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ON COMPLETE REPRESENTATIONS AND MINIMAL COMPLETIONS IN ALGEBRAIC LOGIC, BOTH POSITIVE AND NEGATIVE RESULTS

Abstract

Fix a finite ordinal $n \geq 3$ and let α be an arbitrary ordinal. Let CA_n denote the class of cylindric algebras of dimension n and RA denote the class of relation algebras. Let $\mathsf{PA}_\alpha(\mathsf{PEA}_\alpha)$ stand for the class of polyadic (equality) algebras of dimension α. We reprove that the class CRCA_n of completely representable CA_ns, and the class CRRA of completely representable RAs are not elementary, a result of Hirsch and Hodkinson. We extend this result to any variety V between polyadic algebras of dimension n and diagonal free CA_ns. We show that that the class of completely and strongly representable algebras in V is not elementary either, reproving a result of Bulian and Hodkinson. For relation algebras, we can and will, go further. We show the class CRRA is not closed under $\equiv_{\infty,\omega}$. In contrast, we show that given $\alpha \geq \omega$, and an atomic $\mathfrak{A} \in \mathsf{PEA}_\alpha$, then for any $n < \omega$, $\mathsf{Nr}_n\mathfrak{A}$ is a completely representable PEA_n. We show that for any $\alpha \geq \omega$, the class of completely representable algebras in certain reducts of PA_αs, that happen to be varieties, is elementary. We show that for $\alpha \geq \omega$, the the class of polyadic-cylindric algebras dimension α, introduced by Ferenczi, the completely representable algebras (slightly altering representing algebras) coincide with the atomic ones. In the last algebras cylindrifications commute only one way, in a sense weaker than full fledged commutativity of cylindrifications enjoyed by classical cylindric and polyadic algebras. Finally, we address closure under Dedekind–MacNeille completions for cylindric-like algebras of dimension n and PA_αs for α an infinite ordinal, proving negative results for the first and positive ones for the second.

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1. Introduction

Unless otherwise indicated, $2 < n < \omega$. Lately, it has become fashionable for algebras of relations, such as relation algebras, cylindric algebras due mainly to Tarski and polyadic algebras due to Halmos, to study representations that preserve infinitary meets and joins.

This phenomenon is extensively discussed in [24], where it is shown that it has an affinity with the algebraic notion of complete representations for cylindric like algebras and atom-canonicity in varieties of Boolean algebras with operators (BAOs). a prominent persistence property studied in modal logic.

A completely additive variety V of BAOs is atom-canonical, if whenever $\mathfrak{A} \in \mathsf{V}$ is atomic, then its Dedekind–MacNeille completions, namely, the complex algebra of its atom structure, in symbols $\mathfrak{Cm}\mathsf{At}\mathfrak{A}$ is also in V. The Dedekind–MacNeille completion of a CA_n is often referred to as its minimal Monk completion, since Monk showed that the Dedekind–MacNeille completion of a CA_n is again a CA_n. Here we use *minimal Dedekind–MacNeille completions*, or simply *the Dedekind–MacNeille completions*.

As for complete representations, the typical question is: given an algebra and a set of meets, is there a representation that carries this set of meets to set theoretic intersections? (assuming that our semantics is specified by set algebras, with the concrete Boolean operation of intersection among its basic operations.) When the algebra in question is countable, and we have countably many meets; this is an algebraic version of an omitting types theorem; the representation omits the given set of possibly infinitary meets or non-principal types. When it is only one meet consisting of co-atoms, in an atomic algebra, this representation is a complete one. The correlation of atomicity to complete representations has caused a lot of confusion in the past. It was mistakenly thought for a while, among algebraic logicians, that atomic representable relation and cylindric algebras are completely representable, an error attributed to Lyndon and now referred to as Lyndon's error. For Boolean algebras, however, this is true.

Follows is a crash rundown of known results: For Boolean algebras, the class of completely representable algebras is simply the class of atomic ones, hence is elementary. The class of completely representable polyadic algebras coincide with the class of atomic, completely additive algebras in this class, hence is also elementary [26]. The class CRCA_n of completely representable CA_ns is proved not to be elementary by Hirsch and Hodkinson in [9]. For any pair of ordinal $\alpha < \beta$, $\mathsf{Nr}_\alpha\mathsf{CA}_\beta(\subseteq \mathsf{CA}_\alpha)$ denotes the class of neat α-reducts of CA_βs as defined in [7, Definition 2.2.28]. Neat embeddings and complete representations are linked in [25, Theorem 5.3.6] where it is shown that CRCA_n coincides with the class $\mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_\omega$- on atomic algebra having countably many atoms. Here $\mathbf{S}_c$ denotes the operation of forming *complete* subalgebras, that is to say, given a class of algebras K having a Bolean reduct, then $\mathbf{S}_c\mathsf{K} = \{\mathfrak{B} : (\exists \mathfrak{A} \in \mathsf{K})(\forall X \subseteq \mathfrak{A} \sum^{\mathfrak{A}} X = 1 \implies \sum^{\mathfrak{B}} X = 1\}$ where $\sum$ denotes 'supremum' with the superscript specificying the algebra 'the evaluated supremum' exist in. The analogous result for relation algebras is proved in [8]. The latter result on charecterization of completely representable algebra via neat embedings will be extended below to the infinite dimensional case by defining complete representations via so-called *weak set algebras*.

In [17] it is proved that for any pair of ordinals $\alpha < \beta$, the class $\mathsf{Nr}_\alpha\mathsf{CA}_\beta$ is not elementary. A different model theoretic proof for finite α is given in [25, Theorem 5.4.1]. This result is extended to many cylindric like algebras like Halmos' polyadic algebras with and without equality, and Pinter's substitution algebras [18, 21, 19], cf. [20] for an overview. Below we give a single proof to all cases. The analogous result for relation agebras is proved in [22]. The paper is divided to two parts. Part 1 is devoted to cylindric-like algebras, while Part 2 is devoted to polyadic-like algebras. These two paradigms, the cylindric as opposed to the polyadic, often exhibit conflcting behavior.

Cylindric paradigm:

- In Section 2.1, we give the basic definitions of cylindric and relation algebras. Atomic networks and two player deteministic games between two players $\exists$ Ellosie and $\forall$ belard games characterizing neat embeddings, played on such networks, are defined in Section 2.2. Lemma 2.5 is the main result in Section 2.2. In all games used throughout the paper one of th players has a winning strategy thart can be implemented explicitly using a finite or tranfinite number of rounds and a set of 'nodes' usually finite. There are no draws.

- In Section 3 we reprove a classical result of Hirsch and Hodkinson [9]. Let $2 < n < \omega$. In Section 3.1 we show that the class of completely representable cylindric algebras of dimension n, briefly CRCA_n and the class of completely representable relation algebras, briefly CRRA are not elementary. The proof depends on so called Monk-Maddux relation algebras possessing what Maddux calls cylindric basis [16], cf. Lemma 2.6. We highlight the difference between our proof and the orginal first poof of the result (in print at least) in [9]. The two proofs are conceptually 'disjoint' as is illustrated. Using two player determinisc games between $\exists$ and $\forall$ on pebble paired structures, we go further by showing that CRRA is not closed under $\equiv_\infty$ in Theorem 2.8, thus answering a question posed by Hirsch and Hodkinson in [9, 11].

- Fix $2 < n \leq m \leq \omega$. We study locally classic representations, and locally classic complete representations, referred to as m-square representations or m-clique guarded semantics [10, 27] relating it to neat embeddings via existence of m-dilations and games using m nodes.

- We prove that for any variety V between PEA_n and Pinter's substitution algebras of dimension n (a notion to be made precise), the class $\mathsf{Nr}_n\mathsf{V}_m$ is not elementary for any ordinal $m > n > 1$ unifying the proofs of results established in [18, 21, 17, 25], cf. Theorem 3.5. Our new proof is model-theoretic, resorting to a Fraïssé constuction, analogous to the proof in [25] where the result restricted to only cylindric algebras is proved.

Polyadic paradigm:

- We show that given any atomic $\mathfrak{A} \in \mathsf{PEA}_\alpha$, α an infinite ordinal, we can obtain a plethora of completely representable algebras from $\mathfrak{A}$ for each $n < \omega$, by taking the operation of n neat reduct. In more detail, let $\mathfrak{A} \in \mathsf{PEA}_\alpha$ be atomic, then for any $n < \omega$, any complete subalgebra of $\mathsf{Nr}_n\mathfrak{A}$ is completely representable, cf. Theorem 4.1.

- We show that the class of completely representable algebras, of the variety obtained from polyadic algebras of infinite dimension, by discarding infinitary cylindrications while keeping all substitution operators is elementary, and that the class of polyadic cylindric of infinite dimensional algebras introduced by Ferenczi in [4] is also elementary; in fact in the former case the class of completely representable algebras coincide with the atomic completely additive ones, and in the second case the class of completely representable algebras, are like the case of Boolean algebras, simply the atomic ones, cf. the second part Theorem 4.1.

- Let $2 < n < \omega$. Closure under Dedekind–MacNeille completions, often referred to as *minimal completions* (which is the term used in he title) and Sahlqvist axiomatizability for varieties between QEA_n and Sc_n, where the last denotes the class of Pinter's substituition algebras as defined in [13] and also for the polyadic-like algebras addressed above are approached, cf. Theorems 5.5 and 5.6. Again negative results are obtained in the first case for cylindric-like algebras, while positive results prevail in the second polyadic paradigm, where all substitution operations are available in the signature.

Our results further emphasizes the dichotomy existing between the cylindric paradigm and the polyadic one, a phenomena recurrent in the literature of Tarski's cylindric algebras and Halmos' polyadic algebras, with algebras 'in between' such as Ferenzci's cylindric–polyadic algebras with and without equality, aspiring to share only nice desirable properties of both.

Such properties, some of which are thoroughly investigated below, include (not exclusively) finite axiomatizablity of the variety of representable algebras, the canonicity and atom-canonicity of such varieties, decidability of its equational/ or and universal theory, and the first order definability of the notion of complete representability [4, 5, 6].

2. The cylindric paradigm

2.1. The algebras and some basic concepts

For a set V, $\mathcal{B}(V)$ denotes the Boolean set algebra $\langle \wp(V), \cup, \cap, \sim, \emptyset, V\rangle$. Let U be a set and α an ordinal; α will be the dimension of the algebra. For $s, t \in {}^{\alpha}U$ write $s \equiv_i t$ if $s(j) = t(j)$ for all $j \neq i$. For $X \subseteq {}^{\alpha}U$ and $i, j < \alpha$, let

$$\mathsf{C}_i X = \{s \in {}^{\alpha}U : (\exists t \in X)(t \equiv_i s)\}$$

and
$$\mathsf{D}_{ij} = \{s \in {}^{\alpha}U : s_i = s_j\}.$$

The algebra $\langle \mathcal{B}({}^{\alpha}U), \mathsf{C}_i, \mathsf{D}_{ij}\rangle_{i,j<\alpha}$ is called *the full cylindric set algebra of dimension* α with unit (or greatest element) ${}^{\alpha}U$ referred to as a *cartesian square of dimension* α. Here full refers to the fact that the universe of the algebra is all of $\wp({}^{\alpha}U)$.

Fix an ordinal α. A *cylindric set algebra of dimension* α is a subalgebra of a full cylindric set algebra of the same dimension. The class of cylindric set algebras of dimension α is denoted by Cs_{α}. It is known that the variety gernerated by Cs_{α}, in symbols RCA_{α} denoting the class of *representable cylindric algebras of dimension* α, is the class $\mathbf{SP}\mathsf{Cs}_{\alpha}$ where $\mathbf{S}$ denotes the operation of forming subalgebras and $\mathbf{P}$ is the operation of forming products. Thus the class RCA_{α} is closed under $\mathbf{H}$ (forming homomorphic images). Furthermore, it is known that $\mathsf{RCA}_{\alpha} = \mathbf{I}\mathsf{Gs}_{\alpha}$ where $\mathbf{Gs}_{\alpha}$ is the class of *generalized set algebras of dimension* α and $\mathbf{I}$ is the operation of forming isomorphic images.

An algebra $\mathfrak{A} \in \mathbf{Gs}_{\alpha}$ if it has top element a disjoint union of cartesian squares each of dimension α and all of the the cylindric operations are defined like in the class of set algebras of the same dimension. In particular, the Boolean operations of meet, join and complemenation are the set theoretic operations of intersection, union, and taking complements relative to the top element, respectively. Let α be an ordinal. The (equationally defined) CA_{α} class is obtained from cylindric set algebras by a process of abstraction and is defined by a *finite* schema of equations given in [7, Definition 1.1.1] that holds of course in the more concrete (generalized) set algebras of dimension α.

Definition 2.1. Let $n < \omega$. Then $\mathfrak{A} \in \mathsf{CA}_n$ is *completely representable*, if there exists $\mathfrak{B} \in \mathsf{Gs}_n$ and an isomorphism $f : \mathfrak{A} \to \mathfrak{B}$ such for all $X \subseteq \mathfrak{A}$, $f(\prod X) = \bigcap_{x\in X} f(x)$ whenever $\prod X$ exists.

We consider relation algebras as algebras of the form $\mathcal{R} = \langle R, +, \cdot, -, 1', \smile, ;, \rangle$, where $\langle R, +, \cdot, -\rangle$ is a Boolean algebra $1' \in R$, $\smile$ is a unary operation and ; is a binary operation. A relation algebra is *representable* $\iff$ it is isomorphic to a subalgebra of the form $\langle \wp(X), \cup, \cap, \sim, \smile, \circ, Id\rangle$ where X is an equivalence relation, $1'$ is interpreted as the identity relation, $\smile$ is the operation of forming converses, and the binary operation ; is interpreted as composition of relations. Following standard notation, RA denotes the class of relation algebras. The class RA is a discriminator variety that is finitely axiomatizable, cf. [10, Definition 3.8, Theorems 3.19]. The variety of representable relation algebras is denoted by RRA. It is known that RRA is not finitely axiomatizable; a classical result of Monk using a sequence of called non-representable Lyndon algebars whose ultraproduct is representable. Later this non-finite axiomatizability result was refined considerably by Maddux, Hirsch, Hodkinson and Sagi [12, 16]. We let CRRA and LRRA denote the classes of completely representable RAs, and its elementary closure, namely, the class of RAs satisfying the Lyndon conditions as defined in [10, § 11.3.2], respectively. Complete representability of RAs is defined like the CA case. We denote by CRRA the class of completely representable RAs.

Let α be an ordinal and $\mathfrak{A} \in \mathsf{CA}_\alpha$. For any $i, j, l < \alpha$, let $\mathsf{s}_i^j x = x$ if $i = j$ and $\mathsf{s}_i^j x = \mathsf{c}_j(\mathsf{d}_{ij} \cdot x)$ if $i \neq j$. Let ${}_l\mathsf{s}(i,j)x = \mathsf{s}_i^l \mathsf{s}_j^i \mathsf{s}_l^j x$. In the next definition, in its first item we define the notion of forming α-neat reducts of CA_βs with $\beta > \alpha$, in symbols $\mathfrak{Nr}_\alpha$, and in the second item we define relation algebras obtained from cylindric algebras using the operator $\mathfrak{Nr}_2$.

Definition 2.2.

1. Assume that $\alpha < \beta$ are ordinals and that $\mathfrak{B} \in \mathsf{CA}_\beta$. Then the *$\alpha$–neat reduct* of $\mathfrak{B}$, in symbols $\mathfrak{Nr}_\alpha\mathfrak{B}$, is the algebra obtained from $\mathfrak{B}$, by discarding cylindrifiers and diagonal elements whose indices are in $\beta \setminus \alpha$, and restricting the universe to the set $Nr_\alpha B = \{x \in \mathfrak{B} : \{i \in \beta : \mathsf{c}_i x \neq x\} \subseteq \alpha\}$.

2. Assume that $\alpha \geq 3$. Let $\mathfrak{A} \in \mathsf{CA}_\alpha$. Then $\mathsf{Ra}\mathfrak{A} = \langle Nr_2\mathfrak{A} : +, \cdot, -, ;, \smile, \mathsf{d}_{01}\rangle$ where for any $x, y \in Nr_n\mathfrak{A}$, $x; y = \mathsf{c}_2(\mathsf{s}_2^1 x \cdot \mathsf{s}_2^0 y)$ and $x =_2 \mathsf{s}(0.1)x$

If $\mathfrak{A} \in \mathsf{CA}_3$, $\mathsf{Ra}\mathfrak{A}$, having the same signature as RA may not be a relation algebra as associativiy of the (abstract) composition operation may fail, but for $\alpha \geq 4$, $\mathsf{RaCA}_\beta \subseteq \mathsf{RA}$. relativized to V. By the same token the variety of representable relation algebras is denoted by RRA. It is known that $\mathbf{I}\mathsf{Gs}_\alpha = \mathsf{RCA}_\alpha = \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{\alpha+\omega} = \bigcap_{k\in\omega} \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{\alpha+k}$ and that $\mathsf{RRA} = \mathbf{S}\mathsf{RaCA}_\omega = \bigcap_{\mathsf{k}\in\omega} \mathbf{S}\mathsf{RaCA}_{3+\mathsf{k}}$.

2.2. Neat embeddings and games

From now on, unless otherwise indicated, n is fixed to be a finite ordinal > 2. Let $i < n$. For n–ary sequences $\bar{x}$ and $\bar{y}$, we write $\bar{x} \equiv_i \bar{y} \iff \bar{y}(j) =$

$\bar{x}(j)$ for all $j \neq i$, To define certain games to be used in the sequel, we recall the notions of *atomic networks* and *atomic games* [10, 11]. Let $i < n$. For n–dimensional atomic networks M and N, we write $M \equiv_i N \iff M(\bar{y}) = N(\bar{y})$ for all $\bar{y} \in {}^n(n \sim \{i\})$.

DEFINITION 2.3.

1. Assume that $\mathfrak{A} \in \mathsf{CA}_n$ is atomic and that $m, k \leq \omega$. The *atomic game* $G^m_k(\mathsf{At}\mathfrak{A})$*, or simply* G^m_k, is the game played on atomic networks of $\mathfrak{A}$ using m nodes and having k rounds [11, Definition 3.3.2], where $\forall$ is offered only one move, namely, *a cylindrifier move*:
   Suppose that we are at round $t > 0$. Then $\forall$ picks a previously played network N_t ($\mathsf{nodes}(N_t) \subseteq m$), $i < n$, $a \in \mathsf{At}\mathfrak{A}$, $\bar{x} \in {}^n\mathsf{nodes}(N_t)$, such that $N_t(\bar{x}) \leq \mathsf{c}_i a$. For her response, $\exists$ has to deliver a network M such that $\mathsf{nodes}(M) \subseteq m$, $M \equiv_i N$, and there is $\bar{y} \in {}^n\mathsf{nodes}(M)$ that satisfies $\bar{y} \equiv_i \bar{x}$ and $M(\bar{y}) = a$.

   We write $G_k(\mathsf{At}\mathfrak{A})$, or simply G_k, for $G^m_k(\mathsf{At}\mathfrak{A})$ if $m \geq \omega$.

2. The ω–rounded game $\mathbf{G}^m(\mathsf{At}\mathfrak{A})$ or simply $\mathbf{G}^m$ is like the game $G^m_\omega(\mathsf{At}\mathfrak{A})$ except that $\forall$ has the option to reuse the m nodes in play.

DEFINITION 2.4. Let m be a finite ordinal > 0. An s word is a finite string of substitutions (s^j_i) ($i, j < m$), a c word is a finite string of cylindrifications (c_i), $i < m$; an sc word w, is a finite string of both, namely, of substitutions and cylindrifications. An sc word induces a partial map $\hat{w} : m \to m$:

- $\hat{\epsilon} = Id$,
- $\widehat{w^i_j} = \hat{w} \circ [i|j]$,
- $\widehat{w\mathsf{c}_i} = \hat{w} \upharpoonright (m \smallsetminus \{i\})$.

If $\bar{a} \in {}^{<m-1}m$, we write $\mathsf{s}_{\bar{a}}$, or $\mathsf{s}_{a_0 \ldots a_{k-1}}$, where $k = |\bar{a}|$, for an arbitrary chosen sc word w such that $\hat{w} = \bar{a}$. Such a w exists by [10, Definition 5.23 Lemma 13.29].

In the next theorem $\mathbf{S}_c$ stands for the operation of forming complete subalgebras.

LEMMA 2.5. *Fix finite* $n \geq 3$*. If* $\mathfrak{A} \in \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_m$ *is atomic, then* $\exists$ *has a winning strategy in* $\mathbf{G}^m(\mathsf{At}\mathfrak{A})$.

PROOF: Fix $2 < n < m$. Assume that $\mathfrak{C} \in \mathsf{CA}_m$, $\mathfrak{A} \subseteq_c \mathfrak{Nr}_n\mathfrak{C}$ is an atomic CA_n and N is an $\mathfrak{A}$–network with $\mathsf{nodes}(N) \subseteq m$. Define $N^+ \in \mathfrak{C}$ by (with notation as introducted in Definition 2.4):

$$N^+ = \prod_{i_0,\ldots,i_{n-1}\in\mathsf{nodes}(N)} \mathsf{s}_{i_0,\ldots,i_{n-1}} N(i_0,\ldots,i_{n-1}).$$

For a network N and function θ, the network $N\theta$ is the complete labelled graph with nodes $\theta^{-1}(\mathsf{nodes}(N)) = \{x \in \mathsf{dom}(\theta) : \theta(x) \in \mathsf{nodes}(N)\}$, and labelling defined by

$$(N\theta)(i_0,\ldots,i_{n-1}) = N(\theta(i_0),\theta(i_1),\ldots,\theta(i_{n-1})),$$

for $i_0,\ldots,i_{n-1} \in \theta^{-1}(\mathsf{nodes}(N))$. Then the following hold:

(1): for all $x \in \mathfrak{C}\setminus\{0\}$ and all $i_0,\ldots,i_{n-1} < m$, there is $a \in \mathsf{At}\mathfrak{A}$, such that $\mathsf{s}_{i_0,\ldots,i_{n-1}}a \,.\, x \neq 0$,

(2): for any $x \in \mathfrak{C}\setminus\{0\}$ and any finite set $I \subseteq m$, there is a network N such that $\mathsf{nodes}(N) = I$ and $x \cdot N^+ \neq 0$. Furthermore, for any networks M, N if $M^+ \cdot N^+ \neq 0$, then $M{\restriction}_{\mathsf{nodes}(M)\cap\mathsf{nodes}(N)} = N{\restriction}_{\mathsf{nodes}(M)\cap\mathsf{nodes}(N)}$,

(3): if θ is any partial, finite map $m \to m$ and if $\mathsf{nodes}(N)$ is a proper subset of m, then $N^+ \neq 0 \to (N\theta)^+ \neq 0$. If $i \notin \mathsf{nodes}(N)$, then $\mathsf{c}_i N^+ = N^+$.

Since $\mathfrak{A} \subseteq_c \mathfrak{Nr}_n\mathfrak{C}$, then $\sum^{\mathfrak{C}} \mathsf{At}\mathfrak{A} = 1$. *For (1), s^i_j is a completely additive operator (any $i, j < m$), hence $\mathsf{s}_{i_0,\ldots,i_{n-1}}$ is, too.* So $\sum^{\mathfrak{C}}\{\mathsf{s}_{i_0\ldots,i_{n-1}}a : a \in \mathsf{At}(\mathfrak{A})\} = \mathsf{s}_{i_0\ldots i_{n-1}}\sum^{\mathfrak{C}} \mathsf{At}\mathfrak{A} = \mathsf{s}_{i_0\ldots,i_{n-1}}1 = 1$ for any $i_0,\ldots,i_{n-1} < m$. Let $x \in \mathfrak{C}\setminus\{0\}$. Assume for contradiction that $\mathsf{s}_{i_0\ldots,i_{n-1}}a \cdot x = 0$ for all $a \in \mathsf{At}\mathfrak{A}$. Then $1 - x$ will be an upper bound for $\{\mathsf{s}_{i_0\ldots i_{n-1}}a : a \in \mathsf{At}\mathfrak{A}\}$. But this is impossible because $\sum^{\mathfrak{C}}\{\mathsf{s}_{i_0\ldots,i_{n-1}}a : a \in \mathsf{At}\mathfrak{A}\} = 1$.

To prove the first part of (2), we repeatedly use (1). We define the edge labelling of N one edge at a time. Initially, no hyperedges are labelled. Suppose $E \subseteq \mathsf{nodes}(N) \times \mathsf{nodes}(N) \ldots \times \mathsf{nodes}(N)$ is the set of labelled hyperedges of N (initially $E = \emptyset$) and $x \,.\, \prod_{\bar{c}\in E} \mathsf{s}_{\bar{c}} N(\bar{c}) \neq 0$. Pick $\bar{d}$ such that $\bar{d} \notin E$. Then by (1) there is $a \in \mathsf{At}(\mathfrak{A})$ such that $x \,.\, \prod_{\bar{c}\in E} \mathsf{s}_{\bar{c}} N(\bar{c}) \,.\, \mathsf{s}_{\bar{d}}a \neq 0$. Include the hyperedge $\bar{d}$ in E. We keep on doing this until eventually all hyperedges will be labelled, so we obtain a completely labelled graph N with $N^+ \neq 0$. it is easily checked that N is a network.

For the second part of (2), we proceed contrapositively. Assume that there is $\bar{c} \in \mathsf{nodes}(M) \cap \mathsf{nodes}(N)$ such that $M(\bar{c}) \neq N(\bar{c})$. Since edges are labelled by atoms, we have $M(\bar{c}) \cdot N(\bar{c}) = 0$, so $0 = \mathsf{s}_{\bar{c}}0 = \mathsf{s}_{\bar{c}}M(\bar{c}) \,.\, \mathsf{s}_{\bar{c}}N(\bar{c}) \geq$

$M^+ \cdot N^+$. A piece of notation. For $i < m$, let Id_{-i} be the partial map $\{(k,k) : k \in m \smallsetminus \{i\}\}$. For the first part of (3) (cf. [10, Lemma 13.29] using the notation in *op.cit*), since there is $k \in m \setminus \mathsf{nodes}(N)$, θ can be expressed as a product $\sigma_0\sigma_1 \ldots \sigma_t$ of maps such that, for $s \leq t$, we have either $\sigma_s = Id_{-i}$ for some $i < m$ or $\sigma_s = [i/j]$ for some $i, j < m$ and where $i \notin \mathsf{nodes}(N\sigma_0 \ldots \sigma_{s-1})$. But clearly $(NId_{-j})^+ \geq N^+$ and if $i \notin \mathsf{nodes}(N)$ and $j \in \mathsf{nodes}(N)$, then $N^+ \neq 0 \to (N[i/j])^+ \neq 0$. The required now follows. The last part is straightforward. Using the above proven facts, we are now ready to show that $\exists$ has a winning strategy in $\mathbf{G}^m$. She can always play a network N with $\mathsf{nodes}(N) \subseteq m$, such that $N^+ \neq 0$.
In the initial round, let $\forall$ play $a \in \mathsf{At}\mathfrak{A}$. $\exists$ plays a network N with $N(0, \ldots, n-1) = a$. Then $N^+ = a \neq 0$. Recall that here $\forall$ is offered only one (cylindrifier) move. At a later stage, suppose $\forall$ plays the cylindrifier move, which we denote by $(N, \langle f_0, \ldots, f_{n-2}\rangle, k, b, l)$. He picks a previously played network N, $f_i \in \mathsf{nodes}(N)$, $l < n, k \notin \{f_i : i < n-2\}$, such that $b \leq \mathsf{c}_l N(f_0, \ldots, f_{i-1}, x, f_{i+1}, \ldots, f_{n-2})$ and $N^+ \neq 0$. Let $\bar{a} = \langle f_0 \ldots f_{i-1}, k, f_{i+1}, \ldots f_{n-2}\rangle$. Then by second part of (3) we have that $\mathsf{c}_l N^+ \cdot \mathsf{s}_{\bar{a}} b \neq 0$ and so by first part of (2), there is a network M such that $M^+ \cdot \mathsf{c}_l N^+ \cdot \mathsf{s}_{\bar{a}} b \neq 0$. Hence $M(f_0, \ldots, f_{i-1}, k, f_{i-2}, \ldots, f_{n-2}) = b$, $\mathsf{nodes}(M) = \mathsf{nodes}(N) \cup \{k\}$, and $M^+ \neq 0$, so this property is maintained. $\square$

2.3. The class of completely representable relation and cylindric algebras is not elementary

Let LRRA be the class of relation algebra whose atom structures satisfy the Lyndon condition, and LCA_n denote the class of CA_ns whose atom structures are in $\mathsf{LCAS_n}$ as defined in [11]; i.e those algebras whose atom structures also satisfy the Lyndon conditions for cylindric algebras.

Lemma 2.6. *For any infinite cardinal κ, there exists an atomless $\mathfrak{C} \in \mathsf{CA}_n$ such that for all $2 < n < \omega$, $\mathfrak{Nr}_n\mathfrak{C}$ and $\mathfrak{Ra}\mathsf{CA}_\omega$ are atomic, with $|\mathsf{At}(\mathfrak{Nr}_n\mathfrak{C})| = |\mathsf{At}(\mathfrak{Ra}\mathfrak{C}| = 2^\kappa$, $\mathfrak{Nr}_n\mathfrak{C} \in \mathsf{LCA}_n$ and $\mathfrak{Ra}\mathfrak{C} \in \mathsf{LRRA}$, but neither $\mathfrak{Nr}_n\mathfrak{C}$ nor $\mathfrak{Ra}\mathfrak{C}$ are completely representable.*

Proof: We use the following uncountable version of Ramsey's theorem due to Erdös and Rado: If $r \geq 2$ is finite, k an infinite cardinal, then $exp_r(k)^+ \to (k^+)^{r+1}_k$, where $exp_0(k) = k$ and inductively $exp_{r+1}(k) = 2^{exp_r(k)}$. The above partition symbol describes the following statement. If

f is a coloring of the $r+1$ element subsets of a set of cardinality $exp_r(k)^+$ in k many colors, then there is a homogeneous set of cardinality k^+ (a set, all whose $r+1$ element subsets get the same f-value). We will construct the required $\mathfrak{C} \in \mathsf{CA}_\omega$ from a relation algebra (to be denoted in a while by $\mathfrak{A}$) having an 'ω-dimensional cylindric basis.' in the sense of Maddux [16] To define the relation algebra, we specify its atoms and forbidden triples. Let κ be the given cardinal in the hypothesis of the Theorem. The atoms are Id, $\mathsf{g}_0^i : i < 2^\kappa$ and $\mathsf{r}_j : 1 \leq j < \kappa$, all symmetric. The forbidden triples of atoms are all permutations of (Id, x, y) for $x \neq y$, $(\mathsf{r}_j, \mathsf{r}_j, \mathsf{r}_j)$ for $1 \leq j < \kappa$ and $(\mathsf{g}_0^i, \mathsf{g}_0^{i'}, \mathsf{g}_0^{i^*})$ for $i, i', i^* < 2^\kappa$. Write g_0 for $\{\mathsf{g}_0^i : i < 2^\kappa\}$ and r_+ for $\{\mathsf{r}_j : 1 \leq j < \kappa\}$. Call this atom structure α. Consider the term algebra $\mathfrak{A}$ defined to be the subalgebra of the complex algebra of this atom structure generated by the atoms. We claim that $\mathfrak{A}$, as a relation algebra, has no complete representation, hence any algebra sharing this atom structure is not completely representable, too. Indeed, it is easy to show that if $\mathfrak{A}$ and $\mathfrak{B}$ are atomic relation algebras sharing the same atom structure, so that $\mathsf{At}\mathfrak{A} = \mathsf{At}\mathfrak{B}$, then $\mathfrak{A}$ is completely representable $\iff$ $\mathfrak{B}$ is completely representable.

Assume for contradiction that $\mathfrak{A}$ has a complete representation with base M. Let x, y be points in the representation with $\mathsf{M} \models \mathsf{r}_1(x, y)$. For each $i < 2^\kappa$, there is a point $z_i \in \mathsf{M}$ such that $\mathsf{M} \models \mathsf{g}_0^i(x, z_i) \wedge \mathsf{r}_1(z_i, y)$. Let $Z = \{z_i : i < 2^\kappa\}$. Within Z, each edge is labelled by one of the κ atoms in r_+. The Erdos-Rado theorem forces the existence of three points $z^1, z^2, z^3 \in Z$ such that $\mathsf{M} \models \mathsf{r}_j(z^1, z^2) \wedge \mathsf{r}_j(z^2, z^3) \wedge \mathsf{r}_j(z^3, z_1)$, for some single $j < \kappa$. This contradicts the definition of composition in $\mathfrak{A}$ (since we avoided monochromatic triangles). Let S be the set of all atomic $\mathfrak{A}$-networks N with nodes ω such that $\{\mathsf{r}_i : 1 \leq i < \kappa : \mathsf{r}_i \text{ is the label of an edge in } N\}$ is finite. Then it is straightforward to show S is an amalgamation class, that is for all $M, N \in S$ if $M \equiv_{ij} N$ then there is $L \in S$ with $M \equiv_i L \equiv_j N$, witness [10, Definition 12.8] for notation. Now let X be the set of finite $\mathfrak{A}$-networks N with nodes $\subseteq \kappa$ such that:

1. each edge of N is either (a) an atom of $\mathfrak{A}$ or (b) a cofinite subset of $\mathsf{r}_+ = \{\mathsf{r}_j : 1 \leq j < \kappa\}$ or (c) a cofinite subset of $\mathsf{g}_0 = \{\mathsf{g}_0^i : i < 2^\kappa\}$ and

2. N is 'triangle-closed', i.e. for all $l, m, n \in \mathsf{nodes}(N)$ we have $N(l, n) \leq N(l, m); N(m, n)$. That means if an edge (l, m) is labelled by Id then $N(l, n) = N(m, n)$ and if $N(l, m), N(m, n) \leq \mathsf{g}_0$ then $N(l, n) \cdot \mathsf{g}_0 = 0$ and if $N(l, m) = N(m, n) = \mathsf{r}_j$ (some $1 \leq j < \omega$) then $N(l, n) \cdot \mathsf{r}_j = 0$.

For $N \in X$ let $\widehat{N} \in \mathfrak{Ca}(S)$ be defined by

$$\{L \in S : L(m,n) \leq N(m,n) \text{ for } m,n \in \mathsf{nodes}(N)\}.$$

For $i \in \omega$, let $N{\restriction}_{-i}$ be the subgraph of N obtained by deleting the node i.

Then if $N \in X,\ i < \omega$ then $\widehat{\mathsf{c}_i N} = \widehat{N{\restriction}_{-i}}$. The inclusion $\widehat{\mathsf{c}_i N} \subseteq (\widehat{N{\restriction}_{-i}})$ is clear.

Conversely, let $L \in (\widehat{N{\restriction}_{-i}})$. We seek $M \equiv_i L$ with $M \in \widehat{N}$. This will prove that $L \in \widehat{\mathsf{c}_i N}$, as required. Since $L \in S$ the set $T = \{\mathsf{r}_i \notin L\}$ is infinite. Let T be the disjoint union of two infinite sets $Y \cup Y'$, say. To define the ω-network M we must define the labels of all edges involving the node i (other labels are given by $M \equiv_i L$). We define these labels by enumerating the edges and labeling them one at a time. So let $j \neq i < \kappa$. Suppose $j \in \mathsf{nodes}(N)$. We must choose $M(i,j) \leq N(i,j)$. If $N(i,j)$ is an atom then of course $M(i,j) = N(i,j)$. Since N is finite, this defines only finitely many labels of M. If $N(i,j)$ is a cofinite subset of g_0 then we let $M(i,j)$ be an arbitrary atom in $N(i,j)$. And if $N(i,j)$ is a cofinite subset of r_+ then let $M(i,j)$ be an element of $N(i,j) \cap Y$ which has not been used as the label of any edge of M which has already been chosen (possible, since at each stage only finitely many have been chosen so far). If $j \notin \mathsf{nodes}(N)$ then we can let $M(i,j) = \mathsf{r}_k \in Y$ some $1 \leq k < \kappa$ such that no edge of M has already been labelled by r_k. It is not hard to check that each triangle of M is consistent (we have avoided all monochromatic triangles) and clearly $M \in \widehat{N}$ and $M \equiv_i L$. The labeling avoided all but finitely many elements of Y', so $M \in S$. So $(\widehat{N{\restriction}_{-i}}) \subseteq \widehat{\mathsf{c}_i N}$.

Now let $\widehat{X} = \{\widehat{N} : N \in X\} \subseteq \mathfrak{Ca}(S)$. Then we claim that the subalgebra of $\mathfrak{Ca}(S)$ generated by $\widehat{X}$ is simply obtained from $\widehat{X}$ by closing under finite unions. Clearly all these finite unions are generated by $\widehat{X}$. We must show that the set of finite unions of $\widehat{X}$ is closed under all cylindric operations. Closure under unions is given. For $\widehat{N} \in X$ we have $-\widehat{N} = \bigcup_{m,n \in \mathsf{nodes}(N)} \widehat{N_{mn}}$ where N_{mn} is a network with nodes $\{m,n\}$ and labeling $N_{mn}(m,n) = -N(m,n)$. N_{mn} may not belong to X but it is equivalent to a union of at most finitely many members of $\widehat{X}$. The diagonal $\mathsf{d}_{ij} \in \mathfrak{Ca}(S)$ is equal to $\widehat{N}$ where N is a network with nodes $\{i,j\}$ and labeling $N(i,j) = \mathsf{Id}$. Closure under cylindrification is given. Let $\mathfrak{C}$ be the subalgebra of $\mathfrak{Ca}(S)$ generated by $\widehat{X}$. Then $\mathfrak{A} = \mathfrak{Ra}\mathfrak{C}$.

To see why, each element of $\mathfrak{A}$ is a union of a finite number of atoms, possibly a co–finite subset of g_0 and possibly a co–finite subset of r_+. Clearly $\mathfrak{A} \subseteq \mathfrak{Ra}\mathfrak{C}$. Conversely, each element $z \in \mathfrak{Ra}\mathfrak{C}$ is a finite union $\bigcup_{N\in F} \widehat{N}$, for some finite subset F of X, satisfying $\mathsf{c}_i z = z$, for $i > 1$. Let $i_0, \ldots, i_k$ be an enumeration of all the nodes, other than 0 and 1, that occur as nodes of networks in F. Then, $\mathsf{c}_{i_0} \ldots \mathsf{c}_{i_k} z = \bigcup_{N\in F} \mathsf{c}_{i_0} \ldots \mathsf{c}_{i_k} \widehat{N} = \bigcup_{N\in F} (\widehat{N{\restriction}_{\{0,1\}}}) \in \mathfrak{A}$. So $\mathfrak{Ra}\mathfrak{C} \subseteq \mathfrak{A}$. Thus $\mathfrak{A}$ is the relation algebra reduct of $\mathfrak{C} \in \mathsf{CA}_\omega$, but $\mathfrak{A}$ has no complete representation. Let $n > 2$. Let $\mathfrak{B} = \mathfrak{Nr}_n\mathfrak{C}$. Then $\mathfrak{B} \in \mathsf{Nr}_n\mathsf{CA}_\omega$, is atomic, but has no complete representation for plainly a complete representation of $\mathfrak{B}$ induces one of $\mathfrak{A}$. In fact, because $\mathfrak{B}$ is generated by its two dimensional elements, and its dimension is at least three, its Df reduct is not completely representable.

It remains to show that the ω–dilation $\mathfrak{C}$ is atomless. For any $N \in X$, we can add an extra node extending N to M such that $\emptyset \subsetneq M' \subsetneq N'$, so that N' cannot be an atom in $\mathfrak{C}$. By Lemma 2.5, $\exists$ has a winning strategy in $\mathbf{G}^\omega(\mathsf{At}\mathfrak{B})$. Since infnitely many nodes are in play, then reusing nodes does not make $\mathbf{G}^\omega$ any stronger than the usual ω rounded game G_ω according to [11, Definition 3.3.2]. Thus $\exists$ has a winning strategy in $G_\omega(\mathsf{At}\mathfrak{B})$, *a fortiori*, that $\exists$ has a winning strategy in the k rounded atomic game $G_k(\mathsf{At}\mathfrak{B})$ for all finite $k \in \omega$. By definition; coding winning strategy's in the first order Lyndon conditions, we get $\mathfrak{B} \in \mathsf{LCA}_n$. For relation algebras, we have $\mathfrak{A} \in \mathsf{RaCA}_\omega$ and $\mathfrak{A}$ has no complete representation. The rest is like the CA case, using the Ra analogue of Lemma 2.5, when the dilation is ω-dimensional, namely, $\mathfrak{A} \in \mathbf{S}_c\mathsf{RaCA}_\omega \implies$, and $\exists$ has a winning strategy in $\mathbf{G}^\omega$ with the last notation taken from [8]. $\square$

Corollary 2.7. For $2 < n < \omega$, the classes CRCA_n and CRRA are not elementary.

Proof: $\mathsf{LCA}_n = \mathbf{El}\mathsf{CRCA}_n$, hence $\mathfrak{B} \in \mathbf{El}\mathsf{CRCA}_n \sim \mathsf{CRCA}_n$, so CRCA_n is not elementary. For relation algebras, we use the algebra $\mathfrak{A}$ constructed in the previous Theorem, too. We have $\mathfrak{A} \in \mathsf{RaCA}_\omega$ and $\mathfrak{A}$ has no complete representation. The rest is like the CA case, using the Ra analogue of Lemma 2.5, when the dilation is ω–dimensional, namely, $\mathfrak{A} \in \mathbf{S}_c\mathsf{RaCA}_\omega \implies \exists$ has a winning strategy in F^ω with the last notation taken from [8]. $\square$

The last was proved by Hirsch and Hodkinsdon in [9]. Our proof here is entirely different using so-called Maddux relation algebras by specifying forbidden list of atoms, cf. [16, 10]. These algebars have ω–dimensional cylindric basis. The proof of Hirsch and Hodkinson uses so-called Rainbow constuction. The two proofs are not only distinct but they are conceptually disjoint.

But we can even go further for relation algebras:

Theorem 2.8. *The class* CRRA *is not closed under* $\equiv_{\infty,\omega}$.

Proof: Take $\mathfrak{R}$ to be a symmetric, atomic relation algebra with atoms

$$\mathsf{Id}, \mathsf{r}(i), \mathsf{y}(i), \mathsf{b}(i) : i < \omega.$$

Non-identity atoms have colors, r is red, b is blue, and y is yellow. All atoms are self-converse. The composition of atoms is defined by listing the forbidden triples. The forbidden triples are (Peircean transforms) or permutations of (Id, x, y) for $x \neq y$, and

$$(\mathsf{r}(i), \mathsf{r}(i), \mathsf{r}(j)),\ (\mathsf{y}(i), \mathsf{y}(i), \mathsf{y}(j)),\ (\mathsf{b}(i), \mathsf{b}(i), \mathsf{b}(j))\ \ i \leq j < \omega$$

$\mathfrak{R}$ is the complex algebra over this atom structure. Let α be an ordinal. $\mathfrak{R}^\alpha$ is obtained from $\mathfrak{R}$ by splitting the atom $\mathsf{r}(0)$ into α parts $\mathsf{r}^k(0) : k < \alpha$ and then taking the full complex algebra. In more detail, we put red atoms $r^k(0)$ for $k < \alpha$. In the altered algebra the forbidden triples are $(\mathsf{y}(i), \mathsf{y}(i), \mathsf{y}(j)), (\mathsf{b}(i), \mathsf{b}(i), \mathsf{b}(j)),\ \ i \leq j < \omega$, $(\mathsf{r}(i), \mathsf{r}(i), \mathsf{r}(j)),\ \ 0 < i \leq j < \omega$, $(\mathsf{r}^k(0), \mathsf{r}^l(0), \mathsf{r}(j)),\ \ 0 < j < \omega, k, l < \alpha$, $(\mathsf{r}^k(0), \mathsf{r}^l(0), \mathsf{r}^m(0)),\ \ k, l, m < \alpha$. Now let $\mathfrak{B} = \mathfrak{R}^\omega$ and $\mathfrak{A} = \mathfrak{R}^\mathfrak{n}$ with $\mathfrak{n} \geq 2^{\aleph_0}$. For an ordinal α, $\mathfrak{R}^\alpha$ is as defined in the previous remark. In $\mathfrak{R}^\alpha$, we use the following abbreviations: $r(0) = \sum_{k<\alpha} \mathsf{r}^k(0)$ $\mathsf{r} = \sum_{i<\omega} \mathsf{r}(i)$ $\mathsf{y} = \sum_{i<\omega} \mathsf{y}(i)$ $\mathsf{b} = \sum_{i<\omega} \mathsf{b}(i)$. These suprema exist because they are taken in the complex algebras which are complete. The *index* of $\mathsf{r}(i), \mathsf{y}(i)$ and $\mathsf{b}(i)$ is i and the index of $\mathsf{r}^k(0)$ is also 0. Now let $\mathfrak{B} = \mathfrak{R}^\omega$ and $\mathfrak{A} = \mathfrak{R}^\mathfrak{n}$ with $\mathfrak{n} \geq 2^{\aleph_0}$. We claim that $\mathfrak{B} \in \mathsf{RaCA}_\omega$ and $\mathfrak{A} \equiv \mathfrak{B}$. For the first required, we show that $\mathfrak{B}$ has a cylindric bases by exhibiting a winning strategy for $\exists$ in the cylindric-basis game, which is a simpler version of the hyperbasis game [10, Definition 12.26]. At some stage of the game, let the play so far be $N_0, N_1, \ldots, N_{t-1}$ for some $t < \omega$. We say that an edge (m, n) of an atomic network N is a *diversity edge* if $N(m, n) \cdot \mathsf{Id} = 0$. Each diversity edge of each atomic network in the play has an owner—either $\exists$ or $\forall$, which we will allocate as we define $\exists$'s strategy. If an edge (m, n) belongs to player p then so does the reverse edge (n, m)

and we will only specify one of them. Since our algebra is symmetric, so the label of the reverse edge is equal to the label of the edge, so again need to specify only one. For the next round $\exists$ must define N_t in response to $\forall$'s move. If there is an already played network N_i (some $i < t$) and a finitary map $\sigma : \omega \to \omega$ such that $N_t\sigma$ 'answers' his move, then she lets $N_t = N_i\sigma$. From now on we assume that there is no such N_i and σ. We consider the three types of $\forall$ can make. If he plays an *atom move* by picking an atom a, $\exists$ plays an atomic network N with $N(0, 1) = a$ and for all $x \in \omega \setminus \{1\}$, $N(0, x) = \mathsf{Id}$.

If $\forall$ plays a *triangle move* by picking a previously played N_x (some $x < t$), nodes i, j, k with $k \notin \{i, j\}$ and atoms a, b with $a; b \geq N_x(i, j)$, we know that $a, b \neq 1'$, as we are assuming the $\exists$ cannot play an embedding move (if $a = \mathsf{Id}$, consider N_x and the map $[k/i]$). $\exists$ must play a network $N_t \equiv_k N_x$ such that $N_t(i, k) = a$, $N_t(k, j) = b$. These edges, (i, k) and (k, j), belong to $\forall$ in N_t. All diversity edges not involving k have the same owner in N_t as they did in N_x. And all edges (l, k) for $k \notin \{i, j\}$ belong to $\exists$ in N_x. To label these edges $\exists$ chooses a colour c different than the colours of a, b(we have three colours so this is possible). Then, one at a time, she labels each edge (l, k) by an atom with colour c and a non-zero index which has not yet been used to label any edge of any network played in the game. She does this one edge at a time, each with a new index. There are infinitely many indices to choose, so this can be done.

Finally, $\forall$ can play an amalgamation move by picking M, $N \in \{N_s : s < t\}$, nodes i, j such that $M \equiv_{ij} N$. If there is N_s (some $s < t$) and a map σ : $\mathsf{nodes}(N_s) \to \mathsf{nodes}(M) \cup \mathsf{nodes}(N)$ such that $M \equiv_i N_s\sigma \equiv_j N$ then $\exists$ lets $N_t = N_s\sigma$. Ownership of edges is inherited from N_s. If there is no such N_s and σ then there are two cases. If there are three nodes x, y, z in the 'amalgam' such that $M(j, x)$ and $N(x, i)$ are both red and of the same index, $M(j, y)$, $N(y, i)$ are both yellow and of the same index and $M(j, z)$, $N(z, i)$ are both blue and of the same index, then the new edge (i, j) belongs to $\forall$ in N_t. It will be labelled by either $\mathsf{r}^0(0)$, $\mathsf{b}(0)$ or $\mathsf{y}(0)$ and it it is easy to show that at least one of these will be a consistent choice. Otherwise, if there is no such x, y, z then the new edge (i, j) belongs to $\exists$ in N_t. She chooses a colour c such that there is no x with $M(j, x)$ and $N(x, i)$ both having colour c and the same index. And she chooses a non-zero index for $N_t(i, j)$ which is new to the game (as with triangle moves). If $k \neq k' \in M \cap N$ then (j, k) has the same owner in N_t as it does in M, (k, i) has the same owner in N_t as it does in N and (k, k') belongs to $\exists$ in

N_t if it belongs to $\exists$ in *either* M or N, otherwise it belongs to $\forall$ in N_t. Now the only way $\exists$ could lose, is if $\forall$ played an amalgamation move (M, N, i, j) such that there are $x, y, z \in M \cap N$ such that $M(j, x) = \mathsf{r}^k(0)$, $N(x, i) = \mathsf{r}^{k'}(0)$, $M(j, y) = N(y, i) = \mathsf{b}(0)$ and $M(j, z) = N(z, i) = \mathsf{y}(0)$. But according to $\exists$'s strategy, she never chooses atoms with index 0, so all these edges must have been chosen by $\forall$. This contradiction proves the required.

Now, let $\mathcal{H}$ be an ω-dimensional cylindric basis for $\mathfrak{B}$. Then $\mathfrak{Ca}\mathcal{H} \in \mathsf{CA}_\omega$. Consider the cylindric algebra $\mathfrak{C} = \mathfrak{Sg}^{\mathfrak{Ca}\mathcal{H}}\mathfrak{B}$, the subalgebra of $\mathfrak{Ca}\mathcal{H}$ generated by $\mathfrak{B}$. In principal, new two dimensional elements that were not originally in $\mathfrak{B}$, can be created in $\mathfrak{C}$ using the spare dimensions in $\mathfrak{Ca}(\mathcal{H})$. But next we exclude this possibility. We show that $\mathfrak{B}$ exhausts the 2–dimensional elements of $\mathsf{Ra}\mathfrak{C}$, more concisely, we show that $\mathfrak{B} = \mathsf{Ra}\mathfrak{C}$. For this purpose, we want to find out what are the elements of $\mathfrak{Ca}\mathcal{H}$ that are generated by $\mathfrak{B}$. Let M be a (not necessarily atomic) finite network over $\mathfrak{B}$ whose nodes are a finite subset of ω.

- Define (using the same notation in the proof of Theorem 2.6) $\widehat{M} = \{N \in \mathcal{H} : N \leq M\} \in \mathfrak{Ca}\mathcal{H}$. ($N \leq M$ means that for all $i, j \in M$ we have $N(i, j) \leq M(i, j)$.)
- A *block* is an element of the form $\widehat{M}$ for some finite network M such that
  1. M is triangle-closed, i.e. for all $i, j, k \in M$ we have $M(i, k) \leq M(i, j); M(j, k)$
  2. If x is the label of an irreflexive edge of M then $x = \mathsf{Id}$ or $x \leq \mathsf{r}$ or $x \leq \mathsf{y}$ or $x \leq \mathsf{b}$ (we say x is 'monochromatic'), and $|\{i : x \cdot (\mathsf{r}(i) + \mathsf{y}(i) + \mathsf{b}(i)) \neq 0\}|$ is either $0, 1$ or infinite (we say that the number of indices of x is either $0, 1$ or infinite).

We prove:

1. For any block $\widehat{M}$ and $i < \omega$ we have
$$\mathsf{c}_i\widehat{M} = (M{\restriction}_{\mathsf{dom}(M)\setminus\{i\}})^\frown$$
2. The domain of $\mathfrak{C}$ consists of finite sums of blocks.

$\mathsf{c}_i\widehat{M} \subseteq (M{\restriction}_{\mathsf{dom}(M)\setminus\{i\}})^\frown$ is obvious. If $i \notin M$ the equality is trivial. Let $N \in (M{\restriction}_{\mathsf{dom}(M)\setminus\{i\}})^\frown$, i.e. $N \leq M{\restriction}_{\mathsf{dom}(M)\setminus\{i\}}$. We must show that $N \in \mathsf{c}_i\widehat{M}$

and for this we must find $L \equiv_i N$ with $L \in \widehat{M}$. $L \equiv_i N$ determines every edge of L except those involving i. For each $j \in M$, if the number of indices in $M(i,j)$ is just one, say $M(i,j) = \mathsf{r}(k)$, then let $L(i,j)$ be an arbitrary atom below $\mathsf{r}(k)$. There should be no inconsistencies in the labelling so far defined for L, by triangle-closure for M. For all the other edges (i,j) if $j \in M$ there are infinitely many indices in $M(i,j)$ and if $j \notin M$ then we have an unrestricted choice of atoms for the label. These edges are labelled one at a time and each label is given an atom with a new index, thus avoiding any inconsistencies. This defines $L \equiv_i N$ with $L \in \widehat{M}$. For the second part, we already have seen that the set of finite sums of blocks is closed under cylindrification. We'll show that this set is closed under all the cylindric operations and includes $\mathfrak{B}$. For any $x \in \mathfrak{B}$ and $i,j < \omega$, let N^{ij}_x be the $\mathfrak{B}$-network with two nodes $\{i,j\}$ and labelling $N^{ij}_x(i,i) = N^{ij}_x(j,j) = \mathsf{Id}$, and $N^{ij}(i,j) = x$, $N^{ij}_x(j,i) = \breve{x}$. Clearly N^{ij}_x is triangle closed. And $\widehat{N^{01}_x} = x$. For any $x \in \mathfrak{B}$, we have $x = x \cdot \mathsf{Id} + x \cdot \mathsf{r} + x \cdot \mathsf{y} + x \cdot \mathsf{b}$, so $x = \widehat{N^{01}_{x\cdot\mathsf{Id}}} + \widehat{N^{01}_{x\cdot\mathsf{r}}} + \widehat{N^{01}_{x\cdot\mathsf{y}}} + \widehat{N^{01}_{x\cdot\mathsf{b}}}$ and the labels of these four networks are monochromatic. The first network defines a block and for each of the last three, if the number if indices is infinite then it is a block. If the number of indices is finite then it is a finite union of blocks. So every element of $\mathfrak{B}$ is a finite union of blocks.

For the diagonal elements, $\mathsf{d}_{ij} = \widehat{N^{ij}_{\mathsf{Id}}}$. Closure under sums is obvious. For negation, take a block $\widehat{M}$. Then $-\widehat{M} = \sum_{i,j\in M} \widehat{N^{ij}_{-N(i,j)}}$. As before we can replace $\widehat{N^{ij}_{-N(i,j)}}$ by a finite union of blocks. Thus the set of finite sums of blocks includes $\mathfrak{B}$ and the diagonals and is closed under all the cylindric operations. Since every block is clearly generated from $\mathfrak{B}$ using substitutions and intersection only. It remains to show that $\mathfrak{B} = \mathsf{Ra}\mathfrak{C}$. Take a block $\widehat{M} \in \mathsf{Ra}\mathfrak{C}$. Then $\mathsf{c}_i\widehat{M} = \widehat{M}$ for $2 \le i < \omega$. By the first part of the lemma, $\widehat{M} = \widehat{M\restriction_{\{0,1\}}} \in \mathfrak{B}$.

We finally show that $\exists$ has a winning strategy in an Ehrenfeucht–Fraïssé-game over $(\mathfrak{A}, \mathfrak{B})$ concluding that $\mathfrak{A} \equiv_\infty \mathfrak{B}$. At any stage of the game, if $\forall$ places a pebble on one of $\mathfrak{A}$ or $\mathfrak{B}$, $\exists$ must place a matching pebble, on the other algebra. Let $\bar{a} = \langle a_0, a_1, \ldots, a_{n-1}\rangle$ be the position of the pebbles played so far (by either player) on $\mathfrak{A}$ and let $\bar{b} = \langle b_0, \ldots, b_{n-1}\rangle$ be the the position of the pebbles played on $\mathfrak{B}$. $\exists$ maintains the following properties throughout the game.

- For any atom x (of either algebra) with $x \cdot \mathsf{r}(0) = 0$ then $x \in a_i \iff x \in b_i$.

- $\bar{a}$ induces a finite partition of $\mathsf{r}(0)$ in $\mathfrak{A}$ of 2^n (possibly empty) parts $p_i : i < 2^n$ and $\bar{b}$ induces a partition of $\mathsf{r}(0)$ in $\mathfrak{B}$ of parts $q_i : i < 2^n$. p_i is finite iff q_i is finite and, in this case, $|p_i| = |q_i|$.

Now we show that CRRA is not closed under $\equiv_{\infty,\omega}$. Since $\mathfrak{B} \in \mathsf{RaCA}_\omega$ has countably many atoms, then $\mathfrak{B}$ is completely representable [8, Theorem 29]. For this purpose, we show that $\mathfrak{A}$ is not completely representable. We work with the term algebra, $\mathfrak{TmAt}\mathfrak{A}$, since the latter is completely representable $\iff$ the complex algebra is. Let $\mathsf{r} = \{\mathsf{r}(i) : 1 \leq i < \omega\} \cup \{\mathsf{r}^k(0) : k < 2^{\aleph_0}\}$, $\mathsf{y} = \{\mathsf{y}(i) : i \in \omega\}$, $\mathsf{b}^+ = \{\mathsf{b}(i) : i \in \omega\}$. It is not hard to check every element of $\mathfrak{TmAt}\mathfrak{A} \subseteq \wp(\mathsf{At}\mathfrak{A})$ has the form $F \cup R_0 \cup B_0 \cup Y_0$, where F is a finite set of atoms, R_0 is either empty or a co-finite subset of r, B_0 is either empty or a co–finite subset of b, and Y_0 is either empty or a co–finite subset of y. Using an argument similar to that used in the proof of Lemma 2.6, we show that the existence of a complete representation necessarily forces a monochromatic triangle, that we avoided at the start when defining $\mathfrak{A}$. Let x, y be points in the representation with $M \models \mathsf{y}(0)(x, y)$. For each $i < 2^{\aleph_0}$, there is a point $z_i \in M$ such that $M \models \mathsf{red}(x, z_i) \wedge \mathsf{y}(0)(z_i, y)$ (some red $\mathsf{red} \in \mathsf{r}$). Let $Z = \{z_i : i < 2^{\aleph_0}\}$. Within Z each edge is labelled by one of the ω atoms in y^+ or b^+. The Erdos-Rado theorem forces the existence of three points $z^1, z^2, z^3 \in Z$ such that $M \models \mathsf{y}(j)(z^1, z^2) \wedge \mathsf{y}(j)(z^2, z^3) \wedge \mathsf{y}(j)(z^3, z_1)$, for some single $j < \omega$ or three points $z^1, z^2, z^3 \in Z$ such that $M \models \mathsf{b}(l)(z^1, z^2) \wedge \mathsf{b}(l)(z^2, z^3) \wedge \mathsf{b}(l)(z^3, z_1)$, for some single $l < \omega$. This contradicts the definition of composition in $\mathfrak{A}$ (since we avoided monochromatic triangles). We have proved that CRRA is not closed under $\equiv_{\infty,\omega}$, since $\mathfrak{A} \equiv_{\infty,\omega} \mathfrak{B}$, $\mathfrak{A}$ is not completely representable, but $\mathfrak{B}$ is completely representable. $\square$

3. Other algebras of relations

We shall have the occasion to deal with (in addition to CAs) the following cylindric–like algebras [1]: Df short for diagonal free cylindric algebras, Sc short for Pinter's substitution algebras, $\mathsf{QA}(\mathsf{QEA})$ short for quasi–polyadic (equality) algebras, $\mathsf{PA}(\mathsf{PEA})$ short for polyadic (equality) algebras. For

K any of these classes and α any ordinal, we write K_α for variety of α–dimensional K algebras which can be axiomatized by a finite schema of equations, and RK_α for the class of representable K_αs, which happens to be a variety too (that cannot be axiomatized by a finite schema of equations for $\alpha > 2$ unless $\mathsf{K} = \mathsf{PA}$ and $\alpha \geq \omega$). The standard reference for all the classes of algebras mentioned previously is [7]. We recall the concrete versions of such algebras. Let $\tau : \alpha \to \alpha$ and $X \subseteq {}^\alpha U$, then

$$\mathsf{S}_\tau X = \{s \in {}^\alpha U : s \circ \tau \in X\}.$$

For $i, j \in \alpha$, $[i|j]$ is the replacement on α that sends i to j and is the identity map on $\alpha \sim \{i\}$ while $[i, j]$ is the transposition on α that interchanges i and j.

- A *diagonal free cylindric set algebra of dimension* α is an algebra of the form $\langle \mathfrak{B}({}^\alpha U), \mathsf{C}_i \rangle_{i,j<\alpha}$.
- A *Pinter's substitution set algebra of dimension* α is an algebra of the form
  $\langle \mathfrak{B}({}^\alpha U), \mathsf{C}_i, \mathsf{S}_{[i|j]} \rangle_{i,j<\alpha}$.
- A *quasi-polyadic set algebra of dimension* α is an algebra of the form $\langle \mathfrak{B}({}^\alpha U), \mathsf{C}_i, \mathsf{S}_{[i|j]}, \mathsf{S}_{[i,j]} \rangle_{i,j<\alpha}$.
- A *quasi-polyadic equality set algebra* is an algebra of the form $\langle \mathfrak{B}({}^\alpha U), \mathsf{C}_i, \mathsf{S}_{[i|j]}, \mathsf{S}_{[i,j]}, \mathsf{D}_{ij} \rangle_{i,j<\alpha}$.
- A *polyadic set algebra of dimension* α is an algebra of the form $\langle \mathfrak{B}({}^\alpha U), \mathsf{C}_i, \mathsf{S}_\tau \rangle_{\tau:\alpha\to\alpha}$.
- A *polyadic equality set algebra of dimension* α is an algebra of the form
  $\langle \mathfrak{B}({}^\alpha U), \mathsf{C}_i, \mathsf{S}_\tau \rangle_{\tau:\alpha\to\alpha, i,j<\alpha}$

Let α be an ordinal. For any such abstract class of algebras K_α in the above table, RK_α is defined to be the subdirect product of set algebras of dimension α. For $\alpha < \omega$, $\mathsf{PA}_\alpha(\mathsf{PEA}_\alpha)$ is definitionally equivalent to $\mathsf{QA}_\alpha(\mathsf{QEA}_\alpha)$ which is no longer the case for infinite α where the deviation is largely significant. For example a countable QA_ω has a countable signature, while a countable PA_ω has an uncountable signature having the same cardinality as (substitutions in) ${}^\omega\omega$. The class of completely representable

class	extra non-Boolean operators
Df_α	$\mathsf{c}_i : i < \alpha$
Sc_α	$\mathsf{c}_i, \mathsf{s}_i^j : i, j < \alpha$
CA_α	$\mathsf{c}_i, \mathsf{d}_{ij} : i, j < \alpha$
PA_α	$\mathsf{c}_i, \mathsf{s}_\tau : i < n,\ \tau \in {}^\alpha\alpha$
PEA_α	$\mathsf{c}_i, \mathsf{d}_{ij}, \mathsf{s}_\tau : i, j < n,\ \tau \in {}^\alpha\alpha$
QA_α	$\mathsf{c}_i, \mathsf{s}_i^j, \mathsf{s}_{[i,j]} : i, j < \alpha$
QEA_α	$\mathsf{c}_i, \mathsf{d}_{ij}, \mathsf{s}_i^j, \mathsf{s}_{[i,j]} : i, j < \alpha$

Figure 1. Non-Boolean operators for the classes

K_αs (K any of the above classes) is denoted by CRK_α. For a BAO, $\mathfrak{A}$ say, for any ordinal α, $\mathfrak{Rd}_{ca}\mathfrak{A}$ denotes the cylindric reduct of $\mathfrak{A}$ if it has one, $\mathfrak{Rd}_{sc}\mathfrak{A}$ denotes the Sc reduct of $\mathfrak{A}$ if it has one, and $\mathfrak{Rd}_{df}\mathfrak{A}$ denotes the reduct of $\mathfrak{A}$ obtained by discarding all the operations except for cylindrifications. If $\mathfrak{A}$ is any of the above classes, it is always the case that $\mathfrak{Rd}_{df}\mathfrak{A} \in \mathsf{Df}_\alpha$. If $\mathfrak{A} \in \mathsf{CA}_\alpha$, then $\mathfrak{Rd}_{sc}\mathfrak{A} \in \mathsf{Sc}_\alpha$, and if $\mathfrak{A} \in \mathsf{QEA}_\alpha$ then $\mathfrak{Rd}_{ca}\mathfrak{A} \in \mathsf{CA}_\alpha$. Roughly speaking for an ordinal α, CA_αs are not expansions of Sc_αs, but they are *definitionally equivalent* to expansions of Sc_α, because the s_i^js are term definable in CA_αs by $\mathsf{s}_i^j(x) = \mathsf{c}_i(x \cdot -\mathsf{d}_{ij})$ $(i, j < \alpha)$. This operation reflects algebraically the substitution of the variable v_j for v_i in a formula such that the substitution is free; this can be always done by reindexing bounded variables. In such situation, we say that Scs are *generalized reducts* of CAs. However, CA_αs and QA_α are (real)reducts of QEAs (in the universal algebraic sense), simply obtained by discarding the operations in their signature not in the signature of their common expansion QEA_α.

Definition 3.1. Let α be an ordinal. We say that a variety V is a variety between Df_α and QEA_α if the signature of V expands that of Df_α and is contained in the signature of QEA_α. Furthermore, any equation formulated in the signature of Df_α that holds in V also holds in Sc_α and all equations that hold in V holds in QEA_α.

Proper examples include Sc, CA_α and QA_α (meaning strictly between). Analogously we can define varieties between Sc_α and CA_α or QA_α and QEA_α, and more generally between a class K of BAOs and a generalized reduct of it. Notions like neat reducts generalize verbatim to such algebras, namely, to Dfs and QEAs, and in any variety in between. This stems from the observation that for any pair of ordinals $\alpha < \beta$, $\mathfrak{A} \in \mathsf{QEA}_\beta$ and any non-Boolean extra operation in the signature of QEA_β, f say, if $x \in \mathfrak{A}$ and $\Delta x \subseteq \alpha$, then $\Delta(f(x)) \subseteq \alpha$. Here $\Delta x = \{i \in \beta : \mathsf{c}_i x \neq x\}$ (as defined in the introduction) is referred as *the dimension set* of x; it reflects algebraically the essentially free variables occurring in a formula ϕ. A variable is essentially free in a formula $\Psi \iff$ it is free in every formula equivalent to Ψ.[1] Therefore given a variety V between Sc_β and QEA_β, if $\mathfrak{B} \in \mathsf{V}$ then the algebra $\mathfrak{Nr}_\alpha\mathfrak{B}$ having universe $\{x \in \mathfrak{B} : \Delta x \subseteq \alpha\}$ is closed under all operations in the signature of V.

Definition 3.2. Let $2 < n < \omega$. For a variety V between Df_n and QEA_n, a V *set algebra* is a subalgebra of an algebra, having the same signature as V, of the form $\langle \mathfrak{B}({}^nU), f_i^U\rangle$, say, where f_i^U is identical to the interpretation of f_i in the class of quasi-polyadic equality set algebras. Let $\mathfrak{A}$ be an algebra having the same signature of V; then $\mathfrak{A}$ is *a representable* V *algebra*, or simply *representable* $\iff$ $\mathfrak{A}$ is isomorphic to a subdirect product of V set algebras. We write RV for the class of representable V algebras

It can be proved that the class RV, as defined above, is also closed under $\mathbf{H}$, so that it is a variety.

Proposition 3.3. Let $2 < n < \omega$. Let V be a variety between Df_n and QEA_n. Then RV is not a finitely axiomatizable variety.

Proof: In [15] a sequence $\langle \mathfrak{A}_i : i \in \omega\rangle$ of algebras is constructed such that $\mathfrak{A}_i \in \mathsf{QEA}_n$ and $\mathfrak{Rd}_{df}\mathfrak{A}_n \notin \mathsf{RDf}_n$, but $\Pi_{i\in\omega}\mathfrak{A}_i/F \in \mathsf{RQEA}_n$ for any non principal ultrafilter on ω. An application of Los' Theorem, taking the ultraproduct of V reduct of the $\mathfrak{A}_i$s, finishes the proof. In more detail, let $\mathfrak{Rd}_\mathsf{V}$ denote restricting the signature to that of V. Then $\mathfrak{Rd}_\mathsf{V}\mathfrak{A}_i \notin \mathsf{RV}$ and $\mathfrak{Rd}_\mathsf{V}\Pi_{i\in I}(\mathfrak{A}_i/F) \in \mathsf{RV}$. □

The last result generalizes to infinite dimensions replacing finite axiomatization by axiomatized by a finite schema [7, 13].

[1] It can well happen that a variable is free in formula that is equivalent to another formula in which this same variable is not free.

Theorem 3.4. *Let $2 < n < \omega$. Let V be any variety between Df_n an QEA_n. Then the class of completely representable algebras in V is not elementary.*

Proof: For a complete labelled graph graph N and function θ, the graph $N\theta$ is the complete labelled graph with nodes $\theta^{-1}(\mathsf{nodes}(N)) = \{x \in \mathsf{dom}(\theta) : \theta(x) \in \mathsf{nodes}(N)\}$, and labelling defined by

$$(N\theta)(i_0, \ldots, i_{n-1}) = N(\theta(i_0), \theta(i_1), \ldots, \theta(i_{n-1})),$$

for $i_0, \ldots, i_{n-1} \in \theta^{-1}(\mathsf{nodes}(N))$. We have S is symmetric, that is, if $N \in S$ and $\theta : \omega \to \omega$ is a finitary function, in the sense that $\{i \in \omega : \theta(i) \neq i\}$ is finite, then $N\theta$ is in S. It follows that the complex algebra $\mathfrak{Ca}(S) \in \mathsf{QEA}_\omega$. Thus the algebra $\mathfrak{B}$ can be expanded into a polyadic algebra of dimension n. Also, generated by two dimensional elements, the Df reduct of $\mathfrak{B}$ is not completely representable by [14, Proposition 4.10]. $\square$

In [9] it is proved that the class CRCA_α, where α is an infinite ordinal, is not elementary either. The proof can be generalized to any variety V between CA and QEA. We do not know whether it generalizes to equality free algebras such as Df, Sc and QA for the proof in the infinite dimensional case of CAs in [9] essentially depens on the presence of diagonal elements, namely, only one diagonal $\mathsf{d}_{0,1}$. Recall that $\mathfrak{Rd}_{ca}$ denote the cylindric reduct. One shows that if $\mathfrak{C} \in \mathsf{QEA}_\omega$ is completely representable and $\mathfrak{C} \models \mathsf{d}_{01} < 1$, then $|\mathsf{At}\mathfrak{C}| \geq 2^\omega$. The argument is as follows: Suppose that $\mathfrak{C} \models \mathsf{d}_{01} < 1$. Then there is $s \in h(-\mathsf{d}_{01})$ so that if $x = s_0$ and $y = s_1$, we have $x \neq y$. Fix such x and y. For any $J \subseteq \omega$ such that $0 \in J$, set a_J to be the sequence with ith co-ordinate is x if $i \in J$, and is y if $i \in \omega \setminus J$. By complete representability every a_J is in $h(1^{\mathfrak{C}})$ and so it is in $h(x)$ for some unique atom x, since the representation is an atomic one. Let $J, J' \subseteq \omega$ be distinct sets containing 0. Then there exists $i < \omega$ such that $i \in J$ and $i \notin J'$. So $a_J \in h(\mathsf{d}_{0i})$ and $a'_J \in h(-\mathsf{d}_{0i})$, hence atoms corresponding to different a_J's with $0 \in J$ are distinct. It now follows that $|\mathsf{At}\mathfrak{C}| = |\{J \subseteq \omega : 0 \in J\}| \geq 2^\omega$. Take $\mathfrak{D} \in \mathsf{Pes}_\omega$ with universe $\wp({}^\omega 2)$. Then $\mathfrak{D} \models \mathsf{d}_{01} < 1$ and plainly $\mathfrak{D}$ is completely representable. Using the downward Löwenheim–Skolem–Tarski theorem, take a countable elementary subalgebra $\mathfrak{B}$ of $\mathfrak{D}$. This is possible because the signature of QEA_ω is countable. Then in $\mathfrak{B}$ we have $\mathfrak{B} \models \mathsf{d}_{01} < 1$ because $\mathfrak{B} \equiv \mathfrak{C}$. But

$\mathfrak{Rd}_{ca}\mathfrak{B}$ cannot be completely representable, because if it were then by the above argument, we get that $|\mathsf{At}\mathfrak{Rd}_{ca}\mathfrak{B}| = |\mathsf{At}\mathfrak{B}| \geq 2^\omega$, which is impossible because $\mathfrak{B}$ is countable.

3.1. For $2 < n < \omega$, the class of neat reducts is not elemenatry for any V between Sc_n and QEA_n

THEOREM 3.5. *For any finite $n > 1$, and any uncountable cardinal $\kappa \geq |\alpha|$, there exist completely representable algebras $\mathfrak{A}, \mathfrak{B} \in \mathsf{QEA}_n$, that are set algebras, such that $|\mathfrak{A}| = |\mathfrak{B}| = \kappa$, $\mathfrak{A} \in \mathsf{Nr}_\alpha\mathsf{QEA}_\omega$, $\mathfrak{Rd}_{sc}\mathfrak{B} \notin \mathsf{Nr}_\alpha\mathsf{Sc}_{n+1}$, $\mathfrak{A} \equiv_{\infty,\omega} \mathfrak{B}$ and $\mathsf{At}\mathfrak{A} \equiv_{\omega,\infty} \mathsf{At}\mathfrak{B}$.*

PROOF: Fix $1 < n < \omega$. Let L be a signature consisting of the unary relation symbols $P_0, P_1, \ldots, P_{n-1}$ and uncountably many n–ary predicate symbols. M is as in [25, Lemma 5.1.3], but the tenary relations are replaced by n–ary ones, and we require that the interpretations of the n–ary relations in M are *pairwise disjoint* not only distinct. This can be fixed. In addition to pairwise disjointness of n–ary relations, we require their symmetry, that is, permuting the variables does not change their semantics. In fact the construction is presented this way in [17]. For $u \in {}^nn$, let χ_u be the formula $\bigwedge_{u\in {}^nn} P_{u_i}(x_i)$. We assume that the n–ary relation symbols are indexed by (an uncountable set) I and that there is a binary operation $+$ on I, such that $(I, +)$ is an abelian group, and for distinct $i \neq j \in I$, we have $R_i \circ R_j = R_{i+j}$. For $n \leq k \leq \omega$, let $\mathfrak{A}_k = \{\phi^\mathsf{M} : \phi \in L_k\} (\subseteq \wp({}^k\mathsf{M}))$, where ϕ is taken in the signature L, and $\phi^\mathsf{M} = \{s \in {}^k\mathsf{M} : \mathsf{M} \models \phi[s]\}$.

Let $\mathfrak{A} = \mathfrak{A}_n$, then $\mathfrak{A} \in \mathsf{Pes_n}$ by the added symmetry condition. Also $\mathfrak{A} \cong \mathsf{Nr}_n\mathfrak{A}_\omega$; the isomorphism is given by $\phi^\mathsf{M} \mapsto \phi^\mathsf{M}$. The map is obviously an injective homomorphism; it is surjective, because M (as stipulated in [25, item (1) of lemma 5.1.3]), has quantifier elimination. For $u \in {}^nn$, let $\mathfrak{A}_u = \{x \in \mathfrak{A} : x \leq \chi_u^\mathsf{M}\}$. Then $\mathfrak{A}_u$ is an uncountable and atomic Boolean algebra (atomicity follows from the new disjointness condition) and $\mathfrak{A}_u \cong \mathsf{Cof}(|I|)$, the finite–cofinite Boolean algebra on $|I|$. Define a map $f : \mathfrak{B}\restriction\mathfrak{A} \to \mathbf{P}_{u\in {}^nn}\mathfrak{A}_u$, by $f(a) = \langle a \cdot \chi_u\rangle_{u\in {}^nn+1}$. Let $\mathcal{P}$ denote the structure for the signature of Boolean algebras expanded by constant symbols 1_u, $u \in {}^nn$, d_{ij}, and unary relation symbols $\mathsf{s}_{[i,j]}$ for each $i, j \in n$. Then for each $i < j < n$, there are quantifier free formulas $\eta_i(x, y)$ and $\eta_{ij}(x, y)$ such that $\mathcal{P} \models \eta_i(f(a), b) \iff b = f(\mathsf{c}_i^\mathfrak{A} a)$, and $\mathcal{P} \models \eta_{ij}(f(a), b) \iff b = f(\mathsf{s}_{[i,j]}a)$. The one corresponding to cylindrifiers is exactly like the CA case [25, pp. 113–114]. For substitutions corresponding to transpositions, it is simply $y = \mathsf{s}_{[i,j]}x$. The diagonal elements and the Boolean operations are easy to interpret. Hence, $\mathcal{P}$ is interpretable in $\mathfrak{A}$, and the interpretation is one dimensional and quantifier free. For $v \in {}^nn$, by the Tarski–Skolem downward theorem, let $\mathfrak{B}_v$ be a countable elementary subalgebra of $\mathfrak{A}_v$. (Here we are using the countable signature of PEA_n). Let $S_n(\subseteq {}^nn)$ be the set of permuations in ${}^nn$.

Take $u_1 = (0, 1, 0, \ldots, 0)$ and $u_2 = (1, 0, 0, \ldots, 0) \in {}^nn$. Let $v = \tau(u_1, u_2)$ where $\tau(x, y) = \mathsf{c}_1(\mathsf{c}_0 x \cdot \mathsf{s}_1^0 \mathsf{c}_1 y) \cdot \mathsf{c}_1 x \cdot \mathsf{c}_0 y$. We call τ an approximate witness. It is not hard to show that $\tau(u_1, u_2)$ is actually the composition of u_1 and u_2, so that $\tau(u_1, u_2)$ is the constant zero map; which we denote by $\mathbf{0}$; it is also in ${}^nn$. Clearly for every $i < j < n$, ${\mathsf{s}_{[i,j]}}^{{}^nn}\{\mathbf{0}\} = \mathbf{0} \notin \{u_1, u_2\}$. We can assume without loss that the Boolean reduct of $\mathfrak{A}$ is the following product:

$$\mathfrak{A}_{u_1} \times \mathfrak{A}_{u_2} \times \mathfrak{A}_{\mathbf{0}} \times \mathbf{P}_{u \in V \sim J} \mathfrak{A}_u,$$

where $J = \{u_1, u_2, \mathbf{0}\}$. Let

$$\mathfrak{B} = ((\mathfrak{A}_{u_1} \times \mathfrak{A}_{u_2} \times \mathfrak{B}_{\mathbf{0}} \times \mathbf{P}_{u \in V \sim J} \mathfrak{A}_u), 1_u, \mathsf{d}_{ij}, \mathsf{s}_{[i,j]}x)_{i,j<n},$$

recall that $\mathfrak{B}_{\mathbf{0}} \prec \mathfrak{A}_{\mathbf{0}}$ and $|\mathfrak{B}_{\mathbf{0}}| = \omega$, inheriting the same interpretation. Then by the Feferman–Vaught theorem, we get that $\mathfrak{B} \equiv \mathfrak{A}$.

Now assume for contradiction, that $\mathfrak{Rd}_{sc}\mathfrak{B} = \mathsf{Nr}_n\mathfrak{D}$, with $\mathfrak{D} \in \mathsf{Sc}_{n+1}$. Let $\tau_n(x, y)$, which we call an *n–witness*, be defined by $\mathsf{c}_n(\mathsf{s}_n^1 \mathsf{c}_n x \cdot \mathsf{s}_n^0 \mathsf{c}_n y)$. By a straightforward, but possibly tedious computation, one can obtain $\mathsf{Sc}_{n+1} \models \tau_n(x, y) \le \tau(x, y)$ so that the approximate witness *dominates* the n–witness. The term $\tau(x, y)$ does not use any spare dimensions, and it 'approximates' the term $\tau_n(x, y)$ that uses the spare dimension n. Let $\lambda = |I|$. For brevity, we write $\mathbf{1}_u$ for χ_u^{M}. The algebra $\mathfrak{A}$ can be viewed as splitting the atoms of the atom structure $\mathbf{At} = ({}^nn, \equiv, \equiv_{ij}, D_{ij})_{i,j<n}$ each to uncountably many atoms. We denote $\mathfrak{A}$ by $\mathsf{split}(\mathbf{At}, \mathbf{1}_{\mathbf{0}}, \lambda)$. On the other hand, $\mathfrak{B}$ can be viewed as splitting the same atom structure, each atom – except for one atom that is split into countably many atoms – is also split into uncountably many atoms (the same as in $\mathfrak{A}$). We denote $\mathfrak{B}$ by $\mathsf{split}(\mathbf{At}, \mathbf{1}_{\mathbf{0}}, \omega)$. On the 'global' level, namely, in the complex algebra of the finite (splitted) atom structure ${}^nn$, these two terms are equal, the approximate witness is the n-witness. The complex algebra $\mathfrak{Cm}({}^nn)$ does not 'see' the nth dimension. But in the algebras $\mathfrak{A}$ and $\mathfrak{B}$ (obtained after splitting), the n-witness becomes then a *genuine witness*, not an approximate one. The approximate witness *strictly dominates* the n-witness. The n-witness

using the spare dimension n, detects the cardinality twist that $L_{\infty,\omega}$, *a priori*, first order logic misses out on. If the n-witness were term definable (in the case we have a full neat reduct of an algebra in only one extra dimension), then it takes two uncountable component to an uncountable one, and this is not possible for $\mathfrak{B}$, because in $\mathfrak{B}$, the target component is forced to be countable.

Now for $x \in \mathfrak{B}_{u_1}$ and $y \in \mathfrak{B}_{u_2}$, we have

$$\tau_n^{\mathfrak{D}}(x, y) \leq \tau_n^{\mathfrak{D}}(\chi_{u_1}, \chi_{u_2}) \leq \tau^{\mathfrak{D}}(\chi_{u_1}, \chi_{u_2}) = \chi_{\tau^{\wp({}^nn)}}(u_1, u_2) = \chi_{\tau(u_1,u_2)} = \chi_{\mathbf{0}}.$$

But for $i \neq j \in I$, $\tau_n^{\mathfrak{D}}(R_i^{\mathsf{M}} \cdot \chi_{u_1}, R_j^{\mathsf{M}} \cdot \chi_{u_2}) = R_{i+j}^{\mathsf{M}} \cdot \chi_v$, and so $\mathfrak{B}_{\mathbf{0}}$ will be uncountable, which is impossible. We now show that $\exists$ has a winning strategy in an Ehrenfeucht–Fraïssé back–and–forth game over the now atomic $(\mathfrak{A}, \mathfrak{B})$. At any stage of the game, if $\forall$ places a pebble on one of $\mathfrak{A}$ or $\mathfrak{B}$, $\exists$ must place a matching pebble on the other algebra. Let $\bar{a} = \langle a_0, a_1, \ldots, a_{m-1}\rangle$ be the position of the pebbles played so far (by either player) on $\mathfrak{A}$ and let $\bar{b} = \langle b_0, \ldots, b_{m-1}\rangle$ be the the position of the pebbles played on $\mathfrak{B}$. Denote χ_u^{M}, by $\mathbf{1}_u$. Then $\exists$ has to maintain the following properties throughout the game:

- for any atom x (of either algebra) with $x \cdot \mathbf{1}_{\mathbf{0}} = 0$, , then $x \in a_i$ iff $x \in b_i$,

- $\bar{a}$ induces a finite partition of $\mathbf{1}_{\mathbf{0}}$ in $\mathfrak{A}$ of 2^m (possibly empty) parts $p_i : i < 2^m$ and the $\bar{b}$ induces a partition of 1_u in $\mathfrak{B}$ of parts $q_i : i < 2^m$ such that p_i is finite iff q_i is finite and, in this case, $|p_i| = |q_i|$.

It is easy to see that $\exists$ can maintain these two properties in every round. In this back–and–forth game, $\exists$ will always find a matching pebble, because the pebbles in play are finite. For each $w \in {}^nn$ the component $\mathfrak{B}_w = \{x \in \mathfrak{B} : x \leq \mathbf{1}_v\} (\subseteq \mathfrak{A}_w = \{x \in \mathfrak{A} : x \leq \mathbf{1}_v\})$ contains infinitely many atoms. For any $w \in V$, $|\mathsf{At}\mathfrak{A}_w| = |I|$, while for $u \in V \sim \{\mathbf{0}\}$, $\mathsf{At}\mathfrak{A}_u = \mathsf{At}\mathfrak{B}_u$. For $|\mathsf{At}\mathfrak{B}_{\mathbf{0}}| = \omega$, but it is still an infinite set. Therefore $\mathfrak{A} \equiv_\infty \mathfrak{B}$. It is clear that the above argument works for any $\mathfrak{C}$ such that $\mathsf{At}\mathfrak{C} = \mathsf{At}\mathfrak{B}$, hence $\mathfrak{B} \equiv_{\infty,\omega} \mathfrak{C}$. $\square$

Corollary 3.6. For any $2 < n < \omega$, for any variety V between Sc and QEA and any ordinal $m > n$, the variety $\mathsf{Nr}_n\mathsf{V}_m$ is not elementray [18, 21, 17].

4. Polyadic paradigm, positive results

4.1. Halmos' polyadic algebras of infinite dimension with and witout equality

Throughout this section α is an infinite ordinal. Recall that $\mathsf{PA}_\alpha(\mathsf{PEA}_\alpha)$ denotes the class of polydic algebras of dimension α (with equality) as defined in [7, Definition 5.4.1]. Neat reducts for such algebras are defined in [7, Definition 4.4.16]. For a class K of Boolean algebras with operators, we write K^{ad} for the class of completely aditive algebras in K, and we write $\mathsf{K}\cap\mathbf{At}$ for the class of atomic algebras in K.

THEOREM 4.1. *Let α be an infinite ordinal and $n<\omega$. If $\mathfrak{D}\in\mathsf{PEA}_\alpha$ is atomic, then any complete subalgebra of $\mathfrak{Nr}_n\mathfrak{D}$ is completely representable as a PEA_n. If $\mathfrak{D}\in\mathsf{PA}_\alpha$ is atomic and completely additive and $n\leq\alpha$, then $\mathsf{Nr}_n\mathfrak{A}$ is completely representable. In particular, $\mathbf{S}_c\mathsf{PA}_\alpha^{\mathsf{ad}}\cap\mathbf{At}=\mathsf{PA}_\alpha^{\mathsf{ad}}\cap\mathbf{At}=\mathsf{CRPA}_\alpha$ and the class CRPA_α is elementary.*

PROOF: Assume that $\mathfrak{A}\subseteq_c\mathfrak{Nr}_n\mathfrak{D}$, where $\mathfrak{D}\in\mathsf{PEA}_\alpha$ is atomic. Let $c\in\mathfrak{A}$ be non–zero. We will find a homomorphism $f:\mathfrak{A}\to\wp({}^nU)$ such that $f(c)\neq 0$, and preserves infinitary joins. Assume for the moment (to be proved in a while) that $\mathfrak{A}\subseteq_c\mathfrak{D}$. Then by [10, Lemma 2.16] $\mathfrak{A}$ is atomic because $\mathfrak{D}$ is. For brevity, let $X=\mathsf{At}\mathfrak{A}$. Let $\mathfrak{m}$ be the local degree of $\mathfrak{D}$, $\mathfrak{c}$ its effective cardinality and let β be any cardinal such that $\beta\geq\mathfrak{c}$ and $\sum_{s<\mathfrak{m}}\beta^s=\beta$; such notions are defined in [3]. We can assume that $\mathfrak{D}=\mathfrak{Nr}_\alpha\mathfrak{B}$, with $\mathfrak{B}\in\mathsf{PEA}_\beta$ [7, Theorem 5.4.17]. For any ordinal $\mu\in\beta$, and $\tau\in{}^\mu\beta$, write τ^+ for $\tau\cup Id_{\beta\setminus\mu}(\in{}^\beta\beta)$. Consider the following family of joins evaluated in $\mathfrak{B}$, where $p\in\mathfrak{D}$, $\Gamma\subseteq\beta$ and $\tau\in{}^\alpha\beta$: (*) $\mathsf{c}_{(\Gamma)}p=\sum^{\mathfrak{B}}\{\mathsf{s}_{\tau^+}p:\tau\in{}^\omega\beta,\ \ \tau\restriction\alpha\setminus\Gamma=Id\}$, and (**): $\sum\mathsf{s}_{\tau^+}^{\mathfrak{B}}X=1$. The first family of joins exists [3, Proof of Theorem 6.1], and the second exists, because $\sum^{\mathfrak{A}}X=\sum^{\mathfrak{D}}X=\sum^{\mathfrak{B}}X=1$ and τ^+ is completely additive, since $\mathfrak{B}\in\mathsf{PEA}_\beta$. The last equality of suprema follows from the fact that $\mathfrak{D}=\mathfrak{Nr}_\alpha\mathfrak{B}\subseteq_c\mathfrak{B}$ and the first from the fact that $\mathfrak{A}\subseteq_c\mathfrak{D}$. We prove the former, the latter is exactly the same replacing α and β, by n and α, respectivey, proving that $\mathfrak{Nr}_n\mathfrak{D}\subseteq_c\mathfrak{D}$, hence $\mathfrak{A}\subseteq_c\mathfrak{D}$. We prove that $\mathfrak{Nr}_\alpha\mathfrak{B}\subseteq_c\mathfrak{B}$. Assume that $S\subseteq\mathfrak{D}$ and $\sum^{\mathfrak{B}}S=1$, and for contradiction, that there exists $d\in\mathfrak{B}$ such that $s\leq d<1$ for all $s\in S$. Let $J=\Delta d\setminus\omega$ and take $t=-\mathsf{c}_{(J)}(-d)\in\mathfrak{D}$. Then $\mathsf{c}_{(\beta\setminus\alpha)}t=\mathsf{c}_{(\beta\setminus\alpha)}(-\mathsf{c}_{(J)}(-d))=\mathsf{c}_{(\beta\setminus\alpha)}-\mathsf{c}_{(J)}(-d)=\mathsf{c}_{(\beta\setminus\alpha)}-\mathsf{c}_{(\beta\setminus\alpha)}\mathsf{c}_{(J)}(-d)=-\mathsf{c}_{(\beta\setminus\alpha)}\mathsf{c}_{(J)}(-d)=-\mathsf{c}_{(J)}(-d)=t.$

We have proved that $t \in \mathfrak{D}$. We now show that $s \leq t < 1$ for all $s \in S$, which contradicts $\sum^{\mathfrak{D}} S = 1$. If $s \in S$, we show that $s \leq t$. By $s \leq d$, we have $s \cdot -d = 0$. Hence by $\mathsf{c}_{(J)} s = s$, we get $0 = \mathsf{c}_{(J)}(s \cdot -d) = s \cdot \mathsf{c}_{(J)}(-d)$, so $s \leq -\mathsf{c}_{(J)}(-d)$. It follows th at $s \leq t$ as required. Assume for contradiction that $1 = -\mathsf{c}_{(J)}(-d)$. Then $\mathsf{c}_{(J)}(-d) = 0$, so $-d = 0$ which contradicts that $d < 1$. We have proved that $\sum^{\mathfrak{B}} S = 1$, so $\mathfrak{D} \subseteq_c \mathfrak{B}$. Let F be any Boolean ultrafilter of B generated by an atom below a. We show that F will preserve the family of joins in (*) and (**). One forms nowhere dense sets in the Stone space of $\mathfrak{B}$ corresponding to the aforementioned family of joins as follows: The Stone space of (the Boolean reduct of) $\mathfrak{B}$ has underlying set S, the set of all Boolean ultrafilters of $\mathfrak{B}$. For $b \in \mathfrak{B}$, let N_b be the clopen set $\{F \in S : b \in F\}$. The required nowhere dense sets are defined for $\Gamma \subseteq \beta$, $p \in \mathfrak{D}$ and $\tau \in {}^{\alpha}\beta$ via: $A_{\Gamma,p} = N_{\mathsf{c}_{(\Gamma)}p} \setminus \bigcup_{\tau:\alpha\to\beta} N_{\mathsf{s}_{\tau}+p}$, and $A_{\tau} = S \setminus \bigcup_{x \in X} N_{\mathsf{s}_{\tau}+x}$. The principal ultrafilters are isolated points in the Stone topology, so they lie outside the nowhere dense sets defined above. Hence any such ultrafilter preserve the joins in (*) and (**). Fix a principal ultrafilter F with $a \in F$. Define the equivalence relation E (on β) by setting $iEj \iff \mathsf{d}^{\mathfrak{B}}_{ij} \in F (i, j \in \beta)$. Define $f : \mathfrak{A} \to \wp({}^{n}(\beta/E))$, via $x \mapsto \{\bar{t} \in {}^{n}(\beta/E) : \mathsf{s}^{\mathfrak{B}}_{t \cup Id_{\beta \sim n}} x \in F\}$, where $\bar{t}(i/E) = t(i)$ $(i < n)$ and $t \in {}^{n}\beta$. Let $V = {}^{\beta}\beta^{(Id)}$. To show that f is well defined, it suffices to show that for all $\sigma, \tau \in V$, if $(\tau(i), \sigma(i)) \in E$ for all $i \in \beta$, then for any $x \in \mathfrak{A}$, $\mathsf{s}_{\tau} x \in F \iff \mathsf{s}_{\sigma} x \in F$. We proceed by by induction on $|\{i \in \beta : \tau(i) \neq \sigma(i)\}| (< \omega)$. If $J = \{i \in \beta : \tau(i) \neq \sigma(i)\}$ is empty, the result is obvious. Otherwise assume that $k \in J$. We introduce a helpful piece of notation. For $\eta \in V$, let $\eta(k \mapsto l)$ stand for the η' that is the same as η except that $\eta'(k) = l$. Now take any $\lambda \in \{\eta \in \beta : (\sigma)^{-1}\{\eta\} = (\tau)^{-1}\{\eta\} = \{\eta\}\} \smallsetminus \Delta x$. Recall that $\Delta x = \{i \in \beta : \mathsf{c}_i x \neq x\}$ and that $\beta \setminus \Delta x$ is infinite because $\Delta x \subseteq n$, so such a λ exists. Now we freely use properties of substitutions for cylindric algebras. We have by [7, 1.11.11(i)(iv)] (a) $\mathsf{s}_{\sigma} x = \mathsf{s}^{\lambda}_{\sigma k} \mathsf{s}_{\sigma(k \mapsto \lambda)} x$, and (b) $\mathsf{s}^{\lambda}_{\tau k}(\mathsf{d}_{\lambda,\sigma k} \cdot \mathsf{s}_{\sigma} x) = \mathsf{d}_{\tau k,\sigma k} \mathsf{s}_{\sigma} x$, and (c) $\mathsf{s}^{\lambda}_{\tau k}(\mathsf{d}_{\lambda,\sigma k} \cdot \mathsf{s}_{\sigma(k \mapsto \lambda)} x) = \mathsf{d}_{\tau k,\sigma k} \cdot \mathsf{s}_{\sigma(k \mapsto \tau k)} x$, and finally (d) $\mathsf{d}_{\lambda,\sigma k} \cdot \mathsf{s}^{\lambda}_{\sigma k} \mathsf{s}_{\sigma(k \mapsto \lambda)} x = \mathsf{d}_{\lambda,\sigma k} \cdot \mathsf{s}_{\sigma(k \mapsto \lambda)} x$. Then by (b), (a), (d) and (c), we get,

$\mathsf{d}_{\tau k,\sigma k} \cdot \mathsf{s}_{\sigma} x = \mathsf{s}^{\lambda}_{\tau k}(\mathsf{d}_{\lambda,\sigma k} \cdot \mathsf{s}_{\sigma} x) = \mathsf{s}^{\lambda}_{\tau k}(\mathsf{d}_{\lambda,\sigma k} \cdot \mathsf{s}^{\lambda}_{\sigma k} \mathsf{s}_{\sigma(k \mapsto \lambda)} x) = \mathsf{s}^{\lambda}_{\tau k}(\mathsf{d}_{\lambda,\sigma k} \cdot \mathsf{s}_{\sigma(k \mapsto \lambda)} x) = \mathsf{d}_{\tau k,\sigma k} \cdot \mathsf{s}_{\sigma(k \mapsto \tau k)} x$. But F is a filter and $(\tau k, \sigma k) \in E$, we conclude that $\mathsf{s}_{\sigma} x \in F \iff \mathsf{s}_{\sigma(k \mapsto \tau k)} x \in F$. The conclusion follows from the induction hypothesis. We check only cylindrifications since the other operations are entirely straightforward to handle. Let $k < n$ and $a \in A$.

Let $\bar{\sigma} \in \mathsf{c}_k f(a)$. Then for some $\lambda \in \beta$, we have $\bar{\sigma}(k \mapsto \lambda/E) \in f(a)$ hence $\mathsf{s}_{\sigma^+(k\mapsto\lambda)}a \in F$. It follows from the inclusion $a \leq \mathsf{c}_k a$ that $\mathsf{s}_{\sigma^+(k\mapsto\lambda)}\mathsf{c}_k a \in F$, so $\mathsf{s}_{\sigma^+}\mathsf{c}_k a \in F$. Thus $\mathsf{c}_k f(a) \subseteq f(\mathsf{c}_k a.)$ We prove the other more difficult inclusion that uses the condition (*) of eliminating cylindrifiers. Let $a \in A$ and $k < n$. Let $\bar{\sigma}' \in f\mathsf{c}_k a$ and let $\sigma = \sigma' \cup Id_{\beta\sim n}$. Then $\mathsf{s}_\sigma^{\mathfrak{B}}\mathsf{c}_k a = \mathsf{s}_{\sigma'}^{\mathfrak{B}}\mathsf{c}_k a \in F$. Pick $\lambda \in \{\eta \in \beta : \sigma^{-1}\{\eta\} = \{\eta\}\} \smallsetminus \Delta a$, such a λ exists because Δa is finite, and $|\{i \in \beta : \sigma(i) \neq i\}| < \omega$. Let $\tau = \sigma \restriction \mathfrak{n} \smallsetminus \{k, \lambda\} \cup \{(k,\lambda),(\lambda,k)\}$. Then (in $\mathfrak{B}$):

$$\mathsf{c}_\lambda \mathsf{s}_\tau a = \mathsf{s}_\tau \mathsf{c}_k a = \mathsf{s}_\sigma \mathsf{c}_k a \in F.$$

By the construction of F, there is some $u(\notin \Delta(\mathsf{s}_\tau^{\mathfrak{B}}a))$ such that $\mathsf{s}_u^\lambda \mathsf{s}_\tau a \in F$, so $\mathsf{s}_{\sigma(k\mapsto u)}a \in F$. Hence $\sigma(k \mapsto u) \in f(a)$, from which we get that $\bar{\sigma}' \in \mathsf{c}_k f(a)$. By construction, for every $s \in {}^n(\beta/E)$, there exists $x \in X (= \mathsf{At}\mathfrak{A})$, such that $\mathsf{s}^{\mathfrak{B}}_{s\cup Id_{\beta\sim n}} x \in F$, from which we get $\bigcup_{x\in X} f(x) = {}^n(\beta/E)$ hence f is an atomic, thus a complete representation. If $\mathfrak{A} \in \mathsf{PA}_\alpha$, we do not need to bother about diagonal elements and so the base of the representation will be simply β (as defined above for PEA_α), not β/E, and the desired homomorphism, with $n \leq \alpha$, is defined via $g : \mathfrak{A} \to \wp({}^n\beta)$, via $x \mapsto \mathsf{t} \in {}^n\beta : \mathsf{s}^{\mathfrak{B}}_{t\cup Id_{\beta\sim n}} x \in F\}$. Checking that g preserves the operations and that g is atomic, hence complete, is exactly like the PEA case. For PA_α, atomicity can be expressed by a first order sentence, and complete additivity can be captured by the following continuum many first order formulas, that form a single schema. Let $\mathsf{At}(x)$ be the first order formula expressing that x is an atom. That is $\mathsf{At}(x)$ is the formula $x \neq 0 \wedge (\forall y)(y \leq x \to y = 0 \vee y = x)$. For $\tau \in {}^\alpha\alpha$, let ψ_τ be the formula: $y \neq 0 \to \exists x(\mathsf{At}(x) \wedge \mathsf{s}_\tau x \cdot y \neq 0)$. Let Σ be the set of first order formulas obtained by adding all formulas ψ_τ $(\tau \in {}^\alpha\alpha)$ to the polyadic schema. Then it is not hard to show that $\mathsf{CRPA}_\alpha = \mathbf{Mod}(\Sigma)$. The underlying idea here is that the notion of complete additivity on atomic algebras is definable in $L_{\omega,\omega}$. In more detail: Let $\mathfrak{A} \in \mathsf{CRPA}_\alpha$ with set of atoms X. Then, $\sum_{x\in X} \mathsf{s}_\tau x = 1$ for all $\tau \in {}^\alpha\alpha$. Let $\tau \in {}^\alpha\alpha$. Let a be non-zero, then $a \cdot \sum_{x\in X} \mathsf{s}_\tau x = a \neq 0$, hence there exists $x \in X$, such that $a \cdot \mathsf{s}_\tau x \neq 0$, and so $\mathfrak{A} \models \psi_\tau$. Conversely, let $\mathfrak{A} \models \Sigma$. Then for all $\tau \in {}^\alpha\alpha$, $\sum_{x\in X} \mathsf{s}_\tau x = 1$. Indeed, assume that for some τ, $\sum_{x\in X} \mathsf{s}_\tau x \neq 1$. Let $a = 1 - \sum_{x\in X} \mathsf{s}_\tau x$. Then $a \neq 0$. But then, by assumption, there exists $x' \in X$, such that $\mathsf{s}_\tau x' \cdot a = \mathsf{s}_\tau x' \cdot (1 - \sum_{x\in X} \mathsf{s}_\tau x) = \mathsf{s}_\tau x' - \sum_{x\in X} \mathsf{s}_\tau x \neq 0$, which is impossible. $\square$

4.2. Algebras in between the cylindric and polyadic paradigms; Ferenczi's cylindric-polyadic algebras

We recall the definition of certain reducts of polyadic algebras. By $I \subseteq_\omega J$, we undestand that I is a *finite subset of* J.

DEFINITION 4.2. Let α be an ordinal. By a *cylindric polyadic algebra* of dimension α, or a CPA_α for short, we understand an algebra of the form

$$\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, \mathsf{c}_{(\Gamma)}, \mathsf{s}_\tau \rangle_{\Gamma \subseteq_\omega \alpha, \tau \in {}^\alpha\alpha}$$

where $\mathsf{c}_{(\Gamma)}$ $\Gamma \subseteq_\omega \alpha)$ and s_τ $(\tau \in {}^\alpha\alpha)$ are unary operations on A, such that postulates below hold for $x, y \in A$, $\tau, \sigma \in {}^\alpha\alpha$ and $\Gamma, \Delta \subseteq_\omega \alpha$

1. $\langle A, +, \cdot, -, 0, 1 \rangle$ is a Boolean algebra
2. $\mathsf{c}_{(0)}x = x$
3. $\mathsf{c}_{(\Gamma)}0 = 0$
4. $x \leq \mathsf{c}_{(\Gamma)}x$
5. $\mathsf{c}_{(\Gamma)}(x \cdot \mathsf{c}_{(\Gamma)}y) = \mathsf{c}_{(\Gamma)}x \cdot \mathsf{c}_{(\Gamma)}y$
6. $\mathsf{c}_{(\Gamma)}\mathsf{c}_{(\Delta)}x = \mathsf{c}_{(\Gamma\cup\Delta)}x$
7. s_τ is a Boolean endomorphism
8. $\mathsf{s}_{Id}x = x$
9. $\mathsf{s}_{\sigma\circ\tau} = \mathsf{s}_\sigma \circ \mathsf{s}_\tau$
10. if $\sigma \upharpoonright (\alpha \sim \Gamma) = \tau \upharpoonright (\alpha \sim \Gamma)$, then $\mathsf{s}_\sigma \mathsf{c}_{(\Gamma)}x = \mathsf{s}_\tau \mathsf{c}_{(\Gamma)}x$
11. If $\tau^{-1}\Gamma = \Delta$ and $\tau \upharpoonright \Delta$ is one to one, then $\mathsf{c}_{(\Gamma)}\mathsf{s}_\tau x = \mathsf{s}_\tau \mathsf{c}_{(\Delta)}x$.

The definition of neat reducts for CPA_α is defined as follows: Given any pair of infinite ordinals $\alpha < \beta$ and $\mathfrak{B} \in \mathsf{CPA}_\beta$ then $\mathsf{Nr}_\alpha\mathfrak{B}$ is the CPA_α with domain $Nr_\alpha\mathfrak{B} = \{a \in \mathfrak{B} : \mathsf{c}_i a = a, \forall i \in \beta \sim \alpha\}$ and with all operations except substitutions are those of $\mathfrak{B}$ indexed up to α. As for substitutions, given $\tau \in {}^\alpha\alpha$, and $a \in Nr_\alpha B$, $\mathsf{s}^{\mathfrak{A}}_\tau a = \mathsf{s}^{\mathfrak{B}}_{\bar\tau} a$ where $\bar\tau = \tau \cup Id \upharpoonright \beta \sim \alpha$.

Next we prove that the class of completely representable CPA_βs, β an infinite ordinal, is elementary. This is in sharp contrst to the CA case. The idea of the proof of the next theorem, is simple and in essence the gist of the idea is analogous to the previous proof. Start with an atomic completely additive $\mathfrak{A} \in \mathsf{CPA}_\alpha$. Then $\mathfrak{A}$ neatly embeds into an algebra $\mathfrak{B} \in \mathsf{CPA}_\beta$, having enough spare dimensions $|\beta| > |\alpha|$, called a β- dilation of $\mathfrak{A}$, that is $\mathfrak{A} = \mathsf{Nr}_\alpha\mathfrak{B}$. As it turns out, $\mathfrak{B}$ is also atomic, and by complete additivity the sum of all substituted versions of the set of atoms is the top element in $\mathfrak{B}$. The desired representation is built from any principal ultrafilter thet preserves this set of infinitary joins as well as some infinitary joins that have to do with eliminating cylindrifiers. A principal ultrafilter preserving these sets of joins can always be found because, on the one hand, the set of principal ultrafilters are dense in the Stone space of the Boolean reduct of $\mathfrak{B}$ since the latter is atomic, and on the other hand, finding an ultrafilter preserving the given set joints amounts to finding a principal ultrafilter outside a nowhere dense set corresponding to the infinitary joins. Any such ultrafilter can be used to build the desired representation. But first a definition:

Definition 4.3. A *transformation system* is a quadruple of the form $(\mathfrak{A}, I, G, \mathsf{S})$ where $\mathfrak{A}$ is an algebra of any similarity type, I is a non empty set (we will only be concerned with infinite sets), G is a subsemigroup of $({}^I I, \circ)$ (the operation $\circ$ denotes composition of maps) and S is a homomorphism from G to the semigroup of endomorphisms of $\mathfrak{A}$. Elements of G are called transformations.

Theorem 4.4. *Let α be an infinite ordinal. Let $\mathfrak{A} \in \mathsf{CPA}_\alpha$ be atomic and completely additive. Then $\mathfrak{A}$ has a complete representation.*

Proof: Let $c \in A$ be non-zero. It suffices to find a set U and a homomorphism from $\mathfrak{A}$ into the set algebra with universe $\wp({}^\alpha U)$ that preserves arbitrary suprema whenever they exist and also satisfies that $f(c) \neq 0$. U is called the base of the set algebra. Let $\mathfrak{m}$ be the local degree of $\mathfrak{A}$, $\mathfrak{c}$ its effective cardinality and $\mathfrak{n}$ be any cardinal such that $\mathfrak{n} \geq \mathfrak{c}$ and $\sum_{s<\mathfrak{m}} \mathfrak{n}^s = \mathfrak{n}$. The cardinal $\mathfrak{n}$ will be the base of our desired representation. Substitutions in $\mathfrak{A}$, induce a homomorphism of semigroups $S : {}^\alpha\alpha \to End(\mathfrak{A})$, via $\tau \mapsto \mathsf{s}_\tau$. The operation on both semigroups is composition of maps; the latter is the semigroup of endomorphisms on $\mathfrak{A}$. For any set X, let $F({}^\alpha X, \mathfrak{A})$ be the set of all functions from ${}^\alpha X$ to $\mathfrak{A}$ endowed with Boolean operations defined pointwise and for $\tau \in {}^\alpha\alpha$ and $f \in F({}^\alpha X, \mathfrak{A})$, put $\mathsf{s}_\tau f(x) = f(x \circ \tau)$. This turns $F({}^\alpha X, \mathfrak{A})$ to a transformation system as well that is completely additive The map $H : \mathfrak{A} \to F({}^\alpha\alpha, \mathfrak{A})$ defined by $H(p)(x) = \mathsf{s}_x p$ is easily checked to be an embedding. Assume that $\beta \supseteq \alpha$. Then $K : F({}^{\alpha}\alpha, \mathfrak{A}) \to F({}^{\beta}\alpha, \mathfrak{A})$ defined by $K(f)x = f(x \upharpoonright \alpha)$ is an embedding, too. These facts are straighforward to establish, cf. [3, Theorems 3.1, 3.2]. Call $F({}^{\beta}\alpha, \mathfrak{A})$ a minimal functional dilation of $F({}^{\alpha}\alpha, \mathfrak{A})$. Elements of the big algebra, or the (cylindrifier free) functional dilation, are of form $\mathsf{s}_{\sigma}p$, $p \in F({}^{\beta}\alpha, \mathfrak{A})$ where σ is one to one on α, cf. [3, Theorems 4.3–4.4].

We can assume that $|\alpha| < \mathfrak{n}$. Let $\mathfrak{B}$ be the algebra obtained from $\mathfrak{A}$, by discarding its cylindrifiers, then dilating it to $\mathfrak{n}$ dimensions, that is, taking a minimal functional dilation in $\mathfrak{n}$ dimensions, and then re-defining cylindrifiers and boxes in the bigger algebra, by setting for each $\Gamma \subseteq \mathfrak{n}$:

$$\mathsf{c}_{(\Gamma)}\mathsf{s}^{\mathfrak{B}}_{\sigma}p = \mathsf{s}^{\mathfrak{B}}_{\rho^{-1}}\mathsf{c}^{\mathfrak{A}}_{(\rho(\Gamma)\cap\sigma\alpha)}\mathsf{s}^{\mathfrak{A}}_{(\rho\sigma\upharpoonright\alpha)}p.$$

Here ρ is any permutation such that $\rho \circ \sigma(\alpha) \subseteq \sigma(\alpha)$. The definition is sound, that is, it is independent of ρ, σ, p. Moreover, it agrees with the old cylindrifiers and boxes in $\mathfrak{A}$. Identifying algebras with their transformation systems we have $\mathfrak{A}$ is embeddable in $\mathsf{Nr}_{\alpha}\mathfrak{B}$, via H defined for $f \in \mathfrak{A}$ and $x \in {}^{\mathfrak{n}}\alpha$ by $H(f)x = f(y)$ where $y \in {}^{\alpha}\alpha$ and $x \upharpoonright \alpha = y$; furthermore H so defined exhausts all α dimensional elements of $\mathfrak{B}$ meaning that $\mathfrak{A} = \mathsf{Nr}_{\alpha}\mathfrak{B}$, cf. [3, Theorem 3.10]. The local degree of $\mathfrak{B}$ is the same as that of $\mathfrak{A}$, in particular, each $x \in \mathfrak{B}$ admits a support of cardinality $< \mathfrak{n}$. Also $|\mathfrak{n} \sim \alpha| = |\mathfrak{n}|$ and for all $Y \subseteq A$, we have $\mathfrak{Sg}^{\mathfrak{A}}Y = \mathsf{Nr}_{\alpha}\mathfrak{Sg}^{\mathfrak{B}}Y$. All this can be found in [3], see the proof of Theorem 6.1 therein; in such a proof, $\mathfrak{B}$ is called a minimal dilation of $\mathfrak{A}$, due to the fact that $\mathfrak{B}$ is unique up to isomorphisms that fix $\mathfrak{A}$ pointwise. Clearly $F({}^{\mathfrak{n}}\alpha, \mathfrak{A})$, hence the Boolean reduct of $\mathfrak{B}$, is atomic, because it is isomorphic to a Boolean product of the atomic Boolean reduct of $\mathfrak{A}$. Let $\Gamma \subseteq \alpha$ and $p \in \mathfrak{A}$. Then in $\mathfrak{B}$ we have, see [3, proof of Theorem 6.1]:

$$\mathsf{c}_{(\Gamma)}p = \sum\{\mathsf{s}_{\bar{\tau}}p : \tau \in {}^{\alpha}\mathfrak{n}, \quad \tau \upharpoonright \alpha \sim \Gamma = Id\}. \tag{4.1}$$

Here, and elsewhere throughout the paper, for a transformation τ with domain α and range included in $\mathfrak{n}$, $\bar{\tau} = \tau \cup Id_{\mathfrak{n}\sim\alpha}$. Let X be the set of atoms of $\mathfrak{A}$. Since $\mathfrak{A}$ is atomic, then $\sum^{\mathfrak{A}} X = 1$. By $\mathfrak{A} = \mathsf{Nr}_{\alpha}\mathfrak{B}$, we also have $\sum^{\mathfrak{B}} X = 1$. By complete additivity we have for all $\tau \in {}^{\alpha}\mathfrak{n}$,

$$\sum \mathsf{s}^{\mathfrak{B}}_{\bar{\tau}} X = 1. \tag{4.2}$$

Let S be the Stone space of $\mathfrak{B}$, whose underlying set consists of all Boolean ulltrafilters of $\mathfrak{B}$. Let X^* be the set of principal ultrafilters of $\mathfrak{B}$ (those generated by the atoms). These are isolated points in the Stone topology, and they form a dense set in the Stone topology since $\mathfrak{B}$ is atomic. So we have $X^* \cap T = \emptyset$ for every nowhere dense set T (since principal ultrafilters, which are isolated points in the Stone topology, lie outside nowhere dense sets). For $a \in \mathfrak{B}$, let N_a denote the set of all Boolean ultrafilters containing a. Now for all $\Gamma \subseteq \alpha$, $p \in A$ and $\tau \in {}^\alpha\mathfrak{n}$, we have, by the suprema, evaluated in (1) and (2):

$$G_{\Gamma,p} = N_{\mathsf{c}_{(\Gamma)}p} \sim \bigcup_{\tau\in{}^\alpha\mathfrak{n}} N_{s_{\bar{\tau}}p} \tag{4.3}$$

and

$$G_{X,\tau} = S \sim \bigcup_{x\in X} N_{s_{\bar{\tau}}x}. \tag{4.4}$$

are nowhere dense. Let F be a principal ultrafilter of S containing c. This is possible since $\mathfrak{B}$ is atomic, so there is an atom x below c; just take the ultrafilter generated by x. Then $F \in X^*$, so $F \notin G_{\Gamma,p}$, $F \notin G_{X,\tau}$, for every $\Gamma \subseteq \alpha$, $p \in A$ and $\tau \in {}^\alpha\mathfrak{n}$. Now define for $a \in A$

$$f(a) = \{\tau \in {}^\alpha\mathfrak{n} : \mathsf{s}_{\bar{\tau}}^{\mathfrak{B}} a \in F\}.$$

Then f is a polyadic homomorphism from $\mathfrak{A}$ to the full set algebra with unit ${}^\alpha\mathfrak{n}$, such that $f(c) \neq 0$. We have $f(c) \neq 0$ because $c \in F$, so $Id \in f(c)$. That f is a homomorphism can be proved exactly as in the proof of Theorem 4.1; the preservation of the Boolean operations and substitutions is fairly straightforward. Preservation of cylindrifications is guaranteed by the condition that $F \notin G_{\Gamma,p}$ for all $\Gamma \subseteq \alpha$ and all $p \in A$. (Basically an elimination of cylindrifications, this condition is also used in [3] to prove the main representation result for polyadic algebras.) The proof is complete.

Moreover f is an atomic representation since $F \notin G_{X,\tau}$ for every $\tau \in {}^\alpha\mathfrak{n}$, which means that for every $\tau \in {}^\alpha\mathfrak{n}$, there exists $x \in X$, such that $\mathsf{s}_{\bar{\tau}}^{\mathfrak{B}} x \in F$, and so $\bigcup_{x\in X} f(x) = {}^\alpha\mathfrak{n}$. We conclude that f is a complete representation, since in this case too it can be proved exactly like the CA case that complete and atomic rtepresenations coincide. $\square$

Theorem 4.5. *The class CPA_α is elementary, and it is axiomatizable by a finite schema in first order logic. Furthermore, for any infinite ordinals $\alpha < \beta$, $\mathsf{Nr}_\alpha\mathsf{CPA}_\beta$ is elementary.*

PROOF: Like the proof of Theorem 4.1. □

Also the technique used here adapts without much dificulty to prove completely analagous results for the so-called cylindric-polyadic algebras introduced by Ferenczi in [4] and [5, Definition 6.3.7]. We denote the class of such algebras of dimension α by CPEA_α. For CPEA_α diagonal algebras are present in their signature, so that complete additivity holds anyway. The complete representation in this case is not with respect to square Tarskian semantics, as is the case here, but is relativized to units that are (not necessarily disjoint) unions of Cartesian spaces. The class of such concrete representable algebras of dimension α is denoted by Gp_α. Recall that for a calss $\mathbf{K}$ having a Boolean reduct, we write $\mathbf{K} \cap \mathbf{At}$ for the class of atomic algebras in $\mathbf{K}$.

THEOREM 4.6. *The class of completely representable algebras in* CPEA_α *coincides with* $\mathsf{CPEA}_\alpha \cap \mathbf{At}$*, hence is elementary.*

PROOF: We start with the general idea. then follows a more technical proof. If $\mathfrak{A}$ is atomic, and $\mathfrak{B}$ is the minimal dilation of $\mathfrak{A}$, then $\mathfrak{B}$ is also atomic since its Boolean reduct is simply an infinite product of the atomic $\mathfrak{A}$. This can now be used to show that atomic algebras are completely representable. Like in the above proof, start with an atomic $\mathfrak{A} \in \mathsf{CPEA}_\alpha$. Then $\mathfrak{A}$ is completely additive and it neatly embeds into an algebra $\mathfrak{B}$ having enough spare dimensions, the minimal dilation of $\mathfrak{A}$, that is $\mathfrak{A} = \mathsf{Nr}_\alpha\mathfrak{B}$. As it turns out, $\mathfrak{B}$ is also atomic, and by complete additivity the sum of all all substituted versions of the set of atoms is the top element in $\mathfrak{B}$. The desired representation is built from any principal ultrafilter that preserves this set of infinitary joins as well as some infinitary joins that have to do with eliminating cylindrifiers. A principal ultrafilter preserving these sets of joins can always be found because, on the one hand, the set of principal ultrafilters are dense in the Stone space of the Boolean reduct of $\mathfrak{B}$ since the latter is atomic, and on the other hand, finding an ultrafilter preserving the given set of infinitary joins really amounts to finding a a principal ultrafilter outside a nowhere dense set corresponding to the infinitary joins. The hitherto obtained ultrafilter in $\mathfrak{B}$ can be easily modified to give a so-called *perfect ultrafilter*. One such ultrafilter is found for every non-zero element of $a \in A$ in the dilation $\mathfrak{B}$, containing a, giving an atomic simple representation (model) of $\mathfrak{A}$. Taking the subdirect product of these representations, we get the desired complete representation, whose unit is a disjoint union of units of such simple representations.

More technically, let $c \in \mathfrak{A}$ be non-zero. We will find a $\mathfrak{B} \in \mathsf{Gp}_\alpha$ and a homomorphism from $f : \mathfrak{A} \to \mathfrak{B}$ that preserves arbitrary suprema whenever they exist and also satisfies that $f(c) \neq 0$. Now there exists $\mathfrak{B} \in \mathsf{CPEA}_\mathfrak{n}$, $\mathfrak{n}$ a regular cardinal. such that $\mathfrak{A} \subseteq \mathsf{Nr}_\alpha\mathfrak{B}$ and A generates $\mathfrak{B}$ and we can assume that $|\mathfrak{n} \sim \alpha| = |\mathfrak{n}|$. We also have for all $Y \subseteq A$, we have $\mathfrak{Sg}^{\mathfrak{A}}Y = \mathsf{Nr}_\alpha\mathfrak{Sg}^{\mathfrak{B}}Y$. This dilation also has Boolean reduct isomophic to $F({}^\mathfrak{n}\alpha, \mathfrak{A})$, in particular, it is atomic because $\mathfrak{A}$ is atomic. Also cylindrifiers are defined on this minimal functional dilation exactly like above by restricting to singletions. Let *adm* be the set of admissable substitutions. The transformation τ is admissable if $\mathsf{dom}\tau \subseteq \alpha$ and $\mathsf{rng}\tau \cap \alpha = \emptyset$. Then we have for all $i < \mathfrak{n}$ and $\sigma \in adm$,

$$\mathsf{s}_\sigma\mathsf{c}_i p = \sum \mathsf{s}_\sigma\mathsf{s}_i^j p \tag{4.5}$$

This uses that $\mathsf{c}_k = \sum \mathsf{s}_k^i x$, which is proved like the cylindric case; the proof depends on diagonal elements. Let X be the set of atoms of $\mathfrak{A}$. Since $\mathfrak{A}$ is atomic, then $\sum^{\mathfrak{A}} X = 1$. By $\mathfrak{A} = \mathsf{Nr}_\alpha\mathfrak{B}$, we also have $\sum^{\mathfrak{B}} X = 1$. Because substitutions are completely additive we have for all $\tau \in {}^\alpha\mathfrak{n}$

$$\sum \mathsf{s}_{\bar{\tau}}^{\mathfrak{B}} X = 1. \tag{4.6}$$

Let S be the Stone space of $\mathfrak{B}$, whose underlying set consists of all boolean ulltrafilters of $\mathfrak{B}$, and let F be a principal ultrafilter chosen as before. Let $\mathfrak{B}'$ be the minimal completion of $\mathfrak{B}$. Exists by completey additivity. Take the filter G in $\mathfrak{B}'$ generated by the generator of F and let $F = G \cap \mathfrak{B}$. Then F is a perfect ultrafilter. Because our algebras have diagonal algebras, we have to factor our base by a congruence relation that reflects equality. Define an equivalence relation on $\Gamma = \{i \in \beta : \exists j \in \alpha : \mathsf{c}_i\mathsf{d}_{ij} \in F\}$, via $m \sim n$ iff $\mathsf{d}_{mn} \in F$. Then $\Gamma \subset \alpha$ and the desired representation is defined on a Gp_α with base $\Gamma/\sim$. We omit the details which are the same as in the proof of [27, Theorem 3.4, item 3]. $\square$

5. Related results on minimal Dedekind–MacNeille completions

Unless otherwise indicated, we fix $2 < n < \omega$. In our next Theorem we use rainbow constructions following almost verbatim [9, § 4.3] abeit adding a clause for the polyadic accessibility relations as follows: $[a]T_{ij}[b] \iff a \circ [i,j] = b$ where $a : n \to \Delta$ and $b : n \to \Gamma$ are surjections into complete (finite) coloured graphs Δ an Γ. This allows us to construct n dimensional polyadic equaltiy rainbow atom structures. (Everything else is like the CA_n case dealt with in detail in [9]). However, for the polyadic case, networks should be defined as the cylindric case with an additional symmetry condition:

Definition 5.1. An *n–dimensional atomic network* on an atomic algebra $\mathfrak{A} \in \mathsf{QEA}_n$ is a map $N : {}^n\Delta \to \mathsf{At}\mathfrak{A}$, where Δ is a non–empty finite set of *nodes*, denoted by $\mathsf{nodes}(N)$, satisfying the following consistency conditions for all $i < j < n$:

(i) If $\bar{x} \in {}^n\mathsf{nodes}(N)$ then $N(\bar{x}) \leq \mathsf{d}_{ij} \iff \bar{x}_i = \bar{x}_j$,

(ii) If $\bar{x}, \bar{y} \in {}^n\mathsf{nodes}(N)$, $i < n$ and $\bar{x} \equiv_i \bar{y}$, then $N(\bar{x}) \leq \mathsf{c}_i N(\bar{y})$,

(iii) (Symmetry): if $\bar{x} \in {}^n\mathsf{nodes}(N)$, then $\mathsf{s}_{[i,j]}N(\bar{x}) = N(\bar{x} \circ [i,j])$.

We give a detailed description of the rainbow-like construction we use. Let G be a relational structures. Let $2 < n < \omega$. Then we specify a list of colours from which our algebras are to be constructed:

- greens: g_i $(1 \leq i \leq n-2)$, g_0^i, $i \in \mathsf{G}$,
- whites : $\mathsf{w}_i : i \leq n-2$,
- reds: r_{ij} $i < j \in n$,
- shades of yellow : $\mathsf{y}_S : S$ a finite subset of ω or $S = \omega$.

A *coloured graph* is a graph such that each of its edges is labelled by the colours in the above first three items, greens, whites or reds, and some $n-1$ hyperedges are also labelled by the shades of yellow. Certain coloured graphs will deserve special attention.

Definition 5.2. Let $i \in \mathsf{G}$, and let M be a coloured graph consisting of n nodes $x_0, \ldots, x_{n-2}, z$. We call M *an i-cone* if $M(x_0, z) = \mathsf{g}_0^i$ and for every

$1 \leq j \leq n-2$, $M(x_j, z) = \mathsf{g}_j$, and no other edge of M is coloured green. $(x_0, \ldots, x_{n-2})$ is called the *base of the cone*, z the *apex of the cone* and i the *tint of the cone*.

The rainbow algebra depending on G and n from the class $\mathbf{K}$ consisting of all coloured graphs M such that:

1. M is a complete graph and M contains no triangles (called forbidden triples) of the following types:

$$\begin{array}{rl} (\mathsf{g}, \mathsf{g}^{'}, \mathsf{g}^{*}), (\mathsf{g}_i, \mathsf{g}_i, \mathsf{w}_i) & \text{any } 1 \leq i \leq n-2, \\ (\mathsf{g}_0^j, \mathsf{g}_0^k, \mathsf{w}_0) & \text{any } j, k \in \mathsf{G}, \\ (\mathsf{r}_{ij}, \mathsf{r}_{j'k'}, \mathsf{r}_{i^*k^*}) & \text{unless } |\{(j,k), (j',k'), (j^*,k^*)\}| = 3 \end{array}$$

   and no other triple of atoms is forbidden.

2. If $a_0, \ldots, a_{n-2} \in M$ are distinct, and no edge (a_i, a_j) $i < j < n$ is coloured green, then the sequence $(a_0, \ldots, a_{n-2})$ is coloured a unique shade of yellow. No other $(n-1)$ tuples are coloured shades of yellow. Finally, if $D = \{d_0, \ldots, d_{n-2}, \delta\} \subseteq M$ and $M \upharpoonright D$ is an i cone with apex δ, inducing the order $d_0, \ldots, d_{n-2}$ on its base, and the tuple $(d_0, \ldots, d_{n-2})$ is coloured by a unique shade y_S then $i \in S$.

Let G and n be relational structures as above. Take the set J consisting of all surjective maps $a : n \to \Delta$, where $\Delta \in \mathbf{K}$ and define an equivalence relation $\sim$ on this set relating two such maps iff they essentially define the same graph [9]; the nodes are possibly different but the graph structure is the same. Let At be the atom structure with underlying set $J \sim$. We denote the equivalence class of a by $[a]$. Then define, for $i < j < n$, the accessibility relations corresponding to ijth–diagonal element, and ith–cylindrifier, as follows:

(1) $[a] \in E_{ij}$ iff $a(i) = a(j)$,

(2) $[a]T_i[b]$ iff $a \upharpoonright n \setminus \{i\} = b \upharpoonright n \setminus \{i\}$,

(3) $[a]T_{ij}[b] \iff a \circ [i,j] = b$.

This, as easily checked, defines a QEA_n atom structure. The game $\mathbf{G}^m$ played on networks lifts to a game on coloured graphs like the CA case, that is like the graph games G^m_ω [9], where the number of nodes of graphs played during the ω rounded game does not exceed m, but $\forall$ has the option to re-use nodes. The typical winning strategy for $\forall$ in the graph version of both atomic games is bombarding $\exists$ with cones having a common base and *green* tints until she runs out of (suitable) *reds*, that is to say, reds whose indicies do not match [9, § 4.3].

Let K_n be a variety between Sc_n and QEA_n.

DEFINITION 5.3. A K_n atom structure $\mathbf{At}$ is *weakly representable* if there is an atomic $\mathfrak{A} \in \mathsf{RK}_n$ such that $\mathbf{At} = \mathsf{At}\mathfrak{A}$; it is *strongly representable* if $\mathfrak{Cm}\mathbf{At} \in \mathsf{RK}_n$.

These two notions are distinct, cf. [14] and the following Theorem 5.5. Let $2 < n < m \leq \omega$. The notions of m-square, and m-flat representations are defined and extensively studied in [27, § 5.1]. Let $\mathsf{V} \subseteq \mathsf{W}$ be varieties of Boolean algebras with operators. We say hat V is *atom canonical with respect to* W, if whenever $\mathfrak{A} \in \mathsf{V}$ is atomic, then its Dedekind–MacNeille completion, which is the complex algebra of its atom structure, in symbols $\mathfrak{Cm}\mathsf{At}\mathfrak{A}$ is in W. Let Sc_n denote the class of Pinter's subnstitution algebras as defined in [7] and the appendix of [13] and $\mathfrak{Rd}_{sc}$ denotes the Sc reduct. The following is proved in [27, Lemma 5.7]

LEMMA 5.4. *Let $2 < n < \omega$ and let $\mathfrak{A}$ have signature of CA_n satisfying all axioms except commutativity of cylindrifications. Then $\mathfrak{A}$ has a complete m-square representation $\iff$ $\exists$ has a winning strategy in $G^m_\omega(\mathsf{At}\mathfrak{A})$. The last result extends to any variety V between QEA_n and Sc_n. In particular, $\mathfrak{Rd}_{sc}\mathfrak{A} \notin \mathbf{S}\mathsf{Nr}_n\mathsf{Sc}_m$.*

With these preliminaries out of the way, we are now ready to start digging deeper: The next Theorem generalizes a result proved in [27, Theorem 5.9, Corollary 5.10] for CA_ns to any variety between Sc_n and QEA_n. We use a so called *blow up and blow construction.* This subtle construction may be applied to any two classes $\mathbf{L} \subseteq \mathbf{K}$ of completely additive BAOs. One takes an atomic $\mathfrak{A} \notin \mathbf{K}$ (usually but not always finite), blows it up, by splitting one or more of its atoms each to infinitely many subatoms, obtaining an (infinite) *countable* atomic $\mathfrak{B}b(\mathfrak{A}) \in \mathbf{L}$, such that $\mathfrak{A}$ *is blurred* in $\mathfrak{B}b(\mathfrak{A})$ meaning that $\mathfrak{A}$ *does not* embed in $\mathfrak{B}b(\mathfrak{A})$, but $\mathfrak{A}$ embeds in the Dedekind–MacNeille completion of $\mathfrak{B}b(\mathfrak{A})$, namely, $\mathfrak{Cm}\mathsf{At}\mathfrak{B}b(\mathfrak{A})$. Then any class $\mathbf{M}$ say, between $\mathbf{L}$ and $\mathbf{K}$ that is closed under forming subalgebras will not be atom–canonical, for $\mathfrak{B}b(\mathfrak{A}) \in \mathbf{L}(\subseteq \mathbf{M})$, but $\mathfrak{Cm}\mathsf{At}\mathfrak{B}b(\mathfrak{A}) \notin \mathbf{K}(\supseteq \mathbf{M})$

because $\mathfrak{A} \notin \mathbf{M}$ and $\mathbf{SM} = \mathbf{M}$. We say, in this case, that $\mathbf{L}$ *is not atom–canonical with respect to* $\mathbf{K}$. This method is applied to $\mathbf{K} = \mathsf{SRaCA}_l$, $l \geq 5$ and $\mathbf{L} = \mathsf{RRA}$ in [10, § 17.7] and to $\mathbf{K} = \mathsf{RRA}$ and $\mathbf{L} = \mathsf{RRA} \cap \mathsf{RaCA}_k$ for all $k \geq 3$ in [2], and will applied now below to $\mathbf{K} = \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{t(n)}$ where $t(n) = n(n+1)/2$.

THEOREM 5.5. *Let $2 < n < \omega$. The following propostions 1, 2, and 3 below are true:*

1. *The variety* RRA *is not atom-canonical with respect to* SRaCA_k*, for any $k \geq 6$,*
2. *Let* K *be any variety between* Sc *and* QEA*. Let $t(n) = n(n+1)/2 + 1$. Then* RK_n *is not-atom canonical with respect to* $\mathbf{S}\mathsf{Nr}_n\mathsf{K}_{t(n)}$*. In fact, there is a countable atomic simple $\mathfrak{A} \in \mathsf{RQEA}_n$ such that $\mathfrak{Rd}_{sc}\mathfrak{CmAt}\mathfrak{A}$ does not have an $t(n)$-square,a fortiori $t(n)$-flat, representation.*
3. RDf_n *is not atom-canonical.*

PROOF: For item (1) cf. [11, Lemmata 17.32, 17.34, 17.35, 17.36].

Item (2): The proof is long and uses many ideas in [14]. The proof is divided into four parts:

1. **Blowing up and blurring $\mathfrak{B}_f$ forming a weakly representable atom structure At**: Take the finite rainbow QEA_n, $\mathfrak{B}_f$ where the reds is the complete irreflexive graph n, and the greens are $\{\mathsf{g}_i : 1 \leq i < n-1\} \cup \{\mathsf{g}_0^i : 1 \leq i \leq n(n-1)/2 + 2\}$, endowed with the quasi-polyadic operations. We will show $\mathfrak{Rd}_{\mathsf{K}}\mathfrak{B}_f$ detects that RK_n is not atom-canonical with respect to $\mathbf{S}\mathsf{Nr}_n\mathsf{K}_{t(n)}$ with $t(n)$ as specified in the statement of the theorem. Denote the finite atom structure of $\mathfrak{B}_f$ by $\mathbf{At}_f$; so that $\mathbf{At}_f = \mathsf{At}(\mathfrak{B}_f)$. One then defines a larger the class of coloured graphs like in [14, Definition 2.5]. Let $2 < n < \omega$. Then the colours used are like above except that each red is 'split' into ω many having 'copies' the form r_{ij}^l with $i < j < n$ and $l \in \omega$, with an additional shade of red ρ such that the consistency conditions for the new reds (in addition to the usual rainbow consistency conditions) are as follows:

- $(r_{jk}^i, r_{j'k'}^i, r_{j^*k^*}^{i^*})$ unless $i = i' = i^*$ and $|\{(j,k),(j',k'),(j^*,k^*)\}| = 3$
- (r, ρ, ρ) and $(\mathsf{r}, \mathsf{r}^*, \rho)$, where r, r^* are any reds.

The consistency conditions can be coded in an $L_{\omega,\omega}$ theory T having signture the reds with ρ together with all other colours like in [11, Definition

3.6.9]. The theory T is only a first order theory (not an $L_{\omega_1,\omega}$ theory) because the number of greens is finite which is not the case with [11] where the number of available greens are countably infinite coded by an infinite disjunction. One construct an n-homogeneous model M is as a countable limit of finite models of T using a game played between $\exists$ and $\forall$like in [14, Theorem 2.16]. In the rainbow game $\forall$ challenges $\exists$ with *cones* having green *tints* (g_0^i), and $\exists$ wins if she can respond to such moves. This is the only way that $\forall$ can force a win. $\exists$ has to respond by labelling *appexes* of two succesive cones, having the *same base* played by $\forall$. By the rules of the game, she has to use a red label. She resorts to ρ whenever she is forced a red while using the rainbow reds will lead to an inconsistent triangle of reds; [14, Proposition 2.6, Lemma 2.7]. The number of greens make [14, Lemma 3.10] work with the same proof using only finitely many green and not infinitely many. The winning strategy implemented by $\exists$ using the red label ρ that comes to her rescue whenever she runs out of 'rainbow reds', so she can always and consistently respond with an extended coloured graph.

2. **Representing a term algebra (and its completion) as (generalized) set algebras**: Having M at hand, one constructs two atomic n–dimensional set algebras based on M, sharing the same atom structure and having the same top element. The atoms of each will be the set of coloured graphs, seeing as how, quoting Hodkinson [14] such coloured graphs are 'literally indivisible'. Now L_n and $L^n_{\infty,\omega}$ are taken in the rainbow signature (without ρ). Continuing like in *op.cit*, deleting the one available red shade, set $W = \{\bar{a} \in {}^n\mathsf{M} : \mathsf{M} \models (\bigwedge_{i<j<n} \neg\rho(x_i, x_j))(\bar{a})\}$, and for $\phi \in L^n_{\infty,\omega}$, let $\phi^W = \{s \in W : \mathsf{M} \models_W \phi[s]\}$. Here W is the set of all n–ary assignments in ${}^n\mathsf{M}$, that have no edge labelled by ρ and $\models_W$ is first order emantics with quantifiers relativized to W, cf. [14, §3.2 and Definition 4.1]. Let $\mathfrak{A}$ be the relativized set algebra with domain $\{\varphi^W : \varphi$ a first-order $L_n -$ formula$\}$ and unit W, endowed with the usual concrete cylindric operations read off the connectives. Classical semantics for L_n *rainbow formulas* and their semantics by relativizing to W coincide [14, Proposition 3.13] *but not with respect to $L^n_{\infty,\omega}$ rainbow formulas.* Hence the set algebra $\mathfrak{A}$ is isomorphic to a cylinric set algebra of dimension n having top element ${}^n\mathsf{M}$, so $\mathfrak{A}$ is simple, in fact its Df reduct is simple. Let $\mathfrak{E} = \{\phi^W : \phi \in L^n_{\infty,\omega}\}$ [14, Definition 4.1] with the operations defined like on $\mathfrak{A}$ the usual way. $\mathfrak{Cm}\mathbf{At}$ is a complete CA_n and, so like in [14, Lemma 5.3] we have an isomorphism from $\mathfrak{Cm}\mathbf{At}$ to $\mathfrak{E}$ defined via $X \mapsto \bigcup X$. Since $\mathsf{At}\mathfrak{A} = \mathsf{At}\mathfrak{Tm}(\mathsf{At}\mathfrak{A})$, which we refer to

only by $\mathbf{At}$, and $\mathfrak{TmAtA} \subseteq \mathfrak{A}$, hence $\mathfrak{TmAtA} = \mathfrak{Tm}\mathbf{At}$ is representable. The atoms of $\mathfrak{A}$, $\mathfrak{TmAtA}$ and $\mathfrak{CmAtA} = \mathfrak{Cm}\mathbf{At}$ are the coloured graphs whose edges are *not labelled* by ρ. These atoms are uniquely determined by the interpretion in M of so-called MCA formulas in the rainbow signature of $\mathbf{At}$ as in [14, Definition 4.3].

3. **Embedding $\mathfrak{B}_f$ into $\mathfrak{Cm}(\mathbf{At})$**: Let CRG_f be the class of coloured graphs on $\mathbf{At}_f$ and CRG be the class of coloured graph on $\mathbf{At}$. We can (and will) assume that $\mathsf{CRG}_f \subseteq \mathsf{CRG}$. Write M_a for the atom that is the (equivalence class of the) surjection $a : n \to M$, $M \in \mathsf{CGR}$. Here we identify a with $[a]$; no harm will ensue. We define the (equivalence) relation $\sim$ on $\mathbf{At}$ by $M_b \sim N_a$, $(M, N \in \mathsf{CGR})$:

- $a(i) = a(j) \iff b(i) = b(j)$,
- $M_a(a(i), a(j)) = \mathsf{r}^l \iff N_b(b(i), b(j)) = \mathsf{r}^k$, for some $l, k \in \omega$,
- $M_a(a(i), a(j)) = N_b(b(i), b(j))$, if they are not red,
- $M_a(a(k_0), \ldots, a(k_{n-2})) = N_b(b(k_0), \ldots, b(k_{n-2}))$, whenever defined.

We say that M_a is a *copy of* N_b if $M_a \sim N_b$ (by symmetry N_b is a copy of M_a.) Indeed, the relation 'copy of' is an equivalence relation on $\mathbf{At}$. An atom M_a is called a *red atom*, if M_a has at least one red edge. Any red atom has ω many copies, that are *cylindrically equivalent*, in the sense that, if $N_a \sim M_b$ with one (equivalently both) red, with $a : n \to N$ and $b : n \to M$, then we can assume that $\mathsf{nodes}(N) = \mathsf{nodes}(M)$ and that for all $i < n$, $a \restriction n \sim \{i\} = b \restriction n \sim \{i\}$. In $\mathfrak{Cm}\mathbf{At}$, we write M_a for $\{M_a\}$ and we denote suprema taken in $\mathfrak{Cm}\mathbf{At}$, possibly finite, by $\sum$. Define the map Θ from $\mathfrak{A}_{n+1,n} = \mathfrak{Cm}\mathbf{At}_f$ to $\mathfrak{Cm}\mathbf{At}$, by specifing first its values on $\mathbf{At}_f$, via $M_a \mapsto \sum_j M_a^{(j)}$ where $M_a^{(j)}$ is a copy of M_a. So each atom maps to the suprema of its copies. This map is well-defined because $\mathfrak{Cm}\mathbf{At}$ is complete. We check that Θ is an injective homomorphim. Injectivity is easy. We check preservation of all the CA_n extra Boolean operations.

- Diagonal elements. Let $l < k < n$. Then:

$$M_x \leq \Theta(\mathsf{d}_{lk}^{\mathfrak{Cm}\mathbf{At}_f}) \iff M_x \leq \sum_j \bigcup_{a_l = a_k} M_a^{(j)}$$
$$\iff M_x \leq \bigcup_{a_l = a_k} \sum_j M_a^{(j)}$$
$$\iff M_x = M_a^{(j)} \text{ for some } a : n \to M \text{ such that } a(l) = a(k)$$
$$\iff M_x \in \mathsf{d}_{lk}^{\mathfrak{Cm}\mathbf{At}}.$$

- Cylindrifiers. Let $i < n$. By additivity of cylindrifiers, we restrict our attention to atoms $M_a \in \mathbf{At}_f$ with $a : n \to M$, and $M \in \mathsf{CRG}_f \subseteq \mathsf{CRG}$. Then:

$$\Theta(\mathsf{c}_i^{\mathfrak{Cm}\mathbf{At}_f} M_a) = f(\bigcup_{[c] \equiv_i [a]} M_c) = \bigcup_{[c] \equiv_i [a]} \Theta(M_c)$$
$$= \bigcup_{[c] \equiv_i [a]} \sum_j M_c^{(j)} = \sum_j \bigcup_{[c] \equiv_i [a]} M_c^{(j)} = \sum_j \mathsf{c}_i^{\mathfrak{Cm}\mathbf{At}} M_a^{(j)}$$
$$= \mathsf{c}_i^{\mathfrak{Cm}\mathbf{At}} (\sum_j M_a^{(j)}) = \mathsf{c}_i^{\mathfrak{Cm}\mathbf{At}} \Theta(M_a).$$

- Substitutions: Let $i, k < n$. By additivity of the $\mathsf{s}_{[i,k]}$s, we again restrict ourselves to atoms of the form M_a as specified in the previous items. Now computing we get: $\Theta(\mathsf{s}_{[i,k]}^{\mathfrak{Cm}\mathbf{At}_f} M_a) = \Theta(M_{a \circ [i,k]}) = \sum_j^{\mathfrak{Cm}\mathbf{At}} (M_{a \circ [i,k]}^{(j)}) = \sum_j \mathsf{s}_{[i,k]}^{\mathfrak{Cm}\mathbf{At}} M_a^{(j)} = \mathsf{s}_{[i,k]}^{\mathfrak{Cm}\mathbf{At}} (\sum_j M_a^{(j)}) = \mathsf{s}_{[i,k]}^{\mathfrak{Cm}\mathbf{At}} \Theta(M_a)$.

4. $\forall$ **has a winning strategy in $G^{t(n)}\mathsf{At}(Rd\mathfrak{B}_f)$; and the required result**: It is straightforward to show that $\forall$ has winning strategy first in the Ehrenfeucht–Fraïssé forth private game played between $\exists$ and $\forall$ on the complete irreflexive graphs $n(n-1)/2+2)$ and n in $n(n-1)/2+2$ rounds $\mathsf{EF}_{n(n-1)+2}^{n(n-1)2+2}(n+1, n)$ [11, Definition 16.2] since $n(n-1)/2+2$ is 'longer' than n. Using (any) $p > n$ many pairs of pebbles avalable on the board $\forall$ can win this game in $n+1$ many rounds. For brevity, write $\mathfrak{D} \in \mathsf{Sc}_n$

instead of $\mathfrak{Rd}_{\mathsf{Sc}}\mathfrak{B}$. $\forall$ lifts his winning strategy from the last private Ehrenfeucht–Fraïssé forth game to the graph game on $\mathbf{At}_f = \mathsf{At}(\mathfrak{D})$ [9, p. 841] forcing a win using $t(n)$ nodes. One uses the $n(n-1)/2+2$ green relations in the usual way to force a red clique C, say with $n(n-1)/2+2$. Pick any point $x \in C$. Then there are $> n(n-1)/2$ points y in $C \setminus \{x\}$. There are only $n(n-1)/2$ red relations. So there must be distinct $y, z \in C \setminus \{x\}$ such that (x, y) and (x, z) both have the same red label (it will be some r_{ij}^m for $i < j < n$). But (y, z) is also red, and this contradicts (1.3) above. In more detail, $\forall$ bombards $\exists$ with cones having common base and distinct green tints until $\exists$ is forced to play an inconsistent red triangle (where indicies of reds do not match). He needs $n-1$ nodes as the base of cones, plus $|P|+2$ more nodes, where $P = \{(i,j) : i < j < n\}$ forming a red clique, triangle with two edges satisfying the same r_p^m for $p \in P$. Calculating, we get $t(n) = n-1+n(n-1)/2+2 = n(n+1)/2+1$. By Lemma 2.5, $\mathfrak{D} \notin \mathbf{S}_c\mathsf{Nr}_n\mathsf{Sc}_{t(n)}^{\mathsf{ad}}$ when $2 < n < \omega$). Since $\mathfrak{D}$ is finite, then $\mathfrak{D} \notin \mathbf{S}\mathsf{Nr}_n\mathsf{Sc}_{t(n)}$, because $\mathfrak{D}$ coincides with its canonical extension and for any $\mathfrak{D} \in \mathsf{Sc}_n$, $\mathfrak{D} \in \mathbf{S}\mathsf{Nr}_n\mathsf{Sc}_{t(n)} \implies \mathfrak{D}^+ \in \mathbf{S}_c\mathsf{Nr}_n\mathsf{Sc}_{t(n)}$. To see why, we could omit the superscrpt ad, abbreviating additivity, assume that $\mathfrak{D} \subseteq \mathfrak{Nr}_n\mathfrak{E}^{\mathsf{ad}}$, $\mathfrak{E} \in \mathsf{Sc}_{n+3}$. Let $\mathfrak{E}' = \mathfrak{Sg}^{\mathfrak{E}}\mathfrak{D}$, then $\mathfrak{E}'$ is finite, hence completely additive and $\mathfrak{D} \subseteq \mathfrak{Nr}_n\mathfrak{E}'$. But $\mathfrak{B}_f$ embeds into $\mathfrak{Cm}\mathsf{At}\mathfrak{A}$, hence $\mathfrak{Rd}_{sc}\mathfrak{Cm}\mathsf{At}\mathfrak{A}$ is outside the variety $\mathbf{S}\mathsf{Nr}_n\mathsf{Sc}_{t(n)}$, as well. Since $\mathfrak{Rd}_{sc}\mathfrak{A}$ is completely additive because it is a reduct of a QEA_n, then $\mathfrak{Cm}\mathsf{At}\mathfrak{Rd}_{Sc}\mathfrak{A}$ is the Dedekind–MacNeille completion of $\mathfrak{Rd}_{sc}\mathfrak{A}$. By Lemma 5.4, the required follows. But $\mathfrak{D}$ embeds into $\mathfrak{Rd}_{sc}\mathfrak{Cm}\mathsf{At}\mathfrak{A}$, hence $\mathfrak{Cm}\mathsf{At}\mathfrak{Rd}_{sc}\mathfrak{A}$ is outside the variety $\mathbf{S}\mathsf{Nr}_n\mathsf{Sc}_{t(n)}$, as well.

Now we prove the last item, namely, that RDf_n is not atom-canonical. Using essentially the argument in [7, Lemma 5.1.50, Theorem 5.1.51] by considering closure under infinite intersections instead of intersections, it is enough to show that $\mathfrak{Cm}\mathsf{At}\mathfrak{A}$ is generated by elements whose dimension sets have cardinality $< n$ using infinite unions. We show that for any rainbow atom $[a]$, $a : n \to \Gamma$, Γ a coloured graph, that $[a] = \prod_{i<n} \mathsf{c}_i[a]$. Clearly $\leq$ holds. Assume that $b : n \to \Delta$, Δ a coloured graph, and $[a] \neq [b]$. We show that $[b] \notin \prod_{i<n} \mathsf{c}_i[a]$ by which we will be done. Because a is not equivalent to b, we have one of two possibilities; either $(\exists i, j < n)(\Delta(b(i), b(j) \neq \Gamma(a(i), a(j))$ or $(\exists i_1, \ldots, i_{n-1} < n)(\Delta(b_{i_1}, \ldots, b_{i_{n-1}}) \neq \Gamma(a_{i_1}, \ldots, a_{i_{n-1}}))$. Assume the first possibility (the second is similar): Choose $k \notin \{i, j\}$. This is possible because $n > 2$. Assume for contradiction that $[b] \in \mathsf{c}_k[a]$. Then

$(\forall i,j\in n\setminus\{k\})(\Delta(b(i),b(j))=\Gamma(a(i)a(j)))$. By assumption and the choice of k, $(\exists i,j\in n\setminus k)(\Delta(b(i),b(j))\neq\Gamma(a(i),a(j)))$, contradiction. $\square$

Corollary 5.6. Let $2<n<\omega$, and let $t(n)=n(n+1)/2+1$ and V be any variety between Sc and QEA. Then the following propsitions 1, 2, 3 and 4 are valid:

1. There exists an algebra outside $\mathbf{S}\mathsf{Nr}_n\mathsf{V}_{t(n)}$ with a representable dense subalgebra
2. There exists a countable atomic algebra $\mathfrak{A}\in\mathsf{V}_n$ that is not strongly representable up to $t(n)$.
3. The varieties $\mathbf{S}\mathsf{Nr}_n\mathsf{V}_m$ for any $m\geq t(n)$ are not atom-canonical, *a fortiori* are not closed under Dedekind–MacNeille completions
4. There is an atom structure $\mathbf{At}$ such that $\mathfrak{Tm}\mathbf{At}\in\mathsf{RV}_n$ and $\mathfrak{Cm}\mathbf{At}\notin\mathbf{S}\mathsf{Nr}_n\mathsf{V}_{t(n)}$.

For a class $\mathbf{K}$ of BAOs, let $\mathbf{K}\cap\mathsf{Count}$ denote the class of atomic algebras in $\mathbf{K}$ having countably many atoms.

Proposition 5.7. Let $2<n<\omega$. The following propositions 1,2, and 3 below are valid:

1. For any ordinal $0\leq j$, $\mathsf{RCA}_n\cap\mathsf{Nr}_n\mathsf{CA}_{n+j}\cap\mathsf{Count}$ is not atom-canonical with respect to RCA_n if and only if $j<\omega$,
2. For any ordinal j, $\mathsf{Nr}_n\mathsf{CA}_{n+j}\cap\mathsf{RCA}_n\cap\mathbf{At}\nsubseteq\mathsf{CRCA}_n$,
3. There exists an atomic RCA_n such that its Dedekind–MacNeille (minimal) completion does not embed into its canonical extension.[2]

Proof: 1. One implication follows from [2] where for each $2<n<l<\omega$ an algebra $\mathfrak{A}_l\in\mathsf{RCA}_n\cap\mathsf{Nr}_n\mathsf{CA}_l$ is constructed such that $\mathfrak{Cm}\mathsf{At}\mathfrak{A}_l\notin\mathsf{RCA}_n$, so $\mathfrak{A}_l$ cannot be completely representable. Conversely, for any infinite ordinal j, $\mathsf{Nr}_n\mathsf{CA}_{n+j}=\mathsf{Nr}_n\mathsf{CA}_\omega$ and if $\mathfrak{A}\in\mathsf{Nr}_n\mathsf{CA}_\omega\cap\mathsf{Count}$, then by [24, Theorem 5.3.6], $\mathfrak{A}\in\mathsf{CRCA}_n$, so $\mathfrak{Cm}\mathsf{At}\mathfrak{A}\in\mathsf{RCA}_n$.

[2] In the CA context, the terminology *minimal completion* is misleading because $\mathfrak{A}^+$ is *another completion* of $\mathfrak{A}$; so supposedly the minimal completion of $\mathfrak{A}$ should embed into $\mathfrak{A}^+$, which is not, as we have already seen in Theorem 5.5, always true. Conversely, for an atomic Boolean algebra $\mathfrak{B}$, $\mathfrak{Cm}\mathsf{At}\mathfrak{B}$ always embeds into $\mathfrak{B}^+$ as it should.

2. The case $j < \omega$, follows from the fact that the algebra $\mathfrak{A}_{n+j}$ used in the previous item is in $\mathsf{Nr}_n\mathsf{CA}_{n+j} \cap \mathsf{RCA}_n$ but has no complete representation. For infinite j one uses the construction in Theorem 2.6.

3. Let $\mathfrak{A} = \mathfrak{TmAt}$ be the CA_n as defined in the proof of Theorem 5.5. Since $\mathfrak{Cm}\mathsf{At}\mathfrak{A} \notin \mathsf{RCA}_n$, it does not embed into $\mathfrak{A}^+$, because $\mathfrak{A}^+ \in \mathsf{RCA}_n$ since $\mathfrak{A} \in \mathsf{RCA}_n$ and RCA_n is a canonical variety. $\square$

The strongest result on first order definability is proved by the present author where it is shown that for any class K such that $\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n \subseteq \mathsf{K} \subseteq \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_{n+3}$, we have K is not elementary. This generalizes to any V between Sc_n and QEA_n. For more on connections between atom-canonicity, complete representations with repercussions on omitting types theorems for modal fragments of $L_{\omega,\omega}$, the reader is referred to [29, 28, 23].

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