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Alexej P. Pynko

FOUR-VALUED EXPANSIONS OF DUNN-BELNAP'S LOGIC (I): BASIC CHARACTERIZATIONS

Abstract

Basic results of the paper are that any four-valued expansion L_4 of Dunn-Belnap's logic DB_4 is defined by a unique (up to isomorphism) conjunctive matrix $\mathcal{M}_4$ with exactly two distinguished values over an expansion $\mathfrak{A}_4$ of a De Morgan non-Boolean four-valued diamond, but by no matrix with either less than four values or a single [non-]distinguished value, and has no proper extension satisfying Variable Sharing Property (VSP). We then characterize L_4's having a theorem / inconsistent formula, satisfying VSP and being [inferentially] maximal / subclassical / maximally paraconsistent, in particular, algebraically through $\mathcal{M}_4|\mathfrak{A}_4$'s (not) having certain submatrices|subalgebras.

Likewise, [providing $\mathfrak{A}_4$ is regular / has no three-element subalgebra] L_4 has a proper consistent axiomatic extension if[f] $\mathcal{M}_4$ has a proper paraconsistent / two-valued submatrix [in which case the logic of this submatrix is the only proper consistent axiomatic extension of L_4 and is relatively axiomatized by the *Excluded Middle law* axiom]. As a generic tool (applicable, in particular, to both classically-negative and implicative expansions of DB_4), we also prove that the lattice of axiomatic extensions of the logic of an implicative matrix $\mathcal{M}$ with equality determinant is dual to the distributive lattice of lower cones of the set of all submatrices of $\mathcal{M}$ with non-distinguished values.

Keywords: Propositional logic, logical matrix, Dunn-Belnap's logic, expansion, [bounded] distributive/De Morgan lattice, equality determinant.

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1. Introduction

Dunn-Belnap's four-valued logic (cf. [5] and [3]) arising as the logic of *first-degree entailment* (*FDE*, for short) in relevance logic R has been naturally expanded by additional connectives in [11]. The present paper, equally belonging to General Logic, pursues this line of research in the following *generic* respects in addition to those of functional completeness and both sequential and equational axiomatizations comprehensively explored therein.

First of all, the most natural way of expanding FDE consists in expanding the matrix $\mathcal{DM}_4$ defining FDE by additional connectives. This inevitably raises the question which exactly expansions of FDE are covered by such approach. As we argue here, these are exactly *all four-valued* ones (that excludes E and R). And what is more, any four-valued expansion of FDE is defined by a *unique* expansion of $\mathcal{DM}_4$.

In addition, as a by-product of auxiliary results, we prove that any four-valued expansion of FDE is defined by no matrix with either a unique (non-)distinguished value or less than four values and has no proper extension satisfying *Variable Sharing Property* (*VSP*, for short; cf. [1]), according to which any entailment $\phi \to \psi$ holds only if ϕ and ψ have a propositional variable in common, that is one of the most fundamental peculiarities of FDE, quite independently from whether the expansion itself satisfies VSP. The latter result has been proved for FDE alone in [9] and means, perhaps, a principal maximality of expansions of FDE. In this connection, we find purely algebraic criteria of a FDE expansion's satisfying VSP, being [inferentially] maximal in the sense of not having a proper [inferentially] consistent extension,[1] being a sublogic of a definitional copy of the classical logic and being *maximally* paraconsistent in the sense of [10] (viz., having no proper paraconsistent extension).

After all, we study the issue of axiomatic extensions within the framework of FDE expansions.

The rest of the paper is as follows. The exposition of the material of the paper is entirely self-contained (of course, modulo very basic issues concerning Set Theory, Lattice Theory, Universal Algebra, Model Theory and Mathematical Logic not specified here explicitly, to be found, e.g., in

[1]It is the absence of theorems in FDE, being an inevitable consequence of VSP, that makes "inferential" versions of standard conceptions of consistency and maximality acute within the framework of FDE expansions to be equally void of theorems.

standard mathematical handbooks like [2] and [7]). Section 2 is a concise summary of basic issues underlying the paper, most of which have actually become a part of logical and algebraic folklore. Section 3 is devoted to certain key preliminary issues concerning equality determinants (in the sense of [13]), implicative matrices and De Morgan lattices. In Section 4 we formulate and prove main results of the paper described above. Then, in Section 5 we apply general results of previous two sections to three generic – classically-negative, bilattice and implicative – classes of FDE expansions.

2. Basic issues

Standard notations like img, dom, ker, hom, π_i, Con, et. al., as well as related notions are supposed to be clear.

2.1. Set-theoretical background

We follow the standard convention (among other things, contracting cumbersome finite sequence notations), according to which natural numbers (including 0) are treated as finite ordinals (viz., sets of lesser natural numbers), the ordinal of all them being denoted by ω. The proper class of all ordinals is denoted by ∞. Likewise, functions are viewed as binary relations. In addition, singletons are often identified with their unique elements, unless any confusion is possible.

Given a set S, the set of all subsets of S [of cardinality $\in K \subseteq \infty$][2] is denoted by $\wp_{[K]}(S)$. A subset $T \subseteq S$ is said to be *proper*, if $T \neq S$. Further, given any equivalence relation θ on S, as usual, by ν_θ we denote the function with domain S defined by $\nu_\theta(a) \triangleq \theta[\{a\}]$, for all $a \in S$, in which case $\ker \nu_\theta = \theta$, whereas we set $(T/\theta) \triangleq \nu_\theta[T]$, for every $T \subseteq S$. Next, S-tuples (viz., functions with domain S) are often written in either sequence $\bar{t}$ or vector $\vec{t}$ forms, its s-th component (viz., the value under argument s), where $s \in S$, being written as either t_s or t^s. Given two more sets A and B, any relation $R \subseteq (A \times B)$ (in particular, a mapping $R : A \to B$) determines the equally-denoted relation $R \subseteq (A^S \times B^S)$ (resp., mapping $R : A^S \to B^S$) point-wise, that is, $R \triangleq \{\langle \bar{a}, \bar{b} \rangle \in (A^S \times B^S) \mid \forall s \in S : a_s \; R \; b_s\}$. Likewise, given a set A, an S-tuple $\overline{B}$ of sets and any

[2] As usual, parentheses as well as both square, figure and angle brackets are often used for surrounding a (possibly, multiple) optional content.

$\bar{f} \in (\prod_{s\in S} B_s^A)$, put $(\prod \bar{f}) : A \to (\prod \overline{B}), a \mapsto \langle f_s(a)\rangle_{s\in S}$. (In case $I = 2$, $f_0 \times f_1$ stands for $(\prod \bar{f})$.) Further, a *lower cone of* a $T \subseteq \wp(S)$ is any $L \subseteq T$ such that, for each $X \in L$, $(\wp(X) \cap T) \subseteq L$. Likewise, an *anti-chain of* T is any $A \subseteq T$ such that $\max(A) = A$. (Clearly, in case S is finite, the unary operations $A \mapsto (T \cap \bigcup\{\wp(X) \mid X \in A\})$ and $L \mapsto \max(L)$ on $\wp(\wp(S))$ form inverse to one another bijections between the sets of all anti-chains and all lower cones of T.) Furthermore, set $\Delta_S \triangleq \{\langle a, a\rangle | a \in S\}$, functions of such a kind being referred to as *diagonal*. Finally, given any $R \subseteq S^2$, $\mathrm{Tr}(R) \triangleq \{\langle \pi_0(\pi_0(\bar{r})), \pi_1(\pi_{l-1}(\bar{r}))\rangle | \bar{r} \in R^l, l \in (\omega \setminus 1)\}$ is the least transitive binary relation on S including R, known as the *transitive closure of* R.

2.2. Algebraic background

Unless otherwise specified, abstract algebras are denoted by Fraktur letters (possibly, with indices/prefixes/suffixes), their carriers (viz., underlying sets) being denoted by corresponding Italic letters (with same indices/prefixes/suffixes, if any).

A *(propositional/sentential) language/signature* is any algebraic (viz., functional) signature Σ (to be dealt with by default throughout the paper) constituted by function (viz., operation) symbols of finite arity to be treated as *(propositional/sentential) connectives*. Given any $\alpha \in \wp_{\infty\setminus 1}(\omega)$, put $V_\alpha \triangleq \{x_\beta | \beta \in \alpha\}$, elements of which being viewed as *(propositional/sentential) variables of rank* α. Then, we have the absolutely-free Σ-algebra $\mathfrak{Fm}_\Sigma^\alpha$ freely-generated by the set V_α, referred to as the *formula Σ-algebra of rank* α, its endomorphisms/elements of its carrier $\mathrm{Fm}_\Sigma^\alpha$ (viz., Σ-terms of rank α) being called *(propositional/sentential) Σ-substitutions/-formulas of rank* α. (In general, the reservation "of rank α" is normally omitted, whenever $\alpha = \omega$.) Given a Σ-formula φ, $\mathrm{Var}(\varphi)$ denotes the set of all variables *actually* occurring in φ.

Recall the following useful well-known algebraic fact:

Lemma 2.1. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be Σ-algebras and $h \in \hom(\mathfrak{A}, \mathfrak{B})$. [Suppose $(\operatorname{img} h) = B$.] Then, for every $\vartheta \in \mathrm{Con}(\mathfrak{B})$, $h^{-1}[\vartheta] \in \{\theta \in \mathrm{Con}(\mathfrak{A}) \mid (\ker h) \subseteq \theta\}$ [whereas $h[h^{-1}[\vartheta]] = \vartheta$, while, conversely, for every $\theta \in \mathrm{Con}(\mathfrak{A})$ such that $(\ker h) \subseteq \theta$, $h[\theta] \in \mathrm{Con}(\mathfrak{B})$, whereas $h^{-1}[h[\theta]] = \theta$].*

2.3. Propositional logics and matrices

A *[finitary] Σ-rule* is any couple $\langle \Gamma, \varphi \rangle$, where $(\Gamma \cup \{\varphi\}) \in \wp_{[\omega]}(\mathrm{Fm}^{\omega}_{\Sigma})$, normally written in the standard sequent form $\Gamma \vdash \varphi$, φ/any element of Γ being referred to as the/a *conclusion/premise of* it. A *(substitutional) Σ-instance* of it is then any Σ-rule of the form $\sigma(\Gamma \vdash \varphi) \triangleq (\sigma[\Gamma] \vdash \sigma(\varphi))$, where σ is a Σ-substitution. As usual, Σ-rules without premises are called *Σ-axioms* and are identified with their conclusions. A[n] *[axiomatic] (finitary) Σ-calculus* is any set of (finitary) Σ-rules[-axioms].

A *(propositional/sentential) Σ-logic* (cf., e.g., [6]) is any closure operator C over $\mathrm{Fm}^{\omega}_{\Sigma}$ that is *structural* in the sense that $\sigma[C(X)] \subseteq C(\sigma[X])$, for all $X \subseteq \mathrm{Fm}^{\omega}_{\Sigma}$ and all $\sigma \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{Fm}^{\omega}_{\Sigma})$. A(n) *(in)consistent set of* C is any $X \subseteq \mathrm{Fm}^{\omega}_{\Sigma}$ such that $C(X) \neq (=) \mathrm{Fm}^{\omega}_{\Sigma}$. Then, C is said to be *[inferentially] (in)consistent*, provided $\varnothing[\cup\{x_0\}]$ is a(n in)consistent set of C or, equivalently, in view of the structurality of C, $x_1 \notin (\in) C(\varnothing[\cup\{x_0\}])$. A Σ-rule $\Gamma \vdash \varphi$ is said to be *satisfied in* C, provided $\varphi \in C(\Gamma)$, Σ-axioms satisfied in C being called its *theorems*. A *[proper] extension of* C is any Σ-logic $C' \supseteq C$ [distinct from C], in which case C is said to be a *[proper] sublogic of* C'. Then, an extension C' of C is said to be *axiomatized by* a Σ-calculus $\mathcal{C}$ *relatively to* C, provided it is the least extension of C satisfying each rule of $\mathcal{C}$. Furthermore, an extension C' of C is said to be *axiomatic*, whenever it is relatively axiomatized by an axiomatic Σ-calculus. Next, C is said to be *[inferentially] maximal(ly consistent)*, whenever it is [inferentially] consistent and has no proper [inferentially] consistent extension. Further, C is said to be *$\diamond$-conjunctive*, where $\diamond$ is a (possibly, secondary) binary connective of Σ, provided $C(\phi \diamond \psi) = C(\{\phi, \psi\})$, for all $\phi, \psi \in \mathrm{Fm}^{\omega}_{\Sigma}$, in which case any extension of C is so. Likewise, C is said to be *[maximally] $\wr$-paraconsistent*, where $\wr$ is a unary connective of Σ, provided $x_1 \notin C(\{x_0, \wr x_0\})$ [and C has no proper $\wr$-paraconsistent extension]. In addition, C is said to be *theoremless*, provided $C(\varnothing) = \varnothing$. Finally, *Variable Sharing Property* (*VSP*, for short; cf. [1]) is said to *hold/be satisfied for* C, provided, for all $\phi \in \mathrm{Fm}^{\omega}_{\Sigma}$ and all $\psi \in C(\phi)$, it holds that $(\mathrm{Var}(\phi) \cap \mathrm{Var}(\psi)) \neq \varnothing$, in which case C has neither a theorem nor an inconsistent formula, in view of the finiteness of the set $\mathrm{Var}(\varphi)$, where $\varphi \in \mathrm{Fm}^{\omega}_{\Sigma}$.

A *(logical) Σ-matrix* (cf. [6]) is any couple of the form $\mathcal{A} = \langle \mathfrak{A}, D^{\mathcal{A}} \rangle$, where $\mathfrak{A}$ is a Σ-algebra, called the *underlying algebra of* $\mathcal{A}$, while $D^{\mathcal{A}} \subseteq A$ is called the *truth predicate of* $\mathcal{A}$, elements of which being referred to as *distinguished values of* $\mathcal{A}$. (In general, matrices are denoted by Calligraphic letters [possibly, with indices/prefixes/suffixes], their underlying algebras being denoted by corresponding Fraktur letters [with same indices/prefixes/suffixes, if any].) This is said to be *n-valued/truth[-non]-empty/(in)consistent/false-singular/truth-singular*, where $n \in \omega$, provided $|A| = n/D^{\mathcal{A}} = [\neq]\varnothing/D^{\mathcal{A}} \neq (=)A/|A \setminus D^{\mathcal{A}}| \in 2/|D^{\mathcal{A}}| \in 2$. Next, given any $\Sigma' \subseteq \Sigma$, put $(\mathcal{A}{\upharpoonright}\Sigma') \triangleq \langle \mathfrak{A}{\upharpoonright}\Sigma', D^{\mathcal{A}}\rangle$, in which case $\mathcal{A}$ is said to be a *(Σ-)expansion of* $\mathcal{A}{\upharpoonright}\Sigma'$. (Any notation, being specified for single matrices, is supposed to be extended to classes of matrices member-wise.)

A Σ-matrix $\mathcal{A}$ is said to be *finite/finitely-generated/generated by* a $B \subseteq A$, whenever $\mathfrak{A}$ is so. Then, $\mathcal{A}$ is said to be *K-generated*, where $K \subseteq \infty$, whenever it is generated by a $B \in \wp_K(A)$.

As usual, Σ-matrices are treated as first-order model structures (viz., algebraic systems; cf. [7]) of the first-order signature $\Sigma \cup \{D\}$ with unary predicate D, any [finitary] Σ-rule $\Gamma \vdash \phi$ being viewed as the [first-order] Horn formula $(\bigwedge \Gamma) \to \phi$ under the standard identification of any propositional Σ-formula ψ with the first-order atomic formula $D(\psi)$. Then, the class of all models of a Σ-calculus $\mathcal{C}$ is denoted by $\mathrm{Mod}(\mathcal{C})$. In that case, given any class of Σ-matrices M, $\mathcal{C}$ is said to *axiomatize* $\mathsf{M} \cap \mathrm{Mod}(\mathcal{C})$ *relatively to* M.

Given any $\alpha \in \wp_{\infty\setminus 1}(\omega)$ and any class M of Σ-matrices, we have the closure operator $\mathrm{Cn}^{\alpha}_{\mathsf{M}}$ over $\mathrm{Fm}^{\alpha}_{\Sigma}$ defined by $\mathrm{Cn}^{\alpha}_{\mathsf{M}}(X) \triangleq (\mathrm{Fm}^{\alpha}_{\Sigma} \cap \bigcap\{h^{-1}[D^{\mathcal{A}}] | \mathcal{A} \in \mathsf{M}, h \in \hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{A}), h[X] \subseteq D^{\mathcal{A}}\}$, for all $X \subseteq \mathrm{Fm}^{\alpha}_{\Sigma}$, in which case we have:

$$\mathrm{Cn}^{\alpha}_{\mathsf{M}}(X) = (\mathrm{Fm}^{\alpha}_{\Sigma} \cap \mathrm{Cn}^{\omega}_{\mathsf{M}}(X)), \tag{2.1}$$

because $\hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{A}) = \{h{\upharpoonright}\mathrm{Fm}^{\alpha}_{\Sigma} | h \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})\}$, for any Σ-algebra $\mathfrak{A}$, as $A \neq \varnothing$. (Note that $\mathrm{Cn}^{\alpha}_{\mathsf{M}}(\varnothing) = \varnothing$, whenever M has a truth-empty member.) Then, $\mathrm{Cn}^{\omega}_{\mathsf{M}}$ is a Σ-logic called the one *of* M. Next, a Σ-logic C is said to be *K-defined by* M, where $K \subseteq \infty$, if $(C{\upharpoonright}\wp_K(\mathrm{Fm}^{\omega}_{\Sigma})) = (\mathrm{Cn}^{\omega}_{\mathsf{M}}{\upharpoonright}\wp_K(\mathrm{Fm}^{\omega}_{\Sigma}))$. (As usual, "finitely-" stands for "ω-". Likewise, "∞-" is normally omitted, whenever no confusion is possible.) A Σ-logic C is said to be *[minimally] n-valued*, where $n \in \omega$, whenever it is defined by an n-valued Σ-matrix [but by no m-valued one, where $m \in n$], in which case C is finitary (cf. [6]). A Σ-matrix $\mathcal{A}$ is said to be *$\wr$-paraconsistent*, where $\wr$ is a unary connective of Σ, whenever the logic of $\mathcal{A}$ is so. (Clearly, the logic of any class of matrices is [inferentially] consistent iff the class contains a consistent [truth-non-empty] member.)

Let $\mathcal{A}$ and $\mathcal{B}$ be two Σ-matrices. A *(strict) [surjective] homomorphism from $\mathcal{A}$ [on]to $\mathcal{B}$* is any $h \in \hom(\mathfrak{A}, \mathfrak{B})$ such that $[h[A] = B$ and$]$ $D^{\mathcal{A}} \subseteq (=) h^{-1}[D^{\mathcal{B}}]$, the set of all them being denoted by $\hom^{[\mathrm{S}]}_{(\mathrm{S})}(\mathcal{A}, \mathcal{B})$. Recall that $\forall h \in \hom(\mathfrak{A}, \mathfrak{B}) : [((\operatorname{img} h) = B) \Rightarrow](\hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{B}) \supseteq [=]\{h \circ g | g \in \hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{A})\})$, and so we have:

$$(\exists h \in \hom^{[\mathrm{S}]}_{\mathrm{S}}(\mathcal{A}, \mathcal{B})) \Rightarrow (\mathrm{Cn}^{\alpha}_{\mathcal{B}} \subseteq [=]\, \mathrm{Cn}^{\alpha}_{\mathcal{A}}), \tag{2.2}$$

$$(\exists h \in \hom^{\mathrm{S}}(\mathcal{A}, \mathcal{B})) \Rightarrow (\mathrm{Cn}^{\alpha}_{\mathcal{A}}(\varnothing) \subseteq \mathrm{Cn}^{\alpha}_{\mathcal{B}}(\varnothing)), \tag{2.3}$$

for all $\alpha \in \wp_{\infty \setminus 1}(\omega)$. Then, $\mathcal{A}$ is said to be a *[proper] submatrix of* $\mathcal{B}$, whenever $\Delta_A \in \hom_{\mathrm{S}}(\mathcal{A}, \mathcal{B})$ [and $\mathcal{A} \neq \mathcal{B}$], in which case we set $(\mathcal{B} \restriction A) \triangleq \mathcal{A}$. Injective/bijective strict homomorphisms from $\mathcal{A}$ to $\mathcal{B}$ are referred to as *embeddings/isomorphisms of/from $\mathcal{A}$ into/onto $\mathcal{B}$*, in case of existence of which $\mathcal{A}$ is said to be *embeddable/isomorphic into/to $\mathcal{B}$*.

Let $\mathcal{A}$ be a Σ-matrix. Elements of $\mathrm{Con}(\mathcal{A}) \triangleq \{\theta \in \mathrm{Con}(\mathfrak{A}) | \theta[D^{\mathcal{A}}] \subseteq D^{\mathcal{A}}\} \ni \Delta_A$ are called *congruences of* $\mathcal{A}$. Given any $\varnothing \neq \Theta \subseteq \mathrm{Con}(\mathcal{A}) \subseteq \mathrm{Con}(\mathfrak{A})$, $\mathrm{Tr}(\bigcup \Theta)$, being well-known to be a congruence of $\mathfrak{A}$, is then easily seen to be a congruence of $\mathcal{A}$. Therefore, $\partial(\mathcal{A}) \triangleq (\bigcup \mathrm{Con}(\mathcal{A})) \in \mathrm{Con}(\mathcal{A})$, in which case this is the greatest congruence of $\mathcal{A}$ (it is this fact that justifies using the symbol ∂). Then, $\mathcal{A}$ is said to be *simple/irreducible*, provided $\partial(\mathcal{A}) = \Delta_A$. Given any $\theta \in \mathrm{Con}(\mathfrak{A}[\mathcal{A}])$, we have the *quotient* Σ-matrix $(\mathcal{A}/\theta) \triangleq \langle \mathfrak{A}/\theta, D^{\mathcal{A}}/\theta \rangle$, in which case $\nu_\theta \in \hom^{\mathrm{S}}_{[\mathrm{S}]}(\mathcal{A}, \mathcal{A}/\theta)$. The quotient $\Re(\mathcal{A}) \triangleq (\mathcal{A}/\partial(\mathcal{A}))$ is called the *reduction of* $\mathcal{A}$.

A Σ-matrix $\mathcal{A}$ is said to be a *model of* a Σ-logic C, provided $C \subseteq \mathrm{Cn}^{\omega}_{\mathcal{A}}$, the class of all [irreducible of] them being denoted by $\mathrm{Mod}_{[\Im]}(C)$. Next, $\mathcal{A}$ is said to be $\diamond$*-conjunctive*, where $\diamond$ is a (possibly, secondary) binary connective of Σ, provided $(\{a, b\} \subseteq D^{\mathcal{A}}) \Leftrightarrow ((a \diamond^{\mathfrak{A}} b) \in D^{\mathcal{A}})$, for all $a, b \in A$, that is, $\mathrm{Cn}^{\omega}_{\mathcal{A}}$ is $\diamond$-conjunctive.

Remark 2.2. As an immediate consequence of Lemma 2.1, given any Σ-matrices $\mathcal{A}$ and $\mathcal{B}$ and any $h \in \hom^{[\mathrm{S}]}_{\mathrm{S}}(\mathcal{A}, \mathcal{B})$, for every $\vartheta \in \mathrm{Con}(\mathcal{B})$, $h^{-1}[\vartheta] \in \{\theta \in \mathrm{Con}(\mathcal{A}) \mid (\ker h) \subseteq \theta\}$ [whereas $h[h^{-1}[\vartheta]] = \vartheta$, while, conversely, for every $\theta \in \mathrm{Con}(\mathcal{A})$ such that $(\ker h) \subseteq \theta$, $h[\theta] \in \mathrm{Con}(\mathcal{B})$, whereas $h^{-1}[h[\theta]] = \theta$]. □

By Remark 2.2, we immediately have:

Corollary 2.3. Let $\mathcal{A}$ and $\mathcal{B}$ be Σ-matrices and $h \in \hom_{\mathrm{S}}(\mathcal{A}, \mathcal{B})$. Suppose $\mathcal{A}$ is simple. Then, h is injective.

Remark 2.4 (Matrix Homomorphism Theorem). As an immediate consequence of the Algebra Homomorphism Theorem, given any Σ-matrices $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, any $f \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{A},\mathcal{B})$ and any $g \in \hom^{(\mathrm{S})}_{[\mathrm{S}]}(\mathcal{A},\mathcal{C})$ such that $(\ker f) \subseteq \{=\}(\ker g)$, it holds that $(g \circ f^{-1}) \in \hom^{(\mathrm{S})}_{[\mathrm{S}]}(\mathcal{B},\mathcal{C})$ {is injective}. □

Proposition 2.5. Let $\mathcal{A}$ and $\mathcal{B}$ be two Σ-matrices and $h \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{A},\mathcal{B})$. Then, $\eth(\mathcal{A}) = h^{-1}[\eth(\mathcal{B})]$ and $\eth(\mathcal{B}) = h[\eth(\mathcal{A})]$.

Proof: As $\Delta_B \in \mathrm{Con}(\mathcal{B})$, by Remark 2.2, we have $\ker h = h^{-1}[\Delta_B] \in \mathrm{Con}(\mathcal{A})$, and so $\ker h \subseteq \eth(\mathcal{A})$, in which case, by Remark 2.2, we get:

$$\begin{array}{rcl} h^{-1}[\eth(\mathcal{B})] & \subseteq & \eth(\mathcal{A}), \\ h[h^{-1}[\eth(\mathcal{B})]] & = & \eth(\mathcal{B}), \\ h[\eth(\mathcal{A})] & \subseteq & \eth(\mathcal{B}), \\ h^{-1}[h[\eth(\mathcal{A})]] & = & \eth(\mathcal{A}). \end{array}$$

These collectively imply the equalities to be proved, as required. □

Since, for any equivalence θ on any set A, it holds that $\nu_\theta[\theta] = \Delta_{A/\theta}$, as an immediate consequence of Proposition 2.5, we also have:

Corollary 2.6. Let $\mathcal{A}$ be a Σ-matrix. Then, $\mathcal{A}/\eth(\mathcal{A})$ is simple.

Given a set I and an I-tuple $\overline{\mathcal{A}}$ of Σ-matrices, the Σ-matrix $(\prod_{i\in I} \mathcal{A}_i) \triangleq \langle \prod_{i\in I} \mathfrak{A}_i, (\prod_{i\in I} A_i) \cap \bigcap_{i\in I} \pi_i^{-1}[D^{\mathcal{A}_i}]\rangle$ is called the *direct product of* $\overline{\mathcal{A}}$. (As usual, when $I = 2$, $\mathcal{A}_0 \times \mathcal{A}_1$ stands for the direct product involved. Likewise, if $(\mathrm{img}\,\overline{\mathcal{A}}) \subseteq \{\mathcal{A}\}$, where $\mathcal{A}$ is a Σ-matrix, $\mathcal{A}^I$ stands for the direct product involved.) Any submatrix $\mathcal{B}$ of the direct product involved is referred to as a *subdirect product of* $\overline{\mathcal{A}}$, whenever, for each $i \in I$, $\pi_i[B] = A_i$.

Lemma 2.7 (Subdirect Product Lemma). *Let* M *be a [finite] class of [finite]* Σ*-matrices and* $\mathcal{A}$ *a {truth-non-empty} (simple)* $([\omega\cap](\omega+1))$*-generated model of the logic of* M*. Then, there is some strict surjective homomorphism from a subdirect product of a [finite] tuple constituted by members of* $\mathbf{S}^{\{*\}}_*(\mathsf{M})$ *onto* $\mathcal{A}/\eth(\mathcal{A})$ *(resp., onto* $\mathcal{A}$ *itself).*

Proof: Take any $A' \in \wp_{[\omega\cap](\omega+1)}(A)$ generating $\mathfrak{A}$ and any $a \in A \neq \varnothing$, in which case $A'' \triangleq (A' \cup \{a\}) \in \wp_{([\omega\cap](\omega+1))\setminus 1}(A)$ generates $\mathfrak{A}$, and so $\alpha \triangleq |A''| \in (([\omega\cap](\omega+1)) \setminus 1) \subseteq \wp_{\infty\setminus 1}(\omega)$. Next, take any bijection from

V_α onto A'' to be extended to a surjective $h \in \hom(\mathrm{Fm}^\alpha_\Sigma, \mathfrak{A})$, in which case it is a surjective strict homomorphism from $\mathcal{B} \triangleq \langle \mathrm{Fm}^\alpha_\Sigma, X\rangle$, where $\{\varnothing \neq\}X \triangleq h^{-1}[D^{\mathcal{A}}]$, onto $\mathcal{A}$, and so, by (2.2), $\mathcal{B}$ is a {truth-non-empty} model of the logic of M. Then, applying (2.1) twice, we get $\mathrm{Cn}^\alpha_{\mathsf{M}}(X) \subseteq \mathrm{Cn}^\alpha_{\mathcal{B}}(X) \subseteq X \subseteq \mathrm{Cn}^\alpha_{\mathsf{M}}(X)$. Furthermore, we have the [finite] set $I \triangleq \{\langle h', \mathcal{D}\rangle \mid h' \in \hom(\mathcal{B}, \mathcal{D}), \mathcal{D} \in \mathsf{M}, (\mathrm{img}\, h') \nsubseteq D^{\mathcal{D}}\}$, in which case, for every $i \in I$, we set $h_i \triangleq \pi_0(i)$, and so $\mathcal{C}_i \triangleq (\pi_1(i)\restriction(\mathrm{img}\, h_i))$ is a consistent {truth-non-empty} submatrix of $\pi_1(i) \in \mathsf{M}$. Clearly, $X = \mathrm{Cn}^\alpha_{\mathsf{M}}(X) = (\mathrm{Fm}^\alpha_\Sigma \cap \bigcap_{i\in I} h_i^{-1}[D^{\mathcal{C}_i}])$. Therefore, the mapping $g \triangleq (\prod_{i\in I} h_i) : \mathrm{Fm}^\alpha_\Sigma \to (\prod_{i\in I} C_i)$ is a strict homomorphism from $\mathcal{B}$ to $\prod_{i\in I} \mathcal{C}_i$ such that, for each $i \in I$, $(\pi_i \circ g) = h_i$, in which case $\pi_i[g[\mathrm{Fm}^\alpha_\Sigma]] = h_i[\mathrm{Fm}^\alpha_\Sigma] = C_i$, and so g is a surjective strict homomorphism from $\mathcal{B}$ onto the subdirect product $\mathcal{E} \triangleq ((\prod_{i\in I} \mathcal{C}_i)\restriction(\mathrm{img}\, g))$ of $\overline{\mathcal{C}}$. Put $\theta \triangleq \eth(\mathcal{A})(= \Delta_A)$ and $\mathcal{F} \triangleq (\mathcal{A}/\theta)$. Then, $f \triangleq (\nu_\theta \circ h) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{B}, \mathcal{F})$. Therefore, by Remark 2.2, Proposition 2.5 and Corollary 2.6, we have $(\ker g) = g^{-1}[\Delta_E] \subseteq \eth(\mathcal{B}) = f^{-1}[\Delta_F] = (\ker f)$, in which case, by Remark 2.4, $e \triangleq (f \circ g^{-1}) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{E}, \mathcal{F})$ (and so $(\nu_\theta^{-1} \circ e) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{E}, \mathcal{A})$), as required. □

Given a class M of Σ-matrices, the class of all (truth-non-empty) [consistent] submatrices of members of M is denoted by $\mathbf{S}^{(*)}_{[*]}(\mathsf{M})$. Likewise, the class of all [sub]direct products of tuples (of cardinality $\in K \subseteq \infty$) constituted by members of M is denoted by $\mathbf{P}^{[\mathrm{SD}]}_{(K)}(\mathsf{M})$. Clearly, model classes are closed under $\mathbf{P}$.

THEOREM 2.8. *Let* K *and* M *be classes of* Σ*-matrices,* C *the logic of* M *and* C' *an extension of* C*. Suppose (both* M *and all members of it are finite and)* $[\Re](\mathbf{P}^{\mathrm{SD}}_{(\omega)}(\mathbf{S}_*(\mathsf{M}))) \subseteq \mathsf{K}$ *{in particular,* $[\Re](\mathbf{S}(\mathbf{P}_{(\omega)}(\mathsf{M}))) \subseteq \mathsf{K}$ *⟨in particular,* $\mathsf{K} \supseteq \mathsf{M}$ *is closed under both* $\mathbf{S}$ *and* $\mathbf{P}_{(\omega)}$ *[as well as* $\Re$*]⟩}. Then,* C' *is (finitely-)defined by* $\mathsf{S} \triangleq (\mathrm{Mod}_{[\Im]}(C') \cap \mathsf{K})$.

PROOF: Clearly, $C' \subseteq \mathrm{Cn}^\omega_{\mathsf{S}}$, for $\mathsf{S} \subseteq \mathrm{Mod}(C')$. Conversely, consider any $(\Gamma \cup \{\varphi\}) \in \wp_{(\omega)}(\mathrm{Fm}^\omega_\Sigma)$, in which case (there is some $\alpha' \in (\omega \setminus 1)$ such that $(\Gamma \cup \{\varphi\}) \subseteq \mathrm{Fm}^{\alpha'}_\Sigma$, and so) $(\Gamma \cup \{\varphi\}) \subseteq \mathrm{Fm}^\alpha_\Sigma$, where $\alpha \triangleq ((\alpha' \cap)\omega) \in \wp_{\infty\setminus 1}(\omega)$, such that $\varphi \notin C'(\Gamma)$. Then, by the structurality of C', $\langle \mathfrak{Fm}^\omega_\Sigma, C'(\Gamma)\rangle$ is a model of C' {in particular, of C}, and so is its $(\alpha+1)$-generated (and so ω-generated) submatrix $\mathcal{A} \triangleq \langle \mathfrak{Fm}^\alpha_\Sigma, C'(\Gamma) \cap \mathrm{Fm}^\alpha_\Sigma\rangle$, in view of (2.2), in which case $\varphi \notin \mathrm{Cn}^\alpha_{\mathcal{A}}(\Gamma)$, and so $\varphi \notin \mathrm{Cn}^\omega_{\mathcal{A}}(\Gamma)$, in view of (2.1). Therefore, by Lemma 2.7, there are some $\mathcal{B} \in \mathbf{P}^{\mathrm{SD}}_{(\omega)}(\mathbf{S}_*(\mathsf{M}))$, in which case $\mathcal{D} \triangleq [\Re](\mathcal{B}) \in [\Re](\mathbf{P}^{\mathrm{SD}}_{(\omega)}(\mathbf{S}_*(\mathsf{M}))) \subseteq \mathsf{K}$, and some $g \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{B}, \mathcal{A}/\partial(\mathcal{A}))$. Then, by (2.2), $\mathrm{Cn}^{\omega}_{\mathcal{D}} = \mathrm{Cn}^{\omega}_{\mathcal{A}}$, in which case [by Corollary 2.6] $\mathcal{D} \in \mathsf{S}$, and so $\varphi \notin \mathrm{Cn}^{\omega}_{\mathsf{S}}(\Gamma)$, as required. □

Corollary 2.9. Let M be a class of Σ-matrices and $\mathcal{A}$ an axiomatic Σ-calculus. Then, the axiomatic extension C' of the logic C of M relatively axiomatized by $\mathcal{A}$ is defined by $\mathbf{S}_*(\mathsf{M}) \cap \mathrm{Mod}(\mathcal{A})$.

Proof: Then, $\mathrm{Mod}(C') = (\mathrm{Mod}(C) \cap \mathrm{Mod}(\mathcal{A}))$, and so (2.2), (2.3) and Theorem 2.8 with $\mathsf{K} \triangleq \mathbf{P}^{\mathrm{SD}}_{(\omega)}(\mathbf{S}_*(\mathsf{M})) \subseteq \mathrm{Mod}(C)$, in which case $(\mathrm{Mod}(C') \cap \mathsf{K}) = (\mathrm{Mod}(\mathcal{A}) \cap \mathsf{K}) = \mathbf{P}^{\mathrm{SD}}_{(\omega)}(\mathbf{S}_*(\mathsf{M}) \cap \mathrm{Mod}(\mathcal{A}))$, complete the argument. □

Given any Σ-logic C and any $\Sigma' \subseteq \Sigma$, in which case $\mathrm{Fm}^{\alpha}_{\Sigma} \subseteq \mathrm{Fm}^{\alpha}_{\Sigma'}$ and $\hom(\mathfrak{Fm}^{\alpha}_{\Sigma'}, \mathfrak{Fm}^{\alpha}_{\Sigma'}) = \{h \restriction \mathrm{Fm}^{\alpha}_{\Sigma'} \mid h \in \hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{Fm}^{\alpha}_{\Sigma}), h[\mathrm{Fm}^{\alpha}_{\Sigma'}] \subseteq \mathrm{Fm}^{\alpha}_{\Sigma'}\}$, for all $\alpha \in \wp_{\infty\setminus 1}(\omega)$, we have the Σ'-logic C', defined by $C'(X) \triangleq (\mathrm{Fm}^{\omega}_{\Sigma'} \cap C(X))$, for all $X \subseteq \mathrm{Fm}^{\omega}_{\Sigma'}$, called the *$\Sigma'$-fragment of C*, in which case C is said to be a *(Σ-)expansion of C'*. In that case, given also any class M of Σ-matrices defining C, C' is, in its turn, defined by $\mathsf{M} \restriction \Sigma'$.

2.3.1. Classical matrices and logics

Let $\wr \in \Sigma$ be unary.

A two-valued consistent Σ-matrix $\mathcal{A}$ is said to be *$\wr$-classical*, provided, for all $a \in A$, $(a \in D^{\mathcal{A}}) \Leftrightarrow (\wr^{\mathfrak{A}} a \notin D^{\mathcal{A}})$, in which case it is truth-non-empty, and so both false- and truth-singular, but is not $\wr$-paraconsistent.

A Σ-logic is said to be *$\wr$-[sub]classical*, whenever it is [a sublogic of] the logic of a $\wr$-classical Σ-matrix.

3. Preliminary key issues

3.1. Equality determinants

According to [13], an *equality determinant for* a Σ-matrix $\mathcal{A}$ is any $\Upsilon \subseteq \mathrm{Fm}^{1}_{\Sigma}$ such that any $a, b \in A$ are equal, whenever, for all $\upsilon \in \Upsilon$, $\upsilon^{\mathfrak{A}}(a) \in D^{\mathcal{A}}$ iff $\upsilon^{\mathfrak{A}}(b) \in D^{\mathcal{A}}$.

Example 3.1. $\{x_0\}$ is an equality determinant for any consistent truth-non-empty two-valued (in particular, classical) matrix. □

LEMMA 3.2. *Let $\mathcal{A}$ be a Σ-matrix and Υ an equality determinant for $\mathcal{A}$. Then, $\mathcal{A}$ is simple.*

PROOF: Consider any $\theta \in \mathrm{Con}(\mathcal{A})$ and any $\langle a, b\rangle \in \theta$. Then, for each $\upsilon \in \Upsilon$, $\upsilon^{\mathfrak{A}}(a)\ \theta\ \upsilon^{\mathfrak{A}}(b)$, in which case $(\upsilon^{\mathfrak{A}}(a) \in D^{\mathcal{A}}) \Leftrightarrow (\upsilon^{\mathfrak{A}}(b) \in D^{\mathcal{A}})$, and so $a = b$, as required. $\square$

LEMMA 3.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be Σ-matrices, Υ an equality determinant for $\mathcal{B}$ and $e \in \mathrm{hom_S}(\mathcal{A}, \mathcal{B})$. Suppose e is injective. Then, Υ is an equality determinant for $\mathcal{A}$.*

PROOF: In that case, for all $a \in A$ and every $\upsilon \in \Upsilon$, it holds that $\upsilon^{\mathfrak{A}}(a) \in D^{\mathcal{A}}$ iff $\upsilon^{\mathfrak{B}}(e(a)) = e(\upsilon^{\mathfrak{A}}(a)) \in D^{\mathcal{B}}$, and so the injectivity of e completes the argument. $\square$

3.2. Implicative matrices with equality determinant

Let $\diamond$ and $\veebar$ be (possibly, secondary) binary connectives of Σ.

A Σ-matrix $\mathcal{A}$ is said to be $\diamond$*-implicative/-disjunctive*, provided, for all $a, b \in A$, it holds that $((a \in / \notin D^{\mathcal{A}}) \Rightarrow (b \in D^{\mathcal{A}})) \Leftrightarrow ((a \diamond^{\mathfrak{A}} b) \in D^{\mathcal{A}})$, in which case it is $\vee_\diamond$-disjunctive, where $(x_0 \vee_\diamond x_1) \triangleq ((x_0 \diamond x_1) \diamond x_1)$.

LEMMA 3.4. *Let $\mathcal{A}$ be a finite $\diamond$-implicative and $\veebar$-disjunctive (in particular, $\veebar = \vee_\diamond$) Σ-matrix with equality determinant Υ, $\mathsf{S} \subseteq \mathbf{S}(\mathcal{A})$, $n \triangleq |\mathsf{S}|$ and $\mathcal{B} \in \mathbf{S}_*(\mathcal{A})$. Suppose $\mathcal{B} \notin \mathbf{S}(\mathsf{S})$. Then, there is some Σ-axiom in Fm_Σ^{n+1}, which is true in S but is not true in $\mathcal{B}$.*

PROOF: Take any bijection $\overline{\mathcal{C}} : n \to \mathsf{S}$. Consider any $i \in n$, in which case $B \nsubseteq C_i$, and so there is some $a_i \in (B \setminus C_i) \neq \varnothing$. Define a $\psi_i \in \mathrm{Fm}_\Sigma^2$ as follows. Take any bijection $\bar{c} : m \triangleq |C_i| \to C_i$. By induction on any $j \in (m+1)$, define a $\phi_j \in \mathrm{Fm}_\Sigma^2$ such that, for all $b \in (A \setminus D^{\mathcal{A}})$, it holds that $\phi_j^{\mathfrak{A}}[x_0/a_i, x_1/b] \notin D^{\mathcal{A}}$, while, providing $x_1 \in \mathrm{Var}(\phi_j)$, for all $a \in A$ and all $d \in D^{\mathcal{A}}$, it holds that $\phi_j^{\mathfrak{A}}[x_0/a, x_1/d] \in D^{\mathcal{A}}$, whereas, for all $k \in j$ and all $a \in A$, it holds that $\phi_j^{\mathfrak{A}}[x_0/c_k, x_1/a] \in D^{\mathcal{A}}$, as follows. First, put $\phi_j \triangleq x_1$, if $j = 0$. Otherwise, $(j-1) \in m \subseteq (m+1)$, in which case $c_{j-1} \neq a_i$, for $c_{j-1} \in C_i \not\ni a_i$, and so there is some $\upsilon \in \Upsilon$ such that $\upsilon^{\mathfrak{A}}(a_i) \in D^{\mathcal{A}}$ iff $\upsilon^{\mathfrak{A}}(c_{j-1}) \notin D^{\mathcal{A}}$. Then, set:

$$\phi_j \triangleq \begin{cases} \upsilon \diamond \phi_{j-1} & \text{if } \upsilon^{\mathfrak{A}}(a_i) \in D^{\mathcal{A}}, \\ & \exists a \in A : \phi^{\mathfrak{A}}_{j-1}[x_0/c_{j-1}, x_1/a] \notin D^{\mathcal{A}}, \\ \upsilon \veebar \phi_{j-1} & \text{if } x_1 \notin \mathrm{Var}(\phi_{j-1}), \upsilon^{\mathfrak{A}}(a_i) \notin D^{\mathcal{A}}, \\ & \exists a \in A : \phi^{\mathfrak{A}}_{j-1}[x_0/c_{j-1}, x_1/a] \notin D^{\mathcal{A}}, \\ \varphi_{j-1}[x_1/\upsilon] & \text{if } x_1 \in \mathrm{Var}(\phi_{j-1}), \upsilon^{\mathfrak{A}}(a_i) \notin D^{\mathcal{A}}, \\ & \exists a \in A : \phi^{\mathfrak{A}}_{j-1}[x_0/c_{j-1}, x_1/a] \notin D^{\mathcal{A}}, \\ \phi_{j-1} & \text{otherwise.} \end{cases}$$

In this way, $\psi_i \triangleq \phi_m \in \mathrm{Fm}^2_\Sigma$ is true in $\mathcal{C}_i$, while, for all $b \in (A \setminus D^{\mathcal{A}})$, it holds that $\psi^{\mathfrak{A}}_i[x_0/a_i, x_1/b] \notin D^{\mathcal{A}}$, whereas, providing $x_1 \in \mathrm{Var}(\psi_i)$, for all $a \in A$ and all $d \in D^{\mathcal{A}}$, it holds that $\psi^{\mathfrak{A}}_i[x_0/a, x_1/d] \in D^{\mathcal{A}}$. Finally, by induction on any $l \in (n+1)$, define a $\varphi_l \in \mathrm{Fm}^{l+1}_\Sigma$ such that for all $b \in (A \setminus D^{\mathcal{A}})$, it holds that $\varphi^{\mathfrak{A}}_l[x_{k+1}/a_k, x_0/b]_{k \in l} \notin D^{\mathcal{A}}$, while, providing $x_0 \in \mathrm{Var}(\varphi_l)$, for all $\bar{c} \in A^l$ and all $d \in D^{\mathcal{A}}$, it holds that $\varphi^{\mathfrak{A}}_l[x_0/d, x_{k+1}/c_k]_{k \in l} \in D^{\mathcal{A}}$, whereas, for all $k \in l$, $\mathcal{C}_k \models \varphi_l$, as follows. First, put $\varphi_l \triangleq x_0$, if $l = 0$. Otherwise, $(l-1) \in n \subseteq (n+1)$, so set:

$$\varphi_l \triangleq \begin{cases} \psi_{l-1}[x_1/\varphi_{l-1}, x_0/x_l] & \text{if } x_1 \in \mathrm{Var}(\psi_{l-1}), \mathcal{C}_{l-1} \not\models \varphi_{l-1}, \\ \varphi_{l-1}[x_0/(\psi_{l-1}[x_0/x_l])] & \text{if } x_0 \in \mathrm{Var}(\varphi_{l-1}), \\ & x_1 \notin \mathrm{Var}(\psi_{l-1}), \mathcal{C}_{l-1} \not\models \varphi_{l-1}, \\ \varphi_{l-1} \veebar (\psi_{l-1}[x_0/x_l]) & \text{if } x_0 \notin \mathrm{Var}(\varphi_{l-1}), \\ & x_1 \notin \mathrm{Var}(\psi_{l-1}), \mathcal{C}_{l-1} \not\models \varphi_{l-1}, \\ \varphi_{l-1} & \text{otherwise.} \end{cases}$$

Thus, $\varphi_n \in \mathrm{Fm}^{n+1}_\Sigma$ is true in S but $\mathcal{B} \not\models \varphi_n[x_{i+1}/a_i; x_0/b]_{i \in n}$, where $b \in (B \setminus D^{\mathcal{A}}) \neq \varnothing$, for $\mathcal{B}$ is consistent, as required. □

Since model classes are closed under $\mathbf{S}$ (cf. (2.2)), while any axiomatic extension of a logic is relatively axiomatized by the set of all its theorems, whereas lower cones sets are closed under intersections and unions, combining Corollary 2.9 and Lemma 3.4, we eventually get:

THEOREM 3.5. *Let $\mathcal{A}$ be a finite $\diamond$-implicative Σ-matrix with equality determinant and $\mathsf{S} \triangleq \mathbf{S}_*(\mathcal{A})$. Then, the mappings:*

$$\begin{aligned} C &\mapsto (\mathrm{Mod}(C) \cap \mathsf{S}) = (\mathrm{Mod}(C(\varnothing)) \cap \mathsf{S}), \\ \mathsf{C} &\mapsto \mathrm{Cn}^{\omega}_{\mathsf{C}} \end{aligned}$$

are inverse to one another dual isomorphisms between the lattices of all axiomatic extensions of the logic of $\mathcal{A}$ and of all lower cones of S (under identification of submatrices of $\mathcal{A}$ with the carriers of their underlying algebras), corresponding axiomatic extensions of the logic of $\mathcal{A}$ and lower cones of S having same axiomatic relative axiomatizations, both lattices being distributive. Moreover, for every $\mathsf{M} \subseteq \mathsf{S}$, the logic of M is the axiomatic extension of the logic of $\mathcal{A}$ corresponding to $\mathbf{S}_*(\mathsf{M})$.

It is remarkable that the proof of Lemma 3.4 is constructive, so, in case Σ is finite, it collectively with Theorem 3.5 yield an effective procedure of finding the lattice of axiomatic extensions of the logic of $\mathcal{A}$ collectively with their finite relative axiomatizations and finite anti-chain matrix semantics. In this connection, we should like to highlight that the effective procedure of finding relative axiomatizations of axiomatic extensions to be extracted from the constructive proof of Lemma 3.4 is definitely and obviously much less computationally complex than the straightforward one of direct search among all finite sets of formulas.

3.3. Distributive and De Morgan lattices

Let $\Sigma_{+[01]} \triangleq (\{\wedge, \vee\}[\cup\{\bot, \top\}])$ be the [bounded] lattice signature with binary $\wedge$ (conjunction) and $\vee$ (disjunction) [as well as nullary $\bot$ and $\top$ (falsehood/zero and truth/unit constants, respectively)].

Then, given any Σ-algebra $\mathfrak{A}$ such that $\Sigma_+ \subseteq \Sigma$ and $\mathfrak{A}{\restriction}\Sigma_+$ is a lattice, the partial ordering of $\mathfrak{A}{\restriction}\Sigma_+$ is denoted by $\leqslant^{\mathfrak{A}}$.

Given any $n \in (\omega \setminus 1)$, by $\mathfrak{D}_{n[01]}$ we denote the [bounded] distributive lattice given by the chain n ordered by the natural ordering.

We also deal with the signature $\Sigma_{\sim[01]} \triangleq (\Sigma_{+[01]} \cup \{\sim\})$ with unary $\sim$ (weak negation).

A *[bounded] De Morgan lattice* (cf. [11]; bounded De Morgan lattices are also traditionally called *De Morgan algebras* - cf., e.g., [2]) is any $\Sigma_{\sim[01]}$-algebra $\mathfrak{A}$ such that $\mathfrak{A}{\restriction}\Sigma_{+[01]}$ is a [bounded] distributive lattice (cf. [2]) and the following $\Sigma_\sim$-identities are true in $\mathfrak{A}$:

$$\begin{aligned} {\sim}{\sim}x_0 &\approx x_0, && (3.1)\\ {\sim}(x_0 \vee x_1) &\approx {\sim}x_0 \wedge {\sim}x_1, && (3.2) \end{aligned}$$

the variety of all them being denoted by $[\mathsf{B}]\mathsf{DML}$.

By $\mathfrak{DM}_{4[01]}$ we denote the [bounded] De Morgan lattice such that $(\mathfrak{DM}_{4[01]}{\restriction}\Sigma_{+[01]}) \triangleq \mathfrak{D}^2_{2[01]}$ and $\sim^{\mathfrak{DM}_{4[01]}}\vec{a} \triangleq \langle 1-a_{1-i}\rangle_{i\in 2}$, for all $\vec{a}\in 2^2$. In this connection, we use the following abbreviations going back to [3]:

$$\mathsf{t} \triangleq \langle 1,1\rangle, \qquad \mathsf{f} \triangleq \langle 0,0\rangle, \qquad \mathsf{b} \triangleq \langle 1,0\rangle, \qquad \mathsf{n} \triangleq \langle 0,1\rangle.$$

In addition, set $\mu : 2^2 \to 2^2, \langle a,b\rangle \mapsto \langle b,a\rangle$. Finally, an n-ary operation f on $B \subseteq 2^2$, where $n\in\omega$, is said to be *regular*, provided it is monotonic with respect to the partial ordering $\sqsubseteq$ on 2^2 defined by $(\vec{a} \sqsubseteq \vec{b}) \stackrel{\text{def}}{\Longleftrightarrow} ((a_0 \leqslant b_0)\&(b_1 \leqslant a_1))$, for all $\vec{a},\vec{b}\in 2^2$, in the sense that, for all $\bar{a},\bar{b}\in B^n$ such that $a_i \sqsubseteq b_i$, for each $i\in n$, it holds that $f(\bar{a}) \sqsubseteq f(\bar{b})$.

Remark 3.6. Clearly, $\{\mathsf{b},\mathsf{t}\}$ is a prime filter of $\mathfrak{D}^2_2$, in which case, in particular, $\mathcal{DM}_{4[01]} \triangleq \langle \mathfrak{DM}_{4[01]}, \{\mathsf{b},\mathsf{t}\}\rangle$ is $\wedge$-conjunctive and $\vee$-disjunctive. Moreover, $\{x_0, \sim x_0\}$ is an equality determinant for it. □

Recall also the following well-known algebraic fact:

LEMMA 3.7. *Let $\mathfrak{B}$ be a subalgebra of $\mathfrak{DM}_4$. Then,* $\mathrm{Con}(\mathfrak{B}) \subseteq \{\Delta_B, B^2\}$.

THEOREM 3.8. *Let $\mathfrak{A}$ be a $\Sigma_\sim$-algebra and $(\mathcal{H}\cup\{h\}) \in \wp_\omega(\hom(\mathfrak{A}, \mathfrak{DM}_4))$. Suppose $(\bigcap\{\ker g \mid g\in\mathcal{H}\}) \subseteq (\ker h) \neq A^2$. Then, $(\ker h) = (\ker g)$, for some $g\in\mathcal{H}$.*

PROOF: In that case, combining Lemma 11 and Claim on p. 300 (inside the proof of Lemma 10) of [13] with Remark 3.6, we first conclude that $(\ker g) \subseteq (\ker h)$, for some $g\in\mathcal{H}$, in which case g is a surjective homomorphism from $\mathfrak{A}$ onto the subalgebra $\mathfrak{B} \triangleq (\mathfrak{DM}_4{\restriction}(\mathrm{img}\, g))$ of $\mathfrak{DM}_4$, and so, by the Algebra Homomorphism Theorem, $f \triangleq (h\circ g^{-1}) \in \hom(\mathfrak{B},\mathfrak{DM}_4)$. Hence, by Lemma 2.1, $(\ker f)\in\mathrm{Con}(\mathfrak{B})$. Moreover, $(\ker f)\neq B^2$, for $(\ker h)\neq A^2$. Therefore, by Lemma 3.7, f is injective. Thus, $(\ker h)\subseteq(\ker g)$, as required. □

4. Main results

Fix any language $\Sigma \supseteq \Sigma_{\sim[01]}$ such that either $\{\bot,\top\}\subseteq\Sigma$ or $(\{\bot,\top\}\cap\Sigma)=\varnothing$ and any Σ-algebra $\mathfrak{A}$ such that $(\mathfrak{A}{\restriction}\Sigma_{\sim[01]}) = \mathfrak{DM}_{4[01]}$. Put $\mathcal{A} \triangleq \langle\mathfrak{A},\{\mathsf{b},\mathsf{t}\}]\rangle$. Since [the bounded version of] Dunn-Belnap's four-valued logic [5] (cf. [3]), denoted by $C_{[\mathrm{B}]\mathrm{DB}}$ from now on, is defined by $\mathcal{DM}_{4[01]} = (\mathcal{A}{\restriction}\Sigma_{\sim[01]})$ (cf. [9]), the logic C of $\mathcal{A}$ is a four-valued expansion of $C_{[\mathrm{B}]\mathrm{DB}}$.

A subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ is said to be *specular*, whenever $(\mu{\restriction}B) \in \hom(\mathfrak{B}, \mathfrak{A})$. Likewise, it is said to be *regular*, whenever its primary operations are so, in which case its secondary ones are so as well. (Clearly, $\mathfrak{B}$ is specular/regular, whenever $\mathfrak{A}$ is so. Moreover, $\mathfrak{DM}_{4[01]}$ is both specular and regular.)

4.1. Characteristic matrix expansions

LEMMA 4.1. *Let I be a set, $\overline{\mathcal{C}} \in \mathbf{S}(\mathcal{A})^I$, $\mathcal{B}$ a Σ-matrix and e an embedding of $\mathcal{B}$ into $\prod_{i\in I} \mathcal{C}_i$. Suppose $\{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, $\{I \times \{d\} \mid d \in \{\mathsf{f}, \mathsf{t}\}\} \subseteq e[B]$ and, for each $i \in I$, both $\{\mathsf{f}, \mathsf{b}, \mathsf{t}\} \cup C_i$ forms a regular subalgebra of $\mathfrak{A}$ and either $\mathsf{n} \notin C_i$ or $\mathfrak{A}{\restriction}\{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$ is specular. Then, $(B\dot{+}2) \triangleq ((B \times \{\mathsf{b}\}) \cup \{\langle e^{-1}(I \times \{d\}), d\rangle \mid d \in \{\mathsf{f}, \mathsf{t}\}\})$ forms a subalgebra of $\mathfrak{B} \times (\mathfrak{A}{\restriction}\{\mathsf{f}, \mathsf{b}, \mathsf{t}\})$, in which case $\pi_0{\restriction}(B\dot{+}2)$ is a surjective strict homomorphism from $(\mathcal{B}\dot{+}2) \triangleq ((\mathcal{B} \times (\mathcal{A}{\restriction}\{\mathsf{f}, \mathsf{b}, \mathsf{t}\})){\restriction}(B\dot{+}2))$ onto $\mathcal{B}$.*

PROOF: Consider any $\varsigma \in \Sigma$ of arity $n \in \omega$ and any $\bar{b} \in (B\dot{+}2)^n$. In case $\varsigma^{\mathfrak{A}}(\bar{a}) = \mathsf{b}$, where $\bar{a} \triangleq (\pi_1 \circ \bar{b}) \in \{\mathsf{f}, \mathsf{b}, \mathsf{t}\}^n$, we clearly have $\varsigma^{\mathfrak{B}\times\mathfrak{A}}(\bar{b}) = \langle \varsigma^{\mathfrak{B}}(\pi_0 \circ \bar{b}), \varsigma^{\mathfrak{A}}(\bar{a})\rangle = \langle \varsigma^{\mathfrak{B}}(\pi_0 \circ \bar{b}), \mathsf{b}\rangle \in (B \times \{\mathsf{b}\}) \subseteq (B\dot{+}2)$. Otherwise, since $\{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, we have $\varsigma^{\mathfrak{A}}(\bar{a}) \in \{\mathsf{f}, \mathsf{t}\}$. Put $N \triangleq \{k \in n \mid a_k = \mathsf{b}\}$. Consider any $i \in I$. Put $\bar{c} \triangleq (\pi_i \circ e \circ \pi_0 \circ \bar{b}) \in C_i^n$. Then, for every $j \in (n \setminus N)$, it holds that $c_j = a_j \in \{\mathsf{f}, \mathsf{t}\}$. Hence, $c_j \sqsubseteq a_j$, for all $j \in n$. Therefore, by the regularity of $\mathfrak{A}{\restriction}(\{\mathsf{f}, \mathsf{b}, \mathsf{t}\} \cup C_i)$, we have $\varsigma^{\mathfrak{A}}(\bar{c}) \sqsubseteq \varsigma^{\mathfrak{A}}(\bar{a})$. Consider the following complementary cases:

1. $\mathsf{n} \in C_i$.
   Then, $\mu(a_j) \sqsubseteq c_j$, for all $j \in n$. Therefore, as, in that case, $\mathfrak{A}{\restriction}\{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$ is specular, by the regularity of $(\mathfrak{A}{\restriction}(\{\mathsf{f}, \mathsf{b}, \mathsf{t}\} \cup C_i)) = \mathfrak{A}$, we have $\varsigma^{\mathfrak{A}}(\bar{a}) = \mu(\varsigma^{\mathfrak{A}}(\bar{a})) = \varsigma^{\mathfrak{A}}(\mu \circ \bar{a}) \sqsubseteq \varsigma^{\mathfrak{A}}(\bar{c})$, and so we get $\varsigma^{\mathfrak{A}}(\bar{c}) = \varsigma^{\mathfrak{A}}(\bar{a})$.

2. $\mathsf{n} \notin C_i$.
   Then, $\varsigma^{\mathfrak{A}}(\bar{c}) \in C_i \subseteq \{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$. Therefore, since both f and t are minimal elements of the poset $\{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$ ordered by $\sqsubseteq$, we get $\varsigma^{\mathfrak{A}}(\bar{c}) = \varsigma^{\mathfrak{A}}(\bar{a})$.

Thus, in any case, we have $\varsigma^{\mathfrak{A}}(\bar{c}) = \varsigma^{\mathfrak{A}}(\bar{a})$, and so, since e is an embedding of $\mathfrak{B}$ into $\prod_{i\in I} \mathfrak{C}_i$, we get $\varsigma^{\mathfrak{B}\times\mathfrak{A}}(\bar{b}) = \langle e^{-1}(I \times \{\varsigma^{\mathfrak{A}}(\bar{a})\}), \varsigma^{\mathfrak{A}}(\bar{a})\rangle \in \{\langle e^{-1}(I \times \{d\}), d\rangle \mid d \in \{\mathsf{f}, \mathsf{t}\}\} \subseteq (B\dot{+}2)$, as required. □

Lemma 4.2. *Let $\mathcal{B}$ be a model of C. Suppose either $\{\mathsf{b}\}$ forms a subalgebra of $\mathfrak{A}$ or both $\mathfrak{A}$ is regular and $\{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$ forms a specular subalgebra of $\mathfrak{A}$ (in particular, $\Sigma = \Sigma_{\sim[01]}$), while the rule:*

$$\{x_0, {\sim}x_0\} \vdash (x_1 \vee {\sim}x_1) \tag{4.1}$$

is not true in $\mathcal{B}$. Then, there is some submatrix $\mathcal{D}$ of $\mathcal{B}$ such that $\mathcal{A}$ is isomorphic to $\Re(\mathcal{D})$.

Proof: In that case, there are some $a, b \in B$ such that (4.1) is not true in $\mathcal{B}$ under $[x_0/a, x_1/b]$. Then, in view of (2.2), the submatrix $\mathcal{E}$ of $\mathcal{B}$ generated by $\{a, b\}$ is a finitely-generated model of C, in which (4.1) is not true under $[x_0/a, x_1/b]$ as well. Hence, by Lemma 2.7 with $\mathsf{M} = \{\mathcal{A}\}$, there are some set I, some I-tuple $\overline{\mathcal{C}}$ constituted by submatrices of $\mathcal{A}$, some subdirect product $\mathcal{F}$ of $\overline{\mathcal{C}}$, in which case $(\mathfrak{F}{\restriction}\Sigma_{\sim}) \in \mathsf{DML}$, for $\mathsf{DML} \ni \mathfrak{DM}_4$ is a variety, and some $g \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{F}, \Re(\mathcal{E}))$, in which case, by (2.2), $\mathcal{F}$ is a model of C, in which case it is $\wedge$-conjunctive, for $\mathcal{A}$ is so (cf. Remark 3.6), but is not a model of (4.1), in which case there are some $c, d \in F$ such that $\{c, {\sim}^{\mathfrak{F}}c\} \subseteq D^{\mathcal{F}} \not\ni d \geqslant^{\mathfrak{F}} {\sim}^{\mathfrak{F}}d$. Then, $c = (I \times \{\mathsf{b}\})$, in which case ${\sim}^{\mathfrak{F}}c = c$, and so $(F \setminus D^{\mathcal{F}}) \ni e \triangleq ((c \wedge^{\mathfrak{F}} d) \vee^{\mathfrak{F}} {\sim}^{\mathfrak{F}}d) = {\sim}^{\mathfrak{F}}e \leqslant^{\mathfrak{F}} d$. Hence, $e \in \{\mathsf{b}, \mathsf{n}\}^I$, while $J \triangleq \{i \in I \mid \pi_i(e) = \mathsf{n}\} \neq \varnothing$. Given any $\bar{a} \in A^2$, set $(a_0|a_1) \triangleq ((J \times \{a_0\}) \cup ((I \setminus J) \times \{a_1\})) \in A^I$. In this way, we have:

$$F \ni c = (\mathsf{b}|\mathsf{b}), \tag{4.2}$$

$$F \ni e = (\mathsf{n}|\mathsf{b}), \tag{4.3}$$

$$F \ni (c \wedge^{\mathfrak{F}} e) = (\mathsf{f}|\mathsf{b}), \tag{4.4}$$

$$F \ni (c \vee^{\mathfrak{F}} e) = (\mathsf{t}|\mathsf{b}). \tag{4.5}$$

Consider the following complementary cases:

1. either $\{\mathsf{b}\}$ forms a subalgebra of $\mathfrak{A}$ or $J = I$.
   Then, by (4.2), (4.3), (4.4) and (4.5), $f \triangleq \{\langle x, (x|\mathsf{b})\rangle \mid x \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{F}$, in which case $g' \triangleq (g \circ f) \in \hom_{\mathrm{S}}(\mathcal{A}, \Re(\mathcal{E}))$, and so, by Corollary 2.3, Lemma 3.2 and Remark 3.6, g' is injective. In this way, g' is an isomorphism from $\mathcal{A}$ onto the submatrix $\mathcal{G} \triangleq (\Re(\mathcal{E}){\restriction}(\operatorname{img} g'))$ of $\Re(\mathcal{E})$, and so $h \triangleq {g'}^{-1} \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{G}, \mathcal{A})$.

2. $\{\mathsf{b}\}$ does not form a subalgebra of $\mathfrak{A}$ and $J \neq I$.
   Then, there is some $\varphi \in \mathrm{Fm}^1_{\Sigma}$ such that $\varphi^{\mathfrak{A}}(\mathsf{b}) \neq \mathsf{b}$, in which case

$\phi^{\mathfrak{A}}(\mathsf{b}) = \mathsf{f}$ and $\psi^{\mathfrak{A}}(\mathsf{b}) = \mathsf{t}$, where $\phi \triangleq (x_0 \wedge (\varphi \wedge {\sim}\varphi))$ and $\psi \triangleq (x_0 \vee (\varphi \vee {\sim}\varphi))$, and so, by (4.2), we get:

$$F \ni \phi^{\mathfrak{F}}(c) = (\mathsf{f}|\mathsf{f}), \tag{4.6}$$

$$F \ni \psi^{\mathfrak{F}}(c) = (\mathsf{t}|\mathsf{t}). \tag{4.7}$$

Moreover, in that case, both $\mathfrak{A}$ is regular and $\{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$ forms a specular subalgebra of $\mathfrak{A}$. And what is more, $e' \triangleq \{\langle a', 1 \times \{a'\}\rangle \mid a' \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{A}^1$ such that $\{1 \times \{x\} \mid x \in \{\mathsf{f}, \mathsf{t}\}\} = e'[\{\mathsf{f}, \mathsf{t}\}] \subseteq e'[A]$. In this way, Lemma 4.1 with 1, $\mathcal{A}$ and e' instead of I, $\mathcal{B}$ and e, respectively, used tacitly throughout the rest of the proof, is well-applicable to $\mathcal{A}$. Then, since $J \neq \varnothing \neq (I \setminus J)$, by (4.2), (4.3), (4.4), (4.5), (4.6) and (4.7), we see that $f \triangleq \{\langle\langle x, y\rangle, (x|y)\rangle \mid \langle x, y\rangle \in (A \dotplus 2)\}$ is an embedding of $\mathcal{H} \triangleq (\mathcal{A} \dotplus 2)$ into $\mathcal{F}$, while $h' \triangleq (\pi_0 \restriction (A \dotplus 2)) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{H}, \mathcal{A})$. Then, $g' \triangleq (g \circ f) \in \hom_{\mathrm{S}}(\mathcal{H}, \Re(\mathcal{E}))$, and so g' is a surjective strict homomorphism from $\mathcal{H}$ onto the submatrix $\mathcal{G} \triangleq (\Re(\mathcal{E}) \restriction (\operatorname{img} g'))$ of $\Re(\mathcal{E})$. And what is more, by Lemma 3.2 and Remark 3.6, $\mathcal{A}$ is simple. Hence, by Remark 2.2 and Proposition 2.5, we get $(\ker g') \subseteq \eth(\mathcal{H}) = (\ker h')$. Therefore, by Remark 2.4, $h \triangleq (h' \circ g'^{-1}) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{G}, \mathcal{A})$.

Thus, in any case, there are some submatrix $\mathcal{G}$ of $\mathcal{E}/\theta$, where $\theta \triangleq \eth(\mathcal{E})$, and some $h \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{G}, \mathcal{A})$. Then, $\mathcal{D} \triangleq (\mathcal{E} \restriction \nu_\theta^{-1}[G])$, being a submatrix of $\mathcal{E}$, is so of $\mathcal{B}$, in which case $h'' \triangleq (\nu_\theta \restriction D) \in \hom_{\mathrm{S}}(\mathcal{D}, \mathcal{G})$ is surjective, and so is $h''' \triangleq (h \circ h'') \in \hom_{\mathrm{S}}(\mathcal{D}, \mathcal{A})$. On the other hand, by Lemma 3.2 and Remark 3.6, $\mathcal{A}$ is simple. Hence, by Proposition 2.5, $\vartheta \triangleq \eth(\mathcal{D}) = (\ker h''')$. Therefore, by Remark 2.4, $\nu_\vartheta \circ h'''^{-1}$ is an isomorphism from $\mathcal{A}$ onto $\Re(\mathcal{D})$, as required. □

Corollary 4.3. Let C' be an extension of C. Suppose either $\{\mathsf{b}\}$ forms a subalgebra of $\mathfrak{A}$ or both $\mathfrak{A}$ is regular and $\{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$ forms a specular subalgebra of $\mathfrak{A}$ (in particular, $\Sigma = \Sigma_{\sim[01]}$), while the rule (4.1) is not satisfied in C'. Then, $C' = C$.

Proof: In that case, $(x_1 \vee {\sim}x_1) \notin T \triangleq C'(\{x_0, {\sim}x_0\})$, so, by the structurality of C', $\langle \mathfrak{Fm}^{\omega}_{\Sigma}, T\rangle$ is a model of C' (in particular, of C), in which (4.1) is not true under the diagonal Σ-substitution. In this way, (2.2) and Lemma 4.2 complete the argument. □

Proposition 4.4. Let M be a class of $\Sigma_\sim$-matrices. Suppose C_{DB} is defined by M. Then, there are some $\mathcal{B} \in \mathsf{M}$ and some submatrix $\mathcal{D}$ of $\mathcal{B}$ such that $\mathcal{DM}_4$ is isomorphic to $\mathcal{D}/\eth(\mathcal{D})$.

Proof: Note that the rule (4.1) is not satisfied in C_{DB}, because it is not true in $\mathcal{DM}_4$ under $[x_0/\mathsf{b}, x_1/\mathsf{n}]$. Therefore, as C_{DB} is defined by M, there is some model $\mathcal{B} \in \mathsf{M}$ of C_{DB} not being a model of (4.1), in which case Lemma 4.2 completes the argument. □

Now, we are in a position to argue several interesting corollaries of Proposition 4.4:

Corollary 4.5. Let M be a class of Σ-matrices. Suppose the logic of M is an expansion of C_{DB} (in particular, $\Sigma = \Sigma_\sim$ and the logic of M is C_{DB} itself). Then, some $\mathcal{B} \in \mathsf{M}$ is not truth-/false-singular. In particular, any four-valued expansion of C_{DB} (including C_{DB} itself) is defined by no truth-/false-singular matrix.

Proof: By contradiction. For suppose every member of M is truth-/false-singular. Then, $\mathsf{M}{\restriction}\Sigma_\sim$ is a class of truth-/false-singular $\Sigma_\sim$-matrices defining C_{DB}. Then, by Proposition 4.4, there are some $\mathcal{B} \in (\mathsf{M}{\restriction}\Sigma_\sim)$ and some submatrix $\mathcal{D}$ of $\mathcal{B}$ such that $\mathcal{DM}_4$ is isomorphic to $\mathcal{E} \triangleq (\mathcal{D}/\theta)$, where $\theta \triangleq \eth(\mathcal{D})$, in which case $\mathcal{E}$ is truth-/false-singular, for $\mathcal{D}$ is so, because $\mathcal{B}$ is so/, while $((D/\theta) \setminus (D^{\mathcal{D}}/\theta)) \subseteq ((D \setminus D^{\mathcal{D}})/\theta)$, and so is $\mathcal{DM}_4$. This contradiction completes the argument. □

Corollary 4.6. Any four-valued $\Sigma_\sim$-matrix $\mathcal{B}$ defining C_{DB} is isomorphic to $\mathcal{DM}_4$.

Proof: By Proposition 4.4, there are then some submatrix $\mathcal{D}$ of $\mathcal{B}$ and some isomorphism e from $\mathcal{DM}_4$ onto $\mathcal{D}/\theta$, where $\theta \triangleq \eth(\mathcal{D})$, in which case $4 = |DM_4| = |D/\theta| \leqslant |D| \leqslant |B| = 4$, in which case $4 = |D/\theta| = |D| = |B|$, and so ν_θ is injective, while $D = B$. In this way, $e^{-1} \circ \nu_\theta$ is an isomorphism from $\mathcal{B}$ onto $\mathcal{DM}_4$, as required. □

This, in its turn, enables us to prove:

Theorem 4.7. *Any four-valued Σ-expansion of C_{DB} is defined by a Σ-expansion of $\mathcal{DM}_4$.*

Proof: Let $\mathcal{B}$ be a four-valued Σ-matrix defining an expansion of C_{DB}. Then, $\mathcal{B}{\restriction}\Sigma_\sim$ is a four-valued $\Sigma_\sim$-matrix defining C_{DB} itself. Hence, by

Corollary 4.6, there is an isomorphism e from $\mathcal{B}{\restriction}\Sigma_\sim$ onto $\mathcal{DM}_4$. In that case, e is an isomorphism from $\mathcal{B}$ onto the Σ-expansion $\langle e[\mathfrak{B}], e[D^{\mathcal{B}}]\rangle$ of $\mathcal{DM}_4$. In this way, (2.2) completes the argument. □

Thus, the natural way of construction of four-valued expansions chosen above does exhaust *all* of them. And what is more, any of them is defined by a *unique* expansion of $\mathcal{DM}_4$, as it follows from:

THEOREM 4.8. *Let $\mathcal{B}$ be a Σ-matrix. Suppose $(\mathcal{B}{\restriction}\Sigma_\sim) = \mathcal{DM}_4$ and $\mathcal{B}$ is a model of C (in particular, C is defined by $\mathcal{B}$). Then, $\mathcal{B} = \mathcal{A}$.*

PROOF: In that case, $\mathcal{B}$, being finite, is finitely-generated. In addition, by Lemma 3.2 and Remark 3.6, it is simple. Therefore, as $\mathcal{A}$ is finite, by Lemma 2.7 with $\mathsf{M} = \{\mathcal{A}\}$, there are some finite set I, some I-tuple $\overline{\mathcal{C}}$ constituted by submatrices of $\mathcal{A}$, some subdirect product $\mathcal{D}$ of $\overline{\mathcal{C}}$ and some $g \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D}, \mathcal{B}) \subseteq \hom(\mathfrak{D}{\restriction}\Sigma_\sim, \mathfrak{DM}_4)$, in which case, as $|\operatorname{img} g| = |B| = 4 \neq 1$, $(\bigcap_{i\in I} \ker(\pi_i{\restriction}D)) = \Delta_D \subseteq (\ker g) \neq D^2$, while $\{\pi_i{\restriction}D \mid i \in I\} \in \wp_\omega(\hom(\mathfrak{D}{\restriction}\Sigma_\sim, \mathfrak{DM}_4))$, and so, by Theorem 3.8, there is some $i \in I$ such that $\ker(\pi_i{\restriction}D) = (\ker g)$. Hence, as $(\pi_i{\restriction}D) \in \hom(\mathcal{D}, \mathcal{C}_i)$, by Remark 2.4, $e \triangleq ((\pi_i{\restriction}D) \circ g^{-1}) \in \hom(\mathcal{B}, \mathcal{C}_i) \subseteq \hom(\mathcal{B}, \mathcal{A})$ is injective, in which case $e[\{\mathsf{n}, \mathsf{b}\}] \subseteq \{\mathsf{n}, \mathsf{b}\}$ and $e[\{\mathsf{f}, \mathsf{t}\}] \subseteq \{\mathsf{f}, \mathsf{t}\}$, because $\sim^{\mathfrak{DM}_4} a = a$ iff $a \in \{\mathsf{n}, \mathsf{b}\}$, for all $a \in DM_4$, and so e is diagonal, for $(D^{\mathcal{DM}_4} \cap \{\mathsf{n}, \mathsf{b}\}) = \{\mathsf{b}\}$ and $(D^{\mathcal{DM}_4} \cap \{\mathsf{f}, \mathsf{t}\}) = \{\mathsf{t}\}$. In this way, $\mathcal{B} = \mathcal{A}$, for $B = A$ and $D^{\mathcal{B}} = D^{\mathcal{A}}$, as required. □

In view of Theorem 4.8, $\mathcal{A}$ is said to be *characteristic for/of C*.

COROLLARY 4.9. Let $\Sigma' \supseteq \Sigma$ be a signature and C' a four-valued Σ'-expansion of C. Then, C' is defined by a unique Σ'-expansion of $\mathcal{A}$.

PROOF: Then, by Theorem 4.7, C' is defined by a Σ'-expansion $\mathcal{A}'$ of $\mathcal{DM}_4$, in which case C is defined by the Σ-expansion $\mathcal{A}'{\restriction}\Sigma$ of $\mathcal{DM}_4$, and so $(\mathcal{A}'{\restriction}\Sigma) = \mathcal{A}$, in view of Theorem 4.8. In this way, Theorem 4.8 completes the argument. □

4.1.1. Minimal four-valuedness

As a one more interesting consequence of Proposition 4.4, we have:

THEOREM 4.10. *Let M be a class of Σ-matrices. Suppose the logic of M is an expansion of C_{DB} (in particular, $\Sigma = \Sigma_\sim$ and the logic of M is C_{DB}*

itself). Then, $4 \leqslant |B|$, *for some* $\mathcal{B} \in \mathsf{M}$. *In particular, any four-valued expansion of* C_{DB} *(including* C_{DB} *itself) is minimally four-valued.*

PROOF: In that case, C_{DB} is defined by $\mathsf{M}{\restriction}\Sigma_{\sim}$, and so, by Proposition 4.4, there are some $\mathcal{B} \in \mathsf{M}$ and some submatrix $\mathcal{D}$ of $\mathcal{B}{\restriction}\Sigma_{\sim}$ such that $\mathcal{DM}_4$ is isomorphic to $\mathcal{D}/\theta$, where $\theta \triangleq \partial(\mathcal{D})$. In this way, $4 = |DM_4| = |D/\theta| \leqslant |D| \leqslant |B|$, as required. □

4.2. Variable sharing property

LEMMA 4.11. *C is theorem-less iff* $\{\mathsf{n}\}$ *forms a subalgebra of* $\mathfrak{A}$.

PROOF: First, assume $\{\mathsf{n}\}$ forms a subalgebra of $\mathcal{A}$, in which case $\mathcal{A}{\restriction}\{\mathsf{n}\}$ is a truth-empty submatrix of $\mathcal{A}$, and so C is theorem-less, in view of (2.2).

Conversely, assume $\{\mathsf{n}\}$ does not form a subalgebra of $\mathfrak{A}$. Then, there is some $\varphi \in \mathrm{Fm}^1_\Sigma$ such that $\varphi^{\mathfrak{A}}(\mathsf{n}) \neq \mathsf{n}$, in which case $(\varphi^{\mathfrak{A}}(\mathsf{n}) \vee^{\mathfrak{A}} {\sim}^{\mathfrak{A}}\varphi^{\mathfrak{A}}(\mathsf{n})) \in D^{\mathcal{A}}$, and so $((x_0 \vee {\sim}x_0) \vee (\varphi \vee {\sim}\varphi)) \in C(\varnothing)$, as required. □

LEMMA 4.12. *C has no inconsistent formula iff* $\{\mathsf{b}\}$ *forms a subalgebra of* $\mathfrak{A}$.

PROOF: First, assume $\{\mathsf{b}\}$ does not form a subalgebra of $\mathfrak{A}$. Then, there is some $\varphi \in \mathrm{Fm}^1_\Sigma$ such that $\varphi^{\mathfrak{A}}(\mathsf{b}) \neq \mathsf{b}$, in which case $(\varphi^{\mathfrak{A}}(\mathsf{b}) \wedge^{\mathfrak{A}} {\sim}^{\mathfrak{A}}\varphi^{\mathfrak{A}}(\mathsf{b})) \notin D^{\mathcal{A}}$, and so $((x_0 \wedge {\sim}x_0) \wedge (\varphi \wedge {\sim}\varphi))$ is an inconsistent formula of C.

Conversely, assume $\{\mathsf{b}\}$ forms a subalgebra of $\mathcal{A}$. Let us prove, by contradiction, that C has no inconsistent formula. For suppose some $\varphi \in \mathrm{Fm}^\omega_\Sigma$ is an inconsistent formula of C, in which case $\varphi \in \mathrm{Fm}^\alpha_\Sigma$, for some $\alpha \in (\omega \setminus 1)$, while $x_\alpha \in C(\varphi)$. Let $h \in \hom(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{A})$ extend $(V_\alpha \times \{\mathsf{b}\}) \cup (V_{\omega\setminus\alpha} \times \{\mathsf{f}\})$. Then, $h(\varphi) = \mathsf{b} \in D^{\mathcal{A}}$, whereas $h(x_\alpha) = \mathsf{f} \notin D^{\mathcal{A}}$. This contradiction completes the argument. □

THEOREM 4.13. *The following are equivalent:*

(i) *C satisfies VSP;*

(ii) *C has neither a theorem nor an inconsistent formula;*

(iii) *both* $\{\mathsf{n}\}$ *and* $\{\mathsf{b}\}$ *form subalgebras of* $\mathfrak{A}$.

PROOF: First, (ii) is a particular case of (i). Next, (ii)⇒(iii) is by Lemmas 4.11 and 4.12.

Finally, assume (iii) holds. Consider any $\phi, \psi \in \mathrm{Fm}^{\omega}_{\Sigma}$ such that $V \triangleq \mathrm{Var}(\phi)$ and $\mathrm{Var}(\psi)$ are disjoint. Let $h \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$ extend $(V \times \{\mathsf{b}\}) \cup ((V_\omega \setminus V) \times \{\mathsf{n}\})$. Then, $h(\phi) = \mathsf{b} \in D^{\mathcal{A}}$, whereas $h(\psi) = \mathsf{n} \notin D^{\mathcal{A}}$. Thus, $\psi \notin C(\phi)$, and so (i) holds, as required. □

Corollary 4.14 (cf. Theorem 4.2 of [9] for the case $\Sigma = \Sigma_{\sim}$). *C has no proper extension satisfying VSP.*

Proof: Consider any extension C' of C satisfying VSP, in which case C, being a sublogic of C', does so as well, and so, by Theorem 4.13(i)⇒(iii), $\{\mathsf{b}\}$ forms a subalgebra of $\mathfrak{A}$. Moreover, as C' is $\wedge$-conjunctive, for $\mathcal{A}$ is so (cf. Remark 3.6), (4.1) is not satisfied in C', for $\mathrm{Var}(x_0 \wedge {\sim} x_0) = \{x_0\}$ and $\mathrm{Var}(x_1 \vee {\sim} x_1) = \{x_1\}$ are disjoint. In this way, Corollary 4.3 completes the argument. □

Perhaps, this is the principal *specific* maximality of C in addition to the standard one studied in the next subsection.

4.3. Maximality

Lemma 4.15. *Any proper submatrix $\mathcal{B}$ of $\mathcal{A}$ defines a proper extension C' of C.*

Proof: For consider the following complementary cases:

1. $\mathsf{b} \in B$.
   Then, $\mathsf{n} \notin B$, for $B \neq A$, while $(\mathsf{n} \wedge^{\mathfrak{B}} \mathsf{b}) = \mathsf{f}$, whereas $(\mathsf{n} \vee^{\mathfrak{B}} \mathsf{b}) = \mathsf{t}$. In that case, $(x_0 \vee {\sim} x_0) \in (C'(\varnothing) \setminus C(\varnothing))$.
2. $\mathsf{b} \notin B$.
   Then, $\mathcal{B}$ is not $\sim$-paraconsistent, as opposed to $\mathcal{A}$, and so is C', as opposed to C.

Thus, in any case, $C' \neq C$, as required, in view of (2.2). □

Clearly, $\mathcal{A}$ is consistent (and truth-non-empty), and so C is (inferentially) consistent. In this connection, we have:

Theorem 4.16. *C is [inferentially] maximal iff $\mathcal{A}$ has no proper consistent [truth-non-empty] submatrix.*

Proof: First, consider any proper consistent [truth-non-empty] submatrix $\mathcal{B}$ of $\mathcal{A}$. Then, by Lemma 4.15, the logic C' of $\mathcal{B}$ is a[n inferentially] consistent proper extension of C, and so C is not [inferentially] maximal.

Conversely, assume $\mathcal{A}$ has no proper consistent [truth-non-empty] submatrix. Consider any [inferentially] consistent extension C' of C. Then, $x_0 \notin T \triangleq C'(\varnothing[\cup\{x_1\}])[\ni x_1]$, while, by the structurality of C', $\langle \mathfrak{Fm}_\Sigma^\omega, T\rangle$ is a model of C' (in particular, of C), and so is its consistent [truth-non-empty] finitely-generated submatrix $\mathcal{B} = \langle \mathfrak{Fm}_\Sigma^2, \mathrm{Fm}_\Sigma^2 \cap T\rangle$, in view of (2.2). Hence, by Lemma 2.7 with $\mathsf{M} = \{\mathcal{A}\}$, there are some finite set I, some I-tuple $\overline{\mathcal{C}}$ constituted by consistent [truth-non-empty] submatrices of $\mathcal{A}$, some subdirect product $\mathcal{D}$ of $\overline{\mathcal{C}}$, and some $g \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{D}, \mathcal{B}/\partial(\mathcal{B}))$, in which case, by (2.2), $\mathcal{D}$ is a consistent model of C', and so, in particular, $I \neq \varnothing$. Moreover, for any $i \in I$, as $\mathcal{C}_i$ is consistent [and truth-non-empty] submatrix of $\mathcal{A}$, $\mathcal{C}_i = \mathcal{A}$ is truth non-empty anyway. Hence, by the following claim, both $D \ni a \triangleq (I \times \{\mathsf{f}\})$ and $D \ni b \triangleq (I \times \{\mathsf{t}\})$:

Claim 4.17. Let I be a finite set, $\overline{\mathcal{C}} \in \mathbf{S}_*^*(\mathcal{A})^I$ and $\mathcal{B}$ a subdirect product of $\overline{\mathcal{C}}$. Then, $\{I \times \{\mathsf{f}\}, I \times \{\mathsf{t}\}\} \subseteq B$.

Proof: In that case, $\mathfrak{B}{\restriction}\Sigma_+$ is a finite lattice, so it has both a zero a and a unit b. Consider any $i \in I$. Then, as $\mathcal{C}_i$ is both consistent and truth-non-empty, by the following claim, we have $\{\mathsf{f}, \mathsf{t}\} \subseteq C_i$:

Claim 4.18. Let $\mathcal{D} \in \mathbf{S}_*^*(\mathcal{A})$. Then, $\{\mathsf{f}, \mathsf{t}\} \subseteq D$.

Proof: In that case, we have $(\{\mathsf{f}, \mathsf{n}\} \cap D) \neq \varnothing \neq (\{\mathsf{b}, \mathsf{t}\} \cap D)$. In this way, the fact that $(\mathsf{n} \wedge^{\mathfrak{A}} \mathsf{b}) = \mathsf{f}$, while $\sim^{\mathfrak{A}}\mathsf{f} = \mathsf{t}$, whereas $\sim^{\mathfrak{A}}\mathsf{t} = \mathsf{f}$, completes the argument. $\Box$

Therefore, since $\pi_i[B] = C_i$, there are some $c, d \in B$, such that $\pi_i(c) = \mathsf{f}$ and $\pi_i(d) = \mathsf{t}$, in which case we have $(c \wedge^{\mathfrak{B}} a) = a$ and $(d \vee^{\mathfrak{D}} b) = b$, and so, as $(\pi_i{\restriction}B) \in \hom(\mathfrak{B}{\restriction}\Sigma_+, \mathfrak{C}_i{\restriction}\Sigma_+)$, we eventually get $\pi_i(a) = (\mathsf{f} \wedge^{\mathfrak{A}} \pi_i(a)) = \mathsf{f}$ and $\pi_i(b) = (\mathsf{t} \vee^{\mathfrak{A}} \pi_i(b)) = \mathsf{t}$. Thus, $B \ni a = (I \times \{\mathsf{f}\})$ and $B \ni b = (I \times \{\mathsf{t}\})$, as required. $\Box$

Next, if $\{\mathsf{f}, \mathsf{t}\} \subsetneq A$ [distinct from $\{\mathsf{n}\}$] did form a subalgebra of $\mathfrak{A}$, $\mathcal{A}{\restriction}\{\mathsf{f}, \mathsf{t}\}$ would be a proper consistent [truth-non-empty] submatrix of $\mathcal{A}$. Therefore, there are some $\phi \in \mathrm{Fm}_\Sigma^2$ and $j \in 2$ such that $\phi^{\mathfrak{A}}(\mathsf{f}, \mathsf{t}) = \langle j, 1-j\rangle$. Likewise, if $\{\mathsf{f}, \langle j, 1-j\rangle, \mathsf{t}\} \subsetneq A$ [distinct from $\{\mathsf{n}\}$] did form a subalgebra of $\mathfrak{A}$, $\mathcal{A}{\restriction}\{\mathsf{f}, \langle j, 1-j\rangle, \mathsf{t}\}$ would be a proper consistent [truth-non-empty] submatrix of $\mathcal{A}$. Therefore, there is some $\psi \in \mathrm{Fm}_\Sigma^3$ such that $\psi^{\mathfrak{A}}(\mathsf{f}, \langle j, 1-j\rangle, \mathsf{t}) = \langle 1-j, j\rangle$. In this way, $\{\phi^{\mathfrak{A}}(\mathsf{f}, \mathsf{t}), \psi^{\mathfrak{A}}(\mathsf{f}, \phi^{\mathfrak{A}}(\mathsf{f}, \mathsf{t}), \mathsf{t})\} = \{\mathsf{n}, \mathsf{b}\}$. Then, $D \supseteq \{\phi^{\mathfrak{D}}(a, b), \psi^{\mathfrak{D}}(a, \phi^{\mathfrak{D}}(a, b), b)\} = \{I \times \{\mathsf{n}\}, I \times \{\mathsf{b}\}\}$. Thus,

$\{I \times \{c\} \mid c \in A\} \subseteq D$. Hence, as $I \neq \varnothing$, $\{\langle c, I \times \{c\}\rangle \mid c \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{D}$, in which case, by (2.2), C is an extension of C', and so $C' = C$, as required. $\Box$

4.4. Subclassical expansions

LEMMA 4.19. *Let $\mathcal{B}$ be a (simple) finitely-generated consistent truth-non-empty model of C. Then, the following hold:*

(i) *$\mathcal{B}$ is $\sim$-paraconsistent, if $\sim(x_0 \wedge \sim x_0)$ is true in $\mathcal{B}$ and $\{\mathsf{f}, \mathsf{t}\}$ does not form a subalgebra of $\mathfrak{A}$;*

(ii) *$\mathcal{A}{\upharpoonright}\{\mathsf{f}, \mathsf{t}\}$ is embeddable into $\mathcal{B}/\partial(\mathcal{B})$ (resp., into $\mathcal{B}$ itself), if $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$.*

PROOF: Put $\mathcal{E} \triangleq (\mathcal{B}/\partial(\mathcal{B}))$ (resp., $\mathcal{E} \triangleq \mathcal{B}$). Then, by Lemma 2.7 with $\mathsf{M} = \{\mathcal{A}\}$, there are some finite set I, some I-tuple $\overline{\mathcal{C}}$ constituted by consistent truth-non-empty submatrices of $\mathcal{A}$, some subdirect product $\mathcal{D}$ of $\overline{\mathcal{C}}$ and some $g \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{D}, \mathcal{E})$, in which case, by (2.2), $\mathcal{D}$ is consistent, and so, in particular, $I \neq \varnothing$. Hence, by Claim 4.17, both $D \ni a \triangleq (I \times \{\mathsf{f}\})$ and $D \ni b \triangleq (I \times \{\mathsf{t}\})$. Consider the following respective cases:

(i) $\sim(x_0 \wedge \sim x_0)$ is true in $\mathcal{B}$ and $\{\mathsf{f}, \mathsf{t}\}$ does not form a subalgebra of $\mathfrak{A}$. Then, there is some $\varphi \in \mathrm{Fm}^2_\Sigma$ such that $\varphi^{\mathfrak{A}}(\mathsf{f}, \mathsf{t}) \in \{\mathsf{n}, \mathsf{b}\}$. Take any $i \in I \neq \varnothing$. Then, $\{\mathsf{f}, \mathsf{t}\} = \pi_i[\{a, b\}] \subseteq C_i$. Moreover, $(\pi_i{\upharpoonright}D) \in \hom^{\mathrm{S}}(\mathcal{D}, \mathcal{C}_i)$, in which case, by (2.2) and (2.3), $\mathcal{C}_i$ is a model of $\sim(x_0 \wedge \sim x_0)$, and so $\mathsf{n} \notin C_i$, for $\sim^{\mathfrak{A}}(\mathsf{n} \wedge^{\mathfrak{A}} \sim^{\mathfrak{A}} \mathsf{n}) = \mathsf{n} \notin D^{\mathcal{A}}$. And what is more, $\mathfrak{C}_i$ is a subalgebra of $\mathfrak{A}$. Hence, $\varphi^{\mathfrak{A}}(\mathsf{f}, \mathsf{t}) \in C_i$, and so $\varphi^{\mathfrak{A}}(\mathsf{f}, \mathsf{t}) = \mathsf{b}$, for $\mathsf{n} \notin C_i$. Then, $D \ni c \triangleq \varphi^{\mathfrak{D}}(a, b) = (I \times \{\mathsf{b}\})$, in which case $\sim^{\mathfrak{D}} c = c \in D^{\mathcal{D}}$, and so $\mathcal{D}$, being consistent, is $\sim$-paraconsistent, and so is $\mathcal{B}$, in view of (2.2), as required.

(ii) $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$.
Then, $\mathcal{F} \triangleq (\mathcal{A}{\upharpoonright}\{\mathsf{f}, \mathsf{t}\})$ is $\sim$-classical, and so simple, in view of Example 3.1 and Lemma 3.2. Finally, as $\{I \times \{d\} \mid d \in F\} \subseteq D$ and $I \neq \varnothing$, $e \triangleq \{\langle d, I \times \{d\}\rangle \mid d \in F\}$ is an embedding of $\mathcal{F}$ into $\mathcal{D}$, in which case, $(g \circ e) \in \hom_{\mathrm{S}}(\mathcal{F}, \mathcal{E})$, and so Corollary 2.3 completes the argument. $\Box$

Theorem 4.20. *C is $\sim$-subclassical iff $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, in which case $\mathcal{A}{\upharpoonright}\{\mathsf{f}, \mathsf{t}\}$ is isomorphic to any $\sim$-classical model of C, and so its logic is the only $\sim$-classical extension of C.*

Proof: Let $\mathcal{B}$ be a $\sim$-classical model of C, in which case it is simple (cf. Example 3.1 and Lemma 3.2) and finite (in particular, finitely-generated) but is not $\sim$-paraconsistent.

First, consider any $a \in B$. Then, $\{a, \sim^{\mathfrak{B}} a\} \nsubseteq D^{\mathcal{B}}$, for $\mathcal{B}$ is $\sim$-classical, in which case $(a \wedge^{\mathfrak{B}} \sim^{\mathfrak{B}} a) \notin D^{\mathcal{B}}$, for $\mathcal{B}$ is $\wedge$-conjunctive, because C is so, since $\mathcal{A}$ is so (cf. Remark 3.6), and so $\sim^{\mathfrak{B}}(a \wedge^{\mathfrak{B}} \sim^{\mathfrak{B}} a) \in D^{\mathcal{B}}$, for $\mathcal{B}$ is $\sim$-classical. Thus, $\sim(x_0 \wedge \sim x_0)$ is true in $\mathcal{B}$. Hence, by Lemma 4.19(i), $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$.

Conversely, assume $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, in which case $\mathcal{D} \triangleq (\mathcal{A}{\upharpoonright}\{\mathsf{f}, \mathsf{t}\})$ is a $\sim$-classical model of C, by (2.2), and embeddable into $\mathcal{B}$, by Lemma 4.19(ii), so is isomorphic to $\mathcal{B}$, for $|D| = 2 = |B|$. Then, (2.2) completes the argument. □

In view of Theorem 4.20, the unique $\sim$-classical extension of a $\sim$-subclassical four-valued expansion $\boldsymbol{C}$ of C_{DB} is said to be *characteristic for $\boldsymbol{C}$* and denoted by $\boldsymbol{C}^{\mathrm{PC}}$. Its *specific* maximality feature is as follows:

Theorem 4.21. *Let C' be an inferentially consistent extension of C. Suppose $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$. Then, $\mathcal{A}{\upharpoonright}\{\mathsf{f}, \mathsf{t}\}$ is a model of C'.*

Proof: Then, $x_1 \notin C'(x_0) \ni x_0$, while, by the structurality of C', $\langle \mathfrak{Fm}^{\omega}_{\Sigma}, C'(x_0)\rangle$ is a model of C' (in particular, of C), and so is its consistent truth-non-empty finitely-generated submatrix $\langle \mathfrak{Fm}^{2}_{\Sigma}, \mathrm{Fm}^{2}_{\Sigma} \cap C'(x_0)\rangle$, in view of (2.2). In this way, (2.2) and Lemma 4.19(ii) complete the argument. □

On the other hand, the reservation "inferentially" cannot, generally speaking, be omitted in the formulation of Theorem 4.21, as it ensues from:

Example 4.22. When $\Sigma = \Sigma_{\sim}$, $\{\mathsf{n}\}$ forms a subalgebra of $\mathfrak{A}$, in which case $\mathcal{B} \triangleq (\mathcal{A}{\upharpoonright}\{\mathsf{n}\})$ is a consistent truth-empty submatrix of $\mathcal{A}$, and so, by (2.2), the logic C' of $\mathcal{B}$ is a consistent but inferentially inconsistent extension of C. Then, C' is not subclassical, because any classical logic is inferentially consistent, for any classical matrix is both consistent and truth-non-empty. □

4.5. Axiomatic extensions

LEMMA 4.23. *Suppose $\mathfrak{A}$ is regular and $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of it. Then, so does $\{\mathsf{f},\mathsf{b},\mathsf{t}\}$.*

PROOF: By contradiction. For suppose $\{\mathsf{f},\mathsf{b},\mathsf{t}\}$ does not form a subalgebra of $\mathfrak{A}$, in which case there is some $\varphi \in \mathrm{Fm}^3_\Sigma$ such that $\varphi^\mathfrak{A}(\mathsf{f},\mathsf{b},\mathsf{t}) = \mathsf{n}$. Therefore, as $\mathsf{t} \sqsubseteq \mathsf{b}$, by the regularity of $\mathfrak{A}$ and the reflexivity of $\sqsubseteq$, we get $\varphi^\mathfrak{A}(\mathsf{f},\mathsf{t},\mathsf{t}) \sqsubseteq \mathsf{n}$. Hence, $\varphi^\mathfrak{A}(\mathsf{f},\mathsf{t},\mathsf{t}) = \mathsf{n} \notin \{\mathsf{f},\mathsf{t}\}$. This contradicts to the assumption that $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, as required. $\square$

LEMMA 4.24 (cf. Lemma 4.14 of [12] for the case $B = \{\mathsf{f},\mathsf{t}\}$ and $\Sigma = \Sigma_\sim$). *Let $\mathcal{B} \in \mathbf{S}(\mathcal{A})$. Suppose $B \cup \{\mathsf{b}\}$ forms a regular subalgebra of $\mathfrak{A}$. Then, any Σ-axiom, being true in $\mathcal{B}$, is so in $\mathcal{A}\restriction(B \cup \{\mathsf{b}\})$.*

PROOF: Consider any $\varphi \in \mathrm{Fm}_\Sigma$ not true in $\mathcal{A}\restriction(B \cup \{\mathsf{b}\})$, in which case there is some $h \in \hom(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{A}\restriction(B \cup \{\mathsf{b}\}))$ such that $h(\varphi) \in \{\mathsf{f},\mathsf{n}\}$, and so $h(\varphi) \sqsubseteq \mathsf{f}$. Take any $b \in B \neq \varnothing$. Define a $g : V_\omega \to B$ by setting:

$$g(v) \triangleq \begin{cases} b & \text{if } h(v) = \mathsf{b}, \\ h(v) & \text{otherwise,} \end{cases}$$

for all $v \in V_\omega$. Let $e \in \hom(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{B}) \subseteq \hom(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{A}\restriction(B \cup \{\mathsf{b}\}))$ extend g. Then, $e(v) = g(v) \sqsubseteq h(v)$, for all $v \in V_\omega$, in which case, by the regularity of $\mathfrak{A}\restriction(B \cup \{\mathsf{b}\})$, we have $e(\varphi) \sqsubseteq h(\varphi) \sqsubseteq \mathsf{f}$, and so we eventually get $e(\varphi) \in \{\mathsf{f},\mathsf{n}\}$, as required. $\square$

LEMMA 4.25 (cf. Corollary 5.3 of [9] for the case $\Sigma = \Sigma_\sim$). *Suppose $\{\mathsf{f},\mathsf{b},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$/ $\{\mathsf{f},\mathsf{t}\}[\cup\{\mathsf{b}\}]$ does [not] form a subalgebra of $\mathfrak{A}$. Then, the logic of $\mathcal{A}_{\not{\mathsf{n}}/\not{\mathsf{n}}\not{\mathsf{b}}} \triangleq (\mathcal{A}\restriction(\{\mathsf{f},\mathsf{b},\mathsf{t}\}/\{\mathsf{f},\mathsf{t}\}))$ is the proper consistent axiomatic extension of C relatively axiomatized by*

$$x_1 \vee \sim x_1. \tag{4.8}$$

PROOF: In that case, $(\mathrm{Mod}\,(4.8) \cap \mathbf{S}_*(\mathcal{A})) = \mathbf{S}_*(\mathcal{A}_{\not{\mathsf{n}}/\not{\mathsf{n}}\not{\mathsf{b}}})$. In this way, (2.2), Corollary 2.9, the consistency of $\mathcal{A}_{\not{\mathsf{n}}/\not{\mathsf{n}}\not{\mathsf{b}}}$ and the fact that (4.8) is not true in $\mathcal{A}$ under $[x_1/\mathsf{n}]$ complete the argument. $\square$

THEOREM 4.26. *[Providing $\mathfrak{A}$ is regular/has no three-element subalgebra] C has a proper consistent axiomatic extension if[f] $\{\mathsf{f},\mathsf{b},\mathsf{t}\}/\{\mathsf{f},\mathsf{t}\}$ forms a* *subalgebra of $\mathfrak{A}$ [in which case the logic of $\mathcal{A}_{\not\mathsf{n}/\not\mathsf{n}\not\mathsf{b}}$ is the only proper consistent axiomatic extension of C and is relatively axiomatized by (4.8)].*

Proof: The "if" part is by Lemma 4.25. [Conversely, assume $\mathfrak{A}$ is regular/has no three-element subalgebra. Consider any $\mathcal{A} \subseteq \mathrm{Fm}_\Sigma$ such that the axiomatic extension C' of C relatively axiomatized by $\mathcal{A}$ is both proper, in which case $\mathcal{A} \neq \varnothing$, and consistent, in which case, by Corollary 2.9, C' is the logic of $\mathsf{S} \triangleq (\mathrm{Mod}(\mathcal{A}) \cap \mathbf{S}_*(\mathcal{A}))$, and so $\mathcal{A} \notin \mathsf{S} \neq \varnothing$. Take any $\mathcal{B} \in \mathsf{S}$, in which case it is both consistent and, as $\mathcal{A} \neq \varnothing$, truth-non-empty. Hence, by Claim 4.18, we have $\{\mathsf{f}, \mathsf{t}\} \subseteq B$. Therefore, if n was in B, then $(B \cup \{\mathsf{b}\})$ would be equal to A/B would belong to $\{\{\mathsf{f}, \mathsf{n}, \mathsf{t}\}, A\}$, in which case, by Lemma 4.24/the fact that $\{\mathsf{f}, \mathsf{n}, \mathsf{t}\}$, being three-element, does not form a subalgebra of $\mathfrak{A}$, $\mathcal{A}$ would belong to S. Thus, $B \in \{\{\mathsf{f}, \mathsf{t}\}, \{\mathsf{f}, \mathsf{b}, \mathsf{t}\}\}$. Then, by Lemma 4.23/the fact that $\{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$, being three-element, does not form a subalgebra of $\mathfrak{A}$, we conclude that $\{\mathsf{f}, \mathsf{b}, \mathsf{t}\}/\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$. And what is more, in that case, by Lemma 4.24/the fact that $\{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$, being three-element, does not form a subalgebra of $\mathfrak{A}$, we have $\mathcal{A}_{\not\mathsf{n}/\not\mathsf{n}\not\mathsf{b}} \in \mathsf{S} \subseteq \mathbf{S}_*(\mathcal{A}_{\not\mathsf{n}/\not\mathsf{n}\not\mathsf{b}})$, and so, by (2.2), C' is equal to the logic of $\mathcal{A}_{\not\mathsf{n}/\not\mathsf{n}\not\mathsf{b}}$. In this way, Lemma 4.25 completes the argument.] □

The logic of $\mathcal{DM}_{4[01],\not\mathsf{n}}$ is [the bounded version of] the logic of paradox $LP_{[01]}$ [8] (cf. [10]; viz., in the "unbounded" case, the implication-less fragment of any *paraconsistent* Dunn's $RM\{(2 \cdot n) + 3\}$ {where $n \in \omega$} – cf. [4] and the proof of Corollary 4.15 of [12]). Therefore, in view of the regularity of $\mathfrak{DM}_{4[01]}$, Theorem 4.26 immediately yields:

Corollary 4.27. $LP_{[01]}$ is the only proper consistent axiomatic extension of $C_{[\mathrm{B}]\mathrm{DB}}$ and is relatively axiomatized by (4.8).

In Section 5 we consider more classes of expansions of FDE in this connection.

4.6. Maximal paraconsistency versus paracompleteness

The axiomatic extension of C relatively axiomatized by (4.8) is denoted by C^{EM}. An/A extension/model of C is said to be *paracomplete*, provided it is not that of C^{EM}. Clearly, a submatrix $\mathcal{B}$ of $\mathcal{A}$ is paracomplete/$\sim$-paraconsistent iff $\mathsf{n} \in B$/both $\mathsf{b} \in B$ and $(B \cap \{\mathsf{n}, \mathsf{f}\}) \neq \varnothing$. In particular, $\mathcal{A}$ is both $\sim$-paraconsistent and paracomplete, and so is C.

By $\mathcal{A}_{-\mathsf{n}}$ we denote the submatrix of $\mathcal{A}$ generated by $\{\mathsf{f},\mathsf{b},\mathsf{t}\}$ — this the least $\sim$-paraconsistent submatrix of $\mathcal{A}$, the logic of it being denoted by $C^{-\mathsf{n}}$. (Clearly, $\mathcal{A}_{-\mathsf{n}} = \mathcal{A}_{\not\mathsf{n}}$, whenever $\{\mathsf{f},\mathsf{b},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, and $\mathcal{A}_{-\mathsf{n}} = \mathcal{A}$, otherwise.)

LEMMA 4.28. *Let $\mathcal{B}$ be a $\sim$-paraconsistent model of C. Then, there is some submatrix $\mathcal{D}$ of $\mathcal{B}$ such that $\mathcal{A}_{-\mathsf{n}}$ is embeddable into $\mathcal{D}/\partial(\mathcal{D})$.*

PROOF: In that case, there are some $a \in D^{\mathcal{B}}$ such that $\sim^{\mathfrak{B}} a \in D^{\mathcal{B}}$ and some $b \in (B \setminus D^{\mathcal{B}})$. Then, in view of (2.2), the submatrix $\mathcal{D}$ of $\mathcal{B}$ generated by $\{a, b\}$ is a $\sim$-paraconsistent finitely-generated model of C. Hence, by Lemma 2.7 with $\mathsf{M} = \{\mathcal{A}\}$, there are some finite set I, some I-tuple $\overline{\mathcal{C}}$ constituted by consistent submatrices of $\mathcal{A}$, some subdirect product $\mathcal{E}$ of $\overline{\mathcal{C}}$ and some $g \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{E}, \mathcal{D}/\partial(\mathcal{D}))$. Hence, by (2.2), $\mathcal{E}$ is $\sim$-paraconsistent, in which case it is consistent, and so $I \neq \varnothing$. Take any $a \in D^{\mathcal{E}}$ such that $\sim^{\mathfrak{E}} a \in D^{\mathcal{E}}$. Then, $E \ni a = (I \times \{\mathsf{b}\})$, in which case, for each $i \in I$, $D^{\mathcal{C}_i} \ni \pi_i(a)$, and so $\mathcal{C}_i$ is truth-non-empty. Therefore, by Claim 4.17, we also have both $E \ni b \triangleq (I \times \{\mathsf{f}\})$ and $E \ni c \triangleq (I \times \{\mathsf{t}\})$. Consider the following complementary cases:

1. $\{\mathsf{f},\mathsf{b},\mathsf{t}\}$ does not form a subalgebra of $\mathfrak{A}$.
   Then, $A_{-\mathsf{n}} = A$ and there is some $\varphi \in \mathrm{Fm}^3_\Sigma$ such that $\varphi^{\mathfrak{A}}(\mathsf{f},\mathsf{b},\mathsf{t}) = \mathsf{n}$, in which case $E \ni \varphi^{\mathfrak{E}}(b, a, c) = (I \times \{\varphi^{\mathfrak{A}}(\mathsf{f},\mathsf{b},\mathsf{t})\}) = (I \times \{\mathsf{n}\})$, and so $\{I \times \{d\} \mid d \in A_{-\mathsf{n}}\} \subseteq E$.

2. $\{\mathsf{f},\mathsf{b},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$.
   Then, $A_{-\mathsf{n}} = \{\mathsf{f},\mathsf{b},\mathsf{t}\}$, and so $\{I \times \{d\} \mid d \in A_{-\mathsf{n}}\} \subseteq E$.

Thus, in any case, $\{I \times \{d\} \mid d \in A_{-\mathsf{n}}\} \subseteq E$. Then, as $I \neq \varnothing$, $e \triangleq \{\langle d, I \times \{d\}\rangle \mid d \in A_{-\mathsf{n}}\}$ is an embedding of $\mathcal{A}_{-\mathsf{n}}$ into $\mathcal{E}$, in which case $(g \circ e) \in \hom_{\mathrm{S}}(\mathcal{A}_{-\mathsf{n}}, \mathcal{D}/\partial(\mathcal{D}))$, and so Corollary 2.3, Lemmas 3.2, 3.3 and Remark 3.6 complete the argument. □

THEOREM 4.29. *$\mathcal{A}_{-\mathsf{n}}$ is a model of any $\sim$-paraconsistent extension of C. In particular, $C^{-\mathsf{n}}$ is the greatest $\sim$-paraconsistent extension of C, and so maximally $\sim$-paraconsistent, in which case an extension of C is $\sim$-paraconsistent iff it is a sublogic of $C^{-\mathsf{n}}$.*

Proof: Consider any $\sim$-paraconsistent extension C' of C, in which case $x_1 \notin T \triangleq C'(\{x_0, \sim x_0\})$, and so, by the structurality of C', $\langle \mathfrak{Fm}^{\omega}_{\Sigma}, T\rangle$ is a $\sim$-paraconsistent model of C', and so of C. Then, (2.2) and Lemma 4.28 complete the argument. □

Corollary 4.30 (cf. the reference [Pyn 95b] of [10]). Let $\mathcal{B}$ be a Σ-expansion of $\mathcal{DM}_{4,\not{\mathsf{n}}}$. Then, the logic of $\mathcal{B}$ is maximally $\sim$-paraconsistent.

Proof: In that case, there is clearly a Σ-expansion $\mathcal{A}'$ of $\mathcal{DM}_4$ such that $\mathcal{B}$ is a submatrix of $\mathcal{A}'$, so Theorem 4.29 completes the argument. □

Corollary 4.30 [with $\Sigma = \Sigma_{\sim}$] covers Dunn's $RM3$ [4] [subsumes Theorem 2.1 of [10]].

Theorem 4.31. *The following are equivalent:*

(i) *C is maximally $\sim$-paraconsistent;*

(ii) *$C = C^{-\mathsf{n}}$;*

(iii) *$C^{\mathrm{EM}} \neq C^{-\mathsf{n}}$;*

(iv) *$\{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$ does not form a subalgebra of $\mathfrak{A}$;*

(v) *C^{EM} is not $\sim$-paraconsistent;*

(vi) *C^{EM} is not maximally $\sim$-paraconsistent;*

(vii) *any $\sim$-paraconsistent extension of C is paracomplete;*

(viii) *no expansion of LP is an extension of C;*

(ix) *C^{EM} is not an expansion of LP;*

(x) *$C^{-\mathsf{n}}$ is paracomplete;*

(xi) *$\mathcal{A}$ has no proper $\sim$-paraconsistent submatrix;*

(xii) *any $\sim$-paraconsistent submatrix of $\mathcal{A}$ is paracomplete;*

(xiii) *C^{EM} is either $\sim$-classical, if C is $\sim$-subclassical, or inconsistent, otherwise;*

(xiv) *any consistent non-$\sim$-classical extension of C is paracomplete.*

PROOF: First, (i)⇒(ii) is by (2.2). Next, both (ii)⇒(i), (vi)⇒(iii) and (x)⇒(vii) are by Theorem 4.29. Moreover, (ii)⇒(x) is by the paracompleteness of C. In addition, (xiii)⇒(xiv) is by Theorems 4.20 and 4.21, because any consistent logic with theorems is inferentially consistent.

Further, assume $\{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, in which case $A_{-\mathsf{n}} = A_{\not{\mathsf{n}}}$, and so, by Lemma 4.25, $C^{\mathrm{EM}} = C^{-\mathsf{n}}$ is an expansion of LP. Thus, both (iii)⇒(iv) and (ix)⇒(iv) hold.

Conversely, assume (iv) holds. Let S be the set of all non-paracomplete consistent submatrices of $\mathcal{A}$, in which case, by Corollary 2.9, C^{EM} is defined by S. Consider any $\mathcal{B} \in \mathsf{S}$. Since it is not paracomplete, we have $\mathsf{n} \notin B$, in which case $\mathsf{f} \in B$, for it is consistent, and so $\mathsf{t} = {\sim}^{\mathfrak{A}}\mathsf{f} \in B$. Therefore, by (iv), $\mathsf{b} \notin B$, for $\{\mathsf{f}, \mathsf{t}\} \subseteq B \not\ni \mathsf{n}$. Thus, $B = \{\mathsf{f}, \mathsf{t}\}$. In this way, by Theorem 4.20, either $\mathsf{S} = \{\mathcal{B}\}$, in which case C^{EM} is $\sim$-classical, if C is $\sim$-subclassical, or $\mathsf{S} = \varnothing$, in which case C^{EM} is inconsistent, otherwise. Thus, (xiii) holds.

Furthermore, (xii)⇔(xi)⇔(x)⇔(iv)⇒(ii) are immediate.

Finally, (ix/viii) is a particular case of (viii/vii). Likewise, (vi) is a particular case of (v), while (v) is a particular case of (vii), whereas (vii) is a particular case of (xiv), as required. □

It is Theorem 4.31(i)⇔(iv) that provides a quite useful algebraic criterion of the maximal $\sim$-paraconsistency of C inherited by its four-valued expansions, in view of Corollary 4.9, applications of which are demonstrated in Section 5.

Combining Lemmas 4.23, 4.24, Theorems 4.20, 4.31 and (2.2), we immediately get:

COROLLARY 4.32. Suppose C is $\sim$-subclassical and $\mathfrak{A}$ is regular. Then, C is not maximally $\sim$-paraconsistent and $C^{\mathrm{PC}}(\varnothing) = C^{\mathrm{EM}}(\varnothing)$.

Concluding this subsection, we explore the least non-$\sim$-paraconsistent extension $C^{\mathrm{EM+NP}}$ of C^{EM}, viz., that which is relatively axiomatized by the *Ex Contradictione Quodlibet* rule:

$$\{x_0, {\sim}x_0\} \vdash x_1. \tag{4.9}$$

LEMMA 4.33. *Let I be a finite set, $\overline{\mathcal{C}} \in \{\mathcal{A}, \langle\mathfrak{A}, \{\mathsf{t}, \mathsf{n}\}\rangle, \langle\mathfrak{A}, \{\mathsf{t}\}\rangle\}^I$ and $\mathcal{B}$ a consistent non-$\sim$-paraconsistent submatrix of $\prod_{i\in I} \mathcal{C}_i$. Then,* $\mathrm{hom}(\mathcal{B}, \langle\mathfrak{A}, \{\mathsf{t}\}\rangle) \neq \varnothing$.

Proof: Consider the following complementary cases:

· $\mathcal{B}$ is truth-empty.
Take any $i \in I \neq \varnothing$, for $\mathcal{B}$ is consistent. Then, $h \triangleq (\pi_i \restriction B) \in \hom(\mathfrak{B}, \mathfrak{A})$. Moreover, $D^{\mathcal{B}} = \varnothing \subseteq h^{-1}[\{\mathsf{t}\}]$. Hence, $h \in \hom(\mathcal{B}, \langle \mathfrak{A}, \{\mathsf{t}\}\rangle)$, as required.

· $\mathcal{B}$ is truth-non-empty.
Then, $B \subseteq A^I$ is finite, for both I and A are so, and so is $D^{\mathcal{B}} \subseteq B$. Hence, as $\mathfrak{B} \restriction \Sigma_+$ is a lattice, $D^{\mathcal{B}}$, being non-empty, has a least element a, in which case, as $\mathcal{B}$ is consistent but not $\sim$-paraconsistent, $\sim^{\mathfrak{B}} a \notin D^{\mathcal{B}}$, and so there is some $i \in I$, in which case $h \triangleq (\pi_i \restriction B) \in \hom(\mathcal{B}, \mathcal{C}_i)$, such that $h(\sim^{\mathfrak{B}} a) \notin D^{\mathcal{C}_i}$. If there was some $b \in D^{\mathcal{B}}$ such that $h(b) \neq \mathsf{t}$, we would have $\mathcal{C}_i \in \{\mathcal{A}, \langle \mathfrak{A}, \{\mathsf{t}, \mathsf{n}\}\rangle\}$ and $(\{\mathsf{b}, \mathsf{n}\} \cap D^{\mathcal{C}_i}) \ni h(b) \leqslant^{\mathfrak{A}} h(a) \leqslant^{\mathfrak{A}} h(b)$, for $D^{\mathcal{B}} \ni a \leqslant^{\mathfrak{B}} b$, in which case we would get $h(a) = h(b)$, and so $h(\sim^{\mathfrak{B}} a) = \sim^{\mathfrak{A}} h(a) = \sim^{\mathfrak{A}} h(b) = h(b) \in D^{\mathcal{C}_i}$. Thus, $h \in \hom(\mathcal{B}, \langle \mathfrak{A}, \{\mathsf{t}\}\rangle)$, as required. □

Corollary 4.34. Let I be a finite set, $\overline{\mathcal{C}} \in \{\mathcal{A}, \langle \mathfrak{A}, \{\mathsf{t}, \mathsf{n}\}\rangle, \langle \mathfrak{A}, \{\mathsf{t}\}\rangle\}^I$ and $\mathcal{B}$ a consistent non-$\sim$-paraconsistent non-paracomplete submatrix of $\prod_{i \in I} \mathcal{C}_i$. Then, $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$ and $\hom(\mathcal{B}, \mathcal{A}_{\not{\mathsf{n}}\not{\mathsf{b}}}) \neq \varnothing$.

Proof: Then, by Lemma 4.33, there is some $h \in \hom(\mathcal{B}, \langle \mathfrak{A}, \{\mathsf{t}\}\rangle) \neq \varnothing$, in which case $D \triangleq (\mathrm{img}\, h)$ forms a subalgebra of $\mathfrak{A}$, and so $h \in \hom^{\mathrm{S}}(\mathcal{B}, \mathcal{D})$, where $\mathcal{D} \triangleq (\langle \mathfrak{A}, \{\mathsf{t}\}\rangle \restriction D)$. Hence, by (2.3), $\mathcal{D}$ is not paracomplete. Therefore, as (4.8) is true in $\langle \mathfrak{A}, \{\mathsf{t}\}\rangle$ under neither $[x_1/\mathsf{b}]$ nor $[x_1/\mathsf{n}]$, we have $(D \cap \{\mathsf{b}, \mathsf{n}\}) = \varnothing$. On the other hand, $\mathcal{D}$, being non-paracomplete, is truth-non-empty, for $D \neq \varnothing$. Therefore, $\mathsf{t} \in D$, in which case $\mathsf{f} = \sim^{\mathfrak{A}} \mathsf{t} \in D$, and so $D = \{\mathsf{f}, \mathsf{t}\}$, in which case $\mathcal{D} = (\mathcal{A} \restriction D) = \mathcal{A}_{\not{\mathsf{n}}\not{\mathsf{b}}}$, as required. □

Theorem 4.35. *Suppose C is [not] maximally $\sim$-paraconsistent. Then, $C^{\mathrm{EM+NP}}$ is consistent iff C is $\sim$-subclassical, in which case $C^{\mathrm{EM+NP}}$ is defined by $[\mathcal{A}_{\not{\mathsf{n}}} \times] \mathcal{A}_{\not{\mathsf{n}}\not{\mathsf{b}}}$.*

Proof: First, assume $C^{\mathrm{EM+NP}}$ is consistent, in which case $x_0 \notin T \triangleq C^{\mathrm{EM+NP}}(\varnothing)$, while, by the structurality of $C^{\mathrm{EM+NP}}$, $\langle \mathfrak{Fm}^{\omega}_{\Sigma}, T\rangle$ is a model of $C^{\mathrm{EM+NP}}$ (in particular, of C), and so is its consistent finitely-generated submatrix $\mathcal{B} \triangleq \langle \mathfrak{Fm}^1_{\Sigma}, T \cap \mathrm{Fm}^1_{\Sigma}\rangle$, in view of (2.2). Hence, by Lemma 2.7, there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}(\mathcal{A})^I$, some subdirect product $\mathcal{D}$ of it, in which case this is a submatrix of $\mathcal{A}^I$, and some $h \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D}, \Re(\mathcal{B}))$, in which case, by (2.2), $\mathcal{D}$ is a consistent model of $C^{\mathrm{EM+NP}}$, so it is neither

$\sim$-paraconsistent nor paracomplete. Thus, by Corollary 4.34 and Theorem 4.20, C is $\sim$-subclassical.

Conversely, assume C is $\sim$-subclassical. Consider the following complementary cases:

· C is maximally $\sim$-paraconsistent.
Then, by Theorems 4.20 and 4.31(i)$\Rightarrow$(v,xiii) $C^{\mathrm{EM+NP}} = C^{\mathrm{EM}} = C^{\mathrm{PC}}$ is defined by the consistent $\mathcal{A}_{\not\mathsf{n}\not\mathsf{b}}$, and so, in particular, is consistent, as required.

· C is not maximally $\sim$-paraconsistent.
Then, by Theorem 4.31(iii/iv)$\Rightarrow$(i), C^{EM} is defined by $\mathcal{A}_{-\mathsf{n}} = \mathcal{A}_{\not\mathsf{n}}$. Moreover, by Theorem 4.20, $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, and so of $\mathfrak{A}_{\not\mathsf{n}}$, in which case $\mathcal{A}_{\not\mathsf{n}\not\mathsf{b}}$ is a submatrix of $\mathcal{A}_{\not\mathsf{n}}$, and so, by (2.2), $\mathcal{B} \triangleq (\mathcal{A}_{\not\mathsf{n}} \times \mathcal{A}_{\not\mathsf{n}\not\mathsf{b}})$ is a model of C^{EM}. Moreover, $\{a, \sim^{\mathfrak{A}} a\} \subseteq \{\mathsf{t}\}$, for no $a \in \{\mathsf{f}, \mathsf{t}\}$. Therefore, $\mathcal{B}$ is not $\sim$-paraconsistent, so it is a model of $C^{\mathrm{EM+NP}}$. Conversely, consider any finite set I, any $\overline{\mathcal{C}} \in \mathbf{S}(\mathcal{A}_{\not\mathsf{n}})^I$ and any subdirect product $\mathcal{D} \in \mathrm{Mod}(C^{\mathrm{EM+NP}})$ of $\overline{\mathcal{C}}$, in which case $\mathcal{D}$ is a non-$\sim$-paraconsistent non-paracomplete submatrix of $\mathcal{A}^I$. Put $J \triangleq \hom(\mathcal{D}, \mathcal{B})$. Consider any $a \in (D \setminus D^{\mathcal{D}})$, in which case $\mathcal{D}$ is consistent, and so, by Corollary 4.34, there is some $g \in \hom(\mathcal{D}, \mathcal{A}_{\not\mathsf{n}\not\mathsf{b}}) \neq \varnothing$. Moreover, there is some $i \in I$, in which case $f \triangleq (\pi_i \restriction D) \in \hom(\mathcal{D}, \mathcal{A}_{\not\mathsf{n}})$, such that $f(a) \notin D^{\mathcal{A}_{\not\mathsf{n}}}$. Then, $h \triangleq (f \times g) \in J$ and $h(a) \notin D^{\mathcal{B}}$. In this way, $(\prod \Delta_J) \in \hom_{\mathrm{S}}(\mathcal{D}, \mathcal{B}^J)$. Thus, by (2.2) and Theorem 2.8, $C^{\mathrm{EM+NP}}$ is finitely-defined by the consistent six-valued $\mathcal{B}$, and so is consistent and, being finitary, for both (4.8) and (4.9) are finitary, while the four-valued C is finitary, is defined by $\mathcal{B}$, as required. $\square$

Corollary 4.36 (cf. the last assertion of Theorem 4.13 of [12] for the case $\Sigma = \Sigma_{\sim}$). Let $\mathcal{B}$ be a Σ-expansion of $\mathcal{DM}_{4,\not\mathsf{n}}$. Suppose $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{B}$. Then, the extension of the logic of $\mathcal{B}$ relatively axiomatized by (4.9) is defined by $\mathcal{B} \times (\mathcal{B} \restriction \{\mathsf{f}, \mathsf{t}\})$.

Proof: In that case, there is clearly a Σ-expansion $\mathcal{A}'$ of $\mathcal{DM}_4$ such that $\mathcal{B}$ is a submatrix of $\mathcal{A}'$, so Theorems 4.20, 4.31 and 4.35 complete the argument. $\square$

This is equally applicable to, in particular, $RM3$ [4] and subsumes specific results concerning purely-implicative expansions of $C_{[\mathrm{B}]\mathrm{DB}}$ obtained *ad hoc* in [14] (cf. the last paragraph of Subsection 5.3).

5. Miscellaneous examples

We entirely follow notations of the previous sections.

5.1. Classically-negative expansions

Here, it is supposed that Σ contains a unary connective $\neg$ (classical negation), while $\neg^{\mathfrak{A}}\langle i,j\rangle \triangleq \langle 1-i,1-j\rangle$, for all $i,j\in 2$, in which case $\neg^{\mathfrak{A}}\langle k,1-k\rangle = \langle 1-k,k\rangle$, for each $k\in 2$, and so $\neg^{\mathfrak{A}}$ is not regular, for $\mathsf{b}\not\sqsubseteq \mathsf{n}\sqsubseteq \mathsf{b}$. Then, $\{\mathsf{f},\mathsf{t}\}$ is the only proper subset of A which may form a subalgebra of $\mathfrak{A}$. Thus, by Theorems 4.16, 4.20, 4.26 and 4.31, we have:

Corollary 5.1. *C*:

(i) has no, if it is not $\sim$-subclassical, in which case it is maximal, and, otherwise (in particular, when $\Sigma = (\Sigma_{\sim[01]}\cup\{\neg\})$), a unique proper consistent axiomatic extension, in which case this is equal to $C^{\mathrm{PC}} = C^{\mathrm{EM}}$;

(ii) is maximally $\sim$-paraconsistent.

This provides an application of the "non-regular" particular case of Theorem 4.26. (Another one is provided by the next subsection.) On the other hand, $\mathcal{A}$ is $(\neg x_0 \vee x_1)$-implicative. Therefore, in view of Remark 3.6, Corollary 5.1(i) (but the maximality reservation) equally ensues from Theorem 3.5. After all, Corollary 5.1(ii) provides examples of maximally paraconsistent *four*-valued logics. (Others are provided by the next subsection.)

5.2. Bilattice expansions

Here, it is supposed that Σ contains binary connectives $\sqcap$ and $\sqcup$ (*knowledge* conjunction and disjunction, respectively), while

$$(\langle i,j\rangle(\sqcap/\sqcup)^{\mathfrak{A}}\langle k,l\rangle) \triangleq \langle(\min/\max)(i,k),(\max/\min)(j,l)\rangle,$$

for all $i,j,k,l\in 2$ (cf., e.g., [11]), in which case $(\mathsf{f}(\sqcap/\sqcup)^{\mathfrak{A}}\mathsf{t}) = (\mathsf{n}/\mathsf{b})$, and so, since any non-one-element subalgebra of $\mathfrak{DM}_4$ contains both f and t, $\mathfrak{A}$ has no proper non-one-element subalgebra. Hence, by Theorems 4.16, 4.26 and 4.31, we have:

$$\begin{tikzcd}[column sep=small] & A \arrow[dl, no head] \arrow[dr, no head] & \\ \{\mathsf{f},\mathsf{b},\mathsf{t}\} & & \{\mathsf{f},\mathsf{n},\mathsf{t}\} \\ & \{\mathsf{f},\mathsf{t}\} \arrow[ul, no head] \arrow[ur, no head] & \end{tikzcd}$$

Figure 1. The poset $\mathbf{S}_*(\mathcal{A})$.

Corollary 5.2. *C is inferentially maximal, and so both has no proper consistent axiomatic extension and is maximally $\sim$-paraconsistent.*

This provides both a one more application of the "non-regular" particular case of Theorem 4.26 and more examples of maximally paraconsistent *four*-valued logics. Moreover, it is bilattice expansions that justify studying the maximality issue within the framework of FDE expansions.

5.3. Implicative expansions

Here, it is supposed that Σ contains a binary connective $\supset$ (implication), while:

$$(\vec{a} \supset^{\mathfrak{A}} \vec{b}) \triangleq \begin{cases} \vec{b} & \text{if } a_0 = 1, \\ \mathsf{t} & \text{otherwise,} \end{cases}$$

for all $\vec{a}, \vec{b} \in 2^2$ (cf. [11]), in which case $\mathcal{A}$ is $\supset$-implicative, while $(\mathsf{f} \supset^{\mathfrak{A}} \mathsf{f}) = \mathsf{t}$, whereas $(\mathsf{b} \supset^{\mathfrak{A}} \mathsf{f}) = \mathsf{f}$, and so $\supset^{\mathfrak{A}}$ is not regular, for $\mathsf{t} \not\sqsubseteq \mathsf{f} \sqsubseteq \mathsf{b}$. From now on, it is supposed that $\Sigma = (\Sigma_{\sim[01]} \cup \{\supset\})$ (the opposite case is considered in a similar way *ad hoc*, depending upon which of the four subsets of A depicted at Figure 1 form subalgebras of $\mathfrak{A}$). Moreover, submatrices of $\mathcal{A}$ are identified with the carriers of their underlying algebras. Then, since $\mathcal{DM}_4{\restriction}\{\mathsf{b}\}$ is not consistent, while $(\mathsf{n} \supset^{\mathfrak{A}} \mathsf{n}) = \mathsf{t} \neq \mathsf{n}$, in which case $\{\mathsf{n}\}$ does not form a subalgebra of $\mathfrak{A}$, the poset $\mathbf{S}_*(\mathcal{A})$ forms the diamond depicted at Figure 1, so, in particular, by Theorems 4.16, 4.20 and 4.31, we have:

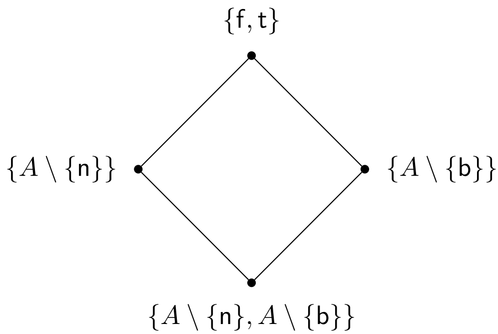

Figure 2. Proper consistent axiomatic extensions of C.

COROLLARY 5.3. C is $\sim$-subclassical but not maximal(ly $\sim$-paraconsistent).

Note that

$$\sim x_1 \supset (x_1 \supset (x_2 \vee \sim x_2)) \tag{5.1}$$

is true in $\{\{\mathsf{f},\mathsf{n},\mathsf{t}\},\{\mathsf{f},\mathsf{b},\mathsf{t}\}\}$ but is not true in $\mathcal{A}$ under $[x_1/\mathsf{b}, x_2/\mathsf{n}]$. Moreover,

$$\sim x_1 \supset (x_1 \supset x_0) \tag{5.2}$$

is true in $\{\mathsf{f},\mathsf{n},\mathsf{t}\}$ but is not true in $\{\mathsf{f},\mathsf{b},\mathsf{t}\}$ under $[x_1/\mathsf{b}, x_0/\mathsf{f}]$. Finally, (4.8) is satisfied in $\{\mathsf{f},\mathsf{b},\mathsf{t}\}$ but is not satisfied in $\{\mathsf{f},\mathsf{n},\mathsf{t}\}$ under $[x_1/\mathsf{n}]$. In this way, by Theorem 3.5 and Remark 3.6, we eventually get:

COROLLARY 5.4. Proper consistent axiomatic extensions of C (given by defining matrix anti-chains) form the diamond depicted at Figure 2 and are relatively axiomatized as follows (actually, according to the constructive proof of Lemma 3.4):

$$\begin{array}{rcl}
\{A\setminus\{\mathsf{n}\},A\setminus\{\mathsf{b}\}\} & : & (5.1), \\
\{A\setminus\{\mathsf{b}\}\} & : & (5.2), \\
\{A\setminus\{\mathsf{n}\}\} & : & (4.8), \\
\{\{\mathsf{f},\mathsf{t}\}\} & : & \{(5.2),(4.8)\}.
\end{array}$$

This, in particular, shows that the optional precondition in the formulation of Theorem 4.26 is essential for the uniqueness of a proper consistent axiomatic extension of C.

Concluding this discussion, recall that the [four-element chain] lattice of *all* extensions of $C^{[\mathrm{EM}]}$ [being a definitional copy of Dunn's $RM3$ [4] in the "unbounded" case] has been found in [14] – taking the general preliminary part of [12] into account – with using an equally automated method but as for merely defining matrices. However, the mentioned study does not at all subsume Corollary 5.4 because of not implying the fact that there is no more proper consistent axiomatic extension of C other than the four ones depicted at Figure 2. This goes without saying that the present study has provided relative axiomatizations *quite effectively*.

6. Conclusions

Aside from the general results and their numerous *generic* illustrative applications, the present paper demonstrates a special value of the conception of equality determinant studied in [13].

And what is more, the methodological algebraic result of Theorem 3.8, in its turn, based upon the apparatus of equality determinant well-advanced in [13], has found more applications within the general topic of FDE expansions, being however beyond the scopes of the present paper and going to be discussed elsewhere.

In general, the topic of [extensions of] expansions of Dunn-Belnap's four-valued logic is too inexhaustible to be studied within a single paper *comprehensively*. The present paper constitutes just a first part of it. Others are going to be presented elsewhere.

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Alexej P. Pynko
National Academy of Sciences of Ukraine
V.M. Glushkov Institute of Cybernetics
Department of Digital Automata Theory (100)
Glushkov prosp. 40
Kiev, 03680, Ukraine
e-mail: pynko@i.ua