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# INTERPOLATION PROPERTY ON VISSER'S FORMAL PROPOSITIONAL LOGIC

**Abstract**

In this paper by using a model-theoretic approach, we prove Craig interpolation property for Formal Propositional Logic, **FPL**, Basic propositional logic, **BPL** and the uniform left-interpolation property for **FPL**. We also show that there are countably infinite extensions of **FPL** with the uniform interpolation property.

*Keywords*: Basic propositional logic, formal propositional logic, layered bisimulation, interpolation.

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## 1. Introduction

A Craig interpolant for formulas $\phi(\vec{q},\vec{p})$ and $\psi(\vec{p},\vec{r})$ where $\vdash \phi \to \psi$, is a formula $\chi(\vec{p})$ such that $\vdash \phi \to \chi$ and $\vdash \chi \to \psi$. The uniform interpolation property is, in a sense, the generalization of the Craig interpolation property. If instead of two formulas, we restrict the interpolant to one formula and a subset of its propositional variables (which are to be the shared variables), we reach a stronger definition: a uniform left-interpolant for $\phi(\vec{q},\vec{p})$ with respect to $\vec{p}$ is a formula $\chi(\vec{p})$ such that for all formulas $\psi(\vec{p},\vec{r})$ with $\vdash \psi \to \phi$, $\chi$ acts as an interpolant for $\phi$ and $\psi$. The uniform

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right-interpolant is defined analogously. A logic whose formulas have both uniform left and right-interpolants is said to satisfy the uniform interpolation property.

It is easy to show that classical propositional logic has the uniform interpolation property. But showing it for intuitionistic propositional logic is highly nontrivial. This was shown first by using a proof theoretic method in [6] and then semantically in [5]. A. Visser in [8] established the result using bisimulation techniques.

The goal of this paper is to establish new interpolation results for Basic propositional logic **BPL** and Formal propositional logic, **FPL**, using the bisimulation techinque of [8]. **BPL** and **FPL** are propositional logics which correspond with modal logics **K4** and **GL** by the Gödel translation, respectively, in the same way that Intuitionistic Propositional Logic **IPL** corresponds with modal logic **S4**. The main difference between **IPL** and **BPL** is that the rule *Modus Ponens* is weakened in **BPL**. We show that **FPL** satisfies the uniform left-interpolation property. The same approach with minor differences leads the Craig interpolation property for Basic propositional logic, **BPL**. We Also show that there are countably infinite extensions of **FPL** with the uniform interpolation property.

The organiztion of the paper is as follows: in the next section we present an overview of the syntax and semantics of **BPL**. Basic model theory for **BPL** including canonical models and layered bisimulation, which are a natural generalization of results known for intuitionistic propositional logic, will be studied in section three. Interpolation properties for formal propositional logic and some of its extensions will be presented in section four.

## 2. Axioms, rules and Kripke models

In this preliminaries section we introduce the most basic concepts and notations we need related to syntax and semantics of basic propositional logic, for more details see [7] and [3, 4].

The language for **BPL** is essentially the same as the language for **IPL**. We build formulas in the standard way from propositional variables, or atoms, using $\top, \bot, \wedge, \vee, \to$. Expressions $\neg\phi$ and $\phi \leftrightarrow \psi$ are usual abbreviations for $\phi \to \bot$ and $(\phi \to \psi) \wedge (\psi \to \phi)$, respectively.

We assume that $p, q, r, \ldots$ range over propositional variables, $\phi, \psi, \chi, \ldots$ range over arbitrary formulas, and $\vec{p}, \vec{q}, \vec{r}, \ldots$ range over finite sets of propositional variables. For $\vec{p}$ and $\vec{q}$, we abbreviate $\vec{p} \cup \vec{q}$ by $\vec{p}, \vec{q}$. $\mathsf{PV}(\phi)$ is the set of propositional variables in $\phi$. $\mathsf{Sub}(\phi)$ is the set of subformulas of $\phi$. For a set of propositional variables $\mathcal{P}$, $\mathcal{L}(\mathcal{P})$ denotes the set of those formulas which only contains propositional variables from $\mathcal{P}$. There are different axiomatizations for **BPL**. The natural deduction system for **BPL** was first introduced by A. Visser in [7]. We choose axiomatization method which was introduced in [3]. A *sequent* is simply an expression of the form $\phi \Rightarrow \psi$, where $\phi$ and $\psi$ are formulae. We write $\phi \Leftrightarrow \psi$ as short for $\phi \Rightarrow \psi$ and $\psi \Rightarrow \phi$.

In the rules below, a single horizontal line means that if the sequents above the line are included, then so are the ones below the line. A double line means the same, but in both directions.

**Table 1.** Sequent calculus of **BPL**

$$\phi \Rightarrow \phi \qquad \phi \Rightarrow \top \qquad \bot \Rightarrow \phi \qquad \phi \wedge (\psi \vee \theta) \Rightarrow (\phi \wedge \psi) \vee (\phi \wedge \theta)$$

$$\frac{\phi \Rightarrow \psi \quad \psi \Rightarrow \theta}{\phi \Rightarrow \theta} \qquad \frac{\phi \Rightarrow \psi \quad \phi \Rightarrow \theta}{\overline{\phi \Rightarrow \psi \wedge \theta}} \qquad \frac{\phi \Rightarrow \psi \quad \theta \Rightarrow \psi}{\overline{\phi \vee \theta \Rightarrow \psi}} \qquad \frac{\phi \wedge \psi \Rightarrow \theta}{\phi \Rightarrow \psi \to \theta}$$

$(\phi \to \psi) \wedge (\psi \to \theta) \Rightarrow \phi \to \theta$
$(\phi \to \psi) \wedge (\phi \to \theta) \Rightarrow \phi \to \psi \wedge \theta$
$(\phi \to \psi) \wedge (\theta \to \psi) \Rightarrow \phi \vee \theta \to \psi$

A *sequent theory* is a set of sequents that includes the sequent axioms and is closed under the closure rules, as given in table 1. A sequent theory $\Sigma$ is *consistent* if $\top \Rightarrow \bot \notin \Sigma$. A theory $\Gamma$ is *schematic* if $\Gamma \vdash \phi \Rightarrow \psi$ implies $\Gamma \vdash \tau\phi \Rightarrow \tau\psi$ for all substitutions $\tau$. A *basic intermediate* logic is a consistent schematic sequent theory. The intuitionistic propositional logic, **IPL**, is **BPL** plus the sequent schema $\top \to \phi \Rightarrow \phi$, and the Formal Propositional logic, **FPL**, is the extension of **BPL** by the Löb's axiom schema, $(\top \to \phi) \to \phi \Rightarrow \top \to \phi$, or equivalently, by Löb's rule:

$$\frac{\phi \wedge (\top \to \psi) \Rightarrow \psi}{\phi \Rightarrow \psi}.$$

The theories **BPL**, **IPL**, **CPL** and **FPL** are all basic intermediate logics.

Sequents $\top \Rightarrow \phi$ are often identified with formulas $\phi$. Given a sequent theory $\Sigma$ we define $F(\Sigma)$ as $\{\phi \mid \top \Rightarrow \phi \in \Sigma\}$. A *f*ormula theory (or simply a *t*heory) is a set of formulas of the form $F(\Sigma)$. The formula theory is *c*onsistent if $\Sigma$ is consistent or, equivalently, if $\bot$ is not an element of the formula theory. Let $\Sigma \cup \{\phi \Rightarrow \psi\}$ be a set of sequents. We say that the $\phi \Rightarrow \psi$ is provable from the $\Sigma$ in the logic **BPL** and we denoted it by $\Sigma \vdash_{\mathbf{BPL}} \phi \Rightarrow \psi$, when the sequent $\phi \Rightarrow \psi$ is provable in the sequent calculus **BPL** augmented by $\phi_i \Rightarrow \psi_i$ for all $\phi_i \Rightarrow \psi_i \in \Sigma$. When $\Sigma$ is empty we simply write $\vdash \phi \Rightarrow \psi$. Also, we use $\Sigma \vdash \phi$ instead of $\Sigma \vdash \top \Rightarrow \phi$.

Proposition 2.1 ([3]). Let $\Sigma$ be a sequent theory. Then:

1. (Functional Completeness) $\Sigma \cup \{\phi\} \vdash \psi \Rightarrow \theta$ if and only if $\Sigma \vdash \phi \wedge \psi \Rightarrow \theta$.
2. (Formalization) $\Sigma \cup \{\phi_1 \Rightarrow \psi_1, ..., \phi_n \Rightarrow \psi_n\} \vdash \phi_0 \Rightarrow \psi_0$ implies $\Sigma \vdash (\phi_1 \to \psi_1) \wedge ... \wedge (\phi_n \to \psi_n) \Rightarrow \phi_0 \to \psi_0$.

$\Sigma$ is called a *faithful* theory if the converse of Proposition 2.1.2, also holds. **IPL** and all of its extensions including Classical Propositional Logic, **CPL**, and **BPL**, **FPL** are examples of faithful theories.

Define the relation $\prec$ on all theories by $\Gamma \prec \Delta$ if and only if for all $\phi, \psi \in \mathcal{L}(\mathcal{P})$ such that both $\Gamma \vdash \phi \to \psi$ and $\Delta \vdash \phi$, we have $\Delta \vdash \psi$.

Proposition 2.2. The relation $\prec$ is transitive, and $\Gamma \prec \Delta$ implies $\Gamma \subseteq \Delta$.

Proof: We first prove the second claim. Suppose $\Gamma \prec \Delta$. If $\phi \in \Gamma$, then $\Gamma \vdash \phi$ which implies, by above Formalization theorem, that $\Gamma \vdash \top \to \phi$ and thus $\Delta \vdash \phi$. Hence $\phi \in \Delta$. So $\Gamma \subseteq \Delta$. For transitivity, suppose that $\Gamma \prec \Delta \prec \Delta'$ are such that $\Gamma \vdash \phi \to \psi$ and $\Delta' \vdash \phi$, for any $\phi, \psi \in \mathcal{L}(\mathcal{P})$. Then $\Gamma \subseteq \Delta \vdash \phi \to \psi$, so $\Delta' \vdash \psi$. Therefore $\Gamma \prec \Delta'$. □

Moving on to the samantics of **BPL**, a *Kripke frame* **F** is a pair $(W, \prec)$ where $W$ is a non-empty set and $\prec$ is a transitive binary relation on $W$. The reflexive closure of $\prec$ is denoted by $\preceq$. Also, for $k, k' \in W$, $k' \succeq k$ means that $k \preceq k'$.

A *Kripke model* based on Kripke frame **F** is a triple $\mathbf{M} = (W, \prec, V)$ where $\mathbf{F} = (W, \prec)$ and the function $V$ assigns to each atoms $p$ of the language of **BPL** a subset $V(p) \subseteq W$ which is upward closed, that is, if $k \in V(p)$ and $k \prec k'$, then $k' \in V(p)$.

Given a Kripke model $\mathbf{M} = (W, \prec, V)$, the notion of a formula $\phi$ being true at a point $k \in W$, written $\mathbf{M}, k \Vdash \phi$ or $k \Vdash \phi$ for short, is like in **IPL**. We extend $\Vdash$ to all sequents. For any sequent $\phi \Rightarrow \psi$, it is defined by

$$k \Vdash \phi \Rightarrow \psi \textit{ if and only if for all } k' \succeq k,\ k' \Vdash \phi \textit{ implies } k' \Vdash \psi.$$

A trivial induction on the complexity of formulas yields that, $k \Vdash \phi$ and $k \prec k'$ implies $k' \Vdash \phi$. So, $k \Vdash \phi$ if and only if $k \Vdash \top \Rightarrow \phi$. A sequent $\phi \Rightarrow \psi$ is *true* in a Kripke model $\mathbf{M}$, written $\mathbf{M} \Vdash \phi \Rightarrow \psi$, if and only if for all $k \in W$, $k \Vdash \phi \Rightarrow \psi$. We often write $\mathbf{M} \Vdash \phi$ as short for $\mathbf{M} \Vdash \top \Rightarrow \phi$. $\phi \Rightarrow \psi$ is *valid* on a Kripke frame $\mathbf{F}$, $\mathbf{F} \Vdash \phi \Rightarrow \psi$, iff $\phi \Rightarrow \psi$ is true on every Kripke model based on $\mathbf{F}$. Let $\mathcal{C}$ be a class of Kripke frames, $\phi \Rightarrow \psi$ is $\mathcal{C}$*-valid*, $\mathcal{C} \Vdash \phi \Rightarrow \psi$, iff $\phi \Rightarrow \psi$ is valid on every Kripke frame in $\mathcal{C}$.

For a set $\Gamma$ of sequents, $\mathbf{M} \Vdash \Gamma$ means that $\mathbf{M} \Vdash \phi \Rightarrow \psi$, for all $\phi \Rightarrow \psi \in \Gamma$. For a set of sequents $\Gamma \cup \{\phi \Rightarrow \psi\}$, the notation $\Gamma \Vdash \phi \Rightarrow \psi$ means that for any Kripke model $\mathbf{M}$, if $\mathbf{M} \Vdash \Gamma$, then $\mathbf{M} \Vdash \phi$.

In the sequel we show a Kripke model by its forcing relation. For $k \in W$, we call $\mathbf{M} = (W, \prec, \Vdash, k)$ *pointed* and it is called *rooted*, with root $k$, if and only if $k \preceq k'$, for all $k' \in W$. Also it is called a tree Kripke model if and only if $\langle W, \prec \rangle$ is a tree. We denote the class of all models, pointed models and rooted models by Mod, Pmod and Rmod, respectively. We denote $W$ by $\mathbf{M}$ when clear from the context. We write $\mathbf{M}(\vec{p})$ for the result of restricting $V$ to $\vec{p}$.

If $\mathbf{M} = (W, \prec, \Vdash)$ is a Kripke model and $w$ a world of $\mathbf{M}$, the *submodel* of $\mathbf{M}$ *generated* by $w$ is the Kripke model $\mathbf{M}[w] := \mathbf{M}' = (W[w], \prec', \Vdash')$ where $W[w] = \{x \in W | w \preceq x\}$, and $\prec'$ and $\Vdash'$ are restrictions of $\prec$ and $\Vdash$ to $W[w]$.

Here we stick to the following characterization of **BPL** and **FPL** models throughout the paper.

Theorem 2.3 ([3]). **BPL** *and* **FPL** *are sound and complete for the class of all irreflexive Kripke models and all conversely well-founded irreflexive Kripke models, respectively.*

The *depth* of a node $k \in W$ is defined inductively by

$$d(k) := \sup\{d(k') + 1 \mid k' \prec k\},\ \textit{where}\ \sup(\emptyset) = 0,$$

and the depth of a model $\mathbf{M}$ is defined as

$$d(\mathbf{M}) := \sup\{d(k) \mid k \in W\}.$$

We notice that $d(\mathbf{M}) = \infty$ is possible. We define inductively $\Box^n\phi$ by $\Box^0\phi := \phi$, $\Box\phi := \top \to \phi$ and $\Box^{n+1}\phi := \Box\Box^n\phi$, for $n \in \omega$. The following extensions of **BPL** were introduced in [3]

- $\mathbf{F}_n := \mathbf{BPL} + \Box^n\bot$, for $n \in \omega$,
- $\mathbf{FPL}_\bot := \mathbf{BPL} + L_\bot$, where $L_\bot := (\Box\bot \to \bot) \to \Box\bot$.

One can see that **BPL** proves $L_\bot \Leftrightarrow \top \to L_\bot$, so $\mathbf{FPL}_\bot$ is faithful. Given a Kripke frame $\langle W, \prec\rangle$, a world $e \in W$ is called an *end-node* if it is maximal with respect to $\preceq$. A Kripke frame $\langle K, \prec\rangle$ with end-nodes is a Kripke frame such that for every $w \in K$ there is some end-node $e \in W$ with $w \preceq e$.

Proposition 2.4 ([3]).

1. $\mathbf{FPL}_\bot$ is sound and complete with respect to the class of all irreflexive Kripke frames with end-nodes,
2. For every $n \geq 1$, the logic $\mathbf{F}_n$ is strongly complete with respect to the class of all irreflexive Kripke models with depth not greater than $n$.

## 3. Basic model theory

In this section, first we briefly review the notion of Henkin construction for basic propositional logic. The results we report on the Henkin model can be found in [4]. However, for the sake of entirety and because of phrasing the results in terms of saturated sets of formulas instead of prime sequent theories and also new relations between saturated sets compare to [4], we decided to present them in full proofs. After which, we recall the notion of bisimulation (and in general, layered bisimulation) between two models. For convenience in our context, this notion has been slightly modified, i.e., the zig and zag conditions hold strictly. In the sequel we need to extend the set of all natural numbers $\omega$ with an extra top element $\infty$. Let $\omega^\infty$ be $\omega \cup \{\infty\}$ which is equipped with the obvious natural ordering $\leq$. We extend addition by, $\infty + \alpha = \alpha + \infty = \infty$ and $\infty - n = \infty$. We let $n$ range over $\omega$, and $\alpha$ range over $\omega^\infty$.

We start by the following lemma which can be proved by induction on the complexity of formulas and distributivity axiom of **BPL**.

LEMMA 3.1. *Let $\phi \in \mathcal{L}(\mathcal{P})$ be a formula. Then it can be written, modulo* **BPL** *provability, as $\bigvee_i \bigwedge_j \phi_{ij}$ where $\phi_{ij}$ is an atom in $\mathcal{P}$, $\top$, $\bot$ or an implication.*

We call $\bigvee_i \bigwedge_j \phi_{ij}$ *the disjunctive normal form* of $\phi$.

A set $X \subseteq \mathcal{L}(\mathcal{P})$ is called $\mathcal{P}$-*adequate* if $\mathcal{P} \subseteq X$ and $X$ is closed under subformulas. We say that a consistent set $\Gamma \subseteq X$ is $X$-saturated if it is $X$-*deductively closed* and $X$-*prime*, i.e.,

- $\Gamma \vdash \phi$ and $\phi \in X$ implies $\phi \in \Gamma$,
- $\Gamma \vdash \phi \vee \psi$ and $\phi \vee \psi \in X$ implies $\Gamma \vdash \phi$ or $\Gamma \vdash \psi$.

We say that a consistent set $\Gamma$ is *prime* exactly when it is $\mathcal{L}(\mathcal{P})$-prime. Given the $\mathcal{P}$-adequate set $X$, let $H_X$ be the collection of all $X$-saturated sets. The Kripe model $\mathbf{H}_X := (H_X, \prec, \Vdash)$ where for every $\Gamma \in H_X$ and every propositional variable $p \in \mathcal{P}$, $\Gamma \Vdash p$ if and only if $p \in \Gamma$ is called *canonical model* over $\mathcal{P}$ with respect to $X$.

PROPOSITION 3.2. Let $X$ be a $\mathcal{P}$-adequate set. For any formula $\phi \in X$ and any $\Gamma \in H_X$, $\mathbf{H}_X, \Gamma \vdash \phi$ if and only if $\Gamma \Vdash \phi$.

PROOF: We complete the proof by induction on the complexity of $\varphi$. We consider the interesting case where $\phi = \psi \to \theta$. Let $\Gamma \vdash \psi \to \theta$ and $\Delta \in H_X$ be such that $\Gamma \prec \Delta$ and $\Delta \Vdash \psi$. By induction hypothesis $\Delta \vdash \psi$ and because of $\Gamma \prec \Delta$ and $\Gamma \vdash \psi \to \theta$ we have $\Delta \vdash \psi \Rightarrow \theta$ and hence $\Delta \vdash \theta$. First, applying induction hypothesis gives $\Delta \Vdash \theta$. And thus $\Gamma \Vdash \psi \to \theta$.

Conversely, suppose that $\Gamma \nvdash \psi \to \theta$. Put $\Gamma_\psi = \{\eta \in X \mid \Gamma \vdash \psi \to \eta\}$. First, we notice that $\Gamma_\psi$ is $X$-deductively closed. Suppose that $\Gamma_\psi \vdash \alpha$, for $\alpha \in X$. Then there exist formulas $\eta_1, \cdots, \eta_i$ such that $\eta_1, \cdots \eta_i \vdash \alpha$. Put $\eta = \eta_1 \wedge \cdots \wedge \eta_i$. Hence, $\Gamma \vdash \psi \to \eta$ and $\vdash \eta \to \alpha$ which implies that $\Gamma \vdash \psi \to \alpha$. Then $\alpha \in \Gamma_\psi$. Next, we show that $\Gamma \prec \Gamma_\psi$. Suppose that $\Gamma \vdash \alpha \to \beta$ and $\Gamma_\psi \vdash \alpha$. Then $\Gamma \vdash \psi \to \alpha$ which implies, by transitivity, that $\Gamma \vdash \psi \to \beta$. Therefore, $\Gamma_\psi \vdash \beta$. Note that, $\Gamma_\psi \nvdash \theta$. Now, Assume that $\Sigma = \{\Delta \mid \Delta$ is a $X$- deductively closed set of formulas with $\Delta \vdash \psi, \Delta \nvdash \theta$ and $\Gamma \prec \Delta\}$. $\Sigma$ is nonempty, since $\Gamma_\psi \in \Sigma$. $(\Sigma, \subseteq)$ satisfies the chain condition for Zorn's lemma. For, suppose that $\{\Delta_i\}_{i \in I}$ is a chain of elements of $\Sigma$ then, one can see that $\bigcup \Delta_i$ is $X$ deductively closed set, $\bigcup \Delta_i \vdash \psi$ and $\bigcup \Delta_i \nvdash \theta$. We only show that $\Gamma \prec \bigcup \Delta_i$. So, suppose that $\Gamma \vdash \gamma \to \delta$ and $\bigcup \Delta_i \vdash \gamma$. Then there exists a $j$ such that $\Delta_j \vdash \gamma$ which implies that $\Delta_j \vdash \delta$, since $\Gamma \prec \Delta_j$. Hence, $\Gamma \prec \bigcup \Delta_i$. Let $\Delta$ be a maximal element of $\Sigma$. $\Delta$ is $X$-saturated. To see that, we need to show that it is $X$-prime. Assume $\alpha \vee \beta \in X$ is such that $\Delta \vdash \alpha \vee \beta, \Delta \nvdash \alpha$ and $\Delta \nvdash \beta$. But $\Gamma \prec \Delta \prec \Gamma_{\Delta,\alpha} := \{\eta \mid \Gamma \vdash \delta \wedge \alpha \to \eta$, for some $\delta \in \Delta\}$ and $\Gamma \prec \Delta \prec \Gamma_{\Delta,\beta}$, then by maximality of $\Delta$ we obtain $\Gamma_{\Delta,\alpha} \vdash \theta$ and $\Gamma_{\Delta,\beta} \vdash \theta$ which implies that $\Gamma \vdash \alpha \wedge \delta_1 \to \theta$ and $\Gamma \vdash \beta \wedge \delta_2 \to \theta$, for some $\delta_1, \delta_2 \in \Delta$. Then $\Gamma \vdash (\alpha \wedge \delta_1) \vee (\beta \wedge \delta_2) \to \theta$. But $\Delta \vdash (\alpha \wedge \delta_1) \vee (\beta \wedge \delta_2)$, then $\Delta \vdash \theta$ which is a contradiction. Hence, we have $\Gamma \prec \Delta$ and $\Delta \nvdash \theta$. Then, by induction hypothesis, $\Gamma \prec \Delta \Vdash \psi$, and $\Delta \nVdash \theta$. So $\Gamma \nVdash \psi \to \theta$. □

Definition 3.3. Let $\mathcal{K}$ be a set of disjoint pointed models for a $X$-saturated set $\Delta$. We define $Glue(H_X[\Delta], \mathcal{K})$ as follows:

- $Glue(\mathbf{H}_X[\Delta], \mathcal{K}) := (H_X[\Delta] \cup (\bigcup_i K_i) \cup \{m\}, \prec)$, where $m$ is a new distinct point, $(K_i, \prec_i, \Vdash_i, k_i)$'s are mutually disjoint pointed models in $\mathcal{K}$ and $\prec$ is defined by:

$$\begin{aligned} \prec \quad = \quad & \prec_i \restriction_{\mathbf{K}_i[k_i]} \cup \prec_{H_X} \restriction_{H_X[\Delta]} \\ & \cup \{(m, y) \ : \ y \in \bigcup_i \mathbf{K}_i[k_i] \setminus \{k_i\} \cup \mathbf{H}_X[\Delta] \setminus \{\Delta\}\}, \end{aligned}$$

- $m \Vdash p$ exactly when $p \in \Delta$.

We would like to notice that in the model $Glue(H_X[\Delta], \mathcal{K})$, $m$ is irreflexive and $m \not\prec k_i$ and $m \not\prec \Delta$ unless $k_i \prec_i k_i$ and $\Delta \prec_{H_X[\Delta]} \Delta$.

Lemma 3.4. *Let $\mathcal{K}$ be a class of pointed models for a $X$-saturated set $\Delta$ and $\phi \in X$. Then $Glue(H_X[\Delta], \mathcal{K}), m \Vdash \phi$ exactly when $\phi \in \Delta$.*

Proof: For atoms the claim is clear. From the construction, conjunction and disjunction are easy due to $X$-saturatedness of $\Delta$. For implication suppose that $\phi = \psi \to \gamma$. If $\psi \to \gamma \in \Delta$ and $m \prec l$, then $l$ must be in one of the models $\mathbf{K}_i[k_i] \setminus \{k_i\}$ or $\mathbb{H}_X[\Delta] \setminus \{\Delta\}$. If $l \in \mathbf{K}_i[k_i] \setminus \{k_i\}$ then, since $\mathbf{K}_i[k_i] \setminus \{k_i\}$ is a model of $\Delta$ we have $l \Vdash \psi \Rightarrow \gamma$ which implies that $m \Vdash \psi \to \gamma$. The case $l \in \mathbb{H}_X[\Delta] \setminus \{\Delta\}$ is obvious.

Conversely, suppose that $m \Vdash \psi \to \gamma$. Then for any $l \in \mathbb{H}_X[\Delta] \setminus \{\Delta\}$ we have $l \Vdash \psi \Rightarrow \gamma$ which implies that $\Delta \Vdash \psi \to \gamma$. But since $\psi \to \gamma \in X$ we have $\psi \to \gamma \in \Delta$. □

Theorem 3.5. *Let $X$ be a $\mathcal{P}$-adequate and $\Delta$ be $X$-saturated. Then $\Delta$ is prime.*

Proof: Suppose that $\Delta \vdash \phi \vee \psi$, for $\phi$ and $\psi \in \mathcal{L}(\mathcal{P})$. Since $X$ is $\mathcal{P}$-adquate, $\phi$ and $\psi$ have disjunctive normal forms: $\phi = \bigvee_i \bigwedge_j \phi_{ij}$ and $\psi = \bigvee_r \bigwedge_s \psi_{rs}$, where $\phi_{ij}, \psi_{rs} \in X$ or they are in implication form. We will show that there exist $i$ or $r$ such that $\Delta \vdash \bigwedge_j \phi_{ij}$ or $\Delta \vdash \bigwedge_s \psi_{rs}$. It is clear that it shows that either $\Delta \vdash \phi$ or $\Delta \vdash \psi$.

Assume, for any $i$ and $r$, that $\Delta \nvdash \bigwedge_j \phi_{ij}$ and $\Delta \nvdash \bigwedge_s \psi_{rs}$. Then there exist $(K_i, k_i)$ and $(L_r, l_r)$ such that $(K_i, k_i) \Vdash \Delta$, $(K_i, k_i) \nVdash \bigwedge_j \phi_{ij}$ and $(L_r, l_r) \Vdash \Delta$, $(L_r, l_r) \nVdash \bigwedge_s \psi_{rs}$. By Lemma 3.4 we have $Glue(H_X[\Delta], \{K_i\} \cup \{L_r\}), m \Vdash \Delta$. Therefore $m \Vdash \phi \vee \psi$, since $\Delta \vdash \phi \vee \psi$. Hence, there exist $i$ or $r$ such that $m \Vdash \bigwedge_j \phi_{ij}$ or $m \Vdash \bigwedge_s \psi_{rs}$. Assume $m \Vdash \bigwedge_j \phi_{ij}$, the other case is similar. Since $(K_i, k_i) \nVdash \bigwedge_j \phi_{ij}$, there are two cases: If $\phi_{ij}$ is an atom then by Lemma 3.4 since $m \Vdash \phi_{ij}$ and $\mathcal{P} \subseteq X$ we have $\phi_{ij} \in \Delta$ and since $(K_i, k_i) \Vdash \Delta$ we would have $(K_i, k_i) \Vdash \phi_{ij}$ which is a contradiction. If $\phi_{ij} = \delta \to \gamma$, for some $\delta$ and $\gamma$. In this case, since $m \Vdash \delta \to \gamma$, then for any $l \succ k_i$ we have $l \Vdash \delta \to \gamma$ which implies that $(K_i, k_i) \Vdash \delta \to \gamma$ which is impossible. □

*Remark* 3.6. We notice that Henkin models can be constructed similarly for **FPL**. Although $\mathbb{H}$ is not an **FPL**-model in this case, what we want from $\mathbb{H}$ in our proof of Lemma 4.8 is for it to be transitive, which it trivially is.

Definition 3.7. We define the complexity measure $i(\phi)$ of a formula $\phi$ recursively as follows:

1. $i(p) = 0$, for each propositional variable $p$;
2. $i(\top) = i(\bot) := 0$;
3. $i(\phi \wedge \psi) = i(\phi \vee \psi) := \max\{i(\phi), i(\psi)\}$;
4. $i(\phi \to \psi) := \max\{i(\phi), i(\psi)\} + 1$.

We define $B_n(\mathcal{P}) := \{\phi \in \mathcal{L}(\mathcal{P}) \mid i(\phi) \leq n\}$ and $B_\infty(\mathcal{P}) := \mathcal{L}(\mathcal{P})$. By induction on $n$ we may prove the following fact:

Fact 3.8. $B_n(\vec{p})$ is finite modulo **BPL**-provable equivalence.

By the above fact, we assume that $B_n(\vec{p})$ is finite from now on.

Definition 3.9. Let $\mathbf{M} = (W, \prec, \Vdash)$ be any Kripke model. For each $X \subseteq \mathcal{L}(\mathcal{P})$, $m \in W$ and $n \in \omega$, we define:

1. $\mathsf{Th}_X(m) = \{\phi \in X \mid m \Vdash \phi\}$;
2. $\mathsf{Th}_n^{\mathcal{P}}(m) = \{\phi \in B_n(\mathcal{P}) \mid m \Vdash \phi\}$;
3. $\mathsf{Th}_X(\langle \mathbf{M}, m\rangle) := \mathsf{Th}_X(m)$ and $\mathsf{Th}(m) := \mathsf{Th}_{\mathcal{L}(\mathcal{P})}(m)$;
4. $\mathsf{Y}_n(m) := \mathsf{Y}_{n,m}(\vec{p}) := \bigwedge \mathsf{Th}_n^{\vec{p}}(m)$;
5. $\mathsf{N}_n(m) := \mathsf{N}_{n,m}(\vec{p}) := \bigvee \{\phi \in B_n(\vec{p}) \mid m \nVdash \phi\}$.

Fact 3.10. $\mathsf{Y}_{n,m}(\vec{p})$ is a prime formula.

Proof: We first note, by definition, that $B_n(\vec{p})$ is closed under subformulas. Next, we show that $\mathsf{Y}_{n,m}(\vec{p})$ is an $B_n(\vec{p})$-saturated. Suppose that $\phi \in B_n(\vec{p})$ and that $\mathsf{Y}_{n,m}(\vec{p}) \vdash \phi$. Then $m \models \phi$ which implies that $\phi \in \mathsf{Y}_{n,m}(\vec{p})$. For $B_n(\vec{p})$-primness suppose that $\phi \vee \psi \in B_n(\vec{p})$ and $\mathsf{Y}_{n,m}(\vec{p}) \vdash \phi \vee \psi$. Then $m \models \phi \vee \psi$ which implies that $m \models \phi$ or $m \models \psi$. Hence, $\phi \in \mathsf{Y}_{n,m}(\vec{p})$ or $\psi \in \mathsf{Y}_{n,m}(\vec{p})$. Therefore, by Theorem 3.5, $\mathsf{Y}_{n,m}(\vec{p})$ is prime. □

Let $\mathbf{M} = (W, \prec, \Vdash)$ and $\mathbf{M}' = (W', \prec', \Vdash')$, be any two $\mathcal{P}$-models. We say a relation $\mathcal{Z} \subseteq W \times \omega^\infty \times W'$ is a *l*ayered $\mathcal{P}$-bisimulation ($l$-bisimulation) between $\mathbf{M}$ and $\mathbf{M}'$ if it satisfies the following three conditions:

1. $(w, \alpha, w') \in \mathcal{Z}$ implies $w \Vdash p$ if and only if $w' \Vdash p$, for all atome $p \in \mathcal{P}$;
2. $(w, \alpha + 1, w') \in \mathcal{Z}$ and $w \prec x$ implies $(w', \alpha, x') \in \mathcal{Z}$, for some $x' \succ' w'$;
3. $(w, \alpha+1, w') \in \mathcal{Z}$ and $w' \prec' x'$ implies $(x, \alpha, x') \in \mathcal{Z}$, for some $x \succ w$.

We call (2) the $zig_{\alpha+1}$-property and (3) $zag_{\alpha+1}$- property. If $\alpha = \infty$, we simply call them the zig- and the zag-property. We write $w\mathcal{Z}_\alpha w'$ for $(w, \alpha, w') \in \mathcal{Z}$ and $w\mathcal{Z}w'$ for $w\mathcal{Z}_\infty w'$. To clarify the definition in the case of $\alpha = \infty$, we rewrite clauses of the above definition, as follows:

1. $(w, \infty, w') \in \mathcal{Z}$ implies $w \Vdash p$ if and only if $w' \Vdash p$, for all atome $p \in \mathcal{P}$;
2. $(w, \infty, w') \in \mathcal{Z}$ and $w \prec x$ implies $(w', \infty, x') \in \mathcal{Z}$, for some $x' \succ' w'$;
3. $(w, \infty, w') \in \mathcal{Z}$ and $w' \prec' x'$ implies $(x, \infty, x') \in \mathcal{Z}$, for some $x \succ w$.

A binary relation $\mathcal{Z}$ between $\mathbf{M}$ and $\mathbf{M}'$ is a *bisimulation* between $\mathbf{M}$ and $\mathbf{N}$ exactly when $\{\langle w, \infty, w'\rangle \mid w\mathcal{Z}w'\}$ is an $l$-bisimulation.

We say $l$-bisimulation $\mathcal{Z}$ is *downward closed* if for any $(w, n, w') \in W \times \omega \times W'$, $(w, n, w') \in \mathcal{Z}$ implies that $(w, m, w') \in \mathcal{Z}$, for all $m \leq n$. Let $PV_{\mathbf{M}}(w) := \{p \in \mathcal{P} : \mathbf{M}, w \Vdash p\}$, we define $w\mathcal{Z}_{\prec 0}w'$ exactly when $PV_{\mathbf{M}}(w) \subseteq PV_{\mathbf{M}'}(w')$; and $w\mathcal{Z}_{\prec\alpha+1}w'$ exactly when $PV_{\mathbf{M}}(w) \subseteq PV_{\mathbf{M}'}(w')$ and for all $x' \succ' w'$ there exists $x \succ w$ with $x\mathcal{Z}_{\alpha}x'$.

We notice that since the set of all $l$-bisimulations between two models $\mathbf{M}$ and $\mathbf{M}'$ are closed under union, then there is always a maximal $l$-bisimulation, $\simeq^{\mathbf{M},\mathbf{M}'}$, which is also downward closed. We will often drop the superscript of $\simeq^{\mathbf{M},\mathbf{M}'}$. In case of $\alpha = \infty$, we will drop the subscript of $\simeq_{\alpha}^{\mathbf{M},\mathbf{M}'}$ (if no confusion is possible). $\mathcal{Z}_{\alpha}$ is *full* if it is both total and surjective as a relation between $\mathbf{M}$ and $\mathbf{M}'$. We say that $\mathbf{M}$ and $\mathbf{M}'$ *$\alpha$-bisimualte (bisimualte)*, or $\mathbf{M} \simeq_{\alpha} \mathbf{M}'$ ($\mathbf{M} \simeq \mathbf{M}'$,) if there is a full $\alpha$- bisimulation (bisimulation) between them. $\mathcal{Z} : \mathbf{M} \simeq_{\alpha} \mathbf{M}'$ means that $\mathcal{Z}$ is a full $\alpha$-bisimulation witnessing that $\mathbf{M} \simeq_{\alpha} \mathbf{M}'$. For a set of propositional variables $\mathcal{Q}$, $\mathbf{M} \simeq_{\alpha,\mathcal{Q}} \mathbf{M}'$ means that $\mathbf{M}$ and $\mathbf{M}'$ $\alpha$-bisimulate with respect to the variables in $\mathcal{Q}$. Note that for rooted models $\mathbf{M}$ and $\mathbf{M}'$ we have $\mathbf{M} \simeq_{\alpha} \mathbf{M}'$ if and only if $r_{\mathbf{M}} \simeq_{\alpha} r_{\mathbf{M}'}$.

We say that $w \in W$ and $w' \in W'$ are $\alpha$-equivalent, written $w \equiv_{\alpha} w'$, exactly when $\mathsf{Th}_{\alpha}(w) = \mathsf{Th}_{\alpha}(w')$. We notice that for $\alpha = \infty$, $w$ and $w'$ are $\alpha$-equivalent if $\mathsf{Th}(w) = \mathsf{Th}(w')$.

THEOREM 3.11. *Let $\mathbf{M} = (W, \prec, \Vdash)$ and $\mathbf{M}' = (W', \prec', \Vdash')$ be any Kripke models, $w \in W, w' \in W'$ and $\alpha \in \omega^{\infty}$. Then $w\mathcal{Z}_{\alpha}w'$ implies $w \equiv_{\alpha} w'$.*

PROOF: The proof is by induction on the complexity of formulas. We only check the case of implication. So suppose that $\phi = \gamma \to \psi$. Suppose $w \nVdash \gamma \to \psi$. Then for some $x \succ w$, $x \Vdash \gamma$ and $x \nVdash \psi$. Notice that $\gamma, \psi \in B_{\alpha-1}(\mathcal{P})$. Moreover, since $w\mathcal{Z}_{\alpha}w'$, then there is $x' \succ' w'$ such that $x\mathcal{Z}_{\alpha-1}x'$. Hence, by induction, we get $x' \Vdash' \gamma$ and $x' \nVdash' \psi$. Therefore, $w' \nVdash' \gamma \to \psi$. By a similar argument, we can prove the reverse implication. □

THEOREM 3.12. *Let $\mathbf{M} = \langle W, \prec, \Vdash\rangle$ and $\mathbf{M}' = \langle W', \prec', \Vdash'\rangle$ be any two Kripke models. For any $w \in W, w' \in W'$ and $n \in \omega$, the following are equivalent:*

1. $\mathsf{Th}_n^{\mathcal{P}}(w) \subseteq \mathsf{Th}_n^{\mathcal{P}}(w')$;
2. *There exists a layered $\mathcal{P}$- bisimulation $\mathcal{Z}$ between $\mathbf{M}$ and $\mathbf{M}'$ such that $w\mathcal{Z}_{\prec n}w'$;*
3. *There exists a downward closed layered $\mathcal{P}$- bisimulation $\mathcal{Z}$ between $\mathbf{M}$ and $\mathbf{M}'$ such that $w\mathcal{Z}_{\prec n}w'$.*

Proof: $(2 \Rightarrow 1)$: We prove that for all $m \in \omega, x \in W$ and $x' \in W'$, if there exists a layered $\mathcal{P}$-bisimulation $\mathcal{Z}$ between $\mathbf{M}$ and $\mathbf{M}'$ such that $x\mathcal{Z}_{\prec m}x'$, then $\mathsf{Th}^{\mathcal{P}}_m(x) \subseteq \mathsf{Th}^{\mathcal{P}}_m(x')$.

Let $m = 0$, then the set $B_0(\mathcal{P})$ is a set of implication-free formulas. By the assumption $PV_{\mathbf{M}}(x) \subseteq PV_{\mathbf{M}'}(x')$, so if $\phi = p$ is a propositional variable, then we have our result. The cases for conjunction and disjunction can be done by induction. Now, Suppose that the statement holds for $m > 0$, and that there exists a layered $\mathcal{P}$-bisimulation $\mathcal{Z}$ between $\mathbf{M}$ and $\mathbf{M}'$ such that $x\mathcal{Z}_{\prec m+1}x'$. By induction on the complexity of given $\phi \in B_{m+1}(\mathcal{P})$ we prove, $x \Vdash \phi$ implies $x' \Vdash' \phi$. We only check that for $\phi := \gamma \to \psi$. We notice that $\gamma, \psi \in B_m(\mathcal{P})$. Suppose that $x' \nVdash' \gamma \to \psi$ then for some $y' \succ' x'$, $y' \Vdash' \gamma$ and $y' \nVdash' \psi$. Since $x\mathcal{Z}_{\prec m+1}x'$, there is a $y \succ x$, such that $y\mathcal{Z}_m y'$. Then, by induction and Theorem 3.11, $y \Vdash \gamma$ and $y \nVdash \psi$. That means $x \nVdash' \gamma \to \psi$ which is a contradiction.

$(1 \Rightarrow 3)$: We prove that for all $m \in \omega, x \in W$ and $x' \in W'$, if $\mathsf{Th}^{\mathcal{P}}_m(x) \subseteq \mathsf{Th}^{\mathcal{P}}_m(x')$, then there exists a layered $\mathcal{P}$- bisimulation $\mathcal{Z}$ between $\mathbf{M}$ and $\mathbf{M}'$ with $w\mathcal{Z}_{\prec m}w'$.

For $m = 0$, put $\mathcal{Z} = \emptyset$ which is obviously downward closed. Now assume that the statement holds for $m > 0$. Suppose $\mathsf{Th}^{\mathcal{P}}_{m+1}(x) \subseteq \mathsf{Th}^{\mathcal{P}}_{m+1}(x')$. Define a relation $\mathcal{Z}$ on $W \times \omega \times W'$ as:

$$w\mathcal{Z}_i w' \text{ if and only if } \mathsf{Th}^{\mathcal{P}}_i(w) = \mathsf{Th}^{\mathcal{P}}_i(w').$$

Clearly, $\mathcal{Z}_i$'s are persistent over atoms. We only show the zig property, suppose $w\mathcal{Z}_i w'$ and $w \prec y$. We want to show that there is $y' \succ' w'$ such that $y\mathcal{Z}_{i-1}y'$. Define $\phi(y) := \mathsf{Y}_{i-1}(y) \to \mathsf{N}_{i-1}(y)$. We have $w \nVdash \phi(y)$ and since $\phi(y) \in \mathcal{B}_i(\mathcal{P})$, $w' \nVdash \phi(y)$. Therefore, for some $y' \succ' w'$ we have $y' \Vdash \mathsf{Y}_{i-1}(y)$, but $y' \nVdash \mathsf{N}_{i-1}(y)$. Hence $y\mathcal{Z}_{i-1}y'$. It remains to show that $x\mathcal{Z}_{\prec m+1}x'$. So assume that $k' \succ' x'$, then $x' \nVdash \phi(k')$. Thus by assumption we have $x \nVdash \phi(k')$ which implies that for some $k \succ x$, $k \Vdash \mathsf{Y}_m(k')$ and $k \nVdash \mathsf{N}_m(k')$. Hence, $k\mathcal{Z}_m k'$. Obviously, $\mathcal{Z}$ is a downward closed $l$-bisimulation.

$(3 \Rightarrow 2)$: Obvious. □

## 4. Interpolation

In this section, we prove the lifting theorem which helps us in establishing the Craig interpolation property. After which, we prove the amalgamation lemma for **FPL** which results in its uniform left-interpolation property.

The proofs are highly influenced by that of similar theorems in [8]. In this section all models are irreflexive unless explicitly mentioned.

Theorem 4.1 (Lifting). *Let* $\mathbf{M} = (W, \prec, \Vdash)$ *be a* $\vec{q}, \vec{p}$*-model and* $\mathbf{M}' = (W', \prec', \Vdash')$ *be a* $\vec{p}, \vec{r}$*- model with* $\mathbf{M}(\vec{p}) \simeq_\alpha \mathbf{M}'(\vec{p})$*. Then there exists* $\vec{q}, \vec{p}, \vec{r}$*-model* $\mathbf{M}'' = (W'', \prec'', \Vdash'')$ *such that* $\mathbf{M}(\vec{q}, \vec{p}) \simeq_\alpha \mathbf{M}''(\vec{q}, \vec{p})$ *and* $\mathbf{M}'(\vec{p}, \vec{r}) \simeq_\alpha \mathbf{M}''(\vec{p}, \vec{r})$*.*

Proof: Let $\mathcal{Z} : \mathbf{M}(\vec{p}) \simeq_\alpha \mathbf{M}'(\vec{p})$. Define $\vec{q}, \vec{p}, \vec{r}$-model $\mathbf{M}''$ as follows:

- $W'' := \{(w, w') \mid (w, \beta, w') \in \mathcal{Z} \text{ for some } \beta\}$;
- $(w, w') \prec'' (v, v')$ exactly when $w \prec v$ and $w' \prec' v'$;
- $(w, w') \Vdash'' s$ exactly when $w \Vdash s$ or $w' \Vdash' s$.

It's easy to see that for $s \in \vec{q}, \vec{p}$ we have $(w, w') \Vdash'' s$ exactly when $w \Vdash s$ and for $s \in \vec{p}, \vec{r}$ we have $(w, w') \Vdash'' s$ exactly when $w' \Vdash' s$. Next, define $\mathcal{Z}'$ by $w\mathcal{Z}'_i(w, w')$ if $w\mathcal{Z}_i w'$ and $\mathcal{Z}''$ by $w'\mathcal{Z}''_i(w, w')$ if $w\mathcal{Z}_i w'$. It's easy to see that $\mathcal{Z}' : \mathbf{M}(\vec{q}, \vec{p}) \simeq_\alpha \mathbf{M}''(\vec{q}, \vec{p})$ and $\mathcal{Z}'' : \mathbf{M}'(\vec{p}, \vec{r}) \simeq_\alpha \mathbf{M}''(\vec{p}, \vec{r})$. □

Corollary 4.2. Let $\mathbf{M}$ be a $\vec{q}, \vec{p}$-model and $\mathbf{M}'$ be a $\vec{p}, \vec{r}$-model with $\mathbf{M}(\vec{p}) \simeq_n \mathbf{M}'(\vec{p})$. Then there exists $\vec{q}, \vec{p}, \vec{r}$-model $\mathbf{M}''$ such that $\mathsf{Th}_n^{(\vec{q},\vec{p})}(\mathbf{M}) = \mathsf{Th}_n^{(\vec{q},\vec{p})}(\mathbf{M}'')$ and $\mathsf{Th}_n^{(\vec{p},\vec{r})}(\mathbf{M}') = \mathsf{Th}_n^{(\vec{p},\vec{r})}(\mathbf{M}'')$.

By the lifting lemma we are ready to prove the Craig interpolation property for **BPL**. The Craig interpolation property for **BPL** was proved in [4]. The proof of the Craig interpolation property for **FPL**, $\mathbf{FPL}_\perp$, **EBPL** and $\mathbf{F}_n$, for $n \in \omega$ are new.

We say a class $\mathcal{C}$ of Kripke models has the *lifting property* if for all models $\mathbf{M}$ and $\mathbf{M}'$ in $\mathcal{C}$, the constructed model $\mathbf{M}''$ in the lifting lemma is also in $\mathcal{C}$.

Theorem 4.3 (Craig Interpolation). *Let* **L** *be a logic over* **BPL** *which is sound and complete with respect to a class* $\mathcal{C}$ *having lifting property. Then* **L** *satisfies the Craig interpolation property.*

Proof: Suppose that $\phi \in B_m(\vec{q}, \vec{p})$ and $\psi \in B_n(\vec{p}, \vec{r})$ are such that $\mathbf{L} \vdash \phi \to \psi$. We show that $\psi_k^*(\vec{p}) := \bigvee\{\chi \in B_k(\vec{p}) \mid \mathbf{L} \vdash \chi \to \psi\}$ is their Craig interpolant, where $k := \max(m, n)$.

Clearly $\mathbf{L} \vdash \psi_k^*(\vec{p}) \to \psi$. If $\mathbf{L} \nvdash \phi \to \psi_k^*(\vec{p})$, there exists a $\vec{q}, \vec{p}$-pointed model $(\mathbf{M}, w)$ such that $w \Vdash \phi$ but $w \nVdash \psi_k^*(\vec{p})$. Let $\mathsf{Y} := \mathsf{Y}_{k,w}((\vec{p}))$ and

$\mathsf{N} := \mathsf{N}_k^{(\vec{p})}(w)$. For contradiction, suppose $\mathsf{Y} \vdash \mathsf{N} \vee \psi$. Note that by Fact 3.10, $\mathsf{Y}$ is prime. So $\mathsf{Y} \vdash \mathsf{N}$ or $\mathsf{Y} \vdash \psi$. Since $\mathsf{Y} \nvdash \mathsf{N}$, it follows that $\mathsf{Y} \vdash \psi$ and hence by definition of $\psi_k^*(\vec{p})$ we have $\mathsf{Y} \vdash \psi_k^*(\vec{p})$ which is a contradiction, since $w \nVdash \psi_k^*(\vec{p})$. So $\mathsf{Y} \nvdash \mathsf{N} \vee \psi$. Then there exists $\vec{q}, \vec{r}$-pointed model $(\mathbf{M}', w')$ such that $w' \Vdash \mathsf{Y}$ but $w' \nVdash \mathsf{N} \vee \psi$. Now, by Theorem 3.12, we have $\mathbf{M}'(\vec{p}) \simeq_k \mathbf{M}(\vec{p})$. Then, by Corollary 4.2, there exists $\vec{p}, \vec{q}, \vec{r}$-model $\mathbf{M}''$ such that $\mathsf{Th}_k^{(\vec{q},\vec{p})}(\mathbf{M}) = \mathsf{Th}_k^{(\vec{q},\vec{p})}(\mathbf{M}'')$ and $\mathsf{Th}_k^{(\vec{p},\vec{r})}(\mathbf{M}') = \mathsf{Th}_k^{(\vec{p},\vec{r})}(\mathbf{M}'')$. In particular, $\mathbf{M}'' \Vdash \phi$ and $\mathbf{M}'' \nVdash \psi$ which is a contradiction. Therefore, $\mathbf{L} \vdash \phi \to \psi_k^*(\vec{p})$. □

Corollary 4.4. **BPL**, **FPL**, and $\mathbf{F}_n$, for $n \in \omega$, have the Craig interpolation property.

Proof: For **BPL** it is trivial. For **FPL**, note that in the lifting lemma, when $\mathbf{M}$ and $\mathbf{M}'$ are conversely well-founded, so will be the constructed model $\mathbf{M}''$. Also, when $\mathbf{M}$ and $\mathbf{M}'$ have depth at most $n$, then $\mathbf{M}''$ also has depth at most $n$. □

The following logic is another interesting extension of **BPL** which behaves very similar to **IPL** [2].

$$\mathbf{EBPL} = \mathbf{BPL} + \top \to \bot \Rightarrow \bot.$$

It was proved in [2, Corollary 3.9] that the logic **EBPL** is sound and complete for the class of finite models with reflexive leaves. Obviously this class of models has the lifting property. Therefore we have the following corollary.

Corollary 4.5. The logic **EBPL** has the Craig interpolation property.

We say a formulas $\phi$ is constant if $V(\phi) = \emptyset$. In the following theorem we show that every faithful extension of basic propositional logic with constant formulas preserves Craig interpolation property.

Theorem 4.6. *Let $X$ be a set of constant fourmulas. If a logic $\mathbf{L}$ has the Craig interpolation property and $\mathbf{L} + X$ is faithful, then $\mathbf{L} + X$ also has Craig interpolation property.*

Proof: Suppose that $\mathbf{L}$ has the Craig interpolation property. Let $\mathbf{L} + X \vdash \phi \to \psi$. Then by faithfulness we have $\mathbf{L} + X \vdash \phi \Rightarrow \psi$. Then there are constant formulas $\theta_1, \cdots, \theta_n$ in $X$ such that in $\mathbf{L}$ we have $\theta_1, \cdots, \theta_n \vdash \phi \Rightarrow$

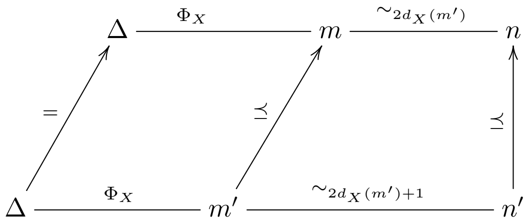


**Figure 1.** Witnessing triple

$\psi$. Put $\theta = \bigwedge \theta_i$, then by Proposition 2.1 we have $\mathbf{L} \vdash \theta \wedge \phi \Rightarrow \psi$ which implies that $\mathbf{L} \vdash \theta \wedge \phi \to \psi$. Now by interpolation property of $\mathbf{L}$, there is a formula $\eta$ in $V(\phi) \cap V(\psi)$ such that $\mathbf{L} \vdash \theta \wedge \phi \to \eta$ and $\mathbf{L} \vdash \eta \to \psi$. Hence,by faithfulness, $\mathbf{L} + X \vdash \phi \to \eta$ and $\mathbf{L} + X \vdash \eta \to \psi$. □

Corollary 4.7. $\mathbf{FPL}_\perp$ has the Craig interpolation property.

The proof of the next lemma is similar to the one used in [8] for **IPL**. However, to show that this proof -especially claim 2- does not work for **BPL** but does work for **FPL**, the details have been provided. In the following lemma, all models are conversely well-founded, i.e., **FPL**- models.

Lemma 4.8 (Amalgamation). *Consider disjoint sets $\vec{q}, \vec{p}$ and $\vec{r}$. Let $X \subseteq \mathcal{L}(\vec{q}, \vec{p})$ be a finite $\mathcal{P}$-adequate set. Let $\langle \mathbf{M}, w_0 \rangle \in \mathsf{Pmod}(\vec{q}, \vec{p})$, $\langle \mathbf{M}', w_0' \rangle \in \mathsf{Pmod}(\vec{p}, \vec{r})$. Let:*

$$\nu := |\{\phi \in X \mid \phi \text{ is a propositional variable or an implicational formula}\}|.$$

*Suppose that $w_0 \simeq_{2\nu+1,\vec{p}} w_0'$. Then there exists a $\vec{q}, \vec{p}, \vec{r}$-model $\langle \mathbf{M}'', w_0'' \rangle$ such that $w_0'' \simeq_{\vec{p},\vec{r}} w_0'$ and $Th_X(w_0'') = Th_X(w_0)$.*

Proof: Let $\mathcal{Z}$ be a downwards closed witness of $w_0 \simeq_{2\nu+1,\vec{p}} w_0'$. Define $\Phi_X : \mathbf{M} \longrightarrow H_X$ by $\Phi_X(w) := \Delta(w) := \{\phi \in X \mid w \Vdash \phi\}$. Define further for $w \in \mathbf{M}$: $d_X(w) = d_{H_X}(\Delta(w))$. Note that $d_X(w) \leq \nu$.

Consider a pair $\langle \Delta, n \rangle$ for $\Delta$ in $H$ and $n$ in $\mathbf{M}'$. We say that $m', m, n'$ is a *witnessing triple* for $\langle \Delta, n \rangle$ if:

$$\Delta = \Delta(m) = \Delta(m'),\ m' \preceq m, n' \preceq' n,\ m' \mathcal{Z}_{2d_X(m')+1} n',\ m \mathcal{Z}_{2d_X(m')} n.$$

The requested model $\mathbf{M}''$ is defined as follows:

- $W'' = \{\langle \Delta, n\rangle \mid \text{there is a witnessing triple for} \langle \Delta, n\rangle\}$,
- $w_0'' := \langle \Delta(w_0), w_0'\rangle$,
- $\langle \Delta, n\rangle \prec'' \langle \Gamma, n'\rangle$ exactly when $\Delta \preceq \Gamma$ and $n \prec' n'$,
- $\langle \Delta, n\rangle \Vdash s$ exactly when $\Delta \Vdash s$ or $n \Vdash s$.

Note that by assumption $w_0 \mathcal{Z}_{2\nu+1} w_0'$ and the fact that $2d_X(w_0) + 1 \leq 2\nu + 1$ we have $w_0 \mathcal{Z}_{2d_X(w_0)+1} w_0'$. So, $w_0, w_0, w_0'$ is a witnessing triple for $w_0''$. Let $m', m, n'$ be a witnessing triple for $\langle \Delta, n\rangle$. For $p \in \vec{p} \cap X$ we have $\Delta \Vdash p$ if and only if $m \Vdash p$ if and only if $n \Vdash p$, and hence $\langle \Delta, n\rangle \Vdash p$ if and only if $\Delta \Vdash p$ if and only if $n \Vdash p$. Also, note that $\mathbf{M}''$ is an **FPL**-model. The following claims prove the lemma.

**Claim 1.** $w_0'' \simeq_{\vec{p},\vec{r}} w_0'$,

**Claim 2.** For $\phi \in X$, $\langle \Delta, n\rangle \Vdash \phi$ exactly when $\phi \in \Delta$.

*Proof of Claim 1:* For $\mathcal{B}$ defined by $\langle \Delta, n\rangle \mathcal{B} n$, by a same argument as [8], we show that it is a bisimulation. Clearly $\mathsf{Th}_{\vec{p},\vec{r}}(\langle \Delta, n\rangle) = \mathsf{Th}_{\vec{p},\vec{r}}(n)$. We only check the *zag*-property of $\mathcal{B}$. Suppose $\langle \Delta, n\rangle \mathcal{B} n \prec m$. We are looking for a pair $\langle \Gamma, m\rangle$ such that $\Delta \preceq \Gamma$. Let $k', k, n'$ be a witnessing triple for $\langle \Delta, n\rangle$. Since $k' \sim_{2d_X(k')+1} n' \preceq m$, there is a $h$ such that $h \prec k'$ and $h \sim_{2d_X(k')} m$. Put, $\Gamma := \Delta(h)$. We need a witnessing triple $k'^*, k^*, n'^*$ for $\langle \Gamma, m\rangle$. If $\Gamma = \Delta$, then put: $k'^* := k', k^* := h, n'^* := n'$, see figure 2.

If $\Gamma \neq \Delta$, then put: $k'^* := h, k^* := h, n'^* := m$. We notice that since $k' \preceq h$, then $\Delta = \Delta(k') \prec \Gamma$ which implies that $d_X(h) < d_X(k')$. Therefore, $2d_X(h) + 1 \leq 2_X(k')$, so $h \sim_{2d_X(k')+1} m$ which implies that $h \sim_{2d_X(k')} m$, because $\mathcal{Z}$ is downward close. Clearly $w_0'' \mathcal{B} w_0'$.

*Proof of Claim 2:* We proceed by induction on the complexity of a formula $\phi \in X$. The cases of atoms, conjuntctions and disjunctions are trivial. Consider $\phi \to \psi \in X$ and the node $\langle \Delta, m\rangle$ with witnessing triple $k', k, m'$. Suppose $\phi \to \psi \notin \Delta$. Since $\Delta = \mathsf{Th}(k)$, then $k \nVdash \phi \to \psi$. So, there is an $h \succ k$ with $h \Vdash \phi$ and $h \nVdash \psi$. Let $h$, by conversely well-foundedness of $\mathbf{M}$, be a maximal in $\mathbf{M}$ with $h \succ k$, $h \Vdash \phi$ and $h \nVdash \psi$. By maximality, we find $h \Vdash \phi \to \psi$. Let $\Gamma := \Delta(h)$. Since $\phi \to \psi \notin \Delta$ and $\phi \to \psi \in \Gamma$, we find $\Delta \prec \Gamma$, which implies that $d_X(k') \geq 1$. Since $k \mathcal{Z}_{2d_X(k')} m$ and $k \prec h$, there is an $n \succ m$ with $h \mathcal{Z}_{2d_X(h)-1} n$. Therefore $h \mathcal{Z}_{2d_X(h)+1} n$. So we can

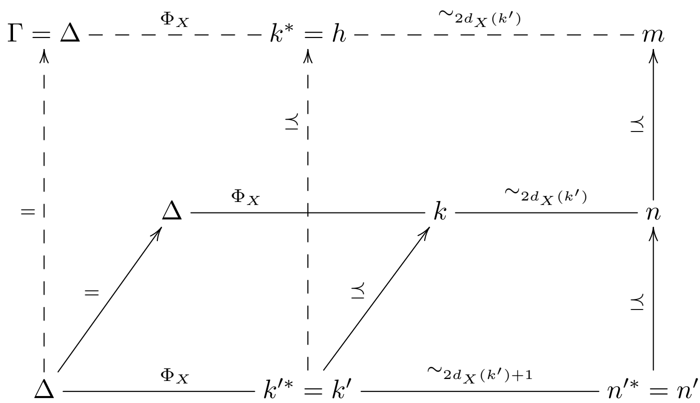


**Figure 2.**

take $h, h, n$ to witness $\langle\Gamma, n\rangle$. Clearly $\langle\Delta, m\rangle \prec'' \langle\Gamma, n\rangle$. By the induction hypothesis, $\langle\Gamma, n\rangle \Vdash \phi$ while $\langle\Gamma, n\rangle \nVdash \psi$, i.e., $\langle\Delta, n\rangle \nVdash \phi \to \psi$.

The other half of the argument, i.e., that $\phi \to \psi \in \Delta$ implies $(\Delta, n) \Vdash \phi \to \psi$, is easy. $\square$

Definition 4.9. Let $\phi(\vec{q}, \vec{p})$ be a formula.

1. A uniform left-interpolant for $\phi(\vec{q}, \vec{p})$ with respect to $\vec{p}$ is a formula $\chi(\vec{p})$ such that for all formulas $\psi(\vec{p}, \vec{r})$ with $\vdash \psi \to \phi$, $\chi$ acts as an interpolant for $\phi$ and $\psi$.

2. A uniform right-interpolant for $\phi(\vec{q}, \vec{p})$ with respect to $\vec{p}$ is a formula $\chi(\vec{p})$ such that for all formulas $\psi(\vec{p}, \vec{r})$ with $\vdash \phi \to \psi$, $\chi$ acts as an interpolant for $\phi$ and $\psi$.

3. A logic whose formulas have both uniform left and right-interpolants is said to satisfy the uniform interpolation property.

Although the Amalgamation lemma is held for FPL models, unlike in intuitionistic logic, we can only prove the uniform left-interpolation property.

Theorem 4.10. **FPL** *has the uniform left-interpolation property.*

Proof: Note that by the Amalgamation lemma, in proof of Craig interpolation for **FPL** we can let $X := sub(\phi)$, and by defining $\nu$ as before, we find that $\phi^*_{2v+1}$ works as Craig interpolant for any given $\psi$ satisfying the conditions. Therefore, $\phi^*_{2v+1}$ is the uniform left-interpolant for $\phi$. □

In the remainder of this section, we prove the uniform interpolation for some extensions of **FPL**. As a matter of fact, we show that countably infinite of such extensions exist.

A logic **L** is said to be *locally tabular* if for any finite set $\mathcal{P}$ of propositional variables, there are only finitely many formulas built from variables in $\mathcal{P}$ up to **L**-provable equivalence.

Theorem 4.11. *If* **L** *is a locally tabular logic over* **BPL** *and has the Craig interpolation property, then* **L** *has the uniform interpolation property.*

Proof: Consider a formula $\phi(\vec{q},\vec{p})$. Let $\Psi = \{\psi(\vec{p},\vec{r}) \mid \mathbf{L} \vdash \psi \to \phi\}$. Consider an effective counting of members of $\Psi$ as $\psi_1, \psi_2, \cdots, \psi_n$. By Craig interpolation, for every $i$ we can find $\chi_i(\vec{p})$ such that $\mathbf{L} \vdash \chi_i \to \phi$ and $\mathbf{L} \vdash \psi_i \to \chi_i$. Now,$\bigvee \chi_i$ works as the uniform left-interpolant of $\phi$ for all $\psi_n$.

For the uniform right-interpolant, let $\Psi = \{\psi(\vec{p},\vec{r}) \mid \mathbf{L} \vdash \phi \to \psi\}$. We can, by locally tabularity, find an effective counting of members of $\Psi$ as $\psi_1, \psi_2, \cdots, \psi_n$. By Craig interpolation, for every $i$ we can find $\chi_i(\vec{p})$ such that $\mathbf{L} \vdash \phi \to \chi_i$ and $\mathbf{L} \vdash \chi_i \to \psi_i$. Therefore $\bigwedge \chi_i$ works as the uniform right-interpolant of $\phi$ for all $\psi_n$. □

The following theorem was proved algebraically in [1, Theorem 2.12].

Theorem 4.12. *For every $n \in \omega$, the logic $\mathbf{F}_n$ is locally tabular.*

Corollary 4.13. The logic $\mathbf{F}_n$, for $n \in \omega$, have the uniform interpolation property.

Proof: Apply Corollary 4.4, Theorem 4.11 and Theorem 4.12. □

We close this paper with the following problem.

**Problem**. Do **BPL**, **FPL**, $\mathbf{FPL}_\bot$ and **EBPL** have the uniform interpolation property?

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