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EXTENDED BCK-IDEAL BASED ON SINGLE-VALUED NEUTROSOPHIC HYPER BCK-IDEALS

Abstract

This paper introduces the concept of single-valued neutrosophic hyper BCK-subalgebras as a generalization and alternative of hyper BCK-algebras and on any given nonempty set constructs at least one single-valued neutrosophic hyper BCK-subalgebra and one a single-valued neutrosophic hyper BCK-ideal. In this study level subsets play the main role in the connection between single-valued neutrosophic hyper BCK-subalgebras and hyper BCK-subalgebras and the connection between single-valued neutrosophic hyper BCK-ideals and hyper BCK-ideals. The congruence and (strongly) regular equivalence relations are the important tools for connecting hyperstructures and structures, so the major contribution of this study is to apply and introduce a (strongly) regular relation on hyper BCK-algebras and to investigate their categorical properties (quasi commutative diagram) via single-valued neutrosophic hyper BCK-ideals. Indeed, by using the single-valued neutrosophic hyper BCK-ideals, we define a congruence relation on (weak commutative) hyper BCK-algebras that under some conditions is strongly regular and the quotient of any (single-valued neutrosophic)hyper BCK-(sub)algebra via this relation is a (single-valued neutrosophic)(hyper BCK-subalgebra) BCK-(sub)algebra.

Keywords: single-valued neutrosophic (hyper)BCK-subalgebra, quasi commutative diagram, extendable single-valued neutrosophic (hyper)BCK-ideal.

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1. Introduction

Theory of neutrosophic set as an extension of classical set, (intuitionistic) fuzzy set [21] and interval-valued (intuitionistic) fuzzy set is introduced by Smarandache for the first time in 2005 [18] and novel concept of neutrosophy theory titled neutro-(hyper)algebra as the development of classical (hyper)algebra and partial-(hyper)algebra [19]. This concept handles problems involving ambiguous, hesitancy, and conflicting data and describes the main tool in modeling unsure hypernetworks in all sciences, see in more detail, accessible single-valued neutrosophic graphs [3], derivable single-valued neutrosophic graphs based on KM-single-valued neutrosophic metric [5] and single-valued neutrosophic directed (hyper)graphs and applications in networks [4], single-valued neutrosophic general machine [17] and a novel similarity measure of single-valued neutrosophic sets based on modified manhattan distance and its applications [22]. Today, in the scope of logical (hyper)algebras, (hyper)BCK-algebras and their generalization such as fuzzy hyper BCK-subalgebras and single-valued neutrosophic hyper BCK-subalgebras are investigated and applied in related interdisciplinary sciences such as inf-hesitant fuzzy ideals in BCK/BCI-algebras [10], length neutrosophic subalgebras of BCK=BCI-algebras [9], fuzzy soft positive implicative hyper BCK-ideals of several types [13], implicative neutrosophic quadruple BCK-Algebras and ideals [15], construction of an HV-K-algebra from a BCK-algebra based on ends lemma [16], and implicative ideals of BCK-algebras based on MBJ-neutrosophic sets [20]. The fundamental relations make an important role in the connection between hyper BCK-subalgebras and BCK-subalgebras and some research is published in this scopes such as on fuzzy quotient, BCK-algebras [2], (semi)topological quotient BCK-algebras [14] and extended fuzzy BCK-subalgebras [23].

Recently in the scope of neutro logical (hyper) algebra Hamidi, et al. introduced the concept of neutro BCK-subalgebras [6] and single-valued neutro hyper BCK-subalgebras [7] as a generalization of BCK-algebras and hyper BCK-subalgebras, respectively and presented the main results in this regard.

Regarding these points, we try to develop the notation of fuzzy hyper BCK-subalgebras to the concept of single-valued neutrosophic hyper BCK-subalgebras and so we want to seek the connection between single-valued neutrosophic BCK-algebras and single-valued neutrosophic hyper BCK-algebras. In this paper, we consider single-valued neutrosophic hyper BCK-ideals and describe the relationship between (BCK-ideals) hyper BCK-ideals and single-valued neutrosophic hyper BCK-ideals. The connection between of category of logical algebras and the category of logical hyperalgebras (as quasi commutative diagram) is based on fundamental relation and this problem is made a motivation to introduce some relation on hyper BCK-subalgebras via the single-valued neutrosophic hyper BCK-subalgebras and single-valued neutrosophic hyper BCK-ideals, it is the main and major contribution of this study. We apply a fundamental relation to any given hyper BCK-algebras and discuss the quotient of single-valued neutrosophic hyper BCK-algebras to the convert of single-valued neutrosophic BCK-algebras and discuss the quotient of single-valued neutrosophic hyper BCK-ideals to the convert of single-valued neutrosophic BCK-ideals. Moreover, applying the concept of single-valued neutrosophic hyper BCK-ideals, we get a congruence relation on (weak commutative) hyper BCK-algebras that the quotient of any given hyper BCK-algebra via this relation is a (hyper BCK-algebra) BCK-algebra. An isomorphism theorem of single-valued neutrosophic hyper BCK-ideals is obtained using the special single-valued neutrosophic hyper BCK-ideals. In the section 3, we investigated on single-valued neutrosophic hyper BCK-subalgebras, especially we converted any given nonempty set to hyper BCK-subalgebra and obtained a family of single-valued neutrosophic hyper BCK-subalgebras. In the section 4, it is presented the concepts of single-valued neutrosophic hyper BCK-ideals, especially any given nonempty set extended to a hyper BCK-algebra with at least a single-valued neutrosophic hyper BCK-ideal.

2. Preliminaries

In this section, we recall some concepts that need to our work.

Definition 2.1. [8] Let $X \neq \emptyset$. Then a universal algebra $(X, \vartheta, 0)$ of type $(2, 0)$ is called a *BCK-algebra*, if $\forall\ x, y, z \in X$:
(BCI-1) $((x\vartheta\ y)\vartheta\ (x\vartheta\ z))\vartheta\ (z\vartheta\ y) = 0$,
(BCI-2) $(x\vartheta\ (x\vartheta\ y))\vartheta\ y = 0$,
(BCI-3) $x\vartheta\ x = 0$,
(BCI-4) $x\vartheta\ y = 0$ and $y\vartheta\ x = 0$ imply $x = y$,
(BCK-5) $0\vartheta\ x = 0$,
where $\vartheta(x, y)$ is denoted by $x\vartheta\ y$.

Definition 2.2. [1, 11] Let $X \neq \emptyset$ and $P^*(X) = \{Y \mid \emptyset \neq Y \subseteq X\}$. Then for a map $\varrho : X^2 \to P^*(X)$ a hyperalgebraic system $(X, \varrho, 0)$ is called a *hyper BCK-algebra*, if $\forall\, x, y, z \in X$:
$(H1)$ $(x\ \varrho\ z)\ \varrho\ (y\ \varrho\ z) \ll x\ \varrho\ y$,
$(H2)$ $(x\ \varrho\ y)\ \varrho\ z = (x\ \varrho\ z)\ \varrho\ y$,
$(H3)$ $x\ \varrho\ X \ll x$,
$(H4)$ $x \ll y$ and $y \ll x$ imply $x = y$,
where $x \ll y$ is defined by $0 \in x\ \varrho\ y$, $\forall\ W, Z \subseteq X$, $W \ll Z \Leftrightarrow \forall\ a \in W\ \exists\ b \in Z$ s.t $a \ll b$, $(W\ \varrho\ Z) = \bigcup\limits_{a\in W, b\in Z} (a\ \varrho\ b)$ and $\varrho(x, y)$ is denoted by $x\varrho\ y$.

We will call X is a *weak commutative* hyper BCK-algebra if, $\forall\ x, y \in X, (x\ \varrho\ (x\ \varrho\ y)) \cap (y\ \varrho\ (y\ \varrho\ x)) \neq \emptyset$.

Theorem 2.3. [11] *Let $(X, \varrho, 0)$ be a hyper BCK-algebra. Then $\forall\ x, y, z \in X$ and $W, Z \subseteq X$,*

(i) $(0\ \varrho\ 0) = 0, 0 \ll x, (0\ \varrho\ x) = 0$, $x \in (x\ \varrho\ 0)$ *and* $(W \ll 0 \Rightarrow W = 0)$,

(ii) $x \ll x$, $x\ \varrho\ y \ll x$ *and* $(y \ll z \Rightarrow x\ \varrho\ z \ll x\ \varrho\ y)$,

(iii) $W\ \varrho\ Z \ll W$, $W \ll W$ *and* $(W \subseteq Z \Rightarrow W \ll Z)$.

Definition 2.4. [18] Let V be a universal set. A neutrosophic subset (NS) X of V is an object having the following form $X = \{(x, T_X(x), I_X(x), F_X(x)) \mid x \in V\}$, or $X : V \to [0, 1] \times [0, 1] \times [0, 1]$ which is characterized by a truth-membership function T_X, an indeterminacy-membership function I_X and a falsity-membership function F_X. There is no restriction on the sum of $T_X(x), I_X(x)$ and $F_X(x)$.

From now on, $\forall\ x, y \in [0, 1]$, consider $T_{min}(x, y) = \min\{x, y\}$ and $S_{max}(x, y) = \max\{x, y\}$ as triangular norm and triangular conorm, respectively.

Definition 2.5. [12] Let $(X, \varrho, 0)$ be a hyper BCK-algebra. A single-valued neutrosophic subset $A = (T_A, I_A, F_A)$ of X is called a single-valued neutrosophic hyper BCK-ideal, if $\forall\ x, y \in X$ it satisfies the following properties:
$(FH1)$ $x \ll y \Rightarrow T_A(x) \geq T_A(y), I_A(x) \geq I_A(y)$ and $F_A(x) \leq F_A(y)$,

$(FH2)$ $T_A(x) \geq T_{min}\{T_A(y), \bigwedge(T_A(x\varrho\, y))\}, I_A(x) \geq T_{min}\{I_A(y), \bigwedge(I_A(x \varrho\, y))\}$ and
$F_A(x) \leq S_{max}\{F_A(y), \bigvee(F_A(x\varrho\, y))\}$.

3. Single-valued neutrosophic hyper BCK-subalgebras

In this section, we make the concept of single-valued neutrosophic hyper BCK-subalgebras as an extension of fuzzy hyper BCK-subalgebras and seek some of their properties.

From now on, consider (X, ϱ) as a hyper BCK-subalgebra.

Definition 3.1. A single-valued neutrosophic subset $A = (T_A, I_A, F_A)$ of (X, ϱ) is called a single-valued neutrosophic hyper BCK-subalgebra of $(X, \varrho, 0)$, if

(i) $\bigwedge(T_A(x\ \varrho\ y)) \geq T_{min}(T_A(x), T_A(y))$;

(ii) $\bigvee(I_A(x\ \varrho\ y)) \leq S_{max}(I_A(x), I_A(y))$;

(iii) $\bigvee(F_A(x\ \varrho\ y)) \leq S_{max}(F_A(x), F_A(y))$.

Theorem 3.2. *Let $A = (T_A, I_A, F_A)$ be a single-valued neutrosophic hyper BCK-subalgebra of $(X, \varrho, 0)$. Then*

(i) $T_A(0) \geq T_A(x)$;

(ii) $\bigwedge(T_A(x\ \varrho\ 0)) = T_A(x)$;

(iii) $\bigwedge(T_A(0\ \varrho\ x)) = T_A(0)$;

Proof: (i) Let $x \in X$. Since $0 \in x\ \varrho\ x$, we get that $T_A(0) \geq \bigwedge(T_A(x\ \varrho\ x)) \geq T_{min}(T_A(x), T_A(x)) = T_A(x)$.

(ii) Let $x \in X$. Since $x \in x\ \varrho\ 0$, we get that $T_A(x) \geq \bigwedge(T_A(x\ \varrho\ 0)) \geq T_{min}(T_A(x), T_A(0)) = T_A(x)$. So $\bigwedge(T_A(x\ \varrho\ 0)) = T_A(x)$.

(iii) Immediate by Theorem 2.3. □

THEOREM 3.3. *Let $A = (T_A, I_A, F_A)$ be a single-valued neutrosophic hyper BCK-subalgebra of X. Then*

(*i*) $I_A(0) \leq I_A(x)$;

(*ii*) $\bigvee(I_A(x \varrho 0)) = I_A(x)$;

(*iii*) $\bigvee(I_A(0 \varrho x)) = I_A(0)$;

PROOF: (*i*) Let $x \in X$. Since $0 \in x \varrho x$, we get that $I_A(0) \leq \bigvee(I_A(x \varrho x)) \leq S_{max}(I_A(x), I_A(x)) = I_A(x)$.

(*ii*) Let $x \in X$. Since $x \in x \varrho 0$, we get that $I_A(x) \leq \bigvee(I_A(x \varrho 0)) \leq S_{max}(I_A(x), I_A(0)) = I_A(x)$. So $\bigvee(I_A(x \varrho 0)) = I_A(x)$.

(*iii*) Immediate by Theorem 2.3. □

COROLLARY 3.4. Let $A = (T_A, I_A, F_A)$ be a single-valued neutrosophic hyper BCK-subalgebra of $(X, \varrho, 0)$. Then

(*i*) $F_A(0) \leq F_A(x)$;

(*ii*) $\bigvee(F_A(x \varrho 0)) = F_A(x)$;

(*iii*) $\bigvee(F_A(0 \varrho x)) = F_A(0)$;

(*iv*) $T_{min}(T_A(x), I_A(0), F_A(0)) \leq T_{min}(T_A(0), I_A(x), F_A(x))$.

THEOREM 3.5. *Let $0 \notin X \neq \emptyset$. Then X converted to a hyper BCK-algebra $(X', \varrho, 0)(X' = X \cup \{0\})$ with at least a single-valued neutrosophic hyper BCK-subalgebra.*

PROOF: Let $x, y \in X'$. Define " ϱ " on X' by $0 \varrho y = 0, x \varrho x = \{0, x\}(x \neq 0)$, else $x \varrho y = x$. Clearly $(X', \varrho, 0)$ is a hyper BCK-algebra. Now, it is easy to see that every single-valued neutrosophic set $A = (T_A, I_A, F_A)$ that $T_A(0) = 1, I_A(0) = F_A(0) = 0$, is a single-valued neutrosophic hyper BCK-subalgebra of X'. □

Let $\mathcal{SVN}h = \{A = (T_A, I_A, F_A) \mid A\}$, whence X is a hyper BCK-algebra, A is a single-valued neutrosophic hyper BCK-subalgebra of X and $|X| \geq 1$.

COROLLARY 3.6. Let $X \neq \emptyset$. Then X can be extended to a hyper BCK-algebra that $|\mathcal{SVN}h| = |\mathbb{R}|$.

PROOF: Let $|X| = 1$. Then (X, ϱ, x) is a hyper BCK-algebra such that $x \ \varrho \ x = X$. Then for a single-valued neutrosophic set $A = (T_A, I_A, F_A)$ by $T_A(x) = I_A(x) = F_A(x) = \alpha$ is a single-valued neutrosophic hyper BCK-subalgebra of X where $\alpha \in [0, 1]$. If $|X| \geq 2$, then by Theorem 3.5, define $A = (T_{A_\alpha}, I_{A_\alpha}, F_{A_\alpha})$ by

$$T_{A_\alpha}(x) = \begin{cases} 1, & \text{if } \ x = 0, \\ \alpha, & \text{if } \ x \neq 0 \end{cases}, I_{A_\alpha}(x) = \begin{cases} 0, & \text{if } \ x = 0, \\ \alpha, & \text{if } \ x \neq 0 \end{cases}$$

and $F_{A_\alpha}(x) = \begin{cases} 0, & \text{if } \ x = 0, \\ \alpha, & \text{if } \ x \neq 0, \end{cases}$ Obviously, $A = (T_{A_\alpha}, I_{A_\alpha}, F_{A_\alpha})$ a single-valued neutrosophic hyper BCK-subalgebra of X and so $|\mathcal{SVN}h| = |[0, 1]|$. $\square$

Let X be a hyper BCK-algebra, $A = (T_A, I_A, F_A)$ a single-valued neutrosophic hyper BCK-subalgebra of X and $\alpha, \beta, \gamma \in [0, \ 1]$. Define $T_A^\alpha = \{x \in X \mid T_A(x) \geq \alpha\}, I_A^\beta = \{x \in X \mid I_A(x) \leq \beta\}$, $F_A^\gamma = \{x \in X \mid F_A(x) \leq \gamma\}$ and $A^{(\alpha,\beta,\gamma)} = \{x \in X \mid T_A(x) \geq \alpha, I_A(x) \leq \beta, F_A(x) \leq \gamma\}$.

THEOREM 3.7. *Let $A = (T_A, I_A, F_A)$ be a single-valued neutrosophic hyper BCK-subalgebra of X. Then*

(*i*) $0 \in A^{(\alpha,\beta,\gamma)} = T_A^\alpha \cap I_A^\beta \cap F_A^\gamma$,

(*ii*) $A^{(\alpha,\beta,\gamma)}$ *is a hyper BCK-subalgebra of X,*

(*iv*) *if $0 \leq \alpha \leq \alpha' \leq 1$, then $T_A^{\alpha'} \subseteq T_A^\alpha, I_A^{\alpha'} \supseteq I_A^\alpha$ and $F_A^{\alpha'} \supseteq F_A^\alpha$.*

PROOF: (*i*) Clearly $A^{(\alpha,\beta,\gamma)} = A^\alpha \cap A^\beta \cap A^\gamma$ and by Theorems 3.2, 3.3, and Corollary 3.4, we get that $0 \in A^{(\alpha,\beta,\gamma)}$.

(*ii*) Let $x, y \in T_A^\alpha$. Then $T_{min}(T_A(x), T_A(y)) \geq \alpha$. Now, for any $z \in x \ \varrho \ y, T_A(z) \geq T_{min}(T_A(x \ \varrho \ y)) \geq T_{min}(T_A(x), T_A(y)) \geq \alpha$. Hence $z \in T_A^\alpha$ and so $x \ \varrho \ y \subseteq T_A^\alpha$. In similar a way $x, y \in I_A^\beta \cap F_A^\gamma$, implies that $x \ \varrho \ y \subseteq (I_A^\beta \cap F_A^\gamma)$. Then $A^{(\alpha,\beta,\gamma)}$ is a hyper BCK-subalgebra of X.

(*iii*) Immediate. $\square$

Corollary 3.8. Let $A = (T_A, I_A, F_A)$ be a single-valued neutrosophic hyper BCK-subalgebra of X. If $0 \leq \alpha \leq \alpha' \leq 1$, then $A^{(\alpha',\alpha,\alpha)}$ is a hyper BCK-subalgebra of $A^{(\alpha,\alpha',\alpha')}$.

Let X be a hyper BCK-algebra, S be a hyper BCK-subalgebra of X and $\alpha, \alpha', \beta, \beta', \gamma, \gamma' \in [0,1]$. Define

$$T_A^{[\alpha,\alpha']}(x) = \begin{cases} \alpha', & \text{if } x \in S, \\ \alpha, & \text{if } x \notin S, \end{cases}, \ I_A^{[\beta,\beta']}(x) = \begin{cases} \beta', & \text{if } x \in S, \\ \beta, & \text{if } x \notin S, \end{cases}, and$$

$F_A^{[\gamma,\gamma']}(x) = \begin{cases} \gamma', & \text{if } x \in S, \\ \gamma, & \text{if } x \notin S, \end{cases}$. Thus we have the following theorem.

Theorem 3.9. *Let X be a hyper BCK-algebra and S be a hyper BCK-subalgebra of X. Then*

(*i*) $T_A^{[\alpha,\alpha']}$ *is a fuzzy hyper BCK-subalgebra of X.*

(*ii*) $I_A^{[\beta,\beta']}$ *is a fuzzy hyper BCK-subalgebra of X.*

(*iii*) $F_A^{[\gamma,\gamma']}$ *is a fuzzy hyper BCK-subalgebra of X.*

(*iv*) $A = (T_A^{[\alpha,\alpha']}, I_A^{[\beta,\beta']}, F_A^{[\gamma,\gamma']})$ *is a single-valued neutrosophic hyper BCK-subalgebra of X.*

Proof: (*i*) Let $x, y \in X$. If $x, y \in S$, since S is a hyper subalgebra of X, we get that $x \varrho y \subseteq S$ and so

$$\bigwedge T_A^{[\alpha,\alpha']}(x \varrho y) \geq \bigwedge T_A^{[\alpha,\alpha']}(S) = \alpha' \geq T_{\min}(T_A^{[\alpha,\alpha']}(x), T_A^{[\alpha,\alpha']}(y)).$$

If ($x \in S$ and $y \notin S$) or ($x \notin S$ and $y \in S$) or ($x \notin S$ and $y \notin S$) then $\bigwedge T_A^{[\alpha,\alpha']}(x \varrho y)) \in \{\alpha, \alpha'\}$. Thus $\bigwedge T_A^{[\alpha,\alpha']}(x \varrho y)) \geq T_{\min}(T_A^{[\alpha,\alpha']}(x), T_A^{[\alpha,\alpha']}(y))$ and so $T_A^{[\alpha,\alpha']}$ is a fuzzy hyper BCK-subalgebra of X.

(*ii*), (*iii*) Are similar to (*i*).

(*iv*) Let $x, y \in X$. If $x, y \in S$, since S is a hyper BCK-subalgebra of X, we get that $x \varrho y \subseteq S$ and so $\bigvee I_A^{[\beta,\beta']}(x \varrho y)) \leq \bigvee I_A^{[\beta,\beta']}(S) = \alpha' \leq S_{\max}(I_A^{[\beta,\beta']}(x), I_A^{[\beta,\beta']}(y))$. If ($x \in S$ and $y \notin S$) or ($x \notin S$ and $y \in S$) or ($x \notin S$ and $y \notin S$) then $\bigvee I_A^{[\beta,\beta']}(x \varrho y)) \in \{\beta, \beta'\}$. Thus

$\bigvee T_A^{[\beta,\beta']}(x\ \varrho\ y)) \leq S_{\max}(I_A^{[\beta,\beta']}(x), I_A^{[\beta,\beta']}(y))$. In similar a way, can see that $\bigvee F_A^{[\gamma,\gamma']}(x\ \varrho\ y)) \leq S_{\max}(F_A^{[\gamma,\gamma']}(x), F_A^{[\gamma,\gamma']}(y))$ an by item (i), $A = (T_A^{[\alpha,\alpha']}, I_A^{[\beta,\beta']}, F_A^{[\gamma,\gamma']})$ is a single-valued neutrosophic hyper BCK-subalgebra of X. □

4. Single-valued neutrosophic hyper BCK-ideals of hyper BCK-algebras

In this section, we extended any given nonempty set to a hyper BCK-algebra with at least a single-valued neutrosophic hyper BCK-ideal and investigate their properties. Also, single-valued neutrosophic hyper BCK-ideals are converted to hyper BCK-ideal via valued cuts. The homomorphisms play the main role in the extension of single-valued neutrosophic hyper BCK-ideals and consequently in the extension of hyper BCK-ideals. A fundamental relation is applied to generate single-valued neutrosophic BCK-ideals from single-valued neutrosophic hyper BCK-ideal and so it is considered their properties of via related diagrams. We consider the (weak commutative) hyper BCK-algebras and define a regular equivalence relation on any given hyper BCK-algebras via single-valued neutrosophic hyper BCK-ideals and prove some isomorphism theorems in this regard, that is the major contribution of this section.

Throughout this work, we denote hyper BCK-algebra $(X, \varrho, 0)$ by X.

PROPOSITION 4.1. Let $(X, \varrho, 0)$ be a hyper BCK-algebra and $A = (T_A, I_A, F_A)$ be a single-valued neutrosophic hyper BCK-ideal. Then

$$S_{max}(T_A(0), I_A(0), F_A(x)) \geq S_{max}(T_A(x), I_A(x), F_A(0)).$$

PROOF: Immediate by definition. □

THEOREM 4.2. *Let $0 \in X$ be an arbitrary set. Then X extended to a hyper BCK-algebra $(X, \varrho, 0)$ with at least a single-valued neutrosophic hyper BCK-ideal.*

PROOF: Let $x, y \in X$. Define " ϱ "on X by Theorem 3.5. Clearly, $(X, \varrho\ , 0)$ is a hyper BCK-algebra. Let $A = (T_A, I_A, F_A)$ be a single-valued neutrosophic set, where $A(0) = (1, 1, 0)$ and $x, y \in X$, then $F_A(0) = 0 \leq F_A(y)$. If $x \neq y$, then

$$F_A(x) \leq S_{max}(F_A(y), F_A(x)) = S_{max}(F_A(y), \bigvee(F_A(x\varrho\, y))).$$

If $0 \neq x = y$, then

$$F_A(x) \leq S_{max}(F_A(y), F_A(x)) = S_{max}(F_A(y), \bigvee(F_A(x\varrho\, y))).$$

In similar a way,

$\forall\ x, y \in X, T_A(x) \geq T_{min}(T_A(y), T_A(x)) = T_{min}(T_A(y), \bigwedge(T_A(x\varrho\, y)))$ and $I_A(x) \geq T_{min}(I_A(y), I_A(x)) = T_{min}(I_A(y), \bigwedge(I_A(x\varrho\, y)))$. Therefore, A is a single-valued neutrosophic hyper BCK-ideal. $\square$

Let $(X, \varrho, 0)$ be a hyper BCK-algebra which is defined in Theorem 4.2 and

$$\mathcal{SVN}hi = \{\mu \mid \mu \text{is a single-valued neutrosophic hyper BCK-ideal on} (X, \varrho, 0)\},$$

then we have the following result.

Corollary 4.3. Let $(X, \varrho, 0)$ be a hyper BCK-algebra. If $|X| \geq 1$, then $|\mathcal{SVN}hi| = |\mathbb{R}|$.

Example 4.4. Let $X = \{-1, -2, -3, -4, -5\} \subseteq \mathbb{Z}$. Then $(X, \varrho\ , -1)$ is a hyper BCK-algebra as follows:

ϱ	-1	-2	-3	-4	-5
-1	$\{-1\}$	$\{-1\}$	$\{-1\}$	$\{-1\}$	$\{-1\}$
-2	$\{-2\}$	$\{-1, -2\}$	$\{-2\}$	$\{-2\}$	$\{-2\}$
-3	$\{-3\}$	$\{-3\}$	$\{-1, -3\}$	$\{-3\}$	$\{-3\}$
-4	$\{-4\}$	$\{-4\}$	$\{-4\}$	$\{-1, -4\}$	$\{-4\}$
-5	$\{-5\}$	$\{-5\}$	$\{-5\}$	$\{-5\}$	$\{-1, -5\}$

Define $A : X \to [0, 1]^3$ by $T_A(x) = I_A(x) = \dfrac{1}{-x}$ and $F_A(x) = \dfrac{1}{x}$. It is easy to see that $A = (T_A, I_A, F_A)$ is a single-valued neutrosophic hyper BCK-ideal.

Theorem 4.5. *Let $(X, \varrho, 0)$ be a hyper BCK-algebra and $A = (T_A, I_A, F_A)$ be a single-valued neutrosophic hyper BCK-ideal of X. Then $\forall\ x, y \in X$ and $Y, Z \subset X$:*

(i) *if* $Y \ll Z$, *then* $\exists\ z \in Z$ *such that* $T_{min}(\bigvee(T_A(Y)), \bigvee(I_A(Y))) \geq T_{min}(T_A(z), I_A(z))$ *and* $\bigwedge(F_A(Y)) \leq F_A(z)$;

(ii) *if* $Y \ll Z$, *then* $T_{min}(\bigvee(T_A(Y)), \bigvee(I_A(Y))) \geq T_{min}(\bigwedge(T_A(Z)), \bigwedge(I_A(Z)))$ *and* $\bigvee(F_A(Z)) \geq \bigwedge(F_A(Y))$;

(iii) $T_{min}(T_A(x), I_A(x)) \leq T_{min}(\bigvee(T_A(x\varrho\, y), \bigvee(I_A(x\varrho\, y))$ *and* $F_A(x) \geq \bigwedge(F_A(x\varrho\, y))$.

(iv) $T_{min}(T_A(x), I_A(x)) \leq T_{min}(\bigwedge(T_A(x\varrho\, y), \bigwedge(I_A(x\varrho\, y))$ *and* $F_A(x) \geq \bigvee(F_A(x\varrho\, y))$.

PROOF: (i) Since $Y \ll Z$, $\forall\ y \in Y$, $\exists\ z \in Z$ such that $y \ll z$. Hence $\bigvee(T_A(Y)) \geq T_A(y) \geq T_A(z)$. In similar a way, $\bigvee(I_A(Y)) \geq I_A(y) \geq I_A(z)$ and so $T_{min}(\bigvee(T_A(Y)), \bigvee(I_A(Y))) \geq T_{min}(T_A(z), I_A(z))$. In addition, $\forall\ y \in Y$, $\exists\ z \in Z$ such that $\bigwedge(F_A(Y)) \leq F_A(y) \leq F_A(z)$.

(ii) Let $Y \ll Z$. Then $\forall\ y \in Y$, $\exists\ z \in Z$ such that $y \ll z$, so $T_A(y) \geq T_A(z), I_A(y) \geq I_A(z)$ and $F_A(y) \leq F_A(z)$. It follows that $\bigvee(T_A(Y)) \geq T_A(y) \geq T_A(z) \geq \bigwedge(T_A(Z))$, $\bigvee(I_A(Y)) \geq I_A(y) \geq I_A(z) \geq \bigwedge(I_A(Z))$ and $\bigvee(F_A(Z)) \geq F_A(z) \geq F_A(y) \geq \bigwedge(F_A(Y))$. Hence $T_{min}(\bigvee(T_A(Y)), \bigvee(I_A(Y))) \geq T_{min}(\bigwedge(T_A(Z)), \bigwedge(I_A(Z)))$ and $\bigvee(F_A(Z)) \geq \bigwedge(F_A(Y))$.

(iii) By Theorem 2.3, $x\varrho\, y \ll x$. Then by (ii), we get that $T_A(x) \leq \bigvee T_A(x\varrho\, y), I_A(x) \leq \bigvee(I_A(x\varrho\, y))$ and $F_A(x) \geq \bigwedge(F_A(x\varrho\, y))$.

(iv) By Theorem 2.3, $x\varrho\, y \ll x$. Then $\forall\ t \in (x\varrho\, y), t \ll x$, we get that $T_A(t) \geq T_A(x)$, so $\bigwedge T_A(x\varrho\, y) \geq T_A(x)$ and similar a way $\bigwedge I_A(x\varrho\, y) \geq I_A(x)$ is obtained. Also $x\varrho\, y \ll x$ implies that $\forall\ t \in (x\varrho\, y), t \ll x$ so $F_A(t) \leq F_A(x)$. Thus $\bigvee(F_A(x\varrho\, y)) \leq F_A(x)$. $\square$

COROLLARY 4.6. Let $(X, \varrho, 0)$ be a hyper BCK-algebra and A be a single-valued neutrosophic hyper BCK-ideal of X. Then $\forall\ x, y \in X$ and $Y, Z \subset X$, get $T_{min}(\bigvee(T_A(Y\ \varrho\ Z)), \bigvee(I_A(Y\ \varrho\ Z))) \geq T_{min}(\bigwedge(T_A(Y)), \bigwedge(I_A(Y)))$ and $\bigvee(F_A(Y)) \geq \bigwedge(F_A(Y\ \varrho\ Z))$.

Let $\alpha, \beta, \gamma \in [0,\ 1]$ and $A = (T_A, I_A, F_A)$ be a single-valued neutrosophic hyper BCK-ideal of X. Define $A^{\lfloor\alpha,\beta,\gamma\rfloor} = T^{(\alpha)} \cap I^{(\beta)} \cap F^{(\gamma)}$, where $T^{(\alpha)} = \{x \in X \mid T_A(x) \geq \alpha\}, I^{(\beta)} = \{x \in X \mid I_A(x) \geq \beta\}$ and $F^{(\gamma)} = \{x \in X \mid F_A(x) \leq \gamma\}$.

Theorem 4.7. *neutrosophic hyper BCK-ideal is a single-valued neutrosophic hyper BCK-ideal. Let $(X, \varrho, 0)$ be a hyper BCK-algebra and $A = (T_A, I_A, F_A)$ be a single-valued neutrosophic hyper BCK-ideal of X such that $T^{(\alpha)}, I^{(\beta)}, F^{(\gamma)} \neq \emptyset$. Then $\forall\ x, y, z \in X$:*

(i) $0 \in A^{\lfloor\alpha,\beta,\gamma\rfloor}$;

(ii) *if* $y \in A^{\lfloor\alpha,\beta,\gamma\rfloor}$ *and* $x \ll y$, *then* $x \in A^{\lfloor\alpha,\beta,\gamma\rfloor}$;

(iii) $(y\varrho\ z) \ll x$ *implies that* $T_A(y) \geq T_{min}(T_A(z), T_A(x)), I_A(y) \geq T_{min}(I_A(z), I_A(x)), F_A(y) \leq S_{max}(F_A(z), F_A(x))$;

(iv) $A^{\lfloor\alpha,\beta,\gamma\rfloor}$ *is a hyper BCK-ideal of X.*

Proof: *(i)* There exists $x \in A^{\lfloor\alpha,\beta,\gamma\rfloor}$ such that $T_A(x) \geq \alpha, I_A(x) \geq \beta$ and $F_A(x) \leq \gamma$. By Corollary 4.1, $T_A(0) \geq T_A(x), I_A(0) \geq I_A(x), F_A(0) \leq F_A(x)$, conclude that $0 \in A^{\lfloor\alpha,\beta,\gamma\rfloor}$.

(ii) Since $x \ll y$, by definition, we get that $T_A(x) \geq T_A(y), I_A(x) \geq I_A(y)$ and $F_A(x) \leq F_A(y)$. Now, $y \in A^{\lfloor\alpha,\beta,\gamma\rfloor}$ implies that $x \in A^{\lfloor\alpha,\beta,\gamma\rfloor}$.

(iii) $(y\varrho\ z) \ll x$ implies that $0 \in (y\varrho\ z)\varrho\ x$, then by Theorem 4.5, we get that $T_A(x) \leq \bigwedge(T_A(y\varrho\ z))$, $I_A(x) \leq \bigwedge(I_A(y\varrho\ z))$ and $F_A(x) \geq \bigvee(F_A(y\varrho\ z))$. Now, A is a single-valued neutrosophic hyper BCK-ideal so

$$T_A(y) \geq T_{min}(T_A(z), \bigwedge(T_A(y\varrho\ z))) \geq T_{min}(T_A(z), T_A(x))$$
$$I_A(y) \geq T_{min}(I_A(z), \bigwedge(I_A(y\varrho\ z))) \geq T_{min}(I_A(z), I_A(x))$$
$$F_A(y) \leq S_{max}(F_A(z), \bigvee(F_A(y\varrho\ z))) \leq S_{max}(F_A(z), F_A(x)).$$

(iv) Let $x, y \in X, x\varrho\ y \ll A^{\lfloor\alpha,\beta,\gamma\rfloor}$ and $y \in A^{(\alpha,\beta,\gamma)}$. Then $T_A(y) \geq \alpha, I_A(y) \geq \beta, F_A(y) \leq \gamma$ and by Theorem 4.5,

$\bigwedge(T_A(x\varrho\ y)) \geq \alpha, \bigwedge(I_A(x\varrho\ y)) \geq \beta$ and $\bigvee(F_A(x\varrho\ y)) \leq \gamma$. Hence

$$T_A(x) \geq T_{min}(T_A(y), \bigwedge(T_A(x\varrho\ y))) \geq T_{min}(\alpha, \alpha) = \alpha$$
$$I_A(x) \geq T_{min}(I_A(y), \bigwedge(I_A(x\varrho\ y))) \geq T_{min}(\beta, \beta) = \beta$$
$$F_A(x) \leq S_{max}(F_A(y), \bigvee(F_A(x\varrho\ y))) \geq S_{max}(\gamma, \gamma) = \gamma.$$

Therefore, $x \in A^{\lfloor \alpha,\beta,\gamma \rfloor}$ and so $A^{\lfloor \alpha,\beta,\gamma \rfloor}$ is a hyper BCK-ideal. □

Let $(X, \varrho, 0)$ be a hyper BCK-algebra. A map $f : X \to X$ is called a homomorphism, if $f(0) = 0$ and $\forall\ x, y \in X, f(x\varrho y) = f(x)\varrho f(y)$. If f be an onto homomorphism and $A = (T_A, I_A, F_A)$ a single-valued neutrosophic subset of X. Define $A_f = (T_{A_f}, I_{A_f}, F_{A_f})$ by

$$A_f(x) = (T_A(f(x)), I_A(f(x)), F_A(f(x)).$$

Thus, have the following theorem.

THEOREM 4.8. *Let $(X, \varrho, 0)$ be a hyper BCK-algebra. Then the single-valued neutrosophic set $A = (T_A, I_A, F_A)$, is a single-valued neutrosophic hyper BCK-ideal of X if and only if $A_f = (T_{A_f}, I_{A_f}, F_{A_f})$ is a single-valued neutrosophic hyper BCK-ideal of X.*

PROOF: Let $A = (T_A, I_A, F_A)$ be a single-valued neutrosophic hyper BCK-ideal of X and $x \in X$. Then

$$T_{A_f}(0) = T_A(f(0)) = T_A(0) \geq T_A(f(x)) = T_{A_f}(x)$$
$$I_{A_f}(0) = I_A(f(0)) = I_A(0) \geq I_A(f(x)) = I_{A_f}(x)$$
$$F_{A_f}(0) = F_A(f(0)) = F_A(0) \leq F_A(f(x)) = F_{A_f}(x)$$

and $\forall\ x, y \in X$,

$$\begin{aligned} T_{A_f}(y) = T_A(f(y)) &\geq T_{min}(T_A(f(x)), \bigwedge(T_A(f(y)\varrho\ f(x)))) \\ &= T_{min}(T_A(f(x)), \bigwedge(T_A(f(y\varrho\ x))) \\ &= T_{min}(T_{A_f}(x), \bigwedge(T_{A_f}(y\varrho\ x))). \end{aligned}$$

In similar a way, $I_{A_f}(y) \geq T_{min}(I_{A_f}(x), \bigwedge(I_{A_f}(y\varrho\ x)))$ and $F_{A_f}(y) \leq S_{miax}(F_{A_f}(x), \bigvee(F_{A_f}(y\varrho\ x)))$ are obtained. Hence $A_f = (T_{A_f}, I_{A_f}, F_{A_f})$ is a single-valued neutrosophic hyper BCK-ideal of X.

Conversely, assume that $A_f = (T_{A_f}, I_{A_f}, F_{A_f})$ is a single-valued neutrosophic hyper BCK-ideal of X and $y \in X$. Since f is onto, $\exists\, x \in X$ such that $f(x) = y$. Then

$$\begin{aligned}
T_A(0) &= T_A(f(0)) = T_{A_f}(0) \geq T_{A_f}(x) = T_A(y)\\
I_A(0) &= I_A(f(0)) = I_{A_f}(0) \geq I_{A_f}(x) = I_A(y)\\
F_A(0) &= F_A(f(0)) = F_{A_f}(0) \leq F_{A_f}(x) = F_A(y).
\end{aligned}$$

Let $x, y \in X$. Then there exists $a, b \in X$ such that $f(a) = x$ and $f(b) = y$. Hence we get that

$$\begin{aligned}
T_A(y) = T_A(f(b)) &= T_{A_f}(b)\\
&\geq T_{min}(T_{A_f}(a), \bigwedge(T_{A_f}(b\varrho\, a)))\\
&= T_{min}(T_A(f(a)), \bigwedge(T_A(f(b\varrho\, a))))\\
&= T_{min}(T_A(f(a)), \bigwedge(T_A(f(b)\varrho\, f(a)))\\
&= T_{min}(T_A(x), \bigwedge(T_A(y\varrho\, x)).
\end{aligned}$$

In similar a way, can see that $I_A(y) \geq T_{min}(I_A(x), \bigwedge(I_A(y\varrho\, x))$ and $F_A(y) \leq S_{max}(F_A(x), \bigvee(F_A(y\varrho\, x))$. Therefore $A = (T_A, I_A, F_A)$ is a single-valued neutrosophic hyper BCK-ideal of X. □

THEOREM 4.9. *Let $(X, \varrho, 0)$ be a hyper BCK-algebra, $A = (T_A, I_A, F_A)$ be a single-valued neutrosophic hyper BCK-ideal of X and $f : X \to X$ be a homomorphism,*

- (i) *if $x \in ker(f)$, then $\forall\ y \in X, T_{min}(T_{A_f}(x), I_{A_f}(x)) \geq T_{min}(T_A(y), I_A(y))$ and $F_{A_f}(x) \leq F_A(y)$.*
- (ii) *if at least one of T_A or I_A or F_A is one to one, then $ker(f)$ is a hyper BCK-ideal.*
- (iii) *if $\exists\, x \in X$ such that $A(x) = (1, 1, 0)$, then $A_{(1,0)} = \{x \in X \mid T_A(x) = I_A(x) = 1, F_A(x) = 0\}$ is a hyper BCK-ideal in X.*
- (iv) *$A_{(0,0)}$ is a single-valued neutrosophic hyper BCK-ideal in X.*

PROOF: (i) Let $x \in ker(f)$. Then, $T_{A_f}(x) = T_A(f(x)) = T_A(0)$, $I_{A_f}(x) = I_A(f(x)) = I_A(0)$ and $F_{A_f}(x) = F_A(f(x)) = F_A(0)$. Thus $\forall\ y \in X$, $T_{A_f}(x) \geq T_A(y), I_{A_f}(x) \geq I_A(y)$ and $F_{A_f}(x) \leq F_A(y)$.

(ii) Clearly $0 \in ker(f)$. Let $y \in ker(f)$ and $x\varrho\, y \ll ker(f)$, where $x, y \in X$. Then $T_{A_f}(y) = T_A(0), I_{A_f}(y) = I_A(0), F_{A_f}(y) = F_A(0)$, $\bigwedge(T_{A_f}(x\varrho\, y)) = T_A(0), \bigwedge(I_{A_f}(x\varrho\, y)) = I_A(0)$ and $\bigvee(F_{A_f}(x\varrho\, y)) = F_A(0)$ So

$$T_{A_f}(x) \geq T_{min}(T_{A_f}(y), \bigwedge(T_{A_f}(x\varrho\, y))) = T_{min}(T_A(0), T_A(0)) = T_A(0)$$

$$I_{A_f}(x) \geq T_{min}(I_{A_f}(y), \bigwedge(I_{A_f}(x\varrho\, y))) = T_{min}(I_A(0), I_A(0)) = I_A(0)$$

$$F_{A_f}(x) \leq S_{max}(F_{A_f}(y), \bigvee(F_{A_f}(x\varrho\, y))) = S_{max}(F_A(0), F_A(0)) = F_A(0).$$

Hence $T_{A_f}(x) = T_A(0), I_{A_f}(x) = I_A(0)$ and $F_{A_f}(x) = F_A(0)$. If if at least one of T_A or I_A or F_A is a one to one map, then $x \in ker(f)$.

(iii) Since there exists $x \in X$ such that $A(x) = (1, 1, 0)$, we get that $1 = T_A(x) \leq T_A(0), 1 = I_A(x) \leq I_A(0)$ and $0 = F_A(x) \geq F_A(0)$. Hence $T_A(0) = I_A(0) = 1, F_A(0) = 0$ and so $0 \in A_{(1,0)}$. Now, let $y \in A_{(1,0)}$ and $x\varrho\, y \ll A_{(1,0)}$, where $x, y \in X$. Then, $T_A(y) = I_A(y) = 1, F_A(y) = 0$, $\bigwedge(T_A(x\varrho\, y)) = \bigwedge(I_A(x\varrho\, y)) = 1$ and $\bigvee(F_A(x\varrho\, y)) = 0$. So

$$T_A(x) \geq T_{min}(T_A(y), \bigwedge(T_A(x\varrho\, y))) = T_{min}(1, 1) = 1$$

$$I_A(x) \geq T_{min}(I_A(y), \bigwedge(I_A(x\varrho\, y))) = T_{min}(1, 1) = 1$$

$$F_A(x) \leq S_{max}(F_A(y), \bigvee(F_A(x\varrho\, y))) = S_{max}(0, 0) = 0.$$

Hence $T_A(x) = I_A(x) = 1, F_A(x) = 0$ and so $x \in A_{(1,0)}$.

(iv) Since $A_{(0,0)} = X$, then the proof is clear. □

THEOREM 4.10. *Let $(X, \varrho, 0)$ be a hyper BCK-algebra, I be a hyper BCK-ideal and $A = (T_A, I_A, F_A), A' = (T_{A'}, I_{A'}, F_{A'})$ be single-valued neutrosophic hyper BCK-ideals of X. Then*

- (i) *$X_A = \{x \in X \mid T_A(x) = T_A(0), I_A(x) = I_A(0), F_A(x) = F_A(0)\}$ is a hyper BCK-ideal of X;*
- (ii) *if $A'(0) = A(0)$, then $X_{A'}\varrho\, X_A = \bigcup_{\substack{a' \in X_{A'} \\ a \in X_A}} (a'\varrho\, a)$, is a hyper BCK-ideal;*

(*iii*) *X_A is a hyper BCK-ideal of $X_A \varrho\, X_{A'}$;*

(*iv*) *if A is restricted to I, then A is a single-valued neutrosophic hyper BCK-ideal of I.*

Proof: (*i*) Let $x, y \in X$ such that $x\varrho\, y \ll X_A$ and $y \in X_A$. Then $T_A(y) = T_A(0), I_A(y) = I_A(0), F_A(y) = F_A(0), \bigwedge(T_A(x\varrho\, y)) = T_A(0)$, $\bigwedge(I_A(x\varrho\, y)) = I_A(0)$ and $\bigvee(F_A(x\varrho\, y)) = F_A(0)$, So $T_A(x) \geq T_{min}\{T_A(y)$, $\bigwedge(T_A(x\varrho\, y))\} = T_A(0), I_A(x) \geq T_{min}\{I_A(y), \bigwedge(I_A(x\varrho\, y))\} = I_A(0)$ and $F_A(x) \leq S_{max}\{F_A(y), \bigvee(F_A(x\varrho\, y))\} = F_A(0)$. So $T_A(x) = T_A(0), I_A(x) = I_A(0), F_A(x) = F_A(0)$, hence $x \in X_A$ and X_A is a hyper BCK-ideal.

(*ii*) Clearly $0 \in X_{A'}\varrho\, X_A$. Let $t, t' \in X$ such that $t'\varrho\, t \ll X_{A'}\varrho\, X_A$ and $t \in X_{A'}\varrho\, X_A$. Then there exist $a' \in X_{A'}$ and $a \in X_A$ such that $t \in a'\varrho\, a$ so by Theorem 4.5,

$$\begin{aligned} & T'_A(t) \geq \bigwedge(T'_A(a'\varrho\, a)) \geq T'_A(a') = T'_A(0), I'_A(t) \geq \bigwedge(I'_A(a'\varrho\, a)) \\ \geq\ & I'_A(a') = I'_A(0) F'_A(t) \leq \bigvee(F'_A(a'\varrho\, a)) \leq F'_A(a') = F'_A(0) \end{aligned}$$

and so

$$\begin{aligned} T'_A(t') &\geq T_{min}(T'_A(t), \bigwedge(T'_A(t'\varrho\, t))) \geq T_{min}(T'_A(t), T'_A(0)) \\ I'_A(t') &\geq T_{min}(I'_A(t), \bigwedge(I'_A(t'\varrho\, t))) \geq T_{min}(I'_A(t), I'_A(0)) \\ F'_A(t') &\leq S_{max}(F'_A(t), \bigwedge(F'_A(t'\varrho\, t))) \geq S_{max}(F'_A(t), F'_A(0)). \end{aligned}$$

Hence $t' \in X_{A'}$ and so $t' \in t'\varrho\, 0 \subseteq X_{A'}\varrho\, X_A$. Therefore $X_{A'}\varrho\, X_A$ is a hyper BCK-ideal in X.

(*iii*) Let $x \in X_A$. Since $x \in x\varrho\, 0$, we get that $x \in X_A \subseteq X_A\varrho\, X_{A'}$ and by (*i*), X_A is a hyper BCK-ideal of $X_A\varrho\, X_{A'}$.

(*iv*) The proof is clear. □

Let X be a hyper BCK-algebra and $x, y \in X$. Then $x\beta y \Leftrightarrow \exists n \in \mathbb{N}, (a_1, \ldots, a_n) \in X^n$ and $\exists u \in \varrho(a_1, \ldots, a_n)$ such that $\{x, y\} \subseteq u$. The relation β is a reflexive and symmetric relation, but not transitive relation. Let $C(\beta)$ be the *transitive closure* of β (the smallest transitive relation such that contains β). Hamidi, et.al in [1], proved that for any given weak commutative hyper BCK-algebra X, $C(\beta)$ is a strongly regular relation on X and $(X/C(\beta), \vartheta, \overline{0})$ is a BCK-algebra, where $C(\beta)(x)\vartheta\ C(\beta)(y) = C(\beta)(x\ \varrho\ y)$ and $\overline{0} = C(\beta)(0)$.

THEOREM 4.11. *Let $(X, \varrho, 0)$ be a hyper BCK-algebra. If $A = (T_A, I_A, F_A)$ is a single-valued neutrosophic hyper BCK-ideal of X, then there exists a single-valued neutrosophic hyper BCK-ideal $\overline{A} = (\overline{T_A}, \overline{I_A}, \overline{F_A})$ of $(X/C(\beta), \vartheta, \overline{0})$ such that $\forall\ x, y \in X$,*

(i) $\overline{A}(\overline{0}) \geq \overline{A}(C(\beta)(x))$;

(ii) $\overline{T_A}(C(\beta)(y)) \geq T_{min}(\overline{T_A}(C(\beta)(x), \bigwedge \big(\overline{T_A}(\vartheta(C(\beta)(y), C(\beta)(x)))\big)$,

(iii) $\overline{I_A}(C(\beta)(y)) \geq T_{min}(\overline{I_A}(C(\beta)(x), \bigwedge \big(\overline{I_A}(\vartheta(C(\beta)(y), C(\beta)(x)))\big)$,

(iv) $\overline{F_A}(C(\beta)(y)) \leq S_{max}(\overline{F_A}(C(\beta)(x), \bigwedge \big(\overline{F_A}(\vartheta(C(\beta)(y), C(\beta)(x)))\big)$.

PROOF: *(i)* We define $\overline{A} : X/C(\beta) \to [0,\ 1]^3$ by $(\overline{T_A}(C(\beta)(t)), \overline{I_A}(C(\beta)(t)), \overline{F_A}(C(\beta)(t))) = (\bigvee\limits_{t\ C(\beta)\ x} T_A(x), \bigvee\limits_{t\ C(\beta)\ x} I_A(x), \bigwedge\limits_{t\ C(\beta)\ x} F_A(x))$, where $x, t \in X$. Consider the following diagram:

$$\begin{tikzcd} X \arrow[r, "T_A"] \arrow[d, "\pi"'] & {[0\ 1]} \\ X/C(\beta) \arrow[ur, "\overline{T_A}"'] & \end{tikzcd}, \quad \begin{tikzcd} X \arrow[r, "I_A"] \arrow[d, "\pi"'] & {[0\ 1]} \\ X/C(\beta) \arrow[ur, "\overline{I_A}"'] & \end{tikzcd}, \quad \begin{tikzcd} X \arrow[r, "F_A"] \arrow[d, "\pi"'] & {[0\ 1]} \\ X/C(\beta) \arrow[ur, "\overline{F_A}"'] & \end{tikzcd}.$$

Firstly we show that $\overline{A}$ is well-defined. Let $t, t', x \in X$ and $C(\beta)(t) = C(\beta)(t')$. Then $t\ C(\beta)\ t'$ and

$$\overline{T_A}(C(\beta)(t)) = \bigvee_{x\ C(\beta)\ t} T_A(x) = \bigvee_{x\ C(\beta)\ t'} T_A(x) = \overline{T_A}(C(\beta)(t'))$$

$$\overline{I_A}(C(\beta)(t)) = \bigvee_{x\ C(\beta)\ t} I_A(x) = \bigvee_{x\ C(\beta)\ t'} I_A(x) = \overline{I_A}(C(\beta)(t'))$$

$$\overline{F_A}(C(\beta)(t)) = \bigwedge_{x\ C(\beta)\ t} F_A(x) = \bigwedge_{x\ C(\beta)\ t'} F_A(x) = \overline{F_A}(C(\beta)(t')).$$

In addition, $\forall\ x, t \in X$, we get that

$$\overline{T_A}(C(\beta)(0)) = \bigvee_{t\ C(\beta)\ 0} T_A(t) = T_A(0) \geq \bigvee_{t\ C(\beta)\ x} T_A(t) = \overline{T_A}(C(\beta)(x))$$

$$\overline{I_A}(C(\beta)(0)) = \bigvee_{t\ C(\beta)\ 0} T_A(t) = I_A(0) \geq \bigvee_{t\ C(\beta)\ x} I_A(t) = \overline{I_A}(C(\beta)(x))$$

$$\overline{F_A}(C(\beta)(0)) = \bigwedge_{t\ C(\beta)\ 0} F_A(t) = F_A(0) \leq \bigwedge_{t\ C(\beta)\ x} F_A(t) = \overline{F_A}(C(\beta)(x)).$$

(ii) Let $x, y \in X$. Since $\forall\ t \in C(\beta)(y)$ and $\forall\ t' \in C(\beta)(x)$,

$$\bigvee_{t\ C(\beta)\ y} T_A(t) \geq T_A(t) \geq T_{min}(T_A(t'), \bigwedge(T_A(t\varrho\ t')))$$

$$\bigvee_{t\ C(\beta)\ y} I_A(t) \geq I_A(t) \geq T_{min}(I_A(t'), \bigwedge(I_A(t\varrho\ t')))$$

$$\bigwedge_{t\ C(\beta)\ y} F_A(t) \leq F_A(t) \leq S_{max}(F_A(t'), \bigwedge(F_A(t\varrho\ t')))$$

we get that

$$\begin{aligned}
\overline{T_A}(C(\beta)(y)) &= \bigvee_{t\ C(\beta)\ y} T_A(t) \\
&\geq \bigvee_{\substack{t'\in C(\beta)(x) \\ t\ C(\beta)\ y}} (T_{min}(T_A(t'), \bigwedge(T_A(t\varrho\ t'))) \\
&\geq T_{min}(\bigvee_{t'\in C(\beta)(x)} T_A(t'), \bigvee_{t'\in C(\beta)(x)} \bigwedge_{t\in C(\beta)(y)} (T_A(t\varrho\ t'))) \\
&\geq T_{min}(\bigvee_{t'\in C(\beta)(x)} T_A(t'), \bigwedge_{m\in\vartheta(C(\beta)(y),C(\beta)(x))} \bigvee(T_A(m)) \\
&\geq T_{min}(\overline{T_A}(C(\beta)(x), \bigwedge(\overline{T_A}(\vartheta(C(\beta)(y), C(\beta)(x)))).
\end{aligned}$$

(iii, iv) Similar to item (ii), can see that
$\overline{I_A}(C(\beta)(y)) \geq T_{min}(\overline{I_A}(C(\beta)(x), \bigwedge(\overline{I_A}(\vartheta(C(\beta)(y), C(\beta)(x))))$ and
$\overline{F_A}(C(\beta)(y)) \leq S_{max}(\overline{F_A}(C(\beta)(x), \bigvee(\overline{F_A}(\vartheta(C(\beta)(y), C(\beta)(x)))).$ □

Let $(Y, \vartheta, 0, \preceq)$ be a BCK-algebra and $B = (T_B, I_B, F_B)$ a single-valued neutrosophic subset of Y. Then $B = (T_B, I_B, F_B)$ is called a single-valued neutrosophic BCK-ideal of Y, if (1); $\forall\ x, y \in Y, x \preceq y \Rightarrow T_A(x) \geq T_A(y), I_A(x) \geq I_A(y)$ and $F_A(x) \leq F_A(y)$,

(2); $T_A(x) \geq T_{min}\{T_A(y), T_A(x\vartheta\ y)\}, I_A(x) \geq T_{min}\{I_A(y), I_A(x\vartheta\ y)\}$ and $F_A(x) \leq S_{max}\{F_A(y), F_A(x\vartheta\ y)\}$.

COROLLARY 4.12. Let $(X, \varrho, 0)$ be a weak commutative hyper BCK-algebra. If $A = (T_A, I_A, F_A)$ is a single-valued neutrosophic hyper BCK-ideal of X, then there exists a single-valued neutrosophic BCK-ideal $B = (T_B, I_B, F_B)$ of BCK-algebra $(X/C(\beta), \vartheta, \overline{0})$, such that $T_B \circ \pi \geq T_A, I_B \circ \pi \geq I_A$, and $F_B \circ \pi \leq F_A$.

PROOF: By Theorem 4.11, consider $B = \overline{T_A}$. For any $x \in X$, since $xC(\beta)x$, we get that $(T_B \circ \pi)(x) = T_B(C(\beta)(x)) = \bigvee\limits_{t\ C(\beta)\ x} T_A(t) \geq T_A(x), (I_B \circ \pi)(x) = I_B(C(\beta)(x)) = \bigvee\limits_{t\ C(\beta)\ x} I_A(t) \geq I_A(x)$ and

$$(F_B \circ \pi)(x) = F_B(C(\beta)(x)) = \bigwedge_{t\ C(\beta)\ x} F_A(t) \leq F_A(x). \qquad \square$$

Example 4.13. Let $X = \{0, b, c, d\}$. Then $A = (T_A, I_A, F_A)$ is a single-valued neutrosophic hyper BCK-subalgebra of hyper BCK-algebra $(X, \varrho, 0)$ as follows:

ϱ	0	b	c	d
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{0\}$	$\{0\}$	$\{0\}$
c	$\{c\}$	$\{c\}$	$\{0\}$	$\{0\}$
d	$\{d\}$	$\{d\}$	$\{c\}$	$\{0, c\}$

and

	0	b	c	d
T_A	1	0.9	0.3	0.3
I_A	1	0.9	0.3	0.3
F_A	0.1	0.15	0.25	0.25

.

Clearly (X, ϱ, A) is not weak commutative and T is a single-valued neutrosophic hyper BCK-ideal. Now we get that $X/C(\beta) = \{C(\beta)(0) = \{0, c\}, C(\beta)(b) = \{b\}, C(\beta)(d) = \{d\}\}$,

ϑ	$C(\beta)(0)$	$C(\beta)(b)$	$C(\beta)(d)$
$C(\beta)(0)$	$C(\beta)(0)$	$C(\beta)(0)$	$C(\beta)(0)$
$C(\beta)(b)$	$C(\beta)(b)$	$C(\beta)(0)$	$C(\beta)(0)$
$C(\beta)(d)$	$C(\beta)(d)$	$C(\beta)(d)$	$C(\beta)(0)$

and

	$C(\beta)(0)$	$C(\beta)(b)$	$C(\beta)(d)$
$\overline{T_A}$	1	0.9	0.3
$\overline{I_A}$	1	0.9	0.3
$\overline{F_A}$	0.1	0.25	0.25

It is easy to see that $(X/C(\beta), \vartheta, C(\beta)(0), \overline{A})$ is a hyper BCK-algebra.

Definition 4.14. Let $(X, \varrho, 0)$ be a hyper BCK-algebra and $A = (T_A, I_A, F_A)$ be a single-valued neutrosophic hyper BCK-ideal of X. For any $x, y \in X$, define binary relations $R^{T_A}, R^{I_A}, R^{F_A}$ on X as follows:

$$xR^{T_A}y \Leftrightarrow T_A(x) \leq T_A(y) \text{ and } \bigwedge(T_A(\varrho(x, y))) \geq T_A(y)$$

$$xR^{I_A}y \Leftrightarrow \ I_A(x) \leq I_A(y) \text{ and } \bigwedge(I_A(\varrho(x, y))) \geq I_A(y)$$

$$xR^{F_A}y \Leftrightarrow \ F_A(x) \geq F_A(y)$$

$$\text{and } \bigvee(F_A(\varrho(x, y))) \leq F_A(y) \text{ and } R = R^{T_A} \cap R^{I_A} \cap R^{F_A}.$$

Theorem 4.15. *Let $(X, \varrho, 0)$ be a hyper BCK-algebra, $A = (T_A, I_A, F_A)$ be a single-valued neutrosophic hyper BCK-ideal of X and $x, y \in X$.*

(i) R is an equivalence relation on X.

(ii) if A is one to one and xRy, then $\forall\ z \in X$ we have $(x\varrho\ z)R(y\varrho\ z)$ and $(z\varrho\ x)R(z\varrho\ y)$.

(iii) if A is one to one, xRy and uRw then $(x\varrho\ u)R(y\varrho\ w)$ $\forall\ u, w \in X$.

Proof: (i) By Theorem 4.5, $T_A(x) \leq \bigwedge(T_A(x\varrho\, x)), I_A(x) \leq \bigvee(I_A(x\varrho\, x))$, $F_A(x) \geq \bigwedge(F_A(x\varrho\ x))$ and so R is a reflexive relation. Let $x, y \in X$ such that xRy. Then $T_A(x) \leq T_A(y), I_A(x) \leq I_A(y)$, $F_A(x) \geq F_A(y)$, $\bigwedge(T_A(\varrho(x, y))) \geq T_A(y), \bigwedge(I_A(\varrho(x, y))) \geq I_A(y)$ and $\bigvee(F_A(\varrho(x, y))) \leq F_A(y)$. Since

$$T_A(x) \geq T_{min}(T_A(y), \bigwedge(T_A(x\varrho\ y))) \geq T_{min}(T_A(y), T_A(y)) = T_A(y)$$

$$I_A(x) \geq T_{min}(I_A(y), \bigwedge(I_A(x\varrho\ y))) \geq T_{min}(I_A(y), I_A(y)) = I_A(y)$$

$$F_A(x) \leq S_{max}(F_A(y), \bigvee(F_A(x\varrho\ y))) \leq S_{max}(F_A(y), F_A(y)) = F_A(y)$$

we get that $T_A(x) = T_A(y), I_A(x) = I_A(y), F_A(x) = F_A(y)$. Using Theorem 4.5, $\bigwedge(T_A(y\varrho\, x)) \geq T_A(y) = T_A(x), \bigwedge(I_A(y\varrho\, x)) \geq I_A(y) = I_A(x)$ and $\bigvee(F_A(y\varrho\, x)) \leq F_A(y) = F_A(x)$ so R is a symmetric relation. Let xRy and yRz. Then $T_A(x) = T_A(y) = T_A(z), I_A(x) = I_A(y) = I_A(z), F_A(x) = F_A(y) = F_A(z)$ and clearly R is a transitive relation.

(ii) Let xRy and $z \in X$. Then by (i), $T_A(x) = T_A(y), I_A(x) = I_A(y), F_A(x) = F_A(y)$ and since A is a one to one map, we have $x = y$. Hence there exists $a \in x\varrho\, z$ and $y \in y\varrho\, z$ such that $T_A(a) \leq T_A(b)$, $\bigwedge(T_A(a\varrho\, b)) \geq T_A(b), I_A(a) \leq I_A(b), \bigwedge(I_A(a\varrho\, b)) \geq I_A(b)$ and $F_A(a) \geq F_A(b), \bigvee(F_A(a\varrho\, b)) \leq F_A(b)$. Therefore $(x\varrho\, z)R(y\varrho\, z)$ and in a similar way get that $(z\varrho\, x)R(z\varrho\, y)$.

(iii) Let xRy and uRw. Then by (ii), $(x\varrho\, u)R(y\varrho\, u)$ and $(y\varrho\, u)R(y\varrho\, w)$. Using the transitivity of R, we get that $(x\varrho\, u)R(y\varrho\, w)$. □

COROLLARY 4.16. Let $(X, \varrho, 0)$ be a hyper BCK-algebra and $A = (T_A, I_A, F_A)$ be a single-valued neutrosophic hyper BCK-ideal of X and $x, y \in X$.

(i) if A is one to one, then R is a congruence relation on X;

(ii) $R(0) = X_A$ and if A is one to one, then $R(0) = \{0\}$;

(iii) if A is one to one, then R is a strongly regular relation on X.

PROOF: (i) Immediate by Theorem 4.15.

(ii) Let $x \in R(0)$. Then by Theorem 4.15, $T_A(x) = T_A(0), I_A(x) = I_A(0), F_A(x) = F_A(0)$ and so $R(0) = X_A$. Since A is one to one, we get that $X_A = \{x \mid T_A(x) = T_A(0), I_A(x) = I_A(0), F_A(x) = F_A(0)\} = \{0\}$.

(iii) Let $x, y, z \in X$ and xRy. Then $x = y$ and so $x\varrho\, z = y\varrho\, z$. Therefore $(x\varrho\, z)\overline{\overline{R}}(y\varrho\, z), (z\varrho\, x)\overline{\overline{R}}(z\varrho\, y)$ and so R is a strongly regular relation. □

THEOREM 4.17. *Let $(X, \varrho, 0)$ be a (weak commutative) hyper BCK-algebra and $A = (T_A, I_A, F_A)$ be a one to one single-valued neutrosophic hyper BCK-ideal of X. Then, $(X/R, \varrho', R(0))$ is a (BCK-algebra) hyper BCK-algebra such that $\forall\, x, y \in X, R(x)\varrho' R(y) = R(x\varrho\, y)$.*

PROOF: By Corollary 4.16, ϱ' is well-defined and the proof is straightforward. □

THEOREM 4.18. *Let $(X, \varrho_1, 0)$ and $(Y, \varrho_2, 0')$ be (weak commutative) hyper BCK-algebras and $A = (T_A, I_A, F_A)$ be a one to one single-valued neutrosophic hyper BCK-ideal of Y. If $f : X \to Y$ is an epimorphism, then*

(i) *A_f is a single-valued neutrosophic hyper BCK-ideal of X;*

(ii) *$X/R_f \cong Y/R$ such that $xR_fx' \Leftrightarrow T_A(f(x)) \leq T_A(f(x')), I_A(f(x)) \leq I_A(f(x')), F_A(f(x)) \geq F_A(f(x')), \bigwedge(T_A(f(x\varrho\ x'))) \geq T_A(f(x'))$, $\bigwedge(I_A(f(x\varrho\ x'))) \geq I_A(f(x'))$ and $\bigvee(F_A(f(x\varrho\ x'))) \leq F_A(f(x'))$, where $x, x' \in X$.*

Proof: (i) Clearly for all $x \in X$, $T_{A_f}(0) = T_A(f(0)) = T_A(0') \geq T_A(f(x)) = T_{A_f}(x), I_{A_f}(0) = I_A(f(0)) = I_A(0') \geq I_A(f(x)) = I_{A_f}(x)$ and $F_{A_f}(0) = F_A(f(0)) = F_A(0') \leq F_A(f(x)) = F_{A_f}(x)$. Let $x, x' \in X$. Since A is a single-valued neutrosophic hyper BCK-ideal of Y, we get that

$$\begin{aligned} T_{A_f}(x) = T_A(f(x)) &\geq T_{min}\{T_A(f(x')), \bigwedge(T_A(f(x)\varrho_2 T_A(f(x')))\} \\ &= T_{min}\{T_A(f(x'), \bigwedge(T_A(f(x\varrho_1 x')))\} \\ &= T_{min}\{T_{A_f}(x'), \bigwedge(T_{A_f}(x\varrho_1 x'))\}. \end{aligned}$$

In similar a way, can see that $I_{A_f}(x) \geq T_{min}\{I_{A_f}(x'), \bigwedge(I_{A_f}(x\varrho_1 x'))\}$ and $I_{A_f}(x) \leq S_{max}\{F_{A_f}(x'), \bigvee$
$(F_{A_f}(x\varrho_1 x'))\}$.

(ii) Since T_A and T_{A_f} are single-valued neutrosophic hyper BCK-ideals of Y, X, respectively, then by Theorem 4.17, $(X/R_f, \varrho\,', R_f(0))$ and $(Y/R, \varrho\,', R(0'))$ are (BCK-algebras) hyper BCK-algebras. Now, define a map $\varphi : X/R_f \to Y/R$ by $\varphi(R_f(x)) = R(f(x))$. Let $x, x' \in X$. Then

$$\begin{aligned} \varphi(R_f^{T_A}(x)) &= \varphi(R_f^{T_A}(x')) \Leftrightarrow f(x)R^{T_A}f(x') \\ &\Leftrightarrow T_A(f(x)) \leq T_A(f(x')), \bigwedge(T_A(f(x)\varrho\ _2 f(x'))) \geq T_A(f(x')) \\ &\Leftrightarrow T_{A_f}(x) \leq T_{A_f}(x') \text{ and } \bigwedge(T_A(f(x\varrho_1 x'))) \geq T_A(f(x')) \\ &\Leftrightarrow T_{A_f}(x) \leq T_{A_f}(x') \text{ and } \bigwedge(T_{A_f}(x\varrho_1 x')) \geq T_{A_f}(x') \\ &\Leftrightarrow R_f^{T_A}(x) = R_f^{T_A}(x'). \end{aligned}$$

In similar a way, $\varphi(R_f^{I_A}(x)) = \varphi(R_f^{I_A}(x')) \Leftrightarrow R_f^{I_A}(x) = R_f^{I_A}(x')$ and $\varphi(R_f^{F_A}(x)) = \varphi(R_f^{F_A}(x')) \Leftrightarrow R_f^{F_A}(x) = R_f^{F_A}(x')$. It follows that $\varphi(R_f(x)) = \varphi(R_f(x')) \Leftrightarrow R_f(x) = R_f(x')$ and hence φ is a well-defined and one to one map. Clearly φ is an epimorphism, and so it is an isomorphism. □

COROLLARY 4.19. (Isomorphism Theorem) Let $(X, \varrho, 0)$ be a hyper BCK-algebra and $A = (T_A, I_A, F_A)$, $A' = (T'_A, I'_A, F'_A)$ be one to one single-valued neutrosophic hyper BCK-ideals of X such that $A(0) = A'(0)$. Then

(i) $A' \cap A$ is a single-valued neutrosophic hyper BCK-ideal of X;

(ii) $(X_A \varrho X_{A'})/R_A \cong X_A/R_{A' \cap A}$.

PROOF: (i) Let $x \in X$. Then

$$(T'_A \cap T_A)(0) = T_{min}(T'_A(0), T_A(0)) \geq T_{min}(T'_A(x), T_A(x)) = (T'_A \cap T_A)(x), (I'_A \cap I_A)(0) = T_{min}(I'_A(0), I_A(0)) \geq T_{min}(I'_A(x), I_A(x)) = (I'_A \cap I_A)(x), (F'_A \cap F_A)(0) = S_{max}(F'_A(0), F_A(0)) \leq S_{max}(F'_A(x), F_A(x)) = (F'_A \cap F_A)(x).$$

Let $x, y \in X$. Then

$$\begin{aligned} & (T'_A \cap T_A)(x) = T_{min}(T'_A(x), T_A(x)) \\ \geq\ & T_{min}[T_{min}[T'_A(y), \bigwedge(T'_A(x\varrho\, y))], T_{min}[T_A(y), \bigwedge(T_A(x\varrho\, y))]] \\ =\ & T_{min}[T_{min}[T'_A(y), T_A(y)], T_{min}[\bigwedge(T'_A(x\varrho\, y)), \bigwedge(T_A(x\varrho\, y))]] \\ =\ & T_{min}[(T'_A \cap T_A)(y), \bigwedge((T'_A \cap T_A)(x\varrho\, y))] \end{aligned}$$

In similar a way can see that $(I'_A \cap I_A)(x) \geq T_{min}[(I'_A \cap I_A)(y), \bigwedge((I'_A \cap I_A)$ and $(F'_A \cap F_A)(x) \leq S_{max}[(F'_A \cap F_A)(y), \bigvee((F'_A \cap F_A)$.

(ii) By Theorem 4.10, $A' \cap A$ is a single-valued neutrosophic hyper BCK-ideal of X_A, then we define $\varphi : X_A/R_{A' \cap A} \to (X_A \varrho\ X_{A'})/R_A$ by $\varphi(R_{A' \cap A}(x)) = R_A(x)$. Let $x, x' \in X_A$ and $R_{A' \cap A}(x) = R_{A' \cap A}(x')$. Then $(A' \cap A)(x) = (A' \cap A)(x')$ and since $A' \cap A$ is one to one, we get that $x = x'$. Hence $R_A(x) = R_A(x')$. Moreover, $\varphi(R_{A' \cap A}(x)\varrho\ 'R_{A' \cap A}(x')) = \varphi(R_{A' \cap A}(x\varrho\ x')) = R_A(x\varrho\ x') = R_A(x)\varrho\ 'R_A(x')$ and so φ is a homomorphism. Clearly φ is bijection and so is an isomorphism. □

Example 4.20. Let $X = \{0, 1, 2, 3, 4, 5\}$. Then $A = (T_A, I_A, F_A)$ is a single-valued neutrosophic hyper BCK-subalgebra of X hyper BCK-algebra $(X, \varrho, 0)$ as follows:

ϱ	0	1	2	3	4	5
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
2	$\{2\}$	$\{2\}$	$\{0,2\}$	$\{2\}$	$\{2\}$	$\{2\}$
3	$\{3\}$	$\{3\}$	$\{3\}$	$\{0,3\}$	$\{3\}$	$\{3\}$
4	$\{4\}$	$\{4\}$	$\{4\}$	$\{4\}$	$\{0,4\}$	$\{0\}$
5	$\{5\}$	$\{5\}$	$\{5\}$	$\{5\}$	$\{5\}$	$\{0,5\}$

and

	0	1	2	3	4	5
T_A	0.72	0.61	0.54	0.34	0.27	0.19
I_A	0.19	0.8	0.2	0.21	0.26	0.25
F_A	0.15	0.28	0.34	0.39	0.48	0.61

(i) If $\alpha = 0.5, \beta = 0.7, \gamma = 0.4$, then $T^{\alpha} = \{0,1,2\}, I^{\beta} = \{0,2,3,4,5\}$, $F^{\gamma} = \{0,1,2,3\}$ and so $A^{(\alpha,\beta,\gamma)} = \{0,2\}$, which is a hyper BCK-subalgebra of $(X, \varrho, 0)$.

(ii) Consider $S = \{0,5\}$, $\alpha = 0.5, \alpha' = 0.7, \beta = 0.6, \beta' = 0.8, \gamma = 0.85$ and $\gamma' = 0.9$. Then

$$T_A^{[\alpha,\alpha']} = \{(0,0.5),(1,0.7),(2,0.7),(3,0.7),(4,0.7),(5,0.5)\}$$
$$I_A^{[\beta,\beta']} = \{(0,0.6),(1,0.8),(2,0.8),(3,0.8),(4,0.8),(5,0.6)\}$$
$$F_A^{[\gamma,\gamma\beta']} = \{(0,0.85),(1,0.9),(2,0.9),(3,0.9),(4,0.9),(5,0.85)\}$$

are fuzzy hyper BCK-subalgebras of X and $A = (T_A^{[\alpha,\alpha']}, I_A^{[\beta,\beta']}, F_A^{[\gamma,\gamma']})$ is a single-valued neutrosophic hyper BCK-subalgebra of X.

(iii) Let $\alpha = 0.3, \beta = 0.1$ and $\gamma = 0.5$. Then $A^{\lfloor\alpha,\beta,\gamma\rfloor} = T^{(\alpha)} \cap I^{(\beta)} \cap F^{(\gamma)} = \{0,1,2,3\} \cap \{0,1,2,3,4,5\} \cap \{0,1,2,3,4\} = \{0,1,2,3\}$. Clearly $2 \in A^{\lfloor\alpha,\beta,\gamma\rfloor}$ and $0 \ll 2$, then $0 \in A^{\lfloor\alpha,\beta,\gamma\rfloor}$.

Example 4.21. Let $X = \{0,1,2,3\}$ and $Y = \{0',a,b,c\}$. Then $A = (T_A, I_A, F_A)$ is a single-valued neutrosophic hyper BCK-ideal of hyper BCK-algebra $(X, \varrho, 0)$ as follows:

ϱ	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$	$\{1\}$
2	$\{2\}$	$\{2\}$	$\{0,2\}$	$\{2\}$
3	$\{3\}$	$\{3\}$	$\{3\}$	$\{0,3\}$

ϱ'	$0'$	a	b	c
$0'$	$\{0'\}$	$\{0'\}$	$\{0'\}$	$\{0'\}$
a	$\{a\}$	$\{0',a\}$	$\{a\}$	$\{a\}$
b	$\{b\}$	$\{b\}$	$\{0',b\}$	$\{b\}$
c	$\{c\}$	$\{c\}$	$\{c\}$	$\{0',c\}$

and

	0	1	2	3
T_A	0.93	0.73	0.13	0.13
I_A	0.87	0.67	0.1	0.05
F_A	0.13	0.23	0.33	0.4

(i) Define $f : Y \to X$ by $f = \{(0', 0), (c, 1), (b, 2), (a, 3)\}$, clearly f is a homomorphism. Hence A_f is a single-valued neutrosophic hyper BCK-ideal of hyper BCK-algebra $(Y, \varrho', 0')$ that is obtained as follows:

	$0'$	a	b	c
T_{A_f}	0.93	0.13	0.13	0.73
I_{A_f}	0.87	0.05	0.1	0.67
F_{A_f}	0.13	0.4	0.33	0.23

(ii) Computations show that $R^{T_A} = \{(x, x), (2, 3), (3, 2) \mid x \in X\}, R^{I_A} = \{(x, x) \mid x \in X\}, R^{F_A} = \{(x, x) \mid x \in X\}$ and so $R = \{(x, x) \mid x \in X\}$ that is a congruence relation. It follows that $R_f = \{(x, x) \mid x \in Y\}$ and so $X/R \cong X \cong Y \cong Y/R_f$.

(iii) Clearly $X_A = R(0) = \{0\}$ and $ker(f) = \{0\}$ that is a trivial (hyper) BCK-ideal. Also for all $x \in ker(f)$ and for all $y \in X, T_{min}(T_{A_f}(x), I_{A_f}(x)) \geq T_{min}(T_A(y), I_A(y))$.

Example 4.22. Let $X = \{0, 1, 2, 3\}$. Then $A = (T_A, I_A, F_A)$ and $A' = (T_{A'}, I_{A'}, F_{A'})$ are single-valued neutrosophic hyper BCK-ideals of hyper BCK-algebra $(X, \varrho, 0)$ as follows:

ϱ	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$	$\{0\}$
2	$\{2\}$	$\{2\}$	$\{0, 2\}$	$\{0\}$
3	$\{3\}$	$\{3\}$	$\{3\}$	$\{0\}$

	0	1	2	3
T_A	0.95	0.85	0.25	0.25
I_A	0.9	0.8	0.2	0.1
F_A	0.15	0.2	0.3	0.35

and

	0	1	2	3
$T_{A'}$	0.95	0.75	0.15	0.15
$I_{A'}$	0.9	0.7	0.1	0.05
$F_{A'}$	0.15	0.25	0.35	0.4

Then

$$\begin{aligned} A \cap A' &= A', X_A = X_{A'} = \{0\}, R^{T_A} = R^{T_{A'}} = \{(x,x),(2,3),(3,2)|x \in X\}, \\ & R^{I_A} = R^{I_{A'}} = \{(x,x),(2,3),(3,2) \mid x \in X\}, R^{F_A} = R^{F_{A'}} = \\ & \{(x,x) \mid x \in X\} \text{ and so } R_A = R^{T_A} \cap R^{I_A} \cap R^{F_A} = R_{A'} = \\ & R^{T_{A'}} \cap R^{I_{A'}} \cap R^{F_{A'}} = \{(x,x) \mid x \in X\}. \end{aligned}$$

It follows that $(X_A \varrho X_{A'}) = \{0\}$ and so $(X_A \varrho X_{A'})/R_A = \{0\}/R_A \cong \{0\}/R_{A'} \cong \{0\}/R_{A' \cap A} = X_A/R_{A' \cap A}$.

5. Conclusion

In some problems in the real world, there are many uncertainties (such as fuzziness, incompatibilities, and randomness), in an expert system, belief system, and information fusion, especially in some scopes of computer sciences such as artificial intelligence. Thus we need to deal with uncertain information and logic establishes the foundations for it, because computer sciences are based on classical logic. The concept of BCK-algebra is one of the important logical algebras that are applied in computer sciences and other networking sciences. In addition, defects in classical algebras that can not work in groups and have limitations can be eliminated with the help of logical hyperalgebra. Thus the concept of hyper BCK-algebra is an important logical hyperalgebra that is applied in the computer sciences and other hypernetworking sciences that some groups of elements must be operated together and have been proposed for semantical hypersystems of logical hypersystems. In addition in some applications such as expert systems, belief systems, and information fusion, we should consider not only the truth membership supported by the evidence but also the falsity-membership against the evidence, which is beyond the scope of fuzzy subsets. Thus the concept of a neutrosophic subset is a powerful general formal framework that generalizes the concept of the classic set and the fuzzy subset is characterized by a truth-membership function, an indeterminacy-membership function, and a falsity-membership function. This assumption is very important in a lot of situations such as information fusion when we try to combine the data from different sensors. In this paper, we consider the collectivity of logical (hyper)BCK-algebras and single-valued neutrosophic hyper BCK-subalgebras to solve some complex real problems dealing with the principles of logical hyperalgebra(one or more groups based on these principles must be combined) and have uncertain information such as complex intelligent hypernetworks and related other sciences. Thus the non-classical mathematics together with the concept of neutrosophic subset, therefore, has nowadays become a useful tool in applications mathematics and complex hypernetworks. Moreover, we can refer to some academic contributions of single-valued neutrosophic subsets such as single-valued neutrosophic directed (hyper)graphs and applications in networks [4], application of single-valued neutrosophic in lifetime in wireless sensor (hyper)network [4], an application of single-valued neutrosophic subsets in social (hyper)networking [4], application of single-valued neutrosophic sets in medical diagnosis, application of neutro hyper BCK-algebras and single-valued neutrosophic hyper BCK-subalgebras in economic hypernetwork [7], and application of neutro hyper BCK-algebras and single-valued neutrosophic hyper BCK-subalgebras in data (hyper)networks [7]. To conclude, we considered the notion of single-valued neutrosophic hyper BCK-ideals and investigated some of their new useful properties. We considered that for any $\alpha \in [0,\ 1]$ there is an algebraic relation between of a single-valued neutrosophic subset hyper BCK-subalgebra, $A = (T_A, I_A, F_A)$ and $A = ({T_A}^{\alpha}, {I_A}^{\alpha}, {F_A}^{\alpha})$. In addition, with respect to the concept of hyper BCK-ideals of given hyper BCK-algebra, is constructed quotient BCK-algebra structures. On any nonempty set, is constructed an extendable single-valued neutrosophic BCK-(ideal)subalgebra and isomorphism theorem of single-valued neutrosophic hyper BCK-ideals is obtained. One of the advantages of this study is the conversion of complex hypernetworks to complex networks in such a way that all the details of the complex hypernetworks are preserved and transferred to the complex networks, but there are some limitations in this work. Although neutrosophic subsets are more flexible and useful as compared to all fuzzy theories, there are some limitations whence we need more than three functions in designing and modeling the real problem with complexity and high dimension. Also, the computations of single-valued neutrosophic hyper BCK-ideals for any given hyper BCK-algebras with large cardinal is hard and so the related mathematical tools such as congruence and strongly relations, nontrivial homomorphisms are complicated. Hence these problems prevent us from having a definite and simple algorithm for our computations.

We wish this research is important for the next studies in logical hyperalgebras. In our future studies, we hope to obtain more results regarding single-valued neutrosophic (hyper)BCK-subalgebras and their applications in handing information regarding various aspects of uncertainty, non-classical mathematics (fuzzy mathematics or great extension and development of classical mathematics) that are considered to be a more powerful technique than classical mathematics.

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