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MODELS OF BOUNDED ARITHMETIC THEORIES AND SOME RELATED COMPLEXITY QUESTIONS

Abstract

In this paper, we study bounded versions of some model-theoretic notions and results. We apply these results to models of bounded arithmetic theories as well as some related complexity questions. As an example, we show that if the theory $\mathrm{S}^1_2(\mathrm{PV})$ has bounded model companion then $\mathrm{NP} = \mathrm{coNP}$. We also study bounded versions of some other related notions such as Stone topology.

Keywords: Bounded arithmetic, complexity theory, existentially closed model, model completeness, model companion, quantifier elimination, Stone topology.

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1. Introduction

The study of first-order theory of arithmetic PA dates back to Hilbert's Program in the foundation of mathematics and Gödel's incompleteness theorems in 1930s. Since then this theory has been a subject of interest for mathematicians, logicians, and computer scientists. In particular, studying various bounded fragments of this theory has been proved to have significant consequences in complexity theory (see e.g., [8], [6] and [9]).

Some important examples of bounded arithmetic theories are the first-order version of Cook's equational theory PV, denoted by PV_1, and Buss's theory S^1_2 and its conservative expansion to the language of PV denoted by $\mathrm{S}^1_2(\mathrm{PV})$ (see [1, 5, 4]). It is known that $\mathrm{S}^1_2(\mathrm{PV})$ is $\forall\Sigma^b_1$-conservative over

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PV_1. The theory PV_1 is the weakest theory of arithmetic we work with in this paper.

In this paper, we also work with a general notion of bounded theory. By a bounded theory, we mean a consistent first-order theory whose language contains a binary relation symbol $\leqslant$, which is a partially ordered relation and has a few more basic properties, and is axiomatized by a set of (universal closures of) bounded formulas (see [11, 2]).

A theory T has *bounded quantifier elimination* if any bounded formula is T-equivalent to a quantifier-free formula. By a Σ_1^b-formula we mean a quantifier-free formula prefixed by a bounded existential quantifier. A model M of T is called *bounded existentially closed* if whenever N is a model of T with $M \subseteq N$, then we have $M \prec^{\Sigma_1^b} N$, i.e. for any Σ_1^b-formula $\varphi(\overline{x})$ with parameters from M, and any $\overline{a} \in M$, if $N \models \varphi(\overline{a})$, then we have $M \models \varphi(\overline{a})$.

A theory T is called *bounded model complete* if whenever $M \subseteq N$ are models of T, then $M \prec^{\Sigma_1^b} N$. Obviously, a theory T is bounded model complete if and only if any model $M \models T$ is a bounded existentially closed model. The following fact is proved in [11]:

Fact 1.1. Let T be a bounded theory. The following are equivalent:

(1) T is bounded model complete.

(2) Every model of T is a bounded existentially closed model of T.

(3) For any Σ_1^b-formula there is a T-equivalent Π_1^b-formula.

(4) For any bounded formula there is a T-equivalent Π_1^b-formula.

If any of the above conditions holds for a theory T, we say that T proves $\mathrm{NP} = \mathrm{coNP}$.

Corollary 1.2. Let T be a bounded theory which is bounded model complete. Then T is $\forall\Sigma_1^b$-axiomatizable.

Proof: See Corollary 2.6 in [11]. □

Propositional logic is closely related to the main open problems in complexity theory. A famous fundamental problem in propositional logic asks whether there is a propositional proof system in which every tautology has a polynomial size proof. By a famous result of Cook and Rechow, NP = coNP if and only if there exists a propositional proof system in which every tautology has a polynomial size proof.

In this paper, we prove some more results concerning bounded model complete theories. We also define the notions of *bounded model companion* and *bounded model completion* in the context of bounded arithmetic and provide some applications around the question NP =? coNP in models of bounded arithmetic. We say that NP = coNP holds in a model M if for each Π_1^b-formula $\varphi(x)$, there is a Σ_1^b-formula $\psi(x)$ such that $M \models \psi \leftrightarrow \varphi$. We also study the notion of *bounded Stone topology* and its applications in the context of bounded arithmetic.

2. NP = coNP in models of bounded arithmetic

Let P be an abstract propositional proof system as described in [9]. By $Prf_P(y, x)$ we mean the PV-formula which states that "y is a proof for x in P". We also show this formula by $y \vdash_P x$. Note that this formula is quantifier-free. Another important formula is $\mathrm{Taut}(x)$ which is a Π_1^b-formula and says that "x is a tautology".

Frege systems and extended Frege systems are special types of propositional proof systems. For more details on the propositional proof systems, specially Frege and Extended Frege systems, see [9], chapter 4.

Definition 2.1. Let P be a proof system.

(*i*) By "P is complete", we mean the PV-sentence

$$\forall x \exists y (\mathrm{Taut}(x) \to y \vdash_P x).$$

(*ii*) By "P is t-bounded", we mean the PV-sentence

$$\forall x \exists y \leq t(x) (\mathrm{Taut}(x) \to y \vdash_P x)$$

where t is a term.

Note that "P is t-bounded" is PV_1-equivalent to a $\forall \Sigma_1^b$-sentence.

Proposition 2.2. Let T be a consistent extension of $\mathrm{S}_2^1(\mathrm{PV})$ and $\varphi(x)$ be a bounded formula. If $T \vdash \forall x \varphi(x)$, then $\mathbb{N} \models \forall x \varphi(x)$.

Proof: See [1], Chapter 8. □

Lemma 2.3. *Let R be a consistent extension of* PV_1 *and* $\varphi(x)$ *be either a* Σ_1^b *or a* Π_1^b *formula. If* $R \vdash \forall x \varphi(x)$*, then* $\mathbb{N} \models \forall x \varphi(x)$.

Proof: Let $R \vdash \forall x \varphi(x)$ but $\mathbb{N} \models \neg\varphi(\overline{n})$ for some tuple $\overline{n} \in \mathbb{N}$. Then, $\mathrm{S}^1_2(\mathrm{PV}) \vdash \neg\varphi(\overline{n})$. In the case that $\varphi(x)$ is a Σ_1^b-formula, $\neg\varphi(\overline{n})$ is a universal sentence and we have $\mathrm{PV}_1 \vdash \neg\varphi(\overline{\mathrm{n}})$ since $\mathrm{S}^1_2(\mathrm{PV})$ and PV_1 have the same universal consequences. In the other case, $\varphi(x)$ is a Π_1^b-formula. We know that $\mathrm{S}^1_2(\mathrm{PV})$ and PV_1 have the same Σ_1^b-theorems. Since the $\neg\varphi(\overline{n})$ is a Σ_1^b-formula, we have $\mathrm{PV}_1 \vdash \neg\varphi(\overline{\mathrm{n}})$. This contradicts consistency of R. $\square$

As mentioned above (Corollary 1.2), if T is a bounded theory which is bounded model complete, then T is $\forall\Sigma_1^b$-axiomatizable. It implies that if the theory $\mathrm{S}^1_2(\mathrm{PV})$ is bounded model complete, then $\mathrm{S}^1_2(\mathrm{PV}) = \mathrm{PV}_1$ as $\mathrm{S}^1_2(\mathrm{PV})$ is $\forall\Sigma_1^b$-conservative over PV_1. Note that if PV_1 is bounded model complete, then obviously $\mathrm{S}^1_2(\mathrm{PV})$ is also bounded model complete.

Theorem 2.4. *The following conditions are equivalent.*

(i) NP = coNP

(ii) $\mathrm{S}^1_2(\mathrm{PV})$ + "*P is t-bounded*" *is consistent for some proof system P and term t.*

(iii) PV_1 + "*P is t-bounded*" *is consistent for some proof system P and some term t.*

Proof: Assume (i), then, by the Cook and Reckhow theorem, there is a proof system P such that any tautology has a polynomial size P-proof. Hence, "P is t-bounded" holds in the standard model for some term t. Therefore, the standard model is a model for $\mathrm{S}^1_2(\mathrm{PV})$ + "P is t-bounded" for some term t. Now assume (ii). Since "P is t-bounded" is a $\forall\Sigma_1^b$-formula, then by Proposition 2.2, we have "P is t-bounded" in $\mathbb{N}$ for some term t. Hence, the standard model satisfies

$$\forall x\ [\mathrm{Taut}(x) \equiv \exists y \leqslant t\ (y \vdash_P x)].$$

Consequently, NP = coNP holds in the standard model.
Finally assume (iii), using Lemma 2.3 and the same argument as above, we can see that NP = coNP holds in $\mathbb{N}$. $\square$

Corollary 2.5. If $\mathrm{S}^1_2(\mathrm{PV})+$"EF is t-bounded" is consistent for some term t, then NP = coNP. Moreover, the same result holds for PV_1 in place of $\mathrm{S}^1_2(\mathrm{PV})$.

Proof: Let $\mathrm{S}^1_2(\mathrm{PV})$ + "EF is t-bounded" be consistent for some term t. Since "EF is t-bounded" is $\forall\Sigma^b_1$, by Proposition 2.2, we have "EF is t-bounded" in $\mathbb{N}$ for some term t. Hence, the standard model satisfies

$$\forall x\ [\mathrm{Taut}(x) \equiv \exists y \leqslant t\ (y \vdash_{\mathrm{EF}} x)].$$

Consequently, NP = coNP holds in the standard model.
Moreover, assuming PV_1 + "EF is t-bounded" is consistent for some term t, using Lemma 2.3 and the same argument as above, we can see that NP = coNP holds in $\mathbb{N}$. □

Lemma 2.6. *We have* NP = coNP *if and only if there is a bounded consistent extension of* $\mathrm{S}^1_2(\mathrm{PV})$ *which is bounded model complete.*

Proof: Let NP = coNP. Consider the theory W introduced by Buss in Proposition 1 in Chapter 8 of [1]. This proposition together with Fact 1.1, imply that W is bounded model complete. Conversely, suppose that T is a consistent extension of $\mathrm{S}^1_2(\mathrm{PV})$ which is bounded model complete. Thus, there exists a Σ^b_1-formula $\varphi(x)$ such that

$$T \vdash \forall x(\mathrm{Taut}(x) \equiv \varphi(x)).$$

Therefore, by Proposition 2.2, $\forall x(\mathrm{Taut}(x) \equiv \varphi(x))$ is true in the standard model. □

Remark 2.7. If PV_1 has a bounded model complete extension T, then by Lemma 2.3 and the same argument as in the proof of Lemma 2.6, we conclude that NP = coNP.

Proposition 2.8. If NP = coNP holds in some model of PV_1, then NP = coNP really.

Proof: Assume that $M \models \mathrm{PV}_1$ satisfies NP = coNP and T is the full Π_1-theory of M, that is the set of all $\forall\Delta_0$-sentences true in M. For each Σ^b_1-formula $\varphi(x)$, there is some Π^b_1-formula, say $\psi(x)$, such that $M \models \varphi \equiv \psi$. Thus, $\varphi \equiv \psi \in T$. Now, using Fact 1.1 and the above remark, we get NP = coNP really. □

Definition 2.9. Let T be a consistent theory.

(*i*) T is called b-complete if for each bounded L-sentence σ either $T \vdash \sigma$ or $T \vdash \neg\sigma$.

(*ii*) A model $M \models T$ is a b-prime model of T, if $N \models T$ implies that there is a b-elementary embedding from M into N (i.e., elementary with respect to bounded formulas).

Clearly, each prime model is b-prime. We note that the converse is not true, since by the MRDP theorem, $\mathbb{N}$ is a b-prime model of PA which is not prime. Also, it is well-known that $\mathbb{N}$ is embeddable in every model of $\mathrm{S}^1_2(\mathrm{PV})$, and this embedding is bounded elementary. So the standard model $\mathbb{N}$ is a b-prime model of $\mathrm{S}^1_2(\mathrm{PV})$.

Note that, a model M of a theory T is said to be algebraically prime iff M is isomorphically embeddable in every model of T, that is, for each $N \models T$, there is a submodel $M_0 \subseteq N$ with $M_0 \cong M$.

Lemma 2.10. *Let T be a bounded model complete theory. If T has an algebraically prime model, then T is b-complete.*

Proof: Let $M \models T$ be an algebraically prime model and σ be a bounded sentence. By Fact 1.1, there is a Σ^b_1-sentence τ such that $T \vdash \sigma \leftrightarrow \tau$. Suppose that $M \models \sigma$. Let $N \models T$ be arbitrary. We have $M \models \tau$. Since $M \prec^{\Sigma^b_1} N$, we have $N \models \tau$ and so $N \models \sigma$. This implies $T \vdash \sigma$. Similarly, if $M \not\models \sigma$, then $T \vdash \neg\sigma$. $\square$

Note that, by [9, Corollary 15.3.10], "EF is complete" in a model $M \models$ PV if and only if any extension of M is Σ^b_1-elementary. This implies that if the theory PV + "EF is complete" is consistent, then it is bounded model complete, because all of its models are bounded existentially closed model.

Proposition 2.11. If the standard model satisfies "EF is t-bounded" for some term t, then PV + "EF is t-bounded" is a b-complete theory.

Proof: By the assumption, PV + "EF *is t-bounded*" is consistent for some term t. By [9, Corollary 15.3.10], this theory is bounded model complete. Since $\mathbb{N}$ is embedded in any model of this theory, by Lemma 2.10 this theory is b-complete. $\square$

Definition 2.12. Let T be a bounded theory and $M \models T$. By Diag(M) one means the set of all quantifier-free $L(M)$-sentences which are true in

M. By BDiag(M), we mean the set of all bounded $L(M)$-sentences which are true in M.

PROPOSITION 2.13. A bounded theory T is a bounded model complete theory if and only if $T \cup \mathrm{BDiag(M)}$ is *b-complete*, for all $M \models T$.

PROOF: Let T be a bounded model complete theory and $M \models T$. Clearly, $T \cup \mathrm{BDiag(M)}$ is bounded model complete. Hence, by Lemma 2.10, $T \cup \mathrm{BDiag(M)}$ is b-complete. On the other hand, suppose $T \cup \mathrm{BDiag(M)}$ is b-complete. Assume that $M, N \models T$ with $M \subseteq N$ and σ is a Σ_1^b-sentence in $L(M)$. Since $N \models T \cup \mathrm{BDiag(M)}$, if $N \models \sigma$, then $T \cup \mathrm{BDiag(M)} \vdash \sigma$ and so $M \models \sigma$. Thus, $M \prec^{\Sigma_1^b} N$. $\square$

3. Model companion of bounded theories

In this section, we introduce bounded versions of the notions of model companion and model completion. We use these notions in the study of bounded arithmetic theories. For more details about model companion and model completion, see [3, 7].

DEFINITION 3.1. Let T be a bounded theory. We say a bounded theory T' is a *bounded model companion* of T if the following two conditions hold.

i) T and T' have the same universal consequences,

ii) T' is bounded model complete.

THEOREM 3.2. *Let* T *be a* $\forall\Sigma_1^b$ *theory and* T' *be its bounded model companion. Then,* M *is a model of* T' *if and only if* M *is a bounded existentially closed model of* T.

PROOF: First, assume that M is a model of T'. By the definition of bounded model companion, M is embeddable in a model N of T and N is embeddable in a model K of T'. We also have M is Σ_1^b-elementarily embedded in K, and so in N. Now, by the assumption, M is a model of T. Moreover, if M is embedded in a model M' of T, then the embedding is Σ_1^b-elementary similarly.

Conversely, let M be a bounded existentially closed model of T. Then, M is Σ_1^b-elementarily embeddable in a model of T', and so M is a model of T'. $\square$

The following theorem is the bounded version of Theorem 3.1.9 in [3].

Theorem 3.3 (Σ_1^b-elementary chain theorem). *Let $\{M_i\}_{i<\lambda}$ be a chain of models with $M_i \prec^{\Sigma_1^b} M_j$ for each $i < j < \lambda$. Then, $M_k \prec^{\Sigma_1^b} M = \bigcup_{i<\lambda} M_i$, for all $k < \lambda$.*

Proof: Assume that $M \models \exists x \leqslant t(\overline{y})\ \varphi(x, \overline{b})$, where $\overline{b}$ is a tuple in M_k. Hence $M \models (a \leqslant t(\overline{c})\ \wedge \varphi(a, \overline{b}))$ for some $\overline{c}, a \in M$. Let $\overline{c}, a \in M_l$ for some $l \geqslant k$. Thus, $M_l \models (a \leqslant t(\overline{c}) \wedge \varphi(a, \overline{b}))$ and so $M_l \models \exists x \leqslant t(\overline{y})\ \varphi(x, \overline{b})$. Since $M_k \prec^{\Sigma_1^b} M_l$, we have $M_k \models \exists x \leqslant t(\overline{y})\ \varphi(x, \overline{b})$. □

Corollary 3.4. If a $\forall\Sigma_1^b$ theory T has a bounded model companion, then this theory is unique up to equivalence.

Proof: Let T^* and T^{**} be model companions of T. Then, T^* and T^{**} are bounded model complete with the same universal consequences. Let M_1 be a model of T^*. There is a chain of models

$$M_1 \subseteq M_2 \subseteq ...$$

such that M_i is a model of T^* for odd i and of T^{**} for even i. Suppose that M is the union of the chain. Now, M_i's form a Σ_1^b-elementary chain for odd i. Using the Σ_1^b-elementary chain theorem, M is a Σ_1^b-elementary extension of M_1. Similarly, M is a Σ_1^b-elementary extension of M_2. Therefore, M_1 is a model of T^{**}. In a similar way, every model of T^{**} is a model of T^*, and so T^* and T^{**} are logically equivalent. □

Theorem 3.5. *Let* T *be a $\forall\Sigma_1^b$ theory. Then, T has a bounded model companion if and only if the class of all bounded existentially closed models of* T *can be axiomatized by a bounded theory.*

Proof: Suppose that the mentioned class is axiomatized by a bounded theory T′. Since every model of T′ is a model of T and every model of T is embeddable in a model of T′ (the proof is similar to the proof of Lemma 3.5.7 in [3]), T and T′ have the same universal consequences. Moreover, if a model M_1 of T′ is embedded in a model M_2 of T′, then this embedding is Σ_1^b-elementary, and so T′ is bounded model complete. The other direction is an immediate consequence of Theorem 3.2. □

Let us now study some applications of the above results in the context of bounded arithmetic theories. The theory PV_1 has a bounded model companion if and only if the class of bounded existentially closed models of

PV_1 is a bounded elementary class (i.e., being axiomatized by a bounded theory). Let $M \models \mathrm{PV}_1$. By definition, M satisfies NP = coNP if any Σ_1^b-formula with possible parameters from M is equivalent in M to a Π_1^b-formula with possible parameters in M. Hence, the main question is the following.

QUESTION 3.6. Is the class of all models of $\mathrm{PV}_1 + \mathrm{NP} = \mathrm{coNP}$ a bounded elementary class?

Assuming NP = coNP, a possible way of axiomatizing the class of bounded existentially closed models of PV_1 is adding the sentence "EF is t-bounded" to PV_1 for some suitable term t. If this sentence is true in some model of PV_1 for some term t, then this theory is consistent and bounded. The remaining question is why this theory has the same universal consequences as PV_1. Or equivalently, why any model of PV_1 is embedded in a model of that sentence. By [9], any model of PV_1 can be embedded in a model of PV_1 in which the mentioned sentence holds for elements greater than some fixed non-standard element. Indeed, the following result shows that the answer to the above question is probably negative.

THEOREM 3.7. *If* PV_1 *has a bounded model companion, then* NP = coNP *really.*

PROOF: Assume that T is the bounded model companion of PV_1. Then

$$T \vdash \forall x(\mathrm{Taut}(x) \equiv \varphi(x))$$

where $\varphi(x)$ is a Σ_1^b-formula. Thus, T is a consistent extension of PV_1 which satisfies $\forall x(\mathrm{Taut}(x) \equiv \varphi(x))$. By Lemma 2.3, this sentence is true in the standard model of natural numbers, and so NP = coNP in the real world. □

In the rest of this section, we study bounded version of the notion model completion.

DEFINITION 3.8. A theory T^* is a bounded model completion of a theory T if T^* is a bounded model companion of T and for every model $M \models T$, $T^* \cup \mathrm{Diag(M)}$ is b-complete.

Lemma 3.9. *A bounded model complete theory T has bounded quantifier elimination if and only if T is a bounded model completion of $T_\forall$.*

Proof: Assume that T is a bounded model completion of $T_\forall$. Let $\varphi(\overline{x})$ be a bounded formula and $\Sigma(\overline{x})$ be the set of all quantifier free consequences of $T + \varphi(\overline{x})$. Also, assume that M realizes $\Sigma(\overline{a})$ and D is the diagram of $(M, \overline{a})$ in the new language $L \cup \{\overline{c}\}$ where $\overline{a} \in M$. Since $T \cup D$ is consistent with $T \cup \Sigma(\overline{c})$, it is consistent with $\varphi(\overline{c})$. As $T \cup D$ is b-complete, we have $T \cup D \models \varphi(\overline{c})$. So $(M, \overline{a})$ is a model of $\varphi(\overline{c})$. Therefore, $T \cup \Sigma(\overline{c}) \models \varphi(\overline{c})$. We can find a sentence $\psi(\overline{c}) \in \Sigma(\overline{c})$ such that $T \models \varphi(\overline{c}) \leftrightarrow \psi(\overline{c})$. Hence, T has quantifier elimination for bounded formulas. The converse is straightforward. □

We showed that if PV_1 has a bounded model companion, then NP = coNP really. The converse is an open problem. In the case of bounded model completion, we have the following result.

Proposition 3.10. If PV_1 has a bounded model completion, then P = NP.

Proof: Let T be a bounded model completion for PV_1. By Lemma 3.9, T has bounded quantifier elimination. Therefore, any Σ^b_1-formula has a T-equivalent quantifier-free formula, i.e. $T \vdash \mathrm{P} = \mathrm{NP}$. On the other hand, $Diag(\mathbb{N}) \cup T$ is a consistent b-complete theory. Let $M \models Diag(\mathbb{N}) \cup T$. It is easy to see that $\mathbb{N} \prec^{\Sigma^b_1} M$. Since P = NP holds in M, it holds in $\mathbb{N}$ too. □

4. Bounded Stone topology

In this section, we study bounded version of the notion of Stone topology. For this, we need to impose some natural conditions on the theories and models we consider which are satisfied by the theories of bounded arithmetic.

Let T be a theory in a language L. Suppose that M is a L-structure and $A \subseteq M$. By L_A, we mean the language obtained by adding constant symbols c_a to L, for each $a \in A$. The structure M can be naturally considered as a L_A-structure by interpreting c_a by a. Let $Th^b_A(M)$ denote the set of all bounded L_A-sentences true in M.

The following definition gives the desired condition.

Definition 4.1.

(i) A model M is said to be t-cofinal (t for term) if the interpretation of the set of all L-terms is cofinal in M.

(ii) A theory T is said to be cofinal in the language L, if for every model $M \models T$ there is t-cofinal model $N \models T$ such that M and N agree on the bounded sentences of L.

It is easy to see that $\mathrm{S}^1_2(\mathrm{PV})$ is a cofinal theory, since it is Σ_1-complete with respect to the standard model.

Definition 4.2. Let T be a bounded theory and p be a set of bounded L-formulas with free variables $v_1, ..., v_n$. We call p a n-ary b-type over T, if $p \cup T$ is satisfiable. Also, the b-type p is b-complete if $\varphi \in p$ or $\neg\varphi \in p$ for each bounded L-formula φ. Moreover, by $BS_n(T)$ we mean the set of all b-complete n-ary b-types over T. Also, Δ_{L_A} denotes the set of all bounded formulas in the language L_A.

Suppose that M is a L-structure and $A \subseteq M$. For $\overline{a} \in M$, let

$$tp_b^M(\overline{a}) = \{\varphi(\overline{v}) \in \Delta_{L_A} : M \models \varphi(\overline{a})\}.$$

If p is a b-type, then there is an elementary extension N of M such that p is realized in N. It is easy to see that a b-type p is b-complete if and only if there exists an elementary extension N of M and $\overline{a} \in N$ such that $p = tp_b^N(\overline{a})$.

Definition 4.3. Assume that φ is a bounded L-formula with free variables $v_1, ..., v_n$. Let

$$[\varphi]_b = \{p \in BS_n(T) : \varphi \in p\}.$$

(i) The bounded Stone topology on $BS_n(T)$ is the topology generated by the sets $[\varphi]_b$.

(ii) A b-complete b-type p is isolated in the bounded Stone topology if $\{p\} = [\varphi]_b$ for some bounded formula φ.

We can easily show (similar to the proof of Proposition 4.1.11 in [10]) that $p \in BS_n(T)$ is isolated if and only if there exists a bounded formula $\varphi(\overline{v})$ such that for all $\psi \in p$, we have

$$T \models \forall\overline{v}(\varphi(\overline{v}) \to \psi(\overline{v})).$$

Definition 4.4. Let $\varphi(\overline{v})$ be a bounded formula such that $T \cup \{\varphi(\overline{v})\}$ is satisfiable, and p be a (not necessarily complete) b-type. We say that φ b-isolates p, if for every $\psi \in p$

$$T \models \forall \overline{v}(\varphi(\overline{v}) \to \psi(\overline{v})).$$

A b-type p is said to be b-isolated if there is some bounded formula φ such that b-isolates p.

Definition 4.5. Let T be a bounded theory.

(i) A theory T is said to be b-atomic, if the set of all b-isolated n-ary b-types p is dense in $BS_n(T)$.

(ii) A model M of T is said to be b-atomic, if $tp_b^M(\overline{a})$ is b-isolated for each $\overline{a} \in M$.

It is easy to see that, a theory T is b-atomic if and only if it has a b-atomic model.

Lemma 4.6. *Let L be a countable language and T be a cofinal b-complete L-theory. Then a model M of T is b-prime if and only if it is countable and b-atomic.*

Proof: Let M be a countable b-atomic model of T. Suppose that $N \models T$ and $a_0, a_1, ...$ is an enumeration of the elements of M. By definition 4.1, there is a sequence $t_0, t_1, ..$ of closed terms such that $a_i \leqslant t_i$. Since M is b-atomic, there is a formula $\theta_i(v_0, ..., v_i)$ that b-isolates the type $tp_b^M(a_0, ..., a_i)$ for each i.

We construct a sequence $j_0 \subseteq j_1 \subseteq ...$ of partial b-elementary maps from M into N, where the domain of j_k is $\{a_0, ..., a_{k-1}\}$. Let $j_0 = \emptyset$. Given j_s, let $j_s(a_i) = b_i$ for $i < s$. Since $M \models \theta_s(a_0, ..., a_s)$ and j_s is a partial b-elementary embedding,

$$N \models \exists v \leqslant t_s \ \theta_s(b_0, ..., b_{s-1}, v).$$

Let $b_s \in N$ such that $N \models \theta_s(b_0, ..., b_s)$. By the assumption, we get

$$tp_b^M(a_0, ..., a_s) = tp_b^N(b_0, ..., b_s).$$

Thus, $j_{s+1} := j_s \cup \{(a_s, b_s)\}$ is a partial b-elementary embedding. Now, $j := \bigcup_{k<\omega} j_k$ is a b-elementary embedding from M into N.
The other direction of the theorem is obvious. □

Corollary 4.7. $S^1_2(PV)$ is a b-atomic theory.

Proof: The theory $S^1_2(PV)$ has the standard model as a b-prime model, and so by Lemma 4.6, $S^1_2(PV)$ is a b-atomic theory. □

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