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# ANALYTIC NON-LABELLED PROOF-SYSTEMS FOR HYBRID LOGIC: OVERVIEW AND A COUPLE OF STRIKING FACTS

## Abstract

This paper is about non-labelled proof-systems for hybrid logic, that is, proof-systems where arbitrary formulas can occur, not just satisfaction statements. We give an overview of such proof-systems, focusing on analytic systems: Natural deduction systems, Gentzen sequent systems and tableau systems. We point out major results and we discuss a couple of striking facts, in particular that non-labelled hybrid-logical natural deduction systems are analytic, but this is not proved in the usual way via step-by-step normalization of derivations.

*Keywords:* Hybrid logic, natural deduction systems, sequent systems, normalization, cut-elimination, analyticity.

## 1. Introduction

In the standard Kripke semantics for modal logic, the truth-value of a formula is relative to points in a set, that is, a formula is evaluated “locally” at a point, where points usually are taken to represent possible worlds, times, locations, persons, or something else. Hybrid modal logics are extended modal logics where it is possible to directly refer to such points in the object language. This means that locality can be handled explicitly, and, crucially, one can formulate statements about what is the case at a particular time or what is the case from the perspective of a specific person. The history of what now is known as hybrid logic goes back to the philosopher Arthur Prior’s work in the 1960s.

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The most basic hybrid logic is obtained by extending ordinary modal logic with *nominals*, which are propositional symbols of a new sort, each interpreted in a restricted way, being true at exactly one point. Most hybrid logics involve further additional machinery; in the present paper we shall consider a kind of operator called *satisfaction operator*. The motivation for adding satisfaction operators is to be able to formalize a statement being true at a particular time, location, or something else. In general, if  $a$  is a nominal and  $\phi$  is an arbitrary formula, then a new formula  $@_a\phi$  can be built, where  $@_a$  is a satisfaction operator. A formula of this form is called a *satisfaction statement*. The formula  $@_a\phi$  expresses that the formula  $\phi$  is true at one particular point, namely the point to which the nominal  $a$  refers.

A bit of terminology: We call a proof-system for hybrid logic *labelled* if all formulas in the system are satisfaction statements. This should not be confused with labelled proof systems for ordinary modal logic, where the labels are part of metalinguistic machinery, modelling the Kripke semantics (of course, there is a close connection between labelled systems for modal logic and labelled systems for hybrid logic, since the outermost satisfaction operators in the hybrid-logical systems can be seen as internalizing metalinguistics labels in the object language). A *non-labelled* proof system for hybrid logic is a system where arbitrary formulas are allowed.

Labelled proof systems for hybrid logic do now constitute a well-established research field with a large number of publications, one reason being that the extra expressive power of hybrid logic remedies a number of deficiencies of proof-systems for ordinary modal logic, for example, proof-systems for wide classes of hybrid logics can be obtained in a uniform way by adding rules or axioms as appropriate (it should be mentioned, though, that uniform proof-systems for ordinary modal logic can be given at the expense of using metalinguistic machinery). However, the present note is about non-labelled proof systems for hybrid logic, which have a less smooth and continuous history than the labelled systems.

The way of reasoning in non-labelled systems is very different from reasoning in labelled systems: Reasoning in non-labelled systems does not directly rely on the global encodings that satisfaction operators make possible, hence, these systems can be considered more in line with the local character of the Kripke semantics for modal logic. In fact, this more local reasoning style enables explicit formalization of reasoning involving shifts between perspectives (what satisfaction operators do), and in partic-

ular, reasoning taking place from different perspectives (embodied in the (*Term*) rule which we shall come back to, see Figure 1). This for example makes non-labelled systems suitable for formalizing the perspectival reasoning taking place in psychological false-belief tests, see the overview in [8]. The reasoning style in non-labelled systems also gives rise to a number of proof-theoretical issues of a more mathematical nature, which will be a main topic of the present paper.

First a definition: A proof-system is analytic if any derivable formula has a derivation satisfying the subformula property, that is, having the property that any formula in the derivation is a subformula of the conclusion or one of the premises. Analyticity is a common success criterion in proof-theory, one reason being that analytic provability facilitates proof-search and therefore is a step towards automated theorem proving. Given completeness, analyticity guarantees that any valid argument (that is, the truth of the premises implies the truth of the conclusion) can be formalized using only subformulas of the premises and the conclusion. The notion of analyticity goes back to G.W. Leibniz (1646–1716) who called a proof analytic if and only if the proof is based on concepts contained in the proven statement, the main aim being to be able to construct a proof by an analysis of the result, as described in the introductory chapter of the book [1].

After having introduced the formal syntax and semantics of hybrid logic in Section 2, in Section 3 we give an overview of different types of non-labelled natural deduction systems, Gentzen sequent systems, and tableau systems, pointing out analyticity results and how such results are proved. In Section 4 we discuss two striking and related facts:

- The natural deduction system of [6] is analytic, but this is not proved in the usual way via normalization of derivations such that normal derivations satisfy the subformula property. In fact, no set of reduction rules serving this purpose is known for the system.
- Seligman's paper [20] sketches a syntactic cut-elimination procedure for a sequent system, but this procedure is complex and highly non-local, that is, some steps in the procedure involve operations on derivations beyond usual local operations like permutation of rules, or replacing a cut on a compound formula by cuts on its subformulas, or in the case of first-order logic, simple non-local operations like substituting of terms for variables.

In Section 5 we then zoom in on an analyticity proof for the natural deduction system of [6], and we consider an extension of this system with rules corresponding to conditions on the accessibility relation. We give an analyticity proof for the extended system, which is a straightforward extension of a proof given in the appendix of [7], which in turn is a sharpened version of a proof given in the book [6] (see Section 5 for a detailed account). Finally, in Section 6 we make a few remarks on further work.

## 2. Formal syntax and semantics of hybrid logic

In what follows we give the formal syntax and semantics of the basic hybrid logic described informally in the introduction. It is assumed that a set of ordinary propositional symbols and a countably infinite set of nominals are given. The sets are assumed to be disjoint. The metavariables  $p, q, r, \dots$  range over ordinary propositional symbols and  $a, b, c, \dots$  range over nominals. Formulas are built from nominals and ordinary propositional symbols using the connectives  $\wedge, \rightarrow, \perp, \Box, \text{and } @_a$ . As usual,  $\neg\phi$  is an abbreviation for  $\phi \rightarrow \perp$  and  $\Diamond\phi$  is an abbreviation for  $\neg\Box\neg\phi$ .

**DEFINITION 2.1.** A *model* for hybrid logic is a tuple  $(W, R, \{V_w\}_{w \in W})$  where

1.  $W$  is a non-empty set;
2.  $R$  is a binary relation on  $W$ ; and
3. for each  $w$ ,  $V_w$  is a function that to each ordinary propositional symbol assigns an element of  $\{0, 1\}$ .

Note that a model for hybrid logic is the same as a model for ordinary modal logic. Given a model  $\mathfrak{M} = (W, R, \{V_w\}_{w \in W})$ , an *assignment* is a function  $g$  that to each nominal assigns an element of  $W$ . The relation  $\mathfrak{M}, g, w \models \phi$  is defined by induction, where  $g$  is an assignment,  $w$  is an element of  $W$ , and  $\phi$  is a formula.

$$\begin{array}{ll}
 \mathfrak{M}, g, w \models p & \text{iff } V_w(p) = 1 \\
 \mathfrak{M}, g, w \models a & \text{iff } w = g(a) \\
 \mathfrak{M}, g, w \models \phi \wedge \psi & \text{iff } \mathfrak{M}, g, w \models \phi \text{ and } \mathfrak{M}, g, w \models \psi \\
 \mathfrak{M}, g, w \models \phi \rightarrow \psi & \text{iff } \mathfrak{M}, g, w \models \phi \text{ implies } \mathfrak{M}, g, w \models \psi
 \end{array}$$

$$\begin{aligned}
\mathfrak{M}, g, w \models \perp & \text{ iff } \text{falsum} \\
\mathfrak{M}, g, w \models \Box\phi & \text{ iff for any } v \in W \text{ such that } wRv, \mathfrak{M}, g, v \models \phi \\
\mathfrak{M}, g, w \models @_a\phi & \text{ iff } \mathfrak{M}, g, g(a) \models \phi
\end{aligned}$$

Validity is defined in the usual way. For further background on hybrid logic, see [5] and the references therein.

### 3. Overview of published results

Non-labelled proof systems for hybrid logic can be split up into two types: Systems where the proof rules for modal operators are taken from standard proof systems for ordinary modal logic, and systems where modal operators are dealt with like in labelled systems, that is, analogous to first-order quantifiers, ranging over accessible worlds in the Kripke semantics. We call the first type of non-labelled systems *mixed*, and the second type *Seligman-style*.

In Table 1 we map out published works on non-labelled natural deduction systems, Gentzen sequent systems and tableau systems for hybrid logic (to the best of our knowledge). We are particularly interested in how completeness and analyticity results are proved, and how general the results are, that is, whether the systems can be extended with axioms or rules coding up classes of conditions on the accessibility relation.<sup>1</sup> Regarding the latter, then we remark that a pure formula is a formula where all propositional symbols are nominals. We shall come back to geometric rules in Section 5.

Note that cut-free sequent systems are usually analytic since the cut-rule is the only rule that introduces a new formula when read from bottom to top. Note also that most tableau systems do not include the cut-rule, hence, such systems are usually analytic. In principle, one could also ask for a syntactic cut-elimination procedure for a tableau system, but this issue is usually not addressed in the more semantically inclined tableau literature, and we shall not address it either in the present note.

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<sup>1</sup>An even more fine-grained classification of the systems would be in terms of the *form* of the proof rules, for example along the lines of [18] as well as further literature in the proof-theoretic semantics tradition, but we shall leave this to further work.

**Table 1.** Non-labelled proof systems for hybrid logic

	Mixed	Seligman-style
ND systems	<p>[14], Subsection 12.4.1, first system Based on ND rules for K Complete via a hybrid-logical axiom system Extendable with pure formulas as axioms Normalization/analyticity not addressed</p>	<p>[20] Complete via an axiom system which is given a Henkin-style completeness proof Normalization/analyticity not addressed</p>
		<p>[14], Subsection 12.4.1, second system Complete via a hybrid-logical axiom system Extendable with pure formulas as axioms Normalization/analyticity not addressed</p>
		<p>[6], Chapter 4, complete via a translation from a labelled ND system Extendable with geometric rules No normalization, but analyticity cf. Section 5 of the present paper</p>
Gentzen sequent systems	<p>[16] Based on sequent rules for K Complete via a sequent system for first-order logic Extendability unknown cf. page 59 Cut-freeness inherited from the first-order system</p>	<p>[20] Complete via a ND system and a direct syntactic cut-elimination proof</p>
		<p>[21] (similar to system in [20]) Complete via a sequent system for first-order logic Cut-freeness inherited from the first-order system</p>
Tableau systems		<p>[3] complete via translation from a labelled tableau system</p>
		<p>[15] (same system as [3]) Semantic completeness Extendable with pure formulas as axioms</p>
		<p>[11, 10] formalizes completeness of [3, 15] in the proof assistant Isabelle/HOL</p>

## 4. The lack of normalization and local syntactic cut-elimination

The lack of normalization for the natural deduction system of the book [6] was already pointed out in the paper [4] (which later became part of the book). In the present section we shall take a closer look at this issue, and make a comparison to sequent systems. We first outline the natural deduction system in question.

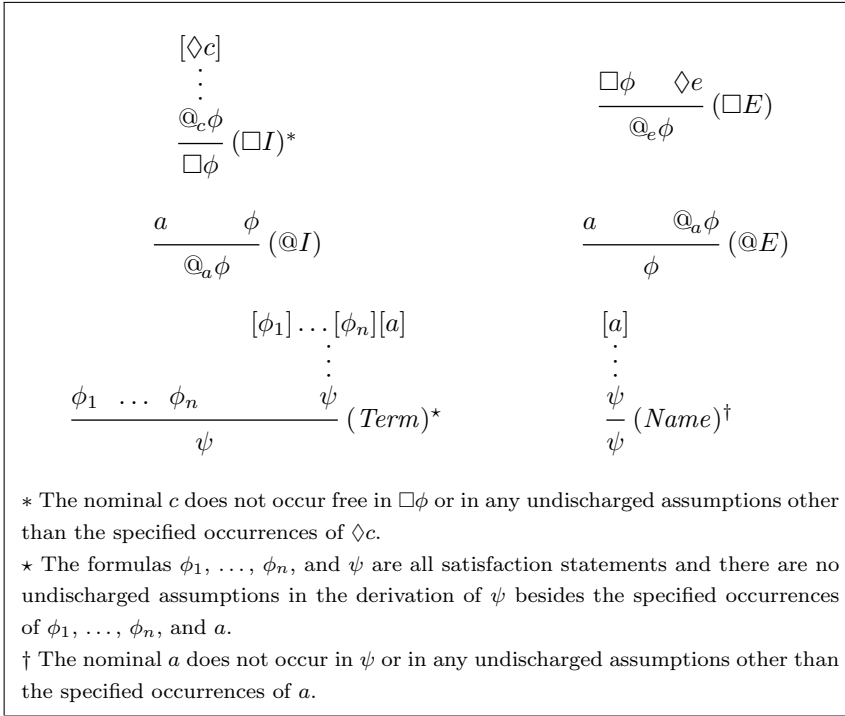
### 4.1. Seligman-style natural deduction for hybrid logic

Natural deduction style proofs are meant to formalize the way human beings actually reason, as pointed out in many places in the literature, and some psychologists have even found experimental support for natural deduction being a mechanism underlying human deductive reasoning, see [19].

Now, Seligman's natural deduction system is obtained by extending the standard natural deduction system for propositional logic (which we omit here) with the rules in Figure 1, cf. Chapter 4 of [6]. The system is a modified version of the system originally published by Jerry Seligman in [20] in the context of situation theory: Rules for modal operators have been added, the rule (*Term*) has been modified, as we shall describe in Subsection 4.2, and moreover, the original system of [20] included rules for substituting co-referring terms (nominals, in the context of hybrid logic), but we do not need substitution rules, as witnessed by the completeness proof in [6].

Natural deduction systems usually have two different kinds of rules for each connective: Rules which introduce a connective and rules which eliminate a connective. Note that the rules ( $@I$ ) and ( $@E$ ) are the introduction and elimination rules for the satisfaction operator. Natural deduction rules allow to make and discharge assumptions; a discharge of assumptions is indicated by putting brackets [...] around the assumptions in question.

The rule (*Term*) in Figure 1 enables hypothetical reasoning about what is the case at a specific possible world, usually different from the actual world. The hypothetical reasoning is formalized by the subderivation delimited by the rule (the vertical line of dots), and the hypothetical world is the world referred to by the nominal discharged by the rule, indicated by  $[a]$  in the (*Term*) rule. This nominal might be called the point-of-view nominal. The side-condition that the assumptions  $\phi_1, \dots, \phi_n$  and the conclusion

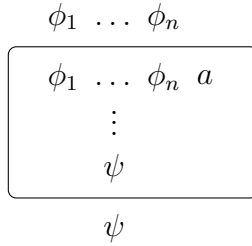


**Figure 1.** Natural deduction rules for hybrid logic

$\psi$  all have to be satisfaction statements, ensures that their truth-values are not affected when the world of evaluation is shifted (the rule would not be sound if that was the case).

The way the  $(Term)$  rule delimits a subderivation is similar to the way subderivations are delimited by proof-boxes in linear logic (introduced by J.-Y. Girard in [12]). Using proof-boxes in the style of linear logic, the  $(Term)$  rule could alternatively be formulated as follows (compare to our formulation in Figure 1).

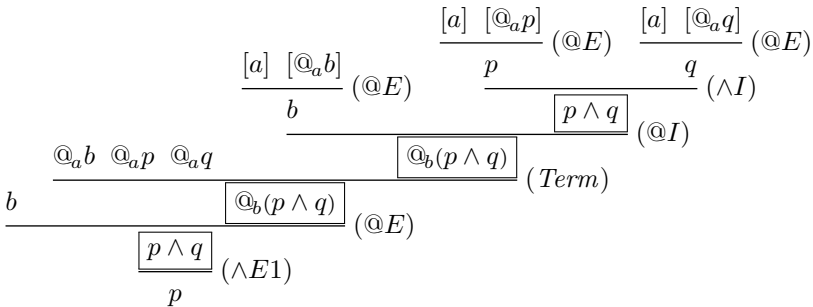




**4.2. What is the problem with the natural deduction system?**

Usually, one wants to equip a natural deduction system with a normalizing set of reduction rules such that normal derivations satisfy the subformula property. Normalization says that any derivation by repeated applications of reduction rules can be rewritten to a derivation which is normal, that is, no reductions apply. From this it follows that the system under consideration is analytic.

However, such normalizing reduction rules are not known for the Seligman-style natural deduction system. Even though Chapter 4 of [6] gives a set of reduction rules with other desirable properties, it ends by exhibiting a derivation where none of the reduction rules in question can be applied, but where the subformula property is not satisfied:



Note that occurrences of the formulas  $p \wedge q$  and  $\@_b(p \wedge q)$  (indicated by putting frames around them) are not subformulas of the end-formula  $p$  or one of the undischarged assumptions,  $b, \@_a b, \@_a p, \@_a q$ . Intuitively, the formula  $p \wedge q$  is a maximum formula (it is introduced and then eliminated), but there are applications of  $(\@I), (Term),$  and  $(\@E)$  sandwiched in be-

tween the introduction and the elimination, preventing application of the standard reduction rule for the  $\wedge$  connective, and moreover, the reduction rule for the satisfaction operator  $@_b$  does not help:

$$\begin{array}{c}
 \begin{array}{c} \vdots \pi_1 \\ \phi \end{array} \quad \begin{array}{c} \vdots \pi_2 \\ \psi \end{array} \\
 \hline
 (\phi \wedge \psi) \\
 \hline
 \phi
 \end{array}
 \rightsquigarrow
 \begin{array}{c}
 \vdots \pi_1 \\
 \phi
 \end{array}
 \quad
 \begin{array}{c}
 \vdots \pi_2 \quad \vdots \pi_3 \\
 b \quad \phi \\
 \hline
 @_b \phi \\
 \hline
 \phi
 \end{array}
 \rightsquigarrow
 \begin{array}{c}
 \vdots \pi_3 \\
 \phi
 \end{array}$$

The natural deduction system is complete and analytic (addressed in Section 5), so there does indeed exist a derivation, in fact, a very simple one, with the displayed premises and conclusion, and satisfying a version of the subformula property, but what is desired is a systematic and local rewriting procedure that removes the occurrences of the formula  $p \wedge q$ .

As explained in Subsection 4.1, the (*Term*) rule can be seen as creating a linear logic style proof-box around a subderivation, as indicated in the alternative formulation of the rule at the end of that subsection. This box does not allow eliminations to be commuted inside the border of the box, and it does not allow introductions to be commuted outside the border. This implies that what intuitively is a maximum formula, might be “stuck” at the border. This in turn prevents proving the usual theorem, stemming from Dag Prawitz’ classic [17], on which the subformula property is based: Any normal derivation can be split up into segments, each consisting of two consecutive parts, read from top to bottom, a part containing only eliminations, followed by a part containing only introductions. Not only is this important theorem the basis of the subformula property, it can also be used to translate normal natural deduction derivations into cut-free Gentzen sequent derivations, thus making cut-elimination a corollary of normalization. It should be remarked that such desirable mathematical results can be obtained for labelled hybrid-logical proof systems, cf. sections 2.2 and 2.4 of the book [6].

A problem analogous to the problem pointed out above, arises in connection with the discharged and reintroduced assumptions  $\phi_1 \dots \phi_n$  in the (*Term*) rule, cf. Figure 1. The original (*Term*) rule of [20] did not involve such explicit substitutions, but we make use of explicit substitutions here because we need the property called *closure under substitution*, which is a prerequisite for rewriting derivations using reduction rules. Closure under substitution says that if a derivation of a formula  $\phi$  is substituted

for an undischarged assumption  $\phi$  in another derivation, that is, the two derivations are composed, then a correct derivation is obtained (in particular, the side-conditions of instances of *(Term)* are not violated). The natural deduction system of the paper [20] does not satisfy closure under substitution, but that paper did not attempt to give reduction rules. See Subsection 4.1.1 of [6] for further elaboration.

We remark that explicit substitutions have been used to solve analogous problems in other logics, for example in connection with a natural deduction system for an intuitionistic version of the modal logic **S4** given in [2]. Also here explicit substitutions are used to prevent that substitutions of derivations cause the violation of the side-condition of a rule, in this case the introduction rule for the  $\Box$  modality. The price payed in the paper [2] is the addition of further reduction rules, allowing certain derivations to be commuted inside the proof-box created by the  $\Box$ -introduction rule, but in the case of that paper, the strategy succeeds, that is, after the addition of further reduction rules, a normalization result can be proved such that normal derivations satisfy the subformula property.

### 4.3. Comparison to sequent systems

As we remarked earlier, the lack of results like normalization and local syntactic cut-elimination procedures appears to be a common problem for non-labelled systems, and moreover, there is often a close connection between normalization in natural deduction systems and syntactic cut-elimination in Gentzen sequent systems, so let us try to convert, rule-by-rule, the example natural deduction derivation of Subsection 4.2 to a derivation in the sequent system of [21] (a bit simpler than the sequent system of [20]):

$$\begin{array}{c}
 \frac{}{a, b, p, q \vdash p} \quad \frac{}{a, b, p, q \vdash q} \wedge R \\
 \frac{}{a, b, p, q \vdash \boxed{p \wedge q}} \wedge I \\
 \frac{}{a, b, @_a p, @_a q \vdash \boxed{p \wedge q}} \vee @L(\text{twice}) \\
 \frac{}{a, b, @_a p, @_a q \vdash \boxed{@_b(p \wedge q)}} \vee @R \\
 \frac{}{a, @_a b, @_a p, @_a q \vdash \boxed{@_b(p \wedge q)}} \vee @L \\
 \frac{}{a, @_a b, @_a p, @_a q \vdash \boxed{@_b(p \wedge q)}} \text{term} \\
 \frac{}{b, p, q \vdash p} \wedge L \\
 \frac{}{b, \boxed{p \wedge q} \vdash p} \wedge E \\
 \frac{}{b, \boxed{@_b(p \wedge q)} \vdash p} \vee @L \\
 \hline
 b, @_a b, @_a p, @_a q \vdash p \quad \text{cut}
 \end{array}$$

See the rules in Figures 6 and 7 of [21] (we have added standard sequent rules for the  $\wedge$  connective, denoted  $\wedge L$  and  $\wedge R$ ). Intuitively, we would like to rewrite the derivation such that a cut on the formula  $p \wedge q$  is obtained, where the left premise is an instance of  $\wedge R$  and the right premise is an instance of  $\wedge L$ , such that the cut formula can be removed in the standard way, cf. for example the cut-elimination proof in Chapter 13 of [13]. However, there does not seem to be a systematic and local way to carry out such a transformation. The sequent system given in [16] also includes the rules  $\vee @L$  and  $\vee @R$  as well as a version of the *term* rule, and if the above derivation is carried out in that system, exactly the same problem arises.

Now, the paper [20] does sketch a syntactic cut-elimination procedure, but this procedure involves complex and highly non-local transformation steps, in particular, see case (b) at page 132 of that paper.

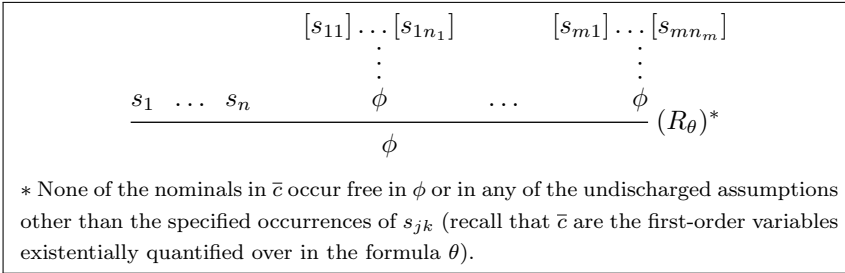
## 5. Analyticity of Seligman-style natural deduction

This section fills a gap in the literature on Seligman-style natural deduction systems: A brief history of this type of systems can be read off from Table 1 in Section 3, and to this we add that analyticity was first proved in an appendix of the paper [7], which was on a completely different issue, namely the application of Seligman-style natural deduction systems to formalize psychological reasoning tests called false-belief tests. In the present section we consider extension of the system with rules corresponding to conditions on the accessibility relation.

In the following we show that analyticity holds when the proof system is extended with appropriate rules for geometric theories, or to be precise, what are called basic geometric theories, cf. Subsection 2.2.1 of [6]. The history of such rules goes back to [22] where they were used in connection with a labelled proof-system for an intuitionistic version of modal logic. A basic geometric theory is a finite number of first-order formulas of the form

$$\forall \bar{a}((S_1 \wedge \dots \wedge S_n) \rightarrow \exists \bar{c} \bigvee_{j=1}^m (S_{j1} \wedge \dots \wedge S_{jn_j}))$$

where  $n, m \geq 0$  and  $n_1, \dots, n_m \geq 1$ , and where the metavariables  $S_k$  and  $S_{jk}$  range over atomic formulas of the forms  $R(a, c)$  and  $a = c$  (of course, the binary predicate  $R$  is to be interpreted as the accessibility relation). To such a formula  $\theta$  is associated the rule  $(R_\theta)$  given in Figure 2, where



**Figure 2.** Natural deduction rules for geometric theories

$s_k = HT(S_k)$  and  $s_{jk} = HT(S_{jk})$ , defined as  $HT(R(a, c)) = @_a \diamond c$  and  $HT(a = c) = @_a c$ .

It is straightforward to check that if a formula  $\theta$  of the form displayed above is a Horn clause (take  $m = 1$ ,  $n_m = 1$ , and the list of variables  $\bar{c}$  to be empty), then the rule  $(R_\theta)$  given in Figure 2 can be replaced by the following simpler rule.

$$\frac{s_1 \quad \dots \quad s_n}{s_{11}}$$

Let  $\mathbf{T}$  be any fixed basic geometric theory. We let  $\mathbf{N}'$  denote the Seligman-style natural deduction system defined in Subsection 4.1 and this system extended with the rules  $\{(R_\theta) \mid \theta \in \mathbf{T}\}$  will be denoted  $\mathbf{N}' + \mathbf{T}$ .

Rules of the form displayed in Figure 2 are also used in the labelled hybrid-logical natural systems of [6], and since all formulas in the rules are satisfaction statements, such rules might seem to be against the spirit of Seligman-style systems, but arguably, this is acceptable since the rules already from the start were intended to code up conditions on the accessibility relations.

In some concrete cases, rules of the form in Figure 2 can be reformulated as rules with fewer satisfaction operators, for example, the first-order condition transitivity

$$\forall a \forall b \forall c ((R(a, b) \wedge R(b, c)) \rightarrow R(a, c))$$

is associated with the left-hand-side rule, which is interderivable with the simpler right-hand-side rule:

$$\frac{\mathbb{Q}_a \diamond b \quad \mathbb{Q}_b \diamond c}{\mathbb{Q}_a \diamond c} \qquad \frac{\diamond b \quad \mathbb{Q}_b \diamond c}{\diamond c}$$

Showing that the two rules can simulate each other is very simple; note that simulating the left-hand-side rule with the right-hand-side rule requires the (*Term*) rule, and the converse direction requires the (*Name*) rule. However, the scope of such simplifications seems to be limited since occurrences of only one satisfaction operator are removed (involving a nominal referring to the actual world).

Given a basic geometric theory  $\mathbf{T}$ , we say that model  $\mathfrak{M}$  for hybrid logic is a  $\mathbf{T}$ -model if and only if  $\mathfrak{M}^* \models \theta$  for every formula  $\theta \in \mathbf{T}$ , where  $\mathfrak{M}^*$  is the obvious first-order model corresponding to the hybrid-logical model  $\mathfrak{M}$ . In other words, a  $\mathbf{T}$ -model is a model where the first-order condition on the accessibility relation expressed by the basic geometric theory  $\mathbf{T}$  is satisfied (so it is really a requirement on the frame on which the model is based).

### 5.1. Completeness and analyticity of the extended system

In the present subsection, we prove a completeness result saying that any valid formula has a derivation in  $\mathbf{N}' + \mathbf{T}$  satisfying a version of the subformula property. This is a straightforward extension of a completeness result for  $\mathbf{N}'$  given in the appendix of [7], which in turn is a sharpened version of a completeness result for  $\mathbf{N}'$  given in Section 4.3 of the book [6] (Theorem 4.1 in [6], which does not involve the subformula property). The proof in this subsection is similar to the proof in the appendix of [7], but we have included the proof anyway to make the present paper more self-contained. However, the reader wanting to follow the details of our proof is advised to obtain a copy of [6].

We first extend the translation  $(\cdot)^\circ$  from  $\mathbf{N}$  to  $\mathbf{N}'$  given in Section 4.3 of [6] such that it translates a derivation  $\pi$  in  $\mathbf{N} + \mathbf{T}$  to a derivation  $\pi^\circ$  in  $\mathbf{N}' + \mathbf{T}$ . The extended version of  $(\cdot)^\circ$  simply translates an instance of a proof-rule ( $R_\theta$ ) to itself. Note that this preserves the property that  $\pi$  and  $\pi^\circ$  have the same end-formulas and the same parcels of undischarged assumptions. The following lemma intuitively says that all formulas in  $\pi^\circ$  stem from  $\pi$ , that is, the translation  $(\cdot)^\circ$  does not add new structure to formulas.

LEMMA 5.1. *Let  $\pi$  be a derivation in  $\mathbf{N} + \mathbf{T}$ . Any formula  $\theta$  occurring in  $\pi^\circ$  has at least one of the following properties.*

1.  $\theta$  occurs in  $\pi$ .
2.  $@_a\theta$  occurs in  $\pi$  for some satisfaction operator  $@_a$ .
3.  $\theta$  is a nominal  $a$  such that some formula  $@_a\psi$  occurs in  $\pi$ .

PROOF: Induction on the structure of the derivation of  $\pi$ . Each case in the translation  $(\cdot)^\circ$  is checked.  $\square$

Note that in Item 1 of the lemma above, the formula  $\theta$  must be a satisfaction statement since since  $\mathbf{N} + \mathbf{T}$  is a labelled system, and hence, all formulas in  $\pi$  are satisfaction statements. In what follows  $@_d\Gamma$  denotes the set of formulas  $\{@_d\xi \mid \xi \in \Gamma\}$ .

**THEOREM 5.2.** *Let  $\pi$  be a normal derivation of  $@_d\phi$  from  $@_d\Gamma$  in  $\mathbf{N} + \mathbf{T}$ . Any formula  $\theta$  occurring in  $\pi^\circ$  has at least one of the following properties.*

1.  $\theta$  is of the form  $@_a\psi$  such that  $\psi$  is a subformula of  $\phi$ , some formula in  $\Gamma$ , or some formula of the form  $c$  or  $\diamond c$ .
2.  $\theta$  is a subformula of  $\phi$ , some formula in  $\Gamma$ , or some formula of the form  $c$  or  $\diamond c$ .
3.  $\theta$  is a nominal.
4.  $\theta$  is of the form  $@_a(p \rightarrow \perp)$  or  $p \rightarrow \perp$  where  $p$  is a subformula of  $\phi$  or some formula in  $\Gamma$ .
5.  $\theta$  is of the form  $@_a\perp$  or  $\perp$ .

PROOF: Follows from Lemma 5.1 above together with Theorem 2.4 (called the quasi-subformula property) in Subsection 2.2.5 of [6].  $\square$

We are now ready to give our main result, which, as indicated above, is a sharpened version of the completeness result given in Theorem 4.1 in Section 4.3 of the book [6].

**THEOREM 5.3.** *Let  $\phi$  be a formula and  $\Gamma$  a set of formulas. The first statement below implies the second statement.*

1. For any  $\mathbf{T}$ -model  $\mathfrak{M}$ , any world  $w$ , and any assignment  $g$ , if  $\mathfrak{M}, g, w \models \xi$  for any formula  $\xi \in \Gamma$ , then  $\mathfrak{M}, g, w \models \phi$ .
2. There exists a derivation of  $\phi$  from  $\Gamma$  in  $\mathbf{N}' + \mathbf{T}$  such that any formula  $\theta$  occurring in the derivation has at least one of the five properties listed in Theorem 5.2.

PROOF: Let  $d$  be a new nominal. It follows that for any  $\mathbf{T}$ -model  $\mathfrak{M}$  and any assignment  $g$ , if, for any formula  $@_d\xi \in @_d\Gamma$ , it is the case that  $\mathfrak{M}, g \models @_d\xi$ , then  $\mathfrak{M}, g \models @_d\phi$ . By completeness of the system  $\mathbf{N} + \mathbf{T}$ , Theorem 2.2 in Subsection 2.2.3 of the book [6], there exists a derivation  $\pi$  of  $@_d\phi$  from  $@_d\Gamma$  in  $\mathbf{N} + \mathbf{T}$ . By normalization, Theorem 2.3 in Subsection 2.2.5 of the book, we can assume that  $\pi$  is normal. Then by Theorem 5.2, the derivation  $\pi^\circ$  satisfies at least one of the five properties listed in the theorem. We now apply the rules ( $@I$ ), ( $@E$ ), and (*Name*) to  $\pi^\circ$  obtaining a derivation of  $\phi$  from  $\Gamma$  in  $\mathbf{N}' + \mathbf{T}$ , also satisfying at least one of the five properties.  $\square$

Note that if the formula occurrence  $\theta$  mentioned in the theorem is not of one of the forms covered by Item 4, and does not have any of a finite number of very simple forms not involving ordinary propositional symbols, then either  $\theta$  is a subformula of  $\phi$  or some formula in  $\Gamma$ , or  $\theta$  is of the form  $@_a\psi$  such that  $\psi$  is a subformula of  $\phi$  or some formula in  $\Gamma$ . This is the version of the subformula property we intended to prove.

We remark that if the proof system  $\mathbf{N}' + \mathbf{T}$  is extended with rules for the standard hybrid-logical binders  $\downarrow$  and  $\forall$ , then the theorem above still holds.

Now, the proof of Theorem 5.3 is via a translation from the labelled proof-system  $\mathbf{N} + \mathbf{T}$ , and this labelled system *can*<sup>2</sup> be equipped with a set of normalizing reduction rules such that normal derivations satisfy a version of the subformula property, and if a normal derivation in the labelled system is translated into the non-labelled system  $\mathbf{N}' + \mathbf{T}$ , then it turns out that the resulting non-labelled derivation satisfies the version of the subformula property in Theorem 5.3. Thus, we have proved analyticity of  $\mathbf{N}' + \mathbf{T}$ , not via a normalization result for this system, but via a normalization result for a labelled system.

The strategy to obtain analyticity of a proof-system via a translation from another system already known to be analytic, has been used elsewhere,

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<sup>2</sup>Thus, it seems that it is easier to find normalizing reduction rules for labelled proof-systems than for non-labelled systems, but even for labelled systems it is not always possible: For a counter-example, see Subsection 9.4.1 of the book [6] which gives a pair of labelled introduction and elimination rules (for a new unary connective  $\sharp$ ), similar to the labelled hybrid-logical introduction and elimination rules for the modal operator  $\Box$ , but when equipped with the obvious reduction rule, it turns out that a derivation  $\pi$  can be defined that cannot be normalized, that is, when the reduction rule is applied to the one and only maximum formula in  $\pi$ , a new copy of the derivation  $\pi$  is created.



for example, Chapter 4 of the book [14] gives two analytic natural deduction systems for classical logic, and analyticity is proved by translation of derivations from another analytic system called KE.<sup>3</sup>

## 6. Conclusion and further work

Several types of proof-systems generalizes the standard formats of natural deduction and Gentzen sequent systems that we have considered in this note, for example hypersequent systems and Nuel Belnap's display logic. Such systems have also been given for modal logics, see [9] and [23], and it is an interesting question whether these systems can shed light on the the issues raised in the present paper, in particular the combination of analyticity and lack of normalization for non-labelled hybrid logical natural deduction systems.

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<sup>3</sup>Thanks to the author of the book, Andrzej Indrzejczak, for mentioning this (personal communication).

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