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SEQUENT CALCULI FOR ORTHOLOGIC WITH STRICT IMPLICATION

Abstract

In this study, new sequent calculi for a minimal quantum logic (**MQL**) are discussed that involve an implication. The sequent calculus **GO** for **MQL** was established by Nishimura, and it is complete with respect to ortho-models (O-models). As **GO** does not contain implications, this study adopts the strict implication and constructs two new sequent calculi $\mathbf{GOI}_1$ and $\mathbf{GOI}_2$ as the expansions of **GO**. Both $\mathbf{GOI}_1$ and $\mathbf{GOI}_2$ are complete with respect to the O-models. In this study, the completeness and decidability theorems for these new systems are proven. Furthermore, some details pertaining to new rules and the strict implication are discussed.

Keywords: Quantum logic, sequent calculus, completeness theorem, implication, orthologic.

1. Introduction

Quantum logic (**QL**) has been introduced in order to manage strange propositions of quantum physics, such as uncertainty principle. Many structures have been studied to represent and analyze such propositions. In particular, *Orthomodular lattices* describe the propositional spaces of quantum physics and have been studied as the main structure of **QL** in the work by Birkhoff and Von Neumann [3]. An orthomodular lattice is based on closed subspaces of a Hilbert space, which is a state space of particles in quantum physics. Instead of these lattices, the *Kripke model* of **QL**, the orthomodular model (*OM-model*), can be used, which also describes a state space

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of quantum particles [12]. *Ortholattices*, which are conceptually simpler than orthomodular lattices, have also been studied. The logic based on ortholattices is a *minimal QL* (**MQL**) or *orthologic*. Moreover, the Kripke model for **MQL**, i.e., ortho-model (*O-model*), also exists [12].

As it is usually studied, **QL** does not contain logical implications and includes only negations, conjunctions, and disjunctions. Several implications in **QL** have been suggested; however, they all have difficulties for varying reasons [12, 14]. Therefore, the deduction systems, such as the Hilbert style axiomatization or sequent calculi that include implications, are not well developed. This problem also holds for **MQL**. In **MQL**, the number of appropriate implications is even smaller than that in **QL**. Therefore, as a part of research to address these problems, this study constructs two new sequent calculi for **MQL** that include rules for specific implication and provides the completeness theorems with respect to O-models.

When the implications are added to **QL** or **MQL**, some problems are encountered. In classical logic, the implication $A \to B$ and $\neg A \vee B$ can be identified. However, in **QL**, if $\neg A \vee B$ is adopted as an implication, critical properties for the implication, such as modus ponens, do not hold. Therefore, in **QL**, many other implications have been considered. Among them, *polynomial implications* that can be defined in terms of connectives $\neg$, $\wedge$ and $\vee$, have been predominantly studied. The polynomial implication *Sasaki arrow* $\neg A \vee (A \wedge B)$ has attracted the most attention in **QL**. In addition to the Sasaki arrow, the *contrapositive Sasaki arrow* $\neg(A \vee B) \vee B$, the *relevance arrow* $(A \wedge B) \vee (\neg A \wedge B) \vee (\neg A \wedge \neg B)$, and two other arrows have been explored [12, 13, 14]. These implications are the only polynomial implications that have suitable properties in terms of the orthomodular lattice and have been studied from both physical and mathematical standpoints [21].

These implications have been investigated in many ways because of their strangeness. The meaning and properties of these implications in quantum physics are associated with the notion of projections [12, 22]. For example, the Sasaki arrow $\neg A \vee (A \wedge B)$ can be translated as "after a measurement of A, if the state is projected to a state which A is true, then B is true." By utilizing this property and embedding the projection relationship in the model, various properties of the Hilbert space can be analyzed using the Kripke model [22]. Recently, these implications have been used in the context of *quantum set theory*, achieving results in the analysis of observed values in quantum mechanics [29]. The algebraic features of these implications have been widely studied in the case of orthomodular lattices and ortholattices [1, 6, 4, 8, 17]. These studies focus on the logical aspects of orthomodular lattices using implications. Furthermore, concepts regarding orthomodular lattices, such as semilattices, have been analyzed, where implications occupy a principal position [10, 9, 11]. Among them, *implication algebras* have been discussed as implication studies that exclude other logical operators [1, 7, 13, 15, 16]. In this field, the properties of orthomodular lattices have been elucidated by analyzing algebraic axioms and conditions for implications. This algebraic research is a purely mathematical study rather than a research related to quantum physics. Few studies on **QL** have employed binary relational models compared with the number of studied on such algebraic studies. Models using binary relations can express the dynamic relations of quantum physics, and some dynamic concepts are closely related to implications. Therefore, research using the Kripke model, such as that proposed in this study, should be conducted.

However, in ortholattices, polynomial implications do not satisfy modus ponens. In this study, the notion of *strict implication* proposed in the literature [12] is adopted for **MQL**, as the strict implication exhibits good mathematical properties, particularly in the Kripke models, and has physically significant meanings. In an ortholattice L, strict implication is defined with some restrictions as follows [12]:

$$a \to b = \bigsqcup \{c \in L \mid c \neq 0 \wedge \forall d((d \neq 0 \wedge c \not\leq d' \wedge d \leq a) \Rightarrow d \leq b)\}$$

where $\leq$ is the order in L, $\sqcup$ is the join, and 0 is the least element. Although this definition seems complicated at the first glance, the definition in the Kripke model corresponding to this definition is clear. This is one reason for adopting the Kripke model in the present study. Intuitively, from a quantum physics viewpoint, the strict implication $A \to B$ can be translated as "after the measurement of any physical quantity, if A is true, then B is true."

Some advantages of the strict implication should be noticed.

- In ortholattices, the Sasaki arrow does not satisfy modus ponens. However, the strict implication satisfies modus ponens in both lattices. Therefore, when **MQL** is considered, the strict implication is more suitable than the Sasaki arrow.

- All material implications are abbreviations of formulas constructed using conjunctions, disjunctions, and negations. However, the strict implication cannot be (finitely) constructed by means of these symbols [12]. Therefore, when the strict implication is added to **MQL**, the descriptive ability of the logic increases.

- The definition of the strict implication in O-models is similar to that of the implication in intuitionistic logic. The deduction rules of the strict implication are similar to those in the sequent calculus **LBP** for the *basic propositional logic* (**BPL**) [20, 31]. Therefore, we can analyze the relationship between **QL** and other logics using this implication.

Although a sequent calculus for **MQL** with the strict implication exists, a sequent is a *labeled* type sequent [23]. From the logic viewpoint, it is important to construct and discuss a simple type of sequent calculus for logic. Furthermore, some deduction systems for **QL** or **MQL** that involve implications are studied; however, they are either not sequent calculi or the implication used in these systems is not a strict implication [5, 28]. Sequent calculi **GO** [25] and **GMQL** [26, 27] have been studied as foundational sequent calculi for **MQL** which only includes $\neg$, $\wedge$ and $\vee$. The present study adopts **GO** for technical reasons, which is presented in Section 6. The rules for the strict implication are added to **GO**, and new calculi $\mathbf{GOI}_1$ and $\mathbf{GOI}_2$ are constructed. This study proves the completeness theorem for these new systems.

Some formulas valid with general implications in other logics are invalid with the strict implication in O-models. For example, $p \to (q \to p)$ is invalid. Therefore, general rules for implications, for example, such as those for the implication in classical logic, cannot be used. As mentioned earlier, this study uses a modified version of the rule for the implication of **LBP** reported in the literature [20]. The implication of **BPL** also does not satisfy some ordinary natures of implication. The semantics of this implication in a Kripke model is the same as that of the strict implication. In other words, $x \models A \to B$ is regarded as "for all y, such that xRy, if $y \models A$, then $y \models B$."

In Sections 2 and 3, some basics and the sequent calculus of **MQL** are presented. In Sections 4 and 5, the new sequent calculi $\mathbf{GOI}_1$ and $\mathbf{GOI}_2$ are constructed and some related theorems are proven. The deduction ability of $\mathbf{GOI}_1$ and $\mathbf{GOI}_2$ is intrinsically the same; however, the rules for the strict implication are different and each has pros and cons. In Section 6, some details regarding the strict implication and rules are discussed.

2. Basics

This study uses language that has a denumerable infinite set of propositional variables, the propositional constant $\perp$, the unary connective $\neg$, and binary connectives $\wedge$ and $\rightarrow$. Formulas are constructed in the usual way. We denote propositional variables by $p, q, \ldots$, formulas by $A, B, C, \ldots$, and finite sets of formulas by $\Gamma, \Delta, \Sigma, \Pi, \ldots$. We use $A \vee B$ as the abbreviation of $\neg(\neg A \wedge \neg B)$.

An *O-frame* is a pair $(W, \perp)$, where W is a nonempty set, and $\perp$ is an irreflexive and symmetric binary relation on W. For traditional reasons, we use the symbol $\perp$ in two ways; one as a relation, the other as a formula. The relation symbol $\perp$ came from the orthogonal relation in the Hilbert space, and the formula symbol $\perp$ denotes the bottom. They can be distinguished by the context.

We write $x \not\perp y$ if not $x \perp y$. We write $x \perp X$ if, for all $y \in X$, $x \perp y$, where $x \in W$ and $X \subseteq W$. Given $X \subseteq W$, we define the set $X^{\perp} = \{x \in W | x \perp X\}$. We say that X is $\perp$-*closed* if $X^{\perp\perp} = X$.

An *O-model* is a triple $(W, \perp, V)$, where $(W, \perp)$ is an O-frame and V is a function assigning each propositional variable p to a $\perp$-closed subset of W.

We define the set $\|A\|$ by induction on the composition of A as follows.

$$\|p\| = V(p)$$

$$\|A \wedge B\| = \|A\| \cap \|B\|$$

$$\|\neg A\| = \|A\|^{\perp}$$

$$\|A \rightarrow B\| = \{x \in W | \text{ for all } y \in W, \text{ if } x \not\perp y \text{ and } y \in \|A\|, y \in \|B\| \ \}$$

$$\|\perp\| = \emptyset$$

A is *true* at x if $x \in \|A\|$ and write $x \models A$. It is easy to evaluate that $\|\neg A\| = \|A \rightarrow \perp\|$ is fulfilled in this definition. Therefore, we regard $\neg A$ as the abbreviation of $A \rightarrow \perp$.

Lemma 2.1. *For all* $\|A\|$, $\|A\|$ *is* $\perp$*-closed.*

Proof: In the cases of $\|p\|$, $\|A \wedge B\|$ and $\|\neg A\|$, see [25]. For all $x \in \|A \to B\|$, $x \perp \{y \in W \mid y \models A \text{ and } y \not\models B\}$. Then, $\{y \in W \mid y \models A \text{ and } y \not\models B\} \in \|A \to B\|^{\perp}$. Therefore, if $z \in \|A \to B\|^{\perp\perp}$ then $z \perp \{y \in W \mid y \models A$ and $y \not\models B\}$. It means $z \in \|A \to B\|$. That is, there is no point z which satisfies $z \notin \|A \to B\|$ and $z \in \|A \to B\|^{\perp\perp}$. Therefore, $\|A \to B\|$ is $\perp$-closed. □

3. Sequent calculus GO

GO is defined below [25].

Axiom: $A \Rightarrow A$

Rules:

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\ (\text{cut}) \qquad \frac{\Gamma \Rightarrow \Delta}{\Pi, \Gamma \Rightarrow \Delta, \Sigma}\ (\text{weakening})$$

$$\frac{A, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ (\wedge\text{L}) \qquad \frac{B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ (\wedge\text{L}) \qquad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\ (\wedge\text{R})$$

$$\frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta}\ (\neg\text{L}) \qquad \frac{A \Rightarrow \Delta}{\neg \Delta \Rightarrow \neg A}\ (\neg\text{R})$$

$$\frac{A, \Gamma \Rightarrow \Delta}{\neg\neg A, \Gamma \Rightarrow \Delta}\ (\neg\neg\text{L}) \qquad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \neg\neg A}\ (\neg\neg\text{R})$$

In [25], Γ, Δ, Π and Σ are defined as probable infinite sets. We restrict these to finite sets because infinite sets are not essential here.

Consider an O-model $(W, \perp, V)$. Sequent $\Gamma \Rightarrow \Delta$ is *false* at $x \in W$ if for all formulas $A \in \Gamma$, $x \models A$, and, for all formulas $B \in \Delta$, $x \not\models B$. If $\Gamma \Rightarrow \Delta$ is not false at x, then it is *true* at x. Sequent $\Gamma \Rightarrow \Delta$ is falsifiable if there exists an O-model $(W, \perp, V)$ and $x \in W$, and $\Gamma \Rightarrow \Delta$ is false at x. If $\Gamma \Rightarrow \Delta$ is unfalsifiable, we say $\Gamma \Rightarrow \Delta$ is *valid*.

Theorem 3.1. *The soundness and completeness theorem for* **GO**. $\Gamma \Rightarrow \Delta$ *is provable in* **GO** *if, and only if, (iff)* $\Gamma \Rightarrow \Delta$ *is valid.*

Proof: See [25]. □

4. Sequent calculus $\mathbf{GOI}_1$

In this section, a sequent calculus including the strict implication is established. The sequent calculus $\mathbf{GOI}_1$ is defined as an expansion of **GO**. The rule ($\to$ R) and axiom $\perp \Rightarrow$ are added to **GO**. The rule ($\to$ R) is the transformation of the rule ($\to$) in [20]. Because this rule ($\to$ R) is complex, using $\mathbf{GOI}_2$ in the next chapter for the main calculus of **MQL** with the strict implication would be better. However, ($\to$ R) is useful to prove the completeness theorem. Therefore, first the details of $\mathbf{GOI}_1$ are shown. The definitions of truth, falsity, and validity of a sequent are identical to that in **GO**.

$$\perp \Rightarrow \quad (\perp)$$

$$\frac{\Gamma_1, A \Rightarrow B, \Delta_1, \Sigma \quad \Gamma_2, A \Rightarrow B, \Delta_2, \Sigma \ \ldots \ \Gamma_{2^n}, A \Rightarrow B, \Delta_{2^n}, \Sigma}{C_1 \to D_1, C_2 \to D_2, \ \ldots \ , C_n \to D_n, \Pi \Rightarrow A \to B, \Lambda} \quad (\to\mathrm{R})$$

where, $0 \leq n$, $\Gamma_i = \{D_j | j \in \gamma(i)\}$, $\Delta_i = \{C_j | j \in \delta(i)\}$, $\langle \delta(i), \gamma(i) \rangle$ is the i-th element of all partitions of $\{1, \ldots, n\}$. Π and Λ are formula sets. Σ is a set of all formulas of the shape $E \to F$ such that E is included in the premise of the lower sequent and F is included in the conclusion of the lower sequent or $\perp$. Therefore, $\Sigma = \{E \to F \ | E \in \{C_1 \to D_1, \ldots, C_n \to D_n, \Pi\}, F \in \{A \to B, \Lambda, \perp\}\}$.

For example, suppose $\Pi = \{I\}$, $\Lambda = \{J, K\}$, then ($\to$R) is as below in the case of $n = 0$, $n = 1$, and $n = 2$.

$$\frac{A \Rightarrow B, I \to (A \to B), I \to J, I \to K, I \to \perp}{I \Rightarrow A \to B, J, K}$$

$$\frac{A \Rightarrow B, C_1, \Sigma \quad D_1, A \Rightarrow B, \Sigma}{C_1 \to D_1, I \Rightarrow A \to B, J, K}$$

where Σ is $\{(C_1 \to D_1) \to (A \to B), (C_1 \to D_1) \to J, (C_1 \to D_1) \to K, (C_1 \to D_1) \to \perp, I \to (A \to B), I \to J, I \to K, I \to \perp\}$.

$$\frac{A \Rightarrow B, C_1, C_2, \Sigma \qquad D_1, A \Rightarrow B, C_2, \Sigma \qquad D_2, A \Rightarrow B, C_1, \Sigma \qquad D_1, D_2, A \Rightarrow B, \Sigma}{C_1 \to D_1, C_2 \to D_2, I \Rightarrow A \to B, J, K}$$

where Σ is $\{(C_1 \to D_1) \to (A \to B), (C_1 \to D_1) \to J, (C_1 \to D_1) \to K, (C_1 \to D_1) \to \bot, (C_2 \to D_2) \to (A \to B), (C_2 \to D_2) \to J, (C_2 \to D_2) \to K, (C_2 \to D_2) \to \bot, I \to (A \to B), I \to J, I \to K, I \to \bot\}$.

In $\mathbf{GOI}_1$, the rule $(\to \mathrm{L})$ is admissible.

$$\frac{\Gamma_1, \Rightarrow \Delta_1, A \qquad B, \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, A \to B \Rightarrow \Delta_1, \Delta_2} \ (\to\mathrm{L})$$

We will prove this lemma in Section 5.

Theorem 4.1. *The soundness theorem for* $\mathbf{GOI}_1$. *If* $\Gamma \Rightarrow \Delta$ *is provable in* $\mathbf{GOI}_1$, $\Gamma \Rightarrow \Delta$ *is valid.*

Proof: Proven by induction on the construction of a proof. For rules in **GO**, the proof is the same as the proof in [25]. For $(\to\mathrm{R})$, we only evaluate $n = 2$. The other cases are similar. For contradiction, suppose all premises of the rule are valid, and there exists O-model $(W, \perp, V)$ and $x \in W$, such that the conclusion of the rule is false at x. Then, as $A \to B$ is false at x, there exists $y \in W$, satisfying $x \not\perp y$, $y \models A$ and $y \not\models B$. Because we assume that $y \models A$ and all premises are valid, from the first premise, B or C_1 or C_2 or one of the formulas in Σ is true at y; however, B is false at y. Now, suppose $E \to F \in \Sigma$. We have $x \models E$ and $x \not\models F$ by assumption and the definition of $(\to\mathrm{R})$. If $y \models E \to F$, from $y \not\perp x$ and $x \models E$, $x \models F$, which is a contradiction. Therefore, for all $E \to F \in \Sigma$, $y \not\models E \to F$. Therefore, C_1 or C_2 is true at y. In the former case, from $x \models C_1 \to D_1$ and $y \models C_1$, $y \models D_1$. From the second premise, B or C_2 or one of the formulas in Σ is true at y. Similarly, the only possibility is C_2; therefore, C_2 is true at y. To continue this method to the end of premises, B or Σ is the only possibility, which is a contradiction. The latter case and cases of the other possibilities are similar to this method. □

To prove the completeness theorem, we define the set Ω as follows. $\Omega(\Gamma \Rightarrow \Delta) = \{$ All subformulas in $\Gamma \cup \Delta\} \cup \{\neg p \mid p$ appear in some formulas in $\Gamma \cup \Delta\} \cup \{\bot\}$. For example, $\Omega(\neg(p \to q) \Rightarrow r \wedge q) = \{\bot, p, q, r, \neg p, \neg q, \neg r, p \to q, r \wedge q, \neg(p \to q)\}$. For each unprovable sequent $\Gamma \Rightarrow \Delta$, we define a *canonical O-model* $(W_c, \perp_c, V_c)$ of $\Gamma \Rightarrow \Delta$ as follows.

W_c: $\{\Gamma_1 \Rightarrow \Delta_1 | \Gamma_1 \Rightarrow \Delta_1$ is unprovable in $\mathbf{GOI}_1$ and $\Gamma_1 \cup \Delta_1 = \Omega(\Gamma \Rightarrow \Delta)\}$

$\perp_c$: $(\Gamma_1 \Rightarrow \Delta_1)\perp(\Gamma_2 \Rightarrow \Delta_2)$ iff for some A and B, at least one of (1) (2) is true. (1) $A \to B \in \Gamma_1$, $A \in \Gamma_2$ and $B \in \Delta_2$. (2) $A \to B \in \Gamma_2$, $A \in \Gamma_1$ and $B \in \Delta_1$.

V_c: assigns p to the set $\{\Gamma_1 \Rightarrow \Delta_1 | p \notin \Delta_1\}$.

LEMMA 4.2. *$(W_c, \perp_c)$ is an O-frame. $V_c(p)$ is $\perp$-closed. Therefore, $(W_c, \perp_c, V_c)$ is an O-model.*

PROOF: If $(\Gamma_1 \Rightarrow \Delta_1)\perp(\Gamma_1 \Rightarrow \Delta_1)$, there is A and B that $A \in \Gamma_1$, $A \to B \in \Gamma_1$ and $B \in \Delta_1$. But $A, A \to B \Rightarrow B$ is proven using ($\to$L); therefore, $\Gamma_1 \Rightarrow \Delta_1$ can be proven, which is a contradiction. Therefore, for every $\Gamma_1 \Rightarrow \Delta_1 \in W_C$, $(\Gamma_1 \Rightarrow \Delta_1) \not\perp (\Gamma_1 \Rightarrow \Delta_1)$. Symmetry is obvious from the definition. If $p \notin \Omega(\Gamma \Rightarrow \Delta)$, $V_c(p) = W_c$. This is clearly $\perp$-closed. If $p \in \Omega(\Gamma \Rightarrow \Delta)$, for every $\Gamma_1 \Rightarrow \Delta_1 \in W_c$, $p \in \Gamma_1$ or $p \in \Delta_1$. Then, $(\Gamma_1 \Rightarrow \Delta_1) \models p$ iff $p \in \Gamma_1$. Therefore, if we can prove the next statement, we can prove this lemma.

For all $(\Gamma_1 \Rightarrow \Delta_1) \in W_c$, if $p \in \Delta_1$, there exists $(\Gamma_2 \Rightarrow \Delta_2) \in W_C$, satisfying $\neg p \in \Gamma_2$ and $(\Gamma_1 \Rightarrow \Delta_1) \not\perp (\Gamma_2 \Rightarrow \Delta_2)$.

For convenience, we prove this statement after the next lemma. □

LEMMA 4.3. *For all canonical O-models and all formulas $A \in \Omega$, A is true at $(\Gamma_1 \Rightarrow \Delta_1)$ if $A \in \Gamma_1$ and A is false at $(\Gamma_1 \Rightarrow \Delta_1)$ if $A \in \Delta_1$.*

PROOF: Proven by induction on the composition of A.

For $A = p$, the proof is obvious from the definition of a canonical O-model.

For $A = B \wedge C$, the proof is the same as in [25].

For $A = \neg B$, the proof is included in $A = B \to C$.

For $A = B \to C$, suppose $B \to C \in \Gamma_1$. Then, for all $(\Gamma_2 \Rightarrow \Delta_2)$ satisfying $B \in \Gamma_2$ and $C \in \Delta_2$, $(\Gamma_1 \Rightarrow \Delta_1)\perp(\Gamma_2 \Rightarrow \Delta_2)$ by the definition of the canonical O-model. Then, by definition of $\to$ and induction hypothesis, $B \to C$ is true at $(\Gamma_1 \Rightarrow \Delta_1)$.

Suppose $B \to C \in \Delta_1$. Because $\Gamma_1 \Rightarrow \Delta_1$ cannot be proven, when we regard this sequent as the lower sequent of the rule ($\to$R), an unprovable sequent $\Gamma_2, B \Rightarrow C, \Delta_2$ exists, which is of the shape of a sequent in the upper sequent of ($\to$R). Then, Γ_2 and Δ_2 distribute all formulas of the shape of $E \to F$ in Γ_1, regarded as a $C_i \to D_i$. If there are formulas in $\Gamma_2, B \Rightarrow C, \Delta_2$ that are excluded in $\Omega(\Gamma \Rightarrow \Delta)$, we delete them from $\Gamma_2, B \Rightarrow C, \Delta_2$ and make a new sequent $\Gamma_3, B \Rightarrow C, \Delta_3$. Then, $\Gamma_3 \cup \{B, C\} \cup \Delta_3 \subseteq \Omega(\Gamma \Rightarrow \Delta)$ and this sequent is still unprovable. This sequent can be expanded to the sequent $\Gamma_4 \Rightarrow \Delta_4 \in W_c$ because for all formulas G, at least one $\Gamma_3, B \Rightarrow C, \Delta_3, G$ or $G, \Gamma_3, B \Rightarrow C, \Delta_3$ is unprovable because of the rule (cut) and because $\Gamma_3, B \Rightarrow C, \Delta_3$ is unprovable. Furthermore, $(\Gamma_1 \Rightarrow \Delta_1) \not\perp (\Gamma_4 \Rightarrow \Delta_4)$ is satisfied because we delete all probability of holding the relation $\perp$ when we construct $\Gamma_2, B \Rightarrow C, \Delta_2$. Therefore, by the definition of $\to$ and induction hypothesis, $B \to C$ is false at $\Gamma_1 \Rightarrow \Delta_1$. $\square$

Now we can prove the statement in Lemma 4.2 using the method of the proof of Lemma 4.3. If $\Gamma_1 \Rightarrow \Delta_1, p$ ($\in W_c$) is unprovable, $\Gamma_1 \Rightarrow \Delta_1, p, \neg\neg p$ is also unprovable. We regard $(p \to \perp) \to \perp$ as $B \to C$ in Lemma 4.3. The same argument for $B \to C \in \Delta_1$ in Lemma 4.3 can be applied. That is, we can find $(\Gamma_4 \Rightarrow \Delta_4) \in W_c$, satisfying $\neg p \in \Gamma_4$, $\perp \in \Delta_4$, and $(\Gamma_1 \Rightarrow \Delta_1, p) \not\perp (\Gamma_4 \Rightarrow \Delta_4)$. If $\neg\neg p$ is included in $\Omega(\Gamma \Rightarrow \Delta)$, $\Gamma_1 \Rightarrow \Delta_1, p, \neg\neg p$ is the same as $\Gamma_1 \Rightarrow \Delta_1, p$ and is included in W_c. If $\neg\neg p$ is excluded in $\Omega(\Gamma \Rightarrow \Delta)$, sequent $\Gamma_4 \Rightarrow \Delta_4$ ($\neg p \in \Gamma_4$), constructed from $\Gamma_1 \Rightarrow \Delta_1, p, \neg\neg p$ is included in W_c, even if $\Gamma_1 \Rightarrow \Delta_1, p, \neg\neg p$ is excluded in W_c. That is, when we make $\Gamma_3, \neg p \Rightarrow \perp, \Delta_3$ from $\Gamma_1 \Rightarrow \Delta_1, p, \neg\neg p$, we eliminate all formulas that are excluded in $\Omega(\Gamma \Rightarrow \Delta)$. Furthermore, it satisfies $(\Gamma_1 \Rightarrow \Delta_1, p) \not\perp (\Gamma_4 \Rightarrow \Delta_4)$ because $\Gamma_1 \Rightarrow \Delta_1, p$ is a part of $\Gamma_1 \Rightarrow \Delta_1, p, \neg\neg p$.

THEOREM 4.4. *The completeness theorem for* $\mathbf{GOI}_1$. *If* $\Gamma \Rightarrow \Delta$ *is valid,* $\Gamma \Rightarrow \Delta$ *is provable in* $\mathbf{GOI}_1$.

PROOF: Suppose $\Gamma \Rightarrow \Delta$ is unprovable. We can make a canonical O-model of $\Gamma \Rightarrow \Delta$. Because (cut) is included in $\mathbf{GOI}_1$, there exists $(\Gamma' \Rightarrow \Delta') \in W_c$, an expansion of $\Gamma \Rightarrow \Delta$. By Lemma 4.3, $\Gamma \Rightarrow \Delta$ is false at $(\Gamma' \Rightarrow \Delta')$. $\square$

5. Sequent calculus $\mathbf{GOI}_2$

We define the sequent calculus $\mathbf{GOI}_2$ as an expansion of **GO**. We add the axioms $(\to \perp)$ and $(\perp)$ and the rule $(\to \mathrm{R})$' to **GO**. The rule $(\to \mathrm{R})$' is similar to the rule $(\to)$ in [20], but there are no contexts in this rule.

$$\bot \Rightarrow \quad (\bot)$$

$$A \Rightarrow (A \to B) \to \bot, B \quad (\to \bot)$$

$$\frac{\Gamma_1, A \Rightarrow B, \Delta_1 \qquad \Gamma_2, A \Rightarrow B, \Delta_2 \quad \ldots \qquad \Gamma_{2^n}, A \Rightarrow B, \Delta_{2^n}}{C_1 \to D_1, C_2 \to D_2, \ldots, C_n \to D_n \Rightarrow A \to B} \ (\to\text{R})\text{'}$$

where $0 \leq n$, $\Gamma_i = \{D_j | j \in \gamma(i)\}$, $\Delta_i = \{C_j | j \in \delta(i)\}$, $\langle \delta(i), \gamma(i) \rangle$ is the i-th element of all partitions of $\{1, \ldots, n\}$.

The rule ($\to$ R)' is a natural expansion of the rule ($\neg R$) in **GO**. That is, if all D_j and B in ($\to$ R)' are $\bot$, it is the same as ($\neg R$) in **GO** because of $A \to \bot \equiv \neg A$.

THEOREM 5.1. *The soundness and completeness theorem for* $\mathbf{GOI_2}$*.* $\Gamma \Rightarrow \Delta$ *is provable in* $\mathbf{GOI_2}$ *iff* $\Gamma \Rightarrow \Delta$ *is valid.*

PROOF: We can prove that all rules of $\mathbf{GOI_1}$ are derivable in $\mathbf{GOI_2}$, and vice versa. The proof of ($\to \bot$) in $\mathbf{GOI_1}$ and ($\to$R) in $\mathbf{GOI_2}$ is explained below. The other cases are obvious.

$$\dfrac{\dfrac{A \to B \Rightarrow A \to B}{A \to B \Rightarrow \bot, A \to B, A \to ((A \to B) \to \bot), A \to \bot} \ (\text{weakening})}{A \Rightarrow (A \to B) \to \bot, B} \ (\to\text{R})$$

Suppose all sequents of upper sequents in ($\to$R) are provable. For example, suppose $n = 2$. Then,

$A \Rightarrow B, C_1, C_2, \Sigma$
$D_1, A \Rightarrow B, C_2, \Sigma$
$D_2, A \Rightarrow B, C_1, \Sigma$
$D_1, D_2, A \Rightarrow B, \Sigma$

are all provable. Now we regard all formulas in Σ as a C_i $(n < i)$. For example, if Σ has three elements, we regard Σ as $\{C_3, C_4, C_5\}$. Furthermore, we define all D_i $(n < i)$ as $D_i = \bot$. Then,

$A \Rightarrow B, C_1, C_2, C_3, C_4, C_5$
$D_1, A \Rightarrow B, C_2, C_3, C_4, C_5$
...
$D_1, D_2, D_3, D_4, D_5, A \Rightarrow B$

are all provable because, if all formulas in $\Sigma = \{C_3,\ C_4,\ C_5\}$ are on the right-hand side, it is obvious from the assumption. If one of a $\{D_3, D_4, D_5\}$

is on the left-hand side and because all are $\perp$, this sequent is provable. We can use all these sequents and use ($\to$R)'. Then, because $E \Rightarrow (E \to F) \to \perp, F$ is provable using ($\to \perp$), use (cut), and prove the lower sequent of ($\to$R). □

THEOREM 5.2. *In* $\mathbf{GOI}_1$ *and* $\mathbf{GOI}_2$*, the rule* ($\to L$) *is admissible.*

PROOF: By using (cut) and because $A, A \to B \Rightarrow B$ is provable in these systems,

$$\dfrac{\dfrac{A \to B \Rightarrow A \to B}{A \to B, \Rightarrow ((A \to B) \to \perp) \to \perp} \quad \dfrac{A \Rightarrow (A \to B) \to \perp, B}{A, ((A \to B) \to \perp) \to \perp \Rightarrow B}}{A, A \to B \Rightarrow B}$$

□

Because the canonical model $(W_c, \perp_c, V_c)$ finite, we can prove the following theorem using the usual method as in **GO**.

THEOREM 5.3. $\mathbf{GOI}_1$ *and* $\mathbf{GOI}_2$ *are decidable. That is, an effective procedure determines whether a sequent* $\Gamma \Rightarrow \Delta$ *is provable in* $\mathbf{GOI}_1$ *and* $\mathbf{GOI}_2$*.*

PROOF: From the construction method of the canonical model $(W_c, \perp_c, V_c)$, built from a sequent $\Gamma \Rightarrow \Delta$, we obtain a finite model for any $\Gamma \Rightarrow \Delta$, and the model's complexity can be bounded by the complexity of formulas and the number of propositional letters in Γ and Δ. Therefore, by evaluating all finite models up to the bound, whether sequent $\Gamma \Rightarrow \Delta$ is valid can be determined. From the soundness and completeness theorem, this method can determine whether $\Gamma \Rightarrow \Delta$ is provable in $\mathbf{GOI}_1$ and $\mathbf{GOI}_2$. □

6. Conclusion and remarks

This study introduced two sequent calculi for **MQL** that involve the strict implication. The rule for the implication in $\mathbf{GOI}_1$ is complicated. On the contrary, the rule for the implication in $\mathbf{GOI}_2$ is less complicated and it is a natural expansion of the rule ($\neg R$). However, the axiom ($\to \perp$) must be included in $\mathbf{GOI}_2$. In both the calculi, the cut-elimination theorem does not hold. In actuality, $p, q \Rightarrow \neg(r \wedge \neg(p \wedge q))$ cannot be proven without (cut), as in **GO** [25]. In other words, based on the rules for

$\mathbf{GOI}_1$ and $\mathbf{GOI}_2$, in the proof of $p, q \Rightarrow \neg(r \wedge \neg(p \wedge q))$, we can only use (weakening), (cut), or ($\rightarrow$R) to deduce $p, q \Rightarrow \neg(r \wedge \neg(p \wedge q))$. However, it is easy to confirm that (weakening) does not work. Additionally, $r \wedge \neg(p \wedge q) \Rightarrow \bot, \neg p, \neg q, p \rightarrow \neg(r \wedge \neg(p \wedge q)), q \rightarrow \neg(r \wedge \neg(p \wedge q))$ can be checked for invalidity. Moreover, it is challenging to construct a sequent calculus for **QL** and **MQL** that satisfies the cut-elimination theorem using an ordinary method. The situation is similar to that in the modal logic **S5**. Both **S5** and **QL** exhibit a symmetric frame. If an attempt is made to construct a canonical model of the **S5**-frame in a stepwise manner, the procedure cannot be stopped because of the symmetry. An effective tool for addressing this problem is an extension of the sequent calculus. Various extensions of sequence calculus for **S5** have been constructed and analyzed [2, 18, 19, 24, 30]. As one of them, labeled sequent calculi or *tree sequent calculi* have been studied. A labeled sequent calculus for **MQL** with the strict implication has been established and is cut-free [23]. It is still an open question whether a normal sequent calculus for **MQL** that satisfies the cut-elimination theorem exists.

In **BPL**, the law of modus ponens does not hold [20]. Modus ponens $A, A \rightarrow B \Rightarrow B$ represents the reflexive condition of relations in frames which is not the nature of frames of **BPL**. Therefore, the rule ($\rightarrow$ L) is not sound in **LBP**. ($\rightarrow$ L) cannot be constructed if only ($\rightarrow$ R)' exists for the implication. In $\mathbf{GOI}_1$ and $\mathbf{GOI}_2$, because other rules or axioms for the implication are included, ($\rightarrow$ L) can be constructed.

Another sequent calculus for **MQL** called **GMQL** [26, 27] is also complete with respect to O-models and exclude implications, similar to **GO**. In **GO**, based on the definition of the truth of a sequent, $\Gamma \Rightarrow \Delta, A, B$ cannot be regarded as $\Gamma \Rightarrow \Delta, A \vee B$ because commas on the right side of the sequent indicate a union of sets. $\|A\| \cup \|B\|$ and $\|A \vee B\|$ are different sets in O-models, and $\|A\| \cup \|B\|$ is not always $\bot$-closed. For example, $\Rightarrow A, \neg A$ cannot be proven in **GO**; however, $\Rightarrow A \vee \neg A$ $(= \Rightarrow \neg(\neg A \wedge \neg\neg A))$ can be proven. In **GMQL**, $\Gamma \Rightarrow \Delta, A, B$ represent $\Gamma \Rightarrow \Delta, A \vee B$. Because the rules in **GMQL** are close to the notion of a lattice, the rules for $\wedge$ and $\vee$ in **GMQL** are symmetric because $\wedge$ and $\vee$ are symmetric in ortholattices. In the case of **GO**, that excludes an implication, this notion of a union of sets is inessential because of the following theorem [20].

THEOREM 6.1. *If $\Gamma \Rightarrow \Delta$ is provable in* **GO** *and Δ is nonempty, then there exists $A \in \Delta$ such that $\Gamma \Rightarrow A$ is provable and all sequents in that proof have at most one formula on the right side.*

When considering the rules for implications, **GMQL** is unsuitable because in the rules for strict implication, the notion of a union of sets on the right side of a sequent is used rather than $\vee$. In the case of $\mathbf{GOI}_1$ and $\mathbf{GOI}_2$, the notion of a union of sets is essential and Theorem 6.1 does not hold in these calculi. This finding can be confirmed by considering the axiom $(\to \bot)$ and the completeness theorem. In other words, both $A \Rightarrow (A \to B) \to \bot$ and $A \Rightarrow B$ are invalid.

In a sense, the axiom $(\to \bot)$ represents the symmetry of the relation in frames. If $\mathbf{GOI}_2$ includes only ($\to$R)' for the strict implication, the symmetry cannot be handled because ($\to$R)' is a part of the sequent calculus reported in the literature [20] which is sound and complete with respect to the frames that do not need to be symmetrical. Assume that in an O-model $(W, \perp, V)$, $x \models A$ and $x \not\models B$, then for all $y \in W$ such that $x \not\perp y$, $y \not\models A \to B$ attributed is the symmetry of $\not\perp$. If $B = \bot$, then the axiom $(\to \bot)$ is $A \Rightarrow \neg\neg A$. When the translation in the literature [12] which translate a formula of **QL** to a formula of modal logic is applied, this sequent corresponds to $A \Rightarrow \Box\Diamond A$, representing the symmetry in the modal logic.

In the rule $(\to)$ in **LBP**, in every left side of the sequent, contexts can be used. Therefore, $p \to (q \to p)$ can be proven in a sequent calculus for **LBP** using $n = 0$ of $(\to)$, which cannot be proven in $\mathbf{GOI}_1$.

$$\frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B}\ (n = 0 \text{ of } (\to) \text{ in } \mathbf{LBP})$$

$$\cfrac{\cfrac{\cfrac{p \Rightarrow p}{q, p \Rightarrow p}}{p \Rightarrow q \to p}}{\Rightarrow p \to (q \to p)}$$

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