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# $n$-FOLD FILTERS OF $EQ$-ALGEBRAS

**Abstract**

In this paper, we apply the notion of $n$-fold filters to the $EQ$-algebras and introduce the concepts of $n$-fold pseudo implicative, $n$-fold implicative, $n$-fold obstinate, $n$-fold fantastic prefilters and filters on an $EQ$-algebra $\mathcal{E}$. Then we investigate some properties and relations among them. We prove that the quotient algebra $\mathcal{E}/F$ modulo an 1-fold pseudo implicative filter of an $EQ$-algebra $\mathcal{E}$ is a good $EQ$-algebra and the quotient algebra $\mathcal{E}/F$ modulo an 1-fold fantastic filter of a good $EQ$-algebra $\mathcal{E}$ is an $IEQ$-algebra.

*Keywords*: $EQ$-algebra, $n$-fold pseudo implicative (implicative, obstinate, fantastic) prefilter, $n$-fold pseudo implicative (implicative, fantastic) $EQ$-algebra.

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## 1. Introduction

Recently, a new class of algebras called $EQ$-algebras has been introduced by Novák in [9]. These algebras are intended to become algebras of truth values for a higher-order fuzzy logic (a fuzzy type theory, FTT). An $EQ$-algebra has three basic binary operations (meet, multiplication and a fuzzy equality) and a top element. The implication is defined from the fuzzy equality "$\sim$" by the formula $a \to b =(a \wedge b) \sim a$. Its implication and multiplication are no more closely tied by the adjunction and so, this algebra generalizes residuated lattice. From the point of view of potential application,

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it seems interesting that unlike Hájek [5], we can have non-commutativity without the necessity to introduce, two kinds of implication. Novák and De Baets in [10] introduced several kinds of $EQ$-algebras. El-Zekey in [4], proved that the class of EQ-algebras is a variety. El-Zekey in [4] introduced prelinear good $EQ$-algebras and proved that a prelinear good $EQ$-algebra is a distributive lattice. Novák and De Baets in [10] defined the concept of prefilter on $EQ$-algebras which is the same as filter of other algebraic structures such as residuated lattices, $MTL$-algebras, and etc. But the binary relation introduced by prefilter is not a congruence relation. To learn more about $EQ$-algebras, the reader can consult [1, 2, 7, 11, 13, 14]. Filter theory plays an important role in studying logical algebras. From a logical point of view, various filters have a natural interpretation as various sets of provable formulas. In this paper, we introduce $n$-fold implicative prefilter, $n$-fold pseudo implicative prefilter, $n$-fold fantastic prefilter, $n$-fold obstinate prefilter in $EQ$-algebra. We prove that the quotient algebra $\mathcal{E}/F$ modulo an 1-fold pseudo implicative filter of an $EQ$-algebra $\mathcal{E}$ is a good $EQ$-algebra and the quotient algebra $\mathcal{E}/F$ modulo an 1-fold fantastic filter of good $EQ$-algebra $\mathcal{E}$ is an involutive $EQ$-algebra. This paper is organized as follows: In Section 2, the basic definitions, special types of $EQ$-algebras and their properties are reviewed. In Section 3, $n$-fold prefilters and $n$-fold pseudo implicative prefilters of $EQ$-algebras and $EQ_n$-algebras are defined and investigated some results about them. We prove that the quotient algebra modulo 1-fold pseudo implicative filter is a good $EQ$-algebra. In Section 4, $n$-fold implicative prefilter of $EQ$-algebra, $n$-fold implicative $EQ$-algebra are studied. We show that in good $EQ$-algebra $\mathcal{E}$ with least element 0, a prefilter $F$ is an $n$-fold implicative prefilter of $\mathcal{E}$ if and only if $\mathcal{E}/F$ is an $n$-fold implicative $EQ$-algebra. In Section 5, $n$-fold obstinate prefilters, and maximal prefilters of $EQ$-algebras are investigated. We show that filter $\{1\}$ is an $n$-fold obstinate filter of residuated $EQ$-algebra $\mathcal{E}$ if and only if every filter of $\mathcal{E}$ is an $n$-fold obstinate filter of $\mathcal{E}$ and in a residuated $EQ$-algebra $\mathcal{E}$, a filter $F$ is an $n$-fold obstinate filter of $\mathcal{E}$ if and only if every filter of quotient algebra $\mathcal{E}/F$ is an $n$-fold obstinate filter of $\mathcal{E}/F$. Finally in Section 6, $n$-fold fantastic prefilters of $EQ$-algebras and $n$-fold fantastic $EQ$-algebras are introduced and studied the relation among the $n$-fold fantastic prefilters and $n$-fold fantastic algebras. Then we prove that in any good $EQ$-algebra, if $F$ is an 1-fold fantastic filter of $\mathcal{E}$, then $\mathcal{E}/F$ is an involutive $EQ$-algebra, and we show that in any residuated $EQ$-algebra with least element, $F$ is an $n$-fold implicative filter of $\mathcal{E}$ if and only if $F$

is an $n$-fold pseudo implicative filter and $n$-fold fantastic filter of $\mathcal{E}$. So we conclude that in any residuated $EQ$-algebra, $\mathcal{E}$ is an $n$-fold implicative $EQ$-algebra if and only if $\mathcal{E}$ is both $n$-fold pseudo implicative $EQ$-algebra and $n$-fold fantastic $EQ$-algebra.

## 2. Preliminaries

In this section, we recollect some definitions and results which will be used in the next sections.

Definition 2.1. [4] An *EQ-algebra* is an algebraic structure $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ of type (2, 2, 2, 0) such that, for all $x, y, z, t \in E$ the following conditions hold:

$(E1)$ $\langle E, \wedge, 1\rangle$ is a commutative idempotent monoid (i.e. $\wedge$-semilattice with top element 1);

$(E2)$ $\langle E, \otimes, 1\rangle$ is a commutative monoid and $\otimes$ is isotone w.r.t. $\leq$, where $x \leq y$ is defined as $x \wedge y = x$;

$(E3)$ $x \sim x = 1$; (reflexivity axiom)

$(E4)$ $((x \wedge y) \sim z) \otimes (t \sim x) \leq z \sim (t \wedge y)$; (substitution axiom)

$(E5)$ $(x \sim y) \otimes (z \sim t) \leq (x \sim z) \sim (y \sim t)$; (congruence axiom)

$(E6)$ $(x \wedge y \wedge z) \sim x \leq (x \wedge y) \sim x$; (monotonicity axiom)

$(E7)$ $x \otimes y \leq x \sim y$. (boundedness axiom)

Proposition 2.2. [10] Let $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ be an $EQ$-algebra. Define $x \to y := (x \wedge y) \sim x$ and $\bar{x} := x \sim 1$. Then, for all $x, y, z, t \in E$ the following properties hold:

$(i)$ $x \otimes y \leq x, y$ and $x \otimes y \leq x \wedge y$;

$(ii)$ $x \leq y \to x$;

$(iii)$ $x \to y \leq (z \to x) \to (z \to y)$ and $x \to y \leq (y \to z) \to (x \to z)$;

$(iv)$ if $x \leq y$, then $x \sim y = y \to x$, $z \to x \leq z \to y$ and $y \to z \leq x \to z$;

$(v)$ $x \to y \leq (x \wedge z) \to (y \wedge z)$.

Definition 2.3. [10] Let $\mathcal{E}$ be an $EQ$-algebra. Then $\mathcal{E}$ is called:

$(i)$ *separated* if $x \sim y = 1$, then $x = y$, for all $x, y \in E$, (in other words $x \sim y = 1$ if and only if $x = y$);

$(ii)$ *good* if $x \sim 1 = x = 1 \sim x$, for all $x \in E$;

$(iii)$ *residuated* if $x \leq y \to z$ if and only if $x \otimes y \leq z$, for all $x, y, z \in E$;

$(iv)$ *involutive (IEQ-algebra)* if $E$ contains 0 and $\neg\neg x = x$, for all $x \in E$, where $\neg x = x \sim 0$;

$(v)$ *lattice ordered* if the poset induced by the underlying semilattice of $\mathcal{E}$ is a lattice;

$(vi)$ a lattice *EQ-algebra ($\ell EQ$-algebra)* if $\mathcal{E}$ is a lattice ordered and for all $x, y, x, t \in E$ the following substitution axiom holds, $((x \vee y) \sim z) \otimes (t \sim x) \leq (z \sim (t \vee y))$.

Proposition 2.4. [10] Each $IEQ$-algebra is a good, separated and $\ell EQ$-algebra.

Proposition 2.5. [4] Let $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ be an $EQ$-algebra. Then, for all $x, y \in E$ the following statements are equivalent:

$(i)$ $\mathcal{E}$ is good;

$(ii)$ $x \otimes (x \sim y) \leq y$;

$(iii)$ $x \otimes (x \to y) \leq y$;

$(iv)$ $1 \to x = x$.

Proposition 2.6. [4] Let $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ be an $EQ$-algebra. Then, for all $x, y, z \in E$ the following statements are equivalent:

$(i)$ $\mathcal{E}$ is residuated;

$(ii)$ $\mathcal{E}$ is good and $x \to y \leq (x \otimes z) \to (y \otimes z)$;

$(iii)$ $\mathcal{E}$ is good and $x \leq y \to (x \otimes y)$;

$(iv)$ $\mathcal{E}$ is separated and $(x \otimes y) \to z = x \to (y \to z)$.

PROPOSITION 2.7. [4] Let $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ be a good $EQ$-algebra. Then, for all $x, y, z \in E$ the following properties hold:

$(i)$ $\mathcal{E}$ is residuated if and only if $x \otimes y \leq z$ implies $x \leq y \to z$;

$(ii)$ $x \leq (x \sim y) \sim y$ and $x \leq (x \to y) \to y$;

$(iii)$ $\mathcal{E}$ is separated;

$(iv)$ $x \to (y \to z) = y \to (x \to z)$;

$(v)$ $x \to (y \to z) \leq (x \otimes y) \to z$.

DEFINITION 2.8. [10] Let $\mathcal{E}$ be an $EQ$-algebra. A nonempty subset $F \subseteq E$ is called a *prefilter* of $\mathcal{E}$, if for all $x, y \in E$,

$(F1)$ $1 \in F$;

$(F2)$ If $x, x \to y \in F$, then $y \in F$.

A prefilter $F$ is said to be a *filter*, if

$(F3)$ $x \to y \in F$ implies $(x \otimes z) \to (y \otimes z) \in F$, for all $x, y, z \in E$.
A proper prefilter $F$ is called a *prime prefilter* of $\mathcal{E}$ if $x \to y \in F$ or $y \to x \in F$, for all $x, y \in E$.

DEFINITION 2.9. [12] A prefilter $F$ of an $EQ$-algebra $\mathcal{E}$ is called *maximal* if and only if it is proper and no prefilter of $\mathcal{E}$ strictly contains $F$ that is, for each prefilter $G$ of $\mathcal{E}$, if $F \subsetneq G$, then $G = E$.

LEMMA 2.10. [3] *Let $F$ be a prefilter of an $EQ$-algebra $\mathcal{E}$. Then, for all $x, y, z \in E$ the following statements hold:*

$(i)$ *If $x \in F$ and $x \leq y$, then $y \in F$;*

$(ii)$ *If $x, x \sim y \in F$, then $y \in F$;*

$(iii)$ *If $x, y \in F$, then $x \wedge y \in F$;*

*Moreover, if $F$ is a filter of $\mathcal{E}$, we have:*

$(iv)$ *If $x, y \in F$, then $x \otimes y \in F$;*

$(v)$ *If $x \to y \in F$ and $y \to z \in F$, then $x \to z \in F$.*

*Remark* 2.11. By Proposition 2.6 and Lemma 2.10, if $\mathcal{E}$ is a residuated $EQ$-algebra, then every prefilter of $\mathcal{E}$ is a filter of $\mathcal{E}$.

Definition 2.12. [8] Let $\mathcal{E}$ be an $EQ$-algebra and $X$ be a nonempty subset of $E$. Then the smallest prefilter of $\mathcal{E}$ which contains $X$, i.e.
$\bigcap\{F \mid \text{F is a prefilter of E such that, } X \subseteq F\}$ is said to be a *prefilter of* $\mathcal{E}$ *generated by* $X$ and is denoted by $\langle X\rangle$. If $a \in E$ and $X = \{a\}$, then we denote by $\langle a\rangle$ the prefilter generated by $\{a\}$ ($\langle a\rangle$ is called principal). For prefilter $F$ and $a \in E$, we denote by $F(a) = \langle F \cup \{a\}\rangle$.
It is clear that $a \in F$ implies $F(a) = F$. We can prove

$$F(a) = \{z \in E \mid a \to z \in F\}$$

and
$\langle X\rangle = \{a \in E \mid x_1 \to (x_2 \to (x_3 \to ...(x_n \to a)...)) = 1, \text{ for some } x_i \in X \text{ and } n \in \mathbb{N}\}$.

Definition 2.13. [6] Let $F$ be a prefilter of an $EQ$-algebra $\mathcal{E}$. Then $F$ is called

($i$) an *implicative prefilter* of $\mathcal{E}$, if for all $x, y, z \in E$,

($F4$) $z \to ((x \to y) \to x) \in F$ and $z \in F$ imply $x \in F$.

($ii$) a *positive implicative prefilter* of $\mathcal{E}$, if for all $x, y, z \in E$,

($F5$) $x \to (y \to z) \in F$ and $x \to y \in F$ imply $x \to z \in F$.

($iii$) a *fantastic prefilter* of $\mathcal{E}$, if for all $x, y \in E$,

($F6$) $y \to x \in F$ implies $((x \to y) \to y) \to x \in F$.

($iv$) an *obstinate prefilter* of $\mathcal{E}$,

($F7$) $x, y \notin F$ imply $x \to y \in F$ and $y \to x \in F$.

Proposition 2.14 ([12]). Let $\mathcal{E}$ be a residuated $EQ$-algebra and $F$ be a fantastic prefilter of $\mathcal{E}$. Then $F$ is an implicative prefilter of $\mathcal{E}$ if and only if $F$ is a positive implicative prefilter of $\mathcal{E}$.

Proposition 2.15 ([12]). Let $\mathcal{E}$ be a residuated $EQ$-algebra and $F$ be a positive implicative prefilter of $\mathcal{E}$. Then $F$ is an implicative prefilter of $E$ if and only if $F$ is a fantastic prefilter of $\mathcal{E}$.

PROPOSITION 2.16 ([12]). Let $\mathcal{E}$ be a good $EQ$-algebra and $F$ be a nonempty subset of $\mathcal{E}$. Then $F$ is an implicative prefilter if and only if $F$ is both a positive implicative prefilter and a fantastic prefilter of $\mathcal{E}$.

Let $F$ be a filter of an $EQ$-algebra $\mathcal{E}$. Then we define a binary relation $\equiv_F$ on $E$ as follows:

$$x \equiv_F y \ \text{ if and only if } \ x \sim y \in F.$$

Then $\equiv_F$ is a congruence relation on $E$. Denote $E/F := \{[x]_F \mid x \in E\}$ and $[x]_F = \{y \in E \mid x \equiv_F y\}$ and define operations $\wedge_F$, $\otimes_F$, $\sim_F$ and relation $\leq_F$ on $E/F$ as follows:

$$[x]_F \wedge_F [y]_F = [x \wedge y]_F, \ \ [x]_F \otimes_F [y]_F = [x \otimes y]_F, \ \ [x]_F \sim_F [y]_F = [x \sim y]_F,$$

$$[x]_F \leq_F [y]_F \ \text{ if and only if } \ x \to y \in F \ \text{ if and only if } \ [x]_F \to_F [y]_F = [1]_F.$$

We write $[x]$ instead of $[x]_F$, for short.

THEOREM 2.17 ([4]). *Let $F$ be a filter of an $\ell EQ$-algebra $\mathcal{E}$. Then the quotient algebra $\mathcal{E}/F = (E/F, \wedge_F, \otimes_F, \sim_F, F)$ is a separated $\ell EQ$-algebra and the mapping $f : x \to [x]_F$ is an epimorphism.*

## 3. $n$-fold pseudo implicative prefilters of $EQ$-algebras

In this section, we introduce the notions of $n$-fold prefilters and $n$-fold pseudo implicative prefilters on $EQ$-algebras and prove some related results. Also, we prove that the quotient algebra modulo by 1-fold pseudo implicative filter is a good $EQ$-algebra.

In what follows, let $n$ denotes a positive integer and for any $x \in E$, $x^n$ denotes $x \otimes x \otimes ... \otimes x$, in which $x$ occurs $n$ times and $x^0 = 1$.

DEFINITION 3.1. Let $\mathcal{E}$ be an $EQ$-algebra. A nonempty subset $F \subseteq E$ is called an *$n$-fold prefilter* of $\mathcal{E}$, if for all $x, y \in E$,

$(i)$ $1 \in F$;

$(ii)$ If $x^n, x^n \to y \in F$, then $y \in F$.

An $n$-fold prefilter $F$ is said to be an *$n$-fold filter* of $\mathcal{E}$, if $F$ satisfies $(F3)$.

Obviously, each prefilter is an $n$-fold prefilter. But the converse is not true.

*Example* 3.2. Let $E = \{0, a, b, c, 1\}$ be a chain such that $0 \le a \le b \le c \le 1$. Define the operations $\wedge, \otimes$ and $\sim$ on $E$ as follows:

| $\otimes$ | 0 | a | b | c | 1 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| a | 0 | 0 | 0 | 0 | a |
| b | 0 | 0 | 0 | 0 | b |
| c | 0 | 0 | 0 | 0 | c |
| 1 | 0 | a | b | c | 1 |

| $\sim$ | 0 | a | b | c | 1 |
|---|---|---|---|---|---|
| 0 | 1 | a | 0 | 0 | 0 |
| a | a | 1 | a | a | a |
| b | 0 | a | 1 | b | b |
| c | 0 | a | b | 1 | c |
| 1 | 0 | a | b | c | 1 |

| $\rightarrow$ | 0 | a | b | c | 1 |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 |
| a | a | 1 | 1 | 1 | 1 |
| b | 0 | a | 1 | 1 | 1 |
| c | 0 | a | b | 1 | 1 |
| 1 | 0 | a | b | c | 1 |

$$x \wedge y = \min\{x, y\}.$$

Then $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ is an $EQ$-algebra. Let $F = \{1, c\}$. Then $F$ is an $n$-fold filter of $\mathcal{E}$, for all $n \in \mathbb{N}$.
Let $F = \{1, a\}$. Then $F$ is a 2-fold prefilter of $\mathcal{E}$. Since $a \in F$ and $a \rightarrow b = 1 \in F$ but $b \notin F$, $F$ is not a prefilter of $\mathcal{E}$. Similarly $F = \{1, b\}$ is a 2-fold prefilter but not a prefilter of $\mathcal{E}$.

Definition 3.3. Let $\mathcal{E}$ be an $EQ$-algebra. A nonempty subset $F \subseteq E$ is called an *$n$-fold pseudo implicative prefilter* of $\mathcal{E}$, if for all $x, y, z \in E$,

($i$) $1 \in F$;

($ii$) $x^n \rightarrow (y \rightarrow z) \in F$ and $x^n \rightarrow y \in F$ imply $x^n \rightarrow z \in F$.

*Example* 3.4. Let $E = \{0, a, b, 1\}$ be a chain such that $0 \le a \le b \le 1$. Define the operations $\wedge, \otimes$ and $\sim$ on $E$ as follows:

| $\otimes$ | 0 | a | b | 1 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| a | 0 | a | a | a |
| b | 0 | a | b | 1 |
| 1 | 0 | a | b | 1 |

| $\sim$ | 0 | a | b | 1 |
|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 |
| a | 0 | 1 | a | a |
| b | 0 | a | 1 | 1 |
| 1 | 0 | a | 1 | 1 |

| $\rightarrow$ | 0 | a | b | 1 |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 |
| a | 0 | 1 | 1 | 1 |
| b | 0 | a | 1 | 1 |
| 1 | 0 | a | 1 | 1 |

$$x \wedge y = \min\{x, y\}.$$

Then $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ is an $EQ$-algebra. Let $F = \{1, b\}$. Then $F$ is an $n$-fold filter and $n$-fold pseudo implicative filter of $\mathcal{E}$, for all $n \geq 2$.
If $F = \{1, a\}$, then $F$ is an $n$-fold pseudo implicative prefilter of $\mathcal{E}$, for all $n \geq 2$. Clearly, $F$ is not a filter of $\mathcal{E}$, since $a^2 = a \in F$ and $a^2 \to b = a \to b = 1 \in F$ but $b \notin F$. In addition, we can see that $F$ is not an $n$-fold filter of $\mathcal{E}$.

Obviously each pseudo implicative prefilter of $\mathcal{E}$ is an $n$-fold pseudo implicative prefilter of $\mathcal{E}$, but the converse is not true.

*Example* 3.5. Let $\mathcal{E}$ be an $EQ$-algebra as in Example 3.2. Suppose $F = \{1, c\}$. Then $F$ is a 2-fold pseudo implicative filter of $\mathcal{E}$. Since $a \to (a \to 0) = 1 \in F$ and $a \to a = 1 \in F$ but $a \to 0 = a \notin F$, we get $F$ is not a pseudo implicative filter of $\mathcal{E}$.

Proposition 3.6. Let $\mathcal{E}$ be a good $EQ$-algebra. Then every $n$-fold pseudo implicative prefilter of $\mathcal{E}$ is an $n$-fold prefilter of $\mathcal{E}$.

Proof: Let $x, y \in E$ such that $x^n, x^n \to y \in F$. Then by goodness, $1^n \to x^n, 1^n \to (x^n \to y) \in F$. Hence $1^n \to y = y \in F$. □

*Example* 3.7. Let $\mathcal{E}$ be the $EQ$-algebra as in Example 3.4. Since $b \sim 1 = 1 \neq b$, we have $\mathcal{E}$ is not good. Suppose $F = \{1, a\}$. Then $F$ is an $n$-fold pseudo implicative filter of $\mathcal{E}$, for all $n \geq 2$. Since $a^n = a \in F$ and $a^n \to b = 1 \in F$ but $b \notin F$, we have $F$ is not an $n$-fold filter of $\mathcal{E}$, for all $n \geq 2$.

Corollary 3.8. Let $\mathcal{E}$ be a good $EQ$-algebra. Then every $n$-fold pseudo implicative prefilter of $\mathcal{E}$ is a prefilter of $\mathcal{E}$.

Proof: Let $F$ be an $n$-fold pseudo implicative prefilter of $\mathcal{E}$, $x \in F$ and $x \to y \in F$. Then $1^n \to (x \to y) \in F$ and $1^n \to x \in F$ and so $1^n \to y = y \in F$. Therefore, $F$ is a prefilter of $\mathcal{E}$. □

Proposition 3.9. Let $\mathcal{E}$ be a good $EQ$-algebra. Then $\{1\}$ is an $n$-fold prefilter of $\mathcal{E}$, for all $n \in \mathbb{N}$.

Proof: Let $x^n \in \{1\}$ and $x^n \to y \in \{1\}$, then $1 \to y = y \in \{1\}$ and so $\{1\}$ is an $n$-fold prefilter of $\mathcal{E}$, for all $n \in \mathbb{N}$. Now, let $x \to y \in \{1\}$. Then $x \leq y$ and so $x \otimes z \leq y \otimes z$, for all $z \in E$. Hence $(x \otimes z) \to (y \otimes z) = 1 \in \{1\}$. Therefore, $\{1\}$ is an $n$-fold filter of $\mathcal{E}$. □

*Example* 3.10. Let $\mathcal{E}$ be the $EQ$-algebra as in Example 3.4. Since $b \sim 1 = 1 \neq b$, we get $\mathcal{E}$ is not a good $EQ$-algebra. Since $1^n = 1 \in \{1\}$ and $1^n \to b = 1 \in \{1\}$ but $b \notin \{1\}$, we get $\{1\}$ is not an $n$-fold filter of $\mathcal{E}$, for all $n \in \mathbb{N}$

In the following theorem, we provide some conditions equivalent to the concept of n-fold pseudo implicative filter.

THEOREM 3.11. *Let $\mathcal{E}$ be a residuated $EQ$-algebra, $F$ be a filter of $\mathcal{E}$ and $n \in \mathbb{N}$. Then, for all $x, y, z \in E$ the following conditions are equivalent:*

($i$) *$F$ is an $n$-fold pseudo implicative filter of $\mathcal{E}$;*

($ii$) *$x^n \to x^{2n} \in F$;*

($iii$) *$x^{n+1} \to y \in F$ implies $x^n \to y \in F$;*

($iv$) *$x^n \to (y \to z) \in F$ implies $(x^n \to y) \to (x^n \to z) \in F$.*

PROOF: $(i) \Longrightarrow (ii)$: By Proposition 2.6($iv$), we have $x^n \to (x^n \to x^{2n}) = x^{2n} \to x^{2n} = 1 \in F$. Since $x^n \to x^n = 1 \in F$ by ($i$), we have $x^n \to x^{2n} \in F$.
$(ii) \Longrightarrow (i)$: Let $x^n \to (y \to z) \in F$ and $x^n \to y \in F$. Then by Propositions 2.6 and 2.5($iii$),

$$\begin{aligned}(x^n \to (y \to z)) \otimes (x^n \to y) \otimes x^{2n} &= (x^n \to (y \to z)) \otimes x^n \otimes (x^n \to y) \otimes x^n \\ &\leq (y \to z) \otimes y \leq z.\end{aligned}$$

Thus by Proposition 2.7($i$), $(x^n \to (y \to z)) \otimes (x^n \to y) \leq x^{2n} \to z$ and so $x^{2n} \to z \in F$. Since by assumption, $F$ is a filter of $\mathcal{E}$, we get $x^{2n} \to z \in F$. Also, by Proposition 2.2($iii$), $x^n \to x^{2n} \leq (x^{2n} \to z) \to (x^n \to z)$. Hence by ($ii$), $(x^{2n} \to z) \to (x^n \to z) \in F$, and so $x^n \to z \in F$. Therefore, $F$ is an $n$-fold pseudo implicative filter of $\mathcal{E}$.
$(ii) \Longrightarrow (iii)$: Let $x^{n+1} \to y \in F$. Then by Proposition 2.6($iv$), we have $x^{n+1} \to y = x^n \to (x \to y) \in F$. Since $x^n \leq x$, we have $x^n \to x = 1 \in F$. Hence by ($i$) or equivalently ($ii$), $x^n \to y \in F$.
$(iii) \Longrightarrow (ii)$: By Proposition 2.6($iv$),

$$x^{n+1} \to (x^{n-1} \to x^{2n}) = x^{2n} \to x^{2n} = 1 \in F.$$

Thus by ($iii$), $x^n \to (x^{n-1} \to x^{2n}) \in F$. Also, we have

$$x^{n+1} \to (x^{n-2} \to x^{2n}) = x^{2n-1} \to x^{2n} = x^n \to (x^{n-1} \to x^{2n}) \in F.$$

Hence by $(iii)$, $x^n \to (x^{n-2} \to x^{2n}) \in F$. By repeating this method $n$ times we get

$$x^n \to (x^0 \to x^{2n}) = x^n \to (1 \to x^{2n}) = x^n \to x^{2n} \in F.$$

$(ii) \Longrightarrow (iv)$: Let $x^n \to (y \to z) \in F$. Then by Propositions 2.2$(iii)$, $(iv)$ and 2.7$(iv)$,

$$\begin{aligned} x^n \to (y \to z) &\leq x^n \to ((x^n \to y) \to (x^n \to z)) \\ &= x^n \to (x^n \to ((x^n \to y) \to z)) \\ &= x^{2n} \to ((x^n \to y) \to z) \in F. \end{aligned}$$

Also, we have $x^{2n} \to ((x^n \to y) \to z)) \leq (x^n \to x^{2n}) \to (x^n \to ((x^n \to y) \to z))$. Thus

$$(x^n \to x^{2n}) \to (x^n \to ((x^n \to y) \to z)) \in F.$$

By $(ii)$, since $x^n \to x^{2n} \in F$, we have

$$x^n \to ((x^n \to y) \to z)) = (x^n \to y) \to (x^n \to z) \in F.$$

$(iv) \Longrightarrow (ii)$: Since $x^n \to (x^n \to x^{2n}) = x^{2n} \to x^{2n} = 1 \in F$ by $(iv)$, we get $(x^n \to x^n) \to (x^n \to x^{2n}) \in F$ and so by goodness, $x^n \to x^{2n} \in F$. □

Proposition 3.12. Let $\mathcal{E}$ be an $EQ$-algebra and $F$ be a prefilter of $\mathcal{E}$. If $F$ is an 1-fold pseudo implicative prefilter of $\mathcal{E}$, then for all $x, y \in E$ and $n \in \mathbb{N}$ the following properties hold:

- $(i)$ $((x^n \wedge (x^n \to y)) \to y) \in F$;
- $(ii)$ $((x^n \otimes (x^n \to y)) \to y) \in F$.

Proof: $(i)$: Let $F$ be an 1-fold pseudo implicative prefilter of $\mathcal{E}$. Since $(x^n \wedge (x^n \to y)) \leq x^n \to y, x^n$, we get $((x^n \wedge (x^n \to y)) \to (x^n \to y) = 1 \in F$ and $(x^n \wedge (x^n \to y)) \to x^n = 1 \in F$. Hence, by assumption $(x^n \wedge (x^n \to y)) \to y \in F$.
$(ii)$: By $(i)$, $(x^n \wedge (x^n \to y) \to y) \in F$. Then by Proposition 2.2$(i)$, $x^n \otimes (x^n \to y) \leq x^n \wedge (x^n \to y)$ and so $(x^n \wedge (x^n \to y)) \to y \leq (x^n \otimes (x^n \to y)) \to y$. Hence, $(x^n \otimes (x^n \to y)) \to y \in F$. □

Corollary 3.13. Let $\mathcal{E}$ be an $EQ$-algebra and $F$ be a prefilter of $\mathcal{E}$. If $F$ is an 1-fold pseudo implicative prefilter of $\mathcal{E}$, then $(1 \to x) \to x \in F$, for all $x \in E$.

Proof: By Proposition 3.12($i$), since $1, x \in E$, we have $(1^n \wedge (1^n \to x)) \to x = (1 \to x) \to x \in F$. $\square$

Theorem 3.14. *Let $\mathcal{E}$ be an EQ-algebra and $F$ be a prefilter of $\mathcal{E}$. If $F$ is an 1-fold pseudo implicative filter of $\mathcal{E}$, then $\mathcal{E}/F$ is a good EQ-algebra.*

Proof: By Theorem 2.17, $\mathcal{E}/F$ is a separated $EQ$-algebra. Then by Corollary 3.13, for any $x \in E$, $(1 \to x) \to x \in F$ and so $[1 \to x] \leq [x]$. Thus $[x] \sim [1] \leq [x]$ and by Proposition 2.2($ii$), $[x] \leq [1] \sim [x]$, that is $[1] \sim [x] = [x]$, for all $[x] \in \mathcal{E}/F$. Therefore, $\mathcal{E}/F$ is a good $EQ$-algebra. $\square$

Theorem 3.15. *Let $\mathcal{E}$ be a residuated EQ-algebra and $F$ be a filter of $\mathcal{E}$. Then the following statements are equivalent:*

($i$) *$F$ is an $n$-fold pseudo implicative filter of $\mathcal{E}$;*

($ii$) *$x^m \to (x \to y) \in F$ implies $x^m \to y \in F$, for all $x, y \in F$ and $m \geq n$.*

Proof: ($i$) $\Longrightarrow$ ($ii$): Let $F$ be an $n$-fold pseudo implicative filter of $\mathcal{E}$ and $x^m \to (x \to y) \in F$, for $x, y \in E$. Since $x^m \leq x$, we have $x^m \to x = 1 \in F$ and so by ($i$), $x^m \to y \in F$.
($ii$) $\Longrightarrow$ ($i$): Let $x^n \to (y \to z) \in F$ and $x^n \to y \in F$. Then by Proposition 2.2($iii$), we have

$$x^n \to (y \to z) \leq ((y \to z) \to (x^n \to z)) \to (x^n \to (x^n \to z)),$$

and $x^n \to y \leq (y \to z) \to (x^n \to z)$. Thus $((y \to z) \to (x^n \to z)) \to (x^n \to (x^n \to z)) \in F$ and $(y \to z) \to (x^n \to z) \in F$ and so $x^n \to (x^n \to z) = x^{2n-1} \to (x \to z) \in F$. By ($ii$), we have $x^{2n-1} \to z \in F$. Since $x^{2n-1} \to z = x^{2n-2} \to (x \to z) \in F$, by ($ii$), we obtain $x^{2n-2} \to z \in F$. By repeating this method, we have $x^n \to z \in F$. Therefore, $F$ is an $n$-fold pseudo implicative filter of $\mathcal{E}$. $\square$

Proposition 3.16. Let $\mathcal{E}$ be a residuated $EQ$-algebra and $F$ be a filter of $\mathcal{E}$. If $F$ is an $n$-fold pseudo implicative filter of $\mathcal{E}$, then $F$ is an $n+1$-fold pseudo implicative filter of $\mathcal{E}$.

Proof: Let $F$ be an $n$-fold pseudo implicative filter of $\mathcal{E}$ and $x, y \in E$ such that $x^{n+2} \to y \in F$. Then by Proposition 2.6($iv$), $x^{n+2} \to y = (x^{n+1} \otimes x) \to y = x^{n+1} \to (x \to y) \in F$. Thus by Theorem 3.15($ii$), $x^{n+1} \to y \in F$ and so $F$ is an $n+1$-fold pseudo implicative filter of $\mathcal{E}$. $\square$

By the following example we show that the converse of Proposition 3.16, is not true.

*Example* 3.17. Let $\mathcal{E}$ be the $EQ$-algebra as in Example 3.2. Suppose $F = \{1, c\}$. Then $F$ is a 2-fold pseudo implicative prefilter of $\mathcal{E}$. Since $a \to (a \to 0) = 1 \in F$ and $a \to a = 1 \in F$ but $a \to 0 = a \notin F$, we get $F$ is not a 1-fold pseudo implicative prefilter of $\mathcal{E}$.

Proposition 3.18. Let $F$ and $G$ be two filters of residuated $EQ$-algebra $\mathcal{E}$ such that $F \subseteq G$. If $F$ is an $n$-fold pseudo implicative filter of $\mathcal{E}$, then $G$ is an $n$-fold pseudo implicative filter of $\mathcal{E}$.

Proof: Let $F$ be an $n$-fold pseudo implicative filter of $\mathcal{E}$. Then by Theorem 3.11$(ii)$, $x^n \to x^{2n} \in F$, for all $x \in E$ and so $x^n \to x^{2n} \in G$. Therefore, $G$ is an $n$-fold pseudo implicative filter of $\mathcal{E}$. □

We now define a sequence of subvarieties of the variety of $EQ$-algebras.

Definition 3.19. Let $\mathcal{E}$ be an $EQ$-algebra. Then $\mathcal{E}$ is called an *$EQ_n$-algebra*, if for all $x, y \in E$, $x^n \to y = x^{n+1} \to y$.

*Example* 3.20. Let $\mathcal{E}$ be the $EQ$-algebra as in Example 3.2. Then $\mathcal{E}$ is an $EQ_n$-algebra, for all $n \geq 2$.

Proposition 3.21. In any residuated $EQ_n$-algebra, $n$-fold filters and $n$-fold pseudo implicative filters coincide.

Proof: By Proposition 3.6, each $n$-fold pseudo implicative filter of $\mathcal{E}$ is an $n$-fold filter of $\mathcal{E}$. Let $F$ be an $n$-fold filter of $\mathcal{E}$ and $x^{n+1} \to y \in F$. Then by assumption, $x^n \to y \in F$ and so by Theorem 3.11, $F$ is an $n$-fold pseudo implicative filter of $\mathcal{E}$. □

Proposition 3.22. Let $\mathcal{E}$ be a residuated $EQ$-algebra. Then $\mathcal{E}$ is an $EQ_n$-algebra if and only if $\{1\}$ is an $n$-fold pseudo implicative filter of $\mathcal{E}$.

Proof: Let $\mathcal{E}$ be an $EQ_n$-algebra and $x^{n+1} \to y \in \{1\}$. Then by Definition 3.19, $x^n \to y \in \{1\}$ and so $\{1\}$ is an $n$-fold pseudo implicative filter of $\mathcal{E}$.

Conversely, let $\{1\}$ be an $n$-fold pseudo implicative filter of $\mathcal{E}$. Since

$$1 = x^{2n} \to x^{n+1} = (x^n \otimes x^n) \to x^{n+1} = x^n \to (x^n \to x^{n+1}) \in \{1\}$$

and $x^n \to x^n = 1 \in \{1\}$, then $x^n \to x^{n+1} \in \{1\}$ and so $x^n \leq x^{n+1}$. On the other hands $x^{n+1} = x^n \otimes x \leq x^n$. Hence $x^n = x^{n+1}$. Therefore, $\mathcal{E}$ is an $EQ_n$-algebra. □

Theorem 3.23. *Let $\mathcal{E}$ be a residuated EQ-algebra. Then the following conditions are equivalent:*

$(i)$ *$\mathcal{E}$ is an $EQ_n$-algebra;*

$(ii)$ *$\{1\}$ is an $n$-fold pseudo implicative filter of $\mathcal{E}$;*

$(iii)$ *Each filter of $\mathcal{E}$ is an $n$-fold pseudo implicative filter of $\mathcal{E}$;*

$(iv)$ *$x^{2n} = x^n$, for all $x \in E$*

Proof: $(i) \Longrightarrow (ii)$: By Proposition 3.22, the proof is clear.
$(ii) \Longrightarrow (iii)$: By Proposition 3.18, the proof is clear.
$(iii) \Longrightarrow (i)$: Since $F = \{1\}$ is a filter of $\mathcal{E}$, by $(iii)$ and Proposition 3.22, we have $(i)$.
$(i) \Longrightarrow (iv)$: By $(i)$ or equivalently $(ii)$ and the proof of Proposition 3.22, $x^n = x^{n+1}$. Thus

$$x^{n+2} = x^{n+1} \otimes x = x^n \otimes x = x^{n+1} = x^n.$$

By repeating this method, we have $x^{2n} = x^n$.
$(iv) \Longrightarrow (i)$: Let $x^{2n} = x^n$. Then $x^n \to x^{2n} = 1$ and so $x^n \to x^{2n} \in \{1\}$. Hence by Theorem 3.11$(ii)$, $\{1\}$ is an $n$-fold pseudo implicative filter of $\mathcal{E}$ and by Proposition 3.22, we have $(i)$. □

Theorem 3.24. *Let $\mathcal{E}$ be a residuated EQ-algebra and $F$ be a filter of $\mathcal{E}$. Then $F$ is an $n$-fold pseudo implicative filter of $\mathcal{E}$ if and only if $\mathcal{E}/F$ is an $EQ_n$-algebra.*

Proof: By Theorem 3.11$(ii)$, $F$ is an $n$-fold pseudo implicative filter of $\mathcal{E}$ if and only if $x^n \to x^{2n} \in F$, for all $x \in E$ if and only if $[x]^n \to [x]^{2n} = [x^n \to x^{2n}] = [1]$ if and only if by Theorem 3.23, $\{[1]\}$ is an $n$-fold pseudo implicative filter of $\mathcal{E}/F$ if and only if $\mathcal{E}/F$ is an $EQ_n$-algebra. □

## 4. $n$-fold implicative prefilters in $EQ$-algebras

In this section, we introduce the concept of an $n$-fold implicative prefilters in $EQ$-algebras and investigate some properties of them. We define an $n$-fold implicative $EQ$-algebra and show that in good $EQ$-algebra $\mathcal{E}$ with least element 0 a prefilter $F$ is an $n$-fold implicative prefilter of $\mathcal{E}$ if and only if $\mathcal{E}/F$ is an $n$-fold implicative $EQ$-algebra.

DEFINITION 4.1. Let $\mathcal{E}$ be an $EQ$-algebra. A nonempty subset $F \subseteq E$ is called an *$n$-fold implicative prefilter* of $\mathcal{E}$, if for all $x, y, z \in E$,

(i) $1 \in F$;

(ii) $z \to ((x^n \to y) \to x) \in F$ and $z \in F$ imply $x \in F$.

Obviously each implicative prefilter is an $n$-fold implicative prefilter (for $n = 1$). But the converse is not true.

*Example* 4.2.

($i$) Let $\mathcal{E}$ be the $EQ$-algebra as in Example 3.2. Suppose $F = \{1, c\}$. Then $F$ is a 2-fold implicative prefilter of $\mathcal{E}$. Since $1 \to ((a \to 0) \to a) = 1 \in F$ and $1 \in F$ but $a \notin F$, we get $F$ is not an implicative prefilter of $\mathcal{E}$.

($ii$) According to Example 3.4, if $F = \{1, a, b\}$, then $F$ is an $n$-fold implicative filter of $\mathcal{E}$, for all $n \in \mathbb{N}$ and $F = \{1, a\}$ is not an $n$-fold implicative filter of $\mathcal{E}$, because $1 \to ((b \to 0) \to b) = 1 \in F$ and $1 \in F$ but $b \notin F$.

PROPOSITION 4.3. Let $\mathcal{E}$ be an $EQ$-algebra and $F$ be an $n$-fold implicative prefilter of $\mathcal{E}$. Then $F$ is an $n$-fold prefilter of $\mathcal{E}$, for all $n \in \mathbb{N}$.

PROOF: Let $x^n \in F$ and $x^n \to y \in F$. Since $y \leq 1 \to y$, we have $x^n \to y \leq x^n \to (1 \to y)$ and so $x^n \to (1 \to y) \in F$. Thus $x^n \to ((y^n \to 1) \to y) \in F$. Since $F$ is an $n$-fold implicative prefilter of $\mathcal{E}$ and $x^n \in F$, we have $y \in F$. Hence $F$ is an $n$-fold prefilter of $\mathcal{E}$, for all $n \in \mathbb{N}$. □

In the next example, we show that the converse of Proposition 4.3 is not true.

*Example* 4.4. Let $\mathcal{E}$ be the $EQ$-algebra as in Example 3.4. Suppose $F = \{1, b\}$. Then $F$ is a 2-fold filter of $\mathcal{E}$. Since $1 \to ((a^2 \to 0) \to a) = 1 \in F$ and $1 \in F$ but $a \notin F$, we get $F$ is not a 2-fold implicative filter of $\mathcal{E}$.

THEOREM 4.5. *Let $\mathcal{E}$ be a good $EQ$-algebra with least element* $0$ *and $F$ be a prefilter of $\mathcal{E}$. Then, for all $x, y \in E$ and $n \in \mathbb{N}$ the following statements are equivalent:*

($i$) *$F$ is an $n$-fold implicative prefilter;*

($ii$) *$(x^n \to 0) \to x \in F$ implies $x \in F$;*

($iii$) *$(x^n \to y) \to x \in F$ implies $x \in F$.*

Proof: $(i) \Longrightarrow (iii)$: Let $F$ be an $n$-fold implicative prefilter of $\mathcal{E}$ and $(x^n \to y) \to x \in F$. Then by goodness we have $1 \to ((x^n \to y) \to x) = (x^n \to y) \to x \in F$. Since $1 \in F$, by $(i)$, we get $x \in F$.
$(iii) \Longrightarrow (ii)$: The proof is clear.
$(ii) \Longrightarrow (i)$: Let $x \to ((y^n \to z) \to y) \in F$ and $x \in F$. Since $F$ is a prefilter of $\mathcal{E}$, we get $(y^n \to z) \to y \in F$. Moreover, from $0 \leq z$, we obtain $y^n \to 0 \leq y^n \to z$ and $(y^n \to z) \to y \leq (y^n \to 0) \to y$. Hence $(y^n \to 0) \to y \in F$. Thus by $(ii)$, $y \in F$. Therefore, $F$ is an $n$-fold implicative prefilter of $\mathcal{E}$. □

Proposition 4.6. Let $\mathcal{E}$ be a good $EQ$-algebra with least element 0. If $F$ is an $n$-fold implicative prefilter of $\mathcal{E}$, then $F$ is an $n+1$-fold implicative prefilter of $\mathcal{E}$.

Proof: Let $F$ be an $n$-fold implicative prefilter of $\mathcal{E}$ such that $(x^{n+1} \to 0) \to x \in F$. Then by Proposition 2.2$(iv)$, from $x^{n+1} \leq x^n$ we have $x^n \to 0 \leq x^{n+1} \to 0$ and so $(x^{n+1} \to 0) \to x \leq (x^n \to 0) \to x$. Since $F$ is prefilter, we have $(x^n \to 0) \to x \in F$ and by assumption $x \in F$. Therefore, $F$ is an $n+1$-fold implicative prefilter of $\mathcal{E}$. □

The next example shows that the converse of Proposition 4.6, is not true.

*Example* 4.7. Let $\mathcal{E}$ be the $EQ$-algebra as in Example 3.2. Suppose $F = \{1, c\}$. Then $F$ is a 2-fold implicative prefilter of $\mathcal{E}$. Since $(a \to 0) \to a = 1 \in F$ but $a \notin F$, we get $F$ is not a 1-implicative prefilter of $\mathcal{E}$.

Theorem 4.8. *Let $\mathcal{E}$ be a residuated $EQ$-algebra. Then each $n$-fold implicative filter of $\mathcal{E}$ is an $n$-fold pseudo implicative filter of $\mathcal{E}$.*

Proof: Let $F$ be an $n$-fold implicative filter of $\mathcal{E}$ and $x^{n+1} \to y \in F$. Then by Propositions 2.2$(iii)$ and 2.7$(iv)$, we have

$$\begin{aligned}
&(x^{n+1} \to y)^n \to (x^n \to y)\\
&= (x^{n+1} \to y)^{n-1} \to ((x^{n+1} \to y) \to (x^n \to y))\\
&= (x^{n+1} \to y)^{n-1} \to ((x^{n+1} \to y) \to (x^{n-1} \to (x \to y))\\
&= (x^{n+1} \to y)^{n-1} \to ((x^{n-1} \to ((x^{n+1} \to y) \to (x \to y)))\\
&= (x^{n+1} \to y)^{n-1} \to (x^{n-1} \to ((x \to (x^n \to y)) \to (x \to y)))\\
&\geq (x^{n+1} \to y)^{n-1} \to (x^{n-1} \to ((x^n \to y) \to y))
\end{aligned}$$

$$= (x^{n+1} \to y)^{n-1} \to ((x^n \to y) \to (x^{n-1} \to y))$$
$$= (x^n \to y) \to ((x^{n+1} \to y)^{n-1} \to (x^{n-1} \to y)).$$

Since $x^{n+1} \to y \leq x^{n+1} \to y = x \to (x^n \to y)$, we have $x \otimes (x^{n+1} \to y) \leq x^n \to y$. Then

$$(x^n \to y) \otimes (x^{n+1} \to y)^{n-1} \otimes x^{n-1} \leq (x^n \to y)^2 \otimes (x^{n+1} \to y)^{n-2} \otimes x^{n-2},$$

Hence
$((x^n \to y)^2 \otimes (x^{n+1} \to y)^{n-2} \otimes x^{n-2}) \to y \leq ((x^n \to y) \otimes (x^{n+1} \to y)^{n-1} \otimes x^{n-1}) \to y$ and so
$(x^n \to y)^2 \to ((x^{n+1} \to y)^{n-2} \to (x^{n-2} \to y)) \leq (x^n \to y) \to ((x^{n+1} \to y)^{n-1} \to (x^{n-1} \to y))$, By (4.1), we have

$$(x^{n+1} \to y)^n \to (x^n \to y) \geq (x^n \to y) \to ((x^{n+1} \to y)^{n-1} \to (x^{n-1} \to y)).$$

Then

$$(x^{n+1} \to y)^n \to (x^n \to y)$$
$$\geq (x^n \to y)^2 \to ((x^{n+1} \to y)^{n-2} \to (x^{n-2} \to y)).$$

Hence, by repeating this method $n$-times we get:

$$(x^{n+1} \to y)^n \to (x^n \to y) \geq (x^n \to y)^2 \to ((x^{n+1} \to y)^{n-2} \to (x^{n-2} \to y))$$
$$\vdots$$
$$\geq (x^n \to y)^n \to ((x^{n+1} \to y)^0 \to (x^0 \to y))$$
$$= (x^n \to y)^n \to (1 \to (1 \to y))$$
$$= (x^n \to y)^n \to y.$$

Thus
$$(x^{n+1} \to y)^n \to (((x^n \to y)^n \to y) \to (x^n \to y)) = 1.$$

Since $F$ is an $n$-fold filter of $\mathcal{E}$ and $x^{n+1} \to y \in F$, we get $((x^n \to y)^n \to y) \to (x^n \to y) \in F$. Hence, by Theorem 4.5(*iii*), $x^n \to y \in F$. Therefore, by Theorem 3.11, $F$ is an $n$-fold pseudo implicative filter of $\mathcal{E}$. $\square$

The following example shows that the converse of Theorem 4.8 is not true.

*Example* 4.9. Let $\mathcal{E}$ be the $EQ$-algebra as in Example 3.4. Suppose $F = \{1, a\}$. Then $F$ is a 2-fold pseudo implicative prefilter of $\mathcal{E}$. Since $(b^2 \to 0) \to b = 1 \in F$ but $b \notin F$, we have $F$ is not a 2-fold implicative prefilter of $\mathcal{E}$.

DEFINITION 4.10. Let $\mathcal{E}$ be an $EQ$-algebra. Then $\mathcal{E}$ is called an *$n$-fold implicative $EQ$-algebra*, if for all $x, y \in E$, $(x^n \to y) \to x = x$.

*Example* 4.11.

(*i*) Let $\mathcal{E}$ be the $EQ$-algebra as in Example 3.2. Then $\mathcal{E}$ is an $n$-fold implicative $EQ$-algebra, for all $n \geq 2$.

(*ii*) Let $\mathcal{E}$ be the $EQ$-algebra as in Example 3.4. Since $(a^n \to 0) \to a = 1 \neq a$, we have $\mathcal{E}$ is not an $n$-fold implicative algebra of $\mathcal{E}$, for all $n \in \mathbb{N}$.

PROPOSITION 4.12. Every $n$-fold implicative $EQ$-algebra is an $n+1$-fold implicative $EQ$-algebra.

PROOF: Let $\mathcal{E}$ be an $n$-fold implicative $EQ$-algebra. Then $(x^n \to y) \to x = x$, for all $x, y \in E$. Since $x^{n+1} = x^n \otimes x \leq x^n$, by Proposition 2.2(*iv*), we have $x^n \to y \leq x^{n+1} \to y$ and so $(x^{n+1} \to y) \to x \leq (x^n \to y) \to x = x$. By Proposition 2.2(*ii*), $x \leq (x^{n+1} \to y) \to x$. Hence $(x^{n+1} \to y) \to x = x$ and so $\mathcal{E}$ is an $n+1$-fold implicative $EQ$-algebra. □

The next example shows that the converse of Proposition 4.12, is not true.

*Example* 4.13. Let $\mathcal{E}$ be the $EQ$-algebra as in Example 3.2. Then $\mathcal{E}$ is a 2-fold implicative $EQ$-algebra. Since $(a \to 0) \to a = 1 \neq a$, we get $\mathcal{E}$ is not a 1-fold implicative $EQ$-algebra.

LEMMA 4.14. *In a good $n$-fold implicative $EQ$-algebra concepts of $n$-fold implicative prefilter and $n$-fold prefilter are coincide.*

PROOF: Let $F$ be an $n$-fold implicative prefilter of $\mathcal{E}$. Then by Proposition 4.3, $F$ is an $n$-fold prefilter of $\mathcal{E}$.

Conversely, let $F$ be an $n$-fold prefilter of $\mathcal{E}$ and $(x^n \to y) \to x \in F$. Then by Definition 4.10, $x \in F$. Hence, $F$ is an $n$-fold implicative prefilter of $\mathcal{E}$. □

Proposition 4.15. Let $\mathcal{E}$ be a good $EQ$-algebra with least element 0. Then the following statements are equivalent:

($i$) $\mathcal{E}$ is an $n$-fold implicative $EQ$-algebra.

($ii$) Every $n$-fold prefilter of $\mathcal{E}$ is an $n$-fold implicative prefilter of $\mathcal{E}$.

($iii$) $\{1\}$ is an $n$-fold implicative prefilter of $\mathcal{E}$.

Proof: $(i) \Longrightarrow (ii)$: By Lemma 4.14, the proof is clear.
$(ii) \Longrightarrow (iii)$: By Proposition 3.9, the proof is clear.
$(iii) \Longrightarrow (i)$: Let $\{1\}$ be an $n$-fold implicative prefilter of $\mathcal{E}$, $x \in E$ and $t = ((x^n \to 0) \to x) \to x$. Then by Propositions 2.2($iii$) and 2.7($iv$), we have:

$$\begin{aligned}
(t^n \to 0) \to t &= (t^n \to 0) \to (((x^n \to 0) \to x) \to x) \\
&= ((x^n \to 0) \to x) \to ((t^n \to 0) \to x) \\
&\geq (t^n \to 0) \to (x^n \to 0) \\
&\geq x^n \to t^n.
\end{aligned}$$

By Proposition 2.2($ii$), $x \leq (x^n \to 0) \to x = t$ and so $x^n \leq t^n$. Hence $(t^n \to 0) \to t = 1 \in \{1\}$. Then by ($iii$), $t = ((x^n \to 0) \to x) \to x \in \{1\}$ and so $(x^n \to 0) \to x \leq x$. By Proposition 2.2($ii$), $x \leq (x^n \to 0) \to x$. Thus $(x^n \to 0) \to x = x$, for all $x \in E$. Therefore, $\mathcal{E}$ is an $n$-fold implicative $EQ$-algebra. □

Theorem 4.16. *Let $\mathcal{E}$ be a good EQ-algebra with least element* 0 *and $F$ be a prefilter of $\mathcal{E}$. Then $F$ is an $n$-fold implicative prefilter of $\mathcal{E}$ if and only if $\mathcal{E}/F$ is an $n$-fold implicative EQ-algebra.*

Proof: Let $F$ be an $n$-fold implicative prefilter of $\mathcal{E}$ and $x \in E$ such that $([x]^n \to [0]) \to [x] = [1]$. Then $(x^n \to 0) \to x \in F$. Thus by Theorem 4.5, $x \in F$, and so $[x] = [1]$. Hence, $\{[1]\}$ is an $n$-fold implicative prefilter of $\mathcal{E}/F$. Therefore, by Proposition 4.15, $\mathcal{E}/F$ is an $n$-fold implicative $EQ$-algebra.

Conversely, let $\mathcal{E}/F$ be an $n$-fold implicative $EQ$-algebra and $x \in E$ such that $(x^n \to 0) \to x \in F$. Then $[x] = ([x]^n \to [0]) \to [x] = [(x^n \to 0) \to x] = [1]$ and so $[x] = [1]$, that is $x \in F$. Therefore, $F$ is an $n$-fold implicative prefilter of $\mathcal{E}$. □

Corollary 4.17. Let $F$ and $G$ be prefilters of good $EQ$-algebra $\mathcal{E}$ with least element $0$ such that $F \subseteq G$ and $F$ be an $n$-fold implicative prefilter of $\mathcal{E}$. Then $G$ is an $n$-fold implicative prefilter of $\mathcal{E}$.

Proof: Let $x \in E$ such that $(x^n \to 0) \to x \in G$. Since $F$ is an $n$-fold implicative prefilter of $\mathcal{E}$, by Theorem 4.16 we have $\mathcal{E}/F$ is an $n$-fold implicative $EQ$-algebra. Then $[(x^n \to 0) \to x] = ([x]^n \to [0]) \to [x] = [x]$ and so $((x^n \to 0) \to x) \to x \in F \subseteq G$. Hence, by assumption, $x \in G$. Therefore, $G$ is an $n$-fold implicative prefilter of $\mathcal{E}$. □

## 5. $n$-fold obstinate prefilters in EQ-algebras

In this section, we introduce the concept of $n$-fold obstinate prefilters in $EQ$-algebras and investigate some properties. We also show that, a filter $\{1\}$ is an $n$-fold obstinate filter of residuated $EQ$-algebra $\mathcal{E}$ if and only if every filter of $\mathcal{E}$ is an $n$-fold obstinate filter of $\mathcal{E}$ and in each residuated $EQ$-algebra $\mathcal{E}$, a filter $F$ is an $n$-fold obstinate filter of $\mathcal{E}$ if and only if every filter of quotient algebra $\mathcal{E}/F$ is an $n$-fold obstinate filter of $\mathcal{E}$.

Definition 5.1. Let $F$ be a prefilter of $EQ$-algebra $\mathcal{E}$. Then $F$ is called an *$n$-fold obstinate prefilter* of $\mathcal{E}$, if $x, y \notin F$ implies $x^n \to y \in F$ and $y^n \to x \in F$.

*Example* 5.2.

(i) Let $\mathcal{E}$ be the $EQ$-algebra as in Example 3.2. Suppose $F = \{1, c\}$. Then $F$ is an $n$-fold obstinate filter of $\mathcal{E}$, for all $n \geq 2$.

(ii) Let $\mathcal{E}$ be the $EQ$-algebra as in Example 3.4. Suppose $F = \{1, b\}$. Then $F$ is a filter and $n$-fold filter of $\mathcal{E}$, for all $n \in \mathbb{N}$. Since $a, 0 \notin F$ and $a^n \to 0 = a \to 0 = 0 \notin F$, we get $F$ is not an $n$-fold obstinate filter of $\mathcal{E}$, for all $n \geq 2$.

Proposition 5.3. Let $\mathcal{E}$ be an $EQ$-algebra. Then every $n$-fold obstinate prefilter is an $n+1$-fold obstinate prefilter of $\mathcal{E}$.

Proof: Let $F$ be an $n$-fold obstinate prefilter of $\mathcal{E}$ and $x, y \notin F$. Then $x^n \to y, y^n \to x \in F$. Since $x^{n+1} \leq x^n$ by Proposition 2.2($ii$), $x^n \to y \leq x^{n+1} \to y$. Thus $x^{n+1} \to y \in F$ and similarly $y^{n+1} \to x \in F$. Therefore, $F$ is an $n+1$-fold prefilter of $\mathcal{E}$. □

The next example shows that the converse of Proposition 5.3, is not true.

*Example* 5.4. Let $\mathcal{E}$ be the $EQ$-algebra as in Example 3.2. Suppose $F = \{1, c\}$. Then $F$ is a 2-fold obstinate filter of $\mathcal{E}$. Since $0, a \notin F$, we have $a \to 0 = a \notin F$. Thus $F$ is not a 1-fold obstinate filter of $\mathcal{E}$.

THEOREM 5.5. *Let $\mathcal{E}$ be an $EQ$-algebra with least element 0 and $F$ be a prefilter of $\mathcal{E}$. Then $F$ is an $n$-fold obstinate prefilter if and only if $x \in F$ or $(\neg(x^n))^m \in F$, for all $x \in E$ and some $m \in \mathbb{N}$.*

PROOF: Let $F$ be an $n$-fold obstinate prefilter of $\mathcal{E}$ such that $x \notin F$. Since $F$ is a filter of $\mathcal{E}$, we have $0 \notin F$. Then $\neg(x^n) = x^n \to 0 \in F$ and $0^n \to x \in F$. Hence, for $m = 1$ we have, $(\neg(x^n))^m \in F$.

Conversely, let $x, y \notin F$. Then $(\neg(x^n))^m \in F$ and $(\neg(y^n))^k \in F$, for some $m, k \in \mathbb{N}$. Thus by Proposition 2.2($i$), $(\neg(x^n))^m \leq \neg(x^n)$ and $(\neg(y^n))^k \leq \neg(y^n)$ and so $\neg(x^n), \neg(y^n) \in F$. By Proposition 2.2($iv$), $x^n \to 0 \leq x^n \to y$ and $y^n \to 0 \leq y^n \to x$. Hence, $x^n \to y, y^n \to x \in F$. Therefore, $F$ is an $n$-fold obstinate prefilter of $\mathcal{E}$. $\square$

THEOREM 5.6. *Let $\mathcal{E}$ be a residuated $EQ$-algebra with least element 0 and $F$ be a filter of $\mathcal{E}$. Then the following statements are equivalent:*

($i$) *$F$ is a maximal filter of $\mathcal{E}$;*

($ii$) *For any $x \notin F$, there exists $n \in \mathbb{N}$ such that $\neg(x^n) \in F$.*

PROOF: ($i$) $\Rightarrow$ ($ii$): Let $F$ be a maximal filter of $\mathcal{E}$ and $x \notin F$. Then $< F \cup \{x\} >= E$ and so $0 \in< F \cup \{x\} >$. Thus $x \to 0 \in F$. Hence, $\neg x \in F$.
($ii$) $\Rightarrow$ ($i$): Let $G$ be a proper filter of $\mathcal{E}$ such that $F \subsetneq G$. Then there exists $x \in G$ such that $x \notin F$. Thus, there exists $n \in \mathbb{N}$ such that $\neg(x^n) \in F$ or $x \to (x \to (...(x \to 0)...)) \in F \subsetneq G$. Since $G$ is a filter of $\mathcal{E}$, we get $0 \in G$. Hence $G = E$, which is a contradiction. Therefore, $F$ is a maximal filter of $\mathcal{E}$. $\square$

COROLLARY 5.7. Let $\mathcal{E}$ be a residuated $EQ$-algebra with least element 0. Then every proper $n$-fold obstinate filter of $\mathcal{E}$ is a maximal filter of $\mathcal{E}$, for all $n \in \mathbb{N}$.

PROOF: Let $F$ be an $n$-fold obstinate filter of $\mathcal{E}$ and $G$ be a filter of $\mathcal{E}$ such that $F \subseteq G \subseteq E$. If $F \neq G$, then there exists $x \in G$ such that $x \notin F$. Since

$0, x \notin F$, by assumption $x^n \to 0 \in F$ and so $\neg(x^n) \in G$. Hence $0 \in G$ and so $G = E$. Therefore, $F$ is a maximal filter of $\mathcal{E}$. $\square$

Proposition 5.8. Let $\mathcal{E}$ be an $EQ$-algebra and $F$ be an $n$-fold obstinate prefilter of $\mathcal{E}$. Then $F$ is an $n$-fold implicative prefilter of $\mathcal{E}$.

Proof: Let $F$ be an $n$-fold obstinate prefilter of $\mathcal{E}$ but not an $n$-fold implicative prefilter of $\mathcal{E}$. Then there exist $x, y \in E$ such that $(x^n \to y) \to x \in F$ but $x \notin F$. Let $y \in F$. Since $y \leq x^n \to y$, we have $x^n \to y \in F$ and so $x \in F$, which is a contradiction. If $y \notin F$, then by assumption $x^n \to y \in F$ and so $x \in F$, which is a contradiction. Therefore, $F$ is an $n$-fold implicative prefilter of $\mathcal{E}$. $\square$

The following example shows that the converse of Proposition 5.8, is not true.

*Example* 5.9. Let $\mathcal{E}$ be the $EQ$-algebra as in Example 3.4. Suppose $F = \{1, a, b\}$. Then $F$ is an $n$-fold implicative filter of $\mathcal{E}$, for all $n \geq 2$. Since $0, a \notin F$ and $a^n \to 0 = 0 \notin F$, we get $F$ is not an $n$-fold obstinate filter of $\mathcal{E}$.

Theorem 5.10. *Let $\mathcal{E}$ be a residuated $EQ$-algebra and $F$ be an $n$-fold obstinate filter of $\mathcal{E}$. Then $F$ is an $n$-fold pseudo implicative filter of $\mathcal{E}$.*

Proof: By Theorem 4.8 and Proposition 5.8, the proof is clear. $\square$

The following example shows that the converse of Theorem 5.10, is not true.

*Example* 5.11. Let $\mathcal{E}$ be the $EQ$-algebra as in Example 3.4. Suppose $F = \{1, a\}$. Then $F$ is an $n$-fold pseudo implicative filter of $\mathcal{E}$. Since $0, b \notin F$ and $b^n \to 0 = 0 \notin F$, we have $F$ is not an $n$-fold obstinate filter of $\mathcal{E}$, for all $n \geq 2$.

Proposition 5.12. Filter $\{1\}$ is an $n$-fold obstinate filter of residuated $EQ$-algebra $\mathcal{E}$ if and only if every filter of $\mathcal{E}$ is an $n$-fold obstinate filter of $\mathcal{E}$.

Proof: Let $F$ be a filter of $\mathcal{E}$ and $x, y \notin F$. Then $x, y \notin \{1\}$ and so $x^n \to y \in \{1\} \subseteq F$ and $y^n \to x \in \{1\} \subseteq F$. Hence, $F$ is an $n$-fold obstinate filter of $\mathcal{E}$. The proof of the converse is clear. $\square$

THEOREM 5.13. *Let $\mathcal{E}$ be a residuated $EQ$-algebra and $F$ be a filter of $\mathcal{E}$. Then $F$ is an $n$-fold obstinate filter of $\mathcal{E}$ if and only if every filter of quotient algebra $\mathcal{E}/F$ is an $n$-fold obstinate filter of $\mathcal{E}/F$.*

PROOF: Let $F$ be an $n$-fold obstinate filter of $\mathcal{E}$ and $x \in E$ such that $[x] \neq [1]$. Then $x \notin F$ and so there exists $m \in \mathbb{N}$ such that $(\neg(x^n))^m \in F$ and so $[(\neg(x^n))^m] \in \{[1]\}$. Hence by Theorem 5.5, $\{[1]\}$ is an $n$-fold obstinate filter of $\mathcal{E}/F$. Therefore, by Proposition 5.12, each filter of the quotient algebra $\mathcal{E}/F$ is an $n$-fold obstinate filter.

Conversely, let every filter of the quotient algebra $\mathcal{E}/F$ be an $n$-fold obstinate filter of $\mathcal{E}/F$ and $x \in E$ such that $x \notin F$. Then $[x] \neq [1]$. Since $\{[1]\}$ is a filter of $\mathcal{E}/F$, by assumption, $\{[1]\}$ is an $n$-fold obstinate filter of $\mathcal{E}$, and so there exists $m \in \mathbb{N}$ such that $[(\neg(x^n))^m] \in \{[1]\}$. Thus $(\neg(x^n))^m \in F$. Hence, by Theorem 5.5, $F$ is an $n$-fold obstinate filter of $\mathcal{E}$. □

## 6. $n$-fold fantastic prefilters in $EQ$-algebras

In this section, we introduce the concept of $n$-fold fantastic prefilters in $EQ$-algebras and investigate some properties about them. Then we prove that in any good $EQ$-algebra, if $F$ is an 1-fold fantastic filter of $\mathcal{E}$ , then $\mathcal{E}/F$ is an $IEQ$-algebra, and we show that in any residuated $EQ$-algebra with least element 0, $F$ is an $n$-fold implicative filter of $\mathcal{E}$ if and only if $F$ is an $n$-fold pseudo implicative filter and $n$-fold fantastic filter of $\mathcal{E}$. So we conclude that in any residuated $EQ$-algebra, $\mathcal{E}$ is an $n$-fold implicative $EQ$-algebra if and only if $\mathcal{E}$ is both $EQ_n$-algebra and $n$-fold fantastic $EQ$-algebra.

DEFINITION 6.1. Let $\mathcal{E}$ be an $EQ$-algebra. A nonempty subset $F \subseteq E$ is called an *$n$-fold fantastic prefilter* of $\mathcal{E}$, if for all $x, y \in E$,

(i) $1 \in F$;

(ii) $z \to (y \to x) \in F$ and $z \in F$, imply $((x^n \to y) \to y) \to x \in F$.

An $n$-fold fantastic prefilter $F$ is said to be an *$n$-fold fantastic filter* if $F$ satisfies in $(F3)$.

*Example* 6.2. $(i)$ Let $\mathcal{E}$ be the $EQ$-algebra as in Example 3.2. Suppose $F = \{1, c\}$. Then $F$ is an $n$-fold fantastic filter of $\mathcal{E}$, for all $n \geq 2$.
$(ii)$ Let $\mathcal{E}$ be the $EQ$-algebra as in Example 3.4. Suppose $F = \{1, b\}$. Since

$1 \to (0 \to a) = 1 \in F$ and $1 \in F$ but $((a^n \to 0) \to 0) \to a = a \notin F$, we get $F$ is not an $n$-fold fantastic prefilter of $\mathcal{E}$, for all $n \in \mathbb{N}$.

THEOREM 6.3. *Let $F$ be a prefilter of good EQ-algebra $\mathcal{E}$. Then $F$ is an $n$-fold fantastic prefilter of $\mathcal{E}$ if and only if $y \to x \in F$ implies $((x^n \to y) \to y) \to x \in F$, for all $x, y \in E$.*

PROOF: Let $F$ be an $n$-fold fantastic prefilter of $\mathcal{E}$ and $y \to x \in F$. Then $1 \to (y \to x) = y \to x \in F$ and $1 \in F$. Hence $((x^n \to y) \to y) \to x \in F$, for all $x, y \in E$.

Conversely, let $z \to (y \to x) \in F$ and $z \in F$. Since $F$ is a prefilter of $\mathcal{E}$, we get $y \to x \in F$ and so $((x^n \to y) \to y) \to x \in F$. Then $F$ is an $n$-fold fantastic prefilter of $\mathcal{E}$. □

PROPOSITION 6.4. Each $n$-fold fantastic prefilter of good $EQ$-algebra $\mathcal{E}$ is an $n$-fold prefilter of $\mathcal{E}$.

PROOF: Let $x^n, x^n \to y \in F$. Then $x^n \to y = x^n \to (1 \to y) \in F$. Since $F$ is an $n$-fold fantastic prefilter of $\mathcal{E}$, we get $((y^n \to 1) \to 1) \to y = y \in F$. □

The next example shows that the converse of Proposition 6.4 is not true and condition of goodness is necessary.

*Example* 6.5. Let $\mathcal{E}$ be the $EQ$-algebra as in Example 3.4.

$(i)$ Suppose $F = \{1, b\}$. Then $F$ is a 2-fold filter of $\mathcal{E}$. Since $1 \to (0 \to a) = 1 \in F$ and $1 \in F$ but $((a^n \to 0) \to 0) \to a = a \notin F$, we get $F$ is not a 2-fold fantastic filter of $\mathcal{E}$.

$(ii)$ Since $b \sim 1 \neq b$, we get $\mathcal{E}$ is not a good $EQ$-algebra. Let $F = \{1, a\}$. Then $F$ is an $n$-fold fantastic filter of $\mathcal{E}$, for all $n \in \mathbb{N}$. Since $a^n = a \in F$ and $a^n \to b = 1 \in F$ but $b \notin F$, we have $F$ is not an $n$-fold filter of $\mathcal{E}$, for all $n \in \mathbb{N}$.

PROPOSITION 6.6. Let $F$ and $G$ be two prefilters of good $EQ$-algebra $\mathcal{E}$ such that $F \subseteq G$. If $F$ is an $n$-fold fantastic prefilter of $\mathcal{E}$, then so is $G$.

PROOF: Let $y \to x \in G$ and $k := (y \to x) \to x$. Then

$$y \to k = y \to ((y \to x) \to x) = (y \to x) \to (y \to x) = 1 \in F.$$

Since $F$ is an $n$-fold fantastic prefilter of $\mathcal{E}$, we have

$$\begin{aligned}(y \to x) \to (((k^n \to y) \to y) \to x) &= ((k^n \to y) \to y) \to ((y \to x) \to x) \\ &= ((k^n \to y) \to y) \to k \in F \subseteq G.\end{aligned}$$

Since $G$ is a filter of $\mathcal{E}$ and $y \to x \in G$, we get $((k^n \to y) \to y) \to x \in G$. Moreover, from $x \leq k = (y \to x) \to x$, we get $k^n \to y \leq x^n \to y$ and so

$$((k^n \to y) \to y) \to x \leq ((x^n \to y) \to y) \to x.$$

Hence $((x^n \to y) \to y) \to x \in G$. Therefore, $G$ is an $n$-fold fantastic prefilter of $\mathcal{E}$. □

Definition 6.7. Let $\mathcal{E}$ be an $EQ$-algebra. Then $\mathcal{E}$ is called an *$n$-fold fantastic $EQ$-algebra*, if for all $x, y \in E$, $((x^n \to y) \to y) \to x = y \to x$.

*Example* 6.8. $(i)$ Let $\mathcal{E}$ be the $EQ$-algebra as in Example 3.2. Then $\mathcal{E}$ is an $n$-fold fantastic $EQ$-algebra, for all $n \geq 2$.
$(ii)$ Let $\mathcal{E}$ be the $EQ$-algebra as in Example 3.4. Since $((a^n \to 0) \to 0) \to a = a \neq 0 \to a = 1$, we have $\mathcal{E}$ is not an $n$-fold fantastic $EQ$-algebra.

Proposition 6.9. Let $\mathcal{E}$ be an $n$-fold fantastic $EQ$-algebra and $F$ be a prefilter of $\mathcal{E}$. Then $F$ is an $n$-fold fantastic prefilter of $\mathcal{E}$

Proof: The proof is clear. □

Theorem 6.10. *Let $\mathcal{E}$ be a residuated $EQ$-algebra $\mathcal{E}$. Then, for all $x, y, z \in E$ the following conditions are equivalent:*

$(i)$ *$\mathcal{E}$ is an $n$-fold fantastic $EQ$-algebra;*

$(ii)$ $(x^n \to y) \to y \leq (y \to x) \to x$;

$(iii)$ *If* $x^n \to z \leq y \to z$ *and* $z \leq x$, *then* $y \leq x$;

$(iv)$ *If* $x^n \to z \leq y \to z$ *and* $z \leq x, y$, *then* $y \leq x$;

$(v)$ *If* $y \leq x$, *then* $(x^n \to y) \to y \leq x$.

Proof: $(i) \Longrightarrow (ii)$: Let $\mathcal{E}$ be an $n$-fold fantastic $EQ$-algebra. Then

$$\begin{aligned}((x^n \to y) \to y) \to ((y \to x) \to x) &= (y \to x) \to (((x^n \to y) \to y) \to x)\\ &= (y \to x) \to (y \to x)\\ &= 1.\end{aligned}$$

Hence by Proposition 2.6, $(x^n \to y) \to y \leq (y \to x) \to x$.
$(ii) \Longrightarrow (i)$: Let $(x^n \to y) \to y \leq (y \to x) \to x$, for all $x, y \in E$. Then

$$\begin{aligned}(y \to x) \to (((x^n \to y) \to y) \to x) &= ((x^n \to y) \to y) \to ((y \to x) \to x)\\ &= 1.\end{aligned}$$

Thus $y \to x \le ((x^n \to y) \to y) \to x$. Also,

$$\begin{aligned}(((x^n \to y) \to y) \to x) \to (y \to x) &\ge y \to ((x^n \to y) \to y)\\ &= (x^n \to y) \to (y \to y)\\ &= (x^n \to y) \to 1\\ &= 1.\end{aligned}$$

Then $(((x^n \to y) \to y) \to x) \to (y \to x) = 1$ and so $(((x^n \to y) \to y) \to x) \le y \to x$. Hence $((x^n \to y) \to y) \to x = y \to x$. Therefore, $\mathcal{E}$ is an $n$-fold fantastic $EQ$-algebra.
$(ii) \Longrightarrow (iii)$: Let $x^n \to z \le y \to z$ and $z \le x$. Then by $(ii)$, we have

$$\begin{aligned}1 = (x^n \to z) \to (y \to z) = y \to ((x^n \to z) \to z) \le y \to ((z \to x) \to x)\\ = y \to (1 \to x)\\ = y \to x.\end{aligned}$$

Thus $y \to x = 1$ and so $y \le x$.
$(iii) \Longrightarrow (iv)$: The proof is clear.
$(iv) \Longrightarrow (v)$: Let $y \le x$. Since $y \le (x^n \to y) \to y$ and

$$\begin{aligned}(x^n \to y) \to (((x^n \to y) \to y) \to y) &= ((x^n \to y) \to y)) \to ((x^n \to y) \to y)\\ &= 1,\end{aligned}$$

we have $x^n \to y \le (((x^n \to y)) \to y) \to y$ and so by $(iv)$, $(x^n \to y) \to y \le x$.
$(v) \Longrightarrow (ii)$: Since $x \le (y \to x) \to x$, by induction we have $((y \to x) \to x)^n \to y \le x^n \to y$ and $(x^n \to y) \to y \le (((y \to x) \to x)^n \to y) \to y$. By Proposition 2.7$(ii)$, we have $y \le (y \to x) \to x$ and by $(v)$ we get

$$(x^n \to y) \to y \le (((y \to x) \to x)^n \to y) \to y \le (y \to x) \to x. \qquad \square$$

Proposition 6.11. Let $\mathcal{E}$ be a residuated $EQ$-algebra. Then $\mathcal{E}$ is an $n$-fold fantastic $EQ$-algebra if and only if $\{1\}$ is an $n$-fold fantastic filter of $\mathcal{E}$.

Proof: Let $\mathcal{E}$ be an $n$-fold fantastic $EQ$-algebra and $y \to x = 1$. Then $((x^n \to y) \to y) \to x = 1$. Hence $\{1\}$ is an $n$-fold fantastic filter of $\mathcal{E}$.

Conversely, let $\{1\}$ be an $n$-fold fantastic filter of $\mathcal{E}$ and $k = (y \to x) \to x$. Then

$$y \to k = y \to ((y \to x) \to x) = (y \to x) \to (y \to x) = 1 \in \{1\},$$

and so $((k^n \to y) \to y) \to k = 1$ that is $(k^n \to y) \to y \leq k$. Since $x \leq k$, we get $k^n \to y \leq x^n \to y$ and $(x^n \to y) \to y \leq (k^n \to y) \to y$. Thus $1 = ((k^n \to y) \to y) \to k \leq ((x^n \to y)) \to y) \to k$. Hence $((x^n \to y) \to y) \to k = 1$. So $((x^n \to y) \to y) \to ((y \to x) \to x) = 1$. Thus $(x^n \to y) \to y \leq (y \to x) \to x$. Therefore, by Theorem 6.10, $\mathcal{E}$ is an $n$-fold fantastic $EQ$-algebra. □

Lemma 6.12. *Each filter of residuated $EQ$-algebra $\mathcal{E}$ is an $n$-fold fantastic filter of $\mathcal{E}$ if and only if $\{1\}$ is an $n$-fold fantastic filter of $\mathcal{E}$.*

Proof: Let $F$ be a filter of $\mathcal{E}$ and $\{1\}$ be an $n$-fold fantastic filter of $\mathcal{E}$. Then by Proposition 6.11, $\mathcal{E}$ is an $n$-fold fantastic $EQ$-algebra and so by Proposition 6.9, $F$ is an $n$-fold fantastic filter of $\mathcal{E}$. The proof of the converse is clear. □

Theorem 6.13. *Let $\mathcal{E}$ be a residuated $EQ$-algebra and $F$ be a filter of $\mathcal{E}$. Then $F$ is an $n$-fold fantastic filter of $\mathcal{E}$ if and only if every filter of $\mathcal{E}/F$ is an $n$-fold fantastic filter of $\mathcal{E}/F$.*

Proof: Let $F$ be an $n$-fold fantastic filter of $\mathcal{E}$ and $[x] \to [y] = [1]$. Then $x \to y \in F$ and so $((y^n \to x) \to x) \to y \in F$. Hence

$$(([y]^n \to [x]) \to [x]) \to [y] = [((y^n \to x) \to x) \to y] = [1].$$

Thus $\{[1]\}$ is an $n$-fold fantastic filter of $\mathcal{E}/F$. By Lemma 6.12, every filter of $\mathcal{E}/F$ is an $n$-fold fantastic filter of $\mathcal{E}/F$.

Conversely, let every filter of $\mathcal{E}/F$ be an $n$-fold fantastic filter of $\mathcal{E}/F$ and let $y \to x \in F$. Then $[y] \to [x] = [y \to x] = [1]$. Since $\{[1]\}$ is an $n$-fold fantastic filter of $\mathcal{E}/F$, we have

$$[((x^n \to y) \to y) \to x] = (([x]^n \to [y]) \to [y]) \to [x] = [1].$$

Hence $((x^n \to y) \to y) \to x \in F$ and so $F$ is an $n$-fold fantastic filter of $\mathcal{E}$. □

Theorem 6.14. *Let $\mathcal{E}$ be a good $EQ$-algebra with least element $0$ and $F$ be a filter of $\mathcal{E}$. If $F$ is an $1$-fold fantastic filter of $\mathcal{E}$, then $\mathcal{E}/F$ is an $IEQ$-algebra.*

Proof: By Theorem 2.17, $\mathcal{E}$ is a good $EQ$-algebra. Since $0 \to x = 1 \in F$, we have

$$((x \to 0) \to 0) \to x = \neg(\neg x) \to x \in F$$

and so $[\neg(\neg x)] \leq [x]$. By Proposition 2.7(*ii*), $[x] \leq [\neg(\neg x)]$. Hence $[\neg(\neg x)] = [x]$ and so $\mathcal{E}/F$ is an $IEQ$-algebra. □

By Theorem 4.8, we see that in residuated $EQ$-algebra such as $\mathcal{E}$, every $n$-fold implicative filter is an $n$-fold pseudo implicative filter, but the converse is not true. Now, we show that under certain conditions an $n$-fold pseudo implicative filter of $\mathcal{E}$ is an $n$-fold implicative filter of $\mathcal{E}$.

Theorem 6.15. *Let $\mathcal{E}$ be a residuated $EQ$-algebra and $F$ be a filter of $\mathcal{E}$. If $F$ is an $n$-fold implicative filter of $\mathcal{E}$, then $F$ is an $n$-fold fantastic filter of $\mathcal{E}$, for all $n \in \mathbb{N}$.*

Proof: Let $F$ be an $n$-fold implicative filter of $\mathcal{E}$ and $y \to x \in F$. Since $x \leq ((x^n \to y) \to y) \to x$, we have $x^n \leq (((x^n \to y) \to y) \to x)^n$ and $(((x^n \to y) \to y) \to x)^n \to y \leq x^n \to y$. Also, we have

$$\begin{aligned} y \to x &\leq ((x^n \to y) \to y) \to ((x^n \to y) \to x) \\ &= (x^n \to y) \to (((x^n \to y) \to y) \to x) \\ &\leq ((((x^n \to y) \to y) \to x)^n \to y) \to (((x^n \to y) \to y) \to x). \end{aligned}$$

Thus

$$((((x^n \to y) \to y) \to x)^n \to y) \to (((x^n \to y) \to y) \to x) \in F$$

and so by Theorem 4.5(*iii*), $((x^n \to y) \to y) \to x \in F$. Therefore, $F$ is an $n$-fold fantastic filter of $\mathcal{E}$. □

Theorem 6.16. *Let $F$ be a filter of residuated $EQ$-algebra $\mathcal{E}$ with least element* 0. *Then $F$ is an $n$-fold implicative filter of $\mathcal{E}$ if and only if $F$ is an $n$-fold pseudo implicative filter and $n$-fold fantastic filter of $\mathcal{E}$.*

Proof: Let $F$ be an $n$-fold pseudo implicative filter and $n$-fold fantastic filter of $\mathcal{E}$ and $(x^n \to 0) \to x \in F$. Since $x^n \to x^{2n} \leq (x^{2n} \to 0) \to (x^n \to 0)$, by Theorem 3.11, we have $x^n \to x^{2n} \in F$ and so $(x^{2n} \to 0) \to (x^n \to 0) \in F$. Also, $F$ is an $n$-fold fantastic filter of $\mathcal{E}$ and $(x^n \to 0) \to x \in F$. Thus

$$((x^n \to (x^n \to 0)) \to (x^n \to 0)) \to x = ((x^{2n} \to 0) \to (x^n \to 0)) \to x \in F.$$

On the other hand, since $(x^{2n} \to 0) \to (x^n \to 0) \in F$ and $F$ is a filter of $\mathcal{E}$, we get $x \in F$. Hence, $F$ is an $n$-fold implicative filter of $\mathcal{E}$.
By Theorems 6.15 and 4.8, the proof of the converse is clear. □

THEOREM 6.17. *Let $\mathcal{E}$ be a residuated $EQ$-algebra. Then $\mathcal{E}$ is an $n$-fold implicative $EQ$-algebra if and only if $\mathcal{E}$ is both $n$-fold pseudo implicative $EQ$-algebra and $n$-fold fantastic $EQ$-algebra.*

PROOF: Let $\mathcal{E}$ be an $n$-fold implicative $EQ$-algebra. Then by Proposition 4.15, $\{1\}$ is an $n$-fold implicative filter of $\mathcal{E}$. Thus by Proposition 3.9, Theorems 6.15 and 4.8, $\{1\}$ is an $n$-fold fantastic filter and pseudo implicative filter of $\mathcal{E}$ and so by Propositions 6.11 and 3.22, $\mathcal{E}$ is both $n$-fold positive implicative $EQ$-algebra and $n$-fold fantastic $EQ$-algebra.

Conversely, let $\mathcal{E}$ be both $n$-fold pseudo implicative $EQ$-algebra and $n$-fold fantastic $EQ$-algebra and $u = x^n \to y$. Then

$$u = x^n \to y = x^{2n} \to y = x^n \to (x^n \to y) = x^n \to u.$$

By Theorem 6.10($ii$), we have

$$((x^n \to y) \to x) \to x = (u \to x) \to x \geq (x^n \to u) \to u = u \to u = 1.$$

Hence $(x^n \to y) \to x \leq x$. By Proposition 2.2($ii$), $x \leq (x^n \to y) \to x$. Thus $(x^n \to y) \to x = x$. Therefore, $\mathcal{E}$ is an $n$-fold implicative $EQ$-algebra. □

## 7. Conclusion

In this paper, the notions of $n$-fold implicative prefilter, $n$-fold pseudo implicative prefilter, $n$-fold fantastic prefilter, $n$-fold obstinate prefilter are introduced and some related results are investigated. At first, equivalent definition of them are studied and the relation between them are investigated. Then by introducing the notions of $n$-fold (pseudo) implicative $EQ$-algebra and $n$-fold fantastic $EQ$-algebra, some related results are studied. In addition, by using the concept of 1-fold pseudo implicative filter of an $EQ$-algebra $\mathcal{E}$, it is shown that $\mathcal{E}/F$ is a good $EQ$-algebra and by using the concept of 1-fold fantastic filter of a good $EQ$-algebra $\mathcal{E}$, it is shown that $\mathcal{E}/F$ is an $IEQ$-algebra.

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## 8. Compliance with ethical standards

**Conflict of interest**: The authors declare that there is no conflict of interest.

**Human and animal rights**: This article does not contain any studies with human participants or animals performed by any of the authors.

## References

[1] N. Akhlaghinia, M. Aaly Kologani, R. A. Borzooei, X. L. Xin, *On the Category of EQ-algebras*, **Bulletin of the Section of Logic**, vol. 50(4) (2021), pp. 397–419, DOI: https://doi.org/10.18778/0138-0680.2021.01.

[2] R. A. Borzooei, B. Ganji Saffar, *States on EQ-Algebras*, **Journal of Intelligent and Fuzzy Systems**, vol. 29 (2015), pp. 209–221, DOI: https://doi.org/10.3233/IFS-151588.

[3] M. El-Zekey, *Representable good EQ-algebras*, **Soft Computing**, vol. 14 (2010), pp. 1011–1023, DOI: https://doi.org/10.1007/s00500-009-0491-4.

[4] M. El-Zekey, V. Novák, R. Mesiar, *On good EQ-algebras*, **Fuzzy Sets and Systems**, vol. 178(1) (2011), pp. 1–23, DOI: https://doi.org/10.1016/j.fss.2011.05.011.

[5] P. Hájek, *Fuzzy logics with noncommutative conjuctions*, **Journal of Logic and Computation**, vol. 13(4) (2003), pp. 469–479, DOI: https://doi.org/10.1093/logcom/13.4.469.

[6] L. Z. Liu, X. Y. Zhang, *Implicative and positive implicative prefilters of EQ-algebras*, **Journal of Intelligent & Fuzzy Systems**, vol. 26(5) (2014), pp. 2087–2097, DOI: https://doi.org/10.3233/IFS-130884.

[7] Z. M. Ma, B. Q. Hu, *Fuzzy EQ-filters of EQ-algebras*, Computer Engineering and Information Science, [in:] **Quantitative Logic and Soft Computing**, World Scientific (2012), pp. 528–535, DOI: https://doi.org/10.1142/9789814401531_0071.

[8] N. Mohtashamniya, L. Torkzadeh, *A prefilter generated by a set in EQ-algebras*, [in:] **46th Annual Iranian Mathematics Conference** (2015), pp. 1–4, URL: https://www.sid.ir/En/Seminar/ViewPaper.aspx?ID=18390.

[9] V. Novák, *EQ-algebra: Primary concepts and properties*, [in:] **International Joint, Czech Republic-Japan & Taiwan-Japan Symposium, Kitakyushu, Japan, August 2006** (2006), pp. 219–223, URL: https://irafm.osu.cz/research_report/101_rep101.pdf.

[10] V. Novák, B. D. Baets, *EQ-algebras*, **Fuzzy Sets and Systems**, vol. 160(20) (2009), pp. 2956–2978, DOI: https://doi.org/10.1016/j.fss.2009.04.010.

[11] W. Wang, X. L. Xin, J. T. Wang, *EQ-algebras with internal states*, **Soft Computing**, vol. 22(20) (2018), pp. 2825–2841, DOI: https://doi.org/10.1007/s00500-017-2754-9.

[12] X. L. Xin, P. F. He, Y. W. Yang, *Characterizations of some fuzzy prefilters (filters) in EQ-algebras*, **The Scientific World Journal**, vol. 2014 (2014), p. 829527, DOI: https://doi.org/10.1155/2014/829527.

[13] X. L. Xin, M. Khan, Y. B. Jun, *Generalized states on EQ-algebras*, **Iranian Journal of Fuzzy Systems**, vol. 16(1) (2019), pp. 159–172, URL: https://www.sid.ir/En/Journal/ViewPaper.aspx?ID=697512.

[14] F. Zebardast, R. A. Borzooei, M. Aaly Kologhani, *Results on equality algebras*, **Information Sciences**, vol. 381 (2017), pp. 270–282, DOI: https://doi.org/10.1016/j.ins.2016.11.027.



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