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AN (α, β)-HESITANT FUZZY SET APPROACH TO IDEAL THEORY IN SEMIGROUPS

Abstract

The aim of this manuscript is to introduce the (α, β)-hesitant fuzzy set and apply it to semigroups. In this paper, as a generalization of the concept of hesitant fuzzy sets to semigroup theory, the concept of (α, β)-hesitant fuzzy subsemigroups of semigroups is introduced, and related properties are discussed. Furthermore, we define and study (α, β)-hesitant fuzzy ideals on semigroups. In particular, we investigate the structure of (α, β)-hesitant fuzzy ideal generated by a hesitant fuzzy ideal in a semigroup. In addition, we also introduce the concepts of (α, β)-hesitant fuzzy semiprime sets of semigroups, and characterize regular semigroups in terms of (α, β)-hesitant fuzzy left ideals and (α, β)-hesitant fuzzy right ideals. Finally, several characterizations of regular and intra-regular semigroups by the properties of (α, β)-hesitant ideals are given.

Keywords: ${}^{\alpha}$-hesitant (${}_{\alpha}$-hesitant) fuzzy set, (α, β)-hesitant fuzzy subsemigroup, (α, β)-hesitant fuzzy ideal, (α, β)-hesitant fuzzy semiprime set, regular semigroup.

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1. Introduction

An (α, β)-hesitant fuzzy set on a semigroup is a generalization of the concept of fuzzy subsets, interval-valued fuzzy sets and hesitant fuzzy sets in semigroups. A hesitant fuzzy set theory is an excellent tool to handle the uncertainty in case of insufficient data. Many authors studied different aspects of hesitant fuzzy sets (see [1, 5, 14, 19, 20]). Also, hesitant

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fuzzy set theory is used in decision making problem etc. (see [12, 16]), and is applied to BCK/BCI-algebras and UP-algebras (see [10, 11, 13]). The notion of interval-valued fuzzy sets has been applied to theory of semigroups [4]. They considered characterizations of left [right] simple, left [right] duo and a semilattice of left [right] simple semigroups. In 2012, Khan, Jun and Abbas [8] characterized regular (resp.intra-regular, simple and semisimple) ordered semigroups by their $(\in, \in \vee q)$-fuzzy interior ideals (resp. $(\in, \in \vee q)$-fuzzy ideals). Also they proved that the an ordered semigroup S is simple if and only if it is $(\in, \in \vee q)$-fuzzy simple. In 2013, Yaqoob [18] characterized regular LA-semigroups by the properties of interval valued intuitionistic fuzzy left ideals [right ideal, generalized bi-ideal and bi-ideal]. In 2014, Jun, Ahn and Muhiuddin [6] applied the notion of hesitant fuzzy soft sets to BCK/BCI-algebras. They introduced the notions of hesitant fuzzy soft subalgebras and (closed) hesitant fuzzy soft ideals and investigated several properties. In 2015, Jun, Lee, and Song [7] introduced the notion of hesitant fuzzy (generalized) bi-ideals on a semigroup, which is a generalization of interval valued fuzzy (generalized) bi-ideals. In 2016, Khan et al. [9] applied the notion of interval-valued fuzzy subsets to ordered semigroups, and proved that the intersection of non-empty class of interval-valued fuzzy interior ideals of an ordered semigroup is also an interval-valued fuzzy interior ideal. In 2017, Tang, Davvaz and Xie [15] defined and studied the completely prime, weakly completely prime and completely semiprime fuzzy quasi-Γ-hyperideals of ordered Γ-semihypergroups, and characterized bi-regular ordered Γ-semihypergroups by the properties of completely semiprime fuzzy quasi-Γ-hyperideals. In 2018, Abbasi et al. [2] gave the concept of hesitant fuzzy ideals and 3-prime hesitant fuzzy ideals in po-semigroup, which is a generalization of fuzzy ideals and 3-prime fuzzy ideals in po-semigroups. In 2019, Arulmozhi, Chinnadurai and Swaminathan [3] introduced the notion of interval valued $\left(\bar{\eta}, \bar{\delta}\right)$-bipolar fuzzy ideal, bi-ideal,interior ideal, $(\in, \in \vee q)$-bipolar fuzzy ideal of ordered Γ-semigroups and established some properties of bipolar fuzzy ideals in terms of $(\bar{\in}, \bar{\in} \vee q)$-bipolar fuzzy ideals. In 2020, Yairayong [17] applied the theory of hesitant fuzzy sets to completely regular semigroups and introduced the notion of hesitant fuzzy semiprime sets and hesitant fuzzy idempotent sets on semigroups, which is a generalization of fuzzy semiprime and fuzzy idempotent sets. He also proved that the every hesitant fuzzy two-sided ideal on a semigroup S is a hesitant fuzzy interior ideal if and only if S is a semisimple semigroup.

The aim of this manuscript is to introduce the (α, β)-hesitant fuzzy set and apply it to semigroups. The rest contents of this paper are arranged as follows. In Section 2, we present the fundamental concepts and properties of $^{\alpha}$-hesitant ($_{\alpha}$-hesitant) fuzzy sets, (α, β)-hesitant fuzzy subsemigroups and (α, β)-hesitant fuzzy ideals, which form the basis of our subsequent discussion. In this regard, we prove that that every hesitant fuzzy set on a semigroup S is (α, β)-hesitant fuzzy left (right, two-sided) ideal if and only if $_{\beta}(\mathcal{S} \odot \mathcal{H}) \subseteq {}^{\alpha}\mathcal{H}(_{\beta}(\mathcal{H} \odot \mathcal{S}) \subseteq {}^{\alpha}\mathcal{H}, {}_{\beta}((\mathcal{S} \odot \mathcal{H}) \cup (\mathcal{H} \odot \mathcal{S})) \subseteq {}^{\alpha}\mathcal{H})$. We prove that the non empty subset of a semigroup S is a subsemigroup (left ideal, right ideal, two-sided ideal) of S if and only if the hesitant fuzzy set on S is the (α, β)-hesitant fuzzy subsemigroup ((α, β)-hesitant fuzzy left ideal, (α, β)-hesitant fuzzy right ideal, (α, β)-hesitant fuzzy two-sided ideal) on S. In Section 3, we define the notions of (α, β)-hesitant fuzzy semiprime sets and equivalent definitions of them. Some related properties of them are obtained. In this paper, we give characterizations of semigroups in terms of (α, β)-hesitant fuzzy ideals, and characterize regular semigroups in terms of (α, β)-hesitant fuzzy left ideals and (α, β)-hesitant fuzzy right ideals. Finally, several characterizations of regular and intra-regular semigroups by the properties of (α, β)-hesitant ideals are given.

2. $^{\alpha}$-hesitant ($_{\alpha}$-hesitant) fuzzy sets

In this section, we present the fundamental concepts and properties of $^{\alpha}$-hesitant ($_{\alpha}$-hesitant) fuzzy sets, (α, β)-hesitant fuzzy subsemigroups and (α, β)-hesitant fuzzy ideals, which form the basis of our subsequent discussion. In this regard, we prove that that every hesitant fuzzy set on a semigroup S is (α, β)-hesitant fuzzy left (right, two-sided) ideal if and only if $_{\beta}(\mathcal{S} \odot \mathcal{H}) \subseteq {}^{\alpha}\mathcal{H}(_{\beta}(\mathcal{H} \odot \mathcal{S}) \subseteq {}^{\alpha}\mathcal{H}, {}_{\beta}((\mathcal{S} \odot \mathcal{H}) \cup (\mathcal{H} \odot \mathcal{S})) \subseteq {}^{\alpha}\mathcal{H})$. We prove that the non empty subset of a semigroup S is a subsemigroup (left ideal, right ideal, two-sided ideal) of S if and only if the hesitant fuzzy set on S is the (α, β)-hesitant fuzzy subsemigroup ((α, β)-hesitant fuzzy left ideal, (α, β)-hesitant fuzzy right ideal, (α, β)-hesitant fuzzy two-sided ideal) on S. These notions will be helpful in later sections.

Let $\mathcal{H} : S \to \mathcal{P}([0, 1])$ be a hesitant fuzzy set on a semigroup S and let α be any element of $\mathcal{P}([0, 1])$. Then the **$^{\alpha}$-hesitant fuzzy set ($_{\alpha}$-hesitant fuzzy set)** on S is defined as ${}^{\alpha}\mathcal{H}_x = \mathcal{H}_x \cup \alpha$ (${}_{\alpha}\mathcal{H}_x = \mathcal{H}_x \cap \alpha$) for all $x \in S$. Next, we define the hesitant fuzzy set over a semigroup S. If $\mathcal{H}_x = [0, 1]$

for all $x \in S$, then it is easy to see that $\mathcal{H}$ is a hesitant fuzzy set on a semigroup S. We denote such type of hesitant fuzzy set $\mathcal{H}$ by $\mathcal{S}$.

The proof of them is straightforward, so we omit it.

LEMMA 2.1. *Let $\mathcal{S}$ be a hesitant fuzzy set on a semigroup S and let α be any element of $\mathcal{P}([0,1])$. The following statements are true.*

1. ${}^{\alpha}\mathcal{S} = \mathcal{S}$.
2. ${}_{\alpha}\mathcal{S} = \alpha$.

Next, we denote by $\mathbf{H}(S)$ the set of all hesitant fuzzy sets on a semigroup S. Let $\mathcal{H}$ and $\mathcal{F}$ be any elements of $\mathbf{H}(S)$. Then, $\mathcal{H}$ is said to be a **subset** of $\mathcal{F}$, denoted by $\mathcal{H} \preceq \mathcal{F}$ if $\mathcal{H}_x \subseteq \mathcal{F}_x$ for all $x \in S$.

Now we are giving some basic properties of hesitant fuzzy subsemigroups on a semigroup S, which will be very helpful in later section.

THEOREM 2.2. *Let $\mathcal{H}$ be a hesitant fuzzy subsemigroup on a semigroup S. Then, the following statements are true:*

1. *If α is an element of $\mathcal{P}([0,1])$, then ${}^{\alpha}\mathcal{H}$ is a hesitant fuzzy subsemigroup on S.*
2. *If α is an element of $\mathcal{P}([0,1])$, then ${}_{\alpha}\mathcal{H}$ is a hesitant fuzzy subsemigroup on S.*

PROOF: 1. Let x and y be any elements of S and let $\alpha \in \mathcal{P}([0,1])$. Then, it is clear that

$$\begin{aligned} {}^{\alpha}\mathcal{H}_{xy} &= \mathcal{H}_{xy} \cup \alpha \\ &\supseteq (\mathcal{H}_x \cap \mathcal{H}_y) \cup \alpha \\ &= (\mathcal{H}_x \cup \alpha) \cap (\mathcal{H}_y \cup \alpha) \\ &= {}^{\alpha}\mathcal{H}_x^y. \end{aligned}$$

This completes the proof.

2. Let x and y be any elements of S and let $\alpha \in \mathcal{P}([0,1])$. Since $\mathcal{H}$ is a hesitant fuzzy subsemigroup on S, we obtain

$$\begin{aligned} {}_{\alpha}\mathcal{H}_{xy} &= \mathcal{H}_{xy} \cap \alpha \\ &\supseteq (\mathcal{H}_x \cap \mathcal{H}_y) \cap \alpha \\ &= (\mathcal{H}_x \cap \alpha) \cap (\mathcal{H}_y \cap \alpha) \\ &= {}_{\alpha}\mathcal{H}_x^y. \end{aligned}$$

From here, we obtain that ${}_{\alpha}\mathcal{H}$ is a hesitant fuzzy subsemigroup on S. □

For a non empty family of a hesitant fuzzy sets $\{\mathcal{H}_i : i \in I\}$, on a semigroup S. The symbols $\bigcup_{i\in I}\mathcal{H}_i$ and $\bigcap_{i\in I}\mathcal{H}_i$ will mean the following hesitant fuzzy sets:

$$\left(\bigcup_{i\in I}\mathcal{H}_i\right)_x = \bigcup_{i\in I}(\mathcal{H}_i)_x$$

and

$$\left(\bigcap_{i\in I}\mathcal{H}_i\right)_x = \bigcap_{i\in I}(\mathcal{H}_i)_x.$$

If I is a finite set, say $I = \{1,2,3,\dots,n\}$, then clearly $\bigcup_{i\in I}\mathcal{H}_i = \mathcal{H}_1 \cup \mathcal{H}_2 \cup \ldots \cup \mathcal{H}_n$ and $\bigcap_{i\in I}\mathcal{H}_i = \mathcal{H}_1 \cap \mathcal{H}_2 \cap \ldots \cap \mathcal{H}_n$ (see [7]).

THEOREM 2.3. *Let $\mathcal{H}$ and $\mathcal{G}$ be two hesitant fuzzy sets of a semigroup S. Then, the following statements are true:*

1. *If α is an element of $\mathcal{P}([0,1])$, then ${}^{\alpha}(\mathcal{H}\cap\mathcal{G}) = {}^{\alpha}\mathcal{H} \cap {}^{\alpha}\mathcal{G}$.*
2. *If α is an element of $\mathcal{P}([0,1])$, then ${}_{\alpha}(\mathcal{H}\cap\mathcal{G}) = {}_{\alpha}\mathcal{H} \cap {}_{\alpha}\mathcal{G}$.*
3. *If α is an element of $\mathcal{P}([0,1])$, then ${}^{\alpha}(\mathcal{H}\cup\mathcal{G}) = {}^{\alpha}\mathcal{H} \cup {}^{\alpha}\mathcal{G}$.*
4. *If α is an element of $\mathcal{P}([0,1])$, then ${}_{\alpha}(\mathcal{H}\cup\mathcal{G}) = {}_{\alpha}\mathcal{H} \cup {}_{\alpha}\mathcal{G}$.*

PROOF: 1. Let x be any element of S. For every $\alpha \in \mathcal{P}([0,1])$ we have

$$\begin{array}{rcl} {}^{\alpha}(\mathcal{H}\cap\mathcal{G})_x & = & (\mathcal{H}\cap\mathcal{H})_x \cup \alpha \\ & = & (\mathcal{H}_x \cap \mathcal{G}_x) \cup \alpha \\ & = & (\mathcal{H}_x \cup \alpha) \cap (\mathcal{G}_x \cup \alpha) \\ & = & {}^{\alpha}\mathcal{H}_x \cap {}^{\alpha}\mathcal{G}_x \\ & = & ({}^{\alpha}\mathcal{H} \cap {}^{\alpha}\mathcal{G})_x\,, \end{array}$$

which implies that ${}^{\alpha}(\mathcal{H}\cap\mathcal{G}) = {}^{\alpha}\mathcal{H} \cap {}^{\alpha}\mathcal{G}$ for all $\alpha \in \mathcal{P}([0,1])$.

2. Let x be any element of S. Then, for every $\alpha \in \mathcal{P}([0,1])$, we have

$$\begin{aligned} {}_{\alpha}(\mathcal{H} \cap \mathcal{G})_x &= (\mathcal{H} \cap \mathcal{H})_x \cap \alpha \\ &= (\mathcal{H}_x \cap \mathcal{G}_x) \cap \alpha \\ &= (\mathcal{H}_x \cap \alpha) \cap (\mathcal{G}_x \cap \alpha) \\ &= {}_{\alpha}\mathcal{H}_x \cap {}_{\alpha}\mathcal{G}_x \\ &= ({}_{\alpha}\mathcal{H} \cap {}_{\alpha}\mathcal{G})_x . \end{aligned}$$

This completes the proof.

3–4. It can be proved similarly to 1. □

Now we introduce the notion of (α, β)-hesitant fuzzy subsemigroups on a semigroup.

Definition 2.4. Let $\mathcal{H}$ be a hesitant fuzzy set on a semigroup S and let α, β be any element of $\mathcal{P}([0,1])$. Then $\mathcal{H}$ is said to be an (α, β)**-hesitant fuzzy subsemigroup** on S if ${}^{\alpha}\mathcal{H}_{xy} \supseteq {}_{\beta}\mathcal{H}_x^y$ for any $x, y \in S$.

Thus every hesitant fuzzy subsemigroup on a semigroup S is an (α, β)-hesitant fuzzy subsemigroup with $\alpha = \emptyset$ and $\beta = [0,1]$. Thus every hesitant fuzzy subsemigroups on S is an (α, β)-hesitant fuzzy subsemigroup on S. However, the converse is not necessarily true as shown in the following example.

Example 2.5. Consider the semigroup $S = \{a, b, c, d\}$ with the following multiplication " $\cdot$ " table below:

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	b
d	a	a	b	c

We define the hesitant fuzzy set $\mathcal{H} : S \to \mathcal{P}([0,1])$ on S as follows:

$$\mathcal{H}_x = \begin{cases} [0.7, 0.8) \cup (0.8, 0.9]\,; & x \in \{a, b\} \\ (0.2, 0.8)\,; & x \in \{c\} \\ \{0, 0.1, 0.2, 0.3\}\,; & \text{otherwise.} \end{cases}$$

Then, as is easily seen, $\mathcal{H}$ is an $([0,3], \{0, 0.1, 0.2\})$-hesitant fuzzy subsemigroup on S, but not a hesitant fuzzy subsemigroup on S. Since

$\mathcal{H}_c^d = \mathcal{H}_d \cap \mathcal{H}_c = \{0, 0.1, 0.2, 0.3\} \cap (0.2, 0.8) = \{0.3\}$, while $\mathcal{H}_{d\cdot c} = \mathcal{H}_b = [0.7, 0.8) \cup (0.8, 0.9]$.

Now we have the following result:

THEOREM 2.6. *If $\mathcal{H}$ and $\mathcal{G}$ are any (α,β)-hesitant fuzzy subsemigroups on a semigroup S, then $\mathcal{H}\cap\mathcal{G}$ is an (α,β)-hesitant fuzzy subsemigroup on S.*

PROOF: Let x and y be any elements of S. Since $\mathcal{H}$ and $\mathcal{G}$ are both (α,β)-hesitant fuzzy subsemigroups on S, we have

$$\begin{aligned}
{}^{\alpha}(\mathcal{H}\cap\mathcal{G})_{xy} &= (\mathcal{H}_{xy}\cap\mathcal{G}_{xy})\cup\alpha \\
&= (\mathcal{H}_{xy}\cup\alpha)\cap(\mathcal{G}_{xy}\cup\alpha) \\
&= {}^{\alpha}\mathcal{H}_{xy}\cap{}^{\alpha}\mathcal{G}_{xy} \\
&\supseteq {}_{\beta}\mathcal{H}_x^y\cap{}_{\beta}\mathcal{G}_x^y \\
&= (\mathcal{H}_x\cap\mathcal{H}_y\cap\beta)\cap(\mathcal{G}_x\cap\mathcal{G}_y\cap\beta) \\
&= (\mathcal{H}_x\cap\mathcal{G}_x)\cap(\mathcal{H}_y\cap\mathcal{G}_y)\cap\beta \\
&= \Big((\mathcal{H}\cap\mathcal{G})_x\cap(\mathcal{H}\cap\mathcal{G})_y\Big)\cap\beta \\
&= {}_{\beta}(\mathcal{H}\cap\mathcal{G})_x^y.
\end{aligned}$$

Therefore, $\mathcal{H}\cap\mathcal{G}$ is an (α,β)-fuzzy subsemigroup on S. □

The following corollary follows from Theorem 2.6 and the definition of (α,β)-hesitant fuzzy subsemigroup on a semigroup S

COROLLARY 2.7. If $\mathcal{H}_i$ is an (α,β)-hesitant fuzzy subsemigroup on a semigroup S for all $i\in I$, then $\bigcap_{i\in I}\mathcal{H}_i$ is an (α,β)-hesitant fuzzy subsemigroup on S.

Let $\mathcal{F}$ and $\mathcal{G}$ be two hesitant fuzzy sets on a semigroup, the **hesitant fuzzy product** (see [7]) of $\mathcal{F}$ and $\mathcal{G}$ is defined to be a hesitant fuzzy set $\mathcal{F}\odot\mathcal{G}$ on S which is given by

$$(\mathcal{F}\odot\mathcal{G})_x = \begin{cases} \bigcup\limits_{x=yz}\mathcal{F}_y\cap\mathcal{G}_z; & \exists y,z\in S, \text{ such that } x=yz \\ \emptyset; & \text{otherwise.}\end{cases}$$

As is well known, the operation "$\odot$" is associative.

Next, we proved that every hesitant fuzzy set on a semigroup S is (α,β)-hesitant fuzzy subsemigroup if and only if ${}_{\beta}(\mathcal{H}\odot\mathcal{H})\subseteq{}^{\alpha}\mathcal{H}$.

Theorem 2.8. *For a hesitant fuzzy set $\mathcal{H}$ on a semigroup S, the following two statements are equivalent:*

1. *$\mathcal{H}$ is an (α, β)-hesitant fuzzy subsemigroup on S.*
2. *${}_{\beta}(\mathcal{H} \odot \mathcal{H}) \subseteq {}^{\alpha}\mathcal{H}$.*

Proof: First assume that $\mathcal{H}$ is an (α, β)-hesitant fuzzy subsemigroup on S. Let x be any element of S such that it is not expressible as product of two elements in S. Observe that ${}_{\beta}(\mathcal{H} \odot \mathcal{H})_x = (\mathcal{H} \odot \mathcal{H})_x \cap \beta = \emptyset \cap \beta = \emptyset \subseteq {}^{\alpha}\mathcal{H}_x$. Otherwise, there exist elements y and z of S such that $x = yz$. Thus, by hypothesis we have

$$\begin{aligned}
{}_{\beta}(\mathcal{H} \odot \mathcal{H})_x &= (\mathcal{H} \odot \mathcal{H})_x \cap \beta \\
&= \left(\bigcup_{x=ab} \mathcal{H}_a \cap \mathcal{H}_b \right) \cap \beta \\
&= \bigcup_{x=ab} (\mathcal{H}_a \cap \mathcal{H}_b \cap \beta) \\
&\subseteq \bigcup_{x=ab} (\mathcal{H}_{ab} \cup \alpha) \\
&= \mathcal{H}_x \cup \alpha \\
&= {}^{\alpha}\mathcal{H}_x.
\end{aligned}$$

Therefore ${}_{\beta}(\mathcal{H} \odot \mathcal{H}) \subseteq {}^{\alpha}\mathcal{H}$.

Conversely, assume that $\mathcal{H}$ is a hesitant fuzzy set on S such that ${}_{\beta}(\mathcal{H} \odot \mathcal{H}) \subseteq {}^{\alpha}\mathcal{H}$. Let x, y and z be any elements of S. Now, choose $x = yz$. Thus, we obtain

$$\begin{aligned}
{}^{\alpha}\mathcal{H}_{yz} &= {}^{\alpha}\mathcal{H}_x \\
&\supseteq {}_{\beta}(\mathcal{H} \odot \mathcal{H})_x \\
&= (\mathcal{H} \odot \mathcal{H})_x \cap \beta \\
&= \left(\bigcup_{x=ab} \mathcal{H}_a \cap \mathcal{H}_b \right) \cap \beta \\
&\supseteq (\mathcal{H}_y \cap \mathcal{H}_z) \cap \beta \\
&= {}_{\beta}\mathcal{H}_y^z.
\end{aligned}$$

Therefore, the proof is completed. □

Now, we can introduce the (α, β)-hesitant fuzzy ideals on a semigroup, in the following manner:

Definition 2.9. Let $\mathcal{H}$ be a hesitant fuzzy set on a semigroup S. Then $\mathcal{H}$ is said to be an (α, β)**-hesitant fuzzy left ideal (**(α, β)**-hesitant right ideal)** on S if ${}^{\alpha}\mathcal{H}_{xy} \supseteq {}_{\beta}\mathcal{H}_y$ $({}^{\alpha}\mathcal{H}_{xy} \supseteq {}_{\beta}\mathcal{H}_x)$ for any $x, y \in S$. A hesitant fuzzy set $\mathcal{H}$ is an (α, β)**-hesitant fuzzy ideal (or** (α, β)**-hesitant fuzzy two-sided ideal)** on S if and only if it is both (α, β)-hesitant fuzzy left and right ideal on$S$.

Remark 2.10. Let $\mathcal{H}$ be a hesitant fuzzy set on a semigroup S. Then, the following statements are true:

1. If $\mathcal{H}$ is an (α, β)-hesitant fuzzy left (right) ideal on S, then $\mathcal{H}$ is an (α, β)-hesitant fuzzy subsemigroup on S.

2. A hesitant fuzzy left ideal (right ideal, ideal) on S is an (α, β)-fuzzy left ideal (right ideal, ideal) with $\alpha = \emptyset$ and $\beta = [0, 1]$. Thus every hesitant fuzzy left ideal (right ideal, ideal) on S is an (α, β)-fuzzy left ideal (right ideal, ideal) on S.

However, the converse is not necessarily true as shown in the following example.

Example 2.11.

1. Consider the semigroup $S = \{a, b, c, d\}$ with the following multiplication " $\cdot$ " table below:

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	b	c	a
c	a	a	a	a
d	a	d	a	a

Now, we define a hesitant fuzzy set $\mathcal{H} : S \to \mathcal{P}([0, 1])$ by

$$\mathcal{H}_x = \begin{cases} [0, 0.85)\,; & x \in \{a, b\} \\ \{0, 0.22, 0.42, 0.52\}\,; & \text{otherwise.} \end{cases}$$

Let $\alpha = (0.52, 0.62]$ and $\beta = \{0.12, 0.22, 0.32, 0.52\}$. Then, as is easily seen, $\mathcal{H}$ is an (α, β)-hesitant fuzzy subsemigroup on S, but $\mathcal{H}$ is not an (α, β)-hesitant fuzzy left (right) ideal on S. Because

$$\begin{array}{rcl} {}^{(0.52,0.62]}\mathcal{H}_{d\cdot b} & = & \mathcal{H}_d \cup (0.52, 0.62] \\ & = & \{0, 0.22, 0.42, 0.52\} \cup (0.5, 0.6] \\ & = & \{0, 0.22, 0.42\} \cup [0.52, 0.62] \\ & \not\supseteq & \{0.12, 0.22, 0.32, 0.52\} \\ & = & [0, 0.8) \cap \{0.12, 0.22, 0.32, 0.52\} \\ & = & \mathcal{H}_b \cap \{0.12, 0.22, 0.32, 0.52\} \\ & = & {}_{\{0.12,0.22,0.32,0.52\}}\mathcal{H}_b. \end{array}$$

2. Suppose that S is the semigroup of Example 2.11 (1). Now, we define a hesitant fuzzy set $\mathcal{F} : S \to \mathcal{P}[0,1]$ by

$$\mathcal{F}_x = \begin{cases} \{0, 0.52, 0.62\}; & x \in \{a, b\} \\ [0, 0.52); & \text{otherwise.} \end{cases}$$

Let $\alpha = [0, 0.72)$ and $\beta = (0.52, 0.62)$. Then, as is easily seen, $\mathcal{F}$ is an (α, β)-hesitant fuzzy ideal on S, but $\mathcal{F}$ is not a hesitant fuzzy ideal on S. Because $\mathcal{F}_{d\cdot b} = \mathcal{F}_d = [0, 0.5) \not\supseteq \{0, 0.5, 0.6\} = \mathcal{F}_b$.

By Theorem 2.2 and Definition 2.9, we immediately obtain the following theorem:

Theorem 2.12. *Let $\mathcal{H}$ be a hesitant fuzzy left (right, two-sided) ideal on a semigroup S. Then the following properties hold.*

1. *If α is an element of $\mathcal{P}([0,1])$, then ${}^{\alpha}\mathcal{H}$ is a hesitant fuzzy left (right, two-sided) ideal on S.*
2. *If α is an element of $\mathcal{P}([0,1])$, then ${}_{\alpha}\mathcal{H}$ is a hesitant fuzzy left (right, two-sided) ideal on S.*

Next, we proved that every hesitant fuzzy set on a semigroup S is (α, β)-hesitant fuzzy left (right, two-sided) ideal if and only if ${}_{\beta}(\mathcal{S} \odot \mathcal{H}) \subseteq {}^{\alpha}\mathcal{H}({}_{\beta}(\mathcal{H} \odot \mathcal{S}) \subseteq {}^{\alpha}\mathcal{H}, {}_{\beta}((\mathcal{S} \odot \mathcal{H}) \cup (\mathcal{H} \odot \mathcal{S})) \subseteq {}^{\alpha}\mathcal{H})$.

Theorem 2.13. *For a hesitant fuzzy set $\mathcal{H}$ on a semigroup S, the following statements are equivalent:*

1. *$\mathcal{H}$ is an (α, β)-hesitant fuzzy left (right, two-sided) ideal on S.*
2. ${}_{\beta}(\mathcal{S} \odot \mathcal{H}) \subseteq {}^{\alpha}\mathcal{H}({}_{\beta}(\mathcal{H} \odot \mathcal{S}) \subseteq {}^{\alpha}\mathcal{H}, {}_{\beta}((\mathcal{S} \odot \mathcal{H}) \cup (\mathcal{H} \odot \mathcal{S})) \subseteq {}^{\alpha}\mathcal{H})$.

Proof: First assume that $\mathcal{H}$ is an (α, β)-hesitant fuzzy left ideal on S. Let x be any element of S such that it is not expressible as product of two elements in S, we can write ${}_\beta(\mathcal{S}\odot\mathcal{H})_x = (\mathcal{S}\odot\mathcal{H})_x \cap \beta = \emptyset \cap \beta = \emptyset \subseteq {}^\alpha\mathcal{H}_x$. Otherwise, there exist elements y and z of S such that $x = yz$. Since $\mathcal{H}$ is an (α,β)-hesitant fuzzy left ideal on S, it follows that

$$\begin{aligned}
{}_\beta(\mathcal{S}\odot\mathcal{H})_x &= (\mathcal{S}\odot\mathcal{H})_x \cap \beta \\
&= \left(\bigcup_{x=ab} \mathcal{S}_a \cap \mathcal{H}_b\right) \cap \beta \\
&= \bigcup_{x=ab} ([0,1] \cap \mathcal{H}_b \cap \beta) \\
&= \bigcup_{x=ab} (\mathcal{H}_b \cap \beta) \\
&\subseteq \bigcup_{x=ab} (\mathcal{H}_{ab} \cup \alpha) \\
&= \mathcal{H}_x \cup \alpha \\
&= {}^\alpha\mathcal{H}_x.
\end{aligned}$$

Therefore ${}_\beta(\mathcal{S}\odot\mathcal{H}) \subseteq {}^\alpha\mathcal{H}$.

Conversely, assume that $\mathcal{H}$ is a hesitant fuzzy set on S such that ${}_\beta(\mathcal{S}\odot\mathcal{H}) \subseteq {}^\alpha\mathcal{H}$. Let y and z be any elements of S. Choose $x \in S$ such that $x = yz$. Since

$$\begin{aligned}
{}^\alpha\mathcal{H}_{yz} &= {}^\alpha\mathcal{H}_x \\
&\supseteq {}_\beta(\mathcal{S}\odot\mathcal{H})_x \\
&= (\mathcal{S}\odot\mathcal{H})_x \cap \beta \\
&= \left(\bigcup_{x=ab} \mathcal{S}_a \cap \mathcal{H}_b\right) \cap \beta \\
&\supseteq (\mathcal{S}_y \cap \mathcal{H}_z) \cap \beta \\
&= ([0,1] \cap \mathcal{H}_z) \cap \beta \\
&= {}_\beta\mathcal{H}_z,
\end{aligned}$$

we obtain $\mathcal{H}$ is an (α,β)-hesitant fuzzy left ideal on S. □

In the following we show that if $\mathcal{H}$ and $\mathcal{G}$ are two (α,β)-hesitant fuzzy left (right, two-sided) ideals on a semigroup, then $\mathcal{H}\cap\mathcal{G}$ and $\mathcal{H}\cup\mathcal{G}$ are (α,β)-hesitant fuzzy left (right, two-sided) ideals on S.

THEOREM 2.14. *Let $\mathcal{H}$ and $\mathcal{G}$ be two (α,β)-hesitant fuzzy left (right, two-sided) ideals on a semigroup S. Then the following statements hold:*

1. *$\mathcal{H}\cap\mathcal{G}$ is an (α,β)-hesitant fuzzy left (right, two-sided) ideal on S.*
2. *$\mathcal{H}\cup\mathcal{G}$ is an (α,β)-hesitant fuzzy left (right, two-sided) ideal on S.*

Proof: 1. The proof follows from Theorem 2.6.

2. By Theorem 2.13, we have

$$\begin{aligned} {}_{\beta}\left(\mathcal{S}\odot(\mathcal{H}\cup\mathcal{G})\right) &= {}_{\beta}\left((\mathcal{S}\odot\mathcal{H})\cup(\mathcal{S}\odot\mathcal{G})\right) \\ &= {}_{\beta}\left(\mathcal{S}\odot\mathcal{H}\right)\cup{}_{\beta}\left(\mathcal{S}\odot\mathcal{G}\right) \\ &\subseteq {}^{\alpha}\mathcal{H}\cup{}^{\alpha}\mathcal{G} \\ &= {}^{\alpha}\left(\mathcal{H}\cap\mathcal{G}\right). \end{aligned}$$

Therefore, we obtain $\mathcal{H}\cup\mathcal{G}$ is an (α,β)-fuzzy left ideal on S. □

The following two corollaries are exactly obtained from Theorem 2.14.

Theorem 2.15. *Let $\mathcal{H}_i$ be an (α,β)-hesitant fuzzy left (right, two-sided) ideal on a semigroup S for all $i\in I$. Then the following statements hold:*

1. $\bigcap\limits_{i\in I}\mathcal{H}_i$ *is an (α,β)-hesitant fuzzy left (right, two-sided) ideal on S.*
2. $\bigcup\limits_{i\in I}\mathcal{H}_i$ *is an (α,β)-hesitant fuzzy left (right, two-sided) ideal on S.*

Now, we define a γ-cut (or γ-level set) of the hesitant fuzzy set $\mathcal{H}$ on a semigroup S and then we present some results in this connection.

Let $\mathcal{H}$ be a hesitant fuzzy set on a semigroup S. For each $\gamma\in\mathcal{P}([0,1])$, the set

$$U\left(\mathcal{H}:\gamma\right)=\{x\in S:\mathcal{H}_x\supseteq\gamma\}$$

is said to be a **γ-cut (or γ-level set)** of $\mathcal{H}$.

In the following, we characterize an (α,β)-hesitant fuzzy subsemigroup ((α,β)-hesitant fuzzy left ideal, (α,β)-hesitant fuzzy right ideal, (α,β)-hesitant fuzzy two-sided ideal) on semigroups in terms γ-level subsemigroups (left ideal, right ideal, two-sided ideal).

Theorem 2.16. *Let $\mathcal{H}$ be a hesitant fuzzy set on a semigroup S. Then the following statements hold:*

1. *For each $\gamma\in\mathcal{P}([0,1])$ such that $\gamma\subseteq\alpha\cup\beta$, the non empty set $U\left({}^{\alpha}\mathcal{H}:\gamma\right)$ is a subsemigroup of S if and only if $\mathcal{H}$ is an (α,β)-hesitant fuzzy subsemigroup on S.*
2. *For each $\gamma\in\mathcal{P}([0,1])$ such that $\gamma\subseteq\alpha\cup\beta$, the non empty set $U\left({}^{\alpha}\mathcal{H}:\gamma\right)$ is a left ideal of S if and only if $\mathcal{H}$ is an (α,β)-hesitant fuzzy left ideal on S.*

3. *For each $\gamma \in \mathcal{P}([0,1])$ such that $\gamma \subseteq \alpha \cup \beta$, the non empty set $U({}^{\alpha}\mathcal{H}:\gamma)$ is a right ideal of S if and only if $\mathcal{H}$ is an (α,β)-hesitant fuzzy right ideal on S.*
4. *For each $\gamma \in \mathcal{P}([0,1])$ such that $\gamma \subseteq \alpha \cup \beta$, the non empty set $U({}^{\alpha}\mathcal{H}:\gamma)$ is an ideal of S if and only if $\mathcal{H}$ is an (α,β)-hesitant fuzzy ideal on S.*

Proof: 1. First assume that the non empty set $U({}^{\alpha}\mathcal{H}:\gamma)$ is a subsemigroup of S. Let x and y be any elements of S. Choose $\gamma \in \mathcal{P}([0,1])$ such that $\gamma = {}^{\alpha}\mathcal{H}_x^y$. Then we have ${}^{\alpha}\mathcal{H}_x \supseteq \gamma$ and ${}^{\alpha}\mathcal{H}_y \supseteq \gamma$, which implies that $x, y \in U({}^{\alpha}\mathcal{H}:\gamma)$. Thus $xy \in U({}^{\alpha}\mathcal{H}:\gamma)$, since $U({}^{\alpha}\mathcal{H}:\gamma)$ is a subsemigroup of S. This implies that ${}^{\alpha}\mathcal{H}_{xy} \supseteq \gamma = {}^{\alpha}\mathcal{H}_x^y \supseteq \mathcal{H}_x^y \cap \beta = {}_{\beta}\mathcal{H}_x^y$ and so $\mathcal{H}$ is an (α,β)-hesitant fuzzy subsemigroup on S.

Conversely, assume that $\mathcal{H}$ is an (α,β)-hesitant fuzzy subsemigroup on S. Let x and y be any elements of S such that $x, y \in U({}^{\alpha}\mathcal{H}:\gamma)$ for all $\gamma \in \mathcal{P}([0,1])$. We obtain that ${}^{\alpha}\mathcal{H}_x \supseteq \gamma$ and ${}^{\alpha}\mathcal{H}_y \supseteq \gamma$. By hypothesis,

$$\begin{array}{rcl} {}^{\alpha}\mathcal{H}_{xy} & = & {}^{\alpha}\mathcal{H}_{xy} \cup \alpha \\ & \supseteq & ({}_{\beta}\mathcal{H}_x^y) \cup \alpha \\ & = & (\mathcal{H}_x \cap \mathcal{H}_y \cap \beta) \cup \alpha \\ & = & (\mathcal{H}_x \cup \alpha) \cap (\mathcal{H}_y \cup \alpha) \cap (\beta \cup \alpha) \\ & = & {}^{\alpha}\mathcal{H}_x \cap {}^{\alpha}\mathcal{H}_y \cap (\beta \cup \alpha) \\ & \supseteq & \gamma \cap \gamma \cap (\beta \cup \alpha) \\ & = & \gamma. \end{array}$$

Therefore $xy \in U({}^{\alpha}\mathcal{H}:\gamma)$ and the theorem is proved.

2–4. The proof is similar to 1. □

Let A be a subset of a semigroup S and let $\delta, \zeta \in \mathcal{P}([0,1])$ such that $\delta \neq \zeta$. Recall that the (δ,ζ)-characteristic function ${\mathcal{C}_{\zeta}^{\delta}}_A$ for a subset A of S is defined by

$$\left({\mathcal{C}_{\zeta}^{\delta}}_A\right)_x = \begin{cases} \delta; & x \in A \\ \zeta; & \text{otherwise.} \end{cases}$$

Observe that if $A = S$, then it is easy to see that ${\mathcal{C}_{\zeta}^{[0,1]}}_S = \mathcal{S}$.

Lemma 2.17. *Let ${\mathcal{C}_{\zeta}^{\delta}}_A$ and ${\mathcal{C}_{\zeta}^{\delta}}_B$ be two (δ,ζ)-characteristic functions on a semigroup S. Then the following statements hold:*

1. ${\mathcal{C}_{\zeta}^{\delta}}_A \cap {\mathcal{C}_{\zeta}^{\delta}}_B = {\mathcal{C}_{\zeta}^{\delta}}_{A\cap B}$.
2. ${\mathcal{C}_{\zeta}^{\delta}}_A \odot {\mathcal{C}_{\zeta}^{\delta}}_B = {\mathcal{C}_{\zeta}^{\delta}}_{AB}$.

Proof: It is straightforward. □

Now, the result follows from fundamental theorem of (δ,ζ)-characteristic function.

Theorem 2.18. *Let A be a subset of a semigroup S and let $\delta,\zeta \in \mathcal{P}([0,1])$ such that $\delta \supset \zeta$. Then the following statements hold:*

1. *If A is a subsemigroup of S, then ${}^{\alpha}\mathcal{C}^{\delta}_{\zeta\, A}$ is an (α,β)-hesitant fuzzy subsemigroup on S.*
2. *If A is a left ideal of S, then ${}^{\alpha}\mathcal{C}^{\delta}_{\zeta\, A}$ is an (α,β)-hesitant fuzzy left ideal on S.*
3. *If A is a right ideal of S, then ${}^{\alpha}\mathcal{C}^{\delta}_{\zeta\, A}$ is an (α,β)-hesitant fuzzy right ideal on S.*
4. *If A is an ideal of S, then ${}^{\alpha}\mathcal{C}^{\delta}_{\zeta\, A}$ is an (α,β)-hesitant fuzzy ideal on S.*

Proof: 1. Let x and y be any elements of S. We consider the following cases:

1. $x,y \in A$.
2. $x \notin A$ or $y \notin A$.

Case 1: Assume that $x,y \in A$. Thus $\left(\mathcal{C}^{\delta}_{\zeta\, A}\right)_x = \delta$ and $\left(\mathcal{C}^{\delta}_{\zeta\, A}\right)_y = \delta$. Since A is a subsemigroup of S, we obtain $xy \in A$, which implies that ${}^{\alpha}\left({}^{\alpha}\mathcal{C}^{\delta}_{\zeta\, A}\right)_{xy} = \left(\mathcal{C}^{\delta}_{\zeta\, A}\right)_{xy} \cup \alpha = \delta \cup \alpha = \left(\left(\mathcal{C}^{\delta}_{\zeta\, A}\right)_x \cap \left(\mathcal{C}^{\delta}_{\zeta\, A}\right)_y\right) \cup \alpha \supseteq \left(\left(\mathcal{C}^{\delta}_{\zeta\, A}\right)_x \cap \left(\mathcal{C}^{\delta}_{\zeta\, A}\right)_y\right) \cap \beta = {}_{\beta}\left(\mathcal{C}^{\delta}_{\zeta\, A}\right)^y_x$.

Case 2: Assume that $x \notin A$ or $y \notin A$. Then we have $\left(\mathcal{C}^{\delta}_{\zeta\, A}\right)_x = \zeta$ or $\left(\mathcal{C}^{\delta}_{\zeta\, A}\right)_y = \zeta$, which implies that

$$\begin{aligned}
{}^{\alpha}\left({}^{\alpha}\mathcal{C}^{\delta}_{\zeta\, A}\right)_{xy} &= \left(\mathcal{C}^{\delta}_{\zeta\, A}\right)_{xy} \cup \alpha \\
&\supseteq \zeta \cup \alpha \\
&= \left({}^{\alpha}\mathcal{C}^{\delta}_{\zeta\, A}\right)_x \cap \left({}^{\alpha}\mathcal{C}^{\delta}_{\zeta\, A}\right)_y \\
&\supseteq \left(\left({}^{\alpha}\mathcal{C}^{\delta}_{\zeta\, A}\right)_x \cap \left({}^{\alpha}\mathcal{C}^{\delta}_{\zeta\, A}\right)_y\right) \cap \beta \\
&= {}_{\beta}\left({}^{\alpha}\mathcal{C}^{\delta}_{\zeta\, A}\right)^y_x.
\end{aligned}$$

Therefore, ${}^{\alpha}\mathcal{C}^{\delta}_{\zeta\, A}$ is an (α, β)-hesitant fuzzy subsemigroup on S.

2–4. The proof is similar to 1. □

The following theorem shows that a non empty subset of a semigroup S is a subsemigroup (left ideal, right ideal, two-sided ideal) of S if and only if the hesitant fuzzy set on S is the (α, β)-hesitant fuzzy subsemigroup ((α, β)-hesitant fuzzy left ideal, (α, β)-hesitant fuzzy right ideal, (α, β)-hesitant fuzzy two-sided ideal) on S.

THEOREM 2.19. *Let A be a subset of a semigroup S and let $\delta, \zeta \in \mathcal{P}([0, 1])$ such that $\delta \supset \zeta$. Then the following properties hold.*

1. *For each $\zeta \cup \alpha \not\supseteq \delta \cup \alpha \subseteq \alpha \cup \beta$, A is a subsemigroup of S if and only if ${}^{\alpha}\mathcal{C}^{\delta}_{\zeta\, A}$ is an (α, β)-hesitant fuzzy subsemigroup on S.*
2. *For each $\zeta \cup \alpha \not\supseteq \delta \cup \alpha \subseteq \alpha \cup \beta$, A is a left ideal of S if and only if ${}^{\alpha}\mathcal{C}^{\delta}_{\zeta\, A}$ is an (α, β)-hesitant fuzzy left ideal on S.*
3. *For each $\zeta \cup \alpha \not\supseteq \delta \cup \alpha \subseteq \alpha \cup \beta$, A is a right ideal of S if and only if ${}^{\alpha}\mathcal{C}^{\delta}_{\zeta\, A}$ is an (α, β)-hesitant fuzzy right ideal on S.*
4. *For each $\zeta \cup \alpha \not\supseteq \delta \cup \alpha \subseteq \alpha \cup \beta$, A is an ideal of S if and only if ${}^{\alpha}\mathcal{C}^{\delta}_{\zeta\, A}$ is an (α, β)-hesitant fuzzy ideal on S.*

PROOF: 1. By Theorem 2.18, the necessity is clear. Now let us show the sufficiency. We suppose now that ${}^{\alpha}\mathcal{C}^{\delta}_{\zeta\, A}$ is an (α, β)-hesitant fuzzy subsemigroup on S. Let x be any element of S such that $x \in A$. Observe that ${}^{\alpha}\left(\mathcal{C}^{\delta}_{\zeta\, A}\right)_x = \left(\mathcal{C}^{\delta}_{\zeta\, A}\right)_x \cup \alpha = \delta \cup \alpha$ implies that $x \in U\left({}^{\alpha}\mathcal{C}^{\delta}_{\zeta\, A} : \delta \cup \alpha\right)$. On the other hand, let x be any element of S such that $x \in U\left({}^{\alpha}\mathcal{C}^{\delta}_{\zeta\, A} : \delta \cup \alpha\right)$. Thus we have ${}^{\alpha}\left(\mathcal{C}^{\delta}_{\zeta\, A}\right)_x \supseteq \delta \cup \alpha$ implies that $x \in A$, since $\zeta \cup \alpha \not\supseteq \delta \cup \alpha$. Therefore $A = U\left({}^{\alpha}\mathcal{C}^{\delta}_{\zeta\, A} : \delta \cup \alpha\right)$ and hence it follows from Theorem 2.16(1) that A is a subsemigroup of S.

2–4. The proof is similar to 1. □

The next result covers some basic properties, which will be useful in the sequel.

Theorem 2.20. *If $\mathcal{H}$ is an (α, β)-hesitant fuzzy right (left) ideal on a semigroup S, then $\mathcal{H} \cup (\mathcal{S} \odot \mathcal{H})$ is an (α, β)-hesitant fuzzy ideal on S.*

Proof: Then, by Theorem 2.13, we have

$$\begin{aligned}
{}_{\beta}(\mathcal{S} \odot (\mathcal{H} \cup (\mathcal{S} \odot \mathcal{H}))) &= {}_{\beta}((\mathcal{S} \odot \mathcal{H}) \cup (\mathcal{S} \odot (\mathcal{S} \odot \mathcal{H}))) \\
&= {}_{\beta}(\mathcal{S} \odot \mathcal{H}) \cup ({}_{\beta}(\mathcal{S} \odot \mathcal{S}) \odot {}_{\beta}\mathcal{H}) \\
&= {}_{\beta}(\mathcal{S} \odot \mathcal{H}) \cup ({}_{\beta}(\mathcal{S} \odot \mathcal{S}) \odot {}_{\beta}\mathcal{H}) \\
&\subseteq {}^{\alpha}(\mathcal{S} \odot \mathcal{H}) \cup ({}^{\alpha}\mathcal{S} \odot {}^{\alpha}\mathcal{H}) \\
&= {}^{\alpha}(\mathcal{S} \odot \mathcal{H}) \cup {}^{\alpha}(\mathcal{S} \odot \mathcal{H}) \\
&\subseteq {}^{\alpha}\mathcal{H} \cup {}^{\alpha}(\mathcal{S} \odot \mathcal{H}) \\
&= {}^{\alpha}(\mathcal{H} \cup (\mathcal{S} \odot \mathcal{H})),
\end{aligned}$$

since $\mathcal{S}$ is an (α, β)-hesitant fuzzy left ideal on S. Thus, we obtain, $\mathcal{H} \cup (\mathcal{S} \odot \mathcal{H})$ is an (α, β)-hesitant fuzzy left ideal on S. Also, we have

$$\begin{aligned}
{}_{\beta}((\mathcal{H} \cup (\mathcal{S} \odot \mathcal{H})) \odot \mathcal{S}) &= {}_{\beta}((\mathcal{H} \odot \mathcal{S}) \cup ((\mathcal{S} \odot \mathcal{H}) \odot \mathcal{S})) \\
&= {}_{\beta}(\mathcal{H} \odot \mathcal{S}) \cup {}_{\beta}((\mathcal{S} \odot \mathcal{H}) \odot \mathcal{S}) \\
&\subseteq {}^{\alpha}\mathcal{H} \cup ({}_{\beta}\mathcal{S} \odot {}_{\beta}(\mathcal{H} \odot \mathcal{S})) \\
&\subseteq {}^{\alpha}\mathcal{H} \cup ({}^{\alpha}\mathcal{S} \odot {}^{\alpha}\mathcal{H}) \\
&= {}^{\alpha}(\mathcal{H} \cup (\mathcal{S} \odot \mathcal{H})),
\end{aligned}$$

since $\mathcal{H}$ is an (α, β)-hesitant fuzzy right ideal on S. Therefore $\mathcal{H} \cup (\mathcal{S} \odot \mathcal{H})$ is an (α, β)-hesitant fuzzy right ideal on S and the proof is completed. □

3. Characterizing regular semigroups

In this section we define the concept of (α, β)-hesitant fuzzy idempotent sets on semigroups and then by using this idea we characterize the regular semigroups in terms of hesitant fuzzy left ideals, hesitant fuzzy right ideals and hesitant fuzzy ideals.

Recall that an element x is said to be **regular** if there exists an element s in a semigroup S such that $x = xsx$. A semigroup S is said to be **regular** if every element of S is regular.

First, we define the operation "$\simeq_{\beta}^{\alpha}$" on a semigroup S. Let α and β be any elements of $\mathcal{P}([0,1])$. We consider two hesitant fuzzy sets $\mathcal{H}$ and $\mathcal{G}$ on a semigroup S. Then we have, $\mathcal{H} \simeq_{\beta}^{\alpha} \mathcal{G}$ if and only if ${}_{\beta}\mathcal{H} \subseteq {}^{\alpha}\mathcal{G}$ and ${}_{\beta}\mathcal{G} \subseteq {}^{\alpha}\mathcal{H}$. A hesitant fuzzy set $\mathcal{H}$ on a semigroup S is said to be (α, β)**-idempotent** if $\mathcal{H} \simeq_{\beta}^{\alpha} \mathcal{H} \odot \mathcal{H}$.

Theorem 3.1. *Every (α,β)-hesitant fuzzy right (left) ideal on a regular semigroup is (α,β)-idempotent.*

Proof: Let $\mathcal{H}$ be an (α,β)-hesitant fuzzy right ideal on a regular semigroup S. Then by Theorem 2.13, we obtain that ${}_{\beta}(\mathcal{H}\odot\mathcal{H}) \subseteq {}_{\beta}(\mathcal{H}\odot\mathcal{S}) \subseteq {}^{\alpha}\mathcal{H}$. On the other hand, let x be any element of S. Then, since S is regular, there exists an element s in S such that $x = xsx$. Therefore

$$\begin{aligned}
{}^{\alpha}(\mathcal{H}\odot\mathcal{H})_x &= (\mathcal{H}\odot\mathcal{H})_x \cup \alpha \\
&= \left(\bigcup_{x=ab} \mathcal{H}_a \cap \mathcal{H}_b\right) \cup \alpha \\
&\supseteq (\mathcal{H}_{xs}\cap\mathcal{H}_x)\cup\alpha \\
&= (\mathcal{H}_{xs}\cup\alpha)\cap(\mathcal{H}_x\cup\alpha) \\
&\supseteq {}^{\alpha}\mathcal{H}_{xs}\cap {}_{\beta}\mathcal{H}_x \\
&\supseteq {}_{\beta}\mathcal{H}_x\cap {}_{\beta}\mathcal{H}_x \\
&= {}_{\beta}\mathcal{H}_x
\end{aligned}$$

i.e., ${}^{\alpha}(\mathcal{H}\odot\mathcal{H}) \supseteq {}_{\beta}\mathcal{H}$ and the proof is now complete. □

By Theorem 3.1, we immediately obtain the following corollary:

Corollary 3.2. Every (α,β)-hesitant fuzzy ideal on a regular semigroup is (α,β)-idempotent.

In the following theorem we give a characterization of a semigroup that is regular in terms of (α,β)-hesitant fuzzy right ideals and (α,β)-hesitant fuzzy left ideals.

Theorem 3.3. *Let ζ and δ be any elements of $\mathcal{P}([0,1])$ such that $\zeta\cup\alpha \not\supseteq \delta\cap\beta$. For a semigroup S, the following statements are equivalent:*

1. *S is regular.*
2. *For every (α,β)-hesitant fuzzy right ideal $\mathcal{H}$ and every (α,β)-hesitant fuzzy left ideal $\mathcal{F}$ on S, $\mathcal{H}\cap\mathcal{F} \simeq^{\alpha}_{\beta} \mathcal{H}\odot\mathcal{F}$.*

Proof: First assume that S is a regular semigroup. Let $\mathcal{H}$ be any (α,β)-hesitant fuzzy right ideal and $\mathcal{F}$ any (α,β)-hesitant fuzzy left ideal on S. Then by Theorem 2.13, we obtain that ${}_{\beta}(\mathcal{H}\odot\mathcal{F}) \subseteq {}_{\beta}(\mathcal{H}\odot\mathcal{S}) \subseteq {}^{\alpha}\mathcal{H}$ and ${}_{\beta}(\mathcal{H}\odot\mathcal{F}) \subseteq {}_{\beta}(\mathcal{S}\odot\mathcal{F}) \subseteq {}^{\alpha}\mathcal{F}$. Consequently we have ${}_{\beta}(\mathcal{H}\odot\mathcal{F}) \subseteq {}^{\alpha}\mathcal{H}\cap{}^{\alpha}\mathcal{F} = {}^{\alpha}(\mathcal{H}\cap\mathcal{F})$. On the other hand, let x be any element of S.

Then, since S is regular, there exists an element s in S such that $x = xsx$. Observe that

$$\begin{aligned}
{}^{\alpha}(\mathcal{H} \odot \mathcal{F})_x &= (\mathcal{H} \odot \mathcal{F})_x \cup \alpha \\
&= \left(\bigcup_{x=ab} \mathcal{H}_a \cap \mathcal{F}_b \right) \cup \alpha \\
&\supseteq (\mathcal{H}_{xs} \cap \mathcal{F}_x) \cup \alpha \\
&= (\mathcal{H}_{xs} \cup \alpha) \cap (\mathcal{F}_x \cup \alpha) \\
&\supseteq {}_{\beta}\mathcal{H}_x \cap (\mathcal{F}_x \cap \beta) \\
&= {}_{\beta}\mathcal{H}_x \cap {}_{\beta}\mathcal{F}_x \\
&= {}_{\beta}(\mathcal{H} \cap \mathcal{F})_x
\end{aligned}$$

implies that ${}^{\alpha}(\mathcal{H} \odot \mathcal{F}) \supseteq {}_{\beta}(\mathcal{H} \cap \mathcal{F})$. Therefore $\mathcal{H} \cap \mathcal{F} \simeq_{\beta}^{\alpha} \mathcal{H} \odot \mathcal{F}$ and so (1) implies (2).

Conversely, assume that (2) holds. Let R and L be any right ideal and any left ideal of S, respectively. In order to see that $R \cap L \subseteq RL$ holds, let x be any element of $R \cap L$. Then by Theorem 2.18, the (δ, ζ)-characteristic functions ${}^{\alpha}\mathcal{C}^{\delta}_{\zeta R}$ and ${}^{\alpha}\mathcal{C}^{\delta}_{\zeta L}$ of R and L is an (α, β)-hesitant fuzzy right ideal and an (α, β)-hesitant fuzzy left ideal on S, respectively. Then it follows from Lemma 2.17, it follows that

$$\begin{aligned}
\left({}^{\alpha}\mathcal{C}^{\delta}_{\zeta RL}\right)_x &= \left(\mathcal{C}^{\delta}_{\zeta RL}\right)_x \cup \alpha \\
&= \left(\mathcal{C}^{\delta}_{\zeta R} \odot \mathcal{C}^{\delta}_{\zeta L}\right)_x \cup \alpha \\
&= {}^{\alpha}\left(\mathcal{C}^{\delta}_{\zeta R} \odot \mathcal{C}^{\delta}_{\zeta L}\right)_x \\
&\supseteq {}_{\beta}\left(\mathcal{C}^{\delta}_{\zeta R} \cap \mathcal{C}^{\delta}_{\zeta L}\right)_x \\
&= {}_{\beta}\left(\mathcal{C}^{\delta}_{\zeta R\cap L}\right)_x \\
&= \left(\mathcal{C}^{\delta}_{\zeta R\cap L}\right)_x \cap \beta \\
&= \delta \cap \beta.
\end{aligned}$$

Hence $x \in RL$ and so $R \cap L \subseteq RL$. Since the inclusion in the other direction always holds, we obtain that $R \cap L = RL$. Therefore S is a regular semigroup and so (2) implies (1). $\square$

Recall that a semigroup S is said to be **right (left) zero** if $xy = y(xy = x)$ for all $x, y \in S$. Now, we can give the main result.

THEOREM 3.4. *Let ζ and δ be any elements of $\mathcal{P}([0,1])$ such that $\zeta \cup \alpha \not\supseteq \delta \cap \beta$. For a regular semigroup S, the following statements are equivalent:*

1. *The set $\mathcal{E}(S)$ of all (α, β)-idempotents of S forms a left (right) zero subsemigroup of S.*
2. *For every (α, β)-hesitant fuzzy left (right) ideal $\mathcal{H}$ on $S, \mathcal{H}_x \simeq_\beta^\alpha \mathcal{H}_y$ for all $x, y \in \mathcal{E}(S)$.*

PROOF: First assume that (1) holds. Let $\mathcal{H}$ be an (α, β)-hesitant fuzzy left ideal on S and let $x, y \in \mathcal{E}(S)$. Since $\mathcal{E}(S)$ is a left zero subsemigroup of S, we have $xy = x$ and $yx = y$. Observe that ${}^\alpha\mathcal{H}_x = {}^\alpha\mathcal{H}_{xy} \supseteq {}_\beta\mathcal{H}_y$ and ${}^\alpha\mathcal{H}_y = {}^\alpha\mathcal{H}_{yx} \supseteq {}_\beta\mathcal{H}_x$, implies that $\mathcal{H}_x \simeq_\beta^\alpha \mathcal{H}_y$ and hence (1) implies (2).

Conversely, assume that (2) holds. Let x be any element of S. Since S is regular, there exists an element $s \in S$ such that $x = xsx$. Now $(xsxs)(xsxs) = (xsx)s(xsx)s = xsxs \in \mathcal{E}(S)$, implies that $\mathcal{E}(S)$ is non empty. Thus it follows from Theorem 2.18(2) that the (δ, ζ)-characteristic function ${}^\alpha\mathcal{C}_{\zeta\, Sy}^{\delta}$ of the principal left ideal Sy of S is an (α, β)-hesitant fuzzy left ideal on S. Clearly, $\left({}^\alpha\mathcal{C}_{\zeta\, Sy}^{\delta}\right)_x \supseteq \left({}_\beta\mathcal{C}_{\zeta\, Sy}^{\delta}\right)_y = \delta \cap \beta$, which implies that $x \in Sy$. Therefore, there exist $s \in S$ such that $x = sy = s(yy) = (sy)y = xy$. Hence $\mathcal{E}(S)$ is a left zero subsemigroup of S and so (2) implies (1). $\square$

By Theorem 3.4, we immediately obtain the following corollary:

COROLLARY 3.5. Let ζ and δ be any elements of $\mathcal{P}([0,1])$ such that $\zeta \cup \alpha \not\supseteq \delta \cap \beta$. For a regular semigroup S, the following statements are equivalent:

1. The set $\mathcal{E}(S)$ of all (α, β)-idempotents of S forms an zero subsemigroup of S.
2. For every (α, β)-hesitant fuzzy ideal $\mathcal{H}$ on $S, \mathcal{H}_x \simeq_\beta^\alpha \mathcal{H}_y$ for all $x, y \in \mathcal{E}(S)$.

Recall that a semigroup S is said to be **left (right) regular** if for each element x of S, there exists an element $s \in S$ such that $x = sx^2(x = x^2s)$.

From the above discussion, we can immediately obtain the following theorems.

Theorem 3.6. *Let ζ and δ be any elements of $\mathcal{P}([0,1])$ such that $\zeta \cup \alpha \not\supseteq \delta \cap \beta$. For a semigroup S, the following conditions are equivalent.*

1. *S is left regular.*
2. *For every (α, β)-hesitant fuzzy left ideal $\mathcal{H}$ on $S, \mathcal{H}_x \simeq_\beta^\alpha \mathcal{H}_{x^2}$ for all $x \in S$.*

Proof: First assume that (1) holds. Let $\mathcal{H}$ be any (α, β)-hesitant fuzzy left ideal on S and let x any element of S. Then, since S is left regular, there exists an element s in S such that $x = sx^2$. Hence we have, ${}^\alpha\mathcal{H}_x = {}^\alpha\mathcal{H}_{sx^2} \supseteq {}_\beta\mathcal{H}_{x^2}$ and ${}^\alpha\mathcal{H}_{x^2} \supseteq {}_\beta\mathcal{H}_x$. Therefore $\mathcal{H}_x \simeq_\beta^\alpha \mathcal{H}_{x^2}$ and so (1) implies (2).

Conversely, assume that (2) holds. Let x be any element of S. Then it follows from Theorem 2.18(2) that the (δ, ζ)-characteristic function ${}^\alpha\mathcal{C}^\delta_{\zeta\, Sx^2}$ of the principal left ideal $x^2 \cup Sx^2$ of S is an (α, β)-hesitant fuzzy left ideal on S. Since $x^2 \in x^2 \cup Sx^2$, we have $\left({}^\alpha\mathcal{C}^\delta_{\zeta\, x^2 \cup Sx^2}\right)_x \supseteq \left({}_\beta\mathcal{C}^\delta_{\zeta\, x^2 \cup Sx^2}\right)_{x^2} = \left(\mathcal{C}^\delta_{\zeta\, x^2 \cup Sx^2}\right)_{x^2} \cap \beta = \delta \cap \beta$. This implies that $x \in x^2 \cup Sx^2$. Hence S is left regular and so (2) implies (1). □

From Theorem 3.6 we can easily obtain the following corollary.

Corollary 3.7. Let ζ and δ be any elements of $\mathcal{P}([0,1])$ such that $\zeta \cup \alpha \not\supseteq \delta \cap \beta$. For a semigroup S, the following conditions are equivalent.

1. S is right regular.
2. For every (α, β)-hesitant fuzzy right ideal $\mathcal{H}$ on $S, \mathcal{H}_x \simeq_\beta^\alpha \mathcal{H}_{x^2}$ for all $x \in S$.

Recall that a subset A of a semigroup S is said to be **semiprime** if for all $x \in S, x^2 \in A$ implies $x \in A$. Now, we give the definition of (α, β)-hesitant fuzzy semiprime set on a semigroup S, which is a generalization of the notion of hesitant fuzzy semiprime sets.

A hesitant fuzzy set $\mathcal{H}$ on a semigroup S is said to be (α, β)**-hesitant fuzzy semiprime** if ${}^\alpha\mathcal{H}_x \supseteq {}_\beta\mathcal{H}_{x^2}$ for all $x \in S$.

THEOREM 3.8. *Let ζ and δ be any elements of $\mathcal{P}([0,1])$ such that $\zeta \cup \alpha \not\supseteq \delta \cap \beta$. If P is a non empty subset of a semigroup S, then the following conditions are equivalent:*

1. *P is semiprime.*
2. *The (δ,ζ)-characteristic function ${}^{\alpha}\mathcal{C}^{\delta}_{\zeta\, P}$ of P is an (α,β)-hesitant fuzzy semiprime.*

PROOF: First assume that P is a semiprime set of S. Let x be any element of S. We consider the following cases:

1. $x^2 \in P$.
2. $x \notin P$ or $y \notin P$.

Case 1: Assume that $x^2 \in P$. Since P is semiprime, we have $x \in P$. Then, we obtain $\left({}^{\alpha}\mathcal{C}^{\delta}_{\zeta\, P}\right)_x = \delta \cup \alpha$ and $\left({}_{\beta}\mathcal{C}^{\delta}_{\zeta\, P}\right)_{x^2} = \delta \cap \beta$.

Case 2: Assume that $x^2 \notin P$. It is easy to see that $\left({}_{\beta}\mathcal{C}^{\delta}_{\zeta\, P}\right)_{x^2} = \zeta \cap \beta$. In any case, we have $\left({}_{\beta}\mathcal{C}^{\delta}_{\zeta\, P}\right)_{x^2} \subseteq \left({}^{\alpha}\mathcal{C}^{\delta}_{\zeta\, P}\right)_x$ for all $x \in S$. Therefore, ${}^{\alpha}\mathcal{C}^{\delta}_{\zeta\, P}$ is an (α,β)-hesitant fuzzy semiprime set on S and hence (1) implies (2).

Conversely, assume that (2) holds. Let x be any element of S such that $x^2 \in P$. Since ${}^{\alpha}\mathcal{C}^{\delta}_{\zeta\, P}$ is an (α,β)-hesitant fuzzy semiprime set on S, it follows that $\left({}^{\alpha}\mathcal{C}^{\delta}_{\zeta\, P}\right)_x \supseteq \left({}_{\beta}\mathcal{C}^{\delta}_{\zeta\, P}\right)_{x^2} = \left(\mathcal{C}^{\delta}_{\zeta\, P}\right)_{x^2} \cap \beta = \delta \cap \beta$. Observe that $\left({}^{\alpha}\mathcal{C}^{\delta}_{\zeta\, P}\right)_x = \delta \cup \alpha$, implies that $x \in P$. Therefore, P is a semiprime set of S and hence (2) implies (1). □

In order to characterize the (α,β)-hesitant fuzzy semiprime set generated by a hesitant fuzzy semiprime set in a semigroup, we need the following theorem.

THEOREM 3.9. *If $\mathcal{H}$ is an (α,β)-hesitant fuzzy subsemigroup on a semigroup S, then the following conditions are equivalent:*

1. *$\mathcal{H}$ is an (α,β)-hesitant fuzzy semiprime set on S.*
2. *For every $x \in S, \mathcal{H}_x \simeq^{\alpha}_{\beta} \mathcal{H}_{x^2}$.*

PROOF: It is clear that (2) implies (1). Assume that $\mathcal{H}$ is an (α,β)-hesitant fuzzy semiprime set on S. Let x be any element of S. Otherwise, we have ${}^{\alpha}\mathcal{H}_x \supseteq {}_{\beta}\mathcal{H}_{x^2}$ and ${}^{\alpha}\mathcal{H}_{x^2} \supseteq {}_{\beta}\mathcal{H}^{x}_{x} = {}_{\beta}\mathcal{H}_x$. Therefore $\mathcal{H}_x \simeq^{\alpha}_{\beta} \mathcal{H}_{x^2}$ and hence (1) implies (2). □

Recall that a semigroup S is said to be **intra-regular** if for each element x of S, there exist elements r and s in S such that $x = rx^2s$. Now we shall characterize the intra-regular semigroups in terms of (α,β)-hesitant fuzzy ideals.

Theorem 3.10. *Let ζ and δ be any elements of $\mathcal{P}([0,1])$ such that $\zeta\cup\alpha \not\supseteq \delta\cap\beta$. If S is a semigroup S, then the following conditions are equivalent:*

1. *S is intra-regular.*
2. *For every (α,β)-hesitant fuzzy ideal on S is (α,β)-hesitant fuzzy semiprime.*
3. *For every (α,β)-hesitant fuzzy ideal $\mathcal{H}$ on S, $\mathcal{H}_x \simeq^{\alpha}_{\beta} \mathcal{H}_{x^2}$ for all $x\in S$.*

Proof: First assume that S is an intra-regular semigroup. Let $\mathcal{H}$ be any (α,β)-hesitant fuzzy ideal on S. Next, let x be any element of S. Then, since S is intra-regular, there exist elements r and s in S such that $x=rx^2s$, which implies that

$$\begin{aligned}
{}^{\alpha}\mathcal{H}_x &= {}^{\alpha}\mathcal{H}_{rx^2s}\cup\alpha\\
&= {}^{\alpha}\mathcal{H}_{rx^2s}\cup\alpha\\
&\supseteq {}_{\beta}\mathcal{H}_{rx^2}\cup\alpha\\
&= (\mathcal{H}_{rx^2}\cap\beta)\cup\alpha\\
&= (\mathcal{H}_{rx^2}\cup\alpha)\cap(\beta\cup\alpha)\\
&= {}^{\alpha}\mathcal{H}_{rx^2}\cap(\beta\cup\alpha)\\
&\supseteq {}_{\beta}\mathcal{H}_{x^2}\cap(\beta\cup\alpha)\\
&= \mathcal{H}_{x^2}\cap\beta\cap(\beta\cup\alpha)\\
&= \mathcal{H}_{x^2}\cap\beta\\
&= {}_{\beta}\mathcal{H}_{x^2}.
\end{aligned}$$

Thus, we obtain ${}^{\alpha}\mathcal{H}_{x^2}\supseteq {}_{\beta}\mathcal{H}_x$, since $\mathcal{H}$ is an (α,β)-hesitant fuzzy ideal. Therefore $\mathcal{H}_x \simeq^{\alpha}_{\beta} \mathcal{H}_{x^2}$ and so (1) implies (3).

Assume that (2) holds. Let x be any element of S. Then it follows from Theorem 2.18(2) that the (δ,ζ)-characteristic function ${}^{\alpha}\mathcal{C}^{\delta}_{\zeta\, x^2\cup Sx^2\cup x^2S\cup Sx^2S}$ of the principal ideal $x^2\cup Sx^2\cup x^2S\cup Sx^2S$ of S is an (α,β)-hesitant fuzzy ideal on S. Therefore, $\left({}^{\alpha}\mathcal{C}^{\delta}_{\zeta\, x^2\cup Sx^2\cup x^2S\cup Sx^2S}\right)_x \supseteq ({}_{\beta}\mathcal{C}^{\delta}_{\zeta\, x^2\cup Sx^2\cup x^2S\cup Sx^2S})_{x^2} = (\mathcal{C}^{\delta}_{\zeta\, x^2\cup Sx^2\cup x^2S\cup Sx^2S})_{x^2}\cap\beta = \delta\cap\beta$, since $x^2\in x^2\cup Sx^2\cup x^2S\cup Sx^2S$. Clearly, $x\in x^2\cup Sx^2\cup x^2S\cup Sx^2S$. Therefore it is easily seen that S is intra-regular and so (2) implies (1). It is clear that (2) and (3) are equivalent. □

Now we characterize the intra-regular semigroup in terms of (α,β)-hesitant fuzzy left ideals and (α,β)-hesitant fuzzy right ideals.

Theorem 3.11. *Let ζ and δ be any elements of $\mathcal{P}([0,1])$ such that $\zeta\cup\alpha \not\supseteq \delta\cap\beta$. If S is a semigroup S, then the following conditions are equivalent:*

1. *S is intra-regular.*
2. *${}_{\beta}(\mathcal{H}\cap\mathcal{F}) \subseteq {}^{\alpha}(\mathcal{H}\odot\mathcal{F})$ for every (α,β)-hesitant fuzzy left ideal $\mathcal{H}$ and every (α,β)-hesitant fuzzy right ideal $\mathcal{F}$ on S.*

Proof: First assume that S is intra-regular. Let $\mathcal{H}$ and $\mathcal{F}$ be any (α,β)-hesitant fuzzy left ideal and any (α,β)-hesitant fuzzy right ideal on S, respectively. Next, let x be any element of S. Then, since S is intra-regular, there exist elements r and s in S such that $x = rx^2s$. Hence we have,

$$\begin{aligned}
{}^{\alpha}(\mathcal{H}\odot\mathcal{F})_x &= (\mathcal{H}\odot\mathcal{F})_x\cup\alpha \\
&= \left(\bigcup_{x=ab}\mathcal{H}_a\cap\mathcal{F}_b\right)\cup\alpha \\
&\supseteq (\mathcal{H}_{rx}\cap\mathcal{F}_{xs})\cup\alpha \\
&= (\mathcal{H}_{rx}\cup\alpha)\cap(\mathcal{F}_{xs}\cup\alpha) \\
&= {}^{\alpha}\mathcal{H}_{rx}\cap{}^{\alpha}\mathcal{F}_{xs} \\
&\supseteq {}_{\beta}\mathcal{H}_x\cap{}_{\beta}\mathcal{F}_x \\
&= {}_{\beta}(\mathcal{H}\cap\mathcal{F})_x,
\end{aligned}$$

which implies that ${}_{\beta}(\mathcal{H}\cap\mathcal{F}) \subseteq {}^{\alpha}(\mathcal{H}\odot\mathcal{F})$. Therefore (1) implies (2).

Conversely, assume that (2) holds. Let L and R be any left ideal and any right ideal of S, respectively. Next, let x be any element of S such that $x\in L\cap R$. Then $x\in L$ and $x\in R$. By Theorem 2.18 ${}^{\alpha}\mathcal{C}^{\delta}_{\zeta\,L}$ and ${}^{\alpha}\mathcal{C}^{\delta}_{\zeta\,R}$ is an (α,β)-hesitant fuzzy left ideal and an (α,β)-hesitant fuzzy right ideal on S, respectively. Then, by Lemma 2.17, we obtain that

$$\begin{aligned}
\left({}^{\alpha}\mathcal{C}^{\delta}_{\zeta\,LR}\right)_x &= \left(\mathcal{C}^{\delta}_{\zeta\,LR}\right)_x\cup\alpha \\
&= \left(\mathcal{C}^{\delta}_{\zeta\,L}\odot\mathcal{C}^{\delta}_{\zeta\,R}\right)_x\cup\alpha \\
&= {}^{\alpha}\left(\mathcal{C}^{\delta}_{\zeta\,L}\odot\mathcal{C}^{\delta}_{\zeta\,R}\right)_x \\
&\supseteq {}_{\beta}\left(\mathcal{C}^{\delta}_{\zeta\,L}\cap\mathcal{C}^{\delta}_{\zeta\,R}\right)_x \\
&= {}_{\beta}\left(\mathcal{C}^{\delta}_{\zeta\,L\cap R}\right)_x \\
&\supseteq \left(\mathcal{C}^{\delta}_{\zeta\,L\cap R}\right)_x\cap\beta \\
&= \delta\cap\beta,
\end{aligned}$$

which means that $x \in LR$. Therefore we obtain that $L \cap R \subseteq LR$ and hence S is intra-regular. Thus (2) implies (1). □

In the following theorem we give a characterization of a semigroup that is both regular and intra-regular in terms of (α, β)-hesitant fuzzy right ideals and (α, β)-hesitant fuzzy left ideals.

THEOREM 3.12. *Let ζ and δ be any elements of $\mathcal{P}([0,1])$ such that $\zeta \cup \alpha \not\supseteq \delta \cap \beta$. If S is a semigroup S, then the following conditions are equivalent:*

1. *S is regular and intra-regular.*
2. *${}_{\beta}(\mathcal{H} \cap \mathcal{F}) \subseteq {}^{\alpha}((\mathcal{H} \odot \mathcal{F}) \cap (\mathcal{F} \odot \mathcal{H}))$ for every (α, β)-hesitant fuzzy right ideal $\mathcal{H}$ and every (α, β)-hesitant fuzzy left ideal $\mathcal{F}$ on S.*

PROOF: First assume that (1) holds. Let $\mathcal{H}$ and $\mathcal{F}$ be any (α, β)-hesitant fuzzy right ideal and any (α, β)-hesitant fuzzy left ideal on S, respectively. Then it follows from Theorems 3.3, 3.11 that ${}_{\beta}(\mathcal{H} \cap \mathcal{F}) \subseteq {}^{\alpha}(\mathcal{H} \odot \mathcal{F})$ and ${}_{\beta}(\mathcal{H} \cap \mathcal{F}) = {}_{\beta}(\mathcal{F} \cap \mathcal{H}) \subseteq {}^{\alpha}(\mathcal{F} \odot \mathcal{H})$. Therefore ${}_{\beta}(\mathcal{H} \cap \mathcal{F}) \subseteq {}^{\alpha}((\mathcal{H} \odot \mathcal{F}) \cap (\mathcal{F} \odot \mathcal{H}))$ and so (1) implies (2).

Conversely, assume that (2) holds. Let $\mathcal{H}$ and $\mathcal{F}$ be any (α, β)-hesitant fuzzy right ideal and any (α, β)-hesitant fuzzy left ideal on S, respectively. We obtain

$$\begin{aligned}
{}_{\beta}(\mathcal{F} \cap \mathcal{H}) &= {}_{\beta}(\mathcal{H} \cap \mathcal{F}) \\
&\subseteq \left(\mathcal{C}^{\delta}_{\zeta L} \odot \mathcal{C}^{\delta}_{\zeta R}\right)_x \cup \alpha \\
&= {}^{\alpha}((\mathcal{H} \odot \mathcal{F}) \cap (\mathcal{F} \odot \mathcal{H})) \\
&\subseteq {}^{\alpha}(\mathcal{F} \odot \mathcal{H}).
\end{aligned}$$

Thus it follows from Theorem 3.11 that S is an intra-regular semigroup. Now, we show that S is a regular semigroup. By the assumption, ${}_{\beta}(\mathcal{H} \cap \mathcal{F}) \subseteq {}^{\alpha}((\mathcal{H} \odot \mathcal{F}) \cap (\mathcal{F} \odot \mathcal{H})) \subseteq {}^{\alpha}(\mathcal{H} \odot \mathcal{F})$. On the other hand, ${}_{\beta}(\mathcal{H} \odot \mathcal{F}) \subseteq {}_{\beta}(\mathcal{S} \odot \mathcal{F}) \subseteq {}^{\alpha}\mathcal{F}$ and ${}_{\beta}(\mathcal{H} \odot \mathcal{F}) \subseteq {}_{\beta}(\mathcal{H} \odot \mathcal{S}) \subseteq {}^{\alpha}\mathcal{H}$, which means that ${}_{\beta}(\mathcal{H} \odot \mathcal{F}) \subseteq {}^{\alpha}\mathcal{H} \cap {}^{\alpha}\mathcal{F} = {}^{\alpha}(\mathcal{H} \cap \mathcal{F})$. Hence it follows from Theorem 3.3 that S is regular. Therefore (2) implies (1). □

4. Conclusion

In study the structure of semigroups, we notice that the (α, β)-hesitant fuzzy sets with special properties always play an important role. The (α, β)-hesitant fuzzy ideals on a semigroup are key tools to describe the algebraic subsystems of a semigroup S. By using the point wise (left, right) ideas and methods, in this paper we defined and studied (α, β)-hesitant fuzzy (left, right) ideals on semigroups. In particular, we introduced the concepts of $^{\alpha}$-hesitant ($_{\alpha}$-hesitant) fuzzy sets, (α, β)-hesitant fuzzy subsemigroups and (α, β)-hesitant fuzzy ideals of semigroups, and characterized regular semigroups in terms of (α, β)-hesitant fuzzy ideals. Furthermore, we prove that the non empty subset of a semigroup S is a subsemigroup (left ideal, right ideal, two-sided ideal) of S if and only if the hesitant fuzzy set on S is the (α, β)-hesitant fuzzy subsemigroup ((α, β)-hesitant fuzzy left ideal, (α, β)-hesitant fuzzy right ideal, (α, β)-hesitant fuzzy two-sided ideal) on S. As an application of the results of this paper, the corresponding results of fuzzy sets. We hope that this work would offer foundation for further study of the theory on semigroups.

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