us consider some cases. Simple sub-cases are omitted.

Case $A = {\sim}{\sim}B$.

$v_1(\Psi({\sim}{\sim}B)) = t \overset{\text{df. }\Psi}{\iff} v_1(\neg(\Psi(B))) = t \overset{\text{t.c. }\neg}{\iff} v_1(\Psi(B)) \neq t \overset{\text{IH}}{\iff} v_2(B) \notin \mathcal{D} \overset{\text{t.c. }\sim}{\iff} v_2({\sim}{\sim}B) \in \mathcal{D}$.

Case $A = {\sim}(B \wedge C)$.

$v_1(\Psi({\sim}(B \wedge C))) = t \overset{\text{df. }\Psi}{\iff} v_1(\Psi({\sim}B) \wedge \Psi({\sim}C))) = t \overset{\text{t.c. }\wedge}{\iff} v_1(\Psi({\sim}B)) = t$ and $v_1(\Psi({\sim}C)) = t \overset{\text{IH}}{\iff} v_2({\sim}B) \in \mathcal{D}$ and $v_2({\sim}C) \in \mathcal{D} \overset{\text{t.c. }\sim}{\iff} v_2(B) \in \{\mathbf{T}, \mathbf{FU}\}$ and $v_2(C) \in \{\mathbf{T}, \mathbf{FU}\} \overset{\text{t.c. }\wedge}{\iff} v_2(B \wedge C) \in \{\mathbf{T}, \mathbf{FU}\} \overset{\text{t.c. }\sim}{\iff} v_2({\sim}(B \wedge C)) \in \mathcal{D}$.

Case $A = {\sim}(B \vee C)$.

$v_1(\Psi({\sim}(B \vee C))) = t \overset{\text{df. }\Psi}{\Longleftrightarrow} v_1(\Psi({\sim}B) \vee \Psi({\sim}C))) = t \overset{\text{t.c. }\vee}{\Longleftrightarrow} v_1(\Psi({\sim}B)) = t$ or $v_1(\Psi({\sim}C)) = t \overset{\text{IH}}{\Longleftrightarrow} v_2({\sim}B) \in \mathcal{D}$ or $v_2({\sim}C) \in \mathcal{D} \overset{\text{t.c. }\sim}{\Longleftrightarrow} v_2(B) \in \{\mathbf{T}, \mathbf{FU}\}$ or $v_2(C) \in \{\mathbf{T}, \mathbf{FU}\} \overset{\text{t.c. }\vee}{\Longleftrightarrow} v_2(B \vee C) \in \{\mathbf{T}, \mathbf{FU}\} \overset{\text{t.c. }\sim}{\Longleftrightarrow} v_1({\sim}(B \vee C)) \in \mathcal{D}$. □

On the other hand, given a classical valuation v^* we can get a $\mathbf{CNL_4^2}$-valuation choosing an arbitrary mapping v such that $v(p_{\frac{i+1}{2}}) \in \mathcal{D}$ when $v^*(p_i) = t$ and $v(p_{\frac{i+1}{2}}) \notin \mathcal{D}$ when $v^*(\Psi(p_i)) = f$ for a an odd integer i, while $v(p_{\frac{i}{2}}) \in \{\mathbf{T}, \mathbf{FU}\}$ when $v^*(p_i) = t$ and $v(p_{\frac{i}{2}}) \notin \{\mathbf{T}, \mathbf{FU}\}$ when $v^*(p_i) = f$ for an even integer i. Thus we obtain an analogue of the previous lemma.

Lemma 4.6. *For every classical valuation v_1 and a formula $A \in \texttt{For}$ there exists a $\mathbf{CNL_4^2}$-valuation v_2 such that*

$$v_1(\Psi(A)) = t \iff v_2(A) \in \mathcal{D}.$$

Proof: Similar to the proof of the lemma 4.5 □

Lemma 4.7. $A \models_{\mathbf{CNL_4^2}} B \iff \Psi(A) \models_{\mathbf{Cl}} \Psi(B)$.

Proof: First assume that $\Psi(A) \models_{\mathbf{Cl}} \Psi(B)$, but $A \not\models_{\mathbf{CNL_4^2}} B$. Then there exists some extended $\mathbf{CNL_4^2}$-valuation v_2 such that $v_2(A) \in \mathcal{D}$ and $v_2(B) \notin \mathcal{D}$. According to lemma 4.5 there exists an extended classical valuation v_1 such that $v_1(\Psi(A)) = t$, but $v_1(\Psi(B)) = f$.

For the other direction suppose that $A \models_{\mathbf{CNL_4^2}} B$, but $\Psi(A) \not\models_{\mathbf{Cl}} \Psi(B)$. Then there exists a classical valuation v such that $v(\Psi(A)) = t$, $v(\Psi(B)) = f$. Using lemma 4.6 we conclude that $A \not\models_{\mathbf{CNL_4^2}} B$. □

Corollary 4.8. Ψ is an embedding of $\mathbf{CNL_4^2}$ into $\mathbf{Cl}$.

To obtain the same result for $\mathbf{CNLL_4^2}$ we need some modification of Ψ. But this time things appear to be far more complicated and, as it seems, there is no simple and elegant translation clauses for the negated $\wedge$ and $\vee$. Nevertheless, technically, it is still possible to define a required function. Let us denote by Ψ' a translation which differs from Ψ in what concerns the images of formulas of the form $\sim(B\wedge C)$ and $\sim(B\vee C)$ and agrees with it in other respects. Namely we put

$$\begin{aligned}\Psi'(\sim(B\wedge C)) = (\neg\Psi'(B)\wedge\Psi'(\sim B))\vee(\neg\Psi'(C)\wedge\Psi'(\sim C))\vee&\\ \vee(\Psi'(\sim B)\wedge\Psi'(\sim C)),&\\ \Psi'(\sim(B\vee C)) = (\Psi'(B)\wedge\Psi'(\sim B))\vee(\Psi'(C)\wedge\Psi'(\sim C))\vee&\\ \vee(\Psi'(\sim B)\wedge\Psi'(\sim C)).&\end{aligned}$$

For this translation we can prove analogues of lemmas 4.5 and 4.6. Let us denote as '*tr*' the right parts of the above equations when they are clear from the context.

Lemma 4.9. *For every extended* $\mathbf{CNLL_4^2}$*-valuation* v_3 *and a formula* $A\in$ `For` *there exists a classical valuation* v_1 *such that*

$$v_1(\Psi'(A)) = t \iff v_3(A)\in\mathcal{D}.$$

Proof: Let us check some crucial cases.

Case $A = \sim(B\wedge C)$.

First we have $v_1(\Psi'(\sim(B\wedge C))) = t \overset{\text{df. }\Psi'}{\Rightarrow} v_1(tr) = t$. Thus any disjunct of tr may be evaluated as t under v_1. Let us inspect all three subcases. We start with $v_1(\neg\Psi'(B)\wedge\Psi'(\sim B)) = t \overset{\text{t.c. }\wedge,\neg}{\Rightarrow} v_1(\Psi'(B)) = f$ and $v_1(\Psi'(\sim B)) = t \overset{\text{IH}}{\Rightarrow} v_3(B)\notin\mathcal{D}$ and $v_3(\sim B)\in\mathcal{D} \overset{\text{t.c.}\sim}{\Rightarrow} v_3(B) = \mathbf{FU} \overset{\text{t.c.}\wedge}{\Rightarrow} v_3(B\wedge C) = \mathbf{FU} \overset{\text{t.c.}\sim}{\Rightarrow} v_3(\sim(B\wedge C)) = \mathbf{T}\in\mathcal{D}$. The second disjunctive sub-case is similar.

Next consider the following implications: $v_1(\Psi'(\sim B)\wedge\Psi'(\sim C)) = t \overset{\text{t.c.}\wedge}{\Rightarrow} v_1(\Psi'(\sim B)) = t$ and $v_1(\Psi'(\sim C)) = t \overset{\text{IH}}{\Rightarrow} v_3(\sim B)\in\mathcal{D}$ and $v_3(\sim C)\in\mathcal{D} \overset{\text{t.c.}\sim}{\Rightarrow} v_3(B)\in\{\mathbf{T},\mathbf{FU}\}$ and $v_3(C)\in\{\mathbf{T},\mathbf{FU}\} \overset{\text{t.c.}\wedge}{\Rightarrow} v_3(B\wedge C)\in\{\mathbf{T},\mathbf{FU}\} \overset{\text{t.c.}\sim}{\Rightarrow} v_3(\sim(B\wedge C))\in\mathcal{D}$.

For the other direction $v_3(\sim(B\wedge C))\in\mathcal{D} \overset{\text{t.c.}\sim}{\Rightarrow} v_3(B\wedge C)\in\{\mathbf{T},\mathbf{FU}\} \overset{\text{t.c.}\wedge}{\Rightarrow}$ (a) $v_3(B) = v_3(C) = \mathbf{T}$ or (b) $v_3(B) = \mathbf{FU}$ or (c) $v_3(C) = \mathbf{FU}$.

Sub-case (a): $v_3(\sim B) = \mathbf{TU}\in\mathcal{D}$ and $v_3(\sim C) = \mathbf{TU}\in\mathcal{D} \overset{\text{IH}}{\Rightarrow} v_1(\Psi'(\sim B)) = t$ and $v_1(\Psi'(\sim C)) = t \overset{\text{t.c.}\wedge}{\Rightarrow} v_1(\Psi'(\sim B)\wedge\Psi'(\sim C)) = t \overset{\text{t.c.}\vee}{\Rightarrow} v_1(tr) = t \overset{\text{df. }\Psi'}{\Rightarrow} v_1(\Psi'(\sim(B\wedge C))) = t$.

Sub-case (b): $v_3(B) = \mathbf{FU} \notin \mathcal{D} \overset{\text{t.c.}\sim}{\Rightarrow} v_3(\sim B) = \mathbf{T} \in \mathcal{D} \overset{\text{IH}}{\Rightarrow} v_1(\Psi'(B)) \neq t$ and $v_1(\Psi'(\sim B)) = t \overset{\text{t.c.}\wedge}{\Rightarrow} v_1(\neg\Psi'(B) \wedge \Psi'(\sim B)) = t \overset{\text{t.c.}\vee}{\Rightarrow} v_1(tr) = t \overset{\text{df.}\Psi'}{\Rightarrow} v_1(\Psi'(\sim(B \wedge C))) = t$. Sub-case (c) is similar.

Case $A = \sim(B \vee C)$.

$v_1(\Psi'(\sim(B \vee C))) = t \overset{\text{df.}\Psi'}{\Rightarrow} v_1(tr) = t$. Again, any disjunct of tr may have the value t under v_1. Consider the following sequence of implications: $v_1(\Psi'(B) \wedge \Psi'(\sim B)) = t \overset{\text{t.c.}\wedge}{\Rightarrow} v_1(\Psi'(B)) = t$ and $v_1(\Psi'(\sim B)) = t \overset{\text{IH}}{\Rightarrow} v_3(B) \in \mathcal{D}$ and $v_3(\sim B) \in \mathcal{D} \overset{\text{t.c.}\sim}{\Rightarrow} v_3(B) = \mathbf{T} \overset{\text{t.c.}\vee}{\Rightarrow} v_3(B \vee C) = \mathbf{T} \overset{\text{t.c.}\sim}{\Rightarrow} v_3(\sim(B \vee C)) = \mathbf{TU} \in \mathcal{D}$. The second disjunctive sub-case is similar, while the third one can be easily seen from the analogues sub-case for $\sim(B \wedge C)$.

For the other direction $v_3(\sim(B \vee C)) \in \mathcal{D} \overset{\text{t.c.}\sim}{\Rightarrow} v_3(B \vee C) \in \{\mathbf{T}, \mathbf{FU}\} \overset{\text{t.c.}\vee}{\Rightarrow}$ (a) $v_3(B) = v_3(C) = \mathbf{FU}$ or (b) $v_3(B) = \mathbf{T}$ or (c) $v_3(C) = \mathbf{T}$.

Sub-case (a): $v_3(\sim B) = \mathbf{T} \in \mathcal{D}$ and $v_3(\sim C) = \mathbf{T} \in \mathcal{D} \overset{\text{IH}}{\Rightarrow} v_1(\Psi'(\sim B)) = t$ and $v_1(\Psi'(\sim C)) = t \overset{\text{t.c.}\wedge}{\Rightarrow} v_1(\Psi'(\sim B) \wedge \Psi'(\sim C)) = t \overset{\text{t.c.}\vee}{\Rightarrow} v_1(tr) = t \overset{\text{df.}\Psi'}{\Rightarrow} v_1(\Psi'(\sim(B \vee C))) = t$.

Sub-case (b): $v_3(B) = \mathbf{T} \in \mathcal{D} \overset{\text{t.c.}\sim}{\Rightarrow} v_3(\sim B) = \mathbf{TU} \in \mathcal{D} \overset{\text{IH}}{\Rightarrow} v_1(\Psi'(B)) = t$ and $v_1(\Psi'(\sim B)) = t \overset{\text{t.c.}\wedge}{\Rightarrow} v_1(\Psi'(B) \wedge \Psi'(\sim B)) = t \overset{\text{t.c.}\vee}{\Rightarrow} v_1(tr) = t \overset{\text{df.}\Psi'}{\Rightarrow} v_1(\Psi'(\sim(B \vee C))) = t$. Sub-case (c) is similar. □

Thus the following two lemmas are readily following.

Lemma 4.10. *For every classical valuation v_1 and a formula $A \in$ `For` there exists a $\mathbf{CNLL_4^2}$-valuation v_3 such that*

$$v_1(\Psi'(A)) = t \iff v_3(A) \in \mathcal{D}.$$

Lemma 4.11. $A \models_{\mathbf{CNLL_4^2}} B \iff \Psi'(A) \models_{\mathbf{Cl}} \Psi'(B)$.

Proof: Similar to the proof of lemma 4.7. □

Corollary 4.12. Ψ' is an embedding of $\mathbf{CNLL_4^2}$ into $\mathbf{Cl}$.

5. Soundness and completeness of $\mathbf{CNL_4^2}$ and $\mathbf{CNLL_4^2}$

5.1. Soundness

Lemma 5.1 (Local Soundness for $\mathbf{CNL_4^2}$). *All axiomatic schemata of $\mathbf{CNL_4^2}$ represent $\mathbf{CNL_4^2}$-valid sequents and the rules of inference preserve $\mathbf{CNL_4^2}$-validity.*

Proof: We need to check each item from the list of axiomatic schemata and inference rules. Let us show a couple of cases. Here, again, a valuation applied to formulas is just an extended valuation function.

Suppose that axiomatic schemata (a7) is invalid, i.e. there a $\mathbf{CNL_4^2}$-valuation v that $v(A \wedge {\sim}^2 A) \in \{\mathbf{T}, \mathbf{TU}\}$ and $v(B) \notin \{\mathbf{T}, \mathbf{TU}\}$ that is $v(B) \in \{\mathbf{F}, \mathbf{FU}\}$. It can be seen that this situation is impossible since $A \wedge {\sim}^2 A$ cannot take its value from the set $\{\mathbf{T}, \mathbf{TU}\}$ at all.

Suppose that the rule (r4) does not preserve validity. This means that there is a valuation v that $A \vDash_{\mathbf{CNL_4^2}} B$, but ${\sim}^2 B \nvDash_{\mathbf{CNL_4^2}} {\sim}^2 A$. From the latter it follows that t $v({\sim}^2 B) \in \{\mathbf{T}, \mathbf{TU}\}$ and $v({\sim}^2 A) \notin \{\mathbf{T}, \mathbf{TU}\}$ which means that $v({\sim}^2 A) \in \{\mathbf{F}, \mathbf{FU}\}$. It is easy to observe that definition of $\sim$ implies $v(A) \in \{\mathbf{T}, \mathbf{TU}\}$ and $v(B) \in \{\mathbf{F}, \mathbf{FU}\}$, but this contradicts to $A \vDash_{\mathbf{CNL_4^2}} B$. Therefore, (r4) preserves validity.

The other cases are similar. □

Theorem 5.2 (Soundness for $\mathbf{CNL_4^2}$). *For any formulas A and B of the language $\mathcal{L}_{cnl}$, the following holds:*

$$A \vdash B \textit{ is } \mathbf{CNL_4^2}\textit{-provable} \Rightarrow A \vDash_{\mathbf{CNL_4^2}} B.$$

Proof: By induction on the length of the proof, using Lemma 5.1. □

Lemma 5.3 (Local Soundness for $\mathbf{CNLL_4^2}$). *All axiomatic schemata of $\mathbf{CNLL_4^2}$ represent $\mathbf{CNLL_4^2}$-valid sequents and the rules of inference preserve $\mathbf{CNLL_4^2}$-validity.*

Proof: Analogously to Lemma 5.1, we show only an example with one axiomatic schemata, because the sets of inference rules of $\mathbf{CNL_4^2}$ and $\mathbf{CNLL_4^2}$ are identical.

Suppose that axiomatic schemata (c8) is invalid, that is there is a valuation v such that $v({\sim}(A \vee B) \wedge {\sim}(A \wedge B)) \in \{\mathbf{T}, \mathbf{TU}\}$ and $v({\sim}A \wedge {\sim}B) \notin \{\mathbf{T}, \mathbf{TU}\}$. The latter means that $v({\sim}A \wedge {\sim}B) \in \{\mathbf{F}, \mathbf{FU}\}$.

(a) Let $v({\sim}(A \vee B) \wedge {\sim}(A \wedge B)) = \mathbf{T}$. According to the definition of conjunction this means that $v({\sim}(A \vee B)) = \mathbf{T}$ and $v({\sim}(A \wedge B)) = \mathbf{T}$. This means that $v(A \vee B) = \mathbf{FU}$ and $v(A \wedge B) = \mathbf{FU}$. The first equation determines $v(A) = \mathbf{FU}$ and $v(B) = \mathbf{FU}$.

    Let $v({\sim}A \wedge {\sim}B) = \mathbf{F}$. This is possible when $v({\sim}A) = \mathbf{F}$ or $v({\sim}B) = \mathbf{F}$. That is $v(A) = \mathbf{TU}$ or $v(B) = \mathbf{TU}$. Each of these cases incompatible with the previous observation.

Let $v(\sim A \wedge \sim B) = \mathbf{FU}$. It takes place when $v(\sim A) = \mathbf{FU}$ or $v(\sim B) = \mathbf{FU}$ which implies $v(A) = \mathbf{F}$ or $v(B) = \mathbf{F}$, impossible again.

(b) Let $v(\sim(A \vee B) \wedge \sim(A \wedge B)) = \mathbf{TU}$. According to the definition of conjunction three cases are to consider, but two of them are identical. Suppose, $v(\sim(A \vee B)) = \mathbf{T}$ and $v(\sim(A \wedge B)) = \mathbf{TU}$. By truth conditions of $\sim$, $v(A \wedge B) = \mathbf{T}$. This means that $v(A) = \mathbf{T}$ and $v(B) = \mathbf{T}$. Inspecting already considered cases when $v(\sim A \wedge \sim B) \in \{\mathbf{F}, \mathbf{FU}\}$ we arrive at impossible valuations. The argument is analogous, when $v(\sim(A \vee B)) = \mathbf{TU}$ and $v(\sim(A \wedge B)) = \mathbf{TU}$. □

THEOREM 5.4 (Soundness for $\mathbf{CNLL^2_4}$). *For any formulas A and B of the language $\mathcal{L}_{cnl}$, the following holds:*

$$A \vdash B \text{ is } \mathbf{CNLL^2_4}\text{-provable } \Rightarrow A \vDash_{\mathbf{CNLL^2_4}} B.$$

PROOF: By induction on the length of the proof, using Lemma 5.3. □

5.2. Completeness

The idea of the completeness theorem proof is based on a technique elaborated by J. M. Dunn for the system of **FDE** (see [9]). This method essentially relies on the notion of a prime theory which is given in the following definition.

DEFINITION 5.5. A $\mathbf{CNL^2_4}$-($\mathbf{CNLL^2_4}$)-theory is the set of formulas α such that for all formulas A and B of the language $\mathcal{L}_{cnl}$,

(1) $A \wedge B \in \alpha$ whenever $A \in \alpha$ and $B \in \alpha$,

(2) $B \in \alpha$ whenever $A \in \alpha$ and $A \vdash B$ is $\mathbf{CNL^2_4}$-($\mathbf{CNLL^2_4}$)-provable.

A $\mathbf{CNL^2_4}$-($\mathbf{CNLL^2_4}$)-theory is prime if $A \vee B \in \alpha$ implies $A \in \alpha$ or $B \in \alpha$. We call a $\mathbf{CNL^2_4}$-($\mathbf{CNLL^2_4}$)-theory α *c-normal* when for each formula A it holds that $A \in \alpha$ if and only if $\sim^2 A \notin \alpha$.

As a first step toward completeness theorems for $\mathbf{CNL^2_4}$ and $\mathbf{CNLL^2_4}$ we prove the Extension Lemma. Note that we use this lemma uniformly for both completeness theorems. So we prove it for the case of $\mathbf{CNL^2_4}$, while proof for another system is the same.

Lemma 5.6 (Extension Lemma). *For all formulas A and B of the language $\mathcal{L}_{cnl}$, if $A \vdash B$ is not $\mathbf{CNL_4^2}$-provable, then there is a c-normal prime theory α such that $A \in \alpha$, $B \notin \alpha$.*

Proof: Suppose that for some formulas A and B, $A \vdash B$ is not $\mathbf{CNL_4^2}$-provable. Let us define $\alpha_0 = \{C \mid A \vdash_{\mathbf{CNL_4^2}} C\}$. α_0 is a theory as it is closed under $\vdash_{\mathbf{CNL_4^2}}$ and $\wedge$ (using the rule (r2)). Next we construct the sequence of theories taking some enumeration of the set `For` $(A_1, A_2, \ldots)$ and define

$$\alpha_{n+1} = \begin{cases} \alpha_n, \text{ if } \alpha_n \cup \{A_{n+1}\} \vdash_{\mathbf{CNL_4^2}} B, \\ \alpha_n \cup \{A_{n+1}\}, \text{ if } \alpha_n \cup \{A_{n+1}\} \nvdash_{\mathbf{CNL_4^2}} B. \end{cases}$$

Let α be the union of all α_n's. First we show that α is a prime theory such that $A \in \alpha$ and $B \notin \alpha$. $A \in \alpha$ by construction. Assume $B \in \alpha$, hence B was added to α_i on i-th stage of construction of the sequence, which is impossible. For the primeness suppose that α is not prime, i. e. $C \vee D \in \alpha$, but $C \notin \alpha$ and $D \notin \alpha$. This means that both extensions $\alpha \cup \{C\}$ and $\alpha \cup \{D\}$ contain B. Then there is a conjunctions of formulas form α, say E, such that $E \wedge C \vdash_{\mathbf{CNL_4^2}} B$ and $E \wedge D \vdash_{\mathbf{CNL_4^2}} B$. From this, using $(r3)$, we derive $(E \wedge C) \vee (E \wedge D) \vdash_{\mathbf{CNL_4^2}} B$. Then, using (a8) and $(r1)$, we have $E \wedge (C \vee D) \vdash_{\mathbf{CNL_4^2}} B$, so $B \in \alpha$.

Finally, α is also c-normal. Indeed, if for some k, $A_k \in \alpha$ and $\sim^2 A_k \in \alpha$, then there is an α_i which contains $A_k \wedge \sim^2 A_k$ as well as B, due to axiom schema $A \wedge \sim^2 A \vdash B$, contrary to the assumption. On the other hand, primeness of α and derivable schema $B \vdash_{\mathbf{CNL_4^2}} A \vee \sim^2 A$ guarantee that for each A_k, one of two formulas, A_k and $\sim^2 A_k$, belongs to α. □

5.3. Completeness for $\mathbf{CNL_4^2}$

Recall that $\mathcal{A}$ denotes the set $\{\mathbf{TU}, \mathbf{F}\}$. We can express our truth-values in terms of $\mathcal{A}$ and $\mathcal{D}$ sets via the following expressions:

$$\begin{aligned} v(A) = \mathbf{T} &\iff v(A) \in \mathcal{D} \text{ and } v(A) \notin \mathcal{A}, \\ v(A) = \mathbf{TU} &\iff v(A) \in \mathcal{D} \text{ and } v(A) \in \mathcal{A}, \\ v(A) = \mathbf{F} &\iff v(A) \notin \mathcal{D} \text{ and } v(A) \in \mathcal{A}, \\ v(A) = \mathbf{FU} &\iff v(A) \notin \mathcal{D} \text{ and } v(A) \notin \mathcal{A}. \end{aligned}$$

It is not difficult to verify the next lemma, having in mind the interpretations of propositional connectives.

LEMMA 5.7. *Let $A, B \in$ `For`, and v be a $\mathbf{CNL_4^2}$-valuation. Then, the following statements hold:*

(1) $v({\sim}A) \in \mathcal{D} \iff v(A) \notin \mathcal{A}$,

(2) $v({\sim}A) \in \mathcal{A} \iff v(A) \in \mathcal{D}$,

(3) $v(A \wedge B) \in \mathcal{D} \iff v(A) \in \mathcal{D}$ *and* $v(B) \in \mathcal{D}$,

(4) $v(A \wedge B) \in \mathcal{A} \iff v(A) \in \mathcal{A}$ *or* $v(B) \in \mathcal{A}$,

(5) $v(A \vee B) \in \mathcal{D} \iff v(A) \in \mathcal{D}$ *or* $v(B) \in \mathcal{D}$,

(6) $v(A \vee B) \in \mathcal{A} \iff v(A) \in \mathcal{A}$ *and* $v(B) \in \mathcal{A}$.

Now we turn to the definition of a $\mathbf{CNL_4^2}$-canonical valuation.

DEFINITION 5.8. For each c-normal prime theory α and propositional variable p we define a $\mathbf{CNL_4^2}$-canonical valuation v^c as a mapping $\mathtt{Var} \mapsto 4\mathcal{Q}$ satisfying the following expressions:

(1) $v^c(p) \in \mathcal{D} \iff p \in \alpha$;

(2) $v^c(p) \in \mathcal{A} \iff {\sim}^3 p \in \alpha$;

We define a unique extension of v^c to the set of all formulas in the usual way and denote this extension by v^c as well. We prove that extended valuation behaves as expected with respect to the c-normal prime theories.

LEMMA 5.9 (Canonical Valuation Lemma for $\mathbf{CNL_4^2}$). *For each c-normal prime theory α, formula A and extended canonical $\mathbf{CNL_4^2}$-valuation v^c the following statements hold:*

(1) $v^c(A) \in \mathcal{D} \iff A \in \alpha$,

(2) $v^c(A) \in \mathcal{A} \iff {\sim}^3 A \in \alpha$.

PROOF: By induction on the structure of a formula A. The base case when A is a propositional variable follows from the definition 5.8. Let us explore the cases for the complex formulas. The induction hypothesis ('IH' in the sequel) claims that lemma is true for their proper subformulas. We also use the two basic properties of theories, namely, their closure under conjunction and the relation $\vdash_{\mathbf{CNL_4^2}}$ throughout the proof.

CASE $A = {\sim}B$.

$v^c({\sim}B) \in \mathcal{D} \overset{\text{lem.5.7}}{\iff} v^c(B) \notin \mathcal{A} \overset{\text{IH}}{\iff} {\sim}^3 B \notin \alpha \overset{c\text{-norm.}}{\iff} {\sim}B \in \alpha$.

$v^c({\sim}B) \in \mathcal{A} \overset{\text{lem.5.7}}{\iff} v^c(B) \in \mathcal{D} \overset{\text{IH}}{\iff} B \in \alpha \overset{\text{(a9),(a10)}}{\iff} {\sim}^4 B \in \alpha$.

CASE $A = B \wedge C$.

$v^c(B \wedge C) \in \mathcal{D} \overset{\text{lem.5.7}}{\iff} v^c(B) \in \mathcal{D}$ and $v^c(C) \in \mathcal{D} \overset{\text{IH}}{\iff} B \in \alpha$ and $C \in \alpha \overset{\text{df.}\alpha,\text{(a1),(a2)}}{\iff} B \wedge C \in \alpha$.

$v^c(B \wedge C) \in \mathcal{A} \overset{\text{lem.5.7}}{\iff} v^c(B) \in \mathcal{A}$ or $v^c(C) \in \mathcal{A} \overset{\text{IH}}{\iff} {\sim}^3 B \in \alpha$ or ${\sim}^3 C \in \alpha \overset{c\text{-norm.}}{\iff} {\sim}B \notin \alpha$ or ${\sim}C \notin \alpha \overset{\text{(a1),(a2),(a5),(b1)}}{\iff} {\sim}(B \wedge C) \notin \alpha \overset{c\text{-norm.}}{\iff} {\sim}^3(B \wedge C) \in \alpha$.

CASE $A = B \vee C$.

$v^c(B \vee C) \in \mathcal{D} \overset{\text{lem.5.7}}{\iff} v^c(B) \in \mathcal{D}$ or $v^c(C) \in \mathcal{D} \overset{\text{IH}}{\iff} B \in \alpha$ or $C \in \alpha \overset{\text{(a3),(a4),prim.}}{\iff} B \vee C \in \alpha$.

$v^c(B \vee C) \in \mathcal{A} \overset{\text{lem.5.7}}{\iff} v^c(B) \in \mathcal{A}$ and $v^c(C) \in \mathcal{A} \overset{\text{IH}}{\iff} {\sim}^3 B \in \alpha$ and ${\sim}^3 C \in \alpha \overset{c\text{-norm.}}{\iff} {\sim}B \notin \alpha$ and ${\sim}C \notin \alpha \overset{\text{(a3),(a4),(a6),(b2),prim.}}{\iff} {\sim}(B \vee C) \notin \alpha \overset{c\text{-norm.}}{\iff} {\sim}^3(B \vee C) \in \alpha$. □

THEOREM 5.10 (Completeness for $\mathbf{CNL}_4^2$). *For any formulas A and B of the language $\mathcal{L}_{cnl}$, the following holds:*

$$A \vDash_{\mathbf{CNL}_4^2} B \Rightarrow A \vdash B \textit{ is } \mathbf{CNL}_4^2\textit{-provable.}$$

PROOF: Suppose $A \vdash B$ is not $\mathbf{CNL}_4^2$-provable. Then, by Lemma 5.6, there is prime theory α such that $A \in \alpha$ and $B \notin \alpha$. Then, by Lemma 5.9, we know that $v^c(A) \in \mathcal{D}$ but $v^c(B) \notin \mathcal{D}$, so $A \nvDash_{\mathbf{CNL}_4^2} B$. □

5.4. Comleteness for $\mathbf{CNLL}_4^2$

Let $\mathcal{B}$ denote the set $\{\mathbf{T}, \mathbf{FU}\}$. The next lemma is rather straightforward consequence of the semantic definitions for the propositional connectives.

LEMMA 5.11. *For any $A, B \in$ `For`, a $\mathbf{CNLL}_4^2$-valuation v the following statements hold:*

(1) $v({\sim}A) \in \mathcal{D} \iff v(A) \in \mathcal{B}$,

(2) $v({\sim}A) \in \mathcal{B} \iff v(A) \notin \mathcal{D}$,

(3) $v(A \wedge B) \in \mathcal{D} \iff v(A) \in \mathcal{D}$ *and* $v(B) \in \mathcal{D}$,

(4) $v(A \wedge B) \in \mathcal{B} \iff v(A), v(B) \in \mathcal{D} \cap \mathcal{B}$ *or* $v(A) \in \mathcal{B} \setminus \mathcal{D}$ *or* $v(B) \in \mathcal{B} \setminus \mathcal{D}$,

(5) $v(A \vee B) \in \mathcal{D} \iff v(A) \in \mathcal{D}$ *or* $v(B) \in \mathcal{D}$,

(6) $v(A \vee B) \in \mathcal{B} \iff v(A), v(B) \in \mathcal{B} \setminus \mathcal{D}$ *or* $v(A) \in \mathcal{D} \cap \mathcal{B}$ *or* $v(B) \in \mathcal{D} \cap \mathcal{B}$.

Definition 5.12. For each c-normal prime theory α and propositional variable p we define a $\mathbf{CNLL_4^2}$-canonical valuation v^c as a mapping $\mathtt{Var} \mapsto 4\mathcal{LQ}$ satisfying the following expressions:

(1) $v^c(p) \in \mathcal{D} \iff p \in \alpha$;

(2) $v^c(p) \in \mathcal{B} \iff {\sim}p \in \alpha$;

Again, we need to extend a canonical valuation to the whole set `For` and prove the canonical valuation lemma.

Lemma 5.13 (Canonical Valuation Lemma for $\mathbf{CNLL_4^2}$). *For each c-normal prime theory α, formula A and extended canonical $\mathbf{CNLL_4^2}$-valuation v^c the following statements hold:*

(1) $v^c(A) \in \mathcal{D} \iff A \in \alpha$,

(2) $v^c(A) \in \mathcal{B} \iff {\sim}A \in \alpha$.

Proof: By induction on the structure of a formula. Propositional variables case immediately follows from the definition of v^c.

Case $A = {\sim}B$.

$v^c({\sim}B) \in \mathcal{D} \overset{\text{lem. 5.11}}{\iff} v^c(B) \in \mathcal{B} \overset{\text{IH}}{\iff} {\sim}B \in \alpha$.

$v^c({\sim}B) \in \mathcal{B} \overset{\text{lem. 5.11}}{\iff} v^c(B) \notin \mathcal{D} \overset{\text{IH}}{\iff} B \notin \alpha \overset{c\text{-norm.}}{\iff} {\sim}^2 B \in \alpha$.

Case $A = B \wedge C$.

$v^c(B \wedge C) \in \mathcal{D} \overset{\text{lem. 5.11}}{\iff} v^c(B) \in \mathcal{D}$ and $v^c(C) \in \mathcal{D} \overset{\text{IH}}{\iff} B \in \alpha$ and $C \in \alpha \overset{\text{df.}\alpha,\text{(a1)},\text{(a2)}}{\iff} B \wedge C \in \alpha$.

($\Rightarrow$) Let $v^c(B \wedge C) \in \mathcal{B}$. By Lemma 5.11 we have to explore three sub-cases. (i) From $[v^c(B) \in \mathcal{B}$ and $v^c(C) \in \mathcal{B}]$ and IH we get ${\sim}B \in \alpha$ and ${\sim}C \in \alpha$, thus by the $\wedge$-closure of α and the axiom scheme (a5), ${\sim}(B \wedge C) \in \alpha$. (ii) If $[v^c(B) \notin \mathcal{D}$ and $v^c(B) \in \mathcal{B}]$ then IH gives $B \notin \alpha$ and ${\sim}B \in \alpha$. By c-normality of α, ${\sim}^2 B \in \alpha$. Thus, by the axiom schema (c3), ${\sim}(B \wedge C) \in \alpha$. (iii) If $[v(C) \notin \mathcal{D}$ and $v(C) \in \mathcal{B}]$ we similarly get ${\sim}C \wedge {\sim}^2 C \in \alpha$, so ${\sim}(C \wedge B) \in \alpha$ and, finally, by (c7), ${\sim}(B \wedge C) \in \alpha$.

($\Leftarrow$) Suppose $\sim(B \wedge C) \in \alpha$, then, by (c2), $\sim B \vee \sim C \in \alpha$, so, by primeness of α, $\sim B \in \alpha$ or $\sim C \in \alpha$. Let us consider the case $\sim B \in \alpha$. According to IH, $v^c(B) \in \mathcal{B}$, but this is not enough to assert $v^c(B \wedge C) \in \mathcal{B}$. So, we should examine the position of B relative to the theory α. Suppose $B \in \alpha$. By the $\wedge$-closure of α, $B \wedge \sim B \in \alpha$. Using the axiom schema (c4) we get $\sim(B \vee C) \in \alpha$ which, along with (c8), and $\wedge$-closure of α again, implies $\sim B \wedge \sim C \in \alpha$, hence $\sim C \in \alpha$. By IH, $v^c(C) \in \mathcal{B}$, so $v^c(B \wedge C) \in \mathcal{B}$. Next assume $B \notin \alpha$. Applying IH we then have $v^c(B) \notin \mathcal{D}$. This means that $v^c(B) = \mathbf{FU}$, so $v^c(B \wedge C) \in \mathcal{B}$. Similarly for $\sim C \in \alpha$.

Case $A = B \vee C$.

$v^c(B \vee C) \in \mathcal{D} \overset{\text{lem. 5.11}}{\iff} v^c(B) \in \mathcal{D}$ or $v^c(C) \in \mathcal{D} \overset{\text{IH}}{\iff} B \in \alpha$ or $C \in \alpha \overset{\text{(a3),(a4),prim.}}{\iff} B \vee C \in \alpha$.

($\Rightarrow$) Assume $v^c(B \vee C) \in \mathcal{B}$. Then, according to Lemma 5.11, we have two disjunctive subcases. First assume [$v^c(B) \in \mathcal{B}$ and $v^c(C) \in \mathcal{B}$]. It is enough to get $\sim B \in \alpha$ and $\sim C \in \alpha$ by IH and then $\sim(B \vee C) \in \alpha$ using (c1). The proof for second subcase is accomplished by the same reasoning.

($\Leftarrow$) Suppose $\sim(B \vee C) \in \alpha$. By the axiom (a6) and primeness of α we then obtain $\sim B \in \alpha$ or $\sim C \in \alpha$. Let us consider the first of the disjunctive sub-cases. From IH it follows that $v^c(B) \in \mathcal{B}$. But to get the required assertion $v^c(B \vee C) \in \mathcal{B}$ we need more information. Applying (c6) and then (c5) to $\sim(B \vee C) \in \alpha$ we get $\sim C \vee B \in \alpha$. Primeness of α and IH give $v^c(C) \in \mathcal{B}$ or $v^c(B) \in \mathcal{D}$. In both of these cases, taking into account $v^c(B) \in \mathcal{B}$, we end with $v^c(B \vee C) \in \mathcal{B}$. Analogues reasoning provides the proof in case when $\sim C \in \alpha$. $\square$

Theorem 5.14 (Completeness for $\mathbf{CNLL_4^2}$). *For any formulas A and B of the language $\mathcal{L}_{cnl}$,*

$$A \vDash_{\mathbf{CNLL_4^2}} B \Rightarrow A \vdash B \textit{ is } \mathbf{CNLL_4^2}\textit{-provable.}$$

Proof: The same as in the previous theorem for $\mathbf{CNL_4^2}$. $\square$

6. Conclusion

Although we have studied probably the most natural logics of paired cyclic negations, the whole picture is still waiting to be explored. Even the framework of the four-valued semantics gives some possible directions for the further investigations. Specifically, one can choose other sets of the designated truth values or combine the different collections of designated and anti-designated truth values. On the other hand, alternative definitions of the consequence relation are also possible. To obtain the more abstract results, paired cyclic negations could be put into more general lattice structures, even not necessary finitely based. Having in mind ability to simulate the other negation-like operations, the potential relationships between logical systems appear to be of the main interest.

References

[1] N. D. Belnap, *A Useful Four-Valued Logic*, [in:] J. M. Dunn, G. Epstein (eds.), **Modern Uses of Multiple-Valued Logic**, Springer Netherlands, Dordrecht (1977), pp. 5–37, DOI: https://doi.org/10.1007/978-94-010-1161-7_2.

[2] N. D. Belnap, *How a Computer Should Think*, [in:] H. Omori, H. Wansing (eds.), **New Essays on Belnap-Dunn Logic**, Springer International Publishing, Cham (2019), pp. 35–53, DOI: https://doi.org/10.1007/978-3-030-31136-0_4.

[3] J. V. Benthem, *Logical dynamics meets logical pluralism?*, **The Australasian Journal of Logic**, vol. 6 (2008), pp. 182–209, DOI: https://doi.org/10.26686/ajl.v6i0.1801.

[4] M. L. D. Chiara, R. Giuntini, R. Greechie, **Reasoning in Quantum Theory: Sharp and Unsharp Quantum Logics**, vol. 22 of Trends in Logic, Springer Science & Business Media (2013), DOI: https://doi.org/https://doi.org/10.1007/978-94-017-0526-4.

[5] M. L. D. Chiara, R. Giuntini, R. Leporini, *Logics from quantum computation*, **International Journal of Quantum Information**, vol. 03(02) (2005), pp. 293–337, DOI: https://doi.org/10.1142/s0219749905000943.

[6] D. Deutsch, A. Ekert, R. Lupacchini, *Machines, logic and quantum physics*, **Bulletin of Symbolic Logic**, vol. 6(3) (2000), pp. 265–283, DOI: https://doi.org/10.2307/421056.

[7] Í. M. L. D'Ottaviano, H. de Araujo Feitosa, *Many-valued logics and translations*, **Journal of Applied Non-Classical Logics**, vol. 9(1) (1999), pp. 121–140, DOI: https://doi.org/10.1080/11663081.1999.10510960.

[8] J. M. Dunn, *Star and perp: Two treatments of negation*, **Philosophical Perspectives**, vol. 7 (1993), pp. 331–357, DOI: https://doi.org/10.2307/2214128.

[9] J. M. Dunn, *Partiality and its dual*, **Studia Logica**, vol. 66 (2000), pp. 5–40, DOI: https://doi.org/10.1023/A:1026740726955.

[10] J. M. Dunn, G. Hardegree, **Algebraic Methods in Philosophical Logic**, Oxford University Press (2001).

[11] H. A. Feitosa, I. M. L. D'Ottaviano, *Conservative translations*, **Annals of Pure and Applied Logic**, vol. 108(1–3) (2001), pp. 205–227, DOI: https://doi.org/10.1016/s0168-0072(00)00046-4.

[12] R. French, **Translational embeddings in modal logic**, Ph.D. thesis, Monash University (2010).

[13] D. M. Gabbay, H. Wansing (eds.), **What is a negation?**, vol. 13 of Applied Logic Series, Springer Netherlands (1999), DOI: https://doi.org/10.1007/978-94-015-9309-0.

[14] L. Humberstone, *Negation by iteration*, **Theoria**, vol. 61(1) (1995), pp. 1–24, DOI: https://doi.org/10.1111/j.1755-2567.1995.tb00489.x.

[15] N. Kamide, *Paraconsistent double negations as classical and intuitionistic negations*, **Studia Logica**, vol. 105(6) (2017), pp. 1167–1191, DOI: https://doi.org/10.1007/s11225-017-9731-2.

[16] S. P. Odintsov, **Constructive Negations and Paraconsistency**, Springer Netherlands (2008), DOI: https://doi.org/10.1007/978-1-4020-6867-6.

[17] H. Omori, K. Sano, *Generalizing functional completeness in Belnap-Dunn logic*, **Studia Logica**, vol. 103(5) (2015), pp. 883–917, DOI: https://doi.org/10.1007/s11225-014-9597-5.

[18] H. Omori, H. Wansing, *On contra-classical variants of Nelson logic N4 and its classical extension*, **The Review of Symbolic Logic**, vol. 11 (2018), pp. 805–820, DOI: https://doi.org/10.1017/s1755020318000308.

[19] F. Paoli, *Bilattice Logics and Demi-Negation*, [in:] H. Omori, H. Wansing (eds.), **New Essays on Belnap–Dunn Logic**, Springer International Publishing, Cham (2019), pp. 233–253, DOI: https://doi.org/10.1007/978-3-030-31136-0_14.

[20] E. Post, *Introduction to a general theory of elementary propositions*, **American Journal of Mathematics**, vol. 43 (1921), pp. 163–185, DOI: https://doi.org/10.2307/2370324.

[21] H. Rasiowa, **An Algebraic Approach to Non-Classical Logics**, vol. 78 of Studies in Logic and Foundations of Mathematics, North-Holland, Amsterdam (1974).

[22] P. Ruet, *Complete set of connectives and complete sequent calculus for Belnap's logic*, *Tech. rep.*, École Normale Supérieure, Logic Colloquium 96, Document LIENS-96–28 (1996).

[23] Y. Shramko, J. M. Dunn, T. Takenaka, *The trilattice of constructive truth values*, **Journal of Logic and Computation**, vol. 11(6) (2001), pp. 761–788, DOI: https://doi.org/10.1093/logcom/11.6.761.

[24] Y. Shramko, H. Wansing, *Some useful 16-valued logics: How a computer network should think*, **Journal of Philosophical Logic**, vol. 34(2) (2005), pp. 121–153, DOI: https://doi.org/10.1007/s10992-005-0556-5.

[25] J. van Benthem, **Logical Dynamics of Information and Interaction**, Cambridge University Press (2011), DOI: https://doi.org/10.1017/cbo9780511974533.

[26] H. Wansing (ed.), **Negation: A notion in focus**, vol. 7 of Perspectives in Analytical Philosophy, W. De Gruyter (1996), DOI: https://doi.org/10.1515/9783110876802.

[27] D. Zaitsev, *A few more useful 8-valued logics for reasoning with tetralattice* $EIGHT_4$, **Studia Logica**, vol. 92(2) (2009), pp. 265–280, DOI: https://doi.org/10.1007/s11225-009-9198-x.

[28] D. Zaitsev, O. Grigoriev, *Two kinds of truth – one logic*, **Logical Investigations**, vol. 17 (2011), pp. 121–139, DOI: https://doi.org/https://doi.org/10.21146/2074-1472-2011-17-0-121-139, (in Russian).

[29] D. Zaitsev, Y. Shramko, *Bi-facial truth: A case for generalized truth values*, **Studia Logica**, vol. 101(6) (2013), pp. 1299–1318, DOI: https://doi.org/10.1007/s11225-013-9534-z.

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Bulletin of the **S**ection of **L**ogic
Volume 51/4 (2022), pp. 535–554
https://doi.org/10.18778/0138-0680.2022.22

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EQUIVALENTIAL ALGEBRAS WITH CONJUNCTION ON DENSE ELEMENTS

Abstract

We study the variety generated by the three-element equivalential algebra with conjunction on the dense elements. We prove the representation theorem which let us construct the free algebras in this variety.

Keywords: Intuitionistic logic, Fregean varieties, equivalential algebras, dense elements.

2020 Mathematical Subject Classification: 08B20, 03G25.

1. Introduction

According to [6], there are only finitely many polynomial clones on a finite algebra which generates a congruence permutable Fregean variety. As we will show in the paper, if a three-element algebra $\mathbf{A}$ generates a congruence permutable Fregean variety, then the universe of $\mathbf{A}$ with the natural order is a chain. Moreover, also the lattice of congruences on $\mathbf{A}$ is a three-element chain. It is known that congruence permutable Fregean varieties are congruence modular, so we can consider in this case the commutator operation. By [6, Corollary 2.8], due to the behavior of the commutator operation on a three-element algebra, we can distinguish four polynomially nonequivalent algebras, that generate congruence permutable Fregean varieties.

Presented by: Janusz Czelakowski
Received: August 3, 2022
Published online: October 25, 2022

Two of them are well known: the three-element equivalential algebra and the three-element Brouwerian semilattice. The equivalential algebras are solvable, so they are of type 2 ([6, p. 606]) in the sense of Tame Congruence Theory of Hobby and McKenzie [4]. However, the Brouwerian semilattices are congruence distributive and so they are of type 3. Equivalential algebras and Brouwerian semilattices have already been carefully studied, both when it comes to the construction of the n-generated free algebras, as well as the cardinality of these algebras for small n and for some subvarieties, see [8, 19, 14, 15] for the equivalential algebras and [9] for the Brouwerian semilattices.

In the other two cases we are dealing with a mixed type. In the first case, we have type 3 at the top of congruence lattice and type 2 at its bottom, see Figure 1. An example of algebra, which meets these conditions is the three-element equivalential algebra with conjunction on the regular elements. The variety generated by this algebra was investigated in [11], where its properties, the representation theorem, the construction of the free algebra and the free spectrum were given.

The aim of this paper is to study the variety generated by the three-element algebra, in which the commutator operation behaves in the opposite way: type 2 is at the top of congruence lattice and type 3 at its bottom. Such structure is the subreduct of the three-element Heyting algebra, with the equivalence operation and the second binary operation which is conjunction on the dense elements.

Both the dense elements as well as the regular elements play an important role in the study of the relation between classical and intuitionistic logic. They appear indirectly in the Glivenko theorem according to which a formula φ is a tautology of classical propositional calculus iff its double negation (i.e. the regularization of φ) is a tautology of intuitionistic propositional calculus. An algebraic version of this theorem refers directly to dense elements: we divide a Heyting algebra by the filter of all dense elements obtaining a Boolean algebra [12, p. 132].

2. Preliminary

Let $\mathbf{A}$ be an algebra. We say that $\mu \in \operatorname{Con}\mathbf{A}$ is **completely meet-irreducible** if $\mu \neq A^2$ and for any family $\{\mu_i : i \in I\} \subseteq \operatorname{Con}\mathbf{A}$ such that $\mu = \bigcap_{i\in I} \mu_i$, we have $\mu = \mu_i$ for some $i \in I$. If μ is completely meet-irreducible, then there exists the unique cover of μ in Con **A**, denoted by μ^+. We will denote by Cm(**A**) the set of all completely meet-irreducible congruences on **A**. Similarly, we can define a **completely join-irreducible** congruence ν and the unique subcover of ν in Con **A**, denoted by ν^-. Let $\Theta(a, b)$ denote the congruence generated by (a, b).

Now, we will recall the most important facts related to the concept of the commutator. At the beginning we need the following definition:

DEFINITION 2.1 ([10, p. 252]). Let α, β, η be congruences of an algebra **A**. We say that α **centralizes** β modulo η, written: $C(\alpha, \beta; \eta)$, iff for all $n \geq 1$, and for every: $t \in \mathrm{Clo}_{n+1} A$, $(a, b) \in \alpha$ and $(c_1, d_1), \ldots, (c_n, d_n) \in \beta$ we have:

$$t(a, c_1, \ldots, c_n) \equiv_\eta t(a, d_1, \ldots, d_n) \text{ iff } t(b, c_1, \ldots, c_n) \equiv_\eta t(b, d_1, \ldots, d_n).$$

DEFINITION 2.2 ([10, p. 252]). For congruences α and β of $\mathbf{A} \in \mathcal{V}$, where $\mathcal{V}$ is a congruence modular variety, we define their **commutator**, denoted $[\alpha, \beta]$, to be the smallest congruence η of **A** for which α centralizes β modulo η, i. e., $\eta = \bigwedge\{\phi : C(\alpha, \beta; \phi)\}$.

DEFINITION 2.3 ([2, p. 35, 47]). Let $\mathbf{A} \in \mathcal{V}$, where $\mathcal{V}$ is a congruence modular variety, $\alpha, \beta \in \mathrm{Con}\, \mathbf{A}$ and $\alpha \leq \beta$. Then:

1. β is called **Abelian over** α if $[\beta, \beta] \leq \alpha$,
2. β is called **Abelian** if $[\beta, \beta] = 0_{\mathbf{A}}$,
3. **A** is called **Abelian** if $[1_{\mathbf{A}}, 1_{\mathbf{A}}] = 0_{\mathbf{A}}$.

We say that an algebra **A** satisfies the condition (**C1**) if $\alpha \wedge [\beta, \beta] = [\alpha \wedge \beta, \beta]$ for all $\alpha, \beta \in \mathrm{Con}\, \mathbf{A}$.

Remark 2.4 ([5, p. 49]). In congruence modular varieties the condition (**C1**) gives $[\alpha, \beta] = (\alpha \wedge [\beta, \beta]) \vee (\beta \wedge [\alpha, \alpha])$, for $\alpha, \beta \in \mathrm{Con}\, \mathbf{A}$, so the commutator operation on congruences of **A** is uniquely determined by the diagonal, i. e., by elements of the form $[\alpha, \alpha]$.

If $\mathbf{A} \in \mathcal{V}$ and $\mathcal{V}$ is a congruence modular variety, we can define the following notion:

DEFINITION 2.5 ([10, p. 252]). The **centralizer** of β modulo α, denoted $(\alpha : \beta)$, is the largest congruence γ of **A** such that γ centralizes β modulo α, i. e., $\gamma = \bigvee\{\phi : C(\phi, \beta; \alpha)\}$.

Now, we give basic information about Fregean varieties.

Definition 2.6 ([6, p. 597]). An algebra **A** with a distinguished constant term 1 is called **Fregean** if **A** is:

1. **1-regular**, i. e., $1/\alpha = 1/\beta$ implies $\alpha = \beta$ for all $\alpha, \beta \in \operatorname{Con} \mathbf{A}$,
2. **congruence orderable**, i. e., $\Theta_{\mathbf{A}}(1, a) = \Theta_{\mathbf{A}}(1, b)$ implies $a = b$ for all $a, b \in A$.

A variety $\mathcal{V}$ is said to be Fregean if all its algebras are Fregean. Natural examples of Fregean varieties are: equivalential algebras, Boolean algebras, Heyting algebras, Brouwerian semillatices or Hilbert algebras. Fregean varieties are closely related with the Fregean logics, see [1].

Congruence orderability allows us to introduce a natural partial order on the universe of **A** in the following way: $a \leq b$ iff $\Theta_{\mathbf{A}}(1, b) \subseteq \Theta_{\mathbf{A}}(1, a)$. Clearly, 1 is the greatest element in this order. From 1-regularity it follows that the Fregean varieties are congruence modular, see [3].

Next, we recall an important theorem, which characterizes subdirectly irreducible algebras in Fregean varieties.

Proposition 2.7 ([16, Proposition 3.1], [6, Lemma 2.1]). Let **A** be an algebra from a Fregean variety $\mathcal{V}$. Then **A** is subdireclty irreducible iff there is the largest non-unit element $*$ in A. Moreover, the monolith μ of **A** has the form $1/\mu = \{*, 1\}$ and all other cosets with respects to μ are one-element.

The Fregean varieties meet the condition (**C1**). Moreover, they satisfy the stronger condition (**SC1**):

Definition 2.8 ([6, p. 602]). If μ is the monolith of a subdirectly irreducible algebra **A** from a Fregean variety then the centralizer $(0 : \mu)$ does not exceed μ.

Definition 2.9. An **equivalential algebra** is an algebra $(A, \leftrightarrow, 1)$ of type $(2, 0)$ that is a subreduct of a Heyting algebra with the binary operation $\leftrightarrow$ given by $x \leftrightarrow y := (x \to y) \wedge (y \to x)$.

In this paper, we adopt the convention of associating to the left and ignoring (or replacing with "$\cdot$") the symbol of equivalence operation. In 1975 J. K. Kabziński and A. Wroński proved that the class **E** of all equivalential algebras is equationally definable by identities: $xxy = y$, $xyzz = (xz)(yz)$, $(xy)(xzz)(xzz) = xy$, and so it forms a variety [8].

We know from [6, p. 598] that $\mathbf{E}$ is congruence permutable. Moreover, the following theorem is true:

THEOREM 2.10 ([6, Theorem 3.8]). *Let $\mathcal{V}$ be a congruence permutable Fregean variety. Then there exists a binary term $\leftrightarrow$ such that for every $\mathbf{A} \in \mathcal{V}$:*

1. $(A, \leftrightarrow, 1)$ *is an equivalential algebra;*

2. $\leftrightarrow$ *is a principal congruence term of* $\mathbf{A}$*, i. e.,* $(a, b) \in \alpha$ *iff* $(1, a \leftrightarrow b) \in \alpha$ *for every* $\alpha \in \mathrm{Con}\,\mathbf{A}$.

If $\mathcal{V}$ is a congruence permutable Fregean variety and $\mathbf{A} \in \mathcal{V}$, then we will denote an equivalential reduct of $\mathbf{A}$ by $\mathbf{A}^e$.

3. The clones of polynomials of a three-element algebra, which generates a congruence permutable Fregean variety

It is known that there exist only two polynomially nonequivalent algebras defined on a two-element set and generating a congruence permutable Fregean variety [6, p. 640]. We examine an analogous situation, but for a three-element set. The first question concerns the number of such polynomially nonequivalent algebras. By Theorem 2.10, for every algebra $\mathbf{A}$ from a congruence permutable Fregean variety there is a binary term $\leftrightarrow$ such as $\mathbf{A}^e$ is an equivalential algebra. In order to answer our question we first need to consider a three-element algebra $\mathbf{A}$ with a universe $\{1, a, b\}$, with the equivalence operation $\leftrightarrow$ and a constant term 1, which is the greatest element in $\mathbf{A}$ in the natural order.

PROPOSITION 3.1. Let $\mathbf{A}$ generate a congruence permutable Fregean variety with a constant term 1 and let $|A| = 3$. Then:

1. A with the natural order is a chain,

2. $(\mathrm{Con}\,\mathbf{A}, \vee, \wedge)$ with the order $\subseteq$ is a three-element chain

PROOF: (1) Let $A = \{1, a, b\}$. Without loss of generality we can assume that $a \leftrightarrow b = a$, since otherwise (i.e. $a \leftrightarrow b = b$) the situation would be analogous. From Theorem 2.10 we have $\Theta(1, a \leftrightarrow b) = \Theta(a, b)$. Thus: $\Theta(1, a) = \Theta(1, a \leftrightarrow b) = \Theta(a, b)$. As $\Theta(1, b) \subseteq \Theta(1, a)$, so from a congruence orderability it follows that $a \leq b$ and consequently $a < b < 1$.

(2) Similarly, from a congruence orderability and inequalities $a < b < 1$ we get: $0_{\mathbf{A}} = \Theta(1, 1) \subsetneq \Theta(1, b) \subsetneq \Theta(1, a) = 1_{\mathbf{A}}$. Thus: $0_{\mathbf{A}} < \Theta(1, b) < 1_{\mathbf{A}}$. This completes the proof because in $\operatorname{Con} \mathbf{A}$ there are only principal congruences. □

Since $\{1, a, b\}$ with the natural order forms a chain, thus we adopt the convention that the smallest element in a three-element chain will be denoted by 0, and the middle element by $*$. We conclude from Proposition 2.7 that an algebra $\mathbf{A}$, which fulfills the assumptions of Proposition 3.1, is a subdirectly irreducible with the monolith $\Theta(1, *)$. Note also, that if a three-element algebra $\mathbf{A}$ comes from a congruence permutable Fregean variety, then $\Theta(x, y) = \Theta^e(x, y)$, for $x, y \in A$.

By [6, Corollary 2.8], the clone of polynomials of a finite algebra from a congruence permutable Fregean variety is uniquely determined by its congruence lattice expanded by the commutator operation, i. e., by the structure $\operatorname{Concom}(\mathbf{A}) := (\operatorname{Con} \mathbf{A}; \wedge, \vee, [\cdot, \cdot])$. Thus, the number of clones of polynomials of $\mathbf{A}$ depends on the behaviour of the commutator operation on a three-element lattice of congruences. There are four such possibilities, shown in the figure below.

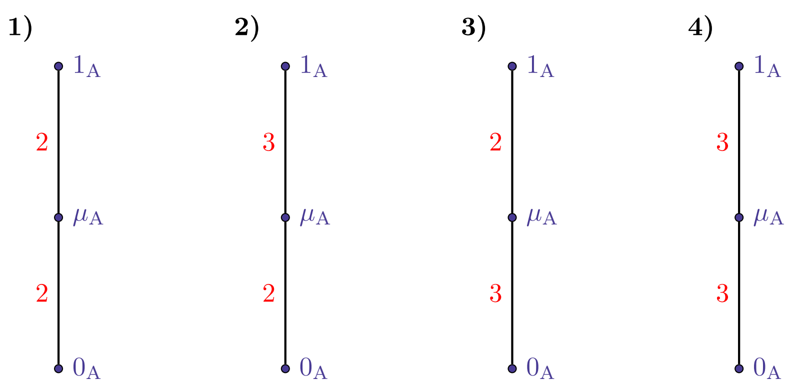

Figure 1.

The number 2 used in the figure means that a congruence α above this number is Abelian over α^-, where α^- denotes the unique subcover of α, i. e., $[\alpha, \alpha] = \alpha^-$. On the other hand, the number 3 means that a congruence α above this number fullfils: $[\alpha, \alpha] = \alpha$. Note that it follows from the condition (**SC1**) that the equality $[1_{\mathbf{A}}, 1_{\mathbf{A}}] = 0_{\mathbf{A}}$ is not possible, because it would lead to a contradiction, i. e., $(0_{\mathbf{A}} : \mu_{\mathbf{A}}) = 1_{\mathbf{A}}$.

An algebra, in which the commutator behaves as in the first case is the three-element equivalential algebra, whereas an algebra, in which the commutator behaves as in the fourth case is the three-element Brouwerian semilattice. An example corresponding to the second case is a three-element equivalential algebra with conjunction on the regular elements, described in [11]. In this article we will give an example of an algebra, in which the commutator behaves as in the third case.

4. Equivalential algebras with conjunction on the dense elements

In Heyting algebras we can consider both the dense elements and the regular elements. An element x is called: **regular** if $(x \to 0) \to 0 = x$, **dense** if $(x \to 0) \to 0 = 1$. The Glivenko theorem mentioned earlier, explains their role in studying of the reducts of the intuitionistic logics. To defined them in Heyting algebras we use the constant 0. In equivalential algebras we can define the regular and dense elements without using this constant. In this situation we say that an element $x \in A$ is regular if $xyy = x$ for all $y \in A$, and it is dense if there is a finite subset $\{y_1, y_2, \ldots, y_n\} \subseteq A$ such that $xy_1y_1y_2y_2 \ldots y_ny_n = 1$. If the equivalential algebra $\mathbf{A}$ is the reduct of the Heyting algebra, then both definitions coincide.

In Heyting algebras we can define an operation of the conjuction on the dense elements. Let us consider a subreduct of the Heyting algebra with the constant 1 and with two binary operation. The first is the equivalence operation, providing the congruence permutability, while the second operation is the conjuction on the dense elements. We will also limit our considerations to the three-element subreduct of the Heyting algebra. From Proposition 3.1 we know that the universe of this algebra with the natural order forms a chain. Finally, we get the following definition.

Definition 4.1. An **equivalential algebra with conjunction on the dense elements** is an algebra $\mathbf{D} := (\{0, *, 1\}, \cdot, d, 1)$ of type $(2, 2, 0)$, which is the reduct of the three-element Heyting algebra $\mathbf{H} = (\{0, *, 1\}, \wedge, \vee, \rightarrow, 0, 1)$ with an order: $0 < * < 1$, the constant 1, the equivalence operation $\cdot$ such that $x \cdot y := (x \rightarrow y) \wedge (y \rightarrow x)$, and an additional binary operation d such that $d(x, y) := x00x \wedge y00y$.

Note that $\mathbf{A}^e$ is an equivalential algebra and d is a binary commutative operation presented in the table below (on the right):

$\cdot$	1	$*$	0
1	1	$*$	0
$*$	$*$	1	0
0	0	0	1

d	1	$*$	0
1	1	$*$	1
$*$	$*$	$*$	$*$
0	1	$*$	1

We denote by $\mathcal{V}(\mathbf{D})$ the variety generated by $\mathbf{D}$. It is easy to see, that $\mathbf{D}$ is a subdirectly irreducible Fregean algebra with the monolith denoted by $\mu_{\mathbf{D}}$. Moreover, $\operatorname{Con} \mathbf{D} = \{0_{\mathbf{D}}, \mu_D, 1_{\mathbf{D}}\}$, where $0_{\mathbf{D}} < \mu_D < 1_{\mathbf{D}}$.

Remark 4.2. $\mathbf{D}$ has two nontrivial subalgebras:

$$\mathbf{2} := (\{1, 0\}, \cdot, d, 1), \text{ where } d \equiv 1,$$

$$\mathbf{2}^{\wedge} := (\{1, *\}, \cdot, d, 1), \text{ where } d(x, y) := x \wedge y.$$

Thus, the algebra $\mathbf{2}$ is a Boolean group and is abelian, while the algebra $\mathbf{2}^{\wedge}$ is a Boolean algebra without zero [18] and is not abelian. Note that $\mathbf{D}/\mu_{\mathbf{D}} \cong \mathbf{2}$, and, consequently, $\mathbf{A} \in HS(\mathbf{D})$ iff $\mathbf{A} \cong \mathbf{2}$ or $\mathbf{A} \cong \mathbf{2}^{\wedge}$ or $\mathbf{A} \cong \mathbf{D}$ for non-trivial $\mathbf{A} \in \mathcal{V}(\mathbf{D})$.

Now, applying [6, Theorem 2.10] we get immediately:

Proposition 4.3. $\mathcal{V}(\mathbf{D})$ is a Fregean variety.

Next, we look at the commutator operation in $\operatorname{Con} \mathbf{D}$.

Proposition 4.4.

1. $[\mu_{\mathbf{D}}, \mu_{\mathbf{D}}] = \mu_{\mathbf{D}}$,
2. $[1_{\mathbf{D}}, 1_{\mathbf{D}}] = \mu_{\mathbf{D}}$,
3. $(0_{\mathbf{D}} : \mu_{\mathbf{D}}) = 0_{\mathbf{D}}$,
4. $(\mu_{\mathbf{D}} : 1_{\mathbf{D}}) = 1_{\mathbf{D}}$.

Proof: (1) From the definition of the commutator we get:

$$1 = d(1,1) \equiv_{[\mu_{\mathbf{D}},\mu_{\mathbf{D}}]} d(1,*) = * \quad \text{iff}$$

$$* = d(*,1) \equiv_{[\mu_{\mathbf{D}},\mu_{\mathbf{D}}]} d(*,*) = *,$$

so $(1,*) \in [\mu_{\mathbf{D}},\mu_{\mathbf{D}}]$, thus $[\mu_{\mathbf{D}},\mu_{\mathbf{D}}] = \mu_{\mathbf{D}}$.

(2) Since $\mathbf{D}/\mu_{\mathbf{D}} \cong \mathbf{2}$, we get immediately from the general property of the commutator operation [2]:

$$\mu_{\mathbf{D}}/\mu_{\mathbf{D}} = [1_{\mathbf{D}}/\mu_{\mathbf{D}}, 1_{\mathbf{D}}/\mu_{\mathbf{D}}] = ([1_{\mathbf{D}}, 1_{\mathbf{D}}] \vee \mu_{\mathbf{D}})/\mu_{\mathbf{D}}.$$

Thus $[1_{\mathbf{D}}, 1_{\mathbf{D}}] \vee \mu_{\mathbf{D}} = \mu_{\mathbf{D}}$, and consequently $[1_{\mathbf{D}}, 1_{\mathbf{D}}] \subseteq \mu_{\mathbf{D}}$. From the equality $[\mu_{\mathbf{D}},\mu_{\mathbf{D}}] = \mu_{\mathbf{D}}$ we get $[1_{\mathbf{D}}, 1_{\mathbf{D}}] \subseteq \mu_{\mathbf{D}} \subseteq [\mu_{\mathbf{D}},\mu_{\mathbf{D}}]$, and therefore $[1_{\mathbf{D}}, 1_{\mathbf{D}}] = \mu_{\mathbf{D}}$.

$(3), (4)$ From Definition 2.3 and (1) and (2) we get that $1_{\mathbf{D}}$ is Abelian over $\mu_{\mathbf{D}}$, and $\mu_{\mathbf{D}}$ is not Abelian in $\mathbf{D}$. Thus, from [5, Lemma 21] we obtain the assertion. □

From the above proposition we get the following result:

Corollary 4.5. The algebra $\mathbf{D}$ is polynomially equivalent neither to the three-element equivalential algebra nor to the three-element Brouwerian semillatice.

Proposition 4.6. There are only three (up to isomorphism) nontrivial subdirectly irreducible algebras in $\mathcal{V}(\mathbf{D})$: $\mathbf{D}, \mathbf{2}, \mathbf{2}^{\wedge}$.

Proof: From Remark 4.2 we know that up to isomorphism the only nontrivial subdirectly irreducible algebras in $HS(\mathbf{D})$ are: $\mathbf{D}, \mathbf{2}, \mathbf{2}^{\wedge}$. Among them only $\mathbf{2}$ has an abelian monolith. Suppose that $\mathbf{B} := \{B, \cdot, d_B, 1\}$ is subdirectly irreducible in $\mathcal{V}(\mathbf{D})$. It follows from [2, Theorem 10.12] that there exists a subdirectly irreducible algebra $\mathbf{A} \in HS(\mathbf{D})$ such that either $\mathbf{B} \cong \mathbf{A}$ or $\mathbf{B}$ and $\mathbf{A}$ have abelian monoliths and $\mathbf{B}/(0_{\mathbf{B}} : \mu_{\mathbf{B}}) \cong \mathbf{A}/(0_{\mathbf{A}} : \mu_{\mathbf{A}})$. Thus $\mathbf{B} \in \{\mathbf{D}, \mathbf{2}, \mathbf{2}^{\wedge}\}$ (up to isomorphism) or $\mathbf{B}$ has an abelian monolith and $\mathbf{B}/(0_{\mathbf{B}} : \mu_{\mathbf{B}}) \cong \mathbf{2}/(0_{\mathbf{2}} : \mu_{\mathbf{2}})$. Assume that the second possibility holds. From (**SC1**) we get $(0_{\mathbf{B}} : \mu_{\mathbf{B}}) = \mu_{\mathbf{B}}$. Thus $\mathbf{B}/\mu_{\mathbf{B}} \cong \mathbf{2}/(0_{\mathbf{2}} : \mu_{\mathbf{2}})$. Since $\mathbf{2}/(0_{\mathbf{2}} : \mu_{\mathbf{2}}) = \mathbf{2}/\mu_{\mathbf{2}}$ is a trivial algebra, it follows from Proposition 2.7 that B with the natural order is the two-element chain, and so $B = \{1, 0\}$. Using identities $d_B(x,1) \approx d_B(x,x) \approx d_B(1,x)$ and $d_B(1,1) \approx 1$, true in $\mathcal{V}(\mathbf{D})$, we get $d_B(1,0) = d_B(0,1) = d_B(0,0)$. Suppose that $d_B(1,0) = 0$, then $d_B(x, y) = x \wedge y$, contrary to the fact that $\mathbf{B}$ has an abelian monolith. Thus $d_B(1, 0) = 1$, and so $d_B \equiv 1$. In consequence $\mathbf{B} \cong \mathbf{2}$, which completes the proof. □

Remark 4.7. It follows from Proposition 4.6 that all subdirectly irreducible algebras in $\mathcal{V}(\mathbf{D})$ belong to $S(\mathbf{D})$. Thus $\mathcal{V}(\mathbf{D}) = SP(\mathbf{D})$. In consequence, a quasivariety generated by $\mathbf{D}$ turns out to be a variety.

5. Frames for the algebras from $\mathcal{V}(\mathbf{D})$

Let $\mathbf{A} \in \mathcal{V}(\mathbf{D})$. Recall, that we denote by $\mathrm{Cm}(\mathbf{A})$ the set of all completely meet-irreducible congruences on $\mathbf{A}$. In this section we describe an additional structure (frame) on $\mathbf{Cm}(\mathbf{A})$. This structure is similar to the frames in the equivalential algebras with conjunction on the regular elements described in [11].

It follows from Proposition 4.6 that $\mu \in \mathrm{Cm}(\mathbf{A})$ iff $\mathbf{A}/\mu \cong \mathbf{k}$, for $\mathbf{k} \in \{\mathbf{D}, \mathbf{2}, \mathbf{2}^\wedge\}$. We use the following notation:

$$\overline{L} := \{\mu \in \mathrm{Cm}(\mathbf{A}) : \mathbf{A}/\mu \cong \mathbf{2}\},$$

$$\underline{L} := \{\mu \in \mathrm{Cm}(\mathbf{A}) : \mathbf{A}/\mu \cong \mathbf{D}\},$$

$$P := \{\mu \in \mathrm{Cm}(\mathbf{A}) : \mathbf{A}/\mu \cong \mathbf{2}^\wedge\},$$

$$L := \overline{L} \cup \underline{L}.$$

Proposition 5.1. Let $\mathbf{A} \in \mathcal{V}(\mathbf{D})$ and $\mu \in \mathrm{Cm}(\mathbf{A})$. If $\mathbf{A}/\mu \cong \mathbf{D}$, then $\mathbf{A}/\mu^+ \cong \mathbf{2}$.

Proof: Let $f : \mathbf{A}/\mu \to \mathbf{2}$ be the function given by $f(1/\mu) = f(*/\mu) = 1$ and $f(0/\mu) = 0$. Therefore f is a surjective homomorphism and $\ker f = \mu^+/\mu$. Thus $(\mathbf{A}/\mu)/(\mu^+/\mu) \cong \mathbf{2}$, and consequently $\mathbf{A}/\mu^+ \cong \mathbf{2}$. □

Corollary 5.2. Let $\mathbf{A} \in \mathcal{V}(\mathbf{D})$ and $\mu \in \mathrm{Cm}(\mathbf{A})$. Then $\mu \in P \cup \overline{L}$ iff $\mu \prec \mathbf{1_A}$ (i. e., $\mu^+ = \mathbf{1_A}$) and $\mu \in \underline{L}$ iff $\mu^+ \in \overline{L}$.

In consequence, the length of the longest chain in $\mathrm{Cm}(\mathbf{A})$ equals two.

Let $\mathbf{A} \in \mathcal{V}(\mathbf{R})$ and $\varphi, \psi \in \mathrm{Cm}(\mathbf{A})$. We introduce a relation on $\mathrm{Cm}(\mathbf{A})$ as follows (see [5, p. 51]):

$$\varphi \sim \psi \text{ iff the intervals } I[\varphi, \varphi^+] \text{ and } I[\psi, \psi^+] \text{ are projective.}$$

It is easy to see, that the relation $\sim$ is an equivalence relation on $\mathrm{Cm}(\mathbf{A})$.

From [17, Lemma 4.2, Corollary 3.7] it follows that the definition of the relation $\sim$ is equivalent to the following definition: $\varphi \sim \psi$ iff $\varphi^+ = \psi^+$ and $\varphi \bullet \psi \in \mathrm{Cm}(\mathbf{A})$, where $\varphi \bullet \psi = (\varphi \div \psi)' \cap \varphi^+$.

Definition 5.3. Let $\mathbf{A} \in \mathcal{V}(\mathbf{D})$. The structure $\mathbf{Cm}(\mathbf{A}) := (\mathrm{Cm}(\mathbf{A}), \leq, \sim)$ is called a **frame** of $\mathbf{A}$, where $\leq$ is the inclusion relation.

First, we show that the relation $\sim$ on $P \cup \underline{L}$ is an identity.

Proposition 5.4. Let $\mathbf{A} \in \mathcal{V}(\mathbf{D})$ and $\mu \in P \cup \underline{L}$. Then $|\mu/_\sim| = 1$.

Proof: Let $\mu \in P$. Then $\mu^+ = 1_\mathbf{A}$. Since $\mathbf{A}/\mu$ is not Abelian, so from [2, Proposition 3.7] we get that $1_\mathbf{A}$ is not Abelian over μ. Thus μ^+ is not Abelian over μ. Let now $\mu \in \underline{L}$. Then $\mathbf{A}/\mu \cong \mathbf{D}$. Since $\mu_\mathbf{D}$ is not Abelian, thus μ^+/μ, the monolith of $\mathbf{A}/\mu$, is also not Abelian. In both cases, from [5, Lemma 21] we have $\mu/_\sim = \{\mu\}$. □

Theorem 5.5. *Let* $\mathbf{A} \in \mathcal{V}(\mathbf{D})$ *and* $\mu \in \overline{L}$. *Then:*
1) $\mu/_\sim = \{\nu \in L : \nu^+ = 1_\mathbf{A}\} = \overline{L}$,
2) $(\mu/_\sim \cup \{1_\mathbf{A}\}, \bullet)$ *forms a Boolean group, where* $\mu_1 \bullet \mu_2 := (\mu_1 \div \mu_2)'$ *for* $\mu_1, \mu_2 \in \mu/_\sim$.

Proof: (1) From [5, Lemma 21] we know that $\mu/_\sim \subseteq \{\nu \in L : \nu^+ = 1_\mathbf{A}\}$. We need to prove the reverse inclusion. Let $\varphi \in \{\nu \in L : \nu^+ = 1_\mathbf{A}\}$ and $\varphi \neq \mu$. First we show that $\mu \bullet \varphi$ is a congruence on $\mathbf{A}$. Since $\mathrm{Cm}(\mathbf{A}) \subseteq \mathrm{Cm}(\mathbf{A}^e)$, see [7, Lemma 4.1], we have $\mu, \varphi \in \mathrm{Cm}(\mathbf{A}^e)$. Thus, from [14, Proposition 3] we get that $\mu \bullet \varphi \in \mathrm{Con}\,\mathbf{A}^e$. Next, we show that the relation $\mu \bullet \varphi$ is compatible with the operation d. Let $(a, b), (e, f) \in \mu \bullet \varphi$. Since the operation $d \equiv 1$ on $\mathbf{A}/\mu \cup \mathbf{A}/\varphi$, we get $d(a, e) \cdot d(b, f) \in 1/\mu$ and $d(a, e) \cdot d(b, f) \in 1/\varphi$. Thus $d(a, e) \cdot d(b, f) \in 1/\mu \wedge \varphi$, and, consequently, $(d(a, e), d(b, f)) \in \mu \wedge \varphi \subseteq \mu \bullet \varphi$. Therefore $\mu \bullet \varphi$ is a congruence. Since $\mu^+ = \varphi^+$, from [17, Corollay 3.7] we get $\mu \sim \varphi$. Thus $\varphi \in \mu/_\sim$, and so $\{\nu \in L : \nu^+ = 1_\mathbf{A}\} \subseteq \mu/_\sim$.

The assertion (2) follows from [17, Theorem 3.6]. □

Summarizing, the equivalence classes of the relation $\sim$ on $\mathrm{Cm}(\mathbf{A})$ take the following form:

1. $\overline{L} \in \mathrm{Cm}(\mathbf{A})/_\sim$,
2. $\mu/_\sim = \{\mu\}$ for all $\mu \in \underline{L} \cup P$.

6. Representation theorem

A maximal proper subalgebra of the Boolean group is called a **hyperplane**. We use this word, because a Boolean group can be interpreted as a vector space over the field $\mathbb{Z}_2$. We will write $Z\uparrow := \{\nu \in \mathrm{Cm}(\mathbf{A}) : \nu \geq \mu \text{ for some } \mu \in Z\}$ and $Z\downarrow := \{\nu \in \mathrm{Cm}(\mathbf{A}) : \nu \leq \mu \text{ for some } \mu \in Z\}$ for $Z \subseteq \mathrm{Cm}(\mathbf{A})$. Let $\mathbf{A} \in \mathcal{V}(\mathbf{D})$. To get the representation theorem, we need to define a family of subsets on the set $\mathrm{Cm}(\mathbf{A})$ called the hereditary sets. This idea came from Słomczyńska, see [14]. The general definition [17, Definition 4.5] works for every algebra $\mathbf{A}$ from a Fregean variety. It is easy to see that in our case this definition takes the following form:

Definition 6.1. Let $\mathbf{A} \in \mathcal{V}(\mathbf{D})$ and $Z \subseteq \mathrm{Cm}(\mathbf{A})$. A set Z is **hereditary** if:

1. $Z = Z\uparrow$,
2. $\overline{L} \subseteq Z$ or $((\overline{L} \cap Z) \cup \{\mathbf{1_A}\}, \bullet)$ is a hyperplane in $(\overline{L} \cup \{\mathbf{1_A}\}, \bullet)$.

We denote by $\mathcal{H}(\mathbf{A})$ the set of all hereditary subsets of $\mathrm{Cm}(\mathbf{A})$.

We define a map M as follows:

$$M : A \ni a \to M(a) := \{\mu \in \mathrm{Cm}(\mathbf{A}) : a \in 1/\mu\},$$

for all $\mathbf{A} \in \mathcal{V}(\mathbf{D})$.

Now, we formulate the representation theorem.

Theorem 6.2. *Let $\mathbf{A} \in \mathcal{V}(\mathbf{D})$ and let $\mathbf{A}$ be finite. Then the map $M : A \ni a \to M(a) := \{\mu \in \mathrm{Cm}(\mathbf{A}) : a \in 1/\mu\}$ is the isomorphism between $\mathbf{A}$ and $(\mathcal{H}(\mathbf{A}), \leftrightarrow, \mathsf{d}, \mathit{1})$, where*

$$Z \leftrightarrow Y := ((Z \div Y)\downarrow)'$$

$$\mathsf{d}(Z, Y) := [Z \cup ((Z\downarrow)' \cap L)] \cap [Y \cup ((Y\downarrow)' \cap L)],$$

$$\mathit{1} := \mathrm{Cm}(A),$$

for $Z, Y \in \mathcal{H}(\mathbf{A})$.

Proof: From [17, Proposition 4.8] we deduce that $M(a)$ is a hereditary set, so the map M is well defined. Next, we conclude from [17, Theorem 4.14] that M is a bijection which preserves the equivalence operation. Clearly, if $Z = \mathrm{Cm}(\mathbf{A})$, then $Z = M(1)$. Thus, it suffices to show that M preserves d.

Of course, we have $\overline{L} \subseteq \mathsf{d}(Z, Y)$ for all $Z, Y \in \mathcal{H}(\mathbf{A})$, and so $\mathsf{d}(Z, Y)$ is a hereditary set. Moreover, $Z \cap Y \subseteq \mathsf{d}(Z, Y)$. We show that

$$M(d(a, b)) = [M(a) \cup ((M(a)\downarrow)' \cap L)] \cap [M(b) \cup ((M(b)\downarrow)' \cap L)],$$

for all $a, b \in A$. We recall that if $\mu \in \overline{L}$, then $d(a, b)/\mu = 1/\mu$, and if $\mu \in P$, then $d(a, b)/\mu = a/\mu \wedge b/\mu$. We show inclusion both ways.

„$\subseteq$" Let $\mu \in M(d(a, b))$. We need consider three cases:

1) $\mu \in \overline{L}$. Then the inclusion is obvious.

2) $\mu \in P$. Then

$$\mu \in M(d(a, b)) \Rightarrow d(a, b) \in 1/\mu \Rightarrow d(a, b)/\mu = 1/\mu \Rightarrow$$

$$a/\mu = 1/\mu \text{ and } b/\mu = 1/\mu \Rightarrow a \in 1/\mu \text{ and } b \in 1/\mu \Rightarrow$$

$$\mu \in M(a) \text{ and } \mu \in M(b) \Rightarrow \mu \in M(a) \cap M(b).$$

3) $\mu \in \underline{L}$. In this situation we get $\mu \in M(d(a, b)) \Rightarrow d(a, b)/\mu = 1/\mu \Rightarrow a/\mu \neq */\mu$ and $b/\mu \neq */\mu$. The following cases are possible:

a) $a/\mu = b/\mu = 1/\mu$. Then $\mu \in M(a) \cap M(b)$.

b) $a/\mu = b/\mu = 0/\mu$. Therefore

$$a/\mu^+ = b/\mu^+ = 0/\mu^+ \Rightarrow a, b \notin 1/\mu^+ \Rightarrow \mu^+ \notin M(a) \text{ and } \mu^+ \notin M(b) \Rightarrow$$

$$\mu \notin M(a)\downarrow \text{ and } \mu \notin M(b)\downarrow \Rightarrow \mu \in (M(a)\downarrow)' \text{ and } \mu \in (M(b)\downarrow)'.$$

Hence $\mu \in [M(a) \cup ((M(a)\downarrow)' \cap L)] \cap [M(b) \cup ((M(b)\downarrow)' \cap L)]$.

c) $a/\mu = 1/\mu$, $b/\mu = 0/\mu$ (or vice versa). Then $a \in 1/\mu$, so $\mu \in M(a)$. Since $b \notin 1/\mu^+$, so $\mu \notin M(b)\downarrow$, and consequently $\mu \in (M(b)\downarrow)'$. Thus

$$\mu \in [M(a) \cup ((M(a)\downarrow)' \cap L)] \cap [M(b) \cup ((M(b)\downarrow)' \cap L)].$$

„$\supseteq$" Let $\mu \in [M(a) \cup ((M(a)\downarrow)' \cap L)] \cap [M(b) \cup ((M(b)\downarrow)' \cap L)]$. Once again we need consider three cases:

1) $\mu \in \overline{L}$. Then $d(a, b)/\mu = 1/\mu \Rightarrow d(a, b) \in 1/\mu \Rightarrow \mu \in M(d(a, b))$.

2) $\mu \in P$. In this case:

$$\mu \in M(a) \cap M(b) \Rightarrow \mu \in M(a) \text{ and } \mu \in M(b) \Rightarrow a/\mu = 1/\mu$$

$$\text{and } b/\mu = 1/\mu \Rightarrow d(a,b)/\mu = 1/\mu \Rightarrow d(a,b) \in 1/\mu \Rightarrow \mu \in M(d(a,b)).$$

3) $\mu \in \underline{L}$. Let us consider the following cases:

a) $\mu \in M(a)$ and $\mu \in M(b)$. Then

$$a, b \in 1/\mu \Rightarrow a/\mu = b/\mu = 1/\mu \Rightarrow d(a,b)/\mu = 1/\mu \Rightarrow$$

$$d(a,b) \in 1/\mu \Rightarrow \mu \in M(d(a,b)).$$

b) $\mu \in M(a)$ and $\mu \in (M(b)\downarrow)'$ (or analogously: $\mu \in (M(a)\downarrow)'$ and $\mu \in M(b)$). Therefore $a/\mu = 1/\mu$ and $b/\mu = 0/\mu$. Then $d(a,b)/\mu = 1/\mu$, so $d(a,b) \in 1/\mu$, and consequently $\mu \in M(d(a,b))$.

c) $\mu \in (M(a)\downarrow)'$ and $\mu \in (M(b)\downarrow)'$. Then $a/\mu = b/\mu = 0/\mu$, so we get as above $d(a,b)/\mu = 1/\mu$. Thus $d(a,b) \in 1/\mu$, and hence $\mu \in M(d(a,b))$. Finally, we conclude that M preserves d, and so M is the isomorphism as claimed. $\square$

Example 6.3. Let $A = \{*, 1\}^3 \cup \{(1,0,0), (0,1,0), (0,0,1), (*,0,0), (0,*,0), (0,0,0)\}$. Thus A is closed under equivalence operation $\cdot$ and $(A, \cdot)$ is the smallest equivalential algebra, which is not a reduct of a Heyting algebra, see [13, Example 3]. Moreover, $d(x,y) \in \{1,*\}^3$ for all $x, y \in \mathbf{D}^3$. Therefore $\mathbf{A} = (A, \cdot, d) \in S(\mathbf{D}^3)$.

Let us consider three subsetes of A: $F_1 := \{*,1\}^3 \cup \{(1,0,0), (*,0,0)\}$, $F_2 := \{*,1\}^3 \cup \{(0,1,0), (0,*,0)\}$ and $F_3 := \{*,1\}^3 \cup \{(0,0,1), (0,0,*)\}$. Then the relations μ_i for $i \in \{1,2,3\}$, defined by: $a \equiv_{\mu_i} b$ iff $ab \in F_i$ for all $a, b \in A$, are congruences of $\mathbf{A}$. Moreover, an easy computation shows that $\mathbf{1}/\mu_i = F_i$ for all $i \in \{1,2,3\}$ (where $\mathbf{1} = (1,1,1)$) and $a/\mu_i = A \setminus F_i$ for all $a \in A \setminus F_i$. Choosing $a = (a_1, a_2, a_3) \in \{0,1\}^3$ for every $i \in \{1,2,3\}$, we get: $d(\mathbf{1}/\mu_i, a/\mu_i) = (d(1,a_1), d(1,a_2), d(1,a_3))/\mu_i = \mathbf{1}/\mu_i$. Therefore $\mathbf{A}/\mu_i \cong \mathbf{2}$.

Next, let us consider 5-element subsets $G_i \subseteq F_i$, for $i \in \{1,2,3\}$: $G_1 := \{(1,x,y) : x,y \in \{1,*\}\} \cup \{(1,0,0)\}$, $G_2 := \{(x,1,y) : x,y \in \{1,*\}\} \cup \{(0,1,0)\}$, $G_3 := \{(x,y,1) : x,y \in \{1,*\}\} \cup \{(0,0,1)\}$. Relations ν_i, which are designated by these subsetes ($a \equiv_{\nu_i} b$ iff $ab \in G_i$), are congruences of $\mathbf{A}$. Moreover, $\mathbf{1}/\nu_i = G_i$ and $c/\nu_i = F_i \setminus G_i$, $a/\nu_i = a/\mu_i$ for all $c \in F_i \setminus G_i$, $a \in A \setminus F_i$. Thus $\mathbf{A}/\nu_i \cong \mathbf{D}$.

Finally, we get that $\mathrm{Cm}(\mathbf{A})$ has the form as shown in Figure 2. It is easy to check that, according to Theorem 6.2, this frame corresponds to the 14-element algebra $\mathbf{A}$. We can also deduce that $\mathbf{A}$ is directly irreducible.

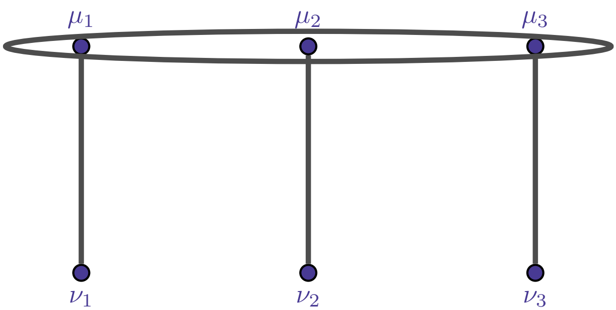

Figure 2.

In general situation, we show that every finite algebra from $\mathcal{V}(\mathbf{D})$ can be naturally decomposed as the direct product of two algebras. Recall that $L = \{\mu \in \mathrm{Cm}(\mathbf{A}) : \mathbf{A}/\mu \cong \mathbf{2}$ or $\mathbf{A}/\mu \cong \mathbf{D}\}$ and $P = \{\mu \in \mathrm{Cm}(\mathbf{A}) : \mathbf{A}/\mu \cong \mathbf{2}^\wedge\}$.

Proposition 6.4. Let $\mathbf{A} \in \mathcal{V}(\mathbf{D})$ be finite. Then:

$$\mathbf{A} \cong \mathbf{A}/_{\bigcap L} \times \mathbf{A}/_{\bigcap P}.$$

Proof: As $\mathbf{A}$ is finite, so $\mathbf{1_A} = \bigvee_{i=1}^n \alpha_i$, where α_i $(i \in \{1, \ldots, n\})$ are join-irreducible congruences. Clearly, $\bigcap L \wedge \bigcap P = \mathrm{Cm}(\mathbf{A}) = 0_{\mathbf{A}}$. We need to prove that $\alpha_i \subseteq \bigcap L \vee \bigcap P$ for all $i \in \{1, \ldots, n\}$. Let $i \in \{1, \ldots, n\}$. Assume that $\alpha_i \nsubseteq \bigcap L$. We show that $\alpha_i \subseteq \bigcap P$. Suppose, contrary to our claim, that there exists $\mu \in P$ such that $\alpha_i \nsubseteq \mu$. Then $\alpha_i \vee \mu = \mathbf{1_A}$ and $\alpha_i \wedge \mu < \alpha_i$. Thus the intervals $I[\alpha_i \wedge \mu, \alpha]$ and $I[\mu, \mathbf{1_A}]$ are projective, and, consequently, $\alpha_i \wedge \mu = \alpha_i^-$. On the other hand, there exists $\nu \in L$ such that $\alpha_i \nsubseteq \nu$ and $\alpha_i \subseteq \nu^+$. Therefore, the intervals $I[\alpha_i^-, \alpha]$ and $I[\nu, \nu^+]$ are projective. Thus, we get $\nu \sim \mu$, a contradiction. $\square$

7. Free algebras in $\mathcal{V}(\mathbf{D})$ – a sketch of construction

Now, we can construct the finitely generated free algebras in $\mathcal{V}(\mathbf{D})$. We will denote by $F_{\mathbf{D}}(n)$ the free n-generated algebra in $\mathcal{V}(\mathbf{D})$ in which $X = \{x_1, x_2, \ldots, x_n\}$ is the n-element set of free generators. Observe that if $\mu \in \mathrm{Cm}(F_{\mathbf{D}}(n))$, then we can identify μ with a map f which sends free generators in $\mathbf{k}$, where $\mathbf{k} \in \{\mathbf{D}, \mathbf{2}, \mathbf{2}^{\wedge}\}$, in such a way that $f^{-1}(\{*\}) \neq \emptyset$. This map can be uniquely extended to a surjective homomorphism $\overline{f}\colon F_{\mathbf{D}}(n) \longrightarrow \mathbf{k}$. It follows that $\ker \overline{f} \in \mathrm{Cm}(F_{\mathbf{D}}(n))$. So, the construction of the frame $\mathrm{Cm}(F_{\mathbf{D}}(n))$ is similar to the construction of the frame of the equivalential algebras with conjunction on the regular elements, described in [11].

This construction proceeds as follows:

1. Each $\mu \in \mathrm{Cm}(F_{\mathbf{D}}(n))$ is labelled by the set indices $\{i : x_i \in X \cap (1/\mu)\} \subseteq \{1, \ldots, n\}$.
2. $\overline{L}$ has $2^n - 1$ elements labelled by all proper subsets of $\{1, \ldots, n\}$ and these elements form only one equivalence class.
3. P has $2^n - 1$ elements also labelled by all proper subsets of $\{1, \ldots, n\}$, but in this case each element forms a one-element equivalence class.
4. If $\mu \in \overline{L}$ is labelled by $S \subsetneq \{1, \ldots, n\}$, then below μ (i. e., in $\underline{L}$) there are elements labelled by all proper subsets of S.
5. Each $\mu \in \underline{L}$ forms a one-element equivalence class.

In the figures below:

a. Each dot denotes an element of the frame.

b. Straight lines denote a partial ordering directed upwards.

c. The equivalence class with more than one element is marked with an ellipse.

d. Each dot that does not lie in an ellipse denotes a one-element equivalence class.

7.1. The frame of $F_{\mathbf{D}}(2)$ – the free algebra in $\mathcal{V}(\mathbf{D})$ with two free generators

The set $\mathrm{Cm}(F_{\mathbf{D}}(2))$ has 8 elements (Figure 3): 5 on the left-hand side (all elements at the top form one equivalence class and the elements at the

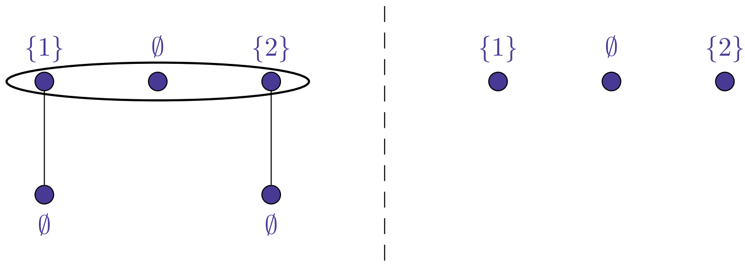

Figure 3. $\mathbf{Cm}(F_{\mathbf{D}}(2))$

bottom form one-element equivalence classes) and 3 on the right-hand side (each in a separate equivalence class). So, there are 9 hereditary sets on the left-hand side and 8 hereditary sets on the right-hand side. Finally, $|F_{\mathbf{D}}(2)| = 9 \cdot 8 = 72$.

7.2. The frame of $F_{\mathbf{D}}(3)$ – the free algebra in $\mathcal{V}(\mathbf{D})$ with three free generators

The set $\mathrm{Cm}(F_{\mathbf{D}}(3))$ has 26 elements (Figure 4): 7 on the left-hand side at the top, 12 on the left-hand side at the bottom, and 7 on the right-hand side. On the left-hand side there are 4536 hereditary sets, and on the right-hand side there are 128 hereditary sets. Finally, $|F_{\mathbf{D}}(3)| = 4536 \cdot 128 = 580608$.

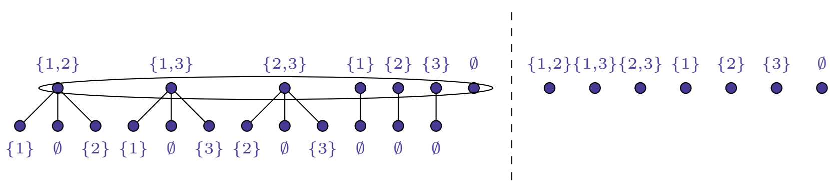

Figure 4. $\mathbf{Cm}(F_{\mathbf{D}}(3))$

Using Theorem 6.2 and the construction above one can also find the formula for the free spectrum. We plan to publish these result in the next article, which will be a continuation of this paper.

References

[1] J. Czelakowski, D. Pigozzi, *Fregean logics*, **Annals of Pure and Applied Logic**, vol. 127(1–3) (2004), pp. 17–76, DOI: https://doi.org/10.1016/j.apal.2003.11.008.

[2] R. Freese, R. McKenzie, **Commutator theory for congruence modular varieties**, vol. 125 of London Mathematical Society Lecture Notes, Cambridge University Press, Cambridge (1987).

[3] J. Hagemann, *On regular and weakly regular congruences*, *Tech. rep.*, TH Darmstadt (1973), preprint no. 75.

[4] D. Hobby, R. McKenzie, **The structure of finite algebras**, vol. 76 of Contemporary Mathematics, American Mathematical Society (1988), DOI: https://doi.org/10.1090/conm/076.

[5] P. M. Idziak, K. Słomczyńska, *Polynomially rich algebras*, **Journal of Pure and Applied Algebra**, vol. 156(1) (2001), pp. 33–68, DOI: https://doi.org/10.1016/S0022-4049(99)00119-X.

[6] P. M. Idziak, K. Słomczyńska, A. Wroński, *Fregean varieties*, **International Journal of Algebra and Computation**, vol. 19(5) (2009), pp. 595–645, DOI: https://doi.org/10.1142/S0218196709005251.

[7] P. M. Idziak, K. Słomczyńska, A. Wroński, *The commutator in equivalential algebras and Fregean varieties*, **Algebra Universalis**, vol. 65(4) (2011), pp. 331–340, DOI: https://doi.org/10.1007/s00012-011-0133-4.

[8] J. K. Kabziński, A. Wroński, *On equivalential algebras*, [in:] G. Epstein, J. M. Dunn, S. C. Shapiro, N. Cocchiarella (eds.), **Proceedings of the 1975 International Symposium on Multipe-Valued Logic, Indiana University, Bloomington, Indiana**, IEEE Computer Society, Long Beach (1975), pp. 419–428, URL: https://apps.dtic.mil/sti/pdfs/ADA045757.pdf.

[9] P. Köhler, *Brouwerian semilattices*, **Transactions of the American Mathematical Society**, vol. 268(1) (1981), pp. 103–126, DOI: https://doi.org/10.1090/S0002-9947-1981-0628448-3.

[10] R. McKenzie, G. McNulty, W. Taylor, **Algebras, Lattices, Varieties: Volume I**, AMS Chelsea Publishing, Providence, Rhode Island (1987).

[11] S. Przybyło, *Equivalential algebras with conjunction on the regular elements*, **Annales Universitatis Paedagogicae Cracoviensis Studia Mathe-**

matica, vol. 20 (2021), pp. 63–75, DOI: https://doi.org/10.2478/aupcsm-2021-0005.

[12] H. Rasiowa, R. Sikorski, **The Mathematics of Metamathematics**, vol. 125, PWN, Warszawa (1963).

[13] K. Słomczyńska, *Equivalential algebras. Part I: representation*, **Algebra Universalis**, vol. 35(4) (1996), pp. 524–547, DOI: https://doi.org/10.1007/BF01243593.

[14] K. Słomczyńska, *Free spectra of linear equivalential algebras*, **The Journal of Symbolic Logic**, vol. 70(4) (2005), pp. 1341–1358, DOI: https://doi.org/10.2178/jsl/1129642128.

[15] K. Słomczyńska, *Free equivalential algebras*, **Annals of Pure and Applied Logic**, vol. 155(2) (2008), pp. 86–96, DOI: https://doi.org/10.1016/j.apal.2008.03.003.

[16] K. Słomczyńska, *Unification and projectivity in Fregean varieties*, **Logic Journal of the IGPL**, vol. 20(1) (2011), pp. 73–93, DOI: https://doi.org/10.1093/jigpal/jzr026.

[17] K. Słomczyńska, *The structure of completely meet irreducible congruences in strongly Fregean algebras*, **Algebra universalis**, vol. 83 (2022), DOI: https://doi.org/10.1007/s00012-022-00787-0, article number: 31.

[18] M. H. Stone, *The theory of representation for Boolean algebras*, **Transactions of the American Mathematical Society**, vol. 40(1) (1936), pp. 37–111, DOI: https://doi.org/10.2307/1989664.

[19] A. Wroński, *On the free equivalential algebra with three generators*, **Bulletin of the Section of Logic**, vol. 22 (1993), pp. 37–39.

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Bulletin of the **S**ection of **L**ogic
Volume 51/4 (2022), pp. 555–645
https://doi.org/10.18778/0138-0680.2022.23

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A LOGIC FOR DUALLY HEMIMORPHIC SEMI-HEYTING ALGEBRAS AND ITS AXIOMATIC EXTENSIONS

Abstract

The variety $\mathbb{DHMSH}$ of dually hemimorphic semi-Heyting algebras was introduced in 2011 by the second author as an expansion of semi-Heyting algebras by a dual hemimorphism. In this paper, we focus on the variety $\mathbb{DHMSH}$ from a logical point of view. The paper presents an extensive investigation of the logic corresponding to the variety of dually hemimorphic semi-Heyting algebras and of its axiomatic extensions, along with an equally extensive universal algebraic study of their corresponding algebraic semantics. Firstly, we present a Hilbert-style axiomatization of a new logic called "Dually hemimorphic semi-Heyting logic" ($\mathcal{DHMSH}$, for short), as an expansion of semi-intuitionistic logic $\mathcal{SI}$ (also called $\mathcal{SH}$) introduced by the first author by adding a weak negation (to be interpreted as a dual hemimorphism). We then prove that it is implicative in the sense of Rasiowa and that it is complete with respect to the variety $\mathbb{DHMSH}$. It is deduced that the logic $\mathcal{DHMSH}$ is algebraizable in the sense of Blok and Pigozzi, with the variety $\mathbb{DHMSH}$ as its equivalent algebraic semantics and that the lattice of axiomatic extensions of $\mathcal{DHMSH}$ is dually isomorphic to the lattice of subvarieties of $\mathbb{DHMSH}$. A new axiomatization for Moisil's logic is also obtained. Secondly, we characterize the axiomatic extensions of $\mathcal{DHMSH}$ in which the "Deduction Theorem" holds. Thirdly, we present several new logics, extending the logic $\mathcal{DHMSH}$, corresponding to several important subvarieties of the variety $\mathbb{DHMSH}$. These include logics corresponding to the varieties generated by two-element, three-element and some four-element dually quasi-De Morgan semi-Heyting algebras, as well as a new axiomatization for the 3-valued Łukasiewicz

Presented by: Andrzej Indrzejczak
Received: February 18, 2022
Published online: December 14, 2022

logic. Surprisingly, many of these logics turn out to be connexive logics, only a few of which are presented in this paper. Fourthly, we present axiomatizations for two infinite sequences of logics namely, De Morgan Gödel logics and dually pseudocomplemented Gödel logics. Fifthly, axiomatizations are also provided for logics corresponding to many subvarieties of regular dually quasi-De Morgan Stone semi-Heyting algebras, of regular De Morgan semi-Heyting algebras of level 1, and of JI-distributive semi-Heyting algebras of level 1. We conclude the paper with some open problems. Most of the logics considered in this paper are discriminator logics in the sense that they correspond to discriminator varieties. Some of them, just like the classical logic, are even primal in the sense that their corresponding varieties are generated by primal algebras.

Keywords: Semi-intuitionistic logic, dually hemimorphic semi-Heyting logic, dually quasi-De Morgan semi-Heyting logic, De Morgan semi-Heyting logic, dually pseudocomplemented semi-Heyting logic, regular dually quasi-De Morgan Stone semi-Heyting algebras of level 1, implicative logic, equivalent algebraic semantics, algebraizable logic, De Morgan Gödel logic, dually pseudocomplemented Gödel logic, Moisil's logic, 3-valued Łukasiewicz logic.

2010 Mathematical Subject Classification: Primary: 03B50, 03G25, 06D20, 06D15; Secondary: 08B26, 08B15, 06D30.

Contents

1. Introduction

Semi-Heyting algebras were introduced by the second author[1], during 1983–84, as a result of his research that went into [24] (still a preprint at the time). Some of the early results were announced in [25]. The first results on these algebras with their proofs, however, were published much later in 2008 (see [28]).

An algebra $\mathbf{L} = \langle L, \vee, \wedge, \to, 0, 1\rangle$ is a semi-Heyting algebra if the following conditions hold:

(SH1) $\langle L, \vee, \wedge, 0, 1\rangle$ is a bounded lattice (with 0 and 1, respectively, as the smallest and largest elements),

(SH2) $x \wedge (x \to y) \approx x \wedge y$,

(SH3) $x \wedge (y \to z) \approx x \wedge [(x \wedge y) \to (x \wedge z)]$,

(SH4) $x \to x \approx 1$.

A semi-Heyting algebra is a Heyting algebra if it satisfies the identity:

(H) $(x \wedge y) \to x \approx 1$.

We will denote the variety of semi-Heyting algebras by $\mathbb{SH}$ and that of Heyting algebras by $\mathbb{H}$. Semi-Heyting algebras share some important properties with Heyting algebras; for instance, semi-Heyting algebras are distributive and pseudocomplemented, with the pseudocomplement $x^* := x \to 0$; the congruences on them are determined by filters and the variety of semi-Heyting algebras is arithmetical. For further results on $\mathbb{SH}$, see [1, 2, 3, 10, 11, 28]. (For algebras closely related to semi-Heyting algebras, see [15, 13].)

It is well known that the variety of Heyting algebras is the equivalent algebraic semantics (in the sense of Blok and Pigozzi) of the intuitionistic propositional logic. In 2011, the first author of this paper defined, in [8], a new logic called "semi-intuitionistic logic" ($\mathcal{SI}$, for short, also called $\mathcal{SH}$) and showed, essentially, that the variety of semi-Heyting algebras is

[1]Parts of this paper were presented by the second author in invited talks at 8th International Conference on Non-Classical Logics: Theory and Applications, Łódź (2016), at Maltsev Meeting, Novosibirsk (2017), and at Asubl (Algebra and Substructural Logics-Take 6) workshop, Cagliari (2018).

an algebraic semantics for this logic and that the intuitionistic logic is an axiomatic extension of it. The axioms of this logic, however, were expressed in a language that was not the same as that of semi-Heyting algebras. In [14], a much simpler, but equivalent, set of axioms for $\mathcal{SI}$ (or $\mathcal{SH}$), was presented in the same language as that of semi-Heyting algebras. The logic $\mathcal{SI}$ as presented in [14] will play a fundamental role in this paper.

In 1942, Moisil [21] (see also [20]) defined a logic called "Logique modale" ($\mathcal{LM}$), an expansion of intuitionistic propositional calculus by a De Morgan negation. He also introduced Heyting algebras endowed with an involution, in [20], as the algebraic models of the logic $\mathcal{LM}$. These algebras were further investigated by Monteiro [22] under the name of symmetric Heyting algebras. In particular, he presented a proof of an algebraic completeness theorem for Moisil's calculus by showing that $\mathcal{LM}$ is complete for the variety of symmetric Heyting algebras.

Independently of the previous work, motivated purely by (universal) algebraic considerations, the second author defined and studied De Morgan Heyting algebras, in [26], by expanding Heyting algebras by a De Morgan operation. Earlier in 1985, he had also introduced (see [24]) the variety of Heyting algebras with a dual pseudocomplementation. Also, in 1987, the concepts of hemimorphism (without name), semi-De Morgan algebra and (lower) quasi-De Morgan algebra were introduced in [27], unifying (and generalizing) the notions of De Morgan operation and pseudocomplementation.

In 2011, motivated by the similarities of the results and proofs in [24] and [26], he introduced in [29] a more general variety of algebras called "dually hemimorphic semi-Heyting algebras"– an expansion of semi-Heyting algebras by a dual hemimorphism, as a common generalization of De Morgan Heyting algebras and dually psedocomplemented Heyting algebras.

Definition 1.1 ([29]). An algebra $\mathbf{A} = \langle A, \vee, \wedge, \rightarrow, ', 0, 1\rangle$ is a dually hemimorphic semi-Heyting algebra (or, semi-Heyting algebra with a dual hemimorphism) if $\mathbf{A}$ satisfies the following conditions:

(D1): $\langle A, \vee, \wedge, \rightarrow, 0, 1\rangle$ is a semi-Heyting algebra,

(D2): $0' \approx 1$,

(D3): $1' \approx 0$,

(D4): $(x \wedge y)' \approx x' \vee y'$ ($\wedge$-De Morgan law).

The unary operation $'$ satisfying (D2)–(D4) is called a dual hemimorphism. The variety of dually hemimorphic semi-Heyting algebras will be denoted by $\mathbb{DHMSH}$.

It is useful to note here that if $a \leq b$ in a $\mathbb{DHMSH}$-algebra, then $a' \geq b'$.

Several important subvarieties of the variety $\mathbb{DHMSH}$, by adding the duals of those given in [27], were introduced in [29], some of which will be recalled in Section 5.

The following problem presents itself naturally.

PROBLEM A: Find a propositional logic in the language $\langle \vee, \wedge, \to, \sim, \bot, \top \rangle$ with the following properties:

(1) It has the variety $\mathbb{DHMSH}$ of dually hemimorphic semi-Heyting algebras as its equivalent algebraic semantics, and

(2) It has Moisil's logic as one of its (axiomatic) extensions (up to equivalence).

The subvariety $\mathbb{DQDSH}$ of $\mathbb{DHMSH}$, consisting of dually quasi-De Morgan semi-Heyting algebras (see item 10 of **LIST 1** in Section 5 for definition), has been intensively investigated in [24, 26, 29, 30, 31, 32, 33, 34, 35, 36]. In Section 8 of [29] (see also [31] and [32]) the following problem was raised:

PROBLEM B: Find Hilbert-type axiomatization for logics corresponding to two-valued, three-valued and four-valued dually quasi-De Morgan semi-Heyting algebras, viewed as logical matrices with $\{1\}$ as the distinguished subset.

In this paper, we focus on the logical aspects of the variety $\mathbb{DHMSH}$ of dually hemimorphic semi-Heyting algebras and many of its subvarieties. The paper presents an extensive investigation of the logic corresponding to the variety of dually hemimorphic semi-Heyting algebras and of its axiomatic extensions, along with an equally extensive universal algebraic study of their corresponding algebraic semantics.

Firstly, we give a solution to **PROBLEM A**. More specifically, we present a Hilbert-style presentation of a new logic called "Dually hemimorphic semi-Heyting logic" ($\mathcal{DHMSH}$, for short), as an expansion of semi-intuitionistic logic presented in [14]. We then prove that it is implicative in the sense of Rasiowa and that it is complete with respect to the variety $\mathbb{DHMSH}$ of dually hemimorphic semi-Heyting algebras. Using the well-known results of Abstract Algebraic Logic we deduce that the logic $\mathcal{DHMSH}$ is algebraizable in the sense of Blok and Pigozzi, with the variety $\mathbb{DHMSH}$ as its equivalent algebraic semantics. It then follows that the lattice of axiomatic extensions of $\mathcal{DHMSH}$ is dually isomorphic to the lattice of subvarieties of $\mathbb{DHMSH}$. As applications of these results, we present several new logics, extending the logic $\mathcal{DHMSH}$, corresponding to some interesting subvarieties (studied in [29]) of the variety of hemimorphic semi-Heyting and Heyting algebras. A new axiomatization for Moisil's logic is also obtained. Secondly, we characterize the axiomatic extensions of $\mathcal{DHMSH}$ in which the "Deduction Theorem" holds. This characterization is further sharpened for the axiomatic extensions of the logic $\mathcal{DQDSH}$. Thirdly, we introduce many morl new logics, extending the logic $\mathcal{DQDSH}$, corresponding to important subvarieties of the variety $\mathbb{DQDSH}$, including some logics corresponding to the varieties generated by two-element, three-element and some four-element dually quasi-De Morgan semi-Heyting algebras, as well as a new axiomatization for the 3-valued Łukasiewicz logic. Many of these logics, to our surprise, turn out to be connexive logics, a few of which are presented in this paper. Fourthly, we present axiomatizations for two infinite sequences of logics, namely De Morgan-Gödel logics and dually pseudocomplemented Gödel logics. Fifthly, axiomatizations are also provided for logics corresponding to many subvarieties of regular dually quasi-De Morgan Stone semi-Heyting algebras of level 1, of Regular De Morgan Semi-Heyting Algebras of level 1 and of JI-distributive semi-Heyting algebras of level 1, studied in [29, 30, 31, 32, 33] (see also [34, 35, 36]). Many of the logics considered in this paper are discriminator logics in the sense that they correspond to discriminator varieties. Some of them, just like the classical logic, are even primal in the sense that their corresponding varieties are generated by primal algebras.

The paper is organized as follows: Section 2 contains definitions, notation and some preliminary results that are needed later in the paper. It includes the axiomatization for semi-intuitionistic logic as presented in [14] which is crucial for the rest of the paper. In Section 3, we present a Hilbert-style axiomatization for the new logic called "Dually hemimorphic semi-Heyting logic" ($\mathcal{DHMSH}$, for short) by expanding the language of semi-intuitionistic logic $\mathcal{SI}$ of [14] by a (weak) negation called *dually* *hemimorphic negation* and by adding new axioms and a new inference rule to the semi-intitionistic logic $\mathcal{SI}$. We then prove that the logic $\mathcal{DHMSH}$ is an implicative logic with respect to the defined connective $\to_H$, where $x \to_H y := x \to (x \wedge y)$. In Section 4, we prove the completeness theorem for the logic $\mathcal{DHMSH}$: The logic $\mathcal{DHMSH}$ is complete with respect to the variety $\mathbb{DHMSH}$ of dually hemimorphic semi-Heyting algebras. In Section 5, we deduce from Abstract Algebraic Logic that the logic $\mathcal{DHMSH}$ is algebraizable, in the sense of Blok and Pigozzi, with the variety $\mathbb{DHMSH}$ as its equivalent algebraic semantics, from which it follows that the lattice of axiomatic extensions of $\mathcal{DHMSH}$ is dually isomorphic to the lattice of subvarieties of $\mathbb{DHMSH}$. These results enable us to present axiomatizations of several extensions of $\mathcal{DHMSH}$ by translating the (equational) axioms of various (known) subvarieties of $\mathbb{DHMSH}$ from [29, 30, 31, 32, 33] (see sections 5 and 8-12) into (propositional) axioms of the corresponding extensions. We also show that Moisil's "logique modale" $\mathcal{LM}$ is equivalent to the logic $\mathcal{DMH}$ corresponding to the variety $\mathbb{DMH}$ of De Morgan Heyting algebras. In Section 6, we characterize the (axiomatic) extensions of $\mathcal{DHMSH}$ in which the "Deduction Theorem" holds. This characterization is further refined for the axiomatic extensions of the logic $\mathcal{DQDSH}$.

Sections 7–12 deal with applications of the results of Section 5 together with the algebraic results proved in [29, 30, 31, 32, 33, 34]. More specifically, in Section 7, we present axiomatizations for some extensions of the logic $\mathcal{DQDSH}$ whose equivalent algebraic semantics are subvarieties of $\mathbb{DQDSH}$ generated by finitely many finite algebras, including two 2-valued logics and twenty 3-valued logics and three 4-valued logics. Then we revisit the 3-valued Łukasiewicz logic and give an alternative axiomatization for it. In fact, we show that the logic corresponding to the 3-element De Morgan Heyting algebra is equivalent to the 3-valued Łukasiewicz logic. Thereafter, we give axiomatizations for extensions of $\mathcal{DQDSH}$ corresponding to the subvarieties of the variety $\mathbb{DQDBSH}$ generated by dually quasi-De Morgan Boolean semi-Heyting algebras, completing the solution to **PROBLEM B** mentioned earlier. We also give some extensions of the logic $\mathcal{DHMSH}$ which fail to possess the disjunction property. Section 8 describes some connections to Connexive Logic by showing that some of these 2-valued, 3-valued and 4-valued logics are, in fact, connexive logics. Section 9 gives axiomatizations for De Morgan Gödel logic and dually pseudocomplemented Gödel logic corresponding to the varieties generated by the De Morgan Heyting chains and the dually pseudocomplemented

Heyting chains, respectively. It also provides axiomatizations for the logics corresponding to their subvarieties. In Section 10, we present axiomatizations for new logics corresponding to several subvarieties of the variety $\mathrm{RDQDStSH}_1$ of regular dually quasi-De Morgan Stone semi-Heyting algebras of level 1. Section 11 presents axiomatizations for logics corresponding to a number of subvarieties of RDMSH1 of regular De Morgan Stone semi-Heyting algebras of level 1, while Section 12 presents axiomatizations for logics corresponding to many subvarieties of JI-distributive linear semi-Heyting algebras of level 1. Section 13 concludes the paper with several open problems for future research.

2. Preliminaries

A language $\mathbf{L}$ is a set of finitary operations (or connectives), each with a fixed arity $n \geq 0$. In this paper, we identify $\perp$ and $\top$ with 0 and 1 respectively and thus consider the languages $\langle \vee, \wedge, \to, \sim, \perp, \top \rangle$ and $\langle \vee, \wedge, \to, ', 0, 1 \rangle$ as the same; however, we frequently use the former in the context of logics and the latter in the context of algebras. For a countably infinite set *Var* of propositional variables, the *formulas* of the language $\mathbf{L}$ are inductively defined as usual. A *logic* (or, *a deductive system*) in the language $\mathbf{L}$ is a pair $\mathcal{L} = \langle \mathbf{L}, \vdash_{\mathcal{L}} \rangle$, where $\vdash_{\mathcal{L}}$ is a substitution-invariant consequence relation on $Fm_{\mathbf{L}}$. We will present logics by means of their "Hilbert style" axioms and inference rules.

The set of formulas $Fm_{\mathbf{L}}$ can be turned into an algebra in the usual way. Throughout the paper, Γ and Δ denote sets of formulas and lower case Greek letters denote formulas. The homomorphisms from the formula algebra $\mathbf{Fm_L}$ into an $\mathbf{L}$-algebra (i.e, an algebra of type $\mathbf{L}$) $\mathbf{A}$ are called *interpretations* (or *valuations*) in A. The set of all such interpretations is denoted by $Hom(\mathbf{Fm_L}, \mathbf{A})$. If $h \in Hom(\mathbf{Fm_L}, \mathbf{A})$ then the *interpretation of a formula* α under h is its image $h\alpha \in A$, while $h\Gamma$ denotes the set $\{h\phi \mid \phi \in \Gamma\}$.

As mentioned earlier, Moisil presented in [21] (see also [20, 22]), a propositional logic called, "Logique modale". We will refer to it as "$\mathcal{LM}$".

Moisil also introduced the variety of Heyting algebras endowed with an involution, in [20], as the algebraic semantics for the logic $\mathcal{LM}$. Monteiro [22] investigated these algebras under the new name of symmetric Heyting algebras. Among other things, he presented a proof of an algebraic completeness theorem for Moisil's calculus $\mathcal{LM}$ by showing that the logic $\mathcal{LM}$ is complete with respect to the variety of symmetric Heyting algebras.

In the next section we will generalize Moisil's logic to a new logic called "dually hemimorphic semi-Heyting logic". As a first step to achieve this goal, we need to present a generalization of intutionistic logic called "Semi-intutionistic logic" which was first introduced by the first author in [8] in the language $\langle \vee, \wedge, \to, \sim \rangle$. We will actually present below the more streamlined version of semi-Intuitionistic logic $\mathcal{SI}$ in the usual language $\langle \vee, \wedge, \to, \perp, \top \rangle$, as first presented in [14] with the intuitionistic logic as an axiomatic extension. To facilitate this presentation, it will be convenient to use $\alpha \to_H \beta$ as an abbreviation for $\alpha \to (\alpha \wedge \beta)$ so that the axioms given are easier to read. Moreover, the operation $\to_H$ plays a crucial role in this section and in the sections that follow. See Lemma 2.4 and Lemma 2.5 for more information about $\to_H$.

Definition 2.1 ([14]). The semi-intuitionistic logic $\mathcal{SI}$ (also called $\mathcal{SH}$) is defined in the language
$\{\vee, \wedge, \to, \perp, \top\}$ and it has the following axioms and the inference rule:

AXIOMS:

(S1): $\alpha \to_H (\alpha \vee \beta)$,

(S2): $\beta \to_H (\alpha \vee \beta)$,

(S3): $(\alpha \to_H \gamma) \to_H [(\beta \to_H \gamma) \to_H ((\alpha \vee \beta) \to_H \gamma)]$,

(S4): $(\alpha \wedge \beta) \to_H \alpha$,

(S5): $(\gamma \to_H \alpha) \to_H [(\gamma \to_H \beta) \to_H (\gamma \to_H (\alpha \wedge \beta))]$,

(S6): $\top$,

(S7): $\perp \to_H \alpha$,

(S8): $((\alpha \wedge \beta) \to_H \gamma) \to_H (\alpha \to_H (\beta \to_H \gamma))$,

(S9): $(\alpha \to_H (\beta \to_H \gamma)) \to_H ((\alpha \wedge \beta) \to_H \gamma)$,

(S10): $(\alpha \to_H \beta) \to_H ((\beta \to_H \alpha) \to_H ((\alpha \to \gamma) \to_H (\beta \to \gamma)))$,

(S11): $(\alpha \to_H \beta) \to_H ((\beta \to_H \alpha) \to_H ((\gamma \to \beta) \to_H (\gamma \to \alpha)))$.

RULE OF INFERENCE:

(SMP): From ϕ and $\phi \to_H \gamma$, deduce γ (semi-Modus Ponens).

The following theorem and the lemma, proved in [14], are useful in later sections.

THEOREM 2.2 ([14], Completeness Theorem). *For all* $\Gamma \cup \{\alpha\} \subseteq Fm$

$$\Gamma \vdash_{\mathcal{SI}} \alpha \textit{ if and only if } \Gamma \models_{\mathbb{SH}} \alpha.$$

LEMMA 2.3 ([14]). *The following statements hold in the logic* $\mathcal{SI}$*:*

1. *If* $\Gamma \vdash_{\mathcal{SI}} \psi$ *then* $\Gamma \vdash_{\mathcal{SI}} \alpha \to_H \psi$,
2. $\vdash_{\mathcal{SI}} \alpha \to_H \alpha$,
3. $\vdash_{\mathcal{SI}} (\alpha \wedge \beta) \to_H \beta$,
4. *If* $\vdash_{\mathcal{SI}} \alpha \to_H \beta$ *then* $\vdash_{\mathcal{SI}} (\alpha \wedge \gamma) \to_H (\beta \wedge \gamma)$ *and* $\vdash_{\mathcal{SI}} (\gamma \wedge \alpha) \to_H (\gamma \wedge \beta)$,
5. $\vdash_{\mathcal{SI}} (\alpha \to_H \beta) \to_H [(\beta \to_H \gamma) \to_H (\alpha \to_H \gamma)]$,
6. *If* $\Gamma \vdash_{\mathcal{SI}} \alpha$ *and* $\Gamma \vdash_{\mathcal{SI}} \beta$ *then* $\Gamma \vdash_{\mathcal{SI}} \alpha \wedge \beta$.

2.1. Some observations about semi-Heyting algebras, in general, and $\to$ and $\to_H$, in particular

A key feature of semi-Heyting algebras is the following:

> Every semi-Heyting algebra $\langle A, \vee, \wedge, \to, 0, 1\rangle$ gives rise, in a natural way, to a Heyting algebra $\langle A, \vee, \wedge, \to_H, 0, 1\rangle$, where $x \to_H y := x \to (x \wedge y)$, for $x, y \in A$ (see [3]).

LEMMA 2.4 ([3]). *Let* $\mathbf{A} = \langle A, \vee, \wedge, \to, 0, 1\rangle$ *be a semi-Heyting algebra and let* $a, b, c \in A$*. Then,*

1. $a \wedge b \leq c$ *if and only if* $a \leq b \to (b \wedge c)$,
2. $(a \to b) \wedge (b \to a) = 1$ *if and only if* $(a \to_H b) \wedge (b \to_H a) = 1$,
3. $a = b$ *if and only if* $(a \to_H b) \wedge (b \to_H a) = 1$,
4. $a \to b \leq a \to_H b$.
5. *The algebra* $\mathbf{A} := \langle A, \vee, \wedge, \to_H, 0, 1\rangle$ *is a Heyting algebra.*

Thus, it follows from the preceding lemma that on the universe of every semi-Heyting algebra, there is a Heyting algebra with $\to_H$ as its implication operation, and $a \to b \leq a \to_H b$, for $a, b \in A$.

The order in a semi-Heyting algebra is determined by $\to_H$ as the following lemma shows.

Lemma 2.5 ([8, Corollary 3.9]). *Let $\mathbf{A} = \langle A, \vee, \wedge, \to, 0, 1\rangle$ be a semi-Heyting algebra and $a, b \in A$. Then, $a \to_H b = 1$ if and only if $a \leq b$.*

It is worth pointing out that the inference rule SMP implies the traditional Modus Ponens (MP) for the connective $\to$ as proved in [14, Lemma 4.3].

On the other hand, it is shown, by an example, in [14] (see pages 313–314) that Modus Ponens **MP** does not imply **SMP**.

Further relevance of the use of $\to_H$ in the axioms of $\mathcal{SH}$ can be seen as follows: Suppose we replace the axiom (S4) by the axiom

(S4') $(\alpha \wedge \beta) \to \alpha$.

and keep the rest of the axioms to form a new list, say, (**LIST 2**) of axioms. Then consider the following algebra:

$\sim$:	0	1	2	3
	1	0	0	0

$\to$:	0	1	2	3
0	1	1	1	1
1	0	1	1	0
2	0	0	1	1
3	0	1	0	1

$\wedge$:	0	1	2	3
0	0	0	0	0
1	0	1	3	3
2	0	2	1	1
3	0	3	2	2

$\vee$:	0	1	2	3
0	0	1	1	1
1	1	1	1	1
2	1	1	1	1
3	1	1	1	1

This algebra satisfies $\phi \approx 1$ for all ϕ in the LIST 2 but its lattice reduct is not distributive, since $2 \wedge (0 \vee 1) \neq (1 \wedge 2) \vee (0 \wedge 2)$.

3. The logic $\mathcal{DHMSH}$: Axioms, Rules, and Rasiowa's Implicativeness

In this section we present a new propositional logic called "dually hemimorphic semi-Heyting logic" denoted by $\mathcal{DHMSH}$ and prove, as a first step toward a completeness theorem, that the logic $\mathcal{DHMSH}$ is implicative in the sense of Rasiowa with respect to the implication $\to_H$.

DEFINITION 3.1. The dually hemimorphic semi-Heyting logic, $\mathcal{DHMSH}$, is defined in the language $\langle \vee, \wedge, \to, \sim, \perp, \top \rangle$ and it has the following axioms and rules of inference:

AXIOMS:
(S1), (S2), ..., (S11) of the logic $\mathcal{SI}$, together with the following three additional axioms:

(S12) $\top \to_H \sim \perp$,

(S13) $\sim \top \to_H \perp$,

(S14) $\sim (\alpha \wedge \beta) \to_H (\sim \alpha \vee \sim \beta)$.

RULES OF INFERENCE:

(SMP) From ϕ and $\phi \to_H \gamma$, deduce γ, (semi-Modus Ponens)

(SCP) From $\phi \to_H \gamma$, deduce $\sim \gamma \to_H \sim \phi$. (semi-Contraposition)

Since the axioms and the inference rule of the logic $\mathcal{SI}$ are included in the logic $\mathcal{DHMSH}$, the following result is immediate.

THEOREM 3.2. *Let $\Gamma \cup \{\alpha\} \subseteq Fm$. If $\Gamma \vdash_{\mathcal{SI}} \alpha$ then $\Gamma \vdash_{\mathcal{DHMSH}} \alpha$.*

The following lemma is needed later.

LEMMA 3.3. *Let $\Gamma \cup \{\alpha, \beta, \gamma, \psi\} \subseteq Fm$. The following statements hold in the logic $\mathcal{DHMSH}$:*

1. *If $\Gamma \vdash_{\mathcal{DHMSH}} \psi$, then $\Gamma \vdash_{\mathcal{DHMSH}} \alpha \to_H \psi$,*
2. $\vdash_{\mathcal{DHMSH}} \alpha \to_H \alpha$,
3. $\vdash_{\mathcal{DHMSH}} (\alpha \wedge \beta) \to_H \beta$,
4. $\Gamma \vdash_{\mathcal{DHMSH}} (\alpha \to_H \beta) \to_H [(\beta \to_H \gamma) \to_H (\alpha \to_H \gamma)]$,

5. $\Gamma \vdash_{\mathcal{DHMSH}} \alpha \wedge \beta$ *if and only if* $\Gamma \vdash_{\mathcal{DHMSH}} \alpha$ *and* $\Gamma \vdash_{\mathcal{DHMSH}} \beta$,
6. *If* $\Gamma \vdash_{\mathcal{DHMSH}} \alpha \to_H \beta$, *then* $\Gamma \vdash_{\mathcal{DHMSH}} (\alpha \wedge \gamma) \to_H (\beta \wedge \gamma)$ *and* $\Gamma \vdash_{\mathcal{DHMSH}} (\gamma \wedge \alpha) \to_H (\gamma \wedge \beta)$.

Proof: Items (1), (2), (3) and (4) follow from Theorem 3.2 and items (1), (2), (3) and (5) of Lemma 2.3, respectively. We have, by Theorem 3.2 and Lemma 2.3 (6), that if $\Gamma \vdash_{\mathcal{DMSH}} \alpha$ and $\Gamma \vdash_{\mathcal{DHMSH}} \beta$ then $\Gamma \vdash_{\mathcal{DHMSH}} \alpha \wedge \beta$. The other half of the item (5) follows easily from axiom (S4), item (3) and (SMP). Finally, Item (6) follows from Lemma 2.3 (4). □

Lemma 3.4. *Let* $\Gamma \cup \{\alpha, \beta, \gamma\} \subseteq Fm$.

1. *If* $\Gamma \vdash_{\mathcal{DHMSH}} \alpha \to_H \beta$ *and* $\Gamma \vdash_{\mathcal{DHMSH}} \beta \to_H \gamma$ *then* $\Gamma \vdash_{\mathcal{DHMSH}} \alpha \to_H \gamma$.
2. $\Gamma, \beta \to_H \alpha \vdash_{\mathcal{DHMSH}} \sim \alpha \to_H \sim \beta$.
3. *If* $\Gamma \vdash_{\mathcal{DHMSH}} \alpha \to_H \beta$, *then* $\Gamma \vdash_{\mathcal{DHMSH}} (\alpha \vee \gamma) \to_H (\beta \vee \gamma)$ *and* $\Gamma \vdash_{\mathcal{DHMSH}} (\gamma \vee \alpha) \to_H (\gamma \vee \beta)$.
4. $\Gamma \vdash_{\mathcal{DHMSH}} (\sim \alpha \vee \sim \beta) \to_H \sim (\alpha \wedge \beta)$.

Proof:

1. This follows from 3.3 (4), using (SMP).
2. This is immediate from (SCP).
3. (a) $\Gamma \vdash_{\mathcal{DHMSH}} \alpha \to_H \beta$ by hypothesis,
   (b) $\Gamma \vdash_{\mathcal{DHMSH}} \beta \to_H (\beta \vee \gamma)$ by (S1),
   (c) $\Gamma \vdash_{\mathcal{DHMSH}} \alpha \to_H (\beta \vee \gamma)$ by (1) in (3a) and (3b),
   (d) $\Gamma \vdash_{\mathcal{DHMSH}} \gamma \to_H (\beta \vee \gamma)$ by (S2),
   (e) $\Gamma \vdash_{\mathcal{DHMSH}} (\alpha \to_H (\beta \vee \gamma)) \to_H [(\gamma \to_H (\beta \vee \gamma)) \to_H ((\alpha \vee \gamma) \to_H (\beta \vee \gamma))]$ by (S3),
   (f) $\Gamma \vdash_{\mathcal{DHMSH}} (\gamma \to_H (\beta \vee \gamma)) \to_H ((\alpha \vee \gamma) \to_H (\beta \vee \gamma))$ by (SMP) in (3c) and (3e),
   (g) $\Gamma \vdash_{\mathcal{DHMSH}} (\alpha \vee \gamma) \to_H (\beta \vee \gamma)$ by (SMP) in (3d) and (3f),
   (h) $\Gamma \vdash_{\mathcal{DHMSH}} \gamma \to_H (\gamma \vee \beta)$ by (S1),
   (i) $\Gamma \vdash_{\mathcal{DHMSH}} \beta \to_H (\gamma \vee \beta)$ by (S2),

(j) $\Gamma \vdash_{\mathcal{DHMSH}} \alpha \to_H (\gamma \vee \beta)$ by (1) and (SMP) in (3a) and (3i),

(k) $\Gamma \vdash_{\mathcal{DHMSH}} (\gamma \to_H (\gamma \vee \beta)) \to_H [(\alpha \to_H (\gamma \vee \beta)) \to_H ((\gamma \vee \alpha) \to_H (\gamma \vee \beta))]$ by (S3),

(l) $\Gamma \vdash_{\mathcal{DHMSH}} (\alpha \to_H (\gamma \vee \beta)) \to_H ((\gamma \vee \alpha) \to_H (\gamma \vee \beta))$ by (SMP) in (3h) and (3k),

(m) $\Gamma \vdash_{\mathcal{DHMSH}} (\gamma \vee \alpha) \to_H (\gamma \vee \beta)$ by (SMP) in (3j) and (3l).

4. (a) $\Gamma \vdash_{\mathcal{DHMSH}} (\alpha \wedge \beta) \to_H \alpha$ by axiom (S4),

(b) $\Gamma \vdash_{\mathcal{DHMSH}} \sim \alpha \to_H \sim (\alpha \wedge \beta)$ by (SCP),

(c) $\Gamma \vdash_{\mathcal{DHMSH}} (\alpha \wedge \beta) \to_H \beta$ by (3.3) and (3),

(d) $\Gamma \vdash_{\mathcal{DHMSH}} \sim \beta \to_H \sim (\alpha \wedge \beta)$ by (SCP),

(e) $\Gamma \vdash_{\mathcal{DHMSH}} (\sim \alpha \to_H \sim (\alpha \wedge \beta)) \to_H [(\sim \beta \to_H \sim (\alpha \wedge \beta)) \to_H ((\sim \alpha \vee \sim \beta) \to_H \sim (\alpha \wedge \beta))]$ by axiom (S3),

(f) $\Gamma \vdash_{\mathcal{DHMSH}} (\sim \beta \to_H \sim (\alpha \wedge \beta)) \to_H ((\sim \alpha \vee \sim \beta) \to_H \sim (\alpha \wedge \beta))$ by (SMP) in (4b), (4e),

(g) $\Gamma \vdash_{\mathcal{DHMSH}} (\sim \alpha \vee \sim \beta) \to_H \sim (\alpha \wedge \beta)$ by (SMP) in (4d) and (4f).

proving the lemma. $\square$

3.1. $\mathcal{DHMSH}$ as an implicative logic in the sense of Rasiowa

In 1974, Rasiowa ([23, page 179]) introduced an important class of logics called "standard systems of implicative extensional propositional calculus" and associated a class of algebras with each of them, by a generalization of the classical Lindenbaum-Tarski process. We will refer to these logics as "implicative logics in the sense of Rasiowa" ("implicative logics", for short). These logics have played a pivotal role in the development of Abstract Algebraic Logic. We now recall the definition of implicative logics. We follow Font [16].

Definition 3.5 ([23, 16]). Let $\mathcal{L}$ be a logic in a language $\mathbf{L}$ that includes a binary connective $\to$, either primitive or defined by a term in exactly two variables. Then $\mathcal{L}$ is called an implicative logic with respect to the binary connective $\to$, if the following conditions are satisfied:

(IL1) $\vdash_{\mathcal{L}} \alpha \to \alpha$,

(IL2) $\alpha \to \beta, \beta \to \gamma \vdash_{\mathcal{L}} \alpha \to \gamma$,

(IL3) For each symbol $f \in \mathbf{L}$ of arity $n \geq 1$,
$$\left\{ \begin{array}{c} \alpha_1 \to \beta_1, \ldots, \alpha_n \to \beta_n, \\ \beta_1 \to \alpha_1, \ldots, \beta_n \to \alpha_n \end{array} \right\} \vdash_{\mathcal{L}} f(\alpha_1, \ldots, \alpha_n) \to f(\beta_1, \ldots, \beta_n),$$

(IL4) $\alpha, \alpha \to \beta \vdash_{\mathcal{L}} \beta$,

(IL5) $\alpha \vdash_{\mathcal{L}} \beta \to \alpha$.

The following lemma was proved in [14, Lemma 4.6].

Lemma 3.6 ([14, Lemma 4.6]). *The logic $\mathcal{SI}$ is implicative with respect to the connective $\to_H$.*

The following theorem follows from Theorem 3.2, Lemma 3.4 (2) and Lemma 3.6.

Theorem 3.7. *The logic $\mathcal{DHMSH}$ is implicative with respect to the connective $\to_H$.*

4. Completeness of $\mathcal{DHMSH}$

Let $\mathbf{L}$ denote the language $\langle \vee, \wedge, \to, \sim, \bot, \top \rangle$. Identities in $\mathbf{L}$ are ordered pairs $\langle \alpha, \beta \rangle$ of $\mathbf{L}$-formulas that will be written in the form $\alpha \approx \beta$. An interpretation h in $\mathbf{A}$ satisfies an identity $\alpha \approx \beta$ if $h\alpha = h\beta$. We denote this satisfaction relation by the notation: $\mathbf{A} \models_h \alpha \approx \beta$. An algebra $\mathbf{A}$ *satisfies the equation* $\alpha \approx \beta$ if all the interpretations in $\mathbf{A}$ satisfy it; in symbols,

$$\mathbf{A} \models \alpha \approx \beta \text{ if and only if } \mathbf{A} \models_h \alpha \approx \beta, \text{ for all } h \in Hom(\mathbf{Fm_L}, \mathbf{A}).$$

A class $\mathbb{K}$ of algebras *satisfies the identity* $\alpha \approx \beta$ when all the algebras in $\mathbb{K}$ satisfy it; i.e.

$$\mathbb{K} \models \alpha \approx \beta \text{ if and only if } \mathbf{A} \models \alpha \approx \beta, \text{ for all } \mathbf{A} \in \mathbb{K}.$$

If $\bar{x}$ is a sequence of variables and h is an interpretation in $\mathbf{A}$, then we write $\bar{a}$ for $h(\bar{x})$. For a class $\mathbb{K}$ of $\mathbf{L}$-algebras, we define the relation $\models_{\mathbb{K}}$ that holds between a set Δ of identities and a single identity $\alpha \approx \beta$ as follows:

$$\Delta \models_{\mathbb{K}} \alpha \approx \beta$$

if and only if

for every $\mathbf{A} \in \mathbb{K}$ and every interpretation $\bar{a}$ of the variables of $\Delta \cup \{\alpha \approx \beta\}$ in $\mathbf{A}$, we have,

$$\text{if } \phi^{\mathbf{A}}(\bar{a}) = \psi^{\mathbf{A}}(\bar{a}), \text{ for every } \phi \approx \psi \in \Delta, \text{ then } \alpha^{\mathbf{A}}(\bar{a}) = \beta^{\mathbf{A}}(\bar{a}).$$

In this case, we say that $\alpha \approx \beta$ is a $\mathbb{K}$-consequence of Δ. The relation $\models_K$ is called the *semantic equational consequence relation determined by* $\mathbb{K}$.

Our goal in this section is to prove that the logic $\mathcal{DHMSH}$ is complete with respect to the variety $\mathbb{DHMSH}$. For this we need the following definition from [23].

Definition 4.1 ([23, Definition 6, page 181], [16, Definition 2.5]). Let $\mathcal{L}$ be an implicative logic in the language $\mathbf{L}$ with an implication connective $\rightarrow$. An algebra $\mathbf{A}$ in the language $\mathbf{L}$ that has an element 1 is called an $\mathcal{L}$-algebra if $\mathbf{A}$ satisfies the following properties:

(LALG1) For all $\Gamma \cup \{\phi\} \subseteq Fm$ and all $h \in Hom(\mathbf{Fm_L}, \mathbf{A})$, if $\Gamma \vdash_{\mathcal{L}} \phi$ and $h\Gamma \subseteq \{1\}$ then $h\phi = 1$,

(LALG2) For all $a, b \in A$, if $a \rightarrow b = 1$ and $b \rightarrow a = 1$ then $a = b$.

The class of $\mathcal{L}$-algebras is denoted by $\mathbf{Alg}^*\mathcal{L}$.

We also need the following result from [14].

Theorem 4.2 ([14, Corollary 4.8]). $\mathbf{Alg}^*\mathcal{SI} = \mathbb{SH}$.

Since $\mathcal{DHMSH}$ is an implicative logic with respect to the binary connective $\rightarrow_H$ by Theorem 3.7, we obtain the following result, in view of [23, Theorem 7.1, p. 222].

Theorem 4.3. *The logic $\mathcal{DHMSH}$ is complete with respect to the class* $\mathsf{Alg}^*\mathcal{DHMSH}$. *In other words,*

for all $\Gamma \cup \{\phi\} \subseteq Fm$, $\Gamma \vdash_{\mathcal{DHMSH}} \phi$ if and only if $\Gamma \models_{Alg^*(\mathcal{DHMSH})} \phi$.

As a last step to complete the proof of algebraic completeness of the logic $\mathcal{DHMSH}$, we need to prove that $\mathbf{Alg}^*\mathcal{DHMSH} = \mathbb{DHMSH}$.

LEMMA 4.4. $\mathbb{DHMSH} = \mathsf{Alg}^*\mathcal{DHMSH}$.

PROOF: First, we wish to prove that $\mathbb{DHMSH} \subseteq \mathsf{Alg}^*\mathcal{DHMSH}$. Let $\mathbf{A} \in \mathbb{DHMSH}$, $\Gamma \cup \{\phi\} \subseteq Fm$ and $h \in Hom(Fm_{\mathbf{L}}, \mathbf{A})$ such that $\Gamma \vdash_{\mathcal{DHMSH}} \phi$ and $h\Gamma \subseteq \{1\}$. We need to verify that $h\phi = 1$.

We will proceed by induction on the length of the proof of $\Gamma \vdash_{\mathcal{DHMSH}} \phi$.

- Assume that ϕ is an axiom.

  If ϕ is one of the axioms (S1) to (S11) then $\vdash_{\mathcal{SI}} \phi$. Hence, by theorem 2.2, $\models_{\mathbb{DHMSH}} \phi$ and so, $h(\phi) = \top$.

  If ϕ is the axiom (S12) then, using (D2), we have $h(\phi) = h(\top \to_H \sim \bot) = 1 \to_H 0' = 1$.

  If ϕ is the axiom (S13) then, using (D3), we get that $h(\phi) = h(\sim \top \to_H \bot) = 0 \to_H 0 = 1$.

  If ϕ is the axiom (S14) then, using (D4), we obtain that $h(\phi) = h(\sim (\alpha \wedge \beta) \to_H (\sim \alpha \vee \sim \beta)) = (h(\alpha) \wedge h(\beta))' \to_H (h(\alpha)' \vee h(\beta)') = (h(\alpha) \wedge h(\beta))' \to_H (h(\alpha) \wedge h(\beta))' = 1$.

- If $\phi \in \Gamma$ then $h(\phi) = \top$ by hypothesis

- Assume now that $\Gamma \vdash_{\mathcal{L}} \phi$ is obtained from an application of (SMP). Then there exist a formula ψ such that $\Gamma \vdash_{\mathcal{L}} \psi$ and $\Gamma \vdash_{\mathcal{L}} \psi \to_H \phi$. By induction, $h(\psi) = 1$ and $h(\psi \to_H \phi) = 1$. Then $1 = h(\psi) \to_H h(\phi) = 1 \to_H h(\phi) = h(\phi)$.

- Assume that $\Gamma \vdash_{\mathcal{L}} \phi$ is the result of an application of the rule (SCP). Then for $\alpha, \beta \in Fm$, $\phi = \sim \beta \to_H \sim \alpha$ and $\Gamma \vdash_{\mathcal{L}} \alpha \to_H \beta$. By induction, $1 = h(\alpha \to_H \beta) = h(\alpha) \to_H h(\beta)$ and, consequently $h(\alpha) \leq h(\beta)$. Then, using condition (D4), $h(\beta)' \leq h(\alpha)'$. Hence $h(\beta)' \to_H h(\alpha)' = 1$. Therefore, $h(\phi) = h(\sim \beta \to_H \sim \alpha) = h(\beta)' \to_H h(\alpha)' = 1$.

Hence, the induction is complete and so, we concludes that $\mathbf{A}$ satisfies (LALG1). It is easy to see that the condition (LALG2) also holds, implying $\mathbf{A} \in \mathsf{Alg}^*\mathcal{DHMSH}$.

Next, we prove the other inclusion. Let $\mathbf{A} = \langle A, \vee, \wedge, \to, ', 0, 1\rangle \in \mathsf{Alg}^*\mathcal{DHMSH}$. Notice that $\langle A, \vee, \wedge, \to, 0, 1\rangle \in \mathsf{Alg}^*\mathcal{SI}$. By Theorem 4.2, $\langle A, \vee, \wedge, \to, 0, 1\rangle \in \mathbb{SH}$. Now, it only remains to show that $\mathbf{A}$ satisfies the conditions (D2) to (D4).

In view of axiom (S12) and (LALG1), we have that $\mathbf{A} \models 1 \to_H 0' \approx 1$. Using (LALG1) and Lemma 3.3 (1), we get $\mathbf{A} \models 0' \to_H 1 \approx 1$. Then by (LALG2), $\mathbf{A} \models 1 \approx 0'$.

In view of axioms (S7) and (S13), together with (LALG1), we have that $\mathbf{A} \models 0 \to_H 1' \approx 1$ and $\mathbf{A} \models 1' \to_H 0 \approx 1$. Then by (LALG2), $\mathbf{A} \models 1' \approx 0$.

By Lemma 3.4 (4) and the condition (LALG1), $\mathbf{A}$ satisfies the identity $(x' \vee y') \to_H (x \wedge y)' \approx 1$. Also, In view of axiom (S14), and (LALG1), $\mathbf{A}$ satisfies the identity $(x \wedge y)' \to_H (x' \vee y') \approx 1$.

Applying (LALG2), the algebra satisfies (D4). Consequently $\mathbf{A} \in \mathbb{DHMSH}$. □

We are now ready to present the completeness theorem for the logic $\mathcal{DHMSH}$.

THEOREM 4.5. *The logic $\mathcal{DHMSH}$ is complete with respect to the variety $\mathbb{DHMSH}$.*

PROOF: From Lemma 4.4 we have $\mathsf{Alg}^*\mathcal{DHMSH} = \mathbb{DHMSH}$. The conclusion follows from Theorem 4.3. □

5. Equivalent algebraic semantics and axiomatic extensions of the logic $\mathcal{DHMSH}$

Our goal in this section is to improve Theorem 4.5 by proving that the logic $\mathcal{DHMSH}$ is algebraizable and $\mathbb{DHMSH}$ is an equivalent algebraic semantics of the logic $\mathcal{DHMSH}$.

Here we first recall some relevant notions and results from Abstract Algebraic Logic (see [6, Section 2.1], [17], or [16]).

DEFINITION 5.1 ([6, Definition 2.2], [16, Definition 3.4]). Let $\langle \mathbf{L}, \vdash_{\mathcal{L}}\rangle$ be a logic (i.e., deductive system) and $\mathbb{K}$ a class of $\mathbf{L}$-algebras. $\mathbb{K}$ is called an "algebraic semantics" for $\langle \mathbf{L}, \vdash_{\mathcal{L}}\rangle$ if $\vdash_{\mathcal{L}}$ can be interpreted in $\models_{\mathbb{K}}$ in the following sense:

There exists a finite set $\delta_i(p) \approx \epsilon_i(p)$, for $i < n$, of identities with a single variable p such that, for all $\Gamma \cup \phi \subseteq Fm$ and each $j < n$,

$$\text{(A)} \qquad \Gamma \vdash_{\mathcal{L}} \phi$$

if and only if

$$\{\delta_i[\psi/p] \approx \epsilon_i[\psi/p] : i < n, \psi \in \Gamma\} \models_K \delta_j[\phi/p] \approx \epsilon_j[\phi/p],$$

where $\delta[\psi/p]$ denotes the formula obtained by the substitution of ψ at every occurrences of p in δ. The identities $\delta_i \approx \epsilon_i$, for $i < n$, are called "defining identities" for $\langle \mathbf{L}, \vdash_{\mathcal{L}} \rangle$ and $\mathbb{K}$.

In what follows, we will use "$\Gamma \mathrel{=\mkern-3mu|\mkern-3mu\models}_{\mathbb{K}} \Delta$" as an abbreviation for "$\Gamma \models_{\mathbb{K}} \Delta$ and $\Delta \models_{\mathbb{K}} \Gamma$." Also, $\delta(\Delta(\phi, \psi))$ denotes the formula obtained by the substitution of the formula $\Delta(\phi, \psi)$ at every occurrence of p in $\delta(p)$.

Definition 5.2 ([6, Definition 2.8], [16, Definition 3.11]). Let $\langle \mathbf{L}, \vdash_{\mathcal{L}} \rangle$ be a logic and $\mathbb{K}$ an algebraic semantics for $\langle \mathbf{L}, \vdash_{\mathcal{L}} \rangle$ with defining identities $\delta_i = \epsilon_i$, for $i < n$.

$\mathbb{K}$ is said to be "equivalent to" $\langle \mathbf{L}, \vdash_{\mathcal{L}} \rangle$ if there exists a finite set $\Delta_j(p, q)$, for $j < m$, of formulas with two variables p, q such that

for every identity $\phi \approx \psi$, for $i < n$, and for $j < m$,

$$\text{(E)} \qquad \phi \approx \psi \mathrel{=\mkern-3mu|\mkern-3mu\models}_{\mathbb{K}} \{\delta_i(\Delta_j(\phi, \psi)) \approx \epsilon_i(\Delta_j(\phi, \psi)) : i < n, j < m\}.$$

The set $\Delta_j(p, q)$, $j < m$, of formulas with two variables, satisfying (E) is called a set of "equivalence formulas" for $\langle \mathbf{L}, \vdash_{\mathcal{L}} \rangle$ and $\mathbb{K}$. A logic $\mathcal{L}$ is said to be "algebraizable" if and only if it has an equivalent algebraic semantics $\mathbb{K}$.

The following theorem, proved in [6], is crucial in what follows.

Theorem 5.3 ([6], [16, Proposition 3.15]). *Every implicative logic $\mathcal{L}$ is algebraizable with respect to the class $Alg^*\mathcal{L}$ and the algebraizability is witnessed by the defining identity $p \approx p \to p$ and the equivalence formulas $\Delta = \{p \to q, q \to p\}$.*

As an immediate consequence of Theorem 5.3, Theorem 3.7 and Theorem 4.5, we obtain the following crucial result.

Corollary 5.4. The logic $\mathcal{DHMSH}$ is algebraizable, and the variety $\mathbb{DHMSH}$ is the equivalent algebraic semantics for $\mathcal{DHMSH}$ with the defining identity $p \approx p \to_H p$ (equivalently, $p \approx 1$) and the equivalence formulas $\Delta = \{p \to_H q, q \to_H p\}$.

Also, we just mention, in passing, the following fact which follows from Theorem 5.3, Lemma 3.6 and Theorem 4.2 about the semi-intuitionistic logic $\mathcal{SH}$.

COROLLARY 5.5. The logic $\mathcal{SI}$ is algebraizable, and the variety $\mathbb{SH}$ is the equivalent algebraic semantics for $\mathcal{SI}$ with the defining identity $p \approx p \to_H p$ and the equivalence formulas $\Delta = \{p \to_H q, q \to_H p\}$.

5.1. Axiomatic extensions of $\mathcal{DHMSH}$

A logic $\mathcal{L}'$ is an *axiomatic extension* of $\mathcal{L}$ if $\mathcal{L}'$ is obtained by adjoining new axioms but keeping the rules of inference the same as in $\mathcal{L}$.

In the sequel, we sometimes use the term "extension" for "axiomatic extension". Let $Ext(\mathcal{L})$ denote the lattice of axiomatic extensions of the logic $\mathcal{L}$ and $\mathbf{L_V}(\mathbb{K})$ denote the lattice of subvarieties of the variety $\mathbb{K}$ of algebras.

The following theorem, due to Blok and Pigozzi, is one of the hallmark accomplishments of Abstract Algebraic Logic.

THEOREM 5.6 ([16, Theorem 3.40]). *Let $\mathcal{L}$ be an algebraizable logic with the variety $\mathbb{K}$ as its equivalent algebraic semantics. Then $Ext(\mathcal{L})$ is dually isomorphic to $\mathbf{L_V}(\mathbb{K})$.*

The following theorem is a consequence of Theorem 5.6, Theorem 4.5 and Corollary 5.4.

THEOREM 5.7 (Isomorphism Theorem for $\mathcal{DHMSH}$). *$Ext(\mathcal{DHMSH})$ is dually isomorphic to $\mathbf{L_V}(\mathbb{DHMSH})$.*

In a similar fashion, the following result is a consequence of Theorem 5.6, Theorem 2.2 and Corollary 5.5.

THEOREM 5.8 (Isomorphism Theorem for $\mathcal{SH}$). *$Ext(\mathcal{SH})$ is dually isomorphic to $\mathbf{L_V}(\mathbb{SH})$.*

The following theorem is an immediate consequence of Theorem 5.7 and plays an important role in the rest of the paper. Let $\mathbf{Mod}(\mathcal{E}) := \{\mathbf{A} \in \mathbb{DHMSH} : \mathbf{A} \models \delta \approx 1,\ \text{for every } \delta \in \mathcal{E}\}$.

THEOREM 5.9. *Let $\mathcal{E}$ be an axiomatic extension of the logic $\mathcal{DHMSH}$. Then:*

(a) *$\mathcal{E}$ is algebraizable with the same equivalence formulas and defining equations as those of the logic $\mathcal{DHMSH}$.*

(b) $\mathbf{Mod}(\mathcal{E})$ *is the equivalent algebraic semantics for $\mathcal{E}$.*

Note that Theorem 5.8 justifies the use of the phrase "the logic corresponding to the subvariety $\mathbb{V}$" of $\mathbb{DHMSH}$.

We now give Hilbert-style axiomatizations for several important extensions of the logic $\mathcal{DHMSH}$. To facilitate the presentation of the extensions of the logic $\mathcal{DHMSH}$, we first list several important subvarieties of the variety $\mathbb{DHMSH}$ of dually hemimorphic semi-Heyting algebras that were introduced (or implicit) in [29].

In the later sections of the paper, we give various applications of the results proved above and the algebraic results proved in [29, 30, 31, 32, 33, 34] (see also [35, 36]).

LIST 1: SOME IMPORTANT SUBVARIETIES OF THE VARIETY $\mathbb{DHMSH}$

1. $\mathbb{DHMH}$: *Dually hemimorphic Heyting algebras* are $\mathbb{DHMSH}$-algebras satisfying the identity:

   (H): $(x \wedge y) \to x \approx 1$.

2. $\mathbb{OCKSH}$: *Ockham semi-Heyting algebras* are $\mathbb{DHMSH}$-algebras satisfying the identity:

   (E1): $(x \vee y)' \approx x' \wedge y'$.

3. $\mathbb{OCKH}$: *Ockham Heyting algebras* are $\mathbb{OCKSH}$-algebras satisfying the identity (H).

4. $\mathbb{D}ms\mathbb{SH}$: *Dually* ms *semi-Heyting algebras* are $\mathbb{OCKSH}$-algebras satisfying the identity:

   (E2): $x'' \leq x$.

5. $\mathbb{D}ms\mathbb{H}$: *Dually* ms *Heyting algebras* are $\mathbb{D}ms\mathbb{SH}$-algebras satisfying the identity (H).

6. $\mathbb{DMSH}$: *De Morgan semi-Heyting algebras* are $\mathbb{OCKSH}$-algebras (or $\mathbb{DHMSH}$-algebras) satisfying the identity

   (E3): $x'' \approx x$.

7. $\mathbb{DMH}$: *De Morgan Heyting algebras* are $\mathbb{DMSH}$-algebras satisfying the identity (H).

8. $\mathbb{DSDSH}$: *Dually semi-De Morgan semi-Heyting algebras* ([27]) are $\mathbb{DHMSH}$-algebras satisfying the identities:

   (E4): $(x \vee y)'' \approx x'' \vee y''$,

   (E5): $x''' \approx x'$.

9. $\mathbb{DSDH}$: *Dually semi-De Morgan Heyting algebras* are $\mathbb{DSDSH}$-algebras satisfying the identity (H).

10. $\mathbb{DQDSH}$: *Dually quasi-De Morgan semi-Heyting algebras* ([27]) are $\mathbb{DSDSH}$-algebras satisfying the identity (E2).

11. $\mathbb{DQDH}$: *Dually quasi-De Morgan Heyting algebras* are $\mathbb{DQDSH}$-algebras satisfying the identity (H).

12. $\mathbb{DPCSH}$: *Dually pseudocomplemented semi-Heyting algebras* are $\mathbb{DQDSH}$-algebras satisfying the identity:

    (E6): $x \vee x' \approx 1$.

13. $\mathbb{DPCH}$: *Dually pseudocomplemented Heyting algebras* are $\mathbb{DPCSH}$-algebras satisfying the identity (H).

14. $\mathbb{BDQDSH}$: *Blended dually quasi-De Morgan semi-Heyting algebras* are $\mathbb{DQDSH}$-algebras satisfying the identity:

    (E7): $(x \vee x^*)' \approx x' \wedge x^{*\prime}$. (Blended $\vee$-De Morgan law)

15. $\mathbb{BDQDH}$: *Blended dually quasi-De Morgan Heyting algebras* are $\mathbb{BDQDSH}$-algebras satisfying the identity (H).

16. $\mathbb{SBDQDSH}$: *Strongly blended dually quasi-De Morgan semi-Heyting algebras* are $\mathbb{DQDSH}$-algebras satisfying the identity:

    (E8): $(x \vee y^*)' \approx x' \wedge y^{*\prime}$. (Strongly blended $\vee$-De Morgan law)

17. $\mathbb{SBDQDH}$: *Strongly blended dually quasi-De Morgan Heyting algebras* are $\mathbb{SBDQDSH}$-algebras satisfying the identity (H).

18. $\mathbb{DQDBSH}$: *Dually quasi-De Morgan Boolean semi-Heyting algebras* are $\mathbb{DQDSH}$-algebras satisfying the identity:

    (E9): $x \vee x^* \approx 1$.

19. $\mathbb{DQDBH}$: *dually quasi-De Morgan Boolean Heyting algebras* are $\mathbb{DQDBSH}$-algebras satisfying the identity (H).

20. $\mathbb{DQS}$t$\mathbb{SH}$: *Dually quasi-Stone semi-Heyting algebras* are $\mathbb{DHMSH}$-algebras satisfying the identities: (E2),

    (E10): $(x \vee y')' \approx x' \wedge y''$, (weak $\vee$-De Morgan law)

    (E11): $x' \wedge x'' \approx 0$. (Dual Stone identity)

21. $\mathbb{DQS}$t$\mathbb{H}$: *Dually quasi-Stone Heyting algebras* are $\mathbb{DQS}$t$\mathbb{SH}$-algebras satisfying the identity (H).

22. $\mathbb{BDQS}$t$\mathbb{SH}$: *Blended dually quasi-Stone semi-Heyting algebras* are $\mathbb{DQS}$t$\mathbb{SH}$-algebras satisfying the identity (E7).

23. $\mathbb{BDQS}$t$\mathbb{H}$: *Blended dually quasi-Stone Heyting algebras* are $\mathbb{BDQS}$t$\mathbb{SH}$-algebras satisfying the identity (H).

24. $\mathbb{SBDQS}$t$\mathbb{SH}$: *Strongly blended dually quasi-Stone semi-Heyting algebras* are $\mathbb{DQS}$t$\mathbb{SH}$-algebras satisfying the identity (E8).

25. $\mathbb{SBDQS}$t$\mathbb{H}$: *Strongly blended dually quasi-Stone Heyting algebras* are $\mathbb{SBDQS}$t$\mathbb{SH}$-algebras satisfying the identity (H).

26. $\mathbb{DS}$t$\mathbb{SH}$: *Dually Stone semi-Heyting algebras* are $\mathbb{DPCSH}$-algebras satisfying the identity (E11).

27. $\mathbb{DS}$t$\mathbb{H}$: *Dually Stone Heyting algebras* are $\mathbb{DS}$t$\mathbb{SH}$-algebras satisfying the identity (H).

28. $\mathbb{DSCSH}$: *Dually semi-complemented semi-Heyting algebras* are $\mathbb{DHMSH}$-algebras satisfying the identity (E6).

29. $\mathbb{DSCH}$: *Dually semi-complemented Heyting algebras* are $\mathbb{DSCSH}$-algebras satisfying the identity (H).

30. $\mathbb{DDPCSH}$: *Dually demi-pseudocomplemented semi-Heyting algebras* are $\mathbb{DSDSH}$-algebras satisfying the identity:

    (E12): $x' \vee x'' \approx 1$.

31. $\mathbb{DDPCH}$: *Dually demi-pseudocomplemented Heyting algebras* are $\mathbb{DDPCSH}$-algebras satisfying the identity (H) (see [27]).

32. $\mathbb{DAPCSH}$: *Dually almost pseudocomplemented semi-Heyting algebras* are $\mathbb{DDPCSH}$-algebras in which $'$ satisfies the identity dual to (E9) (see [27]).

33. $\mathbb{DAPCH}$: *Dually almost pseudocomplemented Heyting algebras* are $\mathbb{DAPCSH}$-algebras in which $'$ satisfies the identity (H).

Next, we present Hilbert-type axiomatizations for the (new) logics that are extensions of $\mathcal{DHMSH}$ and that correspond to the subvarieties of $\mathcal{DHMSH}$ mentioned in **LIST 1**. For the relationships of these logics to the varieties in **LIST 1**, the reader is referred to Theorem 5.10 below.

LIST 2: SOME IMPORTANT EXTENSIONS OF $\mathcal{DHMSH}$

1. $\mathcal{DHMH}$: The dually hemimorphic Heyting logic is the extension of $\mathcal{DHMSH}$ given by

    (A1): $(\alpha \wedge \beta) \to \alpha$.

2. $\mathcal{OCKSH}$: The Ockham semi-Heyting logic is the extension of $\mathcal{DHMSH}$ given by

    (A2): $\sim (\alpha \vee \beta) \to_H (\sim \alpha \wedge \sim \beta)$,

    (A3): $(\sim \alpha \wedge \sim \beta) \to_H \sim (\alpha \vee \beta)$.

3. $\mathcal{OCKH}$: The Ockham Heyting logic is the extension of $\mathcal{OCKSH}$ given by (A1).

4. $\mathcal{D}ms\mathcal{SH}$: The dually ms semi-Heyting logic is the extension of $\mathcal{OCKSH}$ given by

    (A4): $\sim\sim \alpha \to_H \alpha$.

5. $\mathcal{DmsH}$: The dually ms Heyting logic is the extension of $\mathcal{DmsSH}$ given by (A1).

6. $\mathcal{DMSH}$: The De Morgan semi-Heyting logic is the extension of $\mathcal{OCKSH}$ given by (A4) and

   (A5): $\alpha \to_H \sim\sim \alpha$.

7. $\mathcal{DMH}$: The De Morgan Heyting logic is the extension of $\mathcal{DMSH}$ given by (A1).

8. $\mathcal{DSDSH}$: The dually semi-De Morgan semi-Heyting logic is the extension of $\mathcal{DHMSH}$ given by

   (A6): $\sim\sim (\alpha \vee \beta) \to_H (\sim\sim \alpha \vee \sim\sim \beta)$,

   (A7): $(\sim\sim \alpha \vee \sim\sim \beta) \to_H \sim\sim (\alpha \vee \beta)$,

   (A8): $\sim\sim\sim \alpha \to_H \sim \alpha$,

   (A9): $\sim \alpha \to_H \sim\sim\sim \alpha$.

9. $\mathcal{DSDH}$: The dually semi-De Morgan Heyting logic is the extension of $\mathcal{DSDSH}$ given by (A1).

10. $\mathcal{DQDSH}$: The dually quasi-De Morgan semi-Heyting logic is the extension of $\mathcal{DSDSH}$ given by (A4).

11. $\mathcal{DQDH}$: The dually quasi-De Morgan Heyting logic is the extension of $\mathcal{DQDSH}$ given by (A1).

12. $\mathcal{DPCSH}$: The dually pseudocomplemented semi-Heyting logic is the extension of $\mathcal{DQDSH}$ given by

    (A10): $\alpha \vee \sim \alpha$.

13. $\mathcal{DPCH}$: The dually pseudocomplemented Heyting logic is the extension of $\mathcal{DPCSH}$ given by (A1).

14. $\mathcal{BDQDSH}$: The blended dually quasi-De Morgan semi-Heyting logic is the extension of $\mathcal{DQDSH}$ given by

    (A11): $\sim (\alpha \vee (\alpha \to \bot)) \to_H (\sim \alpha \wedge \sim (\alpha \to \bot))$,

    (A12): $(\sim \alpha \wedge \sim (\alpha \to \bot)) \to_H \sim (\alpha \vee (\alpha \to \bot))$.

15. $\mathcal{BDQDH}$: The blended dually quasi-De Morgan Heyting logic is the extension of $\mathcal{BDQDSH}$ given by (A1).

16. $\mathcal{SBDQDSH}$: The strongly blended quasi-De Morgan semi-Heyting logic is the extension of $\mathcal{DQDSH}$ given by

    (A13): $\sim(\alpha \vee (\beta \to \bot)) \to_H (\sim \alpha \wedge \ \sim(\beta \to \bot))$,

    (A14): $(\sim \alpha \wedge \ \sim(\beta \to \bot)) \to_H \ \sim(\alpha \vee (\beta \to \bot))$.

17. $\mathcal{SBDQDH}$: The strongly blended dually quasi-De Morgan Heyting logic is the extension of $\mathcal{SBDQDSH}$ given by (A1).

18. $\mathcal{DQDBSH}$: The dually quasi-De Morgan Boolean semi-Heyting logic is the extension of $\mathcal{DQDSH}$ given by

    (A15): $\alpha \vee (\alpha \to \bot)$.

19. $\mathcal{DQDBH}$: The dually quasi-De Morgan Boolean Heyting logic is the extension of $\mathcal{DQDBSH}$ given by (A1).

20. $\mathcal{DQS}\mathrm{t}\mathcal{SH}$: The dually quasi-Stone semi-Heyting logic is the extension of $\mathcal{DHMSH}$ given by (A4) and the following axioms:

    (A16): $\sim(\alpha \vee \ \sim \beta) \to_H (\sim \alpha \wedge \ \sim\sim \beta)$,

    (A17): $(\sim \alpha \wedge \ \sim\sim \beta) \to_H \ \sim(\alpha \vee \sim \beta)$,

    (A18): $(\sim \alpha \wedge \sim\sim \alpha) \to_H \bot$.

21. $\mathcal{DQS}\mathrm{t}\mathcal{H}$: The dually quasi-Stone Heyting logic is the extension of $\mathcal{DQS}\mathrm{t}\mathcal{SH}$ given by (A1).

22. $\mathcal{BDQS}\mathrm{t}\mathcal{SH}$: The blended dually quasi-Stone semi-Heyting logic is the extension of $\mathcal{DQDSH}$ given by (A11), (A12).

23. $\mathcal{BDQS}\mathrm{t}\mathcal{H}$: The blended dually quasi-Stone Heyting logic is the extension of $\mathcal{BDQS}\mathrm{t}\mathcal{SH}$ given by (A1).

24. $\mathcal{SBDQS}\mathrm{t}\mathcal{SH}$: The strongly blended dually quasi-Stone semi-Heyting logic is the extension of $\mathcal{DQDSH}$ given by (A13), (A14).

25. $\mathcal{SBDQS}\mathrm{t}\mathcal{H}$: The strongly blended dually quasi-Stone Heyting logic is the extension of $\mathcal{SBDQS}\mathrm{t}\mathcal{SH}$ given by (A1).

26. $\mathcal{DS}\mathrm{t}\mathcal{SH}$: The dually Stone semi-Heyting logic is the extension of $\mathcal{DPCSH}$ given by (A18).

27. $\mathcal{DS}\mathrm{t}\mathcal{H}$: The dually Stone Heyting logic is the extension of $\mathcal{DS}\mathrm{t}\mathcal{SH}$ given by (A1).

28. $\mathcal{DSCSH}$: The dually semi-complemented semi-Heyting logic is the extension of $\mathcal{DHMSH}$ given by (A10).

29. $\mathcal{DSCH}$: The dually semi-complemented Heyting logic is the extension of $\mathcal{DSCSH}$ given by (A1).

30. $\mathcal{DDPCSH}$: The dually demi-pseudocomplemented semi-Heyting logic is the extension of $\mathcal{DSDSH}$ given by

    (A19): $\sim \alpha \vee \ \sim\sim \alpha$.

31. $\mathcal{DDPCH}$: The dually demi-pseudocomplemented Heyting logic is the extension of $\mathcal{DDPCSH}$ given by (A1).

32. $\mathcal{DAPCSH}$: *The dually almost pseudocomplemented semi-Heyting logic* is the extension of $\mathcal{DDPCSH}$ given by

    (A20): $\sim\sim \alpha \to_H \alpha$.

33. $\mathcal{DAPCH}$: The dually almost pseudocomplemented Heyting logic is the extension of $\mathcal{DAPCSH}$ given by (A1).

The following theorem which is immediate from Theorem 5.9 describes the correspondence between the logics in **LIST 2** and the varieties in **LIST 1**.

THEOREM 5.10. *Let $\mathbb{V}_i$ be the variety of algebras mentioned in the i-th item of* **LIST 1** *and $\mathcal{V}_i$ be the logic appearing in the i-th item of* **LIST 2**. *Then, the logic $\mathcal{V}_i$ corresponds to the variety $\mathbb{V}_i$ in the sense that $\mathbb{V}_i$ is its equivalent algebraic semantics for $\mathcal{V}_i$.*

In Figure 1, we present a (partial) poset describing the mutual relations among the varieties, whose names end with "SH", mentioned in **PART 1** above. The dual of this poset will show the relations among the logics, whose names end with "$\mathcal{SH}$", presented in **PART 2**, $\mathbb{T}$ being the trivial variety. Note that the links do not necessarily represent covers.

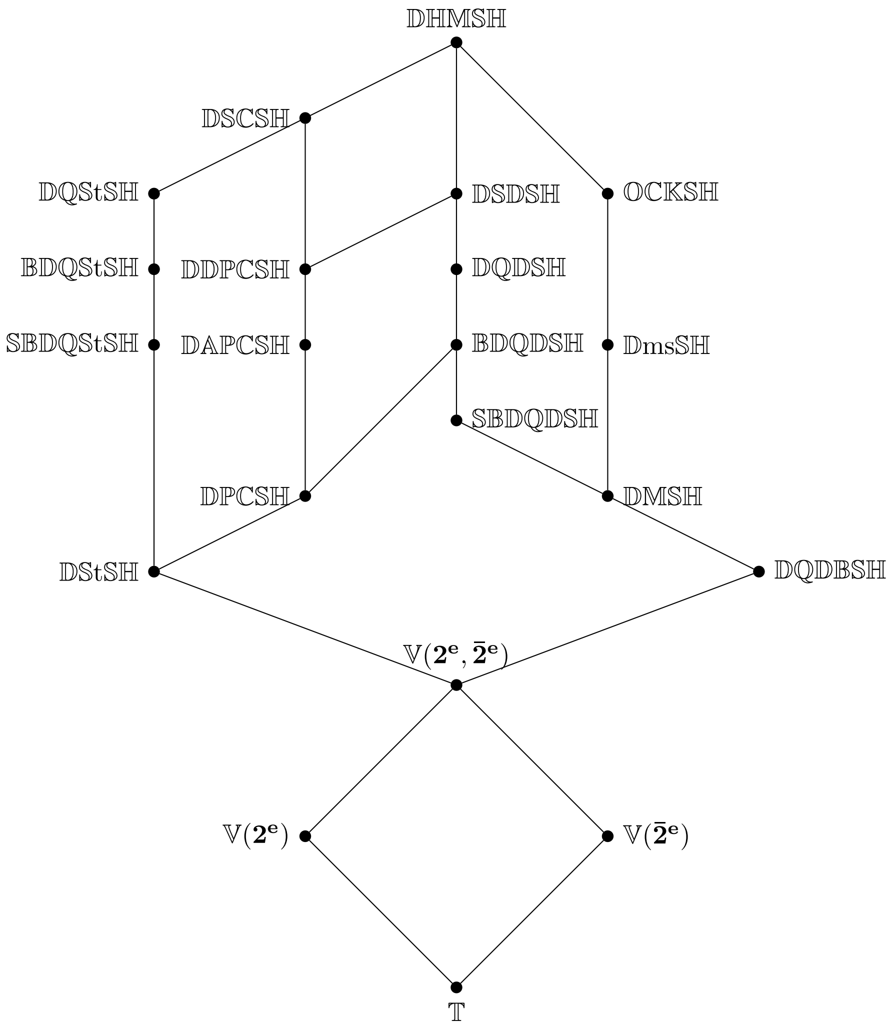


Figure 1. Partial poset of subvarieties of $\mathbb{DHMSH}$

Corollary 5.11. The (Moisil's) logic $\mathcal{LM}$ is equivalent to the logic $\mathcal{DMH}$.

Proof: We know from Moisil's result (see Monteiro [22]) that $\mathcal{LM}$ corresponds to $\mathbb{DMH}$. Also, observe from Theorem 5.10 that the logic $\mathcal{DMH}$ correspond to $\mathbb{DMH}$ as well. □

We just note that there is an 8-element algebra (with heyting reduct) to show that $\mathbb{DPCSH} \not\subset \mathbb{SBDQDSH}$.
Although there has been some investigation of the structre of the lattice of subvarieties of certain subvarietie of the variety $\mathbb{DPCSH}$, the following problem is still open.

PROBLEM 1: Describe the structure of the lattice of subvarieties of the variety $\mathbb{BDQDSH}$.

Similar questions cane be raised about other varieties in the poset of Figure 1, as well.

We now recall some universal algebraic notions (see, for example, [7]) useful in the sequel.

Definition 5.12. Let $\mathbf{A}$ be an algebra. An n-ary function $f : A^n \to A$ is *representable by a term* if there is a term p such that $f(a_1, \ldots, a_n) = p^{\mathbf{A}}(a_1, \ldots, a_n)$, for $a_1, \ldots, a_n \in A$. A finite algebra $\mathbf{A}$ is *primal* if every n-ary function on A, for every $n \geq 1$, is representable by a term.
The *discriminatior* function on a set A is the function $t : A^3 \to A$ defined by

$$t(a, b, c) := \begin{cases} a, & \text{if } a \neq b \\ c, & \text{if } a = b. \end{cases}$$

A ternary term $t(x, y, z)$ representing the discriminator on A is called a *discriminator term* for the algebra $\mathbf{A}$. If a class $\mathbb{K}$ of algebras has a common discriminator term $t(x, y, z)$, then $\mathbb{V}(\mathbb{K})$ is called a discriminator variety. A finite algebra $\mathbf{A}$ with a discriminator term is called *quasiprimal*. An algebra A is semiprimal if

(1) A is quasiprimal,

(2) distinct nontrivial subalgebras of A are not isomorphic,

(3) no subalgebra of A has a proper automorphism.

Definition 5.13. Let $\mathcal{L}$ be an algebraizable logic. We say that $\mathcal{L}$ is a discriminator logic if its equivalent algebraic semantics is a discriminator variety. Furthermore, $\mathcal{L}$ is a primal logic if its equivalent algebraic semantics is a variety generated by a primal algebra. $\mathcal{L}$ is a quasiprimal logic if its equivalent algebraic semantics is a variety generated by a quasiprimal algebra. $\mathcal{L}$ is a semiprimal logic if its equivalent algebraic semantics is a variety generated by a semiprimal algebra.

The classical logic is the first well-known example of a primal logic (as the Boolean algebra **2** is a primal algebra).

Remark 5.14. Since $\mathbb{RDQDS}\mathrm{t}\mathbb{SH}_1$ satisfies (B) (see Section 10), it follows from Corollary 8.2 of [29] that the variety $\mathbb{RDQDS}\mathrm{t}\mathbb{SH}_1$ is a discriminator variety. Thus $\mathcal{RDQDS}\mathrm{t}\mathcal{SH}_1$ is a discriminator logic. Many of the logics considered in the rest of this paper are discriminator logics. We will point them out as they appear.

We conclude this section by noting that the lattice of extensions of the logic $\mathcal{DMH}$ is an interval of the lattice of extensions of $\mathcal{DMSH}$, which, in turn, is an interval in the lattice of extensions of $\mathcal{DHMSH}$.

6. Deduction theorem in the extensions of the logic $\mathcal{DHMSH}$

In this section we first show that the "usual" form of the Deduction Theorem" fails in the logic $\mathcal{DHMSH}$, and then characterize those extensions of $\mathcal{DHMSH}$ where it does hold.

A logic $\mathcal{L}$ is said to have the *Deduction Property* for the connective $\to$ if the following statement holds:

$$\Gamma, \alpha \vdash_{\mathcal{L}} \beta \text{ if and only if } \Gamma \vdash_{\mathcal{L}} \alpha \to \beta,$$

for all $\Gamma \cup \{\alpha, \beta\} \subseteq Fm$.

In the logic $\mathcal{SI}$ the Deduction Property for the conective $\to_H$ is known to hold [8, Theorem 3.18]. But, this property fails in the logic $\mathcal{DHMSH}$, as shown in the following remark.

Remark 6.1. First, we note, by Lemma 3.4 (2), that

$$\phi \to_H \psi \ \vdash_{\mathcal{DHMSH}} \sim\psi \to_H \sim\phi. \tag{6.1}$$

Consider the algebra $\mathbf{L}_1^{dm}$ defined in the second paragraph after Figure 3 in Section 7. Observe that $\mathbf{L}_1^{dm} \not\models_{\mathbb{DHMSH}} (x \to_H y) \to_H (y' \to_H x') \approx 1$ (by taking $x = 1$ and $y = a$). Hence, $\mathbb{DHMSH} \not\models (x \to_H y) \to_H (y' \to_H x') \approx 1$ and therefore, by Theorem 4.5,

$$\nvdash_{\mathcal{DHMSH}} (\phi \to_H \psi) \to_H (\sim \psi \to_H \sim \phi).$$

Thus, the Deduction Property fails in $\mathcal{DHMSH}$, in view of (6.1).

We now wish to characterize the extensions of $\mathcal{DHMSH}$ in which the Deduction Property holds. For this, we need a preliminary lemma.

Lemma 6.2. *Let $\mathcal{E}$ be an extension of the logic $\mathcal{DHMSH}$ such that*

$$\vdash_{\mathcal{E}} (\alpha \to_H \beta) \to_H (\sim \beta \to_H \sim \alpha).$$

Then $\mathcal{E}$ satisfies the Deduction Property for the connective $\to_H$.

Proof: Assume that $\Gamma \cup \{\phi\} \vdash_{\mathcal{E}} \psi$. We shall prove $\Gamma \vdash_{\mathcal{E}} \phi \to_H \psi$ by induction on the proof for ψ. By hypothesis,

$$\vdash_{\mathcal{E}} (\alpha \to_H \beta) \to_H (\sim \beta \to_H \sim \alpha). \tag{6.2}$$

If ψ is an axiom of $\mathcal{E}$ or a formula in Γ, then $\Gamma \vdash_{\mathcal{E}} \psi$. By Lemma 3.3, part (1) we have $\Gamma \vdash_{\mathcal{E}} \phi \to_H \psi$.

Let us assume that $\Gamma \cup \{\phi\} \vdash_{\mathcal{E}} \psi$ is the result of applying the rule (SMP). Then we may assume that there is some formula α such that $\Gamma \cup \{\phi\} \vdash_{\mathcal{E}} \alpha$ and $\Gamma \cup \{\phi\} \vdash_{\mathcal{E}} \alpha \to_H \psi$. So, by inductive hypothesis, we have,

1. $\Gamma \vdash_{\mathcal{E}} \phi \to_H \alpha$,
2. $\Gamma \vdash_{\mathcal{E}} \phi \to_H (\alpha \to_H \psi)$,
3. $\Gamma \vdash_{\mathcal{E}} \phi \to_H \phi$ by Lemma 3.3, part (2),
4. $\Gamma \vdash_{\mathcal{E}} \phi \to_H (\phi \wedge \alpha)$ by (S5) and SMP applied to 1 and 3,
5. $\Gamma \vdash_{\mathcal{E}} (\phi \wedge \alpha) \to_H \psi$ by (S9) and SMP applied to 2,
6. $\Gamma \vdash_{\mathcal{E}} \phi \to_H \psi$ by Lemma 3.3 (4) and SMP applied to 4 and 5.

Assume that $\Gamma \cup \{\phi\} \vdash_{\mathcal{E}} \psi$ is the result of applying the rule (SCP). Hence $\psi = \sim \beta \to_H \sim \alpha$ and $\Gamma \cup \{\phi\} \vdash_{\mathcal{E}} \alpha \to_H \beta$. By induction we have that

1. $\Gamma \vdash_{\mathcal{E}} \phi \to_H (\alpha \to_H \beta)$,
2. $\Gamma \vdash_{\mathcal{E}} (\alpha \to_H \beta) \to_H (\sim \beta \to_H \sim \alpha)$ by (6.2),
3. $\Gamma \vdash_{\mathcal{E}} \phi \to_H (\sim \beta \to_H \sim \alpha)$ by 3.3 (4) and SMP applied to 1 and 2.

For the other implication, we assume that $\Gamma \vdash_{\mathcal{E}} \phi \to_H \psi$. Then $\Gamma \cup \{\phi\} \vdash_{\mathcal{E}} \phi \to_H \psi$. Since $\Gamma \cup \{\phi\} \vdash_{\mathcal{E}} \phi$, we have $\Gamma \cup \{\phi\} \vdash_{\mathcal{E}} \psi$ by (SMP). $\square$

THEOREM 6.3. *The Deduction Property holds in an extension $\mathcal{E}$ of the logic $\mathcal{DHMSH}$ for the connective $\to_H$ if and only if $\mathcal{E} \vdash (\alpha \to_H \beta) \to_H (\sim \beta \to_H \sim \alpha)$.*

PROOF: Let us assume that the Deduction Property holds in $\mathcal{E}$ for the conective $\to_H$. Note that $\alpha \to_H \beta \vdash_{\mathcal{E}} \alpha \to_H \beta$ and $\alpha \to_H \beta \vdash_{\mathcal{E}} \sim \beta \to_H \sim \alpha$ by (SCP). Hence $\vdash_{\mathcal{E}} (\alpha \to_H \beta) \to_H (\sim \beta \to_H \sim \alpha)$ by Deduction Property, or equivalently, $\mathcal{E} \vdash_{\mathcal{DHMSH}} (\alpha \to_H \beta) \to_H (\sim \beta \to_H \sim \alpha)$.

For the converse, let us assume that $\mathcal{E} \vdash (\alpha \to_H \beta) \to_H (\sim \beta \to_H \sim \alpha)$. By Lemma 6.2, the Deduction Property holds in $\mathcal{E}$ for the conective $\to_H$. $\square$

Recall that semi-Heyting algebras are pseudocomplemented with $x^* := x \to 0$ as the pseudocomplement of x. A semi-Heyting algebra $\mathbf{L}$ is a *Stone semi-Heyting algebra* if $\mathbf{L}$ satisfies the Stone identity: $x^* \vee x^{**} \approx 1$. Let $\mathbb{StSH}$ denote the variety of Stone semi-Heyting algebras. Recall also that if $\mathbf{A}$ is a semi-Heyting algebra, then $\langle A, \vee, \wedge, \to_H 0, 1\rangle$ is a Heyting algebra.

LEMMA 6.4. *Let $\mathbf{A} \in \mathbb{DHMSH}$ such that $\mathbf{A} \models (x \to_H y) \to_H (y' \to_H x') \approx 1$. Then*

1. $\mathbf{A} \models x \wedge x' \approx 0$,
2. $\mathbf{A} \models x^* \approx x'$,
3. $\mathbf{A} \models x^* \vee x^{**} \approx 1$.

PROOF: Let $a \in A$.

1. Since $a \to_H (a' \to_H 0) = (1 \to_H a) \to_H (a' \to_H 0) = (1 \to_H a) \to_H (a' \to_H 1') = 1$ in view of hypothesis, we have that $a \wedge (a' \to_H 0) = a \wedge (a \to_H (a' \to_H 0)) = a \wedge 1 = a$. Hence

$$a \wedge (a' \to_H 0) = a. \tag{6.3}$$

Then, $a \wedge a' \overset{(6.3)}{=} a \wedge (a' \to_H 0) \wedge a' = a \wedge (a' \to 0) \wedge a' = a \wedge a' \wedge 0$ $= 0$, proving (1).

2. Observe that $a^* \to_H a' = a^* \to_H (1 \to_H a') = (a \to_H 0) \to_H (1 \to_H a') = (a \to_H 0) \to_H (0' \to_H a') = 1$ by hypothesis. Hence
$$\mathbf{A} \models x^* \leq x'. \tag{6.4}$$
Next, $a' \wedge a^* = a' \wedge (a \to 0) = a' \wedge ((a' \wedge a) \to (a' \wedge 0)) \overset{(1)}{=} a' \wedge (0 \to 0)$ $= a'$. Hence $\mathbf{A} \models x' \leq x^*$. Now, using (6.4) we conclude that $a' = a^*$.

3. $a^* \vee a^{**} \overset{(2)}{=} a' \vee a'' \overset{(ED4)}{=} (a \wedge a')' \overset{(1)}{=} 0' = 1$,

proving the lemma. □

Lemma 6.5. *Let $\mathbf{A} \in \mathbb{DHMSH}$. Then the following conditions are equivalent in the algebra $\mathbf{A}$.*

1. $(x \to_H y) \to_H (y' \to_H x') \approx 1$,
2. $x^* \approx x'$.

Proof: Let $a, b \in A$. Observe that (1) implies (2) from Lemma 6.4. For the converse, suppose $\mathbf{A}$ satisfies the identity (2). Then, using (SH3) and the fact that $\to_H$ is a Heyting implication, we have that $(a \to_H b) \to_H (b' \to_H a') = (a \to_H b) \to_H (b^* \to_H a^*) = (b^* \wedge (a \to_H b)) \to_H a^* = (b^* \wedge (((b^* \wedge a) \to_H (b^* \wedge b)) \to_H a^* = (b^* \wedge ((b^* \wedge a) \to_H 0)) \to_H a^* = (b^* \wedge (a \to_H 0)) \to_H a^* = (b^* \wedge a^*) \to_H a^* = 1$, proving (1). □

Lemma 6.6. *Let $\mathbf{A}$ be a Stone semi-Heyting algebra. Let $\mathbf{A}^e$ be the expansion of $\mathbf{A}$ to the language $\langle \vee, \wedge, \to, ', 0, 1\rangle$, where we define $'$ by: $x' := x^*$. Then*

1. $\mathbf{A}^e \in \mathbb{DHMSH}$ *and satisfies the identity: $x' \approx x^*$,*
2. $\mathbf{A}^e \models (x \vee y)'' \approx x'' \vee y''$.

Proof: The lemma clearly follows from the well-known facts that $\mathbf{A} \models (x \vee y)^* \approx x^* \wedge y^*$ and $\mathbf{A} \models (x \wedge y)^* \approx x^* \vee y^*$. □

We will refer to the algebra $\mathbf{A}^e$ as an "essentially a Stone semi-Heyting algebra".

For $\mathbb{V}$ a subvariety of $\mathbb{S}t\mathbb{SH}$, we let

$$\mathbb{V}^e := \{\mathbf{A}^e : \mathbf{L} \in \mathbb{V}\}.$$

It is clear that $\mathbb{V}^e$ is a subvariety of $\mathbb{DHMSH}$.

We are now ready to present our main result of this section that describes precisely those extensions of the logic $\mathcal{DHMSH}$ that have the Deduction Property. The following theorem is immediate from Theorem 6.3, Lemma 6.5 and Lemma 6.6.

THEOREM 6.7. *The Deduction Property holds in an extension $\mathcal{E}$ of the logic $\mathcal{DHMSH}$ for the connective $\to_H$ if and only if the corresponding variety $\mathbb{E}$ is of the form $\mathbb{V}^e$, where $\mathbb{V} \subseteq \mathbb{S}\mathrm{t}\mathbb{SH}$. .*

6.1. Deduction theorem in the extensions of the logic $\mathcal{DQDSH}$

Recall that the variety $\mathbb{DQDSH}$ of dually quasi-De Morgan semi-Heyting algebras and the corresponding extension $\mathcal{DQDSH}$ of $\mathcal{DHMSH}$ were defined in Section 5.

In this section we show that Theorem 6.7 can be significantly improved for the extensions of the logic $\mathcal{DQDSH}$. In fact, we shall give an explicit description of the extensions of the logic $\mathcal{DQDSH}$ in which the Deduction Property holds.

For this purpose we need the following 2-element semi-Heyting algebras (with $0 < 1$), $\mathbf{2}$ and $\bar{\mathbf{2}}$ which are, up to isomorphism, the only two 2-element algebras in $\mathbb{SH}$.

$$\mathbf{2}: \quad \begin{array}{r|cc} \to: & 0 & 1 \\ \hline 0 & 1 & 1 \\ 1 & 0 & 1 \end{array} \qquad\qquad \bar{\mathbf{2}}: \quad \begin{array}{r|cc} \to: & 0 & 1 \\ \hline 0 & 1 & 0 \\ 1 & 0 & 1 \end{array}$$

Figure 2.

The algebras $\mathbf{2^e}$ and $\bar{\mathbf{2}}^{\mathbf{e}}$ denote the expansions of the semi-Heyting algebras $\mathbf{2}$ and $\bar{\mathbf{2}}$ by the unary operation $'$ defined as follows: $0' = 1$ and $1' = 0$. It is clear that $\mathbf{2^e}$ and $\bar{\mathbf{2}}^{\mathbf{e}}$ are, up to isomorphism, the only two 2-element algebras in $\mathbb{DQDSH}$ (in fact, in $\mathbb{DMSH}$).

Lemma 6.8. *Let $\mathbb{V}$ be a subvariety of $\mathbb{DQDSH}$ such that $\mathbb{V} \models (x \to_H y) \to (y' \to_H x') \approx 1$. Then, $\mathbb{V} \subseteq \mathbb{V}(\mathbf{2^e}, \bar{\mathbf{2}}^{\mathbf{e}})$, where $\mathbb{V}(\mathbf{2^e}, \bar{\mathbf{2}}^{\mathbf{e}})$ denotes the variety generated by $\{\mathbf{2^e}, \bar{\mathbf{2}}^{\mathbf{e}}\}$.*

Proof: The hypothesis and Lemma 6.5 (2) imply that $\mathbb{V} \models x' \approx x^*$. Hence $\mathbb{V} \subseteq \mathbb{V}(\mathbf{2^e}, \bar{\mathbf{2}}^{\mathbf{e}})$ by [29, Theorem 5.11]. □

The following theorem describes precisely those extensions of $\mathcal{DQDSH}$ in which the Deduction Property holds. Let $\mathbb{T}$ denote the trivial variety.

Theorem 6.9. *The Deduction Property holds in a logic $\mathcal{E} \in Ext(\mathcal{DQDSH})$ for $\to_H$ if and only if the corresponding variety is one of the following: $\mathbb{T}$, $\mathbb{V}(\mathbf{2^e})$, $\mathbb{V}(\bar{\mathbf{2}}^{\mathbf{e}})$, $\mathbb{V}(\mathbf{2^e}, \bar{\mathbf{2}}^{\mathbf{e}})$.*

Proof: The theorem is immediate in view of Theorem 5.9, Theorem 6.3, and Lemma 6.8. □

Since $\mathbb{DMSH} \subseteq \mathbb{DQDSH}$ and $\mathbb{DPCSH} \subseteq \mathbb{DQDSH}$ (see [29]), the following corollaries are immediate.

Corollary 6.10. The Deduction Property holds in a logic $\mathcal{E} \in Ext(\mathcal{DMSH})$ for $\to_H$ if and only if the corresponding variety is an element of $\{\mathbb{T}, \mathbb{V}(\mathbf{2^e}), \mathbb{V}(\bar{\mathbf{2}}^{\mathbf{e}}), \mathbb{V}(\mathbf{2^e}, \bar{\mathbf{2}}^{\mathbf{e}})$.

Corollary 6.11. The Deduction Property holds in a logic $\mathcal{E} \in Ext(\mathcal{DPCSH})$ for $\to_H$ if and only if the corresponding variety is either $\mathbb{T}$ or $\mathbb{V}(\mathbf{2^e})$ or $\mathbb{V}(\bar{\mathbf{2}}^{\mathbf{e}})$ or $\mathbb{V}(\mathbf{2^e}, \bar{\mathbf{2}}^{\mathbf{e}})$.

7. Logics in $Ext(\mathcal{DQDSH})$ corresponding to subvarieties of $\mathbb{DQDSH}$ generated by finitely many finite algebras

In this section, as applications of Theorem 5.9 and the algebraic results from [29], we will present several axiomatic extensions of the logic $\mathcal{DQDSH}$ corresponding to subvarieties of $\mathbb{DQDSH}$ generated by finitely many finite algebras, thus providing a solution to **PROBLEM B**.

7.1. 2-valued axiomatic extensions of $\mathcal{DQDSH}$

It was shown in Theorem 6.9 that the Deduction Property holds in an axiomatic extension of the logic $\mathcal{DQDSH}$ if and only if the corresponding variety is a subvariety of $\mathbb{V}(\mathbf{2^e}, \bar{\mathbf{2}}^{\mathbf{e}})$. So, it is only natural to ask for the axiomatizations of the extensions of the logic $\mathcal{DQDSH}$ corresponding to the subvarieties of $\mathbb{V}(\mathbf{2^e}, \bar{\mathbf{2}}^{\mathbf{e}})$.

The variety $\mathbb{V}(\mathbf{2^e}, \bar{\mathbf{2}}^{\mathbf{e}})$ and its only non-trivial proper subvarieties $\mathbb{V}(\mathbf{2^e})$ and $\mathbb{V}(\bar{\mathbf{2}}^{\mathbf{e}})$ were axiomatized in [29, Theorem 5.11]. $\mathbb{V}(\mathbf{2^e}, \bar{\mathbf{2}}^{\mathbf{e}})$ is defined by the identity: $x \leq x'^*$ (equivalently, $x \approx x'^*$), relative to the variety $\mathbb{DQDSH}$. The varieties $\mathbb{V}(\mathbf{2^e})$ and $\mathbb{V}(\bar{\mathbf{2}}^{\mathbf{e}})$ are defined, respectively, by the identities: $0 \to 1 \approx 1$ and $0 \to 1 \approx 0$, relative to $\mathbb{V}(\mathbf{2^e}, \bar{\mathbf{2}}^{\mathbf{e}})$. In view of these observations, we obtain from Theorem 5.9, the following corollaries defining their corresponding logics.

Let $\mathcal{L}(\mathbf{2^e}, \bar{\mathbf{2}}^{\mathbf{e}})$ (or $\mathcal{L}(\mathbb{V}(\mathbf{2^e}, \bar{\mathbf{2}}^{\mathbf{e}}))$ be the extension of the logic $\mathcal{DQDSH}$ corresponding to the variety $\mathbb{V}(\mathbf{2^e}, \bar{\mathbf{2}}^{\mathbf{e}})$. Let $\alpha \Leftrightarrow_H \beta$ denote the formula: $(\alpha \to_H \beta) \wedge (\beta \to_H \alpha)$.

It follows from [29] that $\mathcal{L}(\mathbf{2^e}, \bar{\mathbf{2}}^{\mathbf{e}})$ is a discriminator logic.

Corollary 7.1. The logic $\mathcal{L}(\mathbf{2^e}, \bar{\mathbf{2}}^{\mathbf{e}})$ is defined, as an extension of the logic $\mathcal{DQDSH}$, by the axiom:

$$(\sim \phi \to \perp) \Leftrightarrow_H \phi.$$

Let $\mathcal{L}(\mathbf{2^e})$ (or $\mathcal{L}(\mathbb{V}(\mathbf{2^e}))$) and $\mathcal{L}(\bar{\mathbf{2}}^{\mathbf{e}})$ (or $\mathcal{L}(\mathbb{V}(\bar{\mathbf{2}}^{\mathbf{e}}))$) denote, respectively, the extensions of the logic $\mathcal{L}(\mathbf{2^e}, \bar{\mathbf{2}}^{\mathbf{e}})$ corresponding to the varieties $\mathbb{V}(\mathbf{2^e})$ and $\mathbb{V}(\bar{\mathbf{2}}^{\mathbf{e}})$.

Corollary 7.2. The logic $\mathcal{L}(\mathbf{2^e})$ is defined, as an extension of the logic $\mathcal{L}(2^e, \bar{2}^e)$, by the axiom:

$$\perp \to \top.$$

(We note that $\mathcal{L}(\mathbf{2^e})$ is yet another axiomatization of the classical logic.).

Corollary 7.3. The logic $\mathcal{L}(\bar{\mathbf{2}}^{\mathbf{e}})$ is defined, as an extension of the logic $\mathcal{L}(\mathbf{2^e}, \bar{\mathbf{2}}^{\mathbf{e}})$, by the axiom:

$$(\perp \to \top) \to_H \perp.$$

Remark 7.4. Some features of the logics $\mathcal{L}(\bar{\mathbf{2}}^{\mathbf{e}})$ and $\mathcal{L}(\mathbf{2^e})$:

- The logic $\mathcal{L}(\bar{\mathbf{2}}^{\mathbf{e}})$ is "anti-classical" or "contra-classical" in the sense that the classically provable formula $\perp \to \top$ fails in it.
  (It is somewhat perplexing to us that the intuitionists accept the principle that says, "$False \to True = True$".)

- The logics $\mathcal{L}(\bar{\mathbf{2}}^{\mathbf{e}})$ and $\mathcal{L}(\mathbf{2}^{\mathbf{e}})$ are two of the coatoms in the lattice of extensions of the logic $\mathcal{DMSH}$ and hence that of $\mathcal{DQDSH}$ (and of $\mathcal{DHMSH}$).
- The implication $\to$ in $\mathcal{L}(\bar{\mathbf{2}}^{\mathbf{e}})$ is commutative.
- The logics $\mathcal{L}(\bar{\mathbf{2}}^{\mathbf{e}})$ and $\mathcal{L}(\mathbf{2}^{\mathbf{e}})$ are not only disriminator logics, but, in fact, are primal logics, since $\mathbf{2}^{\mathbf{e}}$ and $(\bar{\mathbf{2}}^{\mathbf{e}})$ are primal algebras.
- The logics $\mathcal{L}(\bar{\mathbf{2}}^{\mathbf{e}})$ and $\mathcal{L}(\mathbf{2}^{\mathbf{e}})$ do not have the Disjunction Property (i.e., if $\alpha \vee \beta$ is provable, then α is provable or β is provable.)

More features of the logic $\mathcal{L}(\bar{\mathbf{2}}^{\mathbf{e}})$ will be given in Remark 8.1 of Section 8.

Remark 7.5. The Deduction Theorem holds only in the three non-trivial logics, namely $\mathcal{L}(\mathbf{2}^{\mathbf{e}})$, $\mathcal{L}(\bar{\mathbf{2}}^{\mathbf{e}})$ and $\mathcal{L}(\mathbf{2}^{\mathbf{e}}, \bar{\mathbf{2}}^{\mathbf{e}})$ in the lattice of extensions of the logic $\mathcal{DQDSH}$, in view of Theorem 6.9.

7.2. 3-valued extensions of the logic $\mathcal{DQDSH}$

It was shown in [28] that there are, up to isomorphism, ten 3-element semi-Heyting algebras whose $\to$ operations are defined in Figure 3 below, where $0 < a < 1$.

Since $0' = 1$ and $1' = 0$, it is easy to see that there are exactly two expansions on each of the above 10 semi-Heyting algebras by a unary operation $'$ so that the expansions are $\mathbb{DQDSH}$-algebras. Ten of these that correspond to $a' = a$ are clearly in $\mathbb{DMSH}$. The other ten, that correspond to $a' = 1$ are in $\mathbb{DPCSH}$.

To put it more precisely, let $\mathbf{L}_i^{dm}$, $i = 1, 2, \ldots, 10$, denote the expansion of $\mathbf{L}_i$ by adding the unary operation $'$ such that $0' = 1$, $1' = 0$, and $a' = a$. Similarly, let $\mathbf{L}_i^{dp}$, $i = 1, 2, \ldots, 10$, denote the expansion of $\mathbf{L}_i$ by adding the unary operation $'$ such that $0' = 1$, $1' = 0$, and $a' = 1$. Then, clearly, $\mathbf{L}_i^{dm} \in \mathbb{DMSH}$ and $\mathbf{L}_i^{dp} \in \mathbb{DPCSH}$.

Let $\mathbf{C}^{dm} := \{\mathbf{L}_i^{dm} : i = 1, 2, \ldots, 10\}$, $\mathbf{C}^{dp} := \{\mathbf{L}_i^{dm} : i = 1, 2, \ldots, 10\}$ and let $\mathbf{C}_{20} := \mathbf{C}^{dm} \cup \mathbf{C}^{dp}$. Thus there are exactly 20 three-element $\mathbb{DQDSH}$-algebras, whose lattice reducts are chains. Let $\mathbb{DQDSHC}^3 := \mathbb{V}(\mathbf{C}_{20})$, the subvariety of $\mathbb{DQDSH}$ generated by all the 20 3-element $\mathbb{DQDSH}$-chains. Also, let $\mathbb{DMSHC}^3 := \mathbb{V}(\mathbf{C}^{dm})$ and let $\mathbb{DPCSHC}^3 := \mathbb{V}(\mathbf{C}^{dp})$.

We shall now present axiomatizations for the logics corresponding to $\mathbb{DQDSHC}^3$, $\mathbb{DMSHC}^3$, and $\mathbb{DPCSHC}^3$.

$\mathbf{L}_1$:

$\to$	0	a	1
0	1	1	1
a	0	1	1
1	0	a	1

$\mathbf{L}_2$:

$\to$	0	a	1
0	1	a	1
a	0	1	1
1	0	a	1

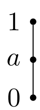
$\mathbf{L}_3$:

$\to$	0	a	1
0	1	1	1
a	0	1	a
1	0	a	1

$\mathbf{L}_4$:

$\to$	0	a	1
0	1	a	1
a	0	1	a
1	0	a	1

$\mathbf{L}_5$:

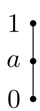


$\to$	0	a	1
0	1	a	a
a	0	1	1
1	0	a	1

$\mathbf{L}_6$:

$\to$	0	a	1
0	1	1	a
a	0	1	1
1	0	a	1

$\mathbf{L}_7$:



$\to$	0	a	1
0	1	a	a
a	0	1	a
1	0	a	1

$\mathbf{L}_8$:

$\to$	0	a	1
0	1	1	a
a	0	1	a
1	0	a	1

$\mathbf{L}_9$:

$\to$	0	a	1
0	1	0	0
a	0	1	1
1	0	a	1

$\mathbf{L}_{10}$:

$\to$	0	a	1
0	1	0	0
a	0	1	a
1	0	a	1

Figure 3.

The following theorem is immediate from [29, Lemma 10.2, Theorem 10.3, Corollary 10.4 and Theorem 11.1]. Let $x^+ := x'^{*'}$. In the rest of the paper, "equational base" is abbreviated to "base".

THEOREM 7.6 ([29]). *A base for* $\mathbb{DQDSHC}^3$*, relative to* $\mathbb{DQDSH}$*, is given by:*

(i) $x^{**} \approx x^{*'}$,

(ii) $x \wedge x^+ \leq y \vee y^*$. (Regularity)

The following theorem follows from Theorem 5.9 and Theorem 7.6.

THEOREM 7.7. *The logic* $\mathcal{DQDSHC}^3$ *corresponding to the variety* $\mathbb{DQDSHC}^3$ *is defined, as an extension of* $\mathcal{DQDSH}$*, by the following axioms:*

- $[(\phi \to \bot) \to \bot] \Leftrightarrow_H \sim(\phi \to \bot)$,
- $[\phi \wedge \sim(\sim\phi \to \bot)] \to_H [\psi \vee (\psi \to \bot)]$.

Since the logic $\mathcal{DQDSHC}^3$ is finitely axiomatized and the corresponding variety $\mathbb{DQDSHC}^3 = \mathbb{V}(\mathbf{C}_{20})$ is finitely generated, the following corollary is immediate.

COROLLARY 7.8. The logic $\mathcal{DQDSHC}^3$ is decidable.

Note also that the logic $\mathcal{DQDSHC}^3$ is a discriminator logic.

7.3. Logics $\mathcal{DMSHC}^3$ and $\mathcal{DPCSHC}^3$

We know from Section 7.2 that $\mathbf{L}_i^{dm} \in \mathbb{DMSH}$ and $\mathbf{L}_i^{dp} \in \mathbf{DPCSH}$, $i = 1, 2, \ldots, 10$, and also that $\mathbb{DMSHC}^3 = \mathbb{V}(\mathbf{C^{dm}})$ and $\mathbb{DPCSHC}^3 = \mathbb{V}(\mathbf{C^{dp}})$.

The following theorem is immediate from Theorem 7.6.

THEOREM 7.9.

(a) *A base for* $\mathbb{DMSHC}^3$*, relative to* $\mathbb{DQDSHC}^3$*, is given by:*

$$x'' \approx x.$$

(b) *A base for* $\mathbb{DPCSHC}^3$*, relative to* $\mathbb{DQDSHC}^3$*, is given by:*

$$x \vee x' \approx 1.$$

Let $\mathcal{DMSHC}^3$ and $\mathcal{DPCSHC}^3$ denote the extensions of the logic $\mathcal{DQDSHC}^3$ corresponding to the varieties $\mathbb{DMSHC}^3$ and $\mathbb{DPCSHC}^3$, respectively. The following theorem is immediate from Theorem 5.9 and Theorem 7.9.

THEOREM 7.10.

1. *$\mathcal{DMSHC}^3$ is defined, as an extension of $\mathcal{DQDSHC}^3$ by the following axiom:*

$$\phi \to_H \sim\sim \phi.$$

2. *$\mathcal{DPCSHC}^3$ is defined, relative to the logic $\mathcal{DQDSHC}^3$ by the following axiom:*

$$\phi \vee \sim \phi.$$

It is clear that the logics $\mathcal{DMSHC}^3$ and $\mathcal{DPCSHC}^3$ are decidable. In view of the above Theorem it is also clear that the logic $\mathcal{DPCSHC}^3$ does not have the Disjunction Property.

7.4. 3-valued extensions of $\mathcal{DMSHC}^3$ and of $\mathcal{DPCSHC}^3$

We are ready to look at the problem of axiomatization for the logics associated with the 20 3-element chains in $\mathbf{C}_{20}$. We need to recall another (algebraic) result from [29] that gives a base for each of 3-chains in $\mathbf{C}^{dm}$ and $\mathbf{C}^{dp}$. To this end, we need the following identities from [29]:

(C1) $x \vee (x \to y) \approx (x \to y)^* \to x$,

(C2) $x \vee [y \to (x \vee y)] \approx (0 \to x) \vee (x \to y)$,

(C3) $x \vee (y \to x) \approx [(x \to y) \to y] \to x$,

(C4) $x \vee (x \to y) \approx x \to [x \vee (y \to 1)]$,

(C5) $(x \to y) \to (0 \to y) \approx x \vee [(x \wedge y) \to 1]$,

(C6) $x^* \vee (x \to y) \approx (x \vee y) \to y$,

(C7) $x \vee (0 \to x) \vee (y \to 1) \approx x \vee [(x \to 1) \to (x \to y)]$,

(C8) $x \vee y \vee (x \to y) \approx x \vee [(x \to y) \to 1]$, (C9) $x \vee [(0 \to y) \to y] \approx x \vee [(x \to 1) \to y]$,

(C10) $x \vee [x \to (y \wedge (0 \to y))] \approx x \to [(x \to y) \to y]$,

(C11) $(0 \to 1)^* = 0$,

(C12) $x \vee y \vee [y \to (y \to x)] \approx x \to [x \vee (0 \to y)]$,

(C13) $x \vee (x \to y) \approx x \vee [(x \to y) \to 1]$,

(C14) $0 \to 1 \approx 0$ (FTF identity),

(C15) $x \to y \approx y \to x$ (commutative identity).

In Theorem 7.11 below, we abbreviate "is a base, relative to $\mathbb{DMSHC}^3$ [$\mathbb{DPCSHC}^3$]" to just "is a base".

The reader should keep in mind that the following theorem is really a simultaneous presentation of two separate theorems (in order to keep the size of the paper within limits). One of the two theorems is regarding $\mathbb{DMSHC}^3$-algebras and the other is about $\mathbb{DPCSHC}^3$-algebras. As an illustration, item (i), when decoded, yields the following two (independent) statements:

(i_{dm}): $\{$(C1)$\}$ is a base, relative to $\mathbb{DMSHC}^3$, for the variety $\mathbf{V}(\mathbf{L_1^{dm}})$,

and

(i_{dp}): $\{$(C1)$\}$ is a base, relative to $\mathbb{DPCHC}^3$, for the variety $\mathbf{V}(\mathbf{L_1^{dp}})$.

A similar remark applies to each of the other items of Theorem 7.11 as well.

Theorem 7.11 is immediate from [29, Theorem 11.2].

THEOREM 7.11.

(i) $\{$(C1)$\}$ *is a base for the variety* $\mathbf{V}(\mathbf{L_1^{dm}})$ [$\mathbf{V}(\mathbf{L_1^{dp}})$],

(ii) $\{$(C2), (C3)$\}$ *is a base for the variety* $\mathbf{V}(\mathbf{L_2^{dm}})$ [$\mathbf{V}(\mathbf{L_2^{dp}})$],

(iii) $\{$(C2), (C4)$\}$ *is a base for the variety* $\mathbf{V}(\mathbf{L_3^{dm}})$ [$\mathbf{V}(\mathbf{L_3^{dp}})$],

(iv) $\{$(C4), (C5)$\}$ *is a base for the variety* $\mathbf{V}(\mathbf{L_4^{dm}})$ [$\mathbf{V}(\mathbf{L_4^{dp}})$],

(v) $\{(C7)\}$ *is a base for the variety* $\mathbf{V}(\mathbf{L_5^{dm}})$ $[\mathbf{V}(\mathbf{L_5^{dp}})]$,

(vi) $\{(C8)\}$ *is a base for the variety* $\mathbf{V}(\mathbf{L_6^{dm}})$ $[\mathbf{V}(\mathbf{L_6^{dp}})]$,

(vii) $\{(C9), (C10)\}$ *is a base for the variety* $\mathbf{V}(\mathbf{L_7^{dm}})$ $[\mathbf{V}(\mathbf{L_7^{dp}})]$,

(viii) $\{(C11), (C12)\}$ *is a base for the variety* $\mathbf{V}(\mathbf{L_8^{dm}})$ $[\mathbf{V}(\mathbf{L_8^{dp}})]$,

(ix) $\{(C6), (C13), (C14)\}$ *is a base for the variety* $\mathbf{V}(\mathbf{L_9^{dm}})$ $[\mathbf{V}(\mathbf{L_9^{dp}})]$,

(x) $\{(C15)\}$ *is a base for the variety* $\mathbf{V}(\mathbf{L_{10}^{dm}})$ $[\mathbf{V}(\mathbf{L_{10}^{dp}})]$.

We are now ready to present the axiomatizations for the logics associated with the 20 3-element chains in $\mathbf{C}_{20}$.

Let $\mathcal{L}(\mathbf{L}_i^{dm})$ (or $\mathcal{L}(\mathbb{V}(\mathbf{L}_i^{dm}))$) denote the extension of the logic $\mathcal{DMSHC}^3$ corresponding to the variety $\mathbb{V}(\mathbf{L}_i^{dm})$, for $i = 1, 2, \cdots, 10$. Also, let $\mathcal{L}(\mathbf{L_i^{dp}})$ (or $\mathcal{L}(\mathbb{V}(\mathbf{L}_i^{dp}))$) denote the extension of the logic $\mathcal{DPCSHC}^3$ corresponding to the variety $\mathbb{V}(\mathbf{L}_i^{dp})$, for $i = 1, 2, \cdots, 10$.

In what follows, "defined, as an extension of the logic $\mathcal{DMSHC}_3$ $[\mathcal{DPCSHC}_3]$, by" is abbreviated to "defined by". The following theorem will follow from Theorem 5.9, Theorem 7.9, Theorem 7.10, and Theorem 7.11.

Theorem 7.12 below is, like Theorem 7.11, a simultaneous presentation of two separate theorems (in order to keep the size of the paper within limits). One of the two theorems is regarding the extensions of $\mathcal{DMSHC}^3$-algebras and the other is about the extensions of $\mathcal{DPCSHC}^3$-algebras.

THEOREM 7.12.

(a) $\mathcal{L}(\mathbf{L}_1^{dm})$ $[\mathcal{L}(\mathbf{L}_1^{dp})]$ *is defined by the following axiom:*

$$[\phi \vee (\phi \to \psi)] \Leftrightarrow_H [((\phi \to \psi) \to \perp) \to \phi].$$

(b) $\mathcal{L}(\mathbf{L}_2^{dm})$ $[\mathcal{L}(\mathbf{L}_2^{dp})]$ *is defined by the following axioms:*

(i) $[\phi \vee \{\psi \to (\phi \vee \psi)\}] \Leftrightarrow_H [(\perp \to \phi) \vee (\phi \to \psi)]$,

(ii) $[\phi \vee (\psi \to \phi)] \Leftrightarrow_H [\{(\phi \to \psi) \to \psi\} \to \phi]$.

(c) $\mathcal{L}(\mathbf{L}_3^{dm})$ $[\mathcal{L}(\mathbf{L}_3^{dp})]$ *is defined by the following axioms:*

(i) $[\phi \vee \{\psi \to (\phi \vee \psi)\}] \Leftrightarrow_H [(\perp \to \phi) \vee (\phi \to \psi)]$,

(ii) $[\phi \vee (\phi \to \psi)] \Leftrightarrow_H [\phi \to \{\phi \vee (\psi \to \top)\}]$.

(d) $\mathcal{L}(\mathbf{L}_4^{dm})$ $[\mathcal{L}(\mathbf{L}_4^{dp})]$ *is defined by the following axioms:*

(i) $[\phi \vee (\phi \to \psi)] \Leftrightarrow_H [\phi \to \{\phi \vee (\psi \to \top)\}]$,

(ii) $[(\phi \to \psi) \to (\perp \to \psi)] \Leftrightarrow_H [\phi \vee \{(\phi \wedge \psi) \to \top\}]$.

(e) $\mathcal{L}(\mathbf{L}_5^{dm})$ $[\mathcal{L}(\mathbf{L}_5^{dp})]$ *is defined by the following axiom:*

$$[\phi \vee (\perp \to \phi) \vee (\psi \to \top)] \Leftrightarrow_H [\phi \vee \{(\phi \to \top) \to (\phi \to \psi)\}].$$

(f) $\mathcal{L}(\mathbf{L}_6^{dm})$ $[\mathcal{L}(\mathbf{L}_6^{dp})]$ *is defined by the following axiom:*

$$[\phi \vee \psi \vee (\phi \to \psi)] \Leftrightarrow_H [\phi \vee \{(\phi \to \psi) \to \top\}].$$

(g) $\mathcal{L}(\mathbf{L}_7^{dm})$ $[\mathcal{L}(\mathbf{L}_7^{dp})]$ *is defined by the following axioms:*

(i) $[\phi \vee \{(\perp \to \psi) \to \psi\}] \Leftrightarrow_H [\phi \vee \{(\phi \to \top) \to \psi\}]$,

(ii) $[(\phi \vee \{\phi \to (\psi \wedge (\perp \to \psi))\}] \Leftrightarrow_H [\phi \to \{(\phi \to \psi) \to \psi\}]$.

(h) $\mathcal{L}(\mathbf{L}_8^{dm})$ $[\mathcal{L}(\mathbf{L}_8^{dp})]$ *is defined by the following axioms:*

(i) $((\perp \to \top) \to \perp) \to_H \perp$,

(ii) $[(\phi \vee \psi \vee \{\psi \to (\psi \to \phi)\}] \Leftrightarrow_H [\phi \to \{\phi \vee (\perp \to \psi)\}]$.

(i) $\mathcal{L}(\mathbf{L}_9^{dm})$ $[\mathcal{L}(\mathbf{L}_9^{dp})]$ *is defined by the following axioms:*

(i) $[\phi^* \vee (\phi \to \psi)] \Leftrightarrow_H [(\phi \vee \psi) \to \psi]$,

(ii) $[\phi \vee (\phi \to \psi)] \Leftrightarrow_H [\phi \vee \{(\phi \to \psi) \to \top\}]$,

(iii) $(\perp \to \top) \to_H \perp$.

(j) $\mathcal{L}(\mathbf{L}_{10}^{dm})$ $[\mathcal{L}(\mathbf{L}_{10}^{dp})]$ *is defined by the following axioms:*

$$(\phi \to \psi) \to_H (\psi \to \phi).$$

Remark 7.13. Some features of these logics:

- The logics $\mathcal{L}(\mathbf{L_i^{dp}}), i \in \{2, 3, \ldots, 10\}$ and the logics $\mathcal{L}(\mathbf{L_i^{dm}}), i \in \{2, 3, \ldots, 10\}$ are, just like the logic $\mathcal{L}(\bar{\mathbf{2}}^{\mathbf{e}})$, "anti-classical" in the sense that the classically provable formula $\perp \to \top$ fails in these logics.
- Each of the logics $\mathcal{L}(\mathbf{L_i^{dp}}), i \in \{5, 6, 7, 8\}$ and $\mathcal{L}(\mathbf{L_i^{dp}}), i \in \{5, 6, 7, 8\}$, just like $\mathcal{L}(\mathbf{2^e})$ and $\mathcal{L}(\bar{\mathbf{2}}^{\mathbf{e}})$, is a coatom in the lattice of extensions of the logic $\mathcal{DQDSH}$.

- Each of the logics $\mathcal{L}(\mathbf{L_i^{dp}}), i \in \{1,2,3,4\}$ and $\mathcal{L}(\mathbf{L_i^{dm}}), i \in \{1,2,3,4\}$ is covered by the coatom $\mathcal{L}(\mathbf{2^e})$, the classical propositional logic, while each of the logics $\mathcal{L}(\mathbf{L_i^{dp}}), i \in \{9,10\}$ and $\mathcal{L}(\mathbf{L_i^{dm}}), i \in \{9,10\}$ is covered by the coatom $\mathcal{L}(\bar{\mathbf{2}}^{\mathbf{e}})$.
- In the logics $\mathcal{L}(\mathbf{L_i^{dp}}), i \in \{9,10\}$ and $\mathcal{L}(\mathbf{L_i^{dm}}), i \in \{9,10\}$, Moreover, in the logics $\mathcal{L}(\mathbf{L_{10}^{dp}})$ and $\mathcal{L}(\mathbf{L_{10}^{dm}})$, the connective $\to$ is commutative.
- The logics $\mathcal{L}(\mathbf{L}_i^{dm})$ and $\mathcal{L}(\mathbf{L_i^{dp}})$, $i = 1, 2, \cdots, 10$, do not have the (DP) as the formula $\alpha^* \vee \alpha^{**}$ is provable in these logics.
- The logics $\mathcal{L}(\mathbf{L_i^{dp}}), i \in \{1,2,3,\cdots,10\}$ and the logics $\mathcal{L}(\mathbf{L_i^{dm}}), i \in \{1,2,3,\cdots,10\}$ are quasiprimal.
- Each of the logics $\mathcal{L}(\mathbf{L_i^{dp}}), i \in \{5,6,7,8\}$ and $\mathcal{L}(\mathbf{L_i^{dp}}), i \in \{5,6,7,8\}$, just like $\mathcal{L}(\mathbf{2^e})$ and $\mathcal{L}(\bar{\mathbf{2}}^{\mathbf{e}})$, is primal.

Further features of some of these logics will be given in Remark 8.1.

We note that all the logics mentioned in this subsection are decidable as their corresponding varieties are easily seen to have the finite model property.

7.5. 3-valued Łukasiewicz Logic revisited

It is worthwhile to point out that the logic $\mathcal{L}(\mathbf{L_1^{dm}})$, defined earlier, has an interesting relationship with the well-known 3-valued Łukasiewicz logic. Let us recall the definition of 3-valued Łukasiewicz algebras.

An algebra $\mathbf{A} = \langle A, \vee, \wedge, ', d_1, d_2, 0, 1\rangle$ is a 3-valued Łukasiewicz algebras if

1. $\langle A, \vee, \wedge, ', 0, 1\rangle$ is a De Morgan algebra,
2. $d_i(x \vee y) = d_i(x) \vee d_i(y)$, for $i = 1, 2$,
3. $d_i(x) \vee (d_i(x))' = 1$, for $i = 1, 2$,
4. $d_i(d_j(x)) = d_j(x)$, for $i = 1, 2$,
5. $d_i(x') = (d_{3-i}(x))'$, for $i = 1, 2$,
6. $d_1(x) \leq d_2(x)$,
7. If $d_1(x) = d_1(y)$ and $d_2(x) = d_2(y)$ then $x = y$.

Let $\mathbf{L} = \langle \{0, a, 1\}, \vee, \wedge,' .d_1, d_2, 0, 1\rangle$ be the algebra such that $\langle \{0, a, 1\}, \vee, \wedge,', 0, 1\rangle$ is a 3-element Kleene algebra with $0 < a < 1$, and d_1 and d_2 are unary operations defined as follows: $d_1(0) = d_1(a) = 0, d_1(1) = 1$, and $d_2(0) = 0$, and $d_2(1) = d_2(a) = 1$. Then it is routine to verify that $\mathbf{L}$ is a 3-valued Łukasiewicz algebra. It is well-known that $\mathbb{V}(\mathbf{L})$ is precisely the variety of all 3-valued Łukasiewicz algebras.

THEOREM 7.14. *The logic $\mathcal{L}(\mathbf{L_1^{dm}})$ is equivalent to the 3-valued* Łukasiewicz *logic.*

PROOF: It suffices to prove that the variety $\mathbb{V}(\mathbf{L_1^{dm}})$ is term-equivalent to the variety $\mathbb{V}$(Ł). Without loss of generality, we can assume that $\mathbf{L_1^{dm}}$ and Ł have the same universe, say $L = \{0, a, 1\}$ with $0 < a < 1$. Given $\mathbf{L_1^{dm}}$, define the unary operations d_1 and d_2 on L by: $d_1(x) = x'^*$ and $d_2 = x^{*\prime}$. Then it is straightforward to verify that $\langle L; \vee, \wedge,', d_1, d_2, 0, 1\rangle =$ Ł. To prove the converse, let us first define the unary function * on $\mathbf{L}$ by: $x^* := d_1((d_2(x))')$. It is routine to verify that * is the pseudocomplement operation on L. Using * we can now define the Katriňák's implication $\to$ by:

$$x \to y := (x^* \vee y^{**}) \wedge [(x \vee x^*)'^{*\prime} \vee x^* \vee y \vee y^*].$$

Then, $\to$ is the Heyting implication (see [26]). Hence, it follows that $\langle L; \vee, \wedge, \to,', 0, 1\rangle = \mathbf{L_1^{dm}}$. The theorem is now proved. □

7.6. 4-valued extensions of $\mathcal{DQDSH}$ with Boolean semi-Heyting reducts

Recall that the variety $\mathbb{DQDSH}$ was defined in Section 5. An algebra $\mathbf{L}$ is a *dually quasi-De Morgan Boolean semi-Heyting algebra* ($\mathbb{DQDBSH}$-algebra, for short) if its term-reduct $\langle L, \vee, \wedge,^*, 0, 1\rangle$ is a Boolean semi-Heyting algebra, that is, $\mathbf{L} \models x \vee x^* \approx 1$. The variety of such algebras is denoted by $\mathbb{DQDBSH}$.

Let $\mathcal{DQDBSH}$ denote the logic corresponding to the variety $\mathbb{DQDBSH}$. The following theorem is now immediate, in view of Theorem 5.9.

THEOREM 7.15. *The logic $\mathcal{DQDBSH}$ is defined, relative to $\mathcal{DQDSH}$ by the following axiom:*

(B) $$\phi \vee (\phi \to \bot).$$

$\mathbf{D_1}$:

$\to$	0	1	a	b
0	1	0	b	a
1	0	1	a	b
a	b	a	1	0
b	a	b	0	1

$\mathbf{D_2}$:

$\to$	0	1	a	b
0	1	1	1	1
1	0	1	a	b
a	b	1	1	b
b	a	1	a	1

$\mathbf{D_3}$:

$\to$	0	1	a	b
0	1	a	1	a
1	0	1	a	b
a	b	a	1	0
b	a	1	a	1

Figure 4.

We note that $\mathcal{DQDBSH}$ is a discriminator logic. In view of the above Theorem it is also clear that the logic $\mathcal{DQDBSH}$ does not have the Disjunction Property.

The concrete description of the lattice of subvarieties of $\mathbb{DQDBSH}$ was given in [29]. We now wish to present the axiomatizations for corresponding extensions of the logic $\mathcal{DQDBSH}$. Toward this end, the following three algebras will be needed.

Figure 4 defines the $\to$ operation on the three 4-element algebras $\mathbf{D_1}$, $\mathbf{D_2}$ and $\mathbf{D_3}$, each of whose lattice reduct is the 4-element Boolean lattice having the universe $\{0, a, b, 1\}$, with b as the complement of a, and $'$ is defined as follows: $a' = a$, $b' = b$, $0' = 1$ and $1' = 0$.

The algebras $\mathbf{D_1}$, $\mathbf{D_2}$, and $\mathbf{D_3}$ are the only simple (=subdirectly irreducible) algebras in $\mathbb{DQDBSH}$.

The following theorem, which follows immediately from [29, Corollary 9.4], reveals the structure of $\mathbb{DQDBSH}$.

Theorem 7.16. $\mathbb{DQDBSH} = \mathbb{V}(\mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}) = \mathbb{DMBSH}$.

The above theorem leads us to the following decidability result, in view of Theorem 5.9.

Corollary 7.17. The logic $\mathcal{DQDBSH}$ is decidable.

We will now turn our attention to the axiomatization of logics corresponding to the varieties generated by these algebras. The following theorem is taken from [29, Theorem 9.5].

Theorem 7.18.

(1) *A base for the variety* $\mathbb{V}(\mathbf{D_1})$*, modulo* $\mathbb{DQDBSH}$*, is given by*

$$0 \to 1 \approx 0.$$

(2) *A base for* $\mathbb{V}(\mathbf{D_2})$*, modulo* $\mathbb{DQDBSH}$*, is given by*

$$0 \to 1 \approx 1.$$

(3) *A base for the variety* $\mathbb{V}(\mathbf{D_3})$*, modulo* $\mathbb{DQDBSH}$*, is given by*

$$(0 \to 1)' \approx 0 \to 1.$$

The following corollary will now follow as an application of Theorem 5.9, Theorem 7.16 and Theorem 7.18.

Let $\mathcal{L}(\mathbf{D_i})$ (or $\mathcal{L}(\mathbb{V}(\mathbf{D_i}))$) denote the extension of the logic $\mathcal{DMBSH}$ corresponding to the variety $\mathbb{V}(\mathbf{D_i})$ for $\mathrm{i} = 1, 2, 3$.

In the rest of this section, "defined, relative to the logic $\mathcal{DMBSH}$, by" is abbreviated to "defined by".

Corollary 7.19.

(1) The logic $\mathcal{L}(\mathbf{D_1})$ is defined by the axiom:

$$(\perp \to \top) \to_H \perp.$$

(2) The logic $\mathcal{L}(\mathbf{D_2})$ is defined by the axiom:

$$\perp \to \top.$$

(3) The logic $\mathcal{L}(\mathbf{D_3})$ is defined by the axiom:

$$\sim (\perp \to \top) \Leftrightarrow_H (\perp \to \top).$$

It is clear that the logics $\mathcal{L}(\mathbf{D_i}), \mathrm{i} \in \{1, 2, 3\}$, are decidable.

Remark 7.20. Some features of the logics $\mathcal{L}(\mathbf{D_i}), \mathrm{i} \in \{1, 2, 3\}$:

(a) The logics $\mathcal{L}(\mathbf{D_i}), \mathrm{i} \in \{1, 3\}$, are anti-classical since the formula $\perp \to \top$ is not provable in each of them.

(b) The logics $\mathcal{L}(\mathbf{D_1})$ and $\mathcal{L}(\mathbf{D_2})$, are covered, respectively, by the coatoms $\mathcal{L}(\bar{\mathbf{2}}^{\mathbf{e}})$ and $\mathcal{L}(\mathbf{2}^{\mathbf{e}})$ in the lattice of extensions of the logic $\mathcal{DQDSH}$.

(c) In the logic $\mathcal{L}(\mathbf{D_1})$, the connective $\to$ is commutative.

(d) The logics $\mathcal{L}(\mathbf{D_1})$ and $\mathcal{L}(\mathbf{D_2})$ are quasiprimal, in the sense that their corresponding varieties are generated by quasiprimal algebras $\mathbf{D_1}$ and $\mathbf{D_2}$ respectively.

(e) The logic $\mathcal{L}(\mathbf{D_3})$, just like $\mathcal{L}(\mathbf{2}^{\mathbf{e}})$ and $\mathcal{L}(\bar{\mathbf{2}}^{\mathbf{e}})$, is a coatom in the lattice of extensions of the logic $\mathcal{DMSH}$ and hence, of $\mathcal{DHMSH}$.

(f) The logic $\mathcal{L}(\mathbf{D_3})$ is primal.

Here is another feature of these algebras, since they have Boolean reducts (i.e., they satisfy the identity: $x \vee x^* \approx 1$).

Theorem 7.21. *The logics $\mathcal{L}(\mathbf{2}^{e})$, $\mathcal{L}(\bar{\mathbf{2}}^{\mathbf{e}})$, $\mathcal{L}(\mathbf{2}, \bar{\mathbf{2}}^{\mathbf{e}})$, $\mathcal{L}(\mathbb{V}(\mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}))$, $\mathcal{L}(\mathbf{D_1})$, $\mathcal{L}(\mathbf{D_2})$, $\mathcal{L}(\mathbf{D_3})$ do not have the disjunction property.*

Further features of some of these and other logics will be given in Remarks 8.1 and 8.2.

8. Connection to connexive logics

The fact that the identity $0 \to 1 \approx 0$ holds in some semi-Heyting algebras led us to consider, in 2020, the possibility that there might be connexive logics arising from semi-Heyting algebras. We noticed in May 2020 that that indeed was the case.

Let $\mathbf{L}$ be a language containing the connective symbols: $\to$ for implication and $\neg$ for negation. A logic $\mathcal{L}$ in $\mathbf{L}$ is a *connexive logic* (see [38], for example) if the following Aristotle's Theses and Boethius' Theses are theorems in $\mathcal{L}$:

Aristotle's Theses:

(AT) $\neg(\neg\alpha \to \alpha)$,

(AT') $\neg(\alpha \to \neg\alpha)$.

Boethius' Theses:

(BT) $(\alpha \to \beta) \to \neg(\alpha \to \neg\beta)$,

(BT') $(\alpha \to \neg\beta) \to \neg(\alpha \to \beta)$.

For more details on the motivation, the origin and the history of connexive logics, see [38] and [18]. Many of the extensions of the logics $\mathcal{SH}$ and $\mathcal{DHMSH}$, to our surprise, turn out to be connexive logics with $\neg\alpha := \alpha \to \bot$. We present a few of these below. (More will be said in the paper [12] which is in preparation.)

Recall that $\bar{\mathbf{2}}$, $\mathbf{L_9}$, and $\mathbf{L_{10}}$, are in $\mathbb{SH}$ and that their corresponging logics $\mathcal{L}(\bar{\mathbf{2}})$, $\mathcal{L}(\mathbf{L_9})$ and $\mathcal{L}(\mathbf{L_{10}})$ are extensions of the semi-intuitionistic logic $\mathcal{SI}$.

Remark 8.1.

(a) The logics $\mathcal{L}(\bar{\mathbf{2}})$ and $\mathcal{L}(\mathbf{L_9})$, which are extensions of the semi-intuitionistic logic $\mathcal{SI}$, are connexive logics since the corresponding varieties $\mathbb{V}(\bar{\mathbf{2}})$ and $\mathbb{V}(\mathbf{L_9})$ satisfy the following identities:

(i) $(x^* \to x)^* \approx 1$,

(ii) $(x \to x^*)^* \approx 1$,

(iii) $(x \to y) \to (x \to y^*)^* \approx 1$,

(iv) $(x \to y^*) \to (x \to y)^* \approx 1$.

(b) (AT) and (AT') are theorems in the logic $\mathcal{L}(\mathbf{L_{10}})$, since the corresponding variety $\mathbb{V}(\mathbf{L_{10}})$ satisfies the identities (i) and (ii), while (BT) and (BT') are not theorems in the logics $\mathcal{L}(\mathbf{L_{10}})$.

(c) Since it is easily seen that $\mathbb{V}(\mathbf{2})$ and $\mathbb{V}(\bar{\mathbf{2}})$ are term-equivalent, it follows that the classical logic $\mathcal{L}(\mathbf{2})$ is equivalent to $\mathcal{L}(\bar{\mathbf{2}})$. Hence the classical logic $\mathcal{L}(\mathbf{2})$ can be viewed as a connexive logic.

Remark 8.2.

(a) The logics $\mathcal{L}(\bar{\mathbf{2}}^{\mathbf{e}})$, $\mathcal{L}(\mathbf{L_9^{dp}})$, $\mathcal{L}(\mathbf{L_9^{dm}})$ and $\mathcal{D}_1$, which are extensions of the logic $\mathcal{DHMSH}$, are connexive logics since it is easily verified that their corresponding varieties $\mathbb{V}(\bar{\mathbf{2}}^{\mathbf{e}})$, $\mathbb{V}(\mathbf{L_9^{dp}})$, $\mathbb{V}(\mathbf{L_9^{dm}})$, and $\mathbb{V}(\mathbf{D}_1)$ satisfy the identities (i)–(iv).

(b) (AT) and (AT') are theorems in the logics $\mathcal{L}(\mathbf{L_{10}^{dp}})$ and $\mathcal{L}(\mathbf{L_{10}^{dm}})$, since the corresponding varieties $\mathbb{V}(\mathbf{L_{10}^{dp}})$ and $\mathbb{V}(\mathbf{L_{10}^{dp}})$ satisfy the identities (i) and (ii), while (BT) and (BT') are not theorems in the logics $\mathcal{L}(\mathbf{L_{10}^{dp}})$ and $\mathcal{L}(\mathbf{L_{10}^{dm}})$.

(c) Since it is easy to see that $\mathbb{V}(\mathbf{L_{1}^{dm}})$ and $\mathbb{V}(\mathbf{L_{9}^{dm}})$ are term-equivalent, it follows that the logic $\mathcal{L}(\mathbf{L_{1}^{dm}})$ is equivalent to $\mathcal{L}(\mathbf{L_{9}^{dm}})$. Hence the logic $\mathcal{L}(\mathbf{L_{1}^{dm}})$can be viewed as a connexive logic. Furthermore, since the logic $\mathcal{L}(\mathbf{L_{1}^{dm}})$ is equivalent to the 3-valued Łukasiewicz logic, it follows that the 3-valued Łukasiewicz logic is a connexive logic.

(d) Since it is easily observed that $\mathbb{V}(\mathbf{D_1})$ and $\mathbb{V}(\mathbf{D_2})$ are term-equivalent, it follows that the logic $\mathcal{L}(\mathbf{D_1})$ is equivalent to $\mathcal{L}(\mathbf{D_2})$. Hence the logic $\mathcal{L}(\mathbf{D_2})$ can be viewed as a connexive logic.

Jarmużek and Malinowski [18] have recently introduced the notion of a "quasi-connexive" logic. A logic is quasi-connexive iff it is not connexive, but at least one of (AT), (AT'), (BT) and (BT') is a theorem in the logic. Thus, in view of the above remark, the logics $\mathcal{L}(\mathbf{L_{10}^{dp}})$ and $\mathcal{L}(\mathbf{L_{10}^{dm}})$, as well as the extension $\mathcal{L}(\mathbf{L_{10}})$ of $\mathcal{SH}$, can be viewed as quasi-connexive logics.

We now mention a few facts about the relationships among the Aristotle's Theses and Boethius' Theses in the logics $\mathcal{SH}$ and $\mathcal{DHMSH}$ whose proofs will appear in the forthcoming paper [12].

THEOREM 8.3. *In the logic $\mathcal{SH}$, and hence in $\mathcal{DHMSH}$,*

(a) (AT) *and* (AT') *are equivalent.*

(b) (AT), (AT') *and* (BT') *are provable from* (BT).

(c) (AT), (AT') *are provable from* (BT'), *but* (BT) *is not.*

(d) *If $\mathbf{A} \in \mathbb{SH}$ satisfies (BT), then $\mathbf{A} \models x \to y^* \approx y \to x^*$.*

THEOREM 8.4. *Let $\mathbf{A} \in \mathbb{RDPCH}$ and define a new operation $\Rightarrow$ on A by: $x \Rightarrow y := (x \to y) \wedge (x^* \to y^*)$. Then $(x \Rightarrow y) \Rightarrow (x \Rightarrow y^*)^* = 1$.*

As an application of Theorem 8.4 and recent results of [4], the following corollary is deduced in [12].

COROLLARY 8.5. There are $2^{\aleph_0}$ axiomatic extensions of the logic $\mathcal{RDPCH}$.

Remark 8.6. We propose that any logic in which the (classically provable) formula $\perp \to \top$ is not provable be included in the family of connexive logics since such a logic would be not only anti-classical but also anti-intuitionistic logic. Accordingly, the logics $\mathcal{L}(\bar{\mathbf{2}}^{\mathbf{e}})$, $\mathcal{L}({\mathbf{L_i}}^{dp}), i = 5, \ldots, 10$, $\mathcal{L}({\mathbf{L_i}}^{dm}), i = 5, \ldots, 10$, $\mathbb{V}(\mathbf{D}_1)$ and $\mathbb{V}(\mathbf{D}_3)$ can be considered as connexive logics.

9. Two infinite chains of extensions of the logic $\mathcal{DQDH}$

Recall that the logic $\mathcal{DQDH}$ corresponds to the variety of dually quasi-De Morgan Heyting algebras. In this section, we present two infinite chains of logics that are extensions of the logic $\mathcal{DQDH}$.

9.1. De Morgan-Gödel logic and its extensions

Recall that the variety $\mathbb{DMH}$ of De Morgan Heyting algebras is the subvariety of $\mathbb{DQDH}$ defined by the axiom: $x'' \approx x$. A De Morgan Heyting algebra whose lattice reduct is a chain is called a *De Morgan Heyting chain*. Let $\mathbb{DMHC}$ denote the subvariety of $\mathbb{DMSH}$ generated by the De Morgan Heyting chains. It is proved in [29, Theorem 12.5] that the lattice of subvarieties of the variety $\mathbb{DMHC}$ is an $\omega + 1$-chain. Let $\mathcal{DMG}$ (or $\mathcal{DMHC}$) denote the extension of the logic $\mathcal{DMH}$, corresponding to $\mathbb{DMHC}$. We will refer to the logic $\mathcal{DMG}$ as "De Morgan-Gödel logic." Then it follows that the lattice of extensions of $\mathcal{DMG}$ is a chain dual to $\omega + 1$.

In this subsection, we present axiomatizations for the logics corresponding to the subvarieties of $\mathbb{DMHC}$. For this purpose, we need the following algebraic result which was proved in [29, Theorem 12.3].

THEOREM 9.1. [29] *A base for* $\mathbb{DMHC}$*, relative to* $\mathbb{DMSH}$*, is given by:*

(1) $x^{*\prime} \approx x^{**}$,

(2) $(x \to y) \vee (y \to x) \approx 1$.

Hence we have the following axiomatization for the logic $\mathcal{DMG}$, relative to the logic $\mathcal{DMSH}$.

COROLLARY 9.2. The logic $\mathcal{DMG}$, relative to the logic $\mathcal{DMSH}$ is defined by:

(i) $\sim(\alpha \to \bot) \Leftrightarrow_H ((\alpha \to \bot) \to \bot)$,

(ii) $(\alpha \to \beta) \vee (\beta \to \alpha)$.

In view of the axiom (ii) it is clear that the logic $\mathcal{DMG}$ does not have the Disjunction Property.

Let $\mathbb{DMHC}_n$ denote the subvariety of $\mathbb{DMHC}$ generated by the n-element $\mathbb{DMH}$-chain, where $n \in \mathbb{N}$ with $n \geq 2$. Let $\mathcal{DMG}_n$ denote the extension of the logic $\mathcal{DMG}$ corresponding to the subvariety $\mathbb{DMHC}_n$ of $\mathbb{DMHC}$ generated by the n-element $\mathbb{DMH}$-chain, where $n \in \omega$ with $n \geq 2$.

Next we will present an axiomatization for the logic $\mathcal{DMG}_n$ for $n \in \mathbb{N}$ with $n \geq 2$.

THEOREM 9.3 ([29]). *Let $n \in \omega$ such that $n \geq 2$. Then $\mathbb{DMHC}_n$ is defined, mod $\mathbb{DMHC}$, by the following axiom:*

$$(\bigvee_{i=1}^{i=n} x_i) \vee [\bigvee_{i=1}^{i=n-1} (x_i \to x_{i+1})] \approx 1.$$

Hence we have the following axiomatization of the logic $\mathcal{DMG}_n$.

COROLLARY 9.4. Let $n \in \omega$ such that $n \geq 2$. Then the logic $\mathcal{DMG}_n$, relative to the logic $\mathcal{DMG}$, is defined by

$$(\bigvee_{i=1}^{i=n} \alpha_i) \vee [\bigvee_{i=1}^{i=n-1} (\alpha_i \to \alpha_{i+1})].$$

In view of the above corollary, it is clear that the logic $\mathcal{DMG}_n$, $n \geq 2$, does not have the Disjunction Property.

9.2. Dually pseudocomplemented Gödel logic and its axiomatic extensions

A $\mathbb{DPCSH}$-algebra $\mathbf{L} = \langle L, \wedge, \vee, \to, ', 0, 1\rangle$, whose lattice reduct is a chain, is called a $\mathbb{DPCSH}$-*chain*. Let $\mathbb{DPCHC}$ denote the subvariety of $\mathbb{DPCH}$ generated by the $\mathbb{DPCH}$-chains. Observe that $\mathbb{DPCHC} = \mathbb{DStHC}$. It was implicit in [29, Section 13] that the lattice of subvarieties of the variety $\mathbb{DPCHC}$

is an $\omega + 1$-chain and was explicitly proved in [33, Theorem 4.7]. We let $\mathcal{DPCG}$ (or $\mathcal{DPCHC}$) denote the extension of the logic $\mathcal{DPCH}$ corresponding to $\mathbb{DPCHC}$. The logic $\mathcal{DPCG}$ will be referred to as "dually pseudocomplemented Gödel logic". It follows from the just mentioned algebraic result that the extensions of $\mathcal{DPCG}$ form a chain dual to $\omega + 1$.

In this subsection, we present axiomatizations for the logics corresponding to the subvarieties of $\mathbb{DPCHC}$. For this purpose, we need the following algebraic result which was proved in [29, Theorem 13.2]. Let $x^+ := x'^{*\prime}$.

Theorem 9.5. *The following identities form a base, mod* $\mathbb{DQDSH}$*, for* $\mathbb{DPCHC}$*:*

(i) $x^+ \approx x'$,

(ii) $(x \to y) \lor (y \to x) \approx 1$.

Corollary 9.6. The logic $\mathcal{DPCG}$ is defined, as an extension of the logic $\mathcal{DQDSH}$ by

(i) $\alpha^+ \Leftrightarrow_H \sim \alpha$, where $\alpha^+ := \sim(\sim \alpha \to \bot)$.

(ii) $(\alpha \to \beta) \lor (\beta \to \alpha)$.

In view of the axiom (ii) it is clear that the logic $\mathcal{DPCG}$ does not have the Disjunction Property.

Let $n \in \omega$ such that $n \geq 2$ and let $\mathbb{DPCHC}_n$ denote the variety generated by the n-element $\mathbb{DPCH}$-chain. Let $\mathcal{DPCG}_n$ denote the extension of the logic $\mathcal{DPCG}$ corresponding to the subvariety $\mathbb{DPCHC}_n$ of $\mathbb{DPCHC}$ generated by the n-element $\mathbb{DPCH}$-chain, where $n \in \omega$ with $n \geq 2$

The following theorem, which follows from [29, Theorem 13.3], gives a base for each subvariety $\mathbb{DPCHC}_n$ of $\mathbb{DPCHC}$.

Theorem 9.7. *Let $n \in \omega$ such that $n \geq 2$. Then,* $\{(\mathrm{A_n})\}$ *is an equational base, mod* $\mathbb{DPCG}$*, for* $\mathbb{DPCG}_n$*, where* (An) *is the following axiom:*

$$\bigvee_{j=1}^{j=n} x_j \lor \bigvee_{j=1}^{j=n-1} (x_j \to x_{j+1}) \approx 1.$$

Corollary 9.8. Let $n \in \omega$ such that $n \geq 2$. Then the logic $\mathcal{DPCG}_n$ is defined as an extension of the logic $\mathcal{DPCG}$ by (Λ_n), where (Λ_n) is the following formula:

$$\bigvee_{j=1}^{j=n} \alpha_j \vee \bigvee_{j=1}^{j=n-1} (\alpha_j \to \alpha_{j+1}).$$

In view of the above corollary, it is clear that the logic $\mathcal{DPCG}_n$, $n \geq 2$, does not have the Disjunction Property.

10. Logics corresponding to subvarieties of regular dually quasi-De Morgan Stone semi-Heyting algebras

In the rest of the paper we will give axiomatizations for more new logics that are extensions of $\mathcal{DQDSH}$, as applications of Theorem 5.9 and the algebraic results from [30, 31, 32, 33, 34]). Recall from Section 7.2 that $\mathbf{C}_{20} = \mathbf{C}^{dm} \cup \mathbf{C}^{dp}$, where $\mathbf{C}^{dm} := \{\mathbf{L}_i^{dm} : i = 1, 2, \ldots, 10\}$, $\mathbf{C}^{dp} := \{\mathbf{L}_i^{dm} : i = 1, 2, \ldots, 10\}$ and that the algebras $\mathbf{L}_i^{dm}$, $\mathbf{L}_i^{dp}$ were defined in Section 8.2 and the three 4-element algebras $\mathbf{D}_1$, $\mathbf{D}_2$ and $\mathbf{D}_3$ were defined in Section 7.6. Recall also that $\mathbb{DQDSHC}^3 = \mathbb{V}(\mathbf{C}_{20})$ which is the subvariety of $\mathbb{DQDSHC}$ generated by all the 20 3-element DQDSH-chains.

The notion of regularity has played an important role in [4, 5, 9, 19, 24, 26, 29, 30, 31, 32, 33, 34, 35, 36, 37].

An algebra $\mathbf{A} \in \mathbb{DQDSH}$ is called *regular* ([29, 30, 31]) if $\mathbf{A}$ satisfies:

$$\text{(R)} \qquad x \wedge x^+ \leq y \vee y^*,$$

where $x^+ := x'^{*\prime}$.

The subvariety of $\mathbb{DQDSH}$ of regular algebras is denoted by $\mathbb{RDQDSH}$. (We caution the reader that the term "regular" was used in [29] to mean something else.)

Observe from Theorem 7.6 that $\mathbb{DQDSHC}^3 \subset \mathbb{RDQDSH}$.

The concept of *level* has played an important role in finding discriminator subvarieties of $\mathbb{DQDSH}$ (see [29, Corollary 8.2]). Here we only need to define $\mathbb{DQDSH}$-algebras of level 1.

An algebra $\mathbf{A} \in \mathbb{DQDSH}$ is of *level* 1 if $\mathbf{A}$ satisfies:

$$x \wedge x'^{*} \approx x \wedge x'^{*} \wedge x'^{*'*}.$$

For the varieties of level 1 considered in the rest of the paper, the above definition of "level 1" is equivalent to the following:

$$x \wedge x'^{*} \approx (x \wedge x'^{*})'^{*}.$$

Let $\mathbb{DQDSH}_1$ denote the variety of $\mathbb{DQDSH}$-algebras of level 1. Let $\mathbb{DQDS}\mathrm{t}\mathbb{SH}$ denote the subvariety of $\mathbb{DQDSH}$ that satisfies the Stone identity:

$$\text{(St)} \qquad x^{*} \ \vee \ x^{**} \approx 1.$$

$\mathbb{DQDS}\mathrm{t}\mathbb{SH}_1$ denotes the subvariety of $\mathbb{DQDS}\mathrm{t}\mathbb{SH}$ of level 1, while $\mathbb{RDQDS}\mathrm{t}\mathbb{SH}_1$ denotes the subvariety of $\mathbb{DQDS}\mathrm{t}\mathbb{SH}_1$ defined by (R).

In this section we present axiomatizations for new logics corresponding to several subvarieties of the variety $\mathbb{RDQDS}\mathrm{t}\mathbb{SH}_1$ of regular dually quasi-De Morgan Stone semi-Heyting algebras of level 1.

In what follows, $\mathcal{V}$ (or $\mathcal{L}(\mathbb{V})$) denotes the logic corresponding to the subvariety $\mathbb{V}$ of $\mathbb{DQDSH}$-algebras.

(Thus, for example, the logic $\mathcal{DQDS}\mathrm{t}\mathcal{SH}_1$ corresponds to the variety $\mathbb{DQDS}\mathrm{t}\mathbb{SH}_1$.)

The following corollary is immediate from the above definitions and Theorem 5.9.

Corollary 10.1.

(a) The logic $\mathcal{DQDS}\mathbf{t}\mathcal{SH}_1$ is defined, as an extension of the logic $\mathcal{DQDSH}$, by the following axioms:

(1) $[\sim \{(\alpha \wedge \ (\sim \alpha)^{*}\}]^{*} \Leftrightarrow_H \ \alpha \wedge \ (\sim \alpha)^{*}$,

(2) $\alpha^{*} \vee \alpha^{**}$.

(b) The logic $\mathcal{RDQDS}\mathrm{t}\mathcal{SH}_1$ is defined, as an extension of the logic $\mathcal{DQDS}\mathrm{t}\mathcal{SH}_1$ by the following axiom:

$$(\alpha \wedge \alpha^{+}) \ \to_H \ (\beta \vee \beta^{*}).$$

The following result is taken from [31, Theorem 3.1].

THEOREM 10.2. $\mathbb{RDQDS}\mathrm{t}\mathbb{SH}_1 = \mathbb{V}(\mathbf{C}_{20} \cup \{\mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3\})$. *In particular,* $\mathbb{RDQDS}\mathrm{t}\mathbb{H}_1 = \mathbb{V}(\{\mathbf{L}_1^{dm}, \mathbf{L}_1^{dp}, \mathbf{D}_2\})$.

The following corollary is immediate from Theorem 10.2 and Theorem 5.9, as the variety $\mathbb{RDQDS}\mathrm{t}\mathbb{SH}_1$ is finitely axiomatized and is generated by a finite set of finite algebras.

COROLLARY 10.3. The logics $\mathcal{RDQDS}\mathrm{t}\mathcal{SH}_1$ and $\mathcal{RDQDS}\mathrm{t}\mathcal{H}_1$ are decidable.

In view of the above corollary, it would be of interest to know if the logic $\mathcal{DQDS}\mathrm{t}\mathcal{SH}_1$ is decidable; in particular, if the logic $\mathcal{DQDS}\mathrm{t}\mathcal{H}_1$ is decidable. This naturally leads us to the following open problem.

PROBLEM 2: Is the variety $\mathbb{DQDS}\mathrm{t}\mathbb{H}_1$ generated by its finite members?

More generally, we can ask the following:

PROBLEM 3: Is the variety $\mathbb{DQDS}\mathrm{t}\mathbb{SH}_1$ generated by its finite members?

Remark 10.4. It was shown in [29] that the variety $\mathbb{RDQDS}\mathrm{t}\mathbb{SH}_1$ is a discriminator variety. Thus $\mathcal{RDQDS}\mathrm{t}\mathcal{SH}_1$ is a discriminator logic.

Recall that $\mathbb{RDMSH}_1$ is the variety of regular De Morgan semi-Heyting algebras of level 1 and $\mathcal{RDMSH}_1$ denotes its corresponding logic. Let $\mathbb{RDMS}\mathrm{t}\mathbb{SH}_1$ and $\mathbb{RDMS}\mathrm{t}\mathbb{H}_1$ denote, respectively, the varieties of regular De Morgan Stone semi-Heyting algebras and regular De Morgan Stone semi-Heyting algebras of level 1. Similarly, the varieties $\mathbb{RDPCS}\mathrm{t}\mathbb{SH}_1$ and $\mathbb{RDPCS}\mathrm{t}\mathbb{H}_1$ denote, respectively, the varieties of regular dually pseudocomplemented Stone semi-Heyting algebras and regular dually pseudocomplemented Stone Heyting algebras. Note that all these varieties are subvarieties of $\mathbb{RDQDS}\mathrm{t}\mathbb{SH}_1$.

Recall $\mathbb{DMSHC}^3 = \mathbb{V}(\mathbf{C^{dm}})$ and $\mathbb{DPCSHC}^3 = \mathbb{V}(\mathbf{C^{dp}})$.

The following corollary is immediate from Theorem 10.2, where "is defined by" means "is defined, as an extension of $\mathcal{RDQDS}\mathrm{t}\mathcal{SH}_1$, by".

Corollary 10.5.

(a) The logic $\mathcal{RDMS}\mathrm{t}\mathcal{SH}_1$ is defined by

$$\alpha \ \to_H \alpha''.$$

(b) The logic $\mathcal{RDPCS}\mathrm{t}\mathcal{SH}_1$ is defined by

$$\alpha \ \vee \ \alpha'.$$

(c) The logic $\mathcal{RDMS}\mathrm{t}\mathcal{H}_1$ is defined by

$$(\alpha \wedge \beta) \to \alpha.$$

(d) The logic $\mathcal{RDPCS}\mathrm{t}\mathcal{H}_1$ is defined by

$$(\alpha \wedge \beta) \to \alpha.$$

The following theorem was recently proved in [34].

Theorem 10.6 ([34, Corollary 3.4]). $\mathbb{DMSH}_1 = \mathbb{DMS}\mathrm{t}\mathbb{SH}_1$. *In particular,* $\mathbb{RDMSH}_1 = \mathbb{RDMS}\mathrm{t}\mathbb{SH}_1$.

The following theorem is immediate from Theorem 10.6 and [31, Corollary 3.4].

Theorem 10.7.

(a) $\mathbb{RDMSH}_1 = \mathbb{RDMS}\mathrm{t}\mathbb{SH}_1 = \mathbb{V}(\mathbf{C}^{dm}) \vee \mathbb{V}(\{\mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}\})$,

(b) $\mathbb{RDMH}_1 = \mathbb{RDMS}\mathrm{t}\mathbb{H}_1 = \mathbb{V}(\{\mathbf{L}_1^{dm}, \mathbf{D}_2\}) = \mathbb{V}(\mathbf{L}_1^{dm}) \vee \mathbb{V}(\mathbf{D}_2)$,

(c) $\mathbb{RDPCS}\mathrm{t}\mathbb{SH}_1 = \mathbb{V}(\mathbf{C}^{\mathrm{dp}})$,

(d) $\mathbb{RDPCS}\mathrm{t}\mathbb{H}_1 = \mathbb{V}(\mathbf{L}_1^{\mathrm{dp}})$.

It is clear from Theorem 10.7 that the logics $\mathcal{RDMSH}_1$ and $\mathcal{RDMS}\mathrm{t}\mathcal{SH}_1$ are equivalent and so are $\mathcal{RDMH}_1$ and $\mathcal{RDMS}\mathrm{t}\mathcal{H}_1$.

The following corollary is immediate from Theorem 10.7.

Corollary 10.8. The logics $\mathcal{RDMSH}_1$ and $\mathcal{RDPCS}\mathrm{t}\mathcal{SH}_1$ are decidable.

Let $\mathbb{RDQD}\mathrm{cm}\mathbb{S}\mathrm{t}\mathbb{SH}_1$ be the subvariety of $\mathbb{RDQDS}\mathrm{t}\mathbb{SH}_1$ defined by the commutative law:

$$x \to y \approx y \to x.$$

Corollary 10.9. The logic $\mathcal{RDQD}cm\mathcal{S}\mathrm{t}\mathcal{H}_1$ is defined, as an extension of $\mathcal{RDQDS}\mathrm{t}\mathcal{SH}_1$, by

$$(\alpha \to \beta) \to_H \ (\beta \to \alpha).$$

The following theorem is an immediate consequence of Theorem 10.2 and Theorem 10.7.

Theorem 10.10 ([31, Corollary 3.5]).

(a) $\mathbb{RDQD}\mathrm{cm}\mathbb{S}\mathrm{t}\mathbb{SH}_1 = \mathbb{V}(\mathbf{L}^{dm}_{10}) \vee \mathbb{V}(\mathbf{L}^{dp}_{10}) \vee \mathbb{V}(\mathbf{D}_1)$,

(b) $\mathbb{RDM}\mathrm{cm}\mathbb{SH}_1 = \mathbb{RDM}\mathrm{cm}\mathbb{S}\mathrm{t}\mathbb{SH}_1 = \mathbb{V}(\{\mathbf{L}^{dm}_{10}, \mathbf{D}_1\})$,

(c) $\mathbb{RDPC}\mathrm{cm}\mathbb{S}\mathrm{t}\mathbb{SH}_1 = \mathbb{V}((\mathbf{L}^{dp}_{10})$,

(d) $\mathbb{RDM}\mathrm{cm}\mathbb{SH}_1 \cap \mathbb{RDPC}\mathrm{cm}\mathbb{S}\mathrm{t}\mathbb{SH}_1 = \mathbb{V}(\bar{2}^e)$.

It follows from the preceding theorem that the logics $\mathcal{RDQD}\mathrm{cm}\mathcal{S}\mathrm{t}\mathcal{SH}_1$, $\mathcal{RDM}\mathrm{cm}\mathcal{SH}_1$ and $\mathcal{RDPC}\mathrm{cm}\mathcal{S}\mathrm{t}\mathcal{SH}_1$ are decidable.

In the rest of this section, unless otherwise stated, the phrase “defined, modulo $\mathbb{RDQDS}\mathrm{t}\mathbb{SH}_1$, by” is abbreviated to the phrase “defined by” in the context of varieties. Similarly, the phrase “defined, as an extension of the logic $\mathcal{RDQDS}\mathrm{t}\mathbb{SH}_1$, by” is also abbreviated to the phrase “defined by” in the case of logics.

The theorems that appear in the rest of this section were proved in [31]. Each of the corollaries appearing below follows from the theorem immediately preceding it and Theorem 5.9.

Theorem 10.11. *The variety* $\mathbf{V}(\{\mathbf{L^{dm}_1}, \mathbf{L^{dp}_1}, \mathbf{L^{dm}_3}, \mathbf{L^{dp}_3}, \mathbf{D_2}\})$ *is defined by the identity:*

$$(x \to y) \to (0 \to y) \approx (x \to y) \to 1.$$

Corollary 10.12. The logic $\mathcal{L}(\mathbb{V}(\mathbf{L^{dm}_1}, \mathbf{L^{dp}_1}, \mathbf{L^{dm}_3}, \mathbf{L^{dp}_3}, \mathbf{D_2}))$ is defined by

$$[(\alpha \to \beta) \to (\perp \to \beta)] \Leftrightarrow_H [(\alpha \to \beta) \to \top].$$

The variety generated by $\mathbf{D}_1$ was axiomatized earlier in Section 7. Here are two more bases for it.

THEOREM 10.13. $\mathbb{V}(\mathbf{D}_1)$ *is defined by*

$$x \to (y \to z) \approx z \to (x \to y).$$

It is also defined by

$$(x \to y) \to (u \to w) \approx (x \to u) \to (y \to w). \qquad \text{(Medial Law)}$$

COROLLARY 10.14. The logic $\mathcal{L}(\mathbb{V}(\mathbf{D_1}))$ is defined by

$$\alpha \to (\beta \to \gamma) \Leftrightarrow_H \gamma \to (\alpha \to \beta).$$

It is also defined by

$$(\alpha \to \beta) \to (\gamma \to \delta) \Leftrightarrow_H ((\alpha \to \gamma) \to (\beta \to \delta)). \qquad \text{(Medial Law)}$$

THEOREM 10.15. *The variety* $\mathbb{V}(\{\mathbf{L_1^{dm}}, \mathbf{L_1^{dp}}, \mathbf{L_2^{dm}}, \mathbf{L_2^{dp}}, \mathbf{D_2}\})$ *is defined by:*

$$y \leq x \to y.$$

It is also defined by:

$$[(x \to y) \to y] \to (x \to y) \approx x \to y.$$

It is also defined by

$$x \to (y \to z) \approx (x \to y) \to (x \to z). \qquad \text{(Left distributive law)}$$

COROLLARY 10.16. The logic $\mathcal{L}(\mathbb{V}(\{\mathbf{L_1^{dm}}, \mathbf{L_1^{dp}}, \mathbf{L_2^{dm}}, \mathbf{L_2^{dp}}, \mathbf{D_2}\})$ is defined by

$$\beta \wedge (\alpha \to \beta) \Leftrightarrow_H \beta.$$

It is also defined by:

$$[(\alpha \to \beta) \to \beta] \to (\alpha \to \beta) \Leftrightarrow_H (\alpha \to \beta).$$

It is also defined by

$$\alpha \to (\beta \to \gamma) \Leftrightarrow_H [(\alpha \to \beta) \to (\alpha \to \gamma).$$

THEOREM 10.17. *The variety* $\mathbb{V}(\{\mathbf{L_1^{dm}}, \mathbf{L_1^{dp}}, \mathbf{L_2^{dm}}, \mathbf{L_2^{dp}}, \mathbf{L_5^{dm}}, \mathbf{L_5^{dp}}, \mathbf{L_6^{dm}}, \mathbf{L_6^{dp}}, \mathbf{D_2}\})$ *is defined by:*

$$[x \to (y \to x)] \to x \approx x.$$

COROLLARY 10.18. The logic $\mathcal{L}(\mathbb{V}(\{\mathbf{L_1^{dm}}, \mathbf{L_1^{dp}}, \mathbf{L_2^{dm}}, \mathbf{L_2^{dp}}, \mathbf{L_5^{dm}}, \mathbf{L_5^{dp}}, \mathbf{L_6^{dm}}, \mathbf{L_6^{dp}}, \mathbf{D_2}\}))$ is defined by

$$[\{\alpha \to (\beta \to \alpha)\} \to \alpha] \Leftrightarrow_H \alpha.$$

THEOREM 10.19. $\mathbb{V}(\{\mathbf{L_1^{dp}}, \mathbf{L_2^{dp}}, \mathbf{L_5^{dp}}, \mathbf{L_6^{dp}}\})$ *is defined by:*

(1) $[x \to (y \to x)] \to x \approx x$,

(2) $x \vee x' \approx 1$.

COROLLARY 10.20. The logic $\mathcal{L}(\mathbb{V}(\{\mathbf{L_1^{dp}}, \mathbf{L_2^{dp}}, \mathbf{L_5^{dp}}, \mathbf{L_6^{dp}}\}))$ is defined by

(1) $[\{\alpha \to (\beta \to \alpha)\} \to \alpha] \Leftrightarrow_H \alpha$,

(2) $\alpha \vee \ \sim \alpha$.

In view of the above corollary, it is clear that the logic in question does not have the Disjunction Property. Recall that $x^+ := x'^{*'}$.

THEOREM 10.21. *The variety generated by the set* $\{\mathbf{L_1^{dm}}, \mathbf{L_1^{dp}}, \mathbf{L_2^{dm}}, \mathbf{L_2^{dp}}, \mathbf{L_3^{dm}}, \mathbf{L_3^{dp}}, \mathbf{L_4^{dm}}, \mathbf{L_4^{dp}}, \mathbf{L_5^{dm}}, \mathbf{L_6^{dm}}, \mathbf{L_7^{dm}}, \mathbf{L_8^{dm}}, \mathbf{D_2}, \mathbf{D_3}\}$ *is defined by the identity:*

$$(0 \to 1)^+ \to (0 \to 1)' \approx 0 \to 1.$$

COROLLARY 10.22. Let $\mathcal{L}_0 = \mathcal{L}((\mathbb{V}(\{\mathbf{L_1^{dm}}, \mathbf{L_1^{dp}}, \mathbf{L_2^{dm}}, \mathbf{L_2^{dp}}, \mathbf{L_3^{dm}}, \mathbf{L_3^{dp}}, \mathbf{L_4^{dm}}, \mathbf{L_4^{dp}}, \mathbf{L_5^{dm}}, \mathbf{L_6^{dm}}, \mathbf{L_7^{dm}}, \mathbf{L_8^{dm}}, \mathbf{D_2}, \mathbf{D_3}\}))$. Then $\mathcal{L}_0$ is defined by

$$[(\bot \to \top)^+ \to \sim (\bot \to \top)] \Leftrightarrow_H (\bot \to \top).$$

THEOREM 10.23. *The variety* $\mathbf{V}(\{\mathbf{L_1^{dp}}, \mathbf{L_2^{dp}}, \mathbf{L_3^{dp}}, \mathbf{L_4^{dp}}\})$ *is defined by the identities:*

(1) $(0 \to 1)^+ \to (0 \to 1)' \approx 0 \to 1$,

(2) $x \vee x' \approx 1$.

COROLLARY 10.24. The logic $\mathcal{L}(\mathbb{V}(\{\mathbf{L_1^{dp}}, \mathbf{L_2^{dp}}, \mathbf{L_3^{dp}}, \mathbf{L_4^{dp}}\}))$ is defined by

(1) $[(\bot \to \top)^+ \to \sim (\bot \to \top)] \Leftrightarrow_H (\bot \to \top)$,

(2) $\alpha \vee \sim \alpha$.

In view of the above corollary, it is clear that the logic in question does not have the Disjunction Property.

THEOREM 10.25. *The variety* $\mathbf{V}(\{\mathbf{L_1^{dm}}, \mathbf{L_2^{dm}}, \mathbf{L_3^{dm}}, \mathbf{L_4^{dm}}, \mathbf{L_5^{dm}}, \mathbf{L_6^{dm}}, \mathbf{L_7^{dm}}, \mathbf{L_8^{dm}}, \mathbf{D_2}, \mathbf{D_3}\})$ *is defined by the identities:*

(1) $(0 \to 1)^+ \to (0 \to 1)' \approx 0 \to 1$,

(2) $x'' \approx x$.

Corollary 10.26. The logic $\mathcal{L}(\mathbb{V}(\{\mathbf{L_1^{dm}}, \mathbf{L_2^{dm}}, \mathbf{L_3^{dm}}, \mathbf{L_4^{dm}}, \mathbf{L_5^{dm}}, \mathbf{L_6^{dm}}, \mathbf{L_7^{dm}}, \mathbf{L_8^{dm}}, \mathbf{D_2}, \mathbf{D_3}\}))$ is defined by

(1) $[(\bot \to \top)^+ \to \sim (\bot \to \top)] \Leftrightarrow_H (\bot \to \top)$,

(2) $\alpha \to_H \sim\sim \alpha$.

Theorem 10.27. *The variety* $\mathbb{V}(\{\mathbf{L_5^{dm}}, \mathbf{L_6^{dm}}, \mathbf{L_7^{dm}}, \mathbf{L_8^{dm}}, \mathbf{D_3}\})$ *is defined by the identity:*

$$(0 \to 1)^+ \to (0 \to 1) \approx (0 \to 1)'.$$

Corollary 10.28. The logic $\mathcal{L}(\mathbb{V}(\{\mathbf{L_5^{dm}}, \mathbf{L_6^{dm}}, \mathbf{L_7^{dm}}, \mathbf{L_8^{dm}}, \mathbf{D_3}\}))$ is defined by

$$[(\bot \to \top)^+ \to (\bot \to \top)] \Leftrightarrow_H \sim (\bot \to \top).$$

$\mathbb{V}(\mathbf{D_3})$ was axiomatized in Section 7. Here is another base for it.

Theorem 10.29. $\mathbb{V}(\mathbf{D_3})$ *is defined by the identities:*

(1) $(0 \to 1)^+ \to (0 \to 1) \approx (0 \to 1)'$,

(2) $x \vee x^* \approx 1$.

Corollary 10.30. The logic $\mathcal{L}(\mathbb{V}(\{\mathbf{D_3}\}))$ is defined by

(1) $(\bot \to \top)^+ \to (\bot \to \top) \Leftrightarrow_H \sim (\bot \to \top)$,

(2) $\alpha \vee \alpha^*$.

In view of the above corollary, it is clear that the logic in question does not have the Disjunction Property.

Theorem 10.31. *The variety generated by the algebras* $\mathbf{L_1^{dm}}, \mathbf{L_2^{dm}}, \mathbf{L_3^{dm}}, \mathbf{L_4^{dm}}$, $\mathbf{D_2}, \mathbf{D_3}$ *is defined by the identities:*

(1) $(0 \to 1)^+ \to (0 \to 1)' \approx 0 \to 1$,

(2) $(0 \to 1)^+ \to (0 \to 1)^{*\prime *} \approx 0 \to 1$,

(3) $x'' \approx x$.

Corollary 10.32. The logic $\mathcal{L}(\mathbb{V}(\{\mathbf{L_1^{dm}}, \mathbf{L_2^{dm}}, \mathbf{L_3^{dm}}, \mathbf{L_4^{dm}}, \mathbf{D_2}, \mathbf{D_3}\}))$ is defined by

(1) $(\bot \to \top)^+ \to \sim (\bot \to \top) \Leftrightarrow_H (\bot \to \top)$,

(2) $[(\bot \to \top)^+ \to (\sim (\bot \to \top)^*)^*] \Leftrightarrow_H (\bot \to \top)$,

(3) $\alpha \to_H \sim\sim \alpha$.

Theorem 10.33. *The variety generated by the algebras* $\mathbf{L_5^{dm}}, \mathbf{L_5^{dp}}, \mathbf{L_6^{dm}}, \mathbf{L_6^{dp}}, \mathbf{L_7^{dm}}, \mathbf{L_7^{dp}}, \mathbf{L_8^{dm}}, \mathbf{L_8^{dp}}, \mathbf{L_9^{dp}}, \mathbf{L_9^{dm}}, \mathbf{L_{10}^{dp}}, \mathbf{L_{10}^{dm}}$, $\mathbf{D_1}, \mathbf{D_3}$ *is defined by the identity:*

$$(0 \to 1)^+ \to (0 \to 1)' \approx (0 \to 1)'.$$

Corollary 10.34. The logic corresponding to the variety generated by $\{\mathbf{L_5^{dm}}, \mathbf{L_5^{dp}}, \mathbf{L_6^{dm}}, \mathbf{L_6^{dp}}, \mathbf{L_7^{dm}}, \mathbf{L_7^{dp}}, \mathbf{L_8^{dm}}, \mathbf{L_8^{dp}}, \mathbf{L_9^{dm}}, \mathbf{L_9^{dp}}, \mathbf{L_{10}^{dm}}, \mathbf{L_{10}^{dp}}, \mathbf{D_1}, \mathbf{D_3}\}$ is defined by

$$[(\perp \to \top)^+ \to \sim (\perp \to \top)] \Leftrightarrow_H \sim (\perp \to \top).$$

Theorem 10.35. *The variety generated by the algebras* $\mathbf{L_5^{dp}}, \mathbf{L_6^{dp}}, \mathbf{L_7^{dp}}, \mathbf{L_8^{dp}}, \mathbf{L_9^{dp}}, \mathbf{L_{10}^{dp}}$ *is defined by the identities:*

(1) $(0 \to 1)^+ \to (0 \to 1)' \approx (0 \to 1)'$,

(2) $x \vee x' \approx 1$.

Corollary 10.36. The logic $\mathcal{L}(\mathbb{V}(\{\mathbf{L_5^{dp}}, \mathbf{L_6^{dp}}, \mathbf{L_7^{dp}}, \mathbf{L_8^{dp}}, \mathbf{L_9^{dp}}, \mathbf{L_{10}^{dp}}\}))$ is defined by

(1) $[(\perp \to \top)^+ \to \sim (\perp \to \top)] \Leftrightarrow_H \sim (\perp \to \top)$,

(2) $\alpha \vee \sim \alpha$.

In view of the above corollary, it is clear that the logic in question does not have the Disjunction Property.

Theorem 10.37. *The variety generated by the algebras* $\mathbf{L_5^{dm}}, \mathbf{L_6^{dm}}, \mathbf{L_7^{dm}}, \mathbf{L_8^{dm}}, \mathbf{L_9^{dm}}, \mathbf{L_{10}^{dm}}$, $\mathbf{D_1}, \mathbf{D_3}$ *is defined by the identities:*

(1) $(0 \to 1)^+ \to (0 \to 1)' \approx (0 \to 1)'$,

(2) $x'' \approx x$.

Corollary 10.38. The logic $\mathcal{L}(\mathbb{V}(\{\mathbf{L_5^{dm}}, \mathbf{L_6^{dm}}, \mathbf{L_7^{dm}}, \mathbf{L_8^{dm}}, \mathbf{L_9^{dm}}, \mathbf{L_{10}^{dm}}, \mathbf{D_1}, \mathbf{D_3}\}))$ is defined by

(1) $[(\perp \to \top)^+ \to \sim (\perp \to \top)] \Leftrightarrow_H \sim (\perp \to \top)$,

(2) $\alpha \to_H \sim\sim \alpha$.

Theorem 10.39. *The variety generated by the algebras* $\mathbf{D_1}, \mathbf{D_3}$ *is defined by the identities:*

(1) $(0 \to 1)^+ \to (0 \to 1)' \approx (0 \to 1)'$,

(2) $x \vee x^* \approx 1$.

Corollary 10.40. The logic $\mathcal{L}(\mathbb{V}(\{\mathbf{D_1}, \mathbf{D_3}\}))$ is defined by

(1) $[(\bot \to \top)^+ \to \sim (\bot \to \top)] \Leftrightarrow_H \sim (\bot \to \top)$,

(2) $\alpha \vee \alpha^*$.

In view of the above corollary, it is clear that the logic in question does not have the Disjunction Property.

Theorem 10.41. *The variety generated by the algebras* $\mathbf{L_5^{dm}}, \mathbf{L_6^{dm}}, \mathbf{L_7^{dm}}, \mathbf{L_8^{dm}}, \mathbf{D_3}$ *is defined by the identities:*

(1) $(0 \to 1)^+ \to (0 \to 1)' \approx (0 \to 1)'$,

(2) $(0 \to 1)^+ \to (0 \to 1)' \approx (0 \to 1)$.

It is also defined by

$$(0 \to 1)' \approx 0 \to 1.$$

Corollary 10.42. The logic $\mathcal{L}(\mathbb{V}(\{\mathbf{L_5^{dm}}, \mathbf{L_6^{dm}}, \mathbf{L_7^{dm}}, \mathbf{L_8^{dm}}, \mathbf{D_3}\}))$ is defined by

(1) $[(\bot \to \top)^+ \to \sim (\bot \to \top)] \Leftrightarrow_H \sim (\bot \to \top)$,

(2) $[(\bot \to \top)^+ \to \sim (\bot \to \top)] \Leftrightarrow_H (\bot \to \top)$.

It is also defined by

$$\sim (\bot \to \top) \Leftrightarrow_H (\bot \to \top).$$

Theorem 10.43. *The variety generated by the algebras* $\mathbf{L_1^{dm}}, \mathbf{L_1^{dp}}, \mathbf{L_2^{dm}}$, $\mathbf{L_2^{dp}}$, $\mathbf{L_3^{dm}}$, $\mathbf{L_3^{dp}}$, $\mathbf{L_4^{dm}}$, $\mathbf{L_4^{dp}}$, $\mathbf{L_5^{dp}}$, $\mathbf{L_6^{dp}}$, $\mathbf{L_7^{dp}}$, $\mathbf{L_8^{dp}}$, $\mathbf{L_9^{dm}}$, $\mathbf{L_9^{dp}}$, $\mathbf{L_{10}^{dm}}$, $\mathbf{L_{10}^{dp}}$, $\mathbf{D_1}$, $\mathbf{D_2}$ *is defined by the identity:*

$$(0 \to 1)' \to (0 \to 1) \approx 0 \to 1.$$

Corollary 10.44. The logic corresponding to the variety generated by the algebras $\mathbf{L_1^{dm}}$, $\mathbf{L_1^{dp}}$, $\mathbf{L_2^{dm}}$, $\mathbf{L_2^{dp}}$, $\mathbf{L_3^{dm}}$, $\mathbf{L_3^{dp}}$, $\mathbf{L_4^{dm}}$, $\mathbf{L_4^{dp}}$, $\mathbf{L_5^{dp}}$, $\mathbf{L_6^{dp}}$, $\mathbf{L_7^{dp}}$, $\mathbf{L_8^{dp}}$, $\mathbf{L_9^{dm}}, \mathbf{L_9^{dp}}, \mathbf{L_{10}^{dm}}, \mathbf{L_{10}^{dp}}, \mathbf{D_1}, \mathbf{D_2}$ is defined by

$$[\sim (\bot \to \top) \to (\bot \to \top)] \Leftrightarrow_H (\bot \to \top).$$

Theorem 10.45. *The variety generated by the algebras* $\mathbf{L_1^{dm}}, \mathbf{L_2^{dm}}, \mathbf{L_3^{dm}}, \mathbf{L_4^{dm}}, \mathbf{L_9^{dm}}, \mathbf{L_{10}^{dm}}$, $\mathbf{D_1}, \mathbf{D_2}$ *is defined by the identities:*

(1) $(0 \to 1)' \to (0 \to 1) \approx 0 \to 1$,

(2) $x'' \approx x$.

Corollary 10.46. The logic $\mathcal{L}(\mathbb{V}(\{\mathbf{L_1^{dm}}, \mathbf{L_2^{dm}}, \mathbf{L_3^{dm}}, \mathbf{L_4^{dm}}, \mathbf{L_9^{dm}}, \mathbf{L_{10}^{dm}}, \mathbf{D_1}, \mathbf{D_2}\}))$ is defined by

(1) $[\sim (\perp \to \top) \to (\perp \to \top)] \Leftrightarrow_H (\perp \to \top)$,

(2) $\alpha \to_H \sim\sim \alpha$.

Theorem 10.47. *The variety generated by the algebras* $\mathbf{D_1}, \mathbf{D_2}$ *is defined by the identities:*

(1) $(0 \to 1)' \to (0 \to 1) \approx 0 \to 1$,

(2) $x \vee x^* \approx 1$.

Corollary 10.48. The logic $\mathcal{L}(\mathbb{V}(\{\mathbf{D_1}, \mathbf{D_2}\}))$ is defined by

(1) $[\sim (\perp \to \top) \to (\perp \to \top)] \Leftrightarrow_H (\perp \to \top)$,

(2) $\alpha \ \vee \ \alpha^*$.

In view of the above corollary, it is clear that the logic in question does not have the Disjunction Property.

Theorem 10.49. *The variety generated by the algebras* $\mathbf{L_1^{dm}}, \mathbf{L_1^{dp}}, \mathbf{L_3^{dm}}, \mathbf{L_3^{dp}}, \mathbf{L_6^{dm}}, \mathbf{L_6^{dp}}, \mathbf{L_8^{dm}}, \mathbf{L_8^{dp}}, \mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}$ *is defined by the identity:*

$$x \vee [y \to (x \vee y)] \approx (0 \to x) \vee x \vee y.$$

Corollary 10.50. The logic $\mathcal{L}(\mathbb{V}(\{\mathbf{L_1^{dm}}, \mathbf{L_1^{dp}}, \mathbf{L_3^{dm}}, \mathbf{L_3^{dp}}, \mathbf{L_6^{dm}}, \mathbf{L_6^{dp}}, \mathbf{L_8^{dm}}, \mathbf{L_8^{dp}}, \mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}\}))$ is defined by

$$[\alpha \vee \{\beta \to (\alpha \vee \beta)\}] \Leftrightarrow_H [(\perp \to \alpha) \vee \alpha \vee \beta].$$

Theorem 10.51. *The variety generated by the algebras* $\mathbf{L_2^{dm}}, \mathbf{L_2^{dp}}, \mathbf{L_5^{dm}}, \mathbf{L_5^{dp}}, \mathbf{D_2}$ *is defined by the identity:*

$$x \vee (y \to x) \approx [(x \to y) \to y] \to x.$$

Corollary 10.52. The logic $\mathcal{L}(\mathbb{V}(\{\mathbf{L_2^{dm}}, \mathbf{L_2^{dp}}, \mathbf{L_5^{dm}}, \mathbf{L_5^{dp}}, \mathbf{D_2}\}))$ is defined by

$$[\alpha \vee (\beta \to \alpha)] \Leftrightarrow_H [\{(\alpha \to \beta) \to \beta\} \to \alpha].$$

Theorem 10.53. *The variety generated by the algebras* $\mathbf{L_3^{dm}}, \mathbf{L_3^{dp}}, \mathbf{L_4^{dm}}, \mathbf{L_4^{dp}}, \mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}$ *is defined by the identity:*

$$x \vee (x \to y) \approx x \to [x \vee (y \to 1)].$$

Corollary 10.54. The logic $\mathcal{L}(\mathbb{V}(\{\mathbf{L_3^{dm}}, \mathbf{L_3^{dp}}, \mathbf{L_4^{dm}}, \mathbf{L_4^{dp}}, \mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}\}))$ is defined by

$$[\alpha \vee (\alpha \to \beta)] \Leftrightarrow_H [\alpha \to \{\alpha \vee (\beta \to \top)\}].$$

Theorem 10.55. *The variety generated by the algebras* $\mathbf{L_5^{dm}}, \mathbf{L_6^{dp}}, \mathbf{L_7^{dm}}, \mathbf{L_8^{dp}}, \mathbf{D_3}$ *is defined by the identity:*

$$(0 \to 1)^* \to (0 \to 1) \approx (0 \to 1)'.$$

Corollary 10.56. The logic $\mathcal{L}(\mathbb{V}(\{\mathbf{L_5^{dm}}, \mathbf{L_6^{dp}}, \mathbf{L_7^{dm}}, \mathbf{L_8^{dp}}, \mathbf{D_3}\}))$ is defined by

$$[(\bot \to \top)^* \to (\bot \to \top)] \Leftrightarrow_H \sim (\bot \to \top).$$

Theorem 10.57. *The variety generated by the algebras* $\mathbf{L_1^{dm}}, \mathbf{L_1^{dp}}, \mathbf{L_2^{dm}}, \mathbf{L_2^{dp}}, \mathbf{L_3^{dm}}, \mathbf{L_3^{dp}}$, $\mathbf{L_4^{dm}}, \mathbf{L_4^{dp}}, \mathbf{D_2}$ *is defined by the identity:*

$$0 \to 1 \approx 1 \text{ (FTT identity).}$$

Corollary 10.58. The logic $\mathcal{L}(\mathbb{V}(\{\mathbf{L_1^{dm}}, \mathbf{L_1^{dp}}, \mathbf{L_2^{dm}}, \mathbf{L_2^{dp}}, \mathbf{L_3^{dm}}, \mathbf{L_3^{dp}}, \mathbf{L_4^{dm}}, \mathbf{L_4^{dp}}, \mathbf{D_2}\}))$ is defined by

$$\bot \to \top \text{ (FTT).}$$

Theorem 10.59. *The variety generated by the algebras* $\mathbf{L_1^{dm}}, \mathbf{L_1^{dp}}, \mathbf{L_3^{dm}}, \mathbf{L_3^{dp}}, \mathbf{L_6^{dm}}, \mathbf{L_6^{dp}}$, $\mathbf{L_8^{dm}}, \mathbf{L_8^{dp}}, \mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}$ *is defined by the identity:*

$$x \vee (y \to x) \approx (x \vee y) \to x.$$

Corollary 10.60. The logic $\mathcal{L}(\mathbb{V}(\{\mathbf{L_1^{dm}}, \mathbf{L_1^{dp}}, \mathbf{L_3^{dm}}, \mathbf{L_3^{dp}}, \mathbf{L_6^{dm}}, \mathbf{L_6^{dp}}, \mathbf{L_8^{dm}}, \mathbf{L_8^{dp}}, \mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}\})$ is defined by

$$[\alpha \vee (\beta \to \alpha)] \Leftrightarrow_H [(\alpha \vee \beta) \to \alpha].$$

Theorem 10.61. *The variety generated by the algebras* $\mathbf{L_1^{dp}}, \mathbf{L_3^{dp}}, \mathbf{L_6^{dp}}$, $\mathbf{L_8^{dp}}$ *is defined by the identities:*

(1) $x \vee (y \to x) \approx (x \vee y) \to x$,

(2) $x \vee x' \approx 1$.

Corollary 10.62. The logic $\mathcal{L}(\mathbb{V}(\{\mathbf{L_1^{dp}}, \mathbf{L_3^{dp}}, \mathbf{L_6^{dp}}, \mathbf{L_8^{dp}}\}))$ is defined by

(1) $[\alpha \vee (\beta \to \alpha)] \Leftrightarrow_H [(\alpha \vee \beta) \to \alpha]$,

(2) $\alpha \vee \sim \alpha$.

In view of the above corollary, it is clear that the logic in question does not have the Disjunction Property.

THEOREM 10.63. *The variety generated by the algebras* $\mathbf{L_1^{dm}}, \mathbf{L_3^{dm}}, \mathbf{L_6^{dm}}$, $\mathbf{L_8^{dm}}, \mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}$ *is defined by the identities:*

(1) $x \vee (y \to x) \approx (x \vee y) \to x$,

(2) $x'' \approx x$.

COROLLARY 10.64. The logic $\mathcal{L}(\mathbb{V}(\{\mathbf{L_1^{dm}}, \mathbf{L_3^{dm}}, \mathbf{L_6^{dm}}, \mathbf{L_8^{dm}}, \mathbf{D_1}, \mathbf{D_2}, \mathbf{D_2}\}))$ is defined by

(1) $[\alpha \vee (\beta \to \alpha)] \Leftrightarrow_H [(\alpha \vee \beta) \to \alpha]$,

(2) $\alpha \to_H \sim\sim \alpha$.

THEOREM 10.65. *The variety generated by the algebras* $\mathbf{L_1^{dm}}, \mathbf{L_1^{dp}}, \mathbf{L_2^{dm}}$, $\mathbf{L_2^{dp}}, \mathbf{L_5^{dm}}, \mathbf{L_5^{dp}}, \mathbf{L_6^{dm}}, \mathbf{L_6^{dp}}, \mathbf{L_9^{dm}}, \mathbf{L_9^{dp}}, \mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}$ *is defined by the identity:*

$$x^* \vee (x \to y) \approx (x \vee y) \to y.$$

COROLLARY 10.66. The logic $\mathcal{L}(\mathbb{V}(\{\mathbf{L_1^{dm}}, \mathbf{L_1^{dp}}, \mathbf{L_2^{dm}}, \mathbf{L_2^{dp}}, \mathbf{L_5^{dm}}, \mathbf{L_5^{dp}}, \mathbf{L_6^{dm}}$, $\mathbf{L_6^{dp}}, \mathbf{L_9^{dm}}, \mathbf{L_9^{dp}}, \mathbf{D_1}, \mathbf{D_2}, \mathbf{D_2}\}))$ is defined by

$$[\alpha^* \vee (\alpha \to \beta)] \Leftrightarrow_H [(\alpha \vee \beta) \to \beta].$$

THEOREM 10.67. $\mathbf{V}(\{\mathbf{L_1^{dp}}, \mathbf{L_2^{dp}}, \mathbf{L_5^{dp}}, \mathbf{L_6^{dp}}, \mathbf{L_9^{dp}}\})$ *is defined by the identity:*

(1) $x^* \vee (x \to y) \approx (x \vee y) \to y$,

(2) $x \vee x' \approx 1$.

COROLLARY 10.68. The logic $\mathcal{L}(\mathbb{V}(\{\mathbf{L_1^{dp}}, \mathbf{L_2^{dp}}, \mathbf{L_5^{dp}}, \mathbf{L_6^{dp}}, \mathbf{L_9^{dp}}\}))$ is defined by

(1) $[\alpha^* \vee (\alpha \to \beta)] \Leftrightarrow_H [(\alpha \vee \beta) \to \beta]$,

(2) $\alpha \vee \sim \alpha$.

In view of the above corollary, it is clear that the logic in question does not have the Disjunction Property.

THEOREM 10.69. *The variety generated by the algebras* $\mathbf{L_1^{dm}}, \mathbf{L_2^{dm}}, \mathbf{L_5^{dm}}$, $\mathbf{L_6^{dm}}, \mathbf{L_9^{dm}}$, $\mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}$ *is defined by the identity:*

(1) $x^* \vee (x \to y) \approx (x \vee y) \to y$,

(2) $x'' \approx x$.

Corollary 10.70. The logic $\mathcal{L}(\mathbb{V}(\{\mathbf{L_1^{dm}}, \mathbf{L_2^{dm}}, \mathbf{L_5^{dm}}, \mathbf{L_6^{dm}}, \mathbf{L_9^{dm}}, \mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}\}))$ is defined by

(1) $[\alpha^* \vee (\alpha \to \beta)] \Leftrightarrow_H [(\alpha \vee \beta) \to \beta]$,

(2) $\alpha \to_H \sim\sim \alpha$.

Theorem 10.71. *The variety generated by the algebras* $\mathbf{L_5^{dm}}, \mathbf{L_5^{dp}}, \mathbf{D_2}$ *is defined by the identity:*

$$x \vee (0 \to x) \vee (y \to 1) \approx x \vee [(x \to 1) \to (x \to y)].$$

Corollary 10.72. The logic $\mathcal{L}(\mathbb{V}(\{\mathbf{L_5^{dm}}, \mathbf{L_5^{dp}}, \mathbf{D_2}\}))$ is defined by

$$[\alpha \vee (\perp \to \alpha) \vee (\beta \to \top)] \Leftrightarrow_H \alpha \vee [(\alpha \to \top) \to (\alpha \to \beta)].$$

Theorem 10.73. *The variety generated by the algebras* $\mathbf{L_6^{dm}}, \mathbf{L_6^{dp}}, \mathbf{D_2}$ *defined by the identity:*

$$x \vee y \vee (x \to y) \approx x \vee [(x \to y) \to 1].$$

Corollary 10.74. The logic $\mathcal{L}(\mathbb{V}(\{\mathbf{L_6^{dm}}, \mathbf{L_6^{dp}}, \mathbf{D_2}\}))$ is defined by

$$\alpha \vee \beta \vee (\alpha \to \beta) \Leftrightarrow_H \alpha \vee [(\alpha \to \beta) \to \top].$$

Theorem 10.75. *The variety generated by the algebras* $\mathbf{L_1^{dm}}, \mathbf{L_1^{dp}}, \mathbf{L_7^{dm}}, \mathbf{L_7^{dp}}, \mathbf{D_2}$ *is defined by the identity:*

$$x \vee [(0 \to y) \to y] \approx x \vee [(x \to 1) \to y].$$

Corollary 10.76. The logic $\mathcal{L}(\mathbb{V}(\{\mathbf{L_1^{dm}}, \mathbf{L_1^{dp}}, \mathbf{L_7^{dm}}, \mathbf{L_7^{dp}}, \mathbf{D_2}\}))$ is defined by

$$[\alpha \vee \{(\perp \to \beta) \to \beta\}] \Leftrightarrow_H [\alpha \vee [(\alpha \to \top) \to \beta].$$

Theorem 10.77. *The variety generated by the algebras* $\mathbf{L_7^{dm}}, \mathbf{L_7^{dp}}, \mathbf{L_8^{dm}}, \mathbf{L_8^{dp}}, \mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}$ *is defined by the identity:*

$$x \ \vee [x \to (y \wedge (0 \to y))] \approx x \to [(x \to y) \to y].$$

Corollary 10.78. The logic $\mathcal{L}(\mathbb{V}(\{\mathbf{L_7^{dm}}, \mathbf{L_7^{dp}}, \mathbf{L_8^{dm}}, \mathbf{L_8^{dp}}, \mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}\}))$ is defined by

$$[\alpha \vee [\alpha \to \{\beta \wedge (\perp \to \beta)\}]] \Leftrightarrow_H [\alpha \to [(\alpha \to \beta) \to \beta].$$

THEOREM 10.79. *The variety generated by the algebras* $\mathbf{L_8^{dm}}, \mathbf{L_8^{dp}}, \mathbf{D_1}, \mathbf{D_2}$, $\mathbf{D_3}$ *is defined by the identity:*

$$x \vee y \vee [y \to (y \to x)] \approx x \to [x \vee (0 \to y)].$$

It is also defined by the identity:

$$x \vee [y \to \{0 \to (y \to x)\}] \approx x \vee y \vee (y \to x).$$

COROLLARY 10.80. The logic $\mathcal{L}(\mathbb{V}(\{\mathbf{L_8^{dm}}, \mathbf{L_8^{dp}}, \mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}\}))$ is defined by

$$[\alpha \vee \beta \vee \{\beta \to (\beta \to \alpha)\}] \Leftrightarrow_H [\alpha \to \{\alpha \vee (0 \to \beta)\}].$$

It is also defined by:

$$[\alpha \vee \{\beta \to (0 \to (\beta \to \alpha))\}] \Leftrightarrow_H [\alpha \vee \beta \vee (\beta \to \alpha)].$$

THEOREM 10.81. *The variety generated by the algebras* $\mathbf{L_7^{dm}}, \mathbf{L_7^{dp}}, \mathbf{L_8^{dm}}$, $\mathbf{L_8^{dp}}, \mathbf{L_9^{dm}}, \mathbf{L_9^{dp}}$, $\mathbf{L_{10}^{dm}}, \mathbf{L_{10}^{dp}}, \mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}$ *is defined by the identity:*

$$x \vee (x \to y) \approx x \vee [(x \to y) \to 1].$$

COROLLARY 10.82. The logic $\mathcal{L}(\mathbb{V}(\{\mathbf{L_7^{dm}}, \mathbf{L_7^{dp}}, \mathbf{L_8^{dm}}, \mathbf{L_8^{dp}}, \mathbf{L_9^{dm}}, \mathbf{L_9^{dp}}, \mathbf{L_{10}^{dm}}, \mathbf{L_{10}^{dp}}, \mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}\}))$ is defined by

$$[\alpha \vee (\alpha \to \beta)] \Leftrightarrow_H [\alpha \vee \{(\alpha \to \beta) \to \top\}].$$

THEOREM 10.83. *The variety generated by the algebras* $\mathbf{2^e}, \mathbf{L_7^{dp}}, \mathbf{L_8^{dp}}, \mathbf{L_9^{dp}}$, $\mathbf{L_{10}^{dp}}$ *is defined by the identities:*

(1) $x \vee (x \to y) \approx x \vee [(x \to y) \to 1]$,

(2) $x \vee x' \approx 1$.

COROLLARY 10.84. The logic $\mathcal{L}(\mathbb{V}(\{\mathbf{2^e}, \mathbf{L_7^{dp}}, \mathbf{L_8^{dp}}, \mathbf{L_9^{dp}}, \mathbf{L_{10}^{dp}}\})$ is defined by

(1) $[\alpha \vee (\alpha \to \beta)] \Leftrightarrow_H [\alpha \vee \{(\alpha \to \beta) \to \top\}]$,

(2) $\alpha \vee \ \sim \alpha$.

In view of the above corollary, it is clear that the logic in question does not have the Disjunction Property.

THEOREM 10.85. *The variety generated by the algebras* $\mathbf{L_7^{dm}}, \mathbf{L_8^{dm}}, \mathbf{L_9^{dm}}$, $\mathbf{L_{10}^{dm}}, \mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}$ *is defined by the identities:*

(1) $x \vee (x \to y) \approx x \vee [(x \to y) \to 1]$,

(2) $x'' \approx x$.

COROLLARY 10.86. The logic $\mathcal{L}(\mathbb{V}(\{\mathbf{L_7^{dm}}, \mathbf{L_8^{dm}}, \mathbf{L_9^{dm}}, \mathbf{L_{10}^{dm}}, \mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}\}))$ is defined by

(1) $[\alpha \vee (\alpha \to \beta)] \Leftrightarrow_H [\alpha \vee \{(\alpha \to \beta) \to \top\}]$,

(2) $\alpha \to_H \sim\sim \alpha$.

THEOREM 10.87. *The variety generated by the algebras* $\mathbf{L_9^{dm}}, \mathbf{L_9^{dp}}, \mathbf{L_{10}^{dm}}, \mathbf{L_{10}^{dp}}, \mathbf{D_1}$ *is defined by the identity:*

$$0 \to 1 \approx 0\ . \qquad \text{(FTF identity)}$$

COROLLARY 10.88. The logic $\mathcal{L}(\mathbb{V}(\{\mathbf{L_9^{dm}}, \mathbf{L_9^{dp}}, \mathbf{L_{10}^{dm}}, \mathbf{L_{10}^{dp}}, \mathbf{D_1}\}))$ is defined by

$$(\perp \to \top) \Leftrightarrow_H \perp. \qquad \text{(FTF)}$$

THEOREM 10.89. *The variety generated by the algebras* $\mathbf{L_{10}^{dm}}, \mathbf{L_{10}^{dp}}, \mathbf{D_1}$ *is defined by the identity:*

$$x \to y \approx y \to x. \qquad \text{(commutative identity)}$$

COROLLARY 10.90. The logic $\mathcal{L}(\mathbb{V}(\{\mathbf{L_{10}^{dm}}, \mathbf{L_{10}^{dp}}, \mathbf{D_1}\}))$ is defined by

$$(\alpha \to \beta) \Leftrightarrow_H (\beta \to \alpha). \qquad \text{(commutativity)}$$

THEOREM 10.91. *The variety* $\mathbb{V}(\mathbf{C}_{20})$ *is defined by*

$$x^* \leq x'.$$

COROLLARY 10.92. The logic $\mathcal{L}(\mathbb{V}(\mathbf{C_{20}}))$ is defined by

$$\alpha^* \to_H \sim \alpha.$$

THEOREM 10.93. *The variety* $\mathbb{V}(\mathbf{D_2})$ *is defined by*

$$(x \to y)^* \approx x \wedge y^*.$$

COROLLARY 10.94. The logic $\mathcal{L}(\mathbb{V}(\mathbf{D_2})$ is defined by

$$(\alpha \to \beta)^* \Leftrightarrow_H \alpha \wedge \beta^*.$$

THEOREM 10.95. *The variety generated by the algebras in* $\{\mathbf{L_i^{dp}} : i = 1, \ldots, 8\} \cup \{\mathbf{L_i^{dm}} : i = 1, \ldots, 8\} \cup \{\mathbf{D_2}\}$ *is defined by the identity:*

$$(x \to y)^* \approx (x \wedge y^*)^{**}.$$

It is also defined by

$$(0 \to 1)^* \approx 0.$$

COROLLARY 10.96. The logic $\mathcal{L}(\mathbb{V}(\{\mathbf{L_i^{dp}} : i = 1, \ldots, 8\} \cup \{\mathbf{L_i^{dm}} : i = 1, \ldots, 8\} \cup \{\mathbf{D_2}\})$ is defined by

$$(\alpha \to \beta)^* \to_H (\alpha \wedge \beta^*)^{**}.$$

It is also defined by

$$(\bot \to \top)^* \Leftrightarrow_H \bot.$$

THEOREM 10.97. *The variety generated by the algebras* $\mathbf{L_i^{dp}}$*,* $i = 1, \ldots, 8$*, is defined by the identities:*

(1) $(x \to y)^* \approx (x \wedge y^*)^{**}$,

(2) $x \vee x' \approx 1$.

COROLLARY 10.98. The logic $\mathcal{L}(\mathbb{V}(\{\mathbf{L_i^{dp}} : \mathbf{i} = \mathbf{1}, \ldots, \mathbf{8}\}))$ is defined by

(1) $(\alpha \to \beta)^* \Leftrightarrow_H (\alpha \wedge \beta^*)^{**}$,

(2) $\alpha \vee \sim \alpha$.

In view of the above corollary, it is clear that the logic in question does not have the Disjunction Property.

THEOREM 10.99. *The variety generated by the algebras* $\mathbf{L_i^{dm}}$*,* $i = 1, \ldots, 8$*, and* $\mathbf{D_2}$ *is defined by the identities:*

(1) $(x \to y)^* \approx (x \wedge y^*)^{**}$,

(2) $x'' \approx x$.

COROLLARY 10.100. The logic $\mathcal{L}(\mathbb{V}(\{\mathbf{L_i^{dm}} : \mathbf{i} = \mathbf{1}, \ldots, \mathbf{8}\} \cup \{\mathbf{D_2}\}))$, is defined by

(1) $(\alpha \to \beta)^* \Leftrightarrow_H (\alpha \wedge \beta^*)^{**}$,

(2) $\alpha \to_H \sim\sim \alpha$.

THEOREM 10.101. *The variety generated by* $\mathbf{L_1^{dm}}, \mathbf{L_1^{dp}}, \mathbf{L_2^{dm}}, \mathbf{L_2^{dp}}, \mathbf{L_5^{dm}}, \mathbf{L_5^{dp}}, \mathbf{L_6^{dm}}, \mathbf{L_6^{dp}}, \mathbf{D_2}$ *is defined by the identity:*

$$x \vee y \leq (x \to y) \to y.$$

COROLLARY 10.102. The logic $\mathcal{L}(\mathbb{V}(\{\mathbf{L_1^{dm}}, \mathbf{L_1^{dp}}, \mathbf{L_2^{dm}}, \mathbf{L_2^{dp}}, \mathbf{L_5^{dm}}, \mathbf{L_5^{dp}}, \mathbf{L_6^{dm}}, \mathbf{L_6^{dp}}, \mathbf{D_2}\}))$ is defined by

$$(\alpha \vee \beta) \to_H [(\alpha \to \beta) \to \beta].$$

THEOREM 10.103. *The variety generated by* $\mathbf{L_1^{dp}}, \mathbf{L_2^{dp}}, \mathbf{L_5^{dp}}, \mathbf{L_6^{dp}}$ *is defined by the identity:*

(1) $x \vee y \leq (x \to y) \to y$,

(2) $x \vee x' \approx 1$.

COROLLARY 10.104. The logic $\mathcal{L}(\mathbb{V}(\{\mathbf{L_1^{dp}}, \mathbf{L_2^{dp}}, \mathbf{L_5^{dp}}, \mathbf{L_6^{dp}}\}))$ is defined by

(1) $(\alpha \vee \beta) \to_H [(\alpha \to \beta) \to \beta]$,

(2) $\alpha \vee \sim \alpha$.

In view of the above corollary, it is clear that the logic in question does not have the Disjunction Property.

THEOREM 10.105. *The variety generated by* $\mathbf{L_1^{dm}}, \mathbf{L_2^{dm}}, \mathbf{L_5^{dm}}, \mathbf{L_6^{dm}}, \mathbf{D_2}$ *is defined by the identity:*

(1) $x \vee y \leq (x \to y) \to y$,

(2) $x'' \approx x$.

COROLLARY 10.106. The logic $\mathcal{L}(\mathbb{V}(\{\mathbf{L_1^{dm}}, \mathbf{L_2^{dm}}, \mathbf{L_5^{dm}}, \mathbf{L_6^{dp}}, \mathbf{D_2}\}))$ is defined by

(1) $(\alpha \vee \beta) \to_H [(\alpha \to \beta) \to \beta]$,

(2) $\alpha \to_H \sim\sim \alpha$.

The variety $\mathbb{V}(\{\mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}\})$ was axiomatized in Theorem 7.16. Here are two more bases for it.

Theorem 10.107. *The variety* $\mathbb{V}(\{\mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}\})$ *is defined by the identity:*

$$x \vee (y \to z) \approx (x \vee y) \to (x \vee z) \qquad \text{(Strong JID)}.$$

It is also defined by the identity:

$$x'^{*'*} \approx x.$$

Corollary 10.108. The logic $\mathcal{L}(\mathbb{V}(\{\mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}\}))$ is defined by

$$(\alpha \vee (\beta \to \gamma)) \Leftrightarrow_H [(\alpha \vee \beta) \to (\alpha \vee \gamma)].$$

It is also defined by the identity:

$$(\sim ((\sim \alpha)^*))^* \Leftrightarrow_H \alpha.$$

Theorem 10.109. *The variety generated by* $\mathbf{L_2^{dm}}, \mathbf{L_2^{dp}}, \mathbf{D_2}$ *is defined by the identity:*

$$(x \to y) \to x \approx x.$$

Corollary 10.110. The logic $\mathcal{L}(\mathbb{V}(\{\mathbf{L_2^{dm}}, \mathbf{L_2^{dp}}, \mathbf{D_2}\}))$ is defined by

$$((\alpha \to \beta) \to \alpha) \Leftrightarrow_H \alpha.$$

$\mathbb{V}(\mathbf{D}_2)$ was axiomatized in Theorem 7.18. Here are some more bases for it.

Theorem 10.111. $\mathbb{V}(\mathbf{D}_2)$ *is defined by the identity:*

$$x \vee y \approx (x \to y) \to y.$$

It is also defined by the identities:

(1) $x \vee (y \to z) \approx (x \vee y) \to (x \vee z)$,

(2) $(x \to y) \to x \approx x$.

It is also defined by the identity:

$$x \vee (x \to y) \approx x \vee ((x \vee y) \to 1).$$

Corollary 10.112. The logic $\mathcal{L}(\mathbb{V}(\mathbf{D}_2))$ is axiomatized by

$$(\alpha \vee \beta) \Leftrightarrow_H ((\alpha \to \beta) \to \beta).$$

This logic has an interesting property in that $\vee$ is definable in terms of $\to$. It is also axiomatized by

(1) $(\alpha \vee (\beta \to \gamma)) \Leftrightarrow_H [(\alpha \vee \beta) \to (\alpha \vee \gamma)]$,

(2) $((\alpha \to \beta) \to \alpha) \Leftrightarrow_H \alpha$.

It is also axiomatized by

$$(\alpha \vee (\alpha \to \beta)) \Leftrightarrow_H [\alpha \vee \{(\alpha \vee \beta) \to \top\}].$$

THEOREM 10.113. *The variety generated by* $\mathbf{L_1^{dm}}, \mathbf{L_1^{dp}}, \mathbf{L_2^{dm}}, \mathbf{L_2^{dp}}, \mathbf{L_9^{dm}}, \mathbf{L_9^{dp}}, \mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}$ *is defined by the identity:*

$$x \to (y \to z) \approx y \to (x \to z).$$

COROLLARY 10.114. The logic $\mathcal{L}(\mathbb{V}(\{\mathbf{L_1^{dm}}, \mathbf{L_1^{dp}}, \mathbf{L_2^{dm}}, \mathbf{L_2^{dp}}, \mathbf{L_9^{dm}}, \mathbf{L_9^{dp}}, \mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}\}))$ is defined by

$$[\alpha \to (\beta \to \gamma)] \Leftrightarrow_H [\beta \to (\alpha \to \gamma)].$$

THEOREM 10.115. *The variety generated by* $\mathbf{L_1^{dm}}, \mathbf{L_1^{dp}}, \mathbf{L_2^{dm}}, \mathbf{L_2^{dp}}, \mathbf{L_5^{dm}}, \mathbf{L_5^{dp}}, \mathbf{D_2}$ *is defined by the identity:*

$$(x \to y) \to z \leq ((y \to x) \to z) \to z.$$

COROLLARY 10.116. The logic $\mathcal{L}(\mathbb{V}(\{\mathbf{L_1^{dm}}, \mathbf{L_1^{dp}}, \mathbf{L_2^{dm}}, \mathbf{L_2^{dp}}, \mathbf{L_5^{dm}}, \mathbf{L_5^{dp}}, \mathbf{D_2}\}))$ is defined by

$$[(\alpha \to \beta) \to \gamma] \to_H [((\beta \to \alpha) \to \gamma) \to \gamma].$$

We note that a new extension of each of the logic defined in this section is obtained by adding the axiom $\alpha'' \Leftrightarrow_H \alpha$, as an extension of the logic $\mathcal{DMSH}$. Similarly, the addition of the axiom: $\alpha \vee \alpha'$ yields new extensions to the logics, over the logic $\mathcal{DPCSH}$, defined in the preceding corollaries.

We conclude this section by remarking that all the logics described in this section are discriminator logics and also are decidable.

11. Logics corresponding to subvarieties of regular De Morgan semi-Heyting algebras of level 1

In this section, we present axiomatizations for logics corresponding to several subvarieties of the variety $\mathbb{RDMSH}_1$ of regular De Morgan semi-Heyting algebras of level 1. The algebraic results mentioned (or referred to) in this section were proved in [32]. Recall that $\mathcal{DMSH}_1$ denotes the logic corresponding to the variety $\mathbb{DMSH}_1$. The following corollary is immediate from Theorem 5.9 and definitions.

In what follows, $\mathcal{V}$ (or $\mathcal{L}(\mathbb{V})$) denotes the logic corresponding to the variety $\mathbb{V}$.

Recall that the variety $\mathbb{DMSH}_1$ was defined in Section 10.

Corollary 11.1.

(a) The logic $\mathcal{DMSH}_1$ is defined, relative to $\mathcal{DMSH}$, by

$$\alpha \ \wedge (\sim \alpha)^* \Leftrightarrow_H \ [\sim (\alpha \ \wedge (\sim \alpha)^*)]^*.$$

(b) The logic $\mathcal{RDMSH}_1$ is defined, relative to $\mathcal{DMSH}_1$, by

$$(\alpha \wedge \alpha^+) \to_H (\beta \vee \beta^*).$$

(c) The logic $\mathcal{RDMH}_1$ is defined, relative to $\mathcal{RDMSH}_1$, by

$$(\alpha \wedge \beta) \to \alpha.$$

(d) The logic $\mathcal{RDM}cm\mathcal{SH}_1$ is defined, relative to $\mathcal{RDMSH}_1$, by

$$(\alpha \to \beta) \to_H (\beta \to \alpha).$$

It follows from Theorem 10.6 that the logic $\mathcal{RDMH}_1$ is decidable. However, the following problems are open.

PROBLEM 4: Is the logic $\mathcal{R}ms\mathcal{H}_1$ decidable?

PROBLEM 5: Is the logic $\mathcal{R}ms\mathcal{SH}_1$ decidable?

Let $\mathbf{L} \in \mathbb{DHMSH}$. We say $\mathbf{L}$ is pseudocommutative if $\mathbf{L}$ satisfies the identity:

$$\text{(PCM)} \qquad x^* \to y^* \approx y^* \to x^*.$$

$\mathbb{RDM}$pcm$\mathbb{SH}$ denotes the variety of regular De Morgan pseudocommutative semi-Heyting algebras.

The following corollary is immediate from Theorem 5.9 and the above definition.

Corollary 11.2. The logic $\mathcal{L}(\mathbb{RDM}\mathrm{pcm}\mathbb{SH}_1)$ is defined by

$$(\alpha^* \to \beta^*) \Leftrightarrow_H (\beta^* \to \alpha^*).$$

Theorem 11.3 ([32]). $\mathbb{RDM}\mathrm{pcm}\mathbb{SH}_1 = \mathbf{V}(\mathbf{L_9^{dm}}, \mathbf{L_{10}^{dm}}, \mathbf{D_1})$.

Corollary 11.4. The logic $\mathcal{L}(\mathbb{RDM}\mathrm{pcm}\mathbb{SH}_1)$ is decidable.

In the rest of this section, unless otherwise stated, the phrase "defined, modulo $\mathbb{RDMSH}_1$, by" is abbreviated to the phrase "defined by" in the context of varieties. Similarly, the phrase "defined, as an extension of the logic $\mathcal{RDMSH}_1$, by" is also abbreviated to the phrase "defined by" in the case of logics.

The theorems that appear in the rest of this section were proved in [32]. Each of the corollaries given below follows from the theorem immediately preceding it and Theorem 5.9.

Here is another axiomatization for $\mathbb{RDM}\mathrm{pcm}\mathbb{SH}$.

Theorem 11.5. *The variety* $\mathbb{RDM}\mathrm{pcm}\mathbb{SH}$ *is defined by*

$$(x \to y)^* \approx (y \to x)^*.$$

Corollary 11.6. The logic $\mathcal{L}(\mathbb{RDM}\mathrm{pcm}\mathbb{SH})$ is defined by

$$(\alpha \to \beta)^* \Leftrightarrow_H (\beta \to \alpha)^*.$$

Theorem 11.7. *The variety* $\mathbb{V}(\mathbf{L_1^{dm}}, \mathbf{L_2^{dm}}, \mathbf{L_3^{dm}}, \mathbf{L_4^{dm}}, \mathbf{D_2}, \mathbf{D_3})$ *is defined by*

$$(0 \to 1)^+ \to\ [\sim \{(0 \to 1)^*\}]^* \approx 0 \to 1.$$

Corollary 11.8. The logic $\mathcal{L}(\mathbb{V}(\mathbf{L_1^{dm}}, \mathbf{L_2^{dm}}, \mathbf{L_3^{dm}}, \mathbf{L_4^{dm}}, \mathbf{D_2}, \mathbf{D_3}))$ is defined by

$$((\bot \to \top)^+ \to\ [\sim \{(\bot \to \top)^*\}]^*) \Leftrightarrow_H (\alpha \to \top).$$

The variety $\mathbf{V}(\mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3})(= \mathbf{DQDBSH})$ was axiomatized earlier. Here are some more bases for $\mathbf{V}(\mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3})$.

Theorem 11.9. *Each of the following identities is a base for the variety* $\mathbf{V}(\mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3})$*:*

(1) $x \to y \approx y^* \to x^*$, (Law of contraposition)

(2) $[\{x \vee (x \to y^*)\} \to (x \to y^*)] \vee (x \vee y^*) = 1.$

Corollary 11.10.

(1) The logic $\mathcal{L}(\mathbb{V}(\mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}))$ is defined by

$$(\alpha \to \beta) \Leftrightarrow_H (\beta^* \to \alpha^*).$$

(2) The logic $\mathcal{L}(\mathbb{V}(\mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}))$ is also defined by

$$[\{\alpha \vee (\alpha \to \beta^*)\} \to (\alpha \to \beta^*)] \vee (\alpha \vee \beta^*).$$

Theorem 11.11. *The variety* $\mathbb{V}(\mathbf{L_1^{dm}}, \mathbf{L_2^{dm}}, \mathbf{L_5^{dm}}, \mathbf{L_6^{dm}}, \mathbf{L_9^{dm}}, \mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3})$ *is defined by*

$$x \to y^* \approx y \to x^*.$$

Corollary 11.12. The logic $\mathcal{L}(\mathbb{V}(\mathbf{L_1^{dm}}, \mathbf{L_2^{dm}}, \mathbf{L_5^{dm}}, \mathbf{L_6^{dm}}, \mathbf{L_9^{dm}}, \mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}))$ is defined by

$$(\alpha \to \beta^*) \Leftrightarrow_H (\beta \to \alpha^*).$$

Theorem 11.13. *The variety* $\mathbf{V}(\mathbf{L_7^{dm}}, \mathbf{L_8^{dm}}, \mathbf{L_9^{dm}}, \mathbf{L_{10}^{dm}}, \mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3})$ *is defined by*

$$x \vee (x \to y) \approx x \vee [(x \to y) \to 1].$$

Corollary 11.14. The logic $\mathcal{L}(\mathbb{V}(\mathbf{L_7^{dm}}, \mathbf{L_8^{dm}}, \mathbf{L_9^{dm}}, \mathbf{L_{10}^{dm}}, \mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}))$ is defined by

$$[\alpha \vee (\alpha \to \beta)] \Leftrightarrow_H [\alpha \vee \{(\alpha \to \beta) \to \top\}],$$

Theorem 11.15. *The variety* $\mathbf{V}(\mathbf{L_7^{dm}}, \mathbf{L_8^{dm}}, \mathbf{D_2})$ *is defined by*

(1) $x \vee (x \to y) \approx x \vee [(x \to y) \to 1]$,

(2) $(0 \to 1)^{**} \approx 1$.

Corollary 11.16. The logic $\mathcal{L}(\mathbb{V}(\mathbf{L_7^{dm}}, \mathbf{L_8^{dm}}, \mathbf{D_2}))$ is defined by

(1) $[\alpha \vee (\alpha \to \beta)] \Leftrightarrow_H [\alpha \vee \{(\alpha \to \beta) \to \top\}]$,

(2) $(\bot \to \top)^{**}$.

Theorem 11.17. *The variety* $\mathbf{V}(\mathbf{2^e}, \mathbf{L_7^{dm}}, \mathbf{L_8^{dm}}, \mathbf{L_9^{dm}}, \mathbf{L_{10}^{dm}})$ *is defined by*

(1) $x \vee (x \to y) \approx x \vee [(x \to y) \to 1]$,

(2) $x^{*\prime} \approx x^{**}$ ($\star$-regular).

We caution the reader that in [29], (2) was referred to as “regular”.

COROLLARY 11.18. The logic $\mathcal{L}(\mathbb{V}(\mathbf{2^e}, \mathbf{L_7^{dm}}, \mathbf{L_8^{dm}}, \mathbf{L_9^{dm}}, \mathbf{L_{10}^{dm}}))$ is defined by

(1) $[\alpha \vee (\alpha \to \beta)] \Leftrightarrow_H [\alpha \vee \{(\alpha \to \beta) \to 1\}]$,

(2) $\sim (\alpha^*) \Leftrightarrow_H \alpha^{**}$.

THEOREM 11.19. *The variety* $\mathbf{V}(\mathbf{2^e}, \mathbf{L_9^{dm}}, \mathbf{L_{10}^{dm}})$ *is defined by*

(1) $x \vee (x \to y) \approx x \vee [(x \to y) \to 1]$,

(2) $x^{*\prime} \approx x^{**}$,

(3) $(0 \to 1) \vee (0 \to 1)^* \approx 1$.

COROLLARY 11.20. The logic $\mathcal{L}(\mathbb{V}(\mathbf{2^e}, \mathbf{L_9^{dm}}, \mathbf{L_{10}^{dm}}))$ is defined by

(1) $[\alpha \vee (\alpha \to \beta)] \Leftrightarrow_H [\alpha \vee \{(\alpha \to \beta) \to \top\}]$,

(2) $\sim (\alpha^*) \Leftrightarrow_H \alpha^{**}$,

(3) $(\perp \to \top) \vee (\perp \to \top)^*$.

THEOREM 11.21. *The variety* $\mathbf{V}(\mathbf{L_9^{dm}}, \mathbf{L_{10}^{dm}})$ *is defined by*

(1) $x \vee (x \to y) \approx x \vee [(x \to y) \to 1]$,

(2) $x^{*\prime} \approx x^{**}$,

(3) $(0 \to 1)^* \approx 1$.

COROLLARY 11.22. The logic $\mathcal{L}(\mathbb{V}(\mathbf{L_9^{dm}}, \mathbf{L_{10}^{dm}}))$ is defined by

(1) $[\alpha \vee (\alpha \to \beta)] \Leftrightarrow_H [\alpha \vee \{(\alpha \to \beta) \to 1\}]$,

(2) $\sim (\alpha^*) \Leftrightarrow_H \alpha^{**}$,

(3) $(\perp \to \top)^*$.

THEOREM 11.23. *The variety* $\mathbf{V}(\mathbf{L_1^{dm}}, \mathbf{L_2^{dm}}, \mathbf{L_3^{dm}}, \mathbf{L_4^{dm}}, \mathbf{L_5^{dm}}, \mathbf{L_6^{dm}}, \mathbf{L_7^{dm}}, \mathbf{L_8^{dm}})$ *is defined by*

(1) $x^{*\prime} \approx x^{**}$,

(2) $(0 \to 1)^{**} \approx 1$.

COROLLARY 11.24. The logic $\mathcal{L}(\mathbb{V}((\mathbf{L_1^{dm}}, \mathbf{L_2^{dm}}, \mathbf{L_3^{dm}}, \mathbf{L_4^{dm}}, \mathbf{L_5^{dm}}, \mathbf{L_6^{dm}}, \mathbf{L_7^{dm}}, \mathbf{L_8^{dm}})$ is defined by

(1) $\sim (\alpha^*) \Leftrightarrow_H \alpha^{**}$.

(2) $(\perp \to \top)^{**}$.

THEOREM 11.25. *The variety* $\mathbf{V}(\mathbf{L_1^{dm}}, \mathbf{L_2^{dm}}, \mathbf{L_3^{dm}}, \mathbf{L_4^{dm}}, \mathbf{D_2})$ *is defined by*

(1) $(0 \to 1) \vee (0 \to 1)^* \approx 1$,

(2) $(0 \to 1)^{**} \approx 1$.

COROLLARY 11.26. The logic $\mathcal{L}(\mathbb{V}((\mathbf{L_1^{dm}}, \mathbf{L_2^{dm}}, \mathbf{L_3^{dm}}, \mathbf{L_4^{dm}}, \mathbf{D_2}))$ is defined by

(1) $(\perp \to \top) \vee (\perp \to \top)^*$,

(2) $(\perp \to \top)^{**}$.

THEOREM 11.27. *The variety* $\mathbf{V}(\mathbf{L_1^{dm}}, \mathbf{L_3^{dm}}, \mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3})$ *is defined by*

(1) $x \vee (y \to x) \approx (x \vee y) \to x$,

(2) $(0 \to 1) \vee (0 \to 1)^* \approx 1$.

COROLLARY 11.28. The logic $\mathcal{L}(\mathbb{V}((\mathbf{L_1^{dm}}, \mathbf{L_3^{dm}}, \mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}))$ is defined by

(1) $[\alpha \vee (\beta \to \alpha)] \Leftrightarrow_H [(\alpha \vee \beta) \to \alpha]$,

(2) $(\perp \to \top) \vee (\perp \to \top)^*$.

THEOREM 11.29. *The variety* $\mathbf{V}(\mathbf{L_1^{dm}}, \mathbf{L_3^{dm}}, \mathbf{D_2})$ *is defined by*

(1) $x \vee (y \to x) \approx (x \vee y) \to x$,

(2) $(0 \to 1) \vee (0 \to 1)^* \approx 1$,

(3) $(0 \to 1)^{**} \approx 1$.

COROLLARY 11.30. The logic $\mathcal{L}(\mathbb{V}((\mathbf{L_1^{dm}}, \mathbf{L_3^{dm}}, \mathbf{D_2}))$ is defined by

(1) $[\alpha \vee (\beta \to \alpha)] \Leftrightarrow_H [(\alpha \vee \beta) \to \alpha]$,

(2) $(\perp \to \top) \vee (\perp \to \top)^*$,

(3) $(\perp \to \top)^{**}$.

THEOREM 11.31. *The variety* $\mathbf{V}(\mathbf{L_1^{dm}}, \mathbf{L_2^{dm}}, \mathbf{L_8^{dm}}, \mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3})$ *is defined by*

$$y \vee (y \to (x \vee y)) \approx (0 \to x) \vee (x \to y).$$

COROLLARY 11.32. The logic $\mathcal{L}(\mathbb{V}((\mathbf{L_1^{dm}}, \mathbf{L_2^{dm}}, \mathbf{L_8^{dm}}, \mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}))$ is defined by

$$[\beta \vee (\beta \to (\alpha \vee \beta))] \Leftrightarrow_H [(\perp \to \alpha) \vee (\alpha \to \beta)].$$

Theorem 11.33. *The variety* $\mathbf{V}(\mathbf{L_8^{dm}}, \mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3})$ *is defined by*

$$x \vee [y \to (0 \to (y \to x))] \approx x \vee y \vee (y \to x).$$

Corollary 11.34. The logic $\mathcal{L}(\mathbb{V}(\mathbf{L_8^{dm}}, \mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}))$ is defined by

$$[\alpha \vee \{\beta \to (\perp \to (\beta \to \alpha))\}] \Leftrightarrow_H [\alpha \vee \beta \vee (\beta \to \alpha)].$$

Theorem 11.35. *The variety* $\mathbb{V}(\mathbf{C}^{dm})$ *is defined by*

$$x \wedge x' \leq y \vee y'. \qquad \text{(Kleene identity)}$$

Corollary 11.36. The logic $\mathcal{L}(\mathbb{V}(\mathbf{C^{dm}}))$ is defined by

$$(\alpha \wedge \ \sim \alpha) \to_H (\beta \vee \ \sim \beta. \qquad \text{(Kleene identity)}$$

Theorem 11.37. *The variety* $\mathbb{V}(\mathbf{L}_{10}^{dm})$ *is defined by*

(1) $x \wedge x' \leq y \vee y'$, (Kleene identity)

(2) $x \to y \approx y \to x$.

Corollary 11.38. The logic $\mathcal{L}(\mathbb{V}(\mathbf{L}_{10}^{dm}))$ is defined by

(1) $(\alpha \wedge \ \sim \alpha) \to_H (\beta \vee \ \sim \beta)$, (Kleene identity)

(2) $\alpha \to \beta \Leftrightarrow_H \beta \to \alpha$.

$\mathbf{V}(\mathbf{D_2})$ was axiomatized in Section 7. Here are some more bases for it, but relative to $\mathbf{RDMH_1}$.

Theorem 11.39. *Each of the following identities is a base for* $\mathbf{V}(\mathbf{D_2})$*, mod* $\mathbf{RDMH_1}$*:*

(1) $[y \to \{0 \to (y \to x)\}] \approx y \vee (y \to x)$.

(2) $x \vee (y \to z) \approx (x \vee y) \to (x \vee z)$.

(3) $[\{x \vee (x \to y^*)\} \to (x \to y^*)] \vee x \vee y^* \approx 1$.

Corollary 11.40. Each of the following axioms defines the logic $\mathcal{L}(\mathbb{V}(\mathbf{D}_2)$, relative to $\mathcal{RDMH}_1$:

(1) $[\beta \to \{\perp \to (\beta \to \alpha)\}] \Leftrightarrow_H [\beta \vee (\beta \to \alpha)]$,

(2) $[\alpha \vee (\beta \to \gamma)] \Leftrightarrow_H [(\alpha \vee \beta) \to (\alpha \vee \gamma)]$,

(3) $[\{\alpha \vee (\alpha \to \beta^*)\} \to (\alpha \to \beta^*)] \vee \alpha \vee \beta^*$.

$\mathbb{V}(\mathbf{D}_1)$ was axiomatized in Section 8. Here are more bases for it. Let $\mathbb{RDM}\mathrm{cm}\mathbb{SH}_1$ denote the subvariety of $\mathbb{RDMSH}_1$ defined by: $x \to y \approx y \to x$.

THEOREM 11.41. *Each of the following identities is an equational base for* $\mathbb{V}(\mathbf{D}_1)$, *mod* $\mathbb{RDM}\mathrm{cm}\mathbb{SH}_1$*:*

(1) $y \vee (y \to (x \vee y)) \approx (0 \to x) \vee (x \to y)$,

(2) $x \vee [y \to (y \to x)^*] \approx x \vee y \vee (y \to x)$,

(3) $[\{x \vee (x \to y^*)\} \to (x \to y^*)] \vee x \vee y^* \approx 1$,

(4) $x \vee (y \to z) \approx (x \vee y) \to (x \vee z)$.

COROLLARY 11.42. Each of the following axioms defines the logic $\mathcal{L}(\mathbb{V}(\mathbf{D}_1))$, relative to $\mathcal{RDM}\mathrm{cm}\mathcal{SH}_1$:

(1) $[\beta \vee (\beta \to (\alpha \vee \beta))] \Leftrightarrow_H [(\perp \to \alpha) \vee (\alpha \to \beta)]$,

(2) $[\alpha \vee \{\beta \to (\beta \to \alpha)^*\}] \Leftrightarrow_H [\alpha \vee \beta \vee (\beta \to \alpha)]$,

(3) $[\{\alpha \vee (\alpha \to \beta^*)\} \to (\alpha \to \beta^*)] \vee \alpha \vee \beta^*$,

(4) $[\alpha \vee (\beta \to \gamma)] \Leftrightarrow_H [(\alpha \vee \beta) \to (\alpha \vee \gamma)]$.

We conclude this section with the remark that all logics introduced in this section are discriminator logics as their corresponding varieties are discriminator varieties.

12. Extensions of the logic $\mathcal{JIDSH}_1$

Algebras closely related to $\mathbb{DS}\mathrm{t}\mathbb{SH}$-algebras, called "JI-distributive semi-Heyting algebras", were introduced in [33].

An algebra $\mathbf{A}$ in $\mathbb{DQDSH}$ is JI-distributive if $\mathbf{A}$ satisfies:

(JID) $$x' \vee (y \to z) \approx (x' \vee y) \to (x' \vee z).$$

((restricted) **J**oin over **I**mplication **D**istributivity).

We note that the identity (JID) is obtained by slightly weakening the identity (Strong JID) that has appeared earlier in Theorem 10.107. Let

$\mathbb{JIDSH}$ denote the variety of JI-distributive $\mathbb{DQDSH}$-algebras and let $\mathbb{JIDSH}_1$ (or $\mathbb{JID}_1$, for short) denote the subvariety of $\mathbb{JIDSH}$ of level 1.

In what follows, $\mathcal{V}$ (or $\mathcal{L}(\mathbb{V})$) denotes the logic corresponding to the subvariety $\mathbb{V}$.

In this section we present axiomatizations of the logics corresponding to the subvarieties of $\mathbb{JIDSH}_1$ which we denote simply by $\mathbb{JID}_1$.

Corollary 12.1. The logic $\mathcal{JID}_1$ corresponding to $\mathbb{JID}_1$ is defined, as an extension of $\mathcal{DQDSH}$, by

(a) $(\sim \alpha \vee (\beta \to \gamma)) \Leftrightarrow_H ((\sim \alpha \vee \beta) \to (\sim \alpha \vee \gamma))$,

(b) $\alpha \ \wedge (\sim \alpha)^* \Leftrightarrow_H \ [\sim (\alpha \ \wedge \ (\sim \alpha)^*)]^*$.

Let $\mathbb{DSt}$ [$\mathbb{DStH}$] denote the variety of dually Stone semi-Heyting [Heyting] algebras. The following theorem was proved in [33, Corollary 5.10].

Theorem 12.2. $\mathbb{JID}_1 = \mathbb{DSt} \vee \mathbb{V}(\mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3})$.

The preceding Theorem leads us naturally to raise the following open problems.

PROBLEM 6: Is the logic $\mathcal{DStH}$ decidable?

We conjecture that the answer to **PROBLEM 5** is in the positive.

More generally, we can ask the following:

PROBLEM 7: Is the logic $\mathcal{DStSH}$ is decidable?

We let $\mathbb{JIDL}_1$ denote the subvariety of $\mathbb{JID}_1$ defined by

$$(x \to y) \vee (y \to x) \approx 1. \tag{L}$$

The results in the rest of this section depend on the corresponding algebraic results of [33]. The relevant results, however, are stated here for the convenience of the reader. The following corollary is immediate from the above definitions in view of Theorem 5.9.

Corollary 12.3. The logic $\mathcal{JIDL}_1$ corresponding to $\mathbb{JIDL}_1$ is defined, modulo $\mathcal{JID}_1$, by

$(\alpha \to \beta) \vee (\beta \to \alpha)$.

Let $\mathbb{DStL}$ denote the subvariety of $\mathbb{DSt}$ defined by the identity (L) and $\mathbb{DStHC}$ denote the subvariety of $\mathbb{DStH}$ generated by its chains.

THEOREM 12.4. [33]. $\mathbb{JIDL}_1 = \mathbb{DStHC} \vee \mathbb{V}(\mathbf{D_2})$.

For $n \in \mathbb{N}$, let $\mathbf{C_n^{dp}}$ denote the n-element $\mathbb{DStH}$-chain (= $\mathbb{DPCH}$-chain) denotes the variety generated by $\mathbf{C_n^{dp}}$. (Note that $\mathbf{C_3^{dp}} = \mathbf{L_1^{dp}}$.)

Since the variety of Boolean algebras is the smallest non-trivial subvariety of $\mathbb{JIDL}_1$, we denote by $\mathbf{L_V}^+(\mathbb{JIDL}_1)$ the latttice of non-trivial subvarieties of $\mathbb{JIDL}_1$.

The following theorem was proved in [33, Corollary 7.1].

THEOREM 12.5.

(1) $\mathbf{L_V}^+(\mathbb{JIDL}_1) \cong [(\omega + \mathbf{1}) \times \mathbf{2}]$, *where* $\times$ *represents the direct product.*

(2) $\mathbb{JIDL}_1$ *and* $\mathbb{DStHC}$ *are the only two elements of infinite height in the lattice* $\mathbf{L_V}^+(\mathbb{JIDL}_1)$.

(3) $\mathbb{V} \in \mathbf{L_V}^+(\mathbb{JIDL}_1)$ *is of finite height if and only if* $\mathbb{V}$ *is either* $\mathbb{V}(\mathbf{D_2})$ *or* $\mathbb{V}(\mathbf{C}_n^{dp})$, *for some* $n \in \mathbb{N} \setminus \{1\}$, *or* $\mathbb{V}(\mathbf{C}_m^{dp}) \vee \mathbb{V}(\mathbf{D_2})$, *for some* $m \in \mathbb{N} \setminus \{1\}$.

The following corollary is immediate from the preceding theorem and Theorem 5.9.

COROLLARY 12.6. The logic $\mathcal{JIDL}_1$ has the finite model property and hence it is decidable.

Bases for all subvarieties of $\mathbb{JIDL}_1$ were given in [33]. The theorems presented below are taken from [33] and each of the corollaries given below follows from the theorem that precedes it and Theorem 5.9.

In the rest of this section, the phrase "defined, modulo $\mathbb{JIDL}_1$, by" is abbreviated to "defined by", in the context of varieties. Similarly, the phrase "defined, as an extension of the logic $\mathcal{JIDL}_1$, by" is also abbreviated to the phrase "defined by" in the case of logics.

The theorems that appear below were proved in [33]. Each of the corollaries given below follows from the theorem immediately preceding it and Theorem 5.9.

THEOREM 12.7. *The variety* $\mathbb{DStHC}$ *is defined by*

$$x \vee x' \approx 1.$$

Corollary 12.8. The logic $\mathcal{DS}\mathrm{t}\mathcal{HC}$ is defined by

$$\alpha \vee \ \sim \alpha.$$

The variety $\mathbb{V}(\mathbf{D}_2)$ was axiomatized earlier. Here is another one.

Theorem 12.9. *The variety* $\mathbb{V}(\mathbf{D}_2)$ *is defined by*

$$x'' \approx x.$$

Corollary 12.10. The logic $\mathcal{L}(\mathbb{V}(\mathbf{D}_2))$ is defined by

$$\alpha \Leftrightarrow_H \sim\sim \alpha.$$

Let $n \in \mathbb{N}$ such that $n \geq 2$.

Theorem 12.11. *The variety* $\mathbb{V}(\mathbf{C}_n^{dp}) \ \vee \ \mathbb{V}(\mathbf{D}_2)$ *is defined by*

(E_n) $x_1 \vee x_2 \vee \cdots \vee x_n \vee (x_1 \to x_2) \vee (x_2 \to x_3) \vee \cdots \vee (x_{n-1} \to x_n) = 1.$

Corollary 12.12. The logic $\mathcal{L}(\mathbb{V}(\mathbf{C_n}^{dp}) \ \vee \ \mathbb{V}(\mathbf{D}_2))$ is defined by

($\mathcal{E}_n$) $\alpha_1 \vee \alpha_2 \vee \cdots \vee \alpha_n \vee (\alpha_1 \to \alpha_2) \vee (\alpha_2 \to \alpha_3) \vee \cdots \vee (\alpha_{n-1} \to \alpha_n).$

Theorem 12.13. *The variety* $\mathbb{V}(\mathbf{C}_n^{dp})$ *is defined by*

(1) $x \vee x' \approx 1,$

(2) $x_1 \vee x_2 \vee \cdots \vee x_n \vee (x_1 \to x_2) \vee (x_2 \to x_3) \vee \cdots \vee (x_{n-1} \to x_n) = 1.$

Corollary 12.14. The logic $\mathcal{L}(\mathbb{V}(\mathbf{C}_n^{dp}))$ is defined by

(a) $\alpha \vee \ \sim \alpha,$

(C_n) $\alpha_1 \vee \alpha_2 \vee \cdots \vee \alpha_n \vee (\alpha_1 \to \alpha_2) \vee (\alpha_2 \to \alpha_3) \vee \cdots \vee (\alpha_{n-1} \to \alpha_n).$

Here are two more axiomatizations for the logic $\mathcal{L}(\mathbb{V}(\mathbf{C_3^{dp}}) \ \vee \ \mathbb{V}(\mathbf{D}_2))$.

Theorem 12.15. *The variety* $\mathbb{V}(\mathbf{C_3^{dp}}) \vee \mathbb{V}(\mathbf{D_2})$ *is defined by*

$$x \wedge x^+ \ \leq \ y \vee y^*. \qquad \text{(Regularity)}$$

It is also defined by

$$x \wedge x' \leq y \vee y^*.$$

Corollary 12.16. The logic $\mathcal{L}(\mathbb{V}(\mathbf{C_3^{dp}}) \ \vee \ \mathbb{V}(\mathbf{D}_2))$ is defined by

$(\alpha \wedge \alpha^+) \to_H (\beta \vee \beta^*)$.

It is also defined by

$(\alpha \wedge \ \sim \alpha) \to_H (\beta \vee \beta^*)$.

Recall that $\mathbf{L_1^{dp}} = \mathbf{C_3^{dp}}$. The logic $\mathcal{L}(\mathbb{V}(\mathbf{L_1^{dp}}))$ is axiomatized in Corollary 7.11. Here is yet another axiomatization for it.

Theorem 12.17. *The variety* $\mathbb{V}(\mathbf{L}_1^{dp})$ *is defined by*

(1) $x \wedge x^+ \leq y \vee y^*$ (Regularity),

(2) $x^{*\prime} = x^{**}$.

Corollary 12.18. The logic $\mathcal{L}(\mathbb{V}(\mathbf{L}_1^{dp}))$ is defined by

(1) $(\alpha \wedge \alpha^+) \to_H (\beta \vee \beta^*)$,

(2) $\sim \alpha^* \to_H \alpha^{**}$.

We note that the extensions of $\mathcal{JIDL}_1$ are all decidable.

We conclude this section with a partial poset of subvarieties of $\mathbb{DQDSH}$ discussed in the last sections (Figure 5). Its dual will give the partial poset of the axiomatic extensions of the logic $\mathcal{DQDSH}$. Note that the links in the poset do not, in general, represent the covers.

13. Concluding remarks and open problems

It is, perhaps, worthwhile to mention here that we know from [29] that every simple algebra in $\mathbb{RDQDS}\mathrm{t}\mathbb{H}_1$ is quasiprimal. Of all the 25 simple algebras in $\mathbb{RDQDS}\mathrm{t}\mathbb{SH}_1$ (Section 7), $\mathbf{2^e}$, $\bar{\mathbf{2}}^{\mathbf{e}}$, and $\mathbf{L_i}, i = 5, 6, 7, 8$, and $\mathbf{D}_3$ are primal algebras and the rest, except $\mathbf{D}_1$ and $\mathbf{D}_2$, are semiprimal algebras.

We will now collect here all the open problems that were mentioned in the earlier sections.

PROBLEM 1: Describe the structure of the lattice of subvarieties of the variety $\mathbb{BDQDSH}$.

PROBLEM 2: Is the variety $\mathbb{DQDS}\mathrm{t}\mathbb{H}_1$ generated by its finite members?

PROBLEM 3: Is the variety $\mathbb{DQDS}\mathrm{t}\mathbb{SH}_1$ generated by its finite members?

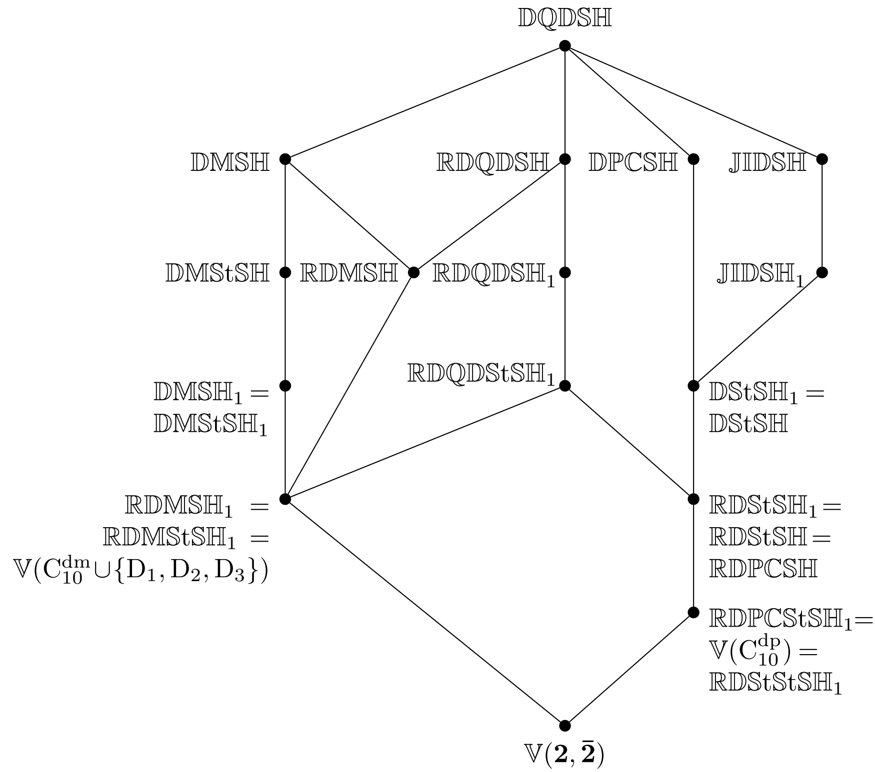


Figure 5. Partial poset of subvarieties of $\mathbb{DQDSH}$

PROBLEM 4: Is the logic $\mathcal{R}ms\mathcal{H}_1$ decidable?

PROBLEM 5: Is the logic $\mathcal{R}ms\mathcal{SH}_1$ decidable?

PROBLEM 6: Is the logic $\mathcal{DS}t\mathcal{H}$ decidable?

PROBLEM 7: Is the logic $\mathcal{DS}t\mathcal{SH}$ decidable?

We will add a few more problems of interest:

PROBLEM 8: Is $\mathbb{RDQDS}\mathrm{t}\mathbb{H} = \mathbb{BRDQDS}\mathrm{t}\mathbb{H}$?

PROBLEM 9: Is $\mathbb{RDQDSH} = \mathbb{BRDQDSH}$?

PROBLEM 10: Determine the subvarieties of $\mathbb{DQDS}\mathrm{t}\mathbb{SH}_1$ that have Amalgamation Property.

PROBLEM 11: Is $\mathbb{DPCSH} = \mathbb{SBDPCSH}$?

We conclude the paper by mentioning a few open-ended problems for future research.

Investigate the extensions of the logic $\mathcal{DHMSH}$ in relation to, among others, the following:

(a) Decidability,

(b) Various interpolation properties,

(c) Beth's Definability property (or equivalently, "the epimorphisms are surjective" property for the corresponding variety),

(d) Disjunction property,

(e) Finite model property,

(f) Finite embeddability,

(g) Structural completeness.

Acknowledgements. The first author wants to thank the institutional support of CONICET (Consejo Nacional de Investigaciones Científicas y Técnicas) and Universidad Nacional del Sur. The authors would like to express their gratitude to Dr. Alex Citkin for his encouraging remarks and insightful comments on an earlier version of this paper. The authors also wish to express their gratitude to the referees for their careful reading of an earlier version of this paper and for their valuable suggestions.

References

[1] M. Abad, J. M. Cornejo, J. P. Diaz Varela, *The variety generated by semi-Heyting chains*, **Soft Computing**, vol. 15(4) (2010), pp. 721–728, DOI: https://doi.org/10.1007/s00500-010-0604-0.

[2] M. Abad, J. M. Cornejo, J. P. Díaz Varela, *The variety of semi-Heyting algebras satisfying the equation* $(0 \to 1)^* \vee (0 \to 1)^{**} \approx 1$, **Reports on Mathematical Logic**, vol. 46 (2011), pp. 75–90.

[3] M. Abad, J. M. Cornejo, J. P. Diaz Varela, *Semi-Heyting Algebras Term-equivalent to Gödel Algebras*, **Order**, vol. 30(2) (2013), pp. 625–642, DOI: https://doi.org/10.1007/s11083-012-9266-0.

[4] M. E. Adams, H. P. Sankappanavar, J. Vaz de Carvalho, *Regular double p-algebras*, **Mathematica Slovaca**, vol. 69(1) (2019), pp. 15–34, DOI: https://doi.org/10.1515/ms-2017-0200.

[5] M. E. Adams, H. P. Sankappanavar, J. Vaz de Carvalho, *Varieties of Regular Pseudocomplemented de Morgan Algebras*, **Order**, vol. 37(3) (2020), pp. 529–557, DOI: https://doi.org/10.1007/s11083-019-09518-y.

[6] W. J. Blok, D. Pigozzi, *Algebraizable logics*, **Memoirs of the American Mathematical Society**, vol. 77(396) (1989), pp. vi+78.

[7] S. Burris, H. P. Sankappanavar, **A course in universal algebra**, vol. 78 of Graduate Texts in Mathematics, Springer-Verlag, New York (1981).

[8] J. M. Cornejo, *Semi-intuitionistic logic*, **Studia Logica**, vol. 98(1–2) (2011), pp. 9–25, DOI: https://doi.org/10.1007/s11225-011-9321-7.

[9] J. M. Cornejo, M. Kinyon, H. P. Sankappanavar, *Regular double p-algebras: A converse to a Katriňák's theorem, and applications* (2022), preprint.

[10] J. M. Cornejo, L. F. Monteiro, H. P. Sankappanavar, I. D. Viglizzo, *A note on chain-based semi-Heyting algebras*, **Mathematical Logic Quarterly**, vol. 66(4) (2020), pp. 409–417, DOI: https://doi.org/10.1002/malq.201900070.

[11] J. M. Cornejo, H. P. Sankappanavar, *Semi-Heyting Algebras and Identities of Associative Type*, **Bulletin of the Section of Logic**, vol. 48(2) (2019), pp. 117–135, DOI: https://doi.org/10.18778/0138-0680.48.2.03.

[12] J. M. Cornejo, H. P. Sankappanavar, *Connexive logics arising from semi-Heyting algebras and from dually hemimorphic semi-Heyting algebras* (2022), in Preparation.

[13] J. M. Cornejo, I. Viglizzo, *Semi-intuitionistic Logic with Strong Negation*, **Studia Logica**, vol. 106(2) (2017), pp. 281–293, DOI: https://doi.org/10.1007/s11225-017-9737-9.

[14] J. M. Cornejo, I. D. Viglizzo, *On Some Semi-Intuitionistic Logics*, **Studia Logica**, vol. 103(2) (2015), pp. 303–344, DOI: https://doi.org/10.1007/s11225-014-9568-x.

[15] J. M. Cornejo, I. D. Viglizzo, *Semi-Nelson Algebras*, **Order**, vol. 35(1) (2018), pp. 23–45, DOI: https://doi.org/10.1007/s11083-016-9416-x.

[16] J. Font, **Abstract Algebraic Logic. An Introductory Textbook**, College Publications, Rickmansworth (2016).

[17] J. M. Font, R. Jansana, D. Pigozzi, *A Survey of Abstract Algebraic Logic*, **Studia Logica**, vol. 74(1/2) (2003), pp. 13–97, DOI: https://doi.org/10.1023/a:1024621922509.

[18] T. Jarmużek, J. Malinowski, *Boolean Connexive Logics: Semantics and tableau approach*, **Logic and Logical Philosophy**, vol. 28 (2019), pp. 427–448, DOI: https://doi.org/10.12775/llp.2019.003.

[19] T. Katriňák, *The structure of distributive double p-algebras. Regularity and congruences*, **Algebra Universalis**, vol. 3(1) (1973), pp. 238–246, DOI: https://doi.org/10.1007/bf02945123.

[20] G. Moisil, **Essais sur les logiques non chrysippiennes**, Éditions de l'Académie Socialiste de Roumanie (1972), URL: https://books.google.com.ar/books?id=pjjQAAAAMAAJ.

[21] G. C. Moisil, *Logique modale. Disquisitiones mathematicae et physicae (Bucharest), vol. 2 (1942), pp. 3–98.*, **Journal of Symbolic Logic**, vol. 13(3) (1948), pp. 162–163, DOI: https://doi.org/10.2307/2267855.

[22] A. A. Monteiro, *Sur les algèbres de Heyting symétriques*, **Portugaliae Mathematica**, vol. 39(1–4) (1980), pp. 1–237, URL: https://eudml.org/doc/115416, special Issue in honor of António Monteiro.

[23] H. Rasiowa, **An algebraic approach to non-classical logics**, Studies in Logic and the Foundations of Mathematics, Vol. 78, North-Holland Publishing Co., Amsterdam (1974).

[24] H. P. Sankappanavar, *Heyting algebras with dual pseudocomplementation*, **Pacific Journal of Mathematics**, vol. 117(2) (1985), pp. 405–415, DOI: https://doi.org/10.2140/pjm.1985.117.405.

[25] H. P. Sankappanavar, *Semi-Heyting algebras*, **Amererican Mathematical Society Abstracts**, (1985), p. 13.

[26] H. P. Sankappanavar, *Heyting algebras with a dual lattice endomorphism*, **Zeitschrift für Mathematische Logik und Grundlagen der Mathe-**

matik, vol. 33(6) (1987), pp. 565–573, DOI: https://doi.org/10.1002/malq.19870330610.

[27] H. P. Sankappanavar, *Semi-De Morgan algebras*, **The Journal of Symbolic Logic**, vol. 52(3) (1987), pp. 712–724, DOI: https://doi.org/10.2307/2274359.

[28] H. P. Sankappanavar, *Semi-Heyting algebras: An abstraction from Heyting algebras*, Actas del Congreso "Dr. Antonio A. R. Monteiro", [in:] **Proceedings of the 9th "Dr. Antonio A. R. Monteiro" Congress (Spanish)**, Univ. Nac. del Sur, Bahía Blanca (2008), pp. 33–66.

[29] H. P. Sankappanavar, *Expansions of semi-Heyting algebras I: Discriminator varieties*, **Studia Logica**, vol. 98(1–2) (2011), pp. 27–81, DOI: https://doi.org/10.1007/s11225-011-9322-6.

[30] H. P. Sankappanavar, *Dually quasi-De Morgan Stone semi-Heyting algebras I. Regularity*, **Categories and General Algebraic Structures with Applications**, vol. 2(1) (2014), pp. 47–64, URL: https://cgasa.sbu.ac.ir/article_6483.html.

[31] H. P. Sankappanavar, *Dually quasi-De Morgan Stone semi-Heyting algebras II. Regularity*, **Categories and General Algebraic Structures with Applications**, vol. 2(1) (2014), pp. 65–82, URL: https://cgasa.sbu.ac.ir/article_6799.html.

[32] H. P. Sankappanavar, *A note on regular De Morgan Stone semi-Heyting algebras*, **Demonstracio Mathematica**, vol. 49(3) (2016), pp. 252–265, DOI: https://doi.org/10.1515/dema-2016-0021.

[33] H. P. Sankappanavar, *JI-distributive dually quasi-De Morgan semi-Heyting and Heyting algebras*, **Scientiae Mathematicae Japonicae**, vol. 82(3) (2019), pp. 245–271, DOI: https://doi.org/10.32219/isms.82.3_245.

[34] H. P. Sankappanavar, *De Morgan semi-Heyting and Heyting algebras*, [in:] K. P. Shum, E. Zelmanov, P. Kolesnikov, S. M. Anita Wong (eds.), **New Trends in Algebras and Combinatorics. Proceeding of the 3rd International Congress in Algebra and Combinatorics ICAC2017, Hong Kong, China, 25–28 August 2017** (2020), pp. 447–457, DOI: https://doi.org/10.1142/9789811215476_0024.

[35] H. P. Sankappanavar, *A few historical glimpses into the interplay between algebra and logic and investigations into Gautama algebras*, [in:] S. Sarukkai, M. K. Chakraborty (eds.), **Handbook of Logical Thought in India**,

Springer, New Delhi (2022), pp. 1–75, DOI: https://doi.org/10.1007/978-81-322-2577-5_54.

[36] H. P. Sankappanavar, *Gautama and Almost Gautama algebras and their associated logics* (2022), preprint.

[37] J. Varlet, *A regular variety of type (2,2,1,1,0,0)*, **Algebra Universalis**, vol. 2(1) (1972), pp. 218–223, DOI: https://doi.org/10.1007/bf02945029.

[38] H. Wansing, *Connexive Logic*, [in:] E. N. Zalta (ed.), **The Stanford Encyclopedia of Philosophy**, Summer 2022 ed., Metaphysics Research Lab, Stanford University (2022).

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