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A SEQUENT SYSTEM WITHOUT IMPROPER DERIVATIONS

Abstract

In the natural deduction system for classical propositional logic given by G. Gentzen, there are some inference rules with assumptions discharged by the rule. D. Prawitz calls such inference rules improper, and others proper. Improper inference rules are more complicated and are often harder to understand than the proper ones.

In the present paper, we distinguish between proper and improper derivations by using sequent systems. Specifically, we introduce a sequent system $\vdash_{\mathbf{Sc}}$ for classical propositional logic with only structural rules, and prove that $\vdash_{\mathbf{Sc}}$ does not allow improper derivations in general. For instance, the sequent $\Rightarrow p \to q$ cannot be derived from the sequent $p \Rightarrow q$ in $\vdash_{\mathbf{Sc}}$. In order to prove the failure of improper derivations, we modify the usual notion of truth valuation, and using the modified valuation, we prove the completeness of $\vdash_{\mathbf{Sc}}$. We also consider whether an improper derivation can be described generally by using $\vdash_{\mathbf{Sc}}$.

Keywords: Sequent system, improper derivation, natural deduction.

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1. Introduction

In the natural deduction system for classical propositional logic given in Gentzen [4], there are some inference rules with assumptions discharged by the rule. For instance, the implication introduction rule and the disjunction elimination rule have such assumptions. Prawitz [7] calls such inference

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rules improper, and others proper. The differences between proper and improper inference rules are also pointed out in Fine [3], Robering [8], and Breckenridge and Magidor [1]. However, there is no description allowing to distinguish them by formal systems. In the present paper, we distinguish between proper and improper derivations by using sequent systems. So, we need to confirm what derivations are proper or improper in sequent systems.

In the following three subsections, we provide some preparations, consider what derivations are proper or improper in sequent systems, and describe our purposes in more detail.

1.1. Preliminaries

Here, we provide some preparations.

Formulas are constructed from $\bot$ (contradiction) and the propositional variables by using logical connectives $\wedge$ (conjunction), $\vee$ (disjunction), and $\rightarrow$ (implication) in the usual way. We use p, q, and r, with or without subscripts, for propositional variables, and ϕ, ψ, and χ, with or without subscripts, for formulas. The set of formulas is denoted by **Wff**. We define $\neg\varphi$ as $\varphi \rightarrow \bot$. We assume $\neg$ to connect formulas stronger than $\wedge$ and $\vee$, which in turn are stronger than $\rightarrow$, and omit those parentheses that can be recovered according to this priority of the connectives. Also, we use U and V, with or without subscripts, for sets of formulas, especially we use Greek letters $\Gamma, \Delta, \cdots$, with or without subscripts, for finite sets of formulas.

A *sequent* is the expression $(\Gamma \Rightarrow \varphi)$. We often write

$$\varphi_1, \cdots, \varphi_i, \Gamma_1, \cdots, \Gamma_j \Rightarrow \varphi$$

instead of

$$(\{\varphi_1, \cdots, \varphi_i\} \cup \Gamma_1 \cup \cdots \cup \Gamma_j \Rightarrow \varphi).$$

We use X, Y, and Z, with or without subscripts, for sequents. The *antecedent* $\mathbf{ant}(\Gamma \Rightarrow \varphi)$ and the *succedent* $\mathbf{suc}(\Gamma \Rightarrow \varphi)$ of a sequent $\Gamma \Rightarrow \varphi$ are defined as

$$\mathbf{ant}(\Gamma \Rightarrow \varphi) = \Gamma \quad \text{and} \quad \mathbf{suc}(\Gamma \Rightarrow \varphi) = \varphi,$$

respectively. We use S and T, with or without subscripts, for sets of sequents.

A *sequent system* is defined as a collection comprising a set **Axi** of sequents and a set **Inf** of inference rules of the form

$$\frac{X_1 \quad \cdots \quad X_n}{X}(I).$$

Specifically, a *proof figure* of X from T in the sequent system is defined by means of the set $\mathbf{Axi} \cup T$ as axioms and **Inf** as inference rules in the usual way. We use $\vdash$, with or without subscripts, for sequent systems and write $T \vdash X$ if there exists a proof figure of X from T in $\vdash$. We often call an expression $T \vdash X$ a *derivation*, and identify the above inference rule (I) with the derivation

$$\{X_1, \cdots, X_n\} \vdash X.$$

We write $\vdash X$ and $T, U \vdash X$ instead of $\emptyset \vdash X$ and $T \cup \{\Rightarrow \phi \mid \phi \in U\} \vdash X$, respectively. Also, we write $T \vdash \Gamma \Rightarrow \Delta$ if $T \vdash \Gamma \Rightarrow \psi$ for every $\psi \in \Delta$. We note

$$T \nvdash \Gamma \Rightarrow \Delta \iff T \nvdash \Gamma \Rightarrow \psi \text{ for some } \psi \in \Delta.$$

We say that $\vdash$ is consistent if $\nvdash \bot$.

For a sequent system for classical propositional logic, we use the system $\vdash_{\mathbf{Gc}}$ which corresponds to the natural deduction system in Gentzen [4] and Prawitz [7]. Specifically, we define the system $\vdash_{\mathbf{Gc}}$ as follows.

Definition 1.1. A proof figure of X from T in $\vdash_{\mathbf{Gc}}$ is defined by means of the following axioms and inference rules.
Axioms:

- $\phi \Rightarrow \phi$,
- $\bot \Rightarrow \phi$,
- members of T.

Inference rules: See Figure 1.

We note that, among the inference rules in Figure 1, there are just three inference rules $(\vee \Rightarrow)$, $(\Rightarrow \rightarrow)$, and (RAA) corresponding to the improper ones in the natural deduction system.

A sequent system $\vdash_{\mathbf{S}(S)}$ is defined as follows.

Definition 1.2. A proof figure of X from T in the system $\vdash_{\mathbf{S}(S)}$ is defined by means of the following axioms and inference rules.

$$\frac{\Gamma \Rightarrow \psi}{\phi, \Gamma \Rightarrow \psi}(\mathrm{w} \Rightarrow) \qquad \frac{\Gamma \to \phi \quad \phi, \Gamma \to \psi}{\Gamma \to \psi}(\mathrm{cut})$$

$$\frac{\phi_1, \phi_2, \Gamma \Rightarrow \psi}{\phi_1 \wedge \phi_2, \Gamma \Rightarrow \psi}(\wedge \Rightarrow) \qquad \frac{\Gamma \Rightarrow \phi_1 \qquad \Gamma \Rightarrow \phi_2}{\Gamma \Rightarrow \phi_1 \wedge \phi_2}(\Rightarrow \wedge)$$

$$\frac{\phi_1, \Gamma \Rightarrow \psi \qquad \phi_2, \Gamma \Rightarrow \psi}{\phi_1 \vee \phi_2, \Gamma \Rightarrow \psi}(\vee \Rightarrow) \qquad \frac{\Gamma \Rightarrow \phi_i}{\Gamma \Rightarrow \phi_1 \vee \phi_2}(\Rightarrow \vee)(i = 1, 2)$$

$$\frac{\Gamma \Rightarrow \phi_1 \qquad \phi_2, \Gamma \Rightarrow \psi}{\phi_1 \to \phi_2, \Gamma \Rightarrow \psi}(\to\Rightarrow) \qquad \frac{\phi_1, \Gamma \Rightarrow \phi_2}{\Gamma \Rightarrow \phi_1 \to \phi_2}(\Rightarrow\to)$$

$$\frac{\neg\phi, \Gamma \Rightarrow \bot}{\Gamma \Rightarrow \phi}(\mathrm{RAA})$$

Figure 1. Inference rules in $\vdash_{\mathbf{Gc}}$

Axioms: members of $S \cup T$,
Inference rules: $(w \Rightarrow)$ and (cut).

We write $\vdash_{\mathbf{Sc}}$ instead of $\vdash_{\mathbf{S}(S)}$ if $S = \{X \mid \vdash_{\mathbf{Gc}} X\}$. It will be shown in section 2 and section 3 that $\vdash_{\mathbf{Sc}}$ distinguishes proper and improper derivations.

The system $\vdash_{\mathbf{S(C)}}$ has only structural rules, and all logical content is put into axiomatic sequents. Such systems has been considered in Hertz [5], Suszko [10], Suszko [11], and Schroeder-Heister [9]. We can also see the works by Hertz and Suszko in Indrzejczak [6]. However, a difference between proper and improper derivations is not discussed there.

1.2. Proper and improper derivations in sequent systems

In the present section, we consider what derivations are proper or improper in sequent systems, especially the derivations among the ones in Figure 1. We consider an derivation

$$\mathcal{D} : \{\Gamma_1 \Rightarrow \phi_1, \cdots, \Gamma_n \Rightarrow \phi_n\} \vdash \Gamma \Rightarrow \phi.$$

We note that improper inference rule has an assumption discharged by the rule. Therefore, $\mathcal{D}$ is proper if $\Gamma_1 \cup \cdots \cup \Gamma_n \subseteq \Gamma$, and so, (w$\rightarrow$), ($\Rightarrow \wedge$), and ($\Rightarrow \vee$) are proper.

We consider the case that $\Gamma_1 \cup \cdots \cup \Gamma_n \not\subseteq \Gamma$ by Fine's description ([3], p. 69) below:

> "A proper inference is one that is meant to valid in the standard way; the conclusion is meant to follow straightforwardly from premisses."

In this point of view, three derivations (cut), ($\wedge \rightarrow$), and ($\rightarrow\Rightarrow$) are proper since the succedent of the lower sequent follow straightforwardly from the antecedent as in Figure 2, where $\Gamma = \{\gamma_1, \cdots, \gamma_m\}$. We note that each figure in Figure 2 is a tree satisfying:

(T1) every leaf is either a member of the antecedent of the lower sequent or an empty node,

(T2) except leaves, every node is a formula,

(T3) the root is the succedent of the lower sequent,

(T4) every branch is either $]\!\!\Longrightarrow$ or $]\!\!\longrightarrow$, where

$$\left.\begin{matrix}\phi_1\\ \vdots\\ \phi_n\end{matrix}\right]\!\!\Longrightarrow \psi \iff (\phi_1, \cdots, \phi_n \Rightarrow \psi) \text{ is an upper sequent,}$$

$$\left.\begin{matrix}\phi_1\\ \vdots\\ \phi_n\end{matrix}\right]\!\!\longrightarrow \psi \iff \vdash_{\mathbf{Gc}} (\phi_1, \cdots, \phi_n \Rightarrow \psi).$$

We can also see such trees for ($\Rightarrow \wedge$) and ($\Rightarrow \vee$) in Figure 3, where $\Gamma = \{\gamma_1, \cdots, \gamma_m\}$.

On the other hand, three derivations ($\vee \Rightarrow$), ($\Rightarrow\rightarrow$), and (RAA) are improper since there is no such tree. More precisely, we have

- $\{\phi_1, \Gamma \Rightarrow \psi\} \not\vdash_{\mathbf{Gc}} (\phi_1 \vee \phi_2, \Gamma \Rightarrow \phi_2)$ and $\{\phi_2, \Gamma \Rightarrow \psi\} \not\vdash_{\mathbf{Gc}} (\phi_1 \vee \phi_2, \Gamma \Rightarrow \phi_1)$ if $(\Gamma, \phi_1, \phi_2, \psi) = (\emptyset, p, q, r)$,

- $\not\vdash_{\mathbf{Gc}} \Gamma \Rightarrow \phi_1$ if $(\Gamma, \phi_1) = (\emptyset, p)$,
- $\not\vdash_{\mathbf{Gc}} \Gamma \Rightarrow \neg\phi$ if $(\Gamma, \phi) = (\emptyset, p)$.

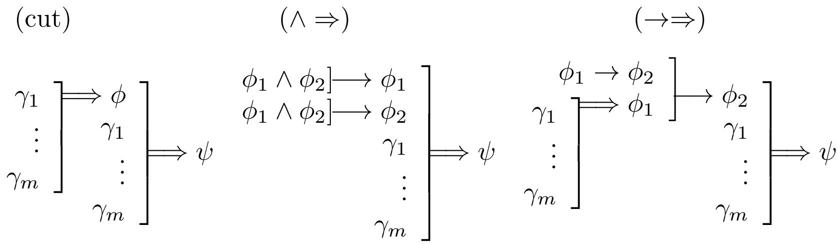

Figure 2. Trees for (cut), $(\wedge \Rightarrow)$, and $(\rightarrow\Rightarrow)$

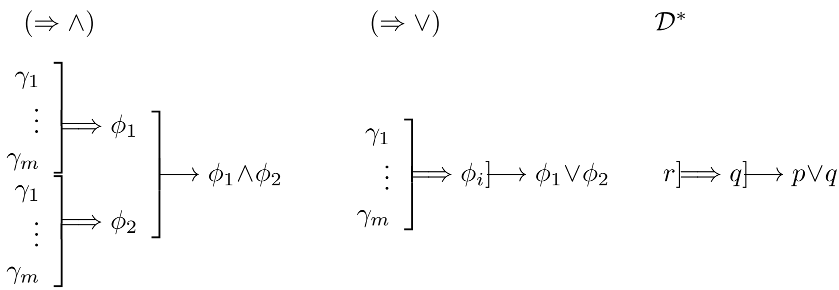

Figure 3. Trees for $(\Rightarrow \wedge)$, $(\Rightarrow \vee)$, and $\mathcal{D}^*$

Consequently, among the derivations in Figure 1, $(\vee \Rightarrow)$, $(\Rightarrow\rightarrow)$, and (RAA) are improper, and the others are proper. In general, $\mathcal{D}$ is proper if $\mathcal{D}$ has a tree satisfying (T1), (T2), (T3), and (T4). Also, for $\psi \in \Gamma_1$, ψ is discharged by $\mathcal{D}$ if the following two conditions hold:

(D1) $\{\Gamma_2 \Rightarrow \phi_2, \cdots, \Gamma_n \Rightarrow \phi_n\} \not\vdash_{\mathbf{Gc}} \Gamma \Rightarrow \psi$,

(D2) $\{\Gamma_2 \Rightarrow \phi_2, \cdots, \Gamma_n \Rightarrow \phi_n\} \not\vdash_{\mathbf{Gc}} \Gamma \Rightarrow \phi$.

Moreover, $\mathcal{D}$ is improper if there exists $\psi \in \Gamma_1$ satisfying (D1) and (D2). Here, we need (D2) since the derivation

$$\mathcal{D}^* : \{p \Rightarrow r, r \Rightarrow q\} \vdash r \Rightarrow p \vee q,$$

which has the tree in Figure 3, should be proper and p in $p \Rightarrow r$ satisfies (D1). We have to note that the meaning of proper and improper derivations has not been clarified yet since there may be a case that the following two conditions hold:

- $\mathcal{D}$ has no tree satisfying (T1), (T2), (T3), and (T4),
- there is no formula $\phi \in \Gamma_i$ satisfying (D1) and (D2).

In section 3, we consider this in more detail by using $\vdash_{\mathbf{Sc}}$.

Here, we also note that the system $\vdash_{\mathbf{S}(S)}$ has only proper structural inference rules, and consequently, it is natural to see that if $T \vdash_{\mathbf{S}(S)} \Gamma \Rightarrow \phi$, then

$$\text{“}\Gamma \Rightarrow \phi \text{ is derived straightforwardly from } T\text{”}. \tag{P1}$$

and the derivation is proper.

1.3. The purposes

In the present paper, we distinguish proper and improper derivations by the sequent system $\vdash_{\mathbf{Sc}}$. Improper derivations are more complicated and are often harder to understand than the proper ones since they have assumptions discharged by the rule and have no tree satisfying (T1), (T2), (T3), and (T4) in the previous subsection. So, if we obtain a system that distinguishes proper and improper derivations, then we know what kind of inference rules are hard to understand. This knowledge is valuable when we teach proof in mathematics education.

In order to distinguish proper and improper derivations, there are two purposes.

One is to prove that our system $\vdash_{\mathbf{Sc}}$ distinguishes the proper and improper derivations among the ones in Figure 1. However, it is not hard to see that the proper derivations in Figure 1 hold in $\vdash_{\mathbf{Sc}}$. So, the main theorem we should prove is as follows.

Theorem 1.3. *None of the improper derivations* $(\vee \Rightarrow)$, $(\Rightarrow\rightarrow)$, *and* (RAA) *holds in* $\vdash_{\mathbf{Sc}}$ *in general.*

We prove the theorem above in the following section by using completeness.

The other is to consider whether an improper derivation can be described generally by using $\vdash_{\mathbf{Sc}}$. As we mentioned in the previous subsection, the description in the subsection is not enough to clarify proper and improper derivations. We consider it in more detail in section 3.

2. Completeness

In the present section, we prove Theorem 1.3. In order to prove the theorem, we modify the usual notion of truth valuation, and using the modified valuation, we prove completeness of the system $\vdash_{\mathbf{Sc}}$. Theorem 1.3 will be obtained as a corollary of the completeness.

The definition of the usual truth valuation is as follows.

Definition 2.1. We say that a mapping $v : \mathbf{Wff} \rightarrow \{\mathsf{t}, \mathsf{f}\}$ is a *truth valuation* if the following conditions hold:

1. $v(\bot) = \mathsf{f}$,
2. $v(\phi \wedge \psi) = \mathsf{t} \iff v(\phi) = v(\psi) = \mathsf{t}$,
3. $v(\phi \vee \psi) = \mathsf{f} \iff v(\phi) = v(\psi) = \mathsf{f}$,
4. $v(\phi \rightarrow \psi) = \mathsf{f} \iff v(\phi) = \mathsf{t}$ and $v(\psi) = \mathsf{f}$.

We use v, with or without subscripts, for truth valuations. We write $v(U) = \mathsf{t}$ if $v(\phi) = \mathsf{t}$ for every $\phi \in U$. Also, we write $v(X) = \mathsf{t}$ if $v(\mathbf{ant}(X)) = \mathsf{f}$ or $v(\mathbf{suc}(X)) = \mathsf{t}$. Moreover, we write $v(T) = \mathsf{t}$ if $v(X) = \mathsf{t}$ for every $X \in T$.

We modify the above definition of truth valuation as follows.

Definition 2.2. Let $\mathbf{v}$ be a set of truth valuations. We define a mapping $\mathbf{v} : \mathbf{Wff} \rightarrow \{\mathsf{t}, \mathsf{f}\}$ as follows:

$$\mathbf{v}(\phi) = \mathsf{t} \iff \text{for every } v \in \mathbf{v},\ v(\phi) = \mathsf{t}.$$

We note that

- $\emptyset(\phi) = \mathsf{t}$,
- $\{v\}(\phi) = v(\phi)$,
- $\{v_1, v_2\}(\phi) = \mathsf{t} \iff v_1(\phi) = v_2(\phi) = \mathsf{t}$.

We write $\mathbf{v}(U)$, $\mathbf{v}(X)$, and $\mathbf{v}(T)$, similarly to $v(U)$, $v(X)$, and $v(T)$, respectively.

The main theorem in the present section is as follows.

THEOREM 2.3. *The following conditions are equivalent:*

(1) $T \vdash_{\mathbf{Sc}} X$,

(2) *for every set* $\mathbf{v}$ *of truth valuations,* $\mathbf{v}(T) = \mathtt{t}$ *implies* $\mathbf{v}(X) = \mathtt{t}$.

In order to prove the above theorem, we provide some preparations. The completeness below can be shown in the usual way. For example, we can refer to Chagrov and Zakharyaschev [2].

LEMMA 2.4.

$$U \vdash_{\mathbf{Gc}} \Rightarrow \phi \iff \textit{for every truth valuation } v, v(U) = \mathtt{t} \textit{ implies } v(\phi) = \mathtt{t}.$$

LEMMA 2.5.

(1) $T \cup \{\Rightarrow \psi\} \vdash_{\mathbf{Gc}} \Gamma \Rightarrow \phi \iff T \vdash_{\mathbf{Gc}} (\psi, \Gamma \Rightarrow \phi)$.

(2) $T \cup \{\Rightarrow \psi\} \vdash_{\mathbf{Sc}} \Gamma \Rightarrow \phi \iff T \vdash_{\mathbf{Sc}} (\psi, \Gamma \Rightarrow \phi)$.

(3) $\vdash_{\mathbf{Gc}} X \iff \vdash_{\mathbf{Sc}} X$.

(4) $U \vdash_{\mathbf{Gc}} X \iff U \vdash_{\mathbf{Sc}} X$.

PROOF: (1), (2), and the direction "$\Longleftarrow$" of (3) can be shown by an induction on a proof figure. The direction "$\Longrightarrow$" of (3) is clear since every member of $\{X \mid \vdash_{\mathbf{Gc}} X\}$ is an axiom of $\vdash_{\mathbf{Sc}}$.

For (4). By (1), (2), and (3), for every finite set U^* of formulas, we have

$$\begin{aligned} U^* \vdash_{\mathbf{Gc}} X &\iff \vdash_{\mathbf{Gc}} (U^*, \mathbf{ant}(X) \Rightarrow \mathbf{suc}(X)) \\ &\iff \vdash_{\mathbf{Sc}} (U^*, \mathbf{ant}(X) \Rightarrow \mathbf{suc}(X)) \\ &\iff U^* \vdash_{\mathbf{Sc}} X. \end{aligned} \tag{4.1}$$

Also, we note that

$$U \vdash_{\mathbf{Gc}} X \iff U^* \vdash_{\mathbf{Gc}} X \text{ for some finite subset } U^* \text{ of } U,$$

and the same equivalence holds in $\vdash_{\mathbf{Sc}}$. Hence, we obtain (4). □

We note that the expression $U \vdash X$ is an abbreviation of $\{\Rightarrow \phi \mid \phi \in U\} \vdash X$. So, none of the improper derivations ($\vee \Rightarrow$), ($\Rightarrow \rightarrow$), and (RAA) can be expressed in the form of $U \vdash X$. On the other hand, some of the proper derivations in Figure 1 can be expressed in the form. For example, the derivation $\{\Rightarrow p\} \vdash \Rightarrow p \vee q$ can be expressed in the form.

By means of this example, we show how a proof figure for $U \vdash_{\mathbf{Gc}} X$ transfer to the one for $U \vdash_{\mathbf{Sc}} X$. Specifically, we show the proof figures, in Table 1, for four derivations occurring in the above (4.1) in Lemma 2.5.

Table 1. Proof figures for derivations in (4.1) in Lemma 2.5

Derivation	Proof figure
$\{\Rightarrow p\} \vdash_{\mathbf{Gc}} \Rightarrow p \vee q$	$\dfrac{\Rightarrow p}{\Rightarrow p \vee q}(\Rightarrow \vee)$
$\vdash_{\mathbf{Gc}} (p \Rightarrow p \vee q)$	$\dfrac{p \Rightarrow p}{p \Rightarrow p \vee q}(\Rightarrow \vee)$
$\vdash_{\mathbf{Sc}} (p \Rightarrow p \vee q)$	$p \Rightarrow p \vee q$
$\{\Rightarrow p\} \vdash_{\mathbf{Sc}} \Rightarrow p \vee q$	$\dfrac{\Rightarrow p \quad p \Rightarrow p \vee q}{\Rightarrow p \vee q}(\mathrm{cut})$

Lemma 2.6. *If* $\vdash_{\mathbf{Sc}} X$*, then for every set* $\mathbf{v}$ *of truth valuations* $\mathbf{v}(X) = \mathsf{t}$.

Proof: By Lemma 2.5 and Lemma 2.4. □

Lemma 2.7. *If* $T \vdash_{\mathbf{Sc}} X$*, then for every set* $\mathbf{v}$ *of truth valuations,* $\mathbf{v}(T) = \mathsf{t}$ *implies* $\mathbf{v}(X) = \mathsf{t}$.

Proof: Suppose that $T \vdash_{\mathbf{Sc}} X$ and $\mathbf{v}(T) = \mathsf{t}$. We show $\mathbf{v}(X) = \mathsf{t}$ by an induction on a proof figure of X from T in $\vdash_{\mathbf{Sc}}$.

Basis. If $X \in T$, then by $\mathbf{v}(T) = \mathsf{t}$, we have $\mathbf{v}(X) = \mathsf{t}$. If $\vdash_{\mathbf{Gc}} X$, then we have $\vdash_{\mathbf{Sc}} X$, and using Lemma 2.6, we have $\mathbf{v}(X) = \mathsf{t}$.

Induction step is clear from

- $\mathbf{v}(\Gamma \Rightarrow \psi) = \mathsf{t}$ implies $\mathbf{v}(\phi, \Gamma \Rightarrow \psi) = \mathsf{t}$,
- $\mathbf{v}(\Gamma \Rightarrow \phi) = \mathbf{v}(\phi, \Gamma \Rightarrow \psi) = \mathsf{t}$ implies $\mathbf{v}(\Gamma \Rightarrow \psi) = \mathsf{t}$. □

Definition 2.8. We call a pair $\langle U, V\rangle$ of sets of formulas T-*consistent* if $T, U \nvdash_{\mathbf{Sc}} \Rightarrow \phi$ for each $\phi \in V$. We call T-consistent pair $\langle U, V\rangle$ *maximal* if $U \cup V = \mathbf{Wff}$.

Lemma 2.9. *If $T \nvdash_{\mathbf{Sc}} \Gamma \Rightarrow \phi$, then there exists a maximal T-consistent pair $\langle U, V\rangle$ satisfying $\Gamma \subseteq U$ and $\phi \in V$.*

Proof: Suppose that $T \nvdash_{\mathbf{Sc}} \Gamma \Rightarrow \phi$. We define U and V as

$$U = \{\chi \mid T \vdash_{\mathbf{Sc}} \Gamma \Rightarrow \chi\} \text{ and } V = \mathbf{Wff} \backslash U.$$

It is sufficient to show the following three conditions:

(1) $\Gamma \subseteq U$ and $\phi \in V$,

(2) (maximarity) $U \cup V = \mathbf{Wff}$,

(3) (consistency) for each formula $\psi \in V$, $T, U \nvdash_{\mathbf{Sc}} \Rightarrow \psi$.

(1) and (2) are clear from the definition. We show (3). Suppose that $\psi \in V$. By the definition of $\vdash_{\mathbf{Sc}}$, we have only to show

(4) for each finite subset U^* of U, $T, U^* \nvdash_{\mathbf{Sc}} \Gamma \Rightarrow \psi$.

In order to show (4), we use an induction on the number of members of U^*. If $U^* \subseteq \Gamma$, then by $T \nvdash_{\mathbf{Sc}} \Gamma \Rightarrow \phi$ and Lemma 2.5, we have (4). Suppose that there exists $\chi \in U^* \backslash \Gamma \subseteq U$. Then by the definition of U, we have

$$T \vdash_{\mathbf{Sc}} \Gamma \Rightarrow \chi,$$

and so,

$$T, U^* \backslash \{\chi\} \vdash_{\mathbf{Sc}} \Gamma \Rightarrow \chi, \qquad (*1)$$

By the induction hypothesis, we have

$$T, U^* \backslash \{\chi\} \nvdash_{\mathbf{Sc}} \Gamma \Rightarrow \psi, \qquad (*2)$$

By $(*1), (*2)$, and cut, we obtain (4). □

Lemma 2.10. *If $T \nvdash_{\mathbf{Sc}} \Gamma \Rightarrow \phi$, then there exists a set $\mathbf{v}$ of truth valuations such that $\mathbf{v}(T) = \mathsf{t}$ and $\mathbf{v}(\Gamma \Rightarrow \phi) = \mathsf{f}$.*

Proof: Suppose that $T \nvdash_{\mathbf{Sc}} \Gamma \Rightarrow \phi$. By Lemma 2.9, there exists a maximal T-consistent pair $\langle U, V\rangle$ satisfying $\Gamma \subseteq U$ and $\phi \in V$. Since $\langle U, V\rangle$ is T-consistent, for each $\psi \in V$, we observe

$$T, U \not\vdash_{\mathbf{Sc}} \Rightarrow \psi.$$

Therefore

$$U \not\vdash_{\mathbf{Sc}} \Rightarrow \psi.$$

Using Lemma 2.5 and Lemma 2.4, there exists a truth valuation v_ψ satisfying

$$v_\psi(U) = \mathsf{t} \text{ and } v_\psi(\psi) = \mathsf{f}.$$

We define $\mathbf{v}$ as

$$\mathbf{v} = \{v_\psi \mid \psi \in V\}.$$

Then we have

$$\mathbf{v}(U) = \mathsf{t} \text{ and } \mathbf{v}(\psi) = \mathsf{f} \text{ for every } \psi \in V,$$

and using $\Gamma \subseteq U$ and $\phi \in V$, we have

$$\mathbf{v}(\Gamma \Rightarrow \phi) = \mathsf{f}.$$

So, we have only to show

(1) $\mathbf{v}(T) = \mathsf{t}$.

Let X be a sequent in T. We divide the cases.

The case that $\mathbf{ant}(X) \not\subseteq U$. By the maximality of $\langle U, V \rangle$, we have $\mathbf{ant}(X) \cap V \neq \emptyset$, and so, $\psi \in \mathbf{ant}(X)$ for some $\psi \in V$, Using $v_\psi(\psi) = \mathsf{f}$, we have $\mathbf{v}(\psi) = \mathsf{f}$. Using $\psi \in \mathbf{ant}(X)$, we obtain $\mathbf{v}(X) = \mathsf{t}$.

The case that $\mathbf{ant}(X) \subseteq U$. Using $X \in T$, we have $T, U \vdash_{\mathbf{Sc}} \Rightarrow \mathbf{suc}(X)$. Since $\langle U, V \rangle$ is T-consistent, we observe $\mathbf{suc}(X) \notin V$. Using maximality of $\langle U, V \rangle$, we have $\mathbf{suc}(X) \in U$, and using $\mathbf{v}(U) = \mathsf{t}$, we have $\mathbf{v}(\mathbf{suc}(X)) = \mathsf{t}$. Hence, we have $\mathbf{v}(X) = \mathsf{t}$.

Hence, we obtain (1). □

By Lemma 2.7 and Lemma 2.10, we obtain Theorem 2.3. Theorem 1.3 is obtained by the following corollary.

Corollary 2.11.

(1) $\{p \Rightarrow q\} \not\vdash_{\mathbf{Sc}} \Rightarrow p \to q$.

(2) $\{p \Rightarrow r, q \Rightarrow r\} \not\vdash_{\mathbf{Sc}} p \vee q \Rightarrow r$.

(3) $\{\neg p \Rightarrow \bot\} \not\vdash_{\mathbf{Sc}} \Rightarrow p$.

PROOF: For (1). We define truth valuations v_1, v_2 as

$$(v_1(p), v_1(q)) = (\mathsf{t}, \mathsf{f}), \quad (v_2(p), v_2(q)) = (\mathsf{f}, \mathsf{f}).$$

Then as in Table 2, we obtain

$$\{v_1, v_2\}(p \Rightarrow q) = \mathsf{t} \text{ and } \{v_1, v_2\}(\Rightarrow p \to q) = \mathsf{f}.$$

Using Theorem 2.3, we obtain (1).

(2) and (3) can be shown similarly using Table 3 and Table 4, respectively. □

Table 2. A truth table for (1)

	p	q	$p \Rightarrow q$	$p \to q$	$\Rightarrow p \to q$
v_1	t	f		f	
v_2	f	f			
$\{v_1, v_2\}$	f		t	f	f

Table 3. A truth table for (2)

	p	q	r	$p \Rightarrow r$	$q \Rightarrow r$	$p \vee q$	$p \vee q \Rightarrow r$
v_1	t	f	f			t	
v_2	f	t				t	
$\{v_1, v_2\}$	f	f	f	t	t	t	f

Table 4. A truth table for (3)

	p	$\neg p$	$\neg p \Rightarrow \bot$	$\Rightarrow p$
v_1	f	t		
v_2	t	f		
$\{v_1, v_2\}$	f	f	t	f

Also, by the fact that classical logic is the maximally consistent logic (cf. Chagrov and Zakharyaschev [2]), we have the following corollary.

COROLLARY 2.12. If $\vdash_{\mathbf{S}(S)}$ is consistent, then

(1) $\{p \Rightarrow q\} \nvdash_{\mathbf{S}(S)} \Rightarrow p \to q$,

(2) $\{p \Rightarrow r, q \Rightarrow r\} \nvdash_{\mathbf{S}(S)} p \vee q \Rightarrow r$,

(3) $\{\neg p \Rightarrow \bot\} \nvdash_{\mathbf{S}(S)} \Rightarrow p$.

3. Improper derivations and the system $\vdash_{\mathbf{Sc}}$

In the present section, we consider whether an improper derivation can be described generally by using our system $\vdash_{\mathbf{Sc}}$. Specifically, we consider a derivation $\mathcal{D} : T \vdash \Gamma \Rightarrow \phi$ and give a precise expression of

$$\text{“}\mathcal{D}\text{ is improper”}, \tag{IP1}$$

assuming that (IP1) is equivalent to

$$\text{“}\mathcal{D}\text{ has some assumptions discharged by }\mathcal{D}\text{”} \tag{IP2}$$

and negation of (P1) in subsection 1.2.

As is described in subsection 1.2, (IP2) follows from the existence of a formula satisfying (D1) and (D2). More generally, we have that (C1) implies (IP2), where (C1) is the following condition.

(C1) There exists $X \in T$ satisfying the following two conditions:

(C1.1) $T \backslash \{X\} \nvdash_{\mathbf{Gc}} \Gamma \Rightarrow \mathbf{ant}(X)$,

(C1.2) $T \backslash \{X\} \nvdash_{\mathbf{Gc}} \Gamma \Rightarrow \phi$.

However, as we also mentioned in subsection 1.2, there may be an improper derivation which does not satisfy (C1). We give such improper derivation in the following example.

Example 3.1. We consider the following two derivations:

$\mathcal{D}_1$: $\{p \Rightarrow \bot, q \Rightarrow \bot\} \vdash \Rightarrow \neg p \vee \neg q$,

$\mathcal{D}_2$: $\{p \Rightarrow \bot, \neg p \Rightarrow \bot\} \vdash \Rightarrow \bot$.

(1) $\mathcal{D}_1$ has two proofs in $\vdash_{\mathbf{Gc}}$. One is to prove

$$\{p \Rightarrow \bot\} \vdash_{\mathbf{Gc}} \Rightarrow \neg p \vee \neg q$$

and the other is to prove

$$\{q \Rightarrow \bot\} \vdash_{\mathbf{Gc}} \Rightarrow \neg p \vee \neg q.$$

If we take the former, then the assumption p in $p \Rightarrow \bot$ is discharged by $\mathcal{D}_1$, and if we take the latter, then the assumption q in $q \Rightarrow \bot$ is.

(2) $\mathcal{D}_2$ also has two proofs. One is to prove

$$\{p \Rightarrow \bot\} \vdash_{\mathbf{Gc}} \Rightarrow \neg p$$

and the other is to prove

$$\{\neg p \Rightarrow \bot\} \vdash_{\mathbf{Gc}} \Rightarrow p.$$

If we take the former, then the assumption p in $p \Rightarrow \bot$ is discharged by $\mathcal{D}_2$, and if we take the latter, then the assmption $\neg p$ in $\neg p \Rightarrow \bot$ is.

So, $\mathcal{D}_1$ must be improper, but it does not satisfy (C1) because of (C1.2). Also, $\mathcal{D}_2$ must be improper, but it does not satisfy (C1) because of (C1.1).

Consequently, in order to give a precise expression of (IP2), (C1) should be modified. Specifically, we consider the following modified condition (C2), and by Example 3.1, it is natural to see that (C2) implies (IP2). We also confirm that $\mathcal{D}_1$ and $\mathcal{D}_2$ satisfy (C2).

(C2) There exists a non-empty subset T' of T satisfying the following two conditions:

(C2.1) $T \backslash T' \nvdash_{\mathbf{Gc}} \Gamma \Rightarrow \mathbf{ant}(X)$ for each $X \in T'$,

(C2.2) $T \backslash T' \nvdash_{\mathbf{Gc}} \Gamma \Rightarrow \phi$.

Now, we consider the condition:

$$T \nvdash_{\mathbf{Sc}} \Gamma \Rightarrow \phi. \tag{C3}$$

In subsection 1.2, we confirmed that the negation of (C3) implies (P1). We have already confirmed that (C2) implies (IP2). Also, we assumed the

$$(\text{C2}) \Longrightarrow (\text{IP2}) \iff (\text{IP1}) \iff \text{the negation of } (\text{P1}) \Longrightarrow (\text{C3})$$

Figure 4. Relations among (C2), (C3), (P1), (IP1), and (IP2)

equivalence among the conditions (IP2), (IP1), and the negation of (P1). We can see these relations in Figure 4.

Therefore, if we show the equivalence between (C2) and (C3), then each of (C2) and (C3) is one of the precise expressions of (IP1). Hence, the remaining to be done is to prove such equivalence, i.e., the following theorem.

THEOREM 3.2. *If* $T \vdash_{\mathbf{Gc}} \Gamma \Rightarrow \phi$, *then the conditions* (C2) *and* (C3) *are equivalent.*

We prove the above theorem, including the derivations that do not hold in $\vdash_{\mathbf{Gc}}$. Specifically, we prove the following lemma. The theorem above is obtained as a corollary of the lemma.

LEMMA 3.3. *The following two conditions are equivalent:*

(1) $T \not\vdash_{\mathbf{Sc}} \Gamma \Rightarrow \phi$,

(2) *there exists a subset* T' *of* T *satisfying the following two conditions:*

(2.1) $T \backslash T' \not\vdash_{\mathbf{Gc}} \Gamma \Rightarrow \mathbf{ant}(X)$ *for each* $X \in T'$,

(2.2) $T \backslash T' \not\vdash_{\mathbf{Gc}} \Gamma \Rightarrow \phi$.

PROOF: For (1) $\Longrightarrow$ (2). Suppose that (1) holds. Then by Theorem 2.3, there exists a set $\mathbf{v}$ of truth valuations such that $\mathbf{v}(T) = \mathbf{v}(\Gamma) = \mathsf{t}$ and $\mathbf{v}(\phi) = \mathsf{f}$. We define T' as

$$T' = \{X \in T \mid \mathbf{v}(\mathbf{ant}(X)) = \mathsf{f}\}.$$

Then we observe $\mathbf{v}(\mathbf{ant}(Y)) = \mathsf{t}$ for every $Y \in T \backslash T'$. Using $\mathbf{v}(T) = \mathsf{t}$, we have

$$\mathbf{v}(\mathbf{suc}(Y)) = \mathsf{t} \text{ for every } Y \in T \backslash T'. \tag{$*1$}$$

We show (2.1). Let X be a sequent in T'. Then we observe $\mathbf{v}(\mathbf{ant}(X)) = \mathsf{f}$, and so, there exists $v_X \in \mathbf{v}$ such that $v_X(\mathbf{ant}(X)) = \mathsf{f}$. Also, by $\mathbf{v}(\Gamma) = \mathsf{t}$, we have $v_X(\Gamma) = \mathsf{t}$. Moreover, by $(*1)$, we have $v_X(\mathbf{suc}(Y)) = \mathsf{t}$ for every $Y \in T \backslash T'$, and so, $v_X(T \backslash T') = \mathsf{t}$. Using Lemma 2.4, we have (2.1).

We show (2.2). By $\mathbf{v}(\phi) = \mathsf{f}$, there exists $v_0 \in \mathbf{v}$ such that $v_0(\phi) = \mathsf{f}$. Also, by $\mathbf{v}(\Gamma) = \mathsf{t}$, we have $v_0(\Gamma) = \mathsf{t}$. Moreover, by $(*1)$, we have $v_0(\mathbf{suc}(Y)) = \mathsf{t}$ for every $Y \in T \backslash T'$, and so, $v_0(T \backslash T') = \mathsf{t}$. Using Lemma 2.4, we have (2.2).

For (2) $\Longrightarrow$ (1). Suppose that (2) holds. Then by (2.1) and Lemma 2.4, for every $X \in T'$, there exists v_X such that

$$v_X(T \backslash T') = v_X(\Gamma) = \mathsf{t} \text{ and } v_X(\mathbf{ant}(X)) = \mathsf{f}.$$

Also, by (2.2) and Lemma 2.4, there exists v_0 such that

$$v_0(T \backslash T') = v_0(\Gamma) = \mathsf{t} \text{ and } v_0(\phi) = \mathsf{f}.$$

We define $\mathbf{v}$ as

$$\mathbf{v} = \{v_0\} \cup \{v_X \mid X \in T'\}.$$

Then we have $\mathbf{v}(T \backslash T') = \mathbf{v}(\Gamma) = \mathsf{t}$ and $\mathbf{v}(\mathbf{ant}(X)) = \mathbf{v}(\phi) = \mathsf{f}$ for every $X \in T'$, and so, we have $\mathbf{v}(T) = \mathbf{v}(\Gamma) = \mathsf{t}$ and $\mathbf{v}(\phi) = \mathsf{f}$. Using Theorem 2.3, we obtain (1). □

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