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A NOTE ON 3×3-VALUED ŁUKASIEWICZ ALGEBRAS WITH NEGATION

Abstract

In 2004, C. Sanza, with the purpose of legitimizing the study of $n \times m$-valued Łukasiewicz algebras with negation (or $NS_{n\times m}$-algebras) introduced 3×3-valued Łukasiewicz algebras with negation. Despite the various results obtained about $NS_{n\times m}$-algebras, the structure of the free algebras for this variety has not been determined yet. She only obtained a bound for their cardinal number with a finite number of free generators. In this note we describe the structure of the free finitely generated $NS_{3\times 3}$-algebras and we determine a formula to calculate its cardinal number in terms of the number of free generators. Moreover, we obtain the lattice $\Lambda(\boldsymbol{NS_{3\times 3}})$ of all subvarieties of $\boldsymbol{NS_{3\times 3}}$ and we show that the varieties of Boolean algebras, three-valued Łukasiewicz algebras and four-valued Łukasiewicz algebras are proper subvarieties of $\boldsymbol{NS_{3\times 3}}$.

Keywords: n-valued Łukasiewicz–Moisil algebras, $n \times m$-valued Łukasiewicz algebras with negation, free algebras, lattice of subvarieties.

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1. Introduction

N. Belnap in [1] introduced four-valued logic, with the purpose of reasoning about incomplete (none) and inconsistent (both) information from different sources. This logical system is well known for the many applications it has found in several fields, for example in the study of deductive data-bases and

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distributed logic programs handling information that may contain conflicts or gaps. Taking into consideration Belnap's four-valued logic, C. Sanza considered an extension from which 3×3-valued Łukasiewicz algebras with negation are obtained as described in [12, 14]. Then in [13] she generalizes this concept defining the $n \times m$-valued Łukasiewicz algebras with negation which constitute a non-trivial generalization of n-valued Łukasiewicz–Moisil algebras ([2, 10, 11]) and a particular case of matrix Łukasiewicz algebras defined by W. Suchoń in [16]. More precisely, $NS_{n\times m}$-algebras rise from matrix Łukasiewicz algebras without the restriction that the endomorphisms be pairwise different and endowed with a De Morgan negation in the following way:

An $n \times m$-valued Łukasiewicz algebra with negation (or $NS_{n\times m}$-algebra), in which n and m are integers, $n \geq 2$, $m \geq 2$, is an algebra $\langle L, \wedge, \vee, \sim, \{\sigma_{ij}\}_{(i,j)\in(n\times m)}, 0, 1\rangle$ where $(n \times m)$ is the cartesian product $\{1, \ldots, n-1\} \times \{1, \ldots, m-1\}$, the reduct $\langle L, \wedge, \vee, \sim, 0, 1\rangle$ is a De Morgan algebra and $\{\sigma_{ij}\}_{(i,j)\in(n\times m)}$ is a family of unary operations on L which fulfills the following conditions:

(T1) $\sigma_{ij}(x \vee y) = \sigma_{ij}x \vee \sigma_{ij}y$,

(T2) $\sigma_{ij}x \wedge \sigma_{(i+1)j}x = \sigma_{ij}x$,

(T3) $\sigma_{ij}x \wedge \sigma_{i(j+1)}x = \sigma_{ij}x$,

(T4) $\sigma_{ij}\sigma_{rs}x = \sigma_{rs}x$,

(T5) $\sigma_{ij} \sim x = \sim \sigma_{(n-i)(m-j)}x$,

(T6) $\sigma_{ij}x \vee \sim \sigma_{ij}x = 1$,

(T7) $$x \wedge \bigwedge_{(i,j)\in(n\times m)} ((\sim \sigma_{ij}x \vee \sigma_{ij}y) \wedge (\sim \sigma_{ij}y \vee \sigma_{ij}x)) =$$
$$y \wedge \bigwedge_{(i,j)\in(n\times m)} ((\sim \sigma_{ij}x \vee \sigma_{ij}y) \wedge (\sim \sigma_{ij}y \vee \sigma_{ij}x)).$$ ([12])

In what follows, we will indicate by $\boldsymbol{NS_{n\times m}}$ the variety of $NS_{n\times m}$-algebras.

By [14, Remark 3.1] we have that every $NS_{2\times m}$-algebra is isomorphic to an m-valued Łukasiewicz–Moisil algebra. It is worth mention that $\boldsymbol{NS_{n\times m}}$ was widely studied in [13, 12, 14, 15, 7, 8].

The notions and results announced here for $NS_{n\times m}$-algebras will be used throughout this article.

Let L be an $NS_{n\times m}$-algebra. A filter F of L is a Stone filter if and only of the hypothesis $x \in F$ implies $\sigma_{11}x \in F$ ([13, Proposition 3.2]). The lattice of all Stone filters of L will be denoted by $\mathcal{F}_S(L)$.

(T8) Let L be an $NS_{n\times m}$-algebra with more than one element and let $Con(L)$ be the lattice of all congruences on L. Then $Con(L) = \{R(F) : F \in \mathcal{F}_S(L)\}$, where $R(F) = \{(x, y) \in L\times L :$ there exists $f \in F$ such that $x \wedge f = y \wedge f\}$. Besides, the lattices $Con(L)$ and $\mathcal{F}_S(L)$ are isomorphic considering the mappings $\theta \longmapsto [1]_\theta$ and $F \longmapsto R(F)$ which are mutually inverse, where $[x]_\theta$ stands for the equivalence class of x modulo θ ([13, Proposition 3.3 and Theorem 3.6]).

(T9) $\boldsymbol{NS_{n\times m}}$ is a discriminator variety ([15, Theorem 3.1]).

(T10) Let L be a non-trivial $NS_{n\times m}$-algebra. Then L is simple if and only if $B(L) = \{0, 1\}$, where $B(L)$ is the set of all Boolean elements of L, ([14, Theorem 5.1]).

(T11) $\boldsymbol{NS_{n\times m}}$ is locally finite ([14, Theorem 5.2]).

Let B be a non trivial Boolean algebra and $x \in B$, we will write x' the Boolean complement of x. Furthermore, we will denote by $B\uparrow^{(n\times m)}= \{f : (n \times m) \longrightarrow B$ such that for arbitraries i, j, $r \leq s$, implies $f(r, j) \leq f(s, j)$ and $f(i, r) \leq f(i, s)\}$. Then

(T12) $\langle B\uparrow^{(n\times m)}, \wedge, \vee, \sim, \{\sigma_{ij}\}_{(i,j)\in(n\times m)}, O, I\rangle$ is an $NS_{n\times m}$-algebra where for each $f \in B\uparrow^{(n\times m)}$ and for $(i, j) \in (n\times m)$, $(\sim f)(i, j) = (f(n-i, m-j))'$, $(\sigma_{rs}f)(i, j) = f(r, s)$, for all $(r, s) \in (n \times m)$, $O(i, j) = 0$, $I(i, j) = 1$ and the remaining operations are defined componentwise ([14, Proposition 3.2]).

(T13) $\mathbf{S}_{n\times m} = \langle\{0, 1\}\uparrow^{(n\times m)}, \wedge, \vee, \sim, \{\sigma_{ij}\}_{(i,j)\in(n\times m)}, O, I\rangle$ generates the variety $\boldsymbol{NS_{n\times m}}$ ([14, Theorem 5.5])

2. Free $\mathbf{NS_{3\times 3}}$-algebras

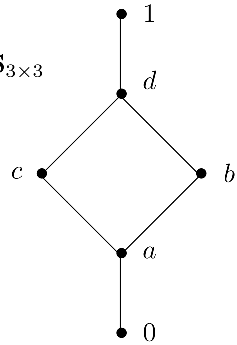
From now on, we will denote by $\mathcal{F}_{3\times 3}(t)$ the free $NS_{3\times 3}$-algebra with a set G of free generators such that $|G| = t$ where t is a cardinal number,

$0 < t < \omega$. The notion of free $NS_{3\times3}$-algebra is the usual one and since $NS_{3\times3}$-algebras are equationally definable, for any cardinal number t, $t > 0$, the free algebra $\mathcal{F}_{3\times3}(t)$ exists and it is unique up to isomorphism ([3]).

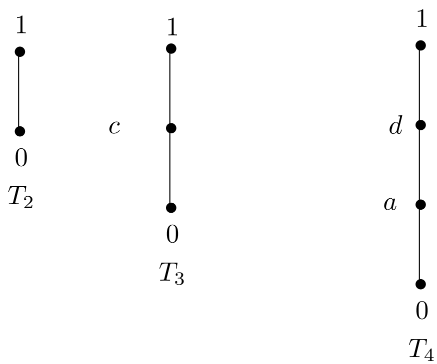
On the other hand, from (T13) we have that $\boldsymbol{NS_{3\times3}}$ is generated by $\mathbf{S}_{3\times3}$ described in [14, p. 85] as follows:

x	$\sim x$	$\sigma_{11}x$	$\sigma_{12}x$	$\sigma_{21}x$	$\sigma_{22}x$
0	1	0	0	0	0
a	d	0	0	0	1
b	b	0	1	0	1
c	c	0	0	1	1
d	a	0	1	1	1
1	0	1	1	1	1

$\mathbf{S}_{3\times3}$



Furthermore, $\mathbf{S}_{3\times3}$ has four non-isomorphic subalgebras: the chains T_2, T_3 and T_4 with $2, 3$ and 4 elements respectively and T_6 which is the algebra itself.



Hence, from the above results and bearing in mind $(T9)$ and $(T11)$ we know that $\mathcal{F}_{3\times3}(t)$ is finite. Furthermore, we have that:

$$\mathcal{F}_{3\times3}(t) \approx T_2^{\alpha_2} \otimes T_3^{\alpha_3} \otimes T_4^{\alpha_4} \otimes T_6^{\alpha_6},$$

where $\alpha_i = |\mathcal{E}_i| = |\{F : F$ is a maximal Stone filter of $\mathcal{F}_{3\times 3}(t)$ and $\mathcal{F}_{3\times 3}(t)/F \approx T_i\}|$, for $i = 2, 3, 4, 6$.

Let us see that

$$\alpha_i = \frac{|Epi(\mathcal{F}_{3\times 3}(t), T_i)|}{|Aut(T_i)|}, \quad i \in \{2, 3, 4, 6\}.$$

where $Epi(\mathcal{F}_{3\times 3}(t), T_i)$ is the set of all epimorphisms from $\mathcal{F}_{3\times 3}(t)$ onto T_i and $Aut(T_i)$ is the set of all automorphisms of T_i.

Let us consider the function $\alpha : Epi(\mathcal{F}_{3\times 3}(t), T_i) \longrightarrow \mathcal{E}_i$ defined by $\alpha(h) = ker(h)$, where $ker(h) = \{x \in \mathcal{F}_{3\times 3}(t) : h(x) = 1\}$. Hence, α is onto. Indeed, for each $F \in \mathcal{E}_i$ let us consider the function $f = \gamma_F \circ q_F$, where q_F is the natural map and γ_F is the $NS_{3\times 3}$-isomorphism from $\mathcal{F}_{3\times 3}(t)/F$ to T_i. Thus, $f \in Epi(\mathcal{F}_{3\times 3}(t), T_i)$ and $ker(f) = F$. Consequently $\alpha(f) = F$. Furthermore, for all $F \in \mathcal{E}_i$ there exists $h' \in Epi(\mathcal{F}_{3\times 3}(t), T_i)$ such that $\alpha(h') = F$. Besides, let us note that $\alpha^{-1}(F) = \{f \in Epi(\mathcal{F}_{3\times 3}(t), T_i) : ker(f) = F\} = \{f \in Epi(\mathcal{F}_{3\times 3}(t), T_i) : ker(f) = ker(h')\} = \{f \in Epi(\mathcal{F}_{3\times 3}(t), T_i) : f = g \circ h', g \in Aut(T_i)\}$. Then, $|\alpha^{-1}(F)| = |Aut(T_i)|$ for $i = 2, 3, 4, 6$.

Besides, observe that $Epi(\mathcal{F}_{3\times 3}(t), T_i)$ and $F^*(G, T_i)$ have the same size, where $F^*(G, T_i)$ is the set of all functions $f : G \longrightarrow T_i$ such that $\overline{f(G)} = T_i$, being $\overline{X}$ the $NS_{3\times 3}$-subalgebra of T_i generated by X.

Indeed, let $\beta : Epi(\mathcal{F}_{3\times 3}(t), T_i) \longrightarrow F^*(G, T_i)$ be the function defined by $\beta(h) = h|_G$ (i.e. β and h agree on G). It is simple to verify that β is injective. Moreover, for each $f \in F^*(G, T_i)$ there is a unique homomorphism $h_f : \mathcal{F}_{3\times 3}(t) \longrightarrow T_i$ such that h_f and f agree on G. Besides, $h_f(\mathcal{F}_{3\times 3}(t)) = h_f(\overline{G}) = \overline{f(G)} = T_i$. Therefore, h is onto and so $Epi(\mathcal{F}_{3\times 3}(t), T_i) = F^*(G, T_i)$.

On the other hand, suppose that $f, g \in Aut(T_i)$ and that there is $x \in T_i$ such that $f(x) \neq g(x)$. Hence, by [13, Theorem 2.7] there is $(s_0, j_0) \in (3\times 3)$ such that $\sigma_{s_0 j_0} f(x) \neq \sigma_{s_0 j_0} g(x)$ and as T_i is a simple $NS_{3\times 3}$-algebra for all $i \in \{2, 3, 4, 6\}$ we have that $\sigma_{sj}(T_i) = B(T_i) = \{0, 1\}$ for all $(s, j) \in (3 \times 3)$. Then, without loss of generality we have that $\sigma_{s_0 j_0} f(x) = 0$ and $\sigma_{s_0 j_0} g(x) = 1$, so $f(\sigma_{s_0 j_0} x) = f(0)$ and $g(\sigma_{s_0 j_0} x) = g(1)$. Since f, g are injective we conclude that $\sigma_{s_0 j_0} x = 0$ and $\sigma_{s_0 j_0} x = 1$, which is a contradiction. Therefore, $|Aut(T_i)| = 1$, $i \in \{2, 3, 4, 6\}$.

Bearing in mind the above results and the fact that T_2, T_3 and T_4 are Łukasiewicz–Moisil algebras of order $n = 2$, $n = 3$ and $n = 4$ respectively, from [4] we have that:

$$\alpha_2 = 2^t, \quad \alpha_3 = 2(3^t - 2^t), \quad \alpha_4 = 4^t - 2^t.$$

Therefore, it only remains to determine α_6. Let us consider the functions $f : \{g_1, g_2, \ldots, g_t\} \longrightarrow T_6$ such that $f(g_i) = b$ and $f(g_j) = c$ for some $i, j \in \{1, \ldots, t\}$, $i \neq j$. If b and c are the image of k generators $1 \leq k \leq t$, then we have that there are $\binom{t}{k} \cdot (2^k - 2) \cdot 4^{t-k}$ different functions f from G to T_6. Hence,

$$\alpha_6 = \sum_{i=1}^{t} \binom{t}{i} \cdot (2^i - 2) \cdot 4^{t-i} = 6^t - 2 \cdot 5^t + 4^t.$$

Then, we have shown

Theorem 2.1. *Let $\mathcal{F}_{3\times3}(t)$ be the free $NS_{3\times3}$-algebra with t generators. Then its cardinality is given by the following formula:*

$$|\mathcal{F}_{3\times3}(t)| = 2^{2^t} \cdot 3^{2(3^t - 2^t)} \cdot 4^{4^t - 2^t} \cdot 6^{6^t - 2 \cdot 5^t + 4^t}.$$

Remark 2.2. By Theorem 2.1 we have that for $t = 1$ and $t = 2$,

$$|\mathcal{F}_{3\times3}(1)| = 2^2 \cdot 3^2 \cdot 4^2 \cdot 6^0 = 576,$$

$$|\mathcal{F}_{3\times3}(2)| = 2^4 \cdot 3^{10} \cdot 4^{12} \cdot 6^2 = 16836317.$$

We will now compare these values with the following bound that C. Sanza determines in [12]:

$$|\mathcal{F}_{n\times m}(t)| \leq |\mathbf{S}_{n\times m}|^{|\mathbf{S}_{n\times m}|^t \cdot K},$$

where K is the number of simple $NS_{n\times m}$-algebras and $|\mathbf{S}_{n\times m}|$ is given by:

$$|\mathbf{S}_{n\times m}| = \begin{cases} m, & \text{if } n = 2 \\ 1 + \sum\limits_{j=2}^{m} |\mathbf{S}_{(n-1)\times j}|, & \text{if } \ n > 2. \end{cases}$$

Then, we have that $|\mathcal{F}_{3\times 3}(t)| \leq 6^{6^t \cdot 4}$

$$|\mathcal{F}_{3\times 3}(1)| \leq 6^{24} = 4,7383813 \cdot 10^{18},$$

$$|\mathcal{F}_{3\times 3}(2)| \leq 6^{144} = 1,131827 \cdot 10^{112}$$

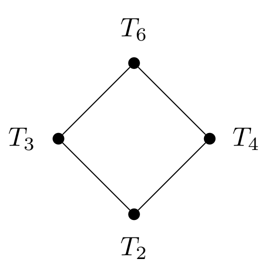
which differ notably from the ones indicated in Remark 2.2.

3. The lattice $\mathbf{\Lambda}(\mathbf{NS_{3\times 3}})$ of all subvarieties of $\mathbf{NS_{3\times 3}}$

If K is a finite set of finite algebras we will denote by $\mathcal{V} = Var(K)$ the variety generated by K. On the other hand, by Jónsson's Lemma ([9]), the lattice $\Lambda(\mathcal{V})$ of all subvarieties of $\mathcal{V}$ is a finite distributive lattice and $\Lambda(\mathcal{V})$ is isomorphic to the lattice $\mathcal{O}(P)$ of order-ideals of the poset P of all join-irreducible elements of $\Lambda(\mathcal{V})$. Again by Jónsson's Lemma, $\mathcal{V}'$ is join-irreducible in $\Lambda(\mathcal{V})$ if and only if there exists some (necessarily finite) subdirectly irreducible algebra $A \in \mathcal{V}$ such that $\mathcal{V}' = Var(\{A\})$. Furthermore, if A and B are subdirectly irreducible algebras of $\mathcal{V}$, then $Var(\{A\}) \subseteq Var(\{B\})$ if and only if $A \in \mathbf{H}(\mathbf{S}(B))$, where $\mathbf{H}(W) = \{C \in \mathcal{V} :$ there exists an epimorphism $p : W \to C\}$ and $\mathbf{S}(Z)$ is the set of all subalgebras of Z.

Taking into account (T10) and (T13) we have that $\mathbf{Si}(\mathbf{NS_{3\times 3}}) = \{T_2, T_3, T_4, T_6\}$ where $\mathbf{Si}(S)$ is the set of all finite subdirectly irreducible $NS_{3\times 3}$-algebras. It is not difficult to see that $\mathbf{H}(\mathbf{S}(A)) = \mathbf{S}(A)$, for all $A \in \mathbf{NS_{3\times 3}}$. Then, $\mathbf{H}(\mathbf{S}(T_2)) = \{T_2\}$, $\mathbf{H}(\mathbf{S}(T_3)) = \{T_2, T_3\}$, $\mathbf{H}(\mathbf{S}(T_4)) = \{T_2, T_4\}$ and $\mathbf{H}(\mathbf{S}(T_6)) = \{T_2, T_3, T_4, T_6\}$.

Then, the poset $(\mathbf{Si}(\mathbf{NS_{3\times 3}}), \leq)$ has the following Hasse diagram:



Let us observe that $\mathcal{V}_2 = Var(T_2)$, $\mathcal{V}_3 = Var(T_3)$, $\mathcal{V}_4 = Var(T_4)$, $\mathcal{V}_5 = Var(\{T_2, T_3, T_4\})$. Clearly $\mathcal{V}_2$ is the variety of Boolean algebras, $\mathcal{V}_3$

is the variety of three-valued Łukasiewicz algebras and $\mathcal{V}_4$ is the variety of four-valued Łukasiewicz algebras.

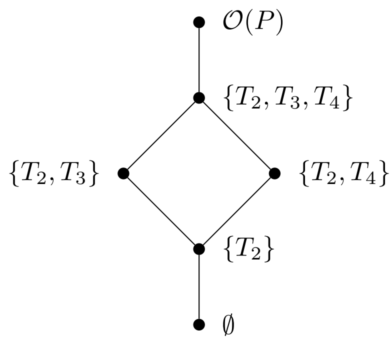
On the other hand, recall that an element x of a complete lattice L is a completely join irreducible (CJI), if $x \leq \bigvee_{i \in I} y_i$ implies $x \leq y_i$ for some $i \in I$. Besides, a finite subdirectly irreducible algebra A in a variety K is a splitting algebra in K if $Var(\{A\})$ is a CJI in $\Lambda(K)$.

Remark 3.1. Taking into account (T9), (T11) and the results established in [5], all finite subdirectly irreducible $NS_{3\times 3}$-algebra is a splitting algebra.

Now, Proposition 3.2 is a direct consequence of Remark 3.1, (T11) and [6, Proposition 2.2].

PROPOSITION 3.2. The natural map from $\Lambda(\mathcal{V})$ to $\mathcal{O}(P)$ is an isomorphism.

Then, we can assert that $\Lambda(\mathbf{NS_{3\times 3}})$ is the following finite distributive lattice:



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