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INTERPOLATION PROPERTY ON VISSER'S FORMAL PROPOSITIONAL LOGIC

Abstract

In this paper by using a model-theoretic approach, we prove Craig interpolation property for Formal Propositional Logic, **FPL**, Basic propositional logic, **BPL** and the uniform left-interpolation property for **FPL**. We also show that there are countably infinite extensions of **FPL** with the uniform interpolation property.

Keywords: Basic propositional logic, formal propositional logic, layered bisimulation, interpolation.

2020 Mathematical Subject Classification: 03F30, 03G25, 03B20.

1. Introduction

A Craig interpolant for formulas $\phi(\vec{q},\vec{p})$ and $\psi(\vec{p},\vec{r})$ where $\vdash \phi \to \psi$, is a formula $\chi(\vec{p})$ such that $\vdash \phi \to \chi$ and $\vdash \chi \to \psi$. The uniform interpolation property is, in a sense, the generalization of the Craig interpolation property. If instead of two formulas, we restrict the interpolant to one formula and a subset of its propositional variables (which are to be the shared variables), we reach a stronger definition: a uniform left-interpolant for $\phi(\vec{q},\vec{p})$ with respect to $\vec{p}$ is a formula $\chi(\vec{p})$ such that for all formulas $\psi(\vec{p},\vec{r})$ with $\vdash \psi \to \phi$, χ acts as an interpolant for ϕ and ψ. The uniform

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right-interpolant is defined analogously. A logic whose formulas have both uniform left and right-interpolants is said to satisfy the uniform interpolation property.

It is easy to show that classical propositional logic has the uniform interpolation property. But showing it for intuitionistic propositional logic is highly nontrivial. This was shown first by using a proof theoretic method in [6] and then semantically in [5]. A. Visser in [8] established the result using bisimulation techniques.

The goal of this paper is to establish new interpolation results for Basic propositional logic **BPL** and Formal propositional logic, **FPL**, using the bisimulation techinque of [8]. **BPL** and **FPL** are propositional logics which correspond with modal logics **K4** and **GL** by the Gödel translation, respectively, in the same way that Intuitionistic Propositional Logic **IPL** corresponds with modal logic **S4**. The main difference between **IPL** and **BPL** is that the rule *Modus Ponens* is weakened in **BPL**. We show that **FPL** satisfies the uniform left-interpolation property. The same approach with minor differences leads the Craig interpolation property for Basic propositional logic, **BPL**. We Also show that there are countably infinite extensions of **FPL** with the uniform interpolation property.

The organiztion of the paper is as follows: in the next section we present an overview of the syntax and semantics of **BPL**. Basic model theory for **BPL** including canonical models and layered bisimulation, which are a natural generalization of results known for intuitionistic propositional logic, will be studied in section three. Interpolation properties for formal propositional logic and some of its extensions will be presented in section four.

2. Axioms, rules and Kripke models

In this preliminaries section we introduce the most basic concepts and notations we need related to syntax and semantics of basic propositional logic, for more details see [7] and [3, 4].

The language for **BPL** is essentially the same as the language for **IPL**. We build formulas in the standard way from propositional variables, or atoms, using $\top, \bot, \wedge, \vee, \rightarrow$. Expressions $\neg\phi$ and $\phi \leftrightarrow \psi$ are usual abbreviations for $\phi \rightarrow \bot$ and $(\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$, respectively.

We assume that $p, q, r, \ldots$ range over propositional variables, $\phi, \psi, \chi, \ldots$ range over arbitrary formulas, and $\vec{p}, \vec{q}, \vec{r}, \ldots$ range over finite sets of propositional variables. For $\vec{p}$ and $\vec{q}$, we abbreviate $\vec{p} \cup \vec{q}$ by $\vec{p}, \vec{q}$. $\mathsf{PV}(\phi)$ is the set of propositional variables in ϕ. $\mathsf{Sub}(\phi)$ is the set of subformulas of ϕ. For a set of propositional variables $\mathcal{P}$, $\mathcal{L}(\mathcal{P})$ denotes the set of those formulas which only contains propositional variables from $\mathcal{P}$. There are different axiomatizations for **BPL**. The natural deduction system for **BPL** was first introduced by A. Visser in [7]. We choose axiomatization method which was introduced in [3]. A *sequent* is simply an expression of the form $\phi \Rightarrow \psi$, where ϕ and ψ are formulae. We write $\phi \Leftrightarrow \psi$ as short for $\phi \Rightarrow \psi$ and $\psi \Rightarrow \phi$.

In the rules below, a single horizontal line means that if the sequents above the line are included, then so are the ones below the line. A double line means the same, but in both directions.

Table 1. Sequent calculus of **BPL**

$$\phi \Rightarrow \phi \qquad \phi \Rightarrow \top \qquad \bot \Rightarrow \phi \qquad \phi \wedge (\psi \vee \theta) \Rightarrow (\phi \wedge \psi) \vee (\phi \wedge \theta)$$

$$\frac{\phi \Rightarrow \psi \quad \psi \Rightarrow \theta}{\phi \Rightarrow \theta} \qquad \frac{\phi \Rightarrow \psi \quad \phi \Rightarrow \theta}{\phi \Rightarrow \psi \wedge \theta}\,(\text{double line}) \qquad \frac{\phi \Rightarrow \psi \quad \theta \Rightarrow \psi}{\phi \vee \theta \Rightarrow \psi}\,(\text{double line}) \qquad \frac{\phi \wedge \psi \Rightarrow \theta}{\phi \Rightarrow \psi \to \theta}$$

$$(\phi \to \psi) \wedge (\psi \to \theta) \Rightarrow \phi \to \theta$$
$$(\phi \to \psi) \wedge (\phi \to \theta) \Rightarrow \phi \to \psi \wedge \theta$$
$$(\phi \to \psi) \wedge (\theta \to \psi) \Rightarrow \phi \vee \theta \to \psi$$

A *sequent theory* is a set of sequents that includes the sequent axioms and is closed under the closure rules, as given in table 1. A sequent theory Σ is *consistent* if $\top \Rightarrow \bot \notin \Sigma$. A theory Γ is *schematic* if $\Gamma \vdash \phi \Rightarrow \psi$ implies $\Gamma \vdash \tau\phi \Rightarrow \tau\psi$ for all substitutions τ. A *basic intermediate* logic is a consistent schematic sequent theory. The intuitionistic propositional logic, **IPL**, is **BPL** plus the sequent schema $\top \to \phi \Rightarrow \phi$, and the Formal Propositional logic, **FPL**, is the extension of **BPL** by the Löb's axiom schema, $(\top \to \phi) \to \phi \Rightarrow \top \to \phi$, or equivalently, by Löb's rule:

$$\frac{\phi \wedge (\top \to \psi) \Rightarrow \psi}{\phi \Rightarrow \psi}.$$

The theories **BPL**, **IPL**, **CPL** and **FPL** are all basic intermediate logics.

Sequents $\top \Rightarrow \phi$ are often identified with formulas ϕ. Given a sequent theory Σ we define $F(\Sigma)$ as $\{\phi \mid \top \Rightarrow \phi \in \Sigma\}$. A *f*ormula theory (or simply a *t*heory) is a set of formulas of the form $F(\Sigma)$. The formula theory is *c*onsistent if Σ is consistent or, equivalently, if $\bot$ is not an element of the formula theory. Let $\Sigma \cup \{\phi \Rightarrow \psi\}$ be a set of sequents. We say that the $\phi \Rightarrow \psi$ is provable from the Σ in the logic **BPL** and we denoted it by $\Sigma \vdash_{\mathbf{BPL}} \phi \Rightarrow \psi$, when the sequent $\phi \Rightarrow \psi$ is provable in the sequent calculus **BPL** augmented by $\phi_i \Rightarrow \psi_i$ for all $\phi_i \Rightarrow \psi_i \in \Sigma$. When Σ is empty we simply write $\vdash \phi \Rightarrow \psi$. Also, we use $\Sigma \vdash \phi$ instead of $\Sigma \vdash \top \Rightarrow \phi$.

Proposition 2.1 ([3]). Let Σ be a sequent theory. Then:

1. (Functional Completeness) $\Sigma \cup \{\phi\} \vdash \psi \Rightarrow \theta$ if and only if $\Sigma \vdash \phi \wedge \psi \Rightarrow \theta$.

2. (Formalization) $\Sigma \cup \{\phi_1 \Rightarrow \psi_1, ..., \phi_n \Rightarrow \psi_n\} \vdash \phi_0 \Rightarrow \psi_0$ implies $\Sigma \vdash (\phi_1 \to \psi_1) \wedge ... \wedge (\phi_n \to \psi_n) \Rightarrow \phi_0 \to \psi_0$.

Σ is called a *faithful* theory if the converse of Proposition 2.1.2, also holds. **IPL** and all of its extensions including Classical Propositional Logic, **CPL**, and **BPL**, **FPL** are examples of faithful theories.

Define the relation $\prec$ on all theories by $\Gamma \prec \Delta$ if and only if for all $\phi, \psi \in \mathcal{L}(\mathcal{P})$ such that both $\Gamma \vdash \phi \to \psi$ and $\Delta \vdash \phi$, we have $\Delta \vdash \psi$.

Proposition 2.2. The relation $\prec$ is transitive, and $\Gamma \prec \Delta$ implies $\Gamma \subseteq \Delta$.

Proof: We first prove the second claim. Suppose $\Gamma \prec \Delta$. If $\phi \in \Gamma$, then $\Gamma \vdash \phi$ which implies, by above Formalization theorem, that $\Gamma \vdash \top \to \phi$ and thus $\Delta \vdash \phi$. Hence $\phi \in \Delta$. So $\Gamma \subseteq \Delta$. For transitivity, suppose that $\Gamma \prec \Delta \prec \Delta'$ are such that $\Gamma \vdash \phi \to \psi$ and $\Delta' \vdash \phi$, for any $\phi, \psi \in \mathcal{L}(\mathcal{P})$. Then $\Gamma \subseteq \Delta \vdash \phi \to \psi$, so $\Delta' \vdash \psi$. Therefore $\Gamma \prec \Delta'$. □

Moving on to the samantics of **BPL**, a *Kripke frame* $\mathbf{F}$ is a pair $(W, \prec)$ where W is a non-empty set and $\prec$ is a transitive binary relation on W. The reflexive closure of $\prec$ is denoted by $\preceq$. Also, for $k, k' \in W$, $k' \succeq k$ means that $k \preceq k'$.

A *Kripke model* based on Kripke frame $\mathbf{F}$ is a triple $\mathbf{M} = (W, \prec, V)$ where $\mathbf{F} = (W, \prec)$ and the function V assigns to each atoms p of the language of **BPL** a subset $V(p) \subseteq W$ which is upward closed, that is, if $k \in V(p)$ and $k \prec k'$, then $k' \in V(p)$.

Given a Kripke model $\mathbf{M} = (W, \prec, V)$, the notion of a formula ϕ being true at a point $k \in W$, written $\mathbf{M}, k \Vdash \phi$ or $k \Vdash \phi$ for short, is like in **IPL**. We extend $\Vdash$ to all sequents. For any sequent $\phi \Rightarrow \psi$, it is defined by

$$k \Vdash \phi \Rightarrow \psi \textit{ if and only if for all } k' \succeq k,\ k' \Vdash \phi \textit{ implies } k' \Vdash \psi.$$

A trivial induction on the complexity of formulas yields that, $k \Vdash \phi$ and $k \prec k'$ implies $k' \Vdash \phi$. So, $k \Vdash \phi$ if and only if $k \Vdash \top \Rightarrow \phi$. A sequent $\phi \Rightarrow \psi$ is *true* in a Kripke model $\mathbf{M}$, written $\mathbf{M} \Vdash \phi \Rightarrow \psi$, if and only if for all $k \in W$, $k \Vdash \phi \Rightarrow \psi$. We often write $\mathbf{M} \Vdash \phi$ as short for $\mathbf{M} \Vdash \top \Rightarrow \phi$. $\phi \Rightarrow \psi$ is *valid* on a Kripke frame $\mathbf{F}$, $\mathbf{F} \Vdash \phi \Rightarrow \psi$, iff $\phi \Rightarrow \psi$ is true on every Kripke model based on $\mathbf{F}$. Let $\mathcal{C}$ be a class of Kripke frames, $\phi \Rightarrow \psi$ is $\mathcal{C}$*-valid*, $\mathcal{C} \Vdash \phi \Rightarrow \psi$, iff $\phi \Rightarrow \psi$ is valid on every Kripke frame in $\mathcal{C}$.

For a set Γ of sequents, $\mathbf{M} \Vdash \Gamma$ means that $\mathbf{M} \Vdash \phi \Rightarrow \psi$, for all $\phi \Rightarrow \psi \in \Gamma$. For a set of sequents $\Gamma \cup \{\phi \Rightarrow \psi\}$, the notation $\Gamma \Vdash \phi \Rightarrow \psi$ means that for any Kripke model $\mathbf{M}$, if $\mathbf{M} \Vdash \Gamma$, then $\mathbf{M} \Vdash \phi$.

In the sequel we show a Kripke model by its forcing relation. For $k \in W$, we call $\mathbf{M} = (W, \prec, \Vdash, k)$ *pointed* and it is called *rooted*, with root k, if and only if $k \preceq k'$, for all $k' \in W$. Also it is called a tree Kripke model if and only if $\langle W, \prec \rangle$ is a tree. We denote the class of all models, pointed models and rooted models by Mod, Pmod and Rmod, respectively. We denote W by $\mathbf{M}$ when clear from the context. We write $\mathbf{M}(\vec{p})$ for the result of restricting V to $\vec{p}$.

If $\mathbf{M} = (W, \prec, \Vdash)$ is a Kripke model and w a world of $\mathbf{M}$, the *submodel* of $\mathbf{M}$ *generated* by w is the Kripke model $\mathbf{M}[w] := \mathbf{M}' = (W[w], \prec', \Vdash')$ where $W[w] = \{x \in W | w \preceq x\}$, and $\prec'$ and $\Vdash'$ are restrictions of $\prec$ and $\Vdash$ to $W[w]$.

Here we stick to the following characterization of **BPL** and **FPL** models throughout the paper.

THEOREM 2.3 ([3]). **BPL** *and* **FPL** *are sound and complete for the class of all irreflexive Kripke models and all conversely well-founded irreflexive Kripke models, respectively.*

The *depth* of a node $k \in W$ is defined inductively by

$$d(k) := \sup\{d(k') + 1 \mid k' \prec k\},\ \textit{where}\ \sup(\emptyset) = 0,$$

and the depth of a model $\mathbf{M}$ is defined as

$$d(\mathbf{M}) := \sup\{d(k) \mid k \in W\}.$$

We notice that $d(\mathbf{M}) = \infty$ is possible. We define inductively $\Box^n\phi$ by $\Box^0\phi := \phi$, $\Box\phi := \top \to \phi$ and $\Box^{n+1}\phi := \Box\Box^n\phi$, for $n \in \omega$. The following extensions of **BPL** were introduced in [3]

- $\mathbf{F}_n := \mathbf{BPL} + \Box^n\bot$, for $n \in \omega$,
- $\mathbf{FPL}_\bot := \mathbf{BPL} + L_\bot$, where $L_\bot := (\Box\bot \to \bot) \to \Box\bot$.

One can see that **BPL** proves $L_\bot \Leftrightarrow \top \to L_\bot$, so $\mathbf{FPL}_\bot$ is faithful. Given a Kripke frame $\langle W, \prec\rangle$, a world $e \in W$ is called an *end-node* if it is maximal with respect to $\preceq$. A Kripke frame $\langle K, \prec\rangle$ with end-nodes is a Kripke frame such that for every $w \in K$ there is some end-node $e \in W$ with $w \preceq e$.

Proposition 2.4 ([3]).

1. $\mathbf{FPL}_\bot$ is sound and complete with respect to the class of all irreflexive Kripke frames with end-nodes,
2. For every $n \geq 1$, the logic $\mathbf{F}_n$ is strongly complete with respect to the class of all irreflexive Kripke models with depth not greater than n.

3. Basic model theory

In this section, first we briefly review the notion of Henkin construction for basic propositional logic. The results we report on the Henkin model can be found in [4]. However, for the sake of entirety and because of phrasing the results in terms of saturated sets of formulas instead of prime sequent theories and also new relations between saturated sets compare to [4], we decided to present them in full proofs. After which, we recall the notion of bisimulation (and in general, layered bisimulation) between two models. For convenience in our context, this notion has been slightly modified, i.e., the zig and zag conditions hold strictly. In the sequel we need to extend the set of all natural numbers ω with an extra top element ∞. Let ω^∞ be $\omega \cup \{\infty\}$ which is equipped with the obvious natural ordering $\leq$. We extend addition by, $\infty + \alpha = \alpha + \infty = \infty$ and $\infty - n = \infty$. We let n range over ω, and α range over ω^∞.

We start by the following lemma which can be proved by induction on the complexity of formulas and distributivity axiom of **BPL**.

LEMMA 3.1. *Let $\phi \in \mathcal{L}(\mathcal{P})$ be a formula. Then it can be written, modulo* **BPL** *provability, as $\bigvee_i \bigwedge_j \phi_{ij}$ where ϕ_{ij} is an atom in $\mathcal{P}$, $\top$, $\bot$ or an implication.*

We call $\bigvee_i \bigwedge_j \phi_{ij}$ *the disjunctive normal form* of ϕ.

A set $X \subseteq \mathcal{L}(\mathcal{P})$ is called $\mathcal{P}$-*adequate* if $\mathcal{P} \subseteq X$ and X is closed under subformulas. We say that a consistent set $\Gamma \subseteq X$ is X-saturated if it is X-*deductively closed* and X-*prime*, i.e.,

- $\Gamma \vdash \phi$ and $\phi \in X$ implies $\phi \in \Gamma$,
- $\Gamma \vdash \phi \vee \psi$ and $\phi \vee \psi \in X$ implies $\Gamma \vdash \phi$ or $\Gamma \vdash \psi$.

We say that a consistent set Γ is *prime* exactly when it is $\mathcal{L}(\mathcal{P})$-prime. Given the $\mathcal{P}$-adequate set X, let H_X be the collection of all X-saturated sets. The Kripe model $\mathbf{H}_X := (H_X, \prec, \Vdash)$ where for every $\Gamma \in H_X$ and every propositional variable $p \in \mathcal{P}$, $\Gamma \Vdash p$ if and only if $p \in \Gamma$ is called *canonical model* over $\mathcal{P}$ with respect to X.

PROPOSITION 3.2. Let X be a $\mathcal{P}$-adequate set. For any formula $\phi \in X$ and any $\Gamma \in H_X$, $\mathbf{H}_X, \Gamma \vdash \phi$ if and only if $\Gamma \Vdash \phi$.

PROOF: We complete the proof by induction on the complexity of φ. We consider the interesting case where $\phi = \psi \to \theta$. Let $\Gamma \vdash \psi \to \theta$ and $\Delta \in H_X$ be such that $\Gamma \prec \Delta$ and $\Delta \Vdash \psi$. By induction hypothesis $\Delta \vdash \psi$ and because of $\Gamma \prec \Delta$ and $\Gamma \vdash \psi \to \theta$ we have $\Delta \vdash \psi \Rightarrow \theta$ and hence $\Delta \vdash \theta$. First, applying induction hypothesis gives $\Delta \Vdash \theta$. And thus $\Gamma \Vdash \psi \to \theta$.

Conversely, suppose that $\Gamma \nvdash \psi \to \theta$. Put $\Gamma_\psi = \{\eta \in X \mid \Gamma \vdash \psi \to \eta\}$. First, we notice that Γ_ψ is X-deductively closed. Suppose that $\Gamma_\psi \vdash \alpha$, for $\alpha \in X$. Then there exist formulas $\eta_1, \cdots, \eta_i$ such that $\eta_1, \cdots \eta_i \vdash \alpha$. Put $\eta = \eta_1 \wedge \cdots \wedge \eta_i$. Hence, $\Gamma \vdash \psi \to \eta$ and $\vdash \eta \to \alpha$ which implies that $\Gamma \vdash \psi \to \alpha$. Then $\alpha \in \Gamma_\psi$. Next, we show that $\Gamma \prec \Gamma_\psi$. Suppose that $\Gamma \vdash \alpha \to \beta$ and $\Gamma_\psi \vdash \alpha$. Then $\Gamma \vdash \psi \to \alpha$ which implies, by transitivity, that $\Gamma \vdash \psi \to \beta$. Therefore, $\Gamma_\psi \vdash \beta$. Note that, $\Gamma_\psi \nvdash \theta$. Now, Assume that $\Sigma = \{\Delta \mid \Delta$ is a X- deductively closed set of formulas with $\Delta \vdash \psi, \Delta \nvdash \theta$ and $\Gamma \prec \Delta\}$. Σ is nonempty, since $\Gamma_\psi \in \Sigma$. $(\Sigma, \subseteq)$ satisfies the chain condition for Zorn's lemma. For, suppose that $\{\Delta_i\}_{i \in I}$ is a chain of elements of Σ then, one can see that $\bigcup \Delta_i$ is X deductively closed set, $\bigcup \Delta_i \vdash \psi$ and $\bigcup \Delta_i \nvdash \theta$. We only show that $\Gamma \prec \bigcup \Delta_i$. So, suppose that $\Gamma \vdash \gamma \to \delta$ and $\bigcup \Delta_i \vdash \gamma$. Then there exists a j such that $\Delta_j \vdash \gamma$ which implies that $\Delta_j \vdash \delta$, since $\Gamma \prec \Delta_j$. Hence, $\Gamma \prec \bigcup \Delta_i$. Let Δ be a maximal element of Σ. Δ is X-saturated. To see that, we need to show that it is X-prime. Assume $\alpha \vee \beta \in X$ is such that $\Delta \vdash \alpha \vee \beta, \Delta \nvdash \alpha$ and $\Delta \nvdash \beta$. But $\Gamma \prec \Delta \prec \Gamma_{\Delta,\alpha} := \{\eta \mid \Gamma \vdash \delta \wedge \alpha \to \eta$, for some $\delta \in \Delta\}$ and $\Gamma \prec \Delta \prec \Gamma_{\Delta,\beta}$, then by maximality of Δ we obtain $\Gamma_{\Delta,\alpha} \vdash \theta$ and $\Gamma_{\Delta,\beta} \vdash \theta$ which implies that $\Gamma \vdash \alpha \wedge \delta_1 \to \theta$ and $\Gamma \vdash \beta \wedge \delta_2 \to \theta$, for some $\delta_1, \delta_2 \in \Delta$. Then $\Gamma \vdash (\alpha \wedge \delta_1) \vee (\beta \wedge \delta_2) \to \theta$. But $\Delta \vdash (\alpha \wedge \delta_1) \vee (\beta \wedge \delta_2)$, then $\Delta \vdash \theta$ which is a contradiction. Hence, we have $\Gamma \prec \Delta$ and $\Delta \nvdash \theta$. Then, by induction hypothesis, $\Gamma \prec \Delta \Vdash \psi$, and $\Delta \nVdash \theta$. So $\Gamma \nVdash \psi \to \theta$. □

Definition 3.3. Let $\mathcal{K}$ be a set of disjoint pointed models for a X-saturated set Δ. We define $Glue(H_X[\Delta], \mathcal{K})$ as follows:

- $Glue(\mathbf{H}_X[\Delta], \mathcal{K}) := (H_X[\Delta] \cup (\bigcup_i K_i) \cup \{m\}, \prec)$, where m is a new distinct point, $(K_i, \prec_i, \Vdash_i, k_i)$'s are mutually disjoint pointed models in $\mathcal{K}$ and $\prec$ is defined by:

$$\begin{aligned}\prec \quad = \quad & \prec_i\restriction_{\mathbf{K}_i[k_i]} \cup \prec_{H_X}\restriction_{H_X[\Delta]} \\ & \qquad \cup \{(m, y) \ : \ y \in \bigcup_i \mathbf{K}_i[k_i] \setminus \{k_i\} \cup \mathbf{H}_X[\Delta] \setminus \{\Delta\}\},\end{aligned}$$

- $m \Vdash p$ exactly when $p \in \Delta$.

We would like to notice that in the model $Glue(H_X[\Delta], \mathcal{K})$, m is irreflexive and $m \not\prec k_i$ and $m \not\prec \Delta$ unless $k_i \prec_i k_i$ and $\Delta \prec_{H_X[\Delta]} \Delta$.

Lemma 3.4. *Let $\mathcal{K}$ be a class of pointed models for a X-saturated set Δ and $\phi \in X$. Then $Glue(H_X[\Delta], \mathcal{K}), m \Vdash \phi$ exactly when $\phi \in \Delta$.*

Proof: For atoms the claim is clear. From the construction, conjunction and disjunction are easy due to X-saturatedness of Δ. For implication suppose that $\phi = \psi \to \gamma$. If $\psi \to \gamma \in \Delta$ and $m \prec l$, then l must be in one of the models $\mathbf{K}_i[k_i] \setminus \{k_i\}$ or $\mathbb{H}_X[\Delta] \setminus \{\Delta\}$. If $l \in \mathbf{K}_i[k_i] \setminus \{k_i\}$ then, since $\mathbf{K}_i[k_i] \setminus \{k_i\}$ is a model of Δ we have $l \Vdash \psi \Rightarrow \gamma$ which implies that $m \Vdash \psi \to \gamma$. The case $l \in \mathbb{H}_X[\Delta] \setminus \{\Delta\}$ is obvious.

Conversely, suppose that $m \Vdash \psi \to \gamma$. Then for any $l \in \mathbb{H}_X[\Delta] \setminus \{\Delta\}$ we have $l \Vdash \psi \Rightarrow \gamma$ which implies that $\Delta \Vdash \psi \to \gamma$. But since $\psi \to \gamma \in X$ we have $\psi \to \gamma \in \Delta$. □

Theorem 3.5. *Let X be a $\mathcal{P}$-adequate and Δ be X-saturated. Then Δ is prime.*

Proof: Suppose that $\Delta \vdash \phi \vee \psi$, for ϕ and $\psi \in \mathcal{L}(\mathcal{P})$. Since X is $\mathcal{P}$-adquate, ϕ and ψ have disjunctive normal forms: $\phi = \bigvee_i \bigwedge_j \phi_{ij}$ and $\psi = \bigvee_r \bigwedge_s \psi_{rs}$, where $\phi_{ij}, \psi_{rs} \in X$ or they are in implication form. We will show that there exist i or r such that $\Delta \vdash \bigwedge_j \phi_{ij}$ or $\Delta \vdash \bigwedge_s \psi_{rs}$. It is clear that it shows that either $\Delta \vdash \phi$ or $\Delta \vdash \psi$.

Assume, for any i and r, that $\Delta \nvdash \bigwedge_j \phi_{ij}$ and $\Delta \nvdash \bigwedge_s \psi_{rs}$. Then there exist (K_i, k_i) and (L_r, l_r) such that $(K_i, k_i) \Vdash \Delta$, $(K_i, k_i) \nVdash \bigwedge_j \phi_{ij}$ and $(L_r, l_r) \Vdash \Delta$, $(L_r, l_r) \nVdash \bigwedge_s \psi_{rs}$. By Lemma 3.4 we have $Glue(H_X[\Delta], \{K_i\} \cup \{L_r\}), m \Vdash \Delta$. Therefore $m \Vdash \phi \vee \psi$, since $\Delta \vdash \phi \vee \psi$. Hence, there exist i or r such that $m \Vdash \bigwedge_j \phi_{ij}$ or $m \Vdash \bigwedge_s \psi_{rs}$. Assume $m \Vdash \bigwedge_j \phi_{ij}$, the other case is similar. Since $(K_i, k_i) \nVdash \bigwedge_j \phi_{ij}$, there are two cases: If ϕ_{ij} is an atom then by Lemma 3.4 since $m \Vdash \phi_{ij}$ and $\mathcal{P} \subseteq X$ we have $\phi_{ij} \in \Delta$ and since $(K_i, k_i) \Vdash \Delta$ we would have $(K_i, k_i) \Vdash \phi_{ij}$ which is a contradiction. If $\phi_{ij} = \delta \to \gamma$, for some δ and γ. In this case, since $m \Vdash \delta \to \gamma$, then for any $l \succ k_i$ we have $l \Vdash \delta \to \gamma$ which implies that $(K_i, k_i) \Vdash \delta \to \gamma$ which is impossible. $\square$

Remark 3.6. We notice that Henkin models can be constructed similarly for **FPL**. Although $\mathbb{H}$ is not an **FPL**-model in this case, what we want from $\mathbb{H}$ in our proof of Lemma 4.8 is for it to be transitive, which it trivially is.

Definition 3.7. We define the complexity measure $i(\phi)$ of a formula ϕ recursively as follows:

1. $i(p) = 0$, for each propositional variable p;
2. $i(\top) = i(\bot) := 0$;
3. $i(\phi \wedge \psi) = i(\phi \vee \psi) := \max\{i(\phi), i(\psi)\}$;
4. $i(\phi \to \psi) := \max\{i(\phi), i(\psi)\} + 1$.

We define $B_n(\mathcal{P}) := \{\phi \in \mathcal{L}(\mathcal{P}) \mid i(\phi) \leq n\}$ and $B_\infty(\mathcal{P}) := \mathcal{L}(\mathcal{P})$. By induction on n we may prove the following fact:

Fact 3.8. $B_n(\vec{p})$ is finite modulo **BPL**-provable equivalence.

By the above fact, we assume that $B_n(\vec{p})$ is finite from now on.

Definition 3.9. Let $\mathbf{M} = (W, \prec, \Vdash)$ be any Kripke model. For each $X \subseteq \mathcal{L}(\mathcal{P})$, $m \in W$ and $n \in \omega$, we define:

1. $\mathsf{Th}_X(m) = \{\phi \in X \mid m \Vdash \phi\}$;
2. $\mathsf{Th}_n^{\mathcal{P}}(m) = \{\phi \in B_n(\mathcal{P}) \mid m \Vdash \phi\}$;
3. $\mathsf{Th}_X(\langle \mathbf{M}, m\rangle) := \mathsf{Th}_X(m)$ and $\mathsf{Th}(m) := \mathsf{Th}_{\mathcal{L}(\mathcal{P})}(m)$;
4. $\mathsf{Y}_n(m) := \mathsf{Y}_{n,m}(\vec{p}) := \bigwedge \mathsf{Th}_n^{\vec{p}}(m)$;
5. $\mathsf{N}_n(m) := \mathsf{N}_{n,m}(\vec{p}) := \bigvee \{\phi \in B_n(\vec{p}) \mid m \nVdash \phi\}$.

Fact 3.10. $\mathsf{Y}_{n,m}(\vec{p})$ is a prime formula.

Proof: We first note, by definition, that $B_n(\vec{p})$ is closed under subformulas. Next, we show that $\mathsf{Y}_{n,m}(\vec{p})$ is an $B_n(\vec{p})$-saturated. Suppose that $\phi \in B_n(\vec{p})$ and that $\mathsf{Y}_{n,m}(\vec{p}) \vdash \phi$. Then $m \models \phi$ which implies that $\phi \in \mathsf{Y}_{n,m}(\vec{p})$. For $B_n(\vec{p})$-primness suppose that $\phi \vee \psi \in B_n(\vec{p})$ and $\mathsf{Y}_{n,m}(\vec{p}) \vdash \phi \vee \psi$. Then $m \models \phi \vee \psi$ which implies that $m \models \phi$ or $m \models \psi$. Hence, $\phi \in \mathsf{Y}_{n,m}(\vec{p})$ or $\psi \in \mathsf{Y}_{n,m}(\vec{p})$. Therefore, by Theorem 3.5, $\mathsf{Y}_{n,m}(\vec{p})$ is prime. $\square$

Let $\mathbf{M} = (W, \prec, \Vdash)$ and $\mathbf{M}' = (W', \prec', \Vdash')$, be any two $\mathcal{P}$-models. We say a relation $\mathcal{Z} \subseteq W \times \omega^\infty \times W'$ is a *l*ayered $\mathcal{P}$-bisimulation (l-bisimulation) between $\mathbf{M}$ and $\mathbf{M}'$ if it satisfies the following three conditions:

1. $(w, \alpha, w') \in \mathcal{Z}$ implies $w \Vdash p$ if and only if $w' \Vdash p$, for all atome $p \in \mathcal{P}$;
2. $(w, \alpha + 1, w') \in \mathcal{Z}$ and $w \prec x$ implies $(w', \alpha, x') \in \mathcal{Z}$, for some $x' \succ' w'$;
3. $(w, \alpha+1, w') \in \mathcal{Z}$ and $w' \prec' x'$ implies $(x, \alpha, x') \in \mathcal{Z}$, for some $x \succ w$.

We call (2) the $zig_{\alpha+1}$-property and (3) $zag_{\alpha+1}$- property. If $\alpha = \infty$, we simply call them the zig- and the zag-property. We write $w\mathcal{Z}_\alpha w'$ for $(w, \alpha, w') \in \mathcal{Z}$ and $w\mathcal{Z}w'$ for $w\mathcal{Z}_\infty w'$. To clarify the definition in the case of $\alpha = \infty$, we rewrite clauses of the above definition, as follows:

1. $(w, \infty, w') \in \mathcal{Z}$ implies $w \Vdash p$ if and only if $w' \Vdash p$, for all atome $p \in \mathcal{P}$;
2. $(w, \infty, w') \in \mathcal{Z}$ and $w \prec x$ implies $(w', \infty, x') \in \mathcal{Z}$, for some $x' \succ' w'$;
3. $(w, \infty, w') \in \mathcal{Z}$ and $w' \prec' x'$ implies $(x, \infty, x') \in \mathcal{Z}$, for some $x \succ w$.

A binary relation $\mathcal{Z}$ between $\mathbf{M}$ and $\mathbf{M}'$ is a *bisimulation* between $\mathbf{M}$ and $\mathbf{N}$ exactly when $\{\langle w, \infty, w'\rangle \mid w\mathcal{Z}w'\}$ is an l-bisimulation.

We say l-bisimulation $\mathcal{Z}$ is *downward closed* if for any $(w, n, w') \in W \times \omega \times W'$, $(w, n, w') \in \mathcal{Z}$ implies that $(w, m, w') \in \mathcal{Z}$, for all $m \leq n$. Let $PV_{\mathbf{M}}(w) := \{p \in \mathcal{P} : \mathbf{M}, w \Vdash p\}$, we define $w\mathcal{Z}_{\prec 0}w'$ exactly when $PV_{\mathbf{M}}(w) \subseteq PV_{\mathbf{M}'}(w')$; and $w\mathcal{Z}_{\prec\alpha+1}w'$ exactly when $PV_{\mathbf{M}}(w) \subseteq PV_{\mathbf{M}'}(w')$ and for all $x' \succ' w'$ there exists $x \succ w$ with $x\mathcal{Z}_{\alpha}x'$.

We notice that since the set of all l-bisimulations between two models $\mathbf{M}$ and $\mathbf{M}'$ are closed under union, then there is always a maximal l-bisimulation, $\simeq^{\mathbf{M},\mathbf{M}'}$, which is also downward closed. We will often drop the superscript of $\simeq^{\mathbf{M},\mathbf{M}'}$. In case of $\alpha = \infty$, we will drop the subscript of $\simeq^{\mathbf{M},\mathbf{M}'}_{\alpha}$ (if no confusion is possible). $\mathcal{Z}_{\alpha}$ is *full* if it is both total and surjective as a relation between $\mathbf{M}$ and $\mathbf{M}'$. We say that $\mathbf{M}$ and $\mathbf{M}'$ *α-bisimualte (bisimualte)*, or $\mathbf{M} \simeq_{\alpha} \mathbf{M}'$ ($\mathbf{M} \simeq \mathbf{M}'$,) if there is a full α- bisimulation (bisimulation) between them. $\mathcal{Z} : \mathbf{M} \simeq_{\alpha} \mathbf{M}'$ means that $\mathcal{Z}$ is a full α-bisimulation witnessing that $\mathbf{M} \simeq_{\alpha} \mathbf{M}'$. For a set of propositional variables $\mathcal{Q}$, $\mathbf{M} \simeq_{\alpha,\mathcal{Q}} \mathbf{M}'$ means that $\mathbf{M}$ and $\mathbf{M}'$ α-bisimulate with respect to the variables in $\mathcal{Q}$. Note that for rooted models $\mathbf{M}$ and $\mathbf{M}'$ we have $\mathbf{M} \simeq_{\alpha} \mathbf{M}'$ if and only if $r_{\mathbf{M}} \simeq_{\alpha} r_{\mathbf{M}'}$.

We say that $w \in W$ and $w' \in W'$ are α-equivalent, written $w \equiv_{\alpha} w'$, exactly when $\mathsf{Th}_{\alpha}(w) = \mathsf{Th}_{\alpha}(w')$. We notice that for $\alpha = \infty$, w and w' are α-equivalent if $\mathsf{Th}(w) = \mathsf{Th}(w')$.

THEOREM 3.11. *Let $\mathbf{M} = (W, \prec, \Vdash)$ and $\mathbf{M}' = (W', \prec', \Vdash')$ be any Kripke models, $w \in W, w' \in W'$ and $\alpha \in \omega^{\infty}$. Then $w\mathcal{Z}_{\alpha}w'$ implies $w \equiv_{\alpha} w'$.*

PROOF: The proof is by induction on the complexity of formulas. We only check the case of implication. So suppose that $\phi = \gamma \to \psi$. Suppose $w \nVdash \gamma \to \psi$. Then for some $x \succ w$, $x \Vdash \gamma$ and $x \nVdash \psi$. Notice that $\gamma, \psi \in B_{\alpha-1}(\mathcal{P})$. Moreover, since $w\mathcal{Z}_{\alpha}w'$, then there is $x' \succ' w'$ such that $x\mathcal{Z}_{\alpha-1}x'$. Hence, by induction, we get $x' \Vdash' \gamma$ and $x' \nVdash' \psi$. Therefore, $w' \nVdash' \gamma \to \psi$. By a similar argument, we can prove the reverse implication. □

THEOREM 3.12. *Let $\mathbf{M} = \langle W, \prec, \Vdash\rangle$ and $\mathbf{M}' = \langle W', \prec', \Vdash'\rangle$ be any two Kripke models. For any $w \in W, w' \in W'$ and $n \in \omega$, the following are equivalent:*

1. $\mathsf{Th}^{\mathcal{P}}_{n}(w) \subseteq \mathsf{Th}^{\mathcal{P}}_{n}(w')$;
2. *There exists a layered $\mathcal{P}$- bisimulation $\mathcal{Z}$ between $\mathbf{M}$ and $\mathbf{M}'$ such that $w\mathcal{Z}_{\prec n}w'$;*
3. *There exists a downward closed layered $\mathcal{P}$- bisimulation $\mathcal{Z}$ between $\mathbf{M}$ and $\mathbf{M}'$ such that $w\mathcal{Z}_{\prec n}w'$.*

Proof: $(2 \Rightarrow 1)$: We prove that for all $m \in \omega, x \in W$ and $x' \in W'$, if there exists a layered $\mathcal{P}$-bisimulation $\mathcal{Z}$ between $\mathbf{M}$ and $\mathbf{M}'$ such that $x\mathcal{Z}_{\prec m}x'$, then $\mathsf{Th}_m^{\mathcal{P}}(x) \subseteq \mathsf{Th}_m^{\mathcal{P}}(x')$.

Let $m = 0$, then the set $B_0(\mathcal{P})$ is a set of implication-free formulas. By the assumption $PV_{\mathbf{M}}(x) \subseteq PV_{\mathbf{M}'}(x')$, so if $\phi = p$ is a propositional variable, then we have our result. The cases for conjunction and disjunction can be done by induction. Now, Suppose that the statement holds for $m > 0$, and that there exists a layered $\mathcal{P}$-bisimulation $\mathcal{Z}$ between $\mathbf{M}$ and $\mathbf{M}'$ such that $x\mathcal{Z}_{\prec m+1}x'$. By induction on the complexity of given $\phi \in B_{m+1}(\mathcal{P})$ we prove, $x \Vdash \phi$ implies $x' \Vdash' \phi$. We only check that for $\phi := \gamma \to \psi$. We notice that $\gamma, \psi \in B_m(\mathcal{P})$. Suppose that $x' \nVdash' \gamma \to \psi$ then for some $y' \succ' x'$, $y' \Vdash' \gamma$ and $y' \nVdash' \psi$. Since $x\mathcal{Z}_{\prec m+1}x'$, there is a $y \succ x$, such that $y\mathcal{Z}_m y'$. Then, by induction and Theorem 3.11, $y \Vdash \gamma$ and $y \nVdash \psi$. That means $x \nVdash' \gamma \to \psi$ which is a contradiction.

$(1 \Rightarrow 3)$: We prove that for all $m \in \omega, x \in W$ and $x' \in W'$, if $\mathsf{Th}_m^{\mathcal{P}}(x) \subseteq \mathsf{Th}_m^{\mathcal{P}}(x')$, then there exists a layered $\mathcal{P}$- bisimulation $\mathcal{Z}$ between $\mathbf{M}$ and $\mathbf{M}'$ with $w\mathcal{Z}_{\prec m}w'$.

For $m = 0$, put $\mathcal{Z} = \emptyset$ which is obviously downward closed. Now assume that the statement holds for $m > 0$. Suppose $\mathsf{Th}_{m+1}^{\mathcal{P}}(x) \subseteq \mathsf{Th}_{m+1}^{\mathcal{P}}(x')$. Define a relation $\mathcal{Z}$ on $W \times \omega \times W'$ as:

$$w\mathcal{Z}_i w' \text{ if and only if } \mathsf{Th}_i^{\mathcal{P}}(w) = \mathsf{Th}_i^{\mathcal{P}}(w').$$

Clearly, $\mathcal{Z}_i$'s are persistent over atoms. We only show the zig property, suppose $w\mathcal{Z}_i w'$ and $w \prec y$. We want to show that there is $y' \succ' w'$ such that $y\mathcal{Z}_{i-1}y'$. Define $\phi(y) := \mathsf{Y}_{i-1}(y) \to \mathsf{N}_{i-1}(y)$. We have $w \nVdash \phi(y)$ and since $\phi(y) \in \mathcal{B}_i(\mathcal{P})$, $w' \nVdash \phi(y)$. Therefore, for some $y' \succ' w'$ we have $y' \Vdash \mathsf{Y}_{i-1}(y)$, but $y' \nVdash \mathsf{N}_{i-1}(y)$. Hence $y\mathcal{Z}_{i-1}y'$. It remains to show that $x\mathcal{Z}_{\prec m+1}x'$. So assume that $k' \succ' x'$, then $x' \nVdash \phi(k')$. Thus by assumption we have $x \nVdash \phi(k')$ which implies that for some $k \succ x$, $k \Vdash \mathsf{Y}_m(k')$ and $k \nVdash \mathsf{N}_m(k')$. Hence, $k\mathcal{Z}_m k'$. Obviously, $\mathcal{Z}$ is a downward closed l-bisimulation.

$(3 \Rightarrow 2)$: Obvious. □

4. Interpolation

In this section, we prove the lifting theorem which helps us in establishing the Craig interpolation property. After which, we prove the amalgamation lemma for **FPL** which results in its uniform left-interpolation property.

The proofs are highly influenced by that of similar theorems in [8]. In this section all models are irreflexive unless explicitly mentioned.

Theorem 4.1 (Lifting). *Let* $\mathbf{M} = (W, \prec, \Vdash)$ *be a* $\vec{q}, \vec{p}$*-model and* $\mathbf{M}' = (W', \prec', \Vdash')$ *be a* $\vec{p}, \vec{r}$*- model with* $\mathbf{M}(\vec{p}) \simeq_\alpha \mathbf{M}'(\vec{p})$*. Then there exists* $\vec{q}, \vec{p}, \vec{r}$*-model* $\mathbf{M}'' = (W'', \prec'', \Vdash'')$ *such that* $\mathbf{M}(\vec{q}, \vec{p}) \simeq_\alpha \mathbf{M}''(\vec{q}, \vec{p})$ *and* $\mathbf{M}'(\vec{p}, \vec{r}) \simeq_\alpha \mathbf{M}''(\vec{p}, \vec{r})$*.*

Proof: Let $\mathcal{Z} : \mathbf{M}(\vec{p}) \simeq_\alpha \mathbf{M}'(\vec{p})$. Define $\vec{q}, \vec{p}, \vec{r}$-model $\mathbf{M}''$ as follows:

- $W'' := \{(w, w') \mid (w, \beta, w') \in \mathcal{Z} \text{ for some } \beta\}$;
- $(w, w') \prec'' (v, v')$ exactly when $w \prec v$ and $w' \prec' v'$;
- $(w, w') \Vdash'' s$ exactly when $w \Vdash s$ or $w' \Vdash' s$.

It's easy to see that for $s \in \vec{q}, \vec{p}$ we have $(w, w') \Vdash'' s$ exactly when $w \Vdash s$ and for $s \in \vec{p}, \vec{r}$ we have $(w, w') \Vdash'' s$ exactly when $w' \Vdash' s$. Next, define $\mathcal{Z}'$ by $w\mathcal{Z}'_i(w, w')$ if $w\mathcal{Z}_i w'$ and $\mathcal{Z}''$ by $w'\mathcal{Z}''_i(w, w')$ if $w\mathcal{Z}_i w'$. It's easy to see that $\mathcal{Z}' : \mathbf{M}(\vec{q}, \vec{p}) \simeq_\alpha \mathbf{M}''(\vec{q}, \vec{p})$ and $\mathcal{Z}'' : \mathbf{M}'(\vec{p}, \vec{r}) \simeq_\alpha \mathbf{M}''(\vec{p}, \vec{r})$. $\square$

Corollary 4.2. Let $\mathbf{M}$ be a $\vec{q}, \vec{p}$-model and $\mathbf{M}'$ be a $\vec{p}, \vec{r}$-model with $\mathbf{M}(\vec{p}) \simeq_n \mathbf{M}'(\vec{p})$. Then there exists $\vec{q}, \vec{p}, \vec{r}$-model $\mathbf{M}''$ such that $\mathsf{Th}_n^{(\vec{q},\vec{p})}(\mathbf{M}) = \mathsf{Th}_n^{(\vec{q},\vec{p})}(\mathbf{M}'')$ and $\mathsf{Th}_n^{(\vec{p},\vec{r})}(\mathbf{M}') = \mathsf{Th}_n^{(\vec{p},\vec{r})}(\mathbf{M}'')$.

By the lifting lemma we are ready to prove the Craig interpolation property for **BPL**. The Craig interpolation property for **BPL** was proved in [4]. The proof of the Craig interpolation property for **FPL**, $\mathbf{FPL}_\perp$, **EBPL** and $\mathbf{F}_n$, for $n \in \omega$ are new.

We say a class $\mathcal{C}$ of Kripke models has the *lifting property* if for all models $\mathbf{M}$ and $\mathbf{M}'$ in $\mathcal{C}$, the constructed model $\mathbf{M}''$ in the lifting lemma is also in $\mathcal{C}$.

Theorem 4.3 (Craig Interpolation). *Let* $\mathbf{L}$ *be a logic over* **BPL** *which is sound and complete with respect to a class* $\mathcal{C}$ *having lifting property. Then* $\mathbf{L}$ *satisfies the Craig interpolation property.*

Proof: Suppose that $\phi \in B_m(\vec{q}, \vec{p})$ and $\psi \in B_n(\vec{p}, \vec{r})$ are such that $\mathbf{L} \vdash \phi \to \psi$. We show that $\psi^*_k(\vec{p}) := \bigvee\{\chi \in B_k(\vec{p}) \mid \mathbf{L} \vdash \chi \to \psi\}$ is their Craig interpolant, where $k := \max(m, n)$.

Clearly $\mathbf{L} \vdash \psi^*_k(\vec{p}) \to \psi$. If $\mathbf{L} \nvdash \phi \to \psi^*_k(\vec{p})$, there exists a $\vec{q}, \vec{p}$-pointed model $(\mathbf{M}, w)$ such that $w \Vdash \phi$ but $w \nVdash \psi^*_k(\vec{p})$. Let $\mathsf{Y} := \mathsf{Y}_{k,w}((\vec{p}))$ and

$\mathsf{N} := \mathsf{N}_k^{(\vec{p})}(w)$. For contradiction, suppose $\mathsf{Y} \vdash \mathsf{N} \vee \psi$. Note that by Fact 3.10, Y is prime. So $\mathsf{Y} \vdash \mathsf{N}$ or $\mathsf{Y} \vdash \psi$. Since $\mathsf{Y} \nvdash \mathsf{N}$, it follows that $\mathsf{Y} \vdash \psi$ and hence by definition of $\psi_k^*(\vec{p})$ we have $\mathsf{Y} \vdash \psi_k^*(\vec{p})$ which is a contradiction, since $w \nVdash \psi_k^*(\vec{p})$. So $\mathsf{Y} \nvdash \mathsf{N} \vee \psi$. Then there exists $\vec{q}, \vec{r}$-pointed model $(\mathbf{M}', w')$ such that $w' \Vdash \mathsf{Y}$ but $w' \nVdash \mathsf{N} \vee \psi$. Now, by Theorem 3.12, we have $\mathbf{M}'(\vec{p}) \simeq_k \mathbf{M}(\vec{p})$. Then, by Corollary 4.2, there exists $\vec{p}, \vec{q}, \vec{r}$-model $\mathbf{M}''$ such that $\mathsf{Th}_k^{(\vec{q},\vec{p})}(\mathbf{M}) = \mathsf{Th}_k^{(\vec{q},\vec{p})}(\mathbf{M}'')$ and $\mathsf{Th}_k^{(\vec{p},\vec{r})}(\mathbf{M}') = \mathsf{Th}_k^{(\vec{p},\vec{r})}(\mathbf{M}'')$. In particular, $\mathbf{M}'' \Vdash \phi$ and $\mathbf{M}'' \nVdash \psi$ which is a contradiction. Therefore, $\mathbf{L} \vdash \phi \to \psi_k^*(\vec{p})$. □

Corollary 4.4. **BPL**, **FPL**, and $\mathbf{F}_n$, for $n \in \omega$, have the Craig interpolation property.

Proof: For **BPL** it is trivial. For **FPL**, note that in the lifting lemma, when $\mathbf{M}$ and $\mathbf{M}'$ are conversely well-founded, so will be the constructed model $\mathbf{M}''$. Also, when $\mathbf{M}$ and $\mathbf{M}'$ have depth at most n, then $\mathbf{M}''$ also has depth at most n. □

The following logic is another interesting extension of **BPL** which behaves very similar to **IPL** [2].

$$\mathbf{EBPL} = \mathbf{BPL} + \top \to \bot \Rightarrow \bot.$$

It was proved in [2, Corollary 3.9] that the logic **EBPL** is sound and complete for the class of finite models with reflexive leaves. Obviously this class of models has the lifting property. Therefore we have the following corollary.

Corollary 4.5. The logic **EBPL** has the Craig interpolation property.

We say a formulas ϕ is constant if $V(\phi) = \emptyset$. In the following theorem we show that every faithful extension of basic propositional logic with constant formulas preserves Craig interpolation property.

Theorem 4.6. *Let X be a set of constant fourmulas. If a logic $\mathbf{L}$ has the Craig interpolation property and $\mathbf{L} + X$ is faithful, then $\mathbf{L} + X$ also has Craig interpolation property.*

Proof: Suppose that $\mathbf{L}$ has the Craig interpolation property. Let $\mathbf{L} + X \vdash \phi \to \psi$. Then by faithfulness we have $\mathbf{L} + X \vdash \phi \Rightarrow \psi$. Then there are constant formulas $\theta_1, \cdots, \theta_n$ in X such that in $\mathbf{L}$ we have $\theta_1, \cdots, \theta_n \vdash \phi \Rightarrow$

$$\begin{tikzcd}
\Delta \arrow[r, dash, "\Phi_X"] & m \arrow[r, dash, "\sim_{2d_X(m')}"] & n \\
\Delta \arrow[u, "="] \arrow[r, dash, "\Phi_X"] & m' \arrow[u, "\preceq"] \arrow[r, dash, "\sim_{2d_X(m')+1}"] & n' \arrow[u, "\preceq"]
\end{tikzcd}$$

Figure 1. Witnessing triple

ψ. Put $\theta = \bigwedge \theta_i$, then by Proposition 2.1 we have $\mathbf{L} \vdash \theta \wedge \phi \Rightarrow \psi$ which implies that $\mathbf{L} \vdash \theta \wedge \phi \to \psi$. Now by interpolation property of $\mathbf{L}$, there is a formula η in $V(\phi) \cap V(\psi)$ such that $\mathbf{L} \vdash \theta \wedge \phi \to \eta$ and $\mathbf{L} \vdash \eta \to \psi$. Hence,by faithfulness, $\mathbf{L} + X \vdash \phi \to \eta$ and $\mathbf{L} + X \vdash \eta \to \psi$. $\square$

COROLLARY 4.7. $\mathbf{FPL}_\perp$ has the Craig interpolation property.

The proof of the next lemma is similar to the one used in [8] for **IPL**. However, to show that this proof -especially claim 2- does not work for **BPL** but does work for **FPL**, the details have been provided. In the following lemma, all models are conversely well-founded, i.e., **FPL**- models.

LEMMA 4.8 (Amalgamation). *Consider disjoint sets $\vec{q}, \vec{p}$ and $\vec{r}$. Let $X \subseteq \mathcal{L}(\vec{q}, \vec{p})$ be a finite $\mathcal{P}$-adequate set. Let $\langle \mathbf{M}, w_0 \rangle \in \mathsf{Pmod}(\vec{q}, \vec{p})$, $\langle \mathbf{M}', w_0' \rangle \in \mathsf{Pmod}(\vec{p}, \vec{r})$. Let:*

$\nu := |\{\phi \in X \mid \phi$ *is a propositional variable or an implicational formula*$\}|$.

Suppose that $w_0 \simeq_{2\nu+1,\vec{p}} w_0'$. Then there exists a $\vec{q}, \vec{p}, \vec{r}$-model $\langle \mathbf{M}'', w_0'' \rangle$ such that $w_0'' \simeq_{\vec{p},\vec{r}} w_0'$ and $Th_X(w_0'') = Th_X(w_0)$.

PROOF: Let $\mathcal{Z}$ be a downwards closed witness of $w_0 \simeq_{2\nu+1,\vec{p}} w_0'$. Define $\Phi_X : \mathbf{M} \longrightarrow H_X$ by $\Phi_X(w) := \Delta(w) := \{\phi \in X \mid w \Vdash \phi\}$. Define further for $w \in \mathbf{M}$: $d_X(w) = d_{H_X}(\Delta(w))$. Note that $d_X(w) \leq \nu$.

Consider a pair $\langle \Delta, n \rangle$ for Δ in H and n in $\mathbf{M}'$. We say that m', m, n' is a *witnessing triple* for $\langle \Delta, n \rangle$ if:

$$\Delta = \Delta(m) = \Delta(m'),\ m' \preceq m, n' \preceq' n,\ m' \mathcal{Z}_{2d_X(m')+1} n',\ m \mathcal{Z}_{2d_X(m')} n.$$

The requested model $\mathbf{M}''$ is defined as follows:

- $W'' = \{\langle \Delta, n\rangle \mid$ there is a witnessing triple for$\langle \Delta, n\rangle\}$,
- $w''_0 := \langle \Delta(w_0), w'_0\rangle$,
- $\langle \Delta, n\rangle \prec'' \langle \Gamma, n'\rangle$ exactly when $\Delta \preceq \Gamma$ and $n \prec' n'$,
- $\langle \Delta, n\rangle \Vdash s$ exactly when $\Delta \Vdash s$ or $n \Vdash s$.

Note that by assumption $w_0 \mathcal{Z}_{2\nu+1} w'_0$ and the fact that $2d_X(w_0) + 1 \leq 2\nu + 1$ we have $w_0 \mathcal{Z}_{2d_X(w_0)+1} w'_0$. So, w_0, w_0, w'_0 is a witnessing triple for w''_0. Let m', m, n' be a witnessing triple for $\langle \Delta, n\rangle$. For $p \in \vec{p} \cap X$ we have $\Delta \Vdash p$ if and only if $m \Vdash p$ if and only if $n \Vdash p$, and hence $\langle \Delta, n\rangle \Vdash p$ if and only if $\Delta \Vdash p$ if and only if $n \Vdash p$. Also, note that $\mathbf{M}''$ is an **FPL**-model. The following claims prove the lemma.

Claim 1. $w''_0 \simeq_{\vec{p},\vec{r}} w'_0$,

Claim 2. For $\phi \in X$, $\langle \Delta, n\rangle \Vdash \phi$ exactly when $\phi \in \Delta$.

Proof of Claim 1: For $\mathcal{B}$ defined by $\langle \Delta, n\rangle \mathcal{B} n$, by a same argument as [8], we show that it is a bisimulation. Clearly $\mathsf{Th}_{\vec{p},\vec{r}}(\langle \Delta, n\rangle) = \mathsf{Th}_{\vec{p},\vec{r}}(n)$. We only check the *zag*-property of $\mathcal{B}$. Suppose $\langle \Delta, n\rangle \mathcal{B} n \prec m$. We are looking for a pair $\langle \Gamma, m\rangle$ such that $\Delta \preceq \Gamma$. Let k', k, n' be a witnessing triple for $\langle \Delta, n\rangle$. Since $k' \sim_{2d_X(k')+1} n' \preceq m$, there is a h such that $h \prec k'$ and $h \sim_{2d_X(k')} m$. Put, $\Gamma := \Delta(h)$. We need a witnessing triple k'^*, k^*, n'^* for $\langle \Gamma, m\rangle$. If $\Gamma = \Delta$, then put: $k'^* := k', k^* := h, n'^* := n'$, see figure 2.

If $\Gamma \neq \Delta$, then put: $k'^* := h, k^* := h, n'^* := m$. We notice that since $k' \preceq h$, then $\Delta = \Delta(k') \prec \Gamma$ which implies that $d_X(h) < d_X(k')$. Therefore, $2d_X(h) + 1 \leq 2_X(k')$, so $h \sim_{2d_X(k')+1} m$ which implies that $h \sim_{2d_X(k')} m$, because $\mathcal{Z}$ is downward close. Clearly $w''_0 \mathcal{B} w'_0$.

Proof of Claim 2: We proceed by induction on the complexity of a formula $\phi \in X$. The cases of atoms, conjuntctions and disjunctions are trivial. Consider $\phi \to \psi \in X$ and the node $\langle \Delta, m\rangle$ with witnessing triple k', k, m'. Suppose $\phi \to \psi \notin \Delta$. Since $\Delta = \mathsf{Th}(k)$, then $k \nVdash \phi \to \psi$. So, there is an $h \succ k$ with $h \Vdash \phi$ and $h \nVdash \psi$. Let h, by conversely well-foundedness of $\mathbf{M}$, be a maximal in $\mathbf{M}$ with $h \succ k$, $h \Vdash \phi$ and $h \nVdash \psi$. By maximality, we find $h \Vdash \phi \to \psi$. Let $\Gamma := \Delta(h)$. Since $\phi \to \psi \notin \Delta$ and $\phi \to \psi \in \Gamma$, we find $\Delta \prec \Gamma$, which implies that $d_X(k') \geq 1$. Since $k \mathcal{Z}_{2d_X(k')} m$ and $k \prec h$, there is an $n \succ m$ with $h \mathcal{Z}_{2d_X(h)-1} n$. Therefore $h \mathcal{Z}_{2d_X(h)+1} n$. So we can

$$\begin{tikzcd}
\Gamma = \Delta \arrow[rr, dashed, dash, "\Phi_X"] & & k^* = h \arrow[rr, dashed, dash, "\sim_{2d_X(k')}"] & & m \\
& \Delta \arrow[rr, dash, "\Phi_X"] & & k \arrow[r, dash, "\sim_{2d_X(k')}"] & n \arrow[u, "\preceq"'] \\
\Delta \arrow[uu, dashed, "="] \arrow[ur, "="] \arrow[rr, dash, "\Phi_X"] & & k'^* = k' \arrow[uu, dashed, "\preceq"] \arrow[ur, "\preceq"] \arrow[rr, dash, "\sim_{2d_X(k')+1}"] & & n'^* = n' \arrow[u, "\preceq"']
\end{tikzcd}$$

Figure 2.

take h, h, n to witness $\langle \Gamma, n \rangle$. Clearly $\langle \Delta, m \rangle \prec'' \langle \Gamma, n \rangle$. By the induction hypothesis, $\langle \Gamma, n \rangle \Vdash \phi$ while $\langle \Gamma, n \rangle \nVdash \psi$, i.e., $\langle \Delta, n \rangle \nVdash \phi \to \psi$.

The other half of the argument, i.e., that $\phi \to \psi \in \Delta$ implies $(\Delta, n) \Vdash \phi \to \psi$, is easy. □

Definition 4.9. Let $\phi(\vec{q}, \vec{p})$ be a formula.

1. A uniform left-interpolant for $\phi(\vec{q}, \vec{p})$ with respect to $\vec{p}$ is a formula $\chi(\vec{p})$ such that for all formulas $\psi(\vec{p}, \vec{r})$ with $\vdash \psi \to \phi$, χ acts as an interpolant for ϕ and ψ.

2. A uniform right-interpolant for $\phi(\vec{q}, \vec{p})$ with respect to $\vec{p}$ is a formula $\chi(\vec{p})$ such that for all formulas $\psi(\vec{p}, \vec{r})$ with $\vdash \phi \to \psi$, χ acts as an interpolant for ϕ and ψ.

3. A logic whose formulas have both uniform left and right-interpolants is said to satisfy the uniform interpolation property.

Although the Amalgamation lemma is held for FPL models, unlike in intuitionistic logic, we can only prove the uniform left-interpolation property.

Theorem 4.10. **FPL** *has the uniform left-interpolation property.*

Proof: Note that by the Amalgamation lemma, in proof of Craig interpolation for **FPL** we can let $X := sub(\phi)$, and by defining ν as before, we find that ϕ^*_{2v+1} works as Craig interpolant for any given ψ satisfying the conditions. Therefore, ϕ^*_{2v+1} is the uniform left-interpolant for ϕ. □

In the remainder of this section, we prove the uniform interpolation for some extensions of **FPL**. As a matter of fact, we show that countably infinite of such extensions exist.

A logic **L** is said to be *locally tabular* if for any finite set $\mathcal{P}$ of propositional variables, there are only finitely many formulas built from variables in $\mathcal{P}$ up to **L**-provable equivalence.

Theorem 4.11. *If* **L** *is a locally tabular logic over* **BPL** *and has the Craig interpolation property, then* **L** *has the uniform interpolation property.*

Proof: Consider a formula $\phi(\vec{q}, \vec{p})$. Let $\Psi = \{\psi(\vec{p}, \vec{r}) \mid \mathbf{L} \vdash \psi \to \phi\}$. Consider an effective counting of members of Ψ as $\psi_1, \psi_2, \cdots, \psi_n$. By Craig interpolation, for every i we can find $\chi_i(\vec{p})$ such that $\mathbf{L} \vdash \chi_i \to \phi$ and $\mathbf{L} \vdash \psi_i \to \chi_i$. Now,$\bigvee \chi_i$ works as the uniform left-interpolant of ϕ for all ψ_n.

For the uniform right-interpolant, let $\Psi = \{\psi(\vec{p}, \vec{r}) \mid \mathbf{L} \vdash \phi \to \psi\}$. We can, by locally tabularity, find an effective counting of members of Ψ as $\psi_1, \psi_2, \cdots, \psi_n$. By Craig interpolation, for every i we can find $\chi_i(\vec{p})$ such that $\mathbf{L} \vdash \phi \to \chi_i$ and $\mathbf{L} \vdash \chi_i \to \psi_i$. Therefore $\bigwedge \chi_i$ works as the uniform right-interpolant of ϕ for all ψ_n. □

The following theorem was proved algebraically in [1, Theorem 2.12].

Theorem 4.12. *For every $n \in \omega$, the logic $\mathbf{F}_n$ is locally tabular.*

Corollary 4.13. The logic $\mathbf{F}_n$, for $n \in \omega$, have the uniform interpolation property.

Proof: Apply Corollary 4.4, Theorem 4.11 and Theorem 4.12. □

We close this paper with the following problem.

Problem. Do **BPL**, **FPL**, $\mathbf{FPL}_\perp$ and **EBPL** have the uniform interpolation property?

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FIRST-ORDER MODAL SEMANTICS AND EXISTENCE PREDICATE

Abstract

In the article we study the existence predicate ε in the context of semantics for first-order modal logic. For a formula φ we define φ^{ε}—the so called *existence relativization*. We point to a gap in the work of Fitting and Mendelsohn [1] concerning the relationship between the truth of φ and φ^{ε} in classes of varying- and constant-domain models. We introduce operations on models which allow us to fill the gap and provide a more general perspective on the issue. As a result we obtain a series of theorems describing the logical connection between the notion of truth of a formula with the existence predicate in constant-domain models and the notion of truth of a formula without the existence predicate in varying-domain models.

Keywords: First-order modal logic, constant-domain model, varying-domain model, existence predicate.

Introduction

Semantic theory for first-order modal logic makes use of two philosophically important notions of varying- and constant-domain models which may shape the discussion about the role of existence predicate in modal logic and the meaning of quantifying over non-existing entities. Models with constant domains correspond to quantifying over merely-possible objects in addition to actually existent entities, while models with varying domains are in consonance with the actualistic interpretation of the quantifier, restraining quantification to that what actually exists. Relationship between

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the two approaches is often being studied via incorporating the existence predicate in the first-order language and examination of the translation of formulas without such a predicate into formulas containing it.

The question whether existence is a property of individuals or even whether it is a property at all has baffled philosophers and logicians for centuries, starting with Immanuel Kant and his *Critique of Pure Reason* in which he argued that existence is not a genuine attribute of things. This idea, defended in its particular form by Frege [2], is built into the very foundation of modern mathematical logic. It manifests itself in the use of the existential quantifier *instead of* the existence predicate. To say that there exists a root of the equation $x^2 - 3x = 0$ is to say that the propositional function '$x^2 - 3x = 0$' is satisfied by *some* number and that is to say that the proposition $\exists x(x^2 - 3x = 0)$ is true.

However, some philosophers, like Alexius Meinong [4], have felt the need for having the existence predicate in addition to the existential quantifier. One obvious way of introducing such a predicate in a first-order language is to define 'x exists' as $\exists y(x = y)$. The problem is that in classical first-order logic individual variables always denote something, and the formula $\exists y(x = y)$ is satisfied in every model. Another possibility is to introduce the existence predicate as a primitive symbol. Assuming the existence predicate is a unary predicate ε the question arises: what does and what does not exist? And this depends on the quantifiers. (For some discussion of these issues you can see [3].) For if the quantifiers quantify over existent objects only, the proposition $\forall x \varepsilon(x)$ is logically true and for any formula φ, $\forall x(\varepsilon(x) \wedge \varphi(x))$ and $\forall x \varphi(x)$ are equivalent, making the existence predicate redundant. If, on the other hand, the scope of quantification includes objects which do not exist but are possible, the existence predicate can do its job and select among all entities those which actually exist. This is exactly the idea standing behind the constant-domain models. Moreover, if the existence predicate seems redundant when quantifiers are actualistic, for then everything exists, but turns out to be useful when quantifiers are possibilistic, surely there must be some kind of connection between these two ways of doing logic. And, indeed, there is.

1. Preliminaries

All crucial definitions and elementary facts can be found in [1]. For readers' convenience let us remind basic concepts. The language with which we will deal is the standard first-order language with individual variables as the only terms with the addition of $\Box$ as the modal operator. We will take $\Box$, $\neg$, $\wedge$ and $\exists$ as primitive.

Two of the most commonly used on the next pages will be notions of *constant-* and *varying-domain* models. We will treat constant-domain models as a special case of varying-domain models (as they actually are). So for us 'model' and 'varying-domain model' will mean pretty much the same.

A (varying-domain) model $\mathcal{M}$ is a four-tuple $(\mathcal{U}, \mathcal{R}, \mathcal{D}, \mathcal{I})$ such that $\mathcal{U}$ is a non-empty set (its elements we will also call 'worlds' or 'points'), $\mathcal{R} \subseteq \mathcal{U} \times \mathcal{U}$ is a binary relation (called the *accessibility relation*), $\mathcal{D}$ is a function which maps elements of $\mathcal{U}$ to non-empty sets—to each element u of $\mathcal{U}$ it assigns a non-empty set $\mathcal{D}(u)$ which we call *a domain of* u, and by $\mathcal{D}(\mathcal{M})$ we mean the sum of all $\mathcal{D}(u)$. $\mathcal{I}$ is an interpretation of predicates. Strictly speaking, $\mathcal{I}$ is a mapping such that $\mathcal{I}(r, u) \subseteq \mathcal{D}(\mathcal{M})^{\tau(r)}$, where r is a predicate and $\tau(r)$ is arity of r.

A *valuation* is a map $v \colon Var \to \mathcal{D}(\mathcal{M})$, where Var is a set of all individual variables. For $a \in \mathcal{D}(\mathcal{M})$ and $x \in Var$, by $v(a/x)$ we mean a valuation such that $v(a/x)(x) = a$ and for any variable y distinct from x, $v(a/x)(y) = v(y)$.

The satisfaction relation $\Vdash$ is defined recursively in the standard way as follows.

Definition 1.1. Take a model $\mathcal{M} = (\mathcal{U}, \mathcal{R}, \mathcal{D}, \mathcal{I}), u \in \mathcal{U}$, valuation v, and predicate r of arity n. For a formula φ we define the expression

$$(\mathcal{M}, u) \Vdash \varphi[v],$$

which we read as φ *is satisfied at* u *in model* $\mathcal{M}$ *under valuation* v:

(i) $(\mathcal{M}, u) \Vdash r(x_1, \ldots, x_n)[v] \iff \langle v(x_1), \ldots, v(x_n)\rangle \in \mathcal{I}(r, u)$,

(ii) $(\mathcal{M}, u) \Vdash \neg\varphi[v] \iff (\mathcal{M}, u) \nVdash \varphi[v]$,

(iii) $(\mathcal{M}, u) \Vdash (\varphi \wedge \psi)[v] \iff (\mathcal{M}, u) \Vdash \varphi[v]$ and $(\mathcal{M}, u) \Vdash \psi[v]$,

(iv) $(\mathcal{M}, u) \Vdash \Box\varphi[v] \iff$ for any $t \in \mathcal{U}$, if $u\mathcal{R}t$, then $(\mathcal{M}, t) \Vdash \varphi[v]$,

(v) $(\mathcal{M}, u) \Vdash \exists x\varphi[v] \iff$ there is $a \in \mathcal{D}(u)$ and $(\mathcal{M}, u) \Vdash \varphi[v(a/x)]$.

A formula φ is satisfied by a class of models K, $K \Vdash \varphi$ in symbols, when $(\mathcal{M}, t) \Vdash \varphi[v]$, for any $\mathcal{M} = (\mathcal{U}, \mathcal{R}, \mathcal{D}, \mathcal{I}) \in K$, any $t \in \mathcal{U}$, and any valuation v. By $\mathbb{VD}$ we denote the class of all (varying-domain) models. Moreover, let $\mathbb{CD}$ stand for the class of all models $\mathcal{M} = (\mathcal{U}, \mathcal{R}, \mathcal{D}, \mathcal{I})$ such that $\mathcal{D}(u) = \mathcal{D}(w)$, for any $u, w \in \mathcal{U}$. Elements of $\mathbb{CD}$ are called *constant-domain models.*

Definition 1.2. Let ε be a unary predicate. Following Fitting and Mendelsohn, for any φ we define φ^ε as follows:

(i) For an atomic formula, $r(x_1, \ldots, x_n)^\varepsilon = r(x_1, \ldots, x_n)$,

(ii) $(\neg\varphi)^\varepsilon = \neg(\varphi)^\varepsilon$,

(iii) $(\varphi \land \psi)^\varepsilon = (\varphi)^\varepsilon \land (\psi)^\varepsilon$,

(iv) $(\Box\varphi)^\varepsilon = \Box(\varphi)^\varepsilon$,

(v) $(\exists x\varphi)^\varepsilon = \exists x(\varepsilon(x) \land \varphi^\varepsilon)$.

2. The construction

In [1] (Proposition 4.8.2.) one can find the claim that

$$\mathbb{VD} \Vdash \varphi \iff \mathbb{CD} \Vdash \varphi^\varepsilon \tag{$\star$}$$

for any sentence φ which does not contain ε. Implication to the left is proven by authors, while the other direction is left to the reader. However, we observed that this implication fails. Indeed, let us consider the sentence:

$$\exists x(r(x) \lor \neg r(x)),$$

where r is an arbitrary unary predicate (distinct from ε). Then we obtain:

$$\begin{aligned}(\exists x(r(x) \lor \neg r(x)))^\varepsilon &= \exists x(\varepsilon(x) \land (r(x) \lor \neg r(x))^\varepsilon) \\ &= \exists x(\varepsilon(x) \land (r(x) \lor \neg r(x))).\end{aligned}$$

Clearly, $\exists x(r(x) \vee \neg r(x))$ is valid in all varying-domain models, however $\exists x(\varepsilon(x) \wedge (r(x) \vee \neg r(x)))$ is not valid in those constant-domain models in which ε is interpreted as empty and this falsifies $(\star)$[1].

Although the implication $\mathbb{VD} \Vdash \varphi \Longrightarrow \mathbb{CD} \Vdash \varphi^{\varepsilon}$ does not hold, we can still prove a weaker version. Before we do it, let us introduce a couple of definitions and facts. If $K \subseteq \mathbb{VD}$, by K_{ε} we denote the class of those models $\mathcal{M} = (\mathcal{U}, \mathcal{R}, \mathcal{D}, \mathcal{I})$ from K such that $\mathcal{I}(\varepsilon, t) \neq \emptyset$, for any $t \in \mathcal{U}$.

Definition 2.1. Let $\mathcal{M} = (\mathcal{U}, \mathcal{R}, \mathcal{D}, \mathcal{I}) \in \mathbb{CD}_{\varepsilon}$ and let w be any object such that $w \notin \mathcal{U}$. We define a model $\mathcal{M}^w \in \mathbb{VD}$ as $\mathcal{M}^w = (\mathcal{U}^w, \mathcal{R}^w, \mathcal{D}^w, \mathcal{I}^w)$, where $\mathcal{U}^w = \mathcal{U} \cup \{w\}$, $\mathcal{R}^w = \mathcal{R}$, and

$$\mathcal{D}^w(t) = \begin{cases} \mathcal{I}(\varepsilon, t) & \text{if } t \neq w, \\ \mathcal{D}(\mathcal{M}) & \text{if } t = w, \end{cases} \quad \text{and} \quad \mathcal{I}^w(r, t) = \begin{cases} \mathcal{I}(r, t) & \text{if } t \neq w, \\ \mathcal{D}(\mathcal{M})^{\tau(r)} & \text{if } t = w. \end{cases}$$

Fact 2.2. $\mathcal{M}^w \in \mathbb{VD}_{\varepsilon}$, for any $\mathcal{M} \in \mathbb{CD}_{\varepsilon}$.

Fact 2.3. Let $\mathcal{S}$ be any proposition of our meta-language (the very language of this paper). For any $\mathcal{U}, t, w$ and $\mathcal{R}$ as in Definition 2.1, the following assertions are equivalent:

(i) *For any* $t \in \mathcal{U}$, such that $u\mathcal{R}t$, $\mathcal{S}$

(ii) *For any* $t \in \mathcal{U} \cup \{w\}$, such that $u\mathcal{R}t$, $\mathcal{S}$

Proof: $(\Longleftarrow)$ Trivial.
$(\Longrightarrow)$ Let $t \in \mathcal{U} \cup \{w\}$. If $t \in \mathcal{U}$, by the assumption, thesis holds. If $t = w$, then, by definition of $\mathcal{R}$, $u\mathcal{R}t$ fails and therefore the thesis holds.

□

Now we can prove the following lemma.

Lemma 2.4. *For any formula* φ *not containing* ε*, model* $\mathcal{M} = (\mathcal{U}, \mathcal{R}, \mathcal{D}, \mathcal{I}) \in \mathbb{CD}_{\varepsilon}$*,* $w \notin \mathcal{U}$*,* $t \in \mathcal{U}$*, and valuation* v*,*

$$(\mathcal{M}, t) \Vdash \varphi^{\varepsilon}[v] \Longleftrightarrow (\mathcal{M}^w, t) \Vdash \varphi[v].$$

Proof: We will prove it inductively.

[1] An error of which Prof. Fitting had been aware before we observed it as he said in personal correspondence, and gratefully offered a suggestion that non-emptyness of the existence predicate is a requirement—an idea which we develop in this article.

For an atomic formula $r(x_1, \ldots, x_n)$ we have:

$$(\mathcal{M}, t) \Vdash r(x_1, \ldots, x_n)^\varepsilon[v] \iff (\mathcal{M}, t) \Vdash r(x_1, \ldots, x_n)[v] \qquad \text{(by 1.2)}$$
$$\iff \langle v(x_1), \ldots, v(x_n)\rangle \in \mathcal{I}(r, t) \qquad \text{(by 1.1)}$$
$$\iff \langle v(x_1), \ldots, v(x_n)\rangle \in \mathcal{I}^w(r, t) \qquad \text{(by 2.1)}$$
$$\iff (\mathcal{M}^w, t) \Vdash r(x_1, \ldots, x_n)[v] \qquad \text{(by 1.1)}$$

Crucial in this step is the fact that interpretations of predicates are the same in the new model for the 'old worlds' *and* that valuations are the same, i.e. every valuation into $\mathcal{M}$ is a valuation into $\mathcal{M}^w$ and vice versa.

For negation we get:

$$(\mathcal{M}, t) \Vdash (\neg\psi)^\varepsilon[v] \iff (\mathcal{M}, t) \Vdash \neg\psi^\varepsilon[v] \qquad \text{(by 1.2)}$$
$$\iff (\mathcal{M}, t) \nVdash \psi^\varepsilon[v] \qquad \text{(by 1.1)}$$
$$\iff (\mathcal{M}^w, t) \nVdash \psi[v] \qquad \text{(induction)}$$
$$\iff (\mathcal{M}^w, t) \Vdash \neg\psi[v] \qquad \text{(by 1.1)}$$

For conjunction we get:

$$(\mathcal{M}, t) \Vdash (\psi \wedge \chi)^\varepsilon[v] \iff (\mathcal{M}, t) \Vdash (\psi^\varepsilon \wedge \chi^\varepsilon)[v] \qquad \text{(by 1.2)}$$
$$\iff (\mathcal{M}, t) \Vdash \psi^\varepsilon[v] \text{ and } (\mathcal{M}, t) \Vdash \chi^\varepsilon[v] \qquad \text{(by 1.1)}$$
$$\iff (\mathcal{M}^w, t) \Vdash \psi[v] \text{ and } (\mathcal{M}^w, t) \Vdash \chi[v] \qquad \text{(induction)}$$
$$\iff (\mathcal{M}^w, t) \Vdash (\psi \wedge \chi)[v] \qquad \text{(by 1.1)}$$

For box we have:

$$(\mathcal{M}, t) \Vdash (\Box\psi)^\varepsilon[v] \iff (\mathcal{M}, t) \Vdash \Box\psi^\varepsilon[v] \qquad \text{(by 1.2)}$$
$$\iff \text{ for any } s \in \mathcal{U}, \text{ if } t\mathcal{R}s, \text{ then } (\mathcal{M}, s) \Vdash \psi^\varepsilon[v] \qquad \text{(by 1.1)}$$
$$\iff \text{ for any } s \in \mathcal{U}, \text{ if } t\mathcal{R}^w s, \text{ then } (\mathcal{M}^w, s) \Vdash \psi[v] \qquad \text{(induction)}$$
$$\iff \text{ for any } s \in \mathcal{U} \cup \{w\}, \text{ if } t\mathcal{R}^w s, \text{ then } (\mathcal{M}^w, s) \Vdash \psi[v] \qquad \text{(by 2.3)}$$
$$\iff (\mathcal{M}^w, s) \Vdash \Box\psi[v] \qquad \text{(by 1.1)}$$

For the quantifier we have:

$$\begin{array}{lll}
(\mathcal{M},t) \Vdash (\exists x\psi)^{\varepsilon}[v] \iff (\mathcal{M},t) \Vdash \exists x(\varepsilon(x) \wedge \psi^{\varepsilon})[v] & & \text{(by 1.2)} \\
\iff \exists_{a\in\mathcal{D}(t)}\ (\mathcal{M},t) \Vdash (\varepsilon(x) \wedge \psi^{\varepsilon})[v(a/x)] & & \text{(by 1.1)} \\
\iff \exists_{a\in\mathcal{D}(t)}\ (\mathcal{M},t) \Vdash \varepsilon(x)[v(a/x)] \text{ and } (\mathcal{M},t) \Vdash \psi^{\varepsilon}[v(a/x)] & & \text{(by 1.1)} \\
\iff \exists_{a\in\mathcal{D}(t)}\ (\mathcal{M},t) \Vdash \varepsilon(x)[v(a/x)] \text{ and } (\mathcal{M}^{w},t) \Vdash \psi[v(a/x)] & & \text{(induction)} \\
\iff \exists_{a\in\mathcal{D}(t)}\ a \in \mathcal{I}(\varepsilon,t) \text{ and } (\mathcal{M}^{w},t) \Vdash \psi[v(a/x)] & & \text{(by 1.1)} \\
\iff \exists_{a\in\mathcal{D}(t)}\ a \in \mathcal{D}^{w}(t) \text{ and } (\mathcal{M}^{w},t) \Vdash \psi[v(a/x)] & & \text{(by 2.1)} \\
\iff \exists_{a\in\mathcal{D}^{w}(t)}\ (\mathcal{M}^{w},t) \Vdash \psi[v(a/x)] & & (\mathcal{M} \in \mathbb{CD}) \\
\iff (\mathcal{M}^{w},t) \Vdash \exists x\psi[v] & & \text{(by 1.1)}
\end{array}$$

□

Now we can state and prove the said weaker version of (⋆).

THEOREM 2.5. *For any formula φ not containing ε, $\mathbb{VD}_{\varepsilon} \Vdash \varphi \Longrightarrow \mathbb{CD}_{\varepsilon} \Vdash \varphi^{\varepsilon}$.*

PROOF: Let $\mathcal{M} = (\mathcal{U},\mathcal{R},\mathcal{D},\mathcal{I}) \in \mathbb{CD}_{\varepsilon}, t \in \mathcal{U}$ and v such that $(\mathcal{M},t) \nVdash \varphi^{\varepsilon}[v]$. Let w be any object such that $w \notin \mathcal{U}$. By Fact 2.2, $\mathcal{M}^{w} \in \mathbb{VD}_{\varepsilon}$, and therefore by Lemma 2.4 we achieve $(\mathcal{M}^{w},t) \nVdash \varphi[v]$. □

3. Conclusion

Let us recall the construction Fitting and Mendelsohn introduced in [1, p. 107].

DEFINITION 3.1. Let $\mathcal{M} = (\mathcal{U},\mathcal{R},\mathcal{D},\mathcal{I}) \in \mathbb{VD}$. Then we define $\mathcal{M}^{\star} = (\mathcal{U}^{\star},\mathcal{R}^{\star},\mathcal{D}^{\star},\mathcal{I}^{\star})$, where $\mathcal{U}^{\star} = \mathcal{U}, \mathcal{R}^{\star} = \mathcal{R}, \mathcal{D}^{\star}(t) = \mathcal{D}(\mathcal{M})$, for any $t \in \mathcal{U}$, and $\mathcal{I}^{\star}(r,t) = \mathcal{I}(r,t)$, for any predicate r distinct from ε, and $\mathcal{I}^{\star}(\varepsilon,t) = \mathcal{D}(t)$, for any $t \in \mathcal{U}$.

FACT 3.2. $\mathcal{M}^{\star} \in \mathbb{CD}_{\varepsilon}$, for any $\mathcal{M} \in \mathbb{VD}$.

LEMMA 3.3 ([1, p. 107]). *For any formula φ not containing ε,*

$$(\mathcal{M},t) \Vdash \varphi[v] \iff (\mathcal{M}^{\star},t) \Vdash \varphi^{\varepsilon}[v].$$

Finally, this allows them to prove the following theorem.

THEOREM 3.4 ([1, Proposition 4.8.2]). *For any formula φ not containing ε, $\mathbb{CD} \Vdash \varphi^\varepsilon \Longrightarrow \mathbb{VD} \Vdash \varphi$.*

The very same construction and the same proof suffice to justify that

FACT 3.5. For any formula φ not containing ε, $\mathbb{CD}_\varepsilon \Vdash \varphi^\varepsilon \Longrightarrow \mathbb{VD} \Vdash \varphi$.

Obviously we have

FACT 3.6. For any formula φ not containing ε, $\mathbb{VD} \Vdash \varphi \Longrightarrow \mathbb{VD}_\varepsilon \Vdash \varphi$.

As a corollary of the above facts and Theorem 2.5 we obtain:

COROLLARY 3.7. For any formula φ not containing ε, the following conditions are equivalent:

(i) $\mathbb{VD} \Vdash \varphi$

(ii) $\mathbb{VD}_\varepsilon \Vdash \varphi$

(iii) $\mathbb{CD}_\varepsilon \Vdash \varphi^\varepsilon$.

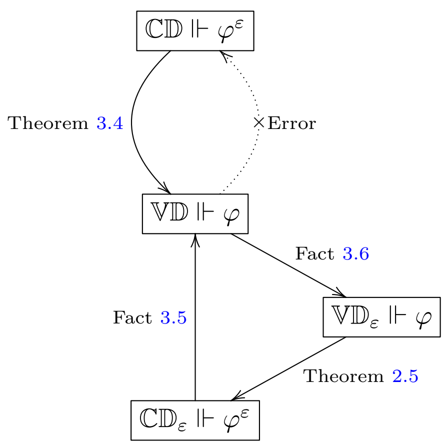

Figure 1. Summary

4. Further results

Corollary 3.7 invites us to asking a natural question: how, if at all, can we 'cut' classes $\mathbb{VD}$, $\mathbb{VD}_\varepsilon$ and $\mathbb{CD}_\varepsilon$ to hold the equivalence? In other words: when $K \Vdash \varphi \iff K \cap \mathbb{VD}_\varepsilon \Vdash \varphi \iff K \cap \mathbb{CD}_\varepsilon \Vdash \varphi^\varepsilon$ holds?

Obviously if $K = \emptyset$, then the equivalence in question is true. But we can do a little better.

We will say that a class of models K is *closed under* $\star$*-operation* (see Definition 3.1), or simply $\star$*-closed*, when for any model $\mathcal{M}$, $\mathcal{M} \in K$ implies $\mathcal{M}^\star \in K$. We will say that K is *closed under adding-new-points-operation*, or *add-closed* for short, when for any $\mathcal{M} = (\mathcal{U}, \mathcal{R}, \mathcal{D}, \mathcal{I}) \in \mathbb{CD}_\varepsilon$, if $\mathcal{M} \in K$, then for some $w \notin \mathcal{U}$, $\mathcal{M}^w \in K$. Finally, we will say that K is *add***-closed* if it is both $\star$- and *add-closed*.

It turns out that operations introduced in Definitions 2.1 and 3.1 provide sufficient conditions for the examined equivalence to hold. Let us decompose the equivalence into conditionals so we can prove the following lemmas.

LEMMA 4.1. *For any formula* φ *not containing* ε *and any* $K \subseteq \mathbb{VD}$*, if* K *is* $\star$*-closed, then* $K \cap \mathbb{CD}_\varepsilon \Vdash \varphi^\varepsilon \Longrightarrow K \cap \mathbb{VD}_\varepsilon \Vdash \varphi$.

PROOF: Suppose $(\mathcal{M}, t) \nVdash \varphi[v]$, for some $\mathcal{M} \in K \cap \mathbb{VD}_\varepsilon$. By Lemma 3.3, $(\mathcal{M}^\star, t) \nVdash \varphi^\varepsilon[v]$. By Fact 3.2, $\mathcal{M}^\star \in \mathbb{CD}_\varepsilon$ and by the assumption that K is $\star$*-closed*, $\mathcal{M}^\star \in K \cap \mathbb{CD}_\varepsilon$. □

It is worth noting that the $\star$-operation does not affect the domain nor the accessibility relation of a model. Therefore if K is a class of models defined by the property of frames[2] on which those models are based, then the implication of Lemma 4.1 holds. Such classes of models, defined by properties of the accessibility relation like reflexivity, transitivity, symmetry etc, are in special interest of logicians, for they give rise to well-behaved and largely explored logical systems.

LEMMA 4.2. *For any formula* φ *not containing* ε *and any* $K \subseteq \mathbb{VD}$*, if* K *is add-closed, then* $K \cap \mathbb{VD}_\varepsilon \Vdash \varphi \Longrightarrow K \cap \mathbb{CD}_\varepsilon \Vdash \varphi^\varepsilon$.

PROOF: Suppose $(\mathcal{M}, t) \nVdash \varphi^\varepsilon[v]$, for some $\mathcal{M} = (\mathcal{U}, \mathcal{R}, \mathcal{D}, \mathcal{I}) \in K \cap \mathbb{CD}_\varepsilon$. By Fact 2.2, $\mathcal{M}^w \in \mathbb{VD}_\varepsilon$ and by Lemma 2.4, $(\mathcal{M}^w, t) \nVdash \varphi$. Moreover, $\mathcal{M}^w \in K$ for some $w \notin \mathcal{U}$, since K is *add–closed*. □

[2]By a *frame* of a model $(\mathcal{U}, \mathcal{R}, \mathcal{D}, \mathcal{I})$ we mean a structure $(\mathcal{U}, \mathcal{R})$.

Lemma 4.3. *For any formula φ not containing ε and any $K \subseteq \mathbb{VD}$, if K is add*-closed, then $K \cap \mathbb{CD}_\varepsilon \Vdash \varphi^\varepsilon \Longrightarrow K \Vdash \varphi$.*

Proof: Suppose $(\mathcal{M}, t) \nVdash \varphi[v]$, for some $\mathcal{M} \in K$. By Lemma 3.3, $(\mathcal{M}^\star, t) \nVdash \varphi^\varepsilon[v]$. By Fact 3.2, $\mathcal{M}^\star \in \mathbb{CD}_\varepsilon$ and by the assumption that K is *add**-closed, $\mathcal{M}^\star \in K \cap \mathbb{CD}_\varepsilon$. $\square$

Let us notice the following trivial facts.

Fact 4.4. For any $K \subseteq \mathbb{VD}$, $K \Vdash \varphi \Longrightarrow K \cap \mathbb{VD}_\varepsilon \Vdash \varphi$.

Fact 4.5. $K \cap \mathbb{VD}_\varepsilon = K_\varepsilon$

The above facts and lemmas entail:

Corollary 4.6. For any formula φ not containing ε and any $K \subseteq \mathbb{VD}$, if K is *add**–closed, then the following conditions are equivalent:

(i) $K \Vdash \varphi$

(ii) $K_\varepsilon \Vdash \varphi$

(iii) $K \cap \mathbb{VD}_\varepsilon \Vdash \varphi$

(iv) $K \cap \mathbb{CD}_\varepsilon \Vdash \varphi^\varepsilon$.

This corollary is a generalization of Corollary 3.7, for if we take $K = \mathbb{VD}$, the assumption of Corollary 4.6 becomes true and we get Corollary 3.7.

$$\begin{tikzcd}
\boxed{K \Vdash \varphi} \arrow[dr, "\text{Fact 4.4}"] & & \\
& \boxed{K \cap \mathbb{VD}_\varepsilon \Vdash \varphi} \arrow[r, leftrightarrow, "\text{Fact 4.5}"'] \arrow[dl, "\text{Lemma 4.2}"] & \boxed{K_\varepsilon \Vdash \varphi} \\
\boxed{K \cap \mathbb{CD}_\varepsilon \Vdash \varphi^\varepsilon} \arrow[uu, "\text{Lemma 4.3}"] & &
\end{tikzcd}$$

Figure 2. Summary

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CATEGORICAL DUALITIES FOR SOME TWO CATEGORIES OF LATTICES: AN EXTENDED ABSTRACT

Abstract

The categorical dualities presented are: (first) for the category of bi-algebraic lattices that belong to the variety generated by the smallest non-modular lattice with complete (0,1)-lattice homomorphisms as morphisms, and (second) for the category of non-trivial (0,1)-lattices belonging to the same variety with (0,1)-lattice homomorphisms as morphisms. Although the two categories coincide on their finite objects, the presented dualities essentially differ mostly but not only by the fact that the duality for the second category uses topology. Using the presented dualities and some known in the literature results we prove that the Q-lattice of any non-trivial variety of (0,1)-lattices is either a 2-element chain or is uncountable and non-distributive.

Keywords: Categorical duality, bi-algebraic lattice, bounded lattice, quasivariety lattice.

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1. Definitions and two key lemmas

Obtaining categorical duality results for certain categories of structures has a long history. The classical examples are the Stone and Priestley dualities

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for bounded distributive lattices and their many extensions for categories of algebras associated with non-classical logics the algebraic parts of which contain distributive lattices. In this note, we present two results of this nature. Each of them goes one step beyond distributivity. The variety of bounded lattices generated by the smallest non-modular lattice is one of the two minimal varieties that extended the variety of bounded distributive latices.

A bi-algebraic lattice is a non-trivial lattice that is algebraic and the lattice dual (by reversing the lattice order) is also algebraic. A (0,1)-lattice is a lattice in which 0 and 1 are the smallest and greatest elements in the lattice and they are included as constants to the signature of the lattice. Lattices of this type are called bounded lattices.

A Q-lattice is the lattice whose elements are the quasivarieties contained in a quasivariety. The lattice order of a Q-lattice is the inclusion. A quasivariety is a class of structures that is closed under the operators $\mathbf{S}$ of forming isomorphic substructures, Cartesian products $\mathbf{P}$, and ultraproducts. A variety is a quasivariety that additionally is closed under the operator of forming homomorphic images.

The lattices N_5 and M_3 each of which has 5 elements are the smallest non-modular and modular but non-distributive lattices, respectively. They are regarded as $(0,1)$-lattices. It is known that the variety of bounded lattices generated by N_5 coincides with $\mathbf{SP}(N_5)$.

For a partially ordered set $\langle X, \leq \rangle$ and subsets Y, Z of X, we write $Y \ll Z$ to mean that for every $y \in Y$ there exists $z \in Z$ such that $y \leq z$.

For a lattice L, an element $a \in L$, and a finite subset X of L with a being below the lattice join in L of the elements of X, it is said that X is a *join cover* of a. If a is not below any element of X, it is said that X is a *non-trivial* join cover of X. A non-trivial join cover X of a in L is said to be *minimal* if, for every non-trivial join cover Y of a in L with $Y \ll X$, it follows that $X \subseteq Y$.

For a fuller account of concepts used in our note we refer to [9] and [11].

The four equations displayed below are valid in N_5 and so they are valid in every lattice belonging to $\mathbf{SP}(N_5)$. They contain the key information for what we need for the functors establishing the presented dualities to be well defined on the objects of the considered categories. What we need is stated in Lemmas 1.1 and 1.2.

The lattice equation D_2 is a particular case of the family of lattice equations D_n, $n \geqslant 2$, which was introduced in [12]. Lattices which satisfy D_n

are called *n-distributive*. In the presence of D_2 the equation C is equivalent to the equation τ'_{21}. The equation τ'_{21} belongs to the family of lattice equations τ'_{nk} constructed in [14].

$$\begin{aligned}
&\mathrm{C}\colon\ x \wedge (y_0 \vee y_1) \wedge (z_0 \vee z_1) = \bigvee_{i<2} \big[x \wedge y_i \wedge (z_0 \vee z_1)\big] \vee \\
&\qquad \vee \bigvee_{i<2} \big[x \wedge z_i \wedge (y_0 \vee y_1)\big] \ \vee\ \vee \bigvee_{i<2} \big[x \wedge \big((y_0 \wedge z_i) \vee (y_1 \wedge z_{1-i})\big)\big];\\
&\mathrm{D}_2\colon\ x \wedge (y_0 \vee y_1 \vee y_2) = \bigvee_{i\leqslant 2} \big[x \wedge \bigvee_{j\neq i} y_j\big];\\
&\mathrm{N}_5^0\colon\ x \wedge (y_0 \vee y_1) = \bigvee_{i<2} \big[x \wedge \big((y_i \wedge x) \vee y_{1-i}\big)\big];\\
&\mathrm{N}_5^1\colon\ x \wedge \big[\big(y_0 \wedge (z_0 \vee z_1)\big) \vee y_1\big] = \big[x \wedge y_0 \wedge (z_0 \vee z_1)\big] \vee \big[x \wedge y_1\big] \vee \\
&\qquad \vee \bigvee_{i<2} \big[x \wedge \big((y_0 \wedge z_i) \vee y_1\big)\big].
\end{aligned}$$

LEMMA 1.1. *For a dually algebraic lattice L, the following conditions are equivalent.*

i) $L \in \mathbf{SP}(N_5)$.

ii) *For every join-irreducible element x of L that is not join-prime, there is a unique minimal non-trivial join cover $\{a, b\}$ of x such that both a and b are join-irreducible and join-prime and, moreover, they satisfy either $a < x$ and $\{x, b\}$ is an antichain or $b < x$ and $\{x, a\}$ is an antichain.*

PROOF (SKETCH): i) implies ii): The equations C and D_2 or, equivalently, τ'_{21} and D_2, by Theorems 3.2 and 3.4 of [14], together imply that every join-irreducible x of L has a unique minimal non-trivial join cover $\{a, b\}$. By minimality of $\{a, b\}$, a and b are join-irreducible. The equations N_5^0 and N_5^1 justify that the unique pair has the remaining properties as stated in ii).

ii) implies i): This implication is an easy consequence of the main result of [3]. It can also be proved without the result of [3] but with some effort. □

In every bi-algebraic lattice, every element is completely join-irreducible or is the lattice join of all completely join-irreducible elements that are below. Moreover, completely join-irreducible elements are compact. Lemma 1.2 stated below follows from Lemma 1.1.

LEMMA 1.2. *For a bi-algebraic lattice L, the following conditions are equivalent.*

i) $L \in \mathbf{SP}(N_5)$.

ii) *For every completely join-irreducible element x of L that is not join-prime, there is a unique minimal non-trivial join cover $\{a, b\}$ of x such that both a and b are completely join-irreducible and join-prime and, moreover, they satisfy either $a < x$ and $\{x, b\}$ is an antichain or $b < x$ and $\{x, a\}$ is an antichain.*

Lemma 1.2 is the key lemma in the construction of the functor $\mathsf{N}\colon \mathbb{B}_5 \to \mathbb{N}_5$ on the objects of $\mathbb{B}_5$ and, consequently, the functor $\mathsf{B}\colon \mathbb{N}_5 \to \mathbb{B}_5$ on the objects of $\mathbb{N}_5$ but after having (discovering) the precise definition of the category $\mathbb{N}_5$. Lemma 1.2 says how to define the function $f\colon Y \to X^2$ which is the most important ingredient in the definition of N_5-space (an object of $\mathbb{N}_5$) that is assigned to L (an object of $\mathbb{B}_5$).

Lemma 1.1 is the key lemma in the construction of the functor $\mathsf{T}\colon \mathbb{L}_5 \to \mathbb{T}_5$ on the objects of $\mathbb{L}_5$ and, consequently, the functor $\mathsf{L}\colon \mathbb{T}_5 \to \mathbb{L}_5$ on the objects of $\mathbb{T}_5$ and again after having (discovering) the precise definition of the category $\mathbb{T}_5$. Lemma 1.1 says how to define the function $f\colon Y(L) \to X(L)^2$ on the spectral N_5-space (an object of $\mathbb{T}_5$) assigned to L (an object of $\mathbb{L}_5$). In defining f, we use the known facts which say that any lattice L embeds into the lattice $F(L)$ of filters on L, $F(L)$ is dually algebraic, and that L and $F(L)$ satisfy the same lattice equations. A detailed description of the correctness of the presented dualities depend on the proof type context.

2. Categories $\mathbb{N}_5$ and $\mathbb{B}_5$

DEFINITION 2.1. A structure $\mathbb{S} = \langle X, Y, \leq, f\rangle$ is an N_5-*space*, if

(s1) $X \cup Y \neq \varnothing$ and $X \cap Y = \varnothing$; moreover, if $Y \neq \varnothing$, then $X \neq \varnothing$;

(s2) $\leq$ is a partial order on $X \cup Y$;

(s3) $f\colon Y \to X^2$ is a function and for all $y \in Y$ with $f(y) = (a, b)$, the following conditions hold:

(a) $a \le y$ and $\{a, b\}$, $\{y, b\}$ are antichains;

(b) if $a, b \le z$ for some $z \in X \cup Y$ then $y \le z$;

(c) if $z \le y$ for some $z \in X \cup Y$ then either $z \le a$ or $z \le b$, or $z \in Y$ and $\{u, v\} \ll \{a, b\}$ where $f(z) = (u, v)$.

Definition 2.2. Let $\mathbb{S} = \langle X, Y, \le, f\rangle$ and $\mathbb{S}' = \langle X', Y', \le', f'\rangle$ be N_5-spaces. A mapping $\varphi\colon \mathbb{S} \to \mathbb{S}'$ is an N_5-*morphism*, if the following conditions hold:

(m1) φ maps $X \cup Y$ into $X' \cup Y' \cup 2X'$, where $2X'$ denotes the collection of 2-element antichains in $\langle X', \le'\rangle$

(m2) if $u, v \in X \cup Y$ are such that $\varphi(u), \varphi(v) \in X' \cup Y'$ and $u \le v$ then $\varphi(u) \le \varphi(v)$;

(m3) for all $x \in X$, $\varphi(x) \in X'$;

(m4) for all $y \in Y$ with $f(y) = (a, b)$, the following holds:

(a) if $\varphi(y) \in X'$, then either $\varphi(y) = \varphi(a)$ or $\varphi(y) \le \varphi(b)$;

(b) if $\varphi(y) \in Y'$, then $f'\big(\varphi(y)\big) = \big(\varphi(a), \varphi(b)\big)$;

(c) if $\varphi(y) \in 2X'$, then $\varphi(y) = \big\{\varphi(a), \varphi(b)\big\}$ and, for every $z \in X \cup Y$ with $y \le z$, one has $\varphi(a), \varphi(b) \le' \varphi(z)$ if $\varphi(z) \in X' \cup Y'$, and $\{\varphi(a), \varphi(b)\} \ll \varphi(z)$ if $\varphi(z) \in 2X'$.

Proposition 2.3. Let $\varphi_0\colon \mathbb{S}_0 \to \mathbb{S}_1$, $\varphi_1\colon \mathbb{S}_1 \to \mathbb{S}_2$ be N_5-morphisms. The composition $\varphi_0 \circ \varphi_1\colon \mathbb{S}_0 \to \mathbb{S}_2$ of φ_0 and φ_1 in $\mathbb{N}_5$ is as follows, where $z \in X_0 \cup Y_0$.

(c1) If $\varphi_0(z) \in X_1 \cup Y_1$, then $\varphi_0 \circ \varphi_1(z) = \varphi_1\varphi_0(z)$.

(c2) If $\varphi_0(z) \in 2X_1$, then

$$\varphi_0 \circ \varphi_1(z) = \begin{cases} \varphi_1\varphi_0(u), & \text{if } \varphi_1\varphi_0(v) \le \varphi_1\varphi_0(u);\\ \varphi_1\varphi_0(v), & \text{if } \varphi_1\varphi_0(u) \le \varphi_1\varphi_0(v);\\ \{\varphi_1\varphi_0(u), \varphi_1\varphi_0(v)\}, & \text{if } \{\varphi_1\varphi_0(u), \varphi_1\varphi_0(v)\} \in 2X_2, \end{cases}$$

where $f(z) = (u, v)$ in $\mathbb{S}_0$.

The two categories $\mathbb{N}_5$ and $\mathbb{B}_5$ are as follows. Objects in $\mathbb{N}_5$ are N_5-spaces; morphisms are N_5-morphisms. Objects in $\mathbb{B}_5$ are bi-algebraic lattices belonging to the variety $\mathbf{SP}(N_5)$; morphisms are complete $(0, 1)$-lattice homomorphisms. In this section, we construct two contravariant functors $\mathsf{B}\colon \mathbb{N}_5 \to \mathbb{B}_5$ and $\mathsf{N}\colon \mathbb{B}_5 \to \mathbb{N}_5$ which establish duality between $\mathbb{N}_5$ and $\mathbb{B}_5$.

Definition 2.4. Let $\mathbb{S} = \langle X, Y, \leq, f\rangle$ be an N_5-space. A subset $I \subseteq X \cup Y$ is an *ideal of* N_5*-space* $\mathbb{S}$ if I is a lower cone with respect to $\leq$ and has the following property:

$$\text{if } f(y) = (a, b) \text{ in } \mathbb{S} \text{ and } a, b \in I \text{ then } y \in I.$$

The set of all ideals of $\mathbb{S}$ forms a complete $(0,1)$-lattice with the lattice operations given by:

$$\textstyle\bigwedge_{i\in I} A_i = \bigcap_{i\in I} A_i;$$

$$\textstyle\bigvee_{i\in I} A_i = \bigcup_{i\in I} A_i \cup \{y \in Y \mid y = f(a, b) \text{ and } a, b \in \bigcup_{i\in I} X \cap A_i\}.$$

The functor $\mathsf{B}\colon \mathbb{N}_5 \to \mathbb{B}_5$ is defined as follows, where $\mathbb{S}$ and $\mathbb{S}'$ are N_5-spaces and $\varphi\colon \mathbb{S} \to \mathbb{S}'$ is an N_5-morphism:

$\mathsf{B}(\mathbb{S})$ is the complete $(0,1)$-lattice defined above;

$\mathsf{B}(\varphi)\colon \mathsf{B}(\mathbb{S}') \to \mathsf{B}(\mathbb{S})$ is defined by $\mathsf{B}(\varphi)(Z') = \varphi^{-1}(Z')$.

Proposition 2.5. The following statements hold.

(1) $\mathsf{B}(\mathbb{S})$ is a bi-algebraic lattice that belongs to $\mathbf{SP}(N_5)$.

(2) $\mathsf{B}(\varphi)\colon \mathsf{B}(\mathbb{S}') \to \mathsf{B}(\mathbb{S})$ is a complete $(0,1)$-lattice homomorphism.

Corollary 2.6. $\mathsf{B}\colon \mathbb{N}_5 \to \mathbb{B}_5$ is a contravariant functor.

For a lattice $L \in \mathbb{B}_5$, let

$$\mathsf{N}(L) = \langle X, Y, \leq, f\rangle,$$

where X is the set of all completely join-irreducible elements of L which are join-prime, Y is the set of all completely join-irreducible elements of L which are not join-prime, $\leq$ is the lattice order in L, and $f\colon Y \to X^2$ is a function that is defined as follows: $f(y) = (a, b)$, where $\{a, b\}$ is the unique pair of elements of X which, by Lemma 1.2, exists for y and, by choice, $a < y$.

For $L, L' \in \mathbb{B}_5$ and a complete lattice $(0,1)$-lattice homomorphism $g\colon L \to L'$, consider the map:

$$\beta_g\colon L' \to L, \quad \beta_g\colon a \mapsto \bigwedge\{b \in L' \mid g(b) = a\}.$$

We note that $g(\beta_g(a)) = a$ for all a of L'. We also note that if a is completely join-irreducible in L', then so is $\beta_g(a)$ but in L.

For a morphism $g\colon L \to L'$ in $\mathbb{B}_5$, we define $\mathsf{N}(g)\colon \mathsf{N}(L') \to \mathsf{N}(L)$ as follows:

$$\mathsf{N}(g)(y) = \begin{cases} \beta_g(y) & \text{if } \beta_g(y) \in X \cup Y; \\ \{\beta_g(a), \beta_g(b)\} & \text{if } \beta_g(y) \in 2X \text{ and } f(y) = (a,b) \end{cases}$$

$\mathsf{N}(L)$ and $\mathsf{N}(g)$ above define the second contravariant functor $\mathsf{N}\colon \mathbb{B}_5 \to \mathbb{N}_5$ justification of which follows from the proposition below.

Proposition 2.7. The following statements hold.

(1) $\mathsf{N}(L) \in \mathbb{N}_5$.

(2) $\mathsf{N}(g)\colon \mathsf{N}(L) \to \mathsf{N}(M)$ is an N_5-morphism.

Corollary 2.8. $\mathsf{N}\colon \mathbb{B}_5 \to \mathbb{N}_5$ is a contravariant functor.

Let $1_{\mathbb{N}_5}$ and $1_{\mathbb{B}_5}$ denote the identity functors within the categories $\mathbb{N}_5$ and $\mathbb{B}_5$, respectively.

Proposition 2.9. The pair NB and $1_{\mathbb{N}_5}$ as well as the pair BN and $1_{\mathbb{B}_5}$ are isomorphic functors.

Corollaries 2.6, 2.8 and Proposition 2.9 justify the following theorem.

Theorem 2.10. *The categories $\mathbb{N}_5$ and $\mathbb{B}_5$ are dually equivalent.*

The following corollary of Theorem 2.10 and the properties of the functors used show an advantage of dualities over algebraic approach if one wants to establish a technically demanding in proof result.

Corollary 2.11. The following statements hold.

(1) N_5-morphisms in $\mathbb{N}_5$ which are onto correspond by duality to one-to-one homomorphisms in $\mathbb{B}_5$ and *vice versa*.

(2) N_5-morphisms in $\mathbb{N}_5$ which are one-to-one correspond by duality to onto homomorphisms in $\mathbb{B}_5$ and *vice versa*.

(3) Disjoint unions of spaces (coproducts in $\mathbb{N}_5$) correspond by duality to Cartesian products in $\mathbb{B}_5$ and *vice versa*.

Let $(\mathbb{N}_5)_{fin}$ and $(\mathbb{B}_5)_{fin}$ denote the full subcategories in $\mathbb{N}_5$ and $\mathbb{B}_5$, respectively, whose objects are finite. From our construction of functors B and N and Theorem 2.10, we obtain

Corollary 2.12. The categories $(\mathbb{N}_5)_{fin}$ and $(\mathbb{B}_5)_{fin}$ are dually equivalent.

3. Categories $\mathbb{T}_5$ and $\mathbb{L}_5$

The two categories in this section $\mathbb{T}_5$ and $\mathbb{L}_5$ are as follows. Objects in $\mathbb{T}_5$ are spectral N_5-spaces; morphisms are spectral N_5-morphisms; see Definitions 3.1 and 3.4 provided below. Objects in $\mathbb{L}_5$ are bounded lattices belonging to the variety $\mathbf{SP}(N_5)$; morphisms are $(0,1)$-lattice homomorphisms. In this section, we construct two contravariant functors $\mathsf{L}\colon \mathbb{T}_5 \to \mathbb{L}_5$ and $\mathsf{T}\colon \mathbb{L}_5 \to \mathbb{T}_5$ which establish duality between $\mathbb{T}_5$ and $\mathbb{L}_5$.

We will consider pairs $(\mathbb{S}, \mathcal{T})$ such that $\mathbb{S} = \langle X, Y, \leq, f\rangle$ is a N_5-space and $\mathcal{T}$ is a topology on $X \cup Y$.

A subset A of $X \cup Y$ is said to be N_5-*compact* in $(\mathbb{S}, \mathcal{T})$ if the following conditions are satisfied:

(i) $A \cap X$ is compact in $(X, \{X \cap Z \mid Z \in \mathcal{T}\})$;

(ii) for every family $\{A_i \mid i \in I\}$ of open sets in $(X \cup Y, \mathcal{T})$, from $A \subseteq \bigcup_{i\in I} A_i$ it follows that $A \subseteq \bigcup_{i\in J} A_i \cup \{y \in Y \mid f(y) = (a,b) \text{ and } a, b \in \bigcup_{i\in J} X \cap A_i\}$ for some finite subset J of I.

We say that a subset A of $X \cup Y$ is f-*closed* in $\mathbb{S}$ if it is an ideal in $\mathbb{S}$; see Definition 2.4.

On the elements of every topological T_0-space with topology $\mathcal{T}$, there is a partial order $\leq_{\mathcal{T}}$ defined as follows: $x \leq_{\mathcal{T}} y$ iff every open set of $\mathcal{T}$ containing x contains y.

Definition 3.1. A pair $(\mathbb{S}, \mathcal{T})$ is said to be a *spectral* N_5-*space* if the following conditions are fulfilled:

(1) $\mathbb{S}$ is a N_5-space, $\mathcal{T}$ is a T_0 topology on $X \cup Y$ the restriction to X of which makes X to be a spectral space, and $X \cup Y$ is N_5-compact in $(\mathbb{S}, \mathcal{T})$.

(2) $\leq = \leq_{\mathcal{T}}^{-1}$.

(3) The collection of all sets that are f-closed in $\mathbb{S}$, open in $(X \cup Y, \mathcal{F})$, and N_5-compact in $(\mathbb{S}, \mathcal{F})$ forms a basis for $(X \cup Y, \mathcal{T})$ that is closed under finite set intersections.

(4) For all sets A and B that are f-closed in $\mathbb{S}$, open in $(X \cup Y, \mathcal{F})$, and N_5-compact in $(\mathbb{S}, \mathcal{F})$, $A \cup B \cup \{y \in Y \mid f(y) = (a, b) \text{ and } a, b \in (A \cup B) \cap X\}$ is open in $(X \cup Y, \mathcal{T})$;

(5) $(\mathbb{S}, \mathcal{T})$ does not have a proper uN_5-extension; see Definition 3.10 provided below.

Remark 3.2. One can show that the set in the conclusion of (4) in Definition 3.1 is N_5-compact in $(\mathbb{S}, \mathcal{T})$ and, obviously, is f-closed in $\mathbb{S}$.

Remark 3.3. The set of all subsets of $X \cup Y$ that are f-closed in $\mathbb{S}$, open in $(X \cup Y, \mathcal{T})$, and N_5-compact in $(\mathbb{S}, \mathcal{T})$ forms a $(0, 1)$-lattice with lattice operations defined by:

$$A \wedge B = A \cap B;$$

$$A \vee B = A \cup B \cup \{y \in Y \mid f(y) = (a, b) \text{ and } a, b \in X \cap (A \cup B)\}.$$

Definition 3.4. For spectral N_5-spaces $(\mathbb{S}, \mathcal{T})$ and $(\mathbb{S}', \mathcal{T}')$ and a map $\varphi \colon \mathbb{S} \to \mathbb{S}'$ that is N_5-morphism, we say that φ is a *spectral N_5-morphism* if, for every N_5-compact open set A in $(\mathbb{S}', \mathcal{T}')$, the set $\varphi^{-1}(A)$ is N_5-compact open in $(\mathbb{S}, \mathcal{T})$.

The functor $\mathsf{L} \colon \mathbb{T}_5 \to \mathbb{L}_5$ is defined as follows, where $(\mathbb{S}, \mathcal{T})$, $(\mathbb{S}', \mathcal{T}')$ are objects of $\mathbb{T}_5$, and $\varphi \colon \mathbb{S} \to \mathbb{S}'$ is a spectral N_5-morphism:

$\mathsf{L}(\mathbb{S}, \mathcal{T})$ is the $(0, 1)$-lattice defined above;

$\mathsf{L}(\varphi) \colon \mathsf{L}(\mathbb{S}', \mathcal{T}') \to \mathsf{L}(\mathbb{S}, \mathcal{T})$ is given by $\mathsf{L}(\varphi)(A') = \varphi^{-1}(A')$.

Proposition 3.5. The following statements hold.

(1) $\mathsf{L}(\mathbb{S}, \mathcal{T})$ forms a lattice which is a $(0, 1)$-sublattice of the ideal lattice of $\mathbb{S}$ $(= \mathsf{B}(\mathbb{S}))$ and so $\mathsf{L}(\mathbb{S}, \mathcal{T})$ belongs to $\mathbf{SP}(N_5)$.

(2) $\mathsf{L}(\varphi) \colon \mathsf{L}(\mathbb{S}', \mathcal{T}') \to \mathsf{L}(\mathbb{S}, \mathcal{T})$ is a $(0, 1)$-lattice homomorphism.

Remark 3.6. Proof of the above proposition does not use the condition (5) of Definition 3.1.

Corollary 3.7. $\mathsf{L}\colon \mathbb{T}_5 \to \mathbb{L}_5$ is a contravariant functor.

In order to construct a contravariant functor $\mathsf{T} : \mathbb{L}_5 \to \mathbb{T}_5$, we now consider, for a $(0,1)$-lattice $L \in \mathbf{SP}(N_5)$, the lattice $F(L)$ of filters on L with the inverse inclusion as the lattice order. The lattices L and $F(L)$ satisfy the same lattice equations. Moreover, $F(L)$ is dually algebraic. This implies that every element of $F(L)$ is join-irreducible (in fact, completely join-irreducible) or is the lattice join in $F(L)$ of the join-irreducible (in fact, completely join-irreducible) elements below it. We say that a filter on L is *join-irreducible* or *join-prime* if the filter regarded as an element of $F(L)$ is join-irreducible or join-prime, respectively.

Let $X(L)$ denote the set of join-prime filters of L and let $Y(L)$ denote the set of join-irreducible filters of L which are not join-prime. Then $S(L) = X(L) \cup Y(L)$ consists of all join-irreducible filters of L. We put $\mathbb{S}(L) = \langle X(L), Y(L), \supseteq, f\rangle$, where $f\colon Y(L) \to X(L)^2$ is a function which is defined as follows: $f(F) = (G, H)$, where $\{G, H\}$ is the unique pair of elements of $X(L)$ which, by Lemma 1.1, exists for F and, by choice, $F \subset G$. One can show that $\mathbb{S}(L)$ is an N_5-space and, consequently, the ideal lattice of $\mathbb{S}(L)(= \mathsf{B}(L))$ is bi-algebraic and belongs to $\mathbf{SP}(N_5)$.

We now enhance $\mathbb{S}(L)$ by a topology and denote it by $\mathcal{T}(L)$. As a consequence, we obtain the pair $(\mathbb{S}(L), \mathcal{T}(L))$.

For $x \in L$, let $I(x) = \{F \in S(L) \mid x \in F\}$ and for $M \subseteq L$, let $I(M) = \bigcup_{x\in M} I(x)$.

Definition 3.8. The open sets of $\mathcal{T}(L)$ are exactly sets of the form $I(M)$, where $M \subseteq L$.

Remark 3.9. Notice that the collection of all sets $I(x)$, $x \in L$, is a multiplicative base for $\mathcal{T}(L)$. This is so because $I(x) \cap I(y) = I(x \wedge y)$. Notice also that $I(x)$ is f-closed in $\mathbb{S}(L)$ and that $\leq_{\mathcal{T}(L)}$ coincides with $\subseteq$ because $\mathcal{T}(L)$ is T_0. Moreover, one can show that the family $\{I(x) \mid x \in L\}$ is exactly the collection of all sets that are f-closed in $\mathbb{S}$, open in $(S(L), \mathcal{T}(L))$, and N_5-compact in $(\mathbb{S}(L), \mathcal{T}(L))$. Also, one can show that $X(L)$ with the topology $\mathcal{T}(L)$ restricted to $X(L)$ is a spectral space. And also, one can show that $(\mathbb{S}(L), \mathcal{T}(L))$ fulfills the condition (5) of Definition 3.1 according to Definition 3.10 that is now given below.

For N_5-spaces $\mathbb{S} = \langle X, Y, \leq, f\rangle$ and $\mathbb{S}' = \langle X', Y', \leq', f'\rangle$ with $X \cup Y \subseteq X' \cup Y'$ and $2X \subseteq 2X'$, we say that $\mathbb{S}$ is an N_5-*subspace* of $\mathbb{S}'$ if the identity map from $X \cup Y \cup 2X$ to $X' \cup Y' \cup 2X'$ is an N_5-morphism.

Definition 3.10. For pairs $(\mathbb{S}, \mathcal{T})$ and $(\mathbb{S}', \mathcal{T}')$ satisfying the conditions (1)–(4) of Definition 3.1 with $\mathbb{S}$ being an N_5-subspace of $\mathbb{S}'$ and $(X \cup Y, \mathcal{T})$ a topological subspace of $(X' \cup Y', \mathcal{T}')$, we say that $(\mathbb{S}', \mathcal{T}')$ is a uN_5-*extension* of $(\mathbb{S}, \mathcal{T})$ if, for every A that is f-closed in $\mathbb{S}'$, N_5-compact in $(\mathbb{S}', \mathcal{T}')$, and open in $(X' \cup Y', \mathcal{T}')$, the following holds:

$$A = \bigcup \{B \in \mathcal{T}' \mid B \cap (X \cup Y) = A\}.$$

Remark 3.11. The notion of uN_5-extension originates from the concept of u-extension considered in a general topological context in [6], see also [7] and [8].

The functor $\mathsf{T}\colon \mathbb{L}_5 \to \mathbb{T}_5$ on objects of $\mathbb{L}_5$ is defined by $\mathsf{T}(L) = (\mathbb{S}(L), \mathcal{T}(L))$ and $\mathsf{T}(g)\colon \mathsf{T}(L') \to \mathsf{T}(L)$ on morphisms $g\colon L \to L'$ of $\mathbb{L}_5$ by

$$\mathsf{T}(g)(F) = \begin{cases} g^{-1}(F), & \text{if } g^{-1}(F) \in X(L) \cup Y(L); \\ \{g^{-1}(G), g^{-1}(H)\}, & \text{if } g^{-1}(F) \in 2X(L) \text{ and } f(F) = (G, H). \end{cases}$$

Proposition 3.12. The following statements hold.

(1) $\mathsf{T}(L)$ is spectral N_5-space.

(2) $\mathsf{T}(g)\colon \mathsf{T}(L') \to \mathsf{T}(L)$ is a spectral N_5-morphism.

Corollary 3.13. $\mathsf{T}\colon \mathbb{L}_5 \to \mathbb{T}_5$ is a contravariant functor.

For $(\mathbb{S}, \mathcal{T}) \in \mathbb{T}_5$ and $L \in \mathbb{L}_5$, we define

$$\tau_{(\mathbb{S},\mathcal{T})}\colon (\mathbb{S}, \mathcal{T}) \to \mathsf{T}(\mathsf{L}(\mathbb{S}, \mathcal{T})) \text{ by } \tau_{(\mathbb{S},\mathcal{T})}(x) = \{A \in \mathsf{L}(\mathbb{S}, \mathcal{T}) \mid x \in A\};$$

$$\rho_L\colon L \to \mathsf{L}(\mathsf{T}(L)) \text{ by } \rho_L(x) = \{F \in \mathbb{S}(L) \mid x \in F\}.$$

Proposition 3.14. The following statements hold.

(1) $\tau_{(\mathbb{S},\mathcal{T})}$ is an N_5-isomorphism on the N_5-space part of $(\mathbb{S}, \mathcal{T})$ and a homeomorphism on the topological part of $(\mathbb{S}, \mathcal{T})$. Moreover, for every morphism $\varphi\colon (\mathbb{S}, \mathcal{T}) \to (\mathbb{S}', \mathcal{T}')$ in $\mathbb{T}_5$, $\mathsf{TL}(\varphi) \circ \tau_{(\mathbb{S},\mathcal{T})} = \tau_{(\mathbb{S}',\mathcal{T}')} \circ \varphi$.

(2) ρ_L is a $(0,1)$-lattice isomorphism. Moreover, for every morphism $f\colon L \to L'$ in $\mathbb{L}_5$, $\mathsf{LT}(f) \circ \rho_L = \rho_{L'} \circ f$.

Remark 3.15. We are now ready to explain the role of the condition (5) of Definition 3.1. In the proof of (1) of Proposition 3.14, it is established first that $\tau_{(\mathbb{S},\mathcal{T})}$ is an embedding in the N_5-space and topological space sense. Next, it is established that $\mathsf{T}(\mathsf{L}(\mathbb{S},\mathcal{T}))$ is a uN_5-extension of the image of $(\mathbb{S},\mathcal{T})$ by $\tau_{(\mathbb{S},\mathcal{T})}$. This, by the condition (5) of Definition 3.1 implies that that $\tau_{(\mathbb{S},\mathcal{T})}$ is surjective.

Corollaries 3.7, 3.13 and Proposition 3.14 justify the following theorem.

THEOREM 3.16. *The categories $\mathbb{L}_5$ and $\mathbb{T}_5$ are dually equivalent.*

COROLLARY 3.17. The categories $(\mathbb{L}_5)_{fin}$ and $(\mathbb{T}_5)_{fin}$ are dually equivalent.

Remark 3.18. Under the assumption that $Y = \varnothing$ in all spectral N_5-spaces, we obtain the category of spectral spaces with spectral morphisms which, as was proved by M. H. Stone in [16], is dually equivalent to the category of bounded distributive lattices with $(0,1)$-lattice homomorphism as morphisms.

4. The Q-lattice of a non-trivial variety of bounded lattices

Let $\mathbb{P}_2$ denote the category whose objects are partially ordered sets with two distinguished constants and morphisms are mappings that preserve partial orders and the distinguished constants.

Theorem 1.5 of [4] says that the category $\mathbb{P}_2$ is universal. An inspection of the proof of this result presented in [4] shows more. It shows that $\mathbb{P}_2$ is finite-to-finite universal. This means that there is a faithful and full functor from the category of undirected graphs with all compatible maps as morphisms to the category $\mathbb{P}_2$ and has the property that it assigns a finite object of $\mathbb{P}_2$ to every finite graph. This in turn means that in the category $\mathbb{P}_2$ there exists a family of finite objects $A_i = \langle X_i, \leq_i, a_i, b_i \rangle$, $i < \omega$, which has the property:

$(*)$ For $i, j < \omega$, there is a morphism of $\mathbb{P}_2$ between A_i and A_j iff $i = j$.

For each $i < \omega$, let y_i be an element not belonging to X_i, and let $A_i^+ = \langle X_i, \{y_i\}, \leq_i^+, f_i \rangle$, where $\leq_i^+ = \leq_i \cup \{(a_i, y_i), (y_i, y_i)\}$ and $f_i(y_i) = (a_i, b_i)$. Each A_i^+ is a finite N_5-space. Corollary 2.11 and the property $(*)$ imply the following: For $I, J \subseteq \omega$, $\mathbf{SP}(\mathsf{F}(A_i^+) \mid i \in I) = \mathbf{SP}(\mathsf{F}(A_i^+) \mid i \in J)$ iff $I = J$. Thus the Q-lattice of the variety of bounded lattices generated by N_5 is uncountable. Without much effort, one can construct a finite N_5-space $\mathbb{S}$ such that the quasivariety generated by $\mathsf{B}(\mathbb{S})$ is a join-irreducible but not join-prime element in the Q-lattice. This means that the Q-lattice of the variety $\mathbf{SP}(N_5)$ is not distributive. On the other hand, by Corollary 1.5 of [2], we know that the variety of bounded lattices generated by M_3 is uncountable and non-distributive. The two lattices N_5 and M_3 are the only lattices which separate lattices from those which are distributive. As the Q-lattice of the variety of bounded distributive lattice is a 2-element chain, the result announced in the abstract is true: The Q-lattice of any nontrivial variety of bounded lattices is either a 2-element chain or is uncountable and non-distributive.

5. Concluding remarks

Our first duality is an extension of the well known due to G. Birkhoff duality for distributive bi-algebraic lattices (assume that $Y = \varnothing$ in the definition of N_5-space). Our second duality is an extension of the Stone topological duality for bounded distributive lattices. The categories of duals of the Stone and the well-known Priestley [15] duality for bounded distributive lattices are equivalent (see [5]). Our work confirms (see also [13]) that a successful attempt of having topological dualities in the categorical (complete) sense for bounded lattices should be focused only on a variety that is generated by a finite lattice and the outcome will be in the style proposed by M. H. Stone in [16]. The key concept in searching for them will be the concept of minimal join cover refinement property and the navigating result will be Theorem 3.4 of [14]. Our original motivation for having the first duality was the open problem independently raised by G. Birkhoff and A. I. Maltsev which asks for a description of the Q-lattices (see [11] or the survey article [1]). Based on our experience, we know that having a good duality helps in contributing to this open problem. However, we do not know what is the real lattice status of the Q-lattice of the variety of bounded lattices generated by N_5. For example, does this Q-lattice satisfy any non-trivial lattice equation?

A result of [10] states that a variety of bounded lattices is universal iff it contains a non-distributive simple lattice. Moreover, it states that the variety is finite-to-finite universal iff the simple lattice is finite. As M_3 is a simple lattice that is not distributive, the variety of bounded lattices generated by M_3 is finite-to-finite universal. The variety of bounded lattices generated by N_5 is not universal for N_5 is not a simple lattice. Our original motivation of having the second duality was to know an answer to the following question: Is the variety of bounded lattices generated by N_5 finite-to-finite universal relative to the variety of bounded distributive lattices? The *relative* means that $(0, 1)$-lattice homomorphisms to bounded distributive lattices are disregarded in the successful construction of a functor from the category of undirected graphs to the category of bounded lattices generated by N_5. We do not know an answer to this well coined by literature question.

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UNIFICATION AND FINITE MODEL PROPERTY FOR LINEAR STEP-LIKE TEMPORAL MULTI-AGENT LOGIC WITH THE UNIVERSAL MODALITY[1]

Abstract

This paper proposes a semantic description of the linear step-like temporal multi-agent logic with the universal modality $\mathcal{LTK}.sl_U$ based on the idea of non-reflexive non-transitive nature of time. We proved a finite model property and projective unification for this logic.

Keywords: Multi-agent system, Kripke semantic, unification, modal logic, non-transitive time, step-like, universal modality, finite model property, p-morphism.

2020 Mathematical Subject Classification: 03B44, 03B42, 03A05, 03B45, 03B70, 03H05.

1. Introduction

Temporal logics have been widely used for more than half a century as an effective tool for describing information processes and calculations [15]. Here the most significant role belongs to logical systems $\mathcal{LTL}$ and $\mathcal{CTL}$,

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that constitute today one of the foundation in the theory of program verification [10]. The interweaving of such logics with multi-agent systems makes it possible to model the intellectual reasoning of various nature sites, including social ones. Examples of such logics are $\mathcal{LTK}$ [9], $\mathcal{LKI}nd$—as a version with the induction axiom [20], $\mathcal{LTK}_r$—as a logic of a reflexive non-transitive temporal relation [16].

In the field of modern approaches to modeling multi-agent systems, there is a lack of a consistent approach: various methods of interaction between agents, modal operators and valuations are proposed, new versions of combining modal systems are chosen. This situation can be partially explained by the fact that suitable (from the idea of natural modeling of processes) combined systems, after a deeper study, turn out to be complex and even lack some useful properties [7]. Of course, this imposes significant restrictions on the applicability of such systems in real information projects [8].

Most temporal logics are built on the idea of reflexive transitive time, what helps make it possible to effectively apply the developed apparatus of modal logics in their study. However, such systems have a lot of weak points when modeling complex systems, in which we are usually required properties of dynamism, indeterminacy, instability of the information transfer process and taking into account possible errors in the translation process.

In addition, the participants of computational process, described as individual agents, whose knowledge is determined by multiple relations, are able to communicate, make decisions under the influence of public opinion of society or their own independent views, accumulate and expand available information and, at the same time, "forget" or "lose" entire segments over time. In this regard, logical systems based on non-transitive, multiple fragmentary relations look promising.

Among other things, the nature of time itself, as a physical process, in many ways remains a mystery to humanity. The argument in favor of its non-transitivity, at least from the point of view of the technical tools available to us for its modeling, is the step-by-step principle of implementing any computational process—when at any moment we only have today's knowledge and know what actions will be taken to move to the next moment of time. From this point of view, it is of interest to study a non-transitive and non-reflective version of temporal logic $\mathcal{LTL}$, in which, taking into account the specified properties of relations, the temporal process is a step-like sequential procedure. Thereby, it seems rational to model such logics using methods of relational semantics.

An adequate approach that allows both to enhance the expressive power of a modal language and to bring some clarity to the process of studying the fundamental properties of a deductive system is the universal modality operator. In the case of non-transitive models it allow us to overcome the limitations associated with the finiteness of the modal degree of formulas and expresses statements that are valid "forever" in temporal systems [14].

One of the important properties of any proposition in logic is its unification, i.e. the ability to transform a formula into a theorem by the substitution of variables. In the case of social models, the unification process actually separates an unconditional true fragment from the general information of arbitrary truth values available to the agent. Among the effective approaches to solving the unification problem, the most important are the method of projective formulas and projective approximation [12], the method for describing complete sets of unifiers in terms of n-characteristic models based on reduced form of formulas [19]. From the standpoint of the social interpretation for the unification problem, it becomes clear that it is also useful to define the boundaries of an wittingly non-unifiable fragment: such an approach was proposed in [18] for extensions of modal logics $\mathcal{S}4$ and $(\mathcal{K}4 + [\Box\bot \equiv \bot])$, and later generalized for a cases of linear transitive temporal logics of knowledge [1, 5].

It is clear that the most important task is to find maximal unifiers that allow to build all the others. However, it is also interesting to find minimal—*ground*—unifiers obtained by a substitution of constants. Often, ground unifiers allow us expressing schemes for constructing maximal and even the most general unifiers [11, 6], although this approach is not always possible [13].

S. I. Bashmakov previously described one non-reflexive non-transitive temporal linear logic with universal modality and proved projective unification [2]. Later, in [3] he announced the possibility of generalizing this result for the case of logic enriched with agent's knowledge relations. In this work, we realized the semantic construction of a linear step-like temporal logic of knowledge with a universal modality, proved the finite model property and projective unification. For this logic, we introduce the notation $\mathcal{LTK}.sl_U$. The term "step-like" is an interpretation of the non-reflective non-transitive nature of the temporal relation given to logic.

As a basic tool for describing and study the logic $\mathcal{LTK}.sl_U$, we use the traditional and well-studied relational Kripke semantics of possible worlds, generalized to the case of temporal multi-agent systems. A key object here is a *$LTK.sl_U$-frame*, represented by a tuple of clustered elements and $(n+3)$ binary relations specified on them.

2. Semantics for $\mathcal{LTK}.sl_U$

There are various approaches to describing temporal logic. We will define the logic under study as a multimodal system with the following semantics.

The alphabet of the language $L^{\mathcal{LTK}.sl_U}$ includes a countable set of propositional variables $Prop = \{p_1, \dots, p_n, \dots\}$, constants $\{\top, \bot\}$, brackets $(,)$, basic Boolean operations and the following set of unary modal operators: $\{N, \Box_e, \Box_1, \dots, \Box_n, \Box_U\}$.

The smallest set containing propositional variables from $Prop$ and closed under connectives from the language $L^{\mathcal{LTK}.sl_U}$ will be standardly denoted by $For(L^{\mathcal{LTK}.sl_U})$.

$LTK.sl_U$-frame is a tuple $F := \langle W, \mathbf{Next}, R_e, R_1, \dots, R_n, R_u \rangle$, where

- $W = \bigcup_{t\in\mathbb{N}} C_t$ is a disjoing union of clusters C_t indexed by natural numbers: $C_{t_1} \cap C_{t_2} = \emptyset$ if $t_1 \neq t_2$;
- **Next** is a (non-reflexive non-transitive) binary relation "next natural number": $\forall a, b \in W : a\mathbf{Next}b \Leftrightarrow \exists t \in \mathbb{N}(a \in C_t \& b \in C_{t+1})$;
- R_e is a binary relation defining equivalence on each cluster:
$$\forall a, b \in W(aR_eb \Longleftrightarrow \exists t \in \mathbb{N}(a, b \in C_t));$$
- $\forall i \in [1, n]\, R_i \subseteq \bigcup_{t\in\mathbb{N}}(C_t)^2$ are an agent's knowledge relations defined on clusters;
- $R_u = W^2$ is a relation of total reachability:
$$\forall a, b \in W : \ aR_ub.$$

A *model* on a $LTK.sl_U$-frame F is a pair $M := \langle F, V \rangle$, where V is a valuation $V : Prop \mapsto 2^W$, where $Prop$ is a countable set of propositional variables. Then $\forall a \in C_t \subset W, \forall t \in \mathbb{N}$ truth conditions of formulas

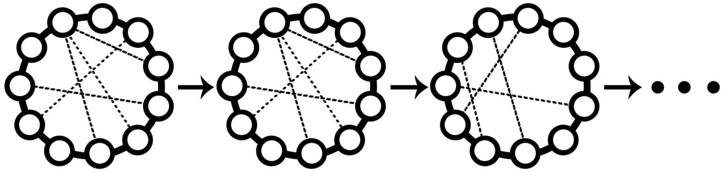

Figure 1. $LTK.sl_U$-frame

containing modal operators are determined in a standard way through the corresponding relations:

- $\langle F, a\rangle \Vdash_V N\varphi \Leftrightarrow \forall b \in C_{t+1} : \langle F, b\rangle \Vdash_V \varphi$;
- $\langle F, a\rangle \Vdash_V \Box_e\varphi \Leftrightarrow \forall b \in C_t : \langle F, b\rangle \Vdash_V \varphi$;
- $\langle F, a\rangle \Vdash_V \Box_i\varphi \Leftrightarrow \forall b \in C_t : a\, R_i\, b \Rightarrow \langle F, b\rangle \Vdash_V \varphi$;
- $\langle F, a\rangle \Vdash_V \Box_U\varphi \Leftrightarrow \forall b \in W : \langle F, b\rangle \Vdash_V \varphi$.

The operator $\Box_U$ is called a *universal modality* and actually sets the truth of a formula on a model; $\Box_e$ is a *Common Knowledge*-operator on each cluster; $\Box_1, \ldots, \Box_n$ are operators of knowledge of agents that they get on each of a frame cluster. We don't impose any special properties on the agent's knowledge, except for the condition that any R_i is a certain limitation of R_e.

We say that a formula φ is *true* in the model $M := \langle F, V\rangle$ (we denote $F \Vdash_V \varphi$) if $V(\varphi) = W$. A formula φ is *valid* on the frame F ($F \Vdash \varphi$) if φ is true in all its models. Finally, φ is *valid* on the class of frames K ($K \Vdash \varphi$), if φ is valid on any frame $F \in K$. Recall that a class of frames is called *characteristic* for a logic $\mathcal{L}$ iff all theorems of a logic are valid on all frames from this class. Let K be the class of all $LTK.sl_U$-frames.

We will call a frame F adequate to a logic $\mathcal{L}$ if for any formula $\varphi \in \mathcal{L}$ it is true that $F \Vdash \varphi$.

A *linear step-like temporal multi-agent logic with universal modality* $\mathcal{LTK}.sl_U$ is a multimodal logic, defined as follows

$$\mathcal{LTK}.sl_U := \{\varphi \in For(L^{\mathcal{LTK}.sl_U}) \mid \forall F \in K : F \Vdash \varphi\}.$$

3. Finite model property of $\mathcal{LTK}.sl_U$

A *modal degree* $d(\alpha)$ of a formula α in $\mathcal{LTK}.sl_U$ is a number of nested non-reflexive non-transitive modal operators N in α:

$$d(p) = 0, p \in Prop; d(\circ\alpha) = d(\alpha), \text{ where } \circ \in \{\neg, \Box_e, \Box_U, \Box_i\};$$
$$d(\alpha \odot \beta) = max(d(\alpha); d(\beta)), \text{ where } \odot \in \{\vee, \wedge\}; d(N\alpha) = d(\alpha) + 1.$$

A *length* $d(\alpha)$ of a formula α of the logic $\mathcal{LTK}.sl_U$ is defined as follows: $l(p) = 0$ where $p \in Prop$; $l(\alpha \circ \beta) = l(\alpha) + l(\beta) + 1$, where $\circ \in \{\wedge, \vee\}$; $l(\bullet\alpha) = l(\alpha) + 1$, where $\bullet \in \{N, \neg, \Box_e, \Box_U, \Box_i\}$.

An important property of logical systems is a *finite model property*, which allows us to operate with simpler finite models, instead of their infinite variants. A logic $\mathcal{L}$ is said to have *finite model property*, if $\mathcal{L}$ is complete with respect to the class of finite frames.

In order to prove that the logic $\mathcal{LTK}.sl_U$ has the finite model property, we define a p-morphic mapping of an infinite $\mathcal{LTK}.sl_U$-model M to a finite-by-time one, and then, using the technique of filtering clusters, we construct a model with clusters of finite cardinality on the p-morphic version. This section proves that such a model will preserve the truth of formulas in our logic.

3.1. p-morphism for $\mathcal{LTK}.\mathbf{sl_U}$

A map f from a frame $F := \langle W, \mathbf{Next}, R_e, R_1, \ldots, R_n, R_u \rangle$ onto a frame $F' := \langle W', R'_e, R'_1, \ldots, R'_n, R'_u \rangle$ is called a p-morphism, if the following conditions hold $\forall a, b \in W \forall R \subseteq \{\mathbf{Next}, R_e, R_1, \ldots, R_n, R_u\}$:

1. $aRb \Rightarrow f(a)R'f(b)$;
2. $f(a)R'f(b) \Rightarrow \exists c \in W[aRc \wedge f(c) = f(b)]$.

Now we define the finite by the time (by the number of clusters) model N as follows:

$$N := \left\langle \bigcup_{j \in [1,k+1]} C_j, \mathbf{Next}', R'_e, R'_1, \ldots, R'_n, R'_u, V' \right\rangle,$$

where for some $\mathcal{LTK}.sl_U$-model $M = \langle W, \mathbf{Next}, R_e, R_1, \ldots, R_n, R_u, V \rangle$ the following conditions are satisfied:

- $\bigcup_{j\in[1,k]} C_j \subset W$ are finite number of clusters, C_{k+1} is a singleton cluster;
- $R'_e, R'_1, \ldots, R'_n$ are given as limitations of corresponding relations $R_e, R_1, \ldots, R_n$ on clusters $\bigcup_{j\in[1,k]} C_j$ supplemented by the following conditions $\forall R \in \{R_e, R_1, \ldots, R_n\}$:

$$\forall a, b \in W \setminus \{C_1, \ldots, C_k\} if\, aRb, then\, a = b \in C_{k+1} \& aR'b;$$

- R'_u coincides on clusters $C_1, \ldots, C_k$ with the relation R_u, and for elements out of these clusters it is given as follows:

$$\forall a \in W \setminus \{C_1, \ldots, C_k\} \forall b \in W\, if\, aR_u b, then\, a \in C_{k+1} \& aR'_u b.$$

- $\mathbf{Next}'$ is defined as follows: $\forall a \in \{C_1, \ldots, C_k\}$ if $a\mathbf{Next}b$, then $b \in \{C_2, \ldots, C_{k+1}\}$, and $\forall a \in W \setminus \{C_1, \ldots, C_k\}$, $\forall b \in W$ if $a\mathbf{Next}b$, then $a, b \in C_{k+1} \& a\mathbf{Next}'b$;
- $V'(p) = V(p) \cap \bigcup_{j\in[1,k]} C_j$ for $p \in Prop$.

To simplify notation, we will denote a finite frame defining such a model N as $F_{fin} := \left\langle \bigcup_{j\in[1,k+1]} C_j, \mathbf{Next}', R'_e, R'_1, \ldots, R'_n, R'_u \right\rangle$. For consistency, we denote an infinite $LTK.sl_U$-frame here as F_{inf}.

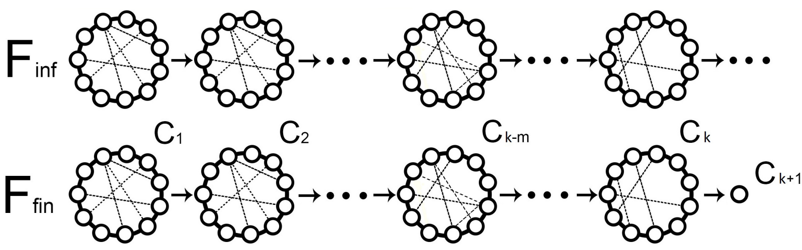

Figure 2. An infinite frame F_{inf} and a finite frame F_{fin}

THEOREM 3.1. *Any F_{fin} is a p-morphic image of F_{inf}.*

PROOF: Let f be a mapping of infinite $LTK.sl_U$-frame

$$F_{inf} = \langle W, \mathbf{Next}, R_e, R_1, \ldots, R_n, R_u \rangle$$

onto a finite

$$F_{fin} = \left\langle \bigcup_{j \in [1,k+1]} C_j, \mathbf{Next}', R'_e, R'_1, \ldots, R'_n, R'_u \right\rangle,$$

given as follows:

1. $\forall x \in \bigcup_{j \in [1,k]} C_j \; f(x) = x$;

2. $\forall x \in W \setminus \bigcup_{j \in [1,k]} C_j \;\; f(x) = y$, where $y \in C_{k+1}$.

Let us prove that the mapping f is a p-morphism. For this, it is necessary to show the correctness of the given mapping, with respect to the points (1.) and (2.) of the definition.

(1.) $\forall a, b \in W$ if $a\mathbf{Next}b$, hence by the definition of **Next**, $a \in C_i$ and $b \in C_{i+1}$. If $b \in \{C_2, \ldots, C_k\}$, then $f(a) = a$, $f(b) = b$ and $f(a)\mathbf{Next}'f(b)$. If $b \in W \setminus \{C_1, \ldots, C_k\}$, then $f(a), f(b) \in C_{k+1}$ and $f(a)\mathbf{Next}'f(b)$.

If aR_eb and $a, b \in C_i \subset \{C_1, \ldots, C_k\}$ then $f(a)R'_ef(b)$. If $C_i \in W \setminus \{C_1, \ldots, C_k\}$, then $f(a) = f(b) = y \in C_{k+1}$.

By virtue of $R_i \subseteq R_e \forall i \in [1, \ldots, n]$, for relations $R'_1, \ldots, R'_n$ proof is similar to R_e.

By definition, $R_u = W^2$ and then $\forall a, b \in \{C_1, \ldots, C_k\} \; f(a)R'_uf(b)$. If $a \in W \setminus \{C_1, \ldots, C_k\}$ or $b \in W \setminus \{C_1, \ldots, C_k\}$, then $f(a) = yR'_uf(b)$ or $f(a)R'_uy = f(b)$. Respectively, $y \in C_{k+1}$.

(2.) $\forall a, b \in W$ if $f(a)\mathbf{Next}'f(b)$, then the following cases are possible:

- if $f(a)\mathbf{Next}'f(b)$, then, by definition $\mathbf{Next}'$, $f(a) \in C_i, f(b) \in C_{i+1}$, where $i + 1 \in [2, \ldots, k]$. In this case $f(a) = a$, and for $f(b)$ two options are possible:

  - when $f(b) \in \{C_2, \ldots, C_k\}$, $b = c$ and $a\mathbf{Next}c$;
  - when $f(b) \in C_{k+1} \& f(a) \in C_k$, as c we take $\forall c \in C_{k+1}$, then $a\mathbf{Next}c$.

- if $f(a), f(b) \in C_{k+1}$, then $a \in C_j, b \in C_{j+1}$(where $\{C_j, C_{j+1}\} \subset W \setminus \{C_1, \ldots, C_k\}$) and then as c we take $\forall x \in C_{j+1}$. In this case $f(c) = f(b) \in C_{k+1} \subset F_{fin}$.

$\forall a, b \in W$ if $f(a)R'_e f(b)$, then two options are possible:

- if $f(a), f(b) \in C_i$, where $i \in \{1, \dots, k\}$, then $aR_e b$, where $a, b \in C_i$, $i \in [1, \dots, k]$. In this case $c \in C_i$;
- if $f(a), f(b) \in C_{k+1}$, then $aR_e b$ and $a, b \in W \setminus \{C_1, \dots, C_k\}$.

In the case of R_i, the proof trivially repeats the reasoning for R_e.

$\forall a, b \in W f(a)R'_u f(b)$, therefore, as c we can take any element of W.

Therefore, any finite frame F_{fin} is a p-morphic image of F_{inf}. $\square$

Now let us prove that any formula refutable on $\mathcal{LTK}.sl_U$-model M is refuted also on N.

Theorem 3.2. *Let $M = \langle F_{inf}, V\rangle$ be an infinite by time $LTK.sl_U$-model, α is an arbitrary formula with the modal degree $d(\alpha) = m, m \in \omega$.*

Then $\forall x \in \bigcup_{j\in[1,k-m]} C_j \subset F_{inf}$ $(m < k)$ it is true:

$$\langle M, x\rangle \nVdash \alpha \Leftrightarrow \langle N, x\rangle \nVdash \alpha,$$

where $N = \langle F_{fin}, V'\rangle = \langle \bigcup_{j\in[1,k+1]} C_j, \mathbf{Next}', R'_e, R'_u, R'_1, \dots, R'_n, V'\rangle$.

Proof: Let us prove that it is true for all formulas in $\mathcal{LTK}.sl_U$. The proof is by induction on the length of the formula α.

The induction base $l(\alpha) = 0$ corresponds to the case $\alpha = p$. Obviously, in this case the modal degree is also equal to 0 and the statement is true $\forall x \in \bigcup_{j\in[1,k]} C_j$.

Suppose the statement of the theorem is true $\forall \beta$: $l(\beta) < t$, i.e. $\langle M, x\rangle \nVdash \beta \Leftrightarrow \langle N, x\rangle \nVdash \beta$. Let us prove for $l(\alpha) = t$.

The cases $\alpha \in \{\neg\varphi, \Box_U\varphi, \Box_e\varphi, \Box_i\varphi, \varphi \vee \psi, \varphi \wedge \psi\}$ satisfy the conditions of inductive hypothesis due to the fact that the modal degree of the formula α is not increased by adding operators $\{\neg, \Box_U, \Box_e, \Box_i\}$ to the subformula φ of less length, and is potentially increased by adding $\{\vee, \wedge\}$ only up to the value of $max(d(\varphi), d(\psi))$, where φ and ψ are also shorter in length (by the definitions of the truth values of such formulas).

$!\alpha = N\varphi$, $l(\varphi) = l(\alpha) - 1$ and $d(\alpha) = d(\varphi) + 1$. By inductive hypothesis, $\langle M, x\rangle \nVdash \varphi \Leftrightarrow \langle N, x\rangle \nVdash \varphi$, where $x \in \bigcup_{j\in[2,k-(m-1)]} C_j$. By the definition of N, it's true, that $\forall x \in C_i\langle M, x\rangle \Vdash N\varphi \Leftrightarrow \forall y \in C_{i+1}$ (i.e. $x\mathbf{Next}y$) $\langle M, y\rangle \Vdash \varphi$, hence, $\exists x \in C_i\langle M, x\rangle \nVdash N\varphi \Leftrightarrow \exists y \in C_{i+1}$ $\langle M, y\rangle \nVdash \varphi$. Then $\forall \hat{x} \in \bigcup_{j\in[1,k-m]} C_j$ $\langle M, \hat{x}\rangle \nVdash N\varphi \Leftrightarrow \langle N, \hat{x}\rangle \nVdash N\varphi$. $\square$

4. Filtration for $\mathcal{LTK}.sl_U$

To build a final finite model that is adequate to our logic, we apply the filtering technique to the frame F_{fin}. Let $M = \langle W, \mathbf{Next}, R_e, R_1, \ldots, R_n, R_u, V\rangle$ be a model, built on the infinite $LTK.sl_U$-frame defined above, $\Phi \subseteq For(L^{\mathcal{LTK}.sl_U})$ is a set of formulas that is closed wrt sub-formulas. We define an equivalence relation $\equiv_\Phi$ on clusters from W as follows: $\forall t \in \mathbb{N}, \forall x, y \in C_t$

$$x \equiv_\Phi y \Longleftrightarrow [\forall \alpha \in \Phi(\langle M, x\rangle \Vdash \alpha \Leftrightarrow \langle M, y\rangle \Vdash \alpha)].$$

In accordance with this definition, below we will use the notation

- $Var(\Phi)$ for a set of all variables of formulas from Φ;
- $[x]_{\equiv_\Phi} := \{y \in W | x \equiv_\Phi y\}$ for equivalence classes;
- $W_\Phi := \{[x]_{\equiv_\Phi} | \forall x \in W\}$ for a set of all such classes;
- $C_{j_\Phi} := \{[x]_{\equiv_\Phi} | \forall x \in C_j \subset F_{fin}\}$, $j \in [1, k+1]$, for each cluster of such classes obtained from each cluster of F_{fin}.

To get only finite clusters, we define a model filtered by a set $\Phi \subseteq For(L^{\mathcal{LTK}.sl_U})$

$$N_\Phi = \left\langle \bigcup_{j \in [1,k+1]} C_{j_\Phi}, \mathbf{Next}'_\Phi, R'_{e_\Phi}, R'_{1_\Phi}, \ldots, R'_{n_\Phi}, R'_{u_\Phi}, V'_\Phi \right\rangle$$

based on a version of model N with a p-morphic frame F_{fin} and additional following filtration of clusters:

1. $\forall p \in Var(\Phi)$ $[V'_\Phi(p) = \{[a]_{\equiv_\Phi} | \langle N, a\rangle \Vdash p\}]$;
2. $\forall a, b \in \{\bigcup_{j \in [1,k+1]} C_j\}$, $\forall R' \in [\mathbf{Next}', R'_e, R'_1, \ldots, R'_n, R'_u]$ $(aR'b \Rightarrow [a]_{\equiv_\Phi} R'_\Phi [b]_{\equiv_\Phi})$;
3. $\forall a, b \in \{\bigcup_{j \in [1,k+1]} C_{j_\Phi}\}$
   (a) $\forall l \in \{e, 1, \ldots, n, u\}$ $([a]_{\equiv_\Phi} R'_{l_\Phi} [b]_{\equiv_\Phi} \Longrightarrow [\forall \Box_l \alpha \subseteq \Phi\ \langle N, a\rangle \Vdash \Box_l \alpha \Rightarrow \langle N, b\rangle \Vdash \alpha])$;
   (b) $[a]_{\equiv_\Phi} \mathbf{Next}'_\Phi [b]_{\equiv_\Phi} \Longrightarrow ([\forall N\alpha \subseteq \Phi\ \langle N, a\rangle \Vdash N\alpha \Rightarrow \langle N, b\rangle \Vdash \alpha])$.

Well-known conditions for building the minimal and maximal filtration can also be applied in our case:

— **the minimal filtration**

$$N_{\Phi}^{min} = \left\langle \bigcup_{j\in[1,k+1]} C_{j_{\Phi}}, \mathbf{Next}_{\Phi}'^{min}, R_{e_{\Phi}}'^{min}, R_{1_{\Phi}}'^{min}, \ldots, R_{n_{\Phi}}'^{min}, R_{u_{\Phi}}'^{min}, V_{\Phi}' \right\rangle,$$

where

- $\forall l \in \{e, 1, \ldots, n, u\}$ $R_{l_{\Phi}}'^{min} = \{([a]_{\equiv_{\Phi}}, [b]_{\equiv_{\Phi}}) | (a, b) \in R_l'\}$,
- $\mathbf{Next}'^{min} = \{([a]_{\equiv_{\Phi}}, [b]_{\equiv_{\Phi}}) | (a, b) \in Next'\}$;

— **the maximal filtration**

$$N_{\Phi}^{max} = \left\langle \bigcup_{j\in[1,k+1]} C_{j_{\Phi}}, \mathbf{Next}_{\Phi}'^{max}, R_{e_{\Phi}}'^{max}, R_{1_{\Phi}}'^{max}, \ldots, R_{n_{\Phi}}'^{max}, R_{u_{\Phi}}'^{max}, V_{\Phi}' \right\rangle,$$

where

- $\forall l \in \{e, 1, \ldots, n, u\} : [a]_{\equiv_{\Phi}} R_{l_{\Phi}}'^{max} [b]_{\equiv_{\Phi}} \Leftrightarrow [\forall \Box_l \alpha \subseteq \Phi(\langle N, a\rangle \Vdash \Box_l \alpha \Rightarrow \langle N, b\rangle \Vdash \alpha)]$,
- $[a]_{\equiv_{\Phi}} \mathbf{Next}_{\Phi}'^{max} [b]_{\equiv_{\Phi}} \Leftrightarrow ([\forall N\alpha \subseteq \Phi \ \langle N, a\rangle \Vdash N\alpha \Rightarrow \langle N, b\rangle \Vdash \alpha])$.

Due to the choice of the set Φ, the finiteness of the number of relations on a frame and all pairwise variants of their intersections, the clusters $C_{t_{\Phi}}$ obtained as a result of the proposed filtration are also will always have finite cardinality. By virtue of the construction of a filtered model, we assume the true following

LEMMA 4.1. *Let $N = \langle \bigcup_{j\in[1,k+1]} C_j, \mathbf{Next}', R_e', R_u', R_1', \ldots, R_n', V'\rangle$ be a p-morphic model N of a $\mathcal{LTK}.sl_U$-model M, $\Phi \subseteq For(L^{\mathcal{LTK}.sl_U})$ is a closed wrt subformulas set of formulas whose modal degree does not exceed m $(m \in \omega, k > m)$,*

$$N_{\Phi} = \left\langle \bigcup_{j\in[1,k+1]} C_{j_{\Phi}}, \mathbf{Next}_{\Phi}', R_{e_{\Phi}}', R_{1_{\Phi}}', \ldots, R_{n_{\Phi}}', R_{u_{\Phi}}', V_{\Phi}' \right\rangle$$

be a filtered variant of the model N to the set Φ. Then $\forall x \in \bigcup_{j\in[1,k-m]} C_j$, $\forall \alpha \in \Phi$:

$$\langle N, x\rangle \nVdash \alpha \Leftrightarrow \langle N_{\Phi}, x\rangle \nVdash \alpha.$$

By virtue of Theorem 2 and Lemma 1, we conclude the finite model property for $\mathcal{LTK}.sl_U$.

5. Unification in $\mathcal{LTK}.sl_U$

5.1. Definitions of unification theory

A formula $\varphi(p_1, \ldots, p_s)$ is called *unifiable* in logic $\mathcal{L}$, if $\exists \sigma : p_i \mapsto \sigma_i$ for every $p_i \in Var(\varphi)$, s.t. $\sigma(\varphi) = \varphi(\sigma_1, \ldots, \sigma_s) \in \mathcal{L}$. A substitution σ is called *unifier* of φ. A *ground* unifier is a constant substitution (i.e. $gu : p_i \mapsto \{\top, \bot\}, \forall p_i \in Var(\varphi)$).

The preorder relation is defined on the set of unifiers: an unifier σ of $\varphi(p_1, \ldots, p_s)$ is called *more general* than σ^1 in $\mathcal{L}$, if there is a substitution γ, s.t. for any p_i: $\sigma^1(p_i) \equiv \gamma(\sigma(p_i)) \in \mathcal{L}$ ($\sigma_1 \preceq \sigma$).

An unifier σ of $\varphi(p_1, \ldots, p_s)$ is said to be *maximal*, if for any other σ^i, either $\sigma^i \preceq \sigma$, or $(\sigma^i \not\preceq \sigma)\&(\sigma \not\preceq \sigma^i)$. If σ is more general than any other, it is called a *most general* (*mgu*, for short).

A set of unifiers CU for a formula φ is called *complete* in $\mathcal{L}$, if for any unifier σ of φ there is $\sigma_1 \in CU$: $\sigma \preceq \sigma^1$.

In general, the existence of infinite sequences of unifiers with respect to a given preorder is possible. If such chains are admissible, the formula (and hence all logic) has a *nullary* unification type. In other cases, when ascending chains are terminated in a finite number of steps, unification is *infinitary* (case of a countable number of maximal unifiers for some formula), *finitary* (case of a finite number of maximal ones for all formulas) or *unitary* (in case of the existence of mgu for all formulas) type.

A formula $\varphi(p_1, \ldots, p_s)$ is called *projective* in $\mathcal{L}$, if there is an unifier τ of φ, s.t. $\Box\varphi \to [p_i \equiv \tau(p_i)] \in \mathcal{L}$ for all $p_i \in Var(\varphi)$. An unifier with such specified properties is called *projective*.

As was proved by S. Ghilardi [12], the projective unifier defines mgu of a formula (and, accordingly, CU). Consequently, having established the projectivity of unification in logic, we will obtain a universal scheme for constructing an mgu and a unitary type of unification. The importance of this approach is reinforced by a corollary from a projective unification, which guarantees almost structural completeness of logic [17].

5.2. Projective unification in $\mathcal{LTK}.sl_U$

To study the unification properties in $\mathcal{LTK}.sl_U$ we need to redefine the notion of a projective formula because of the non-transitive and non-reflective nature of the temporal operator N. Let's do it through the universal modality $\Box_U$: $\varphi(p_1, \dots, p_s)$ is called projective in $\mathcal{LTK}.sl_U$, if there is an unifier τ for φ, s.t. $\Box_U\varphi \to [p_i \equiv \tau(p_i)] \in \mathcal{LTK}.sl_U$ for all $p_i \in Var(\varphi)$.

As the following theorem shows, unifiability of an arbitrary formula $\varphi(p_1, \dots, p_s)$ in $\mathcal{LTK}.sl_U$ can be effectively establish using constant substitutions: $\forall p_i \in Var(\varphi)$ $\sigma(p_i) \in \{\top, \bot\}$.

THEOREM 5.1. *If a formula φ is unifiable in $\mathcal{LTK}.sl_U$, then φ has a ground unifier.*

PROOF: The proof of this theorem is similar to the proof in [4] for the case of pretabular extensions of $\mathcal{S}4$. Here we describe a sketch of the proof and supplement it with some important comments.

Let's take an arbitrary unifiable in $\mathcal{L}$ formula $\varphi(p_1, \dots, p_s)$ and $\delta_1(q_1, \dots, q_r), \dots, \delta_s(q_1, \dots, q_r)$ is its unifier. Then it is true that

$$\delta(\varphi) := \varphi(\delta_1(q_1, \dots, q_r), \dots, \delta_s(q_1, \dots, q_r)) \in \mathcal{L}.$$

Any substitution of variables $q_1, \dots, q_r$ to constants $c_i \in \{\top, \bot\}(i \in [1, r])$ preserves truth values of the formula $\delta(\varphi)$, because of $\delta(\varphi) \in \mathcal{L}$. In particular, $\varphi(gu(p_1), \dots, gu(p_s)) \in \mathcal{L}$, where $gu(p_i) := \delta_i(c_1, \dots, c_r) \in \{\top, \bot\}$, is a partial case of $\delta(\varphi)$. Therefore, any substitution of this form is a ground unifier of φ. To check the existence of such an substitution for arbitrary formula $\psi(p_1, \dots, p_s)$, it suffices to consider no more than 2^s substitutions of $\{\top, \bot\}$ instead of all p_i. If among them there is one s.t. $\psi(gu(p_1), \dots, gu(p_s)) \equiv_{\mathcal{L}} \top$, then ψ is unifiable in $\mathcal{L}$ and gu is its ground unifier. If for all 2^s options $gu(\psi) \notin \mathcal{L}$, then ψ doesn't have a ground unifier and therefore any other unifier in $\mathcal{L}$. $\Box$

We are now ready to prove the main result of this work.

THEOREM 5.2. *Any formula unifiable in $\mathcal{LTK}.sl_U$ is projective.*

PROOF: Let $\varphi(p_1, \dots, p_s)$ be unifiable in $\mathcal{LTK}.sl_U$. Then $\forall p_i \in Var(\varphi)$ we define the following substitution $\sigma(p_i)$:

$$\sigma(p_i) := (\Box_U\varphi \wedge p_i) \vee (\neg\Box_U\varphi \wedge gu(p_i)),$$

where $gu(p_1), \dots, gu(p_s)$ is an arbitrary ground unifier for $\varphi(p_1, \dots, p_s)$.

Let's take an arbitrary $LTK.sl_U$-model $M := \langle F, V \rangle$. If σ is an unifier for φ, then $\sigma(\varphi) \in \mathcal{LTK}.sl_U$ and $\forall x \in F$ $\langle M, x \rangle \Vdash \sigma(\varphi)$. Let us prove that σ is indeed an unifier for φ in $\mathcal{LTK}.sl_U$.

1. If $\forall x \in F : \langle M, x\rangle \Vdash \varphi$, then $\langle M, x\rangle \Vdash_V \Box_U\varphi$ and, therefore, the second disjunctive member will be refuted on x. If $\langle M, x\rangle \Vdash p_i$, then $\langle M, x\rangle \Vdash \Box_U\varphi \wedge p_i$, hence, $\langle M, x\rangle \Vdash \sigma(p_i)$. If $\langle M, x\rangle \Vdash \neg p_i$, then $\langle M, x\rangle \nVdash \Box_U\varphi\wedge p_i$ and, therefore, $\langle M, x\rangle \Vdash \neg\sigma(p_i)$. As a consequence, the truth value $\varphi(p_1, \dots, p_s)$ on an arbitrary element x wrt V coincides with the truth value $\varphi(\sigma(p_1), \dots, \sigma(p_s))$ on the same element with the same valuation V and, in this case, $\langle M, x\rangle \Vdash \sigma(\varphi)$.

2. If $\exists x \in F : \langle M, x\rangle \Vdash \neg\varphi$, then $\langle M, x\rangle \nVdash \Box_U\varphi$. In this case, the second disjunctive member can be valid, but the first is refuted on x. Then truth values of all $\sigma(p_i)$ on x coincide with $gu(p_i)$ (i.e. $\sigma(\varphi) \equiv gu(\varphi)$), and since $\langle M, x\rangle \Vdash gu(\varphi)$ (due to the choice of the ground unifier $gu(\varphi) \in \mathcal{LTK}.sl_U$), again $\langle M, x\rangle \Vdash \sigma(\varphi)$. Hence, $\sigma(\varphi) \in \mathcal{LTK}.sl_U$ for φ unifiable in $\mathcal{LTK}.sl_U$.

Let us prove that $\sigma(\varphi)$ is a projective unifier. By the definition, if $\sigma(p_i)$ is a projective unifier for φ, we obtain the following: $\forall p_i \in Var(\varphi)$

$$\Box_U\varphi \to (p_i \leftrightarrow [(\Box_U\varphi \wedge p_i) \vee (\neg\Box_U\varphi \wedge gu(p_i))]) \in \mathcal{LTK}.sl_U. \tag{5.1}$$

Suppose the opposite: let σ not be a projective unifier and hence 5.1 is refuted at some model. Then it is not difficult to verify that if the premise of the implication is true, it is impossible to refute the conclusion, and therefore we get a contradiction.

Consequently, σ is a projective unifier for φ in $\mathcal{LTK}.sl_U$, so φ is a projective formula. $\Box$

From the proved theorems and mentioned results by S. Ghilardi, hold

Corollary 5.3. Let φ be an arbitrary unifiable formula in $\mathcal{LTK}.sl_U$. Then

1. $\sigma(p_i) := (\Box_U\varphi \wedge p_i) \vee (\neg\Box_U\varphi \wedge gu(p_i))$ is a projective unifier and, hence, mgu for φ;

2. The logic $\mathcal{LTK}.sl_U$ has a unitary unification type.

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CONSTRUCTING A HOOP USING ROUGH FILTERS

Abstract

When it comes to making decisions in vague problems, Rough is one of the best tools to help analyzers. So based on rough and hoop concepts, two kinds of approximations (Lower and Upper) for filters in hoops are defined, and then some properties of them are investigated by us. We prove that these approximations-lower and upper- are interior and closure operators, respectively. Also after defining a hyper operation in hoops, we show that by using this hyper operation, set of all rough filters is monoid. For more study, we define the implicative operation on the set of all rough filters and prove that this set with implication and intersection is made a hoop.

Keywords: Hoop, rough set, rough approximations (lower and upper), rough filter.

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1. Introduction

Pawlak proposed the theory of rough sets in 1982 as a new method for modelling and processing uncertain data. There are different fields such as machine learning, intelligence system, decision making, and etc, in which rough set theory can help to solve some problems. So it has received algebraic researchers attention too, and leads to apply rough set theory in different algebraic systems such as BCK-algebra [13], BCC-algebra [14], MV-algebra [17] and so on.

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Rough set theory includes different concepts some of them which are used in rough controllers are rough relations and rough functions. From algebraic point of view, Iwinski [11] is the first one who algebraically approach to the rough sets. In [16, 4], application of rough set can be seen in groups and semigroups. Till today, relation between rough theory and some algebras are studied, BCK-algebras by Jun [13], and MV-algebras by Rasouli and Davvaz [17]. Bosbach [8] introduced hoop algebra as special groups of monoids: naturally ordered commutative resituated integral monoids. In recent decades, many mathematicians have worked on it and developed structure theory by using the nation of hoop (see [3, 8]). Fuzzy logic and hoops have strong impact on each other results. One of the famous examples is the short proof of the completeness theorem for propositional basic logic introduced by Hájek in [10] which is obtained from the structure theorem of finite basic hoops. There are a lot of areas that hoops are being implemented for algebraic structures such as (see [1, 2, 5, 6, 7]). By considering the impact of rough set theory and since there was no study on the relation between hoop and rough set theory, we decided to apply the rough set theory in hoops. Experience of implementing soft set theory in hoops [6], and the logic used in [15] helped us a lot to have a better view. For this purpose, we defined the concept of the lower and the upper approximations in hoops and then investigated their properties. Also, it is proved that the lower (upper) approximations is an interior operator (closure operator). Moreover, we define a hyper operation on hoop and then we show that by using this operation, the set of all rough filters is a monoid. For more study, we define the implicative operation on the set of all rough filters and prove that this set with implication and intersection is made a hoop.

2. Preliminaries

Some definitions that may be required in the further discussions are reviewed in this part.

A *hoop* [8] is an algebraic structure $\hbar = (\hbar, \odot, \rightarrow, 1)$ of type $(2, 2, 0)$ such that, for all $\kappa, \nu, \delta \in \hbar$ the following conditions hold:

$(HP1)$ $(\hbar, \odot, 1)$ is a commutative monoid,

$(HP2)$ $\kappa \rightarrow \kappa = 1$,

$(HP3)$ $(\kappa \odot \nu) \to \delta = \kappa \to (\nu \to \delta)$,

$(HP4)$ $\kappa \odot (\kappa \to \nu) = \nu \odot (\nu \to \kappa)$.

A relation $\leq$ on hoop $\hbar$ which is defined by $\kappa \leq \nu$ if and only if $\kappa \to \nu = 1$, is a partial order relation on $\hbar$. A hoop $\hbar$ is called *bounded* if there is an element $0 \in \hbar$ such that $0 \leq \kappa$, for all $\kappa \in \hbar$ (see [8]).
Fundamental properties of hoops are provided in the next proposition.

PROPOSITION 2.1 ([8]). Let $\hbar$ be a hoop. Then, for all $\kappa, \nu, \delta \in \hbar$ the following properties hold:

(i) $(\hbar, \leq)$ is a $\wedge$-semilattice with $\kappa \wedge \nu = \kappa \odot (\kappa \to \nu)$;

(ii) $\kappa \odot \nu \leq \delta$ if and only if $\kappa \leq \nu \to \delta$;

(iii) $\kappa \odot \nu \leq \kappa, \nu$;

(iv) $\kappa \leq \nu \to \kappa$;

(v) $1 \to \kappa = \kappa$;

(vi) $\kappa \to 1 = 1$;

(vii) $\nu \leq (\nu \to \kappa) \to \kappa$;

$(viii)$ $\kappa \leq (\kappa \to \nu) \to \kappa$;

(ix) $\kappa \to \nu \leq (\nu \to \delta) \to (\kappa \to \delta)$;

(x) $(\kappa \to \nu) \odot (\nu \to \delta) \leq \kappa \to \delta$;

(xi) $\kappa \leq \nu$ implies $\kappa \odot \delta \leq \nu \odot \delta$, $\delta \to \kappa \leq \delta \to \nu$ and $\nu \to \delta \leq \kappa \to \delta$.

Uninary operation "$\neg$" on a bounded hoop $\hbar$ is defined such that for any $\kappa \in \hbar, \neg\kappa = \kappa \to 0$.
Then for any nonempty subset R of a bounded hoop $\hbar$, consider the sets $\neg R := \{\neg\kappa \in \hbar \mid \kappa \in R\}$ and $DNP(\hbar) := \{\kappa \in \hbar \mid \neg(\neg\kappa) = \kappa\}$.

Double negation property (briefly, DNP of a bounded hoop $\hbar$ is when $DNP(\hbar) = \hbar$.

Proposition 2.2 ([8, 9]). Let $\hbar$ be a bounded hoop. Then, for any $\kappa, \nu \in \hbar$, the following conditions hold:

(i) $\kappa \leq \neg\neg\kappa$ and $\kappa \odot \neg\kappa = 0$

(ii) $\neg\kappa \leq \kappa \to \nu$.

(iii) $\neg\neg\neg\kappa = \neg\kappa$.

(iv) If $\hbar$ has (DNP), then $\kappa \to \nu = \neg\nu \to \neg\kappa$.

(v) If $\hbar$ has (DNP), then $(\kappa \to \nu) \to \nu = (\nu \to \kappa) \to \kappa$.

Let ϱ be an equivalence relation on a hoop $\hbar$ and $\mathcal{P}(\hbar)$ denote the power set of $\hbar$. For all $\kappa \in \hbar$, let $[\kappa]_\varrho$ denote the equivalence class of κ with respect to ϱ. Let ϱ_* and ϱ^* be mappings from $\mathcal{P}(\hbar)$ to $\mathcal{P}(\hbar)$ defined by $\varrho_*(F) = \{\kappa \in \hbar \mid [\kappa]_\varrho \subseteq F\}$ and $\varrho^*(F) = \{\kappa \in \hbar \mid [\kappa]_\varrho \cap F \neq \emptyset\}$, respectively.
The pair $(\hbar, \varrho)$ is called an *approximation space based on* ϱ. A subset F of a hoop $\hbar$ is *definable* if $\varrho_*(F) = \varrho^*(F)$, and *rough* otherwise. The set $\varrho_*(F)$ (resp. $\varrho^*(F)$) is called the *lower* (resp. *upper*) *approximation*. (See [14])

Proposition 2.3 ([14]). Let $(\hbar, \varrho)$ be a ϱ-approximation space. For any $R, M \in \mathcal{P}(\hbar)$, we have

(i) $\varrho_*(R) \subseteq R \subseteq \varrho^*(R)$,

(ii) $\varrho_*(R \cap M) = \varrho_*(R) \cap \varrho_*(M)$,

(iii) $\varrho_*(R) \cup \varrho_*(M) \subseteq \varrho_*(R \cup M)$,

(iv) $\varrho^*(R \cap M) \subseteq \varrho^*(R) \cap \varrho^*(M)$,

(v) $\varrho^*(R) \cup \varrho^*(M) = \varrho^*(R \cup M)$.

(vi) $\varrho_*(\varrho^*(R)) \subseteq \varrho^*(\varrho^*(R))$,

(vii) $\varrho_*(\varrho_*(R)) \subseteq \varrho^*(\varrho_*(R))$,

$(viii)$ $\varrho_*(R^c) = (\varrho^*(R))^c$,

(ix) $\varrho^*(R^c) = (\varrho_*(R))^c$,

(x) $\varrho_*(R) = \emptyset$ for $R \neq \hbar$,

(xi) $\varrho^*(R) = R$ for $R \neq \emptyset$.

(xii) $\varrho_*(R) = R \Leftrightarrow \varrho^*(R^c) = R^c$.

Function $\mathsf{C} : \mathcal{P}(S) \to \mathcal{P}(S)$ on a set S is a *closure operator* [12] if the following conditions are held for all subsets $X, Y \subseteq S$:

(i) $X \subseteq \mathsf{C}(X)$,

(ii) if $X \subseteq Y$, then $\mathsf{C}(X) \subseteq \mathsf{C}(Y)$,

(iii) $\mathsf{C}(\mathsf{C}(X)) = \mathsf{C}(X)$.

Function $\mathfrak{T} : \mathcal{P}(S) \to \mathcal{P}(S)$ on a set S is an *interior operator* [12] in which for all subsets $X, Y \subseteq S$ the following conditions are held:

(i) $\mathfrak{T}(X) \subseteq X$,

(ii) if $X \subseteq Y$, then $\mathfrak{T}(X) \subseteq \mathfrak{T}(Y)$,

(iii) $\mathfrak{T}(\mathfrak{T}(X)) = \mathfrak{T}(X)$.

3. Roughness of filters in hoops

In this section, roughness of hoops is introduced and some properties of it are investigated. Soppose $\digamma$ is a filter of a hoop $\hbar$. We define a relation "$\mathcal{C}_\digamma$" on $\hbar$ for any $\kappa, \nu \in \hbar$ as follows:

$$(\kappa, \nu) \in \mathcal{C}_\digamma \text{ if and only if } \kappa \to \nu \in \digamma \text{ and } \nu \to \kappa \in \digamma.$$

Then $\mathcal{C}_\digamma$ is a congruence relation on $\hbar$. Hence approximation space $(\hbar, \mathcal{C}_\digamma)$ is called an $\digamma$*-approximation space*. The equivalence class of $\kappa \in \hbar$ under $\mathcal{C}_\digamma$ is denoted by $\mathcal{C}_\digamma[\kappa]$.

Let $(\hbar, \mathcal{C}_\digamma)$ be an $\digamma$-approximation space. For any nonempty subset R of $\hbar$, the sets

$$\underline{\mathcal{C}}_\digamma(R) := \{\kappa \in \hbar \mid \mathcal{C}_\digamma[\kappa] \subseteq R\} \text{ and } \overline{\mathcal{C}}_\digamma(R) := \{\kappa \in \hbar \mid \mathcal{C}_\digamma[\kappa] \cap R \neq \emptyset\},$$

are called *lower* and *upper rough approximation*, respectively, of R with respect to the filter $\digamma$.

Example 3.1. Let $\hbar = \{0, \eta, \beta, 1\}$ be a poset such that $0 \leq \eta, \beta \leq 1$. Define the operations $\to$ and $\odot$ on $\hbar$ as follows,

$\to$	0	η	β	1
0	1	1	1	1
η	β	1	β	1
β	η	η	1	1
1	0	η	β	1

$\odot$	0	η	β	1
0	0	0	0	0
η	0	η	0	η
β	0	0	β	β
1	0	η	β	1

Then $(\hbar, \odot, \to, 0, 1)$ is a bounded hoop. Let $F = \{\eta, 1\}$. Then $\mathcal{C}_F[\eta] = \mathcal{C}_F[1] = F$ and $\mathcal{C}_F[\beta] = \mathcal{C}_F[0] = \{0, \beta\}$. Suppose $R = \{\eta, \beta, 1\}$. Then $\underline{\mathcal{C}}_F(R) = \{\eta, 1\}$ and $\overline{\mathcal{C}}_F(R) = \hbar$.

THEOREM 3.2. *If $(\hbar, \mathcal{C}_F)$ is an F-approximation space, then the lower rough approximation operator $\underline{\mathcal{C}}_F$ is an interior operator and the upper rough approximation operator $\overline{\mathcal{C}}_F$ is a closure operator.*

PROOF: Let R be a nonempty subset of $\hbar$ and $\kappa \in \underline{\mathcal{C}}_F(R)$. Then $\mathcal{C}_F[\kappa] \subseteq R$. Since $\kappa \in \mathcal{C}_F[\kappa]$, we have $\kappa \in R$. Hence, $\underline{\mathcal{C}}_F(R) \subseteq R$. If R_1 and R_2 are two subsets of $\hbar$ such that $R_1 \subseteq R_2$ and $\kappa \in \underline{\mathcal{C}}_F(R_1)$, then $\mathcal{C}_F[\kappa] \subseteq R_1$. Thus $\mathcal{C}_F[\kappa] \subseteq R_2$, and so $\kappa \in \underline{\mathcal{C}}_F(R_2)$. Hence, $\underline{\mathcal{C}}_F(R_1) \subseteq \underline{\mathcal{C}}_F(R_2)$. Since $\underline{\mathcal{C}}_F(R) \subseteq R$, we have $\underline{\mathcal{C}}_F(\underline{\mathcal{C}}_F(R)) \subseteq \underline{\mathcal{C}}_F(R)$. Conversely, suppose $\kappa \in \underline{\mathcal{C}}_F(R)$. Then $\mathcal{C}_F[\kappa] \subseteq R$. Let $\delta \in \mathcal{C}_F[\kappa]$. Then $\mathcal{C}_F[\delta] = \mathcal{C}_F[\kappa] \subseteq R$, and so $\delta \in \underline{\mathcal{C}}_F(R)$. Thus, $\mathcal{C}_F[\kappa] \subseteq \underline{\mathcal{C}}_F(R)$. Hence, $\kappa \in \underline{\mathcal{C}}_F(\underline{\mathcal{C}}_F(R))$, and so $\underline{\mathcal{C}}_F(\underline{\mathcal{C}}_F(R)) = \underline{\mathcal{C}}_F(R)$. Therefore, the lower rough approximation operator $\underline{\mathcal{C}}_F$ is an interior operator.

Let R be a nonempty subset of $\hbar$ and $\kappa \in R$. Since $\kappa \in \mathcal{C}_F[\kappa]$, we have $\kappa \in \mathcal{C}_F[\kappa] \cap R \neq \emptyset$. Thus $\kappa \in \overline{\mathcal{C}}_F(R)$. If R_1 and R_2 are two subsets of $\hbar$ such that $R_1 \subseteq R_2$ and $\kappa \in \overline{\mathcal{C}}_F(R_1)$. Then $\mathcal{C}_F[\kappa] \cap R_1 \neq \emptyset$. Thus $\mathcal{C}_F[\kappa] \cap R_2 \neq \emptyset$, and so $\kappa \in \overline{\mathcal{C}}_F(R_2)$. Hence, $\overline{\mathcal{C}}_F(R_1) \subseteq \overline{\mathcal{C}}_F(R_2)$. Since $R \subseteq \overline{\mathcal{C}}_F(R)$, we have $\overline{\mathcal{C}}_F(R) \subseteq \overline{\mathcal{C}}_F(\overline{\mathcal{C}}_F(R))$. Conversely, suppose $\kappa \in \overline{\mathcal{C}}_F(\overline{\mathcal{C}}_F(R))$. Then $\mathcal{C}_F[\kappa] \cap \overline{\mathcal{C}}_F(R) \neq \emptyset$. Let $\delta \in \mathcal{C}_F[\kappa] \cap \overline{\mathcal{C}}_F(R)$. Then $\mathcal{C}_F[\delta] = \mathcal{C}_F[\kappa]$ and $\mathcal{C}_F[\delta] \cap R \neq \emptyset$, and so $\mathcal{C}_F[\kappa] \cap R \neq \emptyset$. Thus, $\kappa \in \overline{\mathcal{C}}_F(R)$. Hence, $\overline{\mathcal{C}}_F(\overline{\mathcal{C}}_F(R)) = \overline{\mathcal{C}}_F(R)$. Therefore, the upper rough approximation operator $\overline{\mathcal{C}}_F$ is a closure operator. □

Let $(\hbar, \mathcal{C}_F)$ be an F-approximation space. A subset R of $\hbar$ is said to be *definable* with respect to F if $\underline{\mathcal{C}}_F(R) = \overline{\mathcal{C}}_F(R)$, and *rough* otherwise.

It is clear that $\emptyset$, $\hbar$ and $\mathcal{C}_{\digamma}[\kappa]$ are definable with respect to $\digamma$ in an $\digamma$-approximation space $(\hbar, \mathcal{C}_{\digamma})$.

Example 3.3. Let $\hbar$ be a hoop as in Example 3.1 and $\digamma = \{\eta, 1\}$. Suppose $R = \{0, \beta\}$. Then $\overline{\mathcal{C}}_{\digamma}(R) = \underline{\mathcal{C}}_{\digamma}(R) = \{0, \beta\}$. Hence R is definable.

THEOREM 3.4. *If $(\hbar, \mathcal{C}_{\digamma})$ is an $\digamma$-approximation space with $\digamma = \{1\}$, then every subset of $\hbar$ is definable with respect to $\digamma$.*

PROOF: Let R be an arbitrary subset of $\hbar$. Since $\digamma = \{1\}$, for all $\kappa \in \hbar$ we have

$$\mathcal{C}_{\digamma}[\kappa] = \{\delta \in \hbar \mid \kappa \to \delta = 1, \delta \to \kappa = 1\} = \{\delta \in \hbar \mid \kappa = \delta\} = \{\kappa\}.$$

Thus,

$$\underline{\mathcal{C}}_{\digamma}(R) = \{\kappa \in \hbar \mid \mathcal{C}_{\digamma}[\kappa] \subseteq R\} = \{\kappa \in \hbar \mid \{\kappa\} \subseteq R\} = R,$$
$$\overline{\mathcal{C}}_{\digamma}(R) = \{\kappa \in \hbar \mid \mathcal{C}_{\digamma}[\kappa] \cap R \neq \emptyset\} = \{\kappa \in \hbar \mid \{\kappa\} \cap R \neq \emptyset\} = R.$$

Therefore, R is definable with respect to $\digamma$. □

For any subsets R and P of a hoop $\hbar$, we define:

$$R \to P = \{\kappa \to \nu \mid \kappa \in R \textit{ and } \nu \in P\},$$
$$R \odot P = \{\kappa \odot \nu \mid \kappa \in R \ \textit{ and } \ \nu \in P\}.$$

PROPOSITION 3.5. If $(\hbar, \mathcal{C}_{\digamma})$ is an $\digamma$-approximation space, then $\overline{\mathcal{C}}_{\digamma}(R) \to \overline{\mathcal{C}}_{\digamma}(P) \subseteq \overline{\mathcal{C}}_{\digamma}(R \to P)$ and $\overline{\mathcal{C}}_{\digamma}(R) \odot \overline{\mathcal{C}}_{\digamma}(P) \subseteq \overline{\mathcal{C}}_{\digamma}(R \odot P)$ for any nonempty subsets R and P of a hoop $\hbar$.

PROOF: If $\delta \in \overline{\mathcal{C}}_{\digamma}(R) \to \overline{\mathcal{C}}_{\digamma}(P)$, then $\delta = \mathfrak{a} \to \mathfrak{b}$ for some $\mathfrak{a} \in \overline{\mathcal{C}}_{\digamma}(R)$ and $\mathfrak{b} \in \overline{\mathcal{C}}_{\digamma}(P)$. It follows that $\mathcal{C}_{\digamma}[\mathfrak{a}] \cap R \neq \emptyset$ and $\mathcal{C}_{\digamma}[\mathfrak{b}] \cap P \neq \emptyset$. Hence, there exist $\kappa \in R$ and $\nu \in P$ such that $\mathcal{C}_{\digamma}[\mathfrak{a}] = \mathcal{C}_{\digamma}[\kappa]$ and $\mathcal{C}_{\digamma}[\mathfrak{b}] = \mathcal{C}_{\digamma}[\nu]$. Since

$$\delta = \mathfrak{a} \to b \in \mathcal{C}_{\digamma}[\mathfrak{a}] \to \mathcal{C}_{\digamma}[\mathfrak{b}] = \mathcal{C}_{\digamma}[\mathfrak{a} \to \mathfrak{b}] = \mathcal{C}_{\digamma}[\kappa \to \nu],$$

we get $\mathcal{C}[\delta] = \mathcal{C}[\kappa \to \nu]$. Moreover since $\kappa \to \nu \in R \to P$ and $\mathcal{C}[\delta] = \mathcal{C}[\kappa \to \nu]$, we get $\mathcal{C}[\delta] \cap (R \to P) \neq \emptyset$, and so $\delta \in \overline{\mathcal{C}}_{\digamma}(R \to P)$. Similarly, we can verify $\overline{\mathcal{C}}_{\digamma}(R) \odot \overline{\mathcal{C}}_{\digamma}(P) \subseteq \overline{\mathcal{C}}_{\digamma}(R \odot P)$. □

DEFINITION 3.6. Let $(\hbar, \mathcal{C}_{\digamma})$ be an $\digamma$-approximation space. A subset R of $\hbar$ is called a lower (resp. upper) rough filter of $\hbar$ if $\underline{\mathcal{C}}_{\digamma}(R)$ (resp., $\overline{\mathcal{C}}_{\digamma}(R)$)

is a filter of $\hbar$. If R is both a lower rough filter and an upper rough filter of $\hbar$, we say R is a *rough filter* of $\hbar$.

Example 3.7. Let $\hbar$ be a hoop as in Example 3.1. Suppose $\digamma = \{\beta, 1\}$. Then $\digamma$ is a filter of $\hbar$. $\mathcal{C}_\digamma[1] = \mathcal{C}_\digamma[\beta] = \{\beta, 1\}$ and $\mathcal{C}_\digamma[0] = \mathcal{C}_\digamma[\eta] = \{0, \eta\}$. If $R = \{\eta, \beta, 1\}$, then $\underline{\mathcal{C}}_\digamma[R] = \{\beta, 1\}$ and $\overline{\mathcal{C}}_\digamma[R] = \hbar$. Hence, R is a rough filter of $\hbar$. If $R = \{\eta, 1\}$ which is a filter of $\hbar$, then $\underline{\mathcal{C}}_\digamma[R] = \emptyset$ and $\overline{\mathcal{C}}_\digamma[R] = \hbar$. Hence R is not a rough filter of $\hbar$.

THEOREM 3.8. *If $(\hbar, \mathcal{C}_\digamma)$ is an $\digamma$-approximation space and R is a nonempty subset of $\hbar$, then*

(*i*) *$\digamma \subseteq R$ if and only if $\digamma \subseteq \underline{\mathcal{C}}_\digamma(R)$.*

(*ii*) *$R \subseteq \digamma$ if and only if $\overline{\mathcal{C}}_\digamma(R) = \digamma$.*

(*iii*) *If G is a filter of $\hbar$, then $\digamma \subseteq \overline{\mathcal{C}}_\digamma(G)$. Also, $\digamma \subseteq G$ if and only if $\underline{\mathcal{C}}_\digamma(G) = G = \overline{\mathcal{C}}_\digamma(G)$.*

(*iv*) *Every filter which is contained in $\digamma$ is an upper rough filter of $\hbar$.*

PROOF: (*i*) Assume that $\digamma \subseteq R$ and $\delta \in \digamma$. Then $\mathcal{C}_\digamma[\delta] = \digamma \subseteq R$ and so $\delta \in \underline{\mathcal{C}}_\digamma(R)$, that is, $\digamma \subseteq \underline{\mathcal{C}}_\digamma(R)$. The converse is clear.

(*ii*) By Proposition 2.3(i), it is clear that if $\overline{\mathcal{C}}_\digamma(R) = \digamma$, then $R \subseteq \digamma$. Suppose $R \subseteq \digamma$ and $\delta \in \overline{\mathcal{C}}_\digamma(R)$. Then $\mathcal{C}_\digamma[\delta] \cap R \neq \emptyset$. Thus $\kappa \in \mathcal{C}_\digamma[\delta] \cap L$. Since $L \subseteq \digamma$, we have $\kappa \in \digamma$ and $\mathcal{C}_\digamma[\delta] = \mathcal{C}_\digamma[\kappa] = \digamma$. Thus $\delta \in \digamma$, which shows that $\overline{\mathcal{C}}_\digamma(R) \subseteq \digamma$. Now, if $\delta \in \digamma$, then $\mathcal{C}_\digamma[\delta] = \digamma$ and so $\mathcal{C}_\digamma[\delta] \cap R = \digamma \cap R = R \neq \emptyset$. Hence $\delta \in \overline{\mathcal{C}}_\digamma(R)$ and so $F \subseteq \overline{\mathcal{C}}_\digamma(R)$. Therefore, $\overline{\mathcal{C}}_\digamma(R) = \digamma$.

(*iii*) Let G be a filter of $\hbar$. If $\nu \in \digamma$, then $\mathcal{C}_\digamma[\nu] = \digamma$ and $1 \in \digamma \cap G = \mathcal{C}_\digamma[\nu] \cap G$ and so $\nu \in \overline{\mathcal{C}}_\digamma(G)$. Hence $\digamma \subseteq \overline{\mathcal{C}}_\digamma(G)$. Assume that $\digamma \subseteq G$. By Proposition 2.3(i), it is clear that $G \subseteq \overline{\mathcal{C}}_\digamma(G)$ and $\underline{\mathcal{C}}_\digamma(G) \subseteq G$. If $\delta \in \overline{\mathcal{C}}_\digamma(G)$, then $\mathcal{C}_\digamma[\delta] \cap G \neq \emptyset$. Hence $\mathcal{C}_\digamma[\delta] = \mathcal{C}_\digamma[\kappa]$, for some $\kappa \in G$. It follows that $\delta \to \kappa \in \digamma \subseteq G$ and $\kappa \to \delta \in \digamma \subseteq G$. Since G is a filter of $\hbar$ and $\kappa \in G$, we have $\delta \in G$ and so $G = \overline{\mathcal{C}}_\digamma(G)$. Let $\nu \in G$. If $\mathfrak{a} \in \mathcal{C}_\digamma[\nu]$, then $\mathfrak{a} \to \nu, \nu \to \mathfrak{a} \in \digamma \subseteq G$. Since G is a filter of $\hbar$, it follows that $\mathfrak{a} \in G$, and so $\mathcal{C}_\digamma[\nu] \subseteq G$ and $\nu \in \underline{\mathcal{C}}_\digamma(G)$. Thus $\underline{\mathcal{C}}_\digamma(G) = G$. Conversely, suppose $\underline{\mathcal{C}}_\digamma(G) = G = \overline{\mathcal{C}}_\digamma(G)$ and $\nu \in \digamma$. Since $1 \in \mathcal{C}_\digamma[\nu] \cap G = \digamma \cap G$, we have $\nu \in \overline{\mathcal{C}}_\digamma(G) = G$. Thus $\digamma \subseteq G$.

(*iv*) It is clear by (ii). $\square$

The following corollary is obtained from Theorem 3.8.

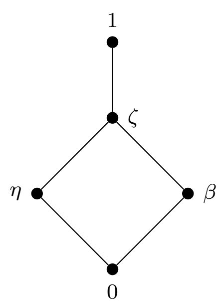
Corollary 3.9. In an $\digamma$-approximation space $(\hbar, \mathcal{C}_\digamma)$, every filter containing $\digamma$ is a rough filter of $\hbar$ and every nonempty subset contained in $\digamma$ is an upper rough filter of $\hbar$.

Proposition 3.10. Let $(\hbar, \mathcal{C}_\digamma)$ be an $\digamma$-approximation space in which $\hbar$ is bounded. Then the upper rough approximation operator $\overline{\mathcal{C}}_\digamma$ satisfies $\neg\overline{\mathcal{C}}_\digamma(R) \subseteq \overline{\mathcal{C}}_\digamma(\neg R)$ for all nonempty subset R of $\hbar$.

Proof: Let $\nu \in \neg\overline{\mathcal{C}}_\digamma(R)$. Then $\nu = \neg\delta$ for some $\delta \in \hbar$ such that $\mathcal{C}_\digamma[\delta] \cap R \neq \emptyset$. Hence there exists $\kappa \in R$ such that $\mathcal{C}_\digamma[\delta] = \mathcal{C}_\digamma[\kappa]$, which implies that $\mathcal{C}_\digamma[\nu] = \mathcal{C}_\digamma[\neg\delta] = \mathcal{C}_\digamma[\neg\kappa]$. Since $\neg\kappa \in \neg R$, we get $\mathcal{C}_\digamma[\nu] \cap \neg R = \mathcal{C}_\digamma[\neg\kappa] \cap \neg R \neq \emptyset$. Hence $\nu \in \overline{\mathcal{C}}_\digamma(\neg R)$. Therefore, $\neg\overline{\mathcal{C}}_\digamma(R) \subseteq \overline{\mathcal{C}}_\digamma(\neg R)$. $\square$

Now by below example we show that the reverse inclusion in Proposition 3.10 is not true, in general.

Example 3.11. Let $\hbar = \{0, \eta, \beta, \zeta, 1\}$ be a poset with the following Hasse diagram. Define the operations $\odot$ and $\to$ on $\hbar$ as follows,



$\to$	0	η	β	ζ	1
0	1	1	1	1	1
η	β	1	β	1	1
β	η	η	1	1	1
ζ	0	η	β	1	1
1	0	η	β	ζ	1

$\odot$	0	η	β	ζ	1
0	0	0	0	0	0
η	0	η	0	η	η
β	0	0	β	β	β
ζ	0	η	β	ζ	ζ
1	0	η	β	ζ	1

Then $(\hbar, \odot, \to, 0, 1)$ is a bounded hoop. Suppose $\digamma = \{\zeta, 1\}$. Then $\mathcal{C}_\digamma[\zeta] = \mathcal{C}_\digamma[1] = \{\zeta, 1\}$, $\mathcal{C}_\digamma[\eta] = \{\eta\}$, $\mathcal{C}_\digamma[\beta] = \{\beta\}$ and $\mathcal{C}_\digamma[0] = \{0\}$.

Thus $\overline{\mathcal{C}}_{\mathcal{F}}(\hbar) = \hbar$ and $\neg\overline{\mathcal{C}}_{\mathcal{F}}(\hbar) = \{0, \eta, \beta, 1\}$. Also, $\neg\hbar = \{0, \eta, \beta, 1\}$ and so $\overline{\mathcal{C}}_{\mathcal{F}}(\neg\hbar) = \hbar$. Hence, $\overline{\mathcal{C}}_{\mathcal{F}}(\neg\hbar) \nsubseteq \neg\overline{\mathcal{C}}_{\mathcal{F}}(\hbar)$.

In the following example we show that lower rough approximation operator $\underline{\mathcal{C}}_{\mathcal{F}}$ does not satisfies in the condition of Proposition 3.10.

Example 3.12. Let $\hbar$ be the hoop as in Example 3.11 and $R = \{\beta, 1\}$. Then $\underline{\mathcal{C}}_{\mathcal{F}}(R) = \{\beta\}$ and so $\neg\underline{\mathcal{C}}_{\mathcal{F}}(R) = \{\eta\}$. Moreover, $\neg R = \{0, \eta\}$ and so $\underline{\mathcal{C}}_{\mathcal{F}}(\neg R) = \{0, \eta\}$. Hence, $\underline{\mathcal{C}}_{\mathcal{F}}(\neg R) \nsubseteq \neg\underline{\mathcal{C}}_{\mathcal{F}}(R)$. Also, if $R = \{0, \eta, \beta\}$, then $\underline{\mathcal{C}}_{\mathcal{F}}(R) = \{0, \eta, \beta\}$ and so $\neg\underline{\mathcal{C}}_{\mathcal{F}}(R) = \{\eta, \beta, 1\}$. Moreover, $\neg R = \{\eta, \beta, 1\}$. Then $\underline{\mathcal{C}}_{\mathcal{F}}(\neg R) = \{\eta, \beta\}$. Hence, $\neg\underline{\mathcal{C}}_{\mathcal{F}}(R) \nsubseteq \underline{\mathcal{C}}_{\mathcal{F}}(\neg R)$. Therefore, lower rough approximation operator $\underline{\mathcal{C}}_{\mathcal{F}}$ does not satisfies in the condition of Proposition 3.10.

Proposition 3.13. If $(\hbar, \mathcal{C}_{\mathcal{F}})$ is an $\mathcal{F}$-approximation space and R is a nonempty subset of $\hbar$, then

(*i*) $DNP(\hbar) \cap \overline{\mathcal{C}}_{\mathcal{F}}(\neg R) \subseteq \neg\overline{\mathcal{C}}_{\mathcal{F}}(\neg(\neg R))$.

(*ii*) $DNP(\hbar) \cap \overline{\mathcal{C}}_{\mathcal{F}}(\neg(R \cap DNP(\hbar))) \subseteq \neg\overline{\mathcal{C}}_{\mathcal{F}}(R)$.

Proof: (*i*) If $\kappa \in DNP(\hbar) \cap \overline{\mathcal{C}}_{\mathcal{F}}(\neg R)$, then $\neg(\neg\kappa) = \kappa$ and since $\kappa \in \overline{\mathcal{C}}_{\mathcal{F}}(\neg R)$, there exists $\nu \in R$ such that $\mathcal{C}_{\mathcal{F}}[\kappa] = \mathcal{C}_{\mathcal{F}}[\neg\nu]$.

It follows that $\mathcal{C}_{\mathcal{F}}[\neg\kappa] \cap \neg(\neg R) = \mathcal{C}_{\mathcal{F}}(\neg[\neg\nu]) \cap \neg(\neg R) \neq \emptyset$, that is, $\neg\kappa \in \overline{\mathcal{C}}_{\mathcal{F}}(\neg(\neg R))$. Hence $\kappa \in \neg\overline{\mathcal{C}}_{\mathcal{F}}(\neg(\neg R))$. Therefore, $DNP(\hbar) \cap \overline{\mathcal{C}}_{\mathcal{F}}(\neg R) \subseteq \neg\overline{\mathcal{C}}_{\mathcal{F}}(\neg(\neg R))$.

(*ii*) Let $\delta \in DNP(\hbar) \cap \overline{\mathcal{C}}_{\mathcal{F}}(\neg(R \cap DNP(\hbar)))$. Then $\neg(\neg\delta) = \delta$ and $\mathcal{C}_{\mathcal{F}}[\delta] \cap \neg(R \cap DNP(\hbar)) \neq \emptyset$. Thus there exists $\kappa \in \mathcal{C}_{\mathcal{F}}[\delta] \cap \neg(R \cap DNP(\hbar))$, it means that, $\mathcal{C}_{\mathcal{F}}[\delta] = \mathcal{C}_{\mathcal{F}}[\kappa]$ and there exists $\nu \in R \cap DNP(\hbar)$ such that $\kappa = \neg\nu$ and so $\mathcal{C}_{\mathcal{F}}[\delta] = \mathcal{C}_{\mathcal{F}}[\neg\nu]$. Then $\mathcal{C}_{\mathcal{F}}[\neg\delta] \cap R = \mathcal{C}_{\mathcal{F}}[\neg(\neg\nu)] \cap R = \mathcal{C}_{\mathcal{F}}[\nu] \cap R \neq \emptyset$, that is, $\delta \in \neg\overline{\mathcal{C}}_{\mathcal{F}}(R)$. Therefore, $DNP(\hbar) \cap \overline{\mathcal{C}}_{\mathcal{F}}(\neg(R \cap DNP(\hbar))) \subseteq \neg\overline{\mathcal{C}}_{\mathcal{F}}(R)$. □

Proposition 3.14. If $\hbar$ is a bounded hoop, then the set $\hbar^{\bigstar} := \{\kappa \in \hbar \mid \neg\kappa = 0\}$ is a filter of $\hbar$.

Proof: Since $\neg 1 = 0$, we have $1 \in \hbar^{\bigstar}$. Consider $\kappa, \nu \in \hbar$ so that $\kappa, \kappa \to \nu \in \hbar^{\bigstar}$. Then $\neg\kappa = 0$ and $\neg(\kappa \to \nu) = 0$. Considering Proposition 2.2(i) and $\nu \leq \neg\neg\nu$, we get $\kappa \to \nu \leq \kappa \to \neg\neg\nu = \neg\nu \to \neg\kappa$. Hence

$$\neg\nu = \neg\neg\neg\nu = \neg(\neg\nu \to 0) = \neg(\neg\nu \to \neg\kappa) \leq \neg(\kappa \to \nu) = 0,$$

and so $\neg\nu = 0$, that is, $\nu \in \hbar^{\bigstar}$. So it is proved that $\hbar^{\bigstar}$ is a filter of $\hbar$. □

Corollary 3.15. If $(\hbar, \mathcal{C}_{\digamma})$ is an $\digamma$-approximation space in which $\hbar$ is bounded, then

$$\digamma \subseteq \overline{\mathcal{C}}_{\digamma}(\hbar^{\bigstar}) \subseteq \digamma^{\bigstar},$$

where $\digamma^{\bigstar} := \{\delta \in \hbar \mid \neg(\neg\delta) \in \digamma\}$.

Proof: By Theorem 3.8(iii) and Proposition 3.14, we know that $\digamma \subseteq \overline{\mathcal{C}}_{\digamma}(\hbar^{\bigstar})$. Let $\kappa \in \overline{\mathcal{C}}_{\digamma}(\hbar^{\bigstar})$. Then $\mathcal{C}_{\digamma}[\kappa] \cap \hbar^{\bigstar} \neq \emptyset$, which implies that there exists $\mathfrak{a} \in \mathcal{C}_{\digamma}[\kappa]$ such that $\neg\mathfrak{a} = 0$. Thus $\mathcal{C}_{\digamma}[0] = \mathcal{C}_{\digamma}[\neg\mathfrak{a}] = \mathcal{C}_{\digamma}[\neg\kappa]$, and so $\neg(\neg\kappa) = \neg\kappa \to 0 \in \digamma$, i.e., $\kappa \in \digamma^{\bigstar}$. $\square$

We provide conditions for a nonempty subset to be definable with respect to a filter of $\hbar$.

Theorem 3.16. *Let $(\hbar, \mathcal{C}_{\digamma})$ be an $\digamma$-approximation space. Then a nonempty subset R of $\hbar$ is definable with respect to $\digamma$ if and only if $\underline{\mathcal{C}}_{\digamma}(R) = R$ or $\overline{\mathcal{C}}_{\digamma}(R) = R$.*

Proof: The necessity is clear. Conversely, suppose $\underline{\mathcal{C}}_{\digamma}(R) = R$. By Proposition 2.3(i), it is clear that $R \subseteq \overline{\mathcal{C}}_{\digamma}(R)$. Suppose $\kappa \in \overline{\mathcal{C}}_{\digamma}(R)$. Then $\mathcal{C}_{\digamma}[\kappa] \cap L \neq \emptyset$. Thus, there exists $\nu \in \mathcal{C}_{\digamma}[\kappa] \cap L$, such that $\mathcal{C}_{\digamma}[\kappa] = \mathcal{C}_{\digamma}[\nu]$. Since $\nu \in R$ and $\underline{\mathcal{C}}_{\digamma}(R) = R$, we have $\nu \in \underline{\mathcal{C}}_{\digamma}(R)$. Then $\mathcal{C}_{\digamma}[\nu] \subseteq R$. Thus, $\mathcal{C}_{\digamma}[\kappa] \subseteq R$, and so $\kappa \in R$. Hence, $\overline{\mathcal{C}}_{\digamma}(R) \subseteq R$ and so $\overline{\mathcal{C}}_{\digamma}(R) = R$. Therefore, $\underline{\mathcal{C}}_{\digamma}(R) = R = \overline{\mathcal{C}}_{\digamma}(R)$ and R is definable. Now, assume that $\overline{\mathcal{C}}_{\digamma}(R) = R$. Obviously, $\underline{\mathcal{C}}_{\digamma}(R) \subseteq R$. For any $\kappa \in R$, let $\delta \in \mathcal{C}_{\digamma}[\kappa]$. Then $\mathcal{C}_{\digamma}[\delta] \cap R = \mathcal{C}_{\digamma}[\kappa] \cap R \neq \emptyset$ and so $\delta \in \overline{\mathcal{C}}_{\digamma}(R) = R$. Hence $\mathcal{C}_{\digamma}[\kappa] \subseteq R$, i.e., $\kappa \in \underline{\mathcal{C}}_{\digamma}(R)$. Then $\underline{\mathcal{C}}_{\digamma}(R) = R = \overline{\mathcal{C}}_{\digamma}(R)$. Therefore, R is definable with respect to $\digamma$. $\square$

Theorem 3.17. *Let $\digamma$ and G be two filters of a hoop $\hbar$. For any nonempty subset $\digamma$ of a hoop $\hbar$, we have*

(*i*) *If $R \subseteq \digamma \cap G$, then $\overline{\mathcal{C}}_{\digamma\cap G}(R) = \overline{\mathcal{C}}_{\digamma}(R) \cap \overline{\mathcal{C}}_{G}(R)$.*

(*ii*) *If R is definable with respect to $\digamma$, then $\overline{\mathcal{C}}_{\digamma\cap G}(R) = \overline{\mathcal{C}}_{\digamma}(R) \cap \overline{\mathcal{C}}_{G}(R)$.*

(*iii*) *If R contains $\digamma$ and G, then $\underline{\mathcal{C}}_{\digamma\cap G}(R) = \underline{\mathcal{C}}_{\digamma}(R) \cap \underline{\mathcal{C}}_{G}(R)$.*

Proof: (*i*) Let $\kappa \in \overline{\mathcal{C}}_{\digamma\cap G}(R)$. Then $\mathcal{C}_{\digamma\cap G}[\kappa] \cap R \neq \emptyset$. Thus there exists $\mathfrak{a} \in \mathcal{C}_{\digamma\cap G}[\kappa] \cap R$. Since $\mathfrak{a} \in R$ and $R \subseteq \digamma \cap G$, we get $\mathfrak{a} \in \digamma$ and $\mathfrak{a} \in G$. Moreover, from $\mathfrak{a} \in \mathcal{C}_{\digamma\cap G}[\kappa]$, we get $\mathfrak{a} \to \kappa, \kappa \to \mathfrak{a} \in \digamma \cap G$.

Since F and G are two filters of $\hbar$, we have $\kappa \in F$ and $\kappa \in G$. Then $\mathcal{C}_F[\kappa] = \mathcal{C}_F[\mathfrak{a}]$, and so $\mathcal{C}_F[\kappa] \cap R \neq \emptyset$. By the similar way, $\mathcal{C}_G[\kappa] \cap R \neq \emptyset$. Hence, $\kappa \in \overline{\mathcal{C}}_F(R) \cap \overline{\mathcal{C}}_G(R)$. Therefore, $\overline{\mathcal{C}}_{F\cap G}(R) \subseteq \overline{\mathcal{C}}_F(R) \cap \overline{\mathcal{C}}_G(R)$.

Conversely, suppose $\kappa \in \overline{\mathcal{C}}_F(R) \cap \overline{\mathcal{C}}_G(R)$. Since $\kappa \in \overline{\mathcal{C}}_F(R)$, we have $\mathcal{C}_F[\kappa] \cap R \neq \emptyset$. Then there exists $\mathfrak{a} \in \mathcal{C}_F[\kappa] \cap R$ such that $\kappa \to \mathfrak{a}, \mathfrak{a} \to \kappa \in F$. By the similar way, there exists $\mathfrak{b} \in \mathcal{C}_G[\kappa] \cap R$ such that $\mathfrak{b} \to \kappa, \kappa \to \mathfrak{b} \in G$. Since $\mathfrak{a}, \mathfrak{b} \in R$, $R \subseteq F \cap G$ and F and G are two filters of $\hbar$, we have $\kappa \in F \cap G$ and $\mathfrak{a}, \mathfrak{b} \in F \cap G$. The $\mathcal{C}_{F\cap G}[\kappa] = \mathcal{C}_{F\cap G}[\mathfrak{a}] = \mathcal{C}_{F\cap G}[\mathfrak{b}]$. Hence, $\mathcal{C}_{F\cap G}[\kappa] \cap R \neq \emptyset$, and so $\kappa \in \overline{\mathcal{C}}_{F\cap G}(R)$. Thus, $\overline{\mathcal{C}}_F(R) \cap \overline{\mathcal{C}}_G(R) \subseteq \overline{\mathcal{C}}_{F\cap G}(R)$. Therefore, $\overline{\mathcal{C}}_F(R) \cap \overline{\mathcal{C}}_G(R) = \overline{\mathcal{C}}_{F\cap G}(R)$.

(ii) Suppose R is definable with respect to F. Then $\overline{\mathcal{C}}_F(R) = R = \underline{\mathcal{C}}_F(R)$. Thus, $\overline{\mathcal{C}}_F(R) \cap \overline{\mathcal{C}}_G(R) = R \cap \overline{\mathcal{C}}_G(R) = R$. Moreover, by defination of upper approximation, we have $R \subseteq \overline{\mathcal{C}}_{F\cap G}(R)$. Now, suppose $\kappa \in \overline{\mathcal{C}}_{F\cap G}(R)$. Then $\mathcal{C}_{F\cap G}[\kappa] \cap R \neq \emptyset$. Let $\nu \in \mathcal{C}_{F\cap G}[\kappa] \cap R$. Since $\nu \in R$ and R is definable with respect to F, we get $\mathcal{C}_F[\nu] \subseteq R$. Also, from $\nu \in \mathcal{C}_{F\cap G}[\kappa]$, we obtain, $\kappa \to \nu, \nu \to \kappa \in F \cap G \subseteq F$. Then $\kappa \in \mathcal{C}_F[\nu] \subseteq R$, and so $\kappa \in R$. Hence, $\overline{\mathcal{C}}_{F\cap G}(R) \subseteq R$. Thus, $\overline{\mathcal{C}}_{F\cap G}(R) = R$. Therefore, $\overline{\mathcal{C}}_{F\cap G}(R) = \overline{\mathcal{C}}_F(R) \cap \overline{\mathcal{C}}_G(R)$.

(iii) Let R be a filter of a hoop $\hbar$ containing F and G and $\kappa \in \underline{\mathcal{C}}_{F\cap G}(R)$. Then $\mathcal{C}_{F\cap G}[\kappa] \subseteq R$, and so $\kappa \in R$. Thus, for any $\mathfrak{a} \in \mathcal{C}_F[\kappa]$, we have $\mathfrak{a} \to \kappa, \kappa \to \mathfrak{a} \in F$. Since R is a filter of $\hbar$ such that $F \subseteq R$ and $\kappa \in R$, we get $\mathfrak{a} \in R$. By the similar way, for any $\mathfrak{b} \in \mathcal{C}_G[\kappa]$, we have $\mathfrak{b} \in R$. Hence, $\mathcal{C}_F[\kappa] \subseteq R$ and $\mathcal{C}_G[\kappa] \subseteq R$. Then $\kappa \in \underline{\mathcal{C}}_F(R)$ and $\kappa \in \underline{\mathcal{C}}_G(R)$, and so $\kappa \in \underline{\mathcal{C}}_F(R) \cap \underline{\mathcal{C}}_G(R)$. Hence, $\underline{\mathcal{C}}_{F\cap G}(R) \subseteq \underline{\mathcal{C}}_F(R) \cap \underline{\mathcal{C}}_G(R)$.

Conversely, suppose $\kappa \in \underline{\mathcal{C}}_F(R) \cap \underline{\mathcal{C}}_G(R)$. Then $\kappa \in \underline{\mathcal{C}}_F(R)$ and $\kappa \in \underline{\mathcal{C}}_G(R)$, $\mathcal{C}_F[\kappa] \subseteq R$, $\mathcal{C}_G[\kappa] \subseteq R$ and so $\kappa \in R$. Let $\nu \in \mathcal{C}_{F\cap G}[\kappa]$. Then by assumption, $\nu \to \kappa, \kappa \to \nu \in F \cap G \subseteq R$. Since $\kappa \to \nu \in R$, $\kappa \in R$ and R is a filter of $\hbar$, we get $\nu \in R$, and so $\mathcal{C}_{F\cap G}[\kappa] \subseteq R$. Thus, $\kappa \in \underline{\mathcal{C}}_{F\cap G}(R)$. Hence, $\underline{\mathcal{C}}_F(R) \cap \underline{\mathcal{C}}_G(R) \subseteq \underline{\mathcal{C}}_{F\cap G}(R)$. Therefore, $\underline{\mathcal{C}}_F(R) \cap \underline{\mathcal{C}}_G(R) = \underline{\mathcal{C}}_{F\cap G}(R)$. □

Lemma 3.18. *Let $f : \hbar \to \mathbb{k}$ be a homomorphism of hoops. Then*

(i) $ker(f) = \{\kappa \in \hbar \mid f(\kappa) = 1\}$ is a filter of $\hbar$.

(ii) If f is an epimorphism such that F is a filter of $\hbar$ and $Kerf \subseteq F$, then $f(F)$ is a filter of $\mathbb{k}$.

Proof: (i) Since f is a homomorphism of hoops, it is clear that $f(1) = 1 \in kerf$. Suppose $\kappa, \nu \in \hbar$ such that $\kappa, \kappa \to \nu \in kerf$. Then $f(\kappa) = f(\kappa \to \nu) = 1$. Since f is a homomorphism of hoop, we have $f(\nu) = 1 \to$

$f(\nu) = f(\kappa) \to f(\nu) = f(\kappa \to \nu) = 1$. Hence, $f(\nu) = 1$, and so $\nu \in kerf$. Therefore, $kerf$ is a filter of $\hbar$.

(ii) Since f is a hoop homomorphism and $\digamma$ is a filter of $\hbar$, it is clear that $1 = f(1) \in f(\digamma)$. Suppose $\kappa, \kappa \to \nu \in f(\digamma)$. Then there are $\mathfrak{a}, \mathfrak{b} \in \digamma$ such that $f(\mathfrak{a}) = \kappa$ and $f(\mathfrak{b}) = \kappa \to \nu$. Since f is onto and $\nu \in \Bbbk$, there exists $\mathfrak{c} \in \hbar$ such that $f(\mathfrak{c}) = \nu$. Thus $f(\mathfrak{b}) = \kappa \to \nu = f(\mathfrak{a}) \to f(\mathfrak{c}) = f(\mathfrak{a} \to \mathfrak{c})$. Thus $\mathfrak{b} \to (\mathfrak{a} \to \mathfrak{c}) \in Kerf \subseteq \digamma$. Since $b \in \digamma$ and $\digamma$ is a filter of $\hbar$, we have $\mathfrak{a} \to \mathfrak{c} \in \digamma$. From $\digamma$ is a filter of $\hbar$, $\mathfrak{a} \in \digamma$ and $\mathfrak{a} \to \mathfrak{c} \in \digamma$, we get $\mathfrak{c} \in \digamma$. Hence, $\nu = f(\mathfrak{c}) \in f(\digamma)$. Therefore, $f(\digamma)$ is a filter of $\Bbbk$. □

THEOREM 3.19. *Let $f : \hbar \to \Bbbk$ be an isomorphism of hoops. Then*

(i) *$f(\overline{\mathcal{C}}_{ker(f)}(R)) = f(R)$ for any nonempty subset R of $\hbar$.*

(ii) *If G is a filter of $\Bbbk$, then $f^{-1}(\overline{\mathcal{C}}_G(f(R)) = \overline{\mathcal{C}}_{f^{-1}(G)}(R)$ for any nonempty subset R of $\hbar$.*

(iii) *Assume that f is onto. If $\digamma$ is a filter of $\hbar$ which contains $ker(f)$, then $f(\overline{\mathcal{C}}_\digamma(R)) = \overline{\mathcal{C}}_{f(\digamma)}(f(R))$ for any nonempty subset R of $\hbar$.*

PROOF: (i) Since by Lemma 3.18, $kerf$ is a filter of $\hbar$, by Proposition 2.3(i), we have $R \subseteq \overline{\mathcal{C}}_{ker(f)}(R)$, and so it is clear that $f(R) \subseteq f(\overline{\mathcal{C}}_{ker(f)}(R))$. Suppose $\nu \in f(\overline{\mathcal{C}}_{ker(f)}(R))$. Then there exists $\kappa \in \overline{\mathcal{C}}_{ker(f)}(R)$ such that $f(\kappa) = \nu$. Since $\kappa \in \overline{\mathcal{C}}_{ker(f)}(R)$, we have $\mathcal{C}_{ker(f)}[\kappa] \cap R \neq \emptyset$. Then there is $\delta \in \mathcal{C}_{ker(f)}[\kappa] \cap R$ such that $\mathcal{C}_{ker(f)}[\kappa] = \mathcal{C}_{ker(f)}[\delta]$ and $\delta \in R$. Thus, $\kappa \to \delta, \delta \to \kappa \in ker(f)$. So, $f(\kappa) \to f(\delta) = f(\delta) \to f(\kappa) = 1$. Hence, $f(\kappa) = f(\delta)$. Since $\delta \in R$, we have $\nu = f(\kappa) = f(\delta) \in f(R)$. Hence, $f(\overline{\mathcal{C}}_{ker(f)}(R)) \subseteq f(R)$. Therefore, $f(\overline{\mathcal{C}}_{ker(f)}(R)) = f(R)$.

(ii) Let $\kappa \in f^{-1}(\overline{\mathcal{C}}_G(f(R))$. Then $f(\kappa) \in \overline{\mathcal{C}}_G(f(R))$, and so $\mathcal{C}_G[f(\kappa)] \cap f(R) \neq \emptyset$. Thus $\nu \in \mathcal{C}_G[f(\kappa)] \cap f(R)$. So $\mathcal{C}_G[f(\kappa)] = \mathcal{C}_G[\nu]$ and $\nu \in f(R)$. Thus, there exists $\delta \in R$ such that $f(\delta) = \nu$, and so $f(\delta) \in \mathcal{C}_G[\nu]$. Then $\mathcal{C}_G[f(\delta)] = \mathcal{C}_G[f(\kappa)]$. Thus, $f(\kappa \to \delta) \in G$ and $f(\delta \to \kappa) \in G$ and so $\kappa \to \delta, \delta \to \kappa \in f^{-1}(G)$. Hence, $\mathcal{C}_{f^{-1}(G)}[\kappa] = \mathcal{C}_{f^{-1}(G)}[\delta]$, and so $\delta \in \mathcal{C}_{f^{-1}(G)}[\kappa] \cap R$. Therefore, $\kappa \in \overline{\mathcal{C}}_{f^{-1}(G)}(R)$. The proof of converse is similar.

(iii) Suppose f is onto and $\digamma$ is a filter of $\hbar$ which contains $ker(f)$. Then by Lemma 3.18, $f(\digamma)$ is a filter of $\Bbbk$. Let $\nu \in f(\overline{\mathcal{C}}_\digamma(R))$. Then there exists $\kappa \in \overline{\mathcal{C}}_\digamma(R)$ such that $\nu = f(\kappa)$. Since $\kappa \in \overline{\mathcal{C}}_\digamma(R)$, we have $\mathcal{C}_\digamma[\kappa] \cap R \neq \emptyset$. Then there exists $\mathfrak{a} \in \mathcal{C}_\digamma[\kappa] \cap R$ such that $f(\mathfrak{a}) \in f(R)$, and $\mathcal{C}_\digamma[\kappa] = \mathcal{C}_\digamma[\mathfrak{a}]$, and so $\kappa \to \mathfrak{a}, \mathfrak{a} \to \kappa \in \digamma$. Thus, $f(\kappa) \to f(\mathfrak{a}), f(\mathfrak{a}) \to f(\kappa) \in f(\digamma)$.

Hence, from $\nu = f(\kappa)$ we get $\mathcal{C}_{f(F)}[\nu] = \mathcal{C}_{f(F)}[f(\kappa)] = \mathcal{C}_{f(F)}[f(\mathfrak{a})]$. So $f(\mathfrak{a}) \in \mathcal{C}_{f(F)}[\nu] \cap f(R) \neq \emptyset$. Then $\nu \in \overline{\mathcal{C}}_{f(F)}[f(R)]$. Therefore, $f(\overline{\mathcal{C}}_F(R)) \subseteq \overline{\mathcal{C}}_{f(F)}(f(R))$.
Conversely, let $\kappa \in \overline{\mathcal{C}}_{f(F)}(f(R))$. Then $\mathcal{C}_{f(F)}[\kappa] \cap f(R) \neq \emptyset$. Since f is onto, there exists $\mathfrak{a} \in \hbar$ such that $f(\mathfrak{a}) = \kappa$. Suppose $\nu \in \mathcal{C}_{f(F)}[\kappa] \cap f(R)$. Then there exist $\mathfrak{b} \in R$ such that $f(\mathfrak{b}) = \nu$. Since $\nu \in \mathcal{C}_{f(F)}[\kappa]$, we have $\nu \to \kappa, \kappa \to \nu \in f(F)$. Thus there are $\mathfrak{m}, \mathfrak{n} \in F$ such that $\nu \to \kappa = f(\mathfrak{m})$ and $\kappa \to \nu = f(\mathfrak{n})$. So $f(\mathfrak{b}) \to f(\mathfrak{a}) = f(\mathfrak{m})$ and $f(\mathfrak{a}) \to f(\mathfrak{b}) = f(\mathfrak{n})$. Since $kerf \subseteq F$ and $\mathfrak{m} \in F$, we get $(\mathfrak{b} \to \mathfrak{a}) \to \mathfrak{m} \in F$ and $\mathfrak{m} \to (\mathfrak{b} \to \mathfrak{a}) \in F$, and so $\mathfrak{b} \to \mathfrak{a} \in F$. By the similar way, $\mathfrak{a} \to \mathfrak{b} \in F$. Thus $\mathcal{C}_F[\mathfrak{a}] = \mathcal{C}_F[\mathfrak{b}]$. Morever, from $\mathfrak{b} \in \mathcal{C}_F[\mathfrak{b}] \cap R$, we get $\mathfrak{b} \in \overline{\mathcal{C}}_F(R)$, and so $\nu \in f(\mathfrak{b}) \in f(\overline{\mathcal{C}}_F(R))$. Hence, $\overline{\mathcal{C}}_{f(F)}(f(R)) \subseteq f(\overline{\mathcal{C}}_F(R))$. Therefore, $f(\overline{\mathcal{C}}_F(R)) = \overline{\mathcal{C}}_{f(F)}(f(R))$. □

We define a hyper operation "$\circledast$" on $\hbar$ as follows:

$$\circledast : \hbar \times \hbar \to \mathcal{P}(\hbar),\ (\kappa, \nu) \mapsto \{\delta \in \hbar \mid \kappa \odot \nu \leq \delta\}.$$

For any $\kappa, \nu \in \hbar$, $\circledast(\kappa, \nu)$ will be denoted by $\kappa \circledast \nu$, that is, $\kappa \circledast \nu := \{\delta \in \hbar \mid \kappa \odot \nu \leq \delta\}$. It is clear that the operation "$\circledast$" is commutative and associative. For any nonempty subsets F and G of a hoop $\hbar$, we define

$$F \circledast G := \bigcup_{\kappa \in F,\, \nu \in G} \kappa \circledast \nu. \tag{3.1}$$

Example 3.20. Let H be the hoop as in Example 3.11. Suppose $F = \{\zeta, 1\}$. Then by routine calculation, it is clear that $\kappa \circledast 0 = \beta \circledast \eta = \hbar$ for any $\kappa \in \hbar$, $\eta \circledast \eta = \eta \circledast 1 = \eta \circledast \zeta = \{\eta, \zeta, 1\}, \beta \circledast \beta = \zeta \circledast \beta = \beta \circledast 1 = \{\beta, \zeta, 1\}, \zeta \circledast \zeta = \zeta \circledast 1 = \{\zeta, 1\}$.
Now, if we consider $K = \{\eta, 1\}$ and $G = \{\zeta\}$, which are two nonempty subsets of $\hbar$, then $K \circledast G := \bigcup_{\kappa \in K,\, \nu \in G} \kappa \circledast \nu = \{\eta, \zeta, 1\}$.

THEOREM 3.21. *If F and G are two filters of $\hbar$, then $F \circledast G$ is the smallest filter of $\hbar$ which contains F and G.*

PROOF: Let F and G be two filters of a hoop $\hbar$. Then $1 \in F$ and $1 \in G$, and so $1 \circledast 1 = \{\kappa \in \hbar \mid 1 = 1 \odot 1 \leq \kappa\} = \{1\}$. Thus $1 \in F \circledast G$. Now, suppose $\kappa, \nu \in \hbar$ such that $\kappa, \kappa \to \nu \in F \circledast G$. Since $F \circledast G := \bigcup_{\mathfrak{a} \in F,\, \mathfrak{b} \in G} \mathfrak{a} \circledast \mathfrak{b}$, there exist $\mathfrak{a}, \mathfrak{c} \in F$ and $\mathfrak{b}, d \in G$ such that $\kappa \in \mathfrak{a} \circledast \mathfrak{b}$ and $\kappa \to \nu \in \mathfrak{c} \circledast d$. Thus

$\kappa \in \mathfrak{a} \circledast \mathfrak{b} = \{\delta \in \hbar \mid \mathfrak{a} \odot \mathfrak{b} \leq \delta\}$ and $\kappa \to \nu \in \mathfrak{c} \circledast d = \{\mathfrak{w} \in \hbar \mid \mathfrak{c} \odot d \leq \mathfrak{w}\}$. So, $\mathfrak{a} \odot \mathfrak{b} \leq \kappa$ and $\mathfrak{c} \odot d \leq \kappa \to \nu$. By Proposition 2.1(vii) and (xi), we have

$$(\mathfrak{a} \odot \mathfrak{c}) \odot (\mathfrak{b} \odot d) \leq \mathfrak{a} \odot \mathfrak{b} \odot \mathfrak{c} \odot d \leq \kappa \odot \mathfrak{c} \odot d \leq \kappa \odot (\kappa \to \nu) \leq \nu.$$

Then $(\mathfrak{a} \odot \mathfrak{c}) \odot (\mathfrak{b} \odot d) \leq \nu$. Since F and G are two filters of $\hbar$, $\mathfrak{a}, \mathfrak{c} \in F$ and $\mathfrak{b}, d \in G$, we have $\mathfrak{a} \odot \mathfrak{c} \in F$ and $\mathfrak{b} \odot d \in G$. Hence $\nu \in (\mathfrak{a} \odot \mathfrak{c}) \circledast (\mathfrak{b} \odot d) \subseteq F \circledast G$, and so $F \circledast G$ is a filter of $\hbar$. Suppose J is a filter of $\hbar$ which contains F and G. If $\kappa \in F \circledast G$, then there are $\mathfrak{a} \in F$ and $\mathfrak{b} \in G$ such that $\kappa \in \mathfrak{a} \circledast \mathfrak{b} = \{\delta \in \hbar \mid \mathfrak{a} \odot \mathfrak{b} \leq \delta\}$. Since J is a filter of $\hbar$ and $F, G \subseteq J$, we get $\mathfrak{a}, \mathfrak{b} \in J$ and so $\mathfrak{a} \odot \mathfrak{b} \in J$. Thus, $\kappa \in J$. Hence, $F \circledast G \subseteq J$. Therefore, $F \circledast G$ is the smallest filter of $\hbar$ which contains F and G. □

PROPOSITION 3.22. Let F be a filter of a hoop $\hbar$. Then for all $R, P \in \mathcal{P}(\hbar) \setminus \{\emptyset\}$, we have:

$$\underline{\mathcal{C}}_F(R) \circledast \underline{\mathcal{C}}_F(P) \subseteq \overline{\mathcal{C}}_F(R \circledast P) \subseteq \overline{\mathcal{C}}_F(R) \circledast \overline{\mathcal{C}}_F(P).$$

PROOF: Let $\kappa \in \underline{\mathcal{C}}_F(R) \circledast \underline{\mathcal{C}}_F(P) = \bigcup_{\mathfrak{a} \in \underline{\mathcal{C}}_F(R),\, \mathfrak{b} \in \underline{\mathcal{C}}_F(P)} \mathfrak{a} \circledast \mathfrak{b}$. Then there exist $\mathfrak{a} \in \underline{\mathcal{C}}_F(R)$ and $\mathfrak{b} \in \underline{\mathcal{C}}_F(P)$ such that $\kappa \in \mathfrak{a} \circledast \mathfrak{b}$. It means $\mathfrak{a} \odot \mathfrak{b} \leq x$. On the other hand, $C_F[\mathfrak{a}] \subseteq L$, *and* $C_F[\mathfrak{b}] \subseteq M$, so $\mathfrak{a} \in L$, *and* $\mathfrak{b} \in M$. Then $\mathfrak{a} \circledast \mathfrak{b} \subseteq L \circledast M = \bigcup_{\kappa \in R,\, \nu \in P} \kappa \circledast \nu$. Now, since $\mathfrak{a} \odot \mathfrak{b} \leq x$ and $\mathfrak{a} \circledast \mathfrak{b} \in R \circledast P$, we get $\kappa \in R \circledast P$. We have $C_F[x] \cap (R \circledast P) \neq \emptyset$. Therefore $\kappa \in \overline{\mathcal{C}}_F(R \circledast P)$. For the second part, let $\kappa \in \overline{\mathcal{C}}_F(R \circledast P)$. Then $\mathcal{C}_F[\kappa] \cap (R \circledast P) \neq \emptyset$. Thus there exists $\nu \in \mathcal{C}_F[\kappa] \cap (R \circledast P)$. Since $\nu \in \mathcal{C}_F[\kappa]$, we have $\mathcal{C}_F[\kappa] = \mathcal{C}_F[\nu]$ and from $\nu \in R \circledast P$, we get that there are $\mathfrak{a} \in R$ and $\mathfrak{b} \in P$ such that $\nu \in \mathfrak{a} \circledast \mathfrak{b}$. Moreover, since $\mathfrak{a} \in \mathcal{C}_F[\mathfrak{a}]$ and $\mathfrak{a} \in R$, we obtain that $\mathfrak{a} \in \mathcal{C}_F[\mathfrak{a}] \cap R$, and so $\mathfrak{a} \in \overline{\mathcal{C}}_F(R)$. By the similar way, $\mathfrak{b} \in \mathcal{C}_F[\mathfrak{b}] \cap P$, and so $\mathfrak{b} \in \overline{\mathcal{C}}_F(P)$. Hence $\mathfrak{a} \circledast \mathfrak{b} \subseteq \overline{\mathcal{C}}_F(R) \circledast \overline{\mathcal{C}}_F(P)$, and so $\nu \in \overline{\mathcal{C}}_F(R) \circledast \overline{\mathcal{C}}_F(P)$. Thus $\mathcal{C}_F[\kappa] \cap (R \circledast P) \subseteq \overline{\mathcal{C}}_F(R) \circledast \overline{\mathcal{C}}_F(P)$. Therefore, $\overline{\mathcal{C}}_F(R \circledast P) \subseteq \overline{\mathcal{C}}_F(R) \circledast \overline{\mathcal{C}}_F(P)$. □

We provide conditions for the equality in Proposition 3.22 to be true.

THEOREM 3.23. *Let F be a filter of a hoop $\hbar$ and R, P are two nonempty subsets of $\hbar$.*

(i) If $R, P \subseteq F$, then $\overline{\mathcal{C}}_F(R \circledast P) = \overline{\mathcal{C}}_F(R) \circledast \overline{\mathcal{C}}_F(P)$.

(ii) If R and P are definable with respect to F, then $\underline{\mathcal{C}}_F(R) \circledast \underline{\mathcal{C}}_F(P) = \underline{\mathcal{C}}_F(R \circledast P)$

Proof: (i) According to the Theorem 3.8(ii), if $R, P \subseteq F$, then $\overline{\mathcal{C}}_F(R) = \overline{\mathcal{C}}_F(P) = F$. Since $R \circledast P = \bigcup_{\kappa \in R, \nu \in P} \{\delta \in \hbar \mid \kappa \odot \nu \leq \delta\}$, $R, P \subseteq F$ and F is a filter of $\hbar$, we obtain $\kappa \odot \nu \in F$, and so $\delta \in F$. Hence $R \circledast P \subseteq F$ which means $\overline{\mathcal{C}}_F(R \circledast P) = F$. Therefore $\overline{\mathcal{C}}_F(R \circledast P) = \overline{\mathcal{C}}_F(R) \circledast \overline{\mathcal{C}}_F(P)$.

(ii) According to Proposition 3.22, we have $\underline{\mathcal{C}}_F(R) \circledast \underline{\mathcal{C}}_F(P) \subseteq \overline{\mathcal{C}}_F(R \circledast P) \subseteq \overline{\mathcal{C}}_F(R) \circledast \overline{\mathcal{C}}_F(P)$. Since R and P are definable with respect to F, we get $R \circledast P \subseteq \overline{\mathcal{C}}_F(R \circledast P) \subseteq R \circledast P$. It implies that $\overline{\mathcal{C}}_F(R \circledast P) = R \circledast P$. Since $\overline{\mathcal{C}}_F(R \circledast P) = R \circledast P$, by Theorem 3.16 we get $\underline{\mathcal{C}}_F(R \circledast P) = R \circledast P$. Therefore, $\underline{\mathcal{C}}_F(R \circledast P) = R \circledast P = \underline{\mathcal{C}}_F(R \circledast P)$. □

Lemma 3.24. *Let $\hbar$ be a linearly ordered hoop and F be a filter of $\hbar$. If $\mathfrak{a} \leq \mathfrak{b}$ and $C_F[\mathfrak{a}] \neq C_F[\mathfrak{b}]$, then for any $u \in C_F[\mathfrak{a}]$ and for any $v \in C_F[\mathfrak{b}]$ we have $u \leq v$.*

Proof: Let $\mathfrak{a} \leq \mathfrak{b}$ and $C_F[\mathfrak{a}] \neq C_F[\mathfrak{b}]$. Suppose that $u \nleq v$. Since $\hbar$ is a linearly ordered hoop, we get $v \leq u$. So $v \to u = 1$. On the other hand, we have $u \in C_F[\mathfrak{a}]$ and so $u \to \mathfrak{a}, \mathfrak{a} \to u \in F$. By Proposition 2.1(ix), we have $v \to u \leq (u \to \mathfrak{a}) \to (v \to \mathfrak{a})$. It implies that $v \to \mathfrak{a} \in F$. Since $v \in C_F[\mathfrak{b}]$ we have $v \to \mathfrak{b}, \mathfrak{b} \to v \in F$. Also, since $\mathfrak{a} \leq \mathfrak{b}$, by Proposition 2.1(xi) we have $\mathfrak{b} \to v \leq \mathfrak{a} \to v$. So $\mathfrak{a} \to v \in F$. Then $v \in C_F[\mathfrak{a}]$ and $v \in C_F[\mathfrak{b}]$, thus $v \in C_F[\mathfrak{a}] \cap C_F[\mathfrak{b}]$. Hence, $C_F[\mathfrak{a}] = C_F[\mathfrak{b}]$, which is a contradiction. Therefore, $v \leq u$. □

Theorem 3.25. *Let $\hbar$ be a linearly ordered hoop, $(\hbar, F)$ be an approximation space and R be a filter of $\hbar$. Then R is an upper rough filter of $\hbar$.*

Proof: If $\mathfrak{a} \leq \mathfrak{b}$ and $\mathfrak{a} \in \overline{\mathcal{C}}_F(R)$, then $C_F[\mathfrak{a}] \cap R \neq \emptyset$. So there is an element $u \in R$ such that $C_F[\mathfrak{a}] = C_F[u]$. If $C_F[\mathfrak{a}] = C_F[\mathfrak{b}]$, then clearly $\mathfrak{b} \in \overline{\mathcal{C}}_F(R)$. If $C_F[\mathfrak{a}] \neq C_F[\mathfrak{b}]$, then by Lemma 3.24, we obtain $u \leq \mathfrak{b}$. Since $u \in R$ and R is a filter of $\hbar$, we get $b \in R$. Thus, $C_F[b] \cap R \neq \emptyset$, and so $\mathfrak{b} \in \overline{\mathcal{C}}_F(R)$.
Let $\mathfrak{a}, \mathfrak{b} \in \overline{\mathcal{C}}_F(R)$. Then $C_F[\mathfrak{a}] \cap R \neq \emptyset$ and $C_F[\mathfrak{b}] \cap R \neq \emptyset$. Hence there exist $u \in C_F[\mathfrak{a}] \cap R$ and $v \in C_F[\mathfrak{b}] \cap R$. Since $C_F[u] = C_F[\mathfrak{a}]$ and $C_F[v] = C_F[\mathfrak{b}]$, $u, v \in R$, and R is a filter of $\hbar$, we have $u \odot v \in R$ and $C_F[u \odot v] = C_F[\mathfrak{a} \odot \mathfrak{b}]$. So $u \odot v \in C_F[\mathfrak{a} \odot \mathfrak{b}] \cap R \neq \emptyset$. Hence $\mathfrak{a} \odot \mathfrak{b} \in \overline{\mathcal{C}}_F(R)$. Therefore, $\overline{\mathcal{C}}_F(R)$ is a filter of $\hbar$. □

Theorem 3.26. *Let F and G be two nonempty subsets of a linearly ordered hoop $\hbar$ and G be a filter of $\hbar$. Then $\overline{\mathcal{C}}_F(R \circledast P) = \overline{\mathcal{C}}_F(R) \circledast \overline{\mathcal{C}}_F(P)$.*

Proof: Let $n \in \overline{\mathcal{C}}_{F}(R) \circledast \overline{\mathcal{C}}_{F}(P)$. Then there are $u \in \overline{\mathcal{C}}_{F}(R)$ and $v \in \overline{\mathcal{C}}_{F}(P)$ such that $n \in u \circledast v$ and so $u \odot v \leq n$. Since $C_F[u] \cap L \neq \emptyset$ and $C_F[v] \cap M \neq \emptyset$, we get that there are $\mathfrak{a} \in L$ and $\mathfrak{b} \in M$ such that $C_F[\mathfrak{a}] = C_F[u]$ and $C_F[\mathfrak{b}] = C_F[v]$. Hence $C_F[\mathfrak{a} \odot \mathfrak{b}] = C_F[u] \odot C_F[v]$, and so, $\mathfrak{a} \odot \mathfrak{b} \in L \odot M$. If $C_F[n] \neq C_F[u \odot v]$, then by Lemma 3.24, since $\mathfrak{a} \odot \mathfrak{b} \in C_F[u \odot v]$, we get $\mathfrak{a} \odot \mathfrak{b} \leq n$. Then $n \in L \circledast M$. Hence $C_F[n] \cap (L \circledast M) \neq \emptyset$. On the other hand, if $C_F[n] = C_F[u \odot v]$, then by hypothesis we get $C_F[n] \cap (L \circledast M) \neq \emptyset$. Thus $n \in \overline{\mathcal{C}}_{F}(R \circledast P)$. Therefore, $\overline{\mathcal{C}}_{F}(R) \circledast \overline{\mathcal{C}}_{F}(P) \subseteq \overline{\mathcal{C}}_{F}(R \circledast P)$. By Proposition 3.22, the proof of converse is clear. □

Theorem 3.27. *The algebraic structure $(\mathcal{F}(\hbar), \circledast)$ is a semilattice, where $\mathcal{F}(\hbar)$ is the set of all filters of a hoop $\hbar$.*

Proof: By Theorem 3.21, it is clear that $(\mathcal{F}(\hbar), \circledast)$ is well-defined. Also, according to definition of operation $\circledast$, we get $(\mathcal{F}(\hbar), \circledast)$ is associative and commutative. It is enough to prove that the operation $\circledast$ is idempotent. For this, let $F \in \mathcal{F}(\hbar)$ and $\delta \in F \circledast F$. Then there exist $\kappa, \nu \in F$ such that $\kappa \odot \nu \leq \delta$. Since F is a filter of $\hbar$, and $\kappa, \nu \in F$, we have $\kappa \odot \nu \in F$ and so $\delta \in F$. Hence $F \circledast F \subseteq F$. Conversely, since $F \in \mathcal{F}(\hbar)$, we have $1 \in F$. Then for any $\kappa \in F$, $1 \odot \kappa \leq \kappa$ and so $\kappa \in F \circledast F$. Thus $F \subseteq F \circledast F$. Hence $F = F \circledast F$. Therefore, $(\mathcal{F}(\hbar), \circledast)$ is a semilattice. □

Corollary 3.28. The algebraic structure $(\mathcal{F}(\hbar), \circledast, \{1\})$ is a commutative monoid.

Let F be a filter of a hoop $\hbar$. Then each filter of $\hbar$ which contains F is rough filter according to Theorem 3.8(iii). The set of all rought filters of hoop $\hbar$ which contain F is denoted by $\mathcal{RF}(\hbar)$.
Let K and G be two filters of $\hbar$. We define the implication relation on $\mathcal{F}(\hbar)$ as follows:

$$K \to G = \{\kappa \in \hbar \mid K \cap \langle \kappa \rangle \subseteq G\}. \tag{3.2}$$

Theorem 3.29. *The set $\mathcal{RF}(\hbar)$ is closed under the operation "$\to$".*

Proof: Let K and G be two filters of $\mathcal{RF}(\hbar)$. Then, $F \subseteq G, K$. Let $\kappa \in F$. Since $F \subseteq K$, we get $\langle \kappa \rangle \subseteq F \subseteq K$ and so $K \cap \langle \kappa \rangle \subseteq K \cap F \subseteq F \subseteq G$. Thus, $K \cap \langle \kappa \rangle \subseteq G$, for any $\kappa \in F$. Hence $F \subseteq K \to G$. Hence, $K \to G \in \mathcal{RF}(\hbar)$. □

Theorem 3.30. *The algebraic structure* $(\mathcal{RF}(\hbar), \cap, \to, \hbar)$ *is a hoop.*

Proof: According to definition of $\cap$, we get $(\mathcal{RF}(\hbar), \cap, \hbar)$ is associative and commutative. So $(\mathcal{RF}(\hbar), \cap, \hbar)$ is a commutative monoid. It is enough to prove that the other properties hold. Since $G \to G = \{\kappa \in \hbar \mid G \cap \langle\kappa\rangle \subseteq G\}$, it is clear that $G \to G = \hbar$. Let $\kappa \in (G \cap K) \to J$. It means $\langle\kappa\rangle \cap (G \cap K) \subseteq J = (\langle\kappa\rangle \cap G) \cap K \subseteq J$. Then $\langle\kappa\rangle \cap G \subseteq K \to J$. Hence, $\kappa \in G \to (K \to J)$. The proof of other side is similar. Moreover, since $G \cap (G \to K) = G \cap \{\kappa \in \hbar \mid G \cap \langle\kappa\rangle \subseteq K\} = \{\kappa \in G \mid \langle\kappa\rangle \subseteq K\}$, we have $G \cap (G \to K) = G \cap K$. By the similar way, we have $K \cap (K \to G) = K \cap G$. Hence $G \cap (G \to K) = K \cap (K \to G)$. Therefore, $(\mathcal{RF}(\hbar), \cap, \to, \hbar)$ is a hoop. □

4. Conclusions and future works

In this paper, by considering the notion of a hoop, the notion of the lower and the upper approximations are introduced and some properties of them are given. Moreover, it is proved that the lower and the upper approximations are an interior operator and a closure operator, respectively. Also, a hyper operation on hoop is defined and then it is shown that the set of all rough filters is a monoid by using this operation. For more study, the implicative operation on the set of all rough filters is introduced and proved that this set with implication and intersection is made a hoop. For the future work, we want to use the notion of soft and rough hoop and introduce soft rough and rough soft on hoops.

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AN (α, β)-HESITANT FUZZY SET APPROACH TO IDEAL THEORY IN SEMIGROUPS

Abstract

The aim of this manuscript is to introduce the (α, β)-hesitant fuzzy set and apply it to semigroups. In this paper, as a generalization of the concept of hesitant fuzzy sets to semigroup theory, the concept of (α, β)-hesitant fuzzy subsemigroups of semigroups is introduced, and related properties are discussed. Furthermore, we define and study (α, β)-hesitant fuzzy ideals on semigroups. In particular, we investigate the structure of (α, β)-hesitant fuzzy ideal generated by a hesitant fuzzy ideal in a semigroup. In addition, we also introduce the concepts of (α, β)-hesitant fuzzy semiprime sets of semigroups, and characterize regular semigroups in terms of (α, β)-hesitant fuzzy left ideals and (α, β)-hesitant fuzzy right ideals. Finally, several characterizations of regular and intra-regular semigroups by the properties of (α, β)-hesitant ideals are given.

Keywords: $^{\alpha}$-hesitant ($_{\alpha}$-hesitant) fuzzy set, (α, β)-hesitant fuzzy subsemigroup, (α, β)-hesitant fuzzy ideal, (α, β)-hesitant fuzzy semiprime set, regular semigroup.

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1. Introduction

An (α, β)-hesitant fuzzy set on a semigroup is a generalization of the concept of fuzzy subsets, interval-valued fuzzy sets and hesitant fuzzy sets in semigroups. A hesitant fuzzy set theory is an excellent tool to handle the uncertainty in case of insufficient data. Many authors studied different aspects of hesitant fuzzy sets (see [1, 5, 14, 19, 20]). Also, hesitant

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fuzzy set theory is used in decision making problem etc. (see [12, 16]), and is applied to BCK/BCI-algebras and UP-algebras (see [10, 11, 13]). The notion of interval-valued fuzzy sets has been applied to theory of semigroups [4]. They considered characterizations of left [right] simple, left [right] duo and a semilattice of left [right] simple semigroups. In 2012, Khan, Jun and Abbas [8] characterized regular (resp.intra-regular, simple and semisimple) ordered semigroups by their $(\in, \in \vee q)$-fuzzy interior ideals (resp. $(\in, \in \vee q)$-fuzzy ideals). Also they proved that the an ordered semigroup S is simple if and only if it is $(\in, \in \vee q)$-fuzzy simple. In 2013, Yaqoob [18] characterized regular LA-semigroups by the properties of interval valued intuitionistic fuzzy left ideals [right ideal, generalized bi-ideal and bi-ideal]. In 2014, Jun, Ahn and Muhiuddin [6] applied the notion of hesitant fuzzy soft sets to BCK/BCI-algebras. They introduced the notions of hesitant fuzzy soft subalgebras and (closed) hesitant fuzzy soft ideals and investigated several properties. In 2015, Jun, Lee, and Song [7] introduced the notion of hesitant fuzzy (generalized) bi-ideals on a semigroup, which is a generalization of interval valued fuzzy (generalized) bi-ideals. In 2016, Khan et al. [9] applied the notion of interval-valued fuzzy subsets to ordered semigroups, and proved that the intersection of non-empty class of interval-valued fuzzy interior ideals of an ordered semigroup is also an interval-valued fuzzy interior ideal. In 2017, Tang, Davvaz and Xie [15] defined and studied the completely prime, weakly completely prime and completely semiprime fuzzy quasi-Γ-hyperideals of ordered Γ-semihypergroups, and characterized bi-regular ordered Γ-semihypergroups by the properties of completely semiprime fuzzy quasi-Γ-hyperideals. In 2018, Abbasi et al. [2] gave the concept of hesitant fuzzy ideals and 3-prime hesitant fuzzy ideals in po-semigroup, which is a generalization of fuzzy ideals and 3-prime fuzzy ideals in po-semigroups. In 2019, Arulmozhi, Chinnadurai and Swaminathan [3] introduced the notion of interval valued $\left(\bar{\eta}, \bar{\delta}\right)$-bipolar fuzzy ideal, bi-ideal,interior ideal, $(\in, \in \vee q)$-bipolar fuzzy ideal of ordered Γ-semigroups and established some properties of bipolar fuzzy ideals in terms of $(\bar{\in}, \bar{\in} \vee q)$-bipolar fuzzy ideals. In 2020, Yairayong [17] applied the theory of hesitant fuzzy sets to completely regular semigroups and introduced the notion of hesitant fuzzy semiprime sets and hesitant fuzzy idempotent sets on semigroups, which is a generalization of fuzzy semiprime and fuzzy idempotent sets. He also proved that the every hesitant fuzzy two-sided ideal on a semigroup S is a hesitant fuzzy interior ideal if and only if S is a semisimple semigroup.

The aim of this manuscript is to introduce the (α, β)-hesitant fuzzy set and apply it to semigroups. The rest contents of this paper are arranged as follows. In Section 2, we present the fundamental concepts and properties of ${}^{\alpha}$-hesitant (${}_{\alpha}$-hesitant) fuzzy sets, (α, β)-hesitant fuzzy subsemigroups and (α, β)-hesitant fuzzy ideals, which form the basis of our subsequent discussion. In this regard, we prove that that every hesitant fuzzy set on a semigroup S is (α, β)-hesitant fuzzy left (right, two-sided) ideal if and only if ${}_{\beta}(\mathcal{S} \odot \mathcal{H}) \subseteq {}^{\alpha}\mathcal{H}({}_{\beta}(\mathcal{H} \odot \mathcal{S}) \subseteq {}^{\alpha}\mathcal{H}, {}_{\beta}((\mathcal{S} \odot \mathcal{H}) \cup (\mathcal{H} \odot \mathcal{S})) \subseteq {}^{\alpha}\mathcal{H})$. We prove that the non empty subset of a semigroup S is a subsemigroup (left ideal, right ideal, two-sided ideal) of S if and only if the hesitant fuzzy set on S is the (α, β)-hesitant fuzzy subsemigroup ((α, β)-hesitant fuzzy left ideal, (α, β)-hesitant fuzzy right ideal, (α, β)-hesitant fuzzy two-sided ideal) on S. In Section 3, we define the notions of (α, β)-hesitant fuzzy semiprime sets and equivalent definitions of them. Some related properties of them are obtained. In this paper, we give characterizations of semigroups in terms of (α, β)-hesitant fuzzy ideals, and characterize regular semigroups in terms of (α, β)-hesitant fuzzy left ideals and (α, β)-hesitant fuzzy right ideals. Finally, several characterizations of regular and intra-regular semigroups by the properties of (α, β)-hesitant ideals are given.

2. ${}^{\alpha}$-hesitant (${}_{\alpha}$-hesitant) fuzzy sets

In this section, we present the fundamental concepts and properties of ${}^{\alpha}$-hesitant (${}_{\alpha}$-hesitant) fuzzy sets, (α, β)-hesitant fuzzy subsemigroups and (α, β)-hesitant fuzzy ideals, which form the basis of our subsequent discussion. In this regard, we prove that that every hesitant fuzzy set on a semigroup S is (α, β)-hesitant fuzzy left (right, two-sided) ideal if and only if ${}_{\beta}(\mathcal{S} \odot \mathcal{H}) \subseteq {}^{\alpha}\mathcal{H}({}_{\beta}(\mathcal{H} \odot \mathcal{S}) \subseteq {}^{\alpha}\mathcal{H}, {}_{\beta}((\mathcal{S} \odot \mathcal{H}) \cup (\mathcal{H} \odot \mathcal{S})) \subseteq {}^{\alpha}\mathcal{H})$. We prove that the non empty subset of a semigroup S is a subsemigroup (left ideal, right ideal, two-sided ideal) of S if and only if the hesitant fuzzy set on S is the (α, β)-hesitant fuzzy subsemigroup ((α, β)-hesitant fuzzy left ideal, (α, β)-hesitant fuzzy right ideal, (α, β)-hesitant fuzzy two-sided ideal) on S. These notions will be helpful in later sections.

Let $\mathcal{H} : S \to \mathcal{P}([0, 1])$ be a hesitant fuzzy set on a semigroup S and let α be any element of $\mathcal{P}([0, 1])$. Then the **${}^{\alpha}$-hesitant fuzzy set (${}_{\alpha}$-hesitant fuzzy set)** on S is defined as ${}^{\alpha}\mathcal{H}_x = \mathcal{H}_x \cup \alpha$ (${}_{\alpha}\mathcal{H}_x = \mathcal{H}_x \cap \alpha$) for all $x \in S$. Next, we define the hesitant fuzzy set over a semigroup S. If $\mathcal{H}_x = [0, 1]$

for all $x \in S$, then it is easy to see that $\mathcal{H}$ is a hesitant fuzzy set on a semigroup S. We denote such type of hesitant fuzzy set $\mathcal{H}$ by $\mathcal{S}$.

The proof of them is straightforward, so we omit it.

LEMMA 2.1. *Let $\mathcal{S}$ be a hesitant fuzzy set on a semigroup S and let α be any element of $\mathcal{P}([0,1])$. The following statements are true.*

1. ${}^{\alpha}\mathcal{S} = \mathcal{S}$.

2. ${}_{\alpha}\mathcal{S} = \alpha$.

Next, we denote by $\mathbf{H}(S)$ the set of all hesitant fuzzy sets on a semigroup S. Let $\mathcal{H}$ and $\mathcal{F}$ be any elements of $\mathbf{H}(S)$. Then, $\mathcal{H}$ is said to be a **subset** of $\mathcal{F}$, denoted by $\mathcal{H} \preceq \mathcal{F}$ if $\mathcal{H}_x \subseteq \mathcal{F}_x$ for all $x \in S$.

Now we are giving some basic properties of hesitant fuzzy subsemigroups on a semigroup S, which will be very helpful in later section.

THEOREM 2.2. *Let $\mathcal{H}$ be a hesitant fuzzy subsemigroup on a semigroup S. Then, the following statements are true:*

1. *If α is an element of $\mathcal{P}([0,1])$, then ${}^{\alpha}\mathcal{H}$ is a hesitant fuzzy subsemigroup on S.*

2. *If α is an element of $\mathcal{P}([0,1])$, then ${}_{\alpha}\mathcal{H}$ is a hesitant fuzzy subsemigroup on S.*

PROOF: 1. Let x and y be any elements of S and let $\alpha \in \mathcal{P}([0,1])$. Then, it is clear that

$$\begin{aligned} {}^{\alpha}\mathcal{H}_{xy} &= \mathcal{H}_{xy} \cup \alpha \\ &\supseteq (\mathcal{H}_x \cap \mathcal{H}_y) \cup \alpha \\ &= (\mathcal{H}_x \cup \alpha) \cap (\mathcal{H}_y \cup \alpha) \\ &= {}^{\alpha}\mathcal{H}_x^y. \end{aligned}$$

This completes the proof.

2. Let x and y be any elements of S and let $\alpha \in \mathcal{P}([0,1])$. Since $\mathcal{H}$ is a hesitant fuzzy subsemigroup on S, we obtain

$$\begin{aligned} {}_{\alpha}\mathcal{H}_{xy} &= \mathcal{H}_{xy} \cap \alpha \\ &\supseteq (\mathcal{H}_x \cap \mathcal{H}_y) \cap \alpha \\ &= (\mathcal{H}_x \cap \alpha) \cap (\mathcal{H}_y \cap \alpha) \\ &= {}_{\alpha}\mathcal{H}_x^y. \end{aligned}$$

From here, we obtain that ${}_{\alpha}\mathcal{H}$ is a hesitant fuzzy subsemigroup on S. $\square$

For a non empty family of a hesitant fuzzy sets $\{\mathcal{H}_i : i \in I\}$, on a semigroup S. The symbols $\bigcup\limits_{i\in I} \mathcal{H}_i$ and $\bigcap\limits_{i\in I} \mathcal{H}_i$ will mean the following hesitant fuzzy sets:

$$\left(\bigcup_{i\in I} \mathcal{H}_i\right)_x = \bigcup_{i\in I} (\mathcal{H}_i)_x$$

and

$$\left(\bigcap_{i\in I} \mathcal{H}_i\right)_x = \bigcap_{i\in I} (\mathcal{H}_i)_x.$$

If I is a finite set, say $I = \{1, 2, 3, \ldots, n\}$, then clearly $\bigcup\limits_{i\in I} \mathcal{H}_i = \mathcal{H}_1 \cup \mathcal{H}_2 \cup \ldots \cup \mathcal{H}_n$ and $\bigcap\limits_{i\in I} \mathcal{H}_i = \mathcal{H}_1 \cap \mathcal{H}_2 \cap \ldots \cap \mathcal{H}_n$ (see [7]).

THEOREM 2.3. *Let $\mathcal{H}$ and $\mathcal{G}$ be two hesitant fuzzy sets of a semigroup S. Then, the following statements are true:*

1. *If α is an element of $\mathcal{P}([0,1])$, then ${}^{\alpha}(\mathcal{H} \cap \mathcal{G}) = {}^{\alpha}\mathcal{H} \cap {}^{\alpha}\mathcal{G}$.*
2. *If α is an element of $\mathcal{P}([0,1])$, then ${}_{\alpha}(\mathcal{H} \cap \mathcal{G}) = {}_{\alpha}\mathcal{H} \cap {}_{\alpha}\mathcal{G}$.*
3. *If α is an element of $\mathcal{P}([0,1])$, then ${}^{\alpha}(\mathcal{H} \cup \mathcal{G}) = {}^{\alpha}\mathcal{H} \cup {}^{\alpha}\mathcal{G}$.*
4. *If α is an element of $\mathcal{P}([0,1])$, then ${}_{\alpha}(\mathcal{H} \cup \mathcal{G}) = {}_{\alpha}\mathcal{H} \cup {}_{\alpha}\mathcal{G}$.*

PROOF: 1. Let x be any element of S. For every $\alpha \in \mathcal{P}([0,1])$ we have

$$\begin{aligned} {}^{\alpha}(\mathcal{H} \cap \mathcal{G})_x &= (\mathcal{H} \cap \mathcal{H})_x \cup \alpha \\ &= (\mathcal{H}_x \cap \mathcal{G}_x) \cup \alpha \\ &= (\mathcal{H}_x \cup \alpha) \cap (\mathcal{G}_x \cup \alpha) \\ &= {}^{\alpha}\mathcal{H}_x \cap {}^{\alpha}\mathcal{G}_x \\ &= ({}^{\alpha}\mathcal{H} \cap {}^{\alpha}\mathcal{G})_x, \end{aligned}$$

which implies that ${}^{\alpha}(\mathcal{H} \cap \mathcal{G}) = {}^{\alpha}\mathcal{H} \cap {}^{\alpha}\mathcal{G}$ for all $\alpha \in \mathcal{P}([0,1])$.

2. Let x be any element of S. Then, for every $\alpha \in \mathcal{P}([0,1])$, we have

$$\begin{aligned} {}_{\alpha}\left(\mathcal{H} \cap \mathcal{G}\right)_{x} &= (\mathcal{H} \cap \mathcal{H})_{x} \cap \alpha \\ &= (\mathcal{H}_{x} \cap \mathcal{G}_{x}) \cap \alpha \\ &= (\mathcal{H}_{x} \cap \alpha) \cap (\mathcal{G}_{x} \cap \alpha) \\ &= {}_{\alpha}\mathcal{H}_{x} \cap {}_{\alpha}\mathcal{G}_{x} \\ &= \left({}_{\alpha}\mathcal{H} \cap {}_{\alpha}\mathcal{G}\right)_{x}. \end{aligned}$$

This completes the proof.

3–4. It can be proved similarly to 1. □

Now we introduce the notion of (α, β)-hesitant fuzzy subsemigroups on a semigroup.

DEFINITION 2.4. Let $\mathcal{H}$ be a hesitant fuzzy set on a semigroup S and let α, β be any element of $\mathcal{P}([0,1])$. Then $\mathcal{H}$ is said to be an (α, β)**-hesitant fuzzy subsemigroup** on S if ${}^{\alpha}\mathcal{H}_{xy} \supseteq {}_{\beta}\mathcal{H}_{x}^{y}$ for any $x, y \in S$.

Thus every hesitant fuzzy subsemigroup on a semigroup S is an (α, β)-hesitant fuzzy subsemigroup with $\alpha = \emptyset$ and $\beta = [0,1]$. Thus every hesitant fuzzy subsemigroups on S is an (α, β)-hesitant fuzzy subsemigroup on S. However, the converse is not necessarily true as shown in the following example.

Example 2.5. Consider the semigroup $S = \{a, b, c, d\}$ with the following multiplication " $\cdot$ " table below:

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	b
d	a	a	b	c

We define the hesitant fuzzy set $\mathcal{H} : S \to \mathcal{P}([0,1])$ on S as follows:

$$\mathcal{H}_{x} = \begin{cases} [0.7, 0.8) \cup (0.8, 0.9]\,; & x \in \{a, b\} \\ (0.2, 0.8)\,; & x \in \{c\} \\ \{0, 0.1, 0.2, 0.3\}\,; & \text{otherwise.} \end{cases}$$

Then, as is easily seen, $\mathcal{H}$ is an $([0,3]\,, \{0, 0.1, 0.2\})$-hesitant fuzzy subsemigroup on S, but not a hesitant fuzzy subsemigroup on S. Since

$\mathcal{H}_c^d = \mathcal{H}_d \cap \mathcal{H}_c = \{0, 0.1, 0.2, 0.3\} \cap (0.2, 0.8) = \{0.3\}$, while $\mathcal{H}_{d\cdot c} = \mathcal{H}_b = [0.7, 0.8) \cup (0.8, 0.9]$.

Now we have the following result:

THEOREM 2.6. *If $\mathcal{H}$ and $\mathcal{G}$ are any (α,β)-hesitant fuzzy subsemigroups on a semigroup S, then $\mathcal{H}\cap\mathcal{G}$ is an (α,β)-hesitant fuzzy subsemigroup on S.*

PROOF: Let x and y be any elements of S. Since $\mathcal{H}$ and $\mathcal{G}$ are both (α,β)-hesitant fuzzy subsemigroups on S, we have

$$\begin{array}{rcl}
{}^{\alpha}\left(\mathcal{H}\cap\mathcal{G}\right)_{xy} & = & (\mathcal{H}_{xy}\cap\mathcal{G}_{xy})\cup\alpha \\
& = & (\mathcal{H}_{xy}\cup\alpha)\cap(\mathcal{G}_{xy}\cup\alpha) \\
& = & {}^{\alpha}\mathcal{H}_{xy}\cap{}^{\alpha}\mathcal{G}_{xy} \\
& \supseteq & {}_{\beta}\mathcal{H}_x^y\cap{}_{\beta}\mathcal{G}_x^y \\
& = & (\mathcal{H}_x\cap\mathcal{H}_y\cap\beta)\cap(\mathcal{G}_x\cap\mathcal{G}_y\cap\beta) \\
& = & (\mathcal{H}_x\cap\mathcal{G}_x)\cap(\mathcal{H}_y\cap\mathcal{G}_y)\cap\beta \\
& = & \Big(\left(\mathcal{H}\cap\mathcal{G}\right)_x\cap\left(\mathcal{H}\cap\mathcal{G}\right)_y\Big)\cap\beta \\
& = & {}_{\beta}\left(\mathcal{H}\cap\mathcal{G}\right)_x^y .
\end{array}$$

Therefore, $\mathcal{H}\cap\mathcal{G}$ is an (α,β)-fuzzy subsemigroup on S. □

The following corollary follows from Theorem 2.6 and the definition of (α,β)-hesitant fuzzy subsemigroup on a semigroup S

COROLLARY 2.7. If $\mathcal{H}_i$ is an (α,β)-hesitant fuzzy subsemigroup on a semigroup S for all $i\in I$, then $\bigcap_{i\in I}\mathcal{H}_i$ is an (α,β)-hesitant fuzzy subsemigroup on S.

Let $\mathcal{F}$ and $\mathcal{G}$ be two hesitant fuzzy sets on a semigroup, the **hesitant fuzzy product** (see [7]) of $\mathcal{F}$ and $\mathcal{G}$ is defined to be a hesitant fuzzy set $\mathcal{F}\odot\mathcal{G}$ on S which is given by

$$\left(\mathcal{F}\odot\mathcal{G}\right)_x = \begin{cases} \bigcup\limits_{x=yz}\mathcal{F}_y\cap\mathcal{G}_z; & \exists y,z\in S, \text{ such that } x=yz \\ \emptyset; & \text{otherwise.} \end{cases}$$

As is well known, the operation "$\odot$" is associative.

Next, we proved that every hesitant fuzzy set on a semigroup S is (α,β)-hesitant fuzzy subsemigroup if and only if ${}_{\beta}(\mathcal{H}\odot\mathcal{H})\subseteq{}^{\alpha}\mathcal{H}$.

Theorem 2.8. *For a hesitant fuzzy set $\mathcal{H}$ on a semigroup S, the following two statements are equivalent:*

1. *$\mathcal{H}$ is an (α, β)-hesitant fuzzy subsemigroup on S.*
2. *${}_{\beta}(\mathcal{H} \odot \mathcal{H}) \subseteq {}^{\alpha}\mathcal{H}$.*

Proof: First assume that $\mathcal{H}$ is an (α, β)-hesitant fuzzy subsemigroup on S. Let x be any element of S such that it is not expressible as product of two elements in S. Observe that ${}_{\beta}(\mathcal{H} \odot \mathcal{H})_x = (\mathcal{H} \odot \mathcal{H})_x \cap \beta = \emptyset \cap \beta = \emptyset \subseteq {}^{\alpha}\mathcal{H}_x$. Otherwise, there exist elements y and z of S such that $x = yz$. Thus, by hypothesis we have

$$\begin{aligned}
{}_{\beta}(\mathcal{H} \odot \mathcal{H})_x &= (\mathcal{H} \odot \mathcal{H})_x \cap \beta \\
&= \left(\bigcup_{x=ab} \mathcal{H}_a \cap \mathcal{H}_b \right) \cap \beta \\
&= \bigcup_{x=ab} (\mathcal{H}_a \cap \mathcal{H}_b \cap \beta) \\
&\subseteq \bigcup_{x=ab} (\mathcal{H}_{ab} \cup \alpha) \\
&= \mathcal{H}_x \cup \alpha \\
&= {}^{\alpha}\mathcal{H}_x.
\end{aligned}$$

Therefore ${}_{\beta}(\mathcal{H} \odot \mathcal{H}) \subseteq {}^{\alpha}\mathcal{H}$.

Conversely, assume that $\mathcal{H}$ is a hesitant fuzzy set on S such that ${}_{\beta}(\mathcal{H} \odot \mathcal{H}) \subseteq {}^{\alpha}\mathcal{H}$. Let x, y and z be any elements of S. Now, choose $x = yz$. Thus, we obtain

$$\begin{aligned}
{}^{\alpha}\mathcal{H}_{yz} &= {}^{\alpha}\mathcal{H}_x \\
&\supseteq {}_{\beta}(\mathcal{H} \odot \mathcal{H})_x \\
&= (\mathcal{H} \odot \mathcal{H})_x \cap \beta \\
&= \left(\bigcup_{x=ab} \mathcal{H}_a \cap \mathcal{H}_b \right) \cap \beta \\
&\supseteq (\mathcal{H}_y \cap \mathcal{H}_z) \cap \beta \\
&= {}_{\beta}\mathcal{H}_y^z.
\end{aligned}$$

Therefore, the proof is completed. $\square$

Now, we can introduce the (α, β)-hesitant fuzzy ideals on a semigroup, in the following manner:

Definition 2.9. Let $\mathcal{H}$ be a hesitant fuzzy set on a semigroup S. Then $\mathcal{H}$ is said to be an (α, β)**-hesitant fuzzy left ideal (**(α, β)**-hesitant right ideal)** on S if ${}^{\alpha}\mathcal{H}_{xy} \supseteq {}_{\beta}\mathcal{H}_{y}$ $({}^{\alpha}\mathcal{H}_{xy} \supseteq {}_{\beta}\mathcal{H}_{x})$ for any $x, y \in S$. A hesitant fuzzy set $\mathcal{H}$ is an (α, β)**-hesitant fuzzy ideal (or** (α, β)**-hesitant fuzzy two-sided ideal)** on S if and only if it is both (α, β)-hesitant fuzzy left and right ideal on$S$.

Remark 2.10. Let $\mathcal{H}$ be a hesitant fuzzy set on a semigroup S. Then, the following statements are true:

1. If $\mathcal{H}$ is an (α, β)-hesitant fuzzy left (right) ideal on S, then $\mathcal{H}$ is an (α, β)-hesitant fuzzy subsemigroup on S.

2. A hesitant fuzzy left ideal (right ideal, ideal) on S is an (α, β)-fuzzy left ideal (right ideal, ideal) with $\alpha = \emptyset$ and $\beta = [0, 1]$. Thus every hesitant fuzzy left ideal (right ideal, ideal) on S is an (α, β)-fuzzy left ideal (right ideal, ideal) on S.

However, the converse is not necessarily true as shown in the following example.

Example 2.11.

1. Consider the semigroup $S = \{a, b, c, d\}$ with the following multiplication "$\cdot$" table below:

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	b	c	a
c	a	a	a	a
d	a	d	a	a

Now, we define a hesitant fuzzy set $\mathcal{H} : S \to \mathcal{P}([0, 1])$ by

$$\mathcal{H}_x = \begin{cases} [0, 0.85)\,; & x \in \{a, b\} \\ \{0, 0.22, 0.42, 0.52\}\,; & \text{otherwise.} \end{cases}$$

Let $\alpha = (0.52, 0.62]$ and $\beta = \{0.12, 0.22, 0.32, 0.52\}$. Then, as is easily seen, $\mathcal{H}$ is an (α, β)-hesitant fuzzy subsemigroup on S, but $\mathcal{H}$ is not an (α, β)-hesitant fuzzy left (right) ideal on S. Because

$$\begin{array}{rcl} {}^{(0.52,0.62]}\mathcal{H}_{d\cdot b} & = & \mathcal{H}_d \cup (0.52, 0.62] \\ & = & \{0, 0.22, 0.42, 0.52\} \cup (0.5, 0.6] \\ & = & \{0, 0.22, 0.42\} \cup [0.52, 0.62] \\ & \not\supseteq & \{0.12, 0.22, 0.32, 0.52\} \\ & = & [0, 0.8) \cap \{0.12, 0.22, 0.32, 0.52\} \\ & = & \mathcal{H}_b \cap \{0.12, 0.22, 0.32, 0.52\} \\ & = & {}_{\{0.12,0.22,0.32,0.52\}}\mathcal{H}_b. \end{array}$$

2. Suppose that S is the semigroup of Example 2.11 (1). Now, we define a hesitant fuzzy set $\mathcal{F} : S \to \mathcal{P}[0,1]$ by

$$\mathcal{F}_x = \begin{cases} \{0, 0.52, 0.62\}\,; & x \in \{a, b\} \\ [0, 0.52)\,; & \text{otherwise.} \end{cases}$$

Let $\alpha = [0, 0.72)$ and $\beta = (0.52, 0.62)$. Then, as is easily seen, $\mathcal{F}$ is an (α,β)-hesitant fuzzy ideal on S, but $\mathcal{F}$ is not a hesitant fuzzy ideal on S. Because $\mathcal{F}_{d\cdot b} = \mathcal{F}_d = [0, 0.5) \not\supseteq \{0, 0.5, 0.6\} = \mathcal{F}_b$.

By Theorem 2.2 and Definition 2.9, we immediately obtain the following theorem:

THEOREM 2.12. *Let $\mathcal{H}$ be a hesitant fuzzy left (right, two-sided) ideal on a semigroup S. Then the following properties hold.*

1. *If α is an element of $\mathcal{P}([0,1])$, then ${}^{\alpha}\mathcal{H}$ is a hesitant fuzzy left (right, two-sided) ideal on S.*
2. *If α is an element of $\mathcal{P}([0,1])$, then ${}_{\alpha}\mathcal{H}$ is a hesitant fuzzy left (right, two-sided) ideal on S.*

Next, we proved that every hesitant fuzzy set on a semigroup S is (α,β)-hesitant fuzzy left (right, two-sided) ideal if and only if ${}_{\beta}(\mathcal{S} \odot \mathcal{H}) \subseteq {}^{\alpha}\mathcal{H}({}_{\beta}(\mathcal{H} \odot \mathcal{S}) \subseteq {}^{\alpha}\mathcal{H}, {}_{\beta}\left((\mathcal{S} \odot \mathcal{H}) \cup (\mathcal{H} \odot \mathcal{S})\right) \subseteq {}^{\alpha}\mathcal{H})$.

THEOREM 2.13. *For a hesitant fuzzy set $\mathcal{H}$ on a semigroup S, the following statements are equivalent:*

1. *$\mathcal{H}$ is an (α,β)-hesitant fuzzy left (right, two-sided) ideal on S.*
2. ${}_{\beta}(\mathcal{S} \odot \mathcal{H}) \subseteq {}^{\alpha}\mathcal{H}({}_{\beta}(\mathcal{H} \odot \mathcal{S}) \subseteq {}^{\alpha}\mathcal{H}, {}_{\beta}\left((\mathcal{S} \odot \mathcal{H}) \cup (\mathcal{H} \odot \mathcal{S})\right) \subseteq {}^{\alpha}\mathcal{H})$.

PROOF: First assume that $\mathcal{H}$ is an (α,β)-hesitant fuzzy left ideal on S. Let x be any element of S such that it is not expressible as product of two elements in S, we can write ${}_\beta(\mathcal{S}\odot\mathcal{H})_x = (\mathcal{S}\odot\mathcal{H})_x \cap \beta = \emptyset \cap \beta = \emptyset \subseteq {}^\alpha\mathcal{H}_x$. Otherwise, there exist elements y and z of S such that $x = yz$. Since $\mathcal{H}$ is an (α,β)-hesitant fuzzy left ideal on S, it follows that

$$\begin{aligned}
{}_\beta(\mathcal{S}\odot\mathcal{H})_x &= (\mathcal{S}\odot\mathcal{H})_x \cap \beta \\
&= \left(\bigcup_{x=ab} \mathcal{S}_a \cap \mathcal{H}_b\right) \cap \beta \\
&= \bigcup_{x=ab} ([0,1] \cap \mathcal{H}_b \cap \beta) \\
&= \bigcup_{x=ab} (\mathcal{H}_b \cap \beta) \\
&\subseteq \bigcup_{x=ab} (\mathcal{H}_{ab} \cup \alpha) \\
&= \mathcal{H}_x \cup \alpha \\
&= {}^\alpha\mathcal{H}_x.
\end{aligned}$$

Therefore ${}_\beta(\mathcal{S}\odot\mathcal{H}) \subseteq {}^\alpha\mathcal{H}$.

Conversely, assume that $\mathcal{H}$ is a hesitant fuzzy set on S such that ${}_\beta(\mathcal{S}\odot\mathcal{H}) \subseteq {}^\alpha\mathcal{H}$. Let y and z be any elements of S. Choose $x \in S$ such that $x = yz$. Since

$$\begin{aligned}
{}^\alpha\mathcal{H}_{yz} &= {}^\alpha\mathcal{H}_x \\
&\supseteq {}_\beta(\mathcal{S}\odot\mathcal{H})_x \\
&= (\mathcal{S}\odot\mathcal{H})_x \cap \beta \\
&= \left(\bigcup_{x=ab} \mathcal{S}_a \cap \mathcal{H}_b\right) \cap \beta \\
&\supseteq (\mathcal{S}_y \cap \mathcal{H}_z) \cap \beta \\
&= ([0,1] \cap \mathcal{H}_z) \cap \beta \\
&= {}_\beta\mathcal{H}_z,
\end{aligned}$$

we obtain $\mathcal{H}$ is an (α,β)-hesitant fuzzy left ideal on S. □

In the following we show that if $\mathcal{H}$ and $\mathcal{G}$ are two (α,β)-hesitant fuzzy left (right, two-sided) ideals on a semigroup, then $\mathcal{H}\cap\mathcal{G}$ and $\mathcal{H}\cup\mathcal{G}$ are (α,β)-hesitant fuzzy left (right, two-sided) ideals on S.

THEOREM 2.14. *Let $\mathcal{H}$ and $\mathcal{G}$ be two (α,β)-hesitant fuzzy left (right, two-sided) ideals on a semigroup S. Then the following statements hold:*

1. *$\mathcal{H}\cap\mathcal{G}$ is an (α,β)-hesitant fuzzy left (right, two-sided) ideal on S.*
2. *$\mathcal{H}\cup\mathcal{G}$ is an (α,β)-hesitant fuzzy left (right, two-sided) ideal on S.*

PROOF: 1. The proof follows from Theorem 2.6.

2. By Theorem 2.13, we have

$$\begin{aligned} {}_{\beta}(\mathcal{S} \odot (\mathcal{H} \cup \mathcal{G})) &= {}_{\beta}((\mathcal{S} \odot \mathcal{H}) \cup (\mathcal{S} \odot \mathcal{G})) \\ &= {}_{\beta}(\mathcal{S} \odot \mathcal{H}) \cup {}_{\beta}(\mathcal{S} \odot \mathcal{G}) \\ &\subseteq {}^{\alpha}\mathcal{H} \cup {}^{\alpha}\mathcal{G} \\ &= {}^{\alpha}(\mathcal{H} \cap \mathcal{G}). \end{aligned}$$

Therefore, we obtain $\mathcal{H} \cup \mathcal{G}$ is an (α, β)-fuzzy left ideal on S. □

The following two corollaries are exactly obtained from Theorem 2.14.

THEOREM 2.15. *Let $\mathcal{H}_i$ be an (α, β)-hesitant fuzzy left (right, two-sided) ideal on a semigroup S for all $i \in I$. Then the following statements hold:*

1. $\bigcap\limits_{i \in I} \mathcal{H}_i$ *is an (α, β)-hesitant fuzzy left (right, two-sided) ideal on S.*
2. $\bigcup\limits_{i \in I} \mathcal{H}_i$ *is an (α, β)-hesitant fuzzy left (right, two-sided) ideal on S.*

Now, we define a γ-cut (or γ-level set) of the hesitant fuzzy set $\mathcal{H}$ on a semigroup S and then we present some results in this connection.

Let $\mathcal{H}$ be a hesitant fuzzy set on a semigroup S. For each $\gamma \in \mathcal{P}([0, 1])$, the set

$$U(\mathcal{H} : \gamma) = \{x \in S : \mathcal{H}_x \supseteq \gamma\}$$

is said to be a **γ-cut (or γ-level set)** of $\mathcal{H}$.

In the following, we characterize an (α, β)-hesitant fuzzy subsemigroup ((α, β)-hesitant fuzzy left ideal, (α, β)-hesitant fuzzy right ideal, (α, β)-hesitant fuzzy two-sided ideal) on semigroups in terms γ-level subsemigroups (left ideal, right ideal, two-sided ideal).

THEOREM 2.16. *Let $\mathcal{H}$ be a hesitant fuzzy set on a semigroup S. Then the following statements hold:*

1. *For each $\gamma \in \mathcal{P}([0, 1])$ such that $\gamma \subseteq \alpha \cup \beta$, the non empty set $U({}^{\alpha}\mathcal{H} : \gamma)$ is a subsemigroup of S if and only if $\mathcal{H}$ is an (α, β)-hesitant fuzzy subsemigroup on S.*
2. *For each $\gamma \in \mathcal{P}([0, 1])$ such that $\gamma \subseteq \alpha \cup \beta$, the non empty set $U({}^{\alpha}\mathcal{H} : \gamma)$ is a left ideal of S if and only if $\mathcal{H}$ is an (α, β)-hesitant fuzzy left ideal on S.*

3. *For each $\gamma \in \mathcal{P}([0,1])$ such that $\gamma \subseteq \alpha \cup \beta$, the non empty set $U\left({}^{\alpha}\mathcal{H} : \gamma\right)$ is a right ideal of S if and only if $\mathcal{H}$ is an (α,β)-hesitant fuzzy right ideal on S.*
4. *For each $\gamma \in \mathcal{P}([0,1])$ such that $\gamma \subseteq \alpha \cup \beta$, the non empty set $U\left({}^{\alpha}\mathcal{H} : \gamma\right)$ is an ideal of S if and only if $\mathcal{H}$ is an (α,β)-hesitant fuzzy ideal on S.*

PROOF: 1. First assume that the non empty set $U\left({}^{\alpha}\mathcal{H} : \gamma\right)$ is a subsemigroup of S. Let x and y be any elements of S. Choose $\gamma \in \mathcal{P}([0,1])$ such that $\gamma = {}^{\alpha}\mathcal{H}_x^y$. Then we have ${}^{\alpha}\mathcal{H}_x \supseteq \gamma$ and ${}^{\alpha}\mathcal{H}_y \supseteq \gamma$, which implies that $x, y \in U\left({}^{\alpha}\mathcal{H} : \gamma\right)$. Thus $xy \in U\left({}^{\alpha}\mathcal{H} : \gamma\right)$, since $U\left({}^{\alpha}\mathcal{H} : \gamma\right)$ is a subsemigroup of S. This implies that ${}^{\alpha}\mathcal{H}_{xy} \supseteq \gamma = {}^{\alpha}\mathcal{H}_x^y \supseteq \mathcal{H}_x^y \cap \beta = {}_{\beta}\mathcal{H}_x^y$ and so $\mathcal{H}$ is an (α,β)-hesitant fuzzy subsemigroup on S.

Conversely, assume that $\mathcal{H}$ is an (α,β)-hesitant fuzzy subsemigroup on S. Let x and y be any elements of S such that $x, y \in U\left({}^{\alpha}\mathcal{H} : \gamma\right)$ for all $\gamma \in \mathcal{P}([0,1])$. We obtain that ${}^{\alpha}\mathcal{H}_x \supseteq \gamma$ and ${}^{\alpha}\mathcal{H}_y \supseteq \gamma$. By hypothesis,

$$\begin{aligned}
{}^{\alpha}\mathcal{H}_{xy} &= {}^{\alpha}\mathcal{H}_{xy} \cup \alpha \\
&\supseteq ({}_{\beta}\mathcal{H}_x^y) \cup \alpha \\
&= (\mathcal{H}_x \cap \mathcal{H}_y \cap \beta) \cup \alpha \\
&= (\mathcal{H}_x \cup \alpha) \cap (\mathcal{H}_y \cup \alpha) \cap (\beta \cup \alpha) \\
&= {}^{\alpha}\mathcal{H}_x \cap {}^{\alpha}\mathcal{H}_y \cap (\beta \cup \alpha) \\
&\supseteq \gamma \cap \gamma \cap (\beta \cup \alpha) \\
&= \gamma.
\end{aligned}$$

Therefore $xy \in U\left({}^{\alpha}\mathcal{H} : \gamma\right)$ and the theorem is proved.

2–4. The proof is similar to 1. □

Let A be a subset of a semigroup S and let $\delta, \zeta \in \mathcal{P}([0,1])$ such that $\delta \neq \zeta$. Recall that the (δ,ζ)-characteristic function ${\mathcal{C}_{\zeta}^{\delta}}_A$ for a subset A of S is defined by

$$\left({\mathcal{C}_{\zeta}^{\delta}}_A\right)_x = \begin{cases} \delta; & x \in A \\ \zeta; & \text{otherwise.} \end{cases}$$

Observe that if $A = S$, then it is easy to see that ${\mathcal{C}_{\zeta}^{[0,1]}}_S = \mathcal{S}$.

LEMMA 2.17. *Let ${\mathcal{C}_{\zeta}^{\delta}}_A$ and ${\mathcal{C}_{\zeta}^{\delta}}_B$ be two (δ,ζ)-characteristic functions on a semigroup S. Then the following statements hold:*

1. ${\mathcal{C}_{\zeta}^{\delta}}_A \cap {\mathcal{C}_{\zeta}^{\delta}}_B = {\mathcal{C}_{\zeta}^{\delta}}_{A \cap B}$.
2. ${\mathcal{C}_{\zeta}^{\delta}}_A \odot {\mathcal{C}_{\zeta}^{\delta}}_B = {\mathcal{C}_{\zeta}^{\delta}}_{AB}$.

Proof: It is straightforward. $\square$

Now, the result follows from fundamental theorem of (δ,ζ)-characteristic function.

Theorem 2.18. *Let A be a subset of a semigroup S and let $\delta,\zeta \in \mathcal{P}([0,1])$ such that $\delta \supset \zeta$. Then the following statements hold:*

1. *If A is a subsemigroup of S, then ${}^{\alpha}\mathcal{C}^{\delta}_{\zeta\,A}$ is an (α,β)-hesitant fuzzy subsemigroup on S.*
2. *If A is a left ideal of S, then ${}^{\alpha}\mathcal{C}^{\delta}_{\zeta\,A}$ is an (α,β)-hesitant fuzzy left ideal on S.*
3. *If A is a right ideal of S, then ${}^{\alpha}\mathcal{C}^{\delta}_{\zeta\,A}$ is an (α,β)-hesitant fuzzy right ideal on S.*
4. *If A is an ideal of S, then ${}^{\alpha}\mathcal{C}^{\delta}_{\zeta\,A}$ is an (α,β)-hesitant fuzzy ideal on S.*

Proof: 1. Let x and y be any elements of S. We consider the following cases:

1. $x,y \in A$.
2. $x \notin A$ or $y \notin A$.

Case 1: Assume that $x,y \in A$. Thus $\left(\mathcal{C}^{\delta}_{\zeta\,A}\right)_x = \delta$ and $\left(\mathcal{C}^{\delta}_{\zeta\,A}\right)_y = \delta$. Since A is a subsemigroup of S, we obtain $xy \in A$, which implies that ${}^{\alpha}\left({}^{\alpha}\mathcal{C}^{\delta}_{\zeta\,A}\right)_{xy} = \left(\mathcal{C}^{\delta}_{\zeta\,A}\right)_{xy} \cup \alpha = \delta \cup \alpha = \left(\left(\mathcal{C}^{\delta}_{\zeta\,A}\right)_x \cap \left(\mathcal{C}^{\delta}_{\zeta\,A}\right)_y\right) \cup \alpha \supseteq \left(\left(\mathcal{C}^{\delta}_{\zeta\,A}\right)_x \cap \left(\mathcal{C}^{\delta}_{\zeta\,A}\right)_y\right) \cap \beta = {}_{\beta}\left(\mathcal{C}^{\delta}_{\zeta\,A}\right)^y_x$.

Case 2: Assume that $x \notin A$ or $y \notin A$. Then we have $\left(\mathcal{C}^{\delta}_{\zeta\,A}\right)_x = \zeta$ or $\left(\mathcal{C}^{\delta}_{\zeta\,A}\right)_y = \zeta$, which implies that

$$\begin{aligned}
{}^{\alpha}\left({}^{\alpha}\mathcal{C}^{\delta}_{\zeta\,A}\right)_{xy} &= \left(\mathcal{C}^{\delta}_{\zeta\,A}\right)_{xy} \cup \alpha \\
&\supseteq \zeta \cup \alpha \\
&= \left({}^{\alpha}\mathcal{C}^{\delta}_{\zeta\,A}\right)_x \cap \left({}^{\alpha}\mathcal{C}^{\delta}_{\zeta\,A}\right)_y \\
&\supseteq \left(\left({}^{\alpha}\mathcal{C}^{\delta}_{\zeta\,A}\right)_x \cap \left({}^{\alpha}\mathcal{C}^{\delta}_{\zeta\,A}\right)_y\right) \cap \beta \\
&= {}_{\beta}\left({}^{\alpha}\mathcal{C}^{\delta}_{\zeta\,A}\right)^y_x.
\end{aligned}$$

Therefore, ${}^{\alpha}\mathcal{C}^{\delta}_{\zeta\,A}$ is an (α,β)-hesitant fuzzy subsemigroup on S.

2–4. The proof is similar to 1. □

The following theorem shows that a non empty subset of a semigroup S is a subsemigroup (left ideal, right ideal, two-sided ideal) of S if and only if the hesitant fuzzy set on S is the (α,β)-hesitant fuzzy subsemigroup ((α,β)-hesitant fuzzy left ideal, (α,β)-hesitant fuzzy right ideal, (α,β)-hesitant fuzzy two-sided ideal) on S.

THEOREM 2.19. *Let A be a subset of a semigroup S and let $\delta,\zeta\in\mathcal{P}([0,1])$ such that $\delta\supset\zeta$. Then the following properties hold.*

1. *For each $\zeta\cup\alpha\not\supseteq\delta\cup\alpha\subseteq\alpha\cup\beta$, A is a subsemigroup of S if and only if ${}^{\alpha}\mathcal{C}^{\delta}_{\zeta\,A}$ is an (α,β)-hesitant fuzzy subsemigroup on S.*
2. *For each $\zeta\cup\alpha\not\supseteq\delta\cup\alpha\subseteq\alpha\cup\beta$, A is a left ideal of S if and only if ${}^{\alpha}\mathcal{C}^{\delta}_{\zeta\,A}$ is an (α,β)-hesitant fuzzy left ideal on S.*
3. *For each $\zeta\cup\alpha\not\supseteq\delta\cup\alpha\subseteq\alpha\cup\beta$, A is a right ideal of S if and only if ${}^{\alpha}\mathcal{C}^{\delta}_{\zeta\,A}$ is an (α,β)-hesitant fuzzy right ideal on S.*
4. *For each $\zeta\cup\alpha\not\supseteq\delta\cup\alpha\subseteq\alpha\cup\beta$, A is an ideal of S if and only if ${}^{\alpha}\mathcal{C}^{\delta}_{\zeta\,A}$ is an (α,β)-hesitant fuzzy ideal on S.*

PROOF: 1. By Theorem 2.18, the necessity is clear. Now let us show the sufficiency. We suppose now that ${}^{\alpha}\mathcal{C}^{\delta}_{\zeta\,A}$ is an (α,β)-hesitant fuzzy subsemigroup on S. Let x be any element of S such that $x\in A$. Observe that ${}^{\alpha}\left(\mathcal{C}^{\delta}_{\zeta\,A}\right)_x=\left(\mathcal{C}^{\delta}_{\zeta\,A}\right)_x\cup\alpha=\delta\cup\alpha$ implies that $x\in U\left({}^{\alpha}\mathcal{C}^{\delta}_{\zeta\,A}:\delta\cup\alpha\right)$. On the other hand, let x be any element of S such that $x\in U\left({}^{\alpha}\mathcal{C}^{\delta}_{\zeta\,A}:\delta\cup\alpha\right)$. Thus we have ${}^{\alpha}\left(\mathcal{C}^{\delta}_{\zeta\,A}\right)_x\supseteq\delta\cup\alpha$ implies that $x\in A$, since $\zeta\cup\alpha\not\supseteq\delta\cup\alpha$. Therefore $A=U\left({}^{\alpha}\mathcal{C}^{\delta}_{\zeta\,A}:\delta\cup\alpha\right)$ and hence it follows from Theorem 2.16(1) that A is a subsemigroup of S.

2–4. The proof is similar to 1. □

The next result covers some basic properties, which will be useful in the sequel.

Theorem 2.20. *If $\mathcal{H}$ is an (α, β)-hesitant fuzzy right (left) ideal on a semigroup S, then $\mathcal{H} \cup (\mathcal{S} \odot \mathcal{H})$ is an (α, β)-hesitant fuzzy ideal on S.*

Proof: Then, by Theorem 2.13, we have

$$\begin{array}{rcl} {}_{\beta}(\mathcal{S} \odot (\mathcal{H} \cup (\mathcal{S} \odot \mathcal{H}))) & = & {}_{\beta}((\mathcal{S} \odot \mathcal{H}) \cup (\mathcal{S} \odot (\mathcal{S} \odot \mathcal{H}))) \\ & = & {}_{\beta}(\mathcal{S} \odot \mathcal{H}) \cup ({}_{\beta}(\mathcal{S} \odot \mathcal{S}) \odot {}_{\beta}\mathcal{H}) \\ & = & {}_{\beta}(\mathcal{S} \odot \mathcal{H}) \cup ({}_{\beta}(\mathcal{S} \odot \mathcal{S}) \odot {}_{\beta}\mathcal{H}) \\ & \subseteq & {}^{\alpha}(\mathcal{S} \odot \mathcal{H}) \cup ({}^{\alpha}\mathcal{S} \odot {}^{\alpha}\mathcal{H}) \\ & = & {}^{\alpha}(\mathcal{S} \odot \mathcal{H}) \cup {}^{\alpha}(\mathcal{S} \odot \mathcal{H}) \\ & \subseteq & {}^{\alpha}\mathcal{H} \cup {}^{\alpha}(\mathcal{S} \odot \mathcal{H}) \\ & = & {}^{\alpha}(\mathcal{H} \cup (\mathcal{S} \odot \mathcal{H})), \end{array}$$

since $\mathcal{S}$ is an (α, β)-hesitant fuzzy left ideal on S. Thus, we obtain, $\mathcal{H} \cup (\mathcal{S} \odot \mathcal{H})$ is an (α, β)-hesitant fuzzy left ideal on S. Also, we have

$$\begin{array}{rcl} {}_{\beta}((\mathcal{H} \cup (\mathcal{S} \odot \mathcal{H})) \odot \mathcal{S}) & = & {}_{\beta}((\mathcal{H} \odot \mathcal{S}) \cup ((\mathcal{S} \odot \mathcal{H}) \odot \mathcal{S})) \\ & = & {}_{\beta}(\mathcal{H} \odot \mathcal{S}) \cup {}_{\beta}((\mathcal{S} \odot \mathcal{H}) \odot \mathcal{S}) \\ & \subseteq & {}^{\alpha}\mathcal{H} \cup ({}_{\beta}\mathcal{S} \odot {}_{\beta}(\mathcal{H} \odot \mathcal{S})) \\ & \subseteq & {}^{\alpha}\mathcal{H} \cup ({}^{\alpha}\mathcal{S} \odot {}^{\alpha}\mathcal{H}) \\ & = & {}^{\alpha}(\mathcal{H} \cup (\mathcal{S} \odot \mathcal{H})), \end{array}$$

since $\mathcal{H}$ is an (α, β)-hesitant fuzzy right ideal on S. Therefore $\mathcal{H} \cup (\mathcal{S} \odot \mathcal{H})$ is an (α, β)-hesitant fuzzy right ideal on S and the proof is completed. □

3. Characterizing regular semigroups

In this section we define the concept of (α, β)-hesitant fuzzy idempotent sets on semigroups and then by using this idea we characterize the regular semigroups in terms of hesitant fuzzy left ideals, hesitant fuzzy right ideals and hesitant fuzzy ideals.

Recall that an element x is said to be **regular** if there exists an element s in a semigroup S such that $x = xsx$. A semigroup S is said to be **regular** if every element of S is regular.

First, we define the operation "$\simeq_{\beta}^{\alpha}$" on a semigroup S. Let α and β be any elements of $\mathcal{P}([0, 1])$. We consider two hesitant fuzzy sets $\mathcal{H}$ and $\mathcal{G}$ on a semigroup S. Then we have, $\mathcal{H} \simeq_{\beta}^{\alpha} \mathcal{G}$ if and only if ${}_{\beta}\mathcal{H} \subseteq {}^{\alpha}\mathcal{G}$ and ${}_{\beta}\mathcal{G} \subseteq {}^{\alpha}\mathcal{H}$. A hesitant fuzzy set $\mathcal{H}$ on a semigroup S is said to be (α, β)**-idempotent** if $\mathcal{H} \simeq_{\beta}^{\alpha} \mathcal{H} \odot \mathcal{H}$.

THEOREM 3.1. *Every (α,β)-hesitant fuzzy right (left) ideal on a regular semigroup is (α,β)-idempotent.*

PROOF: Let $\mathcal{H}$ be an (α,β)-hesitant fuzzy right ideal on a regular semigroup S. Then by Theorem 2.13, we obtain that ${}_{\beta}(\mathcal{H}\odot\mathcal{H})\subseteq{}_{\beta}(\mathcal{H}\odot\mathcal{S})\subseteq{}^{\alpha}\mathcal{H}$. On the other hand, let x be any element of S. Then, since S is regular, there exists an element s in S such that $x=xsx$. Therefore

$$\begin{aligned}
{}^{\alpha}(\mathcal{H}\odot\mathcal{H})_x &= (\mathcal{H}\odot\mathcal{H})_x\cup\alpha\\
&= \left(\bigcup_{x=ab}\mathcal{H}_a\cap\mathcal{H}_b\right)\cup\alpha\\
&\supseteq (\mathcal{H}_{xs}\cap\mathcal{H}_x)\cup\alpha\\
&= (\mathcal{H}_{xs}\cup\alpha)\cap(\mathcal{H}_x\cup\alpha)\\
&\supseteq {}^{\alpha}\mathcal{H}_{xs}\cap{}_{\beta}\mathcal{H}_x\\
&\supseteq {}_{\beta}\mathcal{H}_x\cap{}_{\beta}\mathcal{H}_x\\
&= {}_{\beta}\mathcal{H}_x
\end{aligned}$$

i.e., ${}^{\alpha}(\mathcal{H}\odot\mathcal{H})\supseteq{}_{\beta}\mathcal{H}$ and the proof is now complete. □

By Theorem 3.1, we immediately obtain the following corollary:

COROLLARY 3.2. Every (α,β)-hesitant fuzzy ideal on a regular semigroup is (α,β)-idempotent.

In the following theorem we give a characterization of a semigroup that is regular in terms of (α,β)-hesitant fuzzy right ideals and (α,β)-hesitant fuzzy left ideals.

THEOREM 3.3. *Let ζ and δ be any elements of $\mathcal{P}([0,1])$ such that $\zeta\cup\alpha\not\supseteq\delta\cap\beta$. For a semigroup S, the following statements are equivalent:*

1. *S is regular.*

2. *For every (α,β)-hesitant fuzzy right ideal $\mathcal{H}$ and every (α,β)-hesitant fuzzy left ideal $\mathcal{F}$ on S, $\mathcal{H}\cap\mathcal{F}\simeq^{\alpha}_{\beta}\mathcal{H}\odot\mathcal{F}$.*

PROOF: First assume that S is a regular semigroup. Let $\mathcal{H}$ be any (α,β)-hesitant fuzzy right ideal and $\mathcal{F}$ any (α,β)-hesitant fuzzy left ideal on S. Then by Theorem 2.13, we obtain that ${}_{\beta}(\mathcal{H}\odot\mathcal{F})\subseteq{}_{\beta}(\mathcal{H}\odot\mathcal{S})\subseteq{}^{\alpha}\mathcal{H}$ and ${}_{\beta}(\mathcal{H}\odot\mathcal{F})\subseteq{}_{\beta}(\mathcal{S}\odot\mathcal{F})\subseteq{}^{\alpha}\mathcal{F}$. Consequently we have ${}_{\beta}(\mathcal{H}\odot\mathcal{F})\subseteq{}^{\alpha}\mathcal{H}\cap{}^{\alpha}\mathcal{F}={}^{\alpha}(\mathcal{H}\cap\mathcal{F})$. On the other hand, let x be any element of S.

Then, since S is regular, there exists an element s in S such that $x = xsx$. Observe that

$$\begin{aligned}
{}^{\alpha}(\mathcal{H} \odot \mathcal{F})_x &= (\mathcal{H} \odot \mathcal{F})_x \cup \alpha \\
&= \left(\bigcup_{x=ab} \mathcal{H}_a \cap \mathcal{F}_b \right) \cup \alpha \\
&\supseteq (\mathcal{H}_{xs} \cap \mathcal{F}_x) \cup \alpha \\
&= (\mathcal{H}_{xs} \cup \alpha) \cap (\mathcal{F}_x \cup \alpha) \\
&\supseteq {}_{\beta}\mathcal{H}_x \cap (\mathcal{F}_x \cap \beta) \\
&= {}_{\beta}\mathcal{H}_x \cap {}_{\beta}\mathcal{F}_x \\
&= {}_{\beta}(\mathcal{H} \cap \mathcal{F})_x
\end{aligned}$$

implies that ${}^{\alpha}(\mathcal{H} \odot \mathcal{F}) \supseteq {}_{\beta}(\mathcal{H} \cap \mathcal{F})$. Therefore $\mathcal{H} \cap \mathcal{F} \simeq^{\alpha}_{\beta} \mathcal{H} \odot \mathcal{F}$ and so (1) implies (2).

Conversely, assume that (2) holds. Let R and L be any right ideal and any left ideal of S, respectively. In order to see that $R \cap L \subseteq RL$ holds, let x be any element of $R \cap L$. Then by Theorem 2.18, the (δ, ζ)-characteristic functions ${}^{\alpha}\mathcal{C}^{\delta}_{\zeta R}$ and ${}^{\alpha}\mathcal{C}^{\delta}_{\zeta L}$ of R and L is an (α, β)-hesitant fuzzy right ideal and an (α, β)-hesitant fuzzy left ideal on S, respectively. Then it follows from Lemma 2.17, it follows that

$$\begin{aligned}
\left({}^{\alpha}\mathcal{C}^{\delta}_{\zeta RL}\right)_x &= \left(\mathcal{C}^{\delta}_{\zeta RL}\right)_x \cup \alpha \\
&= \left(\mathcal{C}^{\delta}_{\zeta R} \odot \mathcal{C}^{\delta}_{\zeta L}\right)_x \cup \alpha \\
&= {}^{\alpha}\left(\mathcal{C}^{\delta}_{\zeta R} \odot \mathcal{C}^{\delta}_{\zeta L}\right)_x \\
&\supseteq {}_{\beta}\left(\mathcal{C}^{\delta}_{\zeta R} \cap \mathcal{C}^{\delta}_{\zeta L}\right)_x \\
&= {}_{\beta}\left(\mathcal{C}^{\delta}_{\zeta R\cap L}\right)_x \\
&= \left(\mathcal{C}^{\delta}_{\zeta R\cap L}\right)_x \cap \beta \\
&= \delta \cap \beta.
\end{aligned}$$

Hence $x \in RL$ and so $R \cap L \subseteq RL$. Since the inclusion in the other direction always holds, we obtain that $R \cap L = RL$. Therefore S is a regular semigroup and so (2) implies (1). $\square$

Recall that a semigroup S is said to be **right (left) zero** if $xy = y(xy = x)$ for all $x, y \in S$. Now, we can give the main result.

THEOREM 3.4. *Let ζ and δ be any elements of $\mathcal{P}([0,1])$ such that $\zeta \cup \alpha \not\supseteq \delta \cap \beta$. For a regular semigroup S, the following statements are equivalent:*

1. *The set $\mathcal{E}(S)$ of all (α,β)-idempotents of S forms a left (right) zero subsemigroup of S.*
2. *For every (α,β)-hesitant fuzzy left (right) ideal $\mathcal{H}$ on $S, \mathcal{H}_x \simeq^{\alpha}_{\beta} \mathcal{H}_y$ for all $x, y \in \mathcal{E}(S)$.*

PROOF: First assume that (1) holds. Let $\mathcal{H}$ be an (α,β)-hesitant fuzzy left ideal on S and let $x, y \in \mathcal{E}(S)$. Since $\mathcal{E}(S)$ is a left zero subsemigroup of S, we have $xy = x$ and $yx = y$. Observe that ${}^{\alpha}\mathcal{H}_x = {}^{\alpha}\mathcal{H}_{xy} \supseteq {}_{\beta}\mathcal{H}_y$ and ${}^{\alpha}\mathcal{H}_y = {}^{\alpha}\mathcal{H}_{yx} \supseteq {}_{\beta}\mathcal{H}_x$, implies that $\mathcal{H}_x \simeq^{\alpha}_{\beta} \mathcal{H}_y$ and hence (1) implies (2).

Conversely, assume that (2) holds. Let x be any element of S. Since S is regular, there exists an element $s \in S$ such that $x = xsx$. Now $(xsxs)(xsxs) = (xsx)s(xsx)s = xsxs \in \mathcal{E}(S)$, implies that $\mathcal{E}(S)$ is non empty. Thus it follows from Theorem 2.18(2) that the (δ,ζ)-characteristic function ${}^{\alpha}\mathcal{C}^{\delta}_{\zeta\,Sy}$ of the principal left ideal Sy of S is an (α,β)-hesitant fuzzy left ideal on S. Clearly, $\left({}^{\alpha}\mathcal{C}^{\delta}_{\zeta\,Sy}\right)_x \supseteq \left({}_{\beta}\mathcal{C}^{\delta}_{\zeta\,Sy}\right)_y = \delta \cap \beta$, which implies that $x \in Sy$. Therefore, there exist $s \in S$ such that $x = sy = s(yy) = (sy)y = xy$. Hence $\mathcal{E}(S)$ is a left zero subsemigroup of S and so (2) implies (1). $\square$

By Theorem 3.4, we immediately obtain the following corollary:

COROLLARY 3.5. Let ζ and δ be any elements of $\mathcal{P}([0,1])$ such that $\zeta \cup \alpha \not\supseteq \delta \cap \beta$. For a regular semigroup S, the following statements are equivalent:

1. The set $\mathcal{E}(S)$ of all (α,β)-idempotents of S forms an zero subsemigroup of S.
2. For every (α,β)-hesitant fuzzy ideal $\mathcal{H}$ on $S, \mathcal{H}_x \simeq^{\alpha}_{\beta} \mathcal{H}_y$ for all $x, y \in \mathcal{E}(S)$.

Recall that a semigroup S is said to be **left (right) regular** if for each element x of S, there exists an element $s \in S$ such that $x = sx^2(x = x^2s)$.

From the above discussion, we can immediately obtain the following theorems.

THEOREM 3.6. *Let ζ and δ be any elements of $\mathcal{P}([0,1])$ such that $\zeta \cup \alpha \not\supseteq \delta \cap \beta$. For a semigroup S, the following conditions are equivalent.*

1. *S is left regular.*
2. *For every (α, β)-hesitant fuzzy left ideal $\mathcal{H}$ on $S, \mathcal{H}_x \simeq_\beta^\alpha \mathcal{H}_{x^2}$ for all $x \in S$.*

PROOF: First assume that (1) holds. Let $\mathcal{H}$ be any (α, β)-hesitant fuzzy left ideal on S and let x any element of S. Then, since S is left regular, there exists an element s in S such that $x = sx^2$. Hence we have, ${}^\alpha\mathcal{H}_x = {}^\alpha\mathcal{H}_{sx^2} \supseteq {}_\beta\mathcal{H}_{x^2}$ and ${}^\alpha\mathcal{H}_{x^2} \supseteq {}_\beta\mathcal{H}_x$. Therefore $\mathcal{H}_x \simeq_\beta^\alpha \mathcal{H}_{x^2}$ and so (1) implies (2).

Conversely, assume that (2) holds. Let x be any element of S. Then it follows from Theorem 2.18(2) that the (δ, ζ)-characteristic function ${}^\alpha\mathcal{C}^\delta_{\zeta\, Sx^2}$ of the principal left ideal $x^2 \cup Sx^2$ of S is an (α, β)-hesitant fuzzy left ideal on S. Since $x^2 \in x^2 \cup Sx^2$, we have $\left({}^\alpha\mathcal{C}^\delta_{\zeta\, x^2 \cup Sx^2}\right)_x \supseteq \left({}_\beta\mathcal{C}^\delta_{\zeta\, x^2 \cup Sx^2}\right)_{x^2} = \left(\mathcal{C}^\delta_{\zeta\, x^2 \cup Sx^2}\right)_{x^2} \cap \beta = \delta \cap \beta$. This implies that $x \in x^2 \cup Sx^2$. Hence S is left regular and so (2) implies (1). $\square$

From Theorem 3.6 we can easily obtain the following corollary.

COROLLARY 3.7. Let ζ and δ be any elements of $\mathcal{P}([0,1])$ such that $\zeta \cup \alpha \not\supseteq \delta \cap \beta$. For a semigroup S, the following conditions are equivalent.

1. S is right regular.
2. For every (α, β)-hesitant fuzzy right ideal $\mathcal{H}$ on $S, \mathcal{H}_x \simeq_\beta^\alpha \mathcal{H}_{x^2}$ for all $x \in S$.

Recall that a subset A of a semigroup S is said to be **semiprime** if for all $x \in S, x^2 \in A$ implies $x \in A$. Now, we give the definition of (α, β)-hesitant fuzzy semiprime set on a semigroup S, which is a generalization of the notion of hesitant fuzzy semiprime sets.

A hesitant fuzzy set $\mathcal{H}$ on a semigroup S is said to be (α, β)**-hesitant fuzzy semiprime** if ${}^\alpha\mathcal{H}_x \supseteq {}_\beta\mathcal{H}_{x^2}$ for all $x \in S$.

THEOREM 3.8. *Let ζ and δ be any elements of $\mathcal{P}([0,1])$ such that $\zeta \cup \alpha \not\supseteq \delta \cap \beta$. If P is a non empty subset of a semigroup S, then the following conditions are equivalent:*

1. *P is semiprime.*
2. *The (δ,ζ)-characteristic function ${}^{\alpha}\mathcal{C}^{\delta}_{\zeta\,P}$ of P is an (α,β)-hesitant fuzzy semiprime.*

PROOF: First assume that P is a semiprime set of S. Let x be any element of S. We consider the following cases:

1. $x^2 \in P$.
2. $x \notin P$ or $y \notin P$.

Case 1: Assume that $x^2 \in P$. Since P is semiprime, we have $x \in P$. Then, we obtain $\left({}^{\alpha}\mathcal{C}^{\delta}_{\zeta\,P}\right)_x = \delta \cup \alpha$ and $\left({}_{\beta}\mathcal{C}^{\delta}_{\zeta\,P}\right)_{x^2} = \delta \cap \beta$.

Case 2: Assume that $x^2 \notin P$. It is easy to see that $\left({}_{\beta}\mathcal{C}^{\delta}_{\zeta\,P}\right)_{x^2} = \zeta \cap \beta$. In any case, we have $\left({}_{\beta}\mathcal{C}^{\delta}_{\zeta\,P}\right)_{x^2} \subseteq \left({}^{\alpha}\mathcal{C}^{\delta}_{\zeta\,P}\right)_x$ for all $x \in S$. Therefore, ${}^{\alpha}\mathcal{C}^{\delta}_{\zeta\,P}$ is an (α,β)-hesitant fuzzy semiprime set on S and hence (1) implies (2).

Conversely, assume that (2) holds. Let x be any element of S such that $x^2 \in P$. Since ${}^{\alpha}\mathcal{C}^{\delta}_{\zeta\,P}$ is an (α,β)-hesitant fuzzy semiprime set on S, it follows that $\left({}^{\alpha}\mathcal{C}^{\delta}_{\zeta\,P}\right)_x \supseteq \left({}_{\beta}\mathcal{C}^{\delta}_{\zeta\,P}\right)_{x^2} = \left(\mathcal{C}^{\delta}_{\zeta\,P}\right)_{x^2} \cap \beta = \delta \cap \beta$. Observe that $\left({}^{\alpha}\mathcal{C}^{\delta}_{\zeta\,P}\right)_x = \delta \cup \alpha$, implies that $x \in P$. Therefore, P is a semiprime set of S and hence (2) implies (1). $\square$

In order to characterize the (α,β)-hesitant fuzzy semiprime set generated by a hesitant fuzzy semiprime set in a semigroup, we need the following theorem.

THEOREM 3.9. *If $\mathcal{H}$ is an (α,β)-hesitant fuzzy subsemigroup on a semigroup S, then the following conditions are equivalent:*

1. *$\mathcal{H}$ is an (α,β)-hesitant fuzzy semiprime set on S.*
2. *For every $x \in S, \mathcal{H}_x \simeq^{\alpha}_{\beta} \mathcal{H}_{x^2}$.*

PROOF: It is clear that (2) implies (1). Assume that $\mathcal{H}$ is an (α,β)-hesitant fuzzy semiprime set on S. Let x be any element of S. Otherwise, we have ${}^{\alpha}\mathcal{H}_x \supseteq {}_{\beta}\mathcal{H}_{x^2}$ and ${}^{\alpha}\mathcal{H}_{x^2} \supseteq {}_{\beta}\mathcal{H}^{x}_{x} = {}_{\beta}\mathcal{H}_x$. Therefore $\mathcal{H}_x \simeq^{\alpha}_{\beta} \mathcal{H}_{x^2}$ and hence (1) implies (2). $\square$

Recall that a semigroup S is said to be **intra-regular** if for each element x of S, there exist elements r and s in S such that $x = rx^2s$. Now we shall characterize the intra-regular semigroups in terms of (α, β)-hesitant fuzzy ideals.

THEOREM 3.10. *Let ζ and δ be any elements of $\mathcal{P}([0,1])$ such that $\zeta \cup \alpha \not\supseteq \delta \cap \beta$. If S is a semigroup S, then the following conditions are equivalent:*

1. *S is intra-regular.*
2. *For every (α, β)-hesitant fuzzy ideal on S is (α, β)-hesitant fuzzy semiprime.*
3. *For every (α, β)-hesitant fuzzy ideal $\mathcal{H}$ on S, $\mathcal{H}_x \simeq^{\alpha}_{\beta} \mathcal{H}_{x^2}$ for all $x \in S$.*

PROOF: First assume that S is an intra-regular semigroup. Let $\mathcal{H}$ be any (α, β)-hesitant fuzzy ideal on S. Next, let x be any element of S. Then, since S is intra-regular, there exist elements r and s in S such that $x = rx^2s$, which implies that

$$\begin{aligned}
{}^{\alpha}\mathcal{H}_x &= {}^{\alpha}\mathcal{H}_{rx^2s} \cup \alpha \\
&= {}^{\alpha}\mathcal{H}_{rx^2s} \cup \alpha \\
&\supseteq {}_{\beta}\mathcal{H}_{rx^2} \cup \alpha \\
&= (\mathcal{H}_{rx^2} \cap \beta) \cup \alpha \\
&= (\mathcal{H}_{rx^2} \cup \alpha) \cap (\beta \cup \alpha) \\
&= {}^{\alpha}\mathcal{H}_{rx^2} \cap (\beta \cup \alpha) \\
&\supseteq {}_{\beta}\mathcal{H}_{x^2} \cap (\beta \cup \alpha) \\
&= \mathcal{H}_{x^2} \cap \beta \cap (\beta \cup \alpha) \\
&= \mathcal{H}_{x^2} \cap \beta \\
&= {}_{\beta}\mathcal{H}_{x^2}.
\end{aligned}$$

Thus, we obtain ${}^{\alpha}\mathcal{H}_{x^2} \supseteq {}_{\beta}\mathcal{H}_x$, since $\mathcal{H}$ is an (α, β)-hesitant fuzzy ideal. Therefore $\mathcal{H}_x \simeq^{\alpha}_{\beta} \mathcal{H}_{x^2}$ and so (1) implies (3).

Assume that (2) holds. Let x be any element of S. Then it follows from Theorem 2.18(2) that the (δ, ζ)-characteristic function ${}^{\alpha}\mathcal{C}^{\delta}_{\zeta\, x^2 \cup Sx^2 \cup x^2S \cup Sx^2S}$ of the principal ideal $x^2 \cup Sx^2 \cup x^2S \cup Sx^2S$ of S is an (α, β)-hesitant fuzzy ideal on S. Therefore, $\left({}^{\alpha}\mathcal{C}^{\delta}_{\zeta\, x^2 \cup Sx^2 \cup x^2S \cup Sx^2S}\right)_x \supseteq ({}_{\beta}\mathcal{C}^{\delta}_{\zeta\, x^2 \cup Sx^2 \cup x^2S \cup Sx^2S})_{x^2} = (\mathcal{C}^{\delta}_{\zeta\, x^2 \cup Sx^2 \cup x^2S \cup Sx^2S})_{x^2} \cap \beta = \delta \cap \beta$, since $x^2 \in x^2 \cup Sx^2 \cup x^2S \cup Sx^2S$. Clearly, $x \in x^2 \cup Sx^2 \cup x^2S \cup Sx^2S$. Therefore it is easily seen that S is intra-regular and so (2) implies (1). It is clear that (2) and (3) are equivalent. □

Now we characterize the intra-regular semigroup in terms of (α, β)-hesitant fuzzy left ideals and (α, β)-hesitant fuzzy right ideals.

Theorem 3.11. *Let ζ and δ be any elements of $\mathcal{P}([0,1])$ such that $\zeta \cup \alpha \not\supseteq \delta \cap \beta$. If S is a semigroup S, then the following conditions are equivalent:*

1. *S is intra-regular.*
2. *${}_{\beta}(\mathcal{H} \cap \mathcal{F}) \subseteq {}^{\alpha}(\mathcal{H} \odot \mathcal{F})$ for every (α, β)-hesitant fuzzy left ideal $\mathcal{H}$ and every (α, β)-hesitant fuzzy right ideal $\mathcal{F}$ on S.*

Proof: First assume that S is intra-regular. Let $\mathcal{H}$ and $\mathcal{F}$ be any (α, β)-hesitant fuzzy left ideal and any (α, β)-hesitant fuzzy right ideal on S, respectively. Next, let x be any element of S. Then, since S is intra-regular, there exist elements r and s in S such that $x = rx^2s$. Hence we have,

$$\begin{aligned}
{}^{\alpha}(\mathcal{H} \odot \mathcal{F})_x &= (\mathcal{H} \odot \mathcal{F})_x \cup \alpha \\
&= \left(\bigcup_{x=ab} \mathcal{H}_a \cap \mathcal{F}_b \right) \cup \alpha \\
&\supseteq (\mathcal{H}_{rx} \cap \mathcal{F}_{xs}) \cup \alpha \\
&= (\mathcal{H}_{rx} \cup \alpha) \cap (\mathcal{F}_{xs} \cup \alpha) \\
&= {}^{\alpha}\mathcal{H}_{rx} \cap {}^{\alpha}\mathcal{F}_{xs} \\
&\supseteq {}_{\beta}\mathcal{H}_x \cap {}_{\beta}\mathcal{F}_x \\
&= {}_{\beta}(\mathcal{H} \cap \mathcal{F})_x,
\end{aligned}$$

which implies that ${}_{\beta}(\mathcal{H} \cap \mathcal{F}) \subseteq {}^{\alpha}(\mathcal{H} \odot \mathcal{F})$. Therefore (1) implies (2).

Conversely, assume that (2) holds. Let L and R be any left ideal and any right ideal of S, respectively. Next, let x be any element of S such that $x \in L \cap R$. Then $x \in L$ and $x \in R$. By Theorem 2.18 ${}^{\alpha}\mathcal{C}^{\delta}_{\zeta\, L}$ and ${}^{\alpha}\mathcal{C}^{\delta}_{\zeta\, R}$ is an (α, β)-hesitant fuzzy left ideal and an (α, β)-hesitant fuzzy right ideal on S, respectively. Then, by Lemma 2.17, we obtain that

$$\begin{aligned}
\left({}^{\alpha}\mathcal{C}^{\delta}_{\zeta\, LR}\right)_x &= \left(\mathcal{C}^{\delta}_{\zeta\, LR}\right)_x \cup \alpha \\
&= \left(\mathcal{C}^{\delta}_{\zeta\, L} \odot \mathcal{C}^{\delta}_{\zeta\, R}\right)_x \cup \alpha \\
&= {}^{\alpha}\left(\mathcal{C}^{\delta}_{\zeta\, L} \odot \mathcal{C}^{\delta}_{\zeta\, R}\right)_x \\
&\supseteq {}_{\beta}\left(\mathcal{C}^{\delta}_{\zeta\, L} \cap \mathcal{C}^{\delta}_{\zeta\, R}\right)_x \\
&= {}_{\beta}\left(\mathcal{C}^{\delta}_{\zeta\, L\cap R}\right)_x \\
&\supseteq \left(\mathcal{C}^{\delta}_{\zeta\, L\cap R}\right)_x \cap \beta \\
&= \delta \cap \beta,
\end{aligned}$$

which means that $x \in LR$. Therefore we obtain that $L \cap R \subseteq LR$ and hence S is intra-regular. Thus (2) implies (1). $\square$

In the following theorem we give a characterization of a semigroup that is both regular and intra-regular in terms of (α, β)-hesitant fuzzy right ideals and (α, β)-hesitant fuzzy left ideals.

Theorem 3.12. *Let ζ and δ be any elements of $\mathcal{P}([0,1])$ such that $\zeta \cup \alpha \not\supseteq \delta \cap \beta$. If S is a semigroup S, then the following conditions are equivalent:*

1. *S is regular and intra-regular.*
2. *${}_{\beta}(\mathcal{H} \cap \mathcal{F}) \subseteq {}^{\alpha}((\mathcal{H} \odot \mathcal{F}) \cap (\mathcal{F} \odot \mathcal{H}))$ for every (α, β)-hesitant fuzzy right ideal $\mathcal{H}$ and every (α, β)-hesitant fuzzy left ideal $\mathcal{F}$ on S.*

Proof: First assume that (1) holds. Let $\mathcal{H}$ and $\mathcal{F}$ be any (α, β)-hesitant fuzzy right ideal and any (α, β)-hesitant fuzzy left ideal on S, respectively. Then it follows from Theorems 3.3, 3.11 that ${}_{\beta}(\mathcal{H} \cap \mathcal{F}) \subseteq {}^{\alpha}(\mathcal{H} \odot \mathcal{F})$ and ${}_{\beta}(\mathcal{H} \cap \mathcal{F}) = {}_{\beta}(\mathcal{F} \cap \mathcal{H}) \subseteq {}^{\alpha}(\mathcal{F} \odot \mathcal{H})$. Therefore ${}_{\beta}(\mathcal{H} \cap \mathcal{F}) \subseteq {}^{\alpha}((\mathcal{H} \odot \mathcal{F}) \cap (\mathcal{F} \odot \mathcal{H}))$ and so (1) implies (2).

Conversely, assume that (2) holds. Let $\mathcal{H}$ and $\mathcal{F}$ be any (α, β)-hesitant fuzzy right ideal and any (α, β)-hesitant fuzzy left ideal on S, respectively. We obtain

$$\begin{aligned}
{}_{\beta}(\mathcal{F} \cap \mathcal{H}) &= {}_{\beta}(\mathcal{H} \cap \mathcal{F}) \\
&\subseteq \left(\mathcal{C}^{\delta}_{\zeta L} \odot \mathcal{C}^{\delta}_{\zeta R}\right)_x \cup \alpha \\
&= {}^{\alpha}((\mathcal{H} \odot \mathcal{F}) \cap (\mathcal{F} \odot \mathcal{H})) \\
&\subseteq {}^{\alpha}(\mathcal{F} \odot \mathcal{H}).
\end{aligned}$$

Thus it follows from Theorem 3.11 that S is an intra-regular semigroup. Now, we show that S is a regular semigroup. By the assumption, ${}_{\beta}(\mathcal{H} \cap \mathcal{F}) \subseteq {}^{\alpha}((\mathcal{H} \odot \mathcal{F}) \cap (\mathcal{F} \odot \mathcal{H})) \subseteq {}^{\alpha}(\mathcal{H} \odot \mathcal{F})$. On the other hand, ${}_{\beta}(\mathcal{H} \odot \mathcal{F}) \subseteq {}_{\beta}(\mathcal{S} \odot \mathcal{F}) \subseteq {}^{\alpha}\mathcal{F}$ and ${}_{\beta}(\mathcal{H} \odot \mathcal{F}) \subseteq {}_{\beta}(\mathcal{H} \odot \mathcal{S}) \subseteq {}^{\alpha}\mathcal{H}$, which means that ${}_{\beta}(\mathcal{H} \odot \mathcal{F}) \subseteq {}^{\alpha}\mathcal{H} \cap {}^{\alpha}\mathcal{F} = {}^{\alpha}(\mathcal{H} \cap \mathcal{F})$. Hence it follows from Theorem 3.3 that S is regular. Therefore (2) implies (1). $\square$

4. Conclusion

In study the structure of semigroups, we notice that the (α, β)-hesitant fuzzy sets with special properties always play an important role. The (α, β)-hesitant fuzzy ideals on a semigroup are key tools to describe the algebraic subsystems of a semigroup S. By using the point wise (left, right) ideas and methods, in this paper we defined and studied (α,β)-hesitant fuzzy (left, right) ideals on semigroups. In particular, we introduced the concepts of $^{\alpha}$-hesitant ($_{\alpha}$-hesitant) fuzzy sets, (α,β)-hesitant fuzzy subsemigroups and (α,β)-hesitant fuzzy ideals of semigroups, and characterized regular semigroups in terms of (α,β)-hesitant fuzzy ideals. Furthermore, we prove that the non empty subset of a semigroup S is a subsemigroup (left ideal, right ideal, two-sided ideal) of S if and only if the hesitant fuzzy set on S is the (α,β)-hesitant fuzzy subsemigroup ((α,β)-hesitant fuzzy left ideal, (α,β)-hesitant fuzzy right ideal, (α,β)-hesitant fuzzy two-sided ideal) on S. As an application of the results of this paper, the corresponding results of fuzzy sets. We hope that this work would offer foundation for further study of the theory on semigroups.

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COMPLETE REPRESENTATIONS AND NEAT EMBEDDINGS

Abstract

Let $2 < n < \omega$. Then CA_n denotes the class of cylindric algebras of dimension n, RCA_n denotes the class of representable CA_ns, CRCA_n denotes the class of completely representable CA_ns, and $\mathsf{Nr}_n\mathsf{CA}_\omega(\subseteq \mathsf{CA}_n)$ denotes the class of n-neat reducts of CA_ωs. The elementary closure of the class CRCA_ns ($\mathbf{K_n}$) and the non-elementary class $\mathsf{At}(\mathsf{Nr}_n\mathsf{CA}_\omega)$ are characterized using two-player zero-sum games, where At is the operator of forming atom structures. It is shown that $\mathbf{K_n}$ is not finitely axiomatizable and that it coincides with the class of atomic algebras in the elementary closure of $\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega$ where $\mathbf{S_c}$ is the operation of forming complete subalgebras. For any class $\mathbf{L}$ such that $\mathsf{At}\mathsf{Nr}_n\mathsf{CA}_\omega \subseteq \mathbf{L} \subseteq \mathsf{At}\mathbf{K_n}$, it is proved that $\mathbf{SP}\mathfrak{Cm}\mathbf{L} = \mathsf{RCA}_n$, where $\mathfrak{Cm}$ is the dual operator to At; that of forming complex algebras. It is also shown that any class $\mathbf{K}$ between $\mathsf{CRCA}_n \cap \mathbf{S_d}\mathsf{Nr}_n\mathsf{CA}_\omega$ and $\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_{n+3}$ is not first order definable, where $\mathbf{S_d}$ is the operation of forming dense subalgebras, and that for any $2 < n < m$, any $l \geq n+3$ any any class $\mathbf{K}$ such that $\mathsf{At}(\mathsf{Nr}_n\mathsf{CA}_m \cap \mathsf{CRCA}_n) \subseteq \mathbf{K} \subseteq \mathsf{At}\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_l$, $\mathbf{K}$ is not not first order definable either.

Keywords: Algebraic logic, cylindric algebras, relation algebras, atom-canonicity, combinatorial game theory.

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We follow the notation of [1] which is in conformity with the notation in the monograph [3]. In particular, for any pair of ordinal $\alpha < \beta$, CA_α stands for the class of cylindric algebras of dimension α, RCA_α denotes the class

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of representable CA_αs and $\mathsf{Nr}_\alpha\mathsf{CA}_\beta(\subseteq \mathsf{CA}_\alpha)$ denotes the class of α-neat reducts of CA_βs.

Definition 0.1. Assume that $\alpha < \beta$ are ordinals and that $\mathfrak{B} \in \mathsf{CA}_\beta$. Then the *$\alpha$-neat reduct* of $\mathfrak{B}$, in symbols $\mathfrak{Nr}_\alpha\mathfrak{B}$, is the algebra obtained from $\mathfrak{B}$, by discarding cylindrifiers and diagonal elements whose indices are in $\beta\setminus\alpha$, and restricting the universe to the set

$$Nr_\alpha B = \{x \in \mathfrak{B} : \{i \in \beta : \mathsf{c}_i x \neq x\} \subseteq \alpha\}.$$

It is straightforward to check that $\mathfrak{Nr}_\alpha\mathfrak{B} \in \mathsf{CA}_\alpha$. Let $\alpha < \beta$ be ordinals. If $\mathfrak{A} \in \mathsf{CA}_\alpha$ and $\mathfrak{A} \subseteq \mathfrak{Nr}_\alpha\mathfrak{B}$, with $\mathfrak{B} \in \mathsf{CA}_\beta$, then we say that $\mathfrak{A}$ *neatly embeds* in $\mathfrak{B}$, and that $\mathfrak{B}$ is a *β-dilation of* $\mathfrak{A}$, or simply a *dilation* of $\mathfrak{A}$ if β is clear from context. For $\mathbf{K} \subseteq \mathsf{CA}_\beta$, we write $\mathsf{Nr}_\alpha\mathbf{K}$ for the class $\{\mathfrak{Nr}_\alpha\mathfrak{B} : \mathfrak{B} \in \mathbf{K}\}$. Following [3], Cs_n denotes the class of *cylindric set algebras of dimension* n, and Gs_n denotes the class of *generalized cylindric set algebra of dimension* n; $\mathfrak{C} \in \mathsf{Gs}_n$, if $\mathfrak{C}$ has top element V a disjoint union of cartesian squares, that is $V = \bigcup_{i\in I} {}^nU_i$, I is a non-empty indexing set, $U_i \neq \emptyset$ and $U_i \cap U_j = \emptyset$ for all $i \neq j$. The operations of $\mathfrak{C}$ are defined like in cylindric set algebras of dimension n relativized to V.

Definition 0.2. An algebra $\mathfrak{A} \in \mathsf{CA}_n$ is *completely representable* $\iff$ there exists $\mathfrak{C} \in \mathsf{Gs}_n$, and an isomorphism $f : \mathfrak{A} \to \mathfrak{C}$ such that for all $X \subseteq \mathfrak{A}$, $f(\sum X) = \bigcup_{x\in X} f(x)$, whenever $\sum X$ exists in $\mathfrak{A}$. If $\sum X$ exists in $\mathfrak{A}$, we denote this supremum by $\sum^{\mathfrak{A}} X$. In this case, we say that $\mathfrak{A}$ is *completely representable via* f.

It is known that $\mathfrak{A}$ is completely representable via $f : \mathfrak{A} \to \mathfrak{C}$, where $\mathfrak{C} \in \mathsf{Gs}_n$ has top element V say $\iff$ $\mathfrak{A}$ is atomic and f is *atomic* in the sense that $f(\sum \mathsf{At}\mathfrak{A}) = \bigcup_{x\in\mathsf{At}\mathfrak{A}} f(x) = V$ [5] where $\mathsf{At}\mathfrak{A}$ denotes the set of atoms of $\mathfrak{A}$. We denote the class of completely representable CA_ns by CRCA_n.

For an atomic Boolean algebra with operators $\mathfrak{A}$ say, we may write $\mathsf{At}\mathfrak{A}$ to denote its atom structures, i.e. the set of atoms *expanded with the accessiblity relations corresponding to the non- Boolean operations*- which is a first order structure. In modal logic terminology, this atom structure is nothing more than a *Kripke frame.* It will be clear from context what we mean by $\mathsf{At}\mathfrak{A}$ (either the atom structure of $\mathfrak{A}$ or the set of atoms of $\mathfrak{A}$). No confusion is likely to ensue. We write $\mathfrak{A} \subseteq_d \mathfrak{B}$ if $\mathfrak{A}$ is dense subalgebra of $\mathfrak{B}$. Recall that $\mathfrak{A} \subseteq_d \mathfrak{B}$ if $\mathfrak{A}$ is a subalgebra of B, in symbols $\mathfrak{A} \subseteq \mathfrak{B}$, and for all non-zero $b \in \mathfrak{B}$, there exists a non-zero $a \in \mathfrak{A}$ such that $a \leq b$. Let $\mathbf{S_d}$ denote the class of forming dense subalgebas; that is to say, for a class K of Boolean algebras with operators $\mathbf{S_d}\mathsf{K} = \{\mathfrak{A} : (\exists \mathfrak{B} \in \mathsf{K})(\mathfrak{A} \subseteq_d \mathfrak{B})\}$. Given two Boolean algebras with operators $\mathfrak{A}, \mathfrak{B}$ having the same signature, we write $\mathfrak{A} \subseteq_c \mathfrak{B}$ if $\mathfrak{A}$ is a complete subalgebra of $\mathfrak{B}$ in the sense that for all $X \subseteq A$, if $\sum^{\mathfrak{A}} X = 1$ then $\sum^{\mathfrak{B}} X = 1$.[1] We write $\mathbf{S_c}$ for the operation of forming subalgebras, that is to say for a class K of Boolean algebras with operators, $\mathbf{S_c}\mathsf{K} = \{\mathfrak{A} : (\exists \mathfrak{B} \in \mathsf{K})(\mathfrak{A} \subseteq_c \mathfrak{B})\}$. It is known that the class CRCA_n coincides with the class of atomic algebras in $\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega$ as long as the number of atoms is countable [14, Theorem 5.3.6]. However, unlike ordinary reprsentations, this charactrization using complete neat embeddings does not generalize to the uncountable case. This will be proved below in Theorem 1.16, where an atomic $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_\omega$ having uncountably many atoms but $\mathfrak{A}$ has no complete representation, is constructed.

Define the class LCA_n as follows: $\mathfrak{A} \in \mathsf{LCA}_n \iff \mathfrak{A}$ is atomic and $\exists$ has a winning strategy in $G_k(\mathsf{At}\mathfrak{A})$ for all $k < \omega$, where G_k is the k rounded game defined on atomic networks in [7, Definition 3.3.2] truncated to k rounds. Then this class is elementary, because a winning strategy for $\exists$ in G_k can be coded by a first order sentence; call it ρ_k. Hirsch and Hodkinson study the class of atom structures of this class denoted by LCAS_n on [7, p. 73] that they call atom structures satisfying the 'Lyndon conditions' [7]. In our context, working now on the algebra level, the Lyndon conditions that Hirsch and Hodkinson use can be lifted to the algebra level as first order formulas that are just the ρ_ks.

Layout: Fix $2 < n < \omega$. In the following Section 1, the class $\mathbf{El}\mathsf{CRCA}_n$ is characterized using neat embeddings. It is shown that $\mathbf{El}\mathsf{CRCA}_n$ coincides with the elementary class LCA_n defined by the Lyndon conditions and that $\mathsf{LCA}_n = \mathbf{El}\mathsf{CRCA}_n = \mathbf{El}\mathbf{S_c}\mathsf{Nr}_n(\mathsf{CA}_\omega \cap \mathbf{At}) = (\mathbf{El}\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega) \cap \mathbf{At}$, cf. Theorem 1.4. In particular, $\mathsf{Nr}_n\mathsf{CA}_\omega \subseteq \mathsf{LCA}_n$. We show that LCA_n is not finitely axiomatizable, and we prove that RCA_n is generated by $\mathsf{At}(\mathsf{LCA}_n)$ in the following strong sense $\mathsf{RCA}_n = \mathbf{S}\mathfrak{Cm}\mathsf{At}(\mathsf{LCA}_n)$ and by $\mathsf{At}(\mathsf{Nr}_n\mathsf{CA}_\omega)$ in the weaker sense $\mathsf{RCA}_n = \mathbf{SP}\mathfrak{Cm}\mathsf{At}(\mathsf{Nr}_n\mathsf{CA}_\omega)$, cf. Theorem 1.17. We also show that for any $2 < n < l < m$, there exists an atomic $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_l \cap \mathsf{RCA}_n$ such that its Dedekind–MacNeille completion[2], namely, the complex alge-

[1] This is different from that $\mathfrak{A} \subseteq \mathfrak{B}$ and $\mathfrak{A}$ is complete.

[2] Sometimes referred to as minimal or Monk completion.

bra of its atom structure, in symbols $\mathfrak{Cm}\mathsf{At}\mathfrak{A}$, is outside RCA_n, cf. Theorem 1.12. In Section 2 we continue study atom-canonicity for varieties of cylindric algebras and introduce a new notion of 'degrees of representability' cf. Theorems 2.2, which enables one to measure in a precise sense the degree of representability of a given $\mathfrak{A} \in \mathsf{RCA}_n$; some algebras are more representable than others: Given an atomic algebra $\mathfrak{A} \in \mathsf{RCA}_n$ and $n < m \leq \omega$, then $\mathfrak{A}$ is representable up to m if $\mathfrak{Cm}\mathsf{At}\mathfrak{A} \in \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_m$. In the final Section 4, using certain atomic games, we characterize the non-elementary class $\mathsf{At}(\mathsf{Nr}_n\mathsf{CA}_\omega)$ and it is shown, using such games, that any class $\mathbf{K}$ such that $\mathsf{CRCA}_n \cap \mathbf{S_d}\mathsf{Nr}_n\mathsf{CA}_\omega \subseteq \mathbf{K} \subseteq \mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_{n+3}$, $\mathbf{K}$ is not elementary, cf. Theorem 3.1.

1. Complete representations and the Lyndon conditions

Fix a finite ordinal $n > 2$. For a class $\mathbf{K}$, $\mathsf{El}\mathbf{K}$ denotes its elementary closure. By the Keisler-Shelah Ultrapower Theorem, $\mathsf{ElK} = \mathbf{UpUr}\mathsf{K}$ where $\mathbf{Up}(\mathbf{Ur})$ denotes the operation of forming ultraproducts (ultraroots). For a Boolean algebra $\mathfrak{A}$ and $a \in \mathfrak{A}$, $\mathfrak{Rl}_a\mathfrak{A}$ is the Boolean with universe $\{x \in \mathfrak{A} : x \leq a\}$ and Boolean operations those of $\mathfrak{A}$ relativized to the universe. For a Boolean algebra $\mathfrak{A}$, we write $\mathfrak{A}^+$ to denote its canonical extension.

Definition 1.1. [3, Definition 3.1.2] Let α be an ordinal. A *weak space of dimension* α is a set V of the form $\{s \in {}^\alpha U : |\{i \in \alpha : s_i \neq p_i\}| < \omega\}$ where U is a non-empty set and $p \in {}^\alpha U$. We denote V by ${}^\alpha U^{(p)}$. Following [3], Ws_α denotes the class of weak set algebra of dimension α. The top elements of Ws_αs are weak spaces of dimension α and the operations are defined like in cylindric set algebras of dimension α relativized to the top element.

Observe that when $\alpha < \omega$, $\mathsf{Ws}_\alpha = \mathsf{Cs}_\alpha$. To define certain deterministic games to be used in the sequel, we recall the notions of *atomic networks*, and *atomic games* [6, 7]. Let $i < n$. For n-ary sequences $\bar{x}$ and $\bar{y}$ $\iff$ $\bar{y}(j) = \bar{x}(j)$ for all $j \neq i$.

Definition 1.2. Fix finite $n > 2$ and assume that $\mathfrak{A} \in \mathsf{CA}_n$ is atomic.

(1) An *n-dimensional atomic network* on $\mathfrak{A}$ is a map $N : {}^n\Delta \to \mathsf{At}\mathfrak{A}$, where Δ is a non-empty set of *nodes*, denoted by $\mathsf{nodes}(N)$, satisfying the following consistency conditions for all $i < j < n$:

- If $\bar{x} \in {}^n\mathsf{nodes}(N)$ then $N(\bar{x}) \leq \mathsf{d}_{ij} \iff x_i = x_j$,
- If $\bar{x}, \bar{y} \in {}^n\mathsf{nodes}(N)$, $i < n$ and $\bar{x} \equiv_i \bar{y}$, then $N(\bar{x}) \leq \mathsf{c}_i N(\bar{y})$.

For n-dimensional atomic networks M and N, we write $M \equiv_i N \iff M(\bar{y}) = N(\bar{y})$ for all $\bar{y} \in {}^n(n \sim \{i\})$.

(2) Assume that $m, k \leq \omega$. The *atomic game* $G^m_k(\mathsf{At}\mathfrak{A})$, *or simply* G^m_k, is the game played on atomic networks of $\mathfrak{A}$ using m nodes and having k rounds [7, Definition 3.3.2], where $\forall$ is offered only one move, namely, *a cylindrifier move*: At round zero $\forall$ picks an atom $a \in A$. Then $\exists$ has to respond with a network N and a tuple $\bar{y}$ such that $N(\bar{y}) = a$. Suppose that we are at round $t > 0$. Then $\forall$ picks the played network N_t ($\mathsf{nodes}(N_t) \subseteq m$), $i < n$, $a \in \mathsf{At}\mathfrak{A}$, $x \in {}^n\mathsf{nodes}(N_t)$, such that $N_t(\bar{x}) \leq \mathsf{c}_i a$. For her response, $\exists$ has to deliver a network M such that $\mathsf{nodes}(M) \subseteq m$, $M \equiv_i N$, and there is $\bar{y} \in {}^n\mathsf{nodes}(M)$ that satisfies $\bar{y} \equiv_i \bar{x}$ and $M(\bar{y}) = a$. We write $G_k(\mathsf{At}\mathfrak{A})$, or simply G_k, for $G^m_k(\mathsf{At}\mathfrak{A})$ if $m \geq \omega$.

(3) The ω-rounded game $\mathbf{G}^m(\mathsf{At}\mathfrak{A})$ or simply $\mathbf{G}^m$ is like the game $G^m_\omega(\mathsf{At}\mathfrak{A})$ except that $\forall$ has the bonus to reuse the m nodes in play.[3]

Lemma 1.3. *Let $2 < n < m < \omega$ and assume that $\mathfrak{A} \in \mathsf{CA}_n$ is atomic. If $\mathfrak{A} \in \mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_m$, then $\exists$ has a winning strategy in $\mathbf{G}^m(\mathsf{At}\mathfrak{A})$.*

Proof: [15, Lemma 4.3]. □

For a class K of BAOs, recall that $\mathsf{K} \cap \mathbf{At}$ denotes the class of atomic algebras in K. Let $\mathsf{Fs}_n = \{\mathfrak{A} \in \mathsf{Cs}_n : A = \wp({}^nU) \text{ some non-empty set } U\}$.

Theorem 1.4. *For $2 < n < \omega$ the following hold:*

1. $\mathsf{CRCA}_n \subseteq \mathbf{S_c}\mathsf{Nr}_n(\mathsf{CA}_\omega \cap \mathbf{At}) \cap \mathbf{At} \subseteq \mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At}$,
2. *If $\mathfrak{A} \in \mathsf{CRCA}_n$, then $\exists$ has a winning strategy in $G_\omega(\mathsf{At}\mathfrak{A})$ and $\mathbf{G}^\omega(\mathsf{At}\mathfrak{A})$,*

[3]The games G^m and $\mathbf{G}^m$ are based on a private Ehrenfeucht–Fraïssé deterministic games on two relational structures A and B between two players $\exists$ lloise and $\forall$ belard. Each player chooses a pebble from a particular pebble pair outside the board of the game and places it on one of the structures, A say. The other responds with the other pebble in this pair putting it on the other structure B. The aim of $\exists$ is to show that A and B are alike while the 'spoiler' $\forall$ wants to show that they are different—the 'likeness' here may be measured by existence of isomorphisms between $\mathfrak{A}$ and $\mathfrak{B}$, or partial isomorphisms or elementary equivalence, ... etc. In G^m once $\forall$ has chosen a pebble in his private game Ehrenfeucht–Fraïssé game, he cannot use it again. However, in $\mathbf{G}^m$ the pebbles chosen by $\forall$ always remain outside the board of the play, so that $\forall$ has the option to re-use them in every round of the game. This of course makes it harder for $\exists$ to win.

3. *All reverse inclusions and implications in the previous two items hold, if algebras considered have countably many atoms,*

4. *Non of the classes in the first item is elementary,*

5. $\mathsf{CRCA}_n = \mathbf{S_c}\mathbf{P}\mathsf{Fs}_n$,

6. $\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At} \not\subseteq \mathsf{CRCA}_n$, $\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At} \subsetneq \mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At}$ *and* $\mathsf{CRCA}_n \subsetneq \mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At}$.

7. *Neither of the classes* CRCA_n *and* $\mathbf{S_d}\mathsf{Nr}_n\mathsf{CA}_\omega$ *are contained in each other. In particular,* $\mathbf{S_d}\mathsf{Nr}_n\mathsf{CA}_\omega \subsetneq \mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega$.

Proof: 1. Let $\mathfrak{A} \in \mathsf{CRCA}_n$. Assume that M is the base of a complete representation of $\mathfrak{A}$, whose unit is a generalized cartesian space, that is, $1^{\mathsf{M}} = \bigcup {}^nU_i$, where ${}^nU_i \cap {}^nU_j = \emptyset$ for distinct i and j, in some index set I, that is, we have an isomorphism $t : \mathfrak{B} \to \mathfrak{C}$, where $\mathfrak{C} \in \mathsf{Gs}_n$ has unit 1^{M}, and t preserves arbitrary meets carrying them to set-theoretic intersections. For $i \in I$, let $E_i = {}^nU_i$ and pick an arbitrary $f_i \in {}^\omega U_i$ and let W_i be the ω-dimensional weak space $\{f \in {}^\omega U_i^{(f_i)} : |\{k \in \omega : f(k) \neq f_i(k)\}| < \omega\}$. Identifying set algebras with their domain let $\mathfrak{C}_i = \wp(W_i)$. Then $\mathfrak{C}_i \in \mathsf{Ws}_\omega$ and is atomic; indeed the atoms are the singletons sets $\{f\}$ for $f \in W_i$. Note, for $f, g \in W_i$ ad $i < \omega$ if $f \upharpoonright \omega \sim \{i\} = g \upharpoonright \omega \sim \{i\}$, then $\{f\} \leq C_i\{g\}$.

Let $x \in \mathfrak{Nr}_n\mathfrak{C}_i$, that is $\mathsf{c}_i x = x$ for all $n \leq i < \omega$. Now if $f \in x$ and $g \in W_i$ satisfy $g(k) = f(k)$ for all $k < n$, then $g \in x$ because $|\{n \leq i < \omega : f(i) \neq g(i)\}| < \omega$. Hence $\mathfrak{Nr}_n\mathfrak{C}_i$ is atomic; its atoms are $\{\{g \in W_i : \{g(i) = d : i < n\}, d \in U_i\}$. Define $h_i : \mathfrak{A} \to \mathfrak{Nr}_n\mathfrak{C}_i$ by $h_i(a) = \{f \in W_i : \exists a' \in \mathsf{At}\mathfrak{A}, a' \leq a; (f(i) : i < n) \in t(a')\}$. Let $\mathfrak{D} = \mathbf{P}_i\mathfrak{C}_i$. Let $\pi_i : \mathfrak{D} \to \mathfrak{C}_i$ be the ith projection map. Now clearly $\mathfrak{D}$ is atomic, because it is a product of atomic algebras, and its atoms are $(\pi_i(\beta) : \beta \in \mathsf{At}(\mathfrak{C}_i))$. Now $\mathfrak{A}$ embeds into $\mathfrak{Nr}_n\mathfrak{D}$ via $J : a \mapsto (\pi_i(a) : i \in I)$. If $x \in \mathfrak{Nr}_n\mathfrak{D}$, then for each i, we have $\pi_i(x) \in \mathfrak{Nr}_n\mathfrak{C}_i$, and if x is non-zero, then $\pi_i(x) \neq 0$. By atomicity of $\mathfrak{C}_i$, there is an n-ary tuple y, such that $\{g \in W_i : g(k) = y_k\} \subseteq \pi_i(x)$. It follows that there is an atom of $b \in \mathfrak{A}$, such that $y \in t(b)$. Hence $\{g \in U_i : g(i) = y_i\} \subseteq \pi_i(< x \cdot J(b) >$, so $x \cdot J(b) \neq 0$, and so the embedding is atomic, hence complete. We have shown that $\mathfrak{A} \in \mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At}$, and since $\mathfrak{A}$ is atomic because $\mathfrak{A} \in \mathsf{CRCA}_n$ we are done with the first inclusion. The second inclusion is straightforward since $\mathsf{CA}_\omega \cap \mathbf{At} \subseteq \mathsf{CA}_\omega$.

2. [7, Theorem 3.3.3]. Follows too from the first item taken together with lemma 1.3.

3. Follows by observing that the class $\mathsf{CRCA_n}$ coincides with the class $\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega$ on atomic algebras having countably many atoms, cf. [14, Theorem 5.3.6], taken together with [7, Theorem 3.3.3]. Strictly speaking, in [14] it is shown that the two classes CRCA_n and $\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega$ coincide on countable atomic algebras. One can show that they coincide on the larger class of atomic agebras having countably many atoms by observing that if $\mathfrak{A}$ is an atomic algebra having countably many atoms, then $\mathfrak{Tm}\mathsf{At}\mathfrak{A}$ is countable and $\mathfrak{Tm}\mathsf{At}\mathfrak{A} \in \mathsf{CRCA}_n \iff \mathfrak{A} \in \mathsf{CRCA}_n$.

4. To show that non of the classes in the first item is elementary, let $\mathfrak{D}$ be an atomic RCA_n with countably many atoms that is not completely representable, but is elementary equivalent to some $\mathfrak{B} \in \mathsf{CRCA}_n$. Such algebras exist; see e.g. [5]. Another such algebra is the algebra $\mathfrak{C}_{\mathbb{Z},\mathfrak{N}}$ used in theorem 3.1 below. Then $\mathfrak{D}$ is not in any of the aforementiond classes because it has countably many atoms, and by the first item $\mathfrak{B}$ is in all three classes, proving the required.

5. The inclusion $\subseteq$ is straightforward. Conversely, assume that $\mathfrak{A} \subseteq_c \mathbf{P}_{i\in I}\wp({}^nU_i)$. Then $\mathfrak{B} = \mathbf{P}_{i\in I}\wp({}^nU_i) \cong \wp(V)$, where V is the disjoint union of the ${}^nU_i$, is clearly completely representable. Then since $\mathfrak{A} \subseteq_c \mathfrak{B}$, and so $\mathfrak{A}$ is completely representable, too.

6. First $\nsubseteq$ follows from the construction in [12], cf. corollary 1.16 for more details. Second $\subsetneq$ follows from item (3) of Theorem 2.2. Last $\subsetneq$ follows from the first two parts in this item together with the inclusions in the first item.

7. That $\mathbf{S_d}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At} \nsubseteq \mathsf{CRCA}_n$ follows from the first part of item (6) of theorem 1.4, cf. also corollary 1.16. To show that, conversely $\mathsf{CRCA}_n \nsubseteq \mathbf{S_d}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At}$, we slighty modify the construction in [14, Lemma 5.1.3, Theorem 5.1.4] lifted to any finite $n > 2$. The algebras $\mathfrak{A}$ and $\mathfrak{B}$ constructed in *op. cit.* satisfy that $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_\omega$, $\mathfrak{B} \notin \mathsf{Nr}_n\mathsf{CA}_{n+1}$ and $\mathfrak{A} \equiv \mathfrak{B}$. As they stand, $\mathfrak{A}$ and $\mathfrak{B}$ are not atomic, but it can be fixed that they are atomic, giving the same result, by interpreting the uncountably many n-ary relations in the signature of $\mathfrak{M}$ defined in [14, Lemma 5.1.3] for $n = 3$, which is the base of $\mathfrak{A}$ and $\mathfrak{B}$ to be *disjoint* in $\mathfrak{M}$, not just distinct. In fact the construction is presented in this way in [11]. Let us explain why. We work with $2 < n < \omega$ instead of only $n = 3$. The proof presented in *op. cit.* lifts verbatim to any such n. Let $u \in {}^nn$. Write $\mathbf{1}_u$ for $\chi_u^{\mathfrak{M}}$ (denoted by 1_u (for $n = 3$) in [14, Theorem 5.1.4].) We denote by $\mathfrak{A}_u$ the Boolean

algebra $\mathfrak{Rl}_{\mathbf{1}_u}\mathfrak{A} = \{x \in \mathfrak{A} : x \leq \mathbf{1}_u\}$ and similarly for $\mathfrak{B}$, writing $\mathfrak{B}_u$ short hand for the Boolean algebra $\mathfrak{Rl}_{\mathbf{1}_u}\mathfrak{B} = \{x \in \mathfrak{B} : x \leq \mathbf{1}_u\}$. Using that $\mathfrak{M}$ has quantifier elimination we get, using the same argument in *op. cit.* that $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_\omega$. The property that $\mathfrak{B} \notin \mathsf{Nr}_n\mathsf{CA}_{n+1}$ is also still maintained. To see why consider the substitution operator ${}_n\mathsf{s}(0,1)$ (using one spare dimension) as defined in the proof of [14, Theorem 5.1.4].

Assume for contradiction that $\mathfrak{B} = \mathsf{Nr}_n\mathfrak{C}$, with $\mathfrak{C} \in \mathsf{CA}_{n+1}$. Let $u = (1, 0, 2, \ldots, n-1)$. Then $\mathfrak{A}_u = \mathfrak{B}_u$ and so $|\mathfrak{B}_u| > \omega$. The term ${}_n\mathsf{s}(0,1)$ acts like a substitution operator corresponding to the transposition $[0,1]$; it 'swaps' the first two coordinates. Now one can show that ${}_n\mathsf{s}(0,1)^{\mathfrak{C}}\mathfrak{B}_u \subseteq \mathfrak{B}_{[0,1]\circ u} = \mathfrak{B}_{Id}$, so $|{}_n\mathsf{s}(0,1)^{\mathfrak{C}}\mathfrak{B}_u|$ is countable because $\mathfrak{B}_{Id}$ was forced by construction to be countable. But ${}_n\mathsf{s}(0,1)$ is a Boolean automorpism with inverse ${}_n\mathsf{s}(1,0)$, so that $|\mathfrak{B}_{Id}| = |{}_n\mathsf{s}(0,1)^{\mathfrak{C}}\mathfrak{B}_u| > \omega$, contradiction. One proves that $\mathfrak{A} \equiv \mathfrak{B}$ exactly like in [14]. Take the cardinality κ specifying the signature of $\mathfrak{M}$ to be 2^{2^ω} and assume for contradiction that $\mathfrak{B} \in \mathbf{S_d}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At}$. Then $\mathfrak{B} \subseteq_d \mathfrak{Nr}_n\mathfrak{D}$, for some $\mathfrak{D} \in \mathsf{CA}_\omega$ and $\mathfrak{Nr}_n\mathfrak{D}$ is atomic. For brevity, let $\mathfrak{C} = \mathfrak{Nr}_n\mathfrak{D}$. Then $\mathfrak{B}_{Id} \subseteq_d \mathfrak{Rl}_{Id}\mathfrak{C}$; the last algebra is the Boolean algebra with universe $\{x \in \mathfrak{C} : x \leq Id\}$. Since $\mathfrak{C}$ is atomic, then $\mathfrak{Rl}_{Id}\mathfrak{C}$ is also atomic.

Using the same reasoning as above, we get that $|\mathfrak{Rl}_{Id}\mathfrak{C}| > 2^\omega$ (since $\mathfrak{C} \in \mathsf{Nr}_n\mathsf{CA}_\omega$). By the choice of κ, we get that $|\mathsf{At}\mathfrak{Rl}_{Id}\mathfrak{C}| > \omega$. By $\mathfrak{B} \subseteq_d \mathfrak{C}$, we get that $\mathfrak{B}_{Id} \subseteq_d \mathfrak{Rl}_{Id}\mathfrak{C}$, and that $\mathsf{At}\mathfrak{Rl}_{Id}\mathfrak{C} \subseteq \mathsf{At}\mathfrak{B}_{Id}$, so $|\mathsf{At}\mathfrak{B}_{Id}| \geq |\mathsf{At}\mathfrak{Rl}_{Id}\mathfrak{C}| > \omega$. But by the construction of $\mathfrak{B}$, we have $|\mathfrak{B}_{Id}| = |\mathsf{At}\mathfrak{B}_{Id}| = \omega$, which is a contradiction and we are done. The algebra $\mathfrak{B}$ so constructed is atomic and is outside $\mathbf{S_d}\mathsf{Nr}_n\mathsf{CA}_\omega$. Furthermore, $\mathfrak{B} \in \mathsf{CRCA}_n$ because $\mathfrak{B} \in \mathsf{Gs}_n$ and $\bigcup \mathsf{At}\mathfrak{B} = \bigcup_{u \in {}^nn} \bigcup \mathsf{At}\mathfrak{B}_u = \bigcup_{u \in {}^nn} \mathbf{1}_u = 1^{\mathfrak{B}}$. Thus the identity may establishes a complete representation of $\mathfrak{B}$. $\square$

Here we review and elaborate on the construction in [2] as our first instance of a so-called *blow up and blur construction* in the sense of [16]. This subtle construction may be applied to any two classes $\mathbf{L} \subseteq \mathbf{K}$ of completely additive Boolean algebras with opertors (BAOs). One takes an atomic $\mathfrak{A} \notin \mathbf{K}$ (usually but not always finite), *blows it up*, by *splitting*[4] *one or more of its atoms each to infinitely many subatoms*, obtaining an

[4]The idea of splitting one or more atoms in an algebra to get a (bigger) superalgebra tailored to a certain purpose seems to originate with Henkin [3, p. 378, footnote 1] to be reinvented by Hajnal Andréka as a nutcracker for proving non-finite axiomatizability results for varieties of RCA_n.

(infinite) *countable* atomic $\mathfrak{Bb}(\mathfrak{A}) \in \mathbf{L}$, such that $\mathfrak{A}$ *is blurred* in $\mathfrak{Bb}(\mathfrak{A})$ meaning that $\mathfrak{A}$ *does not* embed in $\mathfrak{Bb}(\mathfrak{A})$, but $\mathfrak{A}$ embeds in the Dedekind–MacNeille completion of $\mathfrak{Bb}(\mathfrak{A})$, namely, $\mathfrak{Cm}\mathsf{At}\mathfrak{Bb}(\mathfrak{A})$. Then any class $\mathbf{M}$ say, between $\mathbf{L}$ and $\mathbf{K}$ that is closed under forming subalgebras will not be atom-canonical, for $\mathfrak{Bb}(\mathfrak{A}) \in \mathbf{L}(\subseteq \mathbf{M})$, but $\mathfrak{Cm}\mathsf{At}\mathfrak{Bb}(\mathfrak{A}) \notin \mathbf{K}(\supseteq \mathbf{M})$ because $\mathfrak{A} \notin \mathbf{M}$ and $\mathbf{SM} = \mathbf{M}$. We say, in this case, that $\mathbf{L}$ *is not atom-canonical with respect to* $\mathbf{K}$. This method is applied to $\mathbf{K} = \mathbf{S}\mathfrak{Ra}\mathsf{CA}_l$, $l \geq 5$ and $\mathbf{L} = \mathsf{RRA}$ in [6, SS 17.7] and to $\mathbf{K} = \mathsf{RRA}$ and $\mathbf{L} = \mathsf{RRA} \cap \mathfrak{Ra}\mathsf{CA}_k$ for all $k \geq 3$ in [2]; the construction in [2] will be generalized below, and will applied below to $\mathbf{K} = \mathsf{SNr}_n\mathsf{CA}_{n(n+1)/2}$ and $\mathbf{L} = \mathsf{RCA}_n$, where $\mathfrak{Ra}$ denotes the operator of forming relation algebra reducts (applied to classes) of CAs, respectively, cf. [3, Definition 5.2.7].

Definition 1.5. Let $\mathfrak{R}$ be an atomic relation algebra. An n-dimensional basic matrix, or simply a matrix on $\mathfrak{R}$, is a map $f : {}^2n \to \mathsf{At}\mathfrak{R}$ satsfying the following two consistency conditions $f(x,x) \leq \mathsf{Id}$ and $f(x,y) \leq f(x,z);f(z,y)$ for all $x,y,z < n$. For any f,g basic matrices and $x,y < m$ we write $f \equiv_{xy} g$ if for all $w,z \in m \setminus \{x,y\}$ we have $f(w,z) = g(w,z)$. We may write $f \equiv_x g$ instead of $f \equiv_{xx} g$.

Definition 1.6. An *n-dimensional cylindric basis* for an atomic relaton algebra $\mathfrak{R}$ is a set $\mathsf{CA}lM$ of n-dimensional matrices on $\mathfrak{R}$ with the following properties:

- If $a,b,c \in \mathsf{At}\mathfrak{R}$ and $a \leq b;c$, then there is an $f \in \mathsf{CA}lM$ with $f(0,1) = a$, $f(0,2) = b$ and $f(2,1) = c$
- For all $f,g \in \mathsf{CA}lM$ and $x,y < n$, with $f \equiv_{xy} g$, there is $h \in \mathsf{CA}lM$ such that $f \equiv_x h \equiv_y g$.

For the next lemma, we refer the reader to [6, Definition 12.11] for the definition of of hyperbasis for relation algebras as well as to [6, Chapter 13, Definitions 13.4, 13.6] for the notions of n-flat and n-square representations for relation algebras ($n > 2$) For a relation algebra $\mathfrak{R}$, recall that $\mathfrak{R}^+$ denotes its canonical extension.

Lemma 1.7. *Let $\mathfrak{R}$ be a relation algebra and $3 < n < \omega$. Then the following hold:*

1. *$\mathfrak{R}^+$ has an n-dimensional infinite basis $\iff$ $\mathfrak{R}$ has an infinite n-square representation.*

2. *$\mathfrak{R}^+$ has an n-dimensional infinite hyperbasis $\iff$ $\mathfrak{R}$ has an infinite n-flat representation.*

PROOF: [6, Theorem 13.46, the equivalence (1) $\iff$ (5) for basis, and the equivalence (7) $\iff$ (11) for hyperbasis]. □

One can construct a CA_n in a natural way from an n-dimensional cylindric basis which can be viewed as an atom structure of a CA_n (like in [6, Definition 12.17] addressing hyperbasis). For an atomic relation algebra $\mathfrak{R}$ and $l > 3$, we denote by $\mathsf{Mat}_n(\mathsf{At}\mathfrak{R})$ the set of all n-dimensional basic matrices on $\mathfrak{R}$. $\mathsf{Mat}_n(\mathsf{At}\mathfrak{R})$ is not always an n-dimensional cylindric basis, but sometimes it is, as will be the case described next. On the other hand, $\mathsf{Mat}_3(\mathsf{At}\mathfrak{R})$ is always a 3-dimensional cylindric basis; a result of Maddux's, so that $\mathfrak{Cm}\mathsf{Mat}_3(\mathsf{At}\mathfrak{R}) \in \mathsf{CA}_3$. The following definition to be used in the sequel is taken from [2]:

DEFINITION 1.8. [2, Definition 3.1] Let $\mathfrak{R}$ be a relation algebra, with non-identity atoms I and $2 < n < \omega$. Assume that $J \subseteq \wp(I)$ and $E \subseteq {}^3\omega$.

1. We say that (J, E) is an *n-blur* for $\mathfrak{R}$, if J is a *complex n-blur* defined as follows:
   (a) Each element of J is non-empty,
   (b) $\bigcup J = I$,
   (c) $(\forall P \in I)(\forall W \in J)(I \subseteq P; W)$,
   (d) $(\forall V_1, \dots V_n, W_2, \dots W_n \in J)(\exists T \in J)(\forall 2 \leq i \leq n)\mathsf{safe}(V_i, W_i, T)$, that is there is for $v \in V_i$, $w \in W_i$ and $t \in T$, we have $v; w \leq t$,
   (e) $(\forall P_2, \dots P_n, Q_2, \dots Q_n \in I)(\forall W \in J) W \cap P_2; Q_n \cap \dots P_n; Q_n \neq \emptyset$.

   and the tenary relation E is an *index blur* defined as in item (ii) of [2, Definition 3.1].

2. We say that (J, E) is a *strong n-blur*, if it (J, E) is an n-blur, such that the complex n-blur satisfies:

$$(\forall V_1, \dots V_n, W_2, \dots W_n \in J)(\forall T \in J)(\forall 2 \leq i \leq n)\mathsf{safe}(V_i, W_i, T).$$

DEFINITION 1.9. An atomic algebra $\mathfrak{A} \in \mathsf{CA}_n$ is *strongly representable* if $\mathfrak{Cm}\mathsf{At}\mathfrak{A} \in \mathsf{RCA}_n$.

LEMMA 1.10. *Let $\mathfrak{A} \in \mathsf{CA}_n$ be completely representable. Then $\mathfrak{A}$ is strongly representable.*

PROOF: Since $\mathfrak{A}$ is completely representable, then it is atomic. Let $f : \mathfrak{A} \to \mathfrak{B}$ be a complete representation of $\mathfrak{A}$ via f, with $\mathfrak{B} \in \mathsf{Gs}_n$. Then one extends f to $\hat{f}$ from $\mathfrak{CmAt}\mathfrak{A}$ to $\mathfrak{B}$ by defining $\hat{f}(a) = \sum^{\mathfrak{CmAt}\mathfrak{A}}_{x \in \mathsf{At}\mathfrak{A}, x \leq a} f(x)$. The last suprema is well defined because $\mathfrak{CmAt}\mathfrak{A}$ is complete. It is easy to check that $\hat{f}$ is an isomorphism and so $\mathfrak{CmAt}\mathfrak{A}$ is isomorphic to $\mathfrak{B}$, hence, by definition, $\mathfrak{CmAt}\mathfrak{A}$ is representable. $\square$

DEFINITION 1.11. A completely additive variety V of BAOs is *atom-canonical* if whenever $\mathfrak{A} \in \mathsf{V}$, then its Dedekind–MacNeille completion, which is the complex algebra of its atom structure, namely, $\mathfrak{CmAt}\mathfrak{A}$, is also in V,

Monk prove that CA_n is atom-canonical; this follows from the fact that CA_n can be axiomatized by positive in the wider sense equations, which are are an instance of Sahlqvist equations. However, the variety RCA_n is not atom-canonical; a result of Hodkinson's [10]. We reprove the last result differently based on the construction in [2].

THEOREM 1.12. *For any $2 < n < l < \omega$, there is an atomic algebra $\mathfrak{B} \in \mathsf{Nr}_n\mathsf{CA}_l \cap \mathsf{RCA}_n$, but $\mathfrak{CmAt}\mathfrak{B} \notin \mathsf{RCA}_n$. In particular, $\mathfrak{B}$ is not completely representable a fortiori $\mathfrak{B}$ is not strongly representable, and RCA_n is not atom-canonical.*

PROOF: Let $2 < n < m \leq \omega$. First we prove the conditionally the non-atom canonicity of $\mathsf{SNr}_n\mathsf{CA}_m$ depending on the existence of a certain finite relation algebra $\mathfrak{R}$ with strong m blur- satisfying a condition that we highlight as we go along. We use the flexible blow up and blur construction used in [2]. The idea is to use $\mathfrak{R}$ in place of the finite Maddux algebras denoted by $\mathfrak{E}_k(2,3)$ on [2, p. 83]. Here $k(< \omega)$ is the number of non-identity atoms and then take it from there to reach the conditions, we move backwards if you like. The required algebra witnessing non-atom canonicity will be obtained by blowing up and blurring $\mathfrak{R}$ in place of the relation algebra $\mathfrak{E}_k(2,3)$ [2].

Our exposition addresses an (abstract) finite relation algebra $\mathfrak{R}$ having an l-blur in the sense of definition [2, Definition 3.1], with $3 \leq l \leq k < \omega$ and k depending on l. Occasionally we use the concrete Maddux algebra $\mathfrak{E}_k(2,3)$ to make certain concepts more tangible. We use the notation in [2]. Let $2 < n \leq l < \omega$. One starts with a finite relation algebra $\mathfrak{R}$ that has only representations, if any, on finite sets (bases), having an l-blur (J, E) as in [2, Definition 3.1] recalled in definition 1.8. After *blowing up and bluring* $\mathfrak{R}$, by splitting each of its atoms into infinitely many, one gets an infinite atomic representable relation algebra $\mathsf{Bb}(\mathfrak{R}, J, E)$ [2, p. 73], whose atom structure $\mathbf{At}$ is weakly but not strongly representable. The atom structure $\mathbf{At}$ is not strongly representable, because $\mathfrak{R}$ is *not blurred* in $\mathsf{Cm}\mathbf{At}$. The finite relation algebra $\mathfrak{R}$ embeds into $\mathfrak{Cm}\mathbf{At}$, so that a representation of $\mathfrak{Cm}\mathbf{At}$, necessarily on an infinite base, induces one of $\mathfrak{R}$ on the same base, which is impossible. The representability of $\mathsf{Bb}(\mathfrak{R}, J, E)$ depend on the properties of the l-blur, which *blurs* $\mathfrak{R}$ *in* $\mathsf{Bb}(\mathfrak{R}, J, E)$. The set of blurs here, namely, J is finite. In the case of $\mathfrak{E}_k(2, 3)$ used in [2], the set of blurs is the set of all subsets of non-identity atoms having the same size $l < \omega$, where $k = f(l) \geq l$ for some recursive function f from $\omega \to \omega$, so that k depends recursively on l.

One (but not the only) way to define the *index blur* $E \subseteq {}^3\omega$ is as follows [13, Theorem 3.1.1]: $E(i, j, k) \iff (\exists p, q, r)(\{p, q, r\} = \{i, j, k\}$ and $r - q = q - p$. This is a concrete instance of an index blur as defined in [2, Definition 3.1(iii)] (recalled in definition 1.8 above), but defined uniformly, it does not depends on the blurs. The underlying set of $\mathbf{At}$, the atom structure of $\mathsf{Bb}(\mathfrak{R}, J, E)$ is the following set consisting of triplets: $At = \{(i, P, W) : i \in \omega, P \in \mathsf{At}\mathfrak{R} \sim \{\mathsf{Id}\}, W \in J\} \cup \{\mathsf{Id}\}$. When $\mathfrak{R} = \mathfrak{E}_k(2, 3)$ (some finite $k > 0$), composition is defined by singling out the following (together with their Peircian transforms), as the consistent triples: (a, b, c) is consistent $\iff$ one of a, b, c is Id and the other two are equal, or if $a = (i, P, S), b = (j, Q, Z), c = (k, R, W)$

$$S \cap Z \cap W \neq \emptyset \implies E(i, j, k) \& |\{P, Q, R\}| \neq 1.$$

(We are avoiding mononchromatic triangles). That is if for $W \in J$, $E^W = \{(i, P, W) : i \in \omega, P \in W\}$, then

$$(i, P, S); (j, Q, Z) = \bigcup \{E^W : S \cap Z \cap W = \emptyset\}$$
$$\bigcup \{(k, R, W) : E(i, j, k), |\{P, Q, R\}| \neq 1\}.$$

More generally, for the $\mathfrak{R}$ as postulated in the hypothesis, composition in $\mathbf{At}$ is defined as follow. First the index blur E can be taken to be like above. Now the triple $((i, P, S), (j, Q, Z), (k, R, W))$ in which no two entries are equal, is consistent if either S, Z, W are *safe*, briefly $\mathsf{safe}(S, Z, W)$, witness item (4) in definition 1.8 (which vacuously hold oif $S \cap Z \cap W = \emptyset$), or $E(i,\ j,\ k)$ and $P\ ;\ Q \leq R$ in $\mathfrak{R}$. This generalizes the above definition of composition, because in $\mathfrak{E}_k(2,\ 3)$, the triple of non-identity atoms $(P,\ Q,\ R)$ is consistent $\Longleftrightarrow$ they do not have the same colour $\Longleftrightarrow |\{P,\ Q,\ R\}| \not\models 1$. Having specified its atom structure, its timely to specfiy the relation algebra $\mathsf{Bb}(\mathfrak{R},\ J,\ E) \subseteq \mathfrak{Cm}\mathbf{At}$. The relation algebra $\mathsf{Bb}(\mathfrak{R},\ J,\ E)$ is $\mathfrak{Tm}\mathbf{At}$ (the term algebra). Its universe is the set $\{X \subseteq H \cup \{\mathsf{Id}\} : X \cap E^W \in \mathsf{Cof}(E^W\ ),$ for all $W \in J\}$, where $\mathsf{Cof}(E^W\ )$ denotes the set of co-finite subsets of E^W , that is subsets of E^W whose complement is infinite, with E^W as defined above. The relation algebra operations lifted from $\mathbf{At}$ the usual way. The algebra $\mathsf{Bb}(\mathfrak{R},\ J,\ E)$ is proved to be representable [2].

For brevity, denote $\mathsf{Bb}(\mathfrak{R}, J, E)$ by $\mathsf{CA}lR$, and its domain by R. For $a \in \mathbf{At}$, and $W \in J$, set $U^a = \{X \in R : a \in X\}$ and $U^W = \{X \in R : |X \cap E^W| \geq \omega\}$. Then the principal ultrafilters of $\mathsf{CA}lR$ are exactly U^a, $a \in H$ and U^W are non-principal ultrafilters for $W \in J$ when E^W is infinite. Let $J' = \{W \in J : |E^W| \geq \omega\}$, and let $\mathsf{Uf} = \{U^a : a \in F\} \cup \{U^W : W \in J'\}$. Uf is the set of ultrafilters of $\mathsf{CA}lR$ which is used as colours to represent $\mathsf{CA}lR$, cf. [2, pp. 75–77]. The representation is built from coloured graphs whose edges are labelled by elements in Uf in a fairly standard step-by-step construction. The step-by-step construction builds in the way coloured graphs, which are basically networks whose edges are labelled by ultrafilters, with non-principal ultrafilters allowed. So such coloured graphs are networks that are *not atomic* because not only principal ultrafilters are allowed as labels. Furthermore, we *cannot restrict our attension to only atomic networks* because we do not want $\mathsf{Bb}(\mathfrak{R}, J, E)$ to be strongly representable, least completely representable. The 'limit' of a sequence of atomic networks constructed in a step-by-step manner, or obtained via winning strategystrategy for $\exists$ in an ω-rounded atomic game, will necessarily produce a complete representation of $\mathsf{Bb}(\mathfrak{R}, J, E)$. But the required representation will be extracted from *a complete representation* of the *canonical extension of* $\mathsf{Bb}(\mathfrak{R}, J, E)$. Nothing wrong with that. A relation algebra $\mathsf{CA}lR$ is representable $\iff$ its canonical extension is representable. A complete representation of the canonical extension of $\mathsf{CA}lR$ induces a representation of $\mathsf{CA}lR$, because $\mathsf{CA}lR$ embeds into its a canonical extension, but the converse is not necessarily true. So here we are proving more than the mere representablity of $\mathfrak{Bb}(\mathfrak{R}, J, E)$, because we are constructing a complete representation of its canonical extension, namely, the algebra $\mathfrak{Cm}\mathbf{Uf}$, where $\mathbf{Uf}$ is the atom structure having domain Uf, with Uf as defined above.

Now we show why the Dedekind–MacNeille completion $\mathfrak{Cm}\mathbf{At}$ is *not* representable. For $P \in I$, let $H^P = \{(i, P, W) : i \in \omega, W \in J, P \in W\}$. Let $P_1 = \{H^P : P \in I\}$ and $P_2 = \{E^W : W \in J\}$. These are two partitions of At. The partition P_2 was used to *represent*, $\mathsf{Bb}(\mathfrak{R}, J, E)$, in the sense that the tenary relation corresponding to composition was defined on $\mathbf{At}$, in a such a way so that the singletons generate the partition $(E^W : W \in J)$ up to "finite deviations." The partition P_1 will now be used to show that $\mathfrak{Cm}(\mathsf{Bb}(\mathfrak{R}, J, E)) = \mathfrak{Cm}(\mathbf{At})$ is *not* representable. This follows by observing that omposition restricted to P_1 satisfies: $H^P; H^Q = \bigcup\{H^Z : Z; P \leq Q \text{ in } \mathfrak{R}\}$ which means that $\mathfrak{R}$ embeds into the complex algebra $\mathfrak{Cm}\mathbf{At}$ prohibiting its representability, because $\mathfrak{R}$ allows only representations having a finite base.

The construction lifts to higher dimensions expressed in CA_ns, $2 < n < \omega$. Because (J, E) is an l-blur, then by [2, Theorem 3.2 9(iii)], $\mathbf{At}_{ca} = \mathsf{Mat}_l(\mathsf{At}\mathsf{Bb}(\mathfrak{R}, J, E))$, the set of l by l basic matrices on $\mathbf{At}$ is an l-dimensional cylindric basis, giving an algebra $\mathfrak{B}_l = \mathsf{Bb}_l(\mathfrak{R}, J, E) \in \mathsf{RCA}_l$. Again $\mathbf{At}_{ca}$ is not strongly representable, for had it been then a representation of $\mathfrak{Cm}\mathbf{At}_{ca}$, induces a representation of $\mathfrak{R}$ on an infinite base, because $\mathfrak{Ra}\mathfrak{Cm}\mathbf{At}_{ca} \supseteq \mathfrak{Cm}\mathbf{At} \supseteq \mathfrak{R}$, and the representability of $\mathfrak{Cm}\mathbf{At}_{ca}$ induces one of $\mathfrak{Ra}\mathfrak{Cm}\mathbf{At}_{ca}$, necessarily having an infinite base. For $2 < n \leq l < \omega$, denote by $\mathfrak{C}_l$ the non-representable Dedekind–MacNeille completion of the algebra $\mathsf{Bb}_l(\mathfrak{R}, J, E) \in \mathsf{RCA}_l$, that is $\mathfrak{C}_l = \mathfrak{Cm}\mathsf{At}(\mathsf{Bb}_l(\mathfrak{R}, J, E)) = \mathfrak{Cm}\mathsf{Mat}_l(\mathbf{At})$. If the l-blur happens to be *strong*, in the sense of definition 1.8 and $n \leq m \leq l$, then we get by [2, item (3), p. 80], that $\mathsf{Bb}_m(\mathfrak{R}, J, E) \cong \mathsf{Nr}_m\mathsf{Bb}_l(\mathfrak{R}, J, E)$. This is proved by defining an embedding $h : \mathfrak{Rd}_m\mathfrak{C}_l \to \mathfrak{C}_m$ via $x \mapsto \{M \upharpoonright m : M \in x\}$ and showing that $h \upharpoonright \mathsf{Nr}_m\mathfrak{C}_l$ is an isomorphism onto $\mathfrak{C}_m$ [2, p. 80]. Surjectiveness uses the condition $(J5)_l$ formulated in the second item of definition 1.8 of strong l-blurness. Without this condition, that is if the l-blur (J, E) is not strong, then still $\mathfrak{C}_m$ and $\mathfrak{C}_l$ can be defined because by definition (J, E) is an t-blur for all $m \leq t \leq l$, so $\mathsf{Mat}_\mathsf{t}(\mathbf{At})$ is a cylindric basis and for $t < l$ $\mathfrak{C}_t$ embeds into $\mathsf{Nr}_m\mathfrak{C}_l$ using the same above map, but this embedding might not be surjective. So for every l, now replacing $\mathfrak{R}$ by the Maddux algebra $\mathfrak{E}_{f(l)}(2, 3)$, the algebra $\mathfrak{A}_l = \mathsf{Nr}_n\mathsf{Bb}_l(\mathfrak{E}_{f(l)}(2, 3)), J, E)$– with $f(l)$ depending recursively on l, having strong l-blur due to the properties of the Maddux algebra $\mathfrak{E}_{f(l)}(2, 3)$, is as required. In other words, and more concisely, we have $\mathfrak{A}_l \in \mathsf{RCA}_n \cap \mathsf{Nr}_n\mathsf{CA}_l$, but $\mathfrak{Cm}\mathsf{At}\mathfrak{A}_l \notin \mathsf{RCA}_n$. $\square$

The following Theorem summarizes the proof of the previous Theorem, generalizes the construction in [2] and says some more new facts. We use the notation $\mathfrak{Bb}(\mathfrak{R}, J, E)$ with atom structure $\mathbf{At}$ obtained by blowing up and blurring $\mathfrak{R}$ with underlying set is denoted by At on [2, p. 73] and is recalled in the previous proof. The algebra $\mathfrak{Bb}_l(\mathfrak{R}, J, E)(\in \mathsf{CA}_l)$ is defined in [2, top of p. 78] and also in the immediately previous proof.

A CA_n atom structure $\mathbf{At}$ is *weakly representable* if there is an atomic $\mathfrak{A} \in \mathsf{RCA}_n$ such that $\mathbf{At} = \mathsf{At}\mathfrak{A}$; recall that it is strongly representable if $\mathfrak{Cm}\mathbf{At} \in \mathsf{RCA}_n$. These two notions are distinct as proved in Theorem 1.12.

THEOREM 1.13. *Let* $2 < n \leq l < m \leq \omega$.

1. *Let $\mathfrak{R}$ be a finite relation algebra with an l-blur (J, E) where J is the l-complex blur and E is the index blur.*

   (a) Let $\mathbf{At}$ be the relation algebra atom structure obtained by blowing up and blurring $\mathfrak{R}$ as specified above. Then the set of l by l-dimensional matrices $\mathbf{At}_{ca} = \mathsf{Mat}_l(\mathbf{At})$ is an l-dimensional cylindric basis, that is a weakly representable atom structure [2, Theorem 3.2]. The algebra $\mathfrak{Bb}_l(\mathfrak{R}, J, E)$ with atom structure $\mathbf{At}_{ra}$ is in RCA_l. Furthermore, $\mathfrak{R}$ embeds into $\mathfrak{Cm}\mathbf{At}$ which embeds into $\mathfrak{Ra}\mathfrak{Cm}(\mathbf{At}_{ca})$.

   (b) If (J, E) is a strong m-blur for $\mathfrak{R}$, then (J, E) is a strong l-blur for $\mathfrak{R}$. Furthermore, $\mathfrak{Bb}_l(\mathfrak{R}, J, E) \cong \mathfrak{Nr}_l\mathfrak{Bb}_m(\mathfrak{R}, J, E)$ and for any $l \leq j \leq m$, $\mathfrak{Bb}(\mathfrak{R}, J, E)$ having atom structure $\mathbf{At}$, is isomorphic to $\mathfrak{Ra}(\mathfrak{Bb}_j(\mathfrak{R}, J, E))$.

2. *For every $n < l$, there is an $\mathfrak{R}$ having a strong l-blur (J, E) but no infinite representations (representations on an infinite base). Hence the atom structures defined in (a) of the previous item (denoted by $\mathbf{At}$ and $\mathbf{At}_{ca}$) for this specific $\mathfrak{R}$ are not strongly representable.*

3. *Let $m < \omega$. If $\mathfrak{R}$ is a finite relation algebra having a strong l-blur, and no m-dimensional hyperbasis, then $l < m$.*

4. *If $n = l < m < \omega$ and $\mathfrak{R}$ is a finite relation algebra with an n blur (J, E) (not necessarily strong) and no infinite m-dimensional hyperbasis, then the algebras $\mathfrak{Cm}\mathsf{At}(\mathfrak{Bb}(\mathfrak{R}, J, E))$ and $\mathfrak{Cm}\mathsf{At}(\mathfrak{Bb}_l(\mathfrak{R}, J, E))$ are outside $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_m$ and $\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_m$, respectively, and the latter two varieties are not atom-canonical.*

PROOF: [2, Lemmata 3.2, 4.2, 4.3]. We start by an outline of (a) of item 1. Let $\mathfrak{R}$ be as in the hypothesis. Let $3 < n \leq l$. We blow up and blur $\mathfrak{R}$. $\mathfrak{R}$ is blown up by splitting all of the atoms each to infinitely many defining an (infinite atoms) structure $\mathbf{At}$. $\mathfrak{R}$ is blurred by using a finite set of blurs (or colours) J. The term algebra denoted in [2] by $\mathfrak{Bb}(\mathfrak{R}, J, E)$) over $\mathbf{At}$, is representable using the finite number of blurs. Such blurs are basically non-principal ultrafilters; they are used as colours together with the principal ultrafilters (the atoms) to represent $\mathfrak{Bb}(\mathfrak{R}, J, E)$. This representation is implemented in step-by-step manner, and in fact this step by step construction adopted in [2] *completely represents the canonical extension* of $\mathfrak{Bb}(\mathfrak{R}, J, E)$. Because (J, E) is a complex set of l-blurs, this atom structure has an l-dimensional cylindric basis, namely, $\mathbf{At}_{ca} = \mathsf{Mat}_l(\mathbf{At})$. The resulting l-dimensional cylindric term algebra $\mathfrak{TmMat}_l(\mathbf{At})$, and an algebra $\mathfrak{C}$ having atom structure $\mathbf{At}_{ca}$ (denoted in [2] by $\mathfrak{Bb}_l(\mathfrak{R}, J, E)$) such that $\mathfrak{TmMat}_l(\mathbf{At}) \subseteq \mathfrak{C} \subseteq \mathfrak{CmMat}_l(\mathbf{At})$ is shown to be representable.
We prove (b) of item (1): Assume that the m-blur (J, E) is strong, then by definition (J, E) is a strong j blur for all $n \leq j \leq m$. Furthermore, by [2, item (3), p. 80], $\mathfrak{Bb}(\mathfrak{R}, J, E) = \mathfrak{Ra}(\mathfrak{Bb}_j(\mathfrak{R}, J, E))$ and $\mathfrak{Bb}_j(\mathfrak{R}, J, E) \cong \mathfrak{Nr}_j\mathfrak{Bb}_m(\mathfrak{R}, J, E)$.

2. Like in [2, Lemma 5.1], one takes $l \geq 2n - 1$, $k \geq (2n-1)l$, $k \in \omega$. The Maddux integral relation algebra $\mathfrak{E}_k(2, 3)$ where k is the number of non-identity atoms is the required $\mathfrak{R}$. In this algebra a triple (a, b, c) of non-identity atoms is consistent $\iff$ $|\{a, b, c\}| \neq 1$, i.e only monochromatic triangles are forbidden.

3. Let (J, E) be the strong l-blur of $\mathfrak{R}$. Assume for contradiction that $m \leq l$. Then we get by [2, item (3), p. 80], that $\mathfrak{A} = \mathfrak{Bb}_n(\mathfrak{R}, J, E) \cong \mathfrak{Nr}_n\mathfrak{Bb}_l(\mathfrak{R}, J, E)$. But the cylindric l-dimensional algebra $\mathfrak{Bb}_l(\mathfrak{R}, J, E)$ is atomic, having atom structure $\mathsf{Mat}_l\mathsf{At}(\mathsf{split}(\mathfrak{R}, J, E))$, so $\mathfrak{A}$ has an atomic l-dilation. So $\mathfrak{A} = \mathfrak{Nr}_n\mathfrak{D}$ where $\mathfrak{D} \in \mathsf{CA}_l$ is atomic. But $\mathfrak{R} \subseteq_c \mathfrak{Ra}\mathfrak{Nr}_n\mathfrak{D} \subseteq_c \mathfrak{Ra}\mathfrak{D}$. By [6, Theorem 13.45 (6) $\iff$ (9)], $\mathfrak{R}$ has a complete l-flat representation, thus it has a complete m-flat representation, because $m < l$ and $l \in \omega$. This is a contradiction.

4. Let $\mathfrak{B} = \mathfrak{Bb}_n(\mathfrak{R}, J, E)$. Then, since (J, E) is an n blur, $\mathfrak{B} \in \mathsf{RCA}_n$. But $\mathfrak{C} = \mathfrak{Cm}\mathsf{At}\mathfrak{B} \notin \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_m$, because $\mathfrak{R} \notin \mathbf{S}\mathfrak{Ra}\mathsf{CA}_m$, $\mathfrak{R}$ embeds into $\mathfrak{Bb}(\mathfrak{R}, J, E)$ which, in turn, embeds into $\mathfrak{Ra}\mathfrak{Cm}\mathsf{At}\mathfrak{B}$. Similarly, $\mathfrak{Bb}(\mathfrak{R}, J, E) \in \mathsf{RRA}$ and $\mathfrak{Cm}(\mathsf{At}\mathfrak{Bb}(\mathfrak{R}, J, E)) \notin \mathbf{S}\mathfrak{Ra}\mathsf{CA}_m$. Hence the alledged varieties are not atom-canonical. $\square$

THEOREM 1.14. *Let $2 < n < \omega$. Then LCA_n is an elementary class that is not finitely axiomatizable.*

PROOF: For each $2 < n \leq l < \omega$, let $\mathfrak{R}_l$ be the finite Maddux algebra $\mathfrak{E}_{f(l)}(2,3)$, as defined on [2, p. 83, S5, in the proof of Theorem 5.1] with l-blur (J_l, E_l) as defined in [2, Definition 3.1] and $f(l) \geq l$ as specified in [2, Lemma 5.1] (denoted by k therein). Let $\mathsf{CA}lR_l = \mathfrak{Bb}(\mathfrak{R}_l, J_l, E_l) \in \mathsf{RRA}$ where $\mathsf{CA}lR_l$ is the relation algebra having atom structure denoted At in [2, p. 73] when the blown up and blurred algebra denoted $\mathfrak{R}_l$ happens to be the finite Maddux algebra $\mathfrak{E}_{f(l)}(2,3)$ and let $\mathfrak{A}_l = \mathfrak{Nr}_n\mathfrak{Bb}_l(\mathfrak{R}_l, J_l, E_l) \in \mathsf{RCA}_n$ as defined in [2, top of p. 80] (with $\mathfrak{R}_l = \mathfrak{E}_{f(l)}(2,3)$). Then $(\mathsf{AtCA}lR_l : l \in \omega \sim n)$, and $(\mathsf{At}\mathfrak{A}_l : l \in \omega \sim n)$ are sequences of weakly representable atom structures that are not strongly representable with a completely representable ultraproduct. $\square$

We have shown that the three classes in the first item of the theorem 1.4 are not elementary and in the last item of *op. cit.* that at least two are distinct. Now we show that their elementary closure coincide with the class LCA_n.

THEOREM 1.15. *Let $2 < n < \omega$. Then:*

$$\begin{aligned}
\mathbf{El}\mathsf{CRCA}_n &= \mathbf{El}[\mathbf{S_c}\mathsf{Nr}_n(\mathsf{CA}_\omega \cap \mathbf{At}) \cap \mathbf{At}] \\
&= \mathbf{El}\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At} \\
&= \mathbf{El}(\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At}) \\
&= \mathsf{LCA}_n.
\end{aligned}$$

PROOF: We show, as claimed, that all the given classes coincide with LCA_n. Assume that $\mathfrak{A} \in \mathsf{LCA}_n$. Take a countable elementary subalgebra $\mathfrak{C}$ of $\mathfrak{A}$. Since LCA_n is elementary, then $\mathfrak{C} \in \mathsf{LCA}_n$, so for $k < \omega$, $\exists$ has a winning strategyρ_k, in $G_k(\mathsf{At}\mathfrak{C})$. Let $\mathfrak{D}$ be a non-principal ultrapower of $\mathfrak{C}$. Then $\exists$ has a winning strategyσ in $G_\omega(\mathsf{At}\mathfrak{D})$ [7, Theorem 3.3.4]. Essentially she uses ρ_k in the k'th component of the ultraproduct so that at each round of $G_\omega(\mathsf{At}\mathfrak{D})$, $\exists$ is still winning in co-finitely many components, this suffices to show she has still not lost. Now one can use an elementary chain argument to construct countable elementary subalgebras $\mathfrak{C} = \mathfrak{A}_0 \preceq \mathfrak{A}_1 \preceq \ldots \preceq \ldots \mathfrak{D}$ in the following way. One defines $\mathfrak{A}_{i+1}$ to be a countable elementary subalgebra of $\mathfrak{D}$ containing $\mathfrak{A}_i$ and all elements of $\mathfrak{D}$ that σ selects in a play of $G_\omega(\mathsf{At}\mathfrak{D})$ in which $\forall$ only chooses elements from $\mathfrak{A}_i$.

Now let $\mathfrak{B} = \bigcup_{i<\omega} \mathfrak{A}_i$. This is a countable elementary subalgebra of $\mathfrak{D}$, hence necessarily atomic, and $\exists$ has a winning strategy in $G_\omega(\mathsf{At}\mathfrak{B})$, so $\mathfrak{B}$ is completely representable.

Thus $\mathfrak{A} \equiv \mathfrak{C} \equiv \mathfrak{B}$, hence $\mathfrak{A} \in \mathbf{El}\mathsf{CRCA}_n$. We have shown that $\mathsf{LCA}_n \subseteq \mathbf{El}\mathsf{CRCA_n}$. If $\mathfrak{A} \in \mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At}$, then by lemma 1.3, $\exists$ has a winning strategy in $\mathbf{G}^\omega(\mathsf{At}\mathfrak{A})$, hence in $G_\omega(\mathsf{At}\mathfrak{A})$, *a fortiori*, in $G_k(\mathsf{At}\mathfrak{A})$ for all $k < \omega$, so $\mathfrak{A} \in \mathsf{LCA}_n$. Since LCA_n is elementary, we get that $\mathbf{El}(\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At}) \subseteq \mathsf{LCA}_n$. But $\mathsf{CRCA}_n \subseteq \mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At}$, hence $\mathsf{LCA}_n = \mathbf{El}\mathsf{CRCA}_n \subseteq \mathbf{El}(\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At}) \subseteq \mathsf{LCA}_n$. Now $\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At}$ $\subseteq \mathbf{El}\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At}$, and the latter class is elementary (if $\mathbf{K}$ is elementary, then $\mathbf{K} \cap \mathbf{At}$ is elementary), so $\mathbf{El}(\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At}) \subseteq \mathbf{El}\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At}$.

Conversely, if $\mathfrak{C}$ is in $\mathbf{El}\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At}$. then $\mathfrak{C}$ is atomic and $\mathfrak{C} \equiv \mathfrak{D}$, for some $\mathfrak{D} \in \mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega$ since $\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega$ is closed under ultraproducts. Hence $\mathfrak{D}$ is atomic because atomicity is a first order property, so $\mathfrak{D} \in \mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At}$, thus $\mathfrak{C} \in \mathbf{El}(\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At})$.

We have shown that $\mathbf{El}\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At} = \mathbf{El}(\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At}) = \mathsf{LCA}_n = \mathbf{El}\mathsf{CRCA}_n$. Finally, by lemma 1.3, $\mathbf{S_c}\mathsf{Nr}_n(\mathsf{CA}_\omega \cap \mathbf{At}) \cap \mathbf{At} \subseteq \mathsf{LCA}_n$, from which it follows that $\mathbf{El}\mathbf{S_c}[\mathsf{Nr}_n(\mathsf{CA}_\omega \cap \mathbf{At}) \cap \mathbf{At}] \subseteq \mathsf{LCA}_n$, since LCA_n is elementary. The other inclusion follows from that, by item (1) of theorem 1.4, $\mathsf{CRCA}_n \subseteq \mathbf{S_c}\mathsf{Nr}_n(\mathsf{CA}_\omega \cap \mathbf{At}) \cap \mathbf{At}$, so $\mathsf{LCA}_n = \mathbf{El}\mathsf{CRCA}_n$ $\subseteq \mathbf{El}[\mathbf{S_c}\mathsf{Nr}_n(\mathsf{CA}_\omega \cap \mathbf{At}) \cap \mathbf{At}]$. We have shown that all classes coincide with LCA_n, which is the elementary closure of CRCA_n, and we are done. □

Corollary 1.16. For each $2 < n < \omega$, there is an atomic algebra $\mathfrak{B} \in \mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{El}\mathsf{CRCA}_n$, that is not completely representable. In particular, CRCA_n is not elementary [5]. Furthermore, each $\mathfrak{A}_n$ is constructed uniformly from one relation algebra.

Proof: In [12], a relation atomic algebra $\mathfrak{R}$ having uncountably many atoms is constructed such that $\mathfrak{R}$ has an ω-dimensional cylindric basis $\mathsf{CA}lH$ (the latter is defined in *opcit*) and $\mathfrak{R}$ is not completely representable. It is shown in [12] that if one takes $\mathfrak{C} = \mathsf{CA}(\mathsf{CA}lH)$, then $\mathfrak{C} \in \mathsf{CA}_\omega$, $\mathfrak{C}$ is atomless, and $\mathfrak{A} = \mathfrak{Ra}\mathfrak{C}$. Now fix $2 < n < \omega$. Then the required CA_n is $\mathfrak{B} = \mathsf{Nr}_n\mathfrak{C}$; $\mathfrak{A}_n$ is atomic and has uncountably many atoms. Furthermore, $\mathfrak{B}$ has no complete representation for a complete representation of $\mathfrak{B}$ induces one of $\mathfrak{A}$. Since $\mathfrak{B} \in \mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At}$, then by theorem 1.15, $\mathfrak{B} \in \mathsf{LCA}_n = \mathbf{El}\mathsf{CRCA}_n$.

For the reader's convenience, we give the details of the above proof. We use the following uncountable version of Ramsey's theorem due to Erdos and Rado: If $r \geq 2$ is finite, k an infinite cardinal, then $exp_r(k)^+ \to (k^+)^{r+1}_k$, where $exp_0(k) = k$ and inductively $exp_{r+1}(k) = 2^{exp_r(k)}$. The above partition symbol describes the following statement. If f is a coloring of the $r+1$ element subsets of a set of cardinality $exp_r(k)^+$ in k many colors, then there is a homogeneous set of cardinality k^+ (a set, all whose $r+1$ element subsets get the same f-value). We will construct the requred $\mathfrak{C} \in \mathsf{CA}_\omega$ from a relation algebra (to be denoted in a while by $\mathfrak{A}$) having an 'ω-dimensional cylindric basis.'

To define the relation algebra, we specify its atoms and forbidden triples. Let κ be the given cardinal in the hypothesis of the Theorem. The atoms are Id, $\mathsf{g}_0^i : i < 2^\kappa$ and $\mathsf{r}_j : 1 \leq j < \kappa$, all symmetric. The forbidden triples of atoms are all permutations of (Id, x, y) for $x \neq y$, $(\mathsf{r}_j, \mathsf{r}_j, \mathsf{r}_j)$ for $1 \leq j < \kappa$ and $(\mathsf{g}_0^i, \mathsf{g}_0^{i'}, \mathsf{g}_0^{i^*})$ for $i, i', i^* < 2^\kappa$. Write g_0 for $\{\mathsf{g}_0^i : i < 2^\kappa\}$ and r_+ for $\{\mathsf{r}_j : 1 \leq j < \kappa\}$. Call this atom structure α.

Consider the term algebra $\mathfrak{A}$ defined to be the subalgebra of the complex algebra of this atom structure generated by the atoms. We claim that $\mathfrak{A}$, as a relation algebra, has no complete representation, hence any algebra sharing this atom structure is not completely representable, too. Indeed, it is easy to show that if $\mathfrak{A}$ and $\mathfrak{B}$ are atomic relation algebras sharing the same atom structure, so that $\mathsf{At}\mathfrak{A} = \mathsf{At}\mathfrak{B}$, then $\mathfrak{A}$ is completely representable $\iff$ $\mathfrak{B}$ is completely representable.

Assume for contradiction that $\mathfrak{A}$ has a complete representation with base M. Let x, y be points in the representation with $\mathsf{M} \models \mathsf{r}_1(x, y)$. For each $i < 2^\kappa$, there is a point $z_i \in \mathsf{M}$ such that $\mathsf{M} \models \mathsf{g}_0^i(x, z_i) \wedge \mathsf{r}_1(z_i, y)$. Let $Z = \{z_i : i < 2^\kappa\}$. Within Z, each edge is labelled by one of the κ atoms in r_+. The Erdos-Rado theorem forces the existence of three points $z^1, z^2, z^3 \in Z$ such that $\mathsf{M} \models \mathsf{r}_j(z^1, z^2) \wedge \mathsf{r}_j(z^2, z^3) \wedge \mathsf{r}_j(z^3, z_1)$, for some single $j < \kappa$. This contradicts the definition of composition in $\mathfrak{A}$ (since we avoided monochromatic triangles).

Let S be the set of all atomic $\mathfrak{A}$-networks N with nodes ω such that $\{\mathsf{r}_i : 1 \leq i < \kappa : \mathsf{r}_i \text{ is the label of an edge in } N\}$ is finite. Then it is straightforward to show S is an amalgamation class, that is for all $M, N \in S$ if $M \equiv_{ij} N$ then there is $L \in S$ with $M \equiv_i L \equiv_j N$, witness [6, Definition 12.8] for notation. We have S is symmetric, that is, if $N \in S$ and $\theta : \omega \to \omega$ is a finitary function, in the sense that $\{i \in \omega : \theta(i) \neq i\}$ is finite, then $N\theta$

is in S. It follows that the complex algebra $\mathsf{CA}(S) \in \mathsf{QEA}_\omega$. Now let X be the set of finite $\mathfrak{A}$-networks N with nodes $\subseteq \kappa$ such that:

1. each edge of N is either (a) an atom of $\mathfrak{A}$ or (b) a cofinite subset of $\mathsf{r}_+ = \{\mathsf{r}_j : 1 \leq j < \kappa\}$ or (c) a cofinite subset of $\mathsf{g}_0 = \{\mathsf{g}_0^i : i < 2^\kappa\}$ and

2. N is 'triangle-closed', i.e. for all $l, m, n \in \mathsf{nodes}(N)$ we have $N(l,n) \leq N(l,m); N(m,n)$. That means if an edge (l,m) is labelled by Id then $N(l,n) = N(m,n)$ and if $N(l,m), N(m,n) \leq \mathsf{g}_0$ then $N(l,n) \cdot \mathsf{g}_0 = 0$ and if $N(l,m) = N(m,n) = \mathsf{r}_j$ (some $1 \leq j < \omega$) then $N(l,n) \cdot \mathsf{r}_j = 0$.

For $N \in X$ let $\widehat{N} \in \mathsf{CA}(S)$ be defined by

$$\{L \in S : L(m,n) \leq N(m,n) \text{ for } m, n \in \mathsf{nodes}(N)\}.$$

For $i \in \omega$, let $N \restriction_{-i}$ be the subgraph of N obtained by deleting the node i. Then if $N \in X,\ i < \omega$ then $\widehat{\mathsf{c}_i N} = \widehat{N \restriction_{-i}}$. The inclusion $\widehat{\mathsf{c}_i N} \subseteq (\widehat{N \restriction_{-i}})$ is clear. Conversely, let $L \in (\widehat{N \restriction_{-i}})$. We seek $M \equiv_i L$ with $M \in \widehat{N}$. This will prove that $L \in \widehat{\mathsf{c}_i N}$, as required. Since $L \in S$ the set $T = \{\mathsf{r}_i \notin L\}$ is infinite. Let T be the disjoint union of two infinite sets $Y \cup Y'$, say. To define the ω-network M we must define the labels of all edges involving the node i (other labels are given by $M \equiv_i L$). We define these labels by enumerating the edges and labeling them one at a time. So let $j \neq i < \kappa$. Suppose $j \in \mathsf{nodes}(N)$. We must choose $M(i,j) \leq N(i,j)$. If $N(i,j)$ is an atom then of course $M(i,j) = N(i,j)$. Since N is finite, this defines only finitely many labels of M. If $N(i,j)$ is a cofinite subset of g_0 then we let $M(i,j)$ be an arbitrary atom in $N(i,j)$. And if $N(i,j)$ is a cofinite subset of r_+ then let $M(i,j)$ be an element of $N(i,j) \cap Y$ which has not been used as the label of any edge of M which has already been chosen (possible, since at each stage only finitely many have been chosen so far). If $j \notin \mathsf{nodes}(N)$ then we can let $M(i,j) = \mathsf{r}_k \in Y$ some $1 \leq k < \kappa$ such that no edge of M has already been labelled by r_k. It is not hard to check that each triangle of M is consistent (we have avoided all monochromatic triangles) and clearly $M \in \widehat{N}$ and $M \equiv_i L$. The labeling avoided all but finitely many elements of Y', so $M \in S$. So $(\widehat{N \restriction_{-i}}) \subseteq \widehat{\mathsf{c}_i N}$.

Now let $\widehat{X} = \{\widehat{N} : N \in X\} \subseteq \mathsf{CA}(S)$. Then we claim that the subalgebra of $\mathsf{CA}(S)$ generated by $\widehat{X}$ is simply obtained from $\widehat{X}$ by closing under finite unions. Clearly all these finite unions are generated by $\widehat{X}$. We must show that the set of finite unions of $\widehat{X}$ is closed under all cylindric operations. Closure under unions is given. For $\widehat{N} \in X$ we have $-\widehat{N} = \bigcup_{m,n\in \mathsf{nodes}(N)} \widehat{N_{mn}}$ where N_{mn} is a network with nodes $\{m, n\}$ and labeling $N_{mn}(m, n) = -N(m, n)$. N_{mn} may not belong to X but it is equivalent to a union of at most finitely many members of $\widehat{X}$. The diagonal $\mathsf{d}_{ij} \in \mathsf{CA}(S)$ is equal to $\widehat{N}$ where N is a network with nodes $\{i, j\}$ and labeling $N(i, j) = \mathsf{Id}$. Closure under cylindrification is given.

Let $\mathfrak{C}$ be the subalgebra of $\mathsf{CA}(S)$ generated by $\widehat{X}$. Then $\mathfrak{A} = \mathfrak{Ra}\mathfrak{C}$. To see why, each element of $\mathfrak{A}$ is a union of a finite number of atoms, possibly a co-finite subset of g_0 and possibly a co-finite subset of r_+. Clearly $\mathfrak{A} \subseteq \mathfrak{Ra}\mathfrak{C}$. Conversely, each element $z \in \mathfrak{Ra}\mathfrak{C}$ is a finite union $\bigcup_{N\in F} \widehat{N}$, for some finite subset F of X, satisfying $\mathsf{c}_i z = z$, for $i > 1$. Let $i_0, \ldots, i_k$ be an enumeration of all the nodes, other than 0 and 1, that occur as nodes of networks in F. Then, $\mathsf{c}_{i_0} \ldots \mathsf{c}_{i_k} z = \bigcup_{N\in F} \mathsf{c}_{i_0} \ldots \mathsf{c}_{i_k} \widehat{N} = \bigcup_{N\in F} (\widehat{N \restriction_{\{0,1\}}}) \in \mathfrak{A}$. So $\mathfrak{Ra}\mathfrak{C} \subseteq \mathfrak{A}$. Thus $\mathfrak{A}$ is the relation algebra reduct of $\mathfrak{C} \in \mathsf{CA}_\omega$, but $\mathfrak{A}$ has no complete representation. Let $n > 2$. Let $\mathfrak{B} = \mathfrak{Nr}_n\mathfrak{C}$. Then $\mathfrak{B} \in \mathsf{Nr}_n\mathsf{CA}_\omega$, is atomic, but has no complete representation for plainly a complete representation of $\mathfrak{B}$ induces one of $\mathfrak{A}$.

By Theorem 1.15 $\mathfrak{B}$ is in $\mathbf{El}\mathsf{CRCA}_n = \mathsf{LCA}_n$. It remains to show that the ω-dilation $\mathfrak{C}$ is atomless. For any $N \in X$, we can add an extra node extending N to M such that $\emptyset \subsetneq M' \subsetneq N'$, so that N' cannot be an atom in $\mathfrak{C}$. $\square$

In the next theorem the inclusions in the third item are valid since by Lemma 1.3, $\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At} \subseteq \mathsf{LCA}_n$ and the last class is elementary.[5]

THEOREM 1.17. *Let $2 < n < \omega$. Then the following hold:*

1. $\mathbf{S}\mathfrak{Cm}\mathsf{LCAS}_n = \mathsf{RCA}_n$,
2. $\mathbf{SP}\mathfrak{Cm}\mathsf{At}(\mathsf{Nr}_n\mathsf{CA}_\omega) = \mathsf{RCA}_n$,
3. *For any class $\mathbf{L}$ such that* $\mathsf{At}(\mathsf{Nr}_n\mathsf{CA}_\omega) \subseteq \mathbf{L} \subseteq \mathsf{LCAS}_n$, $\mathbf{SP}\mathfrak{Cm}\mathbf{L} = \mathsf{RCA}_n$.

[5]The last incusion was implicitly prove in Theorem 1.3. To be more explicit, assume that $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_\omega$ is atomic. Then by lemma 1.3, $\exists$ has a winning strategy in $\mathbf{G}^\omega$, since there are infinitely many nodes, reusing them is superfluous, so $\exists$ has a winning strategyactually in (the harder to win game), $G_\omega(\mathsf{At}\mathfrak{A})$, and so $\exists$ has a winning strategy in all k rounded game $G_k(\mathsf{At}\mathfrak{A})$, so by definition $\mathfrak{A} \in \mathsf{LCA}_n$.

In particular, $\mathbf{SP}\mathfrak{Cm}(\mathbf{El}\mathsf{At}(\mathsf{Nr}_n\mathsf{CA}_\omega)) = \mathsf{RCA}_n$.

PROOF: 1. If $\mathfrak{A} \in \mathsf{RCA}_n$, then $\mathfrak{A}^+$ is completely representable [5], so $\mathsf{At}\mathfrak{A}^+ \in \mathsf{LCAS}_n$. By $\mathfrak{A} \subseteq \mathfrak{A}^+ = \mathfrak{Cm}\mathsf{At}\mathfrak{A}^+$, and $\mathfrak{Cm}\mathsf{At}\mathfrak{A}^+ \in \mathfrak{Cm}\mathsf{LCAS}_n$, we are done.

2. This follows from that $\mathsf{Fs}_n \subseteq \mathfrak{Cm}\mathsf{At}\mathsf{Nr}_n\mathsf{CA}_\omega$. Indeed, suppose that $\mathfrak{A} \in \mathsf{Fs}_n$, then $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_\omega$, hence $\mathsf{At}\mathfrak{A} \in \mathsf{At}\mathsf{Nr}_n\mathsf{CA}_\omega$ and $\mathfrak{A} = \mathfrak{Cm}\mathsf{At}\mathfrak{A} \in \mathfrak{Cm}\mathsf{At}\mathsf{Nr}_n\mathsf{CA}_\omega$. Thus $\mathsf{RCA}_n = \mathbf{SP}\mathsf{Fs}_n \subseteq \mathbf{SP}\mathfrak{Cm}\mathsf{At}\mathsf{Nr}_n\mathsf{CA}_\omega \subseteq \mathbf{SP}\mathfrak{Cm}\mathsf{LCAS}_n \subseteq \mathsf{RCA}_n$.

3. Follows immediately from the previous item. □

2. Atom-canonicity and degrees of representability

In this section, unless otherwise indicated, n is a finite ordinal > 2. We study closure properties of the classes $\mathsf{Nr}_n\mathsf{CA}_m$ $(m > n)$ and CRCA_n. We also introduce several new classes defined via the complex algebra operator $\mathfrak{Cm}$ and the neat reduct operator Nr and study their properties. The most general exposition of CA rainbow constructions is given in [7, Section 6.2, Definition 3.6.9] in the context of constructing atom structures from classes of models. Our models are just coloured graphs [5]. Let G, R be two relational structures. Let $2 < n < \omega$. Then the colours used are:

- greens: g_i $(1 \leq i \leq n-2)$, g_0^i, $i \in \mathsf{G}$,
- whites : $\mathsf{w}_i : i \leq n-2$,
- reds: r_{ij} $(i < j \in n$,
- shades of yellow : $\mathsf{y}_S : S$ a finite subset of ω or $S = \omega$.

A *coloured graph* is a graph such that each of its edges is labelled by the colours in the above first three items, greens, whites or reds, and some $n-1$ hyperedges are also labelled by the shades of yellow. Certain coloured graphs will deserve special attention.

DEFINITION 2.1. Let $i \in \mathsf{G}$, and let M be a coloured graph consisting of n nodes $x_0, \ldots, x_{n-2}, z$. We call M an *i-cone* if $M(x_0, z) = \mathsf{g}_0^i$ and for every $1 \leq j \leq n-2$, $M(x_j, z) = \mathsf{g}_j$, and no other edge of M is coloured green.

$(x_0, \ldots, x_{n-2})$ is called the *base of the cone*, z the *apex of the cone* and i the *tint of the cone.*

The rainbow algebra depending on G and R from the class $\mathbf{K}$ consisting of all coloured graphs M such that:

1. M is a complete graph and M contains no triangles (called forbidden triples) of the following types:

$$(\mathsf{g}, \mathsf{g}^{'}, \mathsf{g}^*), (\mathsf{g}_i, \mathsf{g}_i, \mathsf{w}_i) \qquad \text{any } 1 \leq i \leq n-2, \tag{2.1}$$
$$(\mathsf{g}_0^j, \mathsf{g}_0^k, \mathsf{w}_0) \qquad \text{any } j, k \in \mathsf{G}, \tag{2.2}$$
$$(\mathsf{r}_{ij}, \mathsf{r}_{j'k'}, \mathsf{r}_{i^*k^*}) \qquad \text{unless } |\{(j,k), (j',k'), (j^*,k^*)\}| = 3 \tag{2.3}$$

   and no other triple of atoms is forbidden.

2. If $a_0, \ldots, a_{n-2} \in M$ are distinct, and no edge (a_i, a_j) $i < j < n$ is coloured green, then the sequence $(a_0, \ldots, a_{n-2})$ is coloured a unique shade of yellow. No other $(n-1)$ tuples are coloured shades of yellow. Finally, if $D = \{d_0, \ldots, d_{n-2}, \delta\} \subseteq M$ and $M \upharpoonright D$ is an i cone with apex δ, inducing the order $d_0, \ldots, d_{n-2}$ on its base, and the tuple $(d_0, \ldots, d_{n-2})$ is coloured by a unique shade y_S then $i \in S$.

Let G and R be relational structures as above. Take the set J consisting of all surjective maps $a : n \to \Delta$, where $\Delta \in \mathbf{K}$ and define an equivalence relation $\sim$ on this set relating two such maps iff they essentially define the same graph [5]; the nodes are possibly different but the graph structure is the same. Let At be the atom structure with underlying set $J \sim$. We denote the equivalence class of a by $[a]$. Then define, for $i < j < n$, the accessibility relations corresponding to ijth-diagonal element, and ith-cylindrifier, as follows:

(1) $[a] \in E_{ij}$ iff $a(i) = a(j)$,

(2) $[a]T_i[b]$ iff $a \upharpoonright n \setminus \{i\} = b \upharpoonright n \setminus \{i\}$,

This, as easily checked, defines a CA_n atom structure. The complex CA_n over this atom structure will be denoted by $\mathfrak{A}_{\mathsf{G},\mathsf{R}}$. The dimension of $\mathfrak{A}_{\mathsf{G},\mathsf{R}}$, always finite and > 2, will be clear from context. For rainbow atom structures, there is a one to one correspondence between atomic networks and coloured graphs [5, Lemma 30], so for $2 < n < m \leq \omega$, we use the graph versions of the games G_k^m, $k \leq \omega$, and $\mathbf{G}^m$ played on rainbow atom structures of dimension m [5, pp. 841–842]. The the atomic k rounded game game G^m_k where the number of nodes are limited to n to games on coloured graphs [5, lemma 30]. The game $\mathbf{G}^m$ lifts to a game on coloured graphs, that is like the graph games G^m_ω [5], where the number of nodes of graphs played during the ω rounded game does not exceed m, but $\forall$ has the option to re-use nodes. The typical winning strategy for $\forall$ in the graph version of both atomic games is bombarding $\exists$ with cones having a common base and *green* tints until she runs out of (suitable) *reds*, that is to say, reds whose indicies do not match [5, 4.3]. So roughly if $|G|$ is larger that $|R|$ substantially, then $\forall$ can win; otherwise $\exists$ wins for if there is a winning strategy for $\forall$ it must be implemented as just described. The (complex) rainbow algebra based on G and R is denoted by $\mathfrak{A}_{\mathsf{G},\mathsf{R}}$. The dimension n will always be clear from context.

THEOREM 2.2. *Let $2 < n < \omega$.*

1. *There exists $\mathfrak{A} \in \mathsf{RCA}_n$ such that $\mathfrak{CmAt}\mathfrak{A} \notin \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{t(n)}$, where $t(n) = n(n+1)/2$. Therefore any completely additive variety V such that $\mathsf{RCA}_n \subseteq \mathsf{V} \subseteq \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{t(n)}$ is not atom-canonical.*
2. *There exists $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_l \cap \mathsf{RCA}_n$ such that $\mathfrak{CmAt}\mathfrak{A} \notin \mathsf{RCA}_n$,*
3. *There exists $\mathfrak{B} \in \mathsf{Cs}_n$, $\mathfrak{B} \notin \mathbf{El}\mathsf{Nr}_n\mathsf{CA}_{n+1}$, but $\mathsf{At}\mathfrak{B} \in \mathsf{Nr}_n\mathsf{CA}_\omega$ and $\mathfrak{CmAt}\mathfrak{B} \in \mathsf{Nr}_n\mathsf{CA}_\omega$*

PROOF: 1. The proof of the first item is given in full detail in [16, Theorem 1]; here we give the main ingredients of the proof as another instance of a blow up and blur construction. Take the finite rainbow cylindric algebra $R(\Gamma)$ as defined in [7, Definition 3.6.9], where Γ (the reds) is taken to be the complete irreflexive graph m, and the greens are $\{\mathsf{g}_i : 1 \leq i < n-1\} \cup \{\mathsf{g}^i_0 : 1 \leq i \leq n(n-1)/2\}$ so that G is the complete irreflexive graph $n(n-1)/2$.

Call this finite rainbow n-dimensional cylindric algebra, based on $\mathsf{G} = n(n-1)/2$ and $\mathsf{R} = n$, $\mathsf{CA}_{n(n-1)/2+1,n}$ and denote its finite atom structure by $\mathbf{At_f}$. One then replaces each red colour used in constructing $\mathsf{CA}_{n(n-1)/2,n}$ by infinitely many with superscripts from ω, getting a weakly representable atom structure $\mathbf{At}$, that is, the term algebra $\mathfrak{Tm}\mathbf{At}$ is representable.

The resulting atom structure (with ω-many reds), call it $\mathbf{At}$, is the rainbow atom structure that is like the atom structure of the (atomic set) algebra denoted by $\mathfrak{A}$ in [10, Definition 4.1] except that we have $n(n-1)/2$ greens and not infinitely many as is the case in [10]. Everything else is the same. In particular, the rainbow signature [7, Definition 3.6.9] now consists of $\mathsf{g}_i : 1 \leq i < n-1$, $\mathsf{g}_0^i : 1 \leq i \leq n+1$, $\mathsf{w}_i : i < n-1$, $\mathsf{r}_{kl}^t : k < l < n$, $t \in \omega$, binary relations, and $n-1$ ary relations y_S, $S \subseteq n(n-1)/2$.

There is a shade of red ρ; the latter is a binary relation that is *outside the rainbow signature*. But ρ is used as a label for coloured graphs built during a 'rainbow game', and in fact, $\exists$ can win the rainbow ω-rounded game and she builds an n-homogeneous (coloured graph) model M as indicated in the above outline by using ρ when she is forced a red [10, Proposition 2.6, Lemma 2.7]. Then, it can be shown exactly as in [10], that $\mathfrak{TmAt}$ is representable as a set algebra with unit ${}^n\mathsf{M}$.

We next embed $\mathsf{CA}_{n(n-1)/2,n}$ into the complex algebra $\mathfrak{Cm}\mathbf{At}$, the Dedekind–MacNeille completion of $\mathfrak{Tm}\mathbf{At}$. Let CRG_f denote the class of coloured graphs on $\mathbf{At_f}$ and CRG be the class of coloured graph on $\mathbf{At}$. We can assume that $\mathsf{CRG}_f \subseteq \mathsf{CRG}$. Write M_a for the atom that is the (equivalence class of the) surjection $a : n \to M$, $M \in \mathsf{CRG}$. Here we identify a with $[a]$; no harm will ensue.

We define the (equivalence) relation $\sim$ on At by $M_b \sim N_a$, $(M, N \in \mathsf{CRG}) \iff$ they are everywhere identical except possibly at red edges:

$$M_a(a(i), a(j)) = \mathsf{r}^l \iff N_b(b(i), b(j)) = \mathsf{r}^k, \text{ for some } l, k \in \omega.$$

We say that M_a is a *copy of* N_b if $M_a \sim N_b$. Now we define a map $\Theta : \mathsf{CA}_{n+1,n} = \mathfrak{Cm}\mathbf{At_f}$ to $\mathfrak{Cm}\mathsf{At}$, by specifing first its values on At_f, via $M_a \mapsto \sum_j M_a^{(j)}$; where $M_a^{(j)}$ is a copy of M_a; each atom maps to the suprema of its copies. (If M_a has no red edges, then by $\sum_j M_a^{(j)}$, we understand M_a). This map is extended to $\mathsf{CA}_{n+1,n}$ the obvious way. The map Θ is well defined, because $\mathfrak{Cm}\mathsf{At}$ is complete. It is not hard to show that the map Θ is an injective homomorphim.

One next proves that $\forall$ has a winning strategy for $\exists$ in $\mathbf{G}^{t(n)}\mathsf{At}(\mathsf{CA}_{n(n-1)/2,n})$, where $t(n) = n(n+1)/2 + 1$ using the usual rainbow strategy by bombarding $\exists$ with cones having the same base and distinct green tints. He needs $t(n)$ nodes to implement his winning strategy. In fact, he needs $t(n)$ nodes to force a win in the weaker game $G_\omega^{t(n)}(\mathsf{At}\mathfrak{A}_{n(n-1)/2,n})$ without the need to resue the nodes in play. To see why, first it is straightforward to show that $\forall$ has winning strategy first in the Ehrenfeucht–Fraïssé forth private game played between $\exists$ and $\forall$ on the complete irreflexive graphs $n+1(\leq n(n-1)/2+1)$ and n rounds

$\mathsf{EF}_{n+1}^{n+1}(n+1, n)$ since $n+1$ is 'longer' than n. $\forall$ lifts his winning strategy from the last private Ehrenfeucht–Fraïssé forth game to the graph game on $\mathbf{At}_f = \mathsf{At}(\mathsf{CA}_{n(n-1)/2,n})$ see [5, p. 841] forcing a win using $t(n)$ nodes. One uses the $n(n-1)/2+2$ green relations in the usual way to force a red clique C, say with $n(n-1)/2+2$. Pick any point $x \in C$. Then there are $> n(n-1)/2$ points y in $C \setminus \{x\}$. There are only $n(n-1)/2$ red relations. So there must be distinct y, $z \in C \setminus \{x\}$ such that (x, y) and (x, z) both have the same red label (it will be some $\mathsf{r}_{ij}{}^m$ for $i < j < n$). But (y, z) is also red, and this contradicts the consistency condition of reds. In more detail, $\forall$ bombards $\exists$ with cones having common base and distinct green tints until $\exists$ is forced to play an inconsistent red triangle (where indicies of reds do not match). He needs $n-1$ nodes as the base of cones, plus $|P|+2$ more nodes, where $P = \{(i, j) : i < j < n\}$ forming a red clique, triangle with two edges satisfying the same $r_p{}^m$ for $p \in P$. Calculating, we get $t(n) = n-1+n(n-1)/2+2 = n(n+1)/2+1$. We proved that $\forall$ lifts his winning strategy from the last private game to the graph game on $\mathbf{At}_f = \mathsf{At}(\mathsf{CA}_{n(n-1)/2,n}$ forcing a win using $t(n)$ nodes.

2. This follows from the proof of Theorem 1.12; we give a more streamlined proof. Like before, we use the construction in [2]. Let $\mathfrak{R}$ be a relation algebra, with non-identity atoms I and $2 < n < \omega$. Assume that $J \subseteq \wp(I)$ and $E \subseteq {}^3\omega$. (J, E) is an *n-blur* for $\mathfrak{R}$, if J is a *complex n-blur* and the tenary relation E is an *index blur* defined as in item (ii) of [2, Definition 3.1]. Recall that (J, E) is a *strong n-blur*, if it (J, E) is an n-blur, such that the complex n-blur satisfies: $(\forall V_1, \ldots V_n, W_2, \ldots W_n \in J)(\forall T \in J)(\forall 2 \le i \le n)\mathsf{safe}(V_i, W_i, T)$ (with notation as in [2]). Now let $l \ge 2n-1$, $k \ge (2n-1)l$, $k \in \omega$. One takes the finite integral relation algebra $\mathfrak{R}_l = \mathfrak{E}_k(2, 3)$ where k is the number of non-identity atoms in $\mathfrak{R}_l$. Then $\mathfrak{R}_l$ has a strong l-blur, (J, E) and it can only be represented on a finite basis [2]. Then $\mathfrak{Bb}_n(\mathfrak{R}_l, J, E) = \mathsf{Nr}_n\mathfrak{Bl}_l(\mathfrak{R}_l, J, E)$ has no complete representation, so $\mathfrak{Cm}\mathsf{At}\mathfrak{Bb}_n(\mathfrak{R}_l, J, E)$ is not representable.

3. Let $V = {}^n\mathbb{Q}$ and let $\mathfrak{A} \in \mathsf{Cs}_n$ has universe $\wp(V)$. Then clearly $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_\omega$. To see why, let $W = {}^\omega\mathbb{Q}$ and let $\mathfrak{D} \in \mathsf{Cs}_\omega$ have universe $\wp(W)$. Then the map $\theta : \mathfrak{A} \to \wp(\mathfrak{D})$ defined via $a \mapsto \{s \in W : (s \upharpoonright \alpha) \in a\}$, is an injective homomorphism from $\mathfrak{A}$ into $\mathfrak{Rd}_n\mathfrak{D}$ that is onto $\mathfrak{Nr}_n\mathfrak{D}$. Let y denote the following n-ary relation: $y = \{s \in V : s_0 + 1 = \sum_{i>0} s_i\}$. Let y_s be the singleton containing s, i.e. $y_s = \{s\}$ and $\mathfrak{B} = \mathfrak{Sg}^{\mathfrak{A}}\{y, y_s : s \in y\}$. It is shown in [17] that $\{s\} \in \mathfrak{B}$, for all $s \in V$.

Now $\mathfrak{B}$ and $\mathfrak{A}$ having same top element V, share the same atom structure, namely, the singletons, so $\mathfrak{B} \subseteq_d \mathfrak{A}$ and $\mathfrak{Cm}\mathsf{At}\mathfrak{B} = \mathfrak{A}$. Furthermore, plainly $\mathfrak{A}, \mathfrak{B} \in \mathsf{CRCA}_n$; the identity maps establishes a complete representation for both, since $\bigcup_{s\in V}\{s\} = V$. Since $\mathfrak{B} \subseteq_d \mathfrak{A}$, then $\mathfrak{B} \subseteq_c \mathfrak{A}$, so $\mathfrak{B} \in \mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At}$ because $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_\omega$ is atomic. As proved in [17], $\mathfrak{B} \notin \mathbf{El}\mathsf{Nr}_n\mathsf{CA}_{n+1}(\supseteq \mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At}))$. $\square$

Recall that $\mathbf{S_c}$ denotes the operation of forming complete sublgebras and $\mathbf{S_d}$ denotes the opeartion of forming dense subalgebras. We let $\mathbf{I}$ denote the operation of forming isomorphic images. For any class of BAO, $\mathbf{I}\mathsf{K} \subseteq \mathbf{S_d}\mathsf{K} \subseteq \mathbf{S_c}\mathsf{K}$. (It is not hard to show that for Boolean algebras the inclusion are proper).

Definition 2.3. Let $2 < n \leq l \leq m \leq \omega$. Let $\mathbf{O} \in \{\mathbf{S}, \mathbf{S_d}, \mathbf{S_c}, \mathbf{I}\}$.

1. An algebra $\mathfrak{A} \in \mathsf{CA}_n$ has the $\mathbf{O}$ *neat embedding property up to* m if $\mathfrak{A} \in \mathbf{O}\mathsf{Nr}_n\mathsf{CA}_m$. If $m = \omega$ and $\mathbf{O} = \mathbf{S}$, we say simply that $\mathfrak{A}$ has *the neat embedding property.* (Observe that the last condition is equivalent to that $\mathfrak{A} \in \mathsf{RCA}_n$).
2. An atomic algebra $\mathfrak{A} \in \mathsf{CA_n}$ has the *complex* $\mathbf{O}$ *neat embedding property up to* m, if $\mathfrak{Cm}\mathsf{At}\mathfrak{A} \in \mathbf{O}\mathsf{Nr}_n\mathsf{CA}_m$. The word 'complex' here refers to the *involvement of the complex algebra* in the definition.
3. An atomic algebra $\mathfrak{A} \in \mathsf{RCA}_n$ is *strongly representable up to* l *and* m if $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_l$ and $\mathfrak{Cm}\mathsf{At}\mathfrak{A} \in \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_m$. If $l = n$ and $m = \omega$, we say that $\mathfrak{A}$ is *strongly representable.*
4. Let $\mathbf{L} \subseteq \mathbf{K}$ be subclasses of CA_n. We say that $\mathbf{L}$ is *not atom-canonical relative* $\mathbf{K}$ if there exists an atomic algebra $\mathfrak{A} \in \mathbf{L}$ such that $\mathfrak{Cm}\mathsf{At}\mathfrak{A} \notin \mathbf{K}$. Observe that if $\mathbf{L}$ is not atom-canonical relative to itself, then $\mathbf{L}$ is not atom-canonical.

Example 2.4.

1. The algebra $\mathfrak{A}$ constructed in the third item of theorem 2.2 has the neat embedding property, but not the complex $\mathbf{S}$ neat embedding propery up to m for any $m \geq n(n+1)/2$. In particular, $\mathfrak{A}$ is not strongly representable and $\mathfrak{A}$ lacks a complete representation. Furthermore, the algebra $\mathfrak{A}$ witnesses that RCA_n is not atom-canonical relative to $\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{n+k}$ for any $k \geq n(n+1)/2$.

2. For every $2 < n < l < \omega$, the algebra $\mathfrak{B} = \mathfrak{Bb}_n(\mathfrak{E}_k(2,3), J, E)$ used in the second item of Theorem 2.2 based on Theorem 1.12, where k depends on l and (J, E) is the strong l-blur of the Maddux algebra $\mathfrak{E}_k(2,3)$ as specified in *op. cit.*, is in $\mathsf{Nr}_n\mathsf{CA}_l \cap \mathsf{RCA}_n$, but is not strongly representable up to l and ω. In particular, $\mathfrak{B}$, like $\mathfrak{A}$ in the first item, is also not strongly representable and lacks a complete representation. The algebra $\mathfrak{B}$ witnesses that $\mathsf{RCA}_n \cap \mathsf{Nr}_n\mathsf{CA}_l$ is not atom-canonical relative to RCA_n.

3. The algebra $\mathfrak{B}$ used in the last item of theorem 2.2 has the complex $\mathbf{I}$ neat embedding property up to m for any $m \geq n$ but does not have the $\mathbf{I}$ neat embedding property up to $n+1$, *a fortiori* up to any $m \geq n+1$, cf. the second item of the forthcoming theorem 2.5.

Let $2 < n \leq l \leq m \leq \omega$. Let $\mathbf{O} \in \{\mathbf{S}, \mathbf{S_d}, \mathbf{S_c}, \mathbf{I}\}$. Denote the class of CA_ns having the complex $\mathbf{O}$ neat embedding property up to m by $\mathsf{CNPCA}^{\mathbf{O}}_{n,m}$, and let $\mathsf{RCA}^{\mathbf{O}}_{n,m} := \mathsf{CNPCA}^{\mathbf{O}}_{n,m} \cap \mathsf{RCA}_n$. Denote the class of strongly representable CA_ns up to l and m by $\mathsf{RCA}^{l,m}_n$. Call an algebra $\mathfrak{A} \in \mathsf{CA}_n$ *strongly representable* if $\mathfrak{A}$ is atomic and $\mathsf{At}\mathfrak{A}$ is strongly representable; that is $\mathfrak{Cm}\mathsf{At}\mathfrak{A} \in \mathsf{RCA}_n$. Observe that $\mathsf{RCA}^{n,m}_n = \mathsf{RCA}^{\mathbf{S}}_{n,m}$ and that when $m = \omega$ both classes coincide with the class of strongly representable CA_ns. For a class $\mathbf{K}$ of BAOs, $\mathbf{K} \cap \mathsf{Count}$ denotes the class of countable algebras in $\mathbf{K}$, and recall that $\mathbf{K} \cap \mathbf{At}$ denotes the class of atomic algebras in $\mathbf{K}$.

Theorem 2.5. *Let $2 < n \leq l < m \leq \omega$ and $\mathbf{O} \in \{\mathbf{S}, \mathbf{S_c}, \mathbf{S_d}, \mathbf{I}\}$. Then the following hold:*

1. $\mathsf{RCA}^{\mathbf{O}}_{n,m} \subseteq \mathsf{RCA}^{\mathbf{O}}_{n,l}$ *and* $\mathsf{RCA}^{\mathbf{I}}_{n,l} \subseteq \mathsf{RCA}^{\mathbf{S_d}}_{n,l} \subseteq \mathsf{RCA}^{\mathbf{S_c}}_{n,l} \subseteq \mathsf{RCA}^{\mathbf{S}}_{n,l}$. *The last inclusion is proper for $l \geq n(n+1)/2$,*

2. *For $\mathbf{O} \in \{\mathbf{S}, \mathbf{S_c}, \mathbf{S_d}\}$, $\mathsf{CNPCA}^{\mathbf{O}}_{n,l} \subseteq \mathbf{O}\mathsf{Nr}_n\mathsf{CA}_l$ (that is the complex $\mathbf{O}$ neat embedding property is stronger than the $\mathbf{O}$ neat embedding property), and for $\mathbf{O} = \mathbf{S}$, the inclusion is proper for $l \geq n+3$. But for $\mathbf{O} = \mathbf{I}$, $\mathsf{CNPCA}^{\mathbf{I}}_{n,l} \nsubseteq \mathsf{Nr}_n\mathsf{CA}_l$ (so the complex $\mathbf{I}$ neat embedding property does not imply the $\mathbf{I}$ neat embedding property),*

3. *If $\mathfrak{A}$ is finite, then $\mathfrak{A} \in \mathsf{CNPCA}^{\mathbf{O}}_{n,l} \iff \mathfrak{A} \in \mathbf{O}\mathsf{Nr}_n\mathsf{CA}_l$ and $\mathfrak{A} \in \mathsf{RCA}^{\mathbf{O}}_{n,l} \iff \mathfrak{A} \in \mathsf{RCA}_n \cap \mathbf{O}\mathsf{Nr}_n\mathsf{CA}_l$. Furthermore, for any positive k, $\mathsf{CNPCA}^{\mathbf{O}}_{n,n+k+1} \subsetneq \mathsf{CNPCA}^{\mathbf{O}}_{n,n+k}$, and finally $\mathsf{CNPCA}^{\mathbf{O}}_{n,\omega} \subsetneq \mathsf{RCA}_n$,*

4. $(\exists\mathfrak{A} \in \mathsf{RCA}_n \cap \mathbf{At} \sim \mathsf{CNPCA}^{\mathbf{S}}_{n,l}) \implies \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_k$ *is not atom-canonical for all* $k \geq l$. *In particular,* $\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_k$ *is not atom-canonical for all* $k \geq n+3$*,*

5. *If* $\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_l$ *is atom-canonical, then* $\mathsf{RCA}^{\mathbf{S}}_{n,l}$ *is first order definable. There exists a finite* $k > n+1$*, such that* $\mathsf{RCA}^{\mathbf{S}}_{n,k}$ *is not first order definable.*

6. *Let* $2 < n < l \leq \omega$. *Then* $\mathsf{RCA}_n^{l,\omega} \cap \mathsf{Count} \neq \emptyset \iff l < \omega$.

Proof: 1. The inclusions follow from the definition and the strictness of the last inclusion in this item is witnessed by the algebra $\mathfrak{C} = \mathfrak{C}_{\mathbb{Z},\mathfrak{N}}$ used in Theorem 3.1, since $\mathfrak{C}$ satisfies $\mathfrak{C} = \mathfrak{CmAt}\mathfrak{C} \in \mathsf{RCA}_n$ but $\mathfrak{C} \notin \mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_l$ for $l \geq n+3$.

2. Let $\mathbf{O} \in \{\mathbf{S}, \mathbf{S_c}, \mathbf{S_d}\}$. If $\mathfrak{CmAt}\mathfrak{A} \in \mathbf{O}\mathsf{Nr}_n\mathsf{CA}_l$, then $\mathfrak{A} \subseteq_d \mathfrak{CmAt}\mathfrak{A}$, so $\mathfrak{A} \in \mathbf{S_d}\mathbf{O}\mathsf{Nr}_n\mathsf{CA}_l \subseteq \mathbf{O}\mathsf{Nr}_n\mathsf{CA}_l$. This proves the first part. The strictness of the last inclusion follows from the first part of Theorem 2.2 since the atomic countable algebra $\mathfrak{A}$ constructed in *op. cit.* is in RCA_n, but $\mathfrak{CmAt}\mathfrak{A} \notin \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_l$ for any $l \geq n(n+1)/2$.

For the last non-inclusion in item (2), we use the set algebras $\mathfrak{A}$ and $\mathfrak{B}$ in item (3) of Theorem 2.2. Now $\mathfrak{B} \subseteq_d \mathfrak{A}$, $\mathfrak{A} \in \mathsf{Cs}_n$, and clearly $\mathfrak{CmAt}\mathfrak{B} = \mathfrak{A}(\in \mathsf{Nr}_n\mathsf{CA}_\omega)$. As proved in [17], $\mathfrak{B} \notin \mathbf{El}\mathsf{Nr}_n\mathsf{CA}_{n+1}$, so $\mathfrak{B} \notin \mathsf{Nr}_n\mathsf{CA}_{n+1}(\supseteq \mathsf{Nr}_n\mathsf{CA}_l)$. But $\mathfrak{CmAt}\mathfrak{B} \in \mathsf{Nr}_n\mathsf{CA}_\omega$, hence $\mathfrak{B} \in \mathsf{RCA}^{\mathbf{I}}_{n,l}$. We have shown that $\mathfrak{B} \in \mathsf{RCA}^{\mathbf{I}}_{n,l} \sim \mathsf{Nr}_n\mathsf{CA}_l$, and we are through with the last required in item (2). Here we basically use that $\mathsf{Nr}_n\mathsf{CA}_m$ is not closed under $\mathbf{S_d}$, *a fortiori* under $\mathbf{S_c}$, while, conversely, CRCA_n is closed under $\mathbf{S_c}$ since $\mathbf{S_c}$ is an idempotent operator ($\mathbf{S_c}\mathbf{S_c} = \mathbf{S_c}$), *a fortiori* CRCA_n is closed under $\mathbf{S_d}$.

3. Follows by definition observing that if $\mathfrak{A}$ is finite then $\mathfrak{A} = \mathfrak{CmAt}\mathfrak{A}$. The strictness of the first inclusion follows from the construction in [9] where it shown that for any positive k, there is a *finite algebra* $\mathfrak{A}$ in $\mathsf{Nr}_n\mathsf{CA}_{n+k} \sim \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{n+k+1}$ (witness the appendix for a simplified version of the construction in [9]). The inclusion $\mathsf{CNPCA}^{\mathbf{O}}_{n,\omega} \subseteq \mathsf{RCA}_n$ holds because if $\mathfrak{B} \in \mathsf{CNPCA}^{\mathbf{O}}_{n,\omega}$, then $\mathfrak{B} \subseteq \mathfrak{CmAt}\mathfrak{B} \in \mathbf{O}\mathsf{Nr}_n\mathsf{CA}_\omega \subseteq \mathsf{RCA}_n$. The $\mathfrak{A}$ used in the last item of theorem 2.2 witnesses the strictness of the last inclusion proving the last required in this item.

4. Follows from the definition and the construction used in item (3) of theorem 2.2.

5. Follows from that $\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_l$ is canonical. So if it is atom-canonical too, then $\mathsf{At}(\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_l) = \{\mathfrak{F} : \mathfrak{Cm}\mathfrak{F} \in \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_l\}$, the former class is ele-

mentary [6, Theorem 2.84], and the last class is elementray $\iff \mathsf{RCA}^{\mathbf{S}}_{n,l}$ is elementary. Non-elementarity follows from [7, Corollary 3.7.2] where it is proved that $\mathsf{RCA}^{\mathbf{S}}_{n,\omega}$ is not elementary, together with the fact that $\bigcap_{n<k<\omega}\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_k = \mathsf{RCA}_n$. In more detail, let $\mathfrak{A}_i$ be the sequence of strongly representable CA_ns with $\mathfrak{Cm}\mathsf{At}\mathfrak{A}_i = \mathfrak{A}_i$ and $\mathfrak{A} = \Pi_{i/U}\mathfrak{A}_i$ is not strongly representable. Hence $\mathfrak{Cm}\mathsf{At}\mathfrak{A} \notin \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_\omega = \bigcap_{i\in\omega}\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{n+i}$, so $\mathfrak{Cm}\mathsf{At}\mathfrak{A} \notin \mathbf{S}\mathsf{Nr}_n\mathsf{K}_l$ for all $l > k$, for some $k \in \omega$, $k > n$. But for each such l, $\mathfrak{A}_i \in \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_l(\supseteq \mathsf{RCA}_n)$, so $\mathfrak{A}_i$ is a sequence of algebras such that $\mathfrak{Cm}\mathsf{At}\mathfrak{A}_i = \mathfrak{A}_i \in \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_l$, but $\mathfrak{Cm}(\mathsf{At}(\Pi_{i/U}\mathfrak{A}_i)) = \mathfrak{Cm}\mathsf{At}\mathfrak{A} \notin \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_l$, for all $l \geq k$. That k has to be strictly greater than $n+1$, follows because $\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{n+1}$ is atom-canonical.

6. $\Longleftarrow$: Let $l < \omega$. Then the required follows from theorem 1.12, and item (2) in Theorem 2.2 that there exists a countable $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_l \cap \mathsf{RCA}_n$ such that $\mathfrak{Cm}\mathsf{At}\mathfrak{A} \notin \mathsf{RCA}_n$. Now we prove $\Longrightarrow$: Assume for contradiction that there is an $\mathfrak{A} \in \mathsf{RCA}_n^{\omega,\omega} \cap \mathsf{Count}$. Then by definition $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_\omega$, so by [14, Theorem 5.3.6], we have $\mathfrak{A} \in \mathsf{CRCA}_n$. But this complete representation, induces a(n ordinary) representation of $\mathfrak{Cm}\mathsf{At}\mathfrak{A}$ which is a contradiction. Indeed by Lemma 1.10, if $f : \mathfrak{A} \to \mathfrak{B}$ is a complete representation of $\mathfrak{A}$ via f then one extends f to $\hat{f}$ from $\mathfrak{Cm}\mathsf{At}\mathfrak{A}$ to $\mathfrak{B}$ by defining $\hat{f}(a) = \sum^{\mathfrak{Cm}\mathsf{At}\mathfrak{A}}_{x\in\mathsf{At}\mathfrak{A}, x\leq a} f(x)$. □

3. Non-elementary classes

Still $\mathbf{S_d}$ stands for the operation of forming dense subalgebras and for K a class of BAOs, $\mathbf{S_c}\mathsf{K} = \{\mathfrak{B} : (\exists\mathfrak{A} \in \mathsf{K})(\sum^{\mathfrak{A}} X = 1 \implies \sum^{\mathfrak{B}} X = 1\}$.

THEOREM 3.1. *Let $2 < n < \omega$. Any class between $\mathbf{S_d}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n$ and $\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_{n+3}$ is not first order definable. Furthermore any class between $\mathsf{At}(\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n)$ and $\mathsf{At}(\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_{n+3})$ is not first order definable.*

PROOF: The proof is long and is divided into four parts:

(a) We define an ω-rounded (atomic) game $\mathbf{H}(\alpha)$ played on so-called atomic λ-neat hypernetworks-λ a 'label'.

(b) If α is a countable atom structure, and $\exists$ has a winning strategy in $\mathbf{H}(\alpha)$, then any algebra $\mathfrak{F}$ having atom structure α is completely representable, $\mathfrak{Cm}\alpha \in \mathsf{Nr}_n\mathsf{CA}_\omega$ and $\alpha \in \mathsf{AtNr}_\alpha\mathsf{CA}_\omega$. In fact, there will exist a complete $\mathfrak{D} \in \mathsf{CA}_\omega$ such that $\mathfrak{Cm}\alpha \cong \mathsf{Nr}_n\mathfrak{D}$ and $\alpha \cong \mathsf{AtNr}_n\mathfrak{D}$,

(c) Then the game $\mathbf{H}$ will be applied to the atom structure of a rainbow-like CA_n denoted below by $\mathfrak{C}_{\mathbb{Z},\mathfrak{N}}$. From a winning strategyof $\exists$ in $\mathbf{H}_k(\mathsf{At}\mathfrak{C}_{\mathbb{Z},\mathfrak{N}})$ (where $\mathbf{H}_k$ is $\mathbf{H}$ truncated to k rounds) for all $k \leq \omega$–so that $\mathbf{H}_\omega = \mathbf{H}$– it will follow that $\mathfrak{C}_{\mathbb{Z},\mathfrak{N}} \equiv \mathfrak{Tm}\alpha$ for some completely representable atom structure $\alpha \in \mathsf{At}(\mathsf{Nr}_n\mathsf{CA}_\omega)$, for which $\mathfrak{Cm}\alpha \in \mathsf{Nr}_n\mathsf{CA}_\omega$. On the other hand, we prove that $\forall$ has a winning strategy in $\mathbf{G}^{n+3}(\mathsf{At}\mathfrak{C}_{\mathbb{Z},\mathfrak{N}})$, so by lemma 1.3 $\mathfrak{C}_{\mathbb{Z},\mathfrak{N}} \notin \mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_{n+3}$.

(d) The term algebra $\mathfrak{Tm}\alpha$ will be used to show that any class between $\mathbf{S_d}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n$ and $\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_{n+3}$ is not elementary.

(a) Defining the game $\mathbf{H}_k(k \leq \omega)$ which is $\mathbf{H}$ restricted to k rounds This new game $\mathbf{H}_k$ is stronger than G_k. In $\mathbf{H}_k$ not only the moves are more (which they are), but now the board of the play is different.

Fix $k \leq \omega$. The new game $\mathbf{H}_k$ is played on so-called λ-*neat hypernetworks*, λ a 'hyperlabel' and it has k rounds. These are similar to $m(< n)$-dimensional hypernetworks as defined in item(3) of definition 1.2; they are roughly networks endowed with labelled hyperedges, whose length gets arbitrarily long, but is still finite. Unlike m-dimensional hypernetworks here the lengths of hyperedges are not uniformly bounded. So a hypernetwork of an atomic $\mathfrak{A} \in \mathsf{CA}_n$ has two parts (N^a, N^h) where N^a is network whose n-hyperdges are labelled by atoms of $\mathfrak{A}$ and $N^h :^{<\omega} \mathsf{nodes}(N) \to \Lambda$, where hyperedges get their hyperlabels from a non-empty set (of hyperlabels) Λ.

There is a compatibility condition between N^a and N^h which is a CA analogue of condition (3) in [6, Definition 12.1] formulated for hypernetworks of relation algebras. This condition for hypernetworks as defined in [4], is given in [4, Definition 28]. The form for CAs needed is entirely analogous to the condition in item (3) of definition 1.2. In any such hypernetwork $N = (N^a, N^h)$, there are so-called *short hyperedges* and *long hyperedges* in N^h. The hypernetworks whose short hyperedges are constantly labelled by a hyperlabel $\lambda \in \Lambda$ are called λ-neat hypernetworks. The game $\mathbf{H}$ offers $\forall$ three moves delivered by $\forall$ during the play. There is a *cylindrifier move* analagous to the cylindrifier move in G adapted the obvious way to λ-neat hypernetworks and two more *amalgamation moves*.

First amalgamation move: $\forall$ can play a *transformation move* by picking a previously played λ-neat hypernetwork N and a partial, finite surjection

$\theta : \omega \to \mathsf{nodes}(N)$, this move is denoted (N, θ). $\exists$'s response is mandatory. She must respond with $N\theta$.

Second amalgmation move: $\forall$ can play an *amalgamation move* by picking previously played λ-neat hypernetworks M, N such that $M \restriction_{\mathsf{nodes}(M)\cap\mathsf{nodes}(N)} = N \restriction_{\mathsf{nodes}(M)\cap\mathsf{nodes}(N)}$, and $\mathsf{nodes}(M) \cap \mathsf{nodes}(N) \neq \emptyset$. This move is denoted (M, N). To make a legal response, $\exists$ must play a λ-neat hypernetwork L extending M and N, where $\mathsf{nodes}(L) = \mathsf{nodes}(M) \cup \mathsf{nodes}(N)$.

(b) Forming the required ω-dilation $\mathfrak{D}$ Fix some $a \in \alpha$. The game $\mathbf{H}_\omega$ is designed so that using $\exists$ s winning strategy in the game $\mathbf{H}_\omega(\alpha)$ one can define a nested sequence $M_0 \subseteq M_1, \ldots$ of λ-neat hypernetworks where M_0 is $\exists$'s response to the initial $\forall$-move a, such that: If M_r is in the sequence and $M_r(\bar{x}) \leq \mathsf{c}_i a$ for an atom a and some $i < n$, then there is $s \geq r$ and $d \in \mathsf{nodes}(M_s)$ such that $M_s(\bar{y}) = a$, $\bar{y}_i = d$ and $\bar{y} \equiv_i \bar{x}$. In addition, if M_r is in the sequence and θ is any partial isomorphism of M_r, then there is $s \geq r$ and a partial isomorphism θ^+ of M_s extending θ such that $\mathsf{rng}(\theta^+) \supseteq \mathsf{nodes}(M_r)$ (This can be done using $\exists$'s responses to amalgamation moves). Now let $\mathfrak{M}_a$ be the limit of this sequence, that is $\mathfrak{M}_a = \bigcup M_i$, the labelling of $n-1$ tuples of nodes by atoms, and hyperedges by hyperlabels done in the obvious way using the fact that the M_is are nested. Let L be the signature with one n-ary relation for each $b \in \alpha$, and one k-ary predicate symbol for each k-ary hyperlabel λ. *Now we work in $L_{\infty,\omega}$*. For fixed $f_a \in {}^\omega\mathsf{nodes}(\mathfrak{M}_a)$, let $\mathfrak{U}_a = \{f \in {}^\omega\mathsf{nodes}(\mathfrak{M}_a) : \{i < \omega : g(i) \neq f_a(i)\}$ is finite$\}$. We make $\mathfrak{U}_a$ into the base of an L relativized structure $\mathsf{CA}lM_a$ like in [4, Theorem 29] except that we allow a clause for infinitary disjunctions. In more detail, for $b \in \alpha$, $l_0, \ldots, l_{n-1}, i_0 \ldots, i_{k-1} < \omega$, k-ary hyperlabels λ, and all L-formulas ϕ, ϕ_i, ψ, and $f \in U_a$:

$$\begin{aligned}
\mathsf{CA}lM_a, f \models b(x_{l_0} \ldots, x_{l_{n-1}}) &\iff \mathsf{CA}lM_a(f(l_0), \ldots, f(l_{n-1})) = b,\\
\mathsf{CA}lM_a, f \models \lambda(x_{i_0}, \ldots, x_{i_{k-1}}) &\iff \mathsf{CA}lM_a(f(i_0), \ldots, f(i_{k-1})) = \lambda,\\
\mathsf{CA}lM_a, f \models \neg\phi &\iff \mathsf{CA}lM_a, f \not\models \phi,\\
\mathsf{CA}lM_a, f \models (\bigvee_{i\in I} \phi_i) &\iff (\exists i \in I)(\mathsf{CA}lM_a, f \models \phi_i),\\
\mathsf{CA}lM_a, f \models \exists x_i\phi &\iff \mathsf{CA}lM_a, f[i/m] \models \phi,\\
&\qquad \text{some } m \in \mathsf{nodes}(\mathsf{CA}lM_a).
\end{aligned}$$

For any such L-formula ϕ, write $\phi^{\mathsf{CA}lM_a}$ for $\{f \in \mathfrak{U}_a : \mathsf{CA}lM_a, f \models \phi\}$. Let $D_a = \{\phi^{\mathsf{CA}lM_a} : \phi \text{ is an } L\text{-formula}\}$ and $\mathfrak{D}_a$ be the weak set algebra with universe D_a. Let $\mathfrak{D} = \mathbf{P}_{a\in\alpha}\mathfrak{D}_a$. Then $\mathfrak{D}$ is a generalized *complete* weak set algebra [3, Definition 3.1.2 (iv)]. Now we show that $\alpha \cong \mathsf{At}\mathfrak{Nr}_n\mathfrak{D}$ and $\mathfrak{Cm}\alpha \cong \mathfrak{Nr}_n\mathfrak{D}$. Let $x \in \mathfrak{D}$. Then $x = (x_a : a \in \alpha)$, where $x_a \in \mathfrak{D}_a$. For $b \in \alpha$ let $\pi_b : \mathfrak{D} \to \mathfrak{D}_b$ be the projection map defined by $\pi_b(x_a : a \in \alpha) = x_b$. Conversely, let $\iota_a : \mathfrak{D}_a \to \mathfrak{D}$ be the embedding defined by $\iota_a(y) = (x_b : b \in \alpha)$, where $x_a = y$ and $x_b = 0$ for $b \neq a$. Suppose $x \in \mathfrak{Nr}_n\mathfrak{D}\setminus\{0\}$. Since $x \neq 0$, then it has a non-zero component $\pi_a(x) \in \mathfrak{D}_a$, for some $a \in \alpha$. Assume that $\emptyset \neq \phi(x_{i_0}, \ldots, x_{i_{k-1}})^{\mathfrak{D}_a} = \pi_a(x)$, for some L-formula $\phi(x_{i_0}, \ldots, x_{i_{k-1}})$. We have $\phi(x_{i_0}, \ldots, x_{i_{k-1}})^{\mathfrak{D}_a} \in \mathfrak{Nr}_n\mathfrak{D}_a$. Pick $f \in \phi(x_{i_0}, \ldots, x_{i_{k-1}})^{\mathfrak{D}_a}$ and assume that $\mathsf{CA}lM_a, f \models b(x_0, \ldots x_{n-1})$ for some $b \in \alpha$. We show that $b(x_0, x_1, \ldots, x_{n-1})^{\mathfrak{D}_a} \subseteq \phi(x_{i_0}, \ldots, x_{i_{k-1}})^{\mathfrak{D}_a}$. Take any $g \in b(x_0, x_1 \ldots, x_{n-1})^{\mathfrak{D}_a}$, so that $\mathsf{CA}lM_a, g \models b(x_0, \ldots x_{n-1})$. The map $\{(f(i), g(i)) : i < n\}$ is a partial isomorphism of $\mathsf{CA}lM_a$. Here that short hyperedges are constantly labelled by λ is used. This map extends to a finite partial isomorphism θ of M_a whose domain includes $f(i_0), \ldots, f(i_{k-1})$. Let $g' \in \mathsf{CA}lM_a$ be defined by

$$g'(i) = \begin{cases} \theta(i) & \text{if } i \in \mathsf{dom}(\theta) \\ g(i) & \text{otherwise} \end{cases}$$

We have $\mathsf{CA}lM_a, g' \models \phi(x_{i_0}, \ldots, x_{i_{k-1}})$. But $g'(0) = \theta(0) = g(0)$ and similarly $g'(n-1) = g(n-1)$, so g is identical to g' over n and it differs from g' on only a finite set. Since $\phi(x_{i_0}, \ldots, x_{i_{k-1}})^{\mathfrak{D}_a} \in \mathfrak{Nr}_n\mathfrak{D}_a$, we get that $\mathsf{CA}lM_a, g \models \phi(x_{i_0}, \ldots, x_{i_k})$, so $g \in \phi(x_{i_0}, \ldots, x_{i_{k-1}})^{\mathfrak{D}_a}$ (this can be proved by induction on quantifier depth of formulas). This proves that

$$b(x_0, x_1 \ldots x_{n-1})^{\mathfrak{D}_a} \subseteq \phi(x_{i_0}, \ldots, x_{i_k})^{\mathfrak{D}_a} = \pi_a(x),$$

and so

$$\iota_a(b(x_0, x_1, \ldots x_{n-1})^{\mathfrak{D}_a}) \leq \iota_a(\phi(x_{i_0}, \ldots, x_{i_{k-1}})^{\mathfrak{D}_a}) \leq x \in \mathfrak{D}_a \setminus \{0\}.$$

Now every non-zero element x of $\mathfrak{Nr}_n\mathfrak{D}_a$ is above a non-zero element of the following form $\iota_a(b(x_0, x_1, \ldots, x_{n-1})^{\mathfrak{D}_a})$ (some $a, b \in \alpha$) and these are the atoms of $\mathfrak{Nr}_n\mathfrak{D}_a$. The map defined via $b \mapsto (b(x_0, x_1, \ldots, x_{n-1})^{\mathfrak{D}_a} : a \in \alpha)$ is an isomorphism of atom structures, so that $\alpha \in \mathsf{AtNr}_n\mathsf{CA}_\omega$. Let $X \subseteq \mathfrak{Nr}_n\mathfrak{D}$. Then by completeness of $\mathfrak{D}$, we get that $d = \sum^{\mathfrak{D}} X$ exists. Assume that $i \notin n$, then $\mathsf{c}_i d = \mathsf{c}_i \sum X = \sum_{x\in X} \mathsf{c}_i x = \sum X = d$, because the c_is are completely additive and $\mathsf{c}_i x = x$, for all $i \notin n$, since $x \in \mathfrak{Nr}_n\mathfrak{D}$. We conclude that $d \in \mathfrak{Nr}_n\mathfrak{D}$, hence d is an upper bound of X in $\mathfrak{Nr}_n\mathfrak{D}$. Since $d = \sum_{x\in X}^{\mathfrak{D}} X$ there can be no $b \in \mathfrak{Nr}_n\mathfrak{D}$ $(\subseteq \mathfrak{D})$ with $b < d$ such that b is an upper bound of X for else it will be an upper bound of X in $\mathfrak{D}$. Thus $\sum_{x\in X}^{\mathfrak{Nr}_n\mathfrak{D}} X = d$ We have shown that $\mathfrak{Nr}_n\mathfrak{D}$ is complete. Making the legitimate identification $\mathfrak{Nr}_n\mathfrak{D} \subseteq_d \mathfrak{Cm}\alpha$ by density, we get that $\mathfrak{Nr}_n\mathfrak{D} = \mathfrak{Cm}\alpha$ (since $\mathfrak{Nr}_n\mathfrak{D}$ is complete), hence $\mathfrak{Cm}\alpha \in \mathsf{Nr}_n\mathsf{CA}_\omega$.

Finally, to show that any atomic algebra having atom structure α is completely representable one can reason in one of the two following ways:

One: The game $\mathbf{H}$ is stronger than G and a winning strategyof $\exists$ in $G(\alpha)$ implies that the atom structure α is completely representable, hence any atomic algebra having the atom structure α will be completely representable.

Two: The complex algebra $\mathfrak{Cm}\alpha$ has countably many atoms and is in $\mathsf{Nr}_n\mathsf{CA}_\omega$, so by the third item of theorem 1.4 it is completely representable. Thus, any atomic algebra $\mathfrak{F}$ sharing the atom structure α is also completely representable.

(c) Applying H to a rainbow-like atom structure; excluding first order definability of classes between $\mathbf{S_d}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n$ and $\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_{n+3}$ We apply the new game $\mathbf{H}$ to the rainbow algebra $\mathfrak{C}_{\mathbb{Z},\mathfrak{N}}$ based on the ordered structures $\mathbb{Z}$ and $\mathfrak{N}$. The reds R are the set $\{\mathsf{r}_{ij} : i < j < \omega(= \mathfrak{N})\}$ and the green colours used constitute the set $\{\mathsf{g}_i : 1 \leq i < n-1\} \cup \{\mathsf{g}_0^i : i \in \mathbb{Z}\}$. In complete coloured graphs the forbidden triples are like the usual rainbow constructions based on $\mathbb{Z}$ and $\mathfrak{N}$, but we add a forbidden triple in coloured graphs. The triple $(\mathsf{g}_0^i, \mathsf{g}_0^j, \mathsf{r}_{kl})$ is *forbidden* if $\{(i,k),(j,l)\}$ is not an order preserving partial function from $\mathbb{Z} \to \mathfrak{N}$. In [15], it is shown that $\mathfrak{C}_{\mathbb{Z},\mathfrak{N}} \equiv \mathfrak{B}$ for some countable $\mathfrak{B} \in \mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n$. This is proved by showing that $\exists$ has a winning strategy in $G_k(\mathsf{At}\mathfrak{C}_{\mathbb{Z},\mathfrak{N}})$ for all $k \in \omega$, hence using ultrapowers followed by an elementary chain argument (like the argument used in the proof of theorem 1.15), we get that $\mathfrak{C}_{\mathbb{Z},\mathfrak{N}} \equiv \mathfrak{B}$, and $\exists$ has a winning strategy in $G_\omega(\mathsf{At}\mathfrak{B})$, hence by [7, Theorem 3.3.3] $\mathfrak{B} \in \mathsf{CRCA}_n \subseteq \mathbf{S_c}(\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At})$; the last inclusion follows from the first item of theorem 1.4. With some significantly more effort one can prove more: It can be shown that that $\exists$ can win the game $\mathbf{H}_k(\mathsf{At}\mathfrak{C}_{\mathbb{Z},\mathfrak{N}})$ which is the game $\mathbf{H}$ truncated to k rounds (on the same $\mathfrak{C}_{\mathbb{Z},\mathfrak{N}}$ based on $\mathbb{Z}$ and $\mathfrak{N}$) for all $k < \omega$. Recall that $\mathbf{H}$ is stronger than G hence $\mathbf{H}_k$ is stronger than G_k. Using ultrapowers followed by an elementary chain argument, it follows $\exists$

has a winning strategy in $\mathbf{H}(\alpha)$ for a countable atom structure α, such that $\mathfrak{C}_{\mathbb{Z},\mathfrak{N}} \equiv \mathfrak{Tm}\alpha$. We show that $\forall$ has a winning strategy in the graph version of the game $\mathbf{G}^{n+3}(\mathsf{At}\mathfrak{C}_{\mathbb{Z},\mathfrak{N}})$ played on coloured graphs [5]. The rough idea here, is that, as is the case with winning strategy's of $\forall$ in rainbow constructions, $\forall$ bombards $\exists$ with cones having distinct green tints demanding a red label from $\exists$ to appexes of succesive cones. The number of nodes are limited but $\forall$ has the option to re-use them, so this process will not end after finitely many rounds. The added order preserving condition relating two greens and a red, forces $\exists$ to choose red labels, one of whose indices form a decreasing sequence in $\mathfrak{N}$. In ω many rounds $\forall$ forces a win, so by the first item of lemma 1.3, $\mathfrak{C}_{\mathbb{Z},\mathfrak{N}} \notin \mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_{n+3}$. More rigorously, $\forall$ plays as follows: In the initial round $\forall$ plays a graph M with nodes $0, 1, \ldots, n-1$ such that $M(i,j) = \mathsf{w}_0$ for $i < j < n-1$ and $M(i, n-1) = \mathsf{g}_i$ $(i = 1, \ldots, n-2)$, $M(0, n-1) = \mathsf{g}_0^0$ and $M(0, 1, \ldots, n-2) = \mathsf{y}_{\mathbb{Z}}$. This is a 0 cone. In the following move $\forall$ chooses the base of the cone $(0, \ldots, n-2)$ and demands a node n with $M_2(i,n) = \mathsf{g}_i$ $(i = 1, \ldots, n-2)$, and $M_2(0,n) = \mathsf{g}_0^{-1}$. $\exists$ must choose a label for the edge $(n+1, n)$ of M_2. It must be a red atom r_{mk}, $m, k \in \mathfrak{N}$. Since $-1 < 0$, then by the 'order preserving' condition we have $m < k$. In the next move $\forall$ plays the face $(0, \ldots, n-2)$ and demands a node $n+1$, with $M_3(i,n) = \mathsf{g}_i$ $(i = 1, \ldots, n-2)$, such that $M_3(0, n+2) = \mathsf{g}_0^{-2}$. Then $M_3(n+1, n)$ and $M_3(n+1, n-1)$ both being red, the indices must match. $M_3(n+1, n) = r_{lk}$ and $M_3(n+1, r-1) = r_{km}$ with $l < m \in \mathfrak{N}$. In the next round $\forall$ plays $(0, 1, \ldots n-2)$ and re-uses the node 2 such that $M_4(0,2) = \mathsf{g}_0^{-3}$. This time we have $M_4(n, n-1) = \mathsf{r}_{jl}$ for some $j < l < m \in \mathfrak{N}$. Continuing in this manner leads to a decreasing sequence in $\mathfrak{N}$. We have proved the required.

(d): Putting (a), (b), (c) together We get that $\mathfrak{C}_{\mathbb{Z},\mathfrak{N}} \equiv \mathfrak{Tm}\alpha$, where α is a countable atom structure, such that $\alpha \in \mathsf{At}(\mathsf{Nr}_n\mathsf{CA}_\omega)$, any atomic $\mathfrak{F} \in \mathsf{CA}_n$ having atom structure α is completely representable, and $\mathfrak{Cm}\alpha \in \mathsf{Nr}_n\mathsf{CA}_\omega$. So $\mathfrak{Tm}\alpha \subseteq_d \mathfrak{Cm}\alpha \in \mathsf{Nr}_n\mathsf{CA}_\omega$, $\mathfrak{Tm}\alpha \in \mathsf{CRCA}_n$ and $\mathfrak{C}_{\mathbb{Z},\mathfrak{N}} \notin \mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_{n+3}$. Let K be any class between $\mathbf{S_d}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n$ and $\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_{n+3}$. Then $\mathfrak{C}_{\mathbb{Z},\mathfrak{N}} \notin \mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_{n+3} \supseteq \mathsf{K}$. But $\mathfrak{C}_{\mathbb{Z},\mathfrak{N}} \equiv \mathfrak{Tm}\alpha$, and $\mathfrak{Tm}\alpha \subseteq_d \mathfrak{Cm}\alpha \in \mathsf{Nr}_n\mathsf{CA}_\omega$, and $\mathfrak{Tm}\alpha \in \mathsf{CRCA}_n$, so $\mathfrak{Tm}\alpha \in \mathbf{S_d}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n \subseteq \mathsf{K}$. We have shown that $\mathfrak{Tm}\alpha \equiv \mathfrak{C}_{\mathbb{Z},\mathfrak{N}}$, $\mathfrak{Tm}\alpha \in \mathsf{K}$ but $\mathfrak{C}_{\mathbb{Z},\mathfrak{N}} \notin \mathsf{K}$, and we are done.[6]

[6] Let $m > n$. It is easy to show that if $\mathfrak{D} \in \mathsf{CA}_n$ and $\mathsf{At}\mathfrak{D} \in \mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_m$, then $\mathfrak{D} \in \mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_m$. Since $\alpha \in \mathsf{At}(\mathsf{Nr}_n\mathsf{CA}_\omega)$, by the (contrapositive of the) above obser-

We have also proved that any K between $\mathsf{AtNr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n$ and $\mathsf{At}\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_{n+3}$ is not elementary, because $\alpha \equiv \mathsf{At}\mathfrak{C}_{\mathbb{Z},\mathfrak{N}}$, $\alpha \in \mathsf{At}(\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n)$ but $\mathsf{At}\mathfrak{C}_{\mathbb{Z},\mathfrak{N}} \notin \mathsf{At}(\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_{n+3})$ lest $\mathfrak{C}_{\mathbb{Z},\mathfrak{N}} \in \mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_{n+3}$.[7] $\square$

Remark 3.2. In forming the required ω-dilation $\mathfrak{D}$ we made use of the 'stronger part' of the game $\mathbf{H}$, involving the amalgamation moves on λ-neat hypernetworks, where λ is the constant hyperlabel kept on short hypernetworks to build the ω-dilation $\mathfrak{D}$ which is a *generalized weak set algebra* of dimension ω, that is a set algebra, whose top element is a disjoint union of *weak spaces of dimension* ω; any such weak space is a set of sequences that agree co-finitely with sequences in ${}^\omega U$ (some non-empty set U). This ω-dilation $\mathfrak{D}$ can be (and was) described in a model theoretic framework. Using $\exists$'s winning strategy in $\mathbf{H}$, one builds an ω-dilation $\mathfrak{D}_a$ of $\mathfrak{Tm}\alpha$ for every $a \in \alpha$, based on a structure M_a in some signature specified above. Strictly speaking, M_a is a *weak model* [13, Definition 3.2.1], where assignments are required to agree co-finitely with a fixed sequence in ${}^\omega\mathsf{M}_a$. Thus $\mathfrak{D}_a$ is a weak set algebra of dimension n with base M_a This weak model M_a was taken in a signature L consisting of one n-ary relation for each $b \in \alpha$ and a k-ary relation symbol for each hyperedge of length k labelled by λ.

For $a \in \alpha$, the weak model M_a is the limit of the play $\mathbf{H}_\omega$; in the sense that M_a is the union of the λ-neat hypernetworks on α played during the game $\mathbf{H}_\omega$, with starting point the initial atom a that $\forall$ chose in the first move. Labels for the edges and hyperedges in M_a were defined the obvious way, inherited from the λ-neat hypernetworks played during the game; these are nested so this labelling is well defined, giving an interpretation of only the atomic formulas of L in M_a.

However, there is some freedom here in 'completing' the interpretation. One can use *any extension* $\mathfrak{L}$, *not necessarily a proper one, of* $L_{\omega,\omega}$ as a vehicle for constructing $\mathfrak{D}_a$. The algebra $\mathfrak{D}_a$ constructed above was a weak

vation, $\mathsf{At}\mathfrak{C}_{\mathbb{Z},\mathfrak{N}} \notin \mathsf{At}(\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_{n+3})$, and $\alpha \equiv \mathsf{At}\mathfrak{C}_{\mathbb{Z},\mathfrak{N}}$ because an atom structure of an atomic algebra is interpretable in the algebra, then we have already proved the required. However, if $\mathsf{At}\mathfrak{D} \in \mathsf{At}(\mathsf{Nr}_n\mathsf{CA}_m)$ for some $\mathfrak{D} \in \mathsf{CA}_m$ and some $m > n$ does not imply that $\mathfrak{D} \in \mathsf{Nr}_n\mathsf{CA}_m$, even if the Dedekind–MacNeille completion of $\mathfrak{D}$ is in $\mathsf{Nr}_n\mathsf{CA}_m$, cf. the last item of Theorem 2.2.

[7]There is subtle distinction between $\mathsf{Nr}_n\mathsf{CA}_m$ and the larger $\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_m$ for $1 < n < m \leq \omega$ that we should point out and that is the following: While if $\mathsf{At}\mathfrak{A} \in \mathsf{AtNr}_n\mathsf{K}_m$ *this does not imply that* $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_m$; but on the contrary if $\mathsf{At}\mathfrak{A} \in \mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_m$, then $\mathfrak{A} \in \mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_m$.

set algebra of dimension ω consisting of $\mathfrak{L}$-formulas taken in the signature L. The base of $\mathfrak{D}_a$ is M_a, and the set-theoretic operations of $\mathfrak{D}_a$ are read off the semantics of the connectives avialable in $\mathfrak{L}$. In all cases, as long as $\mathfrak{L}$ contains $L_{\omega,\omega}$ as a fragment, we get that $\mathfrak{Tm}\alpha \subseteq \mathsf{Nr}_n\mathfrak{D}$, where $\mathfrak{D} = \mathbf{P}_{a\in\alpha}\mathfrak{D}_a$. There are three possibilites measuring 'how close' $\mathfrak{Tm}\alpha$ is to $\mathsf{Nr}_n\mathfrak{D}$. We go from the closest to the less close. Either (a) $\mathfrak{Tm}\alpha = \mathsf{Nr}_n\mathfrak{D}$ or (b) $\mathfrak{Tm}\alpha \subseteq_d \mathsf{Nr}_n\mathfrak{D}$ or (c) $\mathfrak{Tm}\alpha \subseteq_c \mathsf{Nr}_n\mathfrak{D}$. It is reasonable to expect that the stronger (the logic) $\mathfrak{L}$ is, the 'more control' α has over the hitherto obtained ω-dilation $\mathfrak{D}$; the closer $\mathfrak{Tm}\alpha$ is to the neat n-reduct of $\mathfrak{D}$ based on $\mathfrak{L}$-formulas.

Suppose we take $\mathfrak{L} = L_{\omega,\omega}$. Then using the fact that in the λ-neat hypernetworks played during the game $\mathbf{H}$ short hyperedges are constantly labelled by λ, one shows that $\alpha \cong \mathsf{AtNr}_n\mathfrak{D}$; the isomorphism defined via $b \mapsto (b^{\mathfrak{D}_a}(x_0, \ldots x_{n-1}) : a \in \alpha)$. But using $\mathfrak{L} = L_{\infty,\omega}$ in the same signature, the resulting algebra $\mathfrak{D}$ which is isomorphic to a generalized ω-dimensionl weak set algebra in the sense of [3, Definition 3.1.2 (iv)] (with top element the disjoint union of top elements of the $\mathfrak{D}_a$) based on the (now) $L_{\infty,\omega}$ weak models M_a taken in the same signature L, $a \in \alpha$, will be *complete*. This is so, because the $\mathfrak{D}_a$s are complete; $\sum_{i\in I}^{\mathfrak{D}_a} \phi_i^{\mathfrak{D}_a} = (\bigvee_{i\in I} \phi_i)^{\mathfrak{D}_a}$. Here $\phi^{\mathfrak{D}_a}$ is the set of all sequences s agreeing co-finitely with a fixed sequence in ${}^{\omega}\mathsf{M}_a$ such that $\mathsf{M}_a, s \models \phi$. So both $\mathfrak{D} = \mathbf{P}_{a\in\alpha}\mathfrak{D}_a$ and its n–neat reduct $\mathsf{Nr}_n\mathfrak{D}$ will be complete. Accordingly, one makes the identification $\mathsf{Nr}_n\mathfrak{D} \subseteq_d \mathfrak{Cm}\alpha$. By density, we get that $\mathsf{Nr}_n\mathfrak{D} = \mathfrak{Cm}\alpha$ (since $\mathsf{Nr}_n\mathfrak{D}$ is complete), hence $\mathfrak{Cm}\alpha \in \mathsf{Nr}_n\mathsf{CA}_\omega$ and so we get (b) (and (c)) since $\mathfrak{Tm}\alpha \subseteq_d \mathfrak{Cm}\alpha$. Also the property that $\alpha \cong \mathsf{AtNr}_n\mathfrak{D}$ is plainly maintained when we passed from $L_{\omega,\omega}$ to $L_{\infty,\omega}$.

For a class $\mathbf{K}$ of algebras, we denote by $\mathbf{K} \cap \mathsf{Count}$ the class of countable algebras in $\mathbf{K}$. Observe that the game $\mathbf{H}_\omega$ 'captures' the class $\mathsf{At}(\mathsf{Nr}_n\mathsf{CA}_\omega) \cap \mathsf{Count}$ in the sense that if α is a countable atom structure and $\exists$ has a winning strategy in $\mathbf{H}_\omega(\alpha)$, then $\alpha \in \mathsf{At}(\mathsf{Nr}_n\mathsf{CA}_\omega)$. Conversely, it can be proved that if $\alpha \in \mathsf{At}(\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{Count})$, then $\exists$ has winning strategy in a game with the same moves as $\mathbf{H}$ but played on networks not λ-neat hypernetworks. However, $\mathbf{H}_\omega$ does not characterize the class $\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At} \cap \mathsf{Count}$ for it can be shown that $\exists$ has a winning strategy in $\mathbf{H}_\omega(\mathsf{At}\mathfrak{B})$ where $\mathfrak{B}$ is the atomic algebra used in item (3) of Theorem 2.2, but $\mathfrak{B} \notin \mathsf{Nr}_n\mathsf{CA}_{n+1} (\supseteq \mathsf{Nr}_n\mathsf{CA}_\omega)$; though (recall that) $\mathsf{At}\mathfrak{B} \in \mathsf{At}(\mathsf{Nr}_n\mathsf{CA}_\omega)$ and $\mathfrak{Cm}\mathsf{At}\mathfrak{B} \in \mathsf{Nr}_n\mathsf{CA}_\omega$. On the other hand, the usual

ω-rounded atomic game G characterizes both the class $\mathsf{CRCA}_n \cap \mathsf{Count}$ and the class $\mathsf{At}(\mathsf{CRCA}_n \cap \mathsf{Count})$ (the class of countable completely representable atom structures), and [7, Theorem 3.3.3].

Corollary 3.3. For any $2 < n < m$, any class K such that

$$\mathsf{At}(\mathsf{Nr}_n\mathsf{CA}_m \cap \mathsf{CRCA}_n) \subseteq \mathsf{K} \subseteq \mathsf{At}\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_{n+3},$$

K is not elementary

Proof: . Let β be he atom structure of $\mathfrak{C}_{\mathbb{Z},\mathfrak{N}}$. Then $\beta \equiv \alpha$ where α is an atom structure such that $\mathfrak{Cm}\alpha \in \mathsf{Nr}_n\mathsf{CA}_\omega$ and $\alpha \in \mathsf{At}(\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n)$. So if K is as in the hypothesis, then $\alpha \in \mathsf{K}$, $\beta \equiv \alpha$, but $\beta \notin \mathsf{At}\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_{n+3} \supseteq \mathsf{K}$. $\square$

Corollary 3.4. Let $2 < n < \omega$ and $k \geq 3$. Then the following classes, together with the intersection of any two of them, the last four taken at the same k, are not elementary: CRCA_n [5], $\mathsf{Nr}_n\mathsf{CA}_{n+k}$ [14, Theorem 5.4.1], $\mathbf{S_d}\mathsf{Nr}_n\mathsf{CA}_{n+k}$, $\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_{n+k}$.

4. Appendix

Theorem 4.1. *Let $2 < m < n < \omega$. For any $k \geq 0$, the variety $\mathbf{S}\mathsf{Nr}_m\mathsf{CA}_{m+k+1}$ is not finitely axiomatizable over the variety $\mathbf{S}\mathsf{Nr}_m\mathsf{CA}_{m+k}$ and RCA_m is not finitely axiomatizable over $\mathbf{S}\mathsf{Nr}_m\mathsf{CA}_{m+l}$ for any $0 < l < \omega$.*

Proof: Fix $2 < m < n < \omega$. Let $\mathfrak{C}(m,n,r)$ be the algebra $\mathsf{CA}(\mathbf{H})$ where $\mathbf{H} = H_m^{n+1}(\mathfrak{A}(n,r),\omega))$, is the CA_m atom structure consisting of all $n+1$-wide m-dimensional wide ω hypernetworks [6, Definition 12.21] on $\mathfrak{A}(n,r)$ as defined in [6, Definition 15.2]. Furthermore, for any $r \in \omega$ and $3 \leq m \leq n < \omega$, $\mathfrak{C}(m,n,r) \in \mathsf{Nr}_m\mathsf{CA}_n$, $\mathfrak{C}(m,n,r) \notin \mathbf{S}\mathsf{Nr}_m\mathsf{CA}_{\mathsf{n}+1}$ and $\Pi_{r/U}\mathfrak{C}(m,n,r) \in \mathsf{RCA}_m$ by [6, Corollaries 15.7, 5.10, Exercise 2, p. 484, Remark 15.13] $\square$

Theorem 4.2. *For $3 \leq m \leq n$ and $r < \omega$ there exists finite algebras $\mathfrak{D}(m,n,r) \in \mathsf{CA}_m$.*

1. $\mathfrak{D}(m,n,r) \in \mathsf{Nr}_m\mathsf{CA}_\mathsf{n}$,
2. $\mathfrak{D}(m,n,r) \notin \mathbf{S}\mathsf{Nr}_m\mathsf{CA}_{n+1}$,
3. *$\Pi_{r/U}\mathfrak{D}(m,n,r)$ is elementarily equivalent to a $\mathfrak{C} \in \mathsf{Nr}_m\mathsf{CA}_{\mathsf{n}+1}$.*

We define the algebras $\mathfrak{D}(m,n,r)$ for $3 \le m \le n < \omega$ and r and then give a sketch of (II) given in detail in [9, pp. 211–215]. We start with.

DEFINITION 4.3. Define a function $\kappa : \omega \times \omega \to \omega$ by $\kappa(x, 0) = 0$ (all $x < \omega$) and $\kappa(x, y+1) = 1 + x \times \kappa(x, y))$ (all $x, y < \omega$). For $n, r < \omega$ let

$$\psi(n,r) = \kappa((n-1)r, (n-1)r) + 1.$$

This is to ensure that $\psi(n,r)$ is sufficiently big compared to n, r for the proof of non-embeddability to work. The second parameter $r < \omega$ may be considered as a finite linear order of length r. For any $n < \omega$ and any linear order r, let

$$\mathfrak{B}(n,r) = \{\mathsf{Id}\} \cup \{a^k(i,j) : i < n-1; j \in r,\ k < \psi(n,r)\}$$

where $\mathsf{Id}, a^k(i,j)$ are distinct objects indexed by k, i, j. (So here every atom $a(i,j)$ is split into $\psi(n,r)$ subatoms). The *forbidden* triples) are:

$$\{(\mathsf{Id}, b, c) : b \neq c \in \mathfrak{B}(n,r)\}$$
$$\cup$$
$$\{(a^k(i,j), a^{k'}(i,j), a^{k^*}(i,j')) : k, k', k^* < \psi(n,r),\ i < n-1,\ j' \le j \in r\}.$$

Let $3 \le m \le n < \omega$. The set of m-basic matrices on $\mathfrak{R}$ is is a QEA_m atom structure $\mathsf{Mat}_m(\mathsf{At}\mathfrak{R})$. $\mathfrak{D}(m,n,r)$ is defined to be the complex algebra of the m-dimensional atom structure $\mathsf{Mat}_m(\mathsf{At}\mathfrak{R})$, that is, $\mathfrak{D}(m,n,r) = \mathfrak{Cm}\mathsf{Mat}_m(\mathsf{At}\mathfrak{R})$. Unlike the algebras $\mathfrak{C}(m,n,r)$ used to prove theorem 4.1, the algebras $\mathfrak{D}(m,n,r)$ are now finite. It is not hard to see that $3 \le m$, $2 \le n$ and $r < \omega$ the algebra $\mathfrak{D}(m,n,r)$ satisfies all of the axioms defining CA_m except, perhaps, the commutativity of cylindrifiers which it satisfies because $\mathsf{Mat}_m(\mathsf{At}\mathfrak{R})$ is a (symmetric) cylindric basis, so that overlapping matrices amalgamate. Furthermore, if $3 \le m \le m'$, then $\mathfrak{D}(m,n,r) \cong \mathsf{Nr}_m\mathfrak{D}(m',n,r)$ via $X \mapsto \{f \in \mathsf{Mat}_{m'}(\mathsf{At}\mathfrak{R}) : f \restriction_{m \times m} \in X\}$.

We give a sketch of proof of 4.2(II), which is the heart and soul of the proof. Assume hoping for a contradiction that $\mathfrak{D}(m,n,r) \subseteq \mathsf{Nr}_m\mathfrak{C}$ for some $\mathfrak{C} \in \mathsf{CA}_{n+1}$, some finite m, n, r. Then for $1 \le t \le n+1$, it can be shown inductively that there must be a 'large set' S_t of distinct elements of $\mathfrak{C}$, satisfying certain inductive assumptions, which we outline next. Here largness depends on t and weakens as t increases; for example S_n has only two elements. For each $s \in S_t$ and $i, j < n+2$ there is an element $\alpha(s,i,j) \in \mathfrak{B}(n,r)$ obtained from s by cylindrifying all dimensions in $(n+1)\setminus\{i,j\}$, then using substitutions to replace i,j by $0,1$. It can be shown that the triple $(\alpha(s,i,j),\alpha(s,j,k),\alpha(s,i,k))$ is consistent (not forbidden). The induction hypothesis says chiefly that $\mathsf{c}_n s$ is constant, for $s\in S_t$, and for $l<n$ there are fixed $i<n-1$, $j<r$ such that for all $s\in S_t$, $\alpha(s,l,n)\leq a(i,j)$. This defines, like in the proof of theorem 15.8 in [7, p. 471] , two functions $I:n\to(n-1)$, $J:n\to r$ such that $\alpha(s,l,n)\leq a(I(l),J(l))$ for all $s\in S_t$. The *rank* $\mathsf{rk}(I,J)$ of (I,J) (as defined in [7, Definition 15.9]) is the sum (over $i<n-1$) of the maximum j with $I(l)=i$, $J(l)=j$ (some $l<n$) or -1 if there is no such j. From S_t one constructs a set S_{t+1} with index functions (I',J'), still relatively large (large in terms of the number of times we need to repeat the induction step) where the same induction hypotheses hold but where $\mathsf{rk}(I',J')>\mathsf{rk}(I,J)$. By repeating this enough times (more than nr times) we obtain a non-empty set T with index functions of rank strictly greater than $(n-1)\times(r-1)$, an impossibility. We sketch the induction step. Since I cannot be injective there must be distinct $l_1,l_2<n$ such that $I(l_1)=I(l_2)$ and $J(l_1)\leq J(l_2)$. We may use l_1 as a "spare dimension" (changing the index functions on l will not reduce the rank). Since $\mathsf{c}_n s$ is constant, we may fix $s_0\in S_{t-1}$ and choose a new element s' below $\mathsf{c}_l s_0\cdot\mathsf{s}_l^n\mathsf{c}_l s$, with certain properties. Let $S_{t+1}=\{s':s\in S_t\setminus\{s_0\}\}$. Re-establishing many of the induction hypotheses for S_{t+1} is not too hard. Also, it can be shown that $J'(l)\geq J(l)$ for all $l<n$. Since $(\alpha(s,i,j),\alpha(s,j,k),\alpha(s,i,k))$ is consistent and by the definition of the forbiden triples either $\mathsf{rng}(I')$ properly extends $\mathsf{rng}(I)$ or there is $l<n$ such that $J'(l)>J(l)$, hence $\mathsf{rk}(I',J')>\mathsf{rk}(I,J)$. The idea of constructing S_{t+1} from S_t is given pictorially on [8, Figure 2, p. 8] in the context of CAs. The essence of the ideas used in [8, 9] is the same. Suppose we are at stage t. Then every $x\in S_t$ gives a set of colours (atoms) denoted in [8] by $x(i,t)$ $(i<t)$. One gets S_{t+1} from S_t by first 'glueing together' any two elements x,z of S_t, using $t+1$ as a spare dimension, first moving the tth co-ordinate of x to $t+1$ forming $\mathsf{s}_{t+1}^t x$. By fixing z and varying x one gets a huge number of different elements. Their $(t,t+1)$th colours cannot be controlled yet; they may not be the same. To get over this hurdle, one uses the pigeon-hole principal to pick the *still large set* S_{t+1} in which the $(t,t+1)$th colour is fixed to be the same. 'Largness' enables one to do so.

We summarize next the essence of the idea used in the solution of [3, Problem 2.12]:

In Figure 2 in [8] there is a top element that is connected by coloured edges to the intermediate elements that are all connected to a bottom element. The number of elements (in this figure) is the number of colours plus one. So one gets the same control as rainbow algebras provided by (the second independent parameter) G. *The key idea here is that the proof of Ramsey in this context does not require an uncontrollable Ramsey number of 'spare dimensions', which were the versions used by Monk and Maddux before proving non finite axiomatizability but only one more than the number of colours used.*

For the above non-representable Monk-style algebras denoted by $\mathfrak{A}(n, r)$, $3 \leq m < n < \omega$ and $r \in \omega$, it is easy to see that $\exists$ cannot win the usual infinite atomic game. But this time one can use 'a hyperbasis game' denoted by $G_r^{m,n+1}$ in [6] with r denoting the number of rounds, to pin point the leask $k > n$ for which $\mathfrak{A}(n, r)$ 'stops to be representable' getting the sharper result we want. The game $G_r^{m,n+1}$ is stronger than G_ω, involving additional amalgamation moves played on $n+1$-dimensional m-wide hypernetworks. One can show that $\forall$ has a winning strategy in $G_r^{m,n+1}(\mathsf{At}\mathfrak{A}(n, r))$, using exactly $n + 1$ nodes (for any $r < \omega$), getting the same control we get from rainbows using the parameter G, and in fact the best possible. This is the approach adopted in [7]. Here $\mathfrak{A}(n, r)$ has an n-dimensional cylindric basis, but no $n+1$-*dimensional hypebasis*. Worthy of note, is that the last condition is strictly stronger than 'not having an $n+1$-dimensional cylindric basis'. Relation algebras having n-dimensional cylindric basis but no $n + 1$-dimensional cylindric basis were constructed by Maddux. We refer to [8] for more. In the proof of theorem 4.1, one uses that $\Pi_{r/U}\mathfrak{C}(m, n, r) \in \mathsf{RCA}_m$. As stated in the last item of theorem 4.2, we do not guarantee that the ultraproduct on r of the $\mathfrak{D}(m, n, r)$s $(2 < m < n < \omega)$ is representable. A standard Lös argument shows that $\Pi_{r/U}\mathfrak{C}(m, n, r) \cong \mathfrak{C}(m, n, \Pi_{r/U}r)$ and $\Pi_{r/U}r$ contains an infinite ascending sequence. Here one extends the definition of ψ by letting $\psi(n, r) = \omega$, for any infinite linear order r. The infinite algebra $\mathfrak{D}(m, n, J) \in \mathbf{El}\mathsf{Nr}_n\mathsf{CA}_{n+1}$ when J is the infinite linear order as above. Since $\Pi_{r/U}r$ is such, then we get $\Pi_{r/U}\mathfrak{D}(m, n, r) \in \mathbf{El}\mathsf{Nr}_m\mathsf{CA}_{n+1}(\subseteq \mathbf{S}\mathsf{Nr}_m\mathsf{CA}_{n+1})$, cf. [9, pp. 216–217]. This suffices to show that for any positive k, the variety $\mathbf{S}\mathsf{Nr}_m\mathsf{CA}_{m+k+1}$ is not finitely axiomatizable over the variety $\mathbf{S}\mathsf{Nr}_m\mathsf{CA}_{m+k}$.

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