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A BENCHMARK SIMILARITY MEASURES FOR FERMATEAN FUZZY SETS

Abstract

In this paper, we utilized triangular conorms (S-norm). The essence of using S-norm is that the similarity order does not change using different norms. In fact, we are investigating for a new conception for calculating the similarity of two Fermatean fuzzy sets. For this purpose, utilizing an S-norm, we first present a formula for calculating the similarity of two Fermatean fuzzy values, so that they are truthful in similarity properties. Following that, we generalize a formula for calculating the similarity of the two Fermatean fuzzy sets which prove truthful in similarity conditions. Finally, various numerical examples have been presented to elaborate this method.

Keywords: Fermatean fuzzy set, similarity measure, S-similarity measure.

1. Introduction

The contemporary decision-making theory has been played a remarkable role in the latest research related to multiple-attribute decision-making problems (MADM) with a variety of applications in engineering and science. In real life, the attribute information is fuzzy in nature, sometimes which cannot be expressed through real numbers. Zadeh put forward an ice-breaking conception known as Fuzzy sets [26], these notions deal with uncertainty and precisely characterized the required uncertain data.

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Atanassov [2] initiated the concept of intuitionist fuzzy set (IFS), which is a generalization of Zadeh's fuzzy sets. Another generalization of fuzzy sets is Pythagorean fuzzy sets (PFSs), which was given by Yager [24] as an efficient expansion of intuitionist fuzzy sets. PFS is also characterized by the degree of membership and the degree of non-membership, whose sum of squares is equal to or less than 1. Senapati and Yager [19] introduced the concept of Fermatean fuzzy set (FFS), which is the extension of Pythagorean fuzzy sets, and it covers more space than Intuitionistic fuzzy sets and Pythagorean fuzzy sets. It is an appropriate way to deal with an ambiguous situation and is characterized by a membership degree and a non-membership degree fulfilling the condition that the cube of its membership degree and non-membership is equal to or less than 1. Senapati and Yager [18] studied subtraction, division, and Fermatean arithmetic mean operations over FFS. Many new operations for FFS were defined by Senapati and Yager. Senapati and Yager [17] introduced four new types of weighted aggregation operators for FFS, namely, Fermatean fuzzy weighted average (FFWA) operator, Fermatean fuzzy weighted geometric (FFWG) operator, Fermatean fuzzy weighted power average (FFWPA) operator, and Fermatean fuzzy weighted power geometric (FFWPG) operator. Recently, Aydemir and Gunduz [3] discussed the TOPSIS method in terms of Dombi aggregation operators based on FFSs and gave a complete overview of FF-sets in the framework of Dombi operations. In [10], the authors have extended FFSs into Hamacher operations and investigated the basic properties of FFSs in Hamacher operations. Some practical examples of real-world scenarios were discussed for the validation of the theory. They [11] extended the work of [10] and proposed a set of new aggregation operators in Fermatean fuzzy environment based upon the Hamacher operations with the prioritization of attributes. They developed develop Fermatean fuzzy Hamacher prioritized average (FFHPA) operator, and Fermatean fuzzy Hamacher prioritized weighted average (FFHPWA) operator, Fermatean fuzzy Hamacher prioritized geometric (FFHPG) operator, and Fermatean fuzzy Hamacher prioritized weighted geometric (FFHPWG) operator. They [22] introduced the concept of Fermatean fuzzy matrices, which are direct extensions of an intuitionistic fuzzy matrix. They also put forward some algebraic operations, such as max-min, min-max, complement, algebraic sum, algebraic product, scalar multiplication (nA), and exponentiation (An). Recently, Liu et al. [12] generalized the notion of intuitionistic linguistic fuzzy sets (ILFS) and introduced the Fermatean fuzzy linguistic set (FFLS) theory by integrating the idea of the linguistic term set with FFS. Besides, an MCDM approach was formulated for solving decision problems with FFL information. Further, Liu et al. [13] defined some new distance and similarity measures between FFLSs based on linguistic scale function (LSF) and utilized them in the development of TODIM and TOPSIS methods. They [23] defined several new aggregation operators (AOs), namely, the Fermatean fuzzy linguistic weighted averaging (FFLWA) operator, the Fermatean fuzzy linguistic weighted geometric (FFLWG) operator, the Fermatean fuzzy linguistic ordered weighted averaging (FFLOWA) operator the Fermatean fuzzy linguistic hybrid averaging (FFLHA) operator, and Fermatean fuzzy linguistic hybrid geometric (FFLHG) operator under the FFL environment. Several properties of these AOs are investigated in detail. Recently, they [20] introduced Fermatean fuzzy Hamacher interactive weighted averaging, Fermatean fuzzy Hamacher interactive ordered weighted averaging, and Fermatean fuzzy Hamacher interactive hybrid weighted averaging operators. They [8] developed the theory for the choice of the most suitable laboratory for the COVID-19 test under the Fermatean fuzzy environment. The effectiveness of a sanitizer in COVID-19 was discussed by Akram et al. [1]. In this paper [9], the emerging concept of the Fermatean fuzzy set is studied in detail and three well-known multi-attribute evaluation methods, namely SAW, ARAS, and VIKOR are extended under Fermatean fuzzy environment. They [5] proposed a novel Fermatean fuzzy entropy measure to describe the fuzziness degree of FFSs. The new Fermatean fuzzy entropy takes into account the uncertainty information and the indeterminacy degree of FFSs. The authors [16] extended the TOPSIS using Fermatean Fuzzy Soft Sets to develop a new approach for solving MCGDM problems. They provided an example to show the computational efficiency of the proposed method. In this paper [21], they defined interval-valued Fermatean fuzzy sets (IVFFSs). They also determined algebraic and aggregation operations for interval-valued Fermatean fuzzy (IVFF) numbers. In this paper [15], the authors have proposed three newly improved score functions for the effective ranking of Fermatean fuzzy sets. They have applied the proposed score function to calculate the separation measure of each alternative from the positive and negative ideal solutions to determine the relative closeness coefficient. In this paper [25], they investigated the properties of continuous Fermatean fuzzy information. They defined the subtraction and division operations of Fermatean fuzzy functions and discussed their properties.

Further, they examined the continuity, derivatives, and differentials of Fermatean fuzzy functions. In this paper [4], they defined an operation on Spherical Fermatean fuzzy matrices and discussed their properties. They also proved some new results on Spherical Fermatean fuzzy matrices. Recently, they [14] introduced a hybrid methodology based on CRITIC and EDAS methods with Fermatean fuzzy sets (FFSs). They developed a new improved generalized score function (IGSF) with its desired properties.

The Similarity measures are important and useful tools for determining the degree of similarity between two objects. The definition of a measure of similarity between PFSs is of the most important topics in PFSs theory and compares the information carried by the PFS. Similarity measures between PFSs are an important tool for the MADMP problem, medical diagnosis, decision making, pattern recognition, machine learning, image processing, and other real-world problems. Recently, some researchers have been involved in the development evaluation of the similarity measure of PFSs and their applications. For example, Zhang [28] developed a new method based on the similarity measure to meet various groups of criteria decision-making problems in a Pythagorean fuzzy environment.

The similarity measures established under the umbrella of Fermatean fuzzy sets are extensively and efficiently utilized in many fields mainly including medical diagnosis, water management systems, pattern recognition, signal detection, and security verification. One of the methods of studying two sets is to calculate the similarity of two sets. Triangular norms and connorms generalize the basic connectives between fuzzy sets, intuitionistic fuzzy sets, Pythagorean fuzzy sets, and Feramatean fuzzy sets. In that role, our goal is to propose similarity measures based on norms for FFSs, and some of the basic properties of the new similarity measures were discussed. The motivation to write this article is that we introduce similarity measure and a new similarity measure for Fermatean fuzzy sets. The advantage of using the S-norm is that the order of similarity does not change using different norms. Also, we propose a multi-criteria group decision-making method based on the new similarity measures. The Numerical results indicate that the proposed methods are reasonable and applicable and also that they are well suited in pattern recognition, linguistic variables, and multicriteria decision-making with FFSs over existing methods.

The rest of the article presented is described as follows: In the "Preliminaries" section, we review some definitions and properties. In "S-similarity measure of Fermatean fuzzy sets section", we propose several new similarity measures for FFSs based on norms. In the "Applications" section, we present various numerical examples. In the "Comparative Analysis" section, we compare the proposed similarity measures with the existing similarity measures. At the end conclusion section is provided.

2. Preliminaries

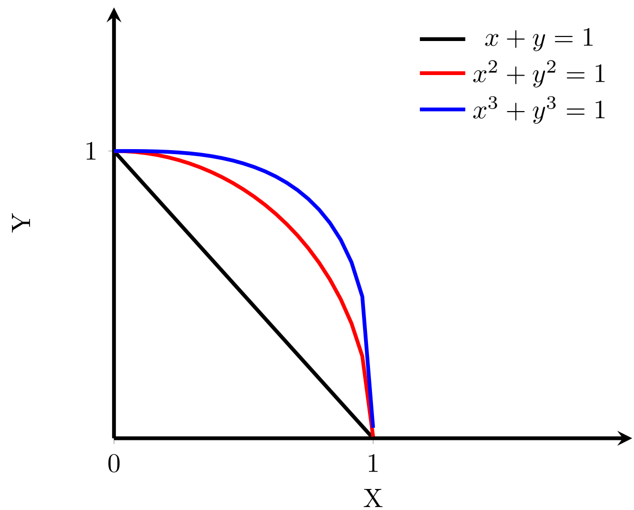
The concept of Intuitionistic fuzzy set brought by Atanassov [2], which is the generalization of the traditional fuzzy set.

Definition 2.1 ([2]). The intuitionistic fuzzy sets defined on a non-empty set X as objects having the form $A = \{\langle x, \zeta_A(x), \eta_A\rangle : x \in X\}$, where the functions $\zeta_A(x) : X \mapsto [0,1]$ and $\eta_A(x) : X \mapsto [0,1]$, denote the degree of membership and the degree of non-membership of each element $x \in X$ to the set A respectively, and $0 \leq \zeta_A(x) + \eta_A(x) \leq 1$ for all $x \in X$.

Definition 2.2 ([24]). Pythagorean fuzzy sets defined on a non-empty set X as objects having the form $P = \{\langle x, \zeta_P(x), \eta_P\rangle : x \in X\}$, where the functions $\zeta_P(x) : X \mapsto [0,1]$ and $\eta_P(x) : X \mapsto [0,1]$, denote the degree of membership and the degree of non-membership of each element $x \in X$ to the set P respectively, and $0 \leq \zeta_P^2(x) + \eta_P^2(x) \leq 1$ for all $x \in X$. For any Pythagorean fuzzy set P and $x \in X$, $\pi_P = \sqrt{1 - (\zeta_P(x))^2 - (\eta_P(x))^2}$ is called the degree of indeterminacy of x to P.

Definition 2.3 ([19]). Fermatean fuzzy sets defined on a non-empty set X as objects having the form $\gamma = \{\langle x, \zeta_\gamma(x), \eta_\gamma\rangle : x \in X\}$, where the functions $\zeta_\gamma(x) : X \mapsto [0,1]$ and $\eta_\gamma(x) : X \mapsto [0,1]$, denote the degree of membership and the degree of non-membership of each element $x \in X$ to the set γ respectively, and $0 \leq \zeta_\gamma^3(x) + \eta_\gamma^3(x) \leq 1$ for all $x \in X$. For any Fermatean fuzzy set γ and $x \in X$, $\pi_\gamma = \sqrt{1 - (\zeta_\gamma(x))^3 - (\eta_\gamma(x))^3}$ is called the degree of indeterminacy of x to γ. For simplicity, we consider the Fermatean fuzzy numbers (FFNs) be the components of the FFS.

From the figure below, it is clear that FFS covers more space, and it is the best tool in dealing with ambiguity compared to Intuitionistic fuzzy set and Pythagorean fuzzy set.



Definition 2.4 ([19]). Let $\gamma_1 = (n_{\gamma_1}, m_{\gamma_1})$ and $\gamma_2 = (n_{\gamma_2}, m_{\gamma_2})$ be two FFNS, a nature quasi-ordering on the FFNs is defined as follows:

$$\gamma_1 \geq \gamma_2 \text{ if and only if } n_{\gamma_1} \geq n_{\gamma_2} \text{ and } m_{\gamma_1} \leq m_{\gamma_2}$$

Definition 2.5 ([6]). Let γ_1, γ_2, and γ_3 be three sets. A real function $s(\cdot,\cdot)$ is called the similarity measure if s satisfies the following properties:

1. $0 \leq s(\gamma_1, \gamma_2) \leq 1$,
2. $s(\gamma_1, \gamma_2) = s(\gamma_2, \gamma_1)$
3. $s(\gamma_1, \gamma_2) = 1 \leftrightarrow \gamma_1 = \gamma_2$
4. If $\gamma_1 \subseteq \gamma_2$, and $\gamma_2 \subseteq \gamma_3$, then $s(\gamma_1, \gamma_3) \leq min\{s(\gamma_1, \gamma_2), s(\gamma_2, \gamma_3)\}$

Definition 2.6 ([7]). A binary operation $S : [0,1] \times [0,1] \mapsto [0,1]$ is a continuous S-norm if it satisfies the following conditions:

(S-1) S is associative and commutative.

(S-2) $S(a, 0) = a$ for all $a \in [0,1]$.

(S-3) $S(a, b) \leq S(c, d)$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0,1]$.

Some of the basic S-norm are as follows:

$$S_1(a,b) = \max\{a,b\}\ S_2(a,b) = \min\{a+b,1\}\ S_3(a,b) = a+b-ab.$$

DEFINITION 2.7 ([19]). Let $\gamma_1 = (n_{\gamma_1}, m_{\gamma_1})$ and $\gamma_2 = (n_{\gamma_2}, m_{\gamma_2})$ be two FFNs. Then the Euclidean distance between γ_1 and γ_2 is:

$$d(\gamma_1,\gamma_2)=\sqrt{\frac{1}{2}[((n_{\gamma_1})^3-(n_{\gamma_2})^3)^2+((m_{\gamma_1})^3-(m_{\gamma_2})^3)^2+((\pi_{\gamma_1})^3-(\pi_{\gamma_2})^3)^2]}.$$

DEFINITION 2.8 ([19]). Let Let $\gamma_1 = (n_{\gamma_1}, m_{\gamma_1})$ and $\gamma_2 = (n_{\gamma_2}, m_{\gamma_2})$ be two FFNs, then

1. $d(\gamma_1, \gamma_2) = d(\gamma_2, \gamma_1)$;
2. $d(\gamma_1, \gamma_2) = 0$ if and only if $\gamma_1 = \gamma_2$;
3. $0 \leq d(\gamma_1, \gamma_2) \leq \sqrt{2}$.

DEFINITION 2.9 ([19]). Let $\gamma = (n_\gamma, m_\gamma)$, $\gamma_1 = (n_{\gamma_1}, m_{\gamma_1})$ and $\gamma_2 = (n_{\gamma_2}, m_{\gamma_2})$ be three FFNs, then their operations are defined as follows:

i $\gamma_1 \cap \gamma_2 = (\min\{n_{\gamma_1}, n_{\gamma_2}\}, \max\{m_{\gamma_1}, m_{\gamma_2}\})$;

ii $\gamma_1 \cup \gamma_2 = (\max\{n_{\gamma_1}, n_{\gamma_2}\}, \min\{m_{\gamma_1}, m_{\gamma_2}\})$;

iii $\gamma^c = (m_\gamma, n_\gamma)$.

3. S-similarity measure of Fermatean fuzzy sets

In this section we introduce a similarity measure and s-similarity measure for FFNS.

DEFINITION 3.1. let $\gamma_i = (\zeta_{\gamma_i}, \eta_{\gamma_i})(i = 1, 2)$ be two FFNs, the similarity measure between γ_1 and γ_2 is defined as follows:

$$sm(\gamma_1,\gamma_2) = \frac{d(\gamma_1,(\gamma_2)^c)}{d(\gamma_1,\gamma_2) + d(\gamma_1,(\gamma_2)^c)} \tag{3.1}$$

where $d(\cdot,\cdot)$ is the distance measure of FFNs and $(\gamma_2)^c = (m_{\gamma_2}, n_{\gamma_2})$ is the complement operation of the FFNs γ_2.

Clearly, by using the definitions of the Fermatean fuzzy distance measure and the complement operation of FFNs, the $sm(\gamma_1, \gamma_2)$ can be described as:

$$\frac{\sqrt{\frac{1}{2}(\Gamma^2 + \Delta^2 + \Theta^2)}}{\sqrt{\frac{1}{2}(\Phi^2 + \Psi^2 + \Theta^2)} + \sqrt{\frac{1}{2}(\Gamma^2 + \Delta^2 + \Theta^2)}}, \tag{3.2}$$

where:

$$\Gamma = (\zeta_{\gamma_1})^3 - (\eta_{\gamma_2})^3,$$

$$\Delta = (\eta_{\gamma_1})^3 - (\zeta_{\gamma_2})^3,$$

$$\Theta = (\pi_{\gamma_1})^3 - (\pi_{\gamma_2})^3,$$

$$\Phi = (\zeta_{\gamma_1})^3 - (\zeta_{\gamma_2})^3,$$

$$\Psi = (\eta_{\gamma_1})^3 - (\eta_{\gamma_2})^3.$$

PROPOSITION 3.2. let $\gamma_i = (\zeta_{\gamma_i}, \eta_{\gamma_i})(i = 1, 2)$ be two FFNs, then $0 \leq sm(\gamma_1, \gamma_2) \leq 1$.

PROOF: Because $0 \leq d(\gamma_1, \gamma_2) \leq \sqrt{2}$ and $0 \leq d(\gamma_1, (\gamma_2)^c) \leq \sqrt{2}$, then

$$sm(\gamma_1, \gamma_2) = \frac{d(\gamma_1, (\gamma_2)^c)}{d(\gamma_1, \gamma_2) + d(\gamma_1, (\gamma_2)^c)} \leq \frac{d(\gamma_1, (\gamma_2)^c)}{d(\gamma_1, (\gamma_2)^c)} = 1. \qquad \square$$

Moreover, by Definition 3.1, it is obvious that $sm(\gamma_1, \gamma_2) \geq 0$. Thus, $0 \leq sm(\gamma_1, \gamma_2) \leq 1$.

PROPOSITION 3.3. let $\gamma_i = (\zeta_{\gamma_i}, \eta_{\gamma_i})(i = 1, 2)$ be two FFNs, then $sm(\gamma_1, \gamma_2) = 1$ if and only if $\gamma_1 = \gamma_2$.

PROOF: If $sm(\gamma_1, \gamma_2) = 1$, then $d(\gamma_1, \gamma_2) = 0$, then we obtain $((\zeta_{\gamma_1})^3 - (\zeta_{\gamma_2})^3)^2 = 0$, $((\eta_{\gamma_1})^3 - (\eta_{\gamma_2})^3)^2 = 0$ and $((\pi_{\gamma_1})^3 - (\pi_{\gamma_2})^3)^2 = 0$. By Definition 2.5, we know that $0 \leq \zeta_{\gamma_1}, \eta_{\gamma_1}, \zeta_{\gamma_2}, \eta_{\gamma_2} \leq 1$, $0 \leq (\zeta_{\gamma_1})^3 + (\eta_{\gamma_1})^3 \leq 1$, and $0 \leq (\zeta_{\gamma_2})^3 + (\eta_{\gamma_2})^3 \leq 1$. Thus, $\zeta_{\gamma_1} = \zeta_{\gamma_2}$, $\eta_{\gamma_1} = \eta_{\gamma_2}$, $\pi_{\gamma_1} = \pi_{\gamma_2}$ i.e., $\gamma_1 = \gamma_2$. On the other hand, if $\gamma_1 = \gamma_2$, by Definition 2.5 we get $\zeta_{\gamma_1} = \zeta_{\gamma_2}$,

$\eta_{\gamma_1} = \eta_{\gamma_2}$. Clearly, $d(\gamma_1, \gamma_2) = 0$, namely, $sm(\gamma_1, \gamma_2) = 1$ which completes the proof. □

PROPOSITION 3.4. let $\gamma_i = (\zeta_{\gamma_i}, \eta_{\gamma_i})(i = 1, 2)$ be two FFNs, then $sm(\gamma_1, \gamma_2) = sm(\gamma_2, \gamma_1)$.

PROOF: By Definition 3.1,

$$
\begin{aligned}
sm(\gamma_2, \gamma_1) =& \frac{d(\gamma_2, (\gamma_1)^c)}{d(\gamma_2, \gamma_1) + d(\gamma_2, (\gamma_1)^c)} \\
=& \frac{\sqrt{\frac{1}{2}((-\Gamma)^2 + (-\Delta)^2 + (-\Theta)^2)}}{\sqrt{\frac{1}{2}((-\Phi)^2 + (-\Psi)^2 + (-\Theta)^2)} + \sqrt{\frac{1}{2}((-\Gamma)^2 + (-\Delta)^2 + (-\Theta)^2)}} \\
=& \frac{\sqrt{\frac{1}{2}(\Gamma^2 + \Delta^2 + \Theta^2)}}{\sqrt{\frac{1}{2}(\Phi^2 + \Psi^2 + \Theta^2)} + \sqrt{\frac{1}{2}(\Gamma^2 + \Delta^2 + \Theta^2)}} \\
=& \frac{d(\gamma_1, (\gamma_2)^c)}{d(\gamma_1, \gamma_2) + d(\gamma_1, (\gamma_2)^c)} = sm(\gamma_1, \gamma_2).
\end{aligned}
$$

Hence it is proved that $sm(\gamma_1, \gamma_2) = sm(\gamma_2, \gamma_1)$. □

DEFINITION 3.5. Let γ_1 and γ_2 be two FFNs, such that $\gamma_1 = (\zeta_{\gamma_1}, \eta_{\gamma_1})$ and $\gamma_2 = (\zeta_{\gamma_2}, \eta_{\gamma_2})$, where $\zeta^3_{\gamma_1} + \eta^3_{\gamma_1} \leq 1$ and $\zeta^3_{\gamma_2} + \eta^3_{\gamma_2} \leq 1$. The s-similarity measure for the FFNs γ_1 and γ_2 represented by $s(\gamma_1, \gamma_2)_S$ is defined as:

$$
s(\gamma_1, \gamma_2)_S = \sqrt[3]{1 - S((\zeta_{\gamma_1} - \zeta_{\gamma_2})^2, (\eta_{\gamma_1} - \eta_{\gamma_2})^2)}, \tag{3.3}
$$

where $S(\cdot, \cdot)$ is a continuous S-norm.

THEOREM 3.6. *Let $\gamma_i = (\zeta_{\gamma_i}, \eta_{\gamma_i})(i = 1, 2, 3)$ be three FFNs. Therefore $s(\cdot, \cdot)_S$ corresponding to the equation (3.3) is satisfied in the following property:*

1. $0 \leq s(\gamma_1, \gamma_2)_S \leq 1$,
2. $s(\gamma_1, \gamma_2)_S = s(\gamma_2, \gamma_1)_S$,
3. $s(\gamma_1, \gamma_2)_S = 1 \leftrightarrow \ \gamma_1 = \gamma_2$,
4. *If $\zeta_{\gamma_1} \leq \zeta_{\gamma_2} \leq \zeta_{\gamma_3}$, and $\eta_{\gamma_1} \geq \eta_{\gamma_2} \geq \eta_{\gamma_3}$, then $s(\gamma_1, \gamma_3)_S \leq$ $\min\{s(\gamma_1, \gamma_2)_S, s(\gamma_2, \gamma_3)_S\}$.*

Proof: We can easily achieve 1, 2 and 3. For 4: If $\zeta_{\gamma_1} \leq \zeta_{\gamma_2} \leq \zeta_{\gamma_3}$, and $\eta_{\gamma_1} \geq \eta_{\gamma_2} \geq \eta_{\gamma_3}$, then

$$(\zeta_{\gamma_1} - \zeta_{\gamma_2})^2 \leq (\zeta_{\gamma_1} - \zeta_{\gamma_3})^2,$$

$$(\zeta_{\gamma_3} - \zeta_{\gamma_2})^2 \leq (\zeta_{\gamma_1} - \zeta_{\gamma_3})^2,$$

$$(\eta_{\gamma_1} - \eta_{\gamma_2})^2 \leq (\eta_{\gamma_1} - \eta_{\gamma_3})^2,$$

$$(\eta_{\gamma_3} - \eta_{\gamma_2})^2 \leq (\eta_{\gamma_1} - \eta_{\gamma_3})^2.$$

Regarding (S-3) of Definition 2.6

$$S((\zeta_{\gamma_1} - \zeta_{\gamma_2})^2, (\eta_{\gamma_1} - \eta_{\gamma_2})^2) \leq S((\zeta_{\gamma_1} - \zeta_{\gamma_3})^2, (\eta_{\gamma_1} - \eta_{\gamma_3})^2),$$

$$S((\zeta_{\gamma_3} - \zeta_{\gamma_2})^2, (\eta_{\gamma_3} - \eta_{\gamma_2})^2) \leq S((\zeta_{\gamma_1} - \zeta_{\gamma_3})^2, (\eta_{\gamma_1} - \eta_{\gamma_3})^2).$$

Hence

$$\sqrt[3]{1 - S((\zeta_{\gamma_1} - \zeta_{\gamma_2})^2, (\eta_{\gamma_1} - \eta_{\gamma_2})^2)} \geq \sqrt[3]{1 - S((\zeta_{\gamma_1} - \zeta_{\gamma_3})^2, (\eta_{\gamma_1} - \eta_{\gamma_3})^2)}$$

$$\sqrt[3]{1 - S((\zeta_{\gamma_3} - \zeta_{\gamma_2})^2, (\eta_{\gamma_3} - \eta_{\gamma_2})^2)} \geq \sqrt[3]{1 - S((\zeta_{\gamma_1} - \zeta_{\gamma_3})^2, (\eta_{\gamma_1} - \eta_{\gamma_3})^2)}.$$

By Definition 3.5

$$s(\gamma_1, \gamma_2)_S \geq s(\gamma_1, \gamma_3)_S, \quad s(\gamma_2, \gamma_3)_S \geq s(\gamma_1, \gamma_3)_S,$$

then

$$s(\gamma_1, \gamma_3)_S \leq \min\{s(\gamma_1, \gamma_2)_S, s(\gamma_2, \gamma_3)_S\},$$ which is the required proof. □

Remark 3.7. $s(\gamma_1, \gamma_2)_{S_2} \leq s(\gamma_1, \gamma_2)_{S_3} \leq s(\gamma_1, \gamma_2)_{S_1}$. It can be easily be proved that

i. $s(\gamma_1, \gamma_2)_{S_1} = \sqrt[3]{1 - \max\{(\zeta_{\gamma_1} - \zeta_{\gamma_2})^2, (\eta_{\gamma_1} - \eta_{\gamma_2})^2\}},$

ii. $s(\gamma_1, \gamma_2)_{S_2} = \sqrt[3]{1 - \min\{(\zeta_{\gamma_1} - \zeta_{\gamma_2})^2 + (\eta_{\gamma_1} - \eta_{\gamma_2})^2, 1\}},$

iii. $s(\gamma_1, \gamma_2)_{S_3} = \sqrt[3]{1 - \{(\zeta_{\gamma_1} - \zeta_{\gamma_2})^2 + (\eta_{\gamma_1} - \eta_{\gamma_2})^2 - (\zeta_{\gamma_1} - \zeta_{\gamma_2})^2(\eta_{\gamma_1} - \eta_{\gamma_2})^2\}}.$

Let $a = (\zeta_{\gamma_1} - \zeta_{\gamma_2})^2$ and $b = (\eta_{\gamma_1} - \eta_{\gamma_2})^2$, then we have $0 \leq a \leq 1, 0 \leq b \leq 1$ and $0 \leq a + b - ab \leq 1, a + b - ab \leq a + b$. Eventually, we have $a + b - ab \leq min\{a + b, 1\}$, that is $s(\gamma_1, \gamma_2)_{S_2} \leq s(\gamma_1, \gamma_2)_{S_3}$.

Moreover, it is obvious that $a \le a+b-ab$ and $b \le a+b-ab$, then $max\{a,b\} \le a+b-ab$ that is $s(\gamma_1,\gamma_2)_{S_3} \le s(\gamma_1,\gamma_2)_{S_1}$.

PROOF: (iii). Let $a=(\zeta_{\gamma_1}-\zeta_{\gamma_2})^2$ and $b=(\eta_{\gamma_1}-\eta_{\gamma_2})^2$, then we have

$$S_3(a,b)=a+b-ab=(\zeta_{\gamma_1}-\zeta_{\gamma_2})^2+(\eta_{\gamma_1}-\eta_{\gamma_2})^2-(\zeta_{\gamma_1}-\zeta_{\gamma_2})^2(\eta_{\gamma_1}-\eta_{\gamma_2})^2.$$

By (3.3), we have

$$s(\gamma_1,\gamma_2)_{S_3}=\sqrt[3]{1-\{(\zeta_{\gamma_1}-\zeta_{\gamma_2})^2+(\eta_{\gamma_1}-\eta_{\gamma_2})^2-(\zeta_{\gamma_1}-\zeta_{\gamma_2})^2(\eta_{\gamma_1}-\eta_{\gamma_2})^2\}}.$$

Hence it is proved. □

DEFINITION 3.8. Let $\kappa_i=(\zeta_i,\eta_i)(i=1,2)$ be two FFSs, a weighted cosine similarity measure between κ_1 and κ_2 is as follows:

$$WFFC^1=\sum_{i=1}^{n} w_i \frac{\zeta^3_{\kappa_1}(x_i)\zeta^3_{\kappa_2}(x_i)+\eta^3_{\kappa_1}(x_i)\eta^3_{\kappa_2}(x_i)}{\sqrt[3]{\zeta_{\kappa_1}(x_i)^4+\eta_{\kappa_1}(x_i)^4}\sqrt[3]{\zeta_{\kappa_2}(x_i)^4+\eta_{\kappa_2}(x_i)^4}}. \tag{3.4}$$

DEFINITION 3.9. Let $\kappa_i=(\zeta_i,\eta_i)(i=1,2)$ be two FFSs, two weighted cosine similarity measure between κ_1 and κ_2 is as follows:

$$WFFCS^1(\kappa_1,\kappa_2)=\sum_{i=1}^{n} w_i \cos\left[\frac{\pi}{2}\left(|\zeta^3_{\kappa_1}(x_i)-\zeta^3_{\kappa_2}(x_i)|\vee|\eta^3_{\kappa_1}(x_i)-\eta^3_{\kappa_2}(x_i)|\right)\right], \tag{3.5}$$

$$WFFCS^2(\kappa_1,\kappa_2)=\sum_{i=1}^{n} w_i \cos\left[\frac{\pi}{4}\left(|\zeta^3_{\kappa_1}(x_i)-\zeta^3_{\kappa_2}(x_i)|\vee|\eta^3_{\kappa_1}(x_i)-\eta^3_{\kappa_2}(x_i)|\right)\right]. \tag{3.6}$$

DEFINITION 3.10. Let $\kappa_i=(\zeta_i,\eta_i)(i=1,2)$ be two FFSs, two weighted cotangent similarity measure between κ_1 and κ_2 is as follows:

$$\begin{aligned}WFFCT^1(\kappa_1,\kappa_2)=\sum_{i=1}^{n} w_i \cot\Bigg[\frac{\pi}{4}+\frac{\pi}{4}\Big(&|\zeta^3_{\kappa_1}(x_i)-\zeta^3_{\kappa_2}(x_i)|\\ &\vee|\eta^3_{\kappa_1}(x_i)-\eta^3_{\kappa_2}(x_i)|\Big)\Bigg],\end{aligned} \tag{3.7}$$

$$WFFCT^2(\kappa_1,\kappa_2) = \sum_{i=1}^{n} w_i \cot\left[\frac{\pi}{4} + \frac{\pi}{8}(|\zeta^3_{\kappa_1}(x_i) - \zeta^3_{\kappa_2}(x_i)| + |\eta^3_{\kappa_1}(x_i) - \eta^3_{\kappa_2}(x_i)|)\right]. \tag{3.8}$$

where $w = \{w_1, w_2, \ldots, w_n\}$ is the weight vector of $x_i (i = 1, 2, \ldots, n)$, with $w_i \in [0, 1] (i = 1, 2, \ldots, n)$, $\sum_{i=1}^{n} w_i = 1$ and the symbol $\vee$ is the maximum operation.

Definition 3.11. Let $U = \{u_1, u_2, \ldots, u_n\}$ denote discourse set and κ_1 and $\kappa_2 \in F(U)$ be two FFSs. $s(\kappa_1, \kappa_2)_S$ denotes the s-similarity measure for the FFSs κ_1 and κ_2.

$$s(\kappa_1, \kappa_2)_S = \sum_{i=1}^{n} w_i s(F_{\kappa_1}(u_i), F_{\kappa_2}(u_i))_S, \tag{3.9}$$

where $w_i > 0$ is the weight of the element $u_i \in U$, $i = 1, 2, \ldots, n$, where $\sum_{i-1}^{n} w_i = 1$ and it depends on what decision maker and $s(\cdot, \cdot)_S$ is defined in (3.3).

Theorem 3.12. *Let $\kappa_i = (\zeta_{\kappa_i}, \eta_{\kappa_i})(i = 1, 2, 3) \in F(U)$ be three FFNs. Therefore $s(\cdot, \cdot)_S$ corresponding to the equation (3.9) is satisfied in the following property:*

1. $0 \le s(\kappa_1, \kappa_2)_S \le 1$,
2. $s(\kappa_1, \kappa_2)_S = s(\kappa_2, \kappa_1)_S$,
3. $s(\kappa_1, \kappa_2)_S = 1 \leftrightarrow \kappa_1 = \kappa_2$,
4. *If* $\kappa_1 \subseteq \kappa_2 \subseteq \kappa_3$, *then* $s(\kappa_1, \kappa_3)_S \le \min\{s(\kappa_1, \kappa_2)_S, s(\kappa_2, \kappa_3)_S\}$.

Proof: It is easy to obtain 1,2 and 3. For 4:

If $\kappa_1 \subseteq \kappa_2 \subseteq \kappa_3$, then $F_{\kappa_1}(u_i) \le F_{\kappa_2}(u_i) \le F_{\kappa_3}(u_i)$; then, with case 4 of Theorem 3.6:

$$s(F_{\kappa_1}(u_i), F_{\kappa_3}(u_i))_S \le \min\{s(F_{\kappa_1}(u_i), F_{\kappa_2}(u_i))_S, s(F_{\kappa_2}(u_i), F_{\kappa_3}(u_i))_S\}.$$

Then

$$\sum_{i=1}^{n} w_i s(F_{\kappa_1}(u_i), F_{\kappa_3}(u_i))_S \le \sum_{i=1}^{n} w_i s(F_{\kappa_1}(u_i), F_{\kappa_2}(u_i))_S,$$

$$\sum_{i=1}^{n} w_i s(F_{\kappa_1}(u_i), F_{\kappa_3}(u_i))_S \leq \sum_{i=1}^{n} w_i s(F_{\kappa_2}(u_i), F_{\kappa_3}(u_i))_S.$$

Moreover, we have

$$s(\kappa_1, \kappa_3)_S \leq s(\kappa_1, \kappa_2)_S, \quad s(\kappa_1, \kappa_3)_S \leq s(\kappa_2, \kappa_3)_S.$$

Finally

$$s(\kappa_1, \kappa_3)_S \leq \min\{s(\kappa_1, \kappa_2)_S, s(\kappa_2, \kappa_3)_S\}.$$

Therefore, $s(\kappa_1, \kappa_3)_S \leq \min\{s(\kappa_1, \kappa_2)_S, s(\kappa_2, \kappa_3)_S\}$ and the proof is completed. □

4. Applications

To provide more insights for the similarity measure for FFSs, in this section, we present two numerical examples.

Example 4.1. Considering a pattern recognition problem about the classification of building materials, three classes of building material are represented by FFSs $\kappa_1, \kappa_2, \kappa_3$ in $X = \{x_1, x_2, x_3\}$, respectively.

$$\kappa_1 = \{(x_1, 0.6, 0.5), (x_2, 0.8, 0.2), (x_3, 0.7, 0.3)\},$$
$$\kappa_2 = \{(x_1, 0.2, 0.4), (x_2, 0.9, 0.3), (x_3, 0.52, 0.55)\},$$
$$\kappa_3 = \{(x_1, 0.66, 0.4), (x_2, 0.8, 0.4), (x_3, 0.6, 0.5)\}.$$

Given another kind of unknown building material $\kappa = \{(x_1, 0.1, 0), (x_2, 0.1, 0), (x_3, 0.1, 0)\}$, our aim is to justify which class the unknown pattern κ belongs to. If we consider the weight of $x_i (i = 1, 2, 3)$, to be 0.5, 0.3, and 0.2, respectively. In Table 1, the similarity measures from κ to $\kappa_i (i = 1, 2, 3)$ are given.

From Table 1, it is clear that the weighted similarity measures between κ_2 and κ are the largest.

Example 4.2. Assume that there are four patterns denoted with FFSs in $X = \{x_1, x_2, x_3, x_4\}$. The four patterns are denoted as follows:

$$\kappa_1 = \{(x_1, 0.6, 0.2), (x_2, 0.3, 0.5), (x_3, 0.7, 0.4), (x_4, 0.4, 0.7)\},$$
$$\kappa_2 = \{(x_1, 0.7, 0.4), (x_2, 0.5, 0), (x_3, 0.9, 0.2), (x_4, 0.3, 0.6)\},$$
$$\kappa_3 = \{(x_1, 0.8, 0.5), (x_2, 0.6, 0), (x_3, 0.3, 0.4), (x_4, 0.5, 0.7)\},$$
$$\kappa_4 = \{(x_1, 0.9, 0.2), (x_2, 0.2, 0.5), (x_3, 0.7, 0), (x_4, 0.1, 0.6)\}.$$

Table 1. Similarity measures between $\kappa_i (i = 1, 2, 3)$ and κ

Similarity measures	(κ_1, κ)	(κ_2, κ)	(κ_3, κ)
$s(\kappa_i, \kappa)_{S_1}$	0.8664	0.8676	0.8625
$s(\kappa_i, \kappa)_{S_2}$	0.7907	0.8247	0.7738
$s(\kappa_i, \kappa)_{S_3}$	0.8163	0.8433	0.8074

Table 2. Similarity measures between $\kappa_i (i = 1, 2, 3, 4)$ and κ

Similarity measures	(κ_1, κ)	(κ_2, κ)	(κ_3, κ)	(κ_4, κ)
$WFFC^1(\kappa_i, \kappa)$	0.0129	0.0187	0.0203	0.0194
$WFFCS^1(\kappa_i, \kappa)$	0.9584	0.9632	0.9108	0.8038
$WFFCS^2(\kappa_i, \kappa)$	0.9839	0.9588	0.9619	0.9427
$WFFCT^1(\kappa_i, \kappa)$	0.7806	0.6897	0.7152	0.5768
$WFFCT^2(\kappa_i, \kappa)$	0.8525	0.8076	0.8001	0.7332
$s(\kappa_i, \kappa)_{S_1}$	*0.9741*	0.9586	0.9580	0.9563
$s(\kappa_i, \kappa)_{S_2}$	*0.9618*	0.9526	0.9419	0.9407
$s(\kappa_i, \kappa)_{S_3}$	*0.9628*	0.9594	0.9445	0.9425

Assume that a sample $\kappa = \{(x_1, 0.5, 0.1), (x_2, 0.5, 0.2), (x_3, 0.4, 0.1), (x_4, 0.4, 0.2)\}$ is given. Given three kinds of mineral fields, each is featured by the content of three minerals and contains one kind of typical hybrid minerals. The four kinds of typical hybrid minerals are represented by FFSs $\kappa_1, \kappa_2, \kappa_3, \kappa_4$ in X, respectively. Given another kind of hybrid mineral κ, to which field does this kind of mineral κ most probably belong to? Our aim is to classify the pattern κ in one of classes $\kappa_i (i = 1, 2, 3, 4)$. For this, the proposed similarity measures which have been computed from κ to $\kappa_i (i = 1, 2, 3, 4)$ are given in Table 2. We consider the weight of $x_i (i = 1, 2, 3, 4)$ to be 0.4,0.3,0.2, and 0.1, respectively.

The results have been added in Table 2 to provide a better view of the comparison results. The italic numbers in Table 2 refer to the most appropriate option. According to the results given in Table 2, it is clear that ranking for similarity measures with different S-norm is as follows:

$$\kappa_4 < \kappa_3 < \kappa_2 < \kappa_1.$$

Meanwhile, similarity measures based on the cosine function between FFSs do not have the same order for ranking.

5. Comparative analysis

In this section, we compare the proposed similarity measures with the existing similarity measures defined in the paper [27].

Definition 5.1. [27]
Let $\kappa_1 = \{\langle x_i, \zeta_{\kappa_1}(x_i), \eta_{\kappa_1}(x_i)|x_i \in X\}$ and $\kappa_2 = \{\langle x_i, \zeta_{\kappa_2}(x_i), \eta_{\kappa_2}(x_i)|x_i \in X\}$ be two FFNs on the domain X, then the similarity measures between κ_1 and κ_2 can be calculated as

$$sm^1(\kappa_1,\kappa_2)=\frac{1}{n}\sum_{i=1}^{n}\frac{\zeta^2_{\kappa_1}(x_i)\zeta^2_{\kappa_2}(x_i)+\eta^2_{\kappa_1}(x_i)\eta^2_{\kappa_2}(x_i)}{\sqrt{\zeta^4_{\kappa_1}(x_i)+\eta^4_{\kappa_1}(x_i)}\sqrt{\zeta^4_{\kappa_2}(x_i)+\eta^4_{\kappa_2}(x_i)}}, \tag{5.1}$$

$$sm^2(\kappa_1,\kappa_2)=\frac{1}{n}\sum_{i=1}^{n}\frac{\zeta^2_{\kappa_1}(x_i)\zeta^2_{\kappa_2}(x_i)+\eta^2_{\kappa_1}(x_i)\eta^2_{\kappa_2}(x_i)+\pi^2_{\kappa_1}(x_i)\pi^2_{\kappa_2}(x_i)}{\sqrt{\zeta^4_{\kappa_1}(x_i)+\eta^4_{\kappa_1}(x_i)+\pi^4_{\kappa_1}(x_i)}\sqrt{\zeta^4_{\kappa_2}(x_i)+\eta^4_{\kappa_2}(x_i)+\pi^4_{\kappa_2}(x_i)}}, \tag{5.2}$$

$$sm^3(\kappa_1,\kappa_2)=\sum_{i=1}^{n}\cos\left[\frac{\pi}{2}\left(|\zeta^2_{\kappa_1}(x_i)-\zeta^2_{\kappa_2}(x_i)|\vee|\eta^2_{\kappa_1}(x_i)-\eta^2_{\kappa_2}(x_i)|\right)\right], \tag{5.3}$$

$$sm^4(\kappa_1,\kappa_2)=\sum_{i=1}^{n}\cos\left[\frac{\pi}{4}\left(|\zeta^2_{\kappa_1}(x_i)-\zeta^2_{\kappa_2}(x_i)|+|\eta^2_{\kappa_1}(x_i)-\eta^2_{\kappa_2}(x_i)|\right)\right], \tag{5.4}$$

$$\begin{aligned}sm^5(\kappa_1,\kappa_2)=\sum_{i=1}^{n}\cos\Bigg[\frac{\pi}{2}(|\zeta^2_{\kappa_1}(x_i)-\zeta^2_{\kappa_2}(x_i)|\\ \vee|\eta^2_{\kappa_1}(x_i)-\eta^2_{\kappa_2}(x_i)|\vee|\pi^2_{\kappa_1}(x_i)-\pi^2_{\kappa_2}(x_i)|)\Bigg],\end{aligned} \tag{5.5}$$

$$\begin{aligned}sm^6(\kappa_1,\kappa_2)=\sum_{i=1}^{n}\cos\Bigg[\frac{\pi}{4}(|\zeta^2_{\kappa_1}(x_i)-\zeta^2_{\kappa_2}(x_i)|+|\eta^2_{\kappa_1}(x_i)\\ -\eta^2_{\kappa_2}(x_i)|+|\pi^2_{\kappa_1}(x_i)-\pi^2_{\kappa_2}(x_i)|)\Bigg].\end{aligned} \tag{5.6}$$

Example 5.2. Let the feature space be $X = \{x_1, x_1, x_3\}$, $\kappa_1 = \{(x_1, 0.2, 0.4), (x_2, 0.7, 0.3), (x_3, 0.4, 0.1)\}$ and $\kappa_2 = \{(x_1, 0.2, 0.4), (x_2, 0.7, 0.3), (x_3, 0.4, 0.13)\}$ are two FFSs on X, respectively. κ_1 and κ_2 are highly similar but not exactly the same. Let the weight of x_i $(i = 1, 2, 3)$ be 0.7, 0.2, and 0.1, respectively. The existing similarity measures and the proposed similarity measures between κ_1 and κ_2 are calculated, respectively. The computed results are as follows:

$$sm^1(\kappa_1,\kappa_2)=1 \quad sm^2(\kappa_1,\kappa_2)=1 \quad sm^3(\kappa_1,\kappa_2)=1$$
$$sm^4(\kappa_1,\kappa_2)=1 \quad sm^5(\kappa_1,\kappa_2)=1 \quad sm^6(\kappa_1,\kappa_2)=1$$
$$s(\kappa_1,\kappa_2)_{S_1}=0.9436 \quad s(\kappa_1,\kappa_2)_{S_2}=0.9436 \quad s(\kappa_1,\kappa_2)_{S_3}=0.9436$$
$$WFFC^1=0.0668 \quad WFFCS^1=0.9999 \quad WFFCS^2=1.$$
$$WFFCT^1=0.9954 \quad WFFCT^2=0.9977$$

The computed results show that existing similarity measures can't distinguish κ_1 and κ_2 from each other, leading to the mistaken conclusion that they're the same. In this case, the existing similarity measures are unreasonable. The proposed similarity measures can distinguish PFSs that are very similar but inconsistent. In conclusion, the new proposed similarity measures are more distinct and can be used to solve a wide range of problems. The MCDM method based on the proposed measures is more accurate and reliable than the MCDM method based on the existing measures in the case of a single decision-maker and multiple decision-makers. The proposed measures are simpler and easier to operate than the existing operators.

6. Conclusion

Since in decision-making process, selecting the best possible course from all of the possible alternatives is a challenging task. We provide an effective approach to select an eligible alternative based on our opinion is its similarity to ideal point. More specifically, we have proposed some similarity measures for FFSs based on S-norms by considering the degree of membership and degree of non-membership. Then, we applied our similarity measures between FFSs to pattern recognition and medical diagnosis using numerical examples. The results show that the proposed similarity measures in FFS environment can suitably handle the real-life decision-making problem. In the future, the application of our proposed similarity measures of FFSs needs to be explored in complex decision making, risk analysis, and many other fields under uncertain environments.

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