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Theofanis Aravanis

AN EPISTEMOLOGICAL STUDY OF THEORY CHANGE

Abstract

Belief Revision is a well-established field of research that deals with how agents rationally change their minds in the face of new information. The milestone of Belief Revision is a general and versatile formal framework introduced by Alchourrón, Gärdenfors and Makinson, known as the AGM paradigm, which has been, to this date, the dominant model within the field. A main shortcoming of the AGM paradigm, as originally proposed, is its lack of any guidelines for relevant change. To remedy this weakness, Parikh proposed a relevance-sensitive axiom, which applies on splittable theories; i.e., theories that can be divided into syntax-disjoint compartments. The aim of this article is to provide an epistemological interpretation of the dynamics (revision) of splittable theories, from the perspective of Kuhn's influential work on the evolution of scientific knowledge, through the consideration of principal belief-change scenarios. The whole study establishes a conceptual bridge between rational belief revision and traditional philosophy of science, which sheds light on the application of formal epistemological tools on the dynamics of knowledge.

Keywords: Belief revision, epistemology, Parikh, relevance, Kuhn, scientific knowledge.

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1. Introduction

A well-established research field that lies at the intersection of Formal Philosophy and Computer Science, and deals with how agents *rationally*

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change their minds in the face of new information, is that of *Belief Revision* [14, 24, 13].[1] Roughly speaking, the process of belief revision can be outlined as follows [14, 24]:

- A rational agent receives *new* information (*epistemic input*).
- In the principal case where the new information *contradicts* her initial beliefs, the agent needs to withdraw some of the old beliefs before she can (consistently) accommodate the new information.
- The agent is, also, obliged to accept the *consequences* that might result from the interaction of the new information with the (remaining) old beliefs.

What makes the problem non-trivial is that several different ways for performing the revision-process may be possible. Suppose, for example, that the beliefs of a rational agent are composed of the following three propositions:

i) All African lions are brown.

ii) The animal Bob encounters is a lion.

iii) The animal Bob encounters comes from Africa.

Along with the above three propositions, the agent is obliged to believe their following immediate consequence:

iv) The animal Bob encounters is brown.

Suppose, now, that the animal Bob encounters turns out to be white. In order for the agent to maintain a *consistent* corpus of beliefs after adding the fact about lion's whiteness, she needs to *revise* her initial beliefs. That is to say, some of her original beliefs must be withdrawn. Clearly, she does not want to give up *all* her beliefs, since this would be an unnecessary loss of valuable information. It is not hard to verify that, in the case described above, there are at least three different ways for performing revision. In general, this can be done in a number of ways. More importantly, the problem of belief revision is that logical considerations alone are not sufficient

[1] To distinguish the research area from the process, we shall use the capitalized term *Belief Revision* for the former, and the lower-case term *belief revision* for the latter.

for choosing which beliefs have to be given up; this has to be decided by means of *extra-logical* structures.

The benchmark of Belief Revision is a general and versatile formal framework introduced by Alchourrón, Gärdenfors and Makinson, known as the *AGM paradigm* (after the initials of its originators), which has been, to this date, the dominant model within the field [1]. The AGM paradigm captures both *axiomatically* and *constructively* the process of *rational* belief revision. Axiomatically, by means of rationality postulates that any rational revision function ought to satisfy, and constructively, by means of extra-logical structures based on preference orderings.

A main shortcoming of the AGM paradigm, as originally introduced, is its insufficiency to capture the notion of *relevance*. To remedy this weakness, Parikh proposed a postulate that supplements the approach of Alchourrón, Gärdenfors and Makinson; the postulate essentially captures a form of *syntactical relevance*, and is typically referred to as axiom (P) [23]. Roughly speaking, axiom (P) states that the revision of a *splittable* theory K—i.e., a theory that can be divided into *syntax-disjoint* compartments referring to mutually *irrelevant* subject matters—by an epistemic input which (syntactically) relates *only* to some compartment of K, should *not affect* any other compartment of K. A central concept that Parikh used for developing his axiom is that of *theory-splitting* [23].

In the present article, we discuss the revision of splittable theories, through the prism of the influential 1970 book "*The Structure of Scientific Revolutions*" by Thomas Kuhn, which studies the *evolution of scientific knowledge* [21]. The task is accomplished by considering a Kuhnian reading of principal belief-change scenarios involving Parikh's concept of theory-splitting; essentially, we argue that these scenarios can be regarded as a reflection of notable phases of scientific development. Our aim is the establishment of a conceptual bridge between rational belief revision and traditional philosophy of science, which will shed light on the application of formal epistemological tools on the dynamics of (any corpus of) knowledge.[2]

The remainder of this article is structured as follows. The next section provides a brief overview of Kuhn's work on philosophy of science. Thereafter, Section 3 sets the formal background for our discussion, followed by

[2]For an excellent book that discusses a variety of perspectives concerning belief revision in the context of scientific enquiry, the interested reader is referred to [22].

Section 4 which presents the AGM paradigm. Section 5 introduces Parikh's notion of relevance, as well as some fundamental definitions based on this notion. Section 6 discusses the revision of splittable theories from an epistemological perspective, and Section 7 reports a collection of representative examples from the history of science that justify the conducted study. A brief concluding section closes the article.

2. Kuhnian epistemology in a nutshell

According to Kuhn, the phases of scientific progress can be summarized as follows [21]:

- Pre-paradigm period.
- Normal science.
- Crisis.
- Scientific revolution.
- New normal science.
- New crisis.

Prior to the formation of a shared paradigm or research consensus (pre-paradigm period), would-be scientists are devoted to the accumulation of random facts and unverified observations. This non-organized activity acquires coherence once the scientific community adopts a unique shared *paradigm*. A paradigm is a distinct set of concepts or thought patterns (including theories and research methodologies), that constitute legitimate contributions to a field. The paradigm is adopted jointly by the members of a scientific community, it establishes the (nature of the) entities of the world (i.e., an *ontological acceptance*), as well as a common language.

Everyone working within a settled paradigm (e.g., Classical Mechanics) is doing *normal science*. In the context of normal science—which Kuhn describes as a puzzle-solving process—scientists are slowly accumulating details in accord with an established broad theory, making the paradigm more coherent and concrete. During this process, the scientific community does not question or challenge the underlying (philosophical and metaphysical) assumptions of the theory.

During the period of normal science, the observable new information (epistemic input) is formulated *within* the language of a particular theory, and its validity depends on the validity of the corresponding theoretical or conceptual context. In this sense, *theories precede observations*. Theories can be formulated—and this is usually the case—prior to the observations that contribute to their justification. Generally, the meaning of a concept is (at least partially) "sculptured" from the role it plays in a theory.

For the normal scientist, anomalies represent challenges to be puzzled out and solved *within* the settled paradigm. At the point where such anomalies cannot be handled within the paradigm, a *crisis* emerges. In case an anomaly (or series of anomalies) persists long enough, and for enough members of the scientific community, the paradigm will itself gradually come under challenge, and perhaps be subjected to a *paradigm shift*, a process often also described as a *scientific revolution*. After the scientific revolution, a new paradigm is established by the majority of the community, which takes the place of the old problematic one, and a period of new normal science begins.[3] It is noteworthy that, according to Kuhn, the language and theories of successive paradigms are *incommensurable*, in the sense that, in principle, they cannot be translated into one another, or rationally evaluated against one another, by means of a formal framework. Kuhn argues that incommensurability constitutes a universal property of scientific revolutions.

Having presented the core principles of Kuhn's work on the evolution of scientific knowledge, we turn to a more analytical tone.

3. Formal background

Throughout this article, we shall be working with a propositional language $\mathcal{L}$, built over a *finite*, non-empty set $\mathcal{P}$ of propositional variables (atoms), using the standard Boolean connectives $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (implication), $\leftrightarrow$ (equivalence), $\neg$ (negation), and governed by *classical propositional logic*.[4] This abstraction is made, mainly, due to the fact that the majority of belief-revision studies, including Parikh's exposition in [23],

[3] Two representative cases of paradigm shifts are the transition from a Ptolemaic Cosmology to a Copernican one, as well as the acceptance of plate-tectonics theory (replacing the idea of continental drift) as an explanation for large-scale geological transformations.

[4] Notice that no zero-ary connectives are considered.

are conducted assuming classical propositional logic, a fact which, in turn, provides easier presentation. Another argument in favour of this convenient formalism, is that, in the context of *Answer Set Programming* (ASP) [11], which constitutes a contemporary formal framework used for modelling the dynamics of a plethora of real-world scientific domains, although a particular scenario is modelled in the syntax of first-order logic, the system ultimately solves a finite *propositional* representation of it (produced through a sophisticated process called *grounding*). We note, lastly, that, even if the essence of our approach easily extends to richer formalisms, a formal account of this issue would be an interesting avenue for future investigation.

A sentence of $\mathcal{L}$ is *contingent* iff it is neither a tautology nor a contradiction. For a set of sentences Γ of $\mathcal{L}$, $Cn(\Gamma)$ denotes the set of all logical consequences of Γ, i.e.,

$$Cn(\Gamma) = \big\{\varphi \in \mathcal{L} : \Gamma \models \varphi\big\},$$

where $\models$ stands for the classical consequence relation. We shall write $Cn(\varphi_1, \ldots, \varphi_n)$ for sentences $\varphi_1, \ldots, \varphi_n$, as an abbreviation of $Cn\big(\{\varphi_1, \ldots, \varphi_n\}\big)$.

An agent's belief corpus shall be modelled by a *theory*, also referred to as a *belief set*. A theory K is any deductively closed set of sentences of $\mathcal{L}$; i.e.,

$$K = Cn(K).$$

The set of all theories is denoted by $\mathbb{K}$. A theory K is *complete* iff, for all sentences $\varphi \in \mathcal{L}$, either $\varphi \in K$ or $\neg\varphi \in K$. For a theory K and a sentence φ of $\mathcal{L}$, the *expansion* of K by φ, denoted by $K + \varphi$, is the deductive closure of the set $K \cup \{\varphi\}$, i.e.,

$$K + \varphi = Cn\big(K \cup \{\varphi\}\big).$$

A *literal* is a propositional variable or its negation. A *possible world* (or, simply, *world*) r is any consistent set of literals, such that, for any propositional variable $p \in \mathcal{P}$, either $p \in r$ or $\neg p \in r$. The set of all possible worlds is denoted by $\mathbb{M}$. For a sentence (or set of sentences) φ of $\mathcal{L}$, $[\varphi]$ is the set of worlds at which φ is true. For a set of worlds $V \subseteq \mathbb{M}$, we denote by $th(V)$ the set of all sentences satisfied by all worlds in V; if $V = \varnothing$, then vacuously $th(V) = \mathcal{L}$. It is not hard to verify that $th(V)$ is a (unique) theory.

4. The AGM paradigm

Within the AGM paradigm [1], the process of belief revision is modelled as a (binary) function $*$, which maps a theory K and a sentence φ to a revised (new) theory $K * \varphi$. *Rational* revision functions, the so-called *AGM revision functions*, respect the *AGM postulates for revision*, listed below.[5]

($\mathbf{K * 1}$) $K * \varphi$ is a theory of $\mathcal{L}$.

($\mathbf{K * 2}$) $\varphi \in K * \varphi$.

($\mathbf{K * 3}$) $K * \varphi \subseteq K + \varphi$.

($\mathbf{K * 4}$) If $\neg\varphi \notin K$, then $K + \varphi \subseteq K * \varphi$.

($\mathbf{K * 5}$) $K * \varphi$ is inconsistent iff φ is inconsistent.

($\mathbf{K * 6}$) If $Cn(\varphi) = Cn(\psi)$, then $K * \varphi = K * \psi$.

($\mathbf{K * 7}$) $K * (\varphi \wedge \psi) \subseteq (K * \varphi) + \psi$.

($\mathbf{K * 8}$) If $\neg\psi \notin K * \varphi$, then $(K * \varphi) + \psi \subseteq K * (\varphi \wedge \psi)$.

The AGM postulates for revision do not suffice to *uniquely* specify the revised belief set $K * \varphi$, given K and φ alone; they simply intend to circumscribe the territory of *all different rational ways* of revising belief sets. For a unique specification of $K * \varphi$, appropriate *extra-logical* tools are required, the so-called *constructive models* for belief revision, the first of which has already been proposed in the seminal work of Alchourrón, Gärdenfors and Makinson [1]. Herein, our focus is on a popular constructive model introduced by Katsuno and Mendelzon, which is based on a special kind of total preorders over possible worlds, called *faithful preorders* [18].

Before discussing the faithful-preorders model, we first recall that a *preorder* over a set V is any reflexive, transitive binary relation on V. A preorder $\preceq$ is called *total* iff, for all $r, r' \in V$, $r \preceq r'$ or $r' \preceq r$. As usual, the strict part of $\preceq$ shall be denoted by $\prec$; namely, $r \prec r'$ iff $r \preceq r'$ and $r' \not\preceq r$. Also, $min(V, \preceq)$ denotes the set of all $\preceq$-minimal elements of V; i.e.,

[5] A detailed discussion on the postulates $(K * 1)$–$(K * 8)$ can be found in [14, Section 3.3] or [24, Section 8.3.1]. It is, also, noteworthy that several concrete "off-the-shelf" revision operators that satisfy $(K * 1)$–$(K * 8)$ have been proposed in the literature; see, indicatively, [12, 26, 7, 4].

$$min(V, \preceq) = \Big\{ r \in V : \text{for all } r' \in V, \text{ if } r' \preceq r, \text{ then } r \preceq r' \Big\}.$$

Definition 4.1 (Faithful Preorder, [18]). A total preorder $\preceq_K$ over $\mathbb{M}$ is faithful to a theory K iff the $\preceq_K$-minimal worlds of $\mathbb{M}$ are those satisfying K; i.e., $min(\mathbb{M}, \preceq_K) = [K]$.

Intuitively, a faithful preorder $\preceq_K$ encodes the *comparative plausibility* of all possible worlds of $\mathbb{M}$, with respect to theory K, so that the more plausible a world is, the lower it appears in the ordering $\preceq_K$.

Definition 4.2 (Faithful Assignment, [18]). A faithful assignment is a function that maps each theory K of $\mathcal{L}$ to a total preorder $\preceq_K$ over $\mathbb{M}$, which is faithful to K.

Katsuno and Mendelzon proceed, then, to the following representation theorem.

Theorem 4.3 ([18]). *A revision operator $*$ satisfies postulates $(K*1)$–$(K*8)$ iff there exists a faithful assignment that maps each theory K of $\mathcal{L}$ to a total preorder $\preceq_K$ over $\mathbb{M}$, such that, for any sentence $\varphi \in \mathcal{L}$:*

$$\textbf{(F}*\textbf{)} \quad K * \varphi = th\Big(min([\varphi], \preceq_K)\Big).$$

Essentially, condition (F$*$) specifies the revised theory $K * \varphi$ as the theory corresponding to the most plausible (with respect to K) φ-worlds. For ease of presentation, we shall consider, in the course of this work, only the principal case of *consistent* belief sets and *contingent* epistemic input.

Gärdenfors and Makinson have introduced another well-known model for constructing AGM revision functions, equivalent to the model of Katsuno and Mendelzon, which is based on the notion of *epistemic entrenchment* [16]. A central aspect of Gärdenfors and Makinson's model is a particular type of total preorder over all beliefs of a theory K, called *epistemic-entrenchment preorder*, which encodes the relative epistemic values of all the sentences *in* K. An investigation by Peppas and Williams of the interconnections between the two aforementioned constructive models revealed that an epistemic-entrenchment preorder, associated with a theory K, suffices to *fully specify* a faithful preorder $\preceq_K$ [25, Theorem 6.3]. On that premises, and given that the definition of an epistemic-entrenchment preorder does *not* require the relative epistemic values of sentences *not belonging* to K, the following remark is true.

Remark 4.4. Let K be a theory of $\mathcal{L}$, and let $\preceq_K$ be a total preorder faithful to K. A rational agent does *not* need to explicitly provide the relative epistemic values of sentences *not belonging* to K (non-beliefs), in order for $\preceq_K$ to be specified.

Before closing this section, we point out an interesting feature of AGM revision functions. In particular, Theorem 4.5, subsequently, shows that there exist AGM revision functions such that, if we "feed" them with the appropriate input, their output is *always confined* to a particular "island" of belief sets.

THEOREM 4.5. *There exist an AGM revision function $*$ and a proper subset Θ of $\mathbb{K}$, such that, for any theory $K \in \Theta$ and any $\varphi \in \mathcal{L}$, $K * \varphi \in \Theta$.*

PROOF: Let Θ be the set of all *complete* theories of $\mathcal{L}$. Clearly, $\Theta \subset \mathbb{K}$. Let $*$ be an AGM revision function such that it assigns (via condition (F$*$)) to *each* theory $K \in \Theta$ the following K-faithful preorder $\preceq_K$ over $\mathbb{M}$:

$$w \ \prec_K \ r_1 \ \prec_K \ r_2 \ \prec_K \ \dots \ ,$$

where w is a world of $\mathbb{M}$ such that $[K] = \{w\}$, and $r_1, r_2, \dots$ is any sequence of all worlds in $\mathbb{M} - \{w\}$.[6] By the construction of $\preceq_K$, it follows that, for *any* sentence $\varphi \in \mathcal{L}$, the set $min([\varphi], \preceq_K)$ is *always a singleton*; that is, the revised belief set $K * \varphi$ is *always* satisfied by exactly one world.[7] Therefore, for any theory $K \in \Theta$ and any $\varphi \in \mathcal{L}$, $K * \varphi \in \Theta$. □

The above result does not only show that there are AGM revision functions that could result in a form of "islanding"; it, also, shows that such functions could "trap" a rational agent into an "omniscience island", in the sense that, if the agent has an opinion about everything (thus, her belief corpus coincides with a complete theory), she will *still* have an opinion about everything, after *any* sequence of revisions.[8]

[6] For any complete theory K, the set $[K]$ is a singleton.

[7] The AGM revision function $*$ implements at theory K a type of revision called *maxichoice* revision [1, 14].

[8] If K is a theory that represents the agent's beliefs, any possible world in $[K]$ is perceived by the agent to be the "actual" world. Hence, the more the agent learns, the fewer possible worlds are compatible with her knowledge. On that premise, a complete theory expresses the beliefs of an omniscient agent.

5. Parikh's notion of relevance

Convincing concrete examples have pointed out that the AGM postulates for revision are insufficient to capture the notion of *relevance*. For instance, the severe *full-meet revision* satisfies the AGM postulates for revision and, at the same time, it discards *all* prior beliefs of a belief set K retaining only the epistemic input φ, in the principal case where φ contradicts K [1]. On that unsatisfactory premise, Parikh proposed a supplementary axiom, named (P) and presented below, that encodes a form of *syntactical relevance* [23].[9]

(P) If $K = Cn(x, y)$, where x, y are sentences of disjoint sublanguages $\mathcal{L}_x, \mathcal{L}_y$, respectively, and $\mathcal{L}_\varphi \subseteq \mathcal{L}_x$, then $K * \varphi = \big(Cn_{\mathcal{L}_x}(x) \diamond \varphi\big) + y$, where $\diamond$ is a local revision operator defined over the sublanguage $\mathcal{L}_x$.

Some remarks on the notation in (P) are in order. For a sentence x of $\mathcal{L}$, $\mathcal{L}_x$ denotes the (unique) *minimal* (sub)language of $\mathcal{L}$ within which x can be expressed; in the limiting case where x is not contingent, $\mathcal{L}_x$ is defined to be the empty set. We note that this definition can be extended to a belief set K, since, given that $\mathcal{P}$ is finite, there exists a sentence $\xi \in \mathcal{L}$ such that $K = Cn(\xi)$; hence, we define $\mathcal{L}_K = \mathcal{L}_\xi$. Moreover, $Cn_{\mathcal{L}_x}(x)$ denotes the deductive closure of x *in* the sublanguage $\mathcal{L}_x$, i.e., $Cn_{\mathcal{L}_x}(x) = Cn(x) \cap \mathcal{L}_x$.

Peppas *et al.* further investigated Parikh's original proposal and concluded that there are, in fact, two distinct interpretations of axiom (P); namely, the *weak* and the *strong* version of (P) [27]. For presenting these two versions of (P), consider first the next two conditions (P1) and (P2) which do not refer to a local revision operator—in (P1), $\overline{\mathcal{L}_x}$ denotes the (sub)language built from the propositional variables that do not appear in $\mathcal{L}_x$, using the standard Boolean connectives (if there are no propositional variables that do not appear in $\mathcal{L}_x$, then $\overline{\mathcal{L}_x}$ is empty).

[9] It is noteworthy that the problem of relevance in the realm of belief change was first highlighted by Gärdenfors in [15]. Although several interpretations of relevance were discussed in that work, the key criterion considered was the following: "*If a belief set K is revised by a sentence φ, then all sentences in K that are irrelevant to the validity of φ should be retained in the revised state of belief*".

(P1) If $K = Cn(x, y)$, $\mathcal{L}_x \cap \mathcal{L}_y = \varnothing$, and $\mathcal{L}_\varphi \subseteq \mathcal{L}_x$, then $(K * \varphi) \cap \overline{\mathcal{L}_x} = K \cap \overline{\mathcal{L}_x}$.

(P2) If $K = Cn(x, y)$, $\mathcal{L}_x \cap \mathcal{L}_y = \varnothing$, and $\mathcal{L}_\varphi \subseteq \mathcal{L}_x$, then $(K * \varphi) \cap \mathcal{L}_x = \big(Cn(x) * \varphi\big) \cap \mathcal{L}_x$.

Condition (P1) corresponds to the weak version of axiom (P). It essentially states that, if a theory K can be expressed in disjoint sublanguages $\mathcal{L}_x$ and $\overline{\mathcal{L}_x}$, then the revision of K by an epistemic input that can be formulated within $\mathcal{L}_x$ should *not* affect the $\overline{\mathcal{L}_x}$-part of K. Appending (P2) to (P1), we get the strong version of axiom (P), according to which the modification of the $\mathcal{L}_x$-part of K is not affected by the $\overline{\mathcal{L}_x}$-part of it. Therefore, in a sense, strong (P) makes the local revision operator $\diamond$ *context-independent*.[10]

In the remainder of this section, we introduce the necessary terminology for our subsequent discussion. To this aim, for a subset Q of the set $\mathcal{P}$ of propositional variables, $\mathcal{L}^Q$ shall denote the sublanguage of $\mathcal{L}$ defined over Q, using the standard Boolean connectives (if Q is empty, then $\mathcal{L}^Q$ is empty).

Definition 5.1 (Splittable/Confined Theory). Let K be a theory of $\mathcal{L}$. We shall say that K is splittable iff, for some sentences $x, y \in \mathcal{L}$, $K = Cn(x, y)$ and $\mathcal{L}_x \cap \mathcal{L}_y = \varnothing$. In the special case where y is a tautology and $\mathcal{L}_x \subset \mathcal{L}$, we shall say that theory K is confined (to the sublanguage $\mathcal{L}_x$ of $\mathcal{L}$) as well.

Essentially, a theory K that is confined to a sublanguage $\mathcal{L}'$ of $\mathcal{L}$ splits between $\mathcal{L}'$ and $\overline{\mathcal{L}'}$, with the $\overline{\mathcal{L}'}$ part being trivial. In this case, K knows nothing about $\overline{\mathcal{L}'}$.[11]

Definition 5.2 (Theory-Splitting, [23]). Let K be a theory of $\mathcal{L}$, and let $Q = \big\{Q_1, Q_2, \ldots, Q_n\big\}$ be a partition of $\mathcal{P}$; i.e., $\bigcup Q = \mathcal{P}$, $Q_i \neq \varnothing$, and $Q_i \cap Q_j = \varnothing$, for all $1 \leqslant i \neq j \leqslant n$. The set Q is a K-splitting iff there exist sentences $\varphi_1 \in \mathcal{L}^{Q_1}, \varphi_2 \in \mathcal{L}^{Q_2}, \ldots, \varphi_n \in \mathcal{L}^{Q_n}$, such that $K = Cn\big(\varphi_1, \varphi_2, \ldots, \varphi_n\big)$.

[10] The characterization of both conditions (P1) and (P2) in the realm of all popular constructive models for belief revision can be found in [3, 6]. For a comprehensive study of important constructive aspects of (P2), the interested reader is referred to [8].

[11] It is assumed that tautologies bear no knowledge.

Parikh showed that, for every theory K, there exists a *unique finest* K-splitting—i.e., one which refines every other K-splitting—hereafter denoted by $\mathcal{F}_K$ [23].[12]

Example 5.3 (Theory-Splitting). Suppose that $\mathcal{P} = \big\{a, b, c, d, e\big\}$, and let K be a splittable theory of $\mathcal{L}$ such that $K = Cn\big(a, b, c \to d\big)$. Then, the finest K-splitting is $\mathcal{F}_K = \big\{\{a\}, \{b\}, \{c, d\}, \{e\}\big\}$. Observe that theory K has no information about propositional variable e, as it is confined to the sublanguage $\mathcal{L}^{\{a,b,c,d\}}$.

Definition 5.4 (Theory-Units, [5, 3, 6]). Let K be a theory of $\mathcal{L}$ which does not contain only tautologies, and let $\big\{F_1, F_2, \ldots, F_n\big\}$ be the finest K-splitting. By the definition of a K-splitting, there exist *contingent* sentences $\chi_1, \chi_2, \ldots, \chi_m$ of $\mathcal{L}$, such that $m \leqslant n$, $\chi_1 \in \mathcal{L}^{F_{i_1}}, \chi_2 \in \mathcal{L}^{F_{i_2}}, \ldots, \chi_m \in \mathcal{L}^{F_{i_m}}$, and $K = Cn\big(\chi_1, \chi_2, \ldots, \chi_m\big)$. The sentences $\chi_1, \chi_2, \ldots, \chi_m$ are the units of K, and the set $\mathcal{U}_K = \big\{\chi_1, \chi_2, \ldots, \chi_m\big\}$ is the unit set of K.[13]

It turns out that each unit of a theory K is unique, modulo logical equivalence. Intuitively, the units of a *splittable* theory K are its "building blocks" which divide K into the *refined* compartments (theories) $Cn(\chi_1), Cn(\chi_2), \ldots, Cn(\chi_m)$. Hence, there is a unique way to think of theory K as being composed of *non-trivial* (since units are *contingent* sentences) *disjoint* compartments, referring to mutually *irrelevant* subject matters.

Remark 5.5. Each unit of a theory K corresponds to a unique element of the finest K-splitting $\mathcal{F}_K$. The converse is true only in case $\mathcal{L}_K = \mathcal{L}$ (i.e., when theory K is not confined to a sublanguage of $\mathcal{L}$); in the case of a confined theory K (where $m < n$), not every element of $\mathcal{F}_K$ corresponds to a unit of K.

Example 5.6 (Theory-Units, Cont'd Example 5.3). The unit set of theory $K = Cn\big(a, b, c \to d\big)$ is $\mathcal{U}_K = \{a, b, c \to d\}$. As Remark 5.5 points out, each unit of K corresponds to a unique element of the finest K-splitting

[12]Kourousias and Makinson in [20] extended this result to a language built over infinitely many propositional variables. We recall, moreover, that a partition Q' refines another partition Q iff, for every $Q'_i \in Q'$, there exists a $Q_j \in Q$, such that $Q'_i \subseteq Q_j$.

[13]The definition of units in [5, 3, 6] is slightly different; herein, a minor modification is made for ease of presentation.

$\mathcal{F}_K = \big\{\{a\}, \{b\}, \{c, d\}, \{e\}\big\}$. Yet, since K is confined to the sublanguage $\mathcal{L}^{\{a,b,c,d\}}$, not every element of $\mathcal{F}_K$ corresponds to a unit of K.

Lastly, the interesting notion of *compartmental coupling* is introduced, which, to the best of our knowledge, has not been formalized elsewhere before.

Definition 5.7 (Compartmental Coupling). Let $*$ be an AGM revision function, and let K be a splittable theory of $\mathcal{L}$. We shall say that the $*$-revision of K by a sentence $\varphi \in \mathcal{L}$ couples compartments of K iff an element of the finest $K * \varphi$-splitting $\mathcal{F}_{K*\varphi}$ contains two distinct propositional variables $a, b \in \mathcal{P}$, which belong to distinct elements F, F' of the finest K-splitting $\mathcal{F}_K$ (i.e., $a \in F$ and $b \in F'$), and at least one of F, F' corresponds to a unit of K.

Roughly speaking, the revision of a theory K by an epistemic input φ couples compartments of K whenever (parts of) two disjoint (refined) compartments of K—at least one of which is *non-trivial*—have been joined into a *single* (refined) compartment in the revised state of belief.[14] Evidently, the coupling of compartments of a theory leads to a change in the *structure* of that theory.

The next concrete examples illustrate (further) features of the above definitions.

Example 5.8 (Revision With Compartmental Coupling, Cont'd 5.6). Consider a sentence $\varphi = \neg a \vee \neg b$, which contradicts theory $K = Cn\big(a, b, c \rightarrow d\big)$, and an AGM revision function $*$ that respects the strong version of axiom (P), such that $K * \varphi = Cn\big(\neg a \vee \neg b, c \rightarrow d\big)$. As $\mathcal{F}_{K*\varphi} = \big\{\{a, b\}, \{c, d\}, \{e\}\big\}$, the $*$-revision of theory K by φ couples compartments of K; this is because the element $\{a, b\}$ of $\mathcal{F}_{K*\varphi}$ contains the propositional variables a and b, which belong to *distinct* elements F, F' of $\mathcal{F}_K$, and each one of F, F' corresponds to a unit of K (recall that $\mathcal{U}_K = \{a, b, c \rightarrow d\}$). Notice, lastly, that, as $*$ respects Parikh's principle and theory K is splittable, the part of K formed by the propositional variables c, d and e remains unaffected during the revision-process.

[14]Notice that compartmental coupling is defined in terms of *finest* splittings of theories; therefore, the compartments involved are, as a matter of fact, *refined* compartments. Furthermore, since, according to Definition 5.7, at least one of $F, F' \in \mathcal{F}_K$ corresponds to a unit (i.e., *contingent* sentence) of the initial theory K, it follows that at least one of the coupled (refined) compartments of K is *non-trivial*.

It is evident from the above example that the revision of a splittable theory by an epistemic input does not necessarily couple *every* compartment of the theory.

Example 5.9 (Revision With Compartmental Coupling). Let $\mathcal{P} = \{a, b, c, d, e, f\}$, $K = Cn(a \leftrightarrow b, c \leftrightarrow d, e \vee f)$, $H = Cn(a \leftrightarrow b, e)$ and $\varphi = (b \vee c) \wedge (\neg e) \wedge (\neg f)$—notice that φ contradicts both K and H. Consider an AGM revision function $*$ such that $K * \varphi = Cn(a, b \vee c, d, \neg e, \neg f)$ and $H * \varphi = Cn(b \vee c, \neg e, \neg f)$. Hence, we have that $\mathcal{F}_K = \{\{a, b\}, \{c, d\}, \{e, f\}\}$ and $\mathcal{F}_{K*\varphi} = \{\{a\}, \{b, c\}, \{d\}, \{e\}, \{f\}\}$, as well as that $\mathcal{F}_H = \{\{a, b\}, \{c\}, \{d\}, \{e\}, \{f\}\}$ and $\mathcal{F}_{H*\varphi} = \{\{a\}, \{b, c\}, \{d\}, \{e\}, \{f\}\}$. Thus, the $*$-revision of K by φ couples compartments of K, since the element $\{b, c\}$ of $\mathcal{F}_{K*\varphi}$ contains the propositional variables b and c, which belong to *distinct* elements F, F' of $\mathcal{F}_K$, and each one of F, F' corresponds to a unit of K. Furthermore, the $*$-revision of H by φ couples compartments of H, since the element $\{b, c\}$ of $\mathcal{F}_{H*\varphi}$ contains the propositional variables b and c, which belong to *distinct* elements F'', F''' of $\mathcal{F}_H$, respectively, and F'' (but *not* F''') corresponds to a unit of H—recall that, for compartmental coupling, Definition 5.7 requires that *at least one* of F'', F''' should correspond to a unit of H.

Example 5.10 (Revision Without Compartmental Coupling). Let $\mathcal{P} = \{a, b, c, d\}$ and let $K = Cn(a, b)$. Then, the finest K-splitting is $\mathcal{F}_K = \{\{a\}, \{b\}, \{c\}, \{d\}\}$, and the unit set of K is $\mathcal{U}_K = \{a, b\}$. Consider, now, a sentence $\varphi = (\neg a) \wedge (\neg b) \wedge (c \vee d)$, which contradicts K, and an AGM revision function $*$ such that $K * \varphi = Cn(\neg a, \neg b, c \vee d)$; hence, $\mathcal{F}_{K*\varphi} = \{\{a\}, \{b\}, \{c, d\}\}$. Notice that the $*$-revision of K by φ does *not* couple compartments of K, since, although the element $\{c, d\}$ of $\mathcal{F}_{K*\varphi}$ contains the propositional variables c and d, which belong to *distinct* elements F, F' of $\mathcal{F}_K$, yet, *none* of F, F' corresponds to a unit of K (as the $\mathcal{L}^{\{c,d\}}$-part of K is trivial).

Coupling of compartments can take place even when the epistemic input is *consistent* with the initial theory; in that case, revision reduces to *expansion*, due to postulates $(K * 3)$–$(K * 4)$. Such a scenario is presented in the subsequent example.

Example 5.11 (Expansion With Compartmental Coupling). Let $\mathcal{P} = \{a, b, c, d\}$, $K = Cn(a \vee b, c \vee d)$ and $\varphi = b \vee c$ — notice that φ is consistent with K. Consider an AGM revision function $*$ such that $K * \varphi = Cn\big(a \vee b, b \vee c, c \vee d\big)$. Then, $\mathcal{F}_K = \big\{\{a, b\}, \{c, d\}\big\}$ and $\mathcal{F}_{K*\varphi} = \big\{\{a, b, c, d\}\big\}$. Thus, the $*$-revision (specifically, $*$-expansion) of K by φ couples compartments of K.

Having introduced the basic concepts of Belief Revision, we turn, in the following sections, to the main contribution of the present article.

6. Evolution of splittable theories from an epistemological perspective

Against the background that has been established so far, this section is devoted to an interpretation of rational belief revision of *splittable* theories from a Kuhnian perspective. In particular, we suggest an epistemological reading of a number of principal belief-change scenarios. Our aim is not an exhaustive investigation of all possible scenarios, but rather an initiation of a discussion between Kuhn and the central figures of Belief Revision.

To this end, let K be a splittable theory of $\mathcal{L}$ such that $K = Cn\big(\chi_1, \chi_2, \dots, \chi_m\big)$, where sentences $\chi_1, \chi_2, \dots, \chi_m$ (with $m \geqslant 2$) are the units of K, and let $\mathcal{F}_K = \big\{F_1, F_2, \dots, F_n\big\}$ be the finest K-splitting.[15] Theory K shall represent the knowledge of a scientific community seen as a single rational agent. Since K can be expressed in disjoint sublanguages, we may assume that it consists of *unrelated refined* compartments, referring to different subject matters. Furthermore, let $*$ be an AGM revision function that the scientific community utilizes as a tool for revision, and assume that $*$ satisfies the *strong* version of axiom (P); i.e., conditions (P1) and (P2).

Given an epistemic input φ, we first distinguish the two cases according to which

$$\mathcal{L}_\varphi \cap \mathcal{L}_K = \varnothing \quad \text{or} \quad \mathcal{L}_\varphi \cap \mathcal{L}_K \neq \varnothing.$$

In the former case, which is abstractly depicted in Figure 1, theory K is necessarily *confined* to a sublanguage of $\mathcal{L}$, and the epistemic input φ is

[15] We assume that the splittable theory K has *at least two* units (i.e., $m \geqslant 2$), so to avoid the trivial case of a theory that refers to a *single* subject matter. Recall moreover that, by definition, $m \leqslant n$.

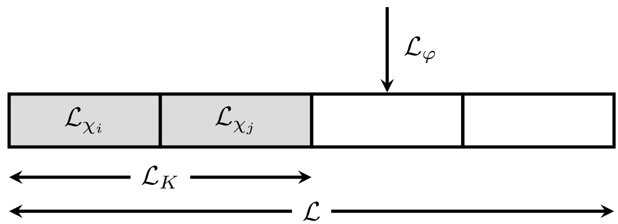

Figure 1. Pre-Paradigm: $K = Cn(\chi_i, \chi_j)$ and $\mathcal{L}_\varphi \cap \mathcal{L}_K = \varnothing$. Each block corresponds to an element of the finest K-splitting.

clearly *consistent* with K (since we have assumed that φ is self-consistent). Hence, according to the AGM postulates for revision $(K*1)$–$(K*8)$, theory K is (set-theoretically) *expanded* by φ; that is,

$$K * \varphi = K + \varphi.$$

In this case, a *pre-paradigm* event takes place, in the context of which new scientific knowledge (that does not contradict initial knowledge) is accumulated, resulting in the formation of new concrete (sub)theories, and, thus, in the establishment of new paradigms.

In the latter case, we further distinguish the two sub-cases according to which $\mathcal{L}_\varphi$ is restricted to *some* sublanguage $\mathcal{L}_{\chi_i}$ (with $1 \leqslant i \leqslant m$) or not. The former case may be regarded as a period of *normal science*, whereas, the latter as a *paradigm shift*. Both these scenarios are formally described in the next two subsections.

6.1. Normal science: $\mathcal{L}_\varphi \subseteq \mathcal{L}_{\chi_i} \subseteq \mathcal{L}_K$ and $\mathcal{L}_\varphi \cap \mathcal{L}_K \neq \varnothing$

During normal science, every new piece of information φ is such that its minimal language $\mathcal{L}_\varphi$ is a subset of a *single* sublanguage $\mathcal{L}_{\chi_i}$; i.e., $\mathcal{L}_\varphi \subseteq \mathcal{L}_{\chi_i} \subseteq \mathcal{L}_K$ (and, of course, $\mathcal{L}_\varphi \cap \mathcal{L}_K \neq \varnothing$). That is to say, the epistemic input φ is related to a *single* (refined) compartment $Cn(\chi_i)$ of K, which in turn refers to a specific subject matter — this is abstractly depicted in Figure 2.

In this case, epistemic inputs, which *contradict* theory $Cn(\chi_i)$, correspond to *anomalies*, that is, challenges to be puzzled out and solved *within* $Cn(\chi_i)$, causing incremental changes. A real-world example of revising

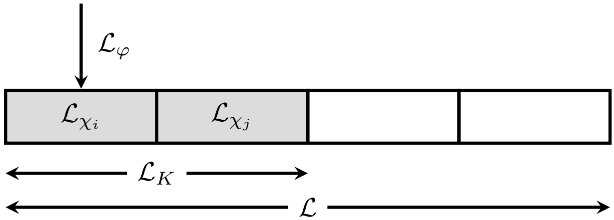

Figure 2. Normal Science: $K = Cn(\chi_i, \chi_j)$, $\mathcal{L}_\varphi \subseteq \mathcal{L}_{\chi_i} \subseteq \mathcal{L}_K$ and $\mathcal{L}_\varphi \cap \mathcal{L}_K \neq \varnothing$.
Each block corresponds to an element of the finest K-splitting.

a scientific theory by new information, which contradicts previous knowledge, concerns an issue of Cosmology, namely, the expansion of the universe. Specifically, Einstein's static universe, which is a relativistic model of the universe proposed by Albert Einstein in 1917, was refuted after the observations of Edwin Hubble in 1929, suggesting an expanding universe.

On the other hand, new information *consistent* with theory $Cn(\chi_i)$ *expands* the present knowledge contained in $Cn(\chi_i)$, without causing loss of existing information. Such an expansion is constituted by the recent (2015) discovery of gravitational waves, which were predicted in 1916 by Einstein.

In any scenario, under the current hypotheses, conditions (P2) and (F$*$) entail, for the $\mathcal{L}_{\chi_i}$-part of the revised theory $K * \varphi$, that:

$$\big(K * \varphi\big) \cap \mathcal{L}_{\chi_i} = \big(Cn(\chi_i) * \varphi\big) \cap \mathcal{L}_{\chi_i} = th\Big(min\big([\varphi], \preceq_{Cn(\chi_i)}\big)\Big) \cap \mathcal{L}_{\chi_i}.$$

Observe that, in order to produce the $\mathcal{L}_{\chi_i}$-part of $K * \varphi$, the agent—which can be viewed as the scientific community as a whole—needs only the preorder $\preceq_{Cn(\chi_i)}$, and not the preorder $\preceq_K$. This, in view of Remark 4.4 of Section 4, allows her to omit comparing the epistemic value of propositions referring to irrelevant subject matters—as the construction of $\preceq_K$ demands—, a requirement that would clearly constitute an epistemological "thorn". For instance, how can a rational agent compare the epistemic value of propositions expressed in the language $\mathcal{L}_{\chi_i}$, referring to the structure of water, with propositions expressed in the language $\mathcal{L}_{\chi_j}$ (with $i \neq j$), referring to monetary economics, in order to build a faithful preorder $\preceq_K$?

As for the $\overline{\mathcal{L}_{\chi_i}}$-part of the revised theory $K * \varphi$, of course, it is equal to $K \cap \overline{\mathcal{L}_{\chi_i}}$, according to condition (P1). Hence, only the φ-relevant compartment of theory K is affected by its $*$-revision by φ. Clearly then, the following remark is true.

Remark 6.1. In normal science, the finest $K * \varphi$-splitting *could be identical* to the finest K-splitting. Furthermore, since we have assumed that the unit set of K contains at least two units (cf. Footnote 15), it follows that the unit set of $K * \varphi$ is, during a period of normal science, *non-singleton.*

We close this subsection with a formal concrete example—which is a variation of Example 5.9 of Section 5—illustrating a normal-science scenario.

Example 6.2 (Normal Science). Suppose that $\mathcal{P} = \{a, b, c, d, e, f\}$ and $K = Cn(a \leftrightarrow b, c \leftrightarrow d, e \vee f)$; thus, $\mathcal{U}_K = \{a \leftrightarrow b, c \leftrightarrow d, e \vee f\}$. Let $\varphi = (\neg a \wedge b) \vee (a \wedge \neg b)$ and $\psi = a \wedge b$ — notice that φ contradicts K, whereas, ψ is consistent with K. Assume that $*$ is an AGM revision function that respects the strong version of axiom (P), such that $K * \varphi = Cn\Big((\neg a \wedge b) \vee (a \wedge \neg b), c \leftrightarrow d, e \vee f\Big)$ and $K * \psi = Cn(a, b, c \leftrightarrow d, e \vee f)$. Hence, we have that $\mathcal{F}_K = \{\{a, b\}, \{c, d\}, \{e, f\}\}$, $\mathcal{F}_{K*\varphi} = \{\{a, b\}, \{c, d\}, \{e, f\}\}$ and $\mathcal{F}_{K*\psi} = \{\{a\}, \{b\}, \{c, d\}, \{e, f\}\}$.

In the above example, each one of the epistemic inputs φ and ψ is *solely* related to the (refined) compartment $Cn(a \leftrightarrow b)$ of the splittable theory K, since $\mathcal{L}_\varphi \subseteq \mathcal{L}_{a\leftrightarrow b}$ and $\mathcal{L}_\psi \subseteq \mathcal{L}_{a\leftrightarrow b}$. Observe that the $*$-revision of K by φ results in a revised theory $K * \varphi$ whose finest splitting is identical to the finest K-splitting (cf. Remark 6.1); this is not the case for the $*$-revision of K by ψ, which leads to a revised theory $K * \psi$ with a different finest splitting. Lastly, as the AGM revision function $*$ respects Parikh's principle, the $\overline{\mathcal{L}_{a\leftrightarrow b}}$-part of K remains unaffected during the revision-process.

6.2. Paradigm shift: $\mathcal{L}_\varphi \nsubseteq \mathcal{L}_{\chi_i}$, $\mathcal{L}_\varphi \cap \mathcal{L}_K \neq \varnothing$ and compartmental coupling

In case $\mathcal{L}_\varphi \cap \mathcal{L}_K \neq \varnothing$, the minimal language $\mathcal{L}_\varphi$ of the epistemic input φ is *not* restricted to a sublanguage $\mathcal{L}_{\chi_i}$ (i.e., $\mathcal{L}_\varphi \nsubseteq \mathcal{L}_{\chi_i}$, for *all* i such that $1 \leqslant i \leqslant m$), and, moreover, the revision of K by φ *couples compartments* of K, over which $\mathcal{L}_\varphi$ spans, a *paradigm shift* takes place in the form of a *scientific revolution*. Note that the coupling of some compartments of theory K is demanded so as to *avoid* characterizing as a paradigm shift a situation in which $\mathcal{L}_\varphi$ spans over multiple (refined) compartments of K, and, yet, the finest $K * \varphi$-splitting is identical to the finest K-splitting; i.e., $\mathcal{F}_{K*\varphi} = \mathcal{F}_K$.[16] In other words, given that $\mathcal{L}_\varphi$ spans over multiple (refined) compartments of K, coupling ensures that some of these (or even all) compartments of K—at least one of which is non-trivial—have been joined in the revised state of belief. This, in turn, implies the following observation.

Remark 6.3. Contrary to the case of normal science (cf. Remark 6.1), in the context of a paradigm shift, it is always true that $\mathcal{F}_{K*\varphi} \neq \mathcal{F}_K$, and the revision in that context changes the *structure* (of non-trivial compartments) of the initial theory K. Furthermore, a paradigm shift may very well lead to revised theories with *singleton* unit sets (cf. Example 5.11 of Section 5).

Against this background, the new information φ corresponds to a challenging anomaly which, in turn, results in a crisis of normal science; this crisis, eventually, leads to a paradigm shift. Formal instances illustrating the current scenario, which is abstractly depicted in Figure 3, are those encoded in Examples 5.8, 5.9 and 5.11 of Section 5.[17] It should come as no surprise that an expansion, such as that encoded in Example 5.11, could give rise to a paradigm shift, since an expansion of a splittable theory by new knowledge may lead, due to compartmental coupling, to a dramatic change in the *structure* (of non-trivial compartments) of the theory (although no information is lost during the expansion). It is, also, noteworthy that Example 5.10 of Section 5 does *not* correspond to a situation of a paradigm shift, since, although the minimal language of the epistemic input in that example spans over multiple (refined) compartments of the initial theory, compartmental coupling (as defined in Definition 5.7) does *not* take place during revision.

[16] Such a scenario is the following: Let $\mathcal{P} = \{a, b\}$, $K = Cn(a, b)$, $\varphi = \neg a \wedge \neg b$, and consider an AGM revision function $*$ such that $K * \varphi = Cn(\neg a, \neg b)$. Clearly then, $\mathcal{L}_\varphi$ spans over the (refined) compartments $Cn(a)$ and $Cn(b)$ of K, and, at the same time, $\mathcal{F}_{K*\varphi} = \mathcal{F}_K = \big\{\{a\}, \{b\}\big\}$.

[17] Historical examples for this scenario are reported in the next section. Recall, moreover, that Example 5.10 of Section 5 refers to a revision-instance in which no coupling of compartments of the initial theory takes place.

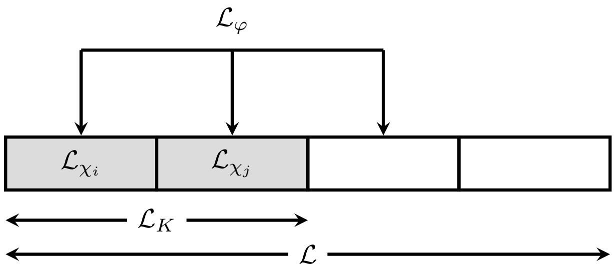

Figure 3. Paradigm Shift: $K = Cn(\chi_i, \chi_j)$, $\mathcal{L}_\varphi \nsubseteq \mathcal{L}_{\chi_i}$, $\mathcal{L}_\varphi \cap \mathcal{L}_K \neq \varnothing$ and compartmental coupling.
Each block corresponds to an element of the finest K-splitting.

6.3. Observations

An interesting discussion on the relation between Kuhnian epistemology and (not necessarily relevance-sensitive) belief change has been conducted by Gärdenfors in [14]. In that work, Gärdenfors argues that a paradigm shift, typically, involves a radical change in the epistemic values of the formulae of a scientific theory, and, conversely, a substantial change in the epistemic values of the formulae of a scientific theory is a strong indication of a scientific revolution [14, p. 88]. This change in the epistemic values of formulae is reflected in a change in the faithful preorders that the scientific community assigns to theories, and, as a consequence, in a change of the AGM revision function that the scientific community utilizes for revision.[18] Such alterations of the revision policy could serve as a means for avoiding "islandings" like that described in Theorem 4.5 (Section 4).

Herein, we supplement the aforementioned view of Gärdenfors by claiming that, in a propositional framework, a scientific revolution could result in a change in the set of propositional variables from which the object language $\mathcal{L}$ is generated, and, vice versa, a change in the set of propositional variables, typically, indicates a scientific revolution. Lastly, a scientific revolution could change the *meaning* (semantics) of a propositional variable

[18] As a matter of fact, a *quantification of the difference* between faithful preorders is, also, feasible, with the aid of well-accepted concepts such as the *Kemeny distance*; given any two total preorders over $\mathbb{M}$, their Kemeny distance is defined as the cardinality of their symmetric difference [19].

(for instance, the word "mass" takes a totally different meaning in the Newtonian than in the Einsteinian framework), a fact which is, in turn, related to the *symbol grounding problem* [17].

We close this section by noting that a formal modelling of the evolution (transition) of scientific theories by means of the AGM paradigm suggests that, contrary to Kuhn's claim (see at the end of Section 2), competing scientific paradigms may be (at least to a certain degree) comparable in a *commensurable* way, a fact which in turn allows for a rational evaluation among them.[19] Consider, for example, a scientific community whose knowledge is represented by theory K_1. Assume that, after a paradigm shift, the knowledge of the scientific community is reflected in a new theory K_2, whereas, after another paradigm shift, the knowledge of the community is reflected in another new theory K_3. Against this background, the AGM paradigm provides the formal guidelines in order for an AGM revision function $*$ to be specified, such that, for two sentences $\varphi_1, \varphi_2 \in \mathcal{L}$ (representing new pieces of information), $K_2 = K_1 * \varphi_1$ and $K_3 = K_2 * \varphi_2$.[20] Since such an AGM revision function $*$ can be defined, the scientific theories of each pair of K_1, K_2, K_3 can be compared through $*$, in the sense that one could, for example, generate theory K_2 from the $*$-revision of K_1 by φ_1; as earlier stated, this capability allows for a rational evaluation among scientific theories.

7. Historical examples of paradigm shifts

Syntheses of *initially unrelated* scientific theories, after a paradigm shift brought about because a new piece of evidence involved concepts from these theories, have happened in the history of science not just once. Indicative such examples are presented subsequently.

- The theory of magnetism—the formulation of which began with Gilbert's careful study of magnetic phenomena in the late 16th century—was initially unrelated to the theory of electricity—formulated, mainly, by Franklin and Coulomb in the late 1700s. The

[19] Although the peculiar nature of the symbol grounding problem involved in a scientific revolution may lead to major difficulties in the comparison.

[20] This is a form of "reverse" belief revision.

first connection between electric and magnetic phenomena was discovered by Hans Christian Ørsted in 1820, when he found that electric currents produce magnetic forces, namely, a piece of information that involves concepts of magnetism *and* electricity. Ørsted's discovery was responsible for the formulation of the combined theory of *Electromagnetism*.

- In the beginning, there was Biology, a discipline that studies life and living organisms, far from Chemistry that studies non-living matter. A dominant principle once biologists believed is imprinted in the following vitalistic view: "Living organisms are fundamentally different from non-living entities because they contain some non-physical element or are governed by different principles than are inanimate things" [10]. Series of strong evidence, however, falsified many vitalistic theories, suggesting that the processes of life are based, in fact, on *chemical* compounds. That is to say, a paradigm shift associated with Biology led to a new discipline that combines a mixture of both Biology and Chemistry, nowadays called *Biochemistry*.

- Classical Mechanics depicted a universe in which objects move in perfectly-determined (non-random) ways. In this context, Probability Theory may be considered, at least to a large extent, irrelevant. Ground-breaking experimental discoveries initiated a paradigm shift that led to a new theory of Physics, named *Quantum Mechanics*, in which the role of pure randomness in physical processes is fundamental.

- Areas of Mathematics such as topology and algebraic geometry, lying at the heart of pure Mathematics and appearing very distant from the Physics frontier, have been dramatically reshaped (in the sense that new discoveries have been emerged) after paradigm shifts associated with fundamental hypotheses of Physics that involved a combination of the aforementioned mathematical principles. This process has led to many hybrid theories, such as the *Topological Quantum Field Theory*, which now form a core of modern research in both Mathematics and Physics [9].

- Connections between two, initially unrelated, theories have not only emerged in the realm of natural sciences. Indicative is the case of Neurobiology and Psychoanalysis, which, at the beginning of the 20th century, seemed completely incompatible. The book by Ansermet and Magistretti, [2], is devoted to the presentation of recent experimental findings—concerning several mechanisms of the brain—that involved a mixture of concepts of both Neurobiology and Psychoanalysis. These observations initiated, in turn, a paradigm shift that revealed a close relation of these two disciplines.

It should be evident that the above points do not refer to arbitrary cases of paradigm shifts, but to cases in which two *initially unrelated* bodies of knowledge (i.e., compartments of a broader scientific theory) *coupled* after a new piece of evidence involved concepts of *both* these bodies. Hence, the presented historical examples respect the hypotheses of the paradigm-shift analysis of Subsection 6.2, and, thus, they come to justify its substance.

8. Conclusion

In this article, the evolution (revision) of splittable theories from an epistemological perspective was investigated. In particular, we have suggested an epistemological reading of principal belief-change scenarios, involving splittable belief corpora, from the perspective of Kuhn's influential work on the evolution of scientific knowledge. Representative examples from the history of science have supported the conducted study, providing historical content to the mathematical contours of the introduced concepts. Our analysis aims at the formation of a conceptual bridge between rational belief revision and traditional philosophy of science, which will shed light on the application of formal epistemological tools in the dynamics of (any corpus of) knowledge.

Despite the brevity of the account in the matter of the epistemology of rational belief change, we hope that the ideas conveyed in this article will become the springboard to future research; for instance, other interesting belief-change scenarios could be explored, against the background of Kuhn's principles, or the presented ones could be further refined.

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References

[1] C. Alchourrón, P. Gärdenfors, D. Makinson, *On the logic of theory change: Partial meet contraction and revision functions*, **Journal of Symbolic Logic**, vol. 50(2) (1985), pp. 510–530, DOI: https://doi.org/10.2307/2274239.

[2] F. Ansermet, P. Magistretti, **Biology of Freedom: Neural Plasticity, Experience, and the Unconscious**, Routledge (2007).

[3] T. Aravanis, **Relevance and Knowledge Dynamics for Intelligent Agents**, Ph.D. thesis, Department of Business Administration, School of Economics & Business, University of Patras, Patras, Greece (2019).

[4] T. Aravanis, *On uniform belief revision*, **Journal of Logic and Computation**, vol. 30 (2020), pp. 1357–1376, DOI: https://doi.org/10.1093/logcom/exaa058.

[5] T. Aravanis, P. Peppas, M.-A. Williams, *Epistemic-entrenchment characterization of Parikh's axiom*, [in:] **Proceedings of the 26th International Joint Conference on Artificial Intelligence (IJCAI 2017)** (2017), pp. 772–778, DOI: https://doi.org/10.24963/ijcai.2017/107.

[6] T. Aravanis, P. Peppas, M.-A. Williams, *Full characterization of Parikh's relevance-sensitive axiom for belief revision*, **Journal of Artificial Intelligence Research**, vol. 66 (2019), pp. 765–792, DOI: https://doi.org/10.1613/jair.1.11838.

[7] T. Aravanis, P. Peppas, M.-A. Williams, *An investigation of parametrized difference revision operators*, **Annals of Mathematics and Artificial Intelligence**, vol. 89 (2019), pp. 7–28, DOI: https://doi.org/10.1007/s10472-019-09625-x.

[8] T. Aravanis, P. Peppas, M.-A. Williams, *A study of possible-worlds semantics of relevance-sensitive belief revision*, **Journal of Logic and Computation**, vol. 30 (2020), pp. 1125–1142, DOI: https://doi.org/10.1093/logcom/exaa029.

[9] M. Atiyah, R. Dijkgraaf, N. Hitchin, *Geometry and physics*, **Philosophical Transactions of the Royal Society A**, vol. 368 (2010), pp. 913–926, DOI: https://doi.org/10.1098/rsta.2009.0227.

[10] W. Bechtel, R. C. Richardson, *Vitalism*, [in:] E. Craig (ed.), **Routledge Encyclopedia of Philosophy**, London: Routledge (1998), DOI: https://doi.org/10.4324/9780415249126-Q109-1.

[11] G. Brewka, T. Eiter, M. Truszczyński, *Answer Set Programming at a glance*, **Communications of the ACM**, vol. 54(12) (2011), pp. 93–103, DOI: https://doi.org/10.1145/2043174.2043195.

[12] M. Dalal, *Investigations into theory of knowledge base revision: Preliminary report*, [in:] **Proceedings of the 7th National Conference of the American Association for Artificial Intelligence (AAAI 1988)**, The AAAI Press, Menlo Park, California (1988), pp. 475–479.

[13] E. Fermé, S. O. Hansson, **Belief Change: Introduction and Overview**, Springer International Publishing (2018), DOI: https://doi.org/10.1007/978-3-319-60535-7.

[14] P. Gärdenfors, **Knowledge in Flux – Modeling the Dynamics of Epistemic States**, MIT Press, Cambridge, Massachusetts (1988).

[15] P. Gärdenfors, *Belief revision and relevance*, [in:] **PSA: Proceedings of the Biennial Meeting of the Philosophy of Science Association** (1990), pp. 349–365, DOI: https://doi.org/10.2307/193079.

[16] P. Gärdenfors, D. Makinson, *Revisions of knowledge systems using epistemic entrenchment*, [in:] **Proceedings of the 2nd Conference on Theoretical Aspects of Reasoning About Knowledge (TARK 1988)**, Morgan Kaufmann, Pacific Grove, California (1988), pp. 83–95.

[17] S. Harnad, *The symbol grounding problem*, **Physica D: Nonlinear Phenomena**, vol. 42 (1990), pp. 335–346, DOI: https://doi.org/10.1016/0167-2789(90)90087-6.

[18] H. Katsuno, A. Mendelzon, *Propositional knowledge base revision and minimal change*, **Artificial Intelligence**, vol. 52(3) (1991), pp. 263–294, DOI: https://doi.org/10.1016/0004-3702(91)90069-V.

[19] J. G. Kemeny, *Mathematics without numbers*, **Daedalus**, vol. 88(4) (1959), pp. 577–591, URL: https://www.jstor.org/stable/20026529.

[20] G. Kourousias, D. Makinson, *Parallel interpolation, splitting, and relevance in belief change*, **Journal of Symbolic Logic**, vol. 72(3) (2007), pp. 994–1002, DOI: https://doi.org/10.2178/jsl/1191333851.

[21] T. S. Kuhn, **The Structure of Scientific Revolutions**, University of Chicago Press, Chicago (1970).

[22] E. J. Olsson, S. Enqvist (eds.), **Belief Revision meets Philosophy of Science**, Springer, Dordrecht (2011), DOI: https://doi.org/10.1007/978-90-481-9609-8.

[23] R. Parikh, *Beliefs, belief revision, and splitting languages*, [in:] L. S. Moss, J. Ginzburg, M. de Rijke (eds.), **Logic, Language and Computation**, vol. 2, CSLI Publications (1999), pp. 266–278.

[24] P. Peppas, *Belief revision*, [in:] F. van Harmelen, V. Lifschitz, B. Porter (eds.), **Handbook of Knowledge Representation**, Elsevier Science (2008), pp. 317–359, DOI: https://doi.org/10.1016/S1574-6526(07)03008-8.

[25] P. Peppas, M.-A. Williams, *Constructive modellings for theory change*, **Notre Dame Journal of Formal Logic**, vol. 36(1) (1995), pp. 120–133, DOI: https://doi.org/10.1305/ndjfl/1040308831.

[26] P. Peppas, M.-A. Williams, *Parametrised difference revision*, [in:] **Proceedings of the 16th International Conference on Principles of Knowledge Representation and Reasoning (KR 2018)** (2018), pp. 277–286.

[27] P. Peppas, M.-A. Williams, S. Chopra, N. Foo, *Relevance in belief revision*, **Artificial Intelligence**, vol. 229 (2015), pp. 126–138, DOI: https://doi.org/10.1016/j.artint.2015.08.007.

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TERNARY RELATIONAL SEMANTICS FOR THE VARIANTS OF BN4 AND E4 WHICH CONTAIN ROUTLEY AND MEYER'S LOGIC B

Abstract

Six interesting variants of the logics BN4 and E4—which can be considered as the 4-valued logics of the relevant conditional and (relevant) entailment, respectively—were previously developed in the literature. All these systems are related to the family of relevant logics and contain Routley and Meyer's basic logic B, which is well-known to be specifically associated with the ternary relational semantics. The aim of this paper is to develop reduced general Routley-Meyer semantics for them. Strong soundness and completeness theorems are proved for each one of the logics.

Keywords: Ternary relational semantics, relevant logics, 4-valued logics, Routley and Meyer's logic B, 4-valued quasi-relevant logics.

1. Introduction

Brady defined in 1982 the system BN4 (cf. [3]), a logic built upon the matrix MBN4. This matrix is the result of a modification of the function $f_\rightarrow$ for the conditional in Smiley's matrix MSm4, which is in its turn a simplification of a matrix which has played an important role in the development of relevant logics (cf. [11, pp. 176, ff.]), i.e., Anderson and Belnap's 8-element

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matrix M_0 (cf. [1]). Smiley's matrix MSm4 is also the matrix characteristic of Anderson and Belnap's First Degree Entailment Logic (FDE; cf. [1, pp. 161–162]). The logic FDE is a well-known core non-classical system among many-valued and relevant logics (about its importance, cf. [8] and references therein) and is equivalent to Belnap and Dunn's logic B4 [2]. It is worth to mention here that, according to Slaney [12, p. 289], BN4 has the truth-functional implication most naturally associated the logic FDE. As a matter of fact, the logic BN4 finds an appealing place in the intersection between 4-valued logics and members in the family of relevant logics. For instance, Meyer *et al.* maintain that "BN4 is the correct logic for the 4-valued situation where the extra values are to be interpreted in the both and neither senses" [7, p. 253]. Accordingly, BN4 can also be seen as an interesting 4-valued extension of Routley and Meyer's basic logic B (cf. [11, chapter 4])—a central system in the family of relevant logics whose significance is briefly explained below—since the logic BN4 was first developed by taking as the starting point the axiomatization of B, as Brady himself stated. As a matter of fact, even though it is tempting to read BN4 as the B(oth) and N(either) 4-valued logic, the label was chosen by Brady because "the system contains the basic system B of Routley *et al.* 1982, Chapter 4, and has a characteristic 4-valued matrix set, one of the values being 'n', representing neither truth nor falsity"(cf. [3, p. 32, note 1]).

Routley and Meyer's relational semantics (R-M semantics) was introduced by the named authors in the early seventies of the past century to model relevant logics, but it was soon noticed to be a highly malleable instrument able of modelling other families of logics. The minimal (non-positive) logic characterized by Routley-Meyer semantics is Sylvan and Plumwood's logic BM [13], which is in fact the result of dropping the double negation axioms ($A \to \neg\neg A$, $\neg\neg A \to A$) from the system B. The basic logic B is also especially significant among relevant logics because is used as a starting system to define a wealth of extensions interpretable with the R-M semantics (cf. [11, chapter 4]).

Robles and Méndez developed the logic E4 [10], another interesting 4-valued logic which was built upon a modification of the conditional function of MBN4 (cf. Definition 2.1) and is also a proper extension of Routley and Meyer's logic B. Robles and Méndez suggested that E4 could be seen as the "4-valued logic of (relevant) entailment" whereas BN4 could be considered as the "4-valued logic of the relevant conditional". The main reason for this is that they believe that E4 is related to BN4 in a similar way to which Anderson and Belnap's logic of entailment E is related to the system R (cf. [1] about the logics E and R). In particular, according to Méndez and Robles, BN4 can intuitively be described as a 4-valued extension of contractionless relevant logic R (RW) and E4 as a 4-valued extension of reductioless logic Er[1].

Although E4 was presented as a companion to BN4 worthy of consideration, Robles and Méndez asserted that E4 might not be the only alternative to BN4 and set out six different variations of the conditional function of MBN4 which could turn out to be possibly interesting 4-valued logics in the family of relevant logics. Some research on the logics built upon these tables has recently been conducted in [6]. In particular, it has been proved that they are the only variants of MBN4 and ME4 which verify Routley and Meyer's logic B. Furthermore, they have been endowed with a Belnap-Dunn semantics. The aim of this paper is to provide a general reduced Routley-Meyer type semantics for those logics in order to connect them to the wide range of logics interpretable with this semantics and specially to the systems BN4 and E4, which have been also already interpreted by the R-M type semantics [9]. It is worth underlining that: (I) validity of formulae depends on a singleton in the reduced general Routley-Meyer semantics (i.e., the set of designated points is limited to a single element); (II) reduced models are generally preferable when there is the possibility to define them (cf. [4, 5]). As a matter of fact, we face some problems when defining reduced models for the majority of the logics here considered given the apparent ineliminability of disjunctive rules. However, these inconvenients are solved according to the methodology suggested in ([4, 5, 11]).

The structure of this paper proceeds as follows. In Section 2, the implicative variants of MBN4 and ME4 which verify Routley and Meyer's logic B are displayed. In Section 3, a basic logic which serves a mere instrumental role—the logic b4—is presented and also extended to each of the logics considered in this paper. Next, reduced general Routley-Meyer semantics for the logic b4 is provided in Section 4. In Section 5 and Section 6, a series of lemmas and notions for extensions of the logic b4—Eb4-logics—are recalled as they were already proved in previous papers ([3, 9]) following the method described in [11, chapter 4]. Finally, completeness theorems for

[1] RW will be the result of dropping the *axiom contraction* $([A \to (A \to B)] \to (A \to B))$ from R and Er the result of dropping the *axiom reductio* $((A \to \neg A) \to \neg A)$ from E. The label Er is Robles and Méndez's.

b4 and its extensions are proved in Section 7 and Section 8, respectively. In particular, in Section 8, the essential postulates of the extensions of b4 considered in this paper are displayed and proved adequate for Eb4-models where their corresponding axiom is valid.

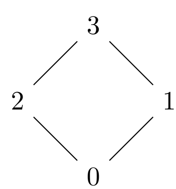
2. Implicative variants of MBN4 and ME4 which verify Routley and Meyer's logic B

In this section, I display the matrices upon which the logics considered in this paper were built, i.e., the implicative variants of MBN4 and ME4 which verify Routley and Meyer's logic B (cf. [11, chapter 4]).

First, the notions of Languages and Logics are fairly standard (cf. [9]). The propositional language $\mathcal{L}$ consists of a denumerable set of propositional variables p_0, p_1..., p_n, ... and some or all of the following connectives $\rightarrow$, $\wedge$, $\vee$, $\neg$. A, B, C, etc. are metalinguistic variables. Logics are formulated as Hilbert-style axiomatic systems. The notions of *proof* and *theorem* are understood as it is customary ($\Gamma \vdash_L A$ means that A is derivable from the set of wffs Γ in the logic L; and $\vdash_L A$ means that A is a theorem of the logic L).

Next, I introduce the matrices upon which systems BN4 and E4 are built.

Definition 2.1 (Brady's matrix MBN4 and Robles and Méndez's matrix ME4). The propositional language $\mathcal{L}$ consists of the connectives $\rightarrow$, $\wedge$, $\vee$ and $\neg$. Brady's matrix MBN4 and Robles and Méndez's matrix ME4 are the structures $< \mathcal{V}, \mathcal{D}, \mathcal{F} >$, where (i) $\mathcal{V}$ is $\{0, 1, 2, 3\}$ and it is partially ordered as shown in the following lattice:



(ii) $\mathcal{D} = \{2, 3\}$; (iii) $\mathcal{F} = \{f_\rightarrow, f_\wedge, f_\vee, f_\neg\}$ where $f_\wedge$ and $f_\vee$ are defined as the glb (or lattice meet) and the lub (or lattice join), respectively. $f_\neg$ is an involution with $f_\neg(0) = 3$, $f_\neg(3) = 0$, $f_\neg(1) = 1$ and $f_\neg(2) = 2$. Tables for $\wedge$, $\vee$ and $\neg$ are now displayed.

$\wedge$	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

$\vee$	0	1	2	3
0	0	1	2	3
1	1	1	3	3
2	2	3	2	3
3	3	3	3	3

	0	1	2	3
$\neg$	3	1	2	0

Finally, $f_\rightarrow$ is defined in each system according to the following tables[2]:

t1 (BN4)

$\rightarrow$	0	1	2	3
0	3	3	3	3
1	1	3	1	3
2	0	1	2	3
3	0	1	0	3

t5 (E4)

$\rightarrow$	0	1	2	3
0	3	3	3	3
1	0	2	0	3
2	0	0	2	3
3	0	0	0	3

Definition 2.2 (Variants of MBN4 and ME4 considered in this paper). Each M ti ($1 \leq i \leq 8$) is a structure $(\mathcal{V}, \mathcal{D}, F)$ where $\mathcal{V}$, $\mathcal{D}$, $f_\wedge$, $f_\vee$ and $f_\neg$ are defined as in MBN4 and ME4 (cf. Definition 2.1) and $f_\rightarrow$ is defined according to the corresponding ti below (t1 and t5 are left out here to refer to BN4 and E4, respectively):

t2

$\rightarrow$	0	1	2	3
0	3	3	3	3
1	0	3	0	3
2	0	0	2	3
3	0	0	0	3

t3

$\rightarrow$	0	1	2	3
0	3	3	3	3
1	1	3	1	3
2	0	0	2	3
3	0	0	0	3

t4

$\rightarrow$	0	1	2	3
0	3	3	3	3
1	0	3	0	3
2	0	1	2	3
3	0	1	0	3

t6

$\rightarrow$	0	1	2	3
0	3	3	3	3
1	0	2	0	3
2	0	1	2	3
3	0	0	0	3

t7

$\rightarrow$	0	1	2	3
0	3	3	3	3
1	0	2	1	3
2	0	0	2	3
3	0	0	0	3

t8

$\rightarrow$	0	1	2	3
0	3	3	3	3
1	0	2	1	3
2	0	1	2	3
3	0	0	0	3

Remark 2.3 (Implicative variants of MBN4 and ME4 which verify Routley and Meyer's logic B). The matrices considered in this paper are the only

[2]From now on, the labels t1 and t5 will be used to refer to the implicative tables of BN4 and E4, respectively, since labels t2-t4 will be used to refer to the implicative tables of the variants of BN4 and, likewise, t6-t8 will be used for those of the implicative variants of E4.

implicative variants of MBN4 (t2–t4) and ME4 (t6–t8) which verify Routley and Meyer's logic B (cf. [11, Chapter 4]). This was already proved as a Proposition in [6, Proposition 3.2].

3. The basic logic b4 and its extensions

The eight logics considered in this paper are developed in this section as implicative extensions of b4. Therefore, the basic logic b4 is a system contained in every Lti-logic ($1 \leq i \leq 8$), i.e., in every logic built upon the matrices characterized by the implicative tables displayed in Section 2 (cf. Definitions 2.1 and 2.2). As a matter of fact, the label b4 is intended to abbreviate "basic logic contained in every companion of BN4 or E4 which includes Routley and Meyer's logic B". In the following sections, b4 will be used as a common ground for the soundness and completeness proofs.

Definition 3.1 (The basic logic b4). The logic b4 is axiomatized with the following axioms and rules ADJ, MP, dMP, dPREF, dSUF, dCON, dCTE displayed below:
Axioms

A1 $A \to A$

A2 $(A \wedge B) \to A$ / $(A \wedge B) \to B$

A3 $[(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)]$

A4 $A \to (A \vee B)$ / $B \to (A \vee B)$

A5 $[(A \to C) \wedge (B \to C)] \to [(A \vee B) \to C]$

A6 $[A \wedge (B \vee C)] \to [(A \wedge B) \vee (A \wedge C)]$

A7 $\neg\neg A \to A$

A8 $A \to \neg\neg A$

A9 $\neg A \to [A \vee (A \to B)]$

A10 $B \to [\neg B \vee (A \to B)]$

A11 $(A \vee \neg B) \vee (A \to B)$

A12 $(A \to B) \vee [(\neg A \wedge B) \to (A \to B)]$

A13 $A \to [B \to [[(A \vee B) \vee \neg(A \vee B)] \vee (A \to B)]]$

Rules of inference

Adjunction: $A,\, B \Rightarrow A \wedge B$

Modus Ponens: $A,\, A \to B \Rightarrow B$

Disjunctive Modus Ponens: $C \vee A,\, C \vee (A \to B) \Rightarrow C \vee B$

Disjunctive prefixing: $C \vee (A \to B) \Rightarrow C \vee [(D \to A) \to (D \to B)]$

Disjunctive suffixing: $C \vee (A \to B) \Rightarrow C \vee [(B \to D) \to (A \to D)]$

Disjunctive Contraposition: $C \vee (A \to B) \Rightarrow C \vee (\neg B \to \neg A)$

Disjunctive Counterexample: $C \vee (A \wedge \neg B) \Rightarrow C \vee \neg(A \to B)$

Remark 3.2 (About b4). b4 is the result of adding the axioms A9–A13 and rules dMP, dPREF, dSUF, dCON and dCTE to Routley and Meyer's basic logic B (cf. [11, Chapter 4]). As a matter of fact, b4 can be seen as an extension of dB (i.e., the disjunctive version of Routley and Meyer's logic B).

Next, I prove some theorems of b4 which will be useful throughout this paper.

Proposition 3.3 (Some theorems and rules of b4). The following theorems and rules are derivable in b4.

T1 $A \leftrightarrow (A \vee A)$

T2 $[(A \to B) \wedge (C \to D)] \to [(A \wedge C) \to (B \wedge D)]$

T3 $[A \vee (B \vee C)] \leftrightarrow [(A \vee B) \vee C]$

T4 $(A \to B) \to [A \to (B \vee C)]$

T5 $(A \to B) \to [(A \wedge C) \to (B \vee D)]$

T6 $\neg(A \wedge B) \leftrightarrow (\neg A \vee \neg B)$

T7 $(\neg A \wedge \neg B) \leftrightarrow \neg(A \vee B)$

T8 $A \to [\neg A \vee (\neg A \to B)]$

T9 $\neg A \to [B \vee [(A \wedge B) \to C]]$

TRAN $A \to B,\, B \to C \Rightarrow A \to C$

SUM $A \to B \Rightarrow (A \vee C) \to (B \vee C)$

Every correspondent non-disjunctive version of the rules of b4 except for the rule MP (i.e., PREF, SUF, CON, CTE) is also a derived rule of b4.

Proof: T1–T7, SUM and TRAN are theorems and rules of the system B. T8 is obtained by A8, A9 and TRAN. T9 can be proved using A2, A9 and rules TRAN, CON and SUM. Finally, rules PREF, SUF, CON and CTE can easily be derived from their disjunctive version plus the rule MP, A4 and T1. $\square$

In the following lines, I introduce the extensions of b4 which I have referred to from the beginning of the section. In the first place, I define the notion of extensions (and expansions) of a propositional logic.

Definition 3.4 (Extensions and expansions of a propositional logic L). Let L be a logic formulated with axioms a_1,..., a_n and rules of derivation r_1,..., r_m. A logic L' includes L iff a_1,..., a_n are theorems of L' and rules r_1,..., r_m are provable in L'. If such were the case, L' would be either an extension of L (i.e., a strengthening of L in the language of L) or an expansion of it (i.e., a strengthening of L in an expansion of the language of L). We shall generally refer to extensions of a logic L by EL-logics.

Definition 3.5 (Extensions of b4 considered in this paper—Lti-logics). We refer by Lti ($1 \leq i \leq 8$) to the eight extensions of b4 considered in this paper, these are, BN4 (Lt1), E4 (Lt5) and the logics characterized by the implicative variants of MBN4 and ME4 (Lt2-Lt4 and Lt6-Lt8, respectively). Each Lti-logic is the result of adding the following axioms (from the list below) to b4:

Lt1 (BN4): A14-A16

Lt2: A17-A23

Lt3: A14, A15, A18, A19, A22–A24

Lt4: A16, A17, A20–A22

Lt5 (E4): A17–A21, A23, A25–A27

Lt6: A17, A20, A21, A23, A26, A28, A29

Lt7: A14, A18, A19, A21, A23, A26, A30

Lt8: A14, A21, A23, A26, A29, A30

Now, I display the list of axioms from which the Lti-logics are built:

A14 $(A \wedge \neg B) \to [(A \wedge \neg B) \to \neg(A \to B)]$

A15 $A \vee [\neg(A \to B) \to A]$

A16 $\neg B \vee [\neg(A \to B) \to \neg B]$

A17 $[A \wedge (A \to B)] \to B$

A18 $[(A \to B) \wedge \neg B] \to \neg A$

A19 $A \to [B \vee \neg(A \to B)]$

A20 $\neg B \to [\neg A \vee \neg(A \to B)]$

A21 $[\neg(A \to B) \wedge \neg A] \to A$

A22 $\neg(A \to B) \to (A \vee \neg B)$

A23 $[\neg(A \to B) \wedge B] \to \neg B$

A24 $B \to \{[B \wedge \neg(A \to B)] \to A]\}$

A25 $(A \to B) \vee \neg(A \to B)$

A26 $(\neg A \vee B) \vee \neg(A \to B)$

A27 $[(A \to B) \wedge (A \wedge \neg B)] \to \neg(A \to B)$

A28 $\neg(A \to B) \vee [(A \wedge \neg B) \to \neg(A \to B)]$

A29 $\{[\neg(A \to B) \wedge \neg A] \to \neg B\} \vee \neg B$

A30 $\{[\neg(A \to B) \wedge B] \to A\} \vee A$

4. Eb4-models in Routley-Meyer ternary relational semantics

In this section, the soundness of b4 is proved, but I establish the setting within which an RM-semantics for any Eb4-logic can be developed. In this sense, some of the facts upon which the soundness proof leans upon are applicable to Eb4-logics in general. First, we underline that the label EL will be used to refer to an extension of the logic L (cf. Definition 3.4). Now, we set the notion of Eb4-models.

Definition 4.1 (Eb4-models). An Eb4-model M is a structure $< T, K, R, *, \vDash>$ where K is a non-empty set, $T \in K$, R is a ternary relation on K and $*$ is a unary operator on K subject (at least) to the following definitions and postulates for all a, b, $c \in K$:

d1 $a \leq b =_{df} RTab$

d2 $a = b =_{df} a \leq b \;\&\; b \leq a$

d3 $R^2abcd =_{df} (\exists x \in K)\ (Rabx \;\&\; Rxcd)$

p1 $a \leq a$

p2 $(a \leq b \;\&\; Rbcd) \Rightarrow Racd$

p3 $R^2Tabc \Rightarrow (\exists x \in K)\ (RTbx \;\&\; Raxc)$

p4 $R^2Tabc \Rightarrow (\exists x \in K)\ (Rabx \;\&\; RTxc)$

p5 $a^{**} \leq a$

p6 $a \leq a^{**}$

p7 $a \leq b \Rightarrow b^* \leq a^*$

p8 RT^*TT^*

p9 $Rabc \Rightarrow (b \leq a^* \text{or } b \leq a)$

p10 $Rabc \Rightarrow (a \leq c \text{ or } a^* \leq c)$

p11 $RTab \Rightarrow (T^* \leq b \text{ or } a \leq T)$

p12 $(RTab \,\&\, R^2Tcde) \Rightarrow (a \leq c^* \text{ or } d \leq c^* \text{ or } c \leq b \text{ or } c \leq e)$

p13 $(Rabc \,\&\, Rcde) \Rightarrow (a \leq c \text{ or } b \leq c \text{ or } c^* \leq c \text{ or } d \leq c \text{ or } b \leq e)$

Finally, $\vDash$ is a valuation relation from K to the set of all wffs such that the following conditions (clauses) are satisfied for every propositional variable p, wffs A, B and $a \in K$:

(i) $(a \leq b \ \& \ a \vDash p) \Rightarrow b \vDash p$

(ii) $a \vDash A \wedge B$ iff $a \vDash A \ \& \ a \vDash B$

(iii) $a \vDash A \vee B$ iff $a \vDash A$ or $a \vDash B$

(iv) $a \vDash A \rightarrow B$ iff for all b, $c \in K$, $(Rabc \ \& \ b \vDash A) \Rightarrow c \vDash B$

(v) $a \vDash \neg A$ iff $a^* \nvDash A$

A structure $< T, K, R, *, \vDash>$ like the latest represent the most basic Eb4-model. On the other hand, other different Eb4-models can be defined adding new postulates to those of the basic Eb4-model presented above (i.e., p1–p13). As a matter of fact, reduced semantics for the Lti-logics displayed in Definition 3.5 (which are indeed Eb4-logics) are defined in Section 8 following the aforementioned method.

Definition 4.2 (b4-models). A b4-model is a Eb4-model with no additional postulate.

In what follows, the definitions of truth, validity and semantic consequence in a class of Eb4-models are defined.

Definition 4.3 (Truth in a class of Eb4-models). Let $\mathfrak{M}$ be a class of Eb4-models and $\mathsf{M} \in \mathfrak{M}$. A wff A is true in M iff $T \vDash A$ in this model.

Definition 4.4 (Validity in a class of Eb4-models). Let $\mathfrak{M}$ be a class of Eb4-models. A wff A is valid in $\mathfrak{M}$ (in symbols, $\vDash A$) iff $T \vDash A$ in all $\mathsf{M} \in \mathfrak{M}$.

Definition 4.5 (Semantic consequence in a class of Eb4-models). Let $\mathfrak{M}$ be a class of Eb4-models. Then, for all $\mathsf{M} \in \mathfrak{M}$ and any set of wffs Γ and wff A: $\Gamma \vDash_{\mathsf{M}} A$ (A is a semantic consequence of Γ in the model M) iff $T \vDash A$ if $T \vDash \Gamma$ ($T \vDash \Gamma$ iff $T \vDash B$ for all $B \in \Gamma$). Then, $\Gamma \vDash_{\mathfrak{M}} A$ (A is a semantic $\mathfrak{M}$-consequence of Γ) iff $\Gamma \vDash_{\mathsf{M}} A$ for all $\mathsf{M} \in \mathfrak{M}$.

Next, we specify what (in our view) constitutes a semantics and when a logic is endowed with a semantics.

Definition 4.6 (Reduced general Routley-Meyer semantics for Eb4-logics). Let L be a Eb4-logic. $\Sigma = \{\mathfrak{M}, \vDash\}$ is a semantics for L iff L is sound and complete with respect to Σ. If this condition is fulfilled, we establish that L-models (i.e., $\mathsf{M} \in \mathfrak{M}$) together with the definition of validity in $\mathfrak{M}$ constitute a Routley-Meyer semantics for L.

The following couple of lemmas is needed to prove that the system b4 is sound w.r.t. the semantics just defined.

Lemma 4.7 (Hereditary condition). *Let $\mathfrak{M}$ be a class of Eb4-models. For any $\mathsf{M} \in \mathfrak{M}$, a, $b \in K$ and wff A: $(a \leq b \,\&\, a \vDash A) \Rightarrow b \vDash A$.*

Proof: By induction on the length of A. The conditional case is proved with p2 and the negation case with p7 and d1. □

Lemma 4.8 (Entailment Lemma). *Let $\mathfrak{M}$ be a class of Eb4-models. For any wffs A, B, we have $\vDash_{\mathfrak{M}} A \to B$ iff $a \vDash A \Rightarrow a \vDash B$ for all $a \in K$ in all $\mathsf{M} \in \mathfrak{M}$.*

Proof: ($\Rightarrow$) By p1 and d1. ($\Leftarrow$) By d1 and Lemma 4.7. □

Proposition 4.9 ($\Gamma \vdash_{b4} A \Rightarrow \Gamma \vDash_{\mathsf{M}} A$). Let $\mathfrak{M}$ be a class of Eb4-models. For any set of wffs Γ, wff A and $\mathsf{M} \in \mathfrak{M}$, if $\Gamma \vdash_{b4} A$, then $\Gamma \vDash_{\mathsf{M}} A$.

Proof: We have to prove three different cases: (i) $A \in \Gamma$; (ii) A is an axiom; (iii) A is derived by means of any rule of b4. In the first place, case (i) is trivial. As for the case (ii), it will be proved that the axioms are valid in any class of Eb4-model. The validity of axioms A1–A8 is proved as in [11, Chapter 4] and that of A9 and A11 as (the validity of A10 and A13, respectively) in [9, p. 12]. Now, A10, A12 and A13 will be proved. We lean upon the Entailment Lemma (Lemma 4.8) and proceed by *reductio ad absurdum*. We also use the Hereditary Condition (Lemma 4.7) and clauses (ii)–(v) in Definition 4.1. Furthermore, the postulates displayed in Definition 4.1 will be used to prove the validity of the named axioms.

(A10) $B \to [\neg B \vee (A \to B)]$ is valid in any Eb4-model. Suppose that there are $a \in K$ in some Eb4-model M and wffs A, B such that (1) $a \vDash B$ but (2) $a \nvDash \neg B \vee (A \to B)$. By clause (iii) of Definition 4.1, (3) $a \nvDash \neg B$ & (4) $a \nvDash A \to B$. By clause (v) of the same definition and 3, (5) $a^* \vDash B$. Then, there are b, $c \in K$ such that (6) $Rabc$, (7) $b \vDash A$, (8) $c \nvDash B$ by 4 and clause (iv). Now, given 6 and p10 ($Rabc \Rightarrow (a \leq c \,\text{or}\, a^* \leq c)$), we have

$a \leq c$ or $a^* \leq c$. Finally, (9) $c \vDash B$ by applying the Hereditary Condition (Lemma 4.7) to either 1 or 5. However, 8 contradicts 9.

(A12) $(A \to B) \vee [(\neg A \wedge B) \to (A \to B)]$ is valid in any Eb4-model. Suppose that there is some Eb4-model M and wffs A, B such that (1) $T \nvDash (A \to B) \vee [(\neg A \wedge B) \to (A \to B)]$. Then, (2) $T \nvDash A \to B$ and (3) $T \nvDash (\neg A \wedge B) \to (A \to B)$. By clause (iv), (4) $RTab$, (5) $a \vDash A$, (6) $b \nvDash B$, for some a, $b \in K$—given 2—and (7) $RTcx$, (8) $c \vDash \neg A \wedge B$ and (9) $x \nvDash A \to B$ for some c, $x \in K$—given 3. Again, by applying clause (iv) to 9, we get (10) $Rxde$, (11) $d \vDash A$ and (12) $e \nvDash B$, for some d, $e \in K$. Similarly, by applying clause (ii) to 8, we obtain (13) $c \vDash \neg A$ (i.e., $c^* \nvDash A$) and $c \vDash B$. Now, given p12 ($(RTab \,\&\, R^2Tcde) \Rightarrow (a \leq c^*$ or $d \leq c^*$ or $c \leq b$ or $c \leq e)$), 4, 7 and 10, we have $a \leq c^*$ or $d \leq c^*$ or $c \leq b$ or $c \leq e$. Let us suppose $a \leq c^*$ or $d \leq c^*$, then we have (14) $c^* \vDash A$—contradicting 13—by applying the Hereditary Condition to either 5 or 11, respectively. Next, let us suppose $c \leq b$, then we get (15) $b \vDash B$ similarly, given 13. However, 15 contradicts 6. Finally, let us suppose $c \leq e$, we get (16) $e \vDash B$ (given 13), which contradicts 12.

(A13) $A \to [B \to [[(A \vee B) \vee \neg(A \vee B)] \vee (A \to B)]]$ is valid in any Eb4-model. Suppose that there is some Eb4-model M and wffs A, B such that (1) $a \vDash A$ but (2) $a \nvDash B \to [[(A \vee B) \vee \neg(A \vee B)] \vee (A \to B)]$. By clause (iv), (3) $Rabc$, (4) $b \vDash B$ and (5) $c \nvDash [(A \vee B) \vee \neg(A \vee B)] \vee (A \to B)$, for b, $c \in K$. Then, we get (6) $c \nvDash A \vee B$ (i.e., $c \nvDash A$ & $c \nvDash B$), (7) $c \nvDash \neg(A \vee B)$ (i.e., $c^* \vDash A \vee B$), (8) $c \nvDash A \to B$, by clause (iii). Now, by clause (iv), we have (9) $d \vDash A$ and (10) $e \nvDash B$ for some d, $e \in K$ such that (11) $Rcde$. Then, $a \leq c$ or $b \leq c$ or $c^* \leq c$ or $d \leq c$ or $b \leq e$ given 3, 11 and p13 ($(Rabc \,\&\, Rcde) \Rightarrow (a \leq c$ or $b \leq c$ or $c^* \leq c$ or $d \leq c$ or $b \leq e)$). Similarly as the proof of A12, we can easily see that a contradiction is reached whatever the case may be given the Hereditary Condition plus 1, 4, 6, 7, 9 and 10.

Case (iii), A is the result of applying a rule of b4, has on its own several subcases. The subcase of ADJ is trivial and the subcases when A is derived by MP and dMP are proved as in [9, Theorems 3.7, 3.10]. The remaining disjunctive rules are proved similarly[3]. As an example, let us prove the subcase when A is the result of applying the rule dCTE. Suppose $\Gamma \vDash_{\mathsf{M}} D \vee (B \wedge \neg C)$ for some wffs B, C and D. Furthermore, suppose $T \vDash \Gamma$.

[3] The reader can also find the proofs for some of these rules displayed in [11, Chapter 4, pp. 336, ff.].

Then, (1) $T \vDash D \vee (B \wedge \neg C)$ and by reductio, (2) $T \nvDash D \vee \neg(B \to C)$. Now, we have (3) $T \vDash D$ or $T \vDash B \wedge \neg C$ and (4) $T \nvDash D$ & $T \nvDash \neg(B \to C)$ (i.e., $T^* \vDash B \to C$) by applying clause (iii) to 1 and 2, respectively. Then, obviously, (5) $T \vDash B \wedge \neg C$ (i.e., $T \vDash B$ & $T^* \nvDash C$). Now, by applying clause (iv) to p8 (RT^*TT^*), 4 and 5, we obtain (6) $T^* \vDash C$. However, 5 contradicts 6. □

Theorem 4.10 (Soundness of b4). *For any set of wffs Γ and wff A: If $\Gamma \vdash_{b4} A$, then $\Gamma \vDash_{b4} A$.*

Proof: It is trivial given Proposition 4.9. □

5. Extension and primeness lemmas

In the present section, we shall introduce the extension lemmas. Firstly, we set the notion of Eb4-theories and several other related notions. We also display a couple of definitions and some lemmas which will be crucial points in the completeness theorem proved in Section 7. In these lemmas, we apply the method developed in "Relevant logics and their rivals I" (cf. [11, Chapter 4]) and followed by Brady (cf. [3, pp. 24–25]). We shall omit some of those proofs since they are similar to Brady's[4] [3].

In the first place, we set some preliminary definitions.

Definition 5.1 (Eb4-theories). Let L be an Eb4-logic. An L-theory $\mathcal{T}$ is a set of wffs closed under Adjunction (Adj) and provable L-entailment (L-ent). That is to say, a set of wffs is closed under Adj iff, whenever A, $B \in \mathcal{T}$, then $A \wedge B \in \mathcal{T}$; a set of wffs is closed under L-ent iff, whenever $A \to B$ is a theorem of L and $A \in \mathcal{T}$, then $B \in \mathcal{T}$.

Definition 5.2 (Types of Eb4-theories). Let L be an Eb4-logic and $\mathcal{T}$ an L-theory. We set (1) $\mathcal{T}$ is prime iff, for wffs A and B, whenever $A \vee B \in \mathcal{T}$, then either $A \in \mathcal{T}$ or $B \in \mathcal{T}$; (2) $\mathcal{T}$ is regular iff $\mathcal{T}$ contains all theorems in L; (3) $\mathcal{T}$ is trivial iff it contains every wff; (4) $\mathcal{T}$ is a-consistent (consistent in an absolute sense) iff $\mathcal{T}$ is not trivial; (5) $\mathcal{T}$ is empty iff it contains no wff.

[4]We shall also follow Robles and Méndez's structure and method for the extension lemmas (cf. [10, pp. 845-847]). In addition, some of the outlined proofs were already given in [6, 9].

Definition 5.3 (Sets of wffs closed under a certain rule). For any wffs A, B, C and D, a set of wffs Γ is closed by: (1) MP iff $A \to B \in \Gamma$ and $A \in \Gamma$, then $B \in \Gamma$; (2) dMP iff $(A \to B) \vee C \in \Gamma$ and $A \vee C \in \Gamma$ then $B \vee C \in \Gamma$; (3) CON iff $A \to B \in \Gamma$, then $\neg B \to \neg A \in \Gamma$; (4) $dCON$ iff $(A \to B) \vee C \in \Gamma$, then $(\neg B \to \neg A) \vee C \in \Gamma$; (5) $PREF$ iff $A \to B \in \Gamma$, then $(C \to A) \to (C \to B) \in \Gamma$; (6) $dPREF$ iff $(A \to B) \vee D \in \Gamma$, then $[(C \to A) \to (C \to B)] \vee D \in \Gamma$; (7) SUF iff $A \to B \in \Gamma$, then $(B \to C) \to (A \to C) \in \Gamma$; (8) $dSUF$ iff $(A \to B) \vee D \in \Gamma$, then $[(B \to C) \to (A \to C)] \vee D \in \Gamma$; (9) CTE iff $A \wedge \neg B \in \Gamma$, then $\neg(A \to B) \in \Gamma$; (10) $dCTE$ iff $(A \wedge \neg B) \vee C \in \Gamma$, then $\neg(A \to B) \vee C \in \Gamma$; (11) MT iff $A \to B \in \Gamma$ and $\neg B \in \Gamma$, then $\neg A \in \Gamma$; (12) $TRAN$ iff $A \to B \in \Gamma$ and $B \to C \in \Gamma$, then $A \to C \in \Gamma$.

Definition 5.4 (Full regularity). Let L be an Eb4-logic, an L-theory $\mathcal{T}$ is fully regular iff it is a regular L-theory (cf. Definitions 5.1 and 5.2) which is closed under the rules of b4 (i.e., MP, dMP, dCON, dPREF, dSUF, dCTE; cf. Definition 5.3).

Proposition 5.5 (Derived rules under which fully regular Eb4-theories are closed). Let L be an Eb4-logic, if $\mathcal{T}$ is a fully regular L-theory, then it is closed under (1) CON, (2) PREF, (3) SUF, (4) CTE, (5) MT and (6) TRAN.

Proof: Cases (1)–(4): by A4 and T1 ($A \leftrightarrow (A \vee A)$) and the fact that $\mathcal{T}$ is fully regular (i.e., closed under dCON, dPREF, dSUF and dCTE, respectively for each case). Cases (5)–(6): by hypothesis, $\mathcal{T}$ is fully regular (therefore closed under MP) and by the fact that $\mathcal{T}$ is closed under CON and SUF (given what has already been proved in cases (1) and (3)), respectively for each case. □

Definition 5.6 (Disjunctive Eb4-derivability). Let L be an Eb4-logic, Γ and Θ be non-empty sets of wffs, Θ is disjunctively derivable from Γ in Eb4 (in symbols, $\Gamma \vdash^d_L \Theta$) iff $A_1 \wedge ... \wedge A_n \vdash_L B_1 \vee ... \vee B_n$ for some wffs $A_1, ..., A_n \in \Gamma$ and $B_1, ..., B_n \in \Theta$.

The following lemma is essential in order to prove the *Extension to maximal sets* lemma (Lemma 5.9).

Lemma 5.7 (Preliminary lemma to the extension lemma). *Let L be an Eb4-logic closed under no other rules than those specified in Definition 5.4. For any wffs A, B_1,..., B_n, if $\{B_1, ..., B_n\} \vdash_L A$, then, for any wff C, $C \vee (B_1 \wedge ... \wedge B_n) \vdash_L C \vee A$.*

Proof: Induction on the length of A (cf. p. 27 in [3] and Lemma 6.2 in [10]). □

Now, the process of extending sets of wffs to maximal sets is required.

Definition 5.8 (Maximal sets). Let L be an Eb4-logic, Γ is an L-maximal set of wffs iff $\Gamma \nvdash^d_L \overline{\Gamma}$ ($\overline{\Gamma}$ is the complement of Γ).

Lemma 5.9 (Extension to maximal sets). *Let L be an Eb4-logic closed under no other rules than those specified in Definition 5.4, Γ and Θ sets of wffs such that $\Gamma \nvdash^d_L \Theta$. Then, there are sets of wffs Γ' and Θ' such that $\Gamma \subseteq \Gamma'$, $\Theta \subseteq \Theta'$, $\Theta' = \overline{\Gamma'}$ and $\Gamma' \nvdash^d_L \Theta'$ (i.e., Γ' is an L-maximal set such that $\Gamma' \nvdash^d_L \Theta'$).*

Proof: Cf. Lemma 9 in [3] and Lemma 6.4 in [10]. □

Finally, *Primeness* Lemma should be proved.

Lemma 5.10 (Primeness). *Let L be an Eb4-logic closed under no other rules than those specified in Definition 5.4. If Γ is an L-maximal set, then it is a fully regular prime L-theory.*

Proof: This Lemma was already provided for the exact same frame of logics in [6, Lemma 7.5]. □

6. Preliminary lemmas to the completeness theorem

Following Routley et al. (cf. [11, Chapter 4]), a series of preliminary lemmas will be proved in order to be used in the completeness proofs for the Lti-logics. We start by presenting the notion of a $\mathcal{T}$-theory.

Definition 6.1 ($\mathcal{T}$-theory). Let L be an Eb4-logic and $\mathcal{T}$ a fully regular prime L-theory (cf. Definition 5.1). A $\mathcal{T}$-theory is a set of formulas closed under Adjunction (Adj.) and $\mathcal{T}$-entailment ($\mathcal{T}$-ent.). That is, a is a $\mathcal{T}$-theory if whenever A, $B \in a$, then $A \wedge B \in a$; and if whenever $A \to B \in \mathcal{T}$ and $A \in a$, then $B \in a$.

Given the fact that $\mathcal{T}$ is regular (i.e., if $\vdash_{Eb4} A \to B$, then $A \to B \in \mathcal{T}$), it is remarkable that any $\mathcal{T}$-theory is an Eb4-theory. Therefore, if $\vdash_L A \to B$ and $A \in a$, then $B \in a$, given that a is closed under $\mathcal{T}$-ent.

Now, some relations on sets of $\mathcal{T}$-theories will be displayed.

Definition 6.2 (The sets K^T, K^C). Let $\mathcal{T}$ be a fully regular and prime Eb4-theory. K^T is the set of all $\mathcal{T}$-theories, and K^C is the set of all a-consistent non-empty and prime $\mathcal{T}$-theories (cf. Definition 5.2).

Definition 6.3 (The relations R^T, R^C and $\vDash^C$). Let $\mathcal{T}$ be a fully regular and prime $\mathcal{T}$-theory and K^T and K^C be defined as in Definition 6.2. R^T is defined on K^T as follows: for all a, b, $c \in K^T$, $R^T abc$ iff for all wffs A, B, $(A \to B \in a \;\&\; A \in b) \Rightarrow B \in c$. Next, R^C is the restriction of R^T to K^C. On the other hand, $\vDash^C$ is defined as follows: for any $a \in K^C$ and wff A, $a \vDash^C A$ iff $A \in a$.

Next, we define a unary operator on K^C.

Definition 6.4 (The operation $*^C$). The unary operation $*^C$ is defined on K^C as follows: for each $a \in K^C$, $a^{*^C} = \{A \mid \neg A \notin a\}$.

Let L be an Eb4-logic, we will use the Extension Lemma (cf. Lemma 5.9) to build a fully regular and prime L-theory $\mathcal{T}$ in Proposition 7.3. For now, we define upon $\mathcal{T}$ the notions of the sets and relations expressed above (Definitions 6.2–6.4) and we define the following notion.

Definition 6.5 (The canonical Eb4-model). Let L be an Eb4-logic, the canonical L-model is the structure $< \mathcal{T}, K^C, R^C, *^C, \vDash^C >$, whose members are understood according to definitions 6.2–6.4.

The canonical Eb4-model will be shown to be an Eb4-model by means of which non-theorems of L are falsified.

Now, some useful lemmas for the completeness theorem developed in Section 7 will be proved. Let us suppose that we are given a fully regular and prime Eb4-theory $\mathcal{T}$ upon which the items K^T, K^C, R^C, $*^C$, $\vDash^C$ are defined as in Definition 6.5. First, we investigate the relations R^T and R^C.

Lemma 6.6 (Defining x for a, b in R^T). *Let a, b be non-empty $\mathcal{T}$-theories. The set $x = \{B \mid \exists A(A \to B \in a\ \&\ A \in b)\}$ is a non-empty $\mathcal{T}$-theory such that $R^T abx$.*

Proof: x is a $\mathcal{T}$-theory: by T2 and the fact that x is closed under PREF. x is non-empty: by A13 (cf. [9, Lemma 5.5]). □

Lemma 6.7 (Extending b in $R^T abc$ to a member in K^C). *Let a and b be non-empty $\mathcal{T}$-theories, and a and c a-consistent, prime $\mathcal{T}$-theories such that $R^T abc$. Then, there is an a-consistent (and non-empty) prime $\mathcal{T}$-theory x such that $b \subseteq x$ and $R^T axc$.*

Proof: By the Kuratowski-Zorn's Lemma and T8 and T9 (cf. [11, pp. 309, ff.] and [9, Lemma 5.6]). □

Lemma 6.8 (Extending a in $R^T abc$ to a member in K^C). *Let a and b be non-empty $\mathcal{T}$-theories and c an a-consistent, prime $\mathcal{T}$-theory such that $R^T abc$. Then, there is an a-consistent (and non-empty) prime $\mathcal{T}$-theory x such that $a \subseteq x$ and $R^T xbc$.*

Proof: Cf. [9, Lemma 5.7]. □

Next, we set a definition to consider in relation to the succeeding lemma, which shows that the relation $\leq^C$ is just a set inclusion relation among a-consistent and non-empty prime $\mathcal{T}$-theories.

Definition 6.9 (The relation $\leq^C$). For any $a, b \in K^C$: $a \leq^C b$ iff $R^C \mathcal{T} ab$.

Lemma 6.10 ($\leq^C$ and $\subseteq$ are coextensive). *For any $a, b \in K^C$: $a \leq^C b$ iff $a \subseteq b$.*

Proof: ($\Rightarrow$) By A1. ($\Leftarrow$) By Definitions 6.2, 6.3 and 6.9 (cf. [9, Lemma 5.9]). □

In relation to the later, we also set the following lemma.

Lemma 6.11 (Extension to prime $\mathcal{T}$-theories). *Let a be a $\mathcal{T}$-theory and A a wff such that $A \notin a$. Then, there is a prime $\mathcal{T}$-theory x such that $a \subseteq x$ and $A \notin x$.*

Proof: By simply applying the Kuratowski-Zorn Lemma as in [11, Chapter 4, pp. 310–311]. □

Throughout the following lemmas, the unary operator $*$ will be investigated[5].

Lemma 6.12 (Primeness of $*$-images). *Let a be a prime $\mathcal{T}$-theory. Then, (1) a^{*^C} is a prime $\mathcal{T}$-theory as well; (2) for any wff A, $\neg A \in a^{*^C}$ iff $A \notin a$.*

Proof: (1) a^* is closed under $\mathcal{T}$-ent, by the fact that $\mathcal{T}$ is closed by CON; a^* is closed under Adj., by T6; a^* is prime, by T7. (2) By A7 and A8. □

Lemma 6.13 ($*^C$ is an operation on K^C). *Let a be an a-consistent and non-empty prime $\mathcal{T}$-theory, then a^{*^C} is an a-consistent and non-empty $\mathcal{T}$-theory as well.*

Proof: In Lemma 6.12, it was already proved that, given our hypothesis, a^* is also a prime $\mathcal{T}$-theory. Next, it is also clear that a^* is a-consistent: there is some wff A such that $A \in a$ (a is non-empty); therefore, $\neg A \notin a^*$ by Lemma 6.12. Similarly, since a is a-consistent, there is some wff A such that $A \notin a$; therefore, $\neg A \in a^*$ by the same lemma. □

Finally, next lemma proves that the relation $\vDash^C$ follows the clauses (i)–(v) in the definition of an Eb4-model (cf. Definition 4.1).

Lemma 6.14 (The relation $\vDash^C$ and clauses (i)–(v)). *For all a, b, $c \in K^C$ and wffs A, B:*

(i) $(a \leq^C b$ *and* $a \vDash^C p) \Rightarrow b \vDash^C p$

(ii) $a \vDash^C A \wedge B$ *iff* $a \vDash^C A$ *and* $a \vDash^C B$

(iii) $a \vDash^C A \vee B$ *iff* $a \vDash^C A$ *or* $a \vDash^C B$

(iv) $a \vDash^C A \to B$ *iff for all* $b, c \in K^C$, $(R^C abc$ *and* $b \vDash^C A) \Rightarrow c \vDash^C B$

(v) $a \vDash^C \neg A$ *iff* $a^* \nvDash^C A$

Proof: (i) Immediate by Lemma 6.10. (ii) By A2 and the fact that a is closed under Adj. (iii) By A4 and primeness of a. (iv) ($\Rightarrow$) Immediate by Definition 6.3; ($\Leftarrow$) By contraposition, we suppose $a \nvDash^C A \to B$ and

[5] The label C on $*^C$ will be omitted throughout the proofs of the following lemmas.

show that there are b, $c \in K^C$ such that $R^C abc$, $A \in b$ and $B \notin c$. We need Lemmas 6.6, 6.7 and 6.8. This kind of proof is already available in the literature (cf. [9, Lemma 5.13]). Finally, (v) is immediate by Definition 6.4. □

7. Completeness of b4

In this section, we shall prove strong completeness of b4 w.r.t. the semantics defined in Section 4. We start by defining the useful concept of "set of consequences of a set of wffs".

Definition 7.1 (The set of consequences of Γ in b4). The set of consequences in b4 of a set of wffs Γ (in symbols $Cn\Gamma[b4]$) is defined as follows: $Cn\Gamma[b4] = \{A \mid \Gamma \vdash_{b4} A\}$.

We note the following remark.

Remark 7.2 (The set of consequences of Γ in b4 is a fully regular theory). It is obvious that for any Γ, $Cn\Gamma[b4]$ contains all theorems of b4 and is closed under the rules of b4. Consequently, it is also closed under b4-entailment.

Proposition 7.3 (The building of $\mathcal{T}$). Let Γ be a set of wffs and A a wff such that $\Gamma \nvdash_{b4} A$. Then, there is a fully regular, a-consistent and prime b4-theory $\mathcal{T}$ such that $\Gamma \subseteq \mathcal{T}$ and $A \notin \mathcal{T}$.

Proof: Suppose $\Gamma \nvdash_{b4} A$ (i.e., $A \notin Cn\Gamma[b4]$ given Remark 7.2). Then, $Cn\Gamma[b4] \nvdash^{d}_{b4} \{A\}$ by Definition 5.6; otherwise $(B_1 \wedge ... \wedge B_n) \vdash_{b4} A$ for some B_1, ..., $B_n \in \Gamma$ and hence A would be in $Cn\Gamma[b4]$ after all. Next, there is some (fully regular and a-consistent) prime b4-theory $\mathcal{T}$ such that $\Gamma \subseteq \mathcal{T}$ (since $\Gamma \subseteq Cn\Gamma[b4]$) and $A \notin \mathcal{T}$, by application of Lemmas 5.7 and 5.9. □

Definition 7.4 (The canonical b4-model). The canonical b4-model is the structure $< \mathcal{T}, K^C, R^C, *^C, \vDash^C >$, where K^C, R^C, $*^C$, $\vDash^C$ are defined upon the b4-theory $\mathcal{T}$ as indicated in Definitions 6.2–6.4.

Once proved that the canonical b4-model is a b4-model, Proposition 7.3 is used to show $\Gamma \nvDash^C A$ in the canonical b4-model, this is, for any set of wffs Γ and wff A such that $\Gamma \nvdash_{b4} A$, it will be shown that A is not a semantic b4-consequence of Γ (cf. Definition 4.5).

In the following paragraphs, it will be proved that the canonical b4-model is indeed a b4-model. Firstly, we prove that b4 postulates hold canonically by using a correspondent axiom scheme or rule from Definition 3.1. In this sense, we shall talk about corresponding postulate (c.p.) to a rule or axiom scheme.

LEMMA 7.5 (b4 postulates hold canonically). *The semantical postulates (p1–p13) hold in the canonical b4-model.*

PROOF: Lemma 6.10 is used to simplify these proofs. p1, p2, p5, p6 and p7 are proved as in [11, Chapter 4] and p9 and p11 as in [9]. Also, in [11, p. 339], the c.p. to the (disjunctive) affixing rule is proved. The c.p. to the (disjunctive) sufixing and preffixing rules (i.e., p3 and p4) can be proved similarly. Let us now prove p8, p10, p12 and p13.

p8 $(R\mathcal{T}^*\mathcal{T}\mathcal{T}^*)$ holds in the canonical b4-model: Suppose that there are wffs A, B such that (1) $A \to B \in \mathcal{T}^*$ (i.e., $\neg(A \to B) \notin \mathcal{T}$) and $A \in \mathcal{T}$. Then, given that $\mathcal{T}$ is closed under CTE, we get (2) $A \wedge \neg B \notin \mathcal{T}$. Therefore, (3) $A \notin \mathcal{T}$ or $\neg B \notin \mathcal{T}$ since $\mathcal{T}$ is also closed under Adj. Finally, we get $\neg B \notin \mathcal{T}$ (i.e., $B \in \mathcal{T}^*$), given 1 and 3.

p10 $Rabc \Rightarrow (a \leq c$ or $a^* \leq c)$ holds in the canonical b4-model: Suppose that there are a, b, $c \in K^C$ and wffs A, B such that (1) $Rabc$ but (2) $A \in a$, $A \notin c$, $B \in a^*$ (i.e., $\neg B \notin a$) and $B \notin c$. Now, let C be a wff such that (3) $C \in b$ (b was non-empty). On the other hand, we have (4) $A \vee B \in a$ by 2, A4 $(A \to (A \vee B))$ and the fact that a is closed under $\mathcal{T}$-ent. Similarly, by A10 in the form $(A \vee B) \to {\boldsymbol .}\neg(A \vee B) \vee [C \to (A \vee B)]$, we get (5) $\neg(A \vee B) \vee [C \to (A \vee B)] \in a$. Therefore, (6) $\neg(A \vee B) \in a$ or $C \to (A \vee B) \in a$ (a is prime). If we suppose $C \to (A \vee B) \in a$, we get $A \vee B \in c$ –given 1 and 3–, contradicting 2 ($A \notin c$ and $B \notin c$). Therefore, let us take $\neg(A \vee B) \in a$. Then, (7) $\neg A \wedge \neg B \in a$ by T7 $(\neg(A \vee B) \to (\neg A \wedge \neg B))$ and (8) $\neg B \in a$ by A2 $((\neg A \wedge \neg B) \to \neg B)$. However, 8 contradicts 2 ($\neg B \notin a$).

p12 $(R\mathcal{T}ab \,\&\, R^2\mathcal{T}cde) \Rightarrow (a \leq c^*$ or $d \leq c^*$ or $c \leq b$ or $c \leq e)$ holds in the canonical b4-model: Suppose that there are a, b, c, d, $e \in K^C$ and wffs A, B, C, D such that (1) $R\mathcal{T}ab$ and $R^2\mathcal{T}cde$ but (2) $A \in a$, $A \notin c^*$(i.e., $\neg A \in c$), $B \in d$, $B \notin c^*$ (i.e., $\neg B \in c$), $C \in c$, $C \notin b$, $D \in c$ and $D \notin e$. By A12 in the form $[(A \vee B) \to (C \wedge D)] \vee \{[\neg(A \vee B) \wedge (C \wedge D)] \to [(A \vee B) \to (C \wedge D)]\}$ and the fact that $\mathcal{T}$ is regular and prime, (3) $(A \vee B) \to (C \wedge D) \in \mathcal{T}$ or $[\neg(A \vee B) \wedge (C \wedge D)] \to [(A \vee B) \to (C \wedge D)] \in \mathcal{T}$. Let us suppose (4) $(A \vee B) \to (C \wedge D) \in \mathcal{T}$. Now, we have (5) $A \vee B \in a$

by A4 ($A \to (A \vee B)$) and 2 ($A \in a$). Thus, (6) $C \wedge D \in b$, given 4, 5 and $R\mathcal{T}ab$ in 1. However, 2 ($C \notin b$) contradicts 6. Let us now suppose (7) $[\neg(A \vee B) \wedge (C \wedge D)] \to [(A \vee B) \to (C \wedge D)] \in \mathcal{T}$. We have (8) $\neg A \wedge \neg B \in c$ by 2 and therefore, (9) $\neg(A \vee B) \in c$ given T7 (cf. Proposition 3.3). Thus, (10) $\neg(A \vee B) \wedge (C \wedge D) \in c$ by 2. On the other hand, by d2 and 1, there is (11) $x \in K^C$ such that $R\mathcal{T}cx$ and $Rxde$. Then, (12) $(A \vee B) \to (C \wedge D) \in x$ by 7 and 10, since $R\mathcal{T}cx$ (by 11). Finally, (13) $C \wedge D \in e$ since we have 12, $Rxde$ (in 11) and $A \vee B \in d$ (by 2). However, 13 contradicts $D \notin e$ in 2.

p13 $(Rabc \,\&\, Rcde) \Rightarrow (a \leq c$ or $b \leq c$ or $c^* \leq c$ or $d \leq c$ or $b \leq e)$ holds in the canonical b4-model: Suppose that there are $a, b, c, d, e \in K^C$ and wffs A, B, C, D, E such that (1) $Rabc$ & $Rcde$ but (2) $A \in a$ & $A \notin c$; $B \in b$ & $B \notin c$; $C \in c^*$ & $C \notin c$; $D \in d$ & $D \notin c$; $E \in b$ & $E \notin e$. Then, we have (3) $B \wedge E \in b$ and also (4) $(A \vee D) \vee C \in a$ by A4 (in the form $A \to [A \vee (D \vee C)]$) and T3. Now, by A13 in the form $[(A \vee D) \vee C] \to \centerdot(B \wedge E) \to \{[[((A \vee D) \vee C) \vee (B \wedge E)] \vee \neg[((A \vee D) \vee C) \vee (B \wedge E)]] \vee [((A \vee D) \vee C) \to (B \wedge E)]\}$ and 4, we obtain (5) $(B \wedge E) \to \{[[((A \vee D) \vee C) \vee (B \wedge E)] \vee \neg[((A \vee D) \vee C) \vee (B \wedge E)]] \vee [((A \vee D) \vee C) \to (B \wedge E)]\} \in a$. Then, (6) $\{[((A \vee D) \vee C) \vee (B \wedge E)] \vee \neg[((A \vee D) \vee C) \vee (B \wedge E)]\} \vee [((A \vee D) \vee C) \to (B \wedge E)] \in c$, given $Rabc$ in 1, 3 and 5. This is, (7) $\{[((A \vee D) \vee C) \vee (B \wedge E)] \vee \neg[((A \vee D) \vee C) \vee (B \wedge E)]\} \in c$ or $[((A \vee D) \vee C) \to (B \wedge E)] \in c$ since c is prime. Let us suppose (8) $[((A \vee D) \vee C) \to (B \wedge E)] \in c$. Similarly as in 4, we have (9) $(A \vee D) \vee C \in d$ by A4 and T3 given 2 ($D \in d$). Then, (10) $B \wedge E \in e$ by 8 and 9 given $Rcde$ in 1. However, 10 contradicts 2 ($E \notin e$). Therefore, we suppose (11) $[((A \vee D) \vee C) \vee (B \wedge E)] \vee \neg[((A \vee D) \vee C) \vee (B \wedge E)] \in c$. Thus, by primeness of c, $[((A \vee D) \vee C) \vee (B \wedge E)] \in c$ or $\neg[((A \vee D) \vee C) \vee (B \wedge E)] \in c$. Suppose (12) $[((A \vee D) \vee C) \vee (B \wedge E)] \in c$, i.e., (13) $(A \vee D) \vee C \in c$ or $(B \wedge E) \in c$, given the fact that c is prime. Now, we have (14) $(A \vee D) \vee C \notin c$ by 2. Therefore, (15) $B \wedge E \in c$ (i.e., $B \in c$ and $E \in c$). However, 2 ($B \notin c$) contradicts 15. Finally, suppose (16) $\neg[((A \vee D) \vee C) \vee (B \wedge E)] \in c$. Then, by T7 and the fact that c is closed by Adj., (17) $\neg((A \vee D) \vee C) \in c$ & $\neg(B \wedge E) \in c$. Now, again by T7, (18) $\neg(A \vee D) \in c$ & $\neg C \in c$. But this contradicts 2 ($\neg C \notin c$). $\square$

Proposition 7.6 (The canonical b4-model is a b4-model). The canonical b4-model is indeed a b4-model.

Proof: Given Definition 7.4 and Proposition 7.3, the proof follows by the fact that $*^C$ is an operation on K^C (Lemma 6.13), the adequacy of the canonical clauses (Lemma 6.14) and the fact that postulates hold canonically (Lemma 7.5). □

Theorem 7.7 (Strong completeness of b4). *For any set of wffs Γ and wff A: if $\Gamma \vDash_{b4} A$, then $\Gamma \vdash_{b4} A$.*

Proof: For some set of wffs Γ and wff A, suppose $\Gamma \nvdash_{b4} A$. By Proposition 7.3, there is a fully regular, a-consistent and prime b4-theory $\mathcal{T}$ such that $\Gamma \subseteq \mathcal{T}$ and $A \notin \mathcal{T}$. Then, following Definition 7.4, the canonical b4-model is defined upon $\mathcal{T}$ and is indeed a b4-model, given Proposition 7.6. Then, $\Gamma \nvDash^C A$ since $\mathcal{T} \vDash^C \Gamma$ but $\mathcal{T} \nvDash^C A$. Therefore, $\Gamma \nvDash_{b4} A$ by Definition 4.5. □

8. Routley-Meyer ternary relational semantics for the Lt*i*-logics

In the present section, we endow the Lt*i*-logics with a Routley-Meyer ternary relational semantics. Given that a Routley-Meyer semantics for an Eb4-logic L is provided when L-models together with the notion of L-validity are defined (cf. Definition 4.6), the idea is to give a semantical postulate corresponding to each one of the axiom schemes A14–A30 in Definition 3.5. Then soundness and completeness theorems for extensions of the logic b4 with any of these schemes are immediate.

Next, I display the list of corresponding postulates (c.p.) to the axiom schemes of the Lt*i*-logics. In general, a postulate pj $(14 \leq j \leq 30)$ will be referred to as corresponding to an axiom scheme Aj iff (1) Aj is true in any Eb4-model M which contains pj and (2) pj is provable in any canonical Eb4-model where Aj is true.

p14 $Rabc \Rightarrow (Rc^*ab^*$ or Rc^*ba^* or Rc^*aa^* or $Rc^*bb^*)$ is the c.p. to A14 $(A \wedge \neg B) \to [(A \wedge \neg B) \to \neg(A \to B)]$

p15 $(RTab$ & $Ra^*cd) \Rightarrow (c \leq T$ or $c \leq b)$ is the c.p. to A15 $A \vee [\neg(A \to B) \to A]$

p16 $(RTab$ & $Ra^*cd) \Rightarrow (T^* \leq d$ or $b^* \leq d)$ is the c.p. to A16 $\neg B \vee [\neg(A \to B) \to \neg B]$

p17 $Raaa$ is the c.p to A17 $[A \wedge (A \to B)] \to B$

p18 Raa^*a^* is the c.p. to A18 $[(A \to B) \wedge \neg B] \to \neg A$

p19 Ra^*aa is the c.p. to A19 $A \to [B \vee \neg(A \to B)]$

p20 $Ra^*a^*a^*$ is the c.p. to A20 $\neg B \to [\neg A \vee \neg(A \to B)]$

p21 $Ra^*bc \Rightarrow (b \leq a$ or $b \leq a^*)$ is the c.p. to A21 $[\neg(A \to B) \wedge \neg A] \to A$

p22 $Ra^*bc \Rightarrow (a^* \leq c$ or $b \leq a)$ is the c.p. to A22 $\neg(A \to B) \to (A \vee \neg B)$

p23 $Ra^*bc \Rightarrow (a \leq c$ or $a^* \leq c)$ is the c.p. to A23 $[\neg(A \to B) \wedge B] \to \neg B$

p24 $(Rabc$ & $Rb^*de) \Rightarrow (a \leq e$ or $b \leq e$ or $d \leq c)$ is the c.p. to A24 $B \to \{[B \wedge \neg(A \to B)] \to A\}$

p25 $RTab \Rightarrow RT^*ab$ is the c.p. to A25 $(A \to B) \vee \neg(A \to B)$

p26 RT^*T^*T is the c.p. to A26 $(\neg A \vee B) \vee \neg(A \to B)$

p27 $Raaa^*$ or Ra^*aa^* is the c.p. to A27 $[(A \to B) \wedge (A \wedge \neg B)] \to \neg(A \to B)$

p28 $RTab \Rightarrow (RT^*aa^*$ or $Rb^*aa^*)$ is the c.p. to A28 $\neg(A \to B) \vee [(A \wedge \neg B) \to \neg(A \to B)]$

p29 $(RTab$ & $Ra^*cd) \Rightarrow (T^* \leq d$ & $b^* \leq d$ & $c \leq a^*)$ is the c.p. to A29 $\{[\neg(A \to B) \wedge \neg A] \to \neg B\} \vee \neg B$

p30 $(RTab$ & $Ra^*cd) \Rightarrow (c \leq T$ or $c \leq b$ or $a \leq d)$ is the c.p. to A30 $\{[\neg(A \to B) \wedge B] \to A\} \vee A$

Next, the notion of an Lti-model $(1 \leq i \leq 8)$ is defined.

Definition 8.1 (Lti-models). An Lti-model ($1 \leq i \leq 8$) M is a structure $< T, K, R, *, \vDash >$ where K, T, R, $*$ and $\vDash$ are defined according to the Definition 4.1 and R is also subject to an additional set of postulates for each Lti-logic:

Lt1-models: p14–p16;

Lt2-models: p17–p23;

Lt3-models: p14, p15, p18, p19, p22–p24;

Lt4-models: p16, p17, p20–p22;

Lt5-models: p17–p21, p23, p25-p27;

Lt6-models: p17, p20, p21, p23, p26, p28, p29;

Lt7-models: p14, p18, p19, p21, p23, p26, p30;

Lt8-models: p14, p21, p23, p26, p29, p30.

The definitions of truth, validity and semantic consequence in the Lti-models are defined as in Definitions 4.3–4.5. Now, given that Lemmas and Propositions were already proved for Eb4-logics and that the Lti-logics are indeed Eb4-logics, it suffices to prove that axioms A14–A30 are valid in Eb4-models where p14–p30 hold.

Proposition 8.2 (Validity of A14–A30). Let $\mathfrak{M}$ be a class of Eb4-models and $\mathsf{M} \in \mathfrak{M}$. Then, for any j ($14 \leq j \leq 30$), Aj is true in M iff pj holds in M.

Proof: We proceed as in Proposition 4.9. A few instances will suffice as an illustration. In particular, we display the proofs for A14, A16, A20, A24 and A29. Proofs for A17 and A18 are already in the main literature [11, Chapter 4] and those for A25 and A27 can be found in [9, p. 13]. The rest of the axiom schemes of the Lti-logics can be proved in a similar way.

(A14) $(A \wedge \neg B) \rightarrow [(A \wedge \neg B) \rightarrow \neg(A \rightarrow B)]$ is valid in any Eb4-model in which p14 holds: Suppose there are $a \in K$ in some Lti-model M and wffs A, B such that $a \vDash A \wedge \neg B$ (i.e., (1) $a \vDash A$ and (2) $a^* \nvDash B$) and (3) $a \nvDash (A \wedge \neg B) \rightarrow \neg(A \rightarrow B)$. Then, we have for some b, $c \in K$, (4) $Rabc$ & (5) $b \vDash A \wedge \neg B$ (i.e., $b \vDash A$ and $b^* \nvDash B$) & (6) $c \nvDash \neg(A \rightarrow B)$ (i.e., $c^* \vDash A \rightarrow B$) by clause (iv) in Definition 4.1. Given 4 and p14 ($Rabc \Rightarrow (Rc^*ab^*$ or Rc^*ba^* or Rc^*aa^* or Rc^*bb^*)), we get (7) Rc^*ab^*or

Rc^*ba^* or Rc^*aa^* or Rc^*bb^*. Now, whatever the case may be, we get a contradiction by applying clause (iv). If we suppose either (8) Rc^*ab^* or (9) Rc^*bb^*, then given 6 and, respectively, 1 and 5 we shall get (10) $b^* \vDash B$, contradicting 5. Similarly, if we suppose either (11) Rc^*ba^* or (12) Rc^*aa^*, we shall get (13) $a^* \vDash B$ again by 6 and, respectively, 5 and 1. However, 2 and 13 are contradictory.

(A16) $\neg B \vee [\neg(A \to B) \to \neg B]$ is valid in any Eb4-model in which p16 holds: Suppose that in some Lti-model M there are wffs A, B such that (1) $T \nvDash \neg B \vee [\neg(A \to B) \to \neg B]$, this is, (2) $T \nvDash \neg B$ (i.e., $T^* \vDash B$) and (3) $T \nvDash \neg(A \to B) \to \neg B$ given clause (iii). Now, by clause (iv), there are some a, $b \in K$ such that (4) $RTab$, (5) $a \vDash \neg(A \to B)$ (i.e., $a^* \nvDash A \to B$) and (6) $b \nvDash \neg B$ (i.e., $b^* \vDash B$). Again by clause (iv) and 5, there are c, $d \in K$ such that (7) Ra^*cd, (8) $c \vDash A$ and (9) $d \nvDash B$. Then, given 4, 7 and p16 (($RTab$ & Ra^*cd) $\Rightarrow$ ($T^* \leq d$ or $b^* \leq d$)), we get (10) $T^* \leq d$ or $b^* \leq d$. Now, given 9, a contradiction follows from any case shown in 10 by applying the Hereditary Condition (Lemma 4.7) to 2 or 6, respectively.

(A20) $\neg B \to [\neg A \vee \neg(A \to B)]$ is valid in any Lti-model in which p20 holds: Suppose there are $a \in K$ in some Lti-model M and wffs A, B such that (1) $a \vDash \neg B$ (i.e., $a^* \nvDash B$) and (2) $a \nvDash \neg A \vee \neg(A \to B)$, this is, (3) $a \nvDash \neg A$ (i.e., $a^* \vDash A$) and (4) $a \nvDash \neg(A \to B)$ (i.e., $a^* \vDash A \to B$). Now, by p20 ($Ra^*a^*a^*$), 3 and 4, we get (5) $a^* \vDash B$, which contradicts 1.

(A24) $B \to \{[B \wedge \neg(A \to B)] \to A\}$ is valid in any Eb4-model in which p24 holds: Suppose there are $a \in K$ in some Lti-model M and wffs A, B such that (1) $a \vDash B$ but (2) $a \nvDash [B \wedge \neg(A \to B)] \to A$. Then, there are some b, $c \in K$ such that (3) $Rabc$, (4) $b \vDash B \wedge \neg(A \to B)$ (i.e., $b \vDash B$ and $b^* \nvDash A \to B$) and (5) $c \nvDash A$. Again, there are some d, $e \in K$ such that (6) Rb^*de, (7) $d \vDash A$ and (8) $e \nvDash B$. Next, by 3, 6 and p24 (($Rabc$ & Rb^*de) $\Rightarrow$ ($a \leq e$ or $b \leq e$ or $d \leq c$)), we get ($a \leq e$ or $b \leq e$ or $d \leq c$). As in the previous proofs, whatever the case may be, a contradiction follows by the Hereditary Condition given 1, 4, 5, 7 and 8.

(A29) $\{[\neg(A \to B) \wedge \neg A] \to \neg B\} \vee \neg B$ is valid in any Eb4-model in which p29 holds: Suppose that in some Lti-model M there are wffs A, B such that (1) $T \nvDash \neg B \vee \{[\neg(A \to B) \wedge \neg A] \to \neg B\}$, this is, (2) $T \nvDash \neg B$ (i.e., $T^* \vDash B$) and (3) $T \nvDash [\neg(A \to B) \wedge \neg A] \to \neg B$. Then, by clause (iv), (4) $RTab$ & (5) $a \vDash \neg(A \to B) \wedge \neg A$ and (6) $b \nvDash \neg B$ (i.e., $b^* \vDash B$) for a, $b \in K$. Now, given 5, we have (7) $a \vDash \neg(A \to B)$ (i.e., $a^* \nvDash A \to B$) and (8) $a \vDash \neg A$ (i.e., $a^* \nvDash A$) by clause (ii). Again by applying clause (iv) to 7, there are c, $d \in K$ such that (9) Ra^*cd, (10) $c \vDash A$ and (11) $d \nvDash B$.

Next, we have (12) ($T^* \leq d$ or $b^* \leq d$ or $c \leq a^*$) given p29 ($(RTab$ & $Ra^*cd) \Rightarrow (T^* \leq d$ or $b^* \leq d$ or $c \leq a^*)$), 4 and 9. Let us suppose $T^* \leq d$ or $b^* \leq d$, then we have (13) $d \vDash B$ –contradicting 11– by applying the Hereditary Condition to either 2 or 6, respectively. Finally, let us suppose $c \leq a^*$, then we get (14) $a^* \vDash A$ similarly, given 10. However, 14 contradicts 8. □

Next, we prove the adequacy of the semantical postulates.

Proposition 8.3 (Proof of p14–p30 in canonical Eb4-models). Let L be an Eb4-logic and for any j ($1 \leq j \leq 30$) let the canonical L$\cup\{Aj\}$-model M be a canonical Eb4-model. Then, pj is provable in M.

Proof: The proof proceeds as that of Lemma 7.5. Proofs for p17 and p18 on the one hand and p25 and p27 on the other hand can be found in [11, Chapter 4] and [9, p. 22], respectively. We prove Proposition 8.3 for the postulates used above in Proposition 8.2. The rest of the postulates can be proved similarly.

p14 $Rabc \Rightarrow (Rc^*ab^*$ or Rc^*ba^* or Rc^*aa^* or $Rc^*bb^*)$ holds in any canonical Eb4-model where A14 is valid. In order to prove p14, A14 in the following form will be used: $[(A \wedge E) \vee (C \wedge G)] \wedge \neg[(D \vee F) \wedge (B \vee H)] \rightarrow .\{[(A \wedge E) \vee (C \wedge G)] \wedge \neg[(D \vee F) \wedge (B \vee H)]\} \rightarrow \neg\{[(A \wedge E) \vee (C \wedge G)] \rightarrow [(D \vee F) \wedge (B \vee H)]\}$. Suppose there are a, b, $c \in K^C$ and wffs A, B, C, D, E, F, G, H such that (1) $Rabc$ but (2) $A \rightarrow B \in c^*$, $A \in a$, $B \notin b^*$, (3) $C \rightarrow D \in c^*$, $C \in b$, $D \notin a^*$, (4) $E \rightarrow F \in c^*$, $E \in a$, $F \notin a^*$ and (5) $G \rightarrow H \in c^*$, $G \in b$, $H \notin b^*$. Given 2 and 4, we have $A \wedge E \in a$ and by A4, (6) $(A \wedge E) \vee (C \wedge G) \in a$. Similarly, given 2 and 3, we have $D \vee F \notin a^*$ (i.e., $\neg(D \vee F) \in a$) and by A4, (7) $\neg(D \vee F) \vee \neg(B \vee H) \in a$, this is, (8) $\neg[(D \vee F) \wedge (B \vee H)] \in a$ by applying T6. Next, we have (9) $[(A \wedge E) \vee (C \wedge G)] \wedge \neg[(D \vee F) \wedge (B \vee H)] \in a$ by 6 and 8. Now, by applying A14 in the aforementioned form, we get (10) $\{[(A \wedge E) \vee (C \wedge G)] \wedge \neg[(D \vee F) \wedge (B \vee H)]\} \rightarrow \neg\{[(A \wedge E) \vee (C \wedge G)] \rightarrow [(D \vee F) \wedge (B \vee H)]\} \in a$. As before, given 2 and 4, we have $C \wedge G \in b$ and by A4, (11) $(A \wedge E) \vee (C \wedge G) \in b$. Similarly, given 1 and 4, we get $B \vee H \notin b^*$ (i.e., $\neg(B \vee H) \in b$) and by A4, (12) $\neg(D \vee F) \vee \neg(B \vee H) \in b$. Whence, (13) $\neg[(D \vee F) \wedge (B \vee H)] \in b$ by T6. Then, we get (14) $[(A \wedge E) \vee (C \wedge G)] \wedge \neg[(D \vee F) \wedge (B \vee H)] \in b$ given 11 and 13. Now we have 1, 10 and 14, therefore (15) $\neg\{[(A \wedge E) \vee (C \wedge G)] \rightarrow [(D \vee F) \wedge (B \vee H)]\} \in c$ (i.e., $[(A \wedge E) \vee (C \wedge G)] \rightarrow [(D \vee F) \wedge (B \vee H)] \notin c^*$). In the following lines, we will get to $[(A \wedge E) \vee (C \wedge G)] \to [(D \vee F) \wedge (B \vee H)] \in c^*$, thus, a contradiction. By 2 ($A \to B \in c^*$ and $E \to F \in c^*$) and T4, we get (16) $A \to (B \vee H) \in c^*$ and (17) $E \to (D \vee F) \in c^*$. Whence (18) $[A \to (B \vee H)] \wedge [E \to (D \vee F)] \in c^*$ and by T2, (19) $(A \wedge E) \to [(B \vee H) \wedge (D \vee F)] \in c^*$. Now, proceeding again as in 16–19, we get (20) $(C \wedge G) \to [(B \vee H) \wedge (D \vee F)] \in c^*$ given 2 ($C \to D \in c^*$ and $G \to H \in c^*$). Finally, given 19 and 20, we get (21) $\{(A \wedge E) \to [(B \vee H) \wedge (D \vee F)]\} \wedge \{(C \wedge G) \to [(B \vee H) \wedge (D \vee F)]\} \in c^*$ and by applying T5, (22) $(A \wedge E) \vee (C \wedge G)] \to [(D \vee F) \wedge (B \vee H)] \in c^*$, as we needed.

p16 $(R\mathcal{T}ab$ & $Ra^*cd) \Rightarrow (\mathcal{T}^* \leq d$ or $b^* \leq d)$ holds in any canonical Eb4-model where A16 is valid. Suppose there are a, b, $c \in K^C$ and wffs A, B such that (1) $R\mathcal{T}ab$ and Ra^*cd but (2) $A \in \mathcal{T}^*$, $A \notin d$, $B \in b^*$ and $B \notin d$. Then, we have (3) $\neg A \notin \mathcal{T}$, whence (4) $\neg(A \vee B) \notin \mathcal{T}$ by A2 and T7. Next, we have for an arbitrary wff C, (5) $\neg(A \vee B) \vee [\neg[C \to (A \vee B)] \to \neg(A \vee B)] \in \mathcal{T}$ by A16. Therefore, (6) $\neg[C \to (A \vee B)] \to \neg(A \vee B) \in \mathcal{T}$ given 4 and the fact that $\mathcal{T}$ is prime. Now, we get (7) $\neg(A \vee B) \notin b$ by 2 ($B \in b^*$, i.e., $\neg B \notin b$), A2 and T7. Lastly, given 1, 6 and 7, we have (8) $\neg[C \to (A \vee B)] \notin a$ (i.e., $C \to (A \vee B) \in a^*$) and on the other hand (9) $A \vee B \notin d$ by 2. Thus, $C \notin c$ (by 1, 9 and 8), contradicting the fact that c is not empty.

p20 $Ra^*a^*a^*$ holds in any canonical Eb4-model where A20 is valid. Suppose that there are $a \in K^C$ and wffs A and B such that (1) $A \to B \in a^*$ ($\neg(A \to B) \notin a$) and (2) $A \in a^*$ (i.e., $\neg A \notin a$), whence (3) $\neg A \vee \neg(A \to B) \notin a$ since a is prime. We have to prove $B \in a^*$. Then, by 3 and A20 ($\neg B \to [\neg A \vee \neg(A \to B)]$), we have (4) $\neg B \notin a$, this is, $B \in a^*$.

p24 $(Rabc$ & $Rb^*de) \Rightarrow (a \leq e$ or $b \leq e$ or $d \leq c)$ holds in any canonical Eb4-model where A24 is valid. Suppose there are a, b, c, d, $e \in K^C$ and wffs A, B, C such that (1) $Rabc$ and Rb^*de but (2) $A \in a$, $A \notin e$, $B \in b$, $B \notin e$, $C \in d$ and $C \notin c$. Then, (3) $A \vee B \in a$ by 2 and A4. Next, using A24 in the form $(A \vee B) \to .[(A \vee B) \wedge \neg[C \to (A \vee B)]] \to C$, we get (4) $[(A \vee B) \wedge \neg[C \to (A \vee B)]] \to C \in a$. Given 1, 2 ($C \notin c$) and 4, we get (5) $(A \vee B) \wedge \neg[C \to (A \vee B)] \notin b$. Thus, (6) $A \vee B \notin b$ or $\neg[C \to (A \vee B)] \notin b$. However, given 2 ($B \in b$) and A4, we clearly have (7) $\neg[C \to (A \vee B)] \notin b$ (i.e., $C \to (A \vee B) \in b^*$). Finally, (8) $A \vee B \in e$ (by 1, 2 ($C \in d$) and 7), contradicting 2 ($A \notin e$ and $B \notin e$).

p29 $(R\mathcal{T}ab$ & $Ra^*cd) \Rightarrow (\mathcal{T}^* \leq d$ or $b^* \leq d$ or $c \leq a^*)$ holds in any canonical Eb4-model where A29 is valid. Suppose there are a, b, c, $d \in K^C$ and wffs A, B, C such that (1) $R\mathcal{T}ab$ and Ra^*cd but (2) $A \in \mathcal{T}^*$, $A \notin d$,

$B \in b^*$, $B \notin d$, $C \in c$ and $C \notin a^*$. Given 2 and A4, we have (3) $A \vee B \in \mathcal{T}^*$ (i.e., $\neg(A \vee B) \notin \mathcal{T}$). Then, by A29 in the form $\neg(A \vee B) \vee \{[\neg[C \to (A \vee B)] \wedge \neg C] \to \neg(A \vee B)\}$, we get (4) $[\neg[C \to (A \vee B)] \wedge \neg C] \to \neg(A \vee B) \in \mathcal{T}$ since $\mathcal{T}$ is regular and prime. As in 3, we have now (5) $A \vee B \in b^*$ (i.e., $\neg(A \vee B) \notin b$) by A4 and 2. Next, we get (6) $\neg[C \to (A \vee B)] \wedge \neg C \notin a$ given 1, 4 and 5. Thus, (7) $\neg[C \to (A \vee B)] \notin a$ or $\neg C \notin a$ (i.e., $C \in a^*$). Therefore, (8) $\neg[C \to (A \vee B)] \notin a$ (i.e., $C \to (A \vee B) \in a^*$) given 2 ($C \notin a^*$). Finally, we get (9) $A \vee B \in d$ (by 1, 2 and 8), contradicting 2. $\square$

9. Conclusion

The variants of BN4 and E4 which contain Routley and Meyer's logic B were developed in [6] as possible alternatives to the systems BN4 and E4. The Lti-logics are clearly related to the family of relevant logics since they enjoy the *quasi relevance property* characteristic of logics such as R-Mingle[6]. Given the position of Lti-logics among members in the family of relevant logics, the ternary relational semantics developed for them in the present paper could be seen as an essential tool to compare them to many other different logics of the said family. This work is also meant to be a detailing of how this kind of semantics works when it comes to 4-valued logics and an additional support to what is shown in [11, Chapter 4]: Routley-Meyer semantics is a malleable and powerful instrument for interpreting non-classical logics.

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References

[1] A. R. Anderson, N. D. Belnap, **Entailment: The Logic of Relevance and Necessity**, vol. I, Princeton University Press (1975).

[2] A. R. Anderson, N. D. Belnap, J. M. Dunn, **Entailment: The Logic of Relevance and Necessity**, vol. II, Princeton University Press (1992).

[6] The reader can find the proof of this and other related properties in [6].

[3] R. T. Brady, *Completeness proofs for the systems RM3 and BN4*, **Logique et Analyse**, vol. 25 (1982), pp. 9–32.

[4] R. T. Brady (ed.), **Relevant Logics and their Rivals**, vol. II, Ashgate (2003).

[5] R. T. Brady, **Universal Logic**, CSLI (2006).

[6] S. M. López, *Belnap-Dunn semantics for the variants of BN4 and E4 which contain Routley and Meyer's logic B*, **Logic and Logical Philosophy**, (Online First), pp. 1–28, DOI: https://doi.org/10.12775/LLP.2021.004.

[7] R. K. Meyer, S. Giambrone, R. T. Brady, *Where Gamma fails*, **Studia Logica**, vol. 43 (1984), pp. 247–256, DOI: https://doi.org/10.1007/BF02429841.

[8] H. Omori, H. Wansing, *40 years of FDE: An Introductory Overview*, **Studia Logica**, vol. 105 (2017), pp. 1021–1049, DOI: https://doi.org/10.1007/s11225-017-9748-6.

[9] G. Robles, J. M. Blanco, S. M. López, J. R. Paradela, M. M. Recio, *Relational Semantics for the 4-valued relevant logics BN4 and E4*, **Logic and Logical Philosophy**, vol. 25(2) (2016), pp. 173–201, DOI: https://doi.org/10.12775/LLP.2016.006.

[10] G. Robles, J. M. Méndez, *A companion to Brady's 4-valued relevant logic BN4: The 4-valued logic of entailment E4*, **Logic Journal of the IGLP**, vol. 24(5) (2016), pp. 838–858, DOI: https://doi.org/10.1093/jigpal/jzw011.

[11] R. Routley, V. Plumwood, R. K. Meyer, R. T. Brady, **Relevant Logics and their Rivals**, vol. I, Ridgeview (1982).

[12] J. K. Slaney, *Relevant Logic and Paraconsistency*, [in:] L. Bertossi, A. Hunter, T. Schaub (eds.), **Inconsistency Tolerance**, vol. 3300 of Lecture Notes in Computer Science, Springer (2005), pp. 270–293, DOI: https://doi.org/10.1007/978-3-540-30597-2_9.

[13] R. Sylvan, V. Plumwood, *Non-normal relevant logics*, [in:] R. T. Brady (ed.), **Relevant Logics and their Rivals**, vol. II, Western Philosophy Series, Ashgate, Aldershot and Burlington (2003), pp. 10–16.

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A NOTE ON THE INTUITIONISTIC LOGIC OF FALSE BELIEF

Abstract

In this paper we analyse logic of false belief in the intuitionistic setting. This logic, studied in its classical version by Steinsvold, Fan, Gilbert and Venturi, describes the following situation: a formula φ is not satisfied in a given world, but we still believe in it (or we think that it should be accepted). Another interpretations are also possible: e.g. that we do not accept φ but it is imposed on us by a kind of council or advisory board. From the mathematical point of view, the idea is expressed by an adequate form of modal operator W which is interpreted in relational frames with neighborhoods. We discuss monotonicity of forcing, soundness, completeness and several other issues. Finally, we mention the fact that it is possible to investigate intuitionistic logics of unknown truths.

Keywords: Intuitionistic modal logic, non-normal modal logic, neighborhood semantics.

2020 Mathematical Subject Classification: 03B45, 03B20, 03A10.

1. Preliminaries

Logic of false belief was studied e.g. by Steinsvold [3], Gilbert and Venturi [2] or Fan [1]. Those authors obtained several interesting results concerning completeness and expressivity. Their propositional systems were based on classical modal logics (i.e. with the law of the excluded middle). As for the semantics, they used relational (Kripke) and neighborhood frames.

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In general, the idea is to describe the following situation: φ is false but it is still believed (see [2]). This concept is expressed by the very definition of forcing: if w is a possible world then $w \Vdash \mathsf{W}\varphi \Leftrightarrow w \nVdash \varphi$ and $V(\varphi) \in \mathcal{N}_w$. The fact that φ is erroneously taken for true, is modelled by the second part of this definition: we see that $V(\varphi)$ is among our neighborhoods.

Other interpretations are also possible. For example, we deny φ (in a given world) but it is imposed on us by a kind of council or advisory board. We are *encouraged* to accept φ, at least in a particular world, because the set of all worlds accepting φ is one of our neighborhoods. This means that $V(\varphi)$ gathers worlds (and thus situations or circumstances) which are similar to our present situation, hence maybe we should rethink our opinion on φ. Also, we can identify possible worlds with different people, accepting (or not) various formulas. Then w-neighborhoods can be considered as (more or less) credible groups of advisors or lustrators.

We can assume that our worlds are pre-ordered and if $w \leq v$, then v accepts at least everything which was previously approved by w, i.e. lower located worlds have certain influence on the upper worlds. In some sense, we get a hierarchy of information and credibility. Now our model becomes intuitionistic: we have persistence of truth (with respect to $\leq$). This approach will be studied in the present paper. We are interested mostly in completeness and monotonicity of forcing. We show several intuitionistic versions of classical false belief systems. We discuss restrictions which can or have to be imposed on neighborhoods. We also point out some subtle limitations and advantages of intuitionistic framework in the context of minimal, maximal and intermediate canonical models. Finally, we make some comments on the intuitionistic logic of unknown truths. Classical systems of this kind are often examined together with logics of false belief. We show some general ideas, difficulties and suppositions.

2. Logic of false belief

2.1. Alphabet and language

Our logic is propositional, without quantifiers. We introduce the alphabet of our language below.

DEFINITION 2.1. $\mathbf{iE}^{\mathsf{W}}$-alphabet consists of:

1. PV which is a fixed denumerable set of propositional variables $p, q, r, s, \dots$
2. Common logical connectives and constants which are $\wedge$, $\vee$, $\perp$ and $\rightarrow$.
3. The only derived connective which is $\neg$ (thus $\neg\varphi$ is a shortcut for $\varphi \rightarrow \perp$).
4. One modal operator: W.

Well-formed formulas are built recursively in a standard style: if φ, ψ are *wff's* then also $\varphi \vee \psi$, $\varphi \wedge \psi$, $\varphi \rightarrow \psi$ and $\mathsf{W}\varphi$. Note that $\Leftarrow, \Rightarrow$ and $\Leftrightarrow$ are used only on the level of meta-language (which is classical).

2.2. Structures and models

Our initial structure is a pre-ordered neighborhood frame (**pn**-frame) defined as follows:

DEFINITION 2.2. **pn**-frame is a triple $F = \langle W, \mathcal{N}, \leq \rangle$ where $\leq$ is a partial order on W and $\mathcal{N}$ is a function from W into $P(P(W))$.

This definition is very general and it does not provide any relationship between $\leq$ and $\mathcal{N}$. In particular, it will not allow us to speak about monotonicity of forcing with respect to modal formulas. Thus, we shall introduce a particular subclass of **pn**-frames.

DEFINITION 2.3.
$\mathbf{iE}^{\mathsf{W}}\mathbf{pn}$-frame is a **pn**-frame with the following additional restriction:

$$[w \leq v, X \in \mathcal{N}_w, v \notin X] \Rightarrow X \in \mathcal{N}_v. \tag{2.1}$$

Having structures with appropriate features, we may introduce the notion of model. The first one is general and can be considered as a pattern for the further development of particular models.

DEFINITION 2.4. A **pn**-model is a quadruple $M = \langle W, \mathcal{N}, \leq, V \rangle$ where $\langle W, \mathcal{N}, \leq \rangle$ is a **pn**-frame and V is a function from PV into $P(W)$ such that: if $w \in V(q)$ and $w \leq v$ then $v \in V(q)$.

Definition 2.5. For every **pn**-model $M = \langle W, \mathcal{N}, \leq, V \rangle$, forcing of formulas in a world $w \in W$ is defined inductively:

1. $w \nVdash \bot$.
2. $w \Vdash q \Leftrightarrow w \in V(q)$ for any $q \in PV$.
3. $w \Vdash \varphi \wedge \psi$ (resp. $\varphi \vee \psi$) $\Leftrightarrow w \Vdash \varphi$ and (resp. or) $w \Vdash \psi$.
4. $w \Vdash \varphi \to \psi \Leftrightarrow v \nVdash \varphi$ or $v \Vdash \psi$ for each $v \in W$ such that $w \leq v$.

We do not make difference between primal valuation and its extended version, using only one symbol V. Let us use shortcut $V(\varphi)$ for $\{z \in W; z \Vdash \varphi\}$.

Remark 2.6. Of course, the definition above allows us to say that $w \Vdash \neg\varphi \Leftrightarrow$ for any $v \geq w$, $v \nVdash \varphi$.

Again, we narrow down our initial definition:

Definition 2.7. An $\mathbf{iE}^{\mathsf{W}}\mathbf{pn}$-model is a **pn**-model with valuation and forcing of non-modal formulas defined just like in Def. 2.5 but with an additional clause:

$w \Vdash \mathsf{W}\varphi \Leftrightarrow w \Vdash \neg\varphi$ and $V(\varphi) \in \mathcal{N}_w$.

2.3. Monotonicity of forcing

Here we prove the following fact:

Theorem 2.8. *In every* $\mathbf{iE}^{\mathsf{W}}\mathbf{pn}$*-model* $M = \langle W, \mathcal{N}, \leq, V \rangle$ *the following holds: if* $w \Vdash \gamma$ *and* $w \leq v$*, then* $v \Vdash \gamma$.

Proof: We shall discuss only the modal case. Assume that $\gamma = \mathsf{W}\varphi$, $w, v \in W$, $w \leq v$ and $w \Vdash \gamma$. Hence, $w \Vdash \neg\varphi$ and $V(\varphi) \in \mathcal{N}_w$. Of course $v \Vdash \neg\varphi$. In particular, it means that $v \notin V(\varphi) \in \mathcal{N}_w$. Now Cond. 2.1 allows us to say that $V(\varphi) \in \mathcal{N}_v$. Hence, $w \Vdash \mathsf{W}\varphi$. □

There is a difference between our definition of W and the one presented by the authors in [1] or [2]. Their operator was defined as follows: $w \Vdash \mathsf{W}\varphi \Leftrightarrow w \nVdash \varphi$ and $V(\varphi) \in \mathcal{N}_w$. However, their framework was classical, so there was no difference between lack of acceptance and acceptance of negation. In our intuitionistic setting, this approach would be problematic: if $w \nVdash \varphi$, then it does not mean that φ is denied in each (or even in one) world v

placed above w. In fact, there is no reason for it: this is the whole difference between classical and intuitionistic negation, at least from the semantical point of view. Hence, we had to modify the interpretation of W.

One could say that there is no need to use pre-orders. Maybe we should express everything in the language of neighborhoods? In fact, we have already established pure neighborhood semantics for intuitionistic modal logics[1]. However, it was applicable for normal modal logics. It was based on the assumption that *minimal* neighborhood (corresponding to the intuitionistic pre-order) is always contained in the *maximal* one (the idea was that forcing of $\Box\varphi$ in a given world w is equivalent with its forcing in any world from $\bigcup N_w$). In case of weak modal logics it would not be relevant. Of course, we may easily replace $\leq$ with neighborhoods but we think that it would lead us to the concept of two neighborhood families (one "intuitionistic" and one "modal"). This can be made but it would be rather a matter of notation and some aesthetic preferences.

Note that the concept of pre-ordered neighborhood model (for weak intuitionistic modal logics) was used (for example) by Dalmonte et al. in [4]. They used pre-order to speak about monotonicity of forcing—and neighborhoods to speak about modalities (well, they used *two* families of neighborhoods but one of them was connected with $\Box$ and the other one with $\Diamond$, none of them modelled pre-order which was, as we said, directly introduced).

2.4. Axiomatization

In this subsection we present sound and complete axiomatization of our basic system.

Definition 2.9. $\mathbf{iE}^{\mathsf{W}}$ is defined as the smallest set of formulas containing **IPC** $\cup$ {`WE`} and closed under the following set of inference rules: {`MP`, `REW`}, where:

1. **IPC** is the set of all intuitionistic axiom schemes and their modal instances (i.e. W-instances).
2. `WE` is the axiom scheme $\mathsf{W}\varphi \to \neg\varphi$.
3. `REW` is the *rule of extensionality*: $\varphi \leftrightarrow \psi \vdash \mathsf{W}\varphi \leftrightarrow \mathsf{W}\psi$.
4. `MP` is the rule *modus ponens*: $\varphi, \varphi \to \psi \vdash \psi$.

[1] See https://arxiv.org/pdf/1707.03859.pdf

The notion of syntactic consequence (i.e. $\vdash$) is rather standard: if Γ is a set of $\mathbf{iE}^{\mathsf{W}}$-formulas, then $w \vdash \varphi$ *iff* φ can be obtained from the finite subset of Γ by using axioms of $\mathbf{iE}^{\mathsf{W}}$ and both inference rules. Clearly, if $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$. The same concept of $\vdash$ will be accepted in the further systems.

The following theorem holds (and is simple to prove):

THEOREM 2.10. $\mathbf{iE}^{\mathsf{W}}$ *is sound with respect to the class of all* $\mathbf{iE}^{\mathsf{W}}\mathbf{pn}$*-frames.*

We can briefly prove completeness of our system with respect to the appropriate class of frames. For brevity, we assume that the reader is aware of the fact that each consistent $\mathbf{iE}^{\mathsf{W}}$-theory can be extended to the prime theory (this is just an intuitionistic version of the well-known Lindenbaum lemma). Hence, we may go directly to the canonical model.

For brevity, we start from the general pattern that will be used many times later.

DEFINITION 2.11. $\mathbf{iL}^{\mathsf{W}}\mathbf{can\text{-}pn}$-model is a triple $\langle W, \leq, \mathcal{N}, V\rangle$ where $\mathbf{L}$ may be any logic expressed in $\mathbf{iE}^{\mathsf{W}}$-language, and:

1. W is the set of all prime theories of the logic $\mathbf{L}$.
2. For every $w, v \in W$ we say that $w \leq v$ *iff* $w \subseteq v$.
3. $\mathcal{N}$ is a function from W into $P(P(W))$.
4. $V : PV \to P(W)$ is a function defined as it follows: $w \in V(q) \Leftrightarrow q \in w$.

Later we shall use the following shortcut: $\widehat{\varphi} = \{z \in W; \varphi \in z\}$. Now we may deal with the first particular case:

DEFINITION 2.12. $\mathbf{iE}^{\mathsf{W}}\mathbf{can\text{-}pn}$-model is an $\mathbf{iL}^{\mathsf{W}}\mathbf{can\text{-}pn}$-model where $\mathbf{L} = \mathbf{iE}^{\mathsf{W}}$ and for every $w \in W$ and for each formula φ:
$\mathcal{N}_w = \{\widehat{\varphi}; \mathsf{W}\varphi \in w\}$.

We need the following lemma:

LEMMA 2.13. $\mathbf{iE}^{\mathsf{W}}\mathbf{can\text{-}pn}$*-model is indeed an* $\mathbf{iE}^{\mathsf{W}}\mathbf{pn}$*-model.*

PROOF: In fact, we must check that the monotonicity holds. Let us assume that $w \subseteq v$ and $v \notin X \in \mathcal{N}_w$. Now $X = \widehat{\varphi}$ for certain φ such that $\mathsf{W}\varphi \in w$. However, w is contained in v, hence $\mathsf{W}\varphi \in v$. Thus $\widehat{\varphi} \in \mathcal{N}_v$. $\square$

Remark 2.14. Note that we did not use the fact that $v \notin X$. Actually, it was not important. For this reason, we may say that $\mathbf{iE}^{\mathsf{W}}\mathbf{can\text{-}pn}$-model satisfies even stronger restriction, namely:

$$[w \leq v, X \in \mathcal{N}_w] \Rightarrow X \in \mathcal{N}_v. \tag{2.2}$$

Clearly, our completeness theorem will be true for this smaller class of frames (models). However, we may be interested not only in narrowing down these classes for which completeness result can be proved, but also in broadening those which are *sufficient* for the monotonicity of forcing.

Our neighborhood function is well-defined. First, if we assume that W is a collection of all prime theories and $\widehat{\varphi} = \widehat{\psi}$, then we can easily prove that $\vdash \varphi \leftrightarrow \psi$ (using only non-modal tools). Second, assume that $\widehat{\varphi} \in \mathcal{N}_w$ and $\widehat{\varphi} = \widehat{\psi}$. If $\widehat{\varphi} \in \mathcal{N}_w$, then $\mathsf{W}\varphi \in w$. But $\varphi \leftrightarrow \psi \in \mathbf{iE}^{\mathsf{W}}$ (as we know from the first point of these considerations). Now, by means of `REW`, $\mathsf{W}\varphi \leftrightarrow \mathsf{W}\psi \in \mathbf{iE}^{\mathsf{W}} \subseteq w$. By `MP`, $\mathsf{W}\psi \in w$.

Our expected theorem about properties of the canonical model is below:

Theorem 2.15. *Let $M = \langle W, \leq, \mathcal{N}, V\rangle$ be a $\mathbf{iE}^{\mathsf{W}}\mathbf{can\text{-}pn}$-model. Then for each γ and for each $w \in W$ the following holds: $w \Vdash \gamma \Leftrightarrow \gamma \in w$.*

Proof: Boolean cases are simple (of course we should remember that implication is intuitionistic). As for the modal case, let us assume that $\gamma = \mathsf{W}\varphi$.

($\Rightarrow$)

Assume that $w \Vdash \gamma$. Hence, $w \Vdash \neg\varphi$ and $V(\varphi) \in \mathcal{N}_w$. By induction hypothesis $\widehat{\varphi} \in \mathcal{N}_w$. But then, by the very definition of canonical neighborhood, $\mathsf{W}\varphi \in w$.

($\Leftarrow$)

Let $\mathsf{W}\varphi \in w$. By means of `WE` and `MP` we infer that $\neg\varphi \in w$. This is Boolean case: we are able to prove in a standard manner[2] that $w \Vdash \neg\varphi$. From the definition of canonical neighborhood we have that $\widehat{\varphi} \in \mathcal{N}_w$. But $\widehat{\varphi} = \{z \in W; \varphi \in z\}$. Now we use induction hypothesis (which is true in any world of our canonical model) to say that $\{z \in W; \varphi \in z\} = \{z \in W; z \Vdash \varphi\}$. But the last set is precisely $V(\varphi)$. Hence $V(\varphi) \in \mathcal{N}_w$. We sum up our results to say that $w \Vdash \mathsf{W}\varphi$. $\Box$

[2]We may use the fact that $\neg\varphi$ can be written as $\varphi \to \bot$.

Now it is easy to formulate theorem about completeness:

THEOREM 2.16. $\mathbf{iE}^{\mathsf{W}}$ *is strongly complete with respect to the class of all* $\mathbf{iE}^{\mathsf{W}}\mathbf{pn}$*-frames; and also with respect to those in which neighborhood function satisfies Cond. 2.2.*

The proof of the theorem above is standard and based on the assumption that w is a theory and $w \nvdash \varphi$. The idea is to show that there is a prime theory v in a canonical model such that $w \subseteq v$ and $w \nVdash \varphi$.

One could say that our logic is not a proper system of "false belief" because it is (in its modal aspect) much weaker than some systems studied in [3], [2] and [1]. This will be discussed in the next subsection.

2.5. Stronger systems of false belief

It is not difficult to add one very natural axiom to our initial kit, namely $\mathtt{WC}$: $(\mathsf{W}\varphi \wedge \mathsf{W}\psi) \to \mathsf{W}(\varphi \wedge \psi)$. Here are necessary definitions:

DEFINITION 2.17. $\mathbf{iEC}^{\mathsf{W}}$ is defined as $\mathbf{iE}^{\mathsf{W}} \cup \{\mathtt{WC}\}$.

DEFINITION 2.18. $\mathbf{iEC}^{\mathsf{W}}\mathbf{pn}$-model is defined as $\mathbf{iE}^{\mathsf{W}}\mathbf{pn}$-model with one additional clause (the one of *closure under binary intersections*):

$$[X, Y \in \mathcal{N}_w] \Rightarrow X \cap Y \in \mathcal{N}_w. \tag{2.3}$$

Canonical model for $\mathbf{iEC}^{\mathsf{W}}$ (i.e. $\mathbf{iEC}^{\mathsf{W}}\mathbf{can\text{-}pn}$-model) is defined exactly in the same way as $\mathbf{iE}^{\mathsf{W}}\mathbf{can\text{-}pn}$-model (but its worlds are prime theories of $\mathbf{iEC}^{\mathsf{W}}$). Thus, we should only prove the following lemma:

LEMMA 2.19. $\mathbf{iEC}^{\mathsf{W}}\mathbf{can\text{-}pn}$*-model is indeed an* $\mathbf{iEC}^{\mathsf{W}}\mathbf{pn}$*-model.*

PROOF: This is simple. Assume that $X, Y \in \mathcal{N}_w$. Hence, $X = \widehat{\varphi}$ and $Y = \widehat{\psi}$ (for certain φ and ψ such that $\mathsf{W}\varphi \in w$ and $\mathsf{W}\psi \in w$). Then $X \cap Y = \widehat{\varphi} \cap \widehat{\psi} = \widehat{\varphi \wedge \psi}$. At the same time, we use axiom $\mathtt{WC}$ to say that $\mathsf{W}(\varphi \wedge \psi) \in w$. Thus $X \cap Y \in \mathcal{N}_w$. $\square$

Now we can say that:

THEOREM 2.20. $\mathbf{iEC}^{\mathsf{W}}$ *is strongly complete with respect to the class of all* $\mathbf{iEC}^{\mathsf{W}}\mathbf{pn}$*-frames (and those* $\mathbf{iEC}^{\mathsf{W}}\mathbf{pn}$*-frames which satisfy Cond. 2.2).*

Let us introduce another system: it will be an intuitionistic version of $\mathbf{M}^{\mathsf{W}}$ studied in [1].[3]

DEFINITION 2.21. $\mathbf{iM}^{\mathsf{W}}$ is defined as $\mathbf{iE}^{\mathsf{W}} \cup \{\texttt{RMW}\}$, where:

1. `RMW` is the rule $\varphi \to \psi \vdash (\mathsf{W}\varphi \land \neg\psi) \to \mathsf{W}\psi)$.

We introduce a new kind of models:

DEFINITION 2.22. $\mathbf{iM}^{\mathsf{W}}\mathbf{pn}$-model is an $\mathbf{iE}^{\mathsf{W}}\mathbf{pn}$-model with one additional clause (the one of *supplementation*):

$$[X \in \mathcal{N}_w, X \subseteq Y] \Rightarrow Y \in \mathcal{N}_w. \tag{2.4}$$

LEMMA 2.23. *$\mathbf{iM}^{\mathsf{W}}$ is sound with respect to the class of all $\mathbf{iM}^{\mathsf{W}}\mathbf{pn}$-models.*

PROOF: We shall check only `RMW`. Assume that $\varphi \to \psi$ is globally true. Suppose that there are $M = \langle W, \leq, \mathcal{N}, V\rangle$ and $w \in W$ such that $w \nVdash (\mathsf{W}\varphi \land \neg\psi) \to \mathsf{W}\psi)$. Hence, there is $v \geq w$ such that $v \Vdash (\mathsf{W}\varphi \land \neg\psi)$ but $v \nVdash \mathsf{W}\psi$. This means that: **i)** $v \Vdash \neg\varphi$, $V(\varphi) \in \mathcal{N}_v$, $v \Vdash \neg\psi$; and **ii)** $v \nVdash \neg\psi$ or $V(\psi) \notin \mathcal{N}_v$. It is not possible that $v \nVdash \neg\psi$. On the other hand, if $\varphi \to \psi$ is globally true, then $V(\varphi) \subseteq V(\psi)$. Supplementation allows us to say that $V(\psi) \in \mathcal{N}_v$. This is contradiction. □

Let us go to the canonical model.

DEFINITION 2.24. $\mathbf{iM}^{\mathsf{W}}\mathbf{can\text{-}pn}$-model is an $\mathbf{iL}^{\mathsf{W}}\mathbf{can\text{-}pn}$-model where $\mathbf{L} = \mathbf{iM}^{\mathsf{W}}$ and for every $w \in W$ and for each formula φ:

$\mathcal{N}_w = \{X \subseteq W;$ there is $Y \in n_w$ such that $Y \subseteq X\}$, where $n_w = \{\widehat{\varphi}; \mathsf{W}\varphi \in w\}$.

We must prove the following lemma:

LEMMA 2.25. *$\mathbf{iM}^{\mathsf{W}}\mathbf{can\text{-}pn}$-model is indeed an $\mathbf{iM}^{\mathsf{W}}\mathbf{pn}$-model.*

PROOF: Let us think about monotonicity condition, namely Cond. 2.1. Assume that $w \subseteq v$, $v \notin X$ and $X \in \mathcal{N}_w$. Now there is $Y \in n_w$ such that $Y \subseteq X$. However, as we already know, function n satisfies Cond. 2.2 which is even stronger than Cond. 2.1. Hence $Y \in n_v$ and thus $X \in \mathcal{N}_v$.

[3] Precisely speaking, Fan used axiom $\mathsf{W}(\varphi \land \psi) \land \neg\psi \to \mathsf{W}\psi$. The rule `REW` can be derived from this axiom and `WE`.

As for the supplementation, it is obvious by the very definition of $\mathbf{iM}^{\mathsf{W}}\mathbf{can\text{-}pn}$-model. Assume that $X \in \mathcal{N}_w$ and $X \subseteq Y$. Then there is $S \in n_w$ such that $S \subseteq X \subseteq Y$. We are ready. $\square$

Remark 2.26. Note that in fact $\mathbf{iM}^{\mathsf{W}}\mathbf{can\text{-}pn}$-model satisfies stronger monotonicity condition, i.e. Cond. 2.2. As in the case of $\mathbf{iE}^{\mathsf{W}}\mathbf{can\text{-}pn}$-model, we did not use the fact that $v \notin X$.

The next theorem is crucial for completeness:

THEOREM 2.27. *Let $M = \langle W, \leq, \mathcal{N}, V\rangle$ be an $\mathbf{iM}^{\mathsf{W}}\mathbf{can\text{-}pn}$-model. Then for each γ and for each $w \in W$ the following holds: $w \Vdash \gamma \Leftrightarrow \gamma \in w$.*

PROOF: We consider the modal case. Assume that $\gamma = \mathsf{W}\varphi$.

($\Rightarrow$)

Let $w \Vdash \mathsf{W}\varphi$. Then $w \Vdash \neg\varphi$ and $V(\varphi) \in \mathcal{N}_w$. Hence $\neg\varphi \in w$ and $\widehat{\varphi} \in \mathcal{N}_w$. Also there is $\widehat{\psi} \in n_w$ such that $\widehat{\psi} \subseteq \widehat{\varphi}$ and $\mathsf{W}\psi \in w$. Now $\vdash \psi \to \varphi$, i.e. this formula is a theorem. Hence, (by means of `RMW`) $\vdash (\mathsf{W}\psi \wedge \neg\varphi) \to \mathsf{W}\varphi$. Assume now that $\mathsf{W}\varphi \notin w$. There are two possible reasons. First: $\mathsf{W}\psi \notin w$ (contradiction). Second: $\neg\varphi \notin w$. But $\neg\varphi \in w$, as we already know.

($\Leftarrow$)

Assume that $\mathsf{W}\varphi \in w$. Now $\neg\varphi \in w$ and then $w \Vdash \neg\varphi$. Then $\widehat{\varphi} \in \mathcal{N}_w$. By induction hypothesis, $V(\varphi) \in \mathcal{N}_w$. Thus, $w \Vdash \mathsf{W}\varphi$. $\square$

THEOREM 2.28. $\mathbf{iM}^{\mathsf{W}}$ *is strongly complete with respect to the class of all* $\mathbf{iM}^{\mathsf{W}}\mathbf{pn}$*-models (and those* $\mathbf{iM}^{\mathsf{W}}\mathbf{pn}$*-models which satisfy Cond. 2.2).*

Let us sum up these results.

DEFINITION 2.29. $\mathbf{iK}^{\mathsf{W}}$ is defined as $\mathbf{iM}^{\mathsf{W}} \cup \{$`WC`$\}$.

DEFINITION 2.30. $\mathbf{iK}^{\mathsf{W}}\mathbf{pn}$-model is an $\mathbf{iE}^{\mathsf{W}}\mathbf{pn}$-model satisfying both supplementation and closure under binary intersections.

DEFINITION 2.31. $\mathbf{iK}^{\mathsf{W}}\mathbf{can\text{-}pn}$-model is defined just like $\mathbf{iM}^{\mathsf{W}}\mathbf{can\text{-}pn}$-model (but W consists of $\mathbf{iK}^{\mathsf{W}}$ prime theories).

THEOREM 2.32. $\mathbf{iK}^{\mathsf{W}}$ *is strongly complete with respect to the class of all supplemented* $\mathbf{iM}^{\mathsf{W}}\mathbf{pn}$*-models closed under binary intersections (and those of them which satisfy Cond. 2.2).*

PROOF: It is enough to check closure under intersections in $\mathbf{iK}^{\mathsf{W}}\mathbf{can\text{-}pn}$-model. Let $X, Y \in \mathcal{N}_w$. Hence, there are $\widehat{\varphi}, \widehat{\psi} \in n_w$ such that $\widehat{\varphi} \subseteq X$ and $\widehat{\psi} \subseteq Y$. Clearly, $\widehat{\varphi} \cap \widehat{\psi} = \widehat{\varphi \wedge \psi} \subseteq X \cap Y$. Also, by means of $\mathtt{WC}$, $\mathsf{W}(\varphi \wedge \psi) \in w$ and thus $X \cap Y \in n_w$. Finally, $X \cap Y \in \mathcal{N}_w$. □

There are some issues which should be discussed. This will be done in the next subsection.

2.6. About canonical models and monotonicity

In general, we borrowed some ideas from [1] and [2]. However, there are some subtle differences. Let us resume the line of thought presented in [1] with respect to the classical version of $\mathbf{iM}^{\mathsf{W}}$, that is $\mathbf{M}^{\mathsf{W}}$.

i) Fan assumed that the canonical model for $\mathbf{M}^{\mathsf{W}}$ is any model based on maximal theories[4] in which $\mathsf{W}\varphi \vee \varphi \in w \Leftrightarrow \widehat{\varphi} \in \mathcal{N}_w$ [*]. Let us define also another condition for the further needs: $\mathsf{W}\varphi \in w \Leftrightarrow \widehat{\varphi} \in \mathcal{N}_w$ [**].

Thus, he has defined the whole family of such models (from the minimal to the maximal one; the former contains precisely proof-sets[5], the latter consists of proof-sets *and* all non-proof-sets).

Whereas we defined our n_w (for any $w \in W$) precisely just like in the minimal model. We said that n_w contains *only* those $\widehat{\varphi}$ for which $\mathsf{W}\varphi \in w$. Also, our line of reasoning was closer to [**] than to [*] but this is not crucial here.

ii) Then Fan introduced the notion of supplemented canonical model M^+ (supplementation of canonical model M, in other words) in which $\mathcal{N}^+_w = \{X \subseteq W$; there is $Y \in \mathcal{N}_w$ such that $Y \subseteq X\}$. He showed that M^+ is indeed canonical: that it satisfies [*]. Due to some reasons, it would be problematic for him to show that M^+ satisfies [**]. It would require typical monotonicity rule $\varphi \to \psi \vdash \mathsf{W}\varphi \to \mathsf{W}\psi$ which is not sound.

Our way is different. We do not say that neighborhood function $\mathcal{N}$ in our $\mathbf{iM}^{\mathsf{W}}\mathbf{can\text{-}pn}$-model is "canonical" in the same sense as n. It would be irrelevant because the definition of n_w leaves no place for any variants: as we said, these are *precisely* proof-sets satisfying certain property. However, maybe it would be sensible to follow Fan directly? Assume that n_w is defined by means of, let us say, clause [**]. It can be [*] also, it does not

[4] With valuation defined as usual.

[5] Such proof-sets $\widehat{\varphi}$ that $\mathsf{W}\varphi \vee \varphi \in w$, of course.

matter: the real problem lies in the persistence of truth. Both [*] and [**] are too vague to force monotonicity (with respect to n and in the sense of Cond. 2.1 or Cond. 2.2). Assume that $w \subseteq v$ and $X \in n_w$. We may also suppose that $v \notin X$. If $X = \widehat{\varphi}$ for certain φ, then we can repeat our actual reasoning. But if $X \neq \widehat{\varphi}$ for any φ, then we cannot say anything special about this fact. Of course, if our model was maximal, then by the very definition X would belong to n_v. But if not, then we would be in a quandary.

It seems that a similar solution to a similar dilemma has been obtained in [4]. Recall that these authors prepared bi-neighborhood semantics for weak intuitionistic modal logics and they also used minimal canonical models. In case of richer logics they used canonical models "equipped with" supplementation (just as our $\mathbf{iM}^{\mathsf{W}}$**can-pn**-model), not the supplementation of previously defined model.

Gilbert and Venturi found different solution than Fan did. They assumed that neighborhoods in canonical model (for the classical version of $\mathbf{iK}^{\mathsf{W}}$) are defined by means of [**]. Then they used *negative* supplementation. This is the following condition:

$$Y \in \mathcal{N}_w, \quad Y \subseteq X, \quad w \notin X \;\Rightarrow\; X \in \mathcal{N}_w$$

In the negative supplementation of canonical model, for any $w \in W$ and for each φ we have:

$$\mathcal{N}_w^+ = \{X \subseteq W;\ \text{there is } Y \in \mathcal{N}_w \text{ such that } Y \subseteq X \text{ and } w \notin X\}.$$

Again, this "feature of negativity" (that is, the assumption that $w \notin X$) is helpful in proving that negative supplementation is indeed canonical. From our point of view, one thing is interesting. Let us reproduce the definition of $\mathbf{iK}^{\mathsf{W}}$**can-pn**-model but with the following definition of neighborhoods:

$$\mathcal{N}_w = \{X \subseteq W;\ \text{there is } Y \in n_w \text{ such that } Y \subseteq X \text{ and } w \notin X\},$$

where $n_w = \{\widehat{\varphi}; \mathsf{W}\varphi \in w\}$.

This is in accordance with our previous considerations. The whole proof of completeness is almost identical. However, there is one noteworthy moment. Let us prove that Cond. 2.1 of monotonicity is satisfied. Let $w \subseteq v$, $X \in \mathcal{N}_w$ and $v \notin X$. There is $Y \in n_w$ such that $Y \subseteq X$. However,

n satisfies Cond. 2.2 (Lem. 2.13 and Rem. 2.14), so $Y \in n_v$. Thus $X \in \mathcal{N}_v$. Note that in this case we prove only that $\mathcal{N}$ satisfies Cond. 2.1 and not necessarily Cond. 2.2. Clearly, we used the assumption that $v \notin X$.

3. Conclusion and future work

In this paper we have discussed several false belief systems based on the intuitionistic core. In general, these results are rather natural. However, it does not mean that all of them are straightforward. We have pointed out some nuances and compared our results with those of other authors.

Logics of false belief are often investigated together with the logics of *unknown truths*. It seems that this line of research is more complicated. Classical versions of these systems were introduced by Steinsvold in [3]. Neighborhood semantics for them has been presented by Gilbert and Venturi in [2]. Some new results on this matter have been later obtained by Fan in [1].

The very idea of *uknown truth* is that a formula is *true* but *not known*. This is expressed by the following interpretation of the modal operator $\bullet$:

$$w \Vdash \bullet\varphi \Leftrightarrow w \Vdash \varphi \text{ and } V(\varphi) \notin \mathcal{N}_w. \tag{3.1}$$

We may also say that φ is accepted (by our agent) but it is not suggested by his "advisory board".

As for the operator $\bullet$, it can be used interchangeably with $\circ$ which is defined as below:

$$w \Vdash \circ\varphi \Leftrightarrow w \nVdash \varphi \text{ or } V(\varphi) \in \mathcal{N}_w \Leftrightarrow \text{ if } w \Vdash \varphi \text{ then } V(\varphi) \in \mathcal{N}_w. \tag{3.2}$$

In a classical setting we may identify $\circ$ with $\neg \bullet \varphi$ and $\bullet$ with $\neg \circ \varphi$. The authors mentioned above have already proved completeness of several systems based on $\circ$ (or, equivalently, $\bullet$).

Things become more complex when our logic is intuitionistic. There are at least three problems: monotonicity of $\circ$[6]; interchangeability of $\bullet$

[6] Assume that $w \Vdash \circ\varphi$. It can mean that $w \nVdash \varphi$. Just as in the case of our earlier operator W, it is not reasonable to expect that φ will be rejected in each $v \geq w$. It would be difficult (if at all possible) to impose an appropriate condition on frames. One possible solution is to replace rejection of φ with an acceptance of $\neg\varphi$. However, this does not give us mutual duality of $\circ$ and $\bullet$, at least not the same as in the classical system.

and $\circ$[7]; soundness and completeness. We shall not discuss these issues here: they deserve more detailed studies which we consider as our future work.

Finally, we think that it would be natural to connect (both classical and intuitionistic) logics of false belief (or / and unknown truths) with some paraconsistent tools. Actually, we think about operators of indeterminacy (N) and ambiguity (M), invented and investigated by Żabski in [5]. Basically, Żabski assumed that valuation V connects each formula φ with 0 or 1. Now $V(N\varphi) = 1 \Leftrightarrow V(\varphi) = 0$ and $V(\neg\varphi) = 0$, while $V(M\varphi) = 1 \Leftrightarrow V(\varphi) = 0$ and $V(\neg\varphi) = 0$. Of course negation is not classical or intuitionistic here. Rather, it is paraconsistent. Note that it makes sense, from the philosophical point of view, to model the following situation: φ is undetermined or ambiguous (e.g. in a given world w), yet it is believed (or suggested by our advisors). We have already made some (unpublished) attempts in this direction, also in the quasi-intuitionistic setting (namely, in paraconsistent models with persistence of truth).

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References

[1] J. Fan, *Notes on neighborhood semantics for logics of unknown truths and false beliefs* (2020), `arXiv:2002.09622`, math.LO.

[2] D. Gilbert, G. Venturi, *Neighborhood Semantics for Logics of Unknown Truths and False Beliefs*, **The Australasian Journal of Logic**, vol. 14(1) (2017), pp. 246–267, DOI: https://doi.org/10.26686/ajl.v14i1.4033.

[3] C. Steinsvold, *Being Wrong: Logics for False Belief*, **Notre Dame Journal of Formal Logic**, vol. 52(3) (2011), pp. 245–253, DOI: https://doi.org/10.1215/00294527-1435438.

[7] One could say that partial or total lack of duality would not be bad in this case. Note that mutual *independence* of $\Box$ and $\Diamond$ is a rather desired property in intuitionistic modal logic. This approach can be transferred to the logics of unknown truths. Moreover, we may always limit our attention to only one of these operators.

[4] N. O. T. Dalmonte, Ch. Grellois, *Intuitionistic non-normal modal logics: A general framework*, **Journal of Philosophical Logic**, vol. 49 (2020), pp. 833–882, DOI: https://doi.org/10.1007/s10992-019-09539-3.

[5] E. Żabski, **Logiki nihilistyczne, czyli teorie prawd "powierzchownych" i "głebokich"**, Oficyna Wydawnicza Politechniki Wrocławskiej, Wrocław (2001).

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SEQUENT CALCULI FOR ORTHOLOGIC WITH STRICT IMPLICATION

Abstract

In this study, new sequent calculi for a minimal quantum logic (**MQL**) are discussed that involve an implication. The sequent calculus **GO** for **MQL** was established by Nishimura, and it is complete with respect to ortho-models (O-models). As **GO** does not contain implications, this study adopts the strict implication and constructs two new sequent calculi $\mathbf{GOI}_1$ and $\mathbf{GOI}_2$ as the expansions of **GO**. Both $\mathbf{GOI}_1$ and $\mathbf{GOI}_2$ are complete with respect to the O-models. In this study, the completeness and decidability theorems for these new systems are proven. Furthermore, some details pertaining to new rules and the strict implication are discussed.

Keywords: Quantum logic, sequent calculus, completeness theorem, implication, orthologic.

1. Introduction

Quantum logic (**QL**) has been introduced in order to manage strange propositions of quantum physics, such as uncertainty principle. Many structures have been studied to represent and analyze such propositions. In particular, *Orthomodular lattices* describe the propositional spaces of quantum physics and have been studied as the main structure of **QL** in the work by Birkhoff and Von Neumann [3]. An orthomodular lattice is based on closed subspaces of a Hilbert space, which is a state space of particles in quantum physics. Instead of these lattices, the *Kripke model* of **QL**, the orthomodular model (*OM-model*), can be used, which also describes a state space

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of quantum particles [12]. *Ortholattices*, which are conceptually simpler than orthomodular lattices, have also been studied. The logic based on ortholattices is a *minimal QL* (**MQL**) or *orthologic*. Moreover, the Kripke model for **MQL**, i.e., ortho-model (*O-model*), also exists [12].

As it is usually studied, **QL** does not contain logical implications and includes only negations, conjunctions, and disjunctions. Several implications in **QL** have been suggested; however, they all have difficulties for varying reasons [12, 14]. Therefore, the deduction systems, such as the Hilbert style axiomatization or sequent calculi that include implications, are not well developed. This problem also holds for **MQL**. In **MQL**, the number of appropriate implications is even smaller than that in **QL**. Therefore, as a part of research to address these problems, this study constructs two new sequent calculi for **MQL** that include rules for specific implication and provides the completeness theorems with respect to O-models.

When the implications are added to **QL** or **MQL**, some problems are encountered. In classical logic, the implication $A \to B$ and $\neg A \vee B$ can be identified. However, in **QL**, if $\neg A \vee B$ is adopted as an implication, critical properties for the implication, such as modus ponens, do not hold. Therefore, in **QL**, many other implications have been considered. Among them, *polynomial implications* that can be defined in terms of connectives $\neg$, $\wedge$ and $\vee$, have been predominantly studied. The polynomial implication *Sasaki arrow* $\neg A \vee (A \wedge B)$ has attracted the most attention in **QL**. In addition to the Sasaki arrow, the *contrapositive Sasaki arrow* $\neg(A \vee B) \vee B$, the *relevance arrow* $(A \wedge B) \vee (\neg A \wedge B) \vee (\neg A \wedge \neg B)$, and two other arrows have been explored [12, 13, 14]. These implications are the only polynomial implications that have suitable properties in terms of the orthomodular lattice and have been studied from both physical and mathematical standpoints [21].

These implications have been investigated in many ways because of their strangeness. The meaning and properties of these implications in quantum physics are associated with the notion of projections [12, 22]. For example, the Sasaki arrow $\neg A \vee (A \wedge B)$ can be translated as "after a measurement of A, if the state is projected to a state which A is true, then B is true." By utilizing this property and embedding the projection relationship in the model, various properties of the Hilbert space can be analyzed using the Kripke model [22]. Recently, these implications have been used in the context of *quantum set theory*, achieving results in the analysis of observed values in quantum mechanics [29]. The algebraic features of these implications have been widely studied in the case of orthomodular lattices and ortholattices [1, 6, 4, 8, 17]. These studies focus on the logical aspects of orthomodular lattices using implications. Furthermore, concepts regarding orthomodular lattices, such as semilattices, have been analyzed, where implications occupy a principal position [10, 9, 11]. Among them, *implication algebras* have been discussed as implication studies that exclude other logical operators [1, 7, 13, 15, 16]. In this field, the properties of orthomodular lattices have been elucidated by analyzing algebraic axioms and conditions for implications. This algebraic research is a purely mathematical study rather than a research related to quantum physics. Few studies on **QL** have employed binary relational models compared with the number of studied on such algebraic studies. Models using binary relations can express the dynamic relations of quantum physics, and some dynamic concepts are closely related to implications. Therefore, research using the Kripke model, such as that proposed in this study, should be conducted.

However, in ortholattices, polynomial implications do not satisfy modus ponens. In this study, the notion of *strict implication* proposed in the literature [12] is adopted for **MQL**, as the strict implication exhibits good mathematical properties, particularly in the Kripke models, and has physically significant meanings. In an ortholattice L, strict implication is defined with some restrictions as follows [12]:

$$a \to b = \bigsqcup \{c \in L \mid c \neq 0 \wedge \forall d((d \neq 0 \wedge c \not\leq d' \wedge d \leq a) \Rightarrow d \leq b)\}$$

where $\leq$ is the order in L, $\sqcup$ is the join, and 0 is the least element. Although this definition seems complicated at the first glance, the definition in the Kripke model corresponding to this definition is clear. This is one reason for adopting the Kripke model in the present study. Intuitively, from a quantum physics viewpoint, the strict implication $A \to B$ can be translated as "after the measurement of any physical quantity, if A is true, then B is true."

Some advantages of the strict implication should be noticed.

- In ortholattices, the Sasaki arrow does not satisfy modus ponens. However, the strict implication satisfies modus ponens in both lattices. Therefore, when **MQL** is considered, the strict implication is more suitable than the Sasaki arrow.

- All material implications are abbreviations of formulas constructed using conjunctions, disjunctions, and negations. However, the strict implication cannot be (finitely) constructed by means of these symbols [12]. Therefore, when the strict implication is added to **MQL**, the descriptive ability of the logic increases.

- The definition of the strict implication in O-models is similar to that of the implication in intuitionistic logic. The deduction rules of the strict implication are similar to those in the sequent calculus **LBP** for the *basic propositional logic* (**BPL**) [20, 31]. Therefore, we can analyze the relationship between **QL** and other logics using this implication.

Although a sequent calculus for **MQL** with the strict implication exists, a sequent is a *labeled* type sequent [23]. From the logic viewpoint, it is important to construct and discuss a simple type of sequent calculus for logic. Furthermore, some deduction systems for **QL** or **MQL** that involve implications are studied; however, they are either not sequent calculi or the implication used in these systems is not a strict implication [5, 28]. Sequent calculi **GO** [25] and **GMQL** [26, 27] have been studied as foundational sequent calculi for **MQL** which only includes $\neg$, $\wedge$ and $\vee$. The present study adopts **GO** for technical reasons, which is presented in Section 6. The rules for the strict implication are added to **GO**, and new calculi $\mathbf{GOI}_1$ and $\mathbf{GOI}_2$ are constructed. This study proves the completeness theorem for these new systems.

Some formulas valid with general implications in other logics are invalid with the strict implication in O-models. For example, $p \to (q \to p)$ is invalid. Therefore, general rules for implications, for example, such as those for the implication in classical logic, cannot be used. As mentioned earlier, this study uses a modified version of the rule for the implication of **LBP** reported in the literature [20]. The implication of **BPL** also does not satisfy some ordinary natures of implication. The semantics of this implication in a Kripke model is the same as that of the strict implication. In other words, $x \models A \to B$ is regarded as "for all y, such that xRy, if $y \models A$, then $y \models B$."

In Sections 2 and 3, some basics and the sequent calculus of **MQL** are presented. In Sections 4 and 5, the new sequent calculi $\mathbf{GOI}_1$ and $\mathbf{GOI}_2$ are constructed and some related theorems are proven. The deduction ability of $\mathbf{GOI}_1$ and $\mathbf{GOI}_2$ is intrinsically the same; however, the rules for the strict implication are different and each has pros and cons. In Section 6, some details regarding the strict implication and rules are discussed.

2. Basics

This study uses language that has a denumerable infinite set of propositional variables, the propositional constant $\perp$, the unary connective $\neg$, and binary connectives $\wedge$ and $\rightarrow$. Formulas are constructed in the usual way. We denote propositional variables by $p, q, \ldots$, formulas by $A, B, C, \ldots$, and finite sets of formulas by $\Gamma, \Delta, \Sigma, \Pi, \ldots$. We use $A \vee B$ as the abbreviation of $\neg(\neg A \wedge \neg B)$.

An *O-frame* is a pair $(W, \perp)$, where W is a nonempty set, and $\perp$ is an irreflexive and symmetric binary relation on W. For traditional reasons, we use the symbol $\perp$ in two ways; one as a relation, the other as a formula. The relation symbol $\perp$ came from the orthogonal relation in the Hilbert space, and the formula symbol $\perp$ denotes the bottom. They can be distinguished by the context.

We write $x \not\perp y$ if not $x \perp y$. We write $x \perp X$ if, for all $y \in X$, $x \perp y$, where $x \in W$ and $X \subseteq W$. Given $X \subseteq W$, we define the set $X^{\perp} = \{x \in W | x \perp X\}$. We say that X is $\perp$-*closed* if $X^{\perp\perp} = X$.

An *O-model* is a triple $(W, \perp, V)$, where $(W, \perp)$ is an O-frame and V is a function assigning each propositional variable p to a $\perp$-closed subset of W.

We define the set $\|A\|$ by induction on the composition of A as follows.

$$\|p\| = V(p)$$

$$\|A \wedge B\| = \|A\| \cap \|B\|$$

$$\|\neg A\| = \|A\|^{\perp}$$

$$\|A \rightarrow B\| = \{x \in W | \text{ for all } y \in W, \text{ if } x \not\perp y \text{ and } y \in \|A\|, y \in \|B\| \ \}$$

$$\|\perp\| = \emptyset$$

A is *true* at x if $x \in \|A\|$ and write $x \models A$. It is easy to evaluate that $\|\neg A\| = \|A \rightarrow \perp\|$ is fulfilled in this definition. Therefore, we regard $\neg A$ as the abbreviation of $A \rightarrow \perp$.

LEMMA 2.1. *For all $\|A\|$, $\|A\|$ is $\perp$-closed.*

PROOF: In the cases of $\|p\|$, $\|A \wedge B\|$ and $\|\neg A\|$, see [25]. For all $x \in \|A \to B\|$, $x \perp \{y \in W \mid y \models A \text{ and } y \not\models B\}$. Then, $\{y \in W \mid y \models A \text{ and } y \not\models B\} \in \|A \to B\|^{\perp}$. Therefore, if $z \in \|A \to B\|^{\perp\perp}$ then $z \perp \{y \in W \mid y \models A$ and $y \not\models B\}$. It means $z \in \|A \to B\|$. That is, there is no point z which satisfies $z \notin \|A \to B\|$ and $z \in \|A \to B\|^{\perp\perp}$. Therefore, $\|A \to B\|$ is $\perp$-closed. □

3. Sequent calculus GO

GO is defined below [25].

Axiom: $A \Rightarrow A$

Rules:

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\ (\text{cut}) \qquad \frac{\Gamma \Rightarrow \Delta}{\Pi, \Gamma \Rightarrow \Delta, \Sigma}\ (\text{weakening})$$

$$\frac{A, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ (\wedge\text{L}) \qquad \frac{B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ (\wedge\text{L}) \qquad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\ (\wedge\text{R})$$

$$\frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta}\ (\neg\text{L}) \qquad \frac{A \Rightarrow \Delta}{\neg \Delta \Rightarrow \neg A}\ (\neg\text{R})$$

$$\frac{A, \Gamma \Rightarrow \Delta}{\neg\neg A, \Gamma \Rightarrow \Delta}\ (\neg\neg\text{L}) \qquad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \neg\neg A}\ (\neg\neg\text{R})$$

In [25], Γ, Δ, Π and Σ are defined as probable infinite sets. We restrict these to finite sets because infinite sets are not essential here.

Consider an O-model $(W, \perp, V)$. Sequent $\Gamma \Rightarrow \Delta$ is *false* at $x \in W$ if for all formulas $A \in \Gamma$, $x \models A$, and, for all formulas $B \in \Delta$, $x \not\models B$. If $\Gamma \Rightarrow \Delta$ is not false at x, then it is *true* at x. Sequent $\Gamma \Rightarrow \Delta$ is falsifiable if there exists an O-model $(W, \perp, V)$ and $x \in W$, and $\Gamma \Rightarrow \Delta$ is false at x. If $\Gamma \Rightarrow \Delta$ is unfalsifiable, we say $\Gamma \Rightarrow \Delta$ is *valid*.

THEOREM 3.1. *The soundness and completeness theorem for* **GO**. *$\Gamma \Rightarrow \Delta$ is provable in* **GO** *if, and only if, (iff) $\Gamma \Rightarrow \Delta$ is valid.*

PROOF: See [25]. □

4. Sequent calculus $\mathbf{GOI}_1$

In this section, a sequent calculus including the strict implication is established. The sequent calculus $\mathbf{GOI}_1$ is defined as an expansion of **GO**. The rule ($\to$ R) and axiom $\perp \Rightarrow$ are added to **GO**. The rule ($\to$ R) is the transformation of the rule ($\to$) in [20]. Because this rule ($\to$ R) is complex, using $\mathbf{GOI}_2$ in the next chapter for the main calculus of **MQL** with the strict implication would be better. However, ($\to$ R) is useful to prove the completeness theorem. Therefore, first the details of $\mathbf{GOI}_1$ are shown. The definitions of truth, falsity, and validity of a sequent are identical to that in **GO**.

$$\perp \Rightarrow \quad (\perp)$$

$$\frac{\mathit{\Gamma}_1, A \Rightarrow B, \mathit{\Delta}_1, \mathit{\Sigma} \quad \mathit{\Gamma}_2, A \Rightarrow B, \mathit{\Delta}_2, \mathit{\Sigma} \ \ldots \ \mathit{\Gamma}_{2^n}, A \Rightarrow B, \mathit{\Delta}_{2^n}, \mathit{\Sigma}}{C_1 \to D_1, C_2 \to D_2, \ \ldots \ , C_n \to D_n, \mathit{\Pi} \Rightarrow A \to B, \mathit{\Lambda}} \quad (\to\mathrm{R})$$

where, $0 \leq n$, $\mathit{\Gamma}_i = \{D_j | j \in \gamma(i)\}$, $\mathit{\Delta}_i = \{C_j | j \in \delta(i)\}$, $\langle \delta(i), \gamma(i) \rangle$ is the i-th element of all partitions of $\{1, \ldots, n\}$. $\mathit{\Pi}$ and $\mathit{\Lambda}$ are formula sets. $\mathit{\Sigma}$ is a set of all formulas of the shape $E \to F$ such that E is included in the premise of the lower sequent and F is included in the conclusion of the lower sequent or $\perp$. Therefore, $\mathit{\Sigma} = \{E \to F \ | E \in \{C_1 \to D_1, \ldots, C_n \to D_n, \mathit{\Pi}\}, F \in \{A \to B, \mathit{\Lambda}, \perp\}\}$.

For example, suppose $\mathit{\Pi} = \{I\}$, $\mathit{\Lambda} = \{J, K\}$, then ($\to$R) is as below in the case of $n = 0$, $n = 1$, and $n = 2$.

$$\frac{A \Rightarrow B, I \to (A \to B), I \to J, I \to K, I \to \perp}{I \Rightarrow A \to B, J, K}$$

$$\frac{A \Rightarrow B, C_1, \mathit{\Sigma} \quad D_1, A \Rightarrow B, \mathit{\Sigma}}{C_1 \to D_1, I \Rightarrow A \to B, J, K}$$

where $\mathit{\Sigma}$ is $\{(C_1 \to D_1) \to (A \to B), (C_1 \to D_1) \to J, (C_1 \to D_1) \to K, (C_1 \to D_1) \to \perp, I \to (A \to B), I \to J, I \to K, I \to \perp\}$.

$$\frac{A \Rightarrow B, C_1, C_2, \mathit{\Sigma} \qquad D_1, A \Rightarrow B, C_2, \mathit{\Sigma} \qquad D_2, A \Rightarrow B, C_1, \mathit{\Sigma} \qquad D_1, D_2, A \Rightarrow B, \mathit{\Sigma}}{C_1 \to D_1, C_2 \to D_2, I \Rightarrow A \to B, J, K}$$

where Σ is $\{(C_1 \to D_1) \to (A \to B), (C_1 \to D_1) \to J, (C_1 \to D_1) \to K, (C_1 \to D_1) \to \bot, (C_2 \to D_2) \to (A \to B), (C_2 \to D_2) \to J, (C_2 \to D_2) \to K, (C_2 \to D_2) \to \bot, I \to (A \to B), I \to J, I \to K, I \to \bot\}$.

In $\mathbf{GOI}_1$, the rule ($\to$ L) is admissible.

$$\frac{\Gamma_1, \Rightarrow \Delta_1, A \qquad B, \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, A \to B \Rightarrow \Delta_1, \Delta_2}\ (\to\mathrm{L})$$

We will prove this lemma in Section 5.

THEOREM 4.1. *The soundness theorem for* $\mathbf{GOI}_1$*. If* $\Gamma \Rightarrow \Delta$ *is provable in* $\mathbf{GOI}_1$*,* $\Gamma \Rightarrow \Delta$ *is valid.*

PROOF: Proven by induction on the construction of a proof. For rules in **GO**, the proof is the same as the proof in [25]. For ($\to$R), we only evaluate $n = 2$. The other cases are similar. For contradiction, suppose all premises of the rule are valid, and there exists O-model $(W, \bot, V)$ and $x \in W$, such that the conclusion of the rule is false at x. Then, as $A \to B$ is false at x, there exists $y \in W$, satisfying $x \not\perp y$, $y \models A$ and $y \not\models B$. Because we assume that $y \models A$ and all premises are valid, from the first premise, B or C_1 or C_2 or one of the formulas in Σ is true at y; however, B is false at y. Now, suppose $E \to F \in \Sigma$. We have $x \models E$ and $x \not\models F$ by assumption and the definition of ($\to$R). If $y \models E \to F$, from $y \not\perp x$ and $x \models E$, $x \models F$, which is a contradiction. Therefore, for all $E \to F \in \Sigma$, $y \not\models E \to F$. Therefore, C_1 or C_2 is true at y. In the former case, from $x \models C_1 \to D_1$ and $y \models C_1$, $y \models D_1$. From the second premise, B or C_2 or one of the formulas in Σ is true at y. Similarly, the only possibility is C_2; therefore, C_2 is true at y. To continue this method to the end of premises, B or Σ is the only possibility, which is a contradiction. The latter case and cases of the other possibilities are similar to this method. □

To prove the completeness theorem, we define the set Ω as follows. $\Omega(\Gamma \Rightarrow \Delta) = \{$ All subformulas in $\Gamma \cup \Delta\} \cup \{\neg p \mid p$ appear in some formulas in $\Gamma \cup \Delta\} \cup \{\bot\}$. For example, $\Omega(\neg(p \to q) \Rightarrow r \wedge q) = \{\bot, p, q, r, \neg p, \neg q, \neg r, p \to q, r \wedge q, \neg(p \to q)\}$. For each unprovable sequent $\Gamma \Rightarrow \Delta$, we define a *canonical O-model* $(W_c, \bot_c, V_c)$ of $\Gamma \Rightarrow \Delta$ as follows.

W_c: $\{\Gamma_1 \Rightarrow \Delta_1 | \Gamma_1 \Rightarrow \Delta_1$ is unprovable in $\mathbf{GOI}_1$ and $\Gamma_1 \cup \Delta_1 = \Omega(\Gamma \Rightarrow \Delta)\}$

$\perp_c$: $(\Gamma_1 \Rightarrow \Delta_1)\perp(\Gamma_2 \Rightarrow \Delta_2)$ iff for some A and B, at least one of (1) (2) is true. (1) $A \to B \in \Gamma_1$, $A \in \Gamma_2$ and $B \in \Delta_2$. (2) $A \to B \in \Gamma_2$, $A \in \Gamma_1$ and $B \in \Delta_1$.

V_c: assigns p to the set $\{\Gamma_1 \Rightarrow \Delta_1 | p \notin \Delta_1\}$.

LEMMA 4.2. *$(W_c, \perp_c)$ is an O-frame. $V_c(p)$ is $\perp$-closed. Therefore, $(W_c, \perp_c, V_c)$ is an O-model.*

PROOF: If $(\Gamma_1 \Rightarrow \Delta_1)\perp(\Gamma_1 \Rightarrow \Delta_1)$, there is A and B that $A \in \Gamma_1$, $A \to B \in \Gamma_1$ and $B \in \Delta_1$. But $A, A \to B \Rightarrow B$ is proven using ($\to$L); therefore, $\Gamma_1 \Rightarrow \Delta_1$ can be proven, which is a contradiction. Therefore, for every $\Gamma_1 \Rightarrow \Delta_1 \in W_C$, $(\Gamma_1 \Rightarrow \Delta_1) \not\perp (\Gamma_1 \Rightarrow \Delta_1)$. Symmetry is obvious from the definition. If $p \notin \Omega(\Gamma \Rightarrow \Delta)$, $V_c(p) = W_c$. This is clearly $\perp$-closed. If $p \in \Omega(\Gamma \Rightarrow \Delta)$, for every $\Gamma_1 \Rightarrow \Delta_1 \in W_c$, $p \in \Gamma_1$ or $p \in \Delta_1$. Then, $(\Gamma_1 \Rightarrow \Delta_1) \models p$ iff $p \in \Gamma_1$. Therefore, if we can prove the next statement, we can prove this lemma.

For all $(\Gamma_1 \Rightarrow \Delta_1) \in W_c$, if $p \in \Delta_1$, there exists $(\Gamma_2 \Rightarrow \Delta_2) \in W_C$, satisfying $\neg p \in \Gamma_2$ and $(\Gamma_1 \Rightarrow \Delta_1) \not\perp (\Gamma_2 \Rightarrow \Delta_2)$.

For convenience, we prove this statement after the next lemma. □

LEMMA 4.3. *For all canonical O-models and all formulas $A \in \Omega$, A is true at $(\Gamma_1 \Rightarrow \Delta_1)$ if $A \in \Gamma_1$ and A is false at $(\Gamma_1 \Rightarrow \Delta_1)$ if $A \in \Delta_1$.*

PROOF: Proven by induction on the composition of A.

For $A = p$, the proof is obvious from the definition of a canonical O-model.

For $A = B \wedge C$, the proof is the same as in [25].

For $A = \neg B$, the proof is included in $A = B \to C$.

For $A = B \to C$, suppose $B \to C \in \Gamma_1$. Then, for all $(\Gamma_2 \Rightarrow \Delta_2)$ satisfying $B \in \Gamma_2$ and $C \in \Delta_2$, $(\Gamma_1 \Rightarrow \Delta_1)\perp(\Gamma_2 \Rightarrow \Delta_2)$ by the definition of the canonical O-model. Then, by definition of $\to$ and induction hypothesis, $B \to C$ is true at $(\Gamma_1 \Rightarrow \Delta_1)$.

Suppose $B \to C \in \Delta_1$. Because $\Gamma_1 \Rightarrow \Delta_1$ cannot be proven, when we regard this sequent as the lower sequent of the rule ($\to$R), an unprovable sequent $\Gamma_2, B \Rightarrow C, \Delta_2$ exists, which is of the shape of a sequent in the upper sequent of ($\to$R). Then, Γ_2 and Δ_2 distribute all formulas of the shape of $E \to F$ in Γ_1, regarded as a $C_i \to D_i$. If there are formulas in $\Gamma_2, B \Rightarrow C, \Delta_2$ that are excluded in $\Omega(\Gamma \Rightarrow \Delta)$, we delete them from $\Gamma_2, B \Rightarrow C, \Delta_2$ and make a new sequent $\Gamma_3, B \Rightarrow C, \Delta_3$. Then, $\Gamma_3 \cup \{B, C\} \cup \Delta_3 \subseteq \Omega(\Gamma \Rightarrow \Delta)$ and this sequent is still unprovable. This sequent can be expanded to the sequent $\Gamma_4 \Rightarrow \Delta_4 \in W_c$ because for all formulas G, at least one $\Gamma_3, B \Rightarrow C, \Delta_3, G$ or $G, \Gamma_3, B \Rightarrow C, \Delta_3$ is unprovable because of the rule (cut) and because $\Gamma_3, B \Rightarrow C, \Delta_3$ is unprovable. Furthermore, $(\Gamma_1 \Rightarrow \Delta_1) \not\perp (\Gamma_4 \Rightarrow \Delta_4)$ is satisfied because we delete all probability of holding the relation $\perp$ when we construct $\Gamma_2, B \Rightarrow C, \Delta_2$. Therefore, by the definition of $\to$ and induction hypothesis, $B \to C$ is false at $\Gamma_1 \Rightarrow \Delta_1$. □

Now we can prove the statement in Lemma 4.2 using the method of the proof of Lemma 4.3. If $\Gamma_1 \Rightarrow \Delta_1, p$ ($\in W_c$) is unprovable, $\Gamma_1 \Rightarrow \Delta_1, p, \neg\neg p$ is also unprovable. We regard $(p \to \perp) \to \perp$ as $B \to C$ in Lemma 4.3. The same argument for $B \to C \in \Delta_1$ in Lemma 4.3 can be applied. That is, we can find $(\Gamma_4 \Rightarrow \Delta_4) \in W_c$, satisfying $\neg p \in \Gamma_4$, $\perp \in \Delta_4$, and $(\Gamma_1 \Rightarrow \Delta_1, p) \not\perp (\Gamma_4 \Rightarrow \Delta_4)$. If $\neg\neg p$ is included in $\Omega(\Gamma \Rightarrow \Delta)$, $\Gamma_1 \Rightarrow \Delta_1, p, \neg\neg p$ is the same as $\Gamma_1 \Rightarrow \Delta_1, p$ and is included in W_c. If $\neg\neg p$ is excluded in $\Omega(\Gamma \Rightarrow \Delta)$, sequent $\Gamma_4 \Rightarrow \Delta_4$ ($\neg p \in \Gamma_4$), constructed from $\Gamma_1 \Rightarrow \Delta_1, p, \neg\neg p$ is included in W_c, even if $\Gamma_1 \Rightarrow \Delta_1, p, \neg\neg p$ is excluded in W_c. That is, when we make $\Gamma_3, \neg p \Rightarrow \perp, \Delta_3$ from $\Gamma_1 \Rightarrow \Delta_1, p, \neg\neg p$, we eliminate all formulas that are excluded in $\Omega(\Gamma \Rightarrow \Delta)$. Furthermore, it satisfies $(\Gamma_1 \Rightarrow \Delta_1, p) \not\perp (\Gamma_4 \Rightarrow \Delta_4)$ because $\Gamma_1 \Rightarrow \Delta_1, p$ is a part of $\Gamma_1 \Rightarrow \Delta_1, p, \neg\neg p$.

THEOREM 4.4. *The completeness theorem for* $\mathbf{GOI}_1$. *If* $\Gamma \Rightarrow \Delta$ *is valid,* $\Gamma \Rightarrow \Delta$ *is provable in* $\mathbf{GOI}_1$.

PROOF: Suppose $\Gamma \Rightarrow \Delta$ is unprovable. We can make a canonical O-model of $\Gamma \Rightarrow \Delta$. Because (cut) is included in $\mathbf{GOI}_1$, there exists $(\Gamma' \Rightarrow \Delta') \in W_c$, an expansion of $\Gamma \Rightarrow \Delta$. By Lemma 4.3, $\Gamma \Rightarrow \Delta$ is false at $(\Gamma' \Rightarrow \Delta')$. □

5. Sequent calculus $\mathbf{GOI}_2$

We define the sequent calculus $\mathbf{GOI}_2$ as an expansion of **GO**. We add the axioms $(\to \perp)$ and $(\perp)$ and the rule $(\to \mathrm{R})$' to **GO**. The rule $(\to \mathrm{R})$' is similar to the rule $(\to)$ in [20], but there are no contexts in this rule.

$$\perp \Rightarrow \quad (\perp)$$

$$A \Rightarrow (A \to B) \to \perp, B \quad (\to \perp)$$

$$\frac{\Gamma_1, A \Rightarrow B, \Delta_1 \qquad \Gamma_2, A \Rightarrow B, \Delta_2 \quad ... \qquad \Gamma_{2^n}, A \Rightarrow B, \Delta_{2^n}}{C_1 \to D_1, C_2 \to D_2, ..., C_n \to D_n \Rightarrow A \to B} \ (\to\text{R})\text{'}$$

where $0 \leq n$, $\Gamma_i = \{D_j | j \in \gamma(i)\}$, $\Delta_i = \{C_j | j \in \delta(i)\}$, $\langle \delta(i), \gamma(i) \rangle$ is the i-th element of all partitions of $\{1, ..., n\}$.

The rule ($\to$ R)' is a natural expansion of the rule ($\neg R$) in **GO**. That is, if all D_j and B in ($\to$ R)' are $\perp$, it is the same as ($\neg R$) in **GO** because of $A \to \perp \equiv \neg A$.

Theorem 5.1. *The soundness and completeness theorem for* $\mathbf{GOI_2}$*.* $\Gamma \Rightarrow \Delta$ *is provable in* $\mathbf{GOI_2}$ *iff* $\Gamma \Rightarrow \Delta$ *is valid.*

Proof: We can prove that all rules of $\mathbf{GOI_1}$ are derivable in $\mathbf{GOI_2}$, and vice versa. The proof of ($\to \perp$) in $\mathbf{GOI_1}$ and ($\to$R) in $\mathbf{GOI_2}$ is explained below. The other cases are obvious.

$$\cfrac{\cfrac{A \to B \Rightarrow A \to B}{A \to B \Rightarrow \perp, A \to B, A \to ((A \to B) \to \perp), A \to \perp} \ (\text{weakening})}{A \Rightarrow (A \to B) \to \perp, B} \ (\to\text{R})$$

Suppose all sequents of upper sequents in ($\to$R) are provable. For example, suppose $n = 2$. Then,

$A \Rightarrow B, C_1, C_2, \Sigma$
$D_1, A \Rightarrow B, C_2, \Sigma$
$D_2, A \Rightarrow B, C_1, \Sigma$
$D_1, D_2, A \Rightarrow B, \Sigma$

are all provable. Now we regard all formulas in Σ as a C_i ($n < i$). For example, if Σ has three elements, we regard Σ as $\{C_3, C_4, C_5\}$. Furthermore, we define all D_i ($n < i$) as $D_i = \perp$. Then,

$A \Rightarrow B, C_1, C_2, C_3, C_4, C_5$
$D_1, A \Rightarrow B, C_2, C_3, C_4, C_5$
...
$D_1, D_2, D_3, D_4, D_5, A \Rightarrow B$

are all provable because, if all formulas in $\Sigma = \{C_3,\ C_4,\ C_5\}$ are on the right-hand side, it is obvious from the assumption. If one of a $\{D_3, D_4, D_5\}$

is on the left-hand side and because all are $\bot$, this sequent is provable. We can use all these sequents and use ($\to$R)'. Then, because $E \Rightarrow (E \to F) \to \bot, F$ is provable using ($\to \bot$), use (cut), and prove the lower sequent of ($\to$R). $\square$

Theorem 5.2. *In* $\mathbf{GOI}_1$ *and* $\mathbf{GOI}_2$*, the rule* ($\to L$) *is admissible.*

Proof: By using (cut) and because $A, A \to B \Rightarrow B$ is provable in these systems,

$$\dfrac{\dfrac{A \to B \Rightarrow A \to B}{A \to B, \Rightarrow ((A \to B) \to \bot) \to \bot} \quad \dfrac{A \Rightarrow (A \to B) \to \bot, B}{A, ((A \to B) \to \bot) \to \bot \Rightarrow B}}{A, A \to B \Rightarrow B}$$

$\square$

Because the canonical model $(W_c, \bot_c, V_c)$ finite, we can prove the following theorem using the usual method as in **GO**.

Theorem 5.3. $\mathbf{GOI}_1$ *and* $\mathbf{GOI}_2$ *are decidable. That is, an effective procedure determines whether a sequent* $\Gamma \Rightarrow \Delta$ *is provable in* $\mathbf{GOI}_1$ *and* $\mathbf{GOI}_2$*.*

Proof: From the construction method of the canonical model $(W_c, \bot_c, V_c)$, built from a sequent $\Gamma \Rightarrow \Delta$, we obtain a finite model for any $\Gamma \Rightarrow \Delta$, and the model's complexity can be bounded by the complexity of formulas and the number of propositional letters in Γ and Δ. Therefore, by evaluating all finite models up to the bound, whether sequent $\Gamma \Rightarrow \Delta$ is valid can be determined. From the soundness and completeness theorem, this method can determine whether $\Gamma \Rightarrow \Delta$ is provable in $\mathbf{GOI}_1$ and $\mathbf{GOI}_2$. $\square$

6. Conclusion and remarks

This study introduced two sequent calculi for **MQL** that involve the strict implication. The rule for the implication in $\mathbf{GOI}_1$ is complicated. On the contrary, the rule for the implication in $\mathbf{GOI}_2$ is less complicated and it is a natural expansion of the rule ($\neg R$). However, the axiom ($\to \bot$) must be included in $\mathbf{GOI}_2$. In both the calculi, the cut-elimination theorem does not hold. In actuality, $p, q \Rightarrow \neg(r \wedge \neg(p \wedge q))$ cannot be proven without (cut), as in **GO** [25]. In other words, based on the rules for

$\mathbf{GOI}_1$ and $\mathbf{GOI}_2$, in the proof of $p, q \Rightarrow \neg(r \wedge \neg(p \wedge q))$, we can only use (weakening), (cut), or ($\to$R) to deduce $p, q \Rightarrow \neg(r \wedge \neg(p \wedge q))$. However, it is easy to confirm that (weakening) does not work. Additionally, $r \wedge \neg(p \wedge q) \Rightarrow \bot, \neg p, \neg q, p \to \neg(r \wedge \neg(p \wedge q)), q \to \neg(r \wedge \neg(p \wedge q))$ can be checked for invalidity. Moreover, it is challenging to construct a sequent calculus for **QL** and **MQL** that satisfies the cut-elimination theorem using an ordinary method. The situation is similar to that in the modal logic **S5**. Both **S5** and **QL** exhibit a symmetric frame. If an attempt is made to construct a canonical model of the **S5**-frame in a stepwise manner, the procedure cannot be stopped because of the symmetry. An effective tool for addressing this problem is an extension of the sequent calculus. Various extensions of sequence calculus for **S5** have been constructed and analyzed [2, 18, 19, 24, 30]. As one of them, labeled sequent calculi or *tree sequent calculi* have been studied. A labeled sequent calculus for **MQL** with the strict implication has been established and is cut-free [23]. It is still an open question whether a normal sequent calculus for **MQL** that satisfies the cut-elimination theorem exists.

In **BPL**, the law of modus ponens does not hold [20]. Modus ponens $A, A \to B \Rightarrow B$ represents the reflexive condition of relations in frames which is not the nature of frames of **BPL**. Therefore, the rule ($\to$ L) is not sound in **LBP**. ($\to$ L) cannot be constructed if only ($\to$ R)' exists for the implication. In $\mathbf{GOI}_1$ and $\mathbf{GOI}_2$, because other rules or axioms for the implication are included, ($\to$ L) can be constructed.

Another sequent calculus for **MQL** called **GMQL** [26, 27] is also complete with respect to O-models and exclude implications, similar to **GO**. In **GO**, based on the definition of the truth of a sequent, $\Gamma \Rightarrow \Delta, A, B$ cannot be regarded as $\Gamma \Rightarrow \Delta, A \vee B$ because commas on the right side of the sequent indicate a union of sets. $\|A\| \cup \|B\|$ and $\|A \vee B\|$ are different sets in O-models, and $\|A\| \cup \|B\|$ is not always $\bot$-closed. For example, $\Rightarrow A, \neg A$ cannot be proven in **GO**; however, $\Rightarrow A \vee \neg A$ ($= \Rightarrow \neg(\neg A \wedge \neg\neg A)$) can be proven. In **GMQL**, $\Gamma \Rightarrow \Delta, A, B$ represent $\Gamma \Rightarrow \Delta, A \vee B$. Because the rules in **GMQL** are close to the notion of a lattice, the rules for $\wedge$ and $\vee$ in **GMQL** are symmetric because $\wedge$ and $\vee$ are symmetric in ortholattices. In the case of **GO**, that excludes an implication, this notion of a union of sets is inessential because of the following theorem [20].

THEOREM 6.1. *If $\Gamma \Rightarrow \Delta$ is provable in* **GO** *and Δ is nonempty, then there exists $A \in \Delta$ such that $\Gamma \Rightarrow A$ is provable and all sequents in that proof have at most one formula on the right side.*

When considering the rules for implications, **GMQL** is unsuitable because in the rules for strict implication, the notion of a union of sets on the right side of a sequent is used rather than $\vee$. In the case of $\mathbf{GOI}_1$ and $\mathbf{GOI}_2$, the notion of a union of sets is essential and Theorem 6.1 does not hold in these calculi. This finding can be confirmed by considering the axiom $(\to \bot)$ and the completeness theorem. In other words, both $A \Rightarrow (A \to B) \to \bot$ and $A \Rightarrow B$ are invalid.

In a sense, the axiom $(\to \bot)$ represents the symmetry of the relation in frames. If $\mathbf{GOI}_2$ includes only ($\to$R)' for the strict implication, the symmetry cannot be handled because ($\to$R)' is a part of the sequent calculus reported in the literature [20] which is sound and complete with respect to the frames that do not need to be symmetrical. Assume that in an O-model $(W, \perp, V)$, $x \models A$ and $x \not\models B$, then for all $y \in W$ such that $x \not\perp y$, $y \not\models A \to B$ attributed is the symmetry of $\not\perp$. If $B = \bot$, then the axiom $(\to \bot)$ is $A \Rightarrow \neg\neg A$. When the translation in the literature [12] which translate a formula of **QL** to a formula of modal logic is applied, this sequent corresponds to $A \Rightarrow \Box\Diamond A$, representing the symmetry in the modal logic.

In the rule $(\to)$ in **LBP**, in every left side of the sequent, contexts can be used. Therefore, $p \to (q \to p)$ can be proven in a sequent calculus for **LBP** using $n = 0$ of $(\to)$, which cannot be proven in $\mathbf{GOI}_1$.

$$\frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B}\ (n = 0 \text{ of } (\to) \text{ in } \mathbf{LBP})$$

$$\dfrac{\dfrac{\dfrac{p \Rightarrow p}{q, p \Rightarrow p}}{p \Rightarrow q \to p}}{\Rightarrow p \to (q \to p)}$$

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References

[1] J. C. Abbott, *Orthoimplication Algebras*, **Studia Logica**, vol. 35(2) (1976), pp. 173–177, DOI: https://doi.org/10.1007/BF02120879.

[2] K. Bednarska, A. Indrzejczak, *Hypersequent Calculi for S5 – The Methods of Cut-elimination*, **Logic and Logical Philosophy**, vol. 24(3) (2015), pp. 277–311, DOI: https://doi.org/10.12775/LLP.2015.018.

[3] G. Birkhoff, J. V. Neumann, *The Logic of Quantum Mechanics*, **Annals of Mathematics**, vol. 37(4) (1936), pp. 823–843, DOI: https://doi.org/10.2307/1968621.

[4] I. Chajda, *The axioms for implication in orthologic*, **Czechoslovak Mathematical Journal**, vol. 58(1) (2008), pp. 735–744, DOI: https://doi.org/10.1007/s10587-008-0002-2.

[5] I. Chajda, J. Cirulis, *An Implicational Logic for Orthomodular Lattices*, **Acta Scientiarum Mathematicarum**, vol. 82(34) (2016), pp. 383–394, DOI: https://doi.org/10.14232/actasm-015-813-6.

[6] I. Chajda, R. Halaš, *An Implication in Orthologic*, **International Journal of Theoretical Physics**, vol. 44(7) (2005), pp. 735–744, DOI: https://doi.org/10.1007/s10773-005-7051-1.

[7] I. Chajda, R. Halaš, H. Länger, *Orthomodular Implication Algebras*, **International Journal of Theoretical Physics**, vol. 40(11) (2001), pp. 1875–1884, DOI: https://doi.org/10.1023/A:1011933018776.

[8] I. Chajda, H. Länger, *Orthomodular Posets Can Be Organized as Conditionally Residuated Structures*, **Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica**, vol. 53(2) (2014), pp. 29–33, URL: https://kma.upol.cz/data/xinha/ULOZISTE/ActaMath/2014/53-2-02.pdf.

[9] I. Chajda, H. Länger, *Orthomodular lattices can be converted into left-residuated l-groupoids*, **Miskolc Mathematical Notes**, vol. 18(2) (2017), pp. 685–689, DOI: https://doi.org/10.18514/mmn.2017.1730.

[10] I. Chajda, H. Länger, *Residuation in orthomodular lattices*, **Topological Algebra and its Applications**, vol. 5(1) (2017), pp. 1–5, DOI: https://doi.org/10.1515/taa-2017-0001.

[11] I. Chajda, H. Länger, *How to introduce the connective implication in orthomodular posets*, **Asian-European Journal of Mathematics**, vol. 14(4) (2021), p. 2150066, DOI: https://doi.org/10.1142/S1793557121500662.

[12] M. L. D. Chiara, R. Giuntini, *Quantum Logics*, [in:] D. M. Gabbay, F. Guenthner (eds.), **Handbook of Philosophical Logic**, Springer Netherlands, Dordrecht (2002), pp. 129–228, DOI: https://doi.org/10.1007/978-94-017-0460-1_2.

[13] P. D. Finch, *Quantum logic as an implication algebra*, **Bulletin of the Australian Mathematical Society**, vol. 2(1) (1970), pp. 101–106, DOI: https://doi.org/10.1017/S0004972700041642.

[14] G. M. Hardegree, *Material Implication in Orthomodular (and Boolean) Lattices*, **Notre Dame Journal of Formal Logic**, vol. 22(2) (1981), pp. 163–182, DOI: https://doi.org/10.1305/ndjfl/1093883401.

[15] G. M. Hardegree, *Quasi-implication algebras, Part I: Elementary theory*, **Algebra Universalis**, vol. 12(1) (1981), pp. 30—47, DOI: https://doi.org/10.1007/BF02483861.

[16] G. M. Hardegree, *Quasi-implication algebras, Part II: Structure theory*, **Algebra Universalis**, vol. 12(1) (1981), pp. 48–65, DOI: https://doi.org/10.1007/BF02483862.

[17] L. Herman, E. L. Marsden, R. Piziak, *Implication Connectives in Orthomodular Lattices*, **Notre Dame Journal of Formal Logic**, vol. 16(3) (1975), pp. 305–328, DOI: https://doi.org/10.1305/ndjfl/1093891789.

[18] A. Indrzejczak, *Two Is Enough – Bisequent Calculus for S5*, [in:] A. Herzig, A. Popescu (eds.), **Frontiers of Combining Systems**, Springer International Publishing, Cham (2019), pp. 277–294, DOI: https://doi.org/10.1007/978-3-030-29007-8_16.

[19] A. Indrzejczak, **Sequents and Trees. An Introduction to the Theory and Applications of Propositional Sequent Calculi**, Studies in Universal Logic, Birkhäuser, Basel (2021), DOI: https://doi.org/10.1007/978-3-030-57145-0.

[20] K. Ishii, R. Kashima, K. Kikuchi, *Sequent Calculi for Visser's Propositional Logics*, **Notre Dame Journal of Formal Logic**, vol. 42(1) (2001), pp. 1–22, DOI: https://doi.org/10.1305/ndjfl/1054301352.

[21] G. Kalmbach, **Orthomodular Lattices**, Academic Press, New York (1983).

[22] T. Kawano, *Advanced Kripke Frame for Quantum Logic*, [in:] L. S. Moss, R. de Queiroz, M. Martinez (eds.), **Logic, Language, Information, and Computation**, Springer Berlin Heidelberg, Berlin, Heidelberg (2018), pp. 237–249, DOI: https://doi.org/10.1007/978-3-662-57669-4_14.

[23] T. Kawano, *Labeled Sequent Calculus for Orthologic*, **Bulletin of the Section of Logic**, vol. 47(4) (2018), pp. 217–232, DOI: https://doi.org/10.18778/0138-0680.47.4.01.

[24] S. Negri, *Proof Theory for Modal Logic*, **Philosophy Compass**, vol. 6(8) (2011), pp. 523–538, DOI: https://doi.org/10.1111/j.1747-9991.2011.00418.x.

[25] H. Nishimura, *Sequential method in quantum logic*, **Journal of Symbolic Logic**, vol. 45(2) (1980), pp. 339–352, DOI: https://doi.org/10.2307/2273194.

[26] H. Nishimura, *Proof Theory for Minimal Quantum Logic I*, **International Journal of Theoretical Physics**, vol. 33(1) (1994), pp. 103–113, DOI: https://doi.org/10.1007/BF00671616.

[27] H. Nishimura, *Proof Theory for Minimal Quantum Logic II*, **International Journal of Theoretical Physics**, vol. 33(7) (1994), pp. 1427–1443, DOI: https://doi.org/10.1007/BF00670687.

[28] H. Ozawa, H. Odera, S. Chitani, *Formal System QL for Quantum Logic*, **The Japan Association for Logical Philosophy, Logical Philosophy Research**, vol. 3 (2003).

[29] M. Ozawa, *Operational Meanings of Orders of Observables Defined through Quantum Set Theories with Different Conditionals*, **Electronic Proceedings in Theoretical Computer Science**, vol. 236 (2017), pp. 127–144, DOI: https://doi.org/10.4204/eptcs.236.9.

[30] F. Poggiolesi, **Gentzen Calculi for Modal Propositional Logic**, vol. 32 of Trends in Logic, Springer, Dordrecht (2011).

[31] A. Visser, *A Propositional Logic with Explicit Fixed Points*, **Studia Logica**, vol. 40(2) (1981), pp. 155–175, DOI: https://doi.org/10.1007/BF01874706.

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A SEQUENT SYSTEM WITHOUT IMPROPER DERIVATIONS

Abstract

In the natural deduction system for classical propositional logic given by G. Gentzen, there are some inference rules with assumptions discharged by the rule. D. Prawitz calls such inference rules improper, and others proper. Improper inference rules are more complicated and are often harder to understand than the proper ones.

In the present paper, we distinguish between proper and improper derivations by using sequent systems. Specifically, we introduce a sequent system $\vdash_{\mathbf{Sc}}$ for classical propositional logic with only structural rules, and prove that $\vdash_{\mathbf{Sc}}$ does not allow improper derivations in general. For instance, the sequent $\Rightarrow p \to q$ cannot be derived from the sequent $p \Rightarrow q$ in $\vdash_{\mathbf{Sc}}$. In order to prove the failure of improper derivations, we modify the usual notion of truth valuation, and using the modified valuation, we prove the completeness of $\vdash_{\mathbf{Sc}}$. We also consider whether an improper derivation can be described generally by using $\vdash_{\mathbf{Sc}}$.

Keywords: Sequent system, improper derivation, natural deduction.

2020 Mathematical Subject Classification: 03B05.

1. Introduction

In the natural deduction system for classical propositional logic given in Gentzen [4], there are some inference rules with assumptions discharged by the rule. For instance, the implication introduction rule and the disjunction elimination rule have such assumptions. Prawitz [7] calls such inference

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rules improper, and others proper. The differences between proper and improper inference rules are also pointed out in Fine [3], Robering [8], and Breckenridge and Magidor [1]. However, there is no description allowing to distinguish them by formal systems. In the present paper, we distinguish between proper and improper derivations by using sequent systems. So, we need to confirm what derivations are proper or improper in sequent systems.

In the following three subsections, we provide some preparations, consider what derivations are proper or improper in sequent systems, and describe our purposes in more detail.

1.1. Preliminaries

Here, we provide some preparations.

Formulas are constructed from $\bot$ (contradiction) and the propositional variables by using logical connectives $\wedge$ (conjunction), $\vee$ (disjunction), and $\to$ (implication) in the usual way. We use p, q, and r, with or without subscripts, for propositional variables, and ϕ, ψ, and χ, with or without subscripts, for formulas. The set of formulas is denoted by **Wff**. We define $\neg\varphi$ as $\varphi \to \bot$. We assume $\neg$ to connect formulas stronger than $\wedge$ and $\vee$, which in turn are stronger than $\to$, and omit those parentheses that can be recovered according to this priority of the connectives. Also, we use U and V, with or without subscripts, for sets of formulas, especially we use Greek letters $\Gamma, \Delta, \cdots$, with or without subscripts, for finite sets of formulas.

A *sequent* is the expression $(\Gamma \Rightarrow \varphi)$. We often write

$$\varphi_1, \cdots, \varphi_i, \Gamma_1, \cdots, \Gamma_j \Rightarrow \varphi$$

instead of

$$(\{\varphi_1, \cdots, \varphi_i\} \cup \Gamma_1 \cup \cdots \cup \Gamma_j \Rightarrow \varphi).$$

We use X, Y, and Z, with or without subscripts, for sequents. The *antecedent* $\mathbf{ant}(\Gamma \Rightarrow \varphi)$ and the *succedent* $\mathbf{suc}(\Gamma \Rightarrow \varphi)$ of a sequent $\Gamma \Rightarrow \varphi$ are defined as

$$\mathbf{ant}(\Gamma \Rightarrow \varphi) = \Gamma \quad \text{and} \quad \mathbf{suc}(\Gamma \Rightarrow \varphi) = \varphi,$$

respectively. We use S and T, with or without subscripts, for sets of sequents.

A *sequent system* is defined as a collection comprising a set **Axi** of sequents and a set **Inf** of inference rules of the form

$$\frac{X_1 \quad \cdots \quad X_n}{X}(I).$$

Specifically, a *proof figure* of X from T in the sequent system is defined by means of the set $\mathbf{Axi} \cup T$ as axioms and **Inf** as inference rules in the usual way. We use $\vdash$, with or without subscripts, for sequent systems and write $T \vdash X$ if there exists a proof figure of X from T in $\vdash$. We often call an expression $T \vdash X$ a *derivation*, and identify the above inference rule (I) with the derivation

$$\{X_1, \cdots, X_n\} \vdash X.$$

We write $\vdash X$ and $T, U \vdash X$ instead of $\emptyset \vdash X$ and $T \cup \{\Rightarrow \phi \mid \phi \in U\} \vdash X$, respectively. Also, we write $T \vdash \Gamma \Rightarrow \Delta$ if $T \vdash \Gamma \Rightarrow \psi$ for every $\psi \in \Delta$. We note

$$T \nvdash \Gamma \Rightarrow \Delta \iff T \nvdash \Gamma \Rightarrow \psi \text{ for some } \psi \in \Delta.$$

We say that $\vdash$ is consistent if $\nvdash \bot$.

For a sequent system for classical propositional logic, we use the system $\vdash_{\mathbf{Gc}}$ which corresponds to the natural deduction system in Gentzen [4] and Prawitz [7]. Specifically, we define the system $\vdash_{\mathbf{Gc}}$ as follows.

Definition 1.1. A proof figure of X from T in $\vdash_{\mathbf{Gc}}$ is defined by means of the following axioms and inference rules.
Axioms:

- $\phi \Rightarrow \phi$,
- $\bot \Rightarrow \phi$,
- members of T.

Inference rules: See Figure 1.

We note that, among the inference rules in Figure 1, there are just three inference rules $(\vee \Rightarrow)$, $(\Rightarrow\rightarrow)$, and (RAA) corresponding to the improper ones in the natural deduction system.

A sequent system $\vdash_{\mathbf{S}(S)}$ is defined as follows.

Definition 1.2. A proof figure of X from T in the system $\vdash_{\mathbf{S}(S)}$ is defined by means of the following axioms and inference rules.

$$\frac{\Gamma \Rightarrow \psi}{\phi, \Gamma \Rightarrow \psi}(\mathrm{w} \Rightarrow) \qquad \frac{\Gamma \to \phi \quad \phi, \Gamma \to \psi}{\Gamma \to \psi}(\mathrm{cut})$$

$$\frac{\phi_1, \phi_2, \Gamma \Rightarrow \psi}{\phi_1 \wedge \phi_2, \Gamma \Rightarrow \psi}(\wedge \Rightarrow) \qquad \frac{\Gamma \Rightarrow \phi_1 \qquad \Gamma \Rightarrow \phi_2}{\Gamma \Rightarrow \phi_1 \wedge \phi_2}(\Rightarrow \wedge)$$

$$\frac{\phi_1, \Gamma \Rightarrow \psi \qquad \phi_2, \Gamma \Rightarrow \psi}{\phi_1 \vee \phi_2, \Gamma \Rightarrow \psi}(\vee \Rightarrow) \qquad \frac{\Gamma \Rightarrow \phi_i}{\Gamma \Rightarrow \phi_1 \vee \phi_2}(\Rightarrow \vee)(i = 1, 2)$$

$$\frac{\Gamma \Rightarrow \phi_1 \qquad \phi_2, \Gamma \Rightarrow \psi}{\phi_1 \to \phi_2, \Gamma \Rightarrow \psi}(\to\Rightarrow) \qquad \frac{\phi_1, \Gamma \Rightarrow \phi_2}{\Gamma \Rightarrow \phi_1 \to \phi_2}(\Rightarrow\to)$$

$$\frac{\neg\phi, \Gamma \Rightarrow \bot}{\Gamma \Rightarrow \phi}(\mathrm{RAA})$$

Figure 1. Inference rules in $\vdash_{\mathbf{Gc}}$

Axioms: members of $S \cup T$,
Inference rules: $(w \Rightarrow)$ and (cut).

We write $\vdash_{\mathbf{Sc}}$ instead of $\vdash_{\mathbf{S}(S)}$ if $S = \{X \mid \vdash_{\mathbf{Gc}} X\}$. It will be shown in section 2 and section 3 that $\vdash_{\mathbf{Sc}}$ distinguishes proper and improper derivations.

The system $\vdash_{\mathbf{S(C)}}$ has only structural rules, and all logical content is put into axiomatic sequents. Such systems has been considered in Hertz [5], Suszko [10], Suszko [11], and Schroeder-Heister [9]. We can also see the works by Hertz and Suszko in Indrzejczak [6]. However, a difference between proper and improper derivations is not discussed there.

1.2. Proper and improper derivations in sequent systems

In the present section, we consider what derivations are proper or improper in sequent systems, especially the derivations among the ones in Figure 1. We consider an derivation

$$\mathcal{D} : \{\Gamma_1 \Rightarrow \phi_1, \cdots, \Gamma_n \Rightarrow \phi_n\} \vdash \Gamma \Rightarrow \phi.$$

We note that improper inference rule has an assumption discharged by the rule. Therefore, $\mathcal{D}$ is proper if $\Gamma_1 \cup \cdots \cup \Gamma_n \subseteq \Gamma$, and so, (w$\rightarrow$), ($\Rightarrow \wedge$), and ($\Rightarrow \vee$) are proper.

We consider the case that $\Gamma_1 \cup \cdots \cup \Gamma_n \not\subseteq \Gamma$ by Fine's description ([3], p. 69) below:

> "A proper inference is one that is meant to valid in the standard way; the conclusion is meant to follow straightforwardly from premisses."

In this point of view, three derivations (cut), ($\wedge \rightarrow$), and ($\rightarrow\Rightarrow$) are proper since the succedent of the lower sequent follow straightforwardly from the antecedent as in Figure 2, where $\Gamma = \{\gamma_1, \cdots, \gamma_m\}$. We note that each figure in Figure 2 is a tree satisfying:

(T1) every leaf is either a member of the antecedent of the lower sequent or an empty node,

(T2) except leaves, every node is a formula,

(T3) the root is the succedent of the lower sequent,

(T4) every branch is either $]\!\Longrightarrow$ or $]\!\longrightarrow$, where

$$\left.\begin{matrix}\phi_1\\ \vdots\\ \phi_n\end{matrix}\right]\Longrightarrow \psi \iff (\phi_1, \cdots, \phi_n \Rightarrow \psi) \text{ is an upper sequent,}$$

$$\left.\begin{matrix}\phi_1\\ \vdots\\ \phi_n\end{matrix}\right]\longrightarrow \psi \iff \vdash_{\mathbf{Gc}} (\phi_1, \cdots, \phi_n \Rightarrow \psi).$$

We can also see such trees for ($\Rightarrow \wedge$) and ($\Rightarrow \vee$) in Figure 3, where $\Gamma = \{\gamma_1, \cdots, \gamma_m\}$.

On the other hand, three derivations ($\vee \Rightarrow$), ($\Rightarrow\rightarrow$), and (RAA) are improper since there is no such tree. More precisely, we have

- $\{\phi_1, \Gamma \Rightarrow \psi\} \nvdash_{\mathbf{Gc}} (\phi_1 \vee \phi_2, \Gamma \Rightarrow \phi_2)$ and $\{\phi_2, \Gamma \Rightarrow \psi\} \nvdash_{\mathbf{Gc}} (\phi_1 \vee \phi_2, \Gamma \Rightarrow \phi_1)$ if $(\Gamma, \phi_1, \phi_2, \psi) = (\emptyset, p, q, r)$,

- $\not\vdash_{\mathbf{Gc}} \Gamma \Rightarrow \phi_1$ if $(\Gamma, \phi_1) = (\emptyset, p)$,
- $\not\vdash_{\mathbf{Gc}} \Gamma \Rightarrow \neg\phi$ if $(\Gamma, \phi) = (\emptyset, p)$.

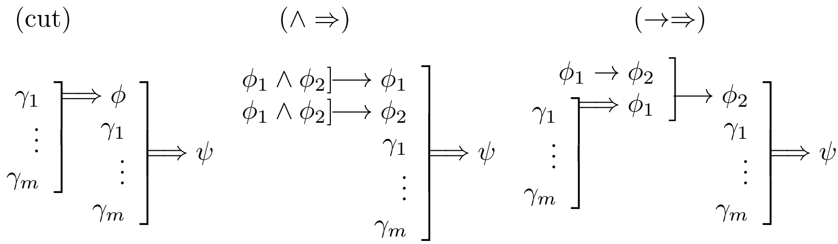

Figure 2. Trees for (cut), $(\wedge \Rightarrow)$, and $(\rightarrow\Rightarrow)$

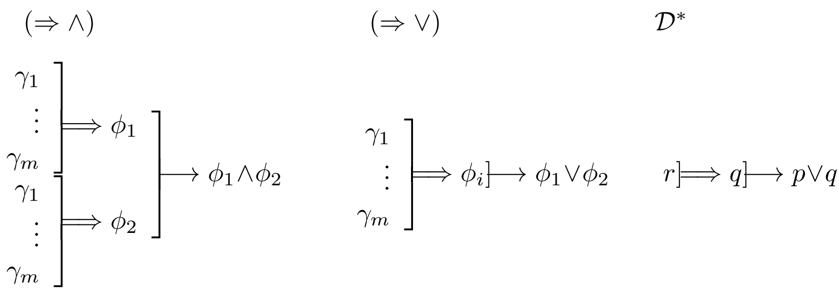

Figure 3. Trees for $(\Rightarrow \wedge)$, $(\Rightarrow \vee)$, and $\mathcal{D}^*$

Consequently, among the derivations in Figure 1, $(\vee \Rightarrow)$, $(\Rightarrow\rightarrow)$, and (RAA) are improper, and the others are proper. In general, $\mathcal{D}$ is proper if $\mathcal{D}$ has a tree satisfying (T1), (T2), (T3), and (T4). Also, for $\psi \in \Gamma_1$, ψ is discharged by $\mathcal{D}$ if the following two conditions hold:

(D1) $\{\Gamma_2 \Rightarrow \phi_2, \cdots, \Gamma_n \Rightarrow \phi_n\} \not\vdash_{\mathbf{Gc}} \Gamma \Rightarrow \psi$,

(D2) $\{\Gamma_2 \Rightarrow \phi_2, \cdots, \Gamma_n \Rightarrow \phi_n\} \not\vdash_{\mathbf{Gc}} \Gamma \Rightarrow \phi$.

Moreover, $\mathcal{D}$ is improper if there exists $\psi \in \Gamma_1$ satisfying (D1) and (D2). Here, we need (D2) since the derivation

$$\mathcal{D}^* : \{p \Rightarrow r, r \Rightarrow q\} \vdash r \Rightarrow p \vee q,$$

which has the tree in Figure 3, should be proper and p in $p \Rightarrow r$ satisfies (D1). We have to note that the meaning of proper and improper derivations has not been clarified yet since there may be a case that the following two conditions hold:

- $\mathcal{D}$ has no tree satisfying (T1), (T2), (T3), and (T4),
- there is no formula $\phi \in \Gamma_i$ satisfying (D1) and (D2).

In section 3, we consider this in more detail by using $\vdash_{\mathbf{Sc}}$.

Here, we also note that the system $\vdash_{\mathbf{S}(S)}$ has only proper structural inference rules, and consequently, it is natural to see that if $T \vdash_{\mathbf{S}(S)} \Gamma \Rightarrow \phi$, then

$$\text{“}\Gamma \Rightarrow \phi \text{ is derived straightforwardly from } T\text{”}. \tag{P1}$$

and the derivation is proper.

1.3. The purposes

In the present paper, we distinguish proper and improper derivations by the sequent system $\vdash_{\mathbf{Sc}}$. Improper derivations are more complicated and are often harder to understand than the proper ones since they have assumptions discharged by the rule and have no tree satisfying (T1), (T2), (T3), and (T4) in the previous subsection. So, if we obtain a system that distinguishes proper and improper derivations, then we know what kind of inference rules are hard to understand. This knowledge is valuable when we teach proof in mathematics education.

In order to distinguish proper and improper derivations, there are two purposes.

One is to prove that our system $\vdash_{\mathbf{Sc}}$ distinguishes the proper and improper derivations among the ones in Figure 1. However, it is not hard to see that the proper derivations in Figure 1 hold in $\vdash_{\mathbf{Sc}}$. So, the main theorem we should prove is as follows.

THEOREM 1.3. *None of the improper derivations* ($\vee \Rightarrow$), ($\Rightarrow\to$), *and* (RAA) *holds in* $\vdash_{\mathbf{Sc}}$ *in general.*

We prove the theorem above in the following section by using completeness.

The other is to consider whether an improper derivation can be described generally by using $\vdash_{\mathbf{Sc}}$. As we mentioned in the previous subsection, the description in the subsection is not enough to clarify proper and improper derivations. We consider it in more detail in section 3.

2. Completeness

In the present section, we prove Theorem 1.3. In order to prove the theorem, we modify the usual notion of truth valuation, and using the modified valuation, we prove completeness of the system $\vdash_{\mathbf{Sc}}$. Theorem 1.3 will be obtained as a corollary of the completeness.

The definition of the usual truth valuation is as follows.

DEFINITION 2.1. We say that a mapping $v : \mathbf{Wff} \to \{\mathsf{t}, \mathsf{f}\}$ is a *truth valuation* if the following conditions hold:

1. $v(\bot) = \mathsf{f}$,
2. $v(\phi \wedge \psi) = \mathsf{t} \iff v(\phi) = v(\psi) = \mathsf{t}$,
3. $v(\phi \vee \psi) = \mathsf{f} \iff v(\phi) = v(\psi) = \mathsf{f}$,
4. $v(\phi \to \psi) = \mathsf{f} \iff v(\phi) = \mathsf{t}$ and $v(\psi) = \mathsf{f}$.

We use v, with or without subscripts, for truth valuations. We write $v(U) = \mathsf{t}$ if $v(\phi) = \mathsf{t}$ for every $\phi \in U$. Also, we write $v(X) = \mathsf{t}$ if $v(\mathbf{ant}(X)) = \mathsf{f}$ or $v(\mathbf{suc}(X)) = \mathsf{t}$. Moreover, we write $v(T) = \mathsf{t}$ if $v(X) = \mathsf{t}$ for every $X \in T$.

We modify the above definition of truth valuation as follows.

DEFINITION 2.2. Let $\mathbf{v}$ be a set of truth valuations. We define a mapping $\mathbf{v} : \mathbf{Wff} \to \{\mathsf{t}, \mathsf{f}\}$ as follows:

$$\mathbf{v}(\phi) = \mathsf{t} \iff \text{for every } v \in \mathbf{v},\ v(\phi) = \mathsf{t}.$$

We note that

- $\emptyset(\phi) = \mathsf{t}$,
- $\{v\}(\phi) = v(\phi)$,
- $\{v_1, v_2\}(\phi) = \mathsf{t} \iff v_1(\phi) = v_2(\phi) = \mathsf{t}$.

We write $\mathbf{v}(U)$, $\mathbf{v}(X)$, and $\mathbf{v}(T)$, similarly to $v(U)$, $v(X)$, and $v(T)$, respectively.

The main theorem in the present section is as follows.

THEOREM 2.3. *The following conditions are equivalent:*

(1) $T \vdash_{\mathbf{Sc}} X$,

(2) *for every set* $\mathbf{v}$ *of truth valuations,* $\mathbf{v}(T) = \mathtt{t}$ *implies* $\mathbf{v}(X) = \mathtt{t}$.

In order to prove the above theorem, we provide some preparations. The completeness below can be shown in the usual way. For example, we can refer to Chagrov and Zakharyaschev [2].

LEMMA 2.4.

$$U \vdash_{\mathbf{Gc}} \Rightarrow \phi \iff \textit{for every truth valuation } v, v(U) = \mathtt{t} \textit{ implies } v(\phi) = \mathtt{t}.$$

LEMMA 2.5.

(1) $T \cup \{\Rightarrow \psi\} \vdash_{\mathbf{Gc}} \Gamma \Rightarrow \phi \iff T \vdash_{\mathbf{Gc}} (\psi, \Gamma \Rightarrow \phi)$.

(2) $T \cup \{\Rightarrow \psi\} \vdash_{\mathbf{Sc}} \Gamma \Rightarrow \phi \iff T \vdash_{\mathbf{Sc}} (\psi, \Gamma \Rightarrow \phi)$.

(3) $\vdash_{\mathbf{Gc}} X \iff \vdash_{\mathbf{Sc}} X$.

(4) $U \vdash_{\mathbf{Gc}} X \iff U \vdash_{\mathbf{Sc}} X$.

PROOF: (1), (2), and the direction "$\Longleftarrow$" of (3) can be shown by an induction on a proof figure. The direction "$\Longrightarrow$" of (3) is clear since every member of $\{X \mid \vdash_{\mathbf{Gc}} X\}$ is an axiom of $\vdash_{\mathbf{Sc}}$.

For (4). By (1), (2), and (3), for every finite set U^* of formulas, we have

$$\begin{aligned} U^* \vdash_{\mathbf{Gc}} X &\iff \vdash_{\mathbf{Gc}} (U^*, \mathbf{ant}(X) \Rightarrow \mathbf{suc}(X)) \\ &\iff \vdash_{\mathbf{Sc}} (U^*, \mathbf{ant}(X) \Rightarrow \mathbf{suc}(X)) \\ &\iff U^* \vdash_{\mathbf{Sc}} X. \end{aligned} \tag{4.1}$$

Also, we note that

$$U \vdash_{\mathbf{Gc}} X \iff U^* \vdash_{\mathbf{Gc}} X \text{ for some finite subset } U^* \text{ of } U,$$

and the same equivalence holds in $\vdash_{\mathbf{Sc}}$. Hence, we obtain (4). □

We note that the expression $U \vdash X$ is an abbreviation of $\{\Rightarrow \phi \mid \phi \in U\} \vdash X$. So, none of the improper derivations $(\vee \Rightarrow)$, $(\Rightarrow \rightarrow)$, and (RAA) can be expressed in the form of $U \vdash X$. On the other hand, some of the proper derivations in Figure 1 can be expressed in the form. For example, the derivation $\{\Rightarrow p\} \vdash \Rightarrow p \vee q$ can be expressed in the form.

By means of this example, we show how a proof figure for $U \vdash_{\mathbf{Gc}} X$ transfer to the one for $U \vdash_{\mathbf{Sc}} X$. Specifically, we show the proof figures, in Table 1, for four derivations occurring in the above (4.1) in Lemma 2.5.

Table 1. Proof figures for derivations in (4.1) in Lemma 2.5

Derivation	Proof figure
$\{\Rightarrow p\} \vdash_{\mathbf{Gc}} \Rightarrow p \vee q$	$\dfrac{\Rightarrow p}{\Rightarrow p \vee q}(\Rightarrow \vee)$
$\vdash_{\mathbf{Gc}} (p \Rightarrow p \vee q)$	$\dfrac{p \Rightarrow p}{p \Rightarrow p \vee q}(\Rightarrow \vee)$
$\vdash_{\mathbf{Sc}} (p \Rightarrow p \vee q)$	$p \Rightarrow p \vee q$
$\{\Rightarrow p\} \vdash_{\mathbf{Sc}} \Rightarrow p \vee q$	$\dfrac{\Rightarrow p \quad p \Rightarrow p \vee q}{\Rightarrow p \vee q}(\mathrm{cut})$

Lemma 2.6. *If $\vdash_{\mathbf{Sc}} X$, then for every set $\mathbf{v}$ of truth valuations $\mathbf{v}(X) = \mathsf{t}$.*

Proof: By Lemma 2.5 and Lemma 2.4. □

Lemma 2.7. *If $T \vdash_{\mathbf{Sc}} X$, then for every set $\mathbf{v}$ of truth valuations, $\mathbf{v}(T) = \mathsf{t}$ implies $\mathbf{v}(X) = \mathsf{t}$.*

Proof: Suppose that $T \vdash_{\mathbf{Sc}} X$ and $\mathbf{v}(T) = \mathsf{t}$. We show $\mathbf{v}(X) = \mathsf{t}$ by an induction on a proof figure of X from T in $\vdash_{\mathbf{Sc}}$.

Basis. If $X \in T$, then by $\mathbf{v}(T) = \mathsf{t}$, we have $\mathbf{v}(X) = \mathsf{t}$. If $\vdash_{\mathbf{Gc}} X$, then we have $\vdash_{\mathbf{Sc}} X$, and using Lemma 2.6, we have $\mathbf{v}(X) = \mathsf{t}$.

Induction step is clear from

- $\mathbf{v}(\Gamma \Rightarrow \psi) = \mathsf{t}$ implies $\mathbf{v}(\phi, \Gamma \Rightarrow \psi) = \mathsf{t}$,
- $\mathbf{v}(\Gamma \Rightarrow \phi) = \mathbf{v}(\phi, \Gamma \Rightarrow \psi) = \mathsf{t}$ implies $\mathbf{v}(\Gamma \Rightarrow \psi) = \mathsf{t}$. □

Definition 2.8. We call a pair $\langle U, V\rangle$ of sets of formulas T-*consistent* if $T, U \nvdash_{\mathbf{Sc}} \Rightarrow \phi$ for each $\phi \in V$. We call T-consistent pair $\langle U, V\rangle$ *maximal* if $U \cup V = \mathbf{Wff}$.

Lemma 2.9. *If $T \nvdash_{\mathbf{Sc}} \Gamma \Rightarrow \phi$, then there exists a maximal T-consistent pair $\langle U, V\rangle$ satisfying $\Gamma \subseteq U$ and $\phi \in V$.*

Proof: Suppose that $T \nvdash_{\mathbf{Sc}} \Gamma \Rightarrow \phi$. We define U and V as

$$U = \{\chi \mid T \vdash_{\mathbf{Sc}} \Gamma \Rightarrow \chi\} \text{ and } V = \mathbf{Wff} \backslash U.$$

It is sufficient to show the following three conditions:

(1) $\Gamma \subseteq U$ and $\phi \in V$,

(2) (maximarity) $U \cup V = \mathbf{Wff}$,

(3) (consistency) for each formula $\psi \in V$, $T, U \nvdash_{\mathbf{Sc}} \Rightarrow \psi$.

(1) and (2) are clear from the definition. We show (3). Suppose that $\psi \in V$. By the definition of $\vdash_{\mathbf{Sc}}$, we have only to show

(4) for each finite subset U^* of U, $T, U^* \nvdash_{\mathbf{Sc}} \Gamma \Rightarrow \psi$.

In order to show (4), we use an induction on the number of members of U^*. If $U^* \subseteq \Gamma$, then by $T \nvdash_{\mathbf{Sc}} \Gamma \Rightarrow \phi$ and Lemma 2.5, we have (4). Suppose that there exists $\chi \in U^* \backslash \Gamma \subseteq U$. Then by the definition of U, we have

$$T \vdash_{\mathbf{Sc}} \Gamma \Rightarrow \chi,$$

and so,

$$T, U^* \backslash \{\chi\} \vdash_{\mathbf{Sc}} \Gamma \Rightarrow \chi, \qquad (*1)$$

By the induction hypothesis, we have

$$T, U^* \backslash \{\chi\} \nvdash_{\mathbf{Sc}} \Gamma \Rightarrow \psi, \qquad (*2)$$

By $(*1), (*2)$, and cut, we obtain (4). □

Lemma 2.10. *If $T \nvdash_{\mathbf{Sc}} \Gamma \Rightarrow \phi$, then there exists a set $\mathbf{v}$ of truth valuations such that $\mathbf{v}(T) = \mathsf{t}$ and $\mathbf{v}(\Gamma \Rightarrow \phi) = \mathsf{f}$.*

Proof: Suppose that $T \nvdash_{\mathbf{Sc}} \Gamma \Rightarrow \phi$. By Lemma 2.9, there exists a maximal T-consistent pair $\langle U, V\rangle$ satisfying $\Gamma \subseteq U$ and $\phi \in V$. Since $\langle U, V\rangle$ is T-consistent, for each $\psi \in V$, we observe

$$T, U \nvdash_{\mathbf{Sc}} \Rightarrow \psi.$$

Therefore

$$U \nvdash_{\mathbf{Sc}} \Rightarrow \psi.$$

Using Lemma 2.5 and Lemma 2.4, there exists a truth valuation v_ψ satisfying

$$v_\psi(U) = \mathsf{t} \text{ and } v_\psi(\psi) = \mathsf{f}.$$

We define $\mathbf{v}$ as

$$\mathbf{v} = \{v_\psi \mid \psi \in V\}.$$

Then we have

$$\mathbf{v}(U) = \mathsf{t} \text{ and } \mathbf{v}(\psi) = \mathsf{f} \text{ for every } \psi \in V,$$

and using $\Gamma \subseteq U$ and $\phi \in V$, we have

$$\mathbf{v}(\Gamma \Rightarrow \phi) = \mathsf{f}.$$

So, we have only to show

(1) $\mathbf{v}(T) = \mathsf{t}$.

Let X be a sequent in T. We divide the cases.

The case that $\mathbf{ant}(X) \not\subseteq U$. By the maximality of $\langle U, V\rangle$, we have $\mathbf{ant}(X) \cap V \neq \emptyset$, and so, $\psi \in \mathbf{ant}(X)$ for some $\psi \in V$, Using $v_\psi(\psi) = \mathsf{f}$, we have $\mathbf{v}(\psi) = \mathsf{f}$. Using $\psi \in \mathbf{ant}(X)$, we obtain $\mathbf{v}(X) = \mathsf{t}$.

The case that $\mathbf{ant}(X) \subseteq U$. Using $X \in T$, we have $T, U \vdash_{\mathbf{Sc}} \Rightarrow \mathbf{suc}(X)$. Since $\langle U, V\rangle$ is T-consistent, we observe $\mathbf{suc}(X) \notin V$. Using maximality of $\langle U, V\rangle$, we have $\mathbf{suc}(X) \in U$, and using $\mathbf{v}(U) = \mathsf{t}$, we have $\mathbf{v}(\mathbf{suc}(X)) = \mathsf{t}$. Hence, we have $\mathbf{v}(X) = \mathsf{t}$.

Hence, we obtain (1). $\square$

By Lemma 2.7 and Lemma 2.10, we obtain Theorem 2.3. Theorem 1.3 is obtained by the following corollary.

Corollary 2.11.

(1) $\{p \Rightarrow q\} \nvdash_{\mathbf{Sc}} \Rightarrow p \to q$.

(2) $\{p \Rightarrow r, q \Rightarrow r\} \nvdash_{\mathbf{Sc}} p \vee q \Rightarrow r$.

(3) $\{\neg p \Rightarrow \bot\} \nvdash_{\mathbf{Sc}} \Rightarrow p$.

PROOF: For (1). We define truth valuations v_1, v_2 as

$$(v_1(p), v_1(q)) = (\mathsf{t}, \mathsf{f}), \quad (v_2(p), v_2(q)) = (\mathsf{f}, \mathsf{f}).$$

Then as in Table 2, we obtain

$$\{v_1, v_2\}(p \Rightarrow q) = \mathsf{t} \text{ and } \{v_1, v_2\}(\Rightarrow p \to q) = \mathsf{f}.$$

Using Theorem 2.3, we obtain (1).

(2) and (3) can be shown similarly using Table 3 and Table 4, respectively. □

Table 2. A truth table for (1)

	p	q	$p \Rightarrow q$	$p \to q$	$\Rightarrow p \to q$
v_1	t	f		f	
v_2	f	f			
$\{v_1, v_2\}$	f		t	f	f

Table 3. A truth table for (2)

	p	q	r	$p \Rightarrow r$	$q \Rightarrow r$	$p \vee q$	$p \vee q \Rightarrow r$
v_1	t	f	f			t	
v_2	f	t				t	
$\{v_1, v_2\}$	f	f	f	t	t	t	f

Table 4. A truth table for (3)

	p	$\neg p$	$\neg p \Rightarrow \bot$	$\Rightarrow p$
v_1	f	t		
v_2	t	f		
$\{v_1, v_2\}$	f	f	t	f

Also, by the fact that classical logic is the maximally consistent logic (cf. Chagrov and Zakharyaschev [2]), we have the following corollary.

COROLLARY 2.12. If $\vdash_{\mathbf{S}(S)}$ is consistent, then

(1) $\{p \Rightarrow q\} \not\vdash_{\mathbf{S}(S)} \Rightarrow p \rightarrow q$,

(2) $\{p \Rightarrow r, q \Rightarrow r\} \not\vdash_{\mathbf{S}(S)} p \vee q \Rightarrow r$,

(3) $\{\neg p \Rightarrow \bot\} \not\vdash_{\mathbf{S}(S)} \Rightarrow p$.

3. Improper derivations and the system $\vdash_{\mathbf{Sc}}$

In the present section, we consider whether an improper derivation can be described generally by using our system $\vdash_{\mathbf{Sc}}$. Specifically, we consider a derivation $\mathcal{D} : T \vdash \Gamma \Rightarrow \phi$ and give a precise expression of

$$\text{“}\mathcal{D}\text{ is improper”}, \tag{IP1}$$

assuming that (IP1) is equivalent to

$$\text{“}\mathcal{D}\text{ has some assumptions discharged by }\mathcal{D}\text{”} \tag{IP2}$$

and negation of (P1) in subsection 1.2.

As is described in subsection 1.2, (IP2) follows from the existence of a formula satisfying (D1) and (D2). More generally, we have that (C1) implies (IP2), where (C1) is the following condition.

(C1) There exists $X \in T$ satisfying the following two conditions:

(C1.1) $T\backslash\{X\} \not\vdash_{\mathbf{Gc}} \Gamma \Rightarrow \mathbf{ant}(X)$,

(C1.2) $T\backslash\{X\} \not\vdash_{\mathbf{Gc}} \Gamma \Rightarrow \phi$.

However, as we also mentioned in subsection 1.2, there may be an improper derivation which does not satisfy (C1). We give such improper derivation in the following example.

Example 3.1. We consider the following two derivations:

$\mathcal{D}_1$: $\{p \Rightarrow \bot, q \Rightarrow \bot\} \vdash \Rightarrow \neg p \vee \neg q$,

$\mathcal{D}_2$: $\{p \Rightarrow \bot, \neg p \Rightarrow \bot\} \vdash \Rightarrow \bot$.

(1) $\mathcal{D}_1$ has two proofs in $\vdash_{\mathbf{Gc}}$. One is to prove

$$\{p \Rightarrow \bot\} \vdash_{\mathbf{Gc}} \Rightarrow \neg p \vee \neg q$$

and the other is to prove

$$\{q \Rightarrow \bot\} \vdash_{\mathbf{Gc}} \Rightarrow \neg p \vee \neg q.$$

If we take the former, then the assumption p in $p \Rightarrow \bot$ is discharged by $\mathcal{D}_1$, and if we take the latter, then the assumption q in $q \Rightarrow \bot$ is.

(2) $\mathcal{D}_2$ also has two proofs. One is to prove

$$\{p \Rightarrow \bot\} \vdash_{\mathbf{Gc}} \Rightarrow \neg p$$

and the other is to prove

$$\{\neg p \Rightarrow \bot\} \vdash_{\mathbf{Gc}} \Rightarrow p.$$

If we take the former, then the assumption p in $p \Rightarrow \bot$ is discharged by $\mathcal{D}_2$, and if we take the latter, then the assmption $\neg p$ in $\neg p \Rightarrow \bot$ is.

So, $\mathcal{D}_1$ must be improper, but it does not satisfy (C1) because of (C1.2). Also, $\mathcal{D}_2$ must be improper, but it does not satisfy (C1) because of (C1.1).

Consequently, in order to give a precise expression of (IP2), (C1) should be modified. Specifically, we consider the following modified condition (C2), and by Example 3.1, it is natural to see that (C2) implies (IP2). We also confirm that $\mathcal{D}_1$ and $\mathcal{D}_2$ satisfy (C2).

(C2) There exists a non-empty subset T' of T satisfying the following two conditions:

(C2.1) $T \backslash T' \nvdash_{\mathbf{Gc}} \Gamma \Rightarrow \mathbf{ant}(X)$ for each $X \in T'$,

(C2.2) $T \backslash T' \nvdash_{\mathbf{Gc}} \Gamma \Rightarrow \phi$.

Now, we consider the condition:

$$T \nvdash_{\mathbf{Sc}} \Gamma \Rightarrow \phi. \tag{C3}$$

In subsection 1.2, we confirmed that the negation of (C3) implies (P1). We have already confirmed that (C2) implies (IP2). Also, we assumed the

$$\text{(C2)} \Longrightarrow \text{(IP2)} \iff \text{(IP1)} \iff \text{the negation of (P1)} \Longrightarrow \text{(C3)}$$

Figure 4. Relations among (C2), (C3), (P1), (IP1), and (IP2)

equivalence among the conditions (IP2), (IP1), and the negation of (P1). We can see these relations in Figure 4.

Therefore, if we show the equivalence between (C2) and (C3), then each of (C2) and (C3) is one of the precise expressions of (IP1). Hence, the remaining to be done is to prove such equivalence, i.e., the following theorem.

THEOREM 3.2. *If* $T \vdash_{\mathbf{Gc}} \Gamma \Rightarrow \phi$, *then the conditions* (C2) *and* (C3) *are equivalent.*

We prove the above theorem, including the derivations that do not hold in $\vdash_{\mathbf{Gc}}$. Specifically, we prove the following lemma. The theorem above is obtained as a corollary of the lemma.

LEMMA 3.3. *The following two conditions are equivalent:*

(1) $T \not\vdash_{\mathbf{Sc}} \Gamma \Rightarrow \phi$,

(2) *there exists a subset* T' *of* T *satisfying the following two conditions:*

(2.1) $T \backslash T' \not\vdash_{\mathbf{Gc}} \Gamma \Rightarrow \mathbf{ant}(X)$ *for each* $X \in T'$,

(2.2) $T \backslash T' \not\vdash_{\mathbf{Gc}} \Gamma \Rightarrow \phi$.

PROOF: For (1) $\Longrightarrow$ (2). Suppose that (1) holds. Then by Theorem 2.3, there exists a set $\mathbf{v}$ of truth valuations such that $\mathbf{v}(T) = \mathbf{v}(\Gamma) = \mathsf{t}$ and $\mathbf{v}(\phi) = \mathsf{f}$. We define T' as

$$T' = \{X \in T \mid \mathbf{v}(\mathbf{ant}(X)) = \mathsf{f}\}.$$

Then we observe $\mathbf{v}(\mathbf{ant}(Y)) = \mathsf{t}$ for every $Y \in T \backslash T'$. Using $\mathbf{v}(T) = \mathsf{t}$, we have

$$\mathbf{v}(\mathbf{suc}(Y)) = \mathsf{t} \text{ for every } Y \in T \backslash T'. \tag{$*1$}$$

We show (2.1). Let X be a sequent in T'. Then we observe $\mathbf{v}(\mathbf{ant}(X)) = \mathsf{f}$, and so, there exists $v_X \in \mathbf{v}$ such that $v_X(\mathbf{ant}(X)) = \mathsf{f}$. Also, by $\mathbf{v}(\Gamma) = \mathsf{t}$, we have $v_X(\Gamma) = \mathsf{t}$. Moreover, by $(*1)$, we have $v_X(\mathbf{suc}(Y)) = \mathsf{t}$ for every $Y \in T \backslash T'$, and so, $v_X(T \backslash T') = \mathsf{t}$. Using Lemma 2.4, we have (2.1).

We show (2.2). By $\mathbf{v}(\phi) = \mathsf{f}$, there exists $v_0 \in \mathbf{v}$ such that $v_0(\phi) = \mathsf{f}$. Also, by $\mathbf{v}(\Gamma) = \mathsf{t}$, we have $v_0(\Gamma) = \mathsf{t}$. Moreover, by $(*1)$, we have $v_0(\mathbf{suc}(Y)) = \mathsf{t}$ for every $Y \in T \backslash T'$, and so, $v_0(T \backslash T') = \mathsf{t}$. Using Lemma 2.4, we have (2.2).

For (2) $\Longrightarrow$ (1). Suppose that (2) holds. Then by (2.1) and Lemma 2.4, for every $X \in T'$, there exists v_X such that

$$v_X(T \backslash T') = v_X(\Gamma) = \mathsf{t} \text{ and } v_X(\mathbf{ant}(X)) = \mathsf{f}.$$

Also, by (2.2) and Lemma 2.4, there exists v_0 such that

$$v_0(T \backslash T') = v_0(\Gamma) = \mathsf{t} \text{ and } v_0(\phi) = \mathsf{f}.$$

We define $\mathbf{v}$ as

$$\mathbf{v} = \{v_0\} \cup \{v_X \mid X \in T'\}.$$

Then we have $\mathbf{v}(T \backslash T') = \mathbf{v}(\Gamma) = \mathsf{t}$ and $\mathbf{v}(\mathbf{ant}(X)) = \mathbf{v}(\phi) = \mathsf{f}$ for every $X \in T'$, and so, we have $\mathbf{v}(T) = \mathbf{v}(\Gamma) = \mathsf{t}$ and $\mathbf{v}(\phi) = \mathsf{f}$. Using Theorem 2.3, we obtain (1). □

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References

[1] W. Breckenridge, O. Magidor, *Arbitrary reference*, **Philosophical Studies: An International Journal for Philosophy in the Analytic Tradition**, vol. 158(3) (2012), pp. 377–400, DOI: https://doi.org/10.1007/s11098-010-9676-z.

[2] A. Chagrov, M. Zakharyaschev, **Modal logic**, Oxford Logic Guides, Oxford University Press, New York (1997).

[3] K. Fine, **Reasoning with arbitrary objects**, Aristotelian Society Series, Basil Blackwell, Oxford (1986).

[4] G. Gentzen, *Untersuchungen über das logisch Schließen*, **Mathematische Zeitschrift**, vol. 39 (1934–1935), pp. 176–210, 405–431, DOI: https://doi.org/10.1007/BF01201353.

[5] P. Hertz, *Über Axiomensysteme für beliebige Satzsysteme*, **Mathematische Annalen**, vol. 101 (1929), pp. 457–514.

[6] A. Indrzejczak, *A Survey of Nonstandard Sequent Calculi*, **Studia Logica**, vol. 102 (2014), pp. 1295–1322, DOI: https://doi.org/10.1007/s11225-014-9567-y.

[7] D. Prawitz, **Natural Deduction: A Proof-Theoretical Study**, Almqvist & Wiksell, Stockholm (1965).

[8] K. Robering, *Ackermann's Implication for Typefree Logic*, **Journal of Logic and Computation**, vol. 11(1) (2001), pp. 5–23, DOI: https://doi.org/10.1093/logcom/11.1.5.

[9] P. Schroeder-Heister, *Resolution and the Origins of Structural Reasoning: Early Proof-Theoretic Ideas of Hertz and Gentzen*, **The Bulletin of Symbolic Logic**, vol. 8(2) (2002), pp. 246–265, DOI: https://doi.org/10.2178/bsl/1182353872.

[10] R. Suszko, *W sprawie logiki bez aksjomatów*, **Kwartalnik Filozoficzny**, vol. 17 (1948), pp. 199–205.

[11] R. Suszko, *Formalna teoria wartości logicznych*, **Studia Logica**, vol. 6 (1957), pp. 145–320, DOI: https://doi.org/10.1007/BF02547932.

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WEAKLY FREE MULTIALGEBRAS

Abstract

In abstract algebraic logic, many systems, such as those paraconsistent logics taking inspiration from da Costa's hierarchy, are not algebraizable by even the broadest standard methodologies, as that of Blok and Pigozzi. However, these logics can be semantically characterized by means of non-deterministic algebraic structures such as Nmatrices, RNmatrices and swap structures. These structures are based on multialgebras, which generalize algebras by allowing the result of an operation to assume a non-empty set of values. This leads to an interest in exploring the foundations of multialgebras applied to the study of logic systems.

It is well known from universal algebra that, for every signature Σ, there exist algebras over Σ which are absolutely free, meaning that they do not satisfy any identities or, alternatively, satisfy the universal mapping property for the class of Σ-algebras. Furthermore, once we fix a cardinality of the generating set, they are, up to isomorphisms, unique, and equal to algebras of terms (or propositional formulas, in the context of logic). Equivalently, the forgetful functor, from the category of Σ-algebras to **Set**, has a left adjoint. This result does not extend to multialgebras. Not only multialgebras satisfying the universal mapping property do not exist, but the forgetful functor $\mathcal{U}$, from the category of Σ-multialgebras to **Set**, does not have a left adjoint.

In this paper we generalize, in a natural way, algebras of terms to multialgebras of terms, whose family of submultialgebras enjoys many properties of the former. One example is that, to every pair consisting of a function, from a submultialgebra of a multialgebra of terms to another multialgebra, and a collection of choices (which selects how a homomorphism approaches indeterminacies), there corresponds a unique homomorphism, what resembles the universal mapping property. Another example is that the multialgebras of terms are generated

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by a set that may be viewed as a strong basis, which we call the ground of the multialgebra. Submultialgebras of multialgebras of terms are what we call weakly free multialgebras. Finally, with these definitions at hand, we offer a simple proof that multialgebras with the universal mapping property for the class of all multialgebras do not exist and that $\mathcal{U}$ does not have a left adjoint.

Keywords: Algebras of terms, universal mapping property, absolutely free algebras, multialgebras, hyperalgebras, non-deterministic algebras, category of multialgebras, non-deterministic semantics.

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1. Introduction

An interesting and fruitful strategy in contemporary formal logic is trying to find an algebraic counterpart for a given logic or family of logics. This is the main goal of the area of mathematical logic known as *algebraic logic*, or *abstract algebraic logic* (AAL) in a more general perspective.

The idea behind (traditional) algebraic logic is to develop an algebraic study of a given class of models (algebras) associated to a given logic. For instance, it can be insightful to study the relationship between Boolean (Heyting, respectively) algebras and propositional classical (intuitionistic, respectively) logic, while an important area of mathematical fuzzy logic deals with the relationship between fuzzy logics and certain classes of residuated lattices. In turn, AAL is more interested in analyzing and classifying the algebraization methods *per se*. As one would expect, the scope of (abstract) algebraic logic is far from being universal: there are important classes of logics which lie outside the usual methods and techniques of AAL.

A good source of examples to this phenomenon can be found in the field of paraconsistency. Because of this, certain classes of paraconsistent logics, as the ones known as *logics of formal inconsistency*,[1] are characterized by means of semantics of non-deterministic character such as non-deterministic matrices, Fidel structures or swap structures (see for instance [3]). Besides giving a semantical characterization, as well as a decision procedure, for these logics, such non-deterministic structures

[1] **LFI**s, introduced in [4] and coming from the tradition of da Costa's approach to paraconsistency ([9])

constitute an interesting object of study by themselves (see, for instance, [3, Chapter 6], [5] and [7]).

It is worth observing that non-deterministic matrix semantics (introduced in [1]) and, more generally, swap structures semantics, are (classes of) *multialgebras* equipped with a subset of designated elements of their domains, what generalizes the very idea of logical matrices. Multialgebras, also known as *hyperalgebras* or *non-deterministic algebras*, introduced in [10], generalize the concept of algebra by replacing operations by *multioperations* (or *hyperoperations*), whose results assume multiple values, that is, a subset of the universe. Here, we will restrict ourselves to multialgebras whose operations cannot return an empty set of values, which is a common requirement when working with non-classical logics and their semantics.

In the realm of universal algebra, it is a well known result ([2]) that there exist algebras $\mathcal{A}$ over a given signature Σ that satisfy the so-called *universal mapping property*, for the class of all Σ-algebras, over some subset X of their universe A. This property says that, for any other Σ-algebra $\mathcal{B}$ with universe B and any function $f : X \to B$, there exists exactly one homomorphism $\overline{f}$ between $\mathcal{A}$ and $\mathcal{B}$ that extends f. Such algebras are called (absolutely) free Σ-algebras generated by X. Moreover, any free Σ-algebra generated by X is isomorphic to the Σ-algebra of terms over X, which will be denoted here by $\mathbf{T}(\Sigma, X)$. Thus, free algebras are unique up to isomorphisms. In the language of categories, the existence of free Σ-algebras means that the forgetful functor $U : \mathbf{Alg}(\Sigma) \to \mathbf{Set}$, from the category of Σ-algebras to the category of sets, has a left adjoint F, associating to a set X any Σ-algebra with the universal mapping property over X (which, as mentioned above, can be taken as being $\mathbf{T}(\Sigma, X)$).

While algebras satisfying the universal mapping property always exist, and are (up to isomorphisms) algebras of terms, the situation is quite different in the context of multialgebras. Indeed, it is well-known that multialgebras satisfying the universal mapping property do not exist, and so the forgetful functor $\mathcal{U} : \mathbf{MAlg}(\Sigma) \to \mathbf{Set}$, from the category of multialgebras over the signature Σ to the category of sets, does not have a left adjoint. This means that any possible "multialgebra of terms" generalizing in some sense the notion of algebra of terms to the category of multialgebras necessarily will not satisfy the universal mapping property. A new proof of this fact will be given in Section 4.

The aim of this paper is proposing a very natural generalization to the category of multialgebras of the concept of algebra of terms by means of a family $\mathcal{FT}(\Sigma, \mathcal{V})$ of multialgebras of terms indexed by the cardinals $\kappa > 0$. Any submultialgebra of a multialgebra of this family satisfies several equivalent characterizations, which are necessarily weaker than the standard characterization of absolutely free algebras by means of the universal mapping property. We propose the novel notion of *weakly free* Σ*-multialgebras* as those multialgebras satisfying any, and therefore all, of these weaker conditions. In particular, all of them are isomorphic to a submultialgebra of a multialgebra in $\mathcal{FT}(\Sigma, \mathcal{V})$ for some $\mathcal{V}$.

This paper is organized as follows: Section 2 proposes a natural notion of multialgebras of (non-deterministic) terms. In Section 3, five equivalent characterizations of the submultialgebras of multialgebras of terms are given, which lead to the notion of weakly free multialgebras. In Section 4 we apply one of the five characterizations obtained in Section 3 to offer a simple proof of the well-known result which states that the category $\mathbf{MAlg}(\Sigma)$ of multialgebras does not have free objects. Finally, some conclusions are provided in Section 5.

2. Multialgebras of non-deterministic terms

This section introduces the first main notion proposed in the paper: multialgebras of (non-deterministic) terms. As we shall see, a generalization to the category of multialgebras of the concept of algebra of terms is attained by means of a family of multialgebras of terms indexed by all the cardinals $\kappa > 0$, instead of considering a single object. This reveals the complexity required for adapting the notion of free objects to the category of multialgebras: all the possible sizes for the outputs of the multioperators, assuming that the outputs consist of sets of terms instead of terms, should be considered. In this sense, κ represents the maximum of such sizes in a given multialgebra of terms. Before introducing the definition itself, some standard notions will be recalled.

A *signature* is a collection $\Sigma = \{\Sigma_n\}_{n\in\mathbb{N}}$ of (possibly empty) pairwise disjoint sets Σ_n. Elements of Σ_n are *functional symbols* of arity n. We will denote by Σ either the collection itself or, when there is no risk of confusion, the set $\bigcup_{n\in\mathbb{N}} \Sigma_n$.

A Σ-*multialgebra*, or *multialgebra*, is a pair $\mathcal{A} = (A, \{\sigma_{\mathcal{A}}\}_{\sigma\in\Sigma})$, where A is a non-empty set (the *universe* of $\mathcal{A}$) and $\{\sigma_{\mathcal{A}}\}_{\sigma\in\Sigma}$ is a collection of functions indexed by $\bigcup_{n\in\mathbb{N}} \Sigma_n$ such that, if $\sigma \in \Sigma_n$, $\sigma_{\mathcal{A}}$ is a function of the form

$$\sigma_{\mathcal{A}} : A^n \to \mathcal{P}(A) \setminus \{\emptyset\},$$

that is, an $n-$ary function from A to the set of non-empty subsets of A.

A *homomorphism* between two Σ-multialgebras $\mathcal{A} = (A, \{\sigma_{\mathcal{A}}\}_{\sigma\in\Sigma})$ and $\mathcal{B} = (B, \{\sigma_{\mathcal{B}}\}_{\sigma\in\Sigma})$ is a function $f : A \to B$ such that, for all $n \in \mathbb{N}$, $\sigma \in \Sigma_n$ and elements $a_1, \dots, a_n \in A$,

$$\{f(a) \ : \ a \in \sigma_{\mathcal{A}}(a_1, \dots, a_n)\} \subseteq \sigma_{\mathcal{B}}(f(a_1), \dots, f(a_n)).$$

When in the previous relation we replace inclusion by equality, we say that f is a *full homomorphism*. To denote that the function f is a homomorphism from $\mathcal{A}$ to $\mathcal{B}$, we write $f : \mathcal{A} \to \mathcal{B}$. If the homomorphism $f : \mathcal{A} \to \mathcal{B}$ is injective, we call it a *monomorphism* and, if it is surjective, we call it an *epimorphism*. A bijective full homomorphism will be called an *isomorphism*.

The class of all Σ-multialgebras, equipped with the homomorphisms between them (where composition and identity homomorphisms are as in the category of sets), becomes the category $\mathbf{MAlg}(\Sigma)$. In this category, the epics are precisely the epimorphisms, while any monomorphism is a monic. In turn, isomorphisms, as defined above, are exactly the isomorphisms in the categorical sense (see, for instance, [5], Section 2). Notice, however, that is not known whether all monics are monomorphisms. Any standard Σ-algebra can be seen as a Σ-multialgebra in which the operators return singletons. It is easy to see that the category of Σ-algebras is a full subcategory of $\mathbf{MAlg}(\Sigma)$.

Given two Σ-multialgebras $\mathcal{A} = (A, \{\sigma_{\mathcal{A}}\}_{\sigma\in\Sigma})$ and $\mathcal{B} = (B, \{\sigma_{\mathcal{B}}\}_{\sigma\in\Sigma})$ such that $B \subseteq A$, we say $\mathcal{B}$ is a *submultialgebra* of $\mathcal{A}$ if the identity function $id : B \to A$ is a homomorphism from $\mathcal{B}$ to $\mathcal{A}$ (being therefore a monic). That is, for every $b_1, \dots, b_n \in B$,

$$\sigma_{\mathcal{B}}(b_1, \dots, b_n) \subseteq \sigma_{\mathcal{A}}(b_1, \dots, b_n).$$

Given a set $\mathcal{V}$ of variables and a signature $\Sigma = \{\Sigma_n\}_{n\in\mathbb{N}}$, the algebra of terms generated by $\mathcal{V}$ over Σ will be denoted by $\mathbf{T}(\Sigma, \mathcal{V})$, and its universe will be denoted by $T(\Sigma, \mathcal{V})$. The set $T(\Sigma, \mathcal{V})$ is the smallest subset X of the set of finite, non-empty sequences over $\mathcal{V} \cup \bigcup_{n\in\mathbb{N}} \Sigma_n$ such that:

1. $\mathcal{V} \cup \Sigma_0 \subseteq X$;

2. $\sigma\alpha_1 \ldots \alpha_n \in X$, whenever $n \geq 1$, $\sigma \in \Sigma_n$ and $\alpha_1, \ldots, \alpha_n$ in X.

The set $T(\Sigma, \mathcal{V})$ becomes the Σ-algebra $\mathbf{T}(\Sigma, \mathcal{V})$ when we define, for any $\sigma \in \Sigma_n$ and terms $\alpha_1, \ldots, \alpha_n$ in $T(\Sigma, \mathcal{V})$,

$$\sigma_{\mathbf{T}(\Sigma,\mathcal{V})}(\alpha_1, \ldots, \alpha_n) = \sigma\alpha_1 \ldots \alpha_n.$$

We define the order (or complexity) $\mathsf{o}(\alpha)$ of a term α of $\mathbf{T}(\Sigma, \mathcal{V})$ as: $\mathsf{o}(\alpha) = 0$ if $\alpha \in \mathcal{V} \cup \Sigma_0$; and $\mathsf{o}(\sigma\alpha_1 \ldots \alpha_\mathsf{n}) = 1 + \max\{\mathsf{o}(\alpha_1), \ldots, \mathsf{o}(\alpha_\mathsf{n})\}$.

Definition 2.1. Given a signature Σ and a cardinal $\kappa > 0$, the *expanded signature* $\Sigma^\kappa = \{\Sigma^\kappa_n\}_{n\in\mathbb{N}}$ is the signature such that $\Sigma^\kappa_n = \Sigma_n \times \kappa$, where we will denote the pair (σ, β) by σ^β for $\sigma \in \Sigma$ and $\beta \in \kappa$.

We demand that κ is greater than zero, which guarantees that, if Σ is non-empty, so is Σ^κ.

Definition 2.2. Given a set of variables $\mathcal{V}$, a signature Σ and a cardinal $\kappa > 0$, we define the *κ-branching Σ-multialgebra of non-deterministic terms*, or simply κ-branching multialgebra of terms, when Σ is obvious from the context, as

$$\mathbf{mT}(\Sigma, \mathcal{V}, \kappa) = (T(\Sigma^\kappa, \mathcal{V}), \{\sigma_{\mathbf{mT}(\Sigma,\mathcal{V},\kappa)}\}_{\sigma\in\Sigma}),$$

with universe $T(\Sigma^\kappa,\mathcal{V})$ and such that, for $\sigma \in \Sigma_n$ and $\alpha_1, \ldots, \alpha_n \in T(\Sigma^\kappa,\mathcal{V})$,

$$\sigma_{\mathbf{mT}(\Sigma,\mathcal{V},\kappa)}(\alpha_1, \ldots, \alpha_n) = \{\sigma^\beta\alpha_1 \ldots \alpha_n \ : \ \beta \in \kappa\}.$$

Let $\mathcal{FT}(\Sigma, \mathcal{V}) = (\mathbf{mT}(\Sigma, \mathcal{V}, \kappa))_{\kappa\geq 1}$ be the family of such multialgebras of terms.

The intuition behind this definition is that connecting given terms $\alpha_1, \ldots, \alpha_n$ with a functional symbol σ can, in a broader interpretation taking into account non-determinism, return many terms with the same general shape, namely $\sigma\alpha_1 \ldots \alpha_n$. All of such terms are constructed with functional symbols σ^β, and the collection of them (for $\beta \in \kappa$) corresponds to the non-deterministic term generated from the given input.

In the general case, not all functional symbols should return the same number κ of generalized terms. Because of this, the submultialgebras of $\mathbf{mT}(\Sigma, \mathcal{V}, \kappa)$ will be considered, where the cardinality of the outputs will vary as long as it is bounded by κ. Here, we will restrict ourselves to the cases where $\Sigma_0 \neq \emptyset$ or $\mathcal{V} \neq \emptyset$, so that $\mathbf{mT}(\Sigma, \mathcal{V}, \kappa)$ is always well defined.

The order of an element α of $\mathbf{mT}(\Sigma, \mathcal{V}, \kappa)$ is, by definition, its order as an element of $T(\Sigma^\kappa, \mathcal{V})$. Notice that, if

$$\sigma_{\mathbf{mT}(\Sigma,\mathcal{V},\kappa)}(\alpha_1, \ldots, \alpha_n) \cap \theta_{\mathbf{mT}(\Sigma,\mathcal{V},\kappa)}(\beta_1, \ldots, \beta_m) \neq \emptyset,$$

then $\sigma = \theta$, $n = m$ and $\alpha_1 = \beta_1$, $\ldots$, $\alpha_n = \beta_m$, since if the intersection is not empty there are $\beta, \gamma \in \kappa$ such that $\sigma^\beta \alpha_1 \ldots \alpha_n = \theta^\gamma \beta_1 \ldots \beta_m$ and by the structure of $T(\Sigma^\kappa, \mathcal{V})$ we find that $\sigma^\beta = \theta^\gamma$.

Example 2.3. The Σ-algebras of terms $\mathbf{T}(\Sigma, \mathcal{V})$, when considered as multialgebras such that $\sigma_{\mathbf{T}(\Sigma,\mathcal{V})}(\alpha_1, \ldots, \alpha_n) = \{\sigma\alpha_1 \ldots \alpha_n\}$, are multialgebras of terms, with $\kappa = 1$. That is, $\mathbf{T}(\Sigma, \mathcal{V})$ and $\mathbf{mT}(\Sigma, \mathcal{V}, 1)$ are isomorphic.

From now on, the cardinal of a set X will be denoted by $|X|$.

Example 2.4. A *directed graph* is a pair (V, A), with V a non-empty set of elements called *vertices* and $A \subseteq V^2$ a set of elements called *arrows*. We say that there is an arrow from u to v, both in V, if $(u, v) \in A$. We say that the n-tuple $(v_1, \ldots, v_n)$ is a *path* between u and v if $u = v_1$, $v = v_n$ and $(v_i, v_{i+1}) \in A$ for every $i \in \{1, \ldots, n-1\}$. We say that $u \in V$ has a *successor* if there exists $v \in V$ such that $(u, v) \in A$, and u has a *predecessor* if there exists $v \in V$ such that $(v, u) \in A$.

A directed graph $F = (V, A)$ is a *forest* if, for any two $u, v \in V$, there exists at most one path between u and v, and a forest is said to have *height* ω if every vertex has a successor. We state that forests of height ω are in bijection with the submultialgebras of the multialgebras of terms over the signature Σ_s with exactly one operator s of arity 1.

Indeed, take as $\mathcal{V}$ the set of elements of F that have no predecessor and define, for $u \in V$,

$$s_{\mathcal{A}}(u) = \{v \in V \ : \ (u, v) \in A\}.$$

It is easy to see that the Σ_s-multialgebra $\mathcal{A} = (V, \{s_{\mathcal{A}}\})$, submultialgebra of $\mathbf{mT}(\Sigma_s, \mathcal{V}, |V|)$, carries the same information that F.

Example 2.5. More generally, a *directed multi-graph* [6], or *directed m-graph*, is a pair (V, A) with V a non-empty set of vertices and A a subset of $V^+ \times V$, where $V^+ = \bigcup_{n \in \mathbb{N} \setminus \{0\}} V^n$ is the set of finite, non-empty, sequences over V. We will say that $(v_1, \ldots, v_n)$ is a *path* between u and v if $u = v_1$,

$v = v_n$ and, for every $i \in \{1, \ldots, n-1\}$, there exists $v_{i_1}, \ldots, v_{i_m}$ such that $((v_{i_1}, \ldots, v_{i_m}), v_{i+1})$ is in A, with $v_i = v_{i_j}$ for some $j \in \{1, \ldots, m\}$.

An *m-forest* is a directed m-graph such that any two elements are connected by at most one path, and an m-forest is said to have *n-height* ω, for $n \in \mathbb{N} \setminus \{0\}$, if, for any $(u_1, \ldots, u_n) \in V^n$, there exists $v \in V$ such that $((u_1, \ldots, u_n), v) \in A$. Finally, we see that every m-forest $F = (V, A)$ with n-height ω, for every $n \in S \subseteq \mathbb{N} \setminus \{\emptyset\}$, is essentially equivalent to the Σ_S-multialgebra $\mathcal{A} = (V, \{\sigma_{\mathcal{A}}\}_{\sigma \in \Sigma_S})$, with

$$\sigma_{\mathcal{A}}(u_1, \ldots, u_n) = \{v \in V \ : \ ((u_1, \ldots, u_n), v) \in A\},$$

for σ of arity n, and Σ_S the signature with exactly one operator of arity n for every $n \in S$. It is not hard to see that $\mathcal{A}$ is a submultialgebra of $\mathbf{mT}(\Sigma_S, \mathcal{V}, |V|)$, with $\mathcal{V}$ the set of elements v of V such that, for no $(u_1, \ldots, u_n) \in V^+$, $((u_1, \ldots, u_n), v) \in A$.

3. Being a submultialgebra of $\mathbf{mT}(\Sigma, \mathcal{V}, \kappa)$ as...

The class of submultialgebras of the members of $\mathcal{FT}(\Sigma, \mathcal{V}) = (\mathbf{mT}(\Sigma, \mathcal{V}, \kappa))_{\kappa \geq 1}$ is proposed to be the generalization of the free Σ-algebras to the category $\mathbf{MAlg}(\Sigma)$ of multialgebras. Because of this, the next step is to characterize the submultialgebras of $\mathbf{mT}(\Sigma, \mathcal{V}, \kappa)$. In this section, five different characterizations of such multialgebras will be found, proving that all of them are equivalent (see Theorem 3.42). From this we arrive to the second main notion proposed in this paper: weakly free multialgebras over Σ.

3.1. ...being cdf-generated

In universal algebra, the algebras of terms $\mathbf{T}(\Sigma, \mathcal{V})$ have the universal mapping property for the class of all Σ-algebras over $\mathcal{V}$. This means that there exists a set, in their case the set of variables $\mathcal{V}$, such that, for every other Σ-algebra $\mathcal{B}$ with universe B and function $f : \mathcal{V} \to B$, there exists a unique homomorphism $\overline{f} : \mathbf{T}(\Sigma, \mathcal{V}) \to \mathcal{B}$ extending f. As we mentioned before, this is no longer true when dealing with multialgebras, but we can define a closely related concept with the aid of what we will call *collections of choices*.

Collections of choices are motivated by the notion of *legal valuations*, first defined in Avron and Lev's seminal paper [1] in the context of non-deterministic logical matrices. A map ν from $\mathbf{T}(\Sigma, \mathcal{V})$ (seen as the algebra of propositional formulas over Σ generated by $\mathcal{V}$) to the universe of a Σ-multialgebra $\mathcal{A}$ is a legal valuation whenever $\nu(\sigma\alpha_1 \dots \alpha_n) \in \sigma_{\mathcal{A}}(\nu(\alpha_1), \dots, \nu(\alpha_n))$, for any connective σ in Σ. Essentially, for any formula $\sigma\alpha_1 \dots \alpha_n$, ν "chooses" a value from all the possible values $\sigma_{\mathcal{A}}(\nu(\alpha_1), \dots, \nu(\alpha_n))$, possible values which depend themselves on the previous choices $\nu(\alpha_1)$, $\dots$, $\nu(\alpha_n)$ performed by ν.

A collection of choices automatizes all these aforementioned choices, what justifies its name.

Definition 3.1. Given multialgebras $\mathcal{A} = (A, \{\sigma_{\mathcal{A}}\}_{\sigma\in\Sigma})$ and $\mathcal{B} = (B, \{\sigma_{\mathcal{B}}\}_{\sigma\in\Sigma})$ over the signature Σ, a *collection of choices* from $\mathcal{A}$ to $\mathcal{B}$ is a collection $C = \{C_n\}_{n\in\mathbb{N}}$ of collections of functions

$$C_n = \left\{C\sigma_{a_1,\dots,a_n}^{b_1,\dots,b_n} \;:\; \sigma \in \Sigma_n,\, a_1, \dots, a_n \in A,\, b_1, \dots, b_n \in B\right\}$$

such that, for $\sigma \in \Sigma_n$, $a_1, \dots, a_n \in A$ and $b_1, \dots, b_n \in B$, $C\sigma_{a_1,\dots,a_n}^{b_1,\dots,b_n}$ is a function of the form

$$C\sigma_{a_1,\dots,a_n}^{b_1,\dots,b_n} : \sigma_{\mathcal{A}}(a_1, \dots, a_n) \to \sigma_{\mathcal{B}}(b_1, \dots, b_n).$$

Example 3.2. If $\mathcal{B}$ is actually an algebra, that is, all its operations return singletons, there exists only one collection of choices from any $\mathcal{A}$ to $\mathcal{B}$. This means that in the classical environment of universal algebras, collections of choices are somewhat irrelevant.

Example 3.3. A directed tree is a directed forest where there exists exactly one element without predecessor. We say that v ramifies from u if there exists an arrow from u to v. Then, for a collection of choices C from T_1 to T_2 ($T_1 = (V_1, A_1)$ and $T_2 = (V_2, A_2)$ are directed trees of height ω, considered as Σ_s-multialgebras) and for every $v \in V_1$ and $u \in V_2$, the function Cs_v^u chooses, for each of the elements that ramify from v, one element that ramifies from u.

Definition 3.4. Given a signature Σ, a Σ-multialgebra $\mathcal{A} = (A, \{\sigma_{\mathcal{A}}\}_{\sigma\in\Sigma})$ is *choice-dependent freely generated* by X if $X \subseteq A$ and, for all Σ-multialgebras $\mathcal{B} = (B, \{\sigma_{\mathcal{B}}\}_{\sigma\in\Sigma})$, all functions $f : X \to B$ and all collections of choices C from $\mathcal{A}$ to $\mathcal{B}$, there is a unique homomorphism $f_C : \mathcal{A} \to \mathcal{B}$ such that:

1. $f_C|_X = f$;

2. for all $\sigma \in \Sigma_n$ and $a_1, \dots, a_n \in A$,

$$f_C|_{\sigma_{\mathcal{A}}(a_1,\dots,a_n)} = C\sigma^{f_C(a_1),\dots,f_C(a_n)}_{a_1,\dots,a_n}.$$

For simplicity, when $\mathcal{A}$ is choice-dependent freely generated by X, we will say that $\mathcal{A}$ is **cdf**-generated by X.

In the next definition, we introduce the concept of ground to indicate what elements of a multialgebra are not "achieved" by its multioperations. To better visualize this definition one can keep in mind that the ground of an algebra of terms is its set of indecomposable terms, that is, variables.

Definition 3.5. Given a Σ-multialgebra $\mathcal{A} = (A, \{\sigma_{\mathcal{A}}\}_{\sigma\in\Sigma})$, we define its *build* as

$$B(\mathcal{A}) = \bigcup\big\{\sigma_{\mathcal{A}}(a_1, \dots, a_n) \ : \ n \in \mathbb{N},\, \sigma \in \Sigma_n,\, a_1, \dots, a_n \in A\big\}.$$

We define the *ground* of $\mathcal{A}$ as

$$G(\mathcal{A}) = A \setminus B(\mathcal{A}).$$

Example 3.6. $B(\mathbf{T}(\Sigma, \mathcal{V})) = T(\Sigma, \mathcal{V}) \setminus \mathcal{V}$ and $G(\mathbf{T}(\Sigma, \mathcal{V})) = \mathcal{V}$.

Example 3.7. If $F = (V, A)$ is a directed forest of height ω, thought as a Σ_s-multialgebra, its ground is the set of elements v in V without predecessors.

Proposition 3.8.

1. If $f : \mathcal{A} \to \mathcal{B}$ is a homomorphism between Σ-multialgebras, then $B(\mathcal{A}) \subseteq f^{-1}(B(\mathcal{B}))$ and $f^{-1}(G(\mathcal{B})) \subseteq G(\mathcal{A})$;

2. If $\mathcal{B}$ is a submultialgebra of $\mathcal{A}$, $B(\mathcal{B}) \subseteq B(\mathcal{A})$ and $G(\mathcal{A}) \cap B \subseteq G(\mathcal{B})$.

Proof:

1. If $a \in B(\mathcal{A})$, there exist $\sigma \in \Sigma_n$ and $a_1, \dots, a_n \in A$ such that $a \in \sigma_{\mathcal{A}}(a_1, \dots, a_n)$. Since $f(\sigma_{\mathcal{A}}(a_1, \dots, a_n)) \subseteq \sigma_{\mathcal{B}}(f(a_1), \dots, f(a_n))$, we find that $f(a) \in \sigma_{\mathcal{B}}(f(a_1), \dots, f(a_n))$ and therefore $f(a) \in B(\mathcal{B})$, meaning that $a \in f^{-1}(B(\mathcal{B}))$. Using that $G(\mathcal{A}) = A \setminus B(\mathcal{A})$ we obtain the second mentioned inclusion.

2. If $b \in B(\mathcal{B})$, there exist $\sigma \in \Sigma_n$ and $b_1, \ldots, b_n \in B$ such that $b \in \sigma_{\mathcal{B}}(b_1, \ldots, b_n)$, and given that $\sigma_{\mathcal{B}}(b_1, \ldots, b_n) \subseteq \sigma_{\mathcal{A}}(b_1, \ldots, b_n)$ we obtain $b \in B(\mathcal{A})$. Using again that $G(\mathcal{A}) = A \setminus B(\mathcal{A})$ we finish the proof. $\square$

From this it also follows that if $f : \mathcal{A} \to \mathcal{B}$ is a homomorphism, $G(\mathcal{B}) \cap f(A)$ is contained in $\{f(a) \,:\, a \in G(\mathcal{A})\}$. Indeed, if b is in $G(\mathcal{B}) \cap f(A)$, any $a \in A$ such that $f(a) = b$ is in $f^{-1}(G(\mathcal{B}))$ and, by the previous proposition, is also in $G(\mathcal{A})$. And therefore b is in $\{f(a) \,:\, a \in G(\mathcal{A})\}$.

Generalizing Example 3.6, we have that $G(\mathbf{mT}(\Sigma, \mathcal{V}, \kappa)) = \mathcal{V}$, or equivalently $B(\mathbf{mT}(\Sigma, \mathcal{V}, \kappa)) = T(\Sigma^{\kappa}, \mathcal{V}) \setminus \mathcal{V}$, what we show by induction. If α is of order 0, either we have $\alpha = \sigma^{\beta}$, for a $\sigma \in \Sigma_0$ and $\beta \in \kappa$, and therefore $\alpha \in B(\mathbf{mT}(\Sigma, \mathcal{V}, \kappa))$; or we have that $\alpha = p \in \mathcal{V}$. In that last case, if there exist $\sigma \in \Sigma_m$ and $\alpha_1, \ldots, \alpha_m \in T(\Sigma^{\kappa}, \mathcal{V})$ such that

$$p \in \sigma_{\mathbf{mT}(\Sigma, \mathcal{V}, \kappa)}(\alpha_1, \ldots, \alpha_m),$$

we have $p = \sigma^{\beta}\alpha_1 \ldots \alpha_m$ for $\beta \in \kappa$, which is absurd given the structure of $T(\Sigma^{\kappa}, \mathcal{V})$, forcing us to conclude that $p \notin B(\mathbf{mT}(\Sigma, \mathcal{V}, \kappa))$. If α is of order $n > 0$, we have that $\alpha = \sigma^{\beta}\alpha_1 \ldots \alpha_m$ for $\sigma \in \Sigma_m$, $\beta \in \kappa$ and $\alpha_1, \ldots, \alpha_m$ of order at most $n - 1$, and therefore we have α in $\sigma_{\mathbf{mT}(\Sigma, \mathcal{V}, \kappa)}(\alpha_1, \ldots, \alpha_m)$, meaning that $\alpha \in B(\mathbf{mT}(\Sigma, \mathcal{V}, \kappa))$.

Definition 3.9. Given a Σ-multialgebra $\mathcal{A} = (A, \{\sigma_{\mathcal{A}}\}_{\sigma \in \Sigma})$ and a set $S \subseteq A$, we define the sets $\langle S \rangle_m$ by induction: $\langle S \rangle_0 = S \cup \bigcup_{\sigma \in \Sigma_0} \sigma_{\mathcal{A}}$; and assuming we have defined $\langle S \rangle_m$, we define

$$\langle S \rangle_{m+1} = \langle S \rangle_m \cup \bigcup \big\{\sigma_{\mathcal{A}}(a_1, \ldots, a_n) \,:\, n \in \mathbb{N},\, \sigma \in \Sigma_n,\, a_1, \ldots, a_n \in \langle S \rangle_m\big\}.$$

The *set generated* by S, denoted by $\langle S \rangle$, is then defined as $\langle S \rangle = \bigcup_{m \in \mathbb{N}} \langle S \rangle_m$.

We say $\mathcal{A}$ is *generated* by S if $\langle S \rangle = A$.

Lemma 3.10. *Every submultialgebra $\mathcal{A}$ of $\boldsymbol{mT}(\Sigma, \mathcal{V}, \kappa)$ is generated by $G(\mathcal{A})$.*

Proof: Suppose a is an element of $\mathcal{A}$ not contained in $\langle G(\mathcal{A}) \rangle$ of minimum order. Given that a cannot belong to $G(\mathcal{A}) \cup \bigcup_{\sigma \in \Sigma_0} \sigma_{\mathcal{A}} = \langle G(\mathcal{A}) \rangle_0$, there exist $n > 0$, $\sigma \in \Sigma_n$ and $a_1, \ldots, a_n \in A$ such that $a \in \sigma_{\mathcal{A}}(a_1, \ldots, a_n)$.

Since $\sigma_{\mathcal{A}}(a_1, \ldots, a_n) \subseteq \sigma_{\mathbf{mT}(\Sigma,X,\kappa)}(a_1, \ldots, a_n)$ we derive that $a_1, \ldots, a_n$ are of smaller order than a. By our hypothesis, there must exist $m_1, \ldots, m_n$ such that $a_j \in \langle G(\mathcal{A})\rangle_{m_j}$ for all $j \in \{1, \ldots, n\}$; taking $m = \max\{m_1, \ldots, m_n\}$ one obtains that $a_1, \ldots, a_n \in \langle G(\mathcal{A})\rangle_m$, and therefore

$$a \in \sigma_{\mathcal{A}}(a_1, \ldots, a_n) \subseteq \langle G(\mathcal{A})\rangle_{m+1},$$

which contradicts our assumption that a is not in $\langle G(\mathcal{A})\rangle$. □

Theorem 3.11. *Every submultialgebra $\mathcal{A}$ of $\boldsymbol{mT}(\Sigma, \mathcal{V}, \kappa)$ is **cdf**-generated by $G(\mathcal{A})$.*

Proof: Let $\mathcal{A} = (A, \{\sigma_{\mathcal{A}}\}_{\Sigma})$ be a submultialgebra of $\mathbf{mT}(\Sigma, \mathcal{V}, \kappa)$, let $\mathcal{B} = (B, \{\sigma_{\mathcal{B}}\}_{\Sigma})$ be any Σ-multialgebra, let $f : G(\mathcal{A}) \to B$ be a function and C a collection of choices from $\mathcal{A}$ to $\mathcal{B}$. We define $f_C : \mathcal{A} \to \mathcal{B}$ by induction on $\langle G(\mathcal{A})\rangle_m$:

1. if $a \in \langle G(\mathcal{A})\rangle_0$ and $a \in G(\mathcal{A})$, we define $f_C(a) = f(a)$;
2. if $a \in \langle G(\mathcal{A})\rangle_0$ and $a \in \sigma_{\mathcal{A}}$, for some $\sigma \in \Sigma_0$, we define $f_C(a) = C\sigma(a)$;
3. if f_C is defined for all elements of $\langle G(\mathcal{A})\rangle_m$, $a_1, \ldots, a_n \in \langle G(\mathcal{A})\rangle_m$ and $\sigma \in \Sigma_n$, for every element $a \in \sigma_{\mathcal{A}}(a_1, \ldots, a_n)$ we define
$$f_C(a) = C\sigma_{a_1,\ldots,a_n}^{f_C(a_1),\ldots,f_C(a_n)}(a).$$

First, we must prove that f_C is well defined. There are two possibly problematic cases to consider for an element $a \in A$:

1. the one in which $a \in G(\mathcal{A})$ and there are $\sigma \in \Sigma_n$ and $a_1, \ldots, a_n \in A$ with $a \in \sigma_{\mathcal{A}}(a_1, \ldots, a_n)$, corresponding to a falling simultaneously in the cases (1) and (2), or (1) and (3) of the definition;
2. and the one where there are $\sigma \in \Sigma_n$, $\theta \in \Sigma_m$, $a_1, \ldots, a_n \in A$ and $b_1, \ldots, b_m \in A$ such that $a \in \sigma_{\mathcal{A}}(a_1, \ldots, a_n)$ and $a \in \theta_{\mathcal{A}}(b_1, \ldots, b_m)$, a situation that corresponds to the cases (2) and (3), (2) and (2),[2] or (3) and (3)[3] occurring simultaneously.

[2] That is, $a \in \langle G(\mathcal{A})\rangle_0$, and $a \in \sigma_{\mathcal{A}}$ and $a \in \theta_{\mathcal{A}}$, for different $\sigma, \theta \in \Sigma_0$, where defining $f_C(a)$ as both $C\sigma(a)$ and $C\theta(a)$ could be impossible.

[3] That is, f_C is defined for all of $\langle G(\mathcal{A})\rangle_k$, $a_1, \ldots, a_n, b_1, \ldots, b_m \in \langle G(\mathcal{A})\rangle_k$, and $a \in \sigma_{\mathcal{A}}(a_1, \ldots, a_n)$ and $a \in \theta_{\mathcal{A}}(b_1, \ldots, b_m)$, for $\sigma \in \Sigma_n$ and $\theta \in \Sigma_m$, meaning it could be impossible to define $f_C(a)$ in a systematic way.

The first case is not possible, since $G(\mathcal{A}) \subseteq A \setminus \sigma_{\mathcal{A}}(a_1, \dots, a_n)$ for every $\sigma \in \Sigma_n$ and $a_1, \dots, a_n \in A$. In the second case, we find that

$$a \in \sigma_{\mathcal{A}}(a_1, \dots, a_n) \cap \theta_{\mathcal{A}}(b_1, \dots, b_m) \subseteq \\ \sigma_{\mathbf{mT}(\Sigma,\mathcal{V},\kappa)}(a_1, \dots, a_n) \cap \theta_{\mathbf{mT}(\Sigma,\mathcal{V},\kappa)}(b_1, \dots, b_m),$$

so $n = m$, $\sigma = \theta$ and $a_1 = b_1, \dots, a_n = b_m$, and therefore $f_C(a)$ is well-defined.

Second, we must prove that f_C is defined over all of A. That is simple, for f_C is defined over all of $\langle G(\mathcal{A})\rangle$ and we established in Lemma 3.10 that $A = \langle G(\mathcal{A})\rangle$.

So $f_C : A \to B$ is a well-defined function. It remains to be shown that it is a homomorphism. So, given $\sigma \in \Sigma_n$ and $a_1, \dots, a_n$, we see that

$$f_C(\sigma_{\mathcal{A}}(a_1, \dots, a_n)) = \left\{C\sigma^{f_C(a_1),\dots,f_C(a_n)}_{a_1,\dots,a_n}(a) \ : \ a \in \sigma_{\mathcal{A}}(a_1, \dots, a_n)\right\} \\ \subseteq \sigma_{\mathcal{B}}(f_C(a_1), \dots, f_C(a_n)),$$

while we also have that f_C clearly extends both f and all $C\sigma^{f_C(a_1,\dots,f_C(a_n)}_{a_1,\dots,a_n}$.

To finish the proof, suppose $g : \mathcal{A} \to \mathcal{B}$ is another homomorphism extending both f and all $C\sigma^{g(a_1,\dots,g(a_n)}_{a_1,\dots,a_n}$. We will prove that $g = f_C$ again by induction on the m of $\langle G(\mathcal{A})\rangle_m$. For $m = 0$, an element $a \in \langle G(\mathcal{A})\rangle_0$ is either in $G(\mathcal{A})$, when we have $g(a) = f(a) = f_C(a)$, or in $\sigma_{\mathcal{A}}$ for a $\sigma \in \Sigma_0$, when $g(a) = C\sigma(a) = f_C(a)$.

Suppose g is equal to f_C in $\langle G(\mathcal{A})\rangle_m$ and take an $a \in \langle G(\mathcal{A})\rangle_{m+1} \setminus \langle G(\mathcal{A})\rangle_m$. Then, there exist $\sigma \in \Sigma_n$ and $a_1, \dots, a_n \in \langle G(\mathcal{A})\rangle_m$ such that $a \in \sigma_{\mathcal{A}}(a_1, \dots, a_n)$ and so

$$g(a) = C\sigma^{g(a_1),\dots,g(a_n)}_{a_1,\dots,a_n}(a) = C\sigma^{f_C(a_1),\dots,f_C(a_n)}_{a_1,\dots,a_n}(a) = f_C(a).$$

This proves that $g = f_C$ and that, in fact, f_C is unique. That is, $\mathcal{A}$ is **cdf**-generated by $G(\mathcal{A})$. □

The proof of the following lemma may be found in Section 2 of [5].

LEMMA 3.12. *Let $\mathcal{A} = (A, \{\sigma_{\mathcal{A}}\}_{\sigma\in\Sigma})$ and $\mathcal{B} = (B, \{\sigma_{\mathcal{B}}\}_{\sigma\in\Sigma})$ be Σ-multialgebras, and let $f : \mathcal{A} \to \mathcal{B}$ be a homomorphism. Then, the structure $\mathcal{C} = (f(A), \{\sigma_{\mathcal{C}}\}_{\sigma\in\Sigma})$ such that*

$$\sigma_{\mathcal{C}}(c_1, \dots, c_n) = \bigcup\{f(\sigma_{\mathcal{A}}(a_1, \dots, a_n)) : f(a_1) = c_1, \dots, f(a_n) = c_n\}$$

is a Σ-submultialgebra of $\mathcal{B}$, while $f:\mathcal{A}\to\mathcal{C}$ is an epimorphism. The Σ-multialgebra $\mathcal{C}$ is known as the direct image of $\mathcal{A}$ trough f.

Theorem 3.13. *If the multialgebra $\mathcal{A}=(A,\{\sigma_{\mathcal{A}}\}_{\sigma\in\Sigma})$ over Σ is **cdf**-generated by X, then $\mathcal{A}$ is isomorphic to a submultialgebra of $\boldsymbol{mT}(\Sigma,X,|A|)$ containing X.*

Proof: Take $f:X\to T(\Sigma^{|A|},X)$ to be the identity map (that is, $f(x)=x$), and take a collection of choices C such that, for $\sigma\in\Sigma_n$, $a_1,\ldots,a_n\in A$ and $\alpha_1,\ldots,\alpha_n\in T(\Sigma^{|A|},X)$,

$$C\sigma_{a_1,\ldots,a_n}^{\alpha_1,\ldots,\alpha_n}:\sigma_{\mathcal{A}}(a_1,\ldots,a_n)\to\sigma_{\mathbf{mT}(\Sigma,X,|A|)}(\alpha_1,\ldots,\alpha_n)$$

is an injective function. Such collection of choices exists since $\sigma_{\mathcal{A}}(a_1,\ldots,a_n)\subseteq A$ and $\sigma_{\mathbf{mT}(\Sigma,X,|A|)}(\alpha_1,\ldots,\alpha_n)$ is of cardinality $|A|$. Since $\mathcal{A}$ is **cdf**-generated by X, there exists a homomorphism $f_C:\mathcal{A}\to\mathbf{mT}(\Sigma,X,|A|)$ extending f and each $C\sigma_{a_1,\ldots,a_n}^{f_C(a_1),\ldots,f_C(a_n)}$.

Let $\mathcal{B}=(f_C(A),\{\sigma_{\mathcal{B}}\}_{\sigma\in\Sigma})$ be the direct image of $\mathcal{A}$ trough f_C, so that $f_C:\mathcal{A}\to\mathcal{B}$ is an epimorphism, what is possible given Lemma 3.12. Notice too that

$$X=X\cap f_C(A)=G(\mathbf{mT}(\Sigma,X,|A|))\cap f_C(A)\subseteq G(\mathcal{B})$$

because $\mathcal{B}$ is a submultialgebra of $\mathbf{mT}(\Sigma,X,|A|)$. Now, take any $g:G(\mathcal{B})\to A$ such that $g(x)=x$, for every $x\in X$, and a collection of choices D from $\mathcal{B}$ to $\mathcal{A}$ such that, for any $\sigma\in\Sigma_n$, $b_1,\ldots,b_n\in f_C(A)$ and $a_1,\ldots,a_n\in A$, the function

$$D\sigma_{b_1,\ldots,b_n}^{a_1,\ldots,a_n}:\sigma_{\mathcal{B}}(b_1,\ldots,b_n)\to\sigma_{\mathcal{A}}(a_1,\ldots,a_n)$$

satisfies the following: if $a\in\sigma_{\mathcal{A}}(a_1,\ldots,a_n)$ is such that $C\sigma_{a_1,\ldots,a_n}^{b_1,\ldots,b_n}(a)\in\sigma_{\mathcal{B}}(b_1,\ldots,b_n)$, then $D\sigma_{b_1,\ldots,b_n}^{a_1,\ldots,a_n}(C\sigma_{a_1,\ldots,a_n}^{b_1,\ldots,b_n}(a))=a$. Given that $C\sigma_{a_1,\ldots,a_n}^{b_1,\ldots,b_n}$ is injective, this condition is well-defined.

Since $\mathcal{B}$ is **cdf**-generated by $G(\mathcal{B})$, we know there exists a homomorphism $g_D:\mathcal{B}\to\mathcal{A}$ extending g and the functions $D\sigma_{b_1,\ldots,b_n}^{g_D(b_1),\ldots,g_D(b_n)}$.

Finally, we take $g_D\circ f_C:\mathcal{A}\to\mathcal{A}$. It extends the injection $id=g\circ f:X\to A$, for which $id(x)=x$. It also extends the collection of choices E defined by

$$E\sigma^{a'_1,\ldots,a'_n}_{a_1,\ldots,a_n} = D\sigma^{a'_1,\ldots,a'_n}_{f_C(a_1),\ldots,f_C(a_n)} \circ C^{f_C(a_1),\ldots,f_C(a_n)}_{a_1,\ldots,a_n} : \\ \sigma_{\mathcal{A}}(a_1,\ldots,a_n) \to \sigma_{\mathcal{A}}(a'_1,\ldots,a'_n),$$

for $\sigma \in \Sigma_n$ and $a_1,\ldots a_n, a'_1,\ldots,a'_n \in A$. This way, $E\sigma^{a_1,\ldots,a_n}_{a_1,\ldots,a_n}$ is the identity on $\sigma_{\mathcal{A}}(a_1,\ldots,a_n)$. Indeed, for any $a \in \sigma_{\mathcal{A}}(a_1,\ldots,a_n)$,

$$C\sigma^{f_C(a_1),\ldots,f_C(a_n)}_{a_1,\ldots,a_n}(a) = f_C(a)$$

by definition of f_C, and, given that $f_C : \mathcal{A} \to \mathcal{B}$ is a homomorphism, $f_C(a)$ belongs to $\sigma_{\mathcal{B}}(f_C(a_1),\ldots,f_C(a_n))$, meaning that $C\sigma^{f_C(a_1),\ldots,f_C(a_n)}_{a_1,\ldots,a_n}(a) \in \sigma_{\mathcal{B}}(f_C(a_1),\ldots,f_C(a_n))$. Then

$$E\sigma^{a_1,\ldots,a_n}_{a_1,\ldots,a_n}(a) = D\sigma^{a_1,\ldots,a_n}_{f_C(a_1),\ldots,f_C(a_n)}(C^{f_C(a_1),\ldots,f_C(a_n)}_{a_1,\ldots,a_n}(a)) = a$$

by the definition of D.

But notice that the identical homomorphism $\mathcal{I} : \mathcal{A} \to \mathcal{A}$ also extends both id and E and, given the uniqueness of such extensions on the definition of being **cdf**-generated, we obtain that $\mathcal{I} = g_D \circ f_C$. The fact that $f_C : \mathcal{A} \to \mathcal{B}$ has a left inverse implies that it is injective, and by definition of $\mathcal{B}$ it is also surjective, meaning that it is a bijective function. Moreover, g_D is the inverse function of f_C. Finally, for $\sigma \in \Sigma_n$ and $a_1,\ldots,a_n \in A$,

$$f_C(\sigma_{\mathcal{A}}(a_1,\ldots,a_n)) \subseteq \sigma_{\mathcal{B}}(f_C(a_1),\ldots,f_C(a_n)),$$

since f_C is a homomorphism. However, given that g_D is also a homomorphism,

$$\begin{aligned} g_D(\sigma_{\mathcal{B}}(f_C(a_1),\ldots,f_C(a_n))) &\subseteq \sigma_{\mathcal{A}}(g_D \circ f_C(a_1),\ldots,g_D \circ f_C(a_n)) \\ &= \sigma_{\mathcal{A}}(a_1,\ldots,a_n), \end{aligned}$$

and by applying f_C to both sides, one obtains

$$\begin{aligned} \sigma_{\mathcal{B}}(f_C(a_1),\ldots,f_C(a_n)) &= f_C(g_D(\sigma_{\mathcal{B}}(f_C(a_1),\ldots,f_C(a_n)))) \\ &\subseteq f_C(\sigma_{\mathcal{A}}(a_1,\ldots,a_n)). \end{aligned}$$

This proves that f_C is a bijective full homomorphism, that is, an isomorphism. $\square$

Notice that, from the proof above, we can see that if $\mathcal{A} = (A, \{\sigma_{\mathcal{A}}\}_{\sigma\in\Sigma})$ is **cdf**-generated by X, then $\mathcal{A}$ is in fact isomorphic to a submultialgebra of $\mathbf{mT}(\Sigma, X, M(\mathcal{A}))$, where

$$M(\mathcal{A}) = \max\big\{|\sigma_{\mathcal{A}}(a_1, \dots, a_n)| \ : \ n \in \mathbb{N}, \sigma \in \Sigma_n, a_1, \dots, a_n \in A\big\}.$$

It is clear that $M(\mathcal{A}) = \kappa$ for the multialgebra $\mathcal{A} = \mathbf{mT}(\Sigma, \mathcal{V}, \kappa)$. The value $M(\mathcal{A})$ has been already regarded in the literature as an important aspect of multialgebras, see [8] (observe, however, that their definition of homomorphism is quite different from ours).

Notice, furthermore, that written in classical terms, the previous Theorems 3.11 and 3.13 state a well known result: an algebra is absolutely free iff it is isomorphic to some algebra of terms over the same signature.

Corollary 3.14. Every **cdf**-generated multialgebra $\mathcal{A}$ is generated by its ground $G(\mathcal{A})$.

Proof: Since every **cdf**-generated multialgebra is isomorphic to a submultialgebra of some $\mathbf{mT}(\Sigma, X, \kappa)$, from 3.13, and every submultialgebra of $\mathbf{mT}(\Sigma, X, \kappa)$ is generated by its ground, the result follows. □

Corollary 3.15. Every **cdf**-generated multialgebra $\mathcal{A}$ is **cdf**-generated by its ground $G(\mathcal{A})$.

Definition 3.16. A Σ-multialgebra $\mathcal{A} = (A, \{\sigma_{\mathcal{A}}\}_{\sigma\in\Sigma})$ is said to be *disconnected* if, for every $\sigma \in \Sigma_n, \theta \in \Sigma_m$, $a_1, \dots, a_n, b_1, \dots, b_m \in A$,

$$\sigma_{\mathcal{A}}(a_1, \dots, a_n) \cap \theta_{\mathcal{A}}(b_1, \dots, b_m) \neq \emptyset$$

implies that $n = m$, $\sigma = \theta$ and $a_1 = b_1, \dots, a_n = b_m$.

Example 3.17. $\mathbf{T}(\Sigma, \mathcal{V})$ is disconnected.

Example 3.18. All directed forests of height ω, when considered as Σ_s-multialgebras, are disconnected, given that no two arrows point to the same element.

It is clear that if $\mathcal{B}$ is a submultialgebra of $\mathcal{A}$ and $\mathcal{A}$ is disconnected, then $\mathcal{B}$ is also disconnected, since if $\sigma_{\mathcal{B}}(a_1, \dots, a_n) \cap \theta_{\mathcal{B}}(b_1, \dots, b_m) \neq \emptyset$, for $a_1, \dots, a_n, b_1, \dots, b_m \in B$, given that $\sigma_{\mathcal{B}}(a_1, \dots, a_n) \subseteq \sigma_{\mathcal{A}}(a_1, \dots, a_n)$ and $\theta_{\mathcal{B}}(b_1, \dots, b_m) \subseteq \theta_{\mathcal{A}}(b_1, \dots, b_m)$, we find that $\sigma_{\mathcal{A}}(a_1, \dots, a_n) \cap \theta_{\mathcal{A}}(b_1, \dots, b_m) \neq \emptyset$ and therefore $n = m$, $\sigma = \theta$ and $a_1 = b_1, \dots, a_n = b_m$.

We noticed before that $\mathbf{mT}(\Sigma, \mathcal{V}, \kappa)$ is disconnected, and by Theorem 3.13 we obtain that every **cdf**−generated algebra is disconnected. This also means something deeper: being disconnected is, in a way, a measure of how free of identities a multialgebra is. After all, the fact that no two multioperations agree on any elements is strongly indicative that the multialgebra does not satisfy any identities.

3.2. . . . being disconnected and generated by its ground

Now, we continue to look at other possible characterizations of the submultialgebras of the multialgebras of terms. One sees that algebras of terms do not have identities, what would partially correspond in our study to the concept of being disconnected. But what is possibly more representative of our intuition for terms is that one starts by defining them from elements that are as simple as possible (variables), and continues indefinitely. The concept of indecomposable element is here replaced by that of being an element of the ground, so one would expect that being generated by it plays some role in what we have defined so far.

Lemma 3.19. *If $\mathcal{A}$ is **cdf**-generated by X, then $X \subseteq G(\mathcal{A})$.*

Proof: If $\mathcal{A}$ is **cdf**-generated by X, then $\mathcal{A}$ is isomorphic to a submultialgebra of $\mathbf{mT}(\Sigma, X, |A|)$ containing X, from Theorem 3.13. Let us assume that $\mathcal{A}$ is equal to this submultialgebra, without loss of generality. Then, $X = G(\mathbf{mT}(\Sigma, X, |A|)) \cap A \subseteq G(\mathcal{A})$. □

Lemma 3.20. *If $\mathcal{A}$ is **cdf**-generated by both X and Y, with $X \subseteq Y$, then $X = Y$.*

Proof: Suppose $X \neq Y$ and let $y \in Y \setminus X$. Take a Σ-multialgebra $\mathcal{B}$, over the same signature as that of $\mathcal{A}$, such that $|B| \geq 2$, and a collection of choices C from $\mathcal{A}$ to $\mathcal{B}$.

Take also two functions $g, h : Y \to B$ such that $g|_X = h|_X$ and $g(y) \neq h(y)$, what is possible since $|B| \geq 2$. Given that $\mathcal{A}$ is **cdf**-generated by Y, there exist unique homomorphisms g_C and h_C extending both g and C, and h and C, respectively.

However, g_C and h_C extend both $g|_X : X \to B$ and C, and since $\mathcal{A}$ is **cdf**-generated by X, we find that $g_C = h_C$. This is not possible, since $g_C(y) \neq h_C(y)$, what must imply that $Y \setminus X = \emptyset$ and therefore $X = Y$. □

Theorem 3.21. *Every* ***cdf****-generated multialgebra* $\mathcal{A}$ *is* ***cdf****-generated only by its ground.*

Proof: From Corollary 3.14, $\mathcal{A}$ is **cdf**-generated by $G(\mathcal{A})$, and from Lemma 3.19, if $\mathcal{A}$ is also **cdf**-generated by X, then $X \subseteq G(\mathcal{A})$. By Lemma 3.20, this implies that $X = G(\mathcal{A})$. □

We have proved so far that if $\mathcal{A}$ is **cdf**-generated, then $\mathcal{A}$ is generated by its ground and disconnected. We would like to prove that this is enough to characterize a **cdf**-generated multialgebra. That is, if $\mathcal{A}$ is generated by its ground and disconnected, then it is **cdf**-generated, exactly by its ground.

The idea is similar to the one we used to prove that all submultialgebras of $\mathbf{mT}(\Sigma, \mathcal{V}, \kappa)$ are **cdf**-generated: take a multialgebra $\mathcal{A}$ that is both generated by its ground $G(\mathcal{A})$, which will be denoted by X, and disconnected, and fix a multialgebra $\mathcal{B}$ over the same signature, a function $f : X \to B$ and a collection of choices C from $\mathcal{A}$ to $\mathcal{B}$.

We define a function $f_C : A \to B$ using induction on the $\langle X\rangle_n$. For $n = 0$, either we have an element $x \in X$, when we define $f_C(x) = f(x)$, or we have $a \in \sigma_{\mathcal{A}}$ for some $\sigma \in \Sigma_0$, when we define $f_C(a) = C\sigma(a)$. Notice that, up to this point, there are no contradictions in this definition, given that an element cannot belong both to X and to a $\sigma_{\mathcal{A}}$, since $X = G(\mathcal{A})$.

Suppose we have successfully defined f_C on $\langle X\rangle_m$ and take an $a \in \sigma_{\mathcal{A}}(a_1, \dots, a_n)$ for $a_1, \dots, a_n \in \langle X\rangle_m$. We then define

$$f_C(a) = C\sigma_{a_1,\dots,a_n}^{f_C(a_1),\dots,f_C(a_n)}(a).$$

Again the function remains well-defined: a cannot belong to X, since $X = G(\mathcal{A})$, and cannot belong to a $\theta_{\mathcal{A}}(b_1, \dots, b_p)$ unless $p = n$, $\theta = \sigma$ and $b_1 = a_1, \dots, b_p = a_n$, since $\mathcal{A}$ is disconnected.

Clearly f_C is a homomorphism, since the image of $\sigma_{\mathcal{A}}(a_1, \dots, a_n)$ under f_C is contained in $\sigma_{\mathcal{B}}(f_C(a_1), \dots, f_C(a_n))$, and f_C extends both f and C.

Lemma 3.22. *If a multialgebra* $\mathcal{A}$ *is both generated by its ground* X *and disconnected,* $\mathcal{A}$ *is* ***cdf****-generated by* X*.*

Proof: It remains for us to show that f_C, as defined above, is the only homomorphism extending f and C. Suppose g is another such homomorphism and we shall proceed yet again by induction.

On $\langle X\rangle_0$, we have that $f_C(x) = f(x) = g(x)$ for all $x \in X$; and for $\sigma \in \Sigma_0$ and $a \in \sigma_{\mathcal{A}}$ we have that

$$f_C(a) = C\sigma(a) = g(a),$$

hence f_C and g coincide on $\langle X\rangle_0$. Suppose that f_C and g are equal on $\langle X\rangle_m$ and take $a \in \sigma_{\mathcal{A}}(a_1, \ldots, a_n)$ for $a_1, \ldots, a_n \in \langle X\rangle_m$. We have by induction hypothesis that

$$f_C(a) = C\sigma_{a_1,\ldots,a_n}^{f_C(a_1),\ldots,f_C(a_n)}(a) = C\sigma_{a_1,\ldots,a_n}^{g(a_1),\ldots,g(a_n)}(a) = g(a),$$

which concludes our proof. □

THEOREM 3.23. *A multialgebra $\mathcal{A}$ is* ***cdf****-generated iff $\mathcal{A}$ is generated by its ground and disconnected.*

It is important to analyze, by means of examples, the differences between the several concepts involved: are there multialgebras that are disconnected but not generated by their grounds? Are there multialgebras that are generated by their grounds but not disconnected? If not, does being generated by its ground imply being disconnected or vice-versa? We show below that this is not the case by providing examples answering positively both previous questions.

Example 3.24. Take the signature Σ_s from Example 2.4. Consider the Σ_s−multialgebra $\mathcal{C} = (\{-1, 1\}, \{s_{\mathcal{C}}\})$ such that $s_{\mathcal{C}}(-1) = \{1\}$ and $s_{\mathcal{C}}(1) = \{-1\}$ (that is, $s_{\mathcal{C}}(x) = \{-x\}$).

We state that $\mathcal{C}$ is disconnected, but not generated by its ground. $\mathcal{C}$ is clearly disconnected since $s_{\mathcal{C}}(-1)\cap s_{\mathcal{C}}(1) = \emptyset$; now, $B(\mathcal{C}) = s_{\mathcal{C}}(-1)\cup s_{\mathcal{C}}(1) = \{-1, 1\}$, and so $G(\mathcal{C}) = \emptyset$.

Since Σ_s has no 0-ary operators and $G(\mathcal{C}) = \emptyset$, it follows that $\langle G(\mathcal{C})\rangle_0 = \emptyset$ and therefore $\langle G(\mathcal{C})\rangle_n = \emptyset$ for every $n \in \mathbb{N}$, meaning that $G(\mathcal{C})$ does not generate $\mathcal{C}$.

Example 3.25. Take again the signature Σ_s with a single unary operator, from Example 2.4. Consider the Σ_s-multialgebra $\mathcal{B} = (\{0, 1\}, \{s_{\mathcal{B}}\})$ such that $s_{\mathcal{B}}(0) = \{1\}$ and $s_{\mathcal{B}}(1) = \{1\}$ (that is, $s_{\mathcal{B}}(x) = \{1\}$).

Then $\mathcal{B}$ is clearly not disconnected, since $s_{\mathcal{B}}(0) \cap s_{\mathcal{B}}(1) = \{1\}$, yet $\mathcal{B}$ is generated by its ground: $B(\mathcal{B}) = \{1\}$ and so $G(\mathcal{B}) = \{0\}$, and we see that $\langle G(\mathcal{B})\rangle_1$ is already $\{0, 1\}$.

$$\begin{tikzcd} -1 \arrow[rr, bend left, "s_{\mathcal{C}}"] & & 1 \arrow[ll, bend left, "s_{\mathcal{C}}"] \end{tikzcd}$$

The Σ_s-multialgebra $\mathcal{C}$

$$\begin{tikzcd} 0 \arrow[r, "s_{\mathcal{B}}"] & 1 \arrow[loop right, "s_{\mathcal{B}}"] \end{tikzcd}$$

The Σ_s-multialgebra $\mathcal{B}$

3.3. ...being disconnected and having a strong basis

We give another characterization of being **cdf**-generated, that is, being disconnected and having a strong basis, in a sense we now define. Remember that Σ-algebras with the universal mapping condition for the entire class of Σ-algebras (i.e. algebras of terms) are easier to be defined than the ones with the universal mapping property for some proper variety. That is why in this article we define only multialgebras of terms. The "strong basis" carries the qualifier "strong" for we hope that, once an adequate generalization of algebras satisfying the universal mapping property for some proper variety is found for the subject of multialgebras, these multialgebras will have minimal, not minimum, generating sets, i.e. basis.

Definition 3.26. We say $B \subseteq A$ is a *strong basis* of the Σ-multialgebra $\mathcal{A} = (A, \{\sigma_{\mathcal{A}}\}_{\sigma\in\Sigma})$ if it is the minimum of the set $\mathcal{G} = \{S \subseteq A \ :\ \langle S\rangle = A\}$ ordered by inclusion.

Example 3.27. The set of variables $\mathcal{V}$ is a strong basis of $\mathbf{T}(\Sigma, \mathcal{V})$.

Example 3.28. The set of elements without predecessor of a directed forest of height ω is a strong basis of the forest, considered as a Σ_s-multialgebra.

Lemma 3.29. *For every subset S of the universe of a Σ-multialgebra $\mathcal{A}$, $G(\mathcal{A}) \cap \langle S\rangle \subseteq S$.*

Proof: Suppose $x \in G(\mathcal{A}) \cap \langle S\rangle$: if $x \notin S$, we will show that x cannot be in $\langle S\rangle$, which contradicts our assumption. Indeed, if $x \notin S$ then

$$x \notin \langle S\rangle_0 = S \cup \bigcup_{\sigma\in\Sigma_0} \sigma_{\mathcal{A}},$$

since $x \notin S$, and $x \in G(\mathcal{A})$ implies that

$$x \in A \setminus B(\mathcal{A}) \subseteq A \setminus \bigcup_{\sigma \in \Sigma_0} \sigma_{\mathcal{A}}.$$

Now, for induction hypothesis, suppose that $x \notin \langle S \rangle_m$. Then,

$$x \notin \langle S \rangle_{m+1} = \langle S \rangle_m \cup \bigcup \big\{\sigma_{\mathcal{A}}(a_1, \ldots, a_n) \ : n \in \mathbb{N}, \sigma \in \Sigma_n, \\ a_1, \ldots, a_n \in \langle S \rangle_m\big\}$$

since $x \notin \langle S \rangle_m$, and $x \in G(\mathcal{A})$ implies that

$$x \in A \setminus B(\mathcal{A}) \subseteq A \setminus \bigcup \big\{\sigma_{\mathcal{A}}(a_1, \ldots, a_n) : n \in \mathbb{N}, \sigma \in \Sigma_n, a_1, \ldots, a_n \in \langle S \rangle_m\big\}.$$

□

THEOREM 3.30. *If the Σ-multialgebra $\mathcal{A}$ has a strong basis B, $G(\mathcal{A}) \subseteq B$.*

PROOF: By Lemma 3.29, $G(\mathcal{A}) = G(\mathcal{A}) \cap A = G(\mathcal{A}) \cap \langle B \rangle \subseteq B$. □

DEFINITION 3.31. If B is a strong basis of a disconnected Σ-multialgebra $\mathcal{A}$, we define the *B-order* of an element $a \in A$ as the natural number

$$o_B(a) = \min\big\{k \in \mathbb{N} \ : \ a \in \langle B \rangle_k\big\}.$$

This is a clear generalization of the order, or complexity, of a term. In fact, the order of a term in $T(\Sigma, \mathcal{V})$ is exactly its $\mathcal{V}$-order.

It is clear that, if $a \in \sigma_{\mathcal{A}}(a_1, \ldots, a_n)$ and $o_B(a) \geq 1$, then $o_B(a_1), \ldots, o_b(a_n) < o_B(a)$. In fact, suppose $m + 1 = o_B(a)$, implying that

$$a \in \langle B \rangle_{m+1} = \langle B \rangle_m \cup \bigcup \big\{\sigma_{\mathcal{A}}(a_1, \ldots, a_n) \ : n \in \mathbb{N}, \sigma \in \Sigma_n, \\ a_1, \ldots, a_n \in \langle B \rangle_m\big\}.$$

Since $m + 1 = \min\{k \in \mathbb{N} \ : \ a \in \langle B \rangle_k\}$, we have that $a \notin \langle B \rangle_m$ and therefore

$$a \in \bigcup \big\{\sigma_{\mathcal{A}}(a_1, \ldots, a_n) \ : \ n \in \mathbb{N}, \sigma \in \Sigma_n, a_1, \ldots, a_n \in \langle B \rangle_m\big\}.$$

Finally, we obtain that there exist $p \in \mathbb{N}$, $\theta \in \Sigma_p$ and $b_1, \ldots, b_p \in \langle B \rangle_m$ such that $a \in \theta_{\mathcal{A}}(b_1, \ldots, b_p)$. Since $a \in \sigma_{\mathcal{A}}(a_1, \ldots, a_n)$, this implies that

$\sigma_{\mathcal{A}}(a_1, \ldots, a_n) \cap \theta_{\mathcal{A}}(b_1, \ldots, b_p) \neq \emptyset$, and therefore $p = n$, $\theta = \sigma$ and $b_1 = a_1$, $\ldots$, $b_p = a_n$, so that $o_B(a_1), \ldots, o_B(a_n) \leq m$.

But what if $a \in \sigma_{\mathcal{A}}(a_1, \ldots, a_n)$, for $n > 0$, and $o_B(a) = 0$, implying $a \in B$? We claim this case cannot occur, for if it does,

$$B^* = \big(B \cup \{a_1, \ldots, a_n\}\big) \setminus \{a\}$$

generates A, while clearly not containing B. We have that $a \in \langle B^* \rangle_1$, since $a_1, \ldots, a_n \in B^*$ and $a \in \sigma_{\mathcal{A}}(a_1, \ldots, a_n)$, and given that $B \setminus \{a\} \subseteq B^*$, it follows that $B \subseteq \langle B^* \rangle_1$, and so $\langle B \rangle_0 \subseteq \langle B^* \rangle_1$.

It is then true that $\langle B \rangle_m \subseteq \langle B^* \rangle_{m+1}$ for every $m \in \mathbb{N}$. Indeed, if this is true for m, let $b \in \langle B \rangle_{m+1}$, and then either $b \in \langle B \rangle_m$, so that $b \in \langle B^* \rangle_{m+1} \subseteq \langle B^* \rangle_{m+2}$, or there exist $\theta \in \Sigma_p$ and $b_1, \ldots, b_p \in \langle B \rangle_m$ such that $b \in \theta_{\mathcal{A}}(b_1, \ldots, b_p)$. In this case, since $\langle B \rangle_m \subseteq \langle B^* \rangle_{m+1}$, we have that

$$b \in \theta_{\mathcal{A}}(b_1, \ldots, b_p) \subseteq \bigcup \big\{\sigma_{\mathcal{A}}(a_1, \ldots, a_n) : n \in \mathbb{N},\ \sigma \in \Sigma_n,\\ a_1, \ldots, a_n \in \langle B^* \rangle_{m+1}\big\} \subseteq \langle B^* \rangle_{m+2},$$

so once again $b \in \langle B^* \rangle_{m+2}$. Since $\langle B \rangle = \bigcup_{m \in \mathbb{N}} \langle B \rangle_m$ equals A, we have that $\langle B^* \rangle$ also equals A, as we previously stated. This is absurd, since B is the minimum of $\{S \subseteq A : \langle S \rangle = A\}$, ordered by inclusion, and $B \not\subseteq B^*$. The conclusion must be that if $a \in \sigma_{\mathcal{A}}(a_1, \ldots, a_n)$ for $n > 0$, then $o_B(a_1), \ldots, o_B(a_n) < o_B(a)$, regardless of the value of $o_B(a)$.

LEMMA 3.32. *If $\mathcal{A}$ is disconnected and has a strong basis B, then $B = G(\mathcal{A})$ and so $\mathcal{A}$ is generated by its ground.*

PROOF: Suppose $a \in B \setminus G(\mathcal{A})$. Since a is in the build of $\mathcal{A}$, there exist $\sigma \in \Sigma_n$ and elements $a_1, \ldots, a_n \in A$ such that $a \in \sigma_{\mathcal{A}}(a_1, \ldots, a_n)$. If $n > 0$, $o_B(a) > o_B(a_1) \geq 0$, which contradicts the fact that $a \in B$ and therefore $o_B(a) = 0$.

If $n = 0$, it is clear that $B^* = B \setminus \{a\}$ is a generating set smaller than B: generating set because, if $\sigma \in \Sigma_0$ and $a \in \sigma_{\mathcal{A}}$, $a \in \bigcup_{\sigma \in \Sigma_0} \sigma_{\mathcal{A}}$ and therefore $B \subseteq \langle B^* \rangle_0$, so that $\langle B \rangle_m \subseteq \langle B^* \rangle_{m+1}$. This is also a contradiction, since B is a strong basis. □

THEOREM 3.33. *$\mathcal{A}$ is generated by its ground and disconnected iff it has a strong basis and it is disconnected.*

PROOF: We already proved, in Lemma 3.32, that if $\mathcal{A}$ is disconnected and has a strong basis B, then it is generated by its ground and disconnected.

Conversely, if $\mathcal{A}$ is disconnected and generated by its ground, first of all it is clearly disconnected.

Now, if $\langle G(\mathcal{A})\rangle = A$, one has that $G(\mathcal{A}) \subseteq S$ for every $S \in \{S \subseteq A : \langle S\rangle = A\}$, by Lemma 3.29. Therefore, the ground is a strong basis. $\square$

Once again, we ask ourselves: does being disconnected imply having a strong basis or vice-versa? We show that this is not the case by providing examples of a multialgebra that is disconnected but does not have a strong basis, and one of a multialgebra that has a strong basis but is not disconnected.

Example 3.34. Take the signature Σ_s and the Σ_s-multialgebra $\mathcal{C}$ from Example 3.24.

We know that $\mathcal{C}$ is disconnected, but we also state that it does not have a strong basis: in fact, we see that the set $\big\{S \subseteq \{-1, 1\} \,:\, \langle S\rangle = \{-1, 1\}\big\}$ is exactly $\big\{\{-1\}, \{1\}, \{-1, 1\}\big\}$, and this set has no minimum.

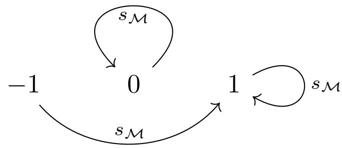
Example 3.35. Take the Σ_s-multialgebra $\mathcal{B}$ from Example 3.25.

As we saw before, $\mathcal{B}$ is not disconnected. However we state that it has a strong basis: $B = \{0\}$ generates $\mathcal{B}$ and, since $\{1\}$ does not generate the multialgebra, we find that B is a minimum generating set.

From these two examples, one could hypothesize that for a multialgebra being generated by its ground is equivalent to having a strong basis. Clearly, being generated by its ground implies having a strong basis, that is, the ground. But as we show in the example below, having a strong basis does not imply being generated by its ground.

Example 3.36. Take the signature Σ_s from Example 2.4, and consider the Σ_s-multialgebra $\mathcal{M} = (\{-1, 0, 1\}, \{s_{\mathcal{M}}\})$ such that $s_{\mathcal{M}}(0) = \{0\}$, $s_{\mathcal{M}}(1) = \{1\}$ and $s_{\mathcal{M}}(-1) = \{1\}$ (that is, $s_{\mathcal{M}}(x) = \{\mathsf{abs}(x)\}$, where $\mathsf{abs}(x)$ denotes the absolute value of x).

We have that $G(\mathcal{M}) = \{-1\}$ and that $\langle\{-1\}\rangle = \{-1, 1\}$, so that $\mathcal{M}$ is not generated by its ground. But we state that $\{-1, 0\}$ is a strong basis. First of all, it clearly generates $\mathcal{M}$. Furthermore, the generating sets of $\mathcal{M}$ are only $\{-1, 0\}$ and $\{-1, 0, 1\}$, so that $\{-1, 0\}$ is in fact the smallest generating set.



The Σ_s-multialgebra $\mathcal{M}$

3.4. . . . being disconnected and chainless

The last equivalence to being a submultialgebra of $\mathbf{mT}(\Sigma, \mathcal{V}, \kappa)$ we give depends on the notion of being chainless, which is rather graph-theoretical in nature. Think of a tree that ramifies ever downward. One can pick any vertex and proceed, against the arrows, upwards until an element without predecessor is reached. More than that, it is not possible to find an infinite path, starting in any one vertex, by always going against the arrows: such a path, if it existed, would be what we shall call a chain. A multialgebra without chains is, very naturally, chainless.

As it was in the case of strong basis, there isn't a parallel concept to being chainless in universal algebra: it seems that this concept is far more natural when dealing with multioperations, although it can be easily applied to algebras if one wishes to do so. Closely related (although not equivalent) to chains are the branches in the formation trees of terms: if allowed to grow infinitely, these would became chains.

Given a permutation $\tau : \{1, \dots, n\} \to \{1, \dots, n\}$ in S_n, the group of permutations on n elements, the action of τ in an n-tuple $(x_1, \dots, x_n) \in X^n$ is given by

$$\tau(x_1, \dots, x_n) = (x_{\tau(1)}, \dots, x_{\tau(n)}).$$

Given $1 \leq i, j \leq n$, we define $[i, j]$ to be the permutation such that $[i, j](i) = j$, $[i, j](j) = i$ and, for $k \in \{1, \dots, n\}$ different from i and j, $[i, j](k) = k$.

Definition 3.37. Given a Σ-multialgebra $\mathcal{A}$, a sequence $\{a_n\}_{n\in\mathbb{N}}$ of elements of A is said to be a *chain* if, for every $n \in \mathbb{N}$, there exist a positive natural number $m_n \in \mathbb{N} \setminus \{0\}$, a functional symbol $\sigma^n \in \Sigma_{m_n}$, a permutation $\tau_n \in S_{m_n}$ and elements $a^n_1, \dots, a^n_{m_n-1} \in A$ such that

$$a_n \in \sigma^n_{\mathcal{A}}(\tau_n(a_{n+1}, a^n_1, \dots, a^n_{m_n-1})).$$

A Σ-multialgebra is said to be *chainless* when it has no chains.

Example 3.38. Take a directed forest of height ω and add a loop to it, that is, choose a vertex v and add an arrow from v to v. Then, $\{a_n\}_{n\in\mathbb{N}}$ such that $a_n = v$, for every $n \in \mathbb{N}$, is a chain.

Example 3.39. $\mathbf{T}(\Sigma, \mathcal{V})$ is chainless.

LEMMA 3.40. *If $\mathcal{A}$ is chainless, then it is generated by its ground.*

PROOF: Suppose that $\mathcal{A}$ is not generated by its ground. Thus, $A \setminus \langle G(\mathcal{A})\rangle$ is not empty, and must therefore contain some element a_0. We create a chain $\{a_n\}_{n\in\mathbb{N}}$ by induction, the case $n = 0$ being already done.

So, suppose we have created a finite sequence of elements $a_0, \dots, a_k \in A \setminus \langle G(\mathcal{A})\rangle$ such that, for each $0 \leq n < k$, there exist a positive integer $m_n \in \mathbb{N}\setminus\{0\}$, a functional symbol $\sigma^n \in \Sigma_{m_n}$, a permutation $\tau_n \in S_{m_n}$ and elements $a_1^n, \dots, a_{m_n-1}^n \in A$ such that

$$a_n \in \sigma^n_{\mathcal{A}}(\tau_n(a_{n+1}, a_1^n, \dots, a_{m_n-1}^n)).$$

Since $a_k \in A\setminus\langle G(\mathcal{A})\rangle$, we have that a_k is not an element of the ground. So, there must exist $m_k \in \mathbb{N}$, a functional symbol $\sigma^k \in \Sigma_{m_k}$ and elements $b_1^k, \dots, b_{m_k}^k \in A$ such that

$$a_k \in \sigma^k_{\mathcal{A}}(b_1^k, \dots, b_{m_k}^k).$$

Now, if all $b_1^k, \dots, b_{m_k}^k$ belonged to $\langle G(\mathcal{A})\rangle$, so would a_k: there must be an element $a_{k+1} \in \{b_1^k, \dots, b_{m_k}^k\}$, say b_l^k, such that $a_{k+1} \in A\setminus\langle G(\mathcal{A})\rangle$. We then define a_i^k as b_j^k, for $j = \min\{i \leq p \leq m_k \ : \ p \neq l\}$ and $i \in \{1, \dots, m_k - 1\}$, and

$$\tau_k = [l-1, l] \circ \cdots \circ [1, 2],$$

and then it is clear that $\{a_n\}_{n\in\mathbb{N}}$ becomes a chain, with the extra condition that $\{a_n\}_{n\in\mathbb{N}} \subseteq A \setminus \langle G(\mathcal{A})\rangle$. Therefore $\mathcal{A}$ is not chainless. $\square$

It becomes clear that a disconnected, chainless multialgebra is, by Lemma 3.40, disconnected and generated by its ground. We state that, in fact, the converse also holds, when we arrive to yet another characterization of being a submultialgebra of $\mathbf{mT}(\Sigma, \mathcal{V}, \kappa)$.

So, suppose $\mathcal{A}$ is disconnected and generated by its ground, and let $\{a_n\}_{n\in\mathbb{N}}$ be a chain in $\mathcal{A}$. Clearly no a_n can belong to the ground, since

$$a_n \in \sigma^n_{\mathcal{A}}(\tau_n(a_{n+1}, a^n_1, \ldots, a^n_{m_n-1})),$$

and therefore $o_{G(\mathcal{A})}(a_{n+1}) < o_{G(\mathcal{A})}(a_n)$, that is, the $G(\mathcal{A})$-order of a_{n+1} is smaller than the $G(\mathcal{A})$-order of a_n. We obtain a contradiction, since if $o_{G(\mathcal{A})}(a_0) = m$, then $o_{G(\mathcal{A})}(a_{m+1}) < 0$, what is impossible. Then, $\mathcal{A}$ must be chainless.

THEOREM 3.41. *$\mathcal{A}$ is generated by its ground and disconnected iff it is chainless and disconnected.*

Finally, Theorems 3.11, 3.12, 3.23, 3.33 and 3.41 can be summarized as follows:

THEOREM 3.42. *Let $\mathcal{A}$ be a Σ-multialgebra. The following statements are equivalent:*

1. *$\mathcal{A}$ is a submultialgebra of some $\boldsymbol{mT}(\Sigma, \mathcal{V}, \kappa)$;*
2. *$\mathcal{A}$ is **cdf**-generated;*
3. *$\mathcal{A}$ is generated by its ground and disconnected;*
4. *$\mathcal{A}$ has a strong basis and is disconnected;*
5. *$\mathcal{A}$ is chainless and disconnected.*

This leads us to the second main notion introduced in the paper:

DEFINITION 3.43. A *weakly free multialgebra* over Σ is a multialgebra over Σ satisfying any of the equivalent conditions of Theorem 3.42.

By definition, weakly free multialgebras over Σ coincide, up-to isomorphisms, with the submultialgebras of the members of the families $\mathcal{FT}(\Sigma, \mathcal{V})$, for some set $\mathcal{V}$ of generators (recall Definition 2.2).

It is important to stress the point that, although not all concepts present in the previous theorem have natural counterparts in universal algebra, by defining them for algebras presented as multialgebras we find that all of the conditions in the theorem are valid only for Σ-algebras of terms. This follows easily from the fact that the only **cdf**-generated algebras are the algebras of terms themselves. That is, weakly free algebras coincide with (absolutely) free algebras. Note that any subalgebra of $\mathbf{T}(\Sigma, X)$ is of the form $\mathbf{T}(\Sigma, Y)$ for some Y. Thus, it can be observed that the generalization in $\mathbf{MAlg}(\Sigma)$ of the collection of subalgebras of $\mathbf{T}(\Sigma, \mathcal{V})$ corresponds to the class of submultialgebras of the members of the family $\mathcal{FT}(\Sigma, \mathcal{V})$. In turn, the meaning of $\mathbf{T}(\Sigma, \mathcal{V})$ itself is generalized in the category $\mathbf{MAlg}(\Sigma)$ of Σ-multialgebras through the class of submultialgebras of the members $\mathcal{A}$ of $\mathcal{FT}(\Sigma, \mathcal{V})$ such that $G(\mathcal{A}) = \mathcal{V}$.

Now, a few examples concerning being chainless, disconnected, having a strong basis and being generated by the ground will be given.

Example 3.44. Take the signature Σ_s from Example 2.4, and consider the Σ_s-multialgebra $\mathcal{Y} = (\mathbb{N} \cup \{a, b\}, \{s_{\mathcal{Y}}\})$ such that $s_{\mathcal{Y}}(n) = \{n + 1\}$, for $n \in \mathbb{N}$, and $s_{\mathcal{Y}}(a) = s_{\mathcal{Y}}(b) = \{0\}$.

We see that $\mathcal{Y}$ is chainless since, given a chain $\{a_n\}_{n\in\mathbb{N}}$, it must be contained in the build of $\mathcal{Y}$, that is, $\mathbb{N}$: but then $a_{n+1} = a_n - 1$, what is a contradiction, since there is only a finite number of elements smaller than a_0. At the same time, $\mathcal{Y}$ is not disconnected, since $s_{\mathcal{Y}}(a) = s_{\mathcal{Y}}(b)$.

$$\begin{tikzcd}
a \arrow[dr, "s_{\mathcal{Y}}"] & & & \\
& 0 \arrow[r, "s_{\mathcal{Y}}"] & 1 \arrow[r, "s_{\mathcal{Y}}"] & \cdots \\
b \arrow[ur, "s_{\mathcal{Y}}"] & & &
\end{tikzcd}$$

The Σ_s-multialgebra $\mathcal{Y}$

Example 3.45. Take the Σ_s-multialgebra $\mathcal{C}$ from Example 3.24.

We know that $\mathcal{C}$ is disconnected, however it is also not chainless: in fact, $\{(-1)^n\}_{n\in\mathbb{N}}$ and $\{(-1)^{n+1}\}_{n\in\mathbb{N}}$ are chains in $\mathcal{C}$.

As we saw, being chainless implies being generated by its ground and having a strong basis. The converse, however, is not true.

Example 3.46. Take the Σ_s-multialgebra $\mathcal{B}$ from Example 3.25.

We have already established that $\mathcal{B}$ has a strong basis and is generated by its ground, $\{0\}$, yet it is not chainless: $\{1\}_{n\in\mathbb{N}}$ is a chain in $\mathcal{B}$.

4. Multialgebras cannot satisfy the universal mapping property

Now, we turn to a somewhat folkloric result: the category of multialgebras does not have free objects. This is equivalent to saying that there do not exist multialgebras satisfying the universal mapping property for the class of all Σ-multialgebras, or better yet, that the forgetful functor from this category to **Set** does not have a left adjoint. Of course, such a result can be stated in various ways, depending on the adopted definition of homomorphism and even on the definition of multialgebra to be considered. So, we offer what we consider to be a simple proof of such result for the category $\mathbf{MAlg}(\Sigma)$ as we have defined it.

DEFINITION 4.1. A Σ-multialgebra $\mathcal{A} = (A, \{\sigma_{\mathcal{A}}\}_{\sigma\in\Sigma})$ satisfies the *universal mapping property* for the class of all Σ-multialgebras, over a set $X \subseteq A$, if, for every Σ-multialgebra $\mathcal{B} = (B, \{\sigma_{\mathcal{B}}\}_{\sigma\in\Sigma})$ and map $f : X \to B$, there exists a unique homomorphism $\overline{f} : \mathcal{A} \to \mathcal{B}$ extending f.

In other words, if $j : X \to A$ is the inclusion, there exists only one homomorphism $\overline{f} : \mathcal{A} \to \mathcal{B}$ commuting the following diagram in **Set**.

$$\begin{tikzcd} & A \arrow[dr, "\overline{f}"] & \\ X \arrow[ur, "j"] \arrow[rr, "f"] & & B \end{tikzcd}$$

PROPOSITION 4.2. If $\mathcal{A}$ and $\mathcal{B}$ satisfy the universal mapping property for the class of all Σ-multialgebras over, respectively, X and Y such that $|X| = |Y|$, then $\mathcal{A}$ and $\mathcal{B}$ are isomorphic.

PROOF: Since X and Y are of the same cardinality, there exist bijective functions $f : X \to Y$ and $g : Y \to X$ inverses of each other. Take the extensions $\overline{f} : \mathcal{A} \to \mathcal{B}$ and $\overline{g} : \mathcal{B} \to \mathcal{A}$ and we have that $\overline{g} \circ \overline{f}$ is a homomorphism extending $g \circ f = id$, the identity on X.

Since the identical homomorphism $Id_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ also extends id, we have that $Id_{\mathcal{A}} = \overline{g} \circ \overline{f}$. In a similar way we have that $Id_{\mathcal{B}} = \overline{f} \circ \overline{g}$; proving both $\overline{f}$ and $\overline{g}$ are full is trivial, so $\mathcal{A}$ and $\mathcal{B}$ are isomorphic. $\square$

This way we can refer ourselves to the single Σ-multialgebra satisfying the universal mapping property over X, up to isomorphisms.

Remember that we have defined $\mathbf{MAlg}(\Sigma)$ as the category whose objects are exactly all Σ-multialgebras and for which, given Σ-multialgebras $\mathcal{A}$ and $\mathcal{B}$, $Hom_{\mathbf{MAlg}(\Sigma)}(\mathcal{A},\mathcal{B})$ is the set of all homomorphisms from $\mathcal{A}$ to $\mathcal{B}$. We will denote by $\mathcal{U} : \mathbf{MAlg}(\Sigma) \to \mathbf{Set}$ the forgetful functor.

LEMMA 4.3. *The functor $F : \boldsymbol{Set} \to \boldsymbol{MAlg}(\Sigma)$, associating a set X with a Σ-multialgebra satisfying the universal mapping property over X, which we will denote FX, and a function $f : X \to Y$ with the only homomorphism $\overline{f} : FX \to FY$ extending f, is a left adjoint of $\mathcal{U}$.*

PROOF: For X a set and $\mathcal{A}$ a Σ-multialgebra with universe A we consider the functions, indexed by pairs consisting of a Σ-multialgebra $\mathcal{A}$ and a set X,

$$\Phi_{\mathcal{A},X} : Hom_{\mathbf{Set}}(X,\mathcal{U}\mathcal{A}) \to Hom_{\mathbf{MAlg}(\Sigma)}(FX,\mathcal{A})$$

taking a map $f : X \to A$ to the only homomorphism $\overline{f} : FX \to \mathcal{A}$ extending f. Each $\Phi_{\mathcal{A},X}$ is clearly a bijection given that FX satisfies the universal mapping property over X.

Now, given sets X and Y, Σ-multialgebras $\mathcal{A}$ and $\mathcal{B}$, a function $f : Y \to X$ and a homomorphism $h : \mathcal{A} \to \mathcal{B}$, we have only to prove that the following diagram commutes in $\mathbf{Set}$.

$$\begin{tikzcd}
Hom_{\mathbf{Set}}(X,\mathcal{U}\mathcal{A}) \arrow[r, "\Phi_{\mathcal{A},X}"] \arrow[d, "Hom(f,\mathcal{U}h)"] & Hom_{\mathbf{MAlg}(\Sigma)}(FX,\mathcal{A}) \arrow[d, "Hom(Ff,h)"] \\
Hom_{\mathbf{Set}}(Y,\mathcal{U}\mathcal{B}) \arrow[r, "\Phi_{\mathcal{B},Y}"] & Hom_{\mathbf{MAlg}(\Sigma)}(FY,\mathcal{B})
\end{tikzcd}$$

So we take a function $g : X \to \mathcal{U}\mathcal{A}$. Taking the upper right side of the diagram we have $\Phi_{\mathcal{A},X}g = \overline{g}$ and $Hom(Ff,h)\overline{g} = h \circ \overline{g} \circ Ff$; on the lower left one, $Hom(f,\mathcal{U}h)g = \mathcal{U}h \circ g \circ f$ and $\Phi_{\mathcal{B},Y}\mathcal{U}h \circ g \circ f = \overline{\mathcal{U}h \circ g \circ f}$.

Now, both $h \circ \overline{g} \circ Ff$ and $\overline{\mathcal{U}h \circ g \circ f}$ are homomorphisms from FY to $\mathcal{B}$ extending $\mathcal{U}h \circ g \circ f : Y \to \mathcal{U}\mathcal{B}$. For the second one this is obvious, for the first we take an element $y \in Y$ and see that

$$h \circ \overline{g} \circ Ff(y) = h \circ \overline{g} \circ f(y) = h \circ g \circ f(y) = \mathcal{U}h \circ g \circ f(y)$$

since, respectively: $Ff = \overline{f}$ (and $\overline{f}$ extends f); $\overline{g}$ extends g (which is defined on $X \ni f(y)$); and $\mathcal{U}h = h$, (considered only as a function between sets).

Given that FY satisfies the universal mapping property over Y, we have that $h \circ \overline{g} \circ Ff = \overline{\mathcal{U}\varphi \circ g \circ f}$ and the diagram in fact commutes. $\square$

THEOREM 4.4. *Given a non-empty signature Σ and a set X, there does not exist a Σ-multialgebra which satisfies the universal mapping property over X.*

PROOF: Suppose that $\mathcal{A} = (A, \{\sigma_{\mathcal{A}}\}_{\sigma\in\Sigma})$ satisfies the universal mapping property over X and let $\mathcal{V}$ be a set that properly contains X, meaning that $\mathcal{V} \neq \emptyset$ and therefore that $\mathbf{T}(\Sigma, \mathcal{V})$ is well defined. Then, take the identity function $j : X \to T(\Sigma, \mathcal{V})$, such that $j(x) = x$ for every $x \in X$, and the homomorphism $\overline{j} : \mathcal{A} \to \mathbf{T}(\Sigma, \mathcal{V})$ extending j.

Now, take the identity function $id : \mathcal{V} \to T(\Sigma^2, \mathcal{V})$ and the collections of choices C and D from $\mathbf{T}(\Sigma, \mathcal{V})$ to $\mathbf{mT}(\Sigma, \mathcal{V}, 2)$ such that, for $\sigma \in \Sigma_n$,

$$C\sigma^{\beta_1,\dots,\beta_n}_{\alpha_1,\dots,\alpha_n}(\sigma\alpha_1\dots\alpha_n) = \sigma^0\beta_1\dots\beta_n$$

and

$$D\sigma^{\beta_1,\dots,\beta_n}_{\alpha_1,\dots,\alpha_n}(\sigma\alpha_1\dots\alpha_n) = \sigma^1\beta_1\dots\beta_n,$$

and consider the only homomorphisms $id_C, id_D : \mathbf{T}(\Sigma, \mathcal{V}) \to \mathbf{mT}(\Sigma, \mathcal{V}, 2)$ extending, respectively, id and C, and id and D, which we know to exist given that $\mathbf{T}(\Sigma, \mathcal{V})$ is **cdf**-generated by $\mathcal{V}$. Since $id_C \circ \overline{j}, id_D \circ \overline{j} : \mathcal{A} \to \mathbf{mT}(\Sigma, \mathcal{V}, 2)$ both extend the function $j' : X \to T(\Sigma^2, \mathcal{V})$ such that $j'(x) = x$ for every $x \in X$ (recalling that $\mathcal{V}$ properly contains X), we have $id_C \circ \overline{j} = id_D \circ \overline{j}$.

Now, if $\alpha \in T(\Sigma, \mathcal{V}) \setminus \mathcal{V}$, we have that there exist $\sigma \in \Sigma_n$, for some $n \in \mathbb{N}$, and elements $\alpha_1, \dots, \alpha_n \in T(\Sigma, \mathcal{V})$ such that $\alpha = \sigma\alpha_1\dots\alpha_n$. In this case,

$$id_C(\alpha) = \sigma^0 id_C(\alpha_1)\dots id_C(\alpha_n) \neq \sigma^1 id_D(\alpha_1)\dots id_D(\alpha_n) = id_D(\alpha),$$

given that the leading functional symbols are distinct. From this, id_C and id_D are always different outside of $\mathcal{V}$.

Since $id_C \circ \overline{j} = id_D \circ \overline{j}$, we must have that $\overline{j}(A) \subseteq \mathcal{V}$, and this is absurd since we are assuming Σ non-empty. Indeed, if $\Sigma_0 \neq \emptyset$, for a $\sigma \in \Sigma_0$ and $a \in \sigma_{\mathcal{A}}$ we have that $\overline{j}(a) = \sigma$ is in $T(\Sigma, \mathcal{V})$, but not in $\mathcal{V}$. If it is another Σ_n which is not empty, given $a \in A$ (which exists since the universes of multialgebras are assumed to be non-empty) we have that, for $b \in \sigma_{\mathcal{A}}(a, \dots, a)$, it holds that $\overline{j}(b) = \sigma(\overline{j}(a), \dots, \overline{j}(a))$, which is not in $\mathcal{V}$.

We must conclude that there are no multialgebras with the universal mapping property. □

Corollary 4.5. The category $\mathbf{MAlg}(\Sigma)$ does not have an initial object.

Proof: We state that, if $\mathcal{A}$ is an initial object, $\mathcal{A}$ has the universal mapping property over $\emptyset$. In fact, for every Σ-multialgebra $\mathcal{B}$ and map $f : \emptyset \to B$, there exists a single homomorphism $!_{\mathcal{B}} : \mathcal{A} \to \mathcal{B}$ extending $f = \emptyset$, that is, the only homomorphism between $\mathcal{A}$ and $\mathcal{B}$. But multialgebras with the universal mapping property do not exist, by Theorem 4.4. This concludes the proof. □

Theorem 4.6. *The forgetful functor $\mathcal{U} : \boldsymbol{MAlg}(\Sigma) \to \boldsymbol{Set}$ does not have a left adjoint.*

Proof: For suppose we have a left adjoint $F : \mathbf{Set} \to \mathbf{MAlg}(\Sigma)$ of $\mathcal{U}$, so that F has a right adjoint and is therefore cocontinuous. Since $\emptyset$ is the initial object in $\mathbf{Set}$, we have that $F\emptyset$ must be an initial object in $\mathbf{MAlg}(\Sigma)$, which does not exist by Corollary 4.5. □

5. Conclusions and future work

The results obtained along the paper indicate that multialgebras of terms constitute a rich topic of study, and deserve to be further analyzed. Their connections to the theories of graphs and of partial orders seem clear, and suggest other properties of these objects, and possibly other characterizations. Multialgebras have been used in order to get satisfactory non-deterministic semantics for some non-classical logics, in particular paraconsistent logics (see, for instance, [3, Chapter 6], [5] and [7]). From the present study, we hope to obtain, with the aid of $\mathbf{mT}(\Sigma, \mathcal{V}, \kappa)$ (now seen as the multialgebra of propositional formulas) and its submultialgebras, new interpretations of existing semantics for logic systems and new semantics altogether. Clearly, decision problems concerning these multialgebras become relevant and need to be addressed.

Finally, in what is possibly the most important open question concerning multialgebras of terms, we refer back to something we have already mentioned in this text. In universal algebra, a Σ-algebra $\mathcal{A}$ has the universal mapping property for a variety $\mathbb{V}$ of Σ-algebras over a subset X of its universe when, for every $\mathcal{B}$ in $\mathbb{V}$ and every function $f : X \to B$, there exists a unique homomorphism $\bar{f}: \mathcal{A} \to \mathcal{B}$ extending f. These are known as the *relatively* free algebras, which can be obtained in a variety from a quotient of $\mathbf{T}(\Sigma, \mathcal{V})$. Some questions naturally arise: are there analogous of **cdf**-generated multialgebras with respect to classes of multialgebras? If so, are they obtained in some reasonable way from the multialgebras of terms?

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References

[1] A. Avron, I. Lev, *Canonical Propositional Gentzen-Type Systems*, [in:] R. Goré, A. Leitsch, T. Nipkow (eds.), **Automated Reasoning**, vol. 2083, Springer (2001), pp. 529–544, DOI: https://doi.org/10.1007/3-540-45744-5_45.

[2] S. Burris, H. P. Sankappanavar, **A Course in Universal Algebra**, vol. 78 of Graduate Texts in Mathematics, Springer New York (1981).

[3] W. A. Carnielli, M. E. Coniglio, **Paraconsistent logic: Consistency, Contradiction and Negation**, vol. 40 of Logic, Epistemology, and the Unity of Science, Springer (2016), DOI: https://doi.org/10.1007/978-3-319-33205-5.

[4] W. A. Carnielli, J. Marcos, *A taxonomy of C-systems*, [in:] W. A. Carnielli, M. E. Coniglio, I. M. L. D'Ottaviano (eds.), **Paraconsistency: The Logical Way to the Inconsistent. Proceedings of the 2nd World Congress on Paraconsistency**, vol. 228 of Lecture Notes in Pure and Applied Mathematics, Marcel Dekker, New York (2002), pp. 1–94, DOI: https://doi.org/10.1201/9780203910139.

[5] M. E. Coniglio, A. Figallo-Orellano, A. C. Golzio, *Non-deterministic algebraization of logics by swap structures*, **Logic Journal of the Interest Group in Pure and Applied Logics**, vol. 28(5) (2020), pp. 1021–1059, DOI: https://doi.org/10.1093/jigpal/jzy072.

[6] M. E. Coniglio, A. Sernadas, C. Sernadas, J. Rasga, *A graph-theoretic account of logics*, **Journal of Logic and Computation**, vol. 19(6) (2009), pp. 1281–1320, DOI: https://doi.org/10.1093/logcom/exp023.

[7] M. E. Coniglio, G. V. Toledo, *A simple decision procedure for da Costa's C_n logics by Restricted Nmatrix semantics* (2020), *arXiv*:2011.10151 [math.LO].

[8] Ǵ. Čupona, R. Madarász, *Free Poly-Algebras*, **Mathematics Series of the University of Novi Sad**, vol. 23(2) (1993), pp. 245–261.

[9] N. C. A. da Costa, *Sistemas formais inconsistentes (Inconsistent formal systems, in Portuguese)*, *Habilitation thesis*, Universidade Federal do Paraná, Curitiba, Brazil (1963), republished by Editora da Universidade Federal do Paraná, Curitiba, Brazil, 1993.

[10] F. Marty, *Sur une généralisation de la notion de groupe*, [in:] **Comptes rendus du huitième congrès des mathématiciens scandinaves**, Stockholm, Sweden (1934), pp. 45–49.

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