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FROM INTUITIONISM TO BROUWER'S MODAL LOGIC

Abstract

We try to translate the intuitionistic propositional logic **INT** into Brouwer's modal logic **KTB**. Our translation is motivated by intuitions behind Brouwer's axiom $p \to \Box\Diamond p$. The main idea is to interpret intuitionistic implication as modal strict implication, whereas variables and other positive sentences remain as they are. The proposed translation preserves fragments of the Rieger-Nishimura lattice which is the Lindenbaum algebra of monadic formulas in **INT**. Unfortunately, **INT** is not embedded by this mapping into **KTB**.

Keywords: Intuitionistic logic, Kripke frames, Brouwer's modal logic.

1. Introduction

Brouwer's modal logic **KTB** is defined as the normal extension of the minimal modal logic **K** with the axioms $T = \Box p \to p$ and $B = p \to \Box\Diamond p$. The set of rules consists of the modus ponens, the rule of uniform substitution and the rule of necessitation. **KTB** is complete with respect to reflexive and symmetric Kripke frames. It has been known since the 1930's when O. Becker [1], and C.I. Lewis and C.H. Langford [5] formulated the strict form of the Brouwerian axiom $p \prec \Box\Diamond p$, and considered the appropriate system of logic. It turned out that the Brouwer system is stronger than the Lewis system **S3** and weaker than **S5**.

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There are some connections between the intuitionistic logic and the axiom B. For instance, let us quote G.E. Hughes and M.J. Cresswell [4, p. 57]:

> As it is known, L. Brouwer is the founder of the intuitionist school of mathematics. The law of double negation does not hold in intuitionistic logic. Exactly it holds that (i) $\vdash_{INT} p \to \neg\neg p$ but (ii) $\nvdash_{INT} \neg\neg p \to p$. Suppose that negation has a stronger meaning – necessarily negative. Hence $\neg p$ may be translated as $\Box\neg p$. The corresponding modal formula to (i) is $p \to \Box\neg\Box\neg p$, which gives us $p \to \Box\Diamond p$ and obviously $\vdash_{KTB} p \to \Box\Diamond p$. If we translate (ii) in this way, we obtain: $\Box\Diamond p \to p$, which is not a thesis even of the system **S5** defined below. Hence $\nvdash_{KTB} \Box\Diamond p \to p$. (..) Thus although the connection with Brouwer is somewhat tenuous, historical usage has continued to associate his name with this formula.

This motivation will be a starting point for our research. We define a function t from the intuitionistic propositional language $\{\to, \wedge, \vee, \bot\}$ into the modal language $\{\to, \wedge, \vee, \Box, \bot\}$. Thus, let us define

Definition 1.1.

$$t(\bot) = \bot, \qquad t(p) = p, \qquad t(\alpha \to \beta) = \Box(t(\alpha) \to t(\beta)),$$
$$t(\alpha \wedge \beta) = t(\alpha) \wedge t(\beta), \qquad t(\alpha \vee \beta) = t(\alpha) \vee t(\beta).$$

The function t will be the desired translation if the equivalence holds:

$$\alpha \in \mathbf{INT} \text{ iff } t(\alpha) \in \mathbf{KTB}.$$

Our translation differs from the standard one (see, for instance, [3, 7]), known as the Gödel-McKinsey-Tarski translation, for which **S4** turns out to be a modal companion of the intuitionistic logic. Note that the Gödel-McKinsey-Tarski translation maps p onto $\Box p$, instead of p, for any propositional variable p. Nevertheless, we have $t(\neg p) = \Box\neg p$ (as $\neg p = p \to \bot$) and $t(\neg\neg p) = \Box\neg\Box\neg p = \Box\Diamond p$.

Suppose a logic **L** is given (in the sequel we deal mainly with **KTB**). We write $\phi =_L \psi$ if both $\phi \to \psi$ and $\psi \to \varphi$ are **L**–valid. We even omit the subscript L, and write $\phi = \psi$ instead of $\phi =_L \psi$, if there is no risk of misunderstanding. It does not mean, however, that we identify

L–equivalent formulas neither we regard any formula as its equivalence class in the the so-called Lindenbaum-Tarski's algebra of **L**.

In our paper we omit definitions of some logical concepts if they can be found in standard text-books on modal logic, e.g., [2, 3]

2. Preliminaries

Our function t translates the intuitionistic law of doubled negation onto Brouwer's axiom:

$$t(p \to \neg\neg p) = p \to \Box\Diamond p.$$

We ask if other intuitionistic theorems are preserved. Let us consider the law of contraposition in the form: $[(p \to q) \wedge \neg q] \to \neg p)$. After applying t we get: $\Box[[\Box(p \to q) \wedge \Box\neg q] \to \Box\neg p]$. We prove that

LEMMA 2.1. $\Box[[\Box(p \to q) \wedge \Box\neg q] \to \Box\neg p] \in$ **KTB**.

PROOF: Suppose that $\Box[[\Box(p \to q) \wedge \Box\neg q] \to \Box\neg p] \notin$ **KTB**. Then exists a KTB-model $\mathfrak{M} = \langle W, R, V\rangle$ and a point $x_1 \in W$ such that:

$$x_1 \models \Box(p \to q) \wedge \Box\neg q \tag{2.1}$$

$$x_1 \not\models \Box\neg p \tag{2.2}$$

From (2.2) there is another point, say x_2 such that $x_1 R x_2$ and $x_2 \not\models \neg p$, which means that $x_2 \models p$. From (2.1) it follows that for all $x_i \in W$ such that $x_1 R x_i$, we have: $x_i \models p \to q$ and $x_i \models \neg q$. Hence it holds also at the point x_2. Then we obtain:

$$x_2 \models (p \to q), \quad x_2 \models p, \quad x_2 \models \neg q\,. \tag{2.3}$$

This is a contradiction. □

On the other hand, one may notice that this contraposition law in the form : $(p \to q) \to (\neg q \to \neg p)$ after translation is not a theorem of **KTB**.

LEMMA 2.2. $\Box[\Box(p \to q) \to \Box(\Box\neg q \to \Box\neg p)] \notin$ **KTB**.

PROOF: Let us consider a KTB-model $\mathfrak{M} = \langle W, R, v\rangle$ such that $W = \{x_1, x_2, x_3\}$, $x_i R x_j$ iff $|i - j| \leq 1$ and $v(p) = \{x_3\}$ and $v(q) = \emptyset$. Then we get $x_2 \models \Box\neg q$ and $x_2 \not\models \Box\neg p$. Hence $x_2 \not\models \Box\neg q \to \Box\neg p$ and $x_1 \not\models$

$\Box(\Box\neg q \to \Box\neg p)$. Also $x_i \models p \to q$ for $i = 1, 2$. Then $x_1 \models \Box(p \to q)$. Hence $x_1 \not\models \Box(p \to q) \to \Box(\Box\neg q \to \Box\neg p)$. $\Box$

From the above, it follows that the law of importation: $[p \to (q \to r)] \to [(p \wedge q) \to r]$ is preserved but the exportation $[(p \wedge q) \to r] \to [p \to (q \to r)]$, is not. The negative results for formulas in two and more variables make us study the monadic fragment of intuitionistic logic. At least, the axiom B is a formula in one variable and B turns out to be the translation of the appropriate intutionistic law. Although the deficiency of the modal analog of the exportation law in **KTB** will be an impediment, we might expect that the monadic language is more fit for our translation.

3. Monadic formulas in KTB

As it is known, see for instance [10], intuitionistic formulas containing only one variable, say p, may be enumerated as follows:

Definition 3.1.

$$\alpha_0 = \bot, \qquad \alpha_1 = p, \qquad \alpha_2 = p \to \bot,$$
$$\alpha_{2n+1} = \alpha_{2n} \vee \alpha_{2n-1}, \qquad \alpha_{2n+2} = \alpha_{2n} \to \alpha_{2n-1}, \quad \text{for any } n \geq 1$$
$$\alpha_\omega = p \to p.$$

Every monadic formula is equivalent in the intuitionistic logic to one of the α_m's. Therefore, the formulas give rise to the so-called Rieger-Nishimura algebra $\mathcal{R}$, which is a single-generated free Heyting algebra (see Figure 1). The order relation in the algebra may be defined as follows:

$$\alpha \leq \beta \qquad \text{iff} \qquad \alpha \to \beta \in INT.$$

Our aim is to check if the algebra is preserved under the translation t or, more specifically, whether the translations of the formulas α_n give rise to the same algebra in the logic **KTB**.

The translations of α_n's do not cover all monadic modal formulas which means that there are monadic modal formulas, for instance $\neg p$ or $\Diamond p$, which are not equivalent to any $t(\alpha_n)$. It will also turn out that the translation t does not preserve the equivalence of (intuitionistic) formulas. We shall start out, however, our considerations with the observation that the "bottom" fragment of the Rieger-Nishimura algebra, consisting of the formulas

$\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4$, is preserved under the translation. Thus, in **KTB**, all "intuitionistic" relations between the formulas $\alpha_0 - \alpha_4$ are preserved:

OBSERVATION 1. $p \wedge \Box\neg p = \Box\Diamond p \wedge \Box\neg p = \Box\left((p \vee \Box\neg p) \to \bot\right) = \bot$
$\Box\Diamond p \wedge (p \vee \Box\neg p) = p$
$\Box(\Box\Diamond p \to \bot) = \Box(\Box\Diamond p \to \Box\neg p) = \Box(p \to \Box\neg p) = \Box((p \vee \Box\neg p) \to \Box\neg p) = \Box(p \to \bot) = \Box\neg p$
$\Box(\Box\neg p \to \bot) = \Box((p \vee \Box\neg p) \to p) = \Box(\Box\neg p \to p) = \Box((p \vee \Box\neg p) \to \Box\Diamond p) = \Box\Diamond p$

Adding $t(\alpha_5) = \Box\Diamond p \vee \Box\neg p$ destroys the $\to$ structure of the algebra. In **KTB**, we do not have $\Box[(\Box\Diamond p \vee \Box\neg p) \to p] = p$ though $\alpha_5 \to \alpha_1$ is

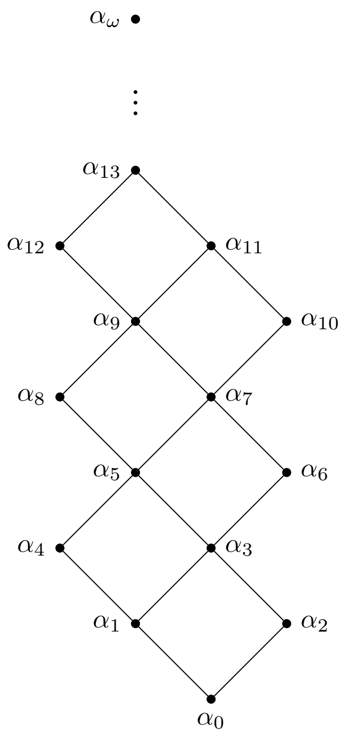

Figure 1

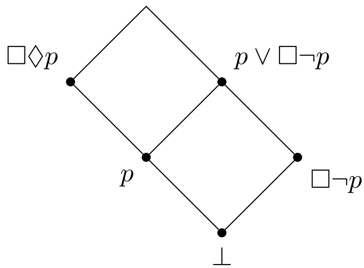

Figure 2

intuitionistically equivalent to α_1. It is clear that we should not expect our translation preserves $\to$. Moreover, it is not true that

$$\varphi =_{INT} \psi \qquad \Rightarrow \qquad t(\varphi) =_{KTB} t(\psi).$$

Let us concentrate on the lattice structure of $\mathcal{R}$ and ask if the Rieger-Nishimura lattice (not Heyting algebra) is preserved under the translation t in **KTB**. Obviously, the fragment of the lattice consisting of $\alpha_0 - \alpha_5$ is preserved. However, even such modified hypothesis turns out to be false as adding $t(\alpha_6) = \Box(\Box\Diamond p \to p)$ to the picture destroys the lattice structure. In the Rieger-Nishimura lattice we have: $\alpha_{2n+3} \wedge \alpha_{2n+4} = \alpha_{2n+1}$, for any $n \geq 0$. We prove that t does not preserve this equation for $n \geq 1$. First, note that:

Lemma 3.2. *$t[(\alpha_{2n+3} \wedge \alpha_{2n+4}) \to \alpha_{2n+1}] \in$ **KTB**, for any $n \geq 1$.*

Proof: We need to show:

$$\{[t(\alpha_{2n+1}) \vee t(\alpha_{2n+2})] \wedge \Box[t(\alpha_{2n+2}) \to t(\alpha_{2n+1})]\} \to t(\alpha_{2n+1}) \in \mathbf{KTB}$$

which is quite obvious. □

Before we prove that

$$t[\alpha_{2n+1} \to (\alpha_{2n+3} \wedge \alpha_{2n+4})] \notin \mathbf{KTB}, \text{ for any } n \geq 1 \tag{3.1}$$

we shall consider the simplest case when $n = 1$.

LEMMA 3.3. $t[\alpha_3 \to (\alpha_5 \wedge \alpha_6)] \notin \mathbf{KTB}$.

PROOF: We shall prove that $t(\alpha_3) \to \{\Box[t(\alpha_4) \to t(\alpha_3)] \wedge t(\alpha_5)\} \notin \mathbf{KTB}$. Let us take the model $\mathfrak{M}_1 = \langle W, R, v\rangle$ where $W = \{x_0, x_1\}$, and R is the total relation on W, and $x_i \models p$ iff $i = 0$.

Then we have $x_0 \models p$, which gives $x_0 \models p \vee \Box\neg p$ and hence $x_0 \models t(\alpha_3)$. Thus, $x_1 \models \Box\Diamond p$ and this means that $x_1 \models t(\alpha_4)$. We have $x_1 \not\models p$ and $x_1 \not\models \Box\neg p$. Hence we get $x_1 \not\models p \vee \Box\neg p$ which shows $x_1 \not\models t(\alpha_3)$. It means that $x_1 \not\models t(\alpha_4) \to t(\alpha_3)$ and $x_0 \not\models \Box[t(\alpha_4) \to t(\alpha_3)]$. Thus, we proved $x_0 \not\models t(\alpha_3) \to \{\Box[t(\alpha_4) \to t(\alpha_3)] \wedge t(\alpha_5)\}$. □

For proving (3.1), we shall define some special KTB-models which are extensions of the above $\mathfrak{M}_1$. Let

DEFINITION 3.4. $\mathfrak{M}_n = \langle W_n, R_n, v_n\rangle$, for $n \geq 2$, where $W_n = \{x_0, x_1, x_2, ..., x_n\}$, R_n is reflexive and symmetric on W_n, and

$$x_0 R x_i \text{ iff } i \neq 1, \quad \text{for any } i \leq n; \tag{3.2}$$

$$x_1 R x_i \text{ iff } i \notin \{0, 3\}, \quad \text{for any } i \leq n; \tag{3.3}$$

$$x_2 R x_i \ , \quad \text{for any } i \leq n; \tag{3.4}$$

$$x_3 R x_i \text{ iff } i \notin \{1, 4\}, \quad \text{for any } i \leq n; \tag{3.5}$$

$$\text{if } 3 < k < n-1, \text{ then } x_k R x_i \text{ iff } i \notin \{k+1, k-1\}, \text{ for any } i \leq n; \tag{3.6}$$

$$\neg x_{n-1} R x_n. \tag{3.7}$$

The valuation v_n is defined: $v_n(p) = \{x_0\}$. See Figure 3.

OBSERVATION 2. If $i \leq n$, then in the model $\mathfrak{M}_n$ it holds that

$$\begin{array}{ccccccc}
x_i \models \Diamond p & \Leftrightarrow & i \neq 1 & x_i \models t(\alpha_2) & \Leftrightarrow & i = 1; \\
x_i \models t(\alpha_3) & \Leftrightarrow & i = 0, 1 & x_i \models t(\alpha_4) & \Leftrightarrow & i = 0, 3; \\
x_i \models t(\alpha_5) & \Leftrightarrow & i = 0, 1, 3 & x_i \models t(\alpha_6) & \Leftrightarrow & i = 1, 4; \\
x_i \models t(\alpha_7) & \Leftrightarrow & i = 0, 1, 3, 4 & x_i \models t(\alpha_8) & \Leftrightarrow & i = 3, 5.
\end{array}$$

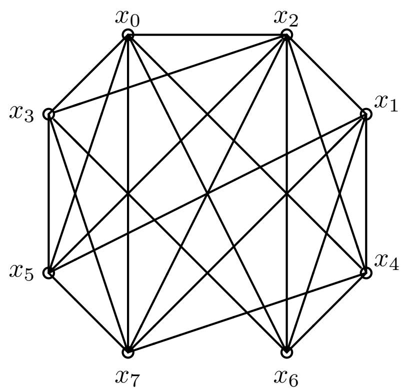

Figure 3. The frame of $\mathfrak{M}_7$

Further:

$$
\begin{array}{lll}
x_i \not\models t(\alpha_4) \to t(\alpha_3) \;\Leftrightarrow\; i = 3; & & x_i \models \Box[t(\alpha_4) \to t(\alpha_3)] \;\Leftrightarrow\; i = 1, 4; \\
x_i \not\models t(\alpha_6) \to t(\alpha_5) \;\Leftrightarrow\; i = 4; & & x_i \models \Box[t(\alpha_6) \to t(\alpha_5)] \;\Leftrightarrow\; i = 3, 5; \\
x_i \not\models t(\alpha_8) \to t(\alpha_7) \;\Leftrightarrow\; i = 5; & & x_i \models \Box[t(\alpha_8) \to t(\alpha_7)] \;\Leftrightarrow\; i = 4, 6.
\end{array}
$$

Then we get:

LEMMA 3.5. *If $2 \leq n \leq k$ and $i \leq n$, then in the model $\mathfrak{M}_k$ it holds that*

(i) $x_i \models t(\alpha_{2n+1})$ *iff* $i \leq n + 1$ *and* $i \neq 2$;

(ii) $x_i \not\models t(\alpha_{2n}) \to t(\alpha_{2n-1})$ *iff* $i = n + 1$;

(iii) $x_i \models \Box[t(\alpha_{2n}) \to t(\alpha_{2n-1})]$ *iff* $i = n$ *or* $i = n + 2$, *(for $n \geq 3$).*

PROOF: We prove it by induction on n. Let $n = 2$. Then, from Observation 1, we get: $x_i \models t(\alpha_5)$ iff $i = 0, 1, 3$. Also $x_i \not\models t(\alpha_4) \to t(\alpha_3)$ iff $i = 3$. Further $x_i \models \Box[t(\alpha_4) \to t(\alpha_3)]$ iff $i = 1$ or $i = 4$. For $n = 3$, from Observation 1, we get $x_i \models t(\alpha_7)$ iff $i = 0, 1, 3, 4$, and $x_i \not\models t(\alpha_6) \to t(\alpha_5)$ iff $i = 4$, and $x_i \models \Box[t(\alpha_6) \to t(\alpha_5)]$ iff $i = 3$ or $i = 5$.

Assume our lemma holds for n and prove it also holds for $n + 1$. We have $t(\alpha_{2n+3}) = t(\alpha_{2n+2}) \vee t(\alpha_{2n+1})$ and $t(\alpha_{2n+2}) = \Box(t(\alpha_{2n}) \to t(\alpha_{2n-1}))$. From our inductive hypothesis (i) and (ii), we get $x_i \models t(\alpha_{2n+3})$ iff $i \leq n+2$ and $i \neq 2$.

Let us consider $t(\alpha_{2n+2}) \to t(\alpha_{2n+1})$. As $t(\alpha_{2n+2}) = \Box[t(\alpha_{2n}) \to t(\alpha_{2n-1})]$, we get $x_i \not\models t(\alpha_{2n+2}) \to t(\alpha_{2n+1})$ iff $i = n+2$, by our inductive hypothesis (i) and (iii).

From the above and the definition of the relation R_n, it follows that $x_i \models \Box[t(\alpha_{2n+2}) \to t(\alpha_{2n+1})]$ iff $i = n+1$ or $i = n+3$. $\Box$

Then we may prove that (3.1) holds.

Lemma 3.6. *For any $n \geq 2$ we get*

$$t[\alpha_{2n+1} \to (\alpha_{2n+3} \wedge \alpha_{2n+4})] \notin \mathbf{KTB}$$

Proof: We take advantage of the model $\mathfrak{M}_n$. From Lemma 3.5 we get $x_n \models t(\alpha_{2n+1})$ and $x_n \not\models t(\alpha_{2n+4})$, for any $n \geq 2$. $\Box$

We see that the Rieger-Nishimura lattice loses, after the translation t, some meets of classes of formulas. Since the joins are preserved by the definition of the translation, we conclude that the obtained structure is a join semi-lattice, only. Figure 3 presents the diagram of the (Rieger-Nishimura) join semi-lattice which is preserved under the translation t. Note that the received structure is infinite as from Lemma 3.5 we get

Corollary 3.7. For any $n \geq 1$, we have $t(\alpha_{2n-1}) \to t(\alpha_{2n+1}) \in \mathbf{KTB}$ and $t(\alpha_{2n+1}) \to t(\alpha_{2n-1}) \notin \mathbf{KTB}$.

We also conclude that the function t is a translation for some classes of formulas.

Corollary 3.8. For any n, $k \geq 1$, we have:

1. $\alpha_{2n-1} \to \alpha_{2k-1} \in \mathbf{INT}$ iff $t(\alpha_{2n-1}) \to t(\alpha_{2k-1}) \in \mathbf{KTB}$,
2. $\alpha_{2n-2} \to \alpha_{2k-1} \in \mathbf{INT}$ iff $t(\alpha_{2n-2}) \to t(\alpha_{2k-1}) \in \mathbf{KTB}$.

3.1. Modal counterpart of Glivienko's theorem

Glivienko's theorem says that the double negation of any classically valid propositional formula is intuitionistically valid. Its analog for the modal logics **S5** and **S4** states that $\alpha \in \mathbf{S5}$ iff $\Diamond\Box\alpha \in \mathbf{S4}$, see [6]. There are other results in this subject e.g Rybakov [8] proved that $\Box\Diamond\alpha \to \Box\Diamond\beta \in \mathbf{K4}$ iff $\Diamond\alpha \to \Diamond\beta \in \mathbf{S5}$. Recently Shapirovsky [9] generalizes Glivenko's translation for logics of arbitrary finite height.

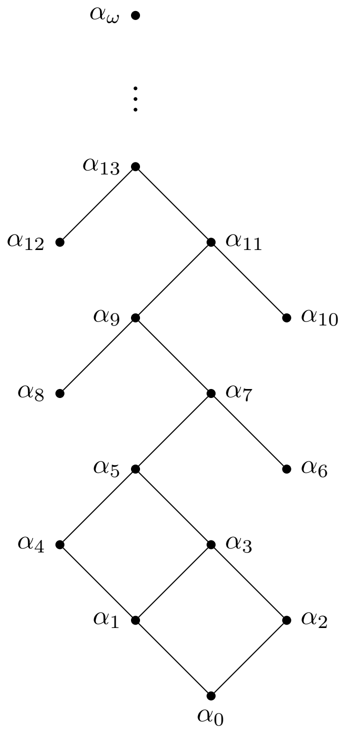

Figure 4

Our approach to Glivienko's theorem is more elementary. The translation t examined in this paper suggests a modal version of this theorem. One could think that it suffices to take $\Box\Diamond\alpha$, instead of the double negation of the classically valid formula α, to obtain the modal version of Glivienko's theorem.

Certainly, it holds for some monadic formulas.

LEMMA 3.9. *For any $n \geq 1$, we have $\Box\Diamond t(\alpha_{2n+1}) \in$ **KTB**.*

PROOF: By Corollary 3.7, it suffices to show that $\Box\Diamond t(\alpha_3) \in$ **KTB** which would be tantamount to prove that $\Diamond(\Box\neg p \vee p) \in$ **KTB**. But in any modal logic $\Diamond(\Box\neg p \vee p) = \Box\Diamond p \to \Diamond p$ and $\Box\Diamond p \to \Diamond p$ is **KT** valid. $\Box$

One could expect that, for any $n \geq 3$, we also have $\Box\Diamond t(\alpha_{2n}) \in \mathbf{KTB}$. But it is not the case. For instance, using the model $\mathfrak{M}_1$ (defined in the proof of Lemma 3.3) one easily shows $\Box\Diamond t(\alpha_6) \notin \mathbf{KTB}$.

3.2. From INT into $KTB.Alt_n$

We may also consider some extensions of the logic **KTB**. Let $\mathbf{KTB.Alt_n}$, for $n \geq 2$, be such an extension with

$$alt_n = \Box p_1 \vee \Box(p_1 \to p_2) \vee \ldots \vee \Box((p_1 \wedge \ldots \wedge p_n) \to p_{n+1}).$$

Logics $\mathbf{KTB.Alt_n}$ are characterized by reflexive and symmetric Kripke frames, in which one point has at most n successors (including itself), see [3, p. 82]. We show that there is a simple correlation between the degree of branching and the possibility of falsifying the formula $t(\alpha_{2n+1})$. Namely, we get:

Lemma 3.10. *For each $n \geq 2$, the model $\mathfrak{M}_n$ is a minimal KTB-model falsifying $t(\alpha_{2n+1})$ (which means that any model falsifying this formula contains $\mathfrak{M}_n$ as a submodel).*

Proof: By induction on n. We construct a KTB-model falsifying $t(\alpha_3)$. Because $t(\alpha_3) = p \vee \Box\neg p$ then in some point x falsifying $t(\alpha_3)$ we have

$$x \not\models p, \tag{3.8}$$

$$x \not\models \Box\neg p. \tag{3.9}$$

From (3.9) we get that $x \models \Diamond p$. Then there must exist a point $x^* \in W$ such that xRx^* and $x^* \models p$. By (3.8) we know that $x^* \neq x$. Then we obtain two-point model which is isomorphic to $\mathfrak{M}_1$.

Before we start doing the induction step, we show how the model rises if we want to falsify the formula $t(\alpha_5)$. Because $t(\alpha_5) = p \vee \Box\neg p \vee \Box\Diamond p$ then at the point x we get (3.8), (3.9) and

$$x \not\models \Box\Diamond p. \tag{3.10}$$

From (3.8) and (3.9) we obtain the existence of another point x^* such that xRx^* and $x^* \models p$. Also $x^* \neq x$. From (3.10) we see that there must exist another point, say $x^{**} \in W$ such that xRx^{**} and $x^{**} \not\models \Diamond p$. Hence $x^{**} \not\models p$ and $x \not\models p$. Also $x^{**} \neq x$ and $x^{**} \neq x^*$. We conclude that $\neg\, x^*Rx^{**}$. Then the falsifying model has to have at least three points.

It is not a cluster and the point x sees two others. Since the situation is analogous to the one described in $\mathfrak{M}_2$ we may substitute: $x := x_2$, $x^* := x_0$ and $x^{**} := x_1$. We really have got a minimal model falsifying $t(\alpha_5)$.

Let us try to falsify the formula $t(\alpha_7) = t(\alpha_5) \vee t(\alpha_6)$. For falsifying $t(\alpha_5)$ we need the model $\mathfrak{M}_2$. Then we try to falsify $t(\alpha_6)$ at x_2 which is $x_2 \not\models \Box[t(\alpha_4) \to t(\alpha_3)]$. Then there must exist a point, say x_3, $x_2 R x_3$, such that $x_3 \not\models t(\alpha_4) \to t(\alpha_3)$ what provides to:

$$x_3 \models \Box\Diamond p, \tag{3.11}$$

$$x_3 \not\models p \vee \Box\neg p. \tag{3.12}$$

Because (3.12) holds then $x_3 \neq x_i$ for $i = 0, 1$. Because of (3.11) we get $x_3 \neq x_2$. We need a successor of x_3 in which p is validated. We may take $x_3 R x_0$. Further, we know that $\neg\, x_3 R x_1$. One should remember that the relation R is reflexive and symmetric. Then we see that the minimal model for falsifying $t(\alpha_7)$ has to have four points with the relations and valuation as in $\mathfrak{M}_3$.

Suppose that our thesis holds for n. Then we know that $\mathfrak{M}_n$ is a minimal KTB-model falsifying $t(\alpha_{2n+1})$ and we take advantage of Observation 1 and Lemma 3.5.

We show that the thesis holds for $n + 1$.

We have $t(\alpha_{2n+3}) = t(\alpha_{2n+1}) \vee t(\alpha_{2n+2})$. We want to falsify the formula at the point x_2. For falsifying $t(\alpha_{2n+1})$ the assumption works and we get a model $\mathfrak{M}_n$ such that $(\mathfrak{M}_n, x_2) \not\models t(\alpha_{2n+1})$. Then we want to get $(\mathfrak{M}_n, x_2) \not\models t(\alpha_{2n+2})$ that is $(\mathfrak{M}_n, x_2) \not\models \Box[t(\alpha_{2n}) \to t(\alpha_{2n-1})]$.

There must exist a new point, say x_{n+1} such that $x_2 R x_{n+1}$ and

$$x_{n+1} \not\models t(\alpha_{2n}) \to t(\alpha_{2n-1}), \tag{3.13}$$

From Lemma 3.5 we know that the point x_{n+1} is a new point different from the others. Also $x_2 R x_{n+1}$. Because $x_{n+1} \models t(\alpha_{2n})$ and $x_n \not\models t(\alpha_{2n-2}) \to t(\alpha_{2n-3})$ then $\neg x_n R x_{n+1}$. We also conclude that x_{n+1} sees all other points x_i for $i \neq n$ because we want to have $x_{n+1} \not\models t(\alpha_{2n})$ for $k < n$.

Then, adding a new point x_{n+1} to $\mathfrak{M}_n$, with the suitable relations, we obtain $\mathfrak{M}_{n+1}$. $\Box$

A correlation between the degree of branching of a frame and the validity of the formula $t(\alpha_{2n+1})$ is as follows:

THEOREM 3.11. *For each* $n \geq 2$, $t(\alpha_{2n+1}) \in \mathbf{KTB.Alt_i}$ *iff* $i \leq n$.

PROOF: If $t(\alpha_{2n+1}) \notin \mathbf{KTB.Alt_i}$ then from Lemma 3.10 we conclude that the minimal model falsifying this formula contains the model $\mathfrak{M}_{n+1}$. In this model (see Definition 3.4) the point x_2 sees all other points (including itself), hence the degree of branching of $\mathfrak{M}_{n+1}$ is equal to $n+1$. Then $i > n$. On the other hand, if $i > n$ then among the models for $\mathbf{KTB.Alt_i}$ is the model $\mathfrak{M}_i$, falsifying $t(\alpha_{2n+1})$. □

One may notice that the formulas $t(\alpha_{2n+1})$, $n \geq 1$ written in one variable, have a similar significance as the formulas alt_n, at least in KTB-frames.

COROLLARY 3.12. $\mathbf{KTB.Alt_i} = \mathbf{KTB} \oplus t(\alpha_{2n+1})$ for any $n \geq 1$.

4. Specific questions

The main problem concerning our translation is the fact that it does not preserve the intuitionistic equivalence of formulas. More specifically, it is not true that

$$\alpha \to \beta \in \mathbf{INT} \qquad \Rightarrow \qquad t(\alpha) \to t(\beta) \in \mathbf{KTB}.$$

We suppose our problem might be solved if we significantly modify our approach. It would be required to opt out from the attempts to define intuitionistic connectives in **KTB** but to translate each formula in its specific way. Technically, it will relay on adding $\Box^k$, for some k, to the predecessor of $t(\alpha) \to t(\beta)$. The number k depends on the difference of modal degrees of the antecedent and consequent of the implication. Let us consider the formula $\Box(t(\alpha_4) \to t(\alpha_3)) \wedge t(\alpha_5) = t(\alpha_3)$ which is not theorem of **KTB** because the reverse implication is not. See Lemma 3.3. The simple implication:

$$(\Box(t(\alpha_4) \to t(\alpha_3)) \wedge t(\alpha_5)) \to t(\alpha_3)$$

which is

$$\{\Box[\Box\Diamond p \to (p \vee \Box\neg p)] \wedge (\Box\Diamond p \vee p \vee \Box\neg p)\} \to (p \vee \Box\neg p)$$

is a theorem of **KTB**. We see that $md\{\Box[\Box\Diamond p \to (p \vee \Box\neg p)] \wedge (\Box\Diamond p \vee p \vee \Box\neg p)\} = 3$ and $md(p \vee \Box\neg p) = 1$. Hence modal degree of the antecedent is larger than the degree of the consequent.

In the reverse implication the situation is opposite and we have $t(\alpha_3) \to [\Box(t(\alpha_4) \to t(\alpha_3)) \wedge t(\alpha_5)] \notin \mathbf{KTB}$. We propose the following strengthening of the above formula.

Since $md\{[\Box[t(\alpha_4) \to t(\alpha_3)] \wedge t(\alpha_5)]\} - md(t(\alpha_3)) = 2$ then we consider the formula $\Box^3 t(\alpha_3) \to [\Box(t(\alpha_4) \to t(\alpha_3)) \wedge t(\alpha_5)]$ and obtain:

LEMMA 4.1. *The formula* $\Box^3 t(\alpha_3) \to [\Box(t(\alpha_4) \to t(\alpha_3)) \wedge t(\alpha_5)]$ *is a theorem of* **KTB**.

PROOF: Suppose that $\Box^3 t(\alpha_3) \to [\Box(t(\alpha_4) \to t(\alpha_3)) \wedge t(\alpha_5)] \notin \mathbf{KTB}$. Then there exists a model $\mathfrak{M} = \langle W, R, v\rangle$ and a point $x \in W$ such that

$$x \models \Box^3 t(\alpha_3) \tag{4.1}$$

$$x \not\models \Box(t(\alpha_4) \to t(\alpha_3)) \wedge t(\alpha_5) \tag{4.2}$$

From (4.2) we know that $x \not\models \Box(t(\alpha_4) \to t(\alpha_3))$ or $x \not\models t(\alpha_5)$.

I. If $x \not\models \Box(t(\alpha_4) \to t(\alpha_3))$ then there is another point, say x_2, xRx_2 such that $x_2 \not\models t(\alpha_4) \to t(\alpha_3)$ what means that

$$x_2 \models t(\alpha_4) \tag{4.3}$$

$$x_2 \not\models t(\alpha_3) \tag{4.4}$$

But from (4.1) and from reflexivity of R we know that $x_2 \models t(\alpha_3)$. This is a contradiction.

II. If $x \not\models t(\alpha_5)$ then since $\alpha_5 = \alpha_3 \vee \alpha_4$ then $x \not\models t(\alpha_3)$ and $x \not\models t(\alpha_4)$. But $x \not\models t(\alpha_3)$ is in contradiction with (4.1). □

Despite the above example, one should not expect the following holds: if $\alpha \to \beta \in \mathbf{INT}$, then

1. if $md(t(\alpha)) > md(t(\beta))$ then $t(\alpha) \to t(\beta) \in \mathbf{KTB}$,
2. if $md(t(\beta)) - md(t(\alpha)) = k \geq 0$ then $\Box^{k+1} t(\alpha) \to t(\beta) \in \mathbf{KTB}$.

We show that this is false (even for formulas in one variable). The counterexample is the formula $\Box\Diamond t(\alpha_6) = \Box[\Box(t(\alpha_6) \to \bot) \to \bot]$. We see that $\Box[\Box(t(\alpha_6) \to \bot) \to \bot] \in \mathbf{KTB}$ iff $\Box(t(\alpha_6) \to \bot) \to \bot \in \mathbf{KTB}$. Obviously $md(\Box(t(\alpha_6) \to \bot)) > md(\bot)$. Let us take the model $\mathfrak{M}_1$, see Definition 3.4. One may easily obtain that $\mathfrak{M}_1 \not\models \Box\Diamond t(\alpha_6)$. Hence $\Box\Diamond t(\alpha_6) \notin \mathbf{KTB}$.

Now, let us consider the implication:

$$t(\alpha) \to t(\beta) \in \mathbf{KTB} \qquad \Rightarrow \qquad \alpha \to \beta \in \mathbf{INT}.$$

We show that it is false. The counterexample is the following formula $\alpha = \neg\neg(T \to p) \to (T \to p)$ which is equivalent to the strong law of doubled negation. Obviously $\alpha \notin \mathbf{INT}$. But we shall prove that:

LEMMA 4.2. $t(\alpha) \in \mathbf{KTB}$.

PROOF: Let us write the formula $t(\alpha) = \Box[\Box\Diamond\Box(T \to p) \to \Box(T \to p)]$. By Brouwer's axiom we have: $\Diamond\Box(T \to p) \to (T \to p) \in \mathbf{KTB}$. Then by the rule of necessitation and the axiom K we obtain $\Box\Diamond\Box(T \to p) \to \Box(T \to p) \in \mathbf{KTB}$. Again by the rule of necessitation we get: $\Box[\Box\Diamond\Box(T \to p) \to \Box(T \to p)] \in \mathbf{KTB}$. $\Box$

5. Conclusions

Since we see that $t(\mathbf{INT}) \not\subset \mathbf{KTB}$, we would like to know what is the image of **INT** by the function t.

As it was mentioned above the formula $\Box\Diamond t(\alpha_6) \notin \mathbf{KTB}$ and moreover the model falsifying it is the model $\mathfrak{M}_1$, see Definition 3.4. Actually, $\mathfrak{M}_1$ is a two-element cluster. One easily conclude that $\Box\Diamond t(\alpha_6) \notin \mathbf{S5}$. Hence it must be $\Box\Diamond t(\alpha_6) \in \mathbf{Triv}$. It means that the least modal logic containing $t(\mathbf{INT})$ is **Triv** which is highly unsatisfactory.

Let us add that we do not decide if there is any other translation from **INT** into **KTB**. We leave this problem open. It seems that the intuitionistic logic is too strong for being translated into any intransitive modal logic.

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